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Analysis of symmetric structures using canonical forms

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SUMMARY

Canonical forms are recently employed in eigenproblems of structural mechanics to calculate the eigenvalues and eigenvectors. In this paper, such applications are extended to the static analysis of symmetric structures. In classical approaches the structure is decomposed into substructures and appropriate boundary conditions are then imposed. Here, a systematic method is developed which does not operate directly on the structural model, but rather on the matrices involved in the analysis of the structures.

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KEY WORDS: structural analysis; canonical forms; symmetry; anti-symmetry; Form II; Form III

1. INTRODUCTION

Symmetry has been widely studied in science and engineering [1–5]. Large eigenvalue problems arise in many scientific and engineering problems [6–8]. While the basic mathematical ideas are independent of the size of matrices, the numerical determination of eigenvalues and eigenvectors requires additional considerations as the dimensions and the sparsity of matrices increase. Special methods are needed for efficient solution of such problems.

Methods are developed for decomposing the graph models of structures in order to calculate the eigenvalues of matrices with special patterns, Reference [9]. The eigenvectors corresponding to such patterns are studied in Reference [10]. The application of these methods in extended to the vibration of mass–spring systems [11] and free vibration of frames [12].

Recently, canonical forms are employed for calculating the eigenvalues and eigenvectors of different problems in structural mechanics. In this paper, these applications are extended to the static analysis of symmetric structures. In traditional methods a given structure is often

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decomposed into substructures and appropriate boundary conditions are then imposed. Here, a method is developed which operates on the matrices involved in the analysis of the structures rather than operating directly on the structural model. The formulation of this paper, clarifies the inter-relation between the classical methods and the properties of canonical forms developed in recent years [9–12].

2. STRUCTURAL SYMMETRY AND CANONICAL FORMS

2.1. Structural symmetry

Consider the symmetric frame of Figure 1. This frame has six degrees of freedom (DOFs), which is altered to a structure with four DOFs (Figure 1(a)) and a structure with five DOFs (Figure 1(b)), depending on the structure being loaded symmetrically or anti-symmetrically, respectively.

Now using the properties of canonical forms presented in References [9–11], we want to have a different look at the problem of analysis of the structure. As an example, we want to find out the answer to some of the following questions:

The effect of symmetry operations on the properties of the stiffness matrices.
Can one predict the changes in the properties of the stiffness matrices.
Can we further reduce the DOFs of the structure.
Can one increase the accuracy of the solution.

For these purposes the following definitions and theorems are needed.

2.2. Definitions: canonical forms

In this section, an $N \times N$ symmetric matrix [**M**] is considered with all entries being real. For three special canonical forms, the eigenvalues of [**M**] are obtained using the properties of its submatrices.

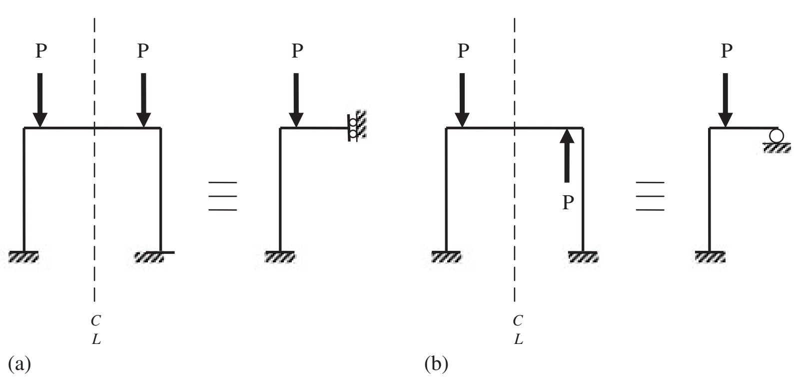

Figure 1. A symmetric and an anti-symmetrically loaded frame: (a) symmetric loading; and (b) anti-symmetric loading.

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2.2.1. Canonical form I. In this case [**M**] has the following pattern:

$$[\mathbf{M}] = \begin{bmatrix} [\mathbf{A}]_{n\times n} & [\mathbf{0}]_{n\times n} \\ [\mathbf{0}]_{n\times n} & [\mathbf{A}]_{n\times n} \end{bmatrix}_{N\times N}$$

with $N = 2n$.

Considering the set of eigenvalues of the submatrix [**A**] as $\{\lambda(\mathbf{A})\}$, the set of eigenvalues of [**M**] can be obtained as

$$\{\lambda(\mathbf{M})\} = \{\lambda(\mathbf{A})\}\bar{\cup}\{\lambda(\mathbf{A})\}$$

Since $\det(\mathbf{M}) = \det(\mathbf{A}) \times \det(\mathbf{A})$, the above relation becomes obvious. The sign $\bar{\cup}$ simply indicates the collection of the eigenvalues of the submatrices.

2.2.2. Canonical form II. For this case, matrix [**M**] can be decomposed into the following form:

$$[\mathbf{M}] = \begin{bmatrix} [\mathbf{A}]_{n\times n} & [\mathbf{B}]_{n\times n} \\ [\mathbf{B}]_{n\times n} & [\mathbf{A}]_{n\times n} \end{bmatrix}_{N\times N}$$

The eigenvalues of [**M**] can be calculated as,

$$\{\lambda(\mathbf{M})\} = \{\lambda(\mathbf{C})\}\bar{\cup}\{\lambda(\mathbf{D})\}$$

where

$$[\mathbf{C}] = [\mathbf{A}] + [\mathbf{B}] \quad \text{and} \quad [\mathbf{D}] = [\mathbf{A}] - [\mathbf{B}]$$

[**C**] and [**D**] are called *condensed submatrices* of [**M**]. The proof of this form can be considered as the special case of the proof for Form III, and it is not repeated for brevity.

2.2.3. Canonical form III. This form has a Form II submatrix augmented by some rows and columns as shown in the following:

$$[\mathbf{M}] = \left[\begin{array}{cccc|cccc}
 & & & & L_{11} & . & . & . & L_{1k} \\
 [\mathbf{A}] & & & [\mathbf{B}] & L_{21} & . & . & . & L_{2k} \\
 & & & & L_{n1} & . & . & . & L_{nk} \\
 & & & & L_{11} & . & . & . & L_{1k} \\
 [\mathbf{B}] & & & [\mathbf{A}] & L_{21} & & & & L_{2k} \\
\hline
 & & & & L_{n1} & . & . & . & L_{nk} \\
C(2n+1,1) & . & & C(2n+1,2n) & C(2n+1,2n+1) & . & . & . & C(2n+1,2n+k) \\
. & . & & . & . & . & . & . & . \\
Z(2n+k,1) & . & & Z(2n+k,2n) & Z(2n+k,2n+1) & . & . & . & Z(2n+k,2n+k)
\end{array}\right]$$

where [**M**] is a $(2n + k) \times (2n + k)$ matrix, with a $2n \times 2n$ submatrix with the pattern of Form II, and k augmented columns and rows. The entries of the augmented columns are repeated in the first and second block for each column, and all the entries of [**M**] are real numbers.

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Now [**D**] is obtained as [**D**] = [**A**] − [**B**], and [**E**] is constructed as the following:

$$[\mathbf{E}]=\left[\begin{array}{cc|cccc} & & L_{11} & . & . & . & L_{1k}\\ & & L_{21} & . & . & . & L_{2k}\\ \multicolumn{2}{c|}{[\mathbf{A}+\mathbf{B}]} & . & . & . & . & .\\ & & L_{n1} & . & . & . & L_{nk}\\ \hline C(2n+1,1)+C(2n+1,n+1) & . & C(2n+1,2n+1) & . & . & . & C(2n+1,2n+k)\\ . & . & . & . & . & . & .\\ Z(2n+k,1)+Z(2n+k,n+1) & . & Z(2n+k,2n+1) & . & . & . & Z(2n+k,2n+k)\end{array}\right]$$

The set of eigenvalues for [**M**] is obtained as

$$\{\lambda(\mathbf{M})\}=\{\lambda(\mathbf{D})\}\bar{\cup}\{\lambda(\mathbf{E})\}$$

2.3. Theorems for canonical forms

Theorem 1 (*Cullen* [*13*])
Consider a matrix of the following form:

$$\mathbf{M}=\begin{bmatrix}\mathbf{M}_{11} & \mathbf{M}_{12}\\ \mathbf{0} & \mathbf{M}_{22}\end{bmatrix}$$

and suppose $\mathbf{M}_{11}^{-1}$ and $\mathbf{M}_{22}^{-1}$ exist, then **M** is non-singular and

$$\mathbf{M}^{-1}=\begin{bmatrix}\mathbf{M}_{11}^{-1} & \mathbf{M}_{11}^{-1}\mathbf{M}_{12}\mathbf{M}_{22}^{-1}\\ \mathbf{0} & \mathbf{M}_{22}^{-1}\end{bmatrix}$$

Proof
By direct calculation

$$\begin{bmatrix}\mathbf{M}_{11} & \mathbf{M}_{12}\\ \mathbf{0} & \mathbf{M}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{M}_{11}^{-1} & \mathbf{M}_{11}^{-1}\mathbf{M}_{12}\mathbf{M}_{22}^{-1}\\ \mathbf{0} & \mathbf{M}_{22}^{-1}\end{bmatrix}$$

$$=\begin{bmatrix}\mathbf{M}_{11}\mathbf{M}_{11}^{-1} & -\mathbf{M}_{11}\mathbf{M}_{11}^{-1}\mathbf{M}_{12}\mathbf{M}_{22}^{-1}+\mathbf{M}_{12}\mathbf{M}_{22}^{-1}\\ \mathbf{0} & \mathbf{M}_{22}\mathbf{M}_{22}^{-1}\end{bmatrix}=\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}=\mathbf{I}$$

and similarly

$$\begin{bmatrix}\mathbf{M}_{11}^{-1} & \mathbf{M}_{11}^{-1}\mathbf{M}_{12}\mathbf{M}_{22}^{-1}\\ \mathbf{0} & \mathbf{M}_{22}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{M}_{11} & \mathbf{M}_{12}\\ \mathbf{0} & \mathbf{M}_{22}\end{bmatrix}=\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}=\mathbf{I} \qquad \square$$

A special case of Theorem 1 which is worth noting is when $\mathbf{M}_{12}=\mathbf{0}$. In this case, $\mathbf{M}=\mathrm{Diag}[\mathbf{M}_{11},\mathbf{M}_{22}]$ is block diagonal and $\mathbf{M}^{-1}=\mathrm{Diag}[\mathbf{M}_{11}^{-1},\mathbf{M}_{22}^{-1}]$ is also block diagonal, where 'Diag' indicates that the considered matrix is block diagonal.

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Theorem 2
For a matrix $\mathbf{R}$ with the Form II and Form III symmetry, the condensed cores (see [9–12] for definitions) of $\mathbf{R}^{-1}$ are the inverse of the condensed cores of $\mathbf{R}$, i.e.

$$\mathbf{R}=\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B} & \mathbf{A}\end{bmatrix} \Rightarrow \mathbf{C}=\mathbf{A}-\mathbf{B} \quad \text{and} \quad \mathbf{D}=\mathbf{A}+\mathbf{B}$$

$$\det \mathbf{R}^{-1}=\det \mathbf{C}^{-1}\times \det \mathbf{D}^{-1}$$

and

$$\text{eigenvalue } \mathbf{R}^{-1}\equiv\{\text{eig }\mathbf{C}^{-1}\cup \text{eig }\mathbf{D}^{-1}\}$$

For the Form III symmetry we have

$$\mathbf{R}=\begin{bmatrix}\mathbf{A} & \mathbf{B} & \mathbf{Q}\\ \mathbf{B} & \mathbf{A} & \mathbf{Q}\\ \mathbf{M} & \mathbf{N} & \mathbf{P}\end{bmatrix} \Rightarrow \mathbf{D}=\mathbf{A}-\mathbf{B} \quad \text{and} \quad \mathbf{E}=\begin{bmatrix}\mathbf{A}+\mathbf{B} & \mathbf{Q}\\ \mathbf{M}+\mathbf{N} & \mathbf{P}\end{bmatrix}$$

$$\det \mathbf{R}^{-1}=\det \mathbf{E}^{-1}\times \det \mathbf{D}^{-1}$$

and

$$\text{eigenvalues of } \mathbf{R}^{-1}\equiv\{\text{eig }\mathbf{E}^{-1}\cup \text{eig }\mathbf{D}^{-1}\}$$

Proof
Using Theorem 1, we have

$$\begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}^{-1}=\begin{bmatrix}\mathbf{A}^{-1} & \mathbf{0}\\ -\mathbf{D}^{-1}\mathbf{C}\mathbf{A}^{-1} & \mathbf{D}^{-1}\end{bmatrix}$$

if

$$\mathbf{R}=\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B} & \mathbf{A}\end{bmatrix}$$

$$\det \mathbf{R}^{-1}=\det\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B} & \mathbf{A}\end{bmatrix}^{-1}=\det\begin{bmatrix}\mathbf{A}+\mathbf{B} & \mathbf{B}+\mathbf{A}\\ \mathbf{B} & \mathbf{A}\end{bmatrix}^{-1}=\det\begin{bmatrix}\mathbf{A}+\mathbf{B} & \mathbf{0}\\ \mathbf{B} & \mathbf{A}-\mathbf{B}\end{bmatrix}^{-1}$$

$$=\det\begin{bmatrix}(\mathbf{A}+\mathbf{B})^{-1} & 0\\ -(\mathbf{A}-\mathbf{B})^{-1}\mathbf{B}(\mathbf{A}+\mathbf{B})^{-1} & (\mathbf{A}-\mathbf{B})^{-1}\end{bmatrix}=\det\underbrace{[\mathbf{A}+\mathbf{B}]}_{\mathbf{D}}{}^{-1}\times\det\underbrace{[\mathbf{A}-\mathbf{B}]}_{\mathbf{C}}{}^{-1}$$

$$=\det \mathbf{C}^{-1}\times \det \mathbf{D}^{-1}$$

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In a similar manner, for the eigenvalues we have

$$
\begin{aligned}
\operatorname{Eig}\mathbf{R}^{-1} \Rightarrow \det[\mathbf{R}^{-1} - \lambda\mathbf{I}_{n\times n}] &= \det\left[\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{bmatrix}^{-1}_{n\times n} - \lambda\mathbf{I}_{n\times n}\right] \\
&= \det\left[\begin{bmatrix} (\mathbf{A}+\mathbf{B})^{-1} & 0 \\ -(\mathbf{A}-\mathbf{B})^{-1}\mathbf{B}(\mathbf{A}+\mathbf{B})^{-1} & (\mathbf{A}-\mathbf{B})^{-1} \end{bmatrix} - \lambda\mathbf{I}_{n\times n}\right] \\
&= \det\begin{bmatrix} (\mathbf{A}+\mathbf{B})^{-1} - \lambda\mathbf{I}_{(n/2)\times n/2} & \mathbf{0} \\ -(\mathbf{A}-\mathbf{B})^{-1}\mathbf{B}(\mathbf{A}+\mathbf{B})^{-1} & (\mathbf{A}-\mathbf{B})^{-1} - \lambda\mathbf{I}_{(n/2)\times n/2} \end{bmatrix} \\
&= \det\left[\underbrace{[\mathbf{A}+\mathbf{B}]}_{D}{}^{-1} - \lambda\mathbf{I}_{(n/2)\times n/2}\right] \times \det\left[\underbrace{[\mathbf{A}-\mathbf{B}]}_{C}{}^{-1} - \lambda\mathbf{I}_{(n/2)\times n/2}\right] \\
&= \det[\mathbf{D}^{-1} - \lambda\mathbf{I}_{(n/2)\times n/2}] \times \det[\mathbf{C}^{-1} - \lambda\mathbf{I}_{(n/2)\times n/2}] = 0
\end{aligned}
$$

$$\det[\mathbf{D}^{-1} - \lambda\mathbf{I}_{(n/2)\times n/2}] = 0 \quad \Rightarrow \quad \operatorname{Eig}\mathbf{D}^{-1}$$

or

$$\Rightarrow \operatorname{Eig}\mathbf{D}^{-1} \cup \operatorname{Eig}\mathbf{C}^{-1}$$

$$\det[\mathbf{C}^{-1} - \lambda\mathbf{I}_{(n/2)\times n/2}] = 0 \Rightarrow \operatorname{Eig}\mathbf{C}^{-1}$$

For the Form III symmetry we have
if

$$\mathbf{R} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{Q} \\ \mathbf{B} & \mathbf{A} & \mathbf{Q} \\ \mathbf{M} & \mathbf{N} & \mathbf{P} \end{bmatrix}$$

then

$$
\begin{aligned}
\det\mathbf{R}^{-1} = \det\begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{Q} \\ \mathbf{B} & \mathbf{A} & \mathbf{Q} \\ \mathbf{M} & \mathbf{N} & \mathbf{P} \end{bmatrix}^{-1} &= \det\begin{bmatrix} \mathbf{A}-\mathbf{B} & \mathbf{B}-\mathbf{A} & \mathbf{0} \\ \mathbf{B} & \mathbf{A} & \mathbf{Q} \\ \mathbf{M} & \mathbf{N} & \mathbf{P} \end{bmatrix}^{-1} \\
&= \det\begin{bmatrix} \mathbf{A}-\mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{B} & \mathbf{A}+\mathbf{B} & \mathbf{Q} \\ \mathbf{M} & \mathbf{M}+\mathbf{N} & \mathbf{P} \end{bmatrix}^{-1} \\
&= \det\begin{bmatrix} [\mathbf{A}-\mathbf{B}] & [\mathbf{0}\ \ \mathbf{0}] \\ \begin{bmatrix} \mathbf{B} \\ \mathbf{M} \end{bmatrix} & \begin{bmatrix} \mathbf{A}+\mathbf{B} & \mathbf{Q} \\ \mathbf{M}+\mathbf{N} & \mathbf{P} \end{bmatrix} \end{bmatrix}^{-1}
\end{aligned}
$$

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$$
= \det \begin{bmatrix} [\mathbf{A}-\mathbf{B}]^{-1} & [\mathbf{0}\ \ \mathbf{0}] \\ \begin{bmatrix} \mathbf{A}+\mathbf{B} & \mathbf{Q} \\ \mathbf{M}+\mathbf{N} & \mathbf{P} \end{bmatrix}^{-1} \times \begin{bmatrix} \mathbf{B} \\ \mathbf{M} \end{bmatrix} \times [\mathbf{A}-\mathbf{B}]^{-1} & \begin{bmatrix} \mathbf{A}+\mathbf{B} & \mathbf{Q} \\ \mathbf{M}+\mathbf{N} & \mathbf{P} \end{bmatrix}^{-1} \end{bmatrix}
$$

$$
= \det \underbrace{[\mathbf{A}-\mathbf{B}]}_{\mathbf{D}}{}^{-1} \times \det \underbrace{\begin{bmatrix} \mathbf{A}+\mathbf{B} & \mathbf{Q} \\ \mathbf{M}+\mathbf{N} & \mathbf{P} \end{bmatrix}^{-1}}_{\mathbf{E}} = \det \mathbf{D}^{-1} \times \det \mathbf{E}^{-1}
$$

The eigenvalues are calculated as

$$
\operatorname{Eig} \mathbf{R}^{-1} \Rightarrow \det[\mathbf{R}^{-1} - \lambda \mathbf{I}_{n\times n}] = \det \left[\begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{Q} \\ \mathbf{B} & \mathbf{A} & \mathbf{Q} \\ \mathbf{M} & \mathbf{N} & \mathbf{P} \end{bmatrix} - 1 - \lambda \mathbf{I} \right]
$$

$$
= \det \left[\begin{bmatrix} [\mathbf{A}-\mathbf{B}]^{-1} & [\mathbf{0}\ \ \mathbf{0}] \\ \begin{bmatrix} \mathbf{A}+\mathbf{B} & \mathbf{Q} \\ \mathbf{M}+\mathbf{N} & \mathbf{P} \end{bmatrix}^{-1} \times \begin{bmatrix} \mathbf{B} \\ \mathbf{M} \end{bmatrix} \times [\mathbf{A}-\mathbf{B}]^{-1} & \begin{bmatrix} \mathbf{A}+\mathbf{B} & \mathbf{Q} \\ \mathbf{M}+\mathbf{N} & \mathbf{P} \end{bmatrix}^{-1} \end{bmatrix} - \lambda \mathbf{I} \right]
$$

$$
= \det \begin{bmatrix} [\mathbf{A}-\mathbf{B}]^{-1} - \lambda \mathbf{I} & [\mathbf{0}\ \ \mathbf{0}] \\ \begin{bmatrix} \mathbf{A}+\mathbf{B} & \mathbf{Q} \\ \mathbf{M}+\mathbf{N} & \mathbf{P} \end{bmatrix}^{-1} \times \begin{bmatrix} \mathbf{B} \\ \mathbf{M} \end{bmatrix} \times [\mathbf{A}-\mathbf{B}]^{-1} & \begin{bmatrix} \mathbf{A}+\mathbf{B} & \mathbf{Q} \\ \mathbf{M}+\mathbf{N} & \mathbf{P} \end{bmatrix}^{-1} - \lambda \mathbf{I} \end{bmatrix}
$$

$$
= \det \left[\underbrace{[\mathbf{A}-\mathbf{B}]}_{\mathbf{D}}{}^{-1} - \lambda \mathbf{I} \right] \times \det \left[\underbrace{\begin{bmatrix} \mathbf{A}+\mathbf{B} & \mathbf{Q} \\ \mathbf{M}+\mathbf{N} & \mathbf{P} \end{bmatrix}}_{\mathbf{E}}{}^{-1} - \lambda \mathbf{I} \right]
$$

$$
= \det[\mathbf{D}^{-1} - \lambda \mathbf{I}] \times \det[\mathbf{E}^{-1} - \lambda \mathbf{I}] = 0
$$

$$
\det[\mathbf{D}^{-1} - \lambda \mathbf{I}] = 0 \quad \Rightarrow \quad \operatorname{Eig} \mathbf{D}^{-1}
$$

or

$$
\Rightarrow \quad \operatorname{Eig} \mathbf{D}^{-1} \cup \operatorname{Eig} \mathbf{E}^{-1}
$$

$$
\det[\mathbf{E}^{-1} - \lambda \mathbf{I}] = 0 \quad \Rightarrow \quad \operatorname{Eig} \mathbf{E}^{-1}
$$

□

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Theorem 1 leads to the following results:

1. If a matrix has the Form II or the Form III symmetry, then its inverse has the same property. The proof is as follows:

$$[\mathbf{R}]^{-1} = \frac{1}{\det \mathbf{R}} \times [\text{adjoint matrix } \mathbf{m}_{ij}]$$

$$\text{CoF } a_{ij} = (-1)^{i+j} \times \mathbf{m}_{ij}$$

The adjoint matrix $\mathbf{m}_{ij}$ has also the Form II or Form III symmetry depending on the original matrix having the Form II or Form III symmetry, respectively. Therefore, the inverse matrix will have the Form II or the Form III symmetry, and has $\mathbf{C}_{\mathbf{R}^{-1}}$ and $\mathbf{D}_{\mathbf{R}^{-1}}$ or $\mathbf{E}_{\mathbf{R}^{-1}}$ and $\mathbf{D}_{\mathbf{R}^{-1}}$ as its condensed cores. Thus,

$$\mathbf{C}_{\mathbf{R}^{-1}} = [\mathbf{C}_{\mathbf{R}}]^{-1} \quad \text{and} \quad \mathbf{D}_{\mathbf{R}^{-1}} = [\mathbf{D}_{\mathbf{R}}]^{-1}$$

or

$$\mathbf{E}_{\mathbf{R}^{-1}} = [\mathbf{E}_{\mathbf{R}}]^{-1} \quad \text{and} \quad \mathbf{D}_{\mathbf{R}^{-1}} = [\mathbf{D}_{\mathbf{R}}]^{-1}$$

Therefore, without calculating $\mathbf{R}^{-1}$, its cores and hence the determinant and eigenvalues can be calculated.

2. For finding the inverse of a matrix with the Form II or Form III symmetry, one can use the inverse of its condensed cores which have smaller dimensions than the main matrix.

$$\mathbf{R} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{bmatrix}$$

$$\mathbf{R}^{-1} = \begin{bmatrix} \mathbf{A}' & \mathbf{B}' \\ \mathbf{B}' & \mathbf{A}' \end{bmatrix} \Rightarrow \mathbf{C}_{\mathbf{R}^{-1}} = \mathbf{A}' - \mathbf{B}' = [\mathbf{C}_{\mathbf{R}}]^{-1} \quad \text{and} \quad \mathbf{D}_{\mathbf{R}^{-1}} = \mathbf{A}' + \mathbf{B}' = [\mathbf{D}_{\mathbf{R}}]^{-1}$$

Leading to

$$\mathbf{A}' = \tfrac{1}{2}([\mathbf{D}_{\mathbf{R}}]^{-1} + [\mathbf{C}_{\mathbf{R}}]^{-1})$$

$$\mathbf{B}' = \tfrac{1}{2}([\mathbf{D}_{\mathbf{R}}]^{-1} - [\mathbf{C}_{\mathbf{R}}]^{-1})$$

As an example, the computational time for calculating the inverse of two 500×500 matrices is 3.9995 s, while the formation of the inverse matrix of 1000×1000 is 13.95 s.

Similarly, for matrices with the Form III symmetry, we have

$$\mathbf{R} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{Q} \\ \mathbf{B} & \mathbf{A} & \mathbf{Q} \\ \mathbf{Q}^{\mathrm{T}} & \mathbf{Q}^{\mathrm{T}} & \mathbf{P} \end{bmatrix}, \quad \mathbf{R}^{-1} = \begin{bmatrix} \mathbf{A}' & \mathbf{B}' & \mathbf{Q}' \\ \mathbf{B}' & \mathbf{A}' & \mathbf{Q}' \\ \mathbf{Q}'^{\mathrm{T}} & \mathbf{Q}'^{\mathrm{T}} & \mathbf{P}' \end{bmatrix}$$

Therefore

$$\mathbf{D}_{\mathbf{R}^{-1}} = \mathbf{A}' - \mathbf{B}' = [\mathbf{D}_{\mathbf{R}}]^{-1} \quad \text{and} \quad \mathbf{E}_{\mathbf{R}^{-1}} = \begin{bmatrix} \mathbf{A}' + \mathbf{B}' & \mathbf{Q}' \\ 2\mathbf{Q}'^{\mathrm{T}} & \mathbf{P}' \end{bmatrix} = [\mathbf{E}_{\mathbf{R}}]^{-1}$$

if

$$\mathbf{G}=\mathbf{A}'+\mathbf{B}' \Rightarrow \mathbf{A}'=\tfrac{1}{2}(\mathbf{G}+[\mathbf{D_R}]^{-1})$$

$$\mathbf{B}'=\tfrac{1}{2}(\mathbf{G}-[\mathbf{D_R}]^{-1})$$

And $\mathbf{Q}'$ and $\mathbf{P}'$ can easily be obtained from $\mathbf{E}_{\mathbf{R}^{-1}}$.

3. FRAMES WITH THE FORM II SYMMETRY

In this section the frames with the Form II are considered. Using appropriate nodal ordering, the stiffness and equivalent joint load matrices will have the following forms:

$$\mathbf{K}=\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B} & \mathbf{A}\end{bmatrix} \quad \text{and} \quad \mathbf{K}^{-1}=\begin{bmatrix}\mathbf{A}' & \mathbf{B}'\\ \mathbf{B}' & \mathbf{A}'\end{bmatrix}$$

Since every general loading can be decomposed into the sum of symmetric and anti-symmetric loadings, therefore we can use superposition as follows:

For a general loading:

$$\mathbf{F}=\begin{bmatrix}\mathbf{P}\\ \mathbf{P}\end{bmatrix}+\begin{bmatrix}\mathbf{R}\\ -\mathbf{R}\end{bmatrix}$$

$$\mathbf{\Delta}=\mathbf{K}^{-1}\times\mathbf{F}=\begin{bmatrix}\mathbf{A}' & \mathbf{B}'\\ \mathbf{B}' & \mathbf{A}'\end{bmatrix}\times\left[\begin{bmatrix}\mathbf{P}\\ \mathbf{P}\end{bmatrix}+\begin{bmatrix}\mathbf{R}\\ -\mathbf{R}\end{bmatrix}\right]=\begin{bmatrix}(\mathbf{A}'+\mathbf{B}')\mathbf{P}\\ (\mathbf{B}'+\mathbf{A}')\mathbf{P}\end{bmatrix}+\begin{bmatrix}(\mathbf{A}'-\mathbf{B}')\mathbf{R}\\ (\mathbf{B}'-\mathbf{A}')\mathbf{R}\end{bmatrix}$$

$$\mathbf{C}'=\mathbf{A}'-\mathbf{B}'=[\mathbf{C}_K]^{-1}=[\mathbf{A}-\mathbf{B}]^{-1}$$

$$\mathbf{D}'=\mathbf{A}'+\mathbf{B}'=[\mathbf{D}_K]^{-1}=[\mathbf{A}+\mathbf{B}]^{-1}$$

resulting in

$$\mathbf{\Delta}_{(n/2)\times 1}=\mathbf{D}^{-1}\times\mathbf{P}+\mathbf{C}^{-1}\times\mathbf{R}$$

$$\mathbf{\Delta}_{(n/2)\times 1}=\mathbf{D}^{-1}\times\mathbf{P}-\mathbf{C}^{-1}\times\mathbf{R}$$

Therefore, for symmetric loading $\mathbf{F}=[{}^{\mathbf{P}}_{\mathbf{P}}]$ and $\mathbf{R}=\mathbf{0}$, leading to $\mathbf{\Delta}_{(n/2)\times 1}=[\mathbf{D}]^{-1}\times\mathbf{P}$, and the displacement for half of the frame is obtained. In this case, the displacements of the other half will be the same the first half of the frame. Therefore, for the symmetric frames with symmetric loading the calculation of the condensed core D of the stiffness matrix is all what is needed for the analysis of the entire frame.

For anti-symmetric loading we have $\mathbf{F}=[{}^{\mathbf{R}}_{-\mathbf{R}}]$ and $\mathbf{P}=0$, leading to $\mathbf{\Delta}_{(n/2)\times 1}=[\mathbf{C}]^{-1}\times\mathbf{R}$.

In this case, the displacements for the second half of the frame are the same as the first half with reverse sign. Therefore, for the frames with the Form II symmetry with anti-symmetric loading, the formation of $\mathbf{C}$ is all that is needed for the analysis of the entire frame.

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Therefore, by the solution of an $(n/2)\times(n/2)$ matrix the analysis of a structure with n DOFs becomes feasible for symmetric or anti-symmetric loading. For general loading, the solution of two $(n/2)\times(n/2)$ matrices will be needed. In any case, the computational effort reduces to a great extent.

In can easily be realized that for the Form II symmetry with any general loading, the present method requires half of the DOFs for solution, while in substructuring with imposing the necessary boundary conditions, one or two DOFs should be added to half of the total DOFs of the structure.

4. FRAMES WITH THE FORM III SYMMETRY

In this section frames with the Form III symmetry are studied. Using appropriate nodal numbering (ordering the DOFs), the stiffness matrix and the equivalent joint load matrix will have the following patterns:

$$\mathbf{K}=\begin{bmatrix}\mathbf{A} & \mathbf{B} & \mathbf{M}\\ \mathbf{B} & \mathbf{A} & \mathbf{M}\\ \mathbf{M}^{\mathrm{T}} & \mathbf{M}^{\mathrm{T}} & \mathbf{N}\end{bmatrix},\quad \mathbf{K}^{-1}=\begin{bmatrix}\mathbf{A}' & \mathbf{B}' & \mathbf{M}'\\ \mathbf{B}' & \mathbf{A}' & \mathbf{M}'\\ \mathbf{M}'^{\mathrm{T}} & \mathbf{M}'^{\mathrm{T}} & \mathbf{N}'\end{bmatrix}$$

For a general loading

$$\mathbf{F}=\begin{bmatrix}\mathbf{P}\\ \mathbf{P}\\ \mathbf{S}\end{bmatrix}+\begin{bmatrix}\mathbf{R}\\ -\mathbf{R}\\ \mathbf{S}\end{bmatrix}$$

$$\boldsymbol{\Delta}=\mathbf{K}^{-1}\times\mathbf{F}=\begin{bmatrix}\mathbf{A}' & \mathbf{B}' & \mathbf{M}'\\ \mathbf{B}' & \mathbf{A}' & \mathbf{M}'\\ \mathbf{M}'^{\mathrm{T}} & \mathbf{M}'^{\mathrm{T}} & \mathbf{N}'\end{bmatrix}\times\left[\begin{bmatrix}\mathbf{P}\\ \mathbf{P}\\ \mathbf{S}\end{bmatrix}+\begin{bmatrix}\mathbf{R}\\ -\mathbf{R}\\ \mathbf{S}\end{bmatrix}\right]$$

$$=\begin{bmatrix}(\mathbf{A}'+\mathbf{B}')\mathbf{P}+\mathbf{M}'\mathbf{S}+(\mathbf{A}'-\mathbf{B}')\mathbf{R}+\mathbf{M}'\mathbf{S}\\ (\mathbf{A}'+\mathbf{B}')\mathbf{P}+\mathbf{M}'\mathbf{S}-(\mathbf{A}'-\mathbf{B}')\mathbf{R}+\mathbf{M}'\mathbf{S}\\ \mathbf{2M}'^{\mathrm{T}}\mathbf{P}+\mathbf{N}'\mathbf{S}+\mathbf{N}'\mathbf{S}\end{bmatrix}$$

$$\mathbf{D}'=\mathbf{D}^{-1}=[\mathbf{A}-\mathbf{B}]^{-1}$$

resulting in

$$\boldsymbol{\Delta}_{(n/2+m)\times 1}=\mathbf{E}^{-1}\times\begin{bmatrix}\mathbf{P}\\ \mathbf{S}\end{bmatrix}+\begin{bmatrix}\mathbf{D}^{-1} & \mathbf{M}'\\ \mathbf{0} & \mathbf{N}'\end{bmatrix}\times\begin{bmatrix}\mathbf{R}\\ \mathbf{S}\end{bmatrix}$$

$$\boldsymbol{\Delta}_{(n/2)\times 1}=[\mathbf{A}'+\mathbf{B}'\ \ \mathbf{M}']\times\begin{bmatrix}\mathbf{P}\\ \mathbf{S}\end{bmatrix}+[\mathbf{D}^{-1}\ \ \mathbf{M}']\times\begin{bmatrix}-\mathbf{R}\\ \mathbf{S}\end{bmatrix}$$

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The above equations can be expressed as
If

$$\mathbf{G} = \mathbf{A}' + \mathbf{B}'$$

then

$$\boldsymbol{\Delta} = \begin{cases} \mathbf{GP} + \mathbf{D}^{-1}\mathbf{R} + 2\mathbf{M}'\mathbf{S} \\ \mathbf{GP} - \mathbf{D}^{-1}\mathbf{R} + 2\mathbf{M}'\mathbf{S} \\ 2\mathbf{M}'^{\mathrm{T}}\mathbf{P} + 2\mathbf{N}'\mathbf{S} \end{cases}$$

Now $\mathbf{G}$, $\mathbf{M}'$ and $\mathbf{N}'$ can be obtained from $\mathbf{E}^{-1}$.
Therefore, for symmetric loading

$$\mathbf{F} = \begin{bmatrix} \mathbf{P} \\ \mathbf{P} \\ \mathbf{S} \end{bmatrix} \quad \text{and} \quad \mathbf{R} = 0$$

leading to

$$\boldsymbol{\Delta}_{(n/2+m)\times 1} = [\mathbf{E}]^{-1} \times \begin{bmatrix} \mathbf{P} \\ \mathbf{S} \end{bmatrix}$$

In this way, the displacements for half of the structure and the displacements of the central nodes are obtained. The displacements for the other half are the same as the first half. Thus, for symmetric frames with the Form II symmetry, the calculation of the inverse of the core E for the stiffness matrix is all what is needed for the analysis of the entire structure.
For anti-symmetric loading:

$$\mathbf{F} = \begin{bmatrix} \mathbf{R} \\ -\mathbf{R} \\ \mathbf{S} \end{bmatrix} \quad \text{and} \quad \mathbf{P} = 0$$

leading to

$$\begin{cases} \boldsymbol{\Delta}_{(n/2+m)\times 1} = \begin{bmatrix} \mathbf{D}^{-1} & \mathbf{M}' \\ 0 & \mathbf{N}' \end{bmatrix} \times \begin{bmatrix} \mathbf{R} \\ \mathbf{S} \end{bmatrix} \\ \boldsymbol{\Delta}_{n/2\times 1} = [\mathbf{D}^{-1} \ \ \mathbf{M}'] \times \begin{bmatrix} -\mathbf{R} \\ \mathbf{S} \end{bmatrix} \end{cases}$$

The above equation can be written as

$$\boldsymbol{\Delta} = \begin{cases} \mathbf{M}'\mathbf{S} + \mathbf{D}^{-1}\mathbf{P} \\ \mathbf{M}'\mathbf{S} - \mathbf{D}^{-1}\mathbf{P} \\ \mathbf{N}'\mathbf{S} \end{cases}$$

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Therefore, for the frame with the Form III symmetry having anti-symmetric loading, to find the inverse of the cores D and E for the stiffness matrix, one can first obtain $\mathbf{M}'$ and $\mathbf{N}'$ and the solution of the entire structure can then be obtained.

Finally, the operations required for the solution of a structure with $(n+m)\times(n+m)$ matrix, results in the inversion of a $(n/2+m)\times(n/2+m)$ matrix for the symmetric loading and two $(n/2+m)\times(n/2+m)$ and $(n/2)\times(n/2)$ matrices for anti-symmetric and general loading cases. In any of these cases, the computational effort will considerably be reduced.

5. NUMERICAL RESULTS

Example 1

The frame shown in Figure 2 is considered and analysis is performed using the present method. This frame is symmetric and the loading is also symmetric. Therefore

$$\mathbf{P}=\begin{bmatrix} 0 \\ -328.5 \\ 492.75 \end{bmatrix}$$

and

$$\mathbf{K}=\frac{EI}{1000}\times\begin{bmatrix} 38.8 & 0 & 103.3 & -11.8 & 0 & 0 \\ 0 & 29.8 & 71.8 & 0 & 15.7 & 71.8 \\ 103.3 & 71.8 & 961.3 & 0 & 71.8 & 218.2 \\ -11.8 & 0 & 0 & 38.8 & 0 & 103.3 \\ 0 & 15.7 & 71.8 & 0 & 29.8 & 71.8 \\ 0 & 71.8 & 218.2 & 103.3 & 71.8 & 961.3 \end{bmatrix}$$

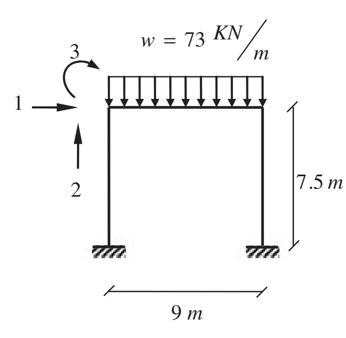

Figure 2. A simple portal frame.

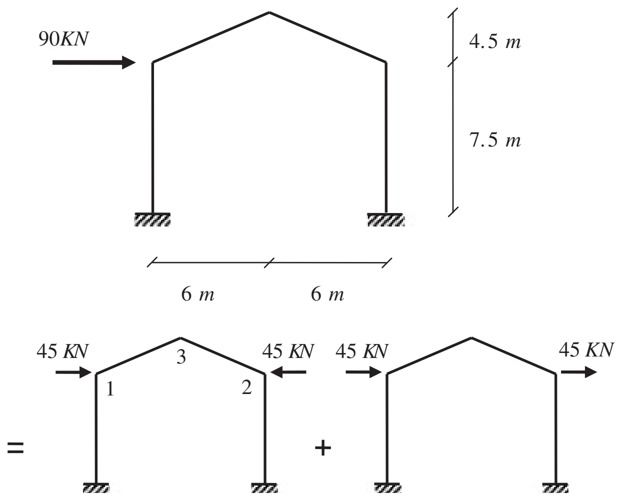

Figure 3. A pitched roof frame and the decomposition of its loading.

Now we find the inverse of the core D

$$\mathbf{D}^{-1} = \frac{1000}{EI} \times \begin{bmatrix} 38.8-11.8 & 0 & 103.3 \\ 0 & 29.8+15.7 & 71.8+71.8 \\ 103.3 & 71.8+71.8 & 961.3+218.2 \end{bmatrix}^{-1} = \mathbf{D}'$$

$$\begin{bmatrix} \Delta_x \\ \Delta_y \\ \theta \end{bmatrix} = \mathbf{D}^{-1} \times \begin{bmatrix} 0 \\ -328.5 \\ 492.75 \end{bmatrix} = \frac{1000}{EI} \times \begin{bmatrix} -17.6752 \text{ m} \\ -21.8002 \text{ m} \\ 4.6198 \text{ rad} \end{bmatrix}$$

Solution of a 3×3 matrix leads to the solution of a frame with a 6×6 matrix.

Example 2
A pitched roof frame is considered as shown in Figure 3. This example is also solved using the present method.

This frame is symmetric and loaded in a general manner. Therefore, in both cases we have

$$\mathbf{P} = \mathbf{R} = \begin{bmatrix} 45 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{S} = [0]$$

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$$\mathbf{K}=\frac{EI}{1000}\times\begin{bmatrix} 45.9 & -6.2 & -41.4 & & & & -18.8 & 6.2 & -61.9\\ -6.2 & 36.5 & -82.9 & & 0 & & 6.2 & -22.4 & 82.9\\ -41.4 & -82.9 & 1049.9 & & & & -61.9 & 82.9 & -262.5\\ & & & 45.9 & -6.2 & -41.4 & -18.8 & 6.2 & -61.9\\ & 0 & & -6.2 & 36.5 & -82.9 & 6.2 & -22.4 & 82.9\\ & & & -41.4 & -82.9 & 1049.9 & -61.9 & 82.9 & -262.5\\ -18.8 & 6.2 & -61.9 & -18.8 & 6.2 & -61.9 & 37.6 & 0 & 123.8\\ 6.2 & -22.4 & 82.9 & 6.2 & -22.4 & 82.9 & 0 & 44.8 & 0\\ -61.9 & 82.9 & -262.5 & -61.9 & 82.9 & -262.5 & 123.8 & 0 & 1049.9 \end{bmatrix}$$

Now the inverse of the cores D and E are calculated as

$$\mathbf{D}^{-1}=\frac{1000}{EI}\times\begin{bmatrix} 45.9 & -6.2 & -41.4\\ -6.2 & 36.5 & -82.9\\ -41.4 & -82.9 & 1049.9 \end{bmatrix}^{-1}=\mathbf{D}'$$

$$\mathbf{E}^{-1}=\frac{1000}{EI}\times\begin{bmatrix} 45.9 & -6.2 & -41.4 & -18.8 & 6.2 & -61.9\\ -6.2 & 36.5 & -82.9 & 6.2 & -22.4 & 82.9\\ -41.4 & -82.9 & 1049.9 & -61.9 & 82.9 & -262.5\\ -37.6 & 12.4 & -123.8 & 37.6 & 0 & 123.8\\ 12.4 & -44.8 & 165.8 & 0 & 44.8 & 0\\ -123.8 & 165.8 & -524.9 & 123.8 & 0 & 1049.9 \end{bmatrix}^{-1}$$

Leading to

$$\mathbf{G}=\frac{1000}{EI}\times\begin{bmatrix} 0.0109 & 0.0761 & -0.0020\\ 0.0761 & 0.0264 & 0.0166\\ -0.0020 & 0.0166 & 0.0004 \end{bmatrix}$$

and

$$\mathbf{M}'^{\mathrm{T}}=\frac{1000}{EI}\times\begin{bmatrix} 0.0146 & 0.645 & 0.0021\\ 0.0402 & -0.0281 & 0.0078\\ -0.0076 & -0.0011 & -0.0016 \end{bmatrix}$$

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Therefore

$$
\begin{cases}
\begin{bmatrix} \Delta_x \\ \Delta_y \\ \theta \end{bmatrix}_1 = \mathbf{GP} + \mathbf{D}^{-1}\mathbf{P} = \dfrac{1000}{EI} \times \begin{bmatrix} 1.5809\ \text{m} \\ 3.7696\ \text{m} \\ -0.0197\ \text{rad} \end{bmatrix} \\
\begin{bmatrix} \Delta_x \\ \Delta_y \\ \theta \end{bmatrix}_2 = \mathbf{GP} - \mathbf{D}^{-1}\mathbf{P} = \dfrac{1000}{EI} \times \begin{bmatrix} -0.5999\ \text{m} \\ 3.0794\ \text{m} \\ -0.1603\ \text{rad} \end{bmatrix} \\
\begin{bmatrix} \Delta_x \\ \Delta_y \\ \theta \end{bmatrix}_3 = 2\mathbf{M}'^{\mathrm{T}}\mathbf{P} = \dfrac{1000}{EI} \times \begin{bmatrix} 1.3140\ \text{m} \\ 3.6180\ \text{m} \\ -0.6840\ \text{rad} \end{bmatrix}
\end{cases}
$$

Therefore, the inversion of two 3×3 and 6×6 matrices results in the solution of the structure which needed the inversion of a 9×9 matrix by the traditional methods.

It should be noted that in symmetric structures with the Form II or Form III symmetry, the formation of the stiffness matrix for the entire structure can be avoided.

As an example, for the Form II symmetry matrix **A** is the stiffness matrix for half of the structure with link beam being fixed at the other end, and **B** is the matrix connecting the DOFs of the symmetry.

$$
\mathbf{B} = \overbrace{\begin{bmatrix} b_{1,n/2+1} & b_{1,n/2+2} & \cdot \\ b_{2,n/2+2} & \cdot & \cdot \\ \cdot & \cdot & \ddots \end{bmatrix}}^{\frac{n}{2}+1,\frac{n}{2}+2,\ldots,n} \left.\vphantom{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}\right\} \begin{matrix} 1 \\ 2 \\ \vdots \\ \frac{n}{2} \end{matrix}
$$

Now the cores C and D and the corresponding inverse can be obtained and the problem can be solved.

$$
\mathbf{D} = \mathbf{A} + \mathbf{B} \quad \text{and} \quad \mathbf{C} = \mathbf{A} - \mathbf{B} \quad \Rightarrow \quad \mathbf{K} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{bmatrix}
$$

For symmetry of Form II, the matrix **A** is the stiffness matrix of the half of the structure obtained by removing the central nodes and adding fixed supports to the end of the members connected to these nodes, and the matrix **B** is a null matrix.

The matrix **M** consists of the relationship between the symmetric DOFs and DOFs of the central nodes, and the matrix **N** is the stiffness matrix corresponding to the DOFs of the central nodes and their relationship with each other (Figure 4).

Similar to the previous problems, the cores D and E and their inverse are obtained and the solution is completed.

$$
\mathbf{D} = \mathbf{A} \quad \text{and} \quad \mathbf{E} = \begin{bmatrix} \mathbf{A} & \mathbf{M} \\ 2\mathbf{M}^{\mathrm{T}} & \mathbf{N} \end{bmatrix}
$$

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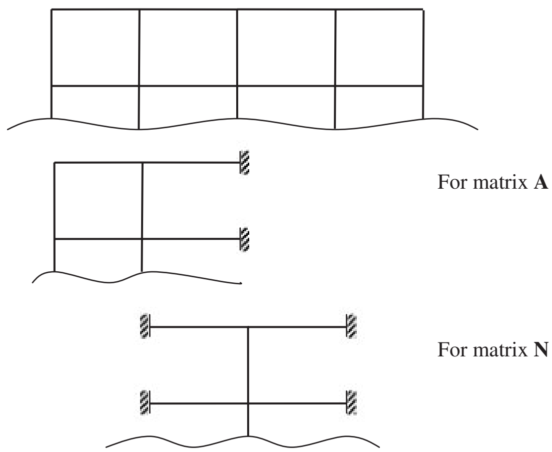

Figure 4. The substructures corresponding to **A** and **N**.

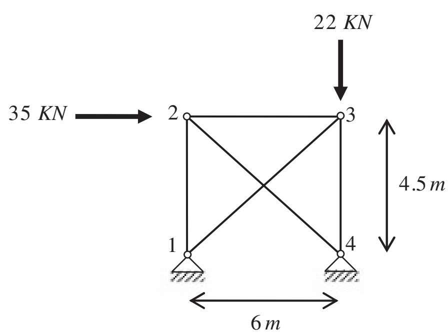

Figure 5. A simple truss and its loading.

leading to

$$\mathbf{K} = \begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{M} \\ \mathbf{0} & \mathbf{A} & \mathbf{M} \\ \mathbf{M}^{\mathrm{T}} & \mathbf{M}^{\mathrm{T}} & \mathbf{N} \end{bmatrix}$$

Example 3

Consider the truss shown in Figure 5. Under the applied loading, the displacements of the nodes 2 and 3 are required.

The loading of the truss can be considered as the sum of two symmetric and anti-symmetric ones as shown in Figure 6.

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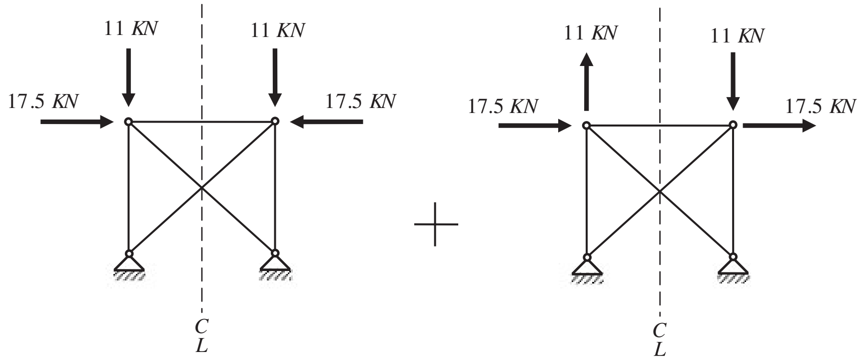

Figure 6. Decomposition of the loading.

Using the method previously described, the matrices **A** and **B** of the stiffness matrix of the truss are calculated as

$$\mathbf{K} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{bmatrix}$$

$$\mathbf{A} = EA \begin{bmatrix} 0.247 & 0.063 \\ 0.063 & 0.266 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = EA \begin{bmatrix} 0.167 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} -17.5 \\ -11 \end{bmatrix} \quad \text{and} \quad \mathbf{R} = \begin{bmatrix} -17.5 \\ 11 \end{bmatrix}$$

$$\mathbf{\Delta}_{(n/2)\times 1} = \begin{bmatrix} \Delta_{2x} \\ \Delta_{2y} \end{bmatrix} = \mathbf{D}^{-1} \times \mathbf{P} + \mathbf{C}^{-1} \times \mathbf{R} = \frac{1}{EA} \begin{bmatrix} 0.247+0.167 & 0.063 \\ 0.063 & 0.266 \end{bmatrix}^{-1} \begin{bmatrix} -17.5 \\ -11 \end{bmatrix}$$

$$+ \frac{1}{EA} \begin{bmatrix} 0.247-0.167 & 0.063 \\ 0.063 & 0.266 \end{bmatrix}^{-1} \begin{bmatrix} -17.5 \\ -11 \end{bmatrix} = \frac{1}{EA} \begin{bmatrix} -338.34 \\ 80.2 \end{bmatrix}$$

$$\mathbf{\Delta}_{(n/2)\times 1} = \begin{bmatrix} \Delta_{3x} \\ \Delta_{3y} \end{bmatrix} = \mathbf{D}^{-1} \times \mathbf{P} - \mathbf{C}^{-1} \times \mathbf{R} = \frac{1}{EA} \begin{bmatrix} 0.247+0.167 & 0.063 \\ 0.063 & 0.266 \end{bmatrix}^{-1} \begin{bmatrix} -17.5 \\ -11 \end{bmatrix}$$

$$- \frac{1}{EA} \begin{bmatrix} 0.247-0.167 & 0.063 \\ 0.063 & 0.266 \end{bmatrix}^{-1} \begin{bmatrix} -17.5 \\ 11 \end{bmatrix} = \frac{1}{EA} \begin{bmatrix} 261.8 \\ -145.8 \end{bmatrix}$$

In this way the displacement of the nodes 2 and 3 are calculated under a general loading by reducing the dimension of the problem.

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Using the classical method of Reference [1] leads to

$$\mathbf{F} = \mathbf{K}\boldsymbol{\Delta}$$

$$\begin{bmatrix} 35 \\ 0 \\ 0 \\ -22 \end{bmatrix} = EA \begin{bmatrix} 0.247 & -0.063 & -0.167 & 0 \\ -0.063 & 0.266 & 0 & 0 \\ -0.167 & 0 & 0.247 & 0.063 \\ 0 & 0 & 0.063 & 0.266 \end{bmatrix} \begin{bmatrix} \Delta_{2x} \\ \Delta_{2y} \\ \Delta_{3x} \\ \Delta_{3y} \end{bmatrix}$$

$$\begin{bmatrix} \Delta_{2x} \\ \Delta_{2y} \\ \Delta_{3x} \\ \Delta_{3y} \end{bmatrix} = \frac{1}{EA} \begin{bmatrix} 338.34 \\ 80.2 \\ 261.8 \\ -145.8 \end{bmatrix}$$

The results of Reference [14] are as follows:

$$\begin{bmatrix} \Delta_{2x} \\ \Delta_{2y} \\ \Delta_{3x} \\ \Delta_{3y} \end{bmatrix} = \frac{1}{EA} \begin{bmatrix} 246.3 \\ 58.4 \\ 190 \\ -106.8 \end{bmatrix}$$

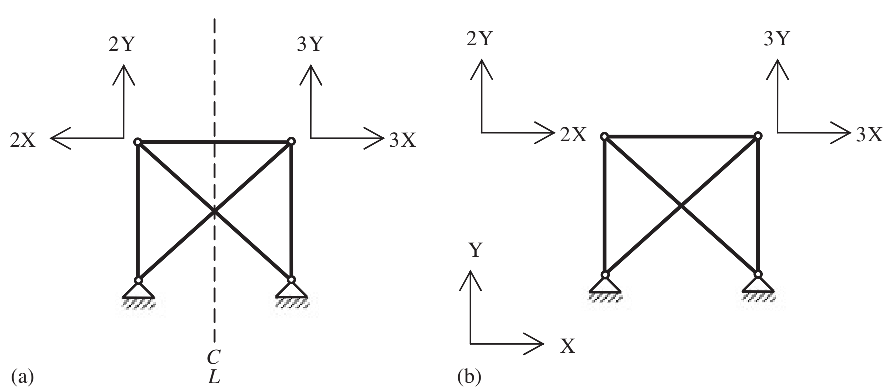

Figure 7. Sign for DOFs in classical method and the present approach.

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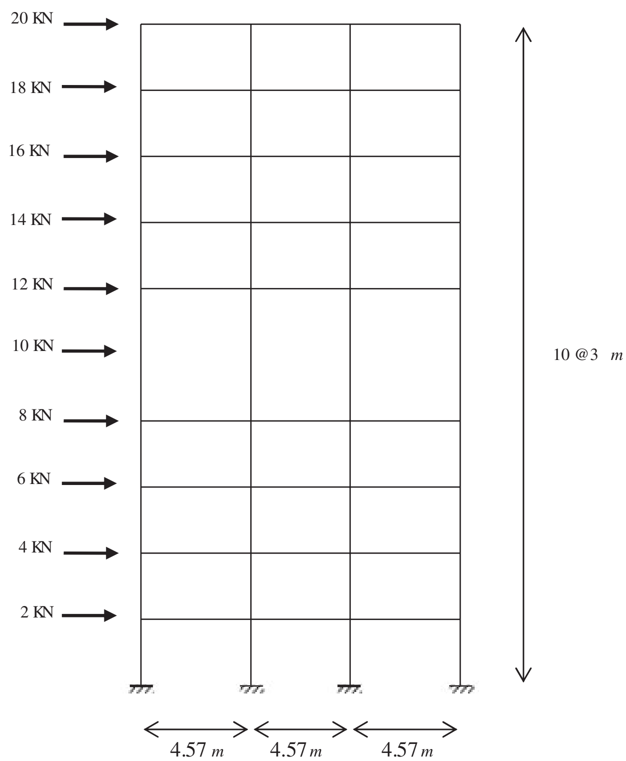

Figure 8. A multi-storey frame and the corresponding loading.

It can be observed between the results are nearly identical for the classical method and the present approach.

$$\begin{bmatrix} \Delta_{2x} \\ \Delta_{2y} \end{bmatrix} = \frac{1}{EA} \begin{bmatrix} -249.561 \\ 59.155 \end{bmatrix}$$

and

$$\begin{bmatrix} \Delta_{3x} \\ \Delta_{3y} \end{bmatrix} = \frac{1}{EA} \begin{bmatrix} 193.123 \\ -107.506 \end{bmatrix}$$

The negative sign for the DOFs specified by $2X$ is because for symmetry the sign shown in Figure 7(a) is employed. In the classical method, the DOFs are considered as shown in Figure 7(b).

Example 4
Consider a 10 storey steel frame as shown in Figure 8. The necessary data are as follows: area $= 0.0167\ \text{m}^2$, $I = 2.223 \times 10^{-4}\ \text{m}^4$, and $E = 2 \times 10^6\ \text{N/cm}^2$.

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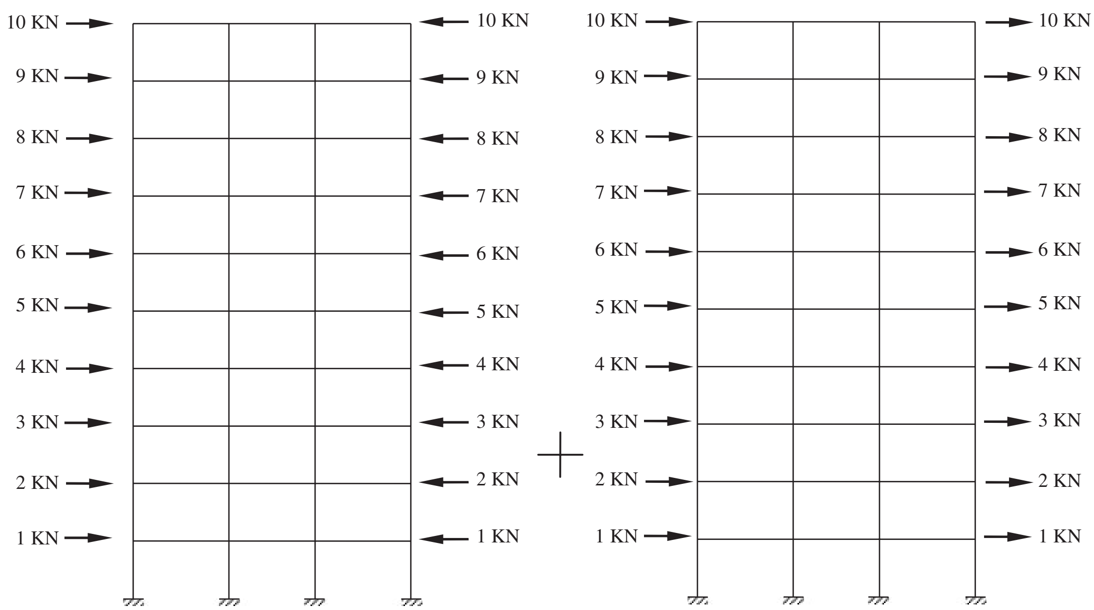

Figure 9. A frame and the decomposition of loading to the sum of symmetric and anti-symmetric cases.

Each node has 3 translation and rotation DOFs, and all together 120 DOFs exists. The frame has the Form II symmetry and the loading can be considered as the sum of a symmetric and anti-symmetric cases as shown in Figure 9.

$$P_{60\times 60} = R_{60\times 60}$$

$$\text{Suppose we have} \begin{cases} \text{for columns:} & EI/l^3 = \alpha,\ EA/l = \gamma \\ \text{for beams:} & EI/l^3 = \beta,\ EA/l = \lambda \end{cases}$$

For static analysis, the loading is considered to have mirror type of symmetry and for the DOFs for both halves of the structure similar DOFs should be considered. As an example, for the present frame we have the DOFs as shown in Figure 10.

The matrix **A** is the stiffness matrix of half the structure which is obtained by fixing the end nodes of the link members, and the matrix **B** is the relationship between the DOFs of the two halves. In this way, the matrix $\mathbf{A}_{60\times 60}$ corresponding to half the structure with 20 nodes and 60 DOFs is obtained. Consider the DOFs of each node as shown in Figure 11.

Considering the repetition of the nodes and with a little care in the relationship between the DOFs, even the pattern of the submatrices of **A** and **B** of the stiffness matrix can

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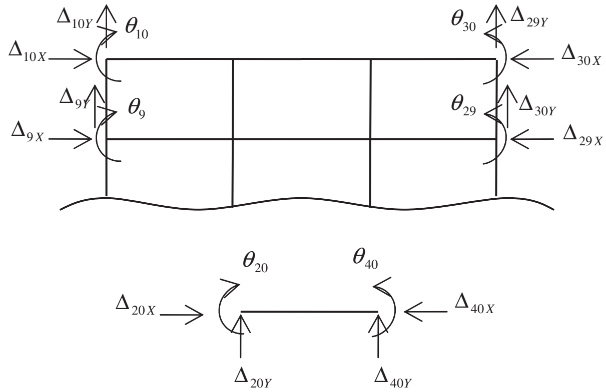

Figure 10. DOFs for the link-beam of the roof.

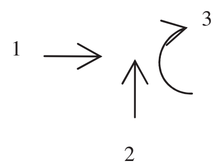

Figure 11. The DOFs for each node.

be recognized.

$$
\mathbf{A}_{60\times 60} = \left[\begin{array}{cccccc|cccccc}
\mathbf{Q} & \mathbf{X} & & & & & \mathbf{S} & & & & & \\
\mathbf{X} & \mathbf{Q} & \mathbf{X} & & \mathbf{0} & & & \mathbf{S} & & & \mathbf{0} & \\
 & \mathbf{X} & \mathbf{Q} & \mathbf{X} & & & & & \mathbf{S} & & & \\
 & & \mathbf{X} & \ddots & & & & & & \ddots & & \\
 & \mathbf{0} & & & \ddots & \mathbf{X} & & \mathbf{0} & & & \ddots & \\
 & & & & \mathbf{X} & \mathbf{Z} & & & & & & \mathbf{S} \\
\hline
\mathbf{S} & & & & & & \mathbf{T} & \mathbf{X} & & & & \\
 & \mathbf{S} & & & \mathbf{0} & & \mathbf{X} & \mathbf{T} & \mathbf{X} & & \mathbf{0} & \\
 & & \mathbf{S} & & & & & \mathbf{X} & \mathbf{T} & \mathbf{X} & & \\
 & & & \ddots & & & & & \mathbf{X} & \ddots & & \\
 & \mathbf{0} & & & \ddots & & & \mathbf{0} & & & \ddots & \mathbf{X} \\
 & & & & & \mathbf{S} & & & & & \mathbf{X} & \mathbf{V}
\end{array}\right]_{20\times 20}
$$

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The following submatrices correspond to the nodes $i = 1, \ldots, 20$.

$$\mathbf{Q}_i = \begin{array}{c} \\ \Delta_{iX} \\ \Delta_{iy} \\ \theta_i \end{array} \begin{array}{c} \begin{array}{ccc} \Delta_{iX} & \Delta_{iy} & \theta_i \end{array} \\ \begin{bmatrix} 24\alpha + \lambda & 0 & 0 \\ 0 & 2\gamma + 12\beta & -6\beta \\ 0 & -6\beta & 8\alpha + 4\beta \end{bmatrix} \end{array}, \quad \mathbf{Z} = \begin{bmatrix} 12\alpha + \lambda & 0 & -6\alpha \\ 0 & \gamma + 12\beta & -6\beta \\ -6\alpha & -6\beta & 4\alpha + 4\beta \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} 24\alpha + 2\lambda & 0 & 0 \\ 0 & 2\gamma + 24\beta & 0 \\ 0 & 0 & 8\alpha + 8\beta \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} 12\alpha + 2\lambda & 0 & -6\alpha \\ 0 & \gamma + 24\beta & 0 \\ -6\alpha & 0 & 4\alpha + 8\beta \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} -12\alpha & 0 & 6\alpha \\ 0 & -\gamma & 0 \\ 6\alpha & 0 & 2\alpha \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} -\lambda & 0 & 0 \\ 0 & -12\beta & 6\beta \\ 0 & 6\beta & 2\beta \end{bmatrix}$$

Similarly for matrix **B** we have

$$\mathbf{B} = \left.\overbrace{\left[\begin{array}{c|c} O_{30\times30} & O_{30\times30} \\ \hline O_{30\times30} & \begin{matrix} L & & & & \\ & L & & O & \\ & & L & & \\ & & & \ddots & \\ & O & & L & \\ & & & & L \end{matrix} \end{array}\right]}^{21,22,23,24,\ldots,39,40}\right\} \begin{matrix} 1 \\ 2 \\ 3 \\ . \\ . \\ . \\ 19 \\ 20 \end{matrix} \quad \mathbf{L} = \begin{array}{c} \begin{array}{ccc} \Delta_{40X} & \Delta_{40Y} & \theta_{40} \end{array} \\ \begin{bmatrix} \lambda & 0 & 0 \\ 0 & -12\beta & 6\beta \\ 0 & 6\beta & -2\beta \end{bmatrix} \end{array} \begin{array}{c} \\ \Delta_{20X} \\ \Delta_{20Y} \\ \theta_{20} \end{array}$$

After the formation of the submatrices **A** and **B**, the formation of the submatrices **C** and **D** is performed as

$$\mathbf{C} = \mathbf{A} - \mathbf{B} \quad \text{and} \quad \mathbf{D} = \mathbf{A} + \mathbf{B}$$

$$\boldsymbol{\Delta}_{(n/2)\times 1} = \mathbf{D}^{-1} \times \mathbf{P} + \mathbf{C}^{-1} \times \mathbf{R} \quad \text{and} \quad \boldsymbol{\Delta}_{(n/2)\times 1} = \mathbf{D}^{-1} \times \mathbf{P} - \mathbf{C}^{-1} \times \mathbf{R}$$

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Table I. The results obtained by the present method.

Roof level	Node number 10	Node number 20	Node number 40	Node number 30
Δ_X	5.23×10^{-2} m	4.99×10^{-2} m	-4.84×10^{-2} m	-4.81×10^{-2} m
Δ_Y	0.085×10^{-2} m	0.053×10^{-2} m	-0.071×10^{-2} m	-0.081×10^{-2} m
θ	0.0003 rad	0.000203 rad	0.0002 rad	0.0003 rad

Table II. The results obtained by SAP 2000.

Roof level	Node number 10	Node number 20	Node number 40	Node number 30
Δ_X	4.71×10^{-2} m	4.7089×10^{-2} m	4.7076×10^{-2} m	4.7072×10^{-2} m
Δ_Y	0.0702×10^{-2} m	0.01×10^{-2} m	-0.0026×10^{-2} m	-0.0777×10^{-2} m
θ	0.00031 rad	0.0002 rad	0.00021 rad	0.00032 rad

Thus, the displacements can be calculated as follows:

$$\mathbf{\Delta}_{1,2,3,\ldots,20} = [\mathbf{D}^{-1} \times \mathbf{P} + \mathbf{C}^{-1} \times \mathbf{P}]_{60\times 60}$$

$$\mathbf{\Delta}_{21,22,23,\ldots,40} = [\mathbf{D}^{-1} \times \mathbf{P} - \mathbf{C}^{-1} \times \mathbf{P}]_{60\times 60}$$

For all the calculations MATLAB 6.1 is used.

As an example, the displacements at the roof level are obtained as shown in Table I.

Using SAP 2000 these displacements are obtained as illustrated in Table II.

The results are nearly identical and no approximation is entered to the calculations. The difference in sign is due to the reason discussed in the previous example.

6. DISCUSSION AND CONCLUSIONS

In order to compare the efficiency of the present method and the methods without using the symmetry, 35 different examples are studied. The diagrams obtained are shown in Figure 12.

For symmetric structures with the Form II symmetry when the axis of the symmetry passes through the members, or for the Form III symmetry when the symmetry axis passes through the nodes, the formation of the entire structure is not needed for the analysis. As an example, for the Form II symmetry the matrix **A** constitutes the stiffness matrix of half of the structure deleting the members crossed by the symmetry axis, and the matrix **B** is the matrix for relating the symmetric DOFs. Once the formation of the cores **D** and **C** is performed and their inverse are obtained, the problem can easily be solved.

For the Form III symmetry, matrix **A** is the stiffness matrix of the half of the structure, obtained by deleting the central nodes and the incident members, and matrix **B** is a null matrix. The matrix

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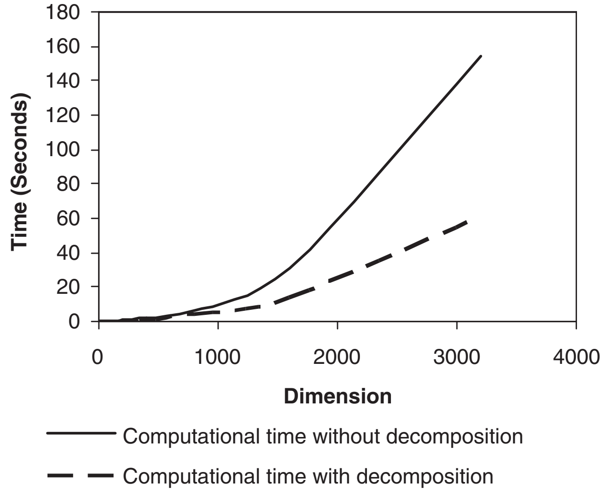

Figure 12. A comparative computational time.

M corresponds to the relationship between the symmetric DOFs and those of the central nodes. Having **D** and **E**, and their inverses, the complete analysis can easily be performed.

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