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Aequationes Mathematicae

Difference equations, Euler's summation formula and Hyers–Ulam stability

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Summary. In connection with the difference equation of the spiral of Theodorus (square root spiral) we develop an approach of Herbert Kociemba who gave an approximation of this spiral using Euler's summation formula (see [7]). We use Hyers–Ulam stability to obtain estimates about the distance between the approximative soution and the exact solution (the normal solution) of the considered difference equation. The presented method can be applied to the general difference equation $f(x+1) = f(x) + c(x)$.

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1. The Theodorus spiral

The spiral of Theodorus (square root spiral) is represented in the complex plane as a special solution of the difference equation of first order

$$f(x+1) = \alpha(x) \cdot f(x), \quad x \in [0, \infty) \tag{1}$$

with the initial condition $f(0) = 1$, where $\alpha(x) = 1 + \frac{i}{\sqrt{x+1}}$.

This specific solution is given by

$$T(x) = \prod_{k=1}^{\infty} \frac{1 + i/\sqrt{k}}{1 + i/\sqrt{x+k}}\ , \quad -1 < x < \infty, \tag{2}$$

see Davis [2] and Gronau [4].

For the very slowly convergent infinite product in (2) we gave in Gronau [4] a representation more appropriate for calculation by personal computers:

$$T(x) = \sqrt{x+1} \cdot e^{i\varphi(x)}, \tag{3}$$

where φ satisfies the difference equation

$$\varphi(x+1) = \varphi(x) + c(x),\ \ x \geq 0, \tag{4}$$

with

$$c(x) = \arctan \frac{1}{\sqrt{x+1}} = \pi/2 - \arctan\sqrt{x+1}.$$

Its unique monotonic solution with $\varphi(0) = 0$ is

$$\varphi(x) = \sum_{k=0}^{\infty} \left(\arctan\sqrt{k+x+1} - \arctan\sqrt{k+1} \right). \tag{5}$$

Thus we get by (3) with φ given by (5) another representation for the Theodorus spiral, i.e.:

$$T(x) = \sqrt{x+1} \cdot e^{i \sum_{k=0}^{\infty} \left(\arctan\sqrt{k+1} - \arctan\sqrt{x+k+1} \right)}. \tag{6}$$

This solution may be called a *normal solution* of (1) in the sense of Nörlund or Krull (see e.g. Gronau [5]).

Remark 1. Of course there are much more solutions of the difference equation (1), depending on an *initial function* prescribed e.g. on the interval $[0, 1)$. To be exact, the following standard result holds: *For any function* $\chi : [0, 1) \to \mathbb{C}$ *there exists one and only one solution of* (1) *which satisfies* $\varphi(x) = \chi(x)$ *for* $0 \le x < 1$.

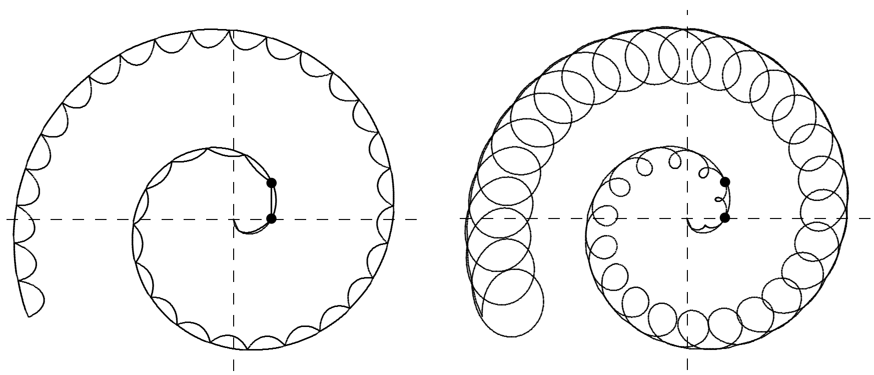

Fig. 1. Thedorus spiral, and other solutions of the difference equation (1). The initial curve χ is between the two bullets: Left: $\chi(x) = i \cdot x$, right: χ is a curly curve.

2. Approximative formula coming from Euler's summation formula

Herbert Kociemba [7] gave an approximation of (3) by

$$\tilde{T}(x) = \sqrt{x+1} \cdot e^{i\tilde{\varphi}(x)},$$

where $\tilde{\varphi}$ is an approximation of φ given by (5)

$$\tilde{\varphi}(x) = -47/48 - 3 \cdot \pi/8 + 1/(6 \cdot \sqrt{x+1}) + 2 \cdot \sqrt{x+1}. \tag{7}$$

With this formula pictures with large arguments can be produced immediately.

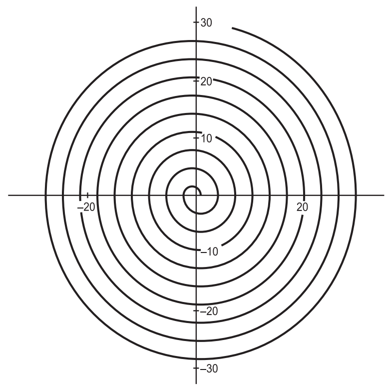

FIG. 2. Theodorus spiral, looking in the large like an Archimedean spiral.

How did Kociemba come to this approximating formula (7)? As it is easy to see we get for any solution φ of (4) and for any natural n

$$\varphi(n) = \varphi(0) + \sum_{k=0}^{n-1} c(k). \tag{8}$$

For the sum in (8) one can use the Euler summation formula (Euler–Maclaurin summation formula) which gives

$$\begin{aligned} \varphi(n) &= \varphi(0) + \int_0^{n-1} c(t)dt + \frac{1}{2}(c(n-1) + c(0)) \\ &\quad + \sum_{j=2}^{m} \frac{(-1)^j B_j}{j!} \left(c^{(j-1)}(n-1) - c^{(j-1)}(0) \right) \\ &\quad + \frac{(-1)^{m-1}}{m!} \int_0^{n-1} c^{(m)}(t) B_m(\{t\}) dt. \end{aligned} \tag{9}$$

Here m is an integer such that the function c is m-times continuously differentiable, the B_j are the Bernoulli numbers, $B_j(t)$ the Bernoulli polynomials and $\{t\}$ is the fractional part of t. Remember that the Bernoulli numbers B_{2j+1} vanish for all $j \geq 1$.

The right-hand side of the above identity (9) is defined also for real n. Thus we can define for real nonnegative x

$$
\begin{aligned}
\Phi(x) \;=\; &\varphi(0) + \int_0^{x-1} c(t)dt + \frac{1}{2}(c(x-1)+c(0)) \\
&+\sum_{j=2}^{m} \frac{(-1)^j B_j}{j!}\left(c^{(j-1)}(x-1) - c^{(j-1)}(0)\right) \\
&+\frac{(-1)^{m-1}}{m!}\int_0^{x-1} c^{(m)}(t) B_m(\{t\})dt, \qquad (10)
\end{aligned}
$$

where the formula for $\Phi(x)$ is the same as (9), with n replaced by x. This function Φ will serve as approximating function of φ, given by (5), as it will be shown below.

3. The difference equation of the approximating function

What is this function Φ good for? We first calculate the difference

$$
\begin{aligned}
\Phi(x+1)-\Phi(x) \;=\; &\int_{x-1}^{x} c(t)dt + \frac{1}{2}(c(x)-c(x-1)) \\
&+\sum_{j=2}^{m} \frac{(-1)^j B_j}{j!}\left(c^{(j-1)}(x) - c^{(j-1)}(x-1)\right) \\
&+\frac{(-1)^{m-1}}{m!}\int_{x-1}^{x} c^{(m)}(t) B_m(\{t\})dt.
\end{aligned}
$$

On the other hand we have the well-known identity (see any book that contains a proof of Euler's summation formula, e.g. [9])

$$
\begin{aligned}
\int_{x-1}^{x} c(t)dt \;=\; &\frac{1}{2}(c(x)+c(x-1)) \\
&-\sum_{j=2}^{m} \frac{(-1)^j B_j}{j!}\left(c^{(j-1)}(x) - c^{(j-1)}(x-1)\right) \\
&-\frac{(-1)^{m-1}}{m!}\int_0^1 c^{(m)}(x-1+t)\cdot B_m(t)dt.
\end{aligned}
$$

So we obtain for Φ the difference equation

$$\Phi(x+1) = \Phi(x) + c(x) + \varepsilon(x), \qquad (11)$$

where

$$\varepsilon(x) = \frac{(-1)^{m-1}}{m!}\int_0^1 c^{(m)}(x-1+t)\cdot \big[B_m(\{t+x\}) - B_m(t)\big]dt. \qquad (12)$$

Note that $\varepsilon(n) = 0$ for all integers n as $\Phi(n) = \varphi(n)$ for all naturals n.

For the Bernoulli polynomials we have an estimate due to Lehmer [8]:

$$\max\left\{|B_m(x)| \;\middle|\; 0 \le x \le 1\right\} \le 4m!/(2\pi)^m. \qquad (13)$$

From this we get the inequality

$$|\Phi(x) - \Phi(x+1) + c(x)| = |\varepsilon(x)| \le \frac{8m!}{(2\pi)^m} \cdot C_m(x), \tag{14}$$

where $C_m(x) = \max\left\{|c^{(m)}(t)| \,\middle|\, x-1 \le t \le x\right\}$.

For $c(x) = \arctan\sqrt{x+1}$, the function $c^{(m)}$ is of the order $O(x^{-1/2-m})$ for $x \to \infty$. Thus in the case of the Theodorus spiral we have $|\varepsilon(x)| = O(x^{-1/2-m})$.

To find estimates of the quality of the approximation we use the theory of Hyers–Ulam stability.

4. Hyers–Ulam stability

Whereas the literature on Hyers–Ulam stability of functional equations is very large, there are relatively few articles concerning the Hyers–Ulam stability of difference equations, specifically. The most appropriate article I found for my purposes is that of Borelli [1], much better than others, (see e.g. Kim [6]). Especially no conditions on the coefficient function of the considered difference equation are supposed. The result of Borelli deals with a general class of functional equations. Adopted to equation

$$f(x+1) = f(x) + c(x), \; x \ge 0, \tag{15}$$

it reads as follows.

Theorem 1 (Borelli)**.** *Suppose that $f : [0,\infty) \to \mathbb{R}$ satisfies*

$$|f(x+1) - f(x) - c(x)| \le \sigma(x), \; x \ge 0, \tag{16}$$

and let the series $\sum_{k=0}^{\infty} \sigma(x+k)$ be convergent for $x \ge 0$. Then the limit

$$g(x) = \lim_{n\to\infty}\left[f(n+x) - \sum_{k=0}^{n-1} c(x+k)\right] \tag{17}$$

exists for all $x \ge 0$ and defines a function g that satisfies

$$g(x+1) = g(x) + c(x), \; x \ge 0.$$

The function g is the unique solution of (15) *which fulfils the inequality*

$$|f(x) - g(x)| \le \sum_{k=0}^{\infty} \sigma(x+k). \tag{18}$$

The proof of this theorem is given by application of standard methods of the theory of Hyers–Ulam stability. (In this connection one should also point the attention to the paper [3] of G. L. Forti.)

It turns out that the exact solution g of (15) is of the form

$$g(x) = f(x) + \sum_{k=0}^{\infty} \varepsilon(x+k),$$

if we define ε by $\varepsilon(x) = f(x+1) - f(x) - c(x)$.

We return now to our approximating function. In the case of the Theodorus spiral, according to (14) the functions $f = \Phi$ and $\sigma(x) = |\varepsilon(x)| = O(x^{-1/2-m})$ so that $|\varepsilon(x)| \leq M\ x^{-1/2-m}$, $x \geq 0$, satisfy the assumptions of Theorem 1, thus it can be applied.

In order to show that the uniquely defined function g, given by (17) in Theorem 1, coincides with φ, the (normal) solution (5) of (4), we will use a lemma which is also valid for a more general term than $c(x) = \arctan \frac{1}{\sqrt{x+1}}$.

Lemma. *Suppose that the limit*

$$\lim_{n\to\infty}\left(\sum_{k=0}^{n} [c(k+x) - c(k)] - x \cdot [c(n+1) - c(n)]\right) \tag{19}$$

exists for $0 < x < 1$. *Then the difference equation* (15) *has, up to a constant, a unique solution* f *satisfying the condition*

$$\lim_{n\to\infty} (f(n+x) - f(n) - x \cdot [c(n+1) - c(n)]) = 0. \tag{20}$$

We will call this uniquely defined solution the normal solution *of* (15).

Sketch of the proof. The proof is similar to that of Theorem 1 in Gronau [5], and is more or less standard. Let f be a solution of (15) which satisfies (20). Then it follows from this for naturals n and $x \in [0, 1)$:

$$f(n+x) - f(n) = f(x) - f(0) + \sum_{k=0}^{n-1} [c(k+x) - c(k)]\,.$$

Subtracting on both sides $x \cdot [c(n+1) - c(n)]$, rearranging, and taking the limits we get for $x \in [0, 1)$

$$f(x) = f(0) + \lim_{n\to\infty}\left(\sum_{k=0}^{n-1}[c(k) - c(k+x)] - x \cdot [c(n+1) - c(n)]\right). \tag{21}$$

Thus f is uniquely determined by (20) on the interval $[0, 1)$. Using Remark 1 (which is valid for equation (15) too) one sees that f is also uniquely defined for all nonnegative reals. On the other hand, by (21) for $0 \leq x < 1$ and Remark 1, a solution of (15) satisfying (20) is given. $\square$

Remark 2. If $\sum_{k=0}^{\infty}[c(k+x)-c(k)]$ converges for $0<x<1$, then the normal solution has the form

$$f(x)=f(0)+\sum_{k=0}^{\infty}[c(k+x)-c(k)] \text{ for } 0\le x<1. \tag{22}$$

If, additionally, $\lim_{n\to\infty} c(n)=0$ then the series (22) converges for all $x\ge 0$ and represents the uniquely defined normal solution of (15), as it is easy to prove.

Now we are able to show that the approximate solution (10) is close to the normal solution of (15).

Theorem 2. *Suppose that the function c is m-times continuously differentiable on $(0,\infty)$ and satisfies the following conditions*

a.) *Condition* (19) *holds for* $0<x<1$,

b.) $\lim_{k\to\infty}[c^{(i)}(k+x)-c^{(i)}(k)]=0$ *for* $0<x<1$, $0\le i\le m$,

c.) $\lim_{k\to\infty}\left(\int_k^{k+x} c(t)dt - x\cdot[c(k+x)-c(k)]\right)=0$ *for* $0<x<1$,

d.) $\sum_{k=0}^{\infty}\varepsilon(k+x)$ *is absolute convergent for* $x\ge 0$, *where* ε *is given by* (12). (This is the case, e.g. if $\sum_{k=0}^{\infty}c^{(m)}(x+m)$ is absolute convergent.)

Then the normal solution f *of* (15) *exists and the function* Φ *given by* (10) *with* $\Phi(0)=f(0)$ *approximates* f. *The inequality*

$$|\phi(x)-f(x)|\le\sum_{k=0}^{\infty}|\varepsilon(x+k)| \quad \textit{for} \quad x\ge 0 \tag{23}$$

holds.

Proof. Let f be the normal solution of (15) which exists according to a.) and the Lemma, Φ defined by (10) with $\Phi(0)=f(0)$, and g be given by Theorem 1. We claim that $g=f$.

Step 1. f satisfies (20).

Step 2. Φ satisfies $\lim_{n\to\infty}(\Phi(n+x)-\Phi(n)-x\cdot[c(n+1)-c(n)])=0$ due to b.)–c.).

Step 3. $g(x)=\lim_{n\to\infty}[\phi(n+x]-\sum_{k=0}^{n-1}c(k+x)]=\Phi(x)+\sum_{k=0}^{\infty}\varepsilon(x+k)$. Hence also g satisfies $\lim_{n\to\infty}[g(n+x)-g(n)-x\cdot[c(n+1)-c(n)]]=0$ and $g(0)=f(0)$.

Step 4. Therefore, due to the Lemma, g is identical with f. Inequality (23) follows from Theorem 1. □

Remark 3. In the case of the spiral of Theodorus, i.e. $c(x)=\arctan\sqrt{x+1}$, the function φ given by (5) is the normal solution of (4). From Theorem 2 we get the estimate

$$|\varphi(x)-\Phi(x)|=O(x^{-1/2-m}). \tag{24}$$

As an example we get for $m = 3$:

$$|\varepsilon(x)| < \frac{5\sqrt{3}}{576} x^{-7/2} = 0.01503 \cdots \cdot x^{-7/3}.$$

For $x \geq 4$ the estimate in (24) yields bounds less than $5 \cdot 10^{-4}$.

The difference between the original spiral formula $T(x) = \sqrt{x+1} \cdot e^{\varphi(x)}$ given by (6) and its approximation by the Euler summation formula $\sqrt{x+1} \cdot e^{\Phi(x)}$ is of order $O(x^{-m})$.

$$\left| T(x) - \sqrt{x+1} \cdot e^{\Phi(x)} \right| = O(x^{-m}).$$

This is less than the resolving power in computer drawings. Some more ambitious estimates are given in Kociemba [7].

5. Another example

As it has been shown this method can be applied for more general linear difference equations than that of the argument of the Theodorus spiral.

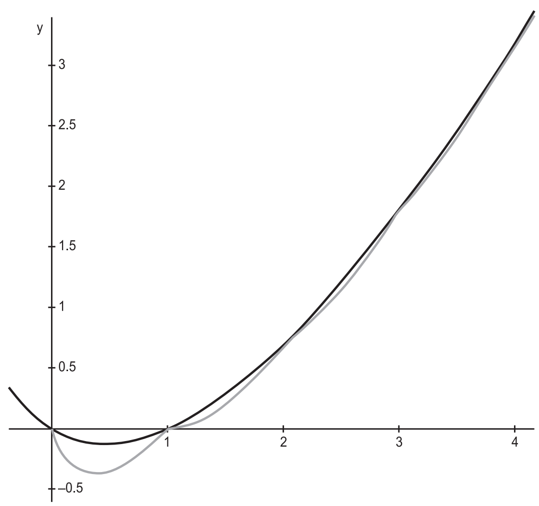

FIG. 3. Approximation of the function $\log \Gamma(x+1)$ obtained with the aid of Euler's summation formula up to degree 1.

As an example we consider the equation

$$f(x+1) = f(x) + \log(x+1), \quad x \geq 0, \quad f(0) = 0,$$

the difference equation of $\log \Gamma(x+1)$. Here the suppositions of Theorem 2 hold.

The approximative function Φ in (10) for $c(x) = \log(x+1)$ is

$$\begin{aligned}\Phi(x) \;=\; & \int_0^{x-1} \log(t+1)dt + \frac{1}{2}\log x \\ & + \sum_{j=2}^{m} \frac{B_j}{j!}(j-2)!\left(\frac{1}{x^{j-1}} - 1\right) \\ & + \frac{1}{m}\int_0^{x-1} \frac{1}{(t+1)^m} B_m(\{t\})dt. \end{aligned} \tag{25}$$

For $m = 1$ we get

$$\Phi(x) = \left(x + \frac{1}{2}\right) \cdot \log x - x + 1 + \int_1^x \frac{\{t\} - 1/2}{t} dt.$$

This yields a nice approximation of $\log\Gamma(x+1)$ as Figure 3 shows.

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