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Pseudosolutions of the Time-Dependent Minimal Surface Problem

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INTRODUCTION

In this work, we are interested in the evolution problem associated with the minimal surface equation. Let Ω be a bounded open set of R^n with boundary Γ; we are looking for a real function u, defined over $\Omega \times [0, T]$ that satisfies:

$$\partial u/\partial t + Au = 0 \quad \text{in} \quad \Omega \times]0, T[\tag{0.1}$$

where

$$Au = -\text{div}(\text{grad } u/(1 + |\text{grad } u|^2)^{1/2}) \tag{0.2}$$

and that satisfies the boundary and initial conditions:

$$u = \Phi \quad \text{on} \quad \Gamma \times [0, T] \tag{0.3}$$

$$u(x, 0) = u_0(x) \quad \text{in} \quad \Omega. \tag{0.4}$$

It is easily seen that the problem (0.1)–(0.4) stated in an appropriate functional setting admits at most one strong solution. On the contrary, the existence of such solutions is not known. Furthermore, simple examples (where Ω can be chosen as the interval $]0,1[$) show that this problem does not have, in some instances, a solution.

Our aim is to give a weakened formulation of problem (0.1)–(0.4) for which we shall obtain the existence of a unique solution, which we shall call a pseudosolution. Then we shall make precise in which sense this pseudosolution verifies Eqs. (0.1)–(0.4). Our approach for defining the pseudosolution of (0.1)–(0.4) is closely related to the definition of the pseudosolutions of the steady-state problem, i.e., the nonparametric Plateau problem (cf. Bombieri, de Giorgi, Miranda [2], Temam [19]). To make the present article self-contained we start Section 1 by a brief description of the concept of pseudosolution for the nonparametric Plateau problem, following [19] and [9].

After these preliminaries, the first paragraph contains the description of our weak formulation of the problem which is based on monotonicity properties of the operator A, some analogies with the nonparametric minimal surface problem (i.e., the stationary case), and the idea of *variational inequalities* (cf.

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Brézis [3], Lions [12]). This section ends with the statement of the existence and uniqueness results.

The second paragraph gives the proofs of existence and uniqueness theorems The proof of the existence result is based on the introduction of the parabolic regularized problems ($\epsilon > 0$):

$$\partial u_\epsilon/\partial t + Au_\epsilon - \epsilon\, \Delta u_\epsilon = 0$$

and on passing to the limit as $\epsilon \searrow 0$.

Sections 3 and 4 make precise in which sense problem (0.1)–(0.4) has thus been solved. First of all we show that the pseudosolution which is defined via a variational inequality is also a solution, in the distributional sense of Eq. (0.1). Next we investigate the boundary behavior of the pseudosolutions: as in the steady-state case it is proved that the boundary condition (0.3) is verified on the part of Γ of positive mean curvature. The behavior of solutions of (0.1)–(0.4) as $t \to +\infty$ is also studied in Section 4.

A preliminary version of this work has been given in [21, 11]. We plan to study, in a later work, analogous problems with Neumann boundary conditions as well as the regularity of the pseudosolutions of problem (0.1)–(0.4).

Plan

1. Formulation of the Problem: Statement of the Existence and Uniqueness Results

1.0. *The Steady-State Case*

Let us assume that $\Omega \subset R^n$ is an open set of class $\mathscr{C}^2$ and that ϕ is a regular function defined over $\partial\Omega$. We consider in this paragraph the nonparametric minimal surface problem which can be formulated in the following way: *Find a real function u such that*:

$$Au = 0 \quad \text{in} \quad \Omega \tag{1.1}$$

$$u = \Phi \quad \text{on} \quad \partial\Omega. \tag{1.2}$$

This problem is equivalent to the variational problem.

$$\text{Find a real function } u \text{ which minimizes} \quad \int_\Omega (1 + |\nabla v|^2)^{1/2}\, dx \text{ among all } v \text{ such that } v = \Phi \text{ on } \partial\Omega. \tag{1.3}$$

It has been shown that, if $\partial\Omega$ is not of nonnegative mean curvature, there exists boundary data Φ (even very regular) such that problems (1.1)–(1.2) or (1.3) do not have solutions neither in the strong sense ($u \in \mathscr{C}^2(\Omega) \cap \mathscr{C}(\bar{\Omega})$, nor in the weak sense ($u \in W^{1,1}(\Omega)$); cf. J. Serrin [16].

It is nevertheless possible to define pseudosolutions,[1] and prove their existence, either by extending the area functional appearing in (1.3) to the class of functions of bounded variations (cf. de Giorgi, Giusti, and Miranda [2], Miranda [15]) or by making use of the relations between problem (1.3) and its dual in the sense of Convex Analysis (cf. Temam [19]). The latter theory gives more precisely the following results (cf. [5, 9, 19]).

PROPOSITION 1.1. *Let Φ be given, satisfying $\Phi \in W^{1,1}(\Omega)$. There exists an analytic function $u \in W^{1,1}(\Omega)$ which is uniquely defined to within an additive constant and called the pseudosolution of* (1.3) *such that*:

$$u \text{ is a solution of the minimal surface equation (1.1)} \tag{1.4}$$

$$\text{every minimizing sequence } (v_m) \text{ of (1.3) in } W^{1,1}(\Omega) \text{ converges to } u \text{ in the following sense:} \tag{1.5}$$

$$v_m \to u \text{ in } L^1(\Omega)/R \qquad \nabla v_m \to \nabla u \text{ in } (L^1_{\mathrm{loc}}(\Omega))^n.$$

The following result, which is established in [5, p. 139] (and makes the connection with de Giorgi's approach for stationary pseudosolutions) shall be of a great importance in the sequel.

PROPOSITION 1.2. *Let Φ be given as above. The pseudosolution u of* (1.3) *is a solution of the following problem*:

Find a real function u which minimizes:

$$\int_\Omega (1 + |\nabla v|^2)^{1/2}\, dx + \int_{\partial\Omega} |\Phi - v|\, ds \tag{1.6}$$

among all the v's in $W^{1,1}(\Omega)$.

[1] Previously referred to as "generalized solutions" in [5, 9, 10, 19].

The pseudosolutions do not satisfy in general the condition (1.2), and this fact is noticeable both in Serrin's counter-examples and in (1.6). However, condition (1.2) is satisfied on those parts of $\partial\Omega$ of positive mean curvature.

PROPOSITION 1.3. (cf. [9, 10]). *Let Ω and ϕ be given as above. Suppose that Γ_1 is a nonempty open subset of $\partial\Omega$, which is a $\mathscr{C}^3$ manifold of nonnegative mean curvature,*[2] *and that Φ is continuous at each point of Γ_1. Then problem* (1.3) *has a unique pseudosolution u which, in addition to the properties stated above, is continuous on Γ_1 and verifies:*

$$u = \Phi \quad on \quad \Gamma_1 .$$

1.1. *The Weak Formulation of the Problem*

Let us assume for the moment that $\Omega \subset R^n$ is an open set of class $\mathscr{C}^2$, and that the boundary and initial data Φ, u_0 are sufficiently regular. Let us assume that u is a classical solution of (0.1)–(0.4) in $Q = \Omega \times]0, T[$ – (say $u \in \mathscr{C}^2(\bar{Q})$). Then let v be a $\mathscr{C}^2(\bar{Q})$ test function which is equal to u on $\Gamma \times]0, T[$; multiplying (0.1) by $v - u$ and integrating in x and t ($x \in \Omega$, $0 < t < s$) we get:[3]

$$\int_0^s (u', v - u)\, dt + \int_0^s \int_\Omega \frac{u_x \cdot (u_x - v_x)}{(1 + u_x^2)^{1/2}}\, dx\, dt = 0.$$

We have written $u_x = \operatorname{grad} u = (\partial u/\partial x_1, \ldots, \partial u/\partial x_n)$, $u_x^2 = |\operatorname{grad} u|^2 = \sum_{i=1}^n |\partial u/\partial x_i|^2$; $(\cdot,\cdot)$ and $|\cdot|$ denote respectively the scalar product and the norm in $L^2(\Omega)$. The function of one real variable

$$s \to \rho(s) = (1 + s^2)^{1/2}$$

being convex, we have, for every t and x

$$(1 + v_x^2)^{1/2} \geqslant (1 + u_x^2)^{1/2} + u_x \cdot (u_x - v_x)/(1 + u_x^2)^{1/2} \tag{1.7}$$

and thus

$$\int_0^s (u', v - u)\, dt + \int_0^s \int_\Omega ((1 + v_x^2)^{1/2} - (1 + u_x^2)^{1/2})\, dx\, dt \geqslant 0. \tag{1.8}$$

Adding to each side of inequality (1.8) the quantity:

$$\int_0^s (v' - u', v - u)\, dt = \tfrac{1}{2} |v(s) - u(s)|^2 - \tfrac{1}{2} |v(0) - u_0|^2$$

[2] The same result holds under less regularity assumptions.

[3] We denote by a $'$ the differentiation with respect to the t variable.

we find:

$$\int_0^s (v', v - u)\, dt + \int_0^s \int_\Omega ((1 + v_x^2)^{1/2} - (1 + u_x^2)^{1/2})\, dx\, dt$$

$$\geqslant \tfrac{1}{2} \mid v(s) - u(s)|^2 - \tfrac{1}{2} \mid v(0) - u_0 \mid^2. \tag{1.9}$$

Conversely, it is easily shown that, if a function $u \in \mathscr{C}^2(\bar{Q})$ verifies $u = \Phi$ on $\Gamma \times]0, T[$ and satisfies (1.9) for all the test functions $v \in \mathscr{C}^2(\bar{Q})$ such that $v = \Phi$ on $\Gamma \times]0, T[$ then u satisfies (0.1) and (0.4).

In a more general fashion, we shall be led to consider functions u (and hence also functions v) that do not verify (0.3). Let us first consider the case where $u \in \mathscr{C}^2(\bar{Q})$, satisfies (0.1)–(0.4) and v belongs to $\mathscr{C}^2(\bar{Q})$ but does not necessarily satisfy (0.3). There exists a sequence $(v_m)_{m \in N}$ of functions in $\mathscr{C}^2(\bar{Q})$, which are equal to Φ on $\Gamma \times [0, T]$, and converge to v in the following sense:

$$v_m \to v \text{ in } L^2(Q), \qquad \partial v_m/\partial t = v'_m \to v' \text{ in } L^2(Q).$$

For each v_m of this sequence we can write (1.9), and we can pass to the lower-limit in (1.9) by imposing additional convergence conditions to the sequence $(v_m)_{m \in N}$ (These conditions shall be discussed later); we find:

$$\int_0^s (v', v - u)\, dt + \int_0^s (e(t, v(t)) - e(t, u(t))\, dt$$

$$\geqslant \tfrac{1}{2} \mid v(s) - u(s)|^2 - \tfrac{1}{2} \mid v(0) - u_0 \mid^2, \tag{1.10}$$

where we have set:

$$e(t, w) = \int_\Omega (1 + w_x^2)^{1/2}\, dx + \int_\Gamma \mid w - \Phi(t)|\, d\Gamma; \tag{1.11}$$

it is readily seen with (0.3) that:

$$e(t, u(t)) = \int_\Omega (1 + u_x(x, t)^2)^{1/2}\, dx.$$

Conversely (cf. Sect. 2), it can be shown that, if a function $u \in \mathscr{C}^2(\bar{Q})$ verifies (1.10) for every $v \in \mathscr{C}^2(\bar{Q})$ then u is a solution of (0.1)–(0.4). We shall use (1.11) as a starting point in the derivation of the weak formulation of problem (0.1)–(0.4). This formulation will be discussed with more details in the sequel; it is closely related to a formulation introduced by J. L. Lions in [12, p. 390] for some evolution variational inequalities.

1.2. *The Functionals e, $\mathscr{E}$*

Unless otherwise stated we shall assume henceforth that Ω is a bounded open set of class $\mathscr{C}^2$ in R^n. We can make more precise the definition of the above-mentioned functional e. Given functions $\phi \in L^1(\Gamma)$ and $u \in L^1(\Omega)$ we set:

$$e(\phi; u) = \operatorname{Sup} \left\{ \int_\Omega \left[\theta_0 + \sum_{i=1}^{n} \frac{\partial \theta_i}{\partial x_i} u \right] dx - \sum_{i=1}^{n} \int_\Gamma \theta_i \nu_i \phi \, d\Gamma \right\} \tag{1.12}$$

where the supremum is being taken among all the $\theta = (\theta_0, ..., \theta_n) \in (\mathscr{C}^\infty(\bar{\Omega}))^{n+1}$ such that $\sum_{i=0}^{n} \theta_i^2 \leqslant 1$. We have denoted $\nu = (\nu_1, ..., \nu_n)$ the unit vector normal to Γ, pointing outwards with respect to Ω. When this will not leave any ambiguity we shall write $e(u)$ instead of $e(\phi; u)$. It is easily checked that the mapping $u \to e(u)$ from $L^1(\Omega)$ into R is convex and lower-semicontinuous. Furthermore, $e(u)$ is finite if and only if $u \in BV(\Omega)$ and the expression of $e(u)$ in this case is given in [21]. For $u \in W^{1,1}(\Omega)$, an easy computation shows that (cf. (1.6)):

$$e(u) = \int_\Omega (1 + u_x^2)^{1/2} \, dx + \int_\Gamma | u - \phi | \, d\Gamma. \tag{1.13}$$

When $\phi \in L^1(\Gamma \times]0, T[)$, $\phi(t) \in L^1(\Gamma)$ for almost all t and we set $e(t, u) = e(\phi(t), u)$. We also define, for $u \in L^1(Q)$ and $s \in [0, T]$:

$$\mathscr{E}(s, u) = \int_0^s e(t, u(t)) \, dt; \qquad \mathscr{E}(u) = \mathscr{E}(T, u). \tag{1.14}$$

The functional $u \to \mathscr{E}(s, u) (s \in [0, T])$ from $L^1(Q)$ into R is convex and l.s.c. Indeed, if this was not true, there would exist a sequence $(u_m)_{m \in N}$ converging to u in $L^1(Q)$ with:

$$\varliminf_{m \to \infty} \mathscr{E}(s, u_m) < \mathscr{E}(s, u);$$

and we might extract a subsequence $(u_{m'})$ such that

$u_{m'}(\cdot, t) \to u(\cdot, t)$ in $L^1(\Omega)$ for almost every t. The functional e being l.s.c. this would imply:

$$\varliminf_{m' \to \infty} \mathscr{E}(s, u_{m'}) = \varliminf_{m' \to \infty} \int_0^s e(t, u_{m'}(t)) \, dt$$

$$\geqslant \int_0^s \varliminf_{m' \to \infty} e(t, u_{m'}(t)) \, dt \geqslant \int_0^s e(t, u(t)) \, dt.$$

Whence the contradiction.

1.3. *Statement of the Existence and Uniqueness Theorems*

It is convenient, but not restrictive at all, to assume that ϕ is given as the restriction to (or the trace on) $\Gamma \times [0, T]$ of a function Φ defined over $\Omega \times [0, T]$. We shall establish the two following results.

THEOREM 1.1. *Suppose Φ, h, u_0 are given such that*

$$\Phi \in H^1(Q) \tag{1.15}$$

$$h \in L^2(Q), \qquad h_x \in L^2(0, T; L^2_{\text{loc}}(\Omega)) \tag{1.16}$$

$$u_0 \in L^2(\Omega) \cap H^1_{\text{loc}}(\Omega) \cap W^{1,1}(\Omega). \tag{1.17}$$

Then there exists a unique function u having the following properties:

$$u \in L^1(0, T; W^{1,1}(\Omega)) \cap L^\infty(0, T; H^1_{\text{loc}}(\Omega)) \tag{1.18}$$

$$u \in \mathscr{C}([0, T]; L^2(\Omega)) \tag{1.19}$$

$$u(0) = u_0 \tag{1.20}$$

$$\begin{gathered} \int_0^s (v' - h, v - u)\, dt + \int_0^s (e(t, v(t)) - e(t, u(t)))\, dt \\ \geqslant \tfrac{1}{2} \mid v(s) - u(s) \mid^2 - \tfrac{1}{2} \mid v(0) - u_0 \mid^2 \\ \forall s \in [0, T],\ \forall v \in L^2(Q) \text{ such that } v' \in L^2(Q),\ v_x \in (L^1(Q))^n \end{gathered} \tag{1.21}$$

Furthermore, if Φ, $h \in L^\infty(Q)$ and $u_0 \in L^\infty(\Omega)$ then $u \in L^\infty(Q)$.

THEOREM 1.2. *Suppose Φ, h, u_0 are given such that*

$$\Phi \in H^1(Q), \qquad \Phi'_x = \partial/\partial t\ \Phi_x \in (L^2(Q))^n \tag{1.22}$$

$$h \in L^2(Q), \qquad h_x \in L^1(0, T; L^2_{\text{loc}}(\Omega)) \tag{1.23}$$

$$u_0 \in H^1(\Omega) \tag{1.24}$$

Then the function u whose existence is given by Theorem 1.1 *above satisfies also the following properties:*

$$u \in L^\infty(0, T; W^{1,1}(\Omega)) \tag{1.25}$$

$$u' \in L^2(Q) \tag{1.26}$$

$$\partial u/\partial t - \operatorname{div}[u_x/(1 + u_x^2)^{1/2}] = h \text{ in } \Omega \times]0, T[\tag{1.27}$$

$$-u_x \cdot \nu/(1 + u_x^2)^{1/2} \in \operatorname{Sgn}(u - \Phi) \text{ a.e. on } \Gamma \times]0, T[. \tag{1.28}$$

Remark 1.1. From (1.26)–(1.27) and a trace Theorem of Lions–Magenes [13], the trace on $\Gamma \times]0, T[$ of $-u_x \cdot \nu/(1 + u_x{}^2)^{1/2}$ is well defined and belongs to $L^\infty(\Gamma \times]0, T[)$ (cf. also a remark in Jouron [7]).

Remark 1.2. When $\partial u/\partial \nu$ is defined on $\Gamma \times]0, T[$ and $u \neq \Phi$, (1.28) implies in particular:

$$\begin{aligned} \partial u/\partial \nu &= +\infty \quad \text{where} \quad u < \Phi \\ \partial u/\partial \nu &= -\infty \quad \text{where} \quad u > \Phi \end{aligned} \tag{1.29}$$

2. Proof of Theorem 1.1

2.1. *The Parabolic-Regularized Problem*

Let $\epsilon > 0$ be fixed; we shall be concerned here with the problem obtained by parabolic regularization of (0.1)–(0.4):

$$\partial u_\epsilon/\partial t - \epsilon \Delta u_\epsilon + A u_\epsilon = h \quad \text{in} \quad Q = \Omega \times]0, T[\tag{2.1}$$

$$u_\epsilon = \Phi \text{ on } \Sigma = \Gamma \times [0, T] \tag{2.2}$$

$$u_\epsilon(x, 0) = u_0(x) \quad \text{in} \quad \Omega. \tag{2.3}$$

(We also introduced here a nonzero right hand side h in Eq. (2.1)). From (1.15)–(1.17) and from standard results on monotone nonlinear equations (cf. for instance Brezis [3], Lions [12]) there exists a unique solution u_ϵ of (2.1)–(2.3) such that, in particular:

$$u_\epsilon \in L^2(0, T; H^1(\Omega)) \cap \mathscr{C}([0, T]; L^2(\Omega)). \tag{2.4}$$

We intend to establish now some a priori estimates for u_ϵ which are *independent* of $\epsilon > 0$.

Lemma 2.1. *As $\epsilon \searrow 0$, the sequence u_ϵ remains in a bounded set of $L^1(0, T; W^{1,1}(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ and $\epsilon^{1/2} u_\epsilon$ is bounded in $L^2(0, T; H^1(\Omega))$.*

Proof. In a standard way, multiplying (2.1) by $u_\epsilon - \phi$ and integrating with respect to x over Ω we find:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \mid u_\epsilon - \Phi \mid^2 + \epsilon \mid u_{\epsilon x} \mid^2 + \int_\Omega \frac{u_{\epsilon x}^2}{(1 + u_{\epsilon x}^2)^{1/2}}\, dx \\ &\quad = (h, u_\epsilon - \Phi) - (\Phi', u_\epsilon - \phi) + \epsilon(u_{\epsilon x}, \Phi_x) + \int_\Omega \frac{u_{\epsilon x} \cdot \Phi_x}{(1 + u_{\epsilon x}^2)^{1/2}}\, dx \\ &\quad \leqslant \tfrac{1}{2}(\mid h \mid^2 + \mid \Phi' \mid^2) + \mid u_\epsilon - \Phi \mid^2 + \frac{\epsilon}{2} \mid u_{\epsilon x} \mid^2 + \frac{\epsilon}{2} \mid \Phi_x \mid^2 + (\text{mes}\, \Omega) \mid \Phi_x \mid. \end{aligned}$$

The result follows.

2.2. *A priori estimates*

We shall establish the following a priori estimate for grad u_ϵ.

LEMMA 2.2. *As* $\epsilon \to 0$, grad u_ϵ *remains bounded in* $L^\infty(0, T; L^2_{\mathrm{loc}}(\Omega))$.

Proof. To simplify further the writing we shall omit in this proof the subscripts ϵ, and we shall write $u_j = \partial u/\partial x_j$, $v = \sum_{j=1}^n u_j^2$. We apply the operator $\sum_{l=1}^n u_l(\partial/\partial x_l)$ to both sides of (2.1); this gives as in [19]:

$$\partial v/\partial t - \epsilon \Delta v - \sum_{i,j=1}^n \partial/\partial x_i\,(a_{ij} v_j) + 2 \sum_{i,l=1}^n \epsilon u_{l,i} u_{l,i} + 2 \sum_{i,j,l=1}^n a_{ij} u_{li} u_{lj} = u_x \cdot h_x \tag{2.5}$$

where $a_{ij} = a_{ij}(u_x)$ and for $\xi = (\xi_1, ..., \xi_n) \in R^n$, $a_{ij}(\xi)$ is defined by

$$a_{ij}(\xi) = \partial^2/\partial\xi_i \partial\xi_j\,(1 + |\,\xi\,|^2)^{1/2}. \tag{2.6}$$

Let $\Omega' \subset\subset \Omega$ be a relatively compact open set of Ω and let $\zeta \in \mathscr{D}(\Omega)$ be a $\mathscr{C}^\infty$ function with compact support in Ω, which takes values in the interval $[0, 1]$ and which is equal to 1 on Ω'. Multiplying (2.5) by ζ^2 and integrating in x over Ω we get:

$$\frac{d}{dt}\left(\int_\Omega v\zeta^2\,dx\right) + 2\epsilon \int_\Omega v_x \zeta \zeta_x\,dx + 2 \sum_{i,j=1}^n \int_\Omega (a_{ij} v_j \zeta \zeta_i + \epsilon u_{ji} u_{ji} \zeta^2)\,dx + 2 \sum_{i,j,l}^n \int_\Omega a_{ij} u_{li} u_{lj} \zeta^2\,dx = 2 \int_\Omega u_x \cdot h_x\,dx.$$

Now, $v_i = 2\sum_{l=1}^n u_l u_{li}$ and:

$$\epsilon \left| \int_\Omega v_x \zeta_x \zeta\,dx \right| \leqslant \epsilon \sum_{l,i=1}^n \int_\Omega u_{li} u_{li} \zeta^2\,dx + \epsilon \int_\Omega v \zeta_x^2\,dx.$$

From [19, Lemma 2.4] we get:

$$2 \sum_{i,j,l} \int_\Omega a_{ij} u_{li} u_{lj} \zeta^2\,dx \geqslant \int_\Omega [|\,\delta u_x\,|^2/(1 + v)^{1/2}]\,\zeta^2\,dx$$

where δ is a first-order differential operator defined in [19], and we also have:

$$2 \left| \sum_{i,j} \int_\Omega a_{ij} v_j \zeta \zeta_i\,dx \right| \leqslant \tfrac{1}{2} \int_\Omega [|\,\delta u_x\,|^2/(1 + v)^{1/2}]\,\zeta^2\,dx + c_1 \int_\Omega \zeta_x^2 (1 + v)^{1/2}\,dx$$

where the constant c_1 depends only on Ω. Finally we have

$$2\int_\Omega u_x \cdot h_x\zeta^2\,dx \leqslant \int_\Omega v\zeta^2\,dx + \int_\Omega h_x{}^2\zeta^2\,dx.$$

Regrouping all these inequalities together and neglecting some unnecessary terms we find:

$$d/\,t\int_\Omega v\zeta^2\,dx \leqslant 2\epsilon\int_\Omega v\zeta_x{}^2\,dx + c_1\int_\Omega (1+v)^{1/2}\,\zeta_x^2\,dx$$
$$+\int_\Omega h_x{}^2\zeta^2\,dx + \int_\Omega v\zeta^2\,dx.$$

Integrating in t and using the fact $|\zeta| \leqslant 1$ and that $|\zeta_x|$ is bounded from above by a quantity which depends only on Ω' we are led to the following:

$$\int_\Omega v(x,s)\,\zeta^2(x)\,dx \leqslant 2\epsilon c_2(\Omega')\int_0^s\int_\Omega v\,dx\,ds + c_3\int_0^s\int_\Omega (1+v)^{1/2}\,dx\,ds$$
$$+\int_0^s\int_\Omega v\zeta^2\,dx\,ds + \int_0^s\int_\Omega h_x{}^2\zeta^2\,dx\,ds,$$
$$\text{for every } s\in[0,T]. \tag{2.7}$$

Using Lemma 2.1, we see that $\int_0^T\int_\Omega(1+v)^{1/2}\,dx\,dt$ and $\epsilon\int_0^T\int_\Omega v\,dx$ are bounded independently of ϵ so that (2.7) gives, with (1.16):

$$\int_\Omega v(x,s)\,\zeta^2(x)\,dx \leqslant d_1 + \int_0^s\int_\Omega v(x,s)\,\zeta^2(x)\,dx\,dt.$$

The quantity d_1 depends only on Ω, Ω' and on the data; Lemma 2.2 is now a straightforward consequence of the above inequality.

We shall now prove the a priori estimate in $L^\infty(Q)$ which implies that $u \in L^\infty(Q$ if the data are bounded.

LEMMA 2.3. *Suppose $u_0 \in L^\infty(\Omega)$ and h, $\Phi \in L^\infty(Q)$. Then u_ϵ is bounded in $L^\infty(Q)$ independently of ϵ.*

Proof. We show that $|u_\epsilon(\cdot,t)|$ is less than $M\,e^{\lambda t}$ where the constants M and λ will be determined later on in the course of this proof. To show that $u_\epsilon(\cdot,t) \leqslant M\,e^{\lambda t}$, we multiply (2.1) by $e^{-\lambda t}(e^{-\lambda t}u_\epsilon - M)_+$ and integrate in x. After some simple calculations we find:

$$\tfrac{1}{2}d/dt\,|(e^{-\lambda t}u_\epsilon - M)_+|^2 \leqslant \int_\Omega (e^{-\lambda t}h - \lambda M)(e^{-\lambda t}u_\epsilon - M)_+\,dx. \tag{2.8}$$

Hence we shall find $d/dt\,|(e^{-\lambda t}u_\epsilon - M)_+|^2 \leqslant 0$ and the lemma will result provided M and λ verify:

$$\begin{aligned} &| u_0 |_{L^\infty(\Omega)} \leqslant M \\ &| h(\cdot, t)| \leqslant \lambda M e^{\lambda t} \text{ a.e } x \in \Omega, \text{ a.e. } t \in [0, T] \end{aligned} \tag{2.9}$$

and a fortiori if M and λ verify:

$$| h |_{L^\infty(Q)} \leqslant \lambda M; \qquad | u_0 |_{L^\infty(\Omega)} \leqslant M, \tag{2.10}$$

i.e., if M, λ are sufficiently large.

2.3. *Passage to the Limit*, $\epsilon \to 0$

(a) Using Lemmas 2.1 and 2.2 we can extract a subsequence $(u_\epsilon)_{\epsilon\to 0}$ such that:

$$\begin{aligned} &u_\epsilon \to u \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ weak-star,} \\ \text{and} \quad & \text{ in } L^\infty(0, T; H^1_{\text{loc}}(\Omega)) \text{ weak-star.} \end{aligned} \tag{2.11}$$

From Eq. (2.1) and Lemma 2.1, $\partial u_\epsilon/\partial t$ is bounded in $L^2(0, T; H^{-1}(\Omega))$ and by the extraction of another subsequence we get:

$$\partial u_\epsilon/\partial t \to \partial u/\partial t \quad \text{weakly in} \quad L^2(0, T; H^{-1}(\Omega)). \tag{2.12}$$

We deduce from this and a result of Strauss [18] (cf. also Temam [20]) that:

$$u_\epsilon(s) \to u(s) \quad \text{weakly in} \quad L^2(\Omega),\ \forall\, s \in [0, T]. \tag{2.13}$$

Now let v be a function of $H^1(Q)$ such that

$$v = \Phi \text{ on } \Sigma = \Gamma \times [0, T]. \tag{2.14}$$

Proceeding as in Section 1.1, we multiply (2.1) by $v - u_\epsilon$ and integrate in x; this gives:

$$\begin{aligned} &(v', v - u_\epsilon) + \epsilon(u_{\epsilon x}, v_x - u_{\epsilon x}) + \left(\frac{u_{\epsilon x}}{(1 + u_{\epsilon x}^2)^{1/2}}, v_x - u_{\epsilon x}\right) \\ &\quad = (v' - u_\epsilon', v - u_\epsilon) + (h, v - u_\epsilon). \end{aligned}$$

We integrate in the t variable from 0 to s; using (1.7) and observing that the nonpositive quantity $-\epsilon\,| u_{\epsilon x} |^2$ can be omited, we get: $(0 \leqslant s \leqslant t)$:

$$\begin{aligned} &\int_0^s (v', v - u_\epsilon)\, dt + \int_0^s \int_\Omega ((1 + v_x{}^2)^{1/2} - (1 + u_{\epsilon x}^2)^{1/2})\, dx\, dt + \epsilon \int_0^s (u_{\epsilon x}, v_x)\, dt \\ &\quad \geqslant \tfrac{1}{2}\, | v(s) - u_\epsilon(s)|^2 - \tfrac{1}{2}\, | v(0) - u_0 |^2 + \int_0^s (h, v - u_\epsilon)\, dt. \end{aligned}$$

From (1.7) we also obtain, since $v = u_\epsilon = \Phi$ on $\sum$:

$$\int_0^s (v' - h, v - u_\epsilon)\, dt + \int_0^s (e(t, v(t)) - e(t, u_\epsilon(t)))\, dt$$

$$\geqslant \tfrac{1}{2} \mid v(s) - u_\epsilon(s)|^2 - \tfrac{1}{2} \mid v(0) - u_0 \mid^2. \tag{2.15}$$

Passing to the lower limit in (2.15), we obtain using (2.11), (2.13) and the l.s.c. of $\mathscr{E}(s, \cdot)$ (cf. Sect. 1.1):

$$\int_0^s (v' - h, v - u)\, dt + \int_0^s (e(t, v(t)) - e(t, u(t)))\, dt$$

$$\geqslant \tfrac{1}{2} \mid v(s) - u(s)|^2 - \tfrac{1}{2} \mid v(0) - u_0 \mid^2; \forall s \in [0, T]. \tag{2.16}$$

This is precisely (1.21) for a v satisfying (2.14).

(b) In a second step we shall extend the class of functions v for which (2.16) is true. Let us consider a function v in $H^1(Q)$ that does not necessarily verify (2.14). For every $\alpha > 0$ sufficiently small, we set

$$\delta_\alpha(x) = \min(d(x)/\alpha, 1)$$

where $d(x) =$ distance from x to Γ, and we set $v_\alpha = \delta_\alpha u + (1 - \delta_\alpha)\Phi$.

It is obvious that $v_\alpha \in H^1(Q)$, $v_\alpha = \Phi$ on $\sum$ and therefore the inequality (2.16) is true for $v = v_\alpha$. As $\alpha \to 0$, $v_\alpha \to v$ in $L^2(Q)$ for the strong topology; it is readily seen that $v_\alpha(s) = v_\alpha(\cdot, s) \to v(s)$ in $L^2(\Omega)$, $\forall s \in]0, T[$. This is sufficient to pass to the lower limit as $\alpha \to 0$ in the inequalities (2.16) corresponding to v_α. We thus obtain (2.16) for any element in $H^1(Q)$ which does not satisfy necessarily (2.14).

(c) To end this proof, we choose $v \in L^2(Q)$ such that $v' \in L^2(Q)$ and $v_x \in (L^1(Q))^n$. There exists a sequence $(v_m)_{m \in N}$ of functions in $H^1(Q)$ such that $v_{mx} \to_{m \to +\infty} v_x$ in $(L^1(Q))^n$. We can write the inequalities (2.16) for each v_m; passing to the lower limit we obtain (2.16) for this v.

2.4. *Proof of* (1.19) *and* (1.20)

To establish the continuity of $t \to u(\cdot, t)$, we shall consider, for $\eta > 0$ the solution u_η of the vector-valued ordinary differential equation:

$$\eta u_\eta' + u_\eta = u \qquad \text{for} \quad 0 < t < T; \qquad u_\eta(0) = u_0 . \tag{2.17}$$

We can take $v = u_\eta$ in (1.21) as we have shown above; we deduce from this that:

$$\tfrac{1}{2} \mid u_\eta(s) - u(s)|^2 \leqslant \int_0^s (e(t, u_\eta(t)) - e(t, u(t)))\, dt - \int_0^s (h, u_\eta - u)\, dt.$$

Using (1.13) and (1.7) written in the following way:

$$(1 + u_{\eta x}^2)^{1/2} - (1 + u_x{}^2)^{1/2} \leqslant u_{\eta x} \cdot (u_x - u_{\eta x})/(1 + u_{\eta x}^2)^{1/2}$$

and observing that $| u_{\eta x}/(1 + u_{\eta x}^2)^{1/2} | \leqslant 1$ we obtain

$$\tfrac{1}{2} \mid u_\eta(s) - u(s)|^2 \leqslant \int_Q \mid u_{\eta x} - u_x \mid dx\, dt + \int_\Sigma \mid u_\eta - u \mid d\Gamma\, dt + \int_0^T \mid h \mid \mid u_\eta - u \mid dt. \tag{2.18}$$

It will be checked hereafter that $u_\eta \to u$ in $L^2(Q)$ and in $L^1(0, T; W^{1,1}(\Omega))$ as $\eta \to 0$. It follows then from (2.18) and from Galliardo's trace theorem [6], that $u_\eta(\cdot, s) \to u(\cdot, s)$ in $L^2(\Omega)$ uniformly with respect to the s variable ($s \in [0, T]$). This implies (1.19) and (1.20) since, for each η:

$$u_\eta \in \mathscr{C}([0, T]; L^2(\Omega)) \qquad \text{and} \qquad u_\eta(0) = u_0 .$$

Let us show, to conclude this proof, that

$$u_\eta \to u \textit{ in } L^2(Q) \textit{ and in } L^1(0, T; W^{1,1}(\Omega)) \textit{ as } \eta \to 0. \tag{2.19}$$

It suffices to observe that:

$$u_\eta(t) = u_0 \exp(-t/\eta) + (u * \rho_\eta)(t) \qquad (t > 0)$$

where the definition of u has been extended by setting $u = 0$ for $t < 0$ and where:

$$\rho_\eta(t) = (1/\eta)\rho(t/\eta);\ \rho(t) = \exp(-t).$$

It is checked in a standard way that if $u \in L^q(0, T; X)$ (where $1 \leqslant q < +\infty$ and X is a Banach space), then $u * \rho_\eta \to u$ in $L^q(0, T; X)$ as $\eta \to 0$. On the other hand, if $u_0 \in X$, $u_0 \exp(-t/\eta) \to 0$ in $L^q(0, T; X)$ as $\eta \to 0$; (2.19) is thus proved.

2.5. *Proof of the Uniqueness*

Let u_1 and u_2 be two solutions of (1.18)–(1.21) and let $w = (u_1 + u_2)/2$. We can define w_η ($\eta > 0$) as in (2.17):

$$\eta w_\eta' + w_\eta = w \quad \text{for} \quad 0 < t < T;\ w_\eta(0) = u_0 .$$

We can take $v = w_\eta$ in the inequalities (1.15) where u is successively replaced by u_1 and u_2; then adding the two inequalities, we find:

$$-2 \int_0^s (h, w_\eta - w)\, dt + \int_0^s (2e(t, w_\eta(t)) - e(t, u_1(t)) - e(t, u_2(t)))\, dt \geqslant \tfrac{1}{2} \mid w_\eta(s) - u_1(s)|^2 + \tfrac{1}{2} \mid w_\eta(s) - u_2(s)|, \tag{2.20}$$

where the term

$$2\int_0^s (w'_\eta, w_\eta - w)\,dt = -2\int_0^s |w'|^2\,dt \leqslant 0$$

has been deleted. Convergence properties analogous to (2.19) will enable us to pass to the limit in (2.20) as $\eta \searrow 0$. Let us first point out that, since $w \in \mathscr{C}([0, T], L^2(\Omega))$ we have, in addition to (2.19):

$$w_\eta(s) \underset{\eta\to 0}{\to} w(s) \text{ in } L^2(\Omega), \qquad \forall s \in]0, T[. \tag{2.21}$$

Passing to the limit we get:

$$\int_0^s (2e(t, w(t)) - e(t, u_1(t)) - e(t, u_2(t)))\,dt \geqslant |u_1(s) - u_2(s)|^2\ \forall s \in [0, T] \tag{2.22}$$

Replacing w by its value $(u_1 + u_2)/2$, we obtain in particular:

$$2\mathscr{E}(s, (u_1 + u_2)/2) \geqslant \mathscr{E}(s, u_1) + \mathscr{E}(s, u_2). \tag{2.23}$$

The opposite inequality is a consequence of the convexity of the mapping: $v \to \mathscr{E}(s, v)$. Hence (2.23) is an equality and this implies with (2.22):

$$|u_1(s) - u_2(s)|^2 \leqslant 0,\ \forall s \in [0, T];$$

whence $u_1 = u_2$. This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

3.1. *An Additional a priori Estimate*

To obtain (1.26) it is sufficient to show that the sequence u_ϵ, given by (2.1)–(2.3) verifies:

$$u'_\epsilon \textit{ remains bounded in } L^2(Q) \textit{ as } \epsilon \to 0. \tag{3.1}$$

We multiply (2.1) by $u'_\epsilon - \Phi'$ and integrate in x and t on $\Omega \times [0, s]$; as $u'_\epsilon - \Phi' = 0$ on $\Gamma \times [0, T]$, this gives:

$$\int_0^s (u'_\epsilon, u'_\epsilon - \Phi')\,dt + \epsilon\int_0^s (u_{\epsilon x}, u'_{\epsilon x} - \Phi'_x)\,dt + \int_0^s\int_\Omega \frac{u_{\epsilon x}\cdot(u'_{\epsilon x} - \Phi'_x)}{(1 + u^2_{\epsilon x})^{1/2}}\,dx\,dt$$
$$= \int_0^s (h, u'_\epsilon - \Phi')\,dt$$

$$\int_0^s |u'_\epsilon|^2\,dt + \frac{\epsilon}{2}|u_{\epsilon x}(s)|^2 + \int_\Omega (1+u_{\epsilon x}^2(s))^{1/2}\,dx$$
$$= \frac{\epsilon}{2}|u_{0x}|^2 + \int_\Omega (1+u_{0x}^2)^{1/2}\,dx$$
$$+ \int_0^s ((u'_\epsilon, \Phi') + \epsilon(u_{\epsilon x}, \Phi'_x))\,dt + \int_0^s\int_\Omega \frac{u_{\epsilon x}\cdot\Phi'_x}{(1+u_{\epsilon x}^2)^{1/2}}\,dx\,dt$$
$$+ \int_0^s (h, u'_\epsilon - \Phi')\,dt \leqslant \frac{\epsilon}{2}|u_{0x}|^2 + \int_\Omega (1+u_{0x}^2)^{1/2}\,dx$$
$$+ \tfrac{1}{2}\int_0^s (|u'_\epsilon|^2 + \epsilon|u_{\epsilon x}|^2)\,dt + \tfrac{1}{2}\int_0^s (|\Phi'|^2 + \epsilon|\Phi'_x|^2 + 2|h|^2)\,dt$$
$$+ \tfrac{1}{4}\int_0^s |u'_\epsilon|^2\,dt - \int_0^s (h, \Phi')\,dt.$$

Finally,

$$\int_0^s |u'_\epsilon|^2\,dt + \int_\Omega (1+u_{\epsilon x}^2)^{1/2}\,dx \leqslant c'_2 \tag{3.2}$$

where the constant c'_2 is independent of ϵ and s and depends only on the data u_0, h, Φ; (3.1) is a consequence of (3.2). From (2.2) and (3.2), $e(s, u_2(s)) \leqslant c'_2$ for all $s \in]0, T[$. As we have already mentionned $u_\epsilon(s) \to u(s)$ weakly in $L^2(\Omega)$ for every s. As $e(s, \cdot)$ is convex and l.s.c. on $L^2(\Omega)$ this implies:

$$e(s, u(s)) \leqslant \varliminf_{\epsilon\to 0} e(s, u_\epsilon(s)) \leqslant c'_2 \tag{3.3}$$

and therefore $u \in L^\infty(0, T; W^{1,1}(\Omega))$; whence (1.25).

3.2. *Further Properties of u*

Suppose that $w \in L^2(Q)$ and that $w' \in L^2(Q)$, $w_x \in (L^1(Q))^n$. It follows from (1.26) that we can set $v = u + \lambda w$ ($\lambda > 0$) in (1.21); after dividing by λ we get:

$$\int_0^s (u' + \lambda w - h, w)\,dt + \int_0^s \frac{e(t, u+\lambda w) - e(t, u)}{\lambda}\,dt \geqslant \tfrac{1}{2}\lambda|w(s)|^2 - \tfrac{1}{2}\lambda|w(0)|^2.$$

Passing to the limit as $\lambda \searrow 0$ this gives:

$$\int_0^s (u' - h, w)\,dt + \lim_{\lambda\searrow 0}[\mathscr{E}(s, u+\lambda w) - \mathscr{E}(s, u)]/\lambda \geqslant 0. \tag{3.4}$$

The functional $\mathscr{E}(s, \cdot)$ being convex we have:

$$\mathscr{E}(s, u+w) - \mathscr{E}(s, u) \geqslant \lim_{\lambda\searrow 0}[\mathscr{E}(s, u+\lambda w) - \mathscr{E}(s, u)]/\lambda$$

and hence:

$$\int_0^s (u' - h, w)\, dt + \mathscr{E}(s, u + w) - \mathscr{E}(s, u) \geqslant 0 \qquad \forall s \in [0, T]. \tag{3.5}$$

Since the set of all w's such that

$$w \in L^2(Q),\ w' \in L^2(Q),\ w_x \in (L^1(Q))^n \tag{3.6}$$

is dense in $L^2(Q)$, it is clear after a passage to the limit, that (3.5) is still true for any w in $L^2(Q)$. Then, this inequality means that

$$u' - h \in \partial\mathscr{E}(s, u) \tag{3.7}$$

where the subdifferential has been taken with respect to the variable u in $L^2(0, s; L^2(\Omega))$, $0 \leqslant s \leqslant T$. The regularity of the solution u (cf. (1.18)–(1.26)) will enable us to show, by making (3.7) more explicit, that u is solution, in the sense of distributions, of the nonlinear differential equation (0.1) in Q. To achieve this we shall write (3.4) with appropriate classes of test functions w.

Let us, first $w \in \mathscr{D}(Q)$ a $\mathscr{C}^\infty$ function with compact support in Q. Then, with (1.13):

$$\frac{\mathscr{E}(s, u + \lambda w) - \mathscr{E}(s, u)}{\lambda} = \int_0^s \int_\Omega \frac{(1 + |u_x + \lambda w_x|^2)^{1/2} - (1 + u_x{}^2)^{1/2}}{\lambda}\, dx\, dt$$

and, using Lebesgue's theorem

$$\lim_{\lambda \searrow 0}[\mathscr{E}(s, u + \lambda w) - \mathscr{E}(s, u)]/\lambda = \int_0^s \int_\Omega u_x \cdot w_x/(1 + u_x{}^2)^{1/2}\, dx\, dt.$$

Thus, (3.4) gives:

$$\int_0^s \int_\Omega (\partial u/\partial t\, w + (u_x \cdot w_x/(1 + u_x{}^2)^{1/2}) - hw)\, dx\, dt \geqslant 0 \tag{3.8}$$

and, as we can as well replace w by $-w$ we have, in fact, an equality. Equation (1.27) follows, in the distributional sense in $Q = \Omega \times]0, T[$

$$\partial u/\partial t + Au = h. \tag{3.9}$$

Let us now write (3.4) with a function w in $\mathscr{C}^\infty(\bar{Q})$; we get:

$$\frac{\mathscr{E}(s, u + \lambda w) - \mathscr{E}(s, u)}{\lambda} = \int_0^s \int_\Omega \frac{(1 + |u_x + \lambda w_x|^2)^{1/2} - (1 + u_x{}^2)^{1/2}}{\lambda}\, dx\, dt$$
$$+ \int_0^s \int_\Gamma \frac{|u + \lambda w - \Phi| - |u - \Phi|}{\lambda}\, d\Gamma\, dt.$$

As $\lambda \searrow 0$,

$$\int_0^s \int_\Omega \frac{(1 + |u_x + \lambda w_x|^2)^{1/2} - (1 + u_x^2)^{1/2}}{\lambda} \, dx\, dt \to \int_0^s \int_\Omega \frac{u_x \cdot w_x}{(1 + u_x^2)^{1/2}} \, dx\, dt.$$

It follows from (1.16) and (1.26) that we can multiply (3.9) by w and integrate in x and t; using the generalized Green formula, (cf. [7]) we obtain:

$$\int_0^s \int_\Omega \frac{u_x \cdot w_x}{(1 + u_x^2)^{1/2}} \, dx\, dt = \int_0^s (h - u', w)\, dt + \int_0^s \int_\Gamma \frac{u_x \cdot \nu}{(1 + u_x^2)^{1/2}} \, d\Gamma\, dt. \tag{3.10}$$

Moreover, for $\lambda \to 0$,

$$\int_0^s \int_\Gamma \frac{|u + \lambda w - \Phi| - |u - \Phi|}{\lambda} \, d\Gamma\, dt \to \int_0^s \int_\Gamma \sigma w \, d\Gamma\, dt$$

where:

$$\sigma \in \operatorname{Sgn}(u - \Phi). \tag{3.11}$$

(In fact, $\sigma = \operatorname{Sgn}(u - \Phi)$ when $u \neq \Phi$; $\sigma = \operatorname{Sgn}(w)$ when $u = \Phi$). Thus we arrive to:

$$\lim_{\lambda \to 0} \frac{\mathscr{E}(s, u + \lambda w) - \mathscr{E}(s, u)}{\lambda}$$
$$= \int_0^s (h - u', w)\, dt + \int_0^s \int_\Gamma \left(\sigma + \frac{u_x \cdot \nu}{(1 + u_x^2)^{1/2}}\right) w \, d\Gamma$$

and inserting this in (3.4) we obtain:

$$\int_0^s \int_\Gamma \left(\sigma + \frac{u_x \cdot \nu}{(1 + u_x^2)^{1/2}}\right) w \, d\Gamma\, dt \geqslant 0. \tag{3.12}$$

Again, this inequality which is true for w and $-w$, is in fact an equality; this implies:

$$\sigma + u_x \cdot \nu/(1 + u_x^2)^{1/2} = 0 \tag{3.13}$$

that is to say:

$$-\frac{u_x \cdot \nu}{(1 + u_x^2)^{1/2}} \in \operatorname{Sgn}(u - \Phi).$$

Remark 3.1. Suppose that $\Phi(x, t) = \Phi(x)$ is independent of t. Then the solution u of (1.18)–(1.21) given by Theorem 1.1 is also the unique solution of the evolution equation associated with the maximal monotone operator $\partial e(\Phi, \cdot)$:

$$du/dt + \partial e(\Phi, u) \ni h;\ u(0) = u_0. \tag{3.14}$$

This setting of the problem (0.1)–(0.4) has been used in Temam [21] to obtain the existence of a unique solution using the nonlinear semigroup theory (cf. Brezis [3]). Moreover the domain of the operator $\partial e(\Phi, \cdot)$ is determined in [21] and $u(t) \in D(\partial e(\Phi, u))$ for a.e. t.[4] However the semigroup approach does not allow us to derive the differential equation (1.27). In the sequel it shall be convenient to use (3.14) instead of the equivalent relation (1.21), and this will also enable us to apply nonlinear semigroup theory to the pseudosolution of (1.21).

4. Behavior of Solutions on Γ and as $t \to +\infty$

The functions Φ which will appear in this paragraph will be independent of the variable t and will satisfy, unless otherwise stated, $\Phi \in H^1(\Omega) \cap L^\infty(\Omega)$. Moreover we shall only consider the case of a zero right-hand side h in (1.27); thus, using Remark 3.1, we may write problem (1.21) in the *equivalent* form:

$$\begin{aligned} du/dt + \partial e(\Phi, u) &\ni 0 \\ u(0) = u_0 &\in H^1(\Omega) \cap L^\infty(\Omega). \end{aligned} \tag{4.1}$$

Regularity assumptions on Φ and u_0 made above ensure that the results of Theorems 1.1 and 1.2 are applicable to u. We shall now prove that the pseudosolution u of the problem (0.1)–(0.4) (i.e. the solution of (4.1) or equivalently (1.21)) satisfies the boundary condition (0.3) on the parts of Γ of positive mean curvature provided the initial data does also verify (0.3). To end this paragraph, we shall prove that the pseudosolution $u(x, t)$ of (0.1)–(0.4) converges to a limit $\tilde{u}(x)$ as $t \to \infty$ in $L^1(\Omega)$, and that $\tilde{u}$ is a pseudosolution of the stationary problem (1.1)–(1.2). Our main tools will be a weak form of the maximum principle suitable to Eq. (4.1) and a result of Bruck [4] concerning the asymptotic behavior of contraction semigroups.

4.1. *A weakened form of the "Maximum Principle"*

Lemma 4.1. *Let u_1 and u_2 be two solutions of*

$$\begin{aligned} du/dt + \partial e(\Phi_i, u) &\ni 0 \\ u_i(0) = u_{i,0} &\in H^1(\Omega) \cap L^\infty(\Omega), \qquad i = 1, 2, \end{aligned} \tag{4.2}$$

where $\Phi_i \in H^1(\Omega) \cap L^\infty(\Omega)$ $(i = 1, 2)$. Suppose that $u_{1,0} \geqslant u_{2,0}$ and $\Phi_1 \geqslant \Phi_2$. Then we have $u_1 \geqslant u_2$ in $\Omega \times]0, T[$.

[4] Here $D(\partial e(\Phi, \cdot))$ denotes the domain of operator ∂e (cf. [3]).

Proof. Equations (1.27) for u_1 and u_2 are true almost everywhere in $\Omega \times$ $]0, T[$; by substracting the two equations corresponding to u_1 and u_2 we get:

$$\frac{\partial}{\partial t}(u_2 - u_1) - \operatorname{div}\left(\frac{u_{2x}}{(1 + u_{2x}^2)^{1/2}} - \frac{u_{1x}}{(1 + u_{1x}^2)^{1/2}}\right) = 0 \text{ in } \Omega. \tag{4.3}$$

Using (1.26), we have almost everywhere in t:

$$\operatorname{div}\left(\frac{u_{ix}}{(1 + u_{ix}^2)^{1/2}}\right) \in L^2(\Omega), \qquad i = 1, 2;$$

hence we can multiply (4.3) by $(u_2 - u_1)_+$ and integrate by parts in Ω using the generalized Green formula (cf. Jouron [7], Temam [21]). We write $\Omega_+(t) = \{x \mid x \in \Omega, u_2(t, x) \geqslant u_1(t, x)\}$ and we get, using a result of G. Stampacchia [17]:

$$\frac{1}{2}\int_\Omega \frac{d}{dt}((u_2 - u_1)_+)^2\, dx + \int_{\Omega_+(t)} \left(\frac{u_{2x}}{(1 + u_{2x}^2)^{1/2}} - \frac{u_{1x}}{(1 + u_{1x}^2)^{1/2}}\right) \cdot (u_{2x} - u_{1x})\, dx$$

$$- \int_\Gamma (u_2 - u_1)_+ \left(\frac{\partial u_2}{\partial \nu_A} - \frac{\partial u_1}{\partial \nu_A}\right) d\Gamma = I_1 + I_2 + I_3 = 0, \tag{4.4}$$

(In the above formula we denote by $\partial u/\partial \nu_A$ the quantity $u_x \cdot \nu/(1 + u_x{}^2)^{1/2}$, where ν stands for the unit vector, normal to Γ and pointing outward with respect to Ω). By a monotonicity argument, the integral I_2 is non-negative; let us examine the integral I_3 : taking (1.28) into account, we have:

$$-u_{ix} \cdot \nu/(1 + u_{ix}^2)^{1/2} |_\Gamma = -\partial u_i/\partial \nu_A \in \operatorname{Sgn}(u_i - \Phi_i) \qquad i = 1, 2.$$

On the set where $u_2 > \Phi_2$, $\partial u_2/\partial \nu_A = -1$ and the integrand $-(u_2 - u_1)_+ \cdot (\partial u_2/\partial \nu_A - \partial u_1/\partial \nu_A)$ is nonnegative; when $u_2 < u_1$ this integrand is zero. It remains simply to settle the case $u_1 \leqslant u_2 \leqslant \Phi_2 \leqslant \Phi_1$. Two possibilities have to be considered: either $u_1 = \Phi_1$ and the integrand is zero since $(u_2 - u_1)_+ = 0$, or $u_1 < \Phi_1$ and then $\partial u_1/\partial \nu_A = 1$ which implies again that the integrand is nonnegative. Hence $I_3 \geqslant 0$. It now follows from (4.4) that I_1 is nonpositive:

$$d/dt \int_\Omega (u_2 - u_1)_+^2\, dx \leqslant 0.$$

The initial condition $u_{1,0} \geqslant u_{2,0}$ gives:

$$\int_\Omega (u_{2,0} - u_{1,0})_+^2\, dx = 0$$

so that for any t:

$$\int_\Omega (u_2 - u_1)_+^2 \, dx \leqslant 0;$$

this ends the proof.

The above proof remains true in the case of super- and subsolutions of (4.1) which we define in the following manner:

DEFINITION. A function u, satisfying (1.17) and (1.24) is called a supersolution (resp., a subsolution) of (4.1) if there exists a nonnegative (resp., nonpositive) function $h \in L^2(Q) \cap L^2(0, T; H^1_{\mathrm{loc}}(\Omega))$ such that

$$du/dt + \partial e(\Phi, u) \ni h. \tag{4.5}$$

Remark 4.1. A so defined super- (sub-) solution u belongs to $L^\infty(0, T; W^{1,1}(\Omega))$ and $u' = \partial u/\partial t \in L^2(Q)$ so that it satisfies Eq. (1.27) in the distributional sense. It must be emphasized that we consider here supersolutions which are in fact pseudosolutions for suitable right-hand sides of Eq. (4.1).

LEMMA 4.2. *Let u_1 be a supersolution of* (4.1) *and u_2 be a subsolution. Suppose that $u_1(0) \geqslant u_2(0)$ and that $\Phi_1 \geqslant \Phi_2$; then we have:*

$$u_1 \geqslant u_2 \quad in \quad \underset{\sim}{Q} = \Omega \times]0, T[.$$

4.2. *Application to Problem* (4.1)

We shall use as super- (sub-) solutions of our problem super- (sub-) solutions of the stationary nonparametric minimal surface problem. As a matter of fact, we shall use local barrier-functions (cf. [9]) which are not defined over the whole open set Ω; hence the proof of Theorem 4.1 will be an extension of the proof of Lemmas 4.1 and 4.2 which will enable us to use functions which are not defined averywhere.

THEOREM 4.1. *Let γ be a subset of $\Gamma = \partial\Omega$ of class $\mathscr{C}^3$ with nonnegative mean curvature. Suppose that $\Phi \in H^1(\Omega) \cap L^\infty(\Omega)$, that Φ is continuous on γ and that the initial data u_0 is continuous on γ and satisfies:*

$$u_0 = \Phi \quad \text{on} \quad \gamma,$$

then the solution u of (4.1) *satisfies $u = \Phi$ on γ for any $t \in [0, T]$ and is continuous on*[5] γ

[5] "f is continuous on γ" means that f has an extension to $\Omega \cup \gamma$ which is continuous at every point of γ.

Proof. (a) Let $M = \max(|\Phi|_{L^\infty(\Gamma)}, |u_0|_{L^\infty(\Omega)})$ and let x_0 be an arbitrary point of γ. For a given $\epsilon > 0$ it is possible by [9] to construct two neighborhoods V_2 and V_1 of x_0 in R^n with $V_2 \Subset V_1$ and a function $x \to \theta(x)$ such that:

The function θ is a stationary supersolution of (0.1) defined over $V_1 \cap \Omega$ which belongs to $\mathscr{C}^2(\overline{V_1 \cap \Omega})$; (4.6)

On $V_1 \cap \partial\Omega : \theta \geqslant \Phi = u_0$ and $\theta(x_0) \leqslant \Phi(x_0) + \epsilon$ (4.7)

On $(V_1 - V_2) \cap \Omega : \theta > M$ and (4.8)
On $V_1 : \theta \geqslant u_0$.

It is a straightforward consequence of Lemma 4.1 that the solution u of (4.1) is such that:

$$|u| \leqslant M \quad \text{on} \quad \Omega \times]0, T[. \tag{4.9}$$

(b) We now go back to the proof of Lemma 4.2 after extending θ over the whole domain Ω as a $\mathscr{C}^2$ function $\bar{\theta}$ such that $\bar{\theta}|_{\Omega - V_1} > M$. As above we set $\Omega_+(t) = \{x \in \Omega \mid u(x, t) \geqslant \bar{\theta}(x)\}$; we observe that $\Omega_+(t) \subset V_2 \Subset V_1$ (from (4.8) and (4.9)). Hence, in $V_1 \cap \Omega$ we have:

$$d/dt\,(u - \theta) - \operatorname{div}\left(\frac{u_x}{(1 + u_x^2)^{1/2}} - \frac{\theta_x}{(1 + \theta_x^2)^{1/2}}\right) \leqslant 0; \tag{4.10}$$

since $V_2 \Subset V_1$, we can multiply this equation by $(u - \theta)_+$ whose support lies in $V_2 \cap \Omega$, and integrate by parts in Ω using the generalized Green formula. Here we have also used the fact that the generalized Green formula can be applied to the pseudosolution u as seen above and to the $\mathscr{C}^2$ extension of $\theta : \bar{\theta}$. We end this step of the proof following the lines of the proof of Lemma 4.1, and noticing that the only nonzero boundary terms are those on $\partial\Omega \cap V_2$ for which the above discussion is still valid.

(c) We conclude this proof by letting $\epsilon \to 0$. Indeed, we have shown above that for every $\epsilon > 0$ there exists a neighborhood V_2 of x_0 and a continuous function θ such that

$$\begin{aligned} &\text{(i)} \quad \theta(x_0) \leqslant \Phi(x_0) + \epsilon \\ &\text{(ii)} \quad \theta(x) \geqslant u(x, t) \text{ a.e. in } V_2 \times]0, T[. \end{aligned} \tag{4.11}$$

We can obtain a similar minoration using subsolutions.

$$\begin{aligned} &\text{(i)} \quad \psi(x) \leqslant u(x, t) \leqslant \theta(x) \text{ a.e. in } V_2 \times]0, T[\\ &\text{(ii)} \quad \theta(x_0) - \epsilon \leqslant \Phi(x_0) \leqslant \psi(x_0) + \epsilon. \end{aligned} \tag{4.12}$$

This implies, ϵ being arbitrarily small that $u(\cdot, t)$ is continuous at x_0 and $u(x_0, t) = \Phi(x_0)$.

In the above proof, we made the assumption that γ is of class $\mathscr{C}^3$; this was required only for the construction of the function θ. However, if γ is only the graph of a lipschitz function from R^{n-1} to R, it is still possible to construct a function θ with similar properties, provided the "exterior mean curvature" of $\partial\Omega$ is strictly positive at each point of γ. This implies the utilization of an approximation of the boundary and has been made explicite in [9]. This remark enables us to state Theorem 4.1 bis whose proof is identical to the proof of Theorem 4.1; the only difference lies in the construction of function θ.

DEFINITION. Suppose that $\partial\Omega$ is locally the graph of a lipschitz mapping from R^{n-1} to R and let x_0 be a point of $\partial\Omega$. We define *the exterior mean curvature* $\tilde{K}$ *of* $\partial\Omega$ *at* x_0 as the (not necessarily finite) least upper bound of the mean curvatures of surfaces σ, passing through x_0, which are exterior to Ω and of class $\mathscr{C}^3$. If there exists no such surface we take $\tilde{K} = -\infty$.

THEOREM 4.1 bis. *We make the same hypotheses as for Theorem* 4.1 *except for* γ: *we suppose only that* γ *is a part of* $\partial\Omega$ *which is locally the graph of a Lipschitz mapping of* R^{n-1} *into R and whose exterior mean curvature is strictly positive. Then the solution u of* (4.1) *is continuous on* γ *and satisfies* $u = \Phi$ *on* γ.

Remark 4.2. If $\partial\Omega$ has an everywhere positive mean curvature· and the boundary and initial data satisfy the requirements in Theorem 4.1 (or 4.1 bis), the solution of (4.1) and (1.21) also satisfies the boundary condition (0.3) $u = \Phi$ on $\partial\Omega \times]0, T[$. In this case we have obtained a *weak solution* of the original problem (0.1)–(0.3)–(0.4).

Remark 4.3. Suppose that we take as initial data u_0, in (4.1), the pseudo-solution of the steady-state problem (1.1)–(1.2) associated to the same boundary data Φ: it follows from Proposition 1.1 and from the application of the maximum principle that u_0 satisfies (1.17) and can therefore be taken as initial data. Now the uniqueness result in Theorem 1.1 implies that the pseudo-solution of (0.1)–(0.3)–(0.4) is in fact a steady-state solution equal to the pseudo-solution of the stationary problem (1.1)–(1.2):

$$u_0(\cdot) = u(t, \cdot).$$

Remark 4.4. Suppose that $\partial\Omega$ is of class $\mathscr{C}^3$ and that at $x_0 \in \partial\Omega$ its mean curvature is strictly negative. It is then possible to construct a regular function Φ (in $\mathscr{C}^\infty(\bar{\Omega})$) and a function u_0 satisfying (1.17) such that the unique pseudo-solution of (0.1)–(0.4) does not satisfy $u = \Phi$ on $]0, T[\times v(x_0)$ where $v(x_0)$ is a neighborhood of x_0 on $\partial\Omega$. This can be done in the following way: construct Φ such that (cf. Serrin [16]) the stationary problem (1.1)–(1.2):

$$\begin{aligned} Au &= 0 \text{ in } \Omega \\ u &= \Phi \text{ on } \partial\Omega \end{aligned} \tag{4.13}$$

has no solution. There exists in fact a pseudosolution $\tilde{u}$ of (4.13) which does not satisfy the boundary condition in a neighborhood $v(x_0)$ of x_0 (cf. Temam [19], Lichnewsky [10, 9]). Then take Φ as boundary data and $\tilde{u}$ as initial data. It follows from Remark 4.3 that $u(x, t) = \tilde{u}(x)$ is the pseudosolution of our problem and does not satisfy the boundary condition.

4.3. *Behavior of the Solutions as $t \to +\infty$*

We shall now prove that $u(\cdot, t)$ converges strongly to a limit $\bar{u}$ as $t \to +\infty$, $\bar{u}$ being a pseudosolution of the steady-state problem (1.1)–(1.2), more precisely *the pseudosolution* if such pseudosolution is unique, and *a specific one* in the case of nonuniqueness.

In fact, the weak convergence to $\bar{u}$ is a consequence of a result of Bruck [4, Theorems 1 and 3(a)] applied to formulation (4.1) of our problem. Let us first recall the semigroup-theoretic arguments we shall need, in the somewhat restricted setting (with respect to [4]), where $\mathscr{A}$ is a maximal monotone operator in the Hilbert space H (cf. Brezis [3]). We shall denote by $(\cdot, \cdot)$ and $\|\cdot\|$ the inner product and norm in H.

PROPOSITION 4.1 (Bruck [4]). *Let $\mathscr{A}$ be the subdifferential $\partial\chi$ of a proper l.s.c. function $\chi : H \to]-\infty, +\infty]$ which assumes a minimum in H.*

Then if $x : [0, \infty[\to H$ is absolutely continuous and satisfies[4]

$$x(t) \in D(\mathscr{A}) \qquad \textit{for all} \quad t \geqslant 0 \tag{4.14}$$

$$dx/dt + \mathscr{A}(x) \ni 0 \quad \textit{almost everywhere} \tag{4.15}$$

$$| dx/dt |_H \in L^\infty(0, \infty). \tag{4.16}$$

Then $x(t)$ has a weak limit $\bar{x}$ in H as $t \to \infty$ and $\bar{x}$ belongs to $\mathscr{A}^{-1}(0)$.

We take $H = L^2(\Omega)$, $\mathscr{A} = \partial e(\Phi, \cdot)$; since (4.16) is a consequence of Brezis [3, Theorem 3.1] we simply have to check that the functional $e(\Phi, \cdot)$ attains its minimum in $L^2(\Omega)$. (But this functional is studied on $L^1(\Omega)$ in [2, 15] and on $W^{1,1}(\Omega)$ in [5]).

LEMMA 4.3. *Suppose $\Phi \in L^\infty(\Omega) \cap W^{1,1}(\Omega)$. Then the functional $e(\Phi, \cdot)$ attains its minimum on $L^2(\Omega)$.*

Proof. Let $(u_m)_{m\in N}$ be a minimizing sequence of $e(\Phi, \cdot)$, we can suppose, without loss of generality, that, for all $m : e(\Phi, u_m) < +\infty$; hence, cf. [21], $u_m \in B.V.(\Omega)$. Now, the functional $e(\Phi, \cdot)$ is continuous over its domain $BV(\Omega)$ (cf. [5, p. 13] for the BV(Ω) topology), so that we can find a minimizing sequence $(v_m)_{m\in N}$, $v_m \in W^{1,1}(\Omega)$ such that

$$1/m + e(\Phi, u_m) \geqslant e(\Phi, v_m).$$

Denote by Ξ the function form R into R such that

$$\begin{aligned}\Xi(x) &= -\mid \Phi \mid_\infty \quad \text{if} \quad x < -\mid \Phi \mid_\infty \\ &= x \quad \text{if} \quad \mid x \mid \leqslant \mid \Phi \mid_\infty \\ &= +\mid \Phi \mid_\infty \quad \text{if} \quad x > \mid \Phi \mid_\infty .\end{aligned}$$

The sequence $w_m = \Xi \circ v_m$ is still a minimizing sequence of $e(\Phi, \cdot)$ (cf. (1.13)) and each w_m belongs to $W^{1,1}(\Omega)$. Moreover this sequence is bounded in $L^\infty(\Omega) \cap W^{1,1}(\Omega)$ and hence relatively compact in $L^1(\Omega)$ using Sobolev's theorem. We may extract a subsequence which converges strongly in $L^1(\Omega)$ to $\bar{u} \in L^1(\Omega) \cap L^\infty(\Omega)$. From Section 1.2 : $e(\Phi, \bar{u}) = \operatorname{Inf}_{u \in L^2(\Omega)} e(\Phi, u)$. As $\bar{u} \in L^2(\Omega)$ the lemma is proved.

Our main result in this section is the following

THEOREM 4.2. *Suppose that* $u_0 \in H^1(\Omega)$ *and that* $\Phi \in H^1(\Omega) \cap L^\infty(\Omega)$. *Then the pseudosolution u of* (0.1)–(0.4) (*i.e. the solution of* (4.1)) *converges as* $t \to +\infty$ *to some limit* $\bar{u}$ *in the following sense*:

$$u(t, \cdot) \to \bar{u} \text{ strongly in } L^1(\Omega) \text{ and weakly in } L^2(\Omega).$$

Moreover $\bar{u}$ *is a solution of the steady-state problem*

$$\partial e(\Phi, \bar{u}) \ni 0$$

i.e., a pseudosolution of (1.1)–(1.2)).

Proof. The existence of a weak limit $\bar{u}$ in $L^2(\Omega)$ of $u(t, \cdot)$ is a consequence of Propositions 4.1 and 4.2; these results also imply that $\partial e(\Phi, \bar{u}) \ni 0$, which means that $\bar{u}$ is a pseudosolution of (1.1)–(1.2) (cf. Temam [21]). We now prove the strong convergence in $L^1(\Omega)$. From Eq. (4.1) we deduce that $e(\Phi, u(t)) \leqslant e(\Phi, u_0)$ for all $t > 0$, and hence $u(t)$ is bounded in $BV(\Omega)$ independent of t. Since $BV(\Omega)$ is compactly imbedded in $L^1(\Omega)$ (cf. Miranda [14]) the result follows.

Remark 4.5. After this work was completed we have been informed that a similar problem has been studied on a geometrical point of view by K. A. Brakke [22].

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