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Exponential sums over products and their L_1-norm

By

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Abstract. We determine the order of magnitude of

$$\int_0^1 \Big| \sum_{uv \leqq x} e^{2\pi i u v \alpha} \Big| d\alpha$$

which turns out to be $\sqrt{x}$.

1. Introduction. Given any sequence of complex numbers c_n, the size of the L_1-norm of the associated exponential sum

$$\sum_{n \leqq x} c_n e(\alpha n)$$

seems to contain some arithmetic information about the coefficients, although its meaning is not very well understood. Very recently, Brüdern, Granville, Perelli, Vaughan and Wooley [4] have established a link between the L_1-norm and the distribution of the coefficients in arithmetic progressions to the effect that if the L_1-norm

$$\int_0^1 \Big| \sum_{n \leqq x} c_n e(\alpha n) \Big| d\alpha$$

is small, then (c_n) must be very evenly distributed in all arithmetic progressions (see [4], Theorem 1.8 for a precise statement). In the same spirit, Vaughan [7] had shown earlier that

$$\int_0^1 \Big| \sum_{p \leqq x} (\log p) e(\alpha p) \Big| d\alpha \gg \sqrt{x}$$

by exploiting the fact that the primes are equidistributed in residue classes $a(\bmod q)$ with $(a,q) = 1$, but do not serve those classes with $(a,q) > 1$.

In the opposite direction, sequences of an arithmetic nature with small L_1-norm have received increased interest in recent years. One always has the "trivial estimate"

$$\int_0^1 \Big| \sum_{n \leqq x} c_n e(\alpha n) \Big| d\alpha \leqq \Big(\int_0^1 \Big| \sum_{n \leqq x} c_n e(\alpha n) \Big|^2 d\alpha \Big)^{\frac{1}{2}} = \Big(\sum_{n \leqq x} |c_n|^2 \Big)^{\frac{1}{2}} \tag{1.1}$$

as a consequence of Schwarz's inequality and Parseval's identity. Work of Brüdern, Granville, Perelli, Vaughan and Wooley [4], Brüdern, Perelli and Wooley [5] and Balog and

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Ruzsa [1] provides a still very small stock of examples where upper bounds superior to those following from (1.1) have been obtained. Leaving aside trivial examples such as arithmetic progressions, it seems that the exponential sum over squarefree numbers is the only example where the order of magnitude of the L_1-norm has been determined. Balog and Ruzsa [1] showed that

$$x^{\frac{1}{3}} \ll \int_0^1 \Big| \sum_{n \leqq x} \mu(n)^2 e(\alpha n) \Big| d\alpha \ll x^{\frac{1}{3}}.$$

In this paper we study the exponential sum with the number of divisors $d(n)$ of the natural number n as coefficients. It turns out that the order of magnitude of the L_1-norm is $\sqrt{x}$ which is surprisingly large for a sequence as well-distributed in residue classes as $d(n)$.

Theorem. *One has*

$$\sqrt{x} \ll \int_0^1 \Big| \sum_{n \leqq x} d(n) e(\alpha n) \Big| d\alpha \ll \sqrt{x}.$$

Note that (1.1) only yields $O(x^{\frac{1}{2}}(\log x)^{\frac{3}{2}})$ for the integral in question.

The ideas underlying the proof of the theorem differ from previous approaches in various aspects. We are able to take advantage of specific properties of the divisor function. A Hardy-Littlewood dissection of the unit interval is the starting point. Converting the exponential sum into a sum over products, it is possible, on each “major arc”, to isolate a subsum which provides the main contribution to the L_1-norm. The same method is applicable to exponential sums over products of bounded height, and yields

$$y \ll \int_0^1 \Big| \sum_{u \leqq y} \sum_{v \leqq y} e(\alpha uv) \Big| d\alpha \ll y.$$

The reader will have no difficulties in adjusting the proof of the Theorem to cover this example.

For further recent work on L_1-norms of exponential sums, see [6, 2].

2. Some preparatory steps. Dirichlet’s classical hyperbola dissection is used to transform the original exponential sum. Writing

$$S(\alpha) = \sum_{u \leqq \sqrt{x}} \sum_{u < v \leqq x/u} e(\alpha uv), \quad T(\alpha) = \sum_{u \leqq \sqrt{x}} e(\alpha u^2),$$

one has

$$\sum_{n \leqq x} d(n) e(\alpha n) = \sum_{uv \leqq x} e(\alpha uv) = 2S(\alpha) + T(\alpha).$$

Note that (1.1) implies that

$$\int_0^1 |T(\alpha)| d\alpha \ll x^{\frac{1}{4}},$$

and by the triangle inequality, we have

$$2|S(\alpha)| - |T(\alpha)| \leqq \Big| \sum_{n \leqq x} d(n) e(\alpha n) \Big| \leqq 2|S(\alpha)| + |T(\alpha)|.$$

Integrating this inequality, it follows that

$$\int_0^1 \Big| \sum_{n \leqq x} d(n)e(\alpha n) \Big| d\alpha = 2\int_0^1 |S(\alpha)| d\alpha + O(x^{\frac{1}{4}}). \tag{2.1}$$

It remains to evaluate the L_1-norm of $S(\alpha)$. To prepare for the proof of the upper bound, we note that for any $Q \geqq 1$ and any real $\alpha \in [Q^{-1}, 1+Q^{-1}]$ there are coprime numbers a, q with $1 \leqq a \leqq q \leqq Q$ and $|q\alpha - a| \leqq Q^{-1}$, by Dirichlet's theorem. Hence,

$$\begin{aligned} \int_0^1 |S(\alpha)| d\alpha &\leqq \sum_{q \leqq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{-1/(qQ)}^{1/(qQ)} \left| S\left(\frac{a}{q} + \beta\right)\right| d\beta \\ &\leqq \sum_{q \leqq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \sum_{d|q} \int_{-1/(qQ)}^{1/(qQ)} \left| S_{q,d}\left(\frac{a}{q} + \beta\right)\right| d\beta \end{aligned} \tag{2.2}$$

where

$$S_{q,d}(\alpha) = \sum_{\substack{u \leqq \sqrt{x} \\ (u,q)=d}} \sum_{u<v \leqq x/u} e(\alpha uv). \tag{2.3}$$

A significant contribution to (2.2) arises only from terms with $d = q$. We therefore single these out and write $U_q(\alpha) = S_{q,q}(\alpha)$ so that

$$U_q(\alpha) = \sum_{\substack{u \leqq \sqrt{x} \\ q|u}} \sum_{u<v \leqq x/u} e(\alpha uv). \tag{2.4}$$

The heart of our method is contained in the following auxiliary estimate.

Lemma. *In the above notation, one has*

$$\sum_{q \leqq 2\sqrt{x}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \sum_{\substack{d|q \\ d<q}} \int_{-1/(2q\sqrt{x})}^{1/(2q\sqrt{x})} \left| S_{q,d}\left(\frac{a}{q} + \beta\right)\right| d\beta \ll (\log x)^3.$$

In (2.2), take $Q = 2\sqrt{x}$. Then, by (2.4) and the Lemma,

$$\int_0^1 |S(\alpha)| d\alpha \leqq \sum_{q \leqq 2\sqrt{x}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{-1/(2q\sqrt{x})}^{1/(2q\sqrt{x})} |U_q(\beta)| d\beta + O\big((\log x)^3\big); \tag{2.5}$$

here it is important to observe that for $q|u$, one has $\dfrac{auv}{q} \in \mathbb{Z}$, and consequently $U_q\left(\dfrac{a}{q} + \beta\right) = U_q(\beta)$.

Before we proceed to deduce the upper bound in the theorem from (2.5), we pause to establish the lemma. In (2.3) the inner sum over v may be evaluated, and yields

$$|S_{q,d}(\alpha)| \leqq \sum_{\substack{u \leqq \sqrt{x} \\ (u,q)=d}} \min\left(\frac{x}{u}, \|\alpha u\|^{-1}\right). \tag{2.6}$$

In the ranges relevant to the statement in the lemma, one has $\alpha = \dfrac{a}{q} + \beta$ with $(a, q) = 1$ and

$|\beta| \leqq (2q\sqrt{x})^{-1}$. For $u \leqq \sqrt{x}$ it follows that $|\beta u| \leqq \dfrac{1}{2q}$, and therefore,

$$\|\alpha u\| = \left\| \frac{au}{q} + \beta u \right\| \geqq \frac{1}{2} \left\| \frac{au}{q} \right\| \geqq \frac{d}{2q}$$

(where $d = (u, q)$). Hence, by (2.6), the left hand side of the inequality proposed in the lemma does not exceed

$$\sum_{q \leqq 2\sqrt{x}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \sum_{\substack{d|q \\ d<q}} \int_{-1/(2q\sqrt{x})}^{1/(2q\sqrt{x})} 2 \sum_{\substack{u \leqq \sqrt{x} \\ (u,q)=d}} \left\| \frac{au}{q} \right\|^{-1} d\beta \ll \sum_{q \leqq 2\sqrt{x}} \frac{1}{q\sqrt{x}} \sum_{\substack{d|q \\ d<q}} \sum_{\substack{u \leqq \sqrt{x} \\ (u,q)=d}} d \log \frac{q}{d}$$

$$\ll \sum_{q \leqq 2\sqrt{x}} \frac{1}{q\sqrt{x}} \sum_{d|q} \sqrt{x} \log \frac{q}{d} \ll (\log x)^3$$

by a succession of elementary estimates, and the lemma follows.

3. The upper bound. Further simplifications of (2.5) can be made by observing that $U_q(\beta) = 0$ unless $q \leqq \sqrt{x}$ (since otherwise the sum in (2.4) is empty). Next, we evaluate the inner sum in (2.4) and then infer from (2.5) that

$$\int_0^1 |S(\alpha)| d\alpha \leqq \sum_{q \leqq \sqrt{x}} q \int_{-1/(2q\sqrt{x})}^{1/(2q\sqrt{x})} \sum_{\substack{u \leqq \sqrt{x} \\ q|u}} \min\left(\frac{x}{u}, \|\beta u\|^{-1}\right) d\beta + O\big((\log x)^3\big).$$

In the sum on the right, we denote the subsum of terms with $\frac{1}{2}\sqrt{x} < q \leqq \sqrt{x}$ by Σ_1, and the remaining part with $q \leqq \frac{1}{2}\sqrt{x}$ by Σ_2. Note that $q > \frac{1}{2}\sqrt{x}, u \leqq \sqrt{x}$ and $q|u$ implies $u = q$. It follows that

$$\Sigma_1 \leqq 2 \sum_{\frac{1}{2}\sqrt{x} < q \leqq \sqrt{x}} q \int_0^{1/(2q\sqrt{x})} \min\left(\frac{x}{q}, \|\beta q\|^{-1}\right) d\beta \leqq \sqrt{x}.$$

For $q \leqq \frac{1}{2}\sqrt{x}$ and $|\beta| \leqq \dfrac{1}{2q\sqrt{x}}, u \leqq \sqrt{x}$ we have

$$\int_{-1/(2q\sqrt{x})}^{1/(2q\sqrt{x})} \min\left(\frac{x}{u}, \|\beta u\|^{-1}\right) d\beta = \frac{2}{u} \int_0^{1/(2q\sqrt{x})} \min\left(x, \beta^{-1}\right) d\beta$$

$$= \frac{2}{u}\left(1 + \int_{1/x}^{1/(2q\sqrt{x})} \beta^{-1} d\beta\right) = \frac{2}{u}\left(1 + \log \frac{\sqrt{x}}{2q}\right)$$

because one always has $\dfrac{1}{2q\sqrt{x}} \geqq \dfrac{1}{x}$. We may now conclude that

$$\Sigma_2 \leqq \sum_{q \leqq \frac{1}{2}\sqrt{x}} \sum_{\substack{u \leqq \sqrt{x} \\ q|u}} \frac{2q}{u}\left(1 + \log \frac{\sqrt{x}}{2q}\right)$$

$$= \sum_{q \leqq \frac{1}{2}\sqrt{x}} \sum_{l \leqq \sqrt{x}/q} \frac{2}{l}\left(1 + \log \frac{\sqrt{x}}{2q}\right)$$

$$\ll \sum_{q \leqq \frac{1}{2}\sqrt{x}} \left(1 + \log \frac{\sqrt{x}}{q}\right)^2 \ll \sqrt{x}$$

Collecting together the previous results, an upper bound of the required strength is now available from the inequality

$$\int_0^1 |S(\alpha)|d\alpha \leqq \Sigma_1 + \Sigma_2 + O\big((\log x)^3\big) \ll \sqrt{x}. \tag{3.1}$$

4. The lower bound. For the lower bound, we recall (2.3) and (2.4). By the triangle inequality, one has

$$|S(\alpha)| \geqq |U_q(\alpha)| - \sum_{\substack{d|q\\ d<q}} |S_{q,d}(\alpha)|.$$

For $1 \leqq a \leqq q \leqq \frac{1}{2}\sqrt{x}, (a,q) = 1$, the intervals $\left|\alpha - \dfrac{a}{q}\right| \leqq \dfrac{1}{8x}$ are pairwise disjoint. By another appeal to the lemma, it follows that

$$\begin{aligned}\int_0^1 |S(\alpha)|d\alpha &\geqq \sum_{q \leqq \frac{1}{2}\sqrt{x}} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{-1/(8x)}^{1/(8x)} \left|S\left(\frac{a}{q}+\beta\right)\right|d\beta \\ &\geqq \sum_{q \leqq \frac{1}{2}\sqrt{x}} \varphi(q) \int_{-1/(8x)}^{1/(8x)} |U_q(\beta)|d\beta - O\big((\log x)^3\big);\end{aligned}$$

here we use again that $U_q\left(\dfrac{a}{q}+\beta\right) = U_q(\beta)$. By (2.4), for $q \leqq \frac{1}{2}\sqrt{x}$,

$$\begin{aligned}U_q(0) &= \sum_{\substack{u \leqq \sqrt{x}\\ q|u}} \left(\frac{x}{u} - u + O(1)\right) = \sum_{l \leqq \sqrt{x}/q} \left(\frac{x}{ql} - ql\right) + O(\sqrt{x}) \\ &= \frac{x}{q}\left(\log\frac{\sqrt{x}}{q} + \gamma - \frac{1}{2}\right) + O(\sqrt{x})\end{aligned}$$

where γ is Euler's constant. In particular, it follow that there exists a constant $c > 0$ such that for all $q \leqq c\sqrt{x}$ and sufficiently large x, one has

$$U_q(0) \geqq \frac{x}{2q}.$$

Once again by (2.4), one has

$$U_q(\beta) = \sum_{n \leqq x} b_n e(\beta n)$$

where b_n denotes the number of products $uv = n$ with $q|u, u \leqq \sqrt{x}$ and $u < v \leqq x/u$. By partial summation, it follows that

$$U_q(\beta) = e(\beta x)U_q(0) - 2\pi i\beta \int_1^x e(\beta y) \sum_{n \leqq y} b_n dy.$$

But

$$0 \leqq \sum_{n \leqq y} b_n \leqq \sum_{n \leqq x} b_n = U_q(0),$$

and therefore, for $|\beta| \leqq \dfrac{1}{8x}$, we conclude that

$$|U_q(\beta)| \geqq U_q(0) - 2\pi|\beta| \int_1^x U_q(0)dy \geqq \tfrac{1}{5}U_q(0).$$

Combining the inequalities obtained so far, it follows that

$$\begin{aligned}\int_0^1 |S(\alpha)|d\alpha &\geqq \sum_{q \leqq \frac{1}{2}\sqrt{x}} \frac{\varphi(q)}{4x}\frac{1}{5}U_q(0) - O\big((\log x)^3\big)\\ &\geqq \frac{1}{20}\sum_{q \leqq c\sqrt{x}} \frac{\varphi(q)}{q} - O\big((\log x)^3\big),\end{aligned}$$

and the lower bound

$$\int_0^1 |S(\alpha)|d\alpha \gg \sqrt{x} \tag{4.1}$$

is now readily inferred from standard results in analytic number theory (see, for example, [3], lemma 5.4.2). The theorem is now available from (2.1), (3.1) and (4.1).

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