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Links of symbolic powers of prime ideals

Hsin-Ju Wang

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Abstract In this paper, we prove the following. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 2$. Suppose that either R is not regular or if R is regular assume that $d \geq 3$. Let $t \geq 2$ be a positive integer. If $\{\alpha_1, \ldots, \alpha_d\}$ is a regular sequence contained in $\mathfrak{m}^t$, then

$$(\alpha_1, \ldots, \alpha_d) : \mathfrak{m}^t \subseteq \mathfrak{m}^t.$$

This result gives an affirmative answer to a conjecture raised by Polini and Ulrich.

1 Introduction

Let R be a Noetherian ring. Two proper ideals I and L of height g are said to be *directly linked*, $I \sim L$, if there exists a regular sequence $\mathbf{z} = z_1, \ldots, z_g \subseteq I \cap L$ such that $I = (\mathbf{z}) : L$ and $L = (\mathbf{z}) : I$. If R is Cohen-Macaulay and I is an unmixed ideal of height g with R_P Gorenstein for every minimal prime P of I, then one can always produce a link $L \sim I$ by choosing a regular sequence $\mathbf{z} = z_1, \ldots, z_g \subset I$ and setting $L = (\mathbf{z}) : I$ ([5]). The linkage class of a proper ideal I is the set of all ideals K obtained by a sequence of links $I = L_0 \sim L_1 \sim \cdots \sim L_n = L$, where $n \geq 0$. The algebraic foundations of linkage theory were established in [4] and [5].

In [6], Polini and Ulrich investigated when an ideal is the unique maximal element of its linkage class, in the sense that it contains every ideal of the class. They showed that if $(R, \mathfrak{m})$ is a Gorenstein local ring of dimension $d \geq 2$, with $d \geq 3$ if R is regular, then every ideal in the linkage class of $\mathfrak{m}^t$ is contained in $\mathfrak{m}^t$ provided that $gr_{\mathfrak{m}}(R)$ is Cohen-Macaulay, or R is a complete intersection, or *ecodim* $R \leq 3$, or $t \leq 3$. They

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H.-J. Wang (✉)
Department of Mathematics, National Chung Cheng University,
Chiayi 621, Taiwan
e-mail: hjwang@math.ccu.edu.tw

pointed out that $\mathfrak{m}^t$ will be the maximal element of its linkage class if the following conjecture holds.

Conjecture 1 Let $(R, \mathfrak{m})$ be a local Gorenstein ring of dimension $d \geq 2$, with $d \geq 3$ if R is regular, and let $t \geq 1$ be an integer; then every ideal directly linked to $\mathfrak{m}^t$ is contained in $\mathfrak{m}^t$.

In the same paper, they also showed that Conjecture 1 is equivalent to the following question: Is a two-dimensional local Gorenstein ring $(R, \mathfrak{m})$ regular if and only if there exists an integer $t \geq 1$ and a regular sequence $\alpha = \alpha_1, \alpha_2$ contained in $\mathfrak{m}^t$ so that $(\alpha) : \mathfrak{m}^t$ is not contained in $\mathfrak{m}^t$?

In this paper, we settle the conjecture in the affirmative by showing the following:

Theorem 1.1 *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 2$, with $d \geq 3$ if R is regular. Let $t \geq 2$ be a positive integer. If $\{\alpha_1, \ldots, \alpha_d\}$ is a regular sequence contained in $\mathfrak{m}^t$, then*

$$(\alpha_1, \ldots, \alpha_d) : \mathfrak{m}^t \subseteq \mathfrak{m}^t.$$

Let R be a Noetherian ring. Recall that an ideal I of R of height g is said to be equi-multiple of reduction number 1 if there exists a g generated ideal J contained in I so that $I^2 = JI$. Ideals that are equi-multiple of reduction number 1 enjoy some nice properties: For example, it is shown in [7] that if R is a Cohen-Macaulay local ring and I is an equi-multiple Cohen-Macaulay ideal of reduction number 1, then the associated graded ring $gr_I(R)$ of I is Cohen-Macaulay. Therefore, it is of great interest to search for such classes of ideals. In [1] and [3], Corso, Polini and Vasconcelos proved that the links of prime ideals in Cohen-Macaulay rings are equi-multiple ideals of reduction number 1 under certain assumptions. Moreover, Corso and Polini [2] addressed the following conjecture.

Conjecture 2 Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and P a prime ideal of height $g \geq 2$. Let $J = (z_1, \ldots, z_g)$ be an ideal generated by a regular sequence contained in $P^{(s)}$, where s is a positive integer ≥ 2. For any $k = 1, \ldots, s$ set $I_k = J : P^{(k)}$. Then

$$I_k^2 = JI_k$$

if one of the following two conditions holds:

(IL_1) R_P is not a regular local ring;
(IL_2) R_P is a regular local ring and two of the z_i's lie in $P^{(s+1)}$.

With the help of Theorem 1.1 and Theorem 3.2, we are able to give a positive answer to Conjecture 2.

Theorem 1.2 *Let R be a Noetherian ring. Let P be a prime ideal of height $g \geq 2$, $J = (z_1, \ldots, z_g)$ an ideal generated by a regular sequence contained in $P^{(k)}$, where $k \geq 2$ is a positive integer. Set $I_k = J : P^{(k)}$. Then $I_k^2 = I_kJ$ if one of the following conditions holds:*

(i) *R_P is not a regular local ring.*
(ii) *R_P is a regular local ring and $g \geq 3$.*
(iii) *R_P is a regular local ring, $g = 2$, and $z_i \in P^{(k+1)}$ for every i.*

2 Main theory

In this section, we study Conjecture 1 stated in Sect. 1. We are able to prove it in a more general form.

Theorem 2.1 *Let* $(R, \mathfrak{m})$ *be a Cohen-Macaulay local ring of dimension* $d \geq 2$. *Let* I *be an ideal of* R *of height 2 and* $\tilde{I} = \sum((\alpha, \beta) : I)$, *where the sum is taken over all regular sequences* $\{\alpha, \beta\}$ *in* I. *Let* t *be a positive integer. If* $\{\alpha, \beta\}$ *is a regular sequence contained in* I^t, *then*

$$(\alpha, \beta) : I^t \subseteq I^{t-1}\tilde{I}.$$

Remark 2.2

(i) $I \subseteq \tilde{I} \subseteq R$.
(ii) Conjecture 1 follows if we set $d = 2$ and $I = \mathfrak{m}$ in Theorem 2.1 and observe that $\tilde{I} = I = \mathfrak{m}$ if R is not regular.
(iii) One can see from the proof of Theorem 2.1 that the assumption that R is Cohen-Macaulay can be removed if the grade of I is 2.

To show Theorem 2.1, we begin with the following lemmas.

Lemma 2.3 *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring. Let* $\{x_1, \ldots, x_{t-1}, y_1, \ldots, y_{t-1}\} \subseteq \mathfrak{m}$ *such that* $\{x_i, y_j\}$ *is a regular sequence* $\forall i, j$, *where* $t \geq 2$. *Then*

$$\cap_{i=1}^{t-1}(x_1 \cdots x_i, y_1 \cdots y_{t-i}) = \langle \{x_1 \cdots x_{i-1} y_1 \cdots y_{t-i} \mid 1 \leq i \leq t\} \rangle .$$

Proof It is enough to show that

$$\cap_{i=1}^{k}(x_1 \cdots x_i, y_1 \cdots y_{t-i}) = (x_1 \cdots x_k) + \langle \{x_1 \cdots x_{i-1} y_1 \cdots y_{t-i} \mid 1 \leq i \leq k\} \rangle$$

for $1 \leq k \leq t-1$. By induction on k, suppose that the equality holds for k. Then

$$\begin{aligned}
&\cap_{i=1}^{k+1}(x_1 \cdots x_i, y_1 \cdots y_{t-i}) \\
&\quad = ((x_1 \cdots x_k) + \langle \{x_1 \cdots x_{i-1} y_1 \cdots y_{t-i} \mid 1 \leq i \leq k\} \rangle) \cap (x_1 \cdots x_{k+1}, y_1 \cdots y_{t-k-1}) \\
&\quad = (x_1 \cdots x_k) \cap (x_1 \cdots x_{k+1}, y_1 \cdots y_{t-k-1}) + \langle \{x_1 \cdots x_{i-1} y_1 \cdots y_{t-i} \mid 1 \leq i \leq k\} \rangle \\
&\quad = (x_1 \cdots x_{k+1}) + \langle \{x_1 \cdots x_{i-1} y_1 \cdots y_{t-i} \mid 1 \leq i \leq k+1\} \rangle.
\end{aligned}$$

□

Lemma 2.4 *Let* $(R, \mathfrak{m})$ *be a Cohen-Macaulay local ring of dimension* $d \geq 2$. *Let* I *be an ideal of* R *of height 2 and* $\tilde{I} = \sum((\alpha, \beta) : I)$, *where the sum is taken over all regular sequences* $\{\alpha, \beta\}$ *in* I. *Let* $t \geq 1$ *and* $\{x_1, \ldots, x_t, y_1, \ldots, y_t\} \subseteq I$ *such that* $\{x_i, y_j\}$ *is a regular sequence* $\forall i, j$. *Then*

$$(x_1 \cdots x_t, y_1 \cdots y_t) : I^t \subseteq I^{t-1}\tilde{I}.$$

Proof We may assume that $t \geq 2$. First observe that

$$(x_1 \cdots x_t, y_1 \cdots y_t) : I^t \subseteq \cap_{i=1}^{t-1}(x_1 \cdots x_i, y_1 \cdots y_{t-i}).$$

This is because

$$\begin{aligned}
(x_1 \cdots x_t, y_1 \cdots y_t) : I^t &\subseteq (x_1 \cdots x_t, y_1 \cdots y_t) : x_{i+1} \cdots x_t y_{t-i+1} \cdots y_t \\
&= (x_1 \cdots x_i, y_1 \cdots y_{t-i})
\end{aligned}$$

for $1 \leq i \leq t-1$.

By Lemma 2.3, every element in $(\prod_i x_i, \prod_i y_i) : I^t$ is of the form

$$\sum_{i=1}^{t} a_i x_1 \cdots x_{i-1} y_1 \cdots y_{t-i}.$$

Therefore, to show the lemma, it is enough to show that $a_i \in \tilde{I}$ for every i. For this, let $z \in I$ then

$$\left(\sum_{j=1}^{t} a_j x_1 \cdots x_{j-1} y_1 \cdots y_{t-j}\right) z x_{i+1} \cdots x_t y_{t-i+2} \cdots y_t \in (x_1 \cdots x_t, y_1 \cdots y_t).$$

Observe that

$$(a_j x_1 \cdots x_{j-1} y_1 \cdots y_{t-j}) z x_{i+1} \cdots x_t y_{t-i+2} \cdots y_t \in (x_1 \cdots x_t, y_1 \cdots y_t)$$

if $j \neq i$. Therefore

$$a_i z x_1 \cdots \hat{x}_i \cdots x_t y_1 \cdots \hat{y}_{t-i+1} \cdots y_t \in (x_1 \cdots x_t, y_1 \cdots y_t).$$

It follows that $a_i z \in (x_i, y_{t-i+1})$, which implies that $a_i \in (x_i, y_{t-i+1}) : I \subseteq \tilde{I}$. □

Proof of Theorem 2.1 We first prove the following statement: If $y_i \in I$ $\forall i$ and $\{\alpha, y_1 \cdots y_t\}$ is a regular sequence contained in I^t, then

$$(\alpha, y_1 \cdots y_t) : I^t \subseteq I^{t-1}\tilde{I}.$$

Observe that the following set generates I^t:

$$S_t = \{z_1 \cdots z_t \mid z_j \in I,\ \{y_i, z_j\} \textit{ is a regular sequence } \forall i, j\}.$$

Let $a \in (\alpha, y_1 \cdots y_t) : I^t$. Let $z_1 \cdots z_t \in S_t$; then there are elements $u(z), v(z) \in R$ such that

$$a z_1 \cdots z_t = u(z)\alpha + v(z)(y_1 \cdots y_t). \tag{1}$$

Therefore if $z_1' \cdots z_t'$ is any element in S_t then

$$a z_1' \cdots z_t' = u(z')\alpha + v(z')(y_1 \cdots y_t). \tag{2}$$

From (1), (2) and the fact that $\{\alpha, y_1 \cdots y_t\}$ is a regular sequence, we see that $u(z) z_1' \cdots z_t' \in (z_1 \cdots z_t, y_1 \cdots y_t)$. Since $z_1' \cdots z_t'$ is an arbitrary element in S_t,

$$u(z) \in (z_1 \cdots z_t, y_1 \cdots y_t) : I^t \subseteq I^{t-1}\tilde{I}$$

by Lemma 2.4. Since $z_1 \cdots z_t$ is an arbitrary element in S_t and $\alpha \in I^t$, there are elements $u, v \in R$ with $u \in I^{t-1}\tilde{I}$ such that $a\alpha = u\alpha + v(y_1 \cdots y_t)$, which implies that $a - u \in (y_1 \cdots y_t)$. It follows that $a \in I^{t-1}\tilde{I}$ and this completes the proof of the statement.

In general, if $\{\alpha, \beta\}$ is a regular sequence contained in I^t, then we can use the same argument as above to conclude that every element in $(\alpha, \beta) : I^t$ is in $I^{t-1}\tilde{I}$. □

Corollary 2.5 *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 2$. Suppose that either R is not regular or if R is regular assume that $d \geq 3$. Let $t \geq 2$ be a positive integer. If $\{\alpha_1, \ldots, \alpha_d\}$ is a regular sequence contained in $\mathfrak{m}^t$, then*

$$(\alpha_1, \ldots, \alpha_d) : \mathfrak{m}^t \subseteq \mathfrak{m}^t.$$

Proof We proceed by induction on d. The case $d = 2$ follows from Theorem 2.1.

Assume that $d \geq 3$. Let $a \in (\alpha_1, \ldots, \alpha_d) : \mathfrak{m}^t$. Since $R/(\alpha_d)$ is Cohen-Macaulay and is not regular, $a \in \mathfrak{m}^t$ $(mod\ (\alpha_d))$ by induction. It follows that $a \in \mathfrak{m}^t$ as $\alpha_d \in \mathfrak{m}^t$. □

The following result will be used in Sect. 3.

Corollary 2.6 *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 2$. Suppose that either R is not regular or if R is regular assume that $d \geq 3$. Let $t \geq 2$ be a positive integer and $\{\alpha_1, \ldots, \alpha_d\}$ be a regular sequence contained in $\mathfrak{m}^t$. Then for every $a \in (\alpha_1, \ldots, \alpha_d) : \mathfrak{m}^t$ and every element $f \in \mathfrak{m}^t$, if $af = \sum_{i=1}^d u_i \alpha_i$ then $u_i \in \mathfrak{m}^t$ for every i.*

Corollary 2.6 follows from the next lemma.

Lemma 2.7 *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 2$. Let $t \geq 2$ be a positive integer. Assume that for any regular sequence $\{\alpha_1, \ldots, \alpha_d\}$ contained in $\mathfrak{m}^t$, $(\alpha_1, \ldots, \alpha_d) : \mathfrak{m}^t \subseteq \mathfrak{m}^t$. Then for every $a \in (\alpha_1, \ldots, \alpha_d) : \mathfrak{m}^t$ and every element $f \in \mathfrak{m}^t$, if $af = \sum_{i=1}^d u_i \alpha_i$ then $u_i \in \mathfrak{m}^t$ for every i.*

Proof It is easy to see that the following set generates the ideal $\mathfrak{m}^t$:

$$S_t = \{z_1 \cdots z_t \mid z_j \in \mathfrak{m},\ \{\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_d, z_j\} \textit{ is a regular sequence } \forall i, j\}.$$

Let $z_1 \cdots z_t \in S_t$; then there are $u_i(z) \in R$ such that

$$a z_1 \cdots z_t = \sum_{i=1}^{d} u_i(z) \alpha_i.$$

From the proof of Theorem 2.1, we see that $u_i(z) \in (\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_d, z_1 \cdots z_t) : \mathfrak{m}^t$ for every i. Therefore $u_i(z) \in \mathfrak{m}^t$ by assumption. This shows that if $f \in \mathfrak{m}^t$ then af can be expressed as $\sum_{i=1}^d u_i \alpha_i$ for some $u_i \in \mathfrak{m}^t$ for every i. If $af = \sum_{i=1}^d u_i' \alpha_i$ for some $u_i' \in R$, then $u_i - u_i' \in (\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_d) \subseteq \mathfrak{m}^t$, it follows that $u_i' \in \mathfrak{m}^t$ for every i. □

3 Ideals of reduction number 1

The goal of this section is to study the following conjecture:

Conjecture Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and P a prime ideal of height $g \geq 2$. Let $J = (z_1, \ldots, z_g)$ be an ideal generated by a regular sequence contained in $P^{(s)}$, where s is a positive integer ≥ 2. For any $k = 1, \ldots, s$ set $I_k = J : P^{(k)}$. Then

$$I_k^2 = J I_k$$

if one of the following two conditions holds:

(IL_1) R_P is not a regular local ring;
(IL_2) R_P is a regular local ring and two of the z_i's lie in $P^{(s+1)}$.

There are some partial answers to this conjecture in [2] and [6]. The following result solves the conjecture completely.

Theorem 3.1 *Let R be a Noetherian ring. Let P be a prime ideal of height $g \geq 2$, $J = (z_1, \ldots, z_g)$ an ideal generated by a regular sequence contained in $P^{(k)}$, where $k \geq 2$ is a positive integer. Set $I_k = J : P^{(k)}$. Then $I_k^2 = I_k J$ if one of the following conditions holds:*

(i) *R_P is not a regular local ring.*
(ii) *R_P is a regular local ring and $g \geq 3$.*
(iii) *R_P is a regular local ring, $g = 2$, and $z_i \in P^{(k+1)}$ for every i.*

To show Theorem 3.1, we need the following result.

Theorem 3.2 *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 2$. Let $k \geq 2$ be a positive integer. If $\mathbf{z} = z_1, \ldots, z_d$ is a regular sequence contained in $\mathfrak{m}^{k+1}$, then*

$$(\mathbf{z}) : \mathfrak{m}^k \subseteq \mathfrak{m}^{2k+1} : \mathfrak{m}^k$$

if depth $gr_{\mathfrak{m}}(R) \geq 2$.

Proof It is not difficult to see that the following set generates the ideal $\mathfrak{m}^k$:

$$S_k = \left\{ \prod_{i=1}^{k} y_i \mid y_i \in \mathfrak{m}, \{y_i^*, y_j^*\} \textit{ is a regular sequence in } gr_{\mathfrak{m}}(R) \right\}.$$

Let $a \in (\mathbf{z}) : \mathfrak{m}^k$ and $f = \prod_{i=1}^{k} y_i \in S_k$; then

$$af = \sum_{i=1}^{d} u_i z_i \tag{3}$$

for some $u_i \in R$. To prove the theorem it is enough to show that $u_i \in \mathfrak{m}^k$ as $z_i \in \mathfrak{m}^{k+1}$ for every i. Without loss of generality, we prove that $u_1 \in \mathfrak{m}^k$.

Choose $y \in \mathfrak{m}$ so that $\{y_j^*, y^*\}$ form a regular sequence in $gr_{\mathfrak{m}}(R)$ for every j. Let $f_j = yf/y_j$; since $f \in S_k$, there are $u_{ij} \in R$ such that

$$af_j = \sum_{i=1}^{d} u_{ij} z_i. \tag{4}$$

From (3) and (4), we obtain

$$(u_1 y - u_{1j} y_j) z_1 \in (z_2, \ldots, z_d).$$

Moreover, since $\{z_1, \ldots, z_d\}$ is a regular sequence and $z_i \in \mathfrak{m}^{k+1}$ by assumption,

$$(u_1 y - u_{1j} y_j) \in \mathfrak{m}^{k+1}.$$

It follows that

$$u_1 \in (y_j) + \mathfrak{m}^k$$

for every j by the choice of y. Now one can conclude that $u_1 \in (y_l \cdots y_k) + \mathfrak{m}^k$ by the facts that $\{y_l^*, y_j^*\}$ is a regular sequence in $gr_{\mathfrak{m}}(R)$ if $j > l$ for $l = k-1, k-2, \ldots,$ in particular, $u_1 \in \mathfrak{m}^k$. □

Corollary 3.3 *Let $(R, \mathfrak{m})$ be a 2-dimensional regular local ring. Let $k \geq 2$ and $\mathbf{z} = z_1, z_2$ be regular sequence contained in $\mathfrak{m}^{k+1}$. Then the following hold.*

(i) $(\mathbf{z}) : \mathfrak{m}^k \subseteq \mathfrak{m}^k$;
(ii) *For every element $a \in (\mathbf{z}) : \mathfrak{m}^k$ and every element $f \in \mathfrak{m}^k$, if $af = \sum_{i=1}^{2} u_i z_i$ then $u_i \in \mathfrak{m}^k$.*

Proof

(i) follows from the fact that $gr_{\mathfrak{m}}(R)$ is Cohen-Macaulay and $\mathfrak{m}^{2k+1} : \mathfrak{m}^k = \mathfrak{m}^{k+1} \subseteq \mathfrak{m}^k$.
(ii) follows from the proof of Theorem 3.2. □

Before proving Theorem 3.1, we need one more lemma.

Lemma 3.4 [2, Lemma 2.1] *Let R be a Noetherian ring. Let H be an ideal of height $g \geq 2$, $J = (z_1, \ldots, z_g)$ an ideal generated by a regular sequence contained in H. Set $I = J : H$. If $I = (a_1, \ldots, a_t)$, $a_i H \subseteq JH$ for every i, and $I^2 \subseteq J$, then $I^2 = IJ$.*

Proof Since $I^2 \subseteq J$, there are $u_{ij}^{(k)} \in R$ such that $a_i a_j = \sum_{k=1}^{g} u_{ij}^{(k)} z_k$. To complete the proof it suffices to show that $u_{ij}^{(k)} \in I$, i.e., $u_{ij}^{(k)} H \subseteq J$ for every k. For this let $b \in H$; then by assumption,

$$\begin{aligned}\textstyle\sum_{k=1}^{g} (u_{ij}^{(k)} b) z_k &= a_i a_j b \\ &\in a_i JH \\ &\subseteq J^2 H \\ &\subseteq (z_1, \ldots, z_g)^2.\end{aligned}$$

Since $z_1, \ldots, z_g$ is a regular sequence (although not necessarily permutable), we see that $u_{ij}^{(k)} b \in (z_1, \ldots, z_g)$ for every k. □

Proof of Theorem 3.1 Let $I = I_k = J : P^{(k)}$. Then $I \subseteq P^{(k)}$ by Corollary 2.5 and Corollary 3.3(i), it follows that $I^2 \subseteq IP^{(k)} \subseteq J$. To complete the proof, by Lemma 3.4, we need only to prove that if $a \in I$, $f \in P^{(k)}$ and $af = \sum_{i=1}^{g} u_i z_i$, then $u_i \in P^{(k)}$ for every i. However this follows by Corollary 2.6 and Corollary 3.3(ii). □

4 The case $t = 1$

We study the links of unmixed radical ideals in a Noetherian ring in this section. The main result Theorem 4.2 will generalize [1, Theorem 2.3], [3, Theorem 2.1] and [6, Theorem 4.1].

Throughout, let R be a Noetherian ring of depth $d \geq g$, H an unmixed radical ideal of height g, and let $\mathbf{z} = z_1, \ldots, z_g \subseteq H$ be a regular sequence. Set $J = (\mathbf{z})$ and $I = J : H$. Suppose that for every minimal prime P of H, one of the following two conditions holds:

(L_1) R_p is not a regular local ring.
(L_2) R_P is a regular local ring of dimension at least 2 and two elements in the sequence $\mathbf{z} = z_1, \ldots, z_g$ lie in the symbolic square $P^{(2)}$.

An easy observation is the following.

Lemma 4.1 [1, Theorem 2.2] *$I \subseteq H$ and $I^2 \subseteq J$.*

Proof To show $I \subseteq H$, it suffices to show that $I \subseteq P$ for every minimal prime P of H. However, if I is not a subset of P then $P_P = J_P$, which contradicts L_1 or L_2. Thus $I \subseteq H$ is fulfilled. Moreover, $I^2 \subseteq IH \subseteq J$. □

Theorem 4.2 *Let R be a Noetherian ring of depth $d \geq g$, H an unmixed radical ideal of height g, and $\mathbf{z} = z_1, \ldots, z_g \subseteq H$ a regular sequence. Set $J = (\mathbf{z})$ and $I = J : H$. Suppose that for every minimal prime P of H, either L_1 or L_2 holds. Then $I^2 = JI$.*

Proof Write $H = P_1 \cap \cdots \cap P_l$. For $1 \leq j \leq l$, let

$$\begin{aligned} T_j &= \cup_i Ass(R/(z_1, \ldots, \hat{z}_i, \ldots, z_g)) \cap Spec(R_{P_j}) \\ &= \cup_i Min(R/(z_1, \ldots, \hat{z}_i, \ldots, z_g)) \cap Spec(R_{P_j}). \end{aligned}$$

Since I is not a subset of every prime in $\cup_j T_j$, we can choose a generating set $\{a_1, \ldots, a_t\}$ of I so that $a_i \notin \cup_j T_j$ for every i.

Now, to finish the proof, it is enough to show that $a_i H \subseteq JH$ for every i by Lemma 3.4. For this, let $b \in H$ and a be any a_i; then there are elements $u_j \in R$ such that

$$ab = \sum_{j=1}^{g} u_j z_j. \tag{5}$$

To show $u_j \in H$, we can prove it locally. Without loss of generality, we will prove that $u_1 \in PR_P$ for every minimal prime P of H.

Let b' be an arbitrary element in H; then there are elements $u'_j \in R$ such that

$$ab' = \sum_{j=1}^{g} u'_j z_j. \tag{6}$$

From (5) and (6), we obtain that

$$a(b'u_1 - bu'_1) \in (z_2, \ldots, z_g).$$

Let P be any minimal prime of H; then $\{a, z_2, \ldots, z_g\}$ is a regular sequence in R_P. Since b' is arbitrary, $u_1 P_P = u_1 H_P \subseteq (b, z_2, \ldots, z_g)R_P$. From the fact that R_P is not regular or at least one element in $\{z_2, \ldots, z_g\}$ is in $P^{(2)}$, we conclude that $u_1 \in PR_P$. □

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