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BOUNDARY-VALUE PROBLEMS FOR FOURTH-ORDER EQUATIONS OF HYPERBOLIC AND COMPOSITE TYPES

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ABSTRACT. Boundary-value problems for fourth-order linear partial differential equations of hyperbolic and composite types are studied. The method of energy inequalities and averaging operators with variable step is used to prove existence and uniqueness theorems for strong solutions. The Riesz theorem on the representation of linear continuous functionals in Hilbert spaces is used to prove the existence and uniqueness theorems for generalized solutions.

CONTENTS

Introduction

For functions $u\colon \mathbb{R}^{n+1} \ni \mathbf{x} = (x_0, x_1, \dots, x_n) \to u(\mathbf{x}) \in \mathbb{R}$, where $\mathbb{R}^{n+1}$ is the $(n+1)$-dimensional Euclidean space of independent variables $\mathbf{x}$, equations of the following kind are considered:

$$\mathcal{L}^{(1)}u \equiv \left(\frac{\partial^2}{\partial x_0^2} - a^2 A\right)\left(\frac{\partial^2}{\partial x_0^2} - b^2 A\right)u + A^{(3)}u = f(\mathbf{x}) \tag{1}$$

and

$$\mathcal{L}^{(2)}u \equiv \left(\frac{\partial^2}{\partial x_0^2} - a^2 A\right)\left(\frac{\partial^2}{\partial x_0^2} + b^2 A\right)u + A^{(3)}u = f(\mathbf{x}), \tag{2}$$

where $A = \sum\limits_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a^{(ij)}\frac{\partial}{\partial x_j}\right)$, $A^{(3)} = \sum\limits_{|\boldsymbol{\alpha}|\le 3} a^{(3)}_{\boldsymbol{\alpha}}(\mathbf{x})D^{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_n)$ is a multi-index, $D^{\boldsymbol{\alpha}} = \dfrac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_0^{\alpha_0} \dots \partial x_n^{\alpha_n}}$, $|\boldsymbol{\alpha}| = \alpha_0 + \ldots + \alpha_n$, the coefficients $a^{(ij)}$ of the operator A form a matrix of a positive quadratic form (see Condition 1.1), $a^2, b^2 \in \mathbb{R}^n$, $a^2 > 0$, and $b^2 > 0$. Under the above conditions, Eq. (1) is hyperbolic with respect to the direction $\boldsymbol{\zeta} = (1, 0, \dots, 0)$ along the axis x_0, while Eq. (2) is of composite type. Boundary-value problems in cylindrical domains are mainly considered for those equations.

A comprehensive literature is devoted to problems for hyperbolic equations. Mixed problems for higher-order equations were also considered. The Cauchy problem for partial differential equations was studied by Kovalevskaya, Petrovskii, Leray, Friedrichs, Ladyzhenskaya, Gårding, Volevich, Gindikin, and others. In [68], the investigation of the Cauchy problem for general hyperbolic equations and

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systems was begun. In [28, 64], further significant results were obtained. In [29], recent investigations of the Cauchy problem are presented. In [46], the Cauchy problem for fourth-order equations with the bi-wave operator at the principal part is considered, where initial-value conditions are posed at any exterior characteristic cone with piecewise base.

Many papers are devoted to mixed problems for hyperbolic equations (see [1, 2, 4, 6, 7, 23–27, 34, 36–41, 43, 44, 46, 47, 55–59, 67, 71, 80–82, 85]); second-order equations are of primary interest.

Seemingly, [30, 31] are the first papers where boundary-value problems for equations of composite type are considered. The following prototype equations of composite type were proposed:

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=0, \qquad \frac{\partial^2}{\partial x\partial y}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=0, \tag{3}$$

where u is a function of the two independent variables $x, y \in \mathbb{R}$. Equations (3) and more general equations of composite type were considered in many papers. In [75], the following boundary-value problem for the first equation in (3) was posed: find a solution regular in the unit disk with given values on the boundary of the disk and on the vertical diameter. In [76], this problem was considered in more general domains with smooth boundaries. In [11], boundary-value problems for the following fourth-order and fifth-order equations were studied:

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=0, \qquad \frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=0.$$

In [13], it is proved that any general third-order linear equation of composite type can be reduced to the canonical type with a principal part of the kind

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right).$$

Equations of the composite type and boundary-value problems for them were studied in many papers afterwards (see, e.g., [3, 5, 13–22, 50, 52–54, 61–63, 69, 70, 72–74, 77, 83]).

1. Setting of the Problem for Equation (1)

Mixed problems for the equation

$$\mathcal{L}^{(1)}u = f(\mathbf{x}), \tag{1.1}$$

where $a^2 \neq b^2$, $x_0 \in (0,T)$, and $\mathbf{x}' = (x_1,\dots,x_n) \in \Omega \subset \mathbb{R}^n$, are considered in the cylindrical domain $Q = (0,T)\times\Omega$ of the variables $\mathbf{x} = (x_0, x_1, \dots, x_n)$.

Condition 1.1. The coefficients $a^{(ij)}$ belong to the class $C^2(\overline{Q})$ of twice continuously differentiable functions on $\overline{Q}$, satisfy the symmetry condition $a^{(ij)} = a^{(ji)}$ for all indices $i,j = 1,\dots,n$, and generate a positive quadratic form

$$A^{(0)}(\xi) = \sum_{i,j=1}^{n} a^{(ij)}(\mathbf{x})\xi_i\xi_j$$

for all $\mathbf{x} \in \overline{Q}$, i.e.,

$$A^{(0)}(\xi) = \sum_{i,j=1}^{n} a^{(ij)}(\mathbf{x})\xi_i\xi_j \geq c^{(0)}|\boldsymbol{\xi}|^2 \tag{1.2}$$

for any $\boldsymbol{\xi} = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n$, where $c^{(0)} > 0$ is a constant.

The boundary ∂Q of the domain Q consists of the lower base $\Omega^{(0)} = \{\mathbf{x} \in \partial Q \mid x_0 = 0\}$, the upper base $\Omega^{(T)} = \{\mathbf{x} \in \partial Q \mid x_0 = T\}$, and the lateral surface $\Gamma = \{\mathbf{x} \in \partial Q \mid 0 < x_0 < T,\ \mathbf{x}' \in \partial\Omega\}$, which is piecewise smooth (here $\partial\Omega$ is the boundary of the domain $\Omega \subset \mathbb{R}^n$). The Ostrogradskii formula is applicable to the domain Q because its geometric structure is sufficiently simple. The initial conditions

$$l_k u \equiv \left.\frac{\partial^k u}{\partial x_0^k}\right|_{x_0=0} = \varphi_k(\mathbf{x}'), \quad k = 0, 1, 2, 3, \tag{1.3}$$

are posed on the lower base $\Omega^{(0)}$ and the homogeneous boundary-value conditions

$$B_1^{(p)} u\Big|_\Gamma = 0 \text{ and } B_2^{(p)} u\Big|_\Gamma = 0 \tag{1.4}$$

are posed on the lateral surface Γ.

The form of the operators $B_i^{(p)}$ $(i = 1, 2)$ depends on the mixed problem under consideration. There are well-posed mixed problems such that n boundary conditions (instead of two) are posed on Γ:

$$B_i^{(k)} u\Big|_\Gamma = 0, \quad i = 1, \ldots, n, \tag{1.5}$$

where $n > 2$.

In [43, 44], mixed problems for Eq. (1.1) are considered for $A = \Delta$, where $\Delta = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator with respect to the independent variables $x_1, \ldots, x_n$, with the following boundary conditions:

$$u|_\Gamma = \left.\frac{\partial^2 u}{\partial \boldsymbol{\nu}^2}\right|_\Gamma = 0, \tag{1.6}$$

$$\left.\frac{\partial u}{\partial \boldsymbol{\nu}}\right|_\Gamma = \left.\frac{\partial^3 u}{\partial \boldsymbol{\nu}^3}\right|_\Gamma = 0, \tag{1.7}$$

$$\left.\frac{\partial^2 u}{\partial x_0^2}\right|_\Gamma = \Delta u|_\Gamma = 0, \tag{1.8}$$

$$u|_\Gamma = \left.\frac{\partial}{\partial \boldsymbol{\nu}}\left(\frac{\partial^2 u}{\partial x_0^2} - c^2 \Delta u\right)\right|_\Gamma = 0, \tag{1.9}$$

$$u|_\Gamma = \left.\left[\frac{\partial}{\partial \boldsymbol{\nu}}\left(\frac{\partial^2 u}{\partial x_0^2} - c^2 \Delta u\right) - \frac{\partial}{\partial x_0}\Delta u\right]\right|_\Gamma = 0, \tag{1.10}$$

$$\left.\frac{\partial u}{\partial \boldsymbol{\nu}}\right|_\Gamma = \left.\left(\frac{\partial^2 u}{\partial x_0^2} - c^2 \Delta u\right)\right|_\Gamma = 0, \tag{1.11}$$

$$\left.\left[(c^2 - a^2 - b^2)\frac{\partial^2 u}{\partial x_0^2} + a^2 b^2 \Delta u\right]\right|_\Gamma = \left.\frac{\partial}{\partial \boldsymbol{\nu}}\Delta u\right|_\Gamma = 0, \tag{1.12}$$

$$\left.\left(\frac{\partial u}{\partial x_0} + \frac{\partial u}{\partial \boldsymbol{\nu}}\right)\right|_\Gamma = \left.\left(\frac{\partial}{\partial x_0} - \frac{\partial}{\partial \boldsymbol{\nu}}\right)\Delta u\right|_\Gamma = 0, \tag{1.13}$$

$$\left.\frac{\partial u}{\partial \boldsymbol{\tau}^{(k)}}\right|_\Gamma = \left.\frac{\partial^2 u}{\partial \boldsymbol{\nu}^2}\right|_\Gamma = 0, \quad k = 1, \ldots, n, \tag{1.14}$$

$$\left.\frac{\partial u}{\partial \boldsymbol{\tau}^{(k)}}\right|_\Gamma = \left.\frac{\partial}{\partial \boldsymbol{\nu}}\left(\frac{\partial^2 u}{\partial x_0^2} - c^2 \Delta u\right)\right|_\Gamma = 0, \quad k = 1, \ldots, n, \tag{1.15}$$

$$\left.\frac{\partial u}{\partial \boldsymbol{\tau}^{(k)}}\right|_\Gamma = \left.\left[\frac{\partial}{\partial \boldsymbol{\nu}}\left(\frac{\partial^2 u}{\partial x_0^2} - c^2 \Delta u\right) - \frac{\partial}{\partial x_0}\Delta u\right]\right|_\Gamma = 0, \quad k = 1, \ldots, n, \tag{1.16}$$

where $\boldsymbol{\nu} = (\nu_0, \nu_1, \ldots, \nu_n)$ is the unit vector of the outer normal to the hypersurface Γ, $\boldsymbol{\tau}^{(k)} = (\tau_0^{(k)}, \ldots, \tau_n^{(k)})$ $(k = 1, \ldots, n)$ are linearly independent vectors tangent to the hypersurface Γ, and $b^2 < c^2 < a^2$.

Thus, taking into account various boundary conditions on Γ given by (1.6)–(1.16), we have eleven mixed problems for Eq. (1.1) with $A = \sum\limits_{i=1}^{n} \dfrac{\partial^2}{\partial x_i^2}$ and with the initial conditions (1.3). Strong solutions of those problems exist and are unique in suitable function spaces defined by energy inequalities.

Consider mixed problems for Eq. (1.1) in cylindrical domains $Q \subset \mathbb{R}^{n+1}$ with the operator

$$A = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a^{ij}(\mathbf{x}) \frac{\partial}{\partial x_j} \right).$$

The problems are posed according to the same scheme as in the case of the Laplace operator. We have to find the unique strong solution u satisfying Eq. (1.1), the initial conditions (1.3), and one of the following conditions on Γ:

$$u|_\Gamma = Au|_\Gamma = 0, \tag{1.17}$$

$$\left.\frac{\partial u}{\partial \mathcal{N}}\right|_\Gamma = \left.\frac{\partial}{\partial \mathcal{N}} Au\right|_\Gamma = 0, \quad \frac{\partial}{\partial \mathcal{N}} = \sum_{i,j=1}^{n} a^{(ij)} \frac{\partial}{\partial x_j} \nu_i, \tag{1.18}$$

$$\left.\frac{\partial u}{\partial x_0}\right|_\Gamma = Au|_\Gamma = 0, \tag{1.19}$$

$$\left.\frac{\partial^2 u}{\partial x_0^2}\right|_\Gamma = Au|_\Gamma = 0, \tag{1.20}$$

$$u|_\Gamma = \left.\frac{\partial}{\partial \mathcal{N}} \left(\frac{\partial^2 u}{\partial x_0^2} - c^2 Au \right)\right|_\Gamma = 0, \tag{1.21}$$

$$u|_\Gamma = \left.\left[\frac{\partial}{\partial \mathcal{N}} \left(\frac{\partial^2 u}{\partial x_0^2} - c^2 Au \right) - \frac{\partial}{\partial x_0} Au \right]\right|_\Gamma = 0, \tag{1.22}$$

$$\left.\frac{\partial u}{\partial \mathcal{N}}\right|_\Gamma = \left.\left(\frac{\partial^2 u}{\partial x_0^2} - c^2 Au \right)\right|_\Gamma = 0, \tag{1.23}$$

$$\left.\left[(c^2 - a^2 - b^2) \frac{\partial^2 u}{\partial x_0^2} + a^2 b^2 Au \right]\right|_\Gamma = \left.\frac{\partial}{\partial \mathcal{N}} Au\right|_\Gamma = 0, \tag{1.24}$$

$$\left.\left(\frac{\partial u}{\partial x_0} + \frac{\partial u}{\partial \mathcal{N}} \right)\right|_\Gamma = \left.\left(\frac{\partial}{\partial x_0} - \frac{\partial}{\partial \mathcal{N}} \right) Au\right|_\Gamma = 0, \tag{1.25}$$

$$\left.\frac{\partial u}{\partial \boldsymbol{\tau}^{(k)}}\right|_\Gamma = Au|_\Gamma = 0, \quad k = 1, \ldots, n, \tag{1.26}$$

$$\left.\frac{\partial u}{\partial \boldsymbol{\tau}^{(k)}}\right|_\Gamma = \left.\frac{\partial}{\partial \mathcal{N}} \left(\frac{\partial^2 u}{\partial x_0^2} - c^2 Au \right)\right|_\Gamma = 0, \quad k = 1, \ldots, n, \tag{1.27}$$

$$\left.\frac{\partial u}{\partial \boldsymbol{\tau}^{(k)}}\right|_\Gamma = \left.\left[\frac{\partial}{\partial \mathcal{N}} \left(\frac{\partial^2 u}{\partial x_0^2} - c^2 Au \right) - \frac{\partial}{\partial x_0} Au \right]\right|_\Gamma = 0, \quad k = 1, \ldots, n. \tag{1.28}$$

Thus, we have twelve mixed problems for Eq. (1.1). The differences are defined by particular boundary conditions (1.s), $s = 17 \ldots 28$, imposed on Γ in addition to the boundary conditions (1.3).

Other boundary-value problems are also possible (see Remark 4.1).

2. Energy Inequalities

Taking into account the boundary conditions (1.17)–(1.28), we represent the mixed problems (1.1), (1.3), and (1.17)–(1.28) in the operator form:

$$\mathbf{L}^{(k)}u = \mathbf{F}, \quad k = 17, \dots, 28, \tag{2.1}$$

where the operators $\mathbf{L}^{(k)}$ are defined by the rule $\mathbf{L}u = \{\mathcal{L}u, \mathbf{l}u\}$, $\mathcal{L}$ is the operator of Eq. (1.1), $\mathbf{l}$ is the operator of the initial conditions, $\mathbf{l}u = \{l_0u, l_1u, l_2u, l_3u\}$, and $\mathbf{F} = \{f, \varphi_0, \varphi_1, \varphi_2, \varphi_3\}$. For any $k \in \{17, \dots, 28\}$, the domain of the operator $\mathbf{L}^{(k)}$ of problem (1.1), (1.3), (1.k) is defined (according to the boundary condition (1.k) on Γ) as follows. The function $u\colon \mathbb{R}^{n+1} \supset \overline{Q} \ni \mathbf{x} \to u(\mathbf{x}) \in \mathbb{R}$ belongs to the domain $\mathcal{D}(\mathbf{L}^{(k)})$ of the operator $\mathbf{L}^{(k)}$ ($u \in \mathcal{D}(\mathbf{L}^{(k)})$) if it belongs to the set $C^4(\overline{Q})$ and satisfies the boundary condition with number (1.k) on Γ, where $C^4(\overline{Q})$ is the set of functions continuously differentiable on $\overline{Q}$ with all their derivatives up to the fourth order, where $Q \subset \mathbb{R}^{n+1}$, $k \in \{17, \dots, 28\}$.

Let $B^{(k)}$, $k \in \{17, \dots, 28\}$, denote the Banach spaces that are the closures of the sets $\mathcal{D}, (\mathbf{L}^{(k)})$ with respect to the norm

$$\|u\|_B = \sup_{0 \le x_0 \le T} \Bigg(\sum_{s=0}^{3} \left\| \frac{\partial^s u}{\partial x_0^s} \right\|_{L_2(\Omega^{(\tau)})} + \sum_{s=0}^{2} \sum_{i=1}^{n} \left\| \frac{\partial^{s+1} u}{\partial x_0^s \partial x_i} \right\|_{L_2(\Omega^{(\tau)})} + \sum_{i=0}^{n} \left\| \frac{\partial}{\partial x_i} Au \right\|_{L_2(\Omega^{(\tau)})} + \|Au\|_{L_2(\Omega^{(\tau)})} \Bigg)(\tau), \tag{2.2}$$

where $\Omega^{(\tau)}$ is the cross-section of the cylindrical domain $Q = (0, T) \times \Omega \subset \mathbb{R}^{n+1}$ by the plane

$$\{\mathbf{x} = (x_0, \mathbf{x}') \mid x_0 = \tau,\ \mathbf{x}' \in \mathbb{R}^n\}$$

and $\|\cdot\|_{L_2(\Omega^{(\tau)})}$ is the norm of the square-integrable (in the Lebesgue sense) functions defined on $\Omega^{(\tau)}$.

The Hilbert spaces of the right-hand sides $\mathbf{F}$ of Eqs. (2.1) are denoted by $\mathbf{H}^{(k)}$,

$$\mathbf{H}^{(k)} = L_2(Q) \times H_0^{(k)}(\Omega^{(0)}) \times H_1^{(k)}(\Omega^{(0)}) \times H_2^{(k)}(\Omega^{(0)}) \times H_3^{(k)}(\Omega^{(0)}). \tag{2.3}$$

Here $L_2(Q)$ is the space of functions square-integrable on Q. The space $H_s^{(k)}$ ($s = 0, 1, 2, 3$, $k = 17, \dots, 28$) is defined as the closure with respect to the norm $\|\cdot\|_{H_s(\Omega^{(0)})}$ of the set $C^{3,(k)}(\Omega^{(0)})$ of functions continuously differentiable with all their derivatives up to the third order (inclusively) defined on $\overline{\Omega^{(0)}}$ and satisfying the corresponding conditions (1.k), $k = 17, \dots, 28$, on the boundary $\partial\Omega^{(0)}$ of the domain $\Omega^{(0)}$ (considered instead of Γ). The Hilbert space $H_0^{(k)}(\Omega^{(0)})$ is the closure of the set $C^{3,(k)}(\overline{\Omega^{(0)}})$ with respect to the norm

$$\|\cdot\|_{H_0(\Omega^{(0)})} = \|\cdot\|_{L_2(\Omega^{(0)})} + \sum_{i=1}^{n} \left\| \frac{\partial \cdot}{\partial x_i} \right\|_{L_2(\Omega^{(0)})} + \|A(0, \mathbf{x}') \cdot\|_{L_2(\Omega^{(0)})} + \left\| \frac{\partial}{\partial x_i} A(0, \mathbf{x}') \cdot \right\|_{L_2(\Omega^{(0)})}, \tag{2.4}$$

where

$$A(0, \mathbf{x}') \cdot = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a^{(ij)}(0, \mathbf{x}') \frac{\partial}{\partial x_j} \cdot \right).$$

Similarly, $H_1^{(k)}(\Omega^{(0)})$ is the closure of the set $C^{3,(k)}(\overline{\Omega^{(0)}})$ with respect to the norm

$$\|\cdot\|_{H_1(\Omega^{(0)})} = \|\cdot\|_{L_2(\Omega^{(0)})} + \sum_{i=1}^{n} \left\| \frac{\partial \cdot}{\partial x_i} \right\|_{L_2(\Omega^{(0)})} + \|A(0, \mathbf{x}') \cdot\|_{L_2(\Omega^{(0)})}; \tag{2.5}$$

$H_2^{(k)}(\Omega^{(0)})$ is the closure of the set $C^{3,(k)}(\overline{\Omega^{(0)}})$ with respect to the norm

$$\|\cdot\|_{H_2(\Omega^{(0)})} = \|\cdot\|_{L_2(\Omega^{(0)})} + \sum_{i=1}^{n} \left\| \frac{\partial \cdot}{\partial x_i} \right\|_{L_2(\Omega^{(0)})}; \tag{2.6}$$

and $H_3^{(k)}(\Omega^{(0)}) = L_2(\Omega^{(0)})$.

Condition 2.1. The boundary $\partial\Omega = \partial\Omega^{(\tau)}$ $(0 \le \tau \le T)$ of the domain Ω is piecewise smooth, and the derivatives of the spaces $B^{(k)}$ and $H_0^{(k)}$ included in the norms (2.2) and (2.4) belong to $L_2(\Omega^{(\tau)})$.

Consider the mixed problems (1.1), (1.3), (1.k), $k \in \{17, \dots, 28\}$, in the case where the operator $A^{(3)}$ in Eq. (1.1) is of the form

$$A^{(3)}u = \sum_{s=0}^{3} a^{(s0)}(\mathbf{x}) \frac{\partial^s u}{\partial x_0^s} + \sum_{s=0}^{2} \sum_{i=1}^{n} a^{(si)}(\mathbf{x}) \frac{\partial^{s+1} u}{\partial x_0^s \partial x_i} + \sum_{i=1}^{n} a^{(0i2)}(\mathbf{x}) \frac{\partial}{\partial x_i} Au + a^{(002)}(\mathbf{x}) Au. \tag{2.7}$$

Theorem 2.1. *Let Conditions* 1.1 *and* 2.1 *be satisfied, the coefficients of the operator $A^{(3)}$ be integrable, and the operator $A^{(3)}$ be of the form* (2.7)*. Let the coefficients $a^{(ij)}$ $(i, j = 1, \dots, n)$ of the operator A not depend on x_0. Then there exists a constant $c > 0$ which does not depend on u and such that the relation*

$$\|u\|_B \le c \|\mathbf{L}^{(k)}u\|_{\mathbf{H}^{(k)}} \tag{2.8}$$

holds for all functions $u \in \mathcal{D}(\mathbf{L}^{(k)})$ and all $k \in \{17, \dots, 28\}$.

Proof. To deduce the energy inequality (2.8) for the operator $\mathcal{L}^{(1)}$, we take the separating operator

$$\mathcal{M} = 2\frac{\partial}{\partial x_0}\left(\frac{\partial^2}{\partial x_0^2} - c^2 A\right),$$

where $b^2 < c^2 < a^2$. The product $\mathcal{L}^{(1)}u\mathcal{M}u$ is transformed as follows:

$$\mathcal{L}^{(1)}u\mathcal{M}u = \frac{\partial}{\partial x_0}(\mathcal{F}u(\mathbf{x})) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\mathcal{G}_i u(\mathbf{x})) + \mathcal{A}^{(3)}(u, u), \tag{2.9}$$

where

$$\begin{aligned}\mathcal{F}u(\mathbf{x}) = {} & \left(\frac{\partial^3 u}{\partial x_0^3}\right)^2 + (a^2 + b^2 - c^2) \sum_{i,j=1}^{n} a^{(ij)} \frac{\partial^3 u}{\partial x_0^2 \partial x_i} \frac{\partial^3 u}{\partial x_0^2 \partial x_j} + [c^2(a^2 + b^2) - a^2 b^2] \frac{\partial}{\partial x_0} Au \\ & + a^2 b^2 c^2 \sum_{i,j=1}^{n} a^{(ij)} \frac{\partial}{\partial x_i} Au \frac{\partial}{\partial x_j} Au - 2c^2 \frac{\partial^3 u}{\partial x_0^3} \frac{\partial}{\partial x_0} Au - 2a^2 b^2 \sum_{i,j=1}^{n} a^{(ij)} \frac{\partial^3}{\partial x_0^2 \partial x_i} \frac{\partial}{\partial x_j} Au,\end{aligned}$$

$$\begin{aligned}\mathcal{G}_i u(\mathbf{x}) = {} & 2[(c^2 - a^2 - b^2) + a^2 b^2] \frac{\partial^3 u}{\partial x_0^3} \sum_{j=1}^{n} a^{(ij)} \frac{\partial^3 u}{\partial x_0^2 \partial x_j} + 2a^2 b^2 \frac{\partial}{\partial x_0} Au \sum_{j=1}^{n} a^{(ij)} \frac{\partial^3 u}{\partial x_0^2 \partial x_j} \\ & - 2a^2 b^2 c^2 \frac{\partial}{\partial x_0} Au \sum_{j=1}^{n} a^{(ij)} \frac{\partial}{\partial x_j} Au,\end{aligned}$$

and $\mathcal{A}^{(3)}(u, u)$ is the quadratic form generated by the function u and its derivatives contained in the operator $A^{(3)}$.

We integrate (2.9) over the domain $Q^\tau = \{x \in Q \mid 0 \le x_0 \le \tau \le T\}$, taking into account the boundary conditions (1.k) corresponding to the operators $\mathbf{L}^{(k)}$, $k \in \{17, \dots, 28\}$. By virtue of those conditions, the following inequality holds:

$$\sum_{i=1}^{n} \int\limits_{\Gamma^\tau} \mathcal{G}_i u(\mathbf{x}) \nu_i \, ds \geq 0, \tag{2.10}$$

where $\Gamma^\tau = (0,\tau)\times\partial\Omega$. The vector of the outer (with respect to the domain Q) normal $\boldsymbol{\nu} = (\nu_0,\dots,\nu_n)$ on Γ^τ is orthogonal to the axis x_0. If it is situated on the hypersurfaces $\Omega^{(0)}\cup\Omega^{(\tau)}$, then it is orthogonal to the coordinate axes $x_1,\dots,x_n$. Taking into account this fact and inequality (2.10), we integrate (2.9) over the domain Q^τ. This yields the inequality

$$\int\limits_{\Omega^{(\tau)}\cup\Omega^{(0)}} \mathcal{F}u(\mathbf{x})\nu_0 \, \mathrm{d}\mathbf{x}' \leq \int\limits_{Q^\tau} \mathcal{L}^{(1)} u \mathcal{M} u \, \mathrm{d}\mathbf{x} - \int\limits_{Q^\tau} \mathcal{A}^{(3)}(u,u)\, \mathrm{d}\mathbf{x}. \tag{2.11}$$

Treating $\mathcal{F}u(\mathbf{x})$ as a quadratic form of $\dfrac{\partial^3 u}{\partial x_0^2 \partial x_i}$, $\dfrac{\partial}{\partial x_i} A u$ $(i = 0,\dots,n)$, and $\dfrac{\partial}{\partial x_j} A u$ $(j = 1,\dots,n)$, we estimate the integral $\int\limits_{\Omega^{(\tau)}} \mathcal{F}u(\tau,\mathbf{x}')\, \mathrm{d}\mathbf{x}'$ from below and the integral $\int\limits_{\Omega^{(0)}} \mathcal{F}u(0,\mathbf{x}')\, \mathrm{d}\mathbf{x}'$ from above. The matrix $[\mathcal{F}]$ of the quadratic form $\mathcal{F}(\tau,\mathbf{x}')$ is as follows:

$$[\mathcal{F}] = \begin{pmatrix} M_0 & 0 & \dots & 0 \\ 0 & M_1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & M_n \end{pmatrix}, \tag{2.12}$$

where

$$M_0 = \begin{pmatrix} 1 & -c^2 \\ -c^2 & c^2(a^2+b^2) - a^2b^2 \end{pmatrix}, \qquad M_i = \begin{pmatrix} a^2+b^2-c^2 & -a^2b^2 \\ -a^2b^2 & a^2b^2c^2 \end{pmatrix}, \quad i = 1,2,\dots,n.$$

If $b^2 < c^2 < a^2$, then one can verify that all principal minors of matrix (2.12) are positive. Therefore, there exists a constant $c^{(1)} > 0$ such that

$$\int\limits_{\Omega^{(\tau)}} \mathcal{F}u(\tau,\mathbf{x}')\, \mathrm{d}\mathbf{x}' \geq c^{(1)} \sum_{i=0}^{n} \left(\left\| \frac{\partial^3 u}{\partial x_0^2 \partial x_i} \right\|^2_{L_2(\Omega^{(\tau)})} + \left\| \frac{\partial}{\partial x_i} A u \right\|^2_{L_2(\Omega^{(\tau)})} \right)(\tau). \tag{2.13}$$

Estimating $\int\limits_{\Omega^{(0)}} \mathcal{F}u(0,\mathbf{x}')\mathrm{d}\mathbf{x}'$ from above, we obtain the inequality

$$\int\limits_{\Omega^{(0)}} \mathcal{F}u(0,\mathbf{x}')\mathrm{d}\mathbf{x}' \leq c^{(2)} \sum_{k=0}^{3} \|l_k u\|^2_{H_k(\Omega^{(0)})}, \quad c^{(2)} > 0. \tag{2.14}$$

By virtue of the Cauchy–Bunyakovsky inequality in the space $L_2(Q^\tau)$ of square-integrable functions, there exists a positive constant $c^{(3)}$ such that

$$\begin{aligned} \left| (\mathcal{L}^{(1)} u, \mathcal{M} u)_{L_2(Q^\tau)} \right| + \left| \int\limits_{Q^\tau} \mathcal{A}(u,u) \mathrm{d}\mathbf{x} \right| &\leq c^{(3)} \Bigg(\|\mathcal{L}^{(1)} u\|^2_{L_2(Q^\tau)} + \sum_{s=0}^{3} \left\| \frac{\partial^s u}{\partial x_0^s} \right\|_{L_2(Q^\tau)} \\ &+ \sum_{s=0}^{2} \sum_{i=1}^{n} \left\| \frac{\partial^{s+1} u}{\partial x_0^s \partial x_i} \right\|^2_{L_2(Q^\tau)} + \sum_{i=0}^{n} \left\| \frac{\partial}{\partial x_i} A u \right\|^2_{L_2(Q^\tau)} + \|Au\|^2_{L_2(Q^\tau)} \Bigg). \end{aligned} \tag{2.15}$$

Using relation (2.9) and estimates (2.10), (2.11), and (2.13)–(2.15), we see that there exists a constant $c^{(4)} > 0$ which does not depend on u and such that the following inequality holds:

$$\sum_{i=0}^{n}\left(\left\|\frac{\partial^3 u}{\partial x_0^2 \partial x_i}\right\|_{L_2(\Omega^{(\tau)})}+\left\|\frac{\partial}{\partial x_i}Au\right\|_{L_2(\Omega^{(\tau)})}\right)(\tau)\le c^{(4)}\|\mathbf{L}^{(k)}u\|_{\mathbf{H}^{(k)}}$$

$$+c^{(4)}\int_0^\tau\left(\sum_{s=0}^{3}\left\|\frac{\partial^s u}{\partial x_0^s}\right\|_{L_2(\Omega^{(t)})}+\sum_{s=0}^{2}\sum_{i=1}^{n}\left\|\frac{\partial^{s+1}u}{\partial x_0^s\partial x_i}\right\|_{L_2(\Omega^{(t)})}+\sum_{i=0}^{n}\left\|\frac{\partial}{\partial x_i}Au\right\|_{L_2(\Omega^{(t)})}+\|Au\|_{L_2(\Omega^{(t)})}\right)(t)\,\mathrm{d}t$$

$$=c^{(4)}\|\mathbf{L}^{(k)}u\|_{\mathbf{H}^{(k)}}+c^{(4)}\int_0^\tau(\mathcal{F}^{(1)}u)(t)\,\mathrm{d}t. \quad (2.16)$$

To apply the Gronwall inequality to (2.16), we include the terms

$$\sum_{s=0}^{2}\left\|\frac{\partial^s u}{\partial x_0^s}\right\|_{L_2(\Omega^{(\tau)})}(\tau),\qquad \sum_{s=0}^{1}\sum_{i=1}^{n}\left\|\frac{\partial^{s+1}u}{\partial x_0^s\partial x_i}\right\|_{L_2(\Omega^{(\tau)})}(\tau),\qquad \|Au\|_{L_2(\Omega^{(\tau)})}(\tau)$$

in the left-hand side of the inequality (2.16). It suffices to integrate the relations

$$\sum_{s=0}^{2}\frac{\partial}{\partial x_0}\left(\frac{\partial^s u}{\partial x_0^s}\right)^2=2\sum_{s=0}^{2}\frac{\partial^{s+1}u}{\partial x_0^{s+1}}\frac{\partial^s u}{\partial x_0^s},$$

$$\sum_{s=0}^{1}\sum_{i=1}^{n}\frac{\partial}{\partial x_0}\left(\frac{\partial^{s+1}u}{\partial x_0^s\partial x_i}\right)^2=2\sum_{s=0}^{1}\sum_{i=1}^{n}\frac{\partial^{s+2}u}{\partial x_0^{s+1}\partial x_i}\frac{\partial^{s+1}u}{\partial x_0^s\partial x_i},$$

$$\frac{\partial}{\partial x_0}(Au)^2=2\frac{\partial}{\partial x_0}Au\,Au$$

over the domain Q^τ, apply the Cauchy–Bunyakovsky inequality to the right-hand sides of the obtained relations, sum up the results, and add inequality (2.16). This yields the inequality

$$(\mathcal{F}^{(1)}u)(\tau)\le c^{(5)}\|\mathbf{L}^{(k)}u\|_{\mathbf{H}^{(k)}}+c^{(5)}\int_0^\tau(\mathcal{F}^{(1)}u)(t)\,\mathrm{d}t. \quad (2.17)$$

Applying the Gronwall inequality to (2.17), we see that

$$(\mathcal{F}^{(1)}u)(\tau)\le c^{(5)}e^{c^{(5)}\tau}\|\mathbf{L}^{(k)}u\|_{\mathbf{H}^{(k)}}\le c^{(5)}e^{c^{(5)}T}\|\mathbf{L}^{(k)}u\|_{\mathbf{H}^{(k)}}. \quad (2.18)$$

The energy inequalities (2.8), $k\in\{17,\dots,28\}$, are easily deduced from (2.18). □

3. Strong Solutions

For any $k\in\{17,\dots,28\}$, problem (1.1), (1.3), (1.k) is treated as the operator equation (2.1) from the Banach space $B^{(k)}$ to the Hilbert space $\mathbf{H}^{(k)}$ with domain $\mathcal{D}(\mathbf{L}^{(k)})$.

Lemma 3.1. *The operator $\mathbf{L}^{(k)}$ from the Banach space $B^{(k)}$ to the Hilbert space $\mathbf{H}^{(k)}$ admits a closure $\overline{\mathbf{L}^{(k)}}$.*

Proof. Let a sequence $\{u^{(m)}\}_{m=1}^\infty$ of functions $u^{(m)}\in\mathcal{D}(\mathbf{L}^{(k)})$ tend to zero with respect to the norm of the space $B^{(k)}$, i.e., $\lim\limits_{m\to\infty}\|u^{(m)}\|_B=0$. Since $\|l_p u^{(m)}\|_{H_p}\le\|u^{(m)}\|_B$, it follows that $\|l_p u^{(m)}\|_{H_p}\to 0$ as $m\to\infty$.

Consider the sequence $\{\mathcal{L}^{(1)}u^{(m)}\}$. For any $v\in C_0^\infty(Q)$, we have

$$(\mathcal{L}^{(1)}u^{(m)},v)_{L_2(Q)}=(u^{(m)},\mathcal{L}^{(1)'}v)_{L_2(Q)}, \quad (3.1)$$

where $C_0^\infty(Q)$ is the set of compactly supported, infinitely differentiable functions defined in the domain Q and $\mathcal{L}^{(1)'}$ is the operator formally adjoint to $\mathcal{L}^{(1)}$. Since $u^{(m)}\to 0$ in $B^{(k)}$, $m\to\infty$, it follows that $u^{(m)} \to 0$ in $L_2(Q)$, $m \to \infty$. It follows from (3.1) that $\mathcal{L}^{(1)} u^{(m)} \to 0$ in $L_2(Q)$, $m \to \infty$, because the set $C_0^\infty(Q)$ is dense in $L_2(Q)$.

Thus, the sequence $\{\mathbf{L}^{(k)} u^{(m)}\}_{m=1}^\infty$ tends to zero as $m \to \infty$ with respect to the norm of the space $\mathbf{H}^{(k)}$, i.e., the operator $\mathbf{L}^{(k)}$ admits the closure $\overline{\mathbf{L}^{(k)}}$ from $B^{(k)}$ to $\mathbf{H}^{(k)}$. □

Let $\mathcal{D}(\overline{\mathbf{L}^{(k)}})$ denote the domain of the operator $\overline{\mathbf{L}^{(k)}}$. The closed operator $\overline{\mathbf{L}^{(k)}}$ obtained by the closure of $\mathbf{L}^{(k)}$ satisfies the energy inequality as well.

Corollary 3.1. *Passing to the limit in inequality* (2.8)*, one can obtain the energy inequality*

$$\|u\|_B \leq c \left\| \overline{\mathbf{L}^{(k)}} \right\|_{\mathbf{H}^{(k)}} \tag{3.2}$$

for the operator $\overline{\mathbf{L}^{(k)}}$*, where the constant* $c > 0$ *does not depend on the function* $u \in \mathcal{D}(\overline{\mathbf{L}^{(k)}})$.

Lemma 3.2. *The following relations are valid for the operators* $\mathbf{L}^{(k)}$ *and* $\overline{\mathbf{L}^{(k)}}$:

(1) $\overline{\mathcal{R}\left(\mathbf{L}^{(k)}\right)} = \mathcal{R}(\overline{\mathbf{L}^{(k)}})$,

(2) $\overline{\mathbf{L}^{(k)-1}} = \overline{\mathbf{L}^{(k)}}^{\,-1}$,

where $\overline{\mathbf{L}^{(k)}}^{\,-1}$ *is the operator inverse to the operator* $\overline{\mathbf{L}^{(k)}}$ *and defined on the range* $\mathcal{R}(\overline{\mathbf{L}^{(k)}})$ *of the operator* $\overline{\mathbf{L}^{(k)}}$, $k \in \{17, \ldots, 28\}$.

Proof. The embedding $\mathcal{R}(\overline{\mathbf{L}^{(k)}}) \subset \overline{\mathcal{R}(\mathbf{L}^{(k)})}$ follows from the definition of closed sets. Let us prove the inverse embedding $\overline{\mathcal{R}(\mathbf{L}^{(k)})} \subset \mathcal{R}(\overline{\mathbf{L}^{(k)}})$. Let $\mathbf{F} \in \overline{\mathcal{R}(\mathbf{L}^{(k)})}$. Then there exists a sequence $\{\mathbf{F}^{(m)}\}_{m=1}^\infty$, $\mathbf{F}^{(m)} \in \mathcal{R}(\mathbf{L}^{(k)})$, converging to $\mathbf{F}$ in $\mathbf{H}^{(k)}$ as $m \to \infty$. Therefore, the sequence $\{\mathbf{F}^{(m)}\}_{m=1}^\infty$ is fundamental and $\mathbf{F}^{(m)} = \mathbf{L}^{(k)} u^{(m)}$, $u^{(m)} \in \mathcal{D}(\mathbf{L}^{(k)})$. It follows from the energy inequality (2.8) that the sequence $\{u^{(m)}\}_{m=1}^\infty$ is fundamental in the space $B^{(k)}$ as well. Since $B^{(k)}$ is a Banach space, it follows that there exists $u \in B^{(k)}$ such that $u^{(m)} \to u$ with respect to the norm of the space $B^{(k)}$. Due to the definition of the operator $\overline{\mathbf{L}^{(k)}}$, this means that $u^{(m)} \in \mathcal{D}(\overline{\mathbf{L}^{(k)}})$ and $\overline{\mathbf{L}^{(k)}} u = \mathbf{F}$, i.e., $\mathbf{F} \in \mathcal{R}(\overline{\mathbf{L}^{(k)}})$. This and the relation $\mathcal{D}(\overline{\mathbf{L}^{(k)-1}}) = \mathcal{D}(\overline{\mathbf{L}^{(k)}}^{\,-1})$ imply assertion (2). □

Definition 3.1. Any solution of the operator equation

$$\overline{\mathbf{L}^{(k)}} u = \mathbf{F} \tag{3.3}$$

is called a *strong solution* of problem (1.1), (1.3), (1.k), $k \in \{17, \ldots, 28\}$.

Corollary 3.2. *It follows from inequality* (3.2) *that problem* (1.1)*,* (1.3)*,* (1.k)*,* $k \in \{17, \ldots, 28\}$*, has at most one strong solution.*

According to assertions of Lemma 3.2, to prove the existence of a strong solution of problem (1.1), (1.3), (1.k) for any $\mathbf{F} \in \mathbf{H}^{(k)}$ in Eq. (3.3), it suffices to prove the density of the range $\mathcal{R}(\mathbf{L}^{(k)})$ of the operator $\mathbf{L}^{(k)}$ in the Hilbert space $\mathbf{H}^{(k)}$.

First, we prove this assertion for the principal part $\mathcal{L}^{(1,0)}$ of the operator $\mathcal{L}^{(1)}$, i.e., for

$$\mathcal{L}^{(1,0)} u = \left(\frac{\partial^2}{\partial x_0^2} - a^2 A\right)\left(\frac{\partial^2}{\partial x_0^2} - b^2 A\right)$$

and

$$\mathbf{L}^{(k,0)} = \{\mathcal{L}^{(1,0)}, l_0, l_1, l_2, l_3\}.$$

Let $\mathcal{D}^{(0)}(\mathbf{L}^{(k,0)})$ denote the set of functions $u \in \mathcal{D}(\mathbf{L}^{(k)})$ such that $l_0 u = l_1 u = l_2 u = l_3 u = 0$. Then the following lemma is true.

Lemma 3.3. *Let Conditions* 1.1 *and* 2.1 *be satisfied. If there exists* $v \in L_2(Q)$ *such that*

$$(\mathcal{L}^{(1,0)}u, v)_{L_2(Q)} = 0 \tag{3.4}$$

for any $u \in \mathcal{D}^{(0)}(\mathbf{L}^{(k,0)})$*, then* $v = 0$ *in* $L_2(Q)$.

To prove Lemma 3.3, we use averaging operators with a variable step (see [8, 10, 32]) $J_{(s)}u$ and $J^*_{(s)}u$ defined by means of the partition of unity for the domain Q.

The domain Q is decomposed into subdomains $Q^{(\varepsilon)}$ as follows. The subdomain $Q^{(\varepsilon)}$ is the set of points $\mathbf{x} \in Q$ such that the distance between any of those points and the boundary is greater than or equal to ε. We also introduce the subdomains $G^{(m)}$, where $G^{(-1)} = \varnothing$, $G^{(0)} = Q^{(1/2)}, \ldots$, $G^{(m)} = Q^{\left(1/2^{m+1}\right)} - Q^{(1/2^m)}$, $m = 1, 2, \ldots$, and $\varnothing$ is the empty set. The subdomains $G^{(m)}$ get concentrated in a neighborhood of the boundary ∂Q as their volumes decrease.

The following partition-of-unity lemma is valid (see [8]).

Lemma 3.4. *There exists a subsequence of nonnegative functions* $\psi^{(m)}\colon \mathbb{R}^{n+1} \ni \mathbf{x} \to \psi^{(m)}(\mathbf{x}) \in \mathbb{R}$, $\psi^{(m)} \in C^\infty(\mathbb{R}^{n+1})$; *such that*

- $\sum\limits_{m=0}^{\infty} \psi^{(m)}(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in Q, \\ 0, & \mathbf{x} \notin Q; \end{cases}$
- $Q = \sum\limits_{m=0}^{\infty} \operatorname{supp} \psi^{(m)}$ *and the multiplicity of the coverage of the set* Q *by the sets* $\operatorname{supp} \psi^{(m)}$ *does not exceed two*;
- $\operatorname{supp} \psi^{(m)} \subset G^{(m-1)} \cup G^{(m)} \cup G^{(m+1)}$, $m = 0, 1, \ldots$;
- *for any multi-index* $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_n)$*, the estimate* $\mathbf{D}^{\boldsymbol{\alpha}}\psi^{(m)}(\mathbf{x}) \leq C^{(\boldsymbol{\alpha})}2^{m|\boldsymbol{\alpha}|}$ *holds, where* $C^{(\boldsymbol{\alpha})}$ *does not depend on* $\boldsymbol{\alpha}$, $|\boldsymbol{\alpha}| = \alpha_0 + \ldots + \alpha_n$, $\operatorname{supp}\psi^{(m)} = \overline{\{\mathbf{x} \mid \psi^{(m)}(\mathbf{x}) \neq 0\}}$*, and* $C^\infty(\mathbb{R}^{n+1})$ *is the set of infinitely differentiable functions in* $\mathbb{R}^{n+1}$.

The averaging operators with a variable step are introduced as follows:

$$J_{(s)}u(\mathbf{x}) = \sum_{m=0}^{\infty} \psi^{(m)}(\mathbf{x}) J_{\delta_{(ms)}} u(\mathbf{x}), \qquad J^*_{(s)}u(\mathbf{x}) = \sum_{m=0}^{\infty} J_{\delta_{(ms)}}(\psi^{(m)}u)(\mathbf{x}), \tag{3.5}$$

where $J_{\delta_{(ms)}}$ are the Sobolev averaging operators (see [78]), $\psi^{(m)}$ are the functions forming the partition of unity, and $\delta_{(ms)} \leq 2^{-m-4}$. The averaging operators (3.5) preserve the boundary conditions for the function u on ∂Q.

Proof of Lemma 3.3. We treat the operator $\mathcal{L}^{(1,0)}$ in (3.4) as the composition $\mathcal{L}^{(1,0)} = \mathcal{L}^{(1,0)}_a \cdot \mathcal{L}^{(1,0)}_b$ of the two operators $\mathcal{L}^{(1,0)}_a = \dfrac{\partial^2}{\partial x_0^2} - a^2 A$ and $\mathcal{L}^{(1,0)}_b = \dfrac{\partial^2}{\partial x_0^2} - b^2 A$. The domain $\mathcal{D}(\mathcal{L}^{(1,0)}_b)$ of the operator $\mathcal{L}^{(1,0)}_b$ coincides with the domain $\mathcal{D}(\mathbf{L}^{(k,0)})$, while the domain $\mathcal{D}(\mathcal{L}^{(1,0)}_a)$ is the range $\mathcal{R}(\mathcal{L}^{(1,0)}_b)$ of the operator $\mathcal{L}^{(1,0)}_b$.

Since $C_0^\infty(Q) \subset \mathcal{D}^{(0)}(\mathbf{L}^{(k)})$, it follows that the domain $\mathcal{D}(\mathcal{L}^{(1,0)}_b)$ is dense in $L_2(Q)$. Let us prove that the range $\mathcal{R}(\mathcal{L}^{(1,0)}_b)$ is a dense set as well. This means that

$$(\mathcal{L}^{(1,0)}_b u, v)_{L_2(Q)} = 0 \tag{3.6}$$

for any u from $\mathcal{D}(\mathcal{L}^{(1,0)}_b)$ if and only if $v = 0$ in $L_2(Q)$.

To prove the latter assertion we substitute $J_{(s)}u$ for the function u in relation (3.6), where $J_{(s)}$ is the averaging operator with a variable step from (3.5). This choice is allowed because $J_{(s)}u \in \mathcal{D}^{(0)}(\mathbf{L}^{(k)})$ for $u \in \mathcal{D}^{(0)}(\mathbf{L}^{(k)})$ by virtue of the properties of the operators $J_{(s)}$ and $J^*_{(s)}$ preserving the boundary conditions for the averaged functions (see [9, 42]). Then relation (3.6) can be represented as follows:

$$(\mathcal{L}_b^{(1,0)} J_{(s)} u, v)_{L_2(Q)} = (J_{(s)} \mathcal{L}_b^{(1,0)} u, v)_{L_2(Q)} + (\mathcal{L}_b^{(1,0)} J_{(s)} u - J_{(s)} \mathcal{L}_b^{(1,0)} u, v)_{L_2(Q)}$$
$$= (\mathcal{L}_b^{(1,0)} u, J_{(s)}^* v)_{L_2(Q)} + (Ku, v)_{L_2(Q)} = 0. \quad (3.7)$$

We represent the commutator K as follows:

$$Ku = \left(\frac{\partial^2}{\partial x_0^2} - b^2 A\right) \sum_{m=0}^{\infty} J_{\delta_{(ms)}} (\psi^{(m)} u)(\mathbf{x}) - \sum_{m=0}^{\infty} \psi^{(m)}(\mathbf{x}) J_{\delta_{(ms)}} \left(\frac{\partial^2 u}{\partial x_0^2} - b^2 A u\right)(\mathbf{x})$$
$$= R^{(0)} u + \sum_{i=0}^{n} K^{(i)} \frac{\partial u}{\partial x_i},$$

where

$$R^{(0)} u = \sum_{m=0}^{\infty} \left[\frac{\partial^2 \psi^{(m)}}{\partial x_0^2} J_{\delta_{(ms)}} u - \sum_{i,j=1}^{n} \left\{ \frac{\partial a^{(ij)}}{\partial x_i} \frac{\partial \psi^{(m)}}{\partial x_j} J_{\delta_{(ms)}} u + a^{(ij)} \frac{\partial^2 \psi^{(m)}}{\partial x_i \partial x_j} J_{\delta_{(ms)}} u \right.\right.$$
$$\left.\left. - \psi^{(m)}(\mathbf{x}) \delta_{(ms)}^{-n-1} \int_Q \frac{\partial}{\partial z_i} \left[\omega\left(\frac{\mathbf{x}-\mathbf{z}}{\delta_{(ms)}}\right) \left(\frac{\partial a^{(ij)}(\mathbf{x})}{\partial x_j} - \frac{\partial a^{(ij)}(\mathbf{z})}{\partial z_j} \right) \right] u(\mathbf{z})\, \mathrm{d}\mathbf{z} \right\} \right],$$

$$K^{(0)} \frac{\partial u}{\partial x_0} = 2 \sum_{m=0}^{\infty} \frac{\partial \psi^{(m)}}{\partial x_0} J_{\delta_{(ms)}} \frac{\partial u}{\partial x_0},$$

and

$$K^{(i)} \frac{\partial u}{\partial x_i} = \sum_{m=0}^{\infty} \sum_{j=1}^{n} \left\{ \psi^{(m)}(\mathbf{x}) \delta_{(ms)}^{-n-1} \int_Q \frac{\partial}{\partial z_j} \left[\omega\left(\frac{\mathbf{x}-\mathbf{z}}{\delta_{(ms)}}\right) \left(a^{(ij)}(\mathbf{x}) - a^{(ij)}(\mathbf{z}) \right) \right] \frac{\partial u(\mathbf{z})}{\partial z_i}\, \mathrm{d}\mathbf{z} \right.$$
$$\left. - 2 a^{(ij)} \frac{\partial \psi^{(m)}}{\partial x_j} J_{\delta_{(ms)}} \frac{\partial u}{\partial x_i} \right\}, \quad i = 1, \ldots, n.$$

Integrating the left-hand side of (3.7) by parts, we obtain the relation

$$\left(u, \mathcal{L}_b^{(1,0)} J_{(s)}^* v\right)_{L_2(Q)} + \left(u, R^{(0)*} v\right)_{L_2(Q)} - \sum_{i=0}^{n} \left(u, \frac{\partial}{\partial x_i} K^{(i)*} v\right)_{L_2(Q)} + \mathcal{M}(u, v; \partial Q) = 0, \qquad (3.8)$$

where $R^{(0)*}$ and $K^{(i)*}$ $(i = 0, \ldots, n)$ are the operators adjoint to the operators $R^{(0)}$ and $K^{(i)}$, respectively, while $\mathcal{M}(u, v; \partial Q)$ is the set of boundary terms which appear after integrating by parts in (3.7).

Varying the function $u \in \mathcal{D}^{(0)}(\mathbf{L}^{(1,k)})$ in relation (3.8), one can prove that (3.8) is satisfied if

$$\left(u, \mathcal{L}_b^{(1,0)} J_{(s)}^* v\right)_{L_2(Q)} + \left(u, R^{(0)*} v\right)_{L_2(Q)} - \sum_{i=0}^{n} \left(u, \frac{\partial}{\partial x_i} K^{(i)*} v\right)_{L_2(Q)} = 0 \qquad (3.9)$$

and

$$\mathcal{M}(u, v; \partial Q) = 0 \qquad (3.10)$$

for any $u \in \mathcal{D}^{(0)}(\mathbf{L}^{(1,k)})$. Using a more detailed form of relation (3.10) (see [37, 42]), one can conclude that the function $J_{(s)}^* v$ satisfies the boundary conditions

$$J_{(s)}^* v\Big|_{x_0=T} = \frac{\partial}{\partial x_0} J_{(s)}^* v\Big|_{x_0=T} = 0, \qquad (3.11)$$

$$J^*_{(s)}v\Big|_\Gamma = 0 \quad \text{or} \quad \frac{\partial}{\partial\mathcal{N}}J^*_{(s)}v\Big|_\Gamma = 0,$$
$$K^{(i)^*}v\Big|_\Gamma = 0, \qquad i = 1,\ldots,n. \tag{3.12}$$

Besides (3.12) other boundary conditions can be imposed on Γ, but conditions (3.12) are sufficient to prove that $v = 0$ in $L_2(Q)$.

Let $\widetilde{Q}^\tau$ denote the following subdomain of Q: $\widetilde{Q}^\tau = \{\mathbf{x} \in Q \mid \tau < x_0 < T\}$. The boundary $\partial\widetilde{Q}^\tau$ consists of the upper base $\Omega^{(T)}$, lower base $\Omega^{(\tau)}$, and lateral surface $\widetilde{\Gamma}^{(\tau)} = \{\mathbf{x} \in \Gamma \mid \tau < x_0 < T\}$.

Since the set $\mathcal{D}^{(0)}(\mathbf{L}^{(k)})$ is dense in $L_2(Q)$, it follows that relation (3.9) holds for any $u \in L_2(Q)$. In particular, in (3.9), we assign

$$u(\mathbf{x}) = \begin{cases} Jv(\mathbf{x}), & \mathbf{x} \in \widetilde{Q}^\tau, \\ 0, & \mathbf{x} \in Q^\tau, \end{cases} \tag{3.13}$$

where $Jv = \int\limits_\tau^{x_0} J^*_{(s)}v(t,\mathbf{x}')\,\mathrm{d}t$, $\mathbf{x}' = (x_1,\ldots,x_n)$. Substituting the function u given by (3.13) into (3.9), yields

$$\left(Jv, \frac{\partial^2}{\partial x_0^2}J^*_{(s)}v\right)_{L_2(\widetilde{Q}^\tau)} - b^2\sum_{i,j=1}^n \left(Jv, \frac{\partial}{\partial x_i}\left(a^{(ij)}\frac{\partial J^*_{(s)}v}{\partial x_j}\right)\right)_{L_2(\widetilde{Q}^\tau)}$$
$$+ \left(Jv, R^{(0)^*}J^*_{(s)}v\right)_{L_2(\widetilde{Q}^\tau)} - \sum_{i=0}^n \left(Jv, \frac{\partial}{\partial x_i}K^{(i)^*}v\right)_{L_2(\widetilde{Q}^\tau)} = 0. \tag{3.14}$$

It follows from the definition of the operator J that $\dfrac{\partial Jv(\mathbf{x})}{\partial x_0} = J^*_{(s)}v$ and $Jv(\tau,\mathbf{x}') = 0$. Taking into account those relations and conditions (3.11) and (3.12), we integrate relation (3.14) by parts. This yields the relation

$$-\left(\frac{\partial}{\partial x_0}Jv, \frac{\partial}{\partial x_0}J^*_{(s)}v\right)_{L_2(\widetilde{Q}^\tau)} + b^2\sum_{i,j=1}^n \left(\frac{\partial}{\partial x_i}Jv, a^{(ij)}\frac{\partial}{\partial x_j}J^*_{(s)}v\right)_{L_2(\widetilde{Q}^\tau)}$$
$$+ \left(Jv, R^{(0)^*}v\right)_{L_2(\widetilde{Q}^\tau)} + \sum_{i=0}^n \left(\frac{\partial}{\partial x_i}Jv, K^{(i)^*}v\right)_{L_2(\widetilde{Q}^\tau)} = 0. \tag{3.15}$$

Using (3.15), we have the relation

$$\left\|J^*_{(s)}v\right\|^2_{L_2(\Omega^{(\tau)})} + b^2\sum_{i,j=1}^n \int\limits_{\Omega^{(T)}} a^{(ij)}(\mathbf{x}')\frac{\partial}{\partial x_i}Jv(T,\mathbf{x}')\frac{\partial}{\partial x_j}Jv(T,\mathbf{x}')\,\mathrm{d}\mathbf{x}' = \mathcal{F}^{(\tau)}(v), \tag{3.16}$$

where

$$\mathcal{F}^{(\tau)}(v) = -2\left(Jv, R^{(0)^*}v\right)_{L_2(\widetilde{Q}^\tau)} - 2\sum_{i=0}^n \left(\frac{\partial}{\partial x_i}Jv, K^{(i)^*}v\right)_{L_2(\widetilde{Q}^\tau)}.$$

Along with the operator J, we introduce the operator $\widetilde{J}$ as follows:

$$\widetilde{J}v(\mathbf{x}) = \int\limits_{x_0}^T J^*_{(s)}v(t,\mathbf{x}')\,\mathrm{d}t.$$

Due to the definitions of J and $\widetilde{J}$, the following relations hold:

$$Jv(\mathbf{x}) + \widetilde{J}v(\mathbf{x}) = \widetilde{J}v(\tau,\mathbf{x}'), \qquad Jv(T,\mathbf{x}') = \widetilde{J}v(\tau,\mathbf{x}').$$

Then relation (3.16) takes the form

$$\left[\left\|J_{(s)}^{*}v\right\|_{L_2(\Omega^{(\tau)})}^{2}+b^2\sum_{i,j=1}^{n}\int\limits_{\Omega^{(\tau)}}a^{(ij)}(\mathbf{x}')\frac{\partial}{\partial x_i}\widetilde{J}v(\tau,\mathbf{x}')\frac{\partial}{\partial x_j}\widetilde{J}v(\tau,\mathbf{x}')\,\mathrm{d}\mathbf{x}'\right](\tau)=\mathcal{F}^{(\tau)}(v). \tag{3.17}$$

Estimating $\mathcal{F}^{(\tau)}(v)$ from above and the left-hand side of (3.17) from below (using condition (1.2)), we deduce the following inequality from (3.17):

$$\begin{aligned}\left(\left\|J_{(s)}^{*}v\right\|_{L_2(\Omega^{(\tau)})}^{2}+\sum_{i=1}^{n}\left\|\frac{\partial}{\partial x_i}\widetilde{J}v\right\|_{L_2(\Omega^{(\tau)})}^{2}\right)(\tau)\le c^{(1)}\Biggl\{\left\|\widetilde{J}v\right\|_{L_2(\widetilde{Q}^{\tau})}^{2}+\sum_{i=1}^{n}\left\|\frac{\partial}{\partial x_i}\widetilde{J}v\right\|_{L_2(\widetilde{Q}^{\tau})}^{2}\\ +\left\|J_{(s)}^{*}v\right\|_{L_2(\widetilde{Q}^{\tau})}^{2}+\|v\|_{L_2(\widetilde{Q}^{\tau})}^{2}+(T-\tau)\left[\left\|\widetilde{J}v\right\|_{L_2(\Omega^{(\tau)})}^{2}+\sum_{i=1}^{n}\left\|\frac{\partial}{\partial x_i}\widetilde{J}v\right\|_{L_2(\Omega^{(\tau)})}^{2}\right](\tau)\Biggr\}.\end{aligned} \tag{3.18}$$

To apply the Gronwall inequality to inequality (3.18), we add the following inequality to (3.18):

$$\left\|\widetilde{J}v\right\|_{L_2(\Omega^{(\tau)})}^{2}\le\left\|J_{(s)}^{*}v\right\|_{L_2(\widetilde{Q}^{\tau})}^{2}+\left\|\widetilde{J}v\right\|_{L_2(\widetilde{Q}^{\tau})}^{2}.$$

This yields the inequality

$$\begin{aligned}&\left(\left\|J_{(s)}^{*}v\right\|_{L_2(\Omega^{(\tau)})}^{2}+\left\|\widetilde{J}v\right\|_{L_2(\Omega^{(\tau)})}^{2}+\sum_{i=1}^{n}\left\|\frac{\partial}{\partial x_i}\widetilde{J}v\right\|_{L_2(\Omega^{(\tau)})}^{2}\right)(\tau)\\ &\quad\le(T-\tau)c^{(1)}e^{2c^{(1)}(T-\tau)}\left(\left\|\widetilde{J}v\right\|_{L_2(\Omega^{(\tau)})}^{2}+\sum_{i=1}^{n}\left\|\frac{\partial}{\partial x_i}\widetilde{J}v\right\|_{L_2(\Omega^{(\tau)})}^{2}\right)+c^{(1)}e^{2c^{(1)}(T-\tau)}\|v\|_{L_2(\widetilde{Q}^{\tau})}^{2}.\end{aligned} \tag{3.19}$$

Let $\xi\in\mathbb{R}$ be a nonnegative number such that

$$2(T-\xi)c^{(1)}e^{2c^{(1)}(T-\xi)}\le 1 \tag{3.20}$$

and $\xi<T$. Then, using (3.19), we have

$$\left\|J_{(s)}^{*}v\right\|_{L_2(\Omega^{(\tau)})}^{2}\le c^{(1)}e^{2c^{(1)}(T-\xi)}\|v\|_{L_2(\widetilde{Q}^{\xi})}^{2} \tag{3.21}$$

for any $\tau\in[\xi,T]$, while (3.21) implies the inequality

$$\sup_{\tau\in[\xi,T]}\left\|J_{(s)}^{*}v\right\|_{L_2(\Omega^{(\tau)})}^{2}\le c^{(1)}e^{2c^{(1)}(T-\xi)}\|v\|_{L_2(\widetilde{Q}^{\xi})}^{2},$$

which means that

$$\left\|J_{(s)}^{*}v\right\|_{L_2(\widetilde{Q}^{\xi})}^{2}\le(T-\xi)c^{(1)}e^{2c^{(1)}(T-\xi)}\|v\|_{L_2(\widetilde{Q}^{\xi})}^{2}.$$

Passing to the limit as $s\to 0$ in the latter inequality and choosing $0<\xi<T$ to satisfy condition (3.20), we obtain that $\|v\|_{L_2(\widetilde{Q}^{\xi})}=0$.

Then relation (3.6) takes the form

$$(\mathcal{L}_b^{(1,0)}u,v)_{L_2(Q^{\xi})}=0. \tag{3.22}$$

We apply this line of reasoning to relation (3.22). For any $T<+\infty$, a finite number of steps is sufficient to prove that $v=0$ in $L_2(Q)$ in relation (3.6), provided that (3.6) holds for any function $u\in\mathcal{D}^{(0)}(\mathbf{L}^{(k,0)})$.

Now consider the set $\{w\}$ of functions $w = \mathcal{L}_b^{(1,0)} u$, where u is an arbitrary function from $\mathcal{D}^{(0)}(\mathbf{L}^{(k,0)})$. Due to the definition of the set $\mathcal{D}^{(0)}(\mathbf{L}^{(k,0)})$, the functions w belong to $C^2(\overline{Q})$ and

$$w|_{x_0=0} = \left.\frac{\partial w}{\partial x_0}\right|_{x_0=0} = 0. \tag{3.23}$$

The functions satisfy at most one condition on Γ:

$$w|_\Gamma = 0 \quad \text{or} \quad \left.\frac{\partial w}{\partial \mathcal{N}}\right|_\Gamma = 0. \tag{3.24}$$

Now consider the operator $\mathcal{L}_a^{(1,0)}$ with domain $\mathcal{D}(\mathcal{L}_a^{(1,0)}) = \mathcal{R}(\mathcal{L}_b^{(1,0)})$. As above, we prove that the relation

$$\left(\mathcal{L}_a^{(1,0)} u, v\right)_{L_2(Q)} = 0$$

holds for any function w from the set $\mathcal{D}\left(\mathcal{L}_a^{(1,0)}\right)$, which is dense in $L_2(Q)$, if and only if $v = 0$ in $L_2(Q)$. Thus, we have proved that the range $\mathcal{R}\left(\mathcal{L}_a^{(1,0)}\right) = \mathcal{R}\left(\mathcal{L}^{(1,0)}\right)$ is dense in $L_2(Q)$. This completes the proof of Lemma 3.3. □

Theorem 3.1. *Let the conditions of Theorem* 2.1 *be satisfied. Then, for any* $\mathbf{F} \in \mathbf{H}^{(k)}$, *there exists a unique strong solution* $u \in B^{(k)}$ *of problem* (1.1), (1.3), (1.k), $k \in \{17, \ldots, 28\}$, *and the estimate*

$$\|u\|_B \le c\|\mathbf{F}\|_{\mathbf{H}^{(k)}} \tag{3.25}$$

holds, where $c > 0$ *is the constant from the energy inequality* (2.8).

Proof. Estimate (3.25) and the uniqueness of the strong solution follow from inequality (3.2).

Due to Lemma 3.2, to prove the existence of a strong solution of problem (1.1), (1.3), (1.k) for any $\mathbf{F} \in \mathbf{H}^{(k)}$, it suffices to prove that the range $\mathcal{R}(\mathbf{L}^{(k)})$ is dense in the space $\mathbf{H}^{(k)}$. First, consider the operator $\mathbf{L}^{(k,0)}$. Let the relation

$$(\mathcal{L}^{(1,0)} u, v)_{L_2(Q)} + \sum_{j=1}^{3} (l_j u, v^{(j)})_{H_j(\Omega^{(0)})} = 0 \tag{3.26}$$

hold for any $u \in \mathcal{D}(\mathbf{L}^{(k)}) = \mathcal{D}(\mathbf{L}^{(k,0)})$. Suppose that it holds for any $u \in \mathcal{D}^{(0)}(\mathbf{L}^{(k,0)}) \subset \mathcal{D}(\mathbf{L}^{(k)})$. Then relation (3.26) takes the form

$$(\mathcal{L}^{(1,0)} u, v)_{L_2(Q)} = 0. \tag{3.27}$$

Due to Lemma 3.3, it follows from (3.27) that $v = 0$ in $L_2(Q)$.

Taking into account that $v = 0$, we represent relation (3.26) in the form

$$\sum_{j=1}^{3} (l_j u, v^{(j)})_{H_j(\Omega^{(0)})} = 0 \tag{3.28}$$

for any $u \in \mathcal{D}(\mathbf{L}^{(k)})$. Since the values $l_j u$ $(j = 0, \ldots, 3)$ are linearly independent and the sets $\{l_j u\}$ are dense in the spaces $H_j(\Omega^{(0)})$ for any $j = 0, 1, 2, 3$, it follows that all the elements $v^{(j)}$ are the zero elements in the spaces $H_j(\Omega^{(0)})$ for any $j = 0, 1, 2, 3$.

To prove the density of the range $\mathcal{R}(\mathbf{L}^{(k)})$ of the operator $\mathbf{L}^{(k)}$, $k \in \{17, \ldots, 28\}$, in the space $\mathbf{H}^{(k)}$, we use the method of continuation with respect to a parameter (see [33]), taking into account that the density of the range $\mathcal{R}(\mathbf{L}^{(k,0)})$ of the operator $\mathbf{L}^{(k,0)}$ in the same space is already proved.

Thus, Theorem 3.1 is proved. □

4. Strong Solutions in Other Function Spaces

It is possible to consider certain mixed problems (1.1), (1.3), (1.k) (e.g., for $k = 17, 18, \dots$) in other spaces. Let $\mathcal{B}^{(k)}$, $k = 17, 18$, denote Banach spaces that are the closures of the sets $\mathcal{D}(\mathbf{L}^{(k)})$ with respect to the norms

$$\|u\|_{\mathcal{B}} = \sup_{0 \le \tau \le T} \sum_{|\boldsymbol{\alpha}| \le 3} \|D^{\boldsymbol{\alpha}}\|_{L_2(\Omega^{(\tau)})} (\tau). \tag{4.1}$$

Let $\boldsymbol{\mathcal{H}}^{(k)}$ denote the Hilbert spaces

$$\boldsymbol{\mathcal{H}}^{(k)} = L_2(Q) \times \mathcal{H}_0^{(k)}(\Omega^{(0)}) \times \mathcal{H}_1^{(k)}(\Omega^{(0)}) \times H_2^{(k)}(\Omega^{(0)}) \times H_3^{(k)}(\Omega^{(0)}),$$

where $\mathcal{H}_0^{(k)}(\Omega^{(0)})$ and $\mathcal{H}_1^{(k)}(\Omega^{(0)})$ are the closures of the sets $C^{3,(k)}(\overline{\Omega^{(0)}})$ with respect to the norms

$$\|\cdot\|_{\mathcal{H}_0^{(k)}(\Omega^{(0)})} = \sum_{|\boldsymbol{\alpha}'| \le 3} \left\| D^{\boldsymbol{\alpha}'} \right\|_{L_2(\Omega^{(0)})}, \qquad \|\cdot\|_{\mathcal{H}_1^{(k)}(\Omega^{(0)})} = \sum_{|\boldsymbol{\alpha}'| \le 2} \left\| D^{\boldsymbol{\alpha}'} \right\|_{L_2(\Omega^{(0)})},$$

$\boldsymbol{\alpha}' = (\alpha_1, \dots, \alpha_n)$, respectively. It is assumed that Condition 2.1 is satisfied. Problem (1.1), (1.3), (1.k), $k \in \{17, 18\}$, is treated as the operator equation (2.1) from the Banach space $\mathcal{B}^{(k)}$ to the Hilbert space $\boldsymbol{\mathcal{H}}^{(k)}$ with the same domain $\mathcal{D}(\mathbf{L}^{(k)})$ as in Sec. 2. Here $A^{(3)}u$ is a third-order differential expression

$$A^{(3)}u = \sum_{|\boldsymbol{\alpha}| \le 3} a^{(\boldsymbol{\alpha})}(\mathbf{x}) \mathbf{D}^{\boldsymbol{\alpha}} u$$

of the general kind. The operator $\mathbf{L}^{(k)}$ from the space $\mathcal{B}^{(k)}$ to the space $\boldsymbol{\mathcal{H}}^{(k)}$ is closable. Its closure is denoted by $\overline{\mathbf{L}^{(k)}}$ again. A solution of Eq. (3.2) for the considered closed operator $\overline{\mathbf{L}^{(k)}} \colon \mathcal{B}^{(k)} \to \boldsymbol{\mathcal{H}}^{(k)}$ is called the *strong solution* of problem (1.1), (1.3), (1.k), $k \in \{17, 18\}$ (in the other function spaces).

Condition 4.1. The coefficients $a^{(\boldsymbol{\alpha})}$ of the operator $A^{(3)}$ are integrable and bounded in the domain Q.

Theorem 4.1. *If Conditions* 1.1, 2.1, *and* 4.1 *are satisfied, then the energy inequality*

$$\|u\|_{\mathcal{B}} \le c \|\mathbf{L}^{(k)}\|_{\boldsymbol{\mathcal{H}}^{(k)}} \tag{4.2}$$

holds for $k = 17, 18$, where u is any function from the domain $\mathcal{D}(\mathbf{L}^{(k)})$ and the constant $c > 0$ does not depend on u.

Proof. As in the proof of Theorem 2.1, we integrate the expression $\mathcal{L}^{(1)}u\mathcal{M}u$ over the domain Q^τ, $\tau \in (0, T)$. This yields the relation

$$\int_{Q^\tau} \mathcal{L}^{(1)}u\mathcal{M}u \, d\mathbf{x} = \int_{Q^\tau} \mathcal{L}^{(1,0)}u\mathcal{M}u \, d\mathbf{x} + \int_{Q^\tau} A^{(3)}u\mathcal{M}u \, d\mathbf{x}, \tag{4.3}$$

where $\mathcal{L}^{(1)} = \mathcal{L}^{(1,0)} + A^{(3)}$.

We represent the expression $\mathcal{L}^{(1,0)}u\mathcal{M}u$ in relation (4.3) in the divergent form:

$$\begin{aligned} \mathcal{L}^{(1,0)}u\mathcal{M}u = {} & \frac{\partial}{\partial x_0}\left(\frac{\partial^3 u}{\partial x_0^3}\right)^2 + \mathcal{A}^{(1)} \sum_{i,j=1}^n \frac{\partial}{\partial x_0}\left(a^{(ij)} \frac{\partial^3 u}{\partial x_0^2 \partial x_i} \frac{\partial^3 u}{\partial x_0^2 \partial x_j}\right) \\ & + (\mathcal{A}^{(2)} + c^4)\gamma \frac{\partial}{\partial x_0}\left(\frac{\partial}{\partial x_0} Au\right)^2 + a^2 b^2 c^2 \gamma \sum_{i,j=1}^n \frac{\partial}{\partial x_0}\left(a^{(ij)} \frac{\partial}{\partial x_i} Au \frac{\partial}{\partial x_j} Au\right) \\ & + (\mathcal{A}^{(2)} + c^4)\varepsilon \sum_{i,j,k,l=1}^n \frac{\partial}{\partial x_0}\left(a^{(ij)} a^{(kl)} \frac{\partial^3 u}{\partial x_0 \partial x_i \partial x_l} \frac{\partial^3 u}{\partial x_0 \partial x_j \partial x_k}\right) \end{aligned}$$

$$
\begin{aligned}
&+ a^2b^2c^2\varepsilon \sum_{i,j,k,l,p,s=1}^{n} \frac{\partial}{\partial x_0}\left(a^{(ij)}a^{(kl)}a^{(ps)}\frac{\partial^3 u}{\partial x_i\partial x_k\partial x_p}\frac{\partial^3 u}{\partial x_j\partial x_l\partial x_s}\right) - 2c^2\frac{\partial}{\partial x_0}\left(\frac{\partial^3 u}{\partial x_0^3}\frac{\partial}{\partial x_0}Au\right)\\
&- 2\mathcal{A}^{(1)}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a^{(ij)}\frac{\partial^3 u}{\partial x_0^3}\frac{\partial^3 u}{\partial x_0^2\partial x_j}\right) + 2a^2b^2\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{\partial^3 u}{\partial x_0^3}a^{(ij)}\frac{\partial}{\partial x_j}Au\right)\\
&- 2a^2b^2\sum_{i,j=1}^{n}\frac{\partial}{\partial x_0}\left(a^{(ij)}\frac{\partial^3 u}{\partial x_0^2\partial x_i}\frac{\partial}{\partial x_j}Au\right) - 2a^2b^2c^2\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a^{(ij)}\frac{\partial}{\partial x_j}Au\frac{\partial}{\partial x_0}Au\right)\\
&+ 2[a^2b^2 + \varepsilon(\mathcal{A}^{(2)} + c^4)]\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a^{(ij)}\frac{\partial^3 u}{\partial x_0^2\partial x_j}\frac{\partial}{\partial x_0}Au\right)\\
&- 2(\mathcal{A}^{(2)} + c^4)\varepsilon\sum_{i,j,k,l=1}^{n}\frac{\partial}{\partial x_k}\left(a^{(ij)}a^{(kl)}\frac{\partial^3 u}{\partial x_0\partial x_i\partial x_l}\frac{\partial^3 u}{\partial x_0^2\partial x_j}\right)\\
&+ 2a^2b^2c^2\varepsilon\sum_{i,j,k,l=1}^{n}\frac{\partial}{\partial x_k}\left(a^{(ij)}a^{(kl)}\frac{\partial^3 u}{\partial x_0\partial x_i\partial x_l}\frac{\partial}{\partial x_j}Au\right)\\
&- 2a^2b^2c^2\varepsilon\sum_{i,j,k,l,p,s=1}^{n}\frac{\partial}{\partial x_p}\left(a^{(ij)}a^{(kl)}a^{(ps)}\frac{\partial^3 u}{\partial x_0\partial x_i\partial x_l}\frac{\partial^3 u}{\partial x_j\partial x_k\partial x_s}\right) + \Phi^{(3)}(u,u), \quad (4.4)
\end{aligned}
$$

where $\gamma,\varepsilon \in (0,1)$, $\gamma+\varepsilon = 1$, $\mathcal{A}^{(1)} = a^2+b^2-c^2$, $\mathcal{A}^{(2)} = a^2c^2+b^2c^2-a^2b^2-c^4$,

$$
\Phi^{(3)}(u,u) = \sum_{|\boldsymbol{\alpha}|,|\boldsymbol{\beta}|\le 3} a^{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\mathbf{x})\mathbf{D}^{\boldsymbol{\alpha}}u\mathbf{D}^{\boldsymbol{\beta}}u,
$$

and the coefficients $a^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ are functions integrable and bounded on $\overline{Q}$ and expressed via $a^{(ij)}$ and their first-order and second-order derivatives. Consider the integral

$$
\int\limits_{Q^\tau}\mathcal{L}^{(1,0)}u\mathcal{M}u\,\mathrm{d}\mathbf{x} = \int\limits_{\partial Q^\tau}\Phi^{(0)}(u,u)\,\mathrm{d}s + \int\limits_{Q^\tau}\Phi^{(3)}(u,u)\,\mathrm{d}\mathbf{x},
$$

where

$$
\begin{aligned}
\Phi^{(0)}(u,u) = &\left(\frac{\partial^3 u}{\partial x_0^3}\right)^2\nu_0 + \mathcal{A}^{(1)}\sum_{i,j=1}^{n}a^{(ij)}\frac{\partial^3 u}{\partial x_0^2\partial x_i}\frac{\partial^3 u}{\partial x_0^2\partial x_j}\nu_0\\
&+ (\mathcal{A}^{(2)}+c^4)\gamma\left(\frac{\partial}{\partial x_0}Au\right)^2\nu_0 + a^2b^2c^2\gamma\sum_{i,j=1}^{n}a^{(ij)}\frac{\partial}{\partial x_i}Au\frac{\partial}{\partial x_j}Au\nu_0\\
&+ (\mathcal{A}^{(2)}+c^4)\varepsilon\sum_{i,j,k,l=1}^{n}a^{(ij)}a^{(kl)}\frac{\partial^3 u}{\partial x_0\partial x_i\partial x_k}\frac{\partial^3 u}{\partial x_0\partial x_j\partial x_l}\nu_0\\
&+ a^2b^2c^2\varepsilon\sum_{i,j,k,l,p,s=1}^{n}a^{(ij)}a^{(kl)}a^{(ps)}\frac{\partial^3 u}{\partial x_i\partial x_k\partial x_p}\frac{\partial^3 u}{\partial x_j\partial x_l\partial x_s}\nu_0 - 2c^2\frac{\partial^3 u}{\partial x_0^3}\frac{\partial}{\partial x_0}Au\nu_0\\
&- 2a^2b^2\sum_{i,j=1}^{n}a^{(ij)}\frac{\partial^3 u}{\partial x_0^2\partial x_i}\frac{\partial}{\partial x_j}Au\nu_0 - 2\mathcal{A}^{(1)}\frac{\partial^3 u}{\partial x_0^3}\frac{\partial^3 u}{\partial\mathcal{N}\partial x_0^2} + 2a^2b^2\frac{\partial^3 u}{\partial x_0^3}\frac{\partial}{\partial\mathcal{N}}Au\\
&- 2a^2b^2c^2\frac{\partial}{\partial x_0}Au\frac{\partial}{\partial\mathcal{N}}Au + 2[a^2b^2+\varepsilon(\mathcal{A}^{(2)}+c^4)]\frac{\partial}{\partial x_0}Au\frac{\partial^3 u}{\partial\mathcal{N}\partial x_0^2}
\end{aligned}
$$

$$
- 2(\mathcal{A}^{(2)} + c^4)\varepsilon \sum_{i,j=1}^{n} a^{(ij)} \frac{\partial^3 u}{\partial \mathcal{N} \partial x_0 \partial x_i} \frac{\partial^3 u}{\partial x_0^2 \partial x_j} + 2a^2 b^2 c^2 \varepsilon \sum_{i,j=1}^{n} a^{(ij)} \frac{\partial^3 u}{\partial \mathcal{N} \partial x_0 \partial x_i} \frac{\partial}{\partial x_j} Au
$$

$$
- 2a^2 b^2 c^2 \varepsilon \sum_{i,j,k,l=1}^{n} a^{(ij)} a^{(kl)} \frac{\partial^3 u}{\partial x_0 \partial x_i \partial x_k} \frac{\partial^3 u}{\partial \mathcal{N} \partial x_j \partial x_l} \tag{4.5}
$$

and $\boldsymbol{\nu} = (\nu_0, \dots, \nu_n)$ is the unit vector of the outer (with respect to the domain Q^τ) normal to the hyperspace $\partial Q^\tau = \Omega^{(0)} \cup \Omega^{(\tau)} \cup \Gamma^\tau$. We treat the integrand $\Phi^{(0)}(u,u)$ as a quadratic form with respect to the third-order derivatives of the function u at the hyperplane $\Omega^{(\tau)}$. Note that if $\mathbf{x} \in \Omega^{(\tau)}$, then $\nu_0 = 1$ and $\nu_i = 0$, $i = 1, \dots, n$. Let us prove that the quadratic form $\Phi^{(0)}(u,u)$ is positive on $\Omega^{(\tau)}$. We denote its principal minors by $M^{(j)}$, $j = 1, \dots, 2 + 2n + n^2 + n^3$, and prove that they all are positive if $\mathbf{x} = (\tau, \mathbf{x}') \in \Omega^{(\tau)}$. The derivatives of $\Phi^{(0)}(u,u)$ are considered in the following order:

$$
\frac{\partial^3 u}{\partial x_0^3}, \quad \frac{\partial}{\partial x_0} Au, \quad \frac{\partial^3 u}{\partial x_0^2 \partial x_1}, \dots, \quad \frac{\partial^3 u}{\partial x_0^2 \partial x_n}, \quad \frac{\partial}{\partial x_1} Au, \dots, \quad \frac{\partial}{\partial x_n} Au,
$$

$$
\underbrace{\frac{\partial^3 u}{\partial x_0 \partial x_1^2}, \frac{\partial^3 u}{\partial x_0 \partial x_1 \partial x_2}, \dots, \frac{\partial^3 u}{\partial x_0 \partial x_{n-1} \partial x_n}, \frac{\partial^3 u}{\partial x_0 \partial x_n^2}}_{n^2},
$$

$$
\underbrace{\frac{\partial^3 u}{\partial x_1^3}, \frac{\partial^3 u}{\partial x_1^2 \partial x_2}, \dots, \frac{\partial^3 u}{\partial x_i \partial x_k \partial x_p}, \dots, \frac{\partial^3 u}{\partial x_n^2 \partial x_{n-1}}, \frac{\partial^3 u}{\partial x_n^3}}_{n^3}.
$$

It follows from (4.5) that the first principal minor $M^{(1)}$ is equal to $\nu_0 = 1$ and the second principal minor $M^{(2)}$ is equal to $\mathcal{A}^{(2)} - \varepsilon(\mathcal{A}^{(2)} + c^4)$. It is easy to verify that if

$$
b^2 < c^2 < a^2, \tag{4.6}
$$

then there exists $c^{(0)} > 0$ such that $M^{(2)} \geq c^{(0)} > 0$ (the number $\varepsilon > 0$ is chosen to be sufficiently small).

Consider the principal minors $M^{(j)}$ for $j = 3, \dots, n+2$. Computing them, we obtain the relations

$$
M^{(j)} = M^{(2)} (\mathcal{A}^{(1)})^{j-2} N^{(j-2)}, \quad j = 3, \dots, n+2, \tag{4.7}
$$

where $N^{(j-2)}$ are the principal minors of the matrix

$$
(a^{(ij)}) = \begin{pmatrix} a^{(11)} & \dots & a^{(1n)} \\ \vdots & \ddots & \vdots \\ a^{(1n)} & \dots & a^{(nn)} \end{pmatrix}.
$$

By virtue of condition (1.2), the principal minors $N^{(j-2)}$ are positive for all $j = 3, \dots, n+2$. Hence, the minors (4.7) are positive.

Computing the principal minors $M^{(j)}$ for $j = n+3, \dots, 2n+2$, we obtain the relation

$$
M^{(j)} = \frac{1}{(\mathcal{A}^{(1)})^{j-n-2}} M^{(n+2)} N^{(j-n-2)} (a^2 b^2)^{j-n-2} [(a^2 c^2 + b^2 c^2 - c^4)\gamma - a^2 b^2]^{j-n-2}. \tag{4.8}
$$

We see that if γ is close to 1 and c^2 satisfies condition (4.6), then all minors $M^{(j)}$ are positive, $j = n+3, \dots, 2n+2$.

It follows from condition (1.2) that the principal minors of the matrices

$$
(a^{(ij)} a^{(kl)})_{i,j,k,l=1}^{n} \tag{4.9}
$$

and

$$
(a^{(ij)} a^{(kl)} a^{(ps)})_{i,j,k,l,p,s=1}^{n} \tag{4.10}
$$

are positive (see [60, Chap. III, Sec. 7]). From (4.5), we see that all the other principal minors for $j = 2n+3, \dots, 2+2n+n^2+n^3$ are determined via the principal minor $M^{(2n+2)}$ multiplied by the principal minors of the matrices (4.9) and (4.10) with the factors $(\mathcal{A}^{(2)} + c^4)\varepsilon$ and $a^2b^2c^2\varepsilon$ to the corresponding powers. Obviously, the obtained products are positive.

Thus, all principal minors of the quadratic form $\Phi^{(0)}(u,u)$ are positive on the upper base $\Omega^{(\tau)}$ of the cylindrical domain Q^τ. Therefore, there exists a constant $c^{(1)} > 0$ such that

$$\int\limits_{\Omega^{(\tau)}} \Phi^{(0)}(u,u)\,\mathrm{d}\mathbf{x}' \geq c^{(1)} \left[\sum_{|\boldsymbol{\alpha}|=3} \int\limits_{\Omega^{(\tau)}} (\mathbf{D}^{\boldsymbol{\alpha}} u)^2(\tau, \mathbf{x}')\,\mathrm{d}\mathbf{x}' + \sum_{i=0}^{n} \int\limits_{\Omega^{(\tau)}} \left(\frac{\partial}{\partial x_i} Au \right)^2 (\tau, \mathbf{x}')\,\mathrm{d}\mathbf{x}' \right]. \tag{4.11}$$

Consider the expression $\Phi^{(0)}(u,u)$ on the lateral surface Γ^τ of the domain Q^τ. For all points $\mathbf{x} \in \Gamma^\tau$, we have $\boldsymbol{\nu}(\mathbf{x}) = (0, \nu_1(\mathbf{x}), \dots, \nu_n(\mathbf{x}))$. Therefore, the following relation holds on Γ^τ:

$$\begin{aligned}
\Phi^{(0)}(u,u) &= -2\mathcal{A}^{(1)} \frac{\partial^3 u}{\partial x_0^3} \frac{\partial^3 u}{\partial \mathcal{N} \partial x_0^2} + 2a^2b^2 \frac{\partial^3 u}{\partial x_0^3} \frac{\partial}{\partial \mathcal{N}} Au - 2a^2b^2c^2 \frac{\partial}{\partial x_0} Au \frac{\partial}{\partial \mathcal{N}} Au \\
&\quad + 2\left[a^2b^2 + \varepsilon(\mathcal{A}^{(2)} + c^4) \right] \frac{\partial}{\partial x_0} Au \frac{\partial^3 u}{\partial \mathcal{N} \partial x_0^2} - 2(\mathcal{A}^{(2)} + c^4)\varepsilon \sum_{i,j=1}^{n} a^{(ij)} \frac{\partial^3 u}{\partial \mathcal{N} \partial x_0 \partial x_i} \frac{\partial^3 u}{\partial x_0^2 \partial x_j} \\
&\quad + 2a^2b^2c^2\varepsilon \sum_{i,j=1}^{n} a^{(ij)} \frac{\partial^3 u}{\partial \mathcal{N} \partial x_0 \partial x_i} \frac{\partial}{\partial x_j} Au - 2a^2b^2c^2\varepsilon \sum_{i,j,k,l=1}^{n} a^{(ij)} a^{(kl)} \frac{\partial^3 u}{\partial x_0 \partial x_i \partial x_k} \frac{\partial^3 u}{\partial \mathcal{N} \partial x_j \partial x_l}.
\end{aligned} \tag{4.12}$$

To obtain the energy inequality, the boundary conditions should be chosen to satisfy inequality (2.10). Then

$$\int\limits_{\Gamma^\tau} \Phi^{(0)}(u,u) \mathrm{d}s \geq 0. \tag{4.13}$$

It is obvious that the first three terms are equal to zero due to the boundary conditions (1.17) and (1.18). A proof is required for the other three terms. There are particular cases where condition (1.17) or condition (1.18) is sufficient to make those terms equal to zero. If those conditions are not sufficient, then they should be complemented in such a way that inequality (1.13) and Lemma 3.3 hold, but this is outside the scope of this paper.

The integral of $\Phi^{(0)}(u,u)$ over $\Omega^{(0)}$ is expressed via the integrals of the given boundary conditions (1.3). Therefore, the inequality

$$\left| \int\limits_{\Omega^{(0)}} \Phi^{(0)}(u,u)(0, \mathbf{x}')\,\mathrm{d}\mathbf{x}' \right| \leq c^{(2)} \sum_{j=0}^{3} \|l_j u\|_{\mathcal{H}_j^{(k)}(\Omega^{(0)})}, \tag{4.14}$$

where $k = 17, 18$ and $\mathcal{H}_j^{(k)}(\Omega^{(0)}) = H_j^{(k)}(\Omega^{(0)})$ for $j = 2, 3$, holds.

Using elementary inequalities, one can easily prove the following estimate:

$$\left| \int\limits_{Q^\tau} \Phi^{(3)}(u,u) \mathrm{d}\mathbf{x} \right| \leq c^{(3)} \sum_{|\boldsymbol{\alpha}| \leq 3} \|\mathbf{D}^{\boldsymbol{\alpha}} u\|_{L_2(Q^\tau)}. \tag{4.15}$$

For $|\boldsymbol{\alpha}| \leq 2$, consider the relations

$$\frac{\partial}{\partial x_0} (\mathbf{D}^{\boldsymbol{\alpha}} u)^2 = 2 \frac{\partial}{\partial x_0} \mathbf{D}^{\boldsymbol{\alpha}} u\, \mathbf{D}^{\boldsymbol{\alpha}} u. \tag{4.16}$$

Integrating (4.16) over the domain Q^τ and using estimates of the type (4.15), we obtain the inequality

$$\sum_{|\boldsymbol{\alpha}|\le 2} \|\mathbf{D}^{\boldsymbol{\alpha}} u(\tau,\mathbf{x}')\|_{L_2(\Omega^{(\tau)})} \le \sum_{|\boldsymbol{\alpha}|\le 3} \|\mathbf{D}^{\boldsymbol{\alpha}} u\|_{L_2(Q^\tau)} + \sum_{j=0}^{3} \|l_j u\|_{\mathcal{H}_j^{(k)}(\Omega^{(0)})}, \quad k = 17, 18. \tag{4.17}$$

Further, we have

$$\left| \int_{Q^\tau} \mathcal{L}^{(1)} u \mathcal{M} u \, \mathrm{d}\mathbf{x} \right| \le \|\mathcal{L}^{(1)} u\|^2_{L_2(Q^\tau)} + \|\mathcal{M} u\|^2_{L_2(Q^\tau)} \le \|\mathcal{L}^{(1)} u\|^2_{L_2(Q)} + c^{(4)} \sum_{|\boldsymbol{\alpha}|\le 3} \|\mathbf{D}^{\boldsymbol{\alpha}} u\|^2_{L_2(Q^\tau)}. \tag{4.18}$$

Combining relations (4.11) and (4.13)–(4.18) and using relation (4.3), we obtain the inequality

$$\sum_{|\boldsymbol{\alpha}|\le 3} \|\mathbf{D}^{\boldsymbol{\alpha}} u(\tau,\mathbf{x}')\|^2_{L_2(\Omega^{(\tau)})} \le c^{(5)} \|\mathbf{L}^{(k)} u\|^2_{\boldsymbol{\mathcal{H}}^{(k)}} + c^{(5)} \sum_{|\boldsymbol{\alpha}|\le 3} \|\mathbf{D}^{\boldsymbol{\alpha}} u\|_{L_2(Q^\tau)}. \tag{4.19}$$

Applying the Gronwall inequality to (4.19), we obtain an inequality which implies the energy inequality (4.2). □

Theorem 4.2. *Let Conditions* 1.1, 2.1, *and* 4.1 *be satisfied. Let* $\mathbf{F} \in \boldsymbol{\mathcal{H}}^{(k)}$, $k = 17, 18$. *Then there exists a unique strong solution* $u \in \mathcal{B}^{(k)}$ *of problem* (1.1), (1.3), (1.k) *and the estimate*

$$\|u\|_{\mathcal{B}} \le c\|\mathbf{F}\|_{\boldsymbol{\mathcal{H}}^{(k)}}$$

holds, where $c > 0$ *is the constant from the energy inequality* (4.2).

The proof of this theorem actually repeats the proof of Theorem 3.1 for the case where the strong solution belongs to the space $\mathcal{B}^{(k)}$ instead of $B^{(k)}$ and $\mathbf{F}$ belongs to the space $\boldsymbol{\mathcal{H}}^{(k)}$ instead of $\mathbf{H}^{(k)}$.

Remark 4.1. If the strong solutions are investigated according to the proposed scheme, then the boundary conditions (1.k), $k \in \{17, \dots, 28\}$, and, perhaps, other boundary conditions on Γ should be chosen to satisfy the following requirements:

(1) inequalities of the type (2.10) and (4.13) should hold on the lateral surfaces;
(2) the corresponding energy inequalities should be proved in suitable spaces;
(3) Lemma 3.3 should be proved.

In [46], a local version of the Cauchy problem for the bi-wave equation is considered for the case where the initial conditions are posed on a hypersurface instead of the hyperplane. The result can be generalized to the case where the operator A is considered instead of the Laplace operator. Furthermore, in problems (1.1), (1.3), (1.k), conditions (1.3) can be replaced by initial conditions posed on a part of a hypersurface instead of the hyperplane.

Problems (1.1), (1.3), (1.k) are considered for the initial conditions (1.3) posed on $\Omega^{(0)}$. Analyzing the reasoning, we see that the existence and uniqueness of strong solutions of Eq. (1.1) with the boundary conditions (1.k), $k \in \{17, \dots, 28\}$, and the initial conditions

$$l_k^{(T)} u = \left. \frac{\partial^k u}{\partial x_0^k} \right|_{x_0 = T} = \varphi_k^{(T)}(\mathbf{x}'), \quad k = 0, 1, 2, 3,$$

are proved almost in the same way.

Remark 4.2. In problems (1.1), (1.3), (1.k), it is assumed that the coefficients $a^{(ij)}$ do not depend on x_0. If the boundary conditions on Γ do not contain the functions $a^{(ij)}$, then $a^{(ij)}$ may depend on x_0. Then the scheme of the proof is not changed. Otherwise, additional reasoning is needed.

5. Boundary-Value Conditions for Fourth-Order Equations of the Composite Type

The setting of the problem. Consider the case where $A^{(3)}$ is a second-order differential operator. Then Eq. (1.1) takes the form

$$\mathcal{L}u \equiv \mathcal{L}^{(0)}u + A^{(2)}u = f(\mathbf{x}), \tag{5.1}$$

where

$$\mathcal{L}^{(0)} = \left(\frac{\partial^2}{\partial x_0^2} - a^2 A\right)\left(\frac{\partial^2}{\partial x_0^2} + b^2 A\right), \qquad A^{(2)} = \sum_{|\boldsymbol{\alpha}|\le 2} a^{(\boldsymbol{\alpha})}(\mathbf{x})D^{\boldsymbol{\alpha}}.$$

Along with Eq. (5.1), we consider the boundary conditions

$$lu = u|_{\Omega^{(0)}} = \varphi(\mathbf{x}'), \tag{5.2}$$

$$\left.\frac{\partial u}{\partial x_0}\right|_{\Omega^{(0)}} = \left.\frac{\partial^2 u}{\partial x_0^2}\right|_{\Omega^{(0)}} = \left.\frac{\partial^2 u}{\partial x_0^2}\right|_{\Omega^{(T)}} = 0 \tag{5.3}$$

and conditions $(1.k)$, $k = 17, 18$.

It is assumed that Conditions 1.1, 2.1, and 4.1 are satisfied, $a^2 < b^2$, and the coefficients $a^{(ij)}$ of the operator A do not depend on x_0.

We represent problem (5.1)–(5.3), $(1.k)$ as the linear operator equation

$$\mathbf{L}^{(k)}u = \mathbf{F}, \tag{5.4}$$

where $\mathbf{L}^{(k)} = (\mathcal{L}, l)$ and $\mathbf{F} = (f, \varphi)$. The domain $\mathcal{D}(\mathbf{L}^{(k)})$ of the operator $\mathbf{L}^{(k)}$ consists of functions from the space $C^4(\overline{Q})$ satisfying the boundary conditions (5.3), $(1.k)$, i.e.,

$$\mathcal{D}(\mathbf{L}^{(17)}) = \left\{u \in C^4(\overline{Q}) \mid \left.\frac{\partial u}{\partial x_0}\right|_{\Omega^{(0)}} = \left.\frac{\partial^2 u}{\partial x_0^2}\right|_{\Omega^{(0)}} = \left.\frac{\partial^2 u}{\partial x_0^2}\right|_{\Omega^{(T)}} = u|_\Gamma = Au|_\Gamma = 0\right\},$$

$$\mathcal{D}(\mathbf{L}^{(18)}) = \left\{u \in C^4(\overline{Q}) \mid \left.\frac{\partial u}{\partial x_0}\right|_{\Omega^{(0)}} = \left.\frac{\partial^2 u}{\partial x_0^2}\right|_{\Omega^{(0)}} = \left.\frac{\partial^2 u}{\partial x_0^2}\right|_{\Omega^{(T)}} = \left.\frac{\partial u}{\partial \mathcal{N}}\right|_\Gamma = \left.\frac{\partial}{\partial \mathcal{N}}Au\right|_\Gamma = 0\right\}.$$

We treat $\mathbf{L}^{(k)}$ as an operator acting from the Banach space $B^{(k)}$ to the Hilbert space $\mathbf{H}^{(k)}$. The Banach space $B^{(k)}$ is the closure of the set $\mathcal{D}(\mathbf{L}^{(k)})$ with respect to the norm

$$\|u\|_B = \sum_{|\boldsymbol{\alpha}|\le 2} \|D^{\boldsymbol{\alpha}}u\|_{L_2(Q)} + \sup_{0<\tau<T} \sum_{|\boldsymbol{\alpha}|\le 1} \|D^{\boldsymbol{\alpha}}u\|_{L_2(\Omega^{(\tau)})}(\tau).$$

The Hilbert space $\mathbf{H}^{(k)}$ is the Cartesian product of the spaces $L_2(Q)$ and $\mathcal{H}_1^{(k)}(\Omega^{(0)})$ (see Sec. 4).

Theorem 5.1. *Suppose that Conditions* 1.1, 2.1, *and* 4.1 *are satisfied,* $a^2 < b^2$, *and the coefficients* $a^{(ij)}$ *of the operator* A *do not depend on* x_0. *Then the operator* $\mathbf{L}^{(k)}$ *of the operator equation* (5.4) *satisfies the energy inequality*

$$\|u\|_B \le c^{(1)}\|\mathbf{L}^{(k)}u\|_{\mathbf{H}^{(k)}} \tag{5.5}$$

for any $u \in \mathcal{D}\left(\mathbf{L}^{(k)}\right)$, *where* $c^{(1)} \in \mathbb{R}$ *is a positive constant independent of the function* u.

Proof. We represent the product $2\mathcal{L}u\mathcal{M}u$, where $\mathcal{M}u = (x_0 - T)\dfrac{\partial u}{\partial x_0}$, in the divergent form:

$$\begin{aligned} &2\mathcal{L}u\mathcal{M}u \\ &= 2\frac{\partial}{\partial x_0}\left(\frac{\partial^3 u}{\partial x_0^3}(x_0 - T)\frac{\partial u}{\partial x_0}\right) - 2\frac{\partial}{\partial x_0}\left(\frac{\partial^2 u}{\partial x_0^2}\frac{\partial u}{\partial x_0}\right) - \frac{\partial}{\partial x_0}\left((x_0 - T)\left(\frac{\partial^2 u}{\partial x_0^2}\right)^2\right) + 3\left(\frac{\partial^2 u}{\partial x_0^2}\right)^2 \\ &\quad + 2(b^2 - a^2)\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a^{(ij)}\frac{\partial^3 u}{\partial x_0^2 \partial x_j}(x_0 - T)\frac{\partial u}{\partial x_0}\right) \end{aligned}$$

$$
\begin{aligned}
&- (b^2 - a^2) \sum_{i,j=1}^{n} \frac{\partial}{\partial x_0} \left(a^{(ij)} \frac{\partial^2 u}{\partial x_0 \partial x_j} (x_0 - T) \frac{\partial^2 u}{\partial x_0 \partial x_i} \right) \\
&+ (b^2 - a^2) \sum_{i,j=1}^{n} a^{(ij)} \frac{\partial^2 u}{\partial x_0 \partial x_j} \frac{\partial^2 u}{\partial x_0 \partial x_i} - 2a^2 b^2 \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a^{(ij)} \frac{\partial}{\partial x_j} Au(x_0 - T) \frac{\partial u}{\partial x_0} \right) \\
&+ 2a^2 b^2 \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(a^{(ij)} Au(x_0 - T) \frac{\partial^2 u}{\partial x_0 \partial x_i} \right) \\
&- a^2 b^2 \frac{\partial}{\partial x_0} ((x_0 - T)(Au)^2) + a^2 b^2 (Au)^2 + 2A_1^{(2)} u \mathcal{M} u, \quad (5.6)
\end{aligned}
$$

where

$$
A_1^{(2)} = \sum_{|\boldsymbol{\alpha}| \le 2} a_1^{(\boldsymbol{\alpha})}(\mathbf{x}) D^{\boldsymbol{\alpha}}
$$

is a second-order differential operator defined via $A^{(2)}$ and additional terms with differential operators of order not exceeding two.

We integrate both sides of (5.6) over the domain Q. Taking into account the boundary conditions (5.2), (5.3), and (1.k), we obtain the relation

$$
\begin{aligned}
2 \int_Q \mathcal{L} u \mathcal{M} u \, \mathrm{d}\mathbf{x} = 3 \int_Q \left(\frac{\partial^2 u}{\partial x_0^2} \right)^2 \mathrm{d}\mathbf{x} + (b^2 - a^2) \int_Q \sum_{i,j=1}^{n} a^{(ij)} \frac{\partial^2 u}{\partial x_0 \partial x_j} \frac{\partial^2 u}{\partial x_0 \partial x_i} \, \mathrm{d}\mathbf{x} - a^2 b^2 T \int_{\Omega^{(0)}} (A\varphi)^2 \, \mathrm{d}\mathbf{x}' \\
+ a^2 b^2 \int_Q (Au)^2 \, \mathrm{d}\mathbf{x} + 2 \int_Q A_1^{(2)} u \mathcal{M} u \, \mathrm{d}\mathbf{x}. \quad (5.7)
\end{aligned}
$$

We estimate the left-hand side of (5.7) from above, using the inequality

$$
2|ab| \le \varepsilon a^2 + \frac{1}{\varepsilon} b^2 \qquad \text{for all } \ \varepsilon > 0. \tag{5.8}
$$

It follows from Condition (1.2) that

$$
\int_Q \sum_{i,j=1}^{n} a^{(ij)} \frac{\partial^2 u}{\partial x_0 \partial x_j} \frac{\partial^2 u}{\partial x_0 \partial x_i} \, \mathrm{d}\mathbf{x} \ge c^{(0)} \sum_{i=1}^{n} \int_Q \left(\frac{\partial^2 u}{\partial x_0 \partial x_i} \right)^2 \mathrm{d}\mathbf{x}.
$$

It is known from the theory of elliptic problems that

$$
\|Au\|_{L_2(Q)}^2 \ge \sum_{i,j=1}^{n} \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L_2(Q)}^2 + \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_{L_2(Q)}^2 - C \|u\|_{L_2(Q)}^2.
$$

We move the term $-a^2 b^2 T \int_{\Omega^{(0)}} (A\varphi)^2 \, \mathrm{d}\mathbf{x}'$ to the left-hand side and estimate it from above by means of inequality (5.8). Thus, passing to the norm of the space $L_2(Q)$, we obtain the inequality

$$
\begin{aligned}
&3 \left\| \frac{\partial^2 u}{\partial x_0^2} \right\|_{L_2(Q)}^2 + (b^2 - a^2) c^{(0)} \sum_{i=1}^{n} \left\| \frac{\partial^2 u}{\partial x_0 \partial x_i} \right\|_{L_2(Q)}^2 \\
&\quad + a^2 b^2 \sum_{i,j=1}^{n} \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L_2(Q)}^2 + a^2 b^2 \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_{L_2(Q)}^2 - a^2 b^2 C \|u\|_{L_2(Q)}^2 + 2 \int_Q A_1^{(2)} u \mathcal{M} u \, \mathrm{d}\mathbf{x} \\
&\qquad \le \frac{T^2}{\varepsilon_2} \|\mathcal{L} u\|_{L_2(Q)}^2 + \varepsilon_2 \left\| \frac{\partial u}{\partial x_0} \right\|_{L_2(Q)}^2 + c^{(1)} \|\varphi\|_{\mathcal{H}_1^{(k)}(\Omega^{(0)})}^2. \quad (5.9)
\end{aligned}
$$

We add the terms $c^{(2)} \sup\limits_{0<\tau<T} \sum\limits_{|\boldsymbol{\alpha}|\le 1} \|D^{\boldsymbol{\alpha}} u\|_{L_2(\Omega^{(\tau)})}(\tau)$, where $c^{(2)} \in \mathbb{R}$ is a sufficiently large constant (see the proof of Theorem 4.1) to the left-hand side of inequality (5.9). Using inequality (5.8), we estimate the integral $2\int\limits_Q A_1^{(2)} u \mathcal{M} u \, d\mathbf{x}$ from above. This yields the energy inequality (5.5). □

It is easy to show that the operator $\mathbf{L}^{(k)}$ admits a closure. The density of the range of the operator $\mathbf{L}^{(k)}$ is proved as follows. The operator $\mathcal{L}^{(0)}$ is treated as the composition of the operators $\mathcal{L}_1^{(0)} = \dfrac{\partial^2}{\partial x_0^2} - a^2 A$ and $\mathcal{L}_2^{(0)} = \dfrac{\partial^2}{\partial x_0^2} + b^2 A$, i.e., $\mathcal{L}^{(0)} = \mathcal{L}_1^{(0)} \mathcal{L}_2^{(0)}$. Let $v = \mathcal{L}_1^{(0)} u$. If $u \in \mathcal{D}^{(0)}(\mathbf{L}^{(k)})$, where $\mathcal{D}^{(0)}(\mathbf{L}^{(k)}) = \big\{u \in \mathcal{D}(\mathbf{L}^{(k)}) \mid lu = 0\big\}$, then the function $v \in C^2(\overline{Q})$ satisfies the boundary-value conditions

$$v|_{\Omega^{(0)}} = v|_\Gamma = 0 \quad \text{or} \quad v|_{\Omega^{(0)}} = \frac{\partial v}{\partial \mathcal{N}}\bigg|_\Gamma = 0$$

depending on conditions (1.k) for the function u. The range of the operator $\mathcal{L}_2^{(0)} - \lambda$, where $\lambda \in \mathbb{R}$ is a sufficiently large number, with the domain

$$\mathcal{D}(\mathcal{L}_2^{(0)} - \lambda) = \{v \in C^2(\overline{Q}) \mid v|_{\Omega^{(0)}} = v|_\Gamma = 0\}$$

or

$$\mathcal{D}(\mathcal{L}_2^{(0)} - \lambda) = \left\{v \in C^2(\overline{Q}) \mid \ v|_{\Omega^{(0)}} = \frac{\partial v}{\partial \mathcal{N}}\bigg|_\Gamma = 0\right\}$$

is dense in $L_2(Q)$ (see [45]). The range of the operator $\mathcal{L}_1^{(0)}$ with the domain $\mathcal{D}(\mathcal{L}_1^{(0)}) = \mathcal{D}^{(0)}(\mathbf{L}^{(k)})$ is also dense in $L_2(Q)$ (see the proof of Lemma 3.3). Hence, the range of the operator $\mathcal{L}^{(0)} - \lambda \mathcal{L}_1^{(0)}$ with the domain $\mathcal{D}(\mathcal{L}^{(0)} - \lambda \mathcal{L}_1^{(0)}) = \mathcal{D}^{(0)}(\mathbf{L}^{(k)})$ is dense in $L_2(Q)$. Therefore, if $w \in L_2(Q)$ and $(\mathcal{L}^{(0)} u - \lambda \mathcal{L}_1^{(0)} u, w)_{L_2(Q)} = 0$ for any $u \in \mathcal{D}^{(0)}(\mathbf{L}^{(k)})$, then $w = 0$ in $L_2(Q)$. Consider the operator $\mathbf{L}^{(k,0)} = (\mathcal{L}^{(0)} - \lambda \mathcal{L}_1^{(0)}, l)$ with the domain $\mathcal{D}(\mathbf{L}^{(k,0)}) = \mathcal{D}(\mathbf{L}^{(k)})$. Let $\mathbf{w} = (w, w_0) \in \mathbf{H}^{(k)}$ be orthogonal to the set $\mathcal{R}(\mathbf{L}^{(k,0)})$. This means that

$$(\mathcal{L}^{(0)} u - \lambda \mathcal{L}_1^{(0)} u, w)_{L_2(Q)} + (lu, w_0)_{\mathcal{H}_1^{(k)}(\Omega)} = 0 \tag{5.10}$$

for any $u \in \mathcal{D}(\mathbf{L}^{(k)})$. In particular, substituting any element of $\mathcal{D}^{(0)}(\mathbf{L}^{(k)})$ for u in (5.10), we obtain the relation $(\mathcal{L}^{(0)} u - \lambda \mathcal{L}_1^{(0)} u, w)_{L_2(Q)} = 0$ for any $u \in \mathcal{D}^{(0)}(\mathbf{L}^{(k)})$. As above, this implies that $w = 0$ in $L_2(Q)$. Then (5.10) takes the form $(lu, w_0)_{\mathcal{H}_1^{(k)}(\Omega)} = 0$ for any $u \in \mathcal{D}(\mathbf{L}^{(k)})$. If $u \in \mathcal{D}(\mathbf{L}^{(k)})$, then $lu = u|_{\Omega^{(0)}} \in C^4(\overline{\Omega^{(0)}})$. Since the set $C^4(\overline{\Omega^{(0)}})$ is dense in $\mathcal{H}_1^{(k)}(\Omega)$, it follows that $w_0 = 0$ in $\mathcal{H}_1^{(k)}(\Omega)$. Thus, the density of the range of the operator $\mathbf{L}^{(k,0)} = (\mathcal{L}^{(0)} - \lambda \mathcal{L}_1^{(0)}, l)$ in the space $\mathbf{H}^{(k)}$ is proved. The proof of the density of the range of the operator $\mathbf{L}^{(k)}$ in $\mathbf{H}^{(k)}$ is concluded by continuation with respect to a parameter (see [33]).

Thus, the following theorem is proved.

Theorem 5.2. *Let the conditions of Theorem* 5.1 *be satisfied. Then, for any* $f \in L_2(Q)$ *and* $\varphi \in \mathcal{H}_1^{(k)}(\Omega)$, $k = 17, 18$, *there exists a unique strong solution* $u \in B^{(k)}$ *of problem* (5.1)–(5.3), (1.k) *and the estimate*

$$\|u\|_B \le c^{(1)} \left(\|f\|_{L_2(Q)} + \|\varphi\|_{\mathcal{H}_1^{(k)}(\Omega)}\right)$$

holds, where the constant $c^{(1)}$ *is the same as in inequality* (5.5).

The existence and uniqueness of the strong solution of this problem for $A = \Delta$, where $\Delta = \sum\limits_{i=1}^{n} \dfrac{\partial^2}{\partial x_i^2}$ is the Laplace operator, and for the conditions $u|_\Gamma = \left.\dfrac{\partial^2 u}{\partial \boldsymbol{\nu}^2}\right|_\Gamma = 0$ on the lateral surface Γ instead of conditions $(1.k)$ are proved in [32]. In [48, 51], other kinds of boundary-value problems for Eq. (5.1) are studied for $A = \Delta$. In general, they all are posed as follows. Along with Eq. (5.1), we consider the boundary conditions

$$l_1^{(0)}u = \varphi_1^{(0)}(\mathbf{x}'), \quad l_2^{(0)}u = \varphi_2^{(0)}(\mathbf{x}'), \quad l_3^{(0)}u = \varphi_3^{(0)}(\mathbf{x}') \ \text{ on } \ \Omega^{(0)},$$
$$l^{(T)}u = \varphi^{(T)}(\mathbf{x}') \ \text{ on } \ \Omega^{(T)},$$
$$l_1^{(\Gamma)}u = \varphi_1^{(\Gamma)}(\mathbf{x}), \quad l_2^{(\Gamma)}u = \varphi_2^{(\Gamma)}(\mathbf{x}) \ \text{ on } \ \Gamma,$$

where

$$l_1^{(0)}u, l_2^{(0)}u, l_3^{(0)}u \in \left\{ u|_{\Omega^{(0)}}, \left.\frac{\partial u}{\partial x_0}\right|_{\Omega^{(0)}}, \left.\frac{\partial u^2}{\partial x_0^2}\right|_{\Omega^{(0)}}, \left.\frac{\partial^3 u}{\partial x_0^3}\right|_{\Omega^{(0)}} \right\}, \quad l^{(T)}u \in \left\{ u|_{\Omega^{(T)}}, \left.\frac{\partial u}{\partial x_0}\right|_{\Omega^{(T)}}, \left.\frac{\partial u^2}{\partial x_0^2}\right|_{\Omega^{(T)}} \right\},$$
$$l_1^{(\Gamma)}u, l_2^{(\Gamma)}u \in \left\{ u|_\Gamma, \left.\frac{\partial u}{\partial \boldsymbol{\nu}}\right|_\Gamma, \left.\frac{\partial u^2}{\partial \boldsymbol{\nu}^2}\right|_\Gamma, \left.\frac{\partial^3 u}{\partial \boldsymbol{\nu}^3}\right|_\Gamma \right\},$$

and $\varphi_1^{(0)}$, $\varphi_2^{(0)}$, $\varphi_3^{(0)}$, $\varphi^{(T)}$, $\varphi_1^{(\Gamma)}$, and $\varphi_2^{(\Gamma)}$ are given functions from suitable spaces, or the boundary conditions

$$l^{(0)}u = \varphi^{(0)}(\mathbf{x}') \ \text{ on } \ \Omega^{(0)},$$
$$l_1^{(T)}u = \varphi_1^{(T)}(\mathbf{x}'), \quad l_2^{(T)}u = \varphi_2^{(T)}(\mathbf{x}'), \quad l_3^{(T)}u = \varphi_3^{(T)}(\mathbf{x}') \ \text{ on } \ \Omega^{(T)},$$
$$l_1^{(\Gamma)}u = \varphi_1^{(\Gamma)}(\mathbf{x}), \quad l_2^{(\Gamma)}u = \varphi_2^{(\Gamma)}(\mathbf{x}) \ \text{ on } \ \Gamma,$$

where

$$l^{(0)}u \in \left\{ u|_{\Omega^{(0)}}, \left.\frac{\partial u}{\partial x_0}\right|_{\Omega^{(0)}}, \left.\frac{\partial u^2}{\partial x_0^2}\right|_{\Omega^{(0)}} \right\}, \quad l_1^{(T)}u, l_2^{(T)}u, l_3^{(T)}u \in \left\{ u|_{\Omega^{(T)}}, \left.\frac{\partial u}{\partial x_0}\right|_{\Omega^{(T)}}, \left.\frac{\partial u^2}{\partial x_0^2}\right|_{\Omega^{(T)}}, \left.\frac{\partial^3 u}{\partial x_0^3}\right|_{\Omega^{(T)}} \right\},$$
$$l_1^{(\Gamma)}u, l_2^{(\Gamma)}u \in \left\{ u|_\Gamma, \left.\frac{\partial u}{\partial \boldsymbol{\nu}}\right|_\Gamma, \left.\frac{\partial u^2}{\partial \boldsymbol{\nu}^2}\right|_\Gamma, \left.\frac{\partial^3 u}{\partial \boldsymbol{\nu}^3}\right|_\Gamma \right\},$$

and $\varphi^{(0)}$, $\varphi_1^{(T)}$, $\varphi_2^{(T)}$, $\varphi_3^{(T)}$, $\varphi_1^{(\Gamma)}$, and $\varphi_2^{(\Gamma)}$ are given functions from suitable spaces. The solvability of those problems is proved by the method of energy inequalities and the averaging operators with a variable step; the Riesz theorem on the representation of linear continuous functionals in Hilbert spaces is used as well.

REFERENCES

1. S. S. Akhiev and S. N. Sanieva, “Problems of Cauchy and Goursat type on composite lines,” *Soviet Math., Dokl.*, **43**, No. 1, 112–116 (1991).
2. S. A. Aldashev, “On problems for a class of multi-dimensional integrodifferential hyperbolic equations,” *Abstr. Third Siberian Congress on Applied and Industrial Mathematics Devoted to the Memory of S. L. Sobolev (1908–1989). Part 4*, Novosibirsk, 137–140 (1998).
3. G. C. Barozzi, “Su alcuni problemi relativi a una equazione lineare a derivate parziali di tipo composito,” *Atti Semin. Mat. Fis. Univ. Modena*, No. 10, 11–30 (1961).
4. J. Bennish, “Mixed initial-boundary value problems for hyperbolic equations with constant coefficients,” *J. Math. Anal. Appl.*, **153**, No. 2., 506–532 (1990).
5. P. E. Berhin, “On a boundary value problem for equations of composite type,” *Soviet Math., Dokl.*, **15**, 161–165 (1974).

6. N. I. Brish, “Classical solution of mixed problems for nonstationary equations,” *Differ. Equ.*, **1**, No. 12, 400–403 (1965).
7. N. I. Brish, “Mixed problems for semi-hyperbolic equations,” *Differ. Uravn.*, **5**, 2200–2215 (1969).
8. V. I. Burenkov, “On the density of infinitely differentiable functions in Sobolev spaces for an arbitrary open set,” *Proc. Steklov Inst. Math.*, **131**, 39–51 (1974).
9. V. I. Burenkov, “Approximation by infinitely differentiable functions with preservation of boundary values,” *Proc. Steklov Inst. Math.*, **180**, 76–79 (1989).
10. V. I. Burenkov, *Sobolev Spaces on Domains*, Teubner, Stuttgart (1998).
11. L. Cattabriga, “Su alcuni problemi per equazioni differenziali di tipo composito,” *Rend. Sem. Mat. Univ. Padova*, **27**, 122–143 (1957).
12. V. V. Dajnyak and V. I. Korzyuk, “A Dirichlet-type problem for a third-order linear differential equation,” *Differ. Equ.*, **23**, 598–602 (1987).
13. R. B. Davis, “A boundary value problem for third-order linear partial differential equations of composite type,” *Proc. Amer. Math. Soc.*, **3**, 751–756 (1952).
14. A. Dzhuraev, *Systems of Equations of Composite Type* [in Russian], Nauka, Moscow (1972).
15. T. D. Dzhuraev, “On a boundary-value problem for a composite-type equation,” *Dokl. Akad. Nauk UzSSR*, No. 4, 5–8 (1962).
16. T. D. Dzhuraev, “On a boundary-value problem for the equation $\frac{\partial}{\partial x}\Delta u = 0$,” *Issled. Differ. Uravn. Primen.*, Tashkent, 46–51 (1963).
17. T. D. Dzhuraev, “On the uniqueness and existence of solutions of certain boundary-value problems for composite-type equations,” *Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk*, No. 3 (1967).
18. T. D. Dzhuraev, “On the uniqueness of the solution of certain boundary-value problems for a class of composite-type equations,” *Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk*, No. 4 (1967).
19. T. D. Dzhuraev and U. Rakhmanov, “On well-posed boundary-value problems of mixed-composite type,” *Differ. Uravn.*, **9**, 32–40 (1973).
20. T. D. Džuraev and Ja. S. Šarifbaev, “Boundary value problems for a class of third order equations,” *Boundary value problems for differential equations*, No. 4, 48–76 (1974).
21. G. I. Èskin, “A boundary-value problem for the equation $\frac{\partial}{\partial t}P\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)u = 0$, where $P\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)$ is an elliptic operator,” *Sibirsk. Mat. Zh.*, **3**, 882–911 (1962).
22. G. I. Èskin, “Boundary-value problems for equations with constant coefficients on the plane,” *Mat. Sb.* (*N.S.*), **59** (**101**), 67–104 (1962).
23. G. Eskin, “Initial-boundary value problem for second order hyperbolic equations with general boundary conditions. I,” *J. d'Analyse Math.*, **40** 43–89 (1981).
24. G. Eskin, “Initial-boundary value problem for second order hyperbolic equations with general boundary conditions. II,” *Comm. Partial Differential Equations*, **10**, No. 10, 1117–1212 (1985).
25. G. Eskin, “Mixed initial-boundary value problem for second hyperbolic equations,” *Comm. Partial Differential Equations*, **12**, No. 5, 503–587 (1987).
26. G. Eskin, “The wave equation in a wedge with general boundary conditions,” *Comm. Partial Differential Equations*, **17**, No. 1-2, 99–160 (1992).
27. K. Friedrichs, “Symmetric hyperbolic linear differential equations,” *Comm. Pure Appl. Math.*, **7**, No. 2, 345–392 (1954).
28. L. Gårding, *Cauchy's Problem for Hyperbolic Equations* [Russian translation], Mir, Moscow (1961).
29. S. G. Gindikin and L. R. Volevich, *Mixed Problem for Partial Differential Equations with Quasihomogeneous Principal Part*, American Mathematical Society, Providence, RI (1996).
30. J. Hadamard, “Propriétés d'une équation linéaire aux dérivées partielles du quatrieme ordre,” *Tohôku Math. J.*, **37**, 133–150 (1933).

31. J. Hadamard, “Equations aux dérivées partielles. Les conditions definies en général. Le cas hyperbolique,” *Enseign. Math.*, **35**, 5–42 (1936).
32. O. A. Konopel'ko, “A boundary-value problem for a fourth-order equation of composite type,” *Vestnik Belarus. Univ. Ser. 1*, No. 3, 73–79 (2008).
33. V. I. Korzyuk, “A mixed problem for certain nonstationary equations with discontinuous coefficients,” *Differ. Uravn.*, **6**, 343–357 (1970).
34. V. I. Korzyuk, “An energy inequality for a boundary value problem for a third-order hyperbolic equation with a wave operator,” *Differ. Equ.*, **27**, No. 6, 719–724 (1991).
35. V. I. Korzyuk, “Averaging operators with a variable step. II,” *Vestnik Belarus. Univ. Ser. 1*, No. 3, 63–65 (1992).
36. V. I. Korzyuk, “The first mixed problem for a second-order linear hyperbolic equation with homogeneous conditions in the case of a nontube domain,” *Differ. Equ.*, **28**, No. 5, 690–698 (1992).
37. V. I. Korzyuk, “The method of energy inequalities and averaging operators,” *Vestnik Belarus. Univ. Ser. 1*, No. 2, 55–71 (1996).
38. V. I. Korzyuk, “A boundary value problem for a third-order Mangeron equation,” *Differ. Equ.*, **33**, No. 12, 1686–1694 (1997).
39. V. I. Korzyuk, “The Goursat problem for hyperbolic linear third order equations,” *Math. Model. Anal.*, **6**, No. 2, 270–279 (2001).
40. V. I. Korzyuk, “A boundary value problem for a hyperbolic equation with a third-order wave operator,” *Differ. Equ.*, **40**, No. 2, 218–226 (2004).
41. V. I. Korzyuk, “Mollifiers with variable step in the theory of boundary problems for partial differential equations,” *Analytic Methods of Analysis and Differential Equations*, *Camb. Sci. Publ.*, Cambridge, 133–154 (2006).
42. V. I. Korzyuk, *Lecture Course of Equations of Mathematical Physics. Part 4* [in Russian], Belarus State Univ., Minsk (2008).
43. V. I. Korzyuk and E. S. Cheb, “A mixed problem for a fourth-order hyperbolic equation,” *Vestsī Nats. Akad. Navuk Belarusī Ser. Fīz.-Mat. Navuk*, No. 2, 9–13 (2004).
44. V. I. Korzyuk and E. S. Cheb, “Mixed problems for the bi-wave equation,” *Vestnik Belarus. Univ. Ser. 1*, No. 1, 63–68 (2005).
45. V. I. Korzyuk and E. S. Cheb, “Boundary-value problems for second-order elliptic equations,” *Proc. Math. Inst.*, **15**, No. 2, 38–47 (2007).
46. V. I. Korzyuk and E. S. Cheb, “The Cauchy problem for a fourth-order equation with a bi-wave operator,” *Differ. Equ.*, **43**, No. 5, 688–695 (2007).
47. V. I. Korzyuk and E. C. Cheb, “Generalized-classical solution of the mixed problems for a hyperbolic equation of the second order,” *Analytic Methods of Analysis and Differential Equations: AMADE 2006*, *Camb. Sci. Publ.*, Cambridge, 165–180 (2008).
48. V. I. Korzyuk, E. S. Cheb, and O. A. Konopel'ko, “On boundary-value problems for fourth-order equations of the hyperbolic and composite types,” *Abstr. Fifth Int. Conf. on Differential and Functional Differential Equations*, Moscow, 102-103 (2008).
49. V. I. Korzyuk and V. V. Daĭnyak, “On a weak solution of a Dirichlet-type problem for a third-order linear differential equation,” *Differ. Equ.*, **28**, No. 6, 857–867 (1992).
50. V. I. Korzyuk and V. V. Daĭnyak, “On the solvability of mixed problems for non-stationary third-order equations,” *Vestnik Belarus. Univ. Ser. 1*, No. 3, 54–60 (2005).
51. V. I. Korzyuk and O. A. Konopel'ko, “Generalized solutions of boundary-value problems in cylindrical domains for a fourth-order equation of the composite type,” *Abstr. Tenth Belarus. Math. Conf. Part 4*, Minsk, 69–71 (2008).
52. A. I. Kozhanov, “A boundary value problem for a class of third order equations,” *Sov. Math., Dokl.*, **20**, 1301–1304 (1979).
53. A. I. Kozhanov, “A mixed problem for a class of nonclassical equations,” *Differ. Equ.*, **15**, 186–191 (1979).

54. A. I. Kozhanov, *On Solvability and Properties of Solutions for a Class of Third-Order Equations*, Preprint No. 11, Math. Inst. Siber. Dept. Acad. Sci. USSR (1986).
55. O. A. Ladyzhenskaya, "On the method of Fourier for the wave equation," *Dokl. Akad. Nauk SSSR* (*N.S.*), **75**, 765–768 (1950).
56. O. A. Ladyzhenskaya, *The Mixed Problem for a Hyperbolic Equation* [in Russian], Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow (1953).
57. O. A. Ladyzhenskaya, "On solvability of the fundamental boundary problems for equations of parabolic and hyperbolic type," *Dokl. Akad. Nauk SSSR* (*N.S.*), **97**, 395–398 (1954).
58. O. A. Ladyzhenskaya, "On the solution of non-stationary operator equations," *Mat. Sb. N.S.*, **39** (**81**), 491–524 (1956).
59. O. A. Ladyzhenskaya, "On non-stationary operator equations and their applications to linear problems of mathematical physics," *Mat. Sb. N.S.*, **45** (**87**), 123–158 (1958).
60. O. A. Ladyzhenskaya and N. N. Uraltseva, *Linear and Quasilinear Equations of Elliptic Type* [in Russian], Nauka, Moscow (1973).
61. N. A. Lar'kin, "Boundary value problems in the large for a class of hyperbolic equations," *Sibirsk. Mat. Zh.*, **18**, No. 6, 1414–1419 (1977).
62. N. A. Lar'kin, "A class of degenerate hyperbolic equations," *Dinamika Sploshn. Sredy*, No. 36, 71–77 (1978).
63. N. A. Lar'kin, "On a class of quasi-linear hyperbolic equations having global solutions," *Sov. Math., Dokl.*, **20**, 28–31 (1979).
64. J. Leray, *Hyperbolic Differential Equations* [Russian translation], Nauka, Moscow (1984).
65. B. M. Levitan and A. È. Mamatov, "Estimation of the Cauchy matrix function for the Dirac system in the case of finite-gap nonperiodic potentials," *Math. Notes*, **53**, No. 3-4 400–409 (1993).
66. M. Miyake and Y. Hashimoto, "Newton polygons and Gevrey indices for linear partial differential operators," *Nagoya Math. J.*, **128**, 15–47 (1992).
67. G. Papi, "A Goursat problem for the wave equation," *Boll. Unione Mat. Ital., VI. Ser., C, Anal. Funz. Appl.*, **5**, No. 1, 139–157 (1986).
68. I. Petrowsky, "Über das Cauchysche Problem für Systeme von partiellen Differentialgleichungen," *Rec. Math. N.S.*, **2**, 815–870 (1937).
69. B. Pini, "Un problema di valori al contorno per un'equazione a derivate parziali del terzo ordine con parte principale di tipo composito," *Rend. Sem. Fac. Sci. Univ. Cagliari*, **27**, 114–135 (1958).
70. S. G. Pyatkov, "A composite equation," *Differ. Equ.*, **16**, 86–90 (1980).
71. R. Sakamoto, "Mixed problems for hyperbolic equations," *J. Math. Kyoto Univ.*, **10**, 403–417 (1970).
72. M. S. Salakhitdinov, "Some correct problems for the equation $y^m u_{xxx} + U_{xyy} = 0$," *Sov. Math., Dokl.*, **4**, 710–713 (1963).
73. M. S. Salakhitdinov, "On the theory of equations of composite type," *Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk*, **9**, No. 3, 710–713 (1965).
74. M. S. Salakhitdinov, *Equations of Mixed-Composite Type* [in Russian], Fan, Tashkent (1974).
75. O. Sjöstrand, "Sur une équation aux dérivées partielles du type composite," *Ark. Mat. Astron. Fys. A.*, **25**, 1–11 (1936).
76. O. Sjöstrand, "Sur une équation aux dérivées partielles du type composite. II," *Ark. Mat. Astron. Fys. A.* **26**, No. 1, 1–10 (1937).
77. M. M. Smirnov, "Some boundary-value problems for an equation of mixed composite type," *Sibirsk. Mat. Zh.*, **5**, 923–928 (1964).
78. S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics* [in Russian], Leningrad. Gos. Univ., Leningrad (1950).
79. A. N. Terekhov, *On certain problems for a nonlinear equation of composite type*, Preprint, Math. Inst. Siber. Dept. Acad. Sci. USSR (1980).

80. V. Thomée, “Estimates of the Friedrichs–Levy type for a hyperbolic equation with three characteristics,” *Math. Scand.*, **3**, 115–123 (1955).
81. V. Thomée, “Estimates of the Friedrichs–Lewy type for mixed problems in the theory of linear hyperbolic differential equations in two independent variables,” *Math. Scand.*, **5**, 93–113 (1957).
82. V. Thomée, “Existence proofs for mixed problems for hyperbolic differential equations in two independent variables by means of the continuity method,” *Math. Scand.*, **6**, 5–32 (1958).
83. V. N. Vragov, “An equation of mixed-composite type,” *Differ. Equ.*, **9**, 132–134 (1974).
84. G. B. Whitham, *Linear and Nonlinear Waves*, Wiley-Interscience, New York–London–Sydney (1974).
85. N. I. Yurchuk and Ya. V. Radyno, “The Cauchy problem for even-order hyperbolic equations,” *Differ. Uravn.*, **12**, No. 2, 331–342 (1976).

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