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Integrals Involving Legendre Functions II

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§ 1. Introductory

Four integrals (8), (9), (10) and (11) below, involving Legendre functions will be evaluated in terms of Appell's function F_4 in section 3, with the help of "Operational Calculus".

Throughout this note the conventional notation $\Phi(p) \doteqdot h(t)$ will be used to represent the classical Laplace's integral

$$\Phi(p) = p \int_0^\infty e^{-pt}\, h(t)\, dt\,. \tag{1}$$

The following results will be required in the proofs.

If

$$\Phi(p) \doteqdot h(t)$$

and

$$\Psi(\nu, p, \lambda) \doteqdot K_\nu(\lambda t)\, h(t)\,,$$

then

$$\begin{cases} \int\limits_0^\infty t^{-\nu}(a + bt + ct^2)^{-1}\, \Phi\left(\dfrac{a + bt + ct^2}{t}\right) dt \\ = 2b^{-1}\,(c/a)^{\nu/2}\, \Psi(\nu, b, 2\sqrt{ac})\,, \end{cases} \tag{2}$$

provided that the integrals are absolutely convergent, $R(a) > 0$ $R(c) > 0$ and $h(t)$ is independent of λ.

If

$$\Phi(p) \doteqdot h(t)$$

and

$$\Psi(\nu, p, \lambda) \doteqdot K_\nu(\lambda t)\, h(t)\,,$$

then

$$\begin{cases} \alpha \int\limits_0^\infty \cosh \nu\theta\, (\alpha + \beta \cosh\theta)^{-1}\, \Phi(\alpha + \beta\cosh\theta)\, d\theta = \Psi(\nu, \alpha, \beta)\,, \end{cases} \tag{3}$$

provided that the integrals are absolutely convergent and $R(\beta) > 0$.

(2) and (3) were recently proved by the author [3, pp. 154, 155].

If $R(\lambda + \mu \pm \nu) > 0$, $R(p \pm \alpha + \beta) > 0$

$$(4) \quad \begin{cases} \int\limits_0^\infty e^{-pt} t^{\lambda-1} I_\mu(\alpha t)\, K_\nu(\beta t)\, dt \\ = \sum\limits_{\nu, -\nu} \dfrac{\Gamma(-\nu)\ \alpha^\mu\, \beta^\nu\ \Gamma(\lambda+\mu+\nu)}{2^{1+\mu+\nu} p^{\lambda+\mu+\nu}\ \Gamma(\mu+1)} \times \\ \times F_4\left[\dfrac{\lambda+\mu+\nu}{2}, \dfrac{\lambda+\mu+\nu+1}{2}; \mu+1, \nu+1; \dfrac{\alpha^2}{p^2}, \dfrac{\beta^2}{p^2}\right]. \end{cases}$$

If $R(\lambda \pm \mu \pm \nu) > 0$, $R(p + \alpha + \beta) > 0$

$$(5) \quad \begin{cases} \int\limits_0^\infty e^{-pt} t^{\lambda-1} K_\mu(\alpha t)\, K_\nu(\beta t)\, dt \\ = \sum\limits_{\mu, -\mu} \sum\limits_{\nu, -\nu} \dfrac{\Gamma(-\nu)\, \Gamma(-\mu)\ \alpha^\mu\, \beta^\nu\ \Gamma(\lambda+\mu+\nu)}{2^{\mu+\nu+2}\ p^{\lambda+\mu+\nu}} \times \\ \times F_4\left[\dfrac{\lambda+\mu+\nu}{2}, \dfrac{\lambda+\mu+\nu+1}{2}; \mu+1, \nu+1; \dfrac{\alpha^2}{p^2}, \dfrac{\beta^2}{p^2}\right]. \end{cases}$$

(4) and (5) can be easily derived from the integral [1, p. 196]

$$(6) \quad \begin{cases} \int\limits_0^\infty t^{\lambda-1} e^{-pt} I_\mu(\alpha t)\, I_\nu(\beta t)\, dt = \dfrac{\alpha^\mu \beta^\nu\, \Gamma(\lambda+\mu+\nu)}{2^{\mu+\nu} p^{\lambda+\mu+\nu}\, \Gamma(\nu+1)\, \Gamma(\mu+1)} \times \\ \times F_4\left[\dfrac{\lambda+\mu+\nu}{2}, \dfrac{\lambda+\mu+\nu+1}{2}; \mu+1, \nu+1; \dfrac{\alpha^2}{p^2}, \dfrac{\beta^2}{p^2}\right] \end{cases}$$

on applying the well-known formula

$$(7) \quad K_\nu(z) = \frac{1}{2} \sum_{\nu, -\nu} \Gamma(-\nu)\, \Gamma(\nu+1)\, I_\nu(z)$$

where the symbol $\sum\limits_{\nu, -\nu}$ indicates that in the expression following it, ν is to be replaced by $-\nu$ and the two expressions are to be added.

In what follows we have used MACROBERT's definition of $Q_n^m(x)$.

§ 2. Integrals

The formulae to be proved are as follows.

If $R(1/2 - m \pm n \pm \nu) > 0$, $R(a) > 0$, $R(c) > 0$

$$(8) \quad \begin{cases} \int\limits_0^\infty t^{-\nu-1} \left\{\left(\dfrac{a+bt+ct^2}{t}\right)^2 - 1\right\}^{\frac{m}{2}} P_{n-1/2}^m\left(\dfrac{a+bt+ct^2}{t}\right) dt \\ = \dfrac{\pi^{-1/2}\ (c/a)^{\nu/2}}{\Gamma(1/2 - m \pm n)} \sum\limits_{n, -n} \sum\limits_{\nu, -\nu} \dfrac{\Gamma(-n)\, \Gamma(-\nu)\, \Gamma(1/2 - m + \nu + n)\, (ac)^{\nu/2}}{b^{\nu+n+1/2-m}\ 2^{1/2+n}} \times \\ \times F_4\left[1/2\,(1/2 - m + \nu + n), 1/2\,(3/2 + \nu + n - m); n+1, \nu+1; \dfrac{1}{b^2}, \dfrac{4ac}{b^2}\right]. \end{cases}$$

If $R(1/2 - m \pm n \pm \nu) > 0$, $R(\beta) > 0$

$$(9)\quad \begin{cases} \int\limits_0^\infty \cosh \nu\,\theta \{(\alpha + \beta \cosh \theta)^2 - 1\}^{\frac{m}{2}} P^m_{n-1/2}(\alpha + \beta \cosh \theta)\, d\theta \\ = \dfrac{\pi^{-1/2}}{\Gamma(1/2 - m \pm n)} \sum\limits_{n, -n} \sum\limits_{\nu, -\nu} \dfrac{\Gamma(-n)\,\Gamma(-\nu)\,\Gamma(1/2 - m + \nu + n)\,\beta^\nu}{\alpha^{\nu+n+1/2-m}\,2^{\nu+n-1/2}} \times \\ \times F_4\left[1/2(1/2 - m + n + \nu), 1/2(3/2 - m + \nu + n); n+1, \nu+1; \dfrac{1}{\alpha^2}, \dfrac{\beta^2}{\alpha^2}\right]. \end{cases}$$

If $R(1/2 + m + n \pm \nu) > 0$, $R(a) > 0$, $R(c) > 0$

$$(10)\quad \begin{cases} \int\limits_0^\infty t^{-\nu-1}\left\{\left(\dfrac{a + bt + ct^2}{t}\right)^2 - 1\right\}^{-\frac{m}{2}} Q^m_{n-1/2}\left(\dfrac{a + bt + ct^2}{t}\right) dt \\ = \dfrac{\pi^{1/2}(c/a)^{\nu/2}}{\Gamma(n+1)\,2^{n+1/2}} \sum\limits_{\nu, -\nu} \dfrac{\Gamma(-\nu)\,(ac)^{\nu/2}\,\Gamma(1/2 + m + n + \nu)}{b^{m+n+\nu+1/2}} \times \\ \times F_4\left[1/2(1/2 + m + n + \nu), 1/2(3/2 + m + n + \nu); n+1, \nu+1; \dfrac{1}{b^2}, \dfrac{4ac}{b^2}\right]. \end{cases}$$

If $R(1/2 + m + n \pm \nu) > 0$, $R(\beta) > 0$

$$(11)\quad \begin{cases} \int\limits_0^\infty \cosh \nu\theta \{(\alpha + \beta \cosh\theta)^2 - 1\}^{-\frac{m}{2}} Q^m_{n-1/2}(\alpha + \beta \cosh\theta)\, d\theta \\ = \dfrac{\Gamma(1/2)}{\Gamma(n+1)} \sum\limits_{\nu, -\nu} \dfrac{\Gamma(-\nu)\,\beta^\nu\,\Gamma(m + n + \nu + 1/2)}{2^{n+\nu+3/2}\,\alpha^{m+n+\nu+1/2}} \times \\ \times F_4\left[1/2(1/2 + m + n + \nu), 1/2(3/2 + m + n + \nu); n+1, \nu+1; \dfrac{1}{\alpha^2}, \dfrac{\beta^2}{\alpha^2}\right]. \end{cases}$$

§ 3. Proofs of the formulae

To prove (8) and (9) we take [1, p. 198]

$$\begin{aligned} h(t) &= t^{-m-1/2} K_n(t) \\ &\doteqdot \sqrt{\frac{\pi}{2}}\, \Gamma(1/2 - m \pm n)\, p(p^2 - 1)^{\frac{m}{2}} P^m_{n-1/2}(p) \\ &= \Phi(p), \end{aligned}$$

where $R(1/2 - m \pm n) > 0$, $R(p + 1) > 0$, then from (5) we have

$$\begin{aligned} K_\nu(\lambda t)\, h(t) &= t^{-m-1/2} K_n(t)\, K_\nu(\lambda t) \\ &\doteqdot \sum_{n, -n} \sum_{\nu, -\nu} \frac{\lambda^\nu\,\Gamma(-n)\,\Gamma(-\nu)\,\Gamma(1/2 - m + \nu + n)}{2^{n+\nu+2}\,p^{\nu+n-1/2-m}} \times \end{aligned}$$

$$\times F_4\left[1/2(1/2 - m + \nu + n), 1/2(3/2 - m + \nu + n); n+1, \nu+1; \frac{1}{p^2}, \frac{\lambda^2}{p^2}\right],$$

where $R(1/2 - m \pm n \pm \nu) > 0$, $R(p + \lambda + 1) > 0$.

Applying (2) and (3) we obtain (8) and (9) respectively.

Formulae (10) and (11) can be proved in a similar way, on using [2, p. 342]

$$\begin{aligned} h(t) &= t^{m-1/2}\, I_n(t) \\ &\doteqdot \sqrt{\frac{2}{\pi}}\, p\,(p^2-1)^{-\frac{m}{2}}\, Q^m_{n-1/2}(p) \\ &= \Phi(p)\,, \end{aligned}$$

where $R(m+n+1/2) > 0$, $R(p) > 1$ and

$$\begin{aligned} K_\nu(\lambda t)\, h(t) &= t^{m-1/2}\, I_n(t)\, K_\nu(\lambda t) \\ &= \sum_{\nu,\,-\nu} \frac{\Gamma(-\nu)\,\lambda^\nu\,\Gamma(m+1/2+n+\nu)}{2^{1+n+\nu}\,p^{m+n+\nu-1/2}\,\Gamma(n+1)} \times \end{aligned}$$

$$\times\, F_4\left[1/2(1/2+m+n+\nu),\, 1/2(3/2+m+n+\nu);\, n+1,\, \nu+1;\, \frac{1}{p^2},\, \frac{\lambda^2}{p^2}\right],$$

where $R(1/2+m+n\pm\nu) > 0$, $R(p+\lambda) > 1$.

The particular cases of (8), (9), (10) and (11) for $m = 0$ were recently obtained by the author [3, pp. 156, 157].

References

[1] ERDELYI, A. et al.: Tables of integral transforms. Vol. 1, New York: McGraw-Hill 1954.

[2] MACROBERT, T. M.: Spherical Harmonics. London: Methuen & Co. 1947.

[3] SAXENA, R. K.: Integrals involving Legendre functions. Math. Ann. 147, 154—157 (1962).

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