MATRIX EQUATIONS OF THE SPATIAL FLIGHT OF AN ASYMMETRIC SOLID

V. V. Kravets UDC 629.73

In the dynamic design of modern airplanes, there is an invariable tendency toward complication of the calculation schemes, generalization of the formulation of the problem, and broad use of computational experiment. The dynamic properties of airplanes have been studied as conditions of the interaction of spatial rotational and translational motion and change in phase coordinates over broad limits, taking account of comprehensive and time-dependent information on the inertial, aerodynamic, and structural parameters and the perturbing factors [5]. The diversity of content of specific engineering problems of airplane dynamics is due to the development of a large number of different forms for writing the differential equations of motion, most of which were given in [3].

Complication of the problems of flight dynamics and the wide use of computational experiments in their solution have stimulated the further search for rational methods of writing the equations of spatial airplane motion [7] and improving the mathematical description, with the aim of convenient computer realization of the problem [2, 8]. In the present work, the choice of the method of writing the equations of motion (the Euler–Lagrange method) [6], and the use of matrix calculus (the matrices called equivalent quaternions by Bellman) [1] and Rodrigues–Hamilton parameters and quasivelocities as variables of integration, conform to the aim of ensuring good adjustment to programming and the possibility of effective use of modern computer software and also serve the purpose of deriving a modern analytical form for the equations of motion. Such a form, while satisfying the aesthetic criteria, would also meet utilitarian needs, for example, the possibility of eliminating mechanical errors and inaccuracies in deriving the equations, writing and debugging the computer programs, etc. Note that, in the traditional form, the equations of motion lack a structural order, are cumbersome and immense, and may include errors that are difficult to detect (see [7], p. 414, Sec. 8.10.5). The expediency of using Euler–Lagrange equations in dynamic problems of systems of solids was noted in [6], where, in particular, the Euler–Lagrange equations describing the spatial motion of a solid were written in vector form. However, the ordering of the equations of motion and their symmetric structure are evident to a greater extent when they are written in matrix form [4], which is also more convenient for direct computer realization. Below, a matrix model of the space flight of an asymmetric solid, based exclusively on four unified equivalent-quaternion matrices, is proposed. The nonlinear dynamic problem of airplane spatial flight in dense layers of the atmosphere under the action of a spatial wind is considered, taking account of inertia–mass asymmetry and perturbing factors due to considerable errors of construction and installation. The initial reference frame adopted is, for example, the initial starting system of axes [5] or the geotopic rectangular horizontal-oriented 11 coordinate system [3], which corresponds to flight above a plane nonrotating earth. The differential equations of motion are written in a noncentered rectangular coordinate system associated with the unchanging part of the airplane [3].

On the basis of a matrix description of the Euler–Lagrange differential equations of motion for a free solid [4], the introduction of unified matrices brings their left-hand sides to the form

$$\left\| \begin{array}{c|c} \Theta & \dfrac{X + {}'X}{2} m \\ \hline \dfrac{X' + {}'X'}{2} m & M \end{array} \right\| \left\| \begin{array}{c} \dot{\omega} \\ \hline \dot{u} \end{array} \right\| +$$

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$$+\left\|\begin{array}{c|c} \dfrac{\Omega+{}'\Omega}{2} & \dfrac{U+{}'U}{2} \\ \hline 0 & \dfrac{\Omega+{}'\Omega}{2} \end{array}\right\| \left\|\begin{array}{c|c} \Theta & \dfrac{X+{}'X}{2}m \\ \hline \dfrac{X'+{}'X'}{2}m & M \end{array}\right\| \left\|\begin{array}{c} \omega \\ \hline u \end{array}\right\|, \tag{1}$$

where m is the mass of the solid; $\Theta$ is a symmetric matrix whose components are the axial $\theta_{11}$, $\theta_{22}$, $\theta_{33}$ and centrifugal $\theta_{12} = \theta_{21}$, $\theta_{13} = \theta_{31}$, $\theta_{23} = \theta_{32}$ moments of inertia. Assuming that the principal central moments of inertia of the solid $J_1$, $J_2$, $J_3$ and the Rodrigues–Hamilton parameters $a_0$, $a_1$, $a_2$, $a_3$ determining the orientation of the principal central axes of inertia are known, the matrix $\Theta$ is expressed in terms of unified matrices

$$\Theta = {}'AAJA'{}'A' + X\frac{X'+{}'X'}{2}m. \tag{2}$$

Here, the unified matrix A takes the form

$$A = \begin{Vmatrix} a_0 & a_1 & a_2 & a_3 \\ -a_1 & a_0 & -a_3 & a_2 \\ -a_2 & a_3 & a_0 & -a_1 \\ -a_3 & -a_2 & a_1 & a_0 \end{Vmatrix}, \tag{3}$$

the matrix 'A' is transposed with respect to A

$${}'A' = \begin{Vmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & a_3 & -a_2 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{Vmatrix}, \tag{4}$$

the matrix 'A is obtained from matrix A by switching the first row and the first column

$$A = \begin{Vmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{Vmatrix}, \tag{5}$$

and the matrix A' is obtained from matrix A by switching the second, third, and fourth rows and the corresponding columns

$$A' = \begin{Vmatrix} a_0 & a_1 & a_2 & a_3 \\ -a_1 & a_0 & a_3 & -a_2 \\ -a_2 & -a_3 & a_0 & a_1 \\ -a_3 & a_2 & -a_1 & a_0 \end{Vmatrix}. \tag{6}$$

The matrix X, identical in structure with the matrix A, has as its components the coordinates of the center of mass of the solid in the system of coupled axes $x_1$, $x_2$, $x_3$, while $x_0 = 0$, i.e., X is the skew-symmetric matrix

$$X = \begin{Vmatrix} 0 & x_1 & x_2 & x_3 \\ -x_1 & 0 & -x_3 & x_2 \\ -x_2 & x_3 & 0 & -x_1 \\ -x_3 & -x_2 & x_1 & 0 \end{Vmatrix}.$$

The matrices J and M take the form

$$J=\left\|\begin{matrix}0&0&0&0\\0&J_1&0&0\\0&0&J_2&0\\0&0&0&J_3\end{matrix}\right\|;\quad M=\left\|\begin{matrix}0&0&0&0\\0&m&0&0\\0&0&m&0\\0&0&0&m\end{matrix}\right\|;$$

0 is a 4 × 4 zero matrix. The projections of the angular-velocity vector of the solid $\omega_1$, $\omega_2$, $\omega_3$ and the pole linear-velocity vector $u_1$, $u_2$, $u_3$ of the coupled coordinate system on the coupled axes are taken as the quasivelocities. The dependence of the quasivelocities taken in terms of unified matrices on the Rodrigues–Hamilton parameters $l_0$, $l_1$, $l_2$, $l_3$ determining the orientation of the solid in inertial space takes the form

$$\omega=2L'\dot{l};\ u=L''L'\dot{z}, \tag{7}$$

where

$$\omega=\left\|\begin{matrix}0\\\omega_1\\\omega_2\\\omega_3\end{matrix}\right\|;\quad \dot{l}=\left\|\begin{matrix}\dot{l}_0\\\dot{l}_1\\\dot{l}_2\\\dot{l}_3\end{matrix}\right\|;\quad u=\left\|\begin{matrix}0\\u_1\\u_2\\u_3\end{matrix}\right\|;\quad \dot{z}=\left\|\begin{matrix}0\\\dot{z}_1\\\dot{z}_2\\\dot{z}_3\end{matrix}\right\|;$$

$\dot{z}_1$, $\dot{z}_2$, $\dot{z}_3$ are the components of the linear pole velocity in the inertial reference frame; $\dot{l}_0$, $\dot{l}_1$, $\dot{l}_2$, $\dot{l}_3$ are the time derivatives of the Rodrigues–Hamilton parameters; $L''L'$ are the matrices of finite rotation on passing from the inertial to the coupled coordinate system. In view of the commutation properties of the given unified matrices, $L''L' = L'L$. The unified matrix denoted by L is composed of the elements $l_0$, $l_1$, $l_2$, $l_3$ and is written in the form

$$L=\left\|\begin{matrix}l_0&l_1&l_2&l_3\\-l_1&l_0&-l_3&l_2\\-l_2&l_3&l_0&-l_1\\-l_3&-l_2&l_1&l_0\end{matrix}\right\|.$$

The unified matrices $\Omega$, $'\Omega$ of quasivelocities $\omega_1$, $\omega_2$, $\omega_3$ are found directly in terms of the unified matrices of Rodrigues–Hamilton parameters according to the formulas

$$1/2\Omega='L'\dot{L};\quad 1/2'\Omega=L''\dot{L} \tag{8}$$

and, correspondingly, for the unified matrices U, 'U of quasivelocities $u_1$, $u_2$, $u_3$

$$U='L'\dot{Z}L;\ 'U='L'\dot{Z}L'. \tag{9}$$

Here, the elements of the unified matrix $\dot{L}$ are the time derivatives of the Rodrigues–Hamilton parameters

$$\dot{L}=\left\|\begin{matrix}\dot{l}_0&\dot{l}_1&\dot{l}_2&\dot{l}_3\\-\dot{l}_1&\dot{l}_0&-\dot{l}_3&\dot{l}_2\\-\dot{l}_2&\dot{l}_3&\dot{l}_0&-\dot{l}_1\\-\dot{l}_3&-\dot{l}_2&\dot{l}_1&\dot{l}_0\end{matrix}\right\|,$$

and the matrices $\Omega$, U, $\dot{Z}$ have as their elements, respectively, 0, $\omega_1$, $\omega_2$, $\omega_3$; 0, $u_1$, $u_2$, $u_3$; 0, $\dot{z}_1$, $\dot{z}_2$, $\dot{z}_3$, i.e., are skew-symmetric.

The right-hand side of the differential equations of motion consists of the sum of block matrices whose form is determined as a function of the composition of the active forces acting on the solid. Below, the use of unified matrices in taking account of some of these forces is illustrated: G, the gravitational force; P, the traction force; Q, the axial aerodynamic force.

The block matrix taking account of the action of the gravitational force on the dynamics of spatial flight depends on the orientation of the solid and the asymmetric position of the center of mass relative to the couples axes and takes the form

$$G\left\|\begin{array}{c}\dfrac{X+'X}{2}\\ \hline E\end{array}\right\| L'\,'L' \left\|\begin{array}{r}0\\0\\0\\-1\end{array}\right\|, \tag{10}$$

where E is a 4 × 4 unit matrix.

The traction force is a following force, whose modulus is assumed to be known; its eccentricity and skewing are taken into account by specifying $x_{r1}$, $x_{r2}$, $x_{r3}$, the coordinates of the point of application of the force in coupled axes, and the Rodrigues—Hamilton parameters $r_0$, $r_1$, $r_2$, $r_3$ determining the direction of action of the traction force relative to the couples coordinate system. The block matrix corresponding to the traction force takes the form

$$P\left\|\begin{array}{c}\dfrac{X_r+'X_r}{2}\\ \hline E\end{array}\right\| R'R \left\|\begin{array}{c}0\\0\\0\\1\end{array}\right\|, \tag{11}$$

where $X_r$ and R are unified matrices composed of the elements 0, $x_{r1}$, $x_{r2}$, $x_{r3}$ and $r_0$, $r_1$, $r_2$, $r_3$

$$X_r=\left\|\begin{array}{cccc}0 & x_{r1} & x_{r2} & x_{r3}\\ -x_{r1} & 0 & -x_{r3} & x_{r2}\\ -x_{r2} & x_{r3} & 0 & -x_{r1}\\ -x_{r3} & -x_{r2} & x_{r1} & 0\end{array}\right\|;\quad R=\left\|\begin{array}{cccc}r_0 & r_1 & r_2 & r_3\\ -r_1 & r_0 & -r_3 & r_2\\ -r_2 & r_3 & r_0 & -r_1\\ -r_3 & -r_2 & r_1 & r_0\end{array}\right\|.$$

The axial aerodynamic force acting in the direction of the longitudinal geometric symmetry axis is taken into account in the equations of motion by matrix terms of the form

$$Q\left\|\begin{array}{c}\dfrac{X_d+'X_d}{2}\\ \hline E\end{array}\right\| D'D \left\|\begin{array}{r}0\\0\\0\\-1\end{array}\right\|. \tag{12}$$

Here, Q is the modulus of the axial force and is calculated from the well-known aerodynamic formula [7]; the unified matrices $X_d$, D

$$X_d=\left\|\begin{array}{cccc}0 & x_{d1} & x_{d2} & x_{d3}\\ -x_{d1} & 0 & -x_{d3} & x_{d2}\\ -x_{d2} & x_{d3} & 0 & -x_{d1}\\ -x_{d3} & -x_{d2} & x_{d1} & 0\end{array}\right\|;\quad D=\left\|\begin{array}{cccc}d_0 & d_1 & d_2 & d_3\\ -d_1 & d_0 & -d_3 & d_2\\ -d_2 & d_3 & d_0 & -d_1\\ -d_3 & -d_2 & d_1 & d_0\end{array}\right\|$$

have the elements 0, $x_{d1}$, $x_{d2}$, $x_{d3}$ which are the coordinates of the pressure center in the coupled axes, taking account of eccentricity of the aerodynamic force, and $d_0$, $d_1$, $d_2$, $d_3$, which are the Rodrigues–Hamilton parameters, determining the orientation of the geometric axes of the solid relative to the coupled axes. The velocity components of the incoming flow referred to the pressure center are determined from the unified matrices in terms of the formula

$$\begin{Vmatrix} 0 \\ v_{d1} \\ v_{d2} \\ v_{d3} \end{Vmatrix} = L''L' \begin{Vmatrix} 0 \\ \dot{z}_1 - w_1 \\ \dot{z}_2 - w_2 \\ \dot{z}_3 - w_3 \end{Vmatrix} + [L''\dot{L} + 'L'\dot{L}] \begin{Vmatrix} 0 \\ x_{d1} \\ x_{d2} \\ x_{d3} \end{Vmatrix}, \tag{13}$$

where $w_1$, $w_2$, $w_3$ are the velocity components of the atmospheric wind at the axis of the inertial coordinate system. The spatial angle of attack reduced to the pressure center is found using unified matrices according to the formula

$$\delta = \arc\cos\left( \frac{\| 0 v_{d1} v_{d2} v_{d3} \|}{\sqrt{v_{d1}^2 + v_{d2}^2 + v_{d3}^2}} D'D \begin{Vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Vmatrix} \right). \tag{14}$$

If the coupled and geometric axes are parallel, the parameters take the values $d_0 = 1$, $d_1 = d_2 = d_3 = 0$ and the resulting formula is simplified:

$$\delta = \arc\cos\left( \frac{v_{d3}}{\sqrt{v_{d1}^2 + v_{d2}^2 + v_{d3}^2}} \right).$$

Analogously, the normal aerodynamic force is taken into account in the equations of motion; this force is coplanar with the local velocity vector of the incoming flow and perpendicular to the longitudinal geometric axis [7]; other active forces acting on the airplane are also taken into account in the same way.

The position and orientation of the solid in space are determined as a result of integrating the matrix equations obtained from Eq. (7) by the inverse transition

$$\dot{l} = \frac{1}{2}\,'L\omega; \; \dot{z} = 'LLu, \tag{15}$$

where the quasivelocity is assumed to be found from the dynamic equations of motion. The first of these matrix equations consists of a system of four linear dependent first-order differential equations, the solution of which is associated with the well-known relation for the Rodrigues–Hamilton parameters

$$l_0^2 + l_1^2 + l_2^2 + l_3^2 = 1. \tag{16}$$

This relation may serve the purpose of monitoring the correctness of integration of the given system of equations. In the second matrix equation, the first row identically vanishes and, therefore, does not lead to superfluous calculations associated with the integration process.

This matrix model of spatial motion of an asymmetric solid includes a closed system of ordinary first-order differential equations in six quasivelocities, three spatial coordinates of the pole, and four Rodrigues–Hamilton parameters of the resulting rotation of the solid related as in Eq. (16). The first-order differential equations reduce directly to Cauchy form, for which there are effective programs for computer integration. The lack of trigonometric functions in the equations of motion permits the elimination of the mathematical singularities intrinsic to these functions and facilitates a reduction in computation time.

The structure of the resulting equations of motion and the symmetry properties of the unified matrices introduced means that the mathematical model is susceptible to easy inspection, and is well suited to programming and effective use of modern computer software.

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