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An alternative characterization of generalized projectors

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Abstract

One of the results of Groß and Trenkler [Linear Algebra Appl. 264 (1997) 463] asserts that a square complex matrix $\mathbf{K}$ is a generalized projector if and only if it is (i) quadripotent, (ii) normal, and (iii) partial isometry. The authors supplemented this statement by proving that condition (iii) in the above characterization can actually be deleted. The purpose of the present note is to show that also an alternative reduction of the set of conditions (i)–(iii) is possible as a consequence of establishing the redundancy of (ii) under the presence of (i) and (iii).

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1. Introduction and statement of the result

Let $\mathbb{C}_{n,n}$ be the set of $n \times n$ complex matrices. The symbols $\mathbf{K}^*$, $\mathrm{r}(\mathbf{K})$, and $\mathrm{tr}(\mathbf{K})$ will denote the conjugate transpose, rank, and trace of $\mathbf{K} \in \mathbb{C}_{n,n}$, respectively. Further, $\mathbf{K}^\dagger$ will stand for the Moore–Penrose inverse of $\mathbf{K}$, i.e., the unique matrix satisfying the equations

$$\mathbf{K}\mathbf{K}^\dagger\mathbf{K} = \mathbf{K}, \quad \mathbf{K}^\dagger\mathbf{K}\mathbf{K}^\dagger = \mathbf{K}^\dagger, \quad \mathbf{K}\mathbf{K}^\dagger = (\mathbf{K}\mathbf{K}^\dagger)^*, \quad \mathbf{K}^\dagger\mathbf{K} = (\mathbf{K}^\dagger\mathbf{K})^*. \tag{1.1}$$

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Moreover, $\mathbb{C}_n^{\mathsf{QP}}$, $\mathbb{C}_n^{\mathsf{PI}}$, and $\mathbb{C}_n^{\mathsf{N}}$ will be the subsets of $\mathbb{C}_{n,n}$ consisting of quadripotent matrices, (square) partial isometries, and normal matrices, i.e.,

$$\mathbb{C}_n^{\mathsf{QP}} = \{\mathbf{K} \in \mathbb{C}_{n,n} : \ \mathbf{K} = \mathbf{K}^4\}, \tag{1.2}$$

$$\mathbb{C}_n^{\mathsf{PI}} = \{\mathbf{K} \in \mathbb{C}_{n,n} : \ \mathbf{K}\mathbf{K}^*\mathbf{K} = \mathbf{K}\} = \{\mathbf{K} \in \mathbb{C}_{n,n} : \ \mathbf{K}^\dagger = \mathbf{K}^*\}, \tag{1.3}$$

$$\mathbb{C}_n^{\mathsf{N}} = \{\mathbf{K} \in \mathbb{C}_{n,n} : \ \mathbf{K}\mathbf{K}^* = \mathbf{K}^*\mathbf{K}\}. \tag{1.4}$$

The present note is concerned with the notion of a generalized projector introduced by Groß and Trenkler [1, p. 465].

Definition. A matrix $\mathbf{K} \in \mathbb{C}_{n,n}$ is said to be generalized projector whenever $\mathbf{K}^2 = \mathbf{K}^*$.

The class of matrices satisfying this defiinition will henceforth be denoted by $\mathbb{C}_n^{\mathsf{GP}}$. Theorem 1 of Groß and Trenkler [1] asserts that

$$\mathbf{K} \in \mathbb{C}_n^{\mathsf{GP}} \ \Leftrightarrow \ \mathbf{K} \in \mathbb{C}_n^{\mathsf{QP}} \cap \mathbb{C}_n^{\mathsf{PI}} \cap \mathbb{C}_n^{\mathsf{N}}. \tag{1.5}$$

The authors supplemented this statement by proving that the condition $\mathbf{K} \in \mathbb{C}_n^{\mathsf{PI}}$ can therein be omitted. Our purpose is to show that also an alternative modification of characterization (1.5), which consists in abandoning the requirement of the normality of $\mathbf{K}$, is possible. Actually, therefore, we aim at establishing the following.

Theorem. *For any* $\mathbf{K} \in \mathbb{C}_{n,n}$, *the statements* (a)–(d) *below are mutually equivalent*:

(a) $\mathbf{K} \in \mathbb{C}_n^{\mathsf{GP}}$,
(b) $\mathbf{K} \in \mathbb{C}_n^{\mathsf{QP}} \cap \mathbb{C}_n^{\mathsf{PI}} \cap \mathbb{C}_n^{\mathsf{N}}$,
(c) $\mathbf{K} \in \mathbb{C}_n^{\mathsf{QP}} \cap \mathbb{C}_n^{\mathsf{N}}$,
(d) $\mathbf{K} \in \mathbb{C}_n^{\mathsf{QP}} \cap \mathbb{C}_n^{\mathsf{PI}}$.

Groß and Trenkler [1] considered also the class $\mathbb{C}_n^{\mathsf{HGP}}$ of hypergeneralized projectors, defined as

$$\mathbb{C}_n^{\mathsf{HGP}} = \{\mathbf{K} \in \mathbb{C}_{n,n} : \ \mathbf{K}^2 = \mathbf{K}^\dagger\}.$$

From the first condition in (1.1) it is clear that $\mathbb{C}_n^{\mathsf{HGP}} \subseteq \mathbb{C}_n^{\mathsf{QP}}$ and hence

$$\mathbb{C}_n^{\mathsf{HGP}} \cap \mathbb{C}_n^{\mathsf{PI}} \subseteq \mathbb{C}_n^{\mathsf{QP}} \cap \mathbb{C}_n^{\mathsf{PI}}. \tag{1.6}$$

Corollary in [1, p. 466] asserts that the intersection $\mathbb{C}_n^{\mathsf{HGP}} \cap \mathbb{C}_n^{\mathsf{PI}}$ represents a characterization of $\mathbb{C}_n^{\mathsf{GP}}$. From part (d) of the theorem above it follows that the implication $\mathbf{K} \in \mathbb{C}_n^{\mathsf{HGP}} \cap \mathbb{C}_n^{\mathsf{PI}} \Rightarrow \mathbf{K} \in \mathbb{C}_n^{\mathsf{GP}}$ included in this characterization is strengthened in the present note in the sense of referring to the intersection on the right-hand side of (1.6) instead of that on the left-hand side.

2. Proof of Theorem

It is obvious that if $\mathbf{K}^2 = \mathbf{K}^*$, then $\mathbf{K}^4 = (\mathbf{K}^*)^2 = (\mathbf{K}^2)^* = (\mathbf{K}^*)^* = \mathbf{K}$, $\mathbf{K}\mathbf{K}^*\mathbf{K} = \mathbf{K}$, and $\mathbf{K}\mathbf{K}^* = \mathbf{K}^3 = \mathbf{K}^*\mathbf{K}$, thus showing that (a) $\Rightarrow$ (b); cf. part (d) $\Rightarrow$ (a) of Theorem 1 in [1]. The implications (b) $\Rightarrow$ (c) and (b) $\Rightarrow$ (d) are trivial, and therefore the proof reduces to establishing that (c) $\Rightarrow$ (a) and (d) $\Rightarrow$ (a). The two lemmas below will be useful in further considerations. The results contained in them are also of independent interest.

Lemma 1. *Let* $\mathbf{T} \in \mathbb{C}_{m,m}$ *be a nonsingular upper triangular matrix with the diagonal entries* $t_{jj},\ j = 1, \ldots, m$. *Moreover, let* $\mathscr{T} = \left\{1, -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}, -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right\}$. *Then*:

(a) $\mathbf{T} \in \mathbb{C}_m^{\mathsf{N}}$ *if and only if* $\mathbf{T}$ *is diagonal*,

(b) $\mathbf{T} \in \mathbb{C}_m^{\mathsf{PI}}$ *if and only if* $\mathbf{T}$ *is diagonal and* $|t_{jj}| = 1$ *for every* $j = 1, \ldots, m$,

(c) *if* $\mathbf{T} \in \mathbb{C}_m^{\mathsf{QP}}$, *then* $t_{jj} \in \mathscr{T}$ *for every* $j = 1, \ldots, m$.

Proof. In view of (1.4), if $\mathbf{T} \in \mathbb{C}_m^{\mathsf{N}}$, then, in particular, the diagonal entries of $\mathbf{T}\mathbf{T}^*$ and $\mathbf{T}^*\mathbf{T}$ expressed as

$$(\mathbf{T}\mathbf{T}^*)_{jj} = \sum_{k=j}^{m} |t_{jk}|^2 \quad \text{and} \quad (\mathbf{T}^*\mathbf{T})_{jj} = \sum_{k=1}^{j} |t_{kj}|^2, \quad j = 1, \ldots, m,$$

must be identical. Consequently, analyzing the equalities $(\mathbf{T}\mathbf{T}^*)_{jj} = (\mathbf{T}^*\mathbf{T})_{jj}$ with $j = 1, \ldots, m-1$ one after the other shows that

$$t_{jk} = 0 \quad \text{for every } j = 1, \ldots, m-1 \text{ and } k = j+1, \ldots, m. \tag{2.1}$$

Since the matrix $\mathbf{T}$ is upper triangular, (2.1) means that it is diagonal, thus establishing the necessity in part (a) of this lemma. The sufificiency is obvious.

Further, on account of the nonsingularity of $\mathbf{T}$, it follows from (1.1) that $\mathbf{T}^\dagger = \mathbf{T}^{-1}$, and then the second characterization of $\mathbb{C}_m^{\mathsf{PI}}$ in (1.3) leads to

$$\mathbf{T} \in \mathbb{C}_m^{\mathsf{PI}} \Leftrightarrow \mathbf{T}\mathbf{T}^* = \mathbf{I}_m = \mathbf{T}^*\mathbf{T}, \tag{2.2}$$

where $\mathbf{I}_m$ denotes the identity matrix of order m. Hence it is seen that $\mathbf{T}$ is normal, which in view of part (a) means that it must be diagonal. Consequently, the additional requirement concerning the diagonal entries of $\mathbf{T}$ in (b) is just a reformulation of the condition on the right-hand side of (2.2). The sufificiency is again obvious.

Part (c) is clear. Since under the assumption that $\mathbf{T}$ is nonsingular the condition $\mathbf{T} = \mathbf{T}^4$ (which according to (1.2) defines $\mathbf{T} \in \mathbb{C}_n^{\mathsf{QP}}$) is equivalent to $\mathbf{T}^3 = \mathbf{I}_m$ and since $\mathbf{T}$ is upper triangular, it follows that the diagonal entries t_{jj} of $\mathbf{T}$ must satisfy $t_{jj}^3 = 1$, i.e., $t_{jj} = 1$ or $t_{jj} = -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$ or $t_{jj} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$, $j = 1, \ldots, m$. $\square$

Lemma 2. *Let* $\mathbf{N} \in \mathbb{C}_{m,m}$ *be an upper triangular matrix with the diagonal entries* $n_{jj} = 0,\ j = 1, \ldots, m$. *Then, for any integer* $a \geqslant 2$,

$$\mathbf{N}^a = \mathbf{N} \Leftrightarrow \mathbf{N} = \mathbf{0}. \tag{2.3}$$

Proof. If $a = 2$, then the condition on the left-hand side in (2.3) means that $\mathbf{N}$ is idempotent. Hence it follows that $\mathrm{r}(\mathbf{N}) = \mathrm{tr}(\mathbf{N}) = 0$, i.e., $\mathbf{N} = \mathbf{0}$. If $a > 2$, then multiplying $\mathbf{N}^a = \mathbf{N}$ by $\mathbf{N}^{a-2}$ yields $\mathbf{N}^{2a-2} = \mathbf{N}^{a-1}$, thus showing that $\mathbf{N}^{a-1}$ is idempotent. Consequently, $\mathrm{r}(\mathbf{N}^{a-1}) = \mathrm{tr}(\mathbf{N}^{a-1}) = 0$, and substituting $\mathbf{N}^{a-1} = \mathbf{0}$ into $\mathbf{N}^a = \mathbf{N}$ yields $\mathbf{N} = \mathbf{0}$. Since the "$\Leftarrow$ part" of (2.3) is trivial, the proof is complete. □

In the remaining part of the proof of Theorem we refer to Schur's unitary triangularization theorem (see, e.g., [2, Theorem 2.3.1]), according to which $\mathbf{K} \in \mathbb{C}_{n,n}$ of rank $\mathrm{r}(\mathbf{K}) = m$ admits a representation

$$\mathbf{K} = \mathbf{U}\begin{pmatrix} \mathbf{T} & \mathbf{X} \\ \mathbf{0} & \mathbf{N} \end{pmatrix}\mathbf{U}^*, \tag{2.4}$$

where $\mathbf{U}$ is a unitary matrix of order n, and $\mathbf{T}$ and $\mathbf{N}$ are upper triangular matrices of order m and $n - m$, respectively, with the diagonal entries t_{jj} $(j = 1, \ldots, m)$ of $\mathbf{T}$ equal to nonzero eigenvalues of $\mathbf{K}$ and the diagonal entries n_{ll} $(l = 1, \ldots, n - m)$ of $\mathbf{N}$ equal to zero. Clearly, the submatrices $\mathbf{X}$, $\mathbf{0}$, and $\mathbf{N}$ are absent in (2.4) when $\mathbf{K}$ is nonsingular. In any such case, the conditions on $\mathbf{K}$ in parts (c) and (d) of Theorem reduce to the analogous conditions on $\mathbf{T}$. Then from Lemma 1 it is seen that

$$\mathbf{T} \in \mathbb{C}_n^{\mathsf{QP}} \cap \mathbb{C}_n^{\mathsf{PI}} \Rightarrow \mathbf{T} \in \mathbb{C}_n^{\mathsf{QP}} \cap \mathbb{C}_n^{\mathsf{N}}, \tag{2.5}$$

the right-hand side of (2.5) further implying that $\mathbf{T}$ is a diagonal matrix with the diagonal entries $t_{jj} \in \mathscr{T}$. From the specification of $\mathscr{T}$ it follows that $t_{jj} = t_{jj}^4 \Leftrightarrow t_{jj}^2 = t_{jj}^{-1} \Leftrightarrow t_{jj}^2 = \bar{t}_{jj}$, $j = 1, \ldots, n$, and hence

$$\mathbf{T} \in \mathbb{C}_n^{\mathsf{QP}} \cap \mathbb{C}_n^{\mathsf{N}} \Rightarrow \mathbf{T}^2 = \mathbf{T}^*,$$

i.e., $\mathbf{T} \in \mathbb{C}_n^{\mathsf{GP}}$. For $\mathbf{K} = \mathbf{UTU}^*$ this is equivalent to $\mathbf{K} \in \mathbb{C}_n^{\mathsf{GP}}$, thus establishing that (d) $\Rightarrow$ (c) $\Rightarrow$ (a).

In the situation where $\mathrm{r}(\mathbf{K}) = m < n$, the quadripotency of a matrix $\mathbf{K}$ represented as in (2.4) entails $\mathbf{T} = \mathbf{T}^4$ and $\mathbf{N} = \mathbf{N}^4$. Since according to Lemma 2 $\mathbf{N} = \mathbf{N}^4$ if and only if $\mathbf{N} = \mathbf{0}$, it is clear from the considerations above that the proof of Theorem will be complete when $\mathbf{X}$ in (2.4) is shown to be the null matrix whenever $\mathbf{K} \in \mathbb{C}_n^{\mathsf{N}}$ or $\mathbf{K} \in \mathbb{C}_n^{\mathsf{PI}}$. But this is really the case. Comparing the south–east submatrices of $\mathbf{U}^*\mathbf{KK}^*\mathbf{U}$ and $\mathbf{U}^*\mathbf{K}^*\mathbf{KU}$ shows that if $\mathbf{N} = \mathbf{0}$, then $\mathbf{K} \in \mathbb{C}_n^{\mathsf{N}}$ implies $\mathbf{X}^*\mathbf{X} = \mathbf{0}$, which is possible merely when $\mathbf{X} = \mathbf{0}$. Similarly, comparing the north–west submatrices of $\mathbf{U}^*\mathbf{KK}^*\mathbf{KU}$ and $\mathbf{U}^*\mathbf{KU}$ shows that if $\mathbf{N} = \mathbf{0}$, then $\mathbf{K} \in \mathbb{C}_n^{\mathsf{PI}}$ implies $(\mathbf{TT}^* + \mathbf{XX}^*)\mathbf{T} = \mathbf{T}$. In view of the nonsingularity of $\mathbf{T}$, this is equivalent to $\mathbf{TT}^* + \mathbf{XX}^* = \mathbf{I}_m$, thus implying

$$\mathrm{tr}(\mathbf{TT}^*) + \mathrm{tr}(\mathbf{XX}^*) = m. \tag{2.6}$$

In view of $\mathbf{T} = \mathbf{T}^4$, it follows from part (c) of Lemma 1 that $t_{jj} \in \mathscr{T}$, and therefore $|t_{jj}| = 1$ for every $j = 1, \ldots, m$. Consequently,

$$\operatorname{tr}(\mathbf{T}\mathbf{T}^*) = \sum_{j=1}^{m} \sum_{k=j}^{m} |t_{jk}|^2 \geqslant m,$$

and therefore (2.6) cannot hold unless $\mathbf{T}$ is diagonal and $\operatorname{tr}(\mathbf{X}\mathbf{X}^*) = 0$, the latter being equivalent to $\mathbf{X} = \mathbf{0}$. This shows that not only supplementing the quadripotency of $\mathbf{K}$ by the normality property alone, but also by the property of being a partial isometry alone is sufficient for forcing $\mathbf{K}$ to be a generalized projector.

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