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ANALYTIC DESCRIPTION OF MATTER-ENHANCED SOLAR NEUTRINO OSCILLATIONS

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The results on the analytic description of the solar matter effects in two-neutrino oscillations of solar neutrinos obtained in the exponential density approximation are reviewed.

Matter-enhanced solar neutrino oscillations[1] and the possible solutions of the solar neutrino problem they imply have been extensively studied in the last four years[1,2] (see also the review articles[3,4]). The relevant basic quantity - the probability that solar electron neutrinos will not transform into neutrinos of different type on their way from the central region to the surface of the Sun and further to the Earth surface $P_{\odot}(t,t_0)$ (t_0 is the time of neutrino production, $t \geqslant t_0$), was calculated in most of the studies by solving numerically the system of evolution equations[5-7] describing the oscillations of solar neutrinos in the Sun. In all studies performed so far the oscillating terms present in $P_{\odot}(t,t_0)$ were either neglected or strongly suppressed by exploiting a very effective but unphysical averaging procedure[1,3] in the calculations of $P_{\odot}(t,t_0)$.

Considerable efforts have been made to find sufficiently accurate and simple analytic description of the solar matter effects in the oscillations of solar neutrinos. It was shown, in particular, that the system of neutrino evolution equations describing neutrino oscillations in matter with nonuniform density distribution[5-7] can be solved exactly if two neutrinos take part in the oscillations and the matter density (or electron number density N_e) changes linearly [8-11], as a "continuous step" function (hyperbolic tangent)[12], or exponentially[13,14] along the neutrino path. Any of the three types of solutions corresponding to the three indicated cases of density variation can be used to calculate the (average) probability $\overline{P}_{\odot}(t,t_0)$. However, the accuracy with which $\overline{P}_{\odot}(t,t_0)$ is reproduced is not the same in the three cases. According to the SSM [15,16], the matter (electron number) density decreases approximately exponentially from the centre to the surface of the Sun. Consequently, it is natural to expect that the solar matter effects in the two-neutrino oscillations of solar neutrinos will be described most precisely by the exponential density solutions of the

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system of neutrino evolution equations. This was shown to be the case for the average probability[17,18], and should certainly be true for the full probability since the oscillating terms present in it are even more sensitive to the way the matter (or electron number) density changes along the neutrino path. From the exact exponential density solution a simple, complete (i.e. containing the oscillating terms) and sufficiently accurate approximate analytic expression for the probability $P_\odot(t,t_0)$ was also derived[13,17-19]. Using this expression can facilitate considerably the interpretation of the data of the current and future solar neutrino experiments in terms of the hypothesis of matter - enhanced neutrino oscillations. In the present paper we shall summarize the results on the analytic description of the solar matter effects in two-neutrino oscillations of solar neutrinos obtained in the exponential density approximation.

We begin by recalling few standard assumptions and well known results in the theory of neutrino oscillations in matter[3,4], which shall be used in our further discussion. Consider two-neutrino oscillations of electron neutrinos. We shall suppose (for concreteness) that ν_e can oscillate into another flavour neutrino, say ν_μ, and that the $\nu_e \rightleftarrows \nu_\mu$ oscillations can undergo a resonance amplification in matter[1,6,3,4], i.e. that $\Delta m^2 \cos 2\theta > 0$, where[4] θ is the neutrino mixing angle in vacuum and $\Delta m^2 = m_2^2 - m_1^2$, $m_{1,2}$ being the masses of the neutrinos with definite mass $\nu_{1,2}$ in vacuum ($\Delta m^2 > 0$, $0 \leq \theta < /4$). The electron number density at which the resonance in $\nu_e \rightleftarrows \nu_\mu$ oscillations can take place in matter is given by [1,3]

$$N_e^{res} = \Delta m^2 \cos 2\theta / (2p\sqrt{2}\, G_F) \qquad (1)$$

Here $p = |\vec{p}|$, where $\vec{p}$ is the neutrino momentum, and the neutrinos $\nu_{1,2}$ are assumed to be stable and relativistic ($E_i = \sqrt{\vec{p}^2 + m_i^2} \cong p + m_i^2/2p$, $i=1,2$).

Numerical studies have shown[1-3] that the effects of solar matter on the solar neutrino oscillations can be substantial for $\sin^2 2\theta \gtrsim 10^{-4}$ and $10^{-8}\,\mathrm{eV}^2 \lesssim \Delta m^2 \lesssim 10^{-4}\,\mathrm{eV}^2$.

According to the SSM[15,16], the N_e distribution is spherically symmetric and N_e changes approximately exponentially along the neutrino path in the Sun. For the radially moving neutrinos we have:

$$N_e(t) = N_e(t_0)\exp(-(t-t_0)/r_0), \quad t \geq t_0 \qquad (2)$$

where $N_e(t_0)$ is the electron number density in the point of neutrino production and r_0 is the scale height. In a large region in the Sun, i.e., for $0.18R_\odot \lesssim r \leq 0.85R_\odot$, where r is the distance from the centre of the Sun and $R_\odot$ is the solar radius ($R_\odot = 6.96\cdot 10^5$ km), one has for the neutrinos moving radially towards the surface[15-18]: $0.085R_\odot \lesssim r_0 \leq 0.12R_\odot$. Outside this region r_0 increases (decreases) monotonically towards the centre (the surface) of the Sun. Approximately 96% of the p-p, 97% of the ^{7}Be and 98% of the ^{8}B neutrinos are produced[15,16] in the central part of the Sun in the spherical regions defined by the inequalities $r \lesssim 0.22R_\odot$, $r \lesssim 0.14R_\odot$ and $r \lesssim 0.10R_\odot$, respectively. In the region of neutrino production one has: $20\ \mathrm{cm}^{-3} \lesssim N_e/N_A \lesssim 100\ \mathrm{cm}^{-3}$, N_A being the Avogadro's number.

The probability $P_\odot(t,t_0)$ as in the case of oscillations in vacuum[4], can be represented as a sum of two terms

$$P_{\odot}(t,t_o)=\bar{P}_{\odot}(t,t_o)+P^{osc}_{\odot}(t,t_o), \quad (3)$$

where $\bar{P}_{\odot}(t,t_o)$ is the average probability to find solar neutrino with momentum $\vec{p}$ at the Earth surface and $P^{osc}_{\odot}(t,t_o)$ is an oscillating term which exhibits periodic dependence on $p/\Delta m^2$, the distance travelled by neutrinos $(t-t_o)$, $N_e(t_o)$, etc. If the neutrino transitions in the Sun are adiabatic [1,3,4],

$$\bar{P}_{\odot}(t,t_o)=\bar{P}_A(t,t_o)=\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta \cos 2\theta_m(t_o), \quad (4)$$

$$P^{osc}_{\odot}(t,t_o)=P^{osc}_A(t,t_o)= \tfrac{1}{2}\sin 2\theta \sin 2\theta_m(t_o) \times \cos\left((\Delta m^2/2p)\int_{t_o}^{t}((1-N_e(t')/N_e^{res})^2\cos^2 2\theta + \sin^2 2\theta)^{1/2}dt'\right), \quad (5)$$

$\theta_m(t)$ being the neutrino mixing angle in matter[5,1]

$$\cos 2\theta_m(t_o)=(1-N_e(t_o)/N_e^{res}) \times ((1-N_e(t_o)/N_e^{res})^2 + tg^2 2\theta)^{-1/2}. \quad (6)$$

In the general case, exact analytic expressions for $\bar{P}_{\odot}(t,t_o)$ and $P^{osc}_{\odot}(t,t_o)$ in terms of confluent hypergeometric functions (the Kummer's functions[21]) have been obtained in refs.[17,19,22] in the exponential density approximation (2) by using the corresponding exact solutions of the system of neutrino evolution equations derived in [13]. As an illustration of the simple and elegant form of these solutions[17,22] we give below the result for the amplitude of the probability to find neutrino ν_μ at time t outside the Sun ($N_e(t)=0$) if the neutrino ν_e has been produced at time t_o in the closer (with respect to the Earth) solar hemisphere:

$$a(\nu_e\to\nu_\mu;t,t_o)=\tfrac{1}{2}\sin 2\theta\,(\phi(a-c,2-c;Z_o) - e^{i(t-t_o)\Delta m^2/2p}\,\phi(a-1,c;Z_o)). \quad (7)$$

Here $\phi(a-c,2-c;Z_o)$ and $\phi(a-1,c;Z_o)$ are the values of two Kummer's functions[21] in the initial point of the neutrino trajectory which depend on the vacuum oscillation parameters and on the scale height r_o,

$$Z_o = ir_o\sqrt{2}\,G_F N_e(t_o), \quad (8)$$

$$a=1+ir_o(\Delta m^2/2p)\sin^2\theta,\; c=1+ir_o\Delta m^2/2p, \quad (9)$$

and the expression (7) is given up to irrelevant overall phase factor. The Kummer's functions appearing in (7) possess the property: $\phi(a';c';o)=1$ provided $a',c' \neq 0,-1,-2,\ldots,a'$ and c' being arbitrary parameters. A complete set of exact formulae needed for the calculation of the probability $P_e(t,t_o)$ in the exponential approximation for $N_e(t)$ can be found in ref.22.

Exploiting the asymptotic series expansions of the confluent hypergeometric functions[21] as well as the results of ref.9 permitted to find rather simple and accurate approximate formulae for the average probability[13,17] $\bar{P}_{\odot}(t,t_o)$ and the oscillating term[19,20] $P^{osc}_{\odot}(t,t_o)$ for neutrinos born in the closer (with respect to the Earth) solar hemisphere. In this case, as was shown in [13,17-19], for fixed $\sin^2 2\theta$ and $N_e(t_o)$

$$\bar{P}_{\odot}(t,t_o)=\bar{P}^{exp}_{\odot}(t,t_o), \quad (10)$$

$$P^{osc}_{\odot}(t,t_o)=P^{osc}_{NA}(t,t_o) \quad (11)$$

for any value of $p/\Delta m^2$ if for the values of $\sin^2 2\theta$ and $N_e(t_o)$ considered

$$4n'_o \gg 1, \quad (12)$$

while in the case

$$4n'_o \lesssim 1 \quad (13)$$

one has:

$$\bar{P}_{\odot}(t,t_o) = \begin{cases} \bar{P}_A(t,t_o), \text{ for } N_e^{res} > N_e(t_o)(1-tg 2\theta)^{-1}, & (14a) \\ \bar{P}_{NA}(t,t_o) \text{ for } |1-N_e(t_o)/N_e^{res}| \lesssim tg 2\theta, & (14b) \\ \bar{P}^{exp}_{\odot}(t,t_o), \text{ for } N_e^{res} < N_e(t_o)(1+tg 2\theta)^{-1}, & (14c) \end{cases}$$

$$P^{osc}_{\odot}(t,t_o) =$$

$P_A^{osc}(t,t_o)$, for $N_e^{res} > N_e(t_o)(1-tg2\theta)^{-1}$ (15a)

$= P_{NA}^{osc}(t,t_o)$, for $N_e^{res} < N_e(t_o)(1+tg2\theta)^{-1}$. (15b)

Here[17,18]

$$4n_o' = 4n_o\Big|_{N_e^{res} = N_e(t_o)} = ((N_e^{res}/|dN_e/dt|_{res})\Big|_{N_e^{res}=N_e(t_o)})\times$$
$$\times(\Delta m^2/2p)(\sin^2 2\theta/\cos 2\theta), \quad (16)$$

is the value of the adiabaticity parameter[1,3,4] when the resonance point coincides with the point of neutrino production, $4n_o$ being the adiabaticity parameter[13,19], and

$$\bar{P}_{\odot}^{exp}(t,t_o) = \tfrac{1}{2} + (\tfrac{1}{2} - P')\cos 2\theta \cos 2\theta_m(t_o), \quad (17)$$

$$P_{NA}^{osc}(t,t_o) = P_A^{osc}(t,t_o) + \textstyle\sum_{i=1}^{3} P_i^{osc}(t,t_o), \quad (18)$$

where $|dN_e/dt|_{res}$ is the derivative of N_e (in the direction of the neutrino momentum) at the resonance point,

$$P' = \frac{\exp(-\pi r_o(1-\cos 2\theta)\Delta m^2/2p) - \exp(-2\pi r_o \Delta m^2/2p)}{1 - \exp(-2\pi r_o \Delta m^2/2p)} \quad (19)$$

is[13,17] (see also [19]) the probability of the $\nu_1^m(t_o) \to \nu_2$ transition in the Sun (the level-crossing probability) for the case of exponentially varying density (2), $\nu_{1,2}^m(t_o)$ being the matter eigenstate neutrinos[3,4] in the point of neutrino production, and $P_i^{osc}(t,t_o)$, $i=1,2,3$, are [19] the oscillating terms in the case of nonadiabatic transitions [3,4]. Explicit analytic expressions for the probability $\bar{P}_{NA}(t,t_o)$ and the oscillating terms $P_i^{osc}(t,t_o)$ can be found in [9] and in [19,20], respectively.

Several remarks are in order. Condition (13) can be realized for the solar neutrinos only for small values of $\sin^2 2\theta$ [17,18], namely, for $\sin^2 2\theta < (4\div 6)\times 10^{-3}$. In general, the upper bound on $\sin^2 2\theta$ which determines the interval of values of $\sin^2 2\theta$ wherein (13) is valid, is inversely proportional to the values of N_e and r_o in the point of neutrino production[17,18]. The number quoted above corresponds to $N_e(t_o)/N_A = 20$ cm^{-3} and $r_o = 0,10R_{\odot}$; it is valid for the neutrinos born at the border of the neutrino production region in the Sun. For the neutrinos born very close to the centre of the Sun, i.e. for $N_e(t_o)/N_A = 98$ cm^{-3}, the bound on $\sin^2 2\theta$ of interest is equal approximately to [17,18] 10^{-4}. This implies that the interval of values of N_e^{res} (or, equivalently, of $p/\Delta m^2$) in which we lack an analytic expression for $P_{\odot}^{osc}(t,t_o)$ and in which $\bar{P}_{\odot}(t,t_o)$ is reproduced by $\bar{P}_{NA}(t,t_o)$ (see eqs. (14a) - (14c) and (15a) - (15b)) is very narrow. It is possible to show also using general arguments that for $\sin^2 2\theta < (4\div 6)\times 10^{-3}$, $P_{\odot}^{osc}(t,t_o)$ does not exceed $5\cdot 10^{-2}$. Therefore the knowledge of $P_{\odot}^{osc}(t,t_o)$ in the indicated interval cannot be expected to be essential. As for the probability $\bar{P}_{NA}(t,t_o)$, being a monotonically increasing function of N_e^{res} (almost a straight line) in this interval, it can be substituted in the practical calculations with its value in the central point of the interval ($N_e^{res} = N_e(t_o)$) given by [17]:

$$\bar{P}_{NA}(t,t_o)\Big|_{N_e^{res}=N_e(t_o)} = \tfrac{1}{2}(1+e^{-\pi n_o'}). \quad (20)$$

Let us add also that, in practice, an accuracy typically not worse than several percent is achieved if one uses $4n_o' \geq (3\div 4)$ and $4n_o' < (2\div 3)$ as inequalities (12) and (13) respectively.

Further, as was shown in [19], for $N_e^{res} \gg N_e(t_o)$, which is realized for the solar neutrinos for $p/\Delta m^2 \ll 6.10^4$ MeV/eV^2, $P_{1,2,3}^{osc}(t,t_o) \to 0$ and $P_A^{osc}(t,t_o)$ reduces to the vacuum oscillating term, while in the limit $\Delta m^2/2p \to 0$ corresponding to extreme nonadiabatic transitions[3,4] ($4n_o \ll 1$), $P_1^{osc}(t,t_o)$ converges to the vacuum oscillating term and $P_A^{osc}(t,t_o) \to 0$.

Using eq.(5) for $P_A^{osc}(t,t_o)$ and the explicit expressions for $P_{1,2,3}^{osc}(t,t_o)$ derived in [19], it was possible to determine the regions of values of the parameters $\sin^2 2\theta$ and $p/\Delta m^2$ for which the oscillating terms can give significant contributions in $P_\odot(t,t_o)$ and can be sufficiently large in magnitude, i.e. for which

$$|P_{i,A}^{osc}(t,t_o)| \geqslant 5.10^{-2}\bar{P}_\odot(t,t_o) \qquad (21)$$

and

$$|P_{i,A}^{osc}(t,t_o)| \geqslant 5.10^{-2}, \; i=1,2,3. \qquad (22)$$

The analysis performed in [19] showed that according to the above criteria the terms $P_{2,3}^{osc}(t,t_o)$ are negligible for all values of $\sin^2 2\theta$ and $p/\Delta m^2$ of interest, and that $P_A^{osc}(t,t_o)$ and $P_1^{osc}(t,t_o)$ can be nonnegligible if $\sin^2 2\theta \geqslant 10^{-2}$ and for $p/\Delta m^2 \leqslant 2.10^6 \text{MeV/eV}^2$ and $p/\Delta m^2 \geqslant 2.10^6 \text{MeV/eV}^2$, respectively. The expression for $P_1^{osc}(t,t_o)$ has the form[19]:

$$P_1^{osc}(t,t_o) = -(P'(1-P'))^{1/2}\sin 2\theta \times \cos 2\theta_m(t_o)\cos(F_{12}+F_{22}) \qquad (23)$$

Here

$$F_{12}+F_{22} = 2f_3+f_1-f_2+r_o(\Delta m^2/2p)\ln|Z_o| - (t-t_o)\Delta m^2/2p, \qquad (24)$$

where

$$f_1 = \arg\Gamma(a-1),\quad f_2 = \arg\Gamma(a-c),\quad f_3 = \arg\Gamma(1-c), \qquad (25)$$

$\Gamma(a-1)$ etc. being the gamma function. The angles $F_{12,22}$ are determined[19] essentially by the phases of the $\nu_{1,2}^m(t_o) \to \nu_2$ transition amplitudes.

As is well known, for $N_e(t_o) \cong N_e^{res}$, $\sin 2\theta_m(t_o) \cong 1$, while if $N_e(t_o) \gg N_e^{res}$, $|\cos 2\theta_m(t_o)| \cong 1$ (see eq.(6)). Both possibilities can be realized in the Sun[3,4]. Since $\max(P'(1-P')) = 1/4$, eqs.(5) and (23) imply that $\max(P_{A,1}^{osc}(t,t_o)) = 1/2\sin 2\theta$, while in vacuum the amplitude of neutrino oscillations cannot be larger than $1/2\sin^2 2\theta$. In such a way, the matter effects can enhance not only the average solar neutrino transition probability but also the amplitude of the solar neutrino oscillations.

Although the results (10) - (20) and (23) - (25) have been derived for neutrinos originating in the closer half of the solar core, they can be used also to describe the oscillations of solar neutrinos born in the far solar hemisphere. If the neutrinos reaching the Earth are produced in the far solar hemisphere and N_e changes adiabatically along the neutrino path between the initial point t_o and the point t_o' located in the closer solar hemisphere in which $N_e(t_o') = N_e(t_o)$, one has[19,20,22]:

$$\bar{P}_\odot(t,t_o) = \bar{P}_\odot(t,t_o'), \qquad (26)$$

$$P_1^{osc}(t,t_o) = P_1^{osc}(t,t_o'), \qquad (27)$$

and

$$P_{NA}^{osc}(t,t_o) \cong P_A^{osc}(t,t_o) + P_1^{osc}(t,t_o'), \qquad (28)$$

where the oscillating terms which do not satisfy the criteria (21)-(22) have been neglected in eq.(28). The indicated possibility will be realized[19,22] if inequality (12) is fulfilled, and for the values of $p/\Delta m^2$ for which $N_e^{res} < N_e(t_o) \times (1+tg2\theta)^{-1}$ if (13) takes place. Under these conditions the probability $P_\odot(t,t_o)$ is determined by eqs.(3),(26)-(28) and by eqs.(10)-(11) and (14c) and (15b), respectively. The oscillating terms $P_{A,1}^{osc}(t,t_o)$ can be nonnegligible (according to the criteria (21) - (22)) again only for $\sin^2 2\theta \geqslant 10^{-2}$.

Eqs.(26) - (28) together with eqs. (10) - (20) and (23) - (25) are not sufficient to determine $P_\odot(t,t_0)$ for neutrinos crossing two resonance layers located in the region of neutrino production if (13) is valid and the change of N_e between the points t_0 and $t_0'(N_e(t_0)=N_e(t_0'))$ is nonadiabatic. In this case one can use the exact results [13,22] for the relevant probability amplitudes derived in the exponential density approximation to calculate $P_\odot(t,t_0)$ or just expression (17) in which the factor P' is substituted[23] by $2P'(1-P')$. (The accuracy of the latter approximation has not been studied.) It should be added, however, that the transitions of solar neutrinos under discussion can occur only for a very small range of values of $\sin^2 2\theta$ and Δm^2 of interest, and moreover, only for $\sin^2 2\theta < (4-6).10^{-3}$. For instance, for the neutrinos born in the far solar hemisphere at distance $r=0.1R_\odot$ from the solar centre ($N_e(t_0)/N_A = 65.5$ cm^{-3}) such transitions can take place for[20] $6.5.10^4$ MeV/eV$^2 \leqslant p/\Delta m^2 \leqslant 10^5$ MeV/eV^2 and $\sin^2 2\theta < 7.10^{-4}|\cos\beta|$, where β is the angle between the direction of the neutrino momentum and the solar radius through the resonance point.

How accurate are the analytic formulae for $P_\odot(t,t_0)$ presented and discussed above? There are two major potential sources of errors in the calculation of $P_\odot(t,t_0)$ based on the analytic expressions derived in the exponential density approximation (2) for N_e. Eq.(2) with one fixed value of the scale height r_0 cannot describe with high precision[18] the change of N_e along the path of the neutrinos, moving radially in the Sun, predicted by the SSM (see the discussion following eq.(2)). At the same time the factor P' (eq.(19)) which determines the probability of the nonadiabatic transitions of solar neutrinos depends on r_0 exponentially. In addition, the variation of N_e along the neutrino path in the Sun with the distance $(t-t_0)$ travelled by the neutrinos is not given by eq.(2) for a large fraction of the solar neutrinos which do not traverse the Sun along the radial direction. Nevertheless, expressions (10) and (14a) - (14c) were shown[17,18,24] to reproduce with high precision (typically better than several percent) the average probability $\bar{P}_\odot(t,t_0)$ calculated numerically using the SSM prediction [15,16] for $N_e(t_0)$ and the variation of N_e along the neutrino path in the Sun if for given $N_e^{res} \leqslant N_e(t_0)$ ($N_e^{res} > N_e(t_0)$) the value of r_0 entering into the formulae for $\bar{P}_\odot^{exp}(t,t_0)$ and $\bar{P}_{NA}(t,t_0)$ is chosen to coincide with the value of the ratio $(-N_e(t)/dN_e/dt)$ in the layer of the Sun in which $N_e = N_e^{res}$ ($N_e = N_e(t_0)$), where the derivative of $N_e(t)$ should be taken in the direction of the neutrino momentum. A somewhat simpler but less accurate (for $\sin^2 2\theta < 4.10^{-3}$) description of the matter-enhanced solar neutrino oscillations is achieved by using eqs.(3), (10), (11), (17) - (19), (23) - (25), (26) - (28) only and the prescription for r_0 determination given above. It should be added also that[13,17,18] in contrast to the expression for the average probability $\bar{P}_\odot(t,t_0)$ suggested in ref.23 (see also[9]) on the basis of the linear approximation for N_e and the Landau-Zener result[8,25] for the level-crossing, $\bar{P}_\odot^{exp}(t,t_0)$ (eqs.(17) and (19)) describes correctly the extreme nonadiabatic transitions of solar neutrinos.

Finally, the study of the effects of various physically realistic averagings on the matter-enhanced oscillations of solar neutrinos performed in ref.20 showed that the averaging over the region of neutrino production renders the oscillations described by the adiabatic term $P_A^{osc}(t,t_o)$ (eq.(5)) essentially unobservable. This implies that the term $P_A^{osc}(t,t_o)$ can be neglected in all practical calculations of the effects of oscillations on the solar neutrino flux. (A detailed discussion of the effects of averaging on the oscillating term $P_1^{osc}(t,t_o)$ can be found in ref.20.)

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