SOME RESULTS ON NEAREST POINTS IN NORMED LINEAR SPACES

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Received Oct. 31, 2004.

Abstract

We will focus on some results that we hope to give an algorithm for constructing the best approximations in some types of normed linear spaces. Also some results on best approximation will be obtained.

Key words *approximatively compact, proximinality, Chebyshevity, minimizing sequence*

AMS(2000) subject classification 41A17, 46B25, 41A50, 41A52

1 Introduction and Preliminaries

The problem of finding the shortest distance and nearest points in normed linear spaces is an important task in many applications, such as:

1. Solution to an over-determined system of equations[2,7].
2. Best least squares polynomial approximation to a function[2,7].
3. Some control problems[7].

Many papers have been published on this subject and some algorithms are given for these purposes[3,5,2].

In this paper we are going to give two efficient algorithms for finding the shortest distances and nearest points in normed linear spaces.

In this section we review some elementary definitions, also we assume that X is a normed linear space and M is a nonempty subset of X. A point $y_0 \in M$ is said to be a best approximation for $x \in X$ if

$$\|x - y_0\| = d(x, M) = \inf\{\|x - y\| : y \in M\}.$$

The set of all best approximations of $x \in X$ in M is denoted by $P_M(x)$, in other words

$$P_M(x) = \{y \in M : \|x - y\| = d(x, M)\}.$$

If $P_M(x)$ is non-empty for every $x \in X$, then M is called a proximinal set, and if each $x \in X$ has a unique best approximation in M, then M is called a Chebyshev subset of X. If X is a finite dimensional space, then each non-empty closed subset of X is proximinal. Also if M is a compact subset of an arbitrary normed linear space X, then M is proximinal[7].

A nonempty subset M of X is called approximatively compact if given any $x \in X$, each sequence $\{g_n\}$ in M with $\|x - g_n\| \longrightarrow d(x, M)$ has a convergent subsequence to a point of M. A sequence $\{g_n\}$ in M satisfying $\|x - g_n\| \longrightarrow d(x, M)$ is called a minimizing sequence for x. Thus M is approximatively compact if each minimizing sequence in M has a subsequence converging to a point of M.

Every closed convex subset of a Hilbert space is a (approximatively compact) Chebyshev set. Also every compact set in a normed linear space is approximatively compact[2].

As usual, the diameter of the set M is denoted by diam M, and

$$\bar{B}(x_0, r) = \{y \in X : \|x_0 - y\| \leq r\}$$

is the closed ball of radius $r > 0$ with center at $x_0 \in X$.

The space X is said to be strictly convex, if the relation

$$\|x + y\| = \|x\| + \|y\|, \qquad 0 \neq x, \quad 0 \neq y \in X$$

imply the existence of a $c \neq 0$ such that $y = cx$.

Every Hibert space is strictly convex.

Lemma 1.1[7]. *A non-empty closed convex subset of a speace X is a Chebyshev set if and only if X is strictly convex and reflexive .*

Finally we state a lemma, which will be proved very easily. This lemma will be needed in proving our main results.

Lemma 1.2. *Let (X, d) be a metric space, and $\{B_n\}_{n \geq 1}$ be a sequence of compact subsets of X. If $B_{n+1} \subseteq B_n$ and $\bigcap_{n=1}^{\infty} B_n = \{x_0\}$ for some $x_0 \in X$, then*

$$\lim_{n \to \infty} \text{diam} B_n = 0.$$

Proof. Since $B_{n+1} \subseteq B_n$ for all $n \geq 1$, we have $0 \leq$diam $B_{n+1} \leq$diam B_n. Therefore the sequence $\{\text{diam} B_n\}_{n \geq 1}$ converges to $\beta_0 = \inf\{\text{diam} B_n | n \in \mathbb{N}\}$. We will prove that $\beta_0 = 0$.

Suppose that $\beta_0 > 0$, then we can choose α_0 such that $0 < \alpha_0 < \beta_0$, and thus diam $B_n > \alpha_0$ for all n. Therefore for any natural n there exist x_n and y_n in B_n such that $d(x_n, y_n) > \alpha_0$. For any $n \in \mathbb{N}$, then at least one of the numbers $d(x_n, x_0)$ or $d(y_n, x_0)$ exceeds $\alpha_0/2$. In each of these two cases, we can find a sequence $\{z_n\}_{n\geq 1}$ in X with $z_n \in B_n$ and $d(z_n, x_0) > \alpha_0/2$ for all $n \geq 1$. Since $\{z_n\}_{n\geq 1}$ is in B_1 and B_1 is compact, there exist a subsequence $\{z_{n_k}\}_{k\geq 1}$ and a $z_0 \in B_1$ such that $\lim_{k\to\infty} z_{n_k} = z_0$ and $d(z_0, x_0) \geq \alpha_0/2$. Also, for each set B_n, there exists a $l_n \in \mathbb{N}$ such that $z_{n_m} \in B_n$ for all $m \geq l_n$. Now since B_n is closed, $z_0 \in B_n$ for all $n \geq 1$. So $z_0 \neq x_0$ and $\{z_0, x_0\} \subseteq \bigcap_{n=1}^{\infty} B_n$, thus $z_0 \in B_n$ for all $n \geq 1$. This is a contradiction.

2 Main Results

Now we can state and prove some statements about the best approximations in normed linear spaces and give some results in the direction of giving our algorithms.

Theorem 2.1. *Suppose M is a closed subset of a normed linear space X. If $x \in X\backslash M$, then there exist a minimizing sequence $\{g_k\}_{k\geq 1}$ in M, a decreasing sequence $\{\alpha_k\}_{k\geq 1}$ and an increasing sequence $\{\beta_k\}_{k\geq 1}$ of positive real numbers such that $\beta_k < \alpha_k$ and*

$$\beta_k < \|x - g_k\| \leq d(x, M) \leq \alpha_k,$$

and

$$\alpha_k - \beta_k = \frac{1}{2^{k-1}}(\alpha_1 - \beta_1)$$

for all $k \in \mathbb{N}$.

Proof. Suppose y_1 is an arbitrary point in M and $\alpha_1 = \|x - y_1\|$. Then $B(x, \alpha_1) \bigcap M \neq \emptyset$. Choose $g_1 \in B(x, \alpha_1) \bigcap M$. Now since M is closed and $x \in X\backslash M$, we can choose a positive real number β_1 such that $0 < \beta_1 < \alpha_1$ and $B(x, \beta_1) \bigcap M = \emptyset$. Assume that $\gamma_1 = \dfrac{\alpha_1 + \beta_1}{2}$. Then we can consider two cases:

1) If $B(x, \gamma_1) \bigcap M \neq \emptyset$, choose $g_2 \in B(x, \gamma_1) \bigcap M$ and put $\alpha_2 = \gamma_1$ *and* $\beta_2 = \beta_1$. In this case $\beta_2 = \beta_1 < \|x - g_2\| \leq \alpha_2 = \gamma_1$.

2) If $B(x, \gamma_1) \bigcap M = \emptyset$, put $\alpha_2 = \alpha_1$, $\beta_2 = \gamma_1$ and $g_2 := g_1$. Also in this case $\beta_2 = \gamma_1 < \|x - g_2\| \leq \alpha_2 = \alpha_1$.

Continuing this process k-times, we find positive real numbers, $\alpha_1, \beta_1, \alpha_2, \beta_2, ..., \alpha_k, \beta_k$, such that for $1 \leq i \leq k$, $0 < \beta_i < \alpha_i$, $g_k \in B(x, \alpha_i) \bigcap M$, $B(x, \beta_i) \bigcap M = \emptyset$ and also $\beta_i < \|x - g_k\| \leq \alpha_i$ for all $1 \leq i \leq k$.

Now put $\gamma_k = \dfrac{\alpha_k + \beta_k}{2}$, and consider the following two cases:

1) If $B(x, \gamma_k) \bigcap M \neq \emptyset$, put $\alpha_{k+1} = \gamma_k$ and $\beta_{k+1} = \beta_k$ and choose $g_{k+1} \in B(x, \gamma_k) \bigcap M$.

2) If $B(x, \gamma_k) \bigcap M = \emptyset$, put $\alpha_{k+1} = \alpha_k$, $\beta_{k+1} = \gamma_k$ and $g_{k+1} = g_k$.

In each of the above cases we have for all $k \geq 1$, $\beta_k \leq \beta_{k+1} < \alpha_{k+1} \leq \alpha_k$,

$$g_k \in B(x, \alpha_k) \bigcap M, \quad B(x, \beta_k) \bigcap M = \emptyset.$$

Moreover, we have

$$|\alpha_{k+1} - \beta_{k+1}| = \frac{1}{2}|\alpha_k - \beta_k|. \tag{1}$$

It follows that for all $k \geq 1$,

$$(\alpha_k - \beta_k) = \frac{1}{2^{k-1}}(\alpha_1 - \beta_1). \tag{2}$$

Now since $d(x, M) \leq \alpha_k$, then

$$\beta_k < \|x - g_k\| \leq d(x, M) \leq \alpha_k. \tag{3}$$

From (2) and (3), we have

$$\lim_{k \to \infty} \|x - g_k\| = \lim_{k \to \infty} \alpha_k = \lim_{k \to \infty} \beta_k = d(x, M).$$

This theorem suggests an "algorithm" for computing the distance of x from a closed set M not containing it. It may be briefly described as follows:

Algorithm 1.

Suppose $(X, \|.\|)$ is a normed space, M is a closed subset of X and $x \in X \backslash M$. This algorithm approximates $d(x, M)$ arbitrarily good.

Let $\epsilon > 0$ be given. Then

1. Put $k = 1$, choose $0 < \beta_k < \alpha_k$ such that $B(x, \alpha_k) \bigcap M \neq \emptyset$ and $B(x, \beta_k) \bigcap M = \emptyset$.
2. If $|\alpha_k - \beta_k| < \epsilon$, put $d(x, M) := \alpha_k$, and stop.
3. Put $\gamma_k = \dfrac{\alpha_k + \beta_k}{2}$.
4. If $B(x, \gamma_k) \bigcap M = \emptyset$, put $\alpha_{k+1} := \alpha_k$, $\beta_{k+1} := \gamma_k$, $k := k + 1$, then go to step 2.
5. Else, put $\alpha_{k+1} := \gamma_k$, $\beta_{k+1} := \beta_k$, $k := k + 1$, and then go to step 3.

Theorem 2.2. *Suppose $(X, \|.\|)$ is a normed space, M is a closed subset of X and $x \in X \backslash M$. If $\{r_n\}_{n \geq 1}$ is a decreasing sequence of positive real numbers, which converges to $d(x, M)$, then*

$$P_M(x) = \bigcap_{n=1}^{\infty} (B(x, r_n) \bigcap M).$$

Proof. First suppose $g \in P_M(x)$. Then for every $n \in \mathbb{N}$ we have $\|x - g\| = d(x, M) \leq r_n$ and $g \in M$. It follows that $g \in \bigcap_{n=1}^{\infty}(B(x, r_n) \bigcap M)$.

Conversely, suppose $g \in \bigcap_{n=1}^{\infty}(B(x, r_n) \bigcap M)$. Then for every $n \in \mathbb{N}$ we have $\|x - g\| \leq r_n$. So tending n to ∞, it follows that $\|x - g\| \leq d(x, M)$, also $\|x - g\| \geq d(x, M)$. Therefore $\|x - g\| = d(x, M)$. Hence $g \in P_M(x)$.

Corollary 2.3. *If M is an approximatively compact subset of a normed linear space X, then we can construct a minimizing sequence $\{g_n\}_{n\geq 1}$ in M which converges to $g_0 \in P_M(x)$.*

In the following theorem, we construct a sequence in M which is convergent to a best approximation.

Theorem 2.4. *Let X be a strictly convex and reflexive normed space and M be a compact convex subset of X. If $x \in X\backslash M$, then there exist a g_0 in M and a minimizing sequence $\{g_n\}_{n\geq 1}$ such that $g_n \neq g_0$.*

$$\lim_{n\to\infty} g_n = g_0$$

and

$$\|x - g_{n+1}\| \leq \|x - g_n\|$$

for all $n \in \mathbb{N}$.

Proof. Suppose $\{\alpha_n\}_{n\geq 1}$ be as in Theorem 2.1 and put $B_n := B(x,\alpha_n)\bigcap M$ for $n \in \mathbb{N}$. Then $B_{n+1} \subseteq B_n$ for all $n \geq 1$, and as a closed convex subset of a strictly convex and reflexive space M is a Chebyshev set. Therefore from Theorem 2.2, $\bigcap_{n=1}^{\infty} B_n = \{g_0\}$ for some $g_0 \in M$. Now from Lemma 1.2, $\lim\limits_{n\to\infty}$ diam $B_n = 0$, thus if we choose an arbitrary $g_n \in B(x,\alpha_n)\bigcap M$, it follows that $\|g_n - g_0\| \leq$diam B_n. Therefore,

$$\lim_{n\to\infty} g_n = g_0$$

and

$$\|x - g_{n+1}\| = \alpha_{n+1} \leq \|x - g_n\| = \alpha_n$$

and also

$$\|x - g_0\| = \lim_{n\to\infty} \|x - g_n\| = \lim_{n\to\infty} \alpha_n = d(x, M).$$

Now, having proved the necessary theorems, we can give the algorithm for construction of the best approximation:

Algorithm 2.

Let X be a strictly convex and reflexive normed space, M is a compact convex subset of X and $x \in X\backslash M$. This algorithm approximates a best approximation for x arbitrarily good.

Let $\epsilon > 0$ be given. Then

1. Put $k = 1$, choose $0 < \beta_k < \alpha_k$ such that $g_1 \in B(x,\alpha_k)\bigcap M \neq \emptyset$ and $B(x,\beta_k)\bigcap M = \emptyset$.

2. If diam $(B(x,\alpha_k)\bigcap M) < \epsilon$, put $g_0 := g_k$ and stop.

3. Put $\gamma_k = \dfrac{\alpha_k + \beta_k}{2}$.

4. If $g_0 \in B(x,\gamma_k)\bigcap M \neq \emptyset$, then put $g_k := g_0$, $\alpha_{k+1} := \gamma_k$, $\beta_{k+1} := \beta_k$ and $k := k+1$ then go to step 2.

5. Else put $\alpha_{k+1} := \alpha_k$, $\beta_{k+1} := \gamma_k$ and $k := k+1$ then go to step 3.

Example 2.5. Suppose $X = C([0,1])$ is the space of continuous functions on $[0,1]$, with the uniform norm $\|f\|_\infty = \sup\{|f(t)| : 0 \le t \le 1\}$, and put $M = \{f \in X | f(0) = 0\}$.

Suppose $f(x) = e^x$. We will apply Algorithm 1, and obtain $d(f, M)$.

1. If $\alpha_1 = e - 1$, then $g_1(x) = x \in B(e^x, \alpha_1) \cap M$. Also if $\beta_1 = \frac{1}{2}$, then $B(e^x, \beta_1) \cap M = \emptyset$.

2. Put $\gamma_1 = \dfrac{\alpha_1 + \beta_1}{2} = \dfrac{e - 0.5}{2} > 1.1$. Then $g_2(x) = 3x \in B(e^x, 1.1) \cap M$, so $\alpha_2 = 1.1$ and $\beta_2 = 0.5$.

3. Put $\gamma_2 = \dfrac{\alpha_2 + \beta_2}{2} = \dfrac{1.1 + 0.5}{2} = 0.8$. Then $B(e^x, 0.8) \cap M = \emptyset$, so $\alpha_3 = 1.1$ and $\beta_3 = 0.8$.

4. Put $\gamma_3 = \dfrac{\alpha_3 + \beta_3}{2} = \dfrac{0.8 + 1.1}{2} = 9.5$. Then $B(e^x, 9.5) \cap M = \emptyset$, so $\alpha_4 = 1.1$ and $\beta_4 = 0.95$. If we continue this process, we obtain $d(f, M) \approx 1$.

Note that, in this example we can obtain the best approximation and nearest point to f by elements of M directly. Consider $g = f - f(0)$ and we have

$$\|f - g\|_\infty = |f(0)| = |f(0) - h(0)| \le \sup |f(x) - h(x)| = \|f - h\|_\infty,$$

for all $h \in M$. Therefore $g \in P_M(f)$ and $d(f, M) = |f(0)|$.

Suppose $(X, \|\ \|)$ is a normed linear space, and M is a closed nonempty subset of X. We denote the boundary and the closed convex hull of M by $\partial(M)$ and $co(M)$ respectively. Note that if M is not convex and $\partial(M) = \partial(co(M))$, then $P_M(x) = P_{co(M)}(x)$. Moreover, note that if M is not compact, but $M \cap B(x, \alpha)$ is compact for some $\alpha > 0$, then Algorithm 2 will be Applicable to M.

Example 2.6. Let X be $\mathbf{R}^3$ with Euclidean norm and

$$M = \{(x_1, x_2, x_3) \in \mathbf{R}^3 | \ x_1 + x_2 + x_3 = 2\}$$

be a plane in $\mathbf{R}^3$. Put $x_0 = (0,0,0)$. We will apply Algorithm 2, and obtain the best approximation for x_0 in M. Suppose $\epsilon = 10^{-1}$ is given.

1. If $\alpha_1 = 3$, then $g_1 = (1, 1, 0) \in B(x_0, \alpha_1) \cap M$. Also if $\beta_1 = 1$, then $B(x_0, \beta_1) \cap M = \emptyset$. Put $\gamma_1 = 2$.

2. Since $g_2 = (1.2, 0.4, 0.4) \in B(x_0, \gamma_1) \cap M$ and $\operatorname{diam}(B(x_0, \alpha_1) \bigcap M) \ge \|g_2 - g_1\| > \epsilon$. Put $\alpha_2 = 2$, $\beta_2 = 1$ and $\gamma_2 = \frac{3}{2}$.

3. Since $g_3 = (0.9, 0.6, 0.5) \in B(x_0, \gamma_2) \cap M$ and $\operatorname{diam}(B(x_0, \alpha_2) \bigcap M) \ge \|g_3 - g_2\| > \epsilon$. Put $\alpha_3 = \frac{3}{2}$, $\beta_3 = 1$ and $\gamma_3 = \frac{5}{4}$.

4. Since $B(x_0, \gamma_3) \cap M = \emptyset$. Put $\alpha_4 = \frac{3}{2}$, $\beta_4 = \frac{5}{4}$, $g_4 = (0.9, 0.6, 0.5)$ and $\gamma_4 = \frac{11}{8}$.

5. Since $g_5 = (0.7, 0.7, 0.6) \in B(x_0, \gamma_4) \cap M$ and $\operatorname{diam}(B(x_0, \alpha_4) \bigcap M) \ge \|g_5 - g_4\| > \epsilon$. Put

$\alpha_5 = \frac{11}{8}$, $\beta_5 = \frac{5}{4}$ and $\gamma_5 = \frac{21}{16}$.

6. Since $B(x_0, \gamma_5) \cap M = \emptyset$. Put $\alpha_6 = \frac{11}{8}$, $\beta_6 = \frac{21}{16}$, $g_6 = (0.7, 0.7, 0.6)$ and $\gamma_6 = \frac{43}{32}$.

7. Since $g_7 = (0.65, 0.65, 0.7) \in B(x_0, \gamma_6) \cap M$ and $\operatorname{diam}(B(x_0, \alpha_6) \bigcap M) \geq \|g_7 - g_6\| > \epsilon$. Put $\alpha_7 = \frac{43}{32}$, $\beta_7 = \frac{21}{16}$ and $\gamma_7 = \frac{85}{64}$.

8. Since $B(x_0, \gamma_7) \cap M = \emptyset$. Put $\alpha_8 = \frac{43}{32}$, $\beta_8 = \frac{85}{64}$, $g_6 = (0.65, 0.65, 0.7)$ and $\gamma_8 = \frac{171}{128}$.

9. If we choose $g_9 = (0.7, 0.65, 0.65) \in B(x_0, \gamma_8) \cap M$, then $\|g_9 - g_8\| < \epsilon$. Therefore $g_0 = (0.7, 0.65, 0.65)$ can be a best approximation for x_0. But we can continue this process and approximate a best approximation for x_0 arbitrarily good.

Example 2.7. Let X be $\mathbf{R}^3$ with Euclidean norm and

$$M = \{(x_1, x_2, x_3) \in \mathbf{R}^3 | \ \frac{x_1^2}{2} + \frac{x_2^2}{4} + \frac{x_3^2}{2} = 1\}$$

be an ellipse in $\mathbf{R}^3$. If $x_0 \in X \backslash M$, we can apply Algorithm 2, and obtain the best approximation for x_0 in M.

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