Control Dynamical Systems

by

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1. Control Dynamical Systems; Geometrical Concepts and a Physical Example

A control dynamical system is a differential system

$$(\mathscr{S}) \qquad \frac{dx}{dt} = \dot{x} = F(x, t, u)$$

depending on a control parameter u. The state x is a real n-vector, that is, a point in the real number space R^n or possibly in some other n-manifold W, and the time t lies in R^1. For each admissible controller $u(t)$, say a real m-vector function with small magnitude restraints as described later, and for given initial data $x = x_0$ at $t = t_0$, there exists a unique solution or response

$$x(t) = \varphi([u], x_0, t_0; t).$$

In control theory $u(t)$ is selected to steer $x(t)$ along some prescribed path or to display some desired behavior. Thus the control is introduced for some specific purpose. In this article we take a different viewpoint, that control represents a natural uncertainty in the mathematical description of a physical dynamical system. This uncertainty corresponds to an intrinsic noise or measurement roughness that tends to blur out the microscopic complexities and pathologies of the solution curve family and enables us to discover the basic qualitative features of the dynamical system without the distraction of the fine structure. In this sense the study of control dynamical systems has the same philosophical motivation as the theory of structural stability [2], and has the same technical apparatus as the theory of contingent or multivalued differential equations [1].

In this paper we develop a qualitative theory of control dynamical systems on a compact manifold W, where the free motion (with $u(t) \equiv 0$) is conservative or measure-preserving. We prove a general periodicity theorem, Theorem 1, that every initial point lies on a controlled periodic orbit, which generalizes the Poincaré-Carathéodory recurrence theorem. Also we present a general transitivity theorem, Theorem 2, that the set of attainability from any initial state is the entire space W, which can be interpreted as a global controllability theorem.

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In order to motivate our theory of control dynamical systems, we consider a rather fanciful astronautical situation. Imagine a derelict spaceship adrift without power in the solar system, or perhaps in some other astronomical system. The position, velocity, and future course of the derelict are known with precision as it follows its inertial trajectory through the gravitational field in space. We must capture and rescue the derelict since it contains something of ultimate human worth—say, a proof of Fermat's last theorem—but time is not pressing.

Accordingly, a robot rescue ship is dispatched to steer through space with a limited, but essentially eternal, engine. The rescue ship must steer a controlled trajectory through the conservative gravitational field and capture the derelict. Our mathematical theory asserts that such a capture is always possible (under reasonable hypotheses stated later), even though the rescue may take a very long time.

Since the steering engine of the rescue ship has a very small limited thrust, the robot must program some strategy of either tracking or leading the trajectory of the derelict. In either case the orbital revolutions of the planets of the system introduce periodic gravitational fluctuations that must be considered.

The theory presented here guarantees the existence of an appropriate capture strategy, matching positions and velocities of the rescue ship and the derelict. No constructive control is described, and no practical rules of space navigation are advanced. The mathematical theory should be considered as a contribution to topological dynamics and is only a rarefied abstraction of the physical rescue problem (for instance, all collisions and direct interference with planets are ignored).

Let the real 6-vector $x(t)$ denote the position and velocity at time t of the rescue ship in the inertial phase space of the astronomical system or world W. The time-varying gravitational force field in W is the known vector function $f(x, t)$. Then the Hamiltonian differential equations of motion of the pursuit or rescue ship are

$$\dot{x} = f(x, t) + Bu,$$

where $u(t)$ is the controlling thrust and B is a constant matrix appropriate to the geometry of the engine. The controlling thrust $u(t)$ is an arbitrarily chosen (say, piecewise continuous) vector function subject to the bound

$$|u(t)| \leq \epsilon,$$

for a prescribed positive number ϵ (here the norm of a vector or matrix is the sum of the absolute values of all the components). Then for such an admissible controller $u(t)$ and a given initial point $x_0 \in W$ at time t_0, the trajectory $x(t)$ of the pursuit ship is determined by the equations of motion.

The free motion (without engine thrust) of the pursuit ship is a solution $x(t)$ of the 6-vector differential system

$$\dot{x} = f(x, t),$$

where $f(x, t)$ is in class C^1 (continuous with continuous first partial derivatives) for $x \in W$ and t in the real line R^1. Here we have assumed that the gravitational field is entirely determined by the stars and other major bodies of our astronomical system and that the infinitesimal (constant) masses of the two spaceships do not affect this field.

We shall assume that the astronomical system is strictly periodic (with period normalized to the value 1), so

$$f(x, t+1) \equiv f(x, t).$$

Also assume that the free motion in W is measure-preserving, corresponding to the conservative nature of the gravitational field, according to the assertion

$$\operatorname{div}_x f(x, t) \equiv 0.$$

Further we assume controllability about each free motion, that is, the control enters the dynamics effectively as described later. We shall also require that the world W is compact, which can be asserted for various physical systems by prescribing a priori bounds on the energy and angular momentum of the derelict ship.

These assumptions can be easily verified for appropriate Hamiltonian equations of motion of a spaceship through a periodic, conservative gravitational field. A detailed study of the analytical and astronomical aspects of these assumptions has been developed in the reference by Markus and Sell [3]. In that paper the capture of the derelict was proved. Namely, given the known derelict trajectory $Q(t)$ for $t \geq 0$ in W, and given an initial point $\hat{x}_0 \in W$ at $t = t_0$, there exists an admissible control thrust $\hat{u}(t)$, with $|\hat{u}(t)| \leq \epsilon$ on some finite duration $t_0 \leq t \leq \tau$, such that the corresponding response $\hat{x}(t) = \varphi([\hat{u}], \hat{x}_0, t_0; t)$ of $\dot{x} = f(x, t) + B\hat{u}(t)$ captures $Q(t)$ at the finite time τ, that is, $\hat{x}(\tau) = Q(\tau)$.

2. Controlled Periodicity for Conservative Systems

Let us consider a dynamical system involving a control process

$$(\mathscr{S}) \qquad \dot{x} = F(x, t, u).$$

We assume that $(\mathscr{S})$ is defined and in class C^1 for x in a compact differentiable n-manifold† W, t in R^1, and $|u| \leq \epsilon$ for a prescribed positive ϵ. We assume the *periodicity hypothesis*

$$(\mathrm{p}) \qquad F(x, t, u) \equiv F(x, t+1, u),$$

and then the usual theory of differential equations applies and guarantees the existence of a unique (absolutely continuous) solution

$$x(t) = \varphi([u], x_0, t_0; t) \text{ in } W \text{ on } t_0 \leq t \leq t_1$$

† A differentiable n-manifold W is a separable metrizable topological space that admits a covering by local coordinate systems $x = (x^1, \cdots, x^n)$, which are C^∞ differentiably interrelated on any overlap of local charts. Technically, $(\mathscr{S})$ is defined by a tangent (contravariant) vector field on W for each (t, u). That is, $(\mathscr{S})$ corresponds to a C^1 cross-section map from $W \times R^1 \times (|u| \leq \epsilon)$ into the tangent bundle of W. Each such differentiable manifold admits a (many) Riemann metric line element $ds^2 = g_{ij}(x)\, dx^i\, dx^j$ and a corresponding volume element $dv = \sqrt{\det |g_{ij}|}\, dx^1 \wedge \cdots \wedge dx^n$ that define a metric and a measure on W.

for each piecewise continuous controller $u(t)$,

$$|u(t)| \leq \epsilon \text{ on } t_0 \leq t \leq t_1,$$

that is, for each *admissible controller* $u(t)$ and each initial datum $x = x_0$ at $t = t_0$.

For simplicity of discussion assume that W is a compact Riemannian manifold with the usual concepts of distance and volume. We shall assume that the *conservation hypothesis* holds for the free motion

$$\text{(c)} \qquad \operatorname{div}_x F(x, t, 0) \equiv 0.$$

Then the transformation

$$\mathscr{T} : x_0 \to \varphi([0], x_0, 0; 1)$$

is a measure-preserving C^1-diffeomorphism (homeomorphism with $\mathscr{T}$ and $\mathscr{T}^{-1}$ of class C^1) of the finite measure space W onto itself. In this case the Carathéodory recurrence theorem [4] asserts that the Poisson (past and future) stable points of $\mathscr{T}$ are dense in W.

We shall say that the differential system

$$(\mathscr{S}) \qquad \dot{x} = F(x, t, u)$$

satisfies the controllability hypothesis in case: For each $(x_0, t_0) \in W \times R^1$ and each $\tau > 0$ there exists a number $\eta > 0$ such that the attainable set

$$\mathscr{A}_\tau(x_0, t_0) = \{\varphi([u], x_0, t_0; t_0 + \tau) | \text{ all admissible } u(t)\}$$

contains the open ball in W of radius η with center at $\varphi([0], x_0, t_0; t_0 + \tau)$. The controllability hypothesis asserts that a full η-ball neighborhood of the free motion endpoint can be attained by using admissible controllers, that is, piecewise continuous $u(t)$ with

$$|u(t)| \leq \epsilon \text{ on } t_0 \leq t \leq t_0 + \tau.$$

We further declare that $(\mathscr{S})$ satisfies the *uniform controllability hypothesis*

$$\text{(u)} \qquad \eta = \eta(t_0, \tau)$$

in case: $\eta > 0$ can be chosen independently of $x_0 \in W$.

THEOREM 1. *Consider the control differential system*

$$(\mathscr{S}) \qquad \dot{x} = F(x, t, u)$$

in C^1 for x in a compact Riemannian manifold W, t in R^1, and $|u| \leq \epsilon$. Assume:

(p) *Periodicity hypothesis* $F(x, t, u) \equiv F(x, t+1, u)$

(c) *Conservation hypothesis* $\operatorname{div}_x F(x, t, 0) \equiv 0$

(u) *Uniform controllability hypothesis with* $\eta = \eta(t_0, \tau) > 0$.

Then for each initial point $x_0 \in W$ at t_0, there exists a piecewise continuous controller $u(t)$ with

$$|u(t)| \leq \epsilon \text{ on } t \geq t_0$$

and with integral period $K > 0$, such that the controlled motion

$$x(t) = \varphi([u], x_0, t_0; t)$$

has period K.

Proof. Consider the free trajectory initiating at x_0 when $t = t_0$ (and for convenience we shift the time scale so that $t_0 = 0$), which we denote by

$$\varphi_0(x_0; t) = \varphi([0], x_0, 0; t).$$

We mark the points in W:

$$x_- = \varphi_0(x_0; -1) \text{ and } x_+ = \varphi_0(x_0; 1).$$

Let $\mathscr{U}$ be the ball of radius $\eta_0/2$ centered at x_0, where $\eta_0 = \eta(0, 1) > 0$ as designated in the uniform controllability hypothesis. Take a ball $\mathscr{U}_-$ centered at x_- and so small that the free motion carries $\mathscr{U}_-$ into $\mathscr{U}$, so $\varphi([0], \mathscr{U}_-, -1; 0)$ is in $\mathscr{U}$. If x_0 can be steered to some point in $\mathscr{U}_-$ in some positive integral time $K-1$, then the uniform controllability hypothesis and the periodicity hypothesis guarantee that the required controller $u(t)$ with period K exists, and the theorem is proved.

Take an open ball $\mathscr{V}_+$ centered at x_+ so small that

$$\mathscr{V}_+ \subset \varphi([0], \mathscr{U}_-, -1; 1)$$

and the radius of $\mathscr{V}_+$ is less than $\eta_0/2$. Consider the measure-preserving map

$$\mathscr{T}: W \to W: x \to \varphi([0], x, 0; 1)$$

and let $P \in \mathscr{V}_+$ be a Poisson stable point for $\mathscr{T}$.

We now proceed to construct the admissible controller $u(t)$ on $0 \leq t \leq K$. Take an admissible controller $u_1(t)$ on $0 \leq t \leq 1$ steering x_0 to P. Then define $u_2(t) \equiv 0$ on $t > 1$ until the free motion carries the Poisson stable point P back to $\mathscr{T}^{K+1} P \in \mathscr{V}_+$ for some integer $K > 2$. But then $\mathscr{T}^{K+1} P \in \mathscr{U}_-$ and is attained from x_0 by the admissible controller

$$u(t) = \begin{cases} u_1(t) \text{ on } 0 \leq t \leq 1 \\ u_2(t) \text{ on } 1 \leq t \leq K-1. \end{cases}$$

Finally define $u_3(t)$ on $K-1 \leq t \leq K$ to steer $\mathscr{T}^{K-1}P$ forward to $x_0 \in \mathscr{U}$. Then $u(t)$ on $0 \leq t \leq K$ is the concatenation of $u_1(t)$, $u_2(t)$, and $u_3(t)$; and $u(t)$ is thereafter continued on $t \geq 0$ to have period K, as required.

3. Controlled Transitivity and Global Controllability

For a conservative control dynamical system $(\mathscr{S})$ we shall show that any initial state $x_0 \in W$ at $t = t_0$ can be steered to an arbitrary point in W. Thus we find that $(\mathscr{S})$ is control transitive or globally controllable in that every two points of W can be joined by a trajectory.

LEMMA. *Consider the control differential system*

$$(\mathscr{S}) \qquad \dot{x} = F(x, t, u)$$

in C^1 for x in a compact Riemannian manifold W, t in R^1, and $|u| \leq \epsilon$. Assume:

(p) *Periodicity hypothesis* $F(x, t, u) \equiv F(x, t+1, u)$
(c) *Conservation hypothesis* $\text{div}_x F(x, t, 0) \equiv 0$
(u) *Uniform controllability hypothesis with* $\eta = \eta(t_0, \tau) > 0$.

Consider the sets of attainability from x_0 *at* $t_0 = 0$,

$$\mathscr{A}_k(x_0) = \{\varphi([u], x_0, 0; k) \mid \textit{all admissible } u(t)\} \textit{ for } k = 1, 2, 3, \cdots.$$

Then

$$W = \bigcup_{k=1}^{\infty} \mathscr{A}_k(x_0).$$

Proof. Suppose $\mathscr{A}_\infty(x_0) = \bigcup_{k=1}^{\infty} \mathscr{A}_k(x_0)$ were not all W but had a boundary point P in W. Then there would exist an attainable point Q_1 in some $\mathscr{A}_{k_1}(x_0)$ with Q_1 within a distance of $\frac{1}{8}\,\eta(0, 1)$ of P. By shutting off the power (setting $u \equiv 0$) for a very short time interval I_1 ending at k_1, x_0 is steered to Q_2 within a distance of $\frac{1}{8}\,\eta(0, 1)$ of Q_1. But by using an arbitrary admissible control for a final subinterval I_2 of I_1, we can steer x_0 to an open set and hence a Poisson stable point Q_3 of

$$\mathscr{T}: W \to W: x \to \varphi([0], x, 0; 1)$$

with

$$\text{distance}\,(Q_3, P) < \tfrac{1}{2}\,\eta(0, 1), \text{ and } Q_3 \in \mathscr{A}_{k_1}(x_0).$$

Next use the free motion for $t > k_1$ until Q_3 returns to a point $\mathscr{T}^{k_2}Q_3$ within a distance of $\frac{1}{4}\,\eta(0, 1)$ of Q_3. Then, if we write $L = k_1 + k_2 \geq 2$,

$$\mathscr{T}^{-1}\mathscr{T}^{k_2}Q_3 \in \mathscr{A}_{L-1}(x_0) \text{ and distance } (\mathscr{T}^{k_2}Q_3, P) < \tfrac{3}{4}\,\eta(0, 1).$$

By the periodicity and the uniform controllability, we find that P lies in the interior of $\mathscr{A}_L(x_0)$. This contradicts the selection of P as a boundary point of $\mathscr{A}_\infty(x_0)$. Therefore $W = \mathscr{A}_\infty(x_0)$, as required.

THEOREM 2. *Consider the control differential system*

$$(\mathscr{S}) \qquad \dot{x} = F(x, t, u)$$

in C^1 *for* x *in a compact Riemannian manifold* W, t *in* R^1, *and* $|u| \leq \epsilon$. *Assume:*

(p) *Periodicity hypothesis* $F(x, t, u) \equiv F(x, t+1, u)$
(c) *Conservation hypothesis* $\operatorname{div}_x F(x, t, 0) \equiv 0$
(u) *Uniform controllability hypothesis with* $\eta = \eta(t_0, \tau) > 0$.

Then there exists a positive integer N *such that for each pair of points* x_1 *and* x_2 *in* W, *there exists a piecewise continuous controller* $u(t)$ *with*

$$|u(t)| \leq \epsilon \textit{ on } 0 \leq t \leq N$$

steering x_1 *to* x_2, *that is,*

$$\varphi([u], x_1, 0; N) = x_2.$$

Proof. Choose a finite set of points $P_1, P_2, \cdots, P_l$ in W from which every point of W is attainable in unit time, that is,

$$\bigcup_{i=1}^{l} \mathscr{A}_1(P_i) = W.$$

Such a choice of points P_i is possible since the interiors of the attainable sets

$\mathscr{A}_1(P)$, for $P \in W$, form an open covering of the compact space W. Further, we can enlarge the set $\{P_i\}$ to a finite set such that:

$$\text{each } \mathscr{A}_1(P), \text{ for } P \in W, \text{ contains some } P_i.$$

This is possible since the set of points of W that can be steered to a target point $Q \in W$ in $0 \leq t \leq 1$ contains an open set, and we select a corresponding finite subcovering of W leading to a finite set of target points $Q_1, \cdots, Q_m$. We augment the original finite ensemble $\{P_i\}$ by $Q_1, \cdots, Q_m$ and denote the entire union by $P_1, \cdots, P_l$.

By Theorem 1 each point P_i lies on a controlled periodic orbit with integral period $K_i \geq 1$. Then each of $P_1, \cdots, P_l$ has the common integral period $K = K_1 \cdot K_2 \cdots K_l$. We now consider the dynamical system $(\mathscr{S})$ to have basic period K, that is, we note that

$$F(x, t, u) \equiv F(x, t+K, u).$$

By the preceding lemma each P_i can be steered to any P_j in an integral multiple $N_{ij} \geq 1$ of the basic period K, that is, in $k \leq t \leq N_{ij}K+k$ for any integer k. Let

$$N_1 = \max_{1 \leq i, j \leq l} N_{ij}$$

and then P_i can be steered to P_j in duration $N_1 K$, because if $N_{ij} < N_1$, the trajectory from P_i to P_j can be extended by periodic loops of duration K that are based at P_j.

Finally we define

$$N = 1 + N_1 K + 1.$$

For any given points x_1 and x_2 in W, we can steer x_1 to some P_i in $0 \leq t \leq 1$. Then continue to control P_i to some appropriate P_j (from which x_2 can be reached in unit time) in time $1 \leq t \leq 1+N_1K$. Then steer P_j to the desired endpoint x_2 in $1+N_1K \leq t \leq N$. Thus any two points x_1 and x_2 in W can be joined by a response to an admissible controller in $0 \leq t \leq N$, as required.

In conclusion we remark that the derelict capture posed in Section 1 is guaranteed by Theorem 2. In particular, the capture can be effected in the time duration N which depends only on the system $(\mathscr{S})$ in W, but is independent of the initial state $\hat{x}_0$ of the rescue ship and the derelict trajectory $Q(t)$.

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