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The scrambling index of primitive digraphs☆

Bolian Liu*, Yufei Huang

School of Mathematical Science, South China Normal University, Guangzhou, 510631, PR China

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ABSTRACT

In 2009, Akelbek and Kirkland introduced a useful parameter called the scrambling index of a primitive digraph D, which is the smallest positive integer k such that for every pair of vertices u and v, there is a vertex w such that we can get to w from u and v in D by directed walks of length k. In this paper, we obtain some new upper bounds for the scrambling index of primitive digraphs. Moreover, the maximum index problem, the extremal matrix problem and the index set problem for the scrambling index of various classes of primitive digraphs (e.g. primitive digraphs with d loops, minimally strong digraphs, nearly decomposable digraphs, micro-symmetric digraphs, etc.) are settled, respectively.

1. Introduction

Let $D = (V, E)$ be a digraph with vertex set $V = V(D)$ and arc set $E = E(D)$. Loops are permitted but multiple arcs are not. A $u \to v$ walk in D is a sequence of vertices $u, u_1, \ldots, u_t, v \in V(D)$ and a sequence of arcs $(u, u_1), \ldots, (u_t, v) \in E(D)$, where the vertices and arcs are not necessarily distinct. A path is a walk with distinct vertices. A cycle is a closed $u \to u$ walk with distinct vertices except for $u = v$. The length of a walk W is the number of arcs in W, and is denoted by $|W|$. The length of a shortest cycle in D is called the girth of D. The notation $u \xrightarrow{k} v$ is used to indicate that there is a $u \to v$ walk of length k. The distance from u to v in D, is the length of a shortest walk from u to v, and denoted by $d(u, v)$. Let C_p denote a cycle of length p.

For a digraph D with n vertices, the adjacency matrix of D is defined to be the $n \times n$ matrix $A(D) = (a_{ij})$, where $a_{ij} = 1$ if there is an arc from i to j, and $a_{ij} = 0$ otherwise. Conversely, we can define the associated digraph $D(A)$ of an $n \times n$ Boolean matrix A. For a positive integer l, the lth power of D, denoted by D^l, is the digraph on the same vertex set and with an arc from i to j if and only if $i \xrightarrow{l} j$ in D. It is easy to see that $[D(A)]^l = D(A^l)$. A digraph D is primitive if there exists some positive integer k such that $u \xrightarrow{k} v$ for every pair $u, v \in V(D)$. The smallest such k is called the exponent of D, denoted by $\exp(D)$. Let P_n denote the set of all primitive digraphs of order n [1]. It is well known that D is primitive if and only if D is strongly connected and the greatest common divisor of the lengths of its cycles is 1 [2].

By using the definition of coefficients of ergodicity, Akelbek and Kirkland [3] provided an attainable upper bound on the second largest modulus of eigenvalues of a primitive matrix that makes use of the so-called scrambling index. The scrambling index of a primitive digraph D is the smallest positive integer k such that for every pair of vertices u and v, there exists a vertex w such that $u \xrightarrow{k} w$ and $v \xrightarrow{k} w$ in D. It is denoted by $k(D)$. For $u, v \in V(D)$ and $u \neq v$, the local scrambling index of u and v is the number

$$k_{u,v}(D) = \min\left\{k \mid u \xrightarrow{k} w \text{ and } v \xrightarrow{k} w,\ \text{for some } w \in V(D)\right\}.$$

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* Corresponding author.
E-mail address: liubl@scnu.edu.cn (B. Liu).

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Obviously,

$$k(D) = \max_{u,v \in V(D)} \{k_{u,v}(D)\} \quad \text{and} \quad k(D) \le \exp(D).$$

We would like to mention that $k(D)$ and $\exp(D)$ can be also defined by matrices. In fact, $k(D)$ is the smallest positive integer k such that any two rows of $[A(D)]^k$ have at least one positive element in a coincident position, and $\exp(D)$ is the smallest positive integer k such that $[A(D)]^k = J$, where J denote the all 1's matrix. Hence

$$k(A(D)) = k(D) \quad \text{and} \quad \exp(A(D)) = \exp(D).$$

In the study of the exponent, the maximum index problem (MIP), the extremal matrix problem (EMP) and the index set problem (ISP) are the three main problems. For surveys on the exponents of various classes of primitive digraphs, see e.g. [1,2]. Note that the scrambling index gives another characterization of primitivity. It is natural to consider the MIP, EMP and ISP for the scrambling index of various classes of primitive digraphs. Observe that the scrambling index is the competition index [4] in the case of primitive digraphs. In [4], Cho and Kim gave an upper bound on the competition index of a primitive digraph D of order n and girth s. In [3,5], Akelbek and Kirkland improved the upper bound given by Cho and Kim, and then they settled the MIP and EMP for the scrambling index of primitive digraphs of order n. In [6], Kim investigated the competition index of tournaments. In [7], the MIP, EMP and ISP for the scrambling index of primitive symmetric digraphs are studied, respectively. In [8], the authors obtained a bound on the scrambling index of a primitive digraph using Boolean rank.

In Section 2 of this paper, some new upper bounds of $k(D)$ for a primitive digraph D are obtained. In Section 3, we settle the MIP, EMP and ISP for the scrambling index of various classes of primitive digraphs, e.g. primitive digraphs with d loops ($1 \le d \le n$), primitive minimally strong digraphs, primitive doubly symmetric digraphs, r-indecomposable digraphs ($r \ge 1$), fully indecomposable digraphs, nearly decomposable digraphs, primitive circulant digraphs, primitive micro-symmetric digraphs, and so on.

2. Bounds on the scrambling index

Let A be the adjacency matrix of a primitive digraph D of order n. Let A^T denote the transposed matrix of A. Let k be an integer, $1 \le k \le n$. The k-point exponent of A, denoted by $\exp(A, k)$, is the smallest power of A for which there are k rows with no zero entry [9,1,2]. It is known that $k(A) \le \exp(A)$. Now we show a relationship between $k(A)$ and $\exp(A^T, 1)$ as follows.

Proposition 2.1. $k(D) = k(A) \le \exp(A^T, 1) = \exp(D(A^T), 1)$.

Proof. Note that $\exp(A^T, 1)$ is the smallest power of A^T for which there is 1 row with no zero entry. Since $(A^{\exp(A^T,1)})^T = (A^T)^{\exp(A^T,1)}$, there is 1 column with no zero entry for $A^{\exp(A^T,1)}$. By the definition of the scrambling index, the inequality $k(A) \le \exp(A^T, 1)$ holds. $\square$

For certain classes of primitive digraphs, by the inequality in Proposition 2.1, the MIP for the scrambling index can be easily settled. For example, let S_n denote the set of all primitive symmetric digraphs of order n [2], and let T_n denote the set of all primitive tournaments of order n [4,1]. It is known that $\exp(D, 1) \le n - 1$ for $D \in S_n$ [2]. Moreover, $A(D) = [A(D)]'$ since $D \in S_n$ is symmetric. Then we have the following result in [7]:

$$k(D) \le n - 1 \quad \text{for } D \in S_n.$$

Since $\exp(D, 1) \le 3$ [10,2], and $D([A(D)]') \in T_n$ for $D \in T_n$ $(n > 6)$, the result in [4] can be obtained immediately:

$$k(D) \le 3 \quad \text{for } D \in T_n \ (n > 6).$$

Let D be a primitive digraph. If R is a set of distinct lengths of the elementary cycles in D, then let $d_R(i, j)$ denote the length of the shortest walk from i to j which meets at least one circuit of each length of R [11].

Lemma 2.2 ([11]). *Let $D \in P_n$ and $s_1, \ldots, s_t$ $(s_1 > \cdots > s_t \ge 1, t \ge 2)$ are relatively prime lengths of circuits of D (that is, $\gcd(s_1, \ldots, s_t) = 1$). Then for $h \ge d_R(i, j) + \phi(s_1, \ldots, s_t)$, there exists a walk of length h from i to j, where $R = \{s_1, \ldots, s_t\}$, and $\phi(s_1, \ldots, s_t)$ is the Frobenius number of $s_1, \ldots, s_t$.*

Theorem 2.3. *Let $D \in P_n$ and $R = \{s_1, \ldots, s_t\}$ $(t \ge 2)$ be a set of distinct lengths of cycles $C_{s_1}, \ldots, C_{s_t}$ in D, where $s_1 > \cdots > s_t \ge 1$ and $\gcd(s_1, \ldots, s_t) = 1$. Then*

$$k(D) \le \sum_{i=1}^{t} (n - s_i) + \phi(s_1, \ldots, s_t) + \left\lfloor \frac{s_t}{2} \right\rfloor.$$

Proof. Let u, v be any pair of vertices of D. Let $u_1 \in V(C_{s_1})$ such that

$$d_1 = d(u, u_1) = \min\{d(u, x_1) \mid x_1 \in V(C_{s_1})\}.$$

Then $d_1 \leq n + 1 - s_1 - 1 = n - s_1$. Let $u_2 \in V(C_{s_2})$ such that

$$d_1 = d(u_1, u_2) = \min\{d(u_1, x_2) \mid x_2 \in V(C_{s_2})\}.$$

Then $d_2 \leq n - s_2$. Similarly we have $d_i \leq n - s_i$ $(i = 3, 4, \dots, t)$. Thus

$$d_R(u, u_t) \leq \sum_{i=1}^{t} d_i \leq \sum_{i=1}^{t} (n - s_i), \quad \text{where } u_t \in V(C_{s_t}).$$

Analogously, there exists a vertex $v_t \in V(C_{s_t})$ such that

$$d_R(v, v_t) \leq \sum_{i=1}^{t} (n - s_i).$$

By Lemma 2.2, for each $h \geq \sum_{i=1}^{t}(n - s_i) + \phi(s_1, \dots, s_t)$, there exists a walk of length h from u to u_t (resp. from v to v_t). Moreover, $u_t, v_t \in V(C_{s_t})$, hence $d(u_t, v_t) \leq \lfloor \frac{s_t}{2} \rfloor$. Without loss of generality, we may assume that $u_t \xrightarrow{l} v_t$, where $0 \leq l \leq \lfloor \frac{s_t}{2} \rfloor$. Hence we have

$$u \xrightarrow{\sum_{i=1}^{t}(n-s_i)+\phi(s_1,\dots,s_t)} u_t \xrightarrow{l} v_t \quad \text{and} \quad v \xrightarrow{\sum_{i=1}^{t}(n-s_i)+\phi(s_1,\dots,s_t)+l} v_t.$$

Then $k_{u,v} \leq \sum_{i=1}^{t}(n - s_i) + \phi(s_1, \dots, s_t) + l$ for any $u, v \in V(D)$. Consequently,

$$k(D) \leq \sum_{i=1}^{t}(n - s_i) + \phi(s_1, \dots, s_t) + \left\lfloor \frac{s_t}{2} \right\rfloor.$$

This completes the proof of Theorem 2.3. $\square$

Lemma 2.4 ([2]). *Let $s_1 > \cdots > s_t \geq 1$ be integers with $\gcd(s_1, \dots, s_t) = 1$. Then*

$$\phi(s_1, s_2, \dots, s_t) \leq (s_1 - 1)(s_t - 1), \quad \text{where } t \geq 2.$$

By Theorem 2.3 and Lemma 2.4, we obtain that

Corollary 2.5. *Let $D \in P_n$ and $R = \{s_1, \dots, s_t\}$ $(t \geq 2)$ be a set of distinct lengths of cycles $C_{s_1}, \dots, C_{s_t}$ in D, where $s_1 > \cdots > s_t \geq 1$ and $\gcd(s_1, \dots, s_t) = 1$. Then*

$$k(D) \leq \sum_{i=1}^{t}(n - s_i) + (s_1 - 1)(s_t - 1) + \left\lfloor \frac{s_t}{2} \right\rfloor.$$

Notice that in Theorem 2.3, if cycles $C_{s_1}, \dots, C_{s_t}$ in D share some common vertices, then the upper bound of $k(D)$ can be improved as follows.

Theorem 2.6. *Let $D \in P_n$ and $R = \{s_1, \dots, s_t\}$ $(t \geq 2)$ be a set of distinct lengths of cycles $C_{s_1}, \dots, C_{s_t}$ such that $s_1 > \cdots > s_t \geq 1$, $\gcd(s_1, \dots, s_t) = 1$, and $|\bigcap_{i=1}^{t} V(C_{s_i})| = p (\leq s_t)$. Then*

$$k(D) \leq n - p + \phi(s_1, \dots, s_t) + \left\lfloor \frac{s_t}{2} \right\rfloor.$$

Proof. Let u, v be any pair of vertices of D. Let $u_1 \in \bigcap_{i=1}^{t} V(C_{s_i})$ such that

$$d_1 = d(u, u_1) = \min\left\{ d(u, x_1) \mid x_1 \in \bigcap_{i=1}^{t} V(C_{s_i}) \right\}.$$

Then $d_R(u, u_1) = d_1 \leq n + 1 - p - 1 = n - p$. Similarly, there exists a vertex $v_t \in V(C_{s_t})$ such that $d_R(v, v_t) \leq n - p$. By Lemma 2.2, for each $h \geq n - p + \phi(s_1, \dots, s_t)$, there exists a walk of length h from u to u_t (resp. from v to v_t). Since $u_t, v_t \in V(C_{s_t})$, $d(u_t, v_t) \leq \lfloor \frac{s_t}{2} \rfloor$. Without loss of generality, suppose $u_t \xrightarrow{l} v_t$, where $0 \leq l \leq \lfloor \frac{s_t}{2} \rfloor$. Hence

$$u \xrightarrow{n-p+\phi(s_1,\dots,s_t)} u_t \xrightarrow{l} v_t \quad \text{and} \quad v \xrightarrow{n-p+\phi(s_1,\dots,s_t)+l} v_t.$$

Therefore, $k(D) = \max_{u,v \in V(D)}\{k_{u,v}(D)\} \leq n - p + \phi(s_1, \dots, s_t) + \lfloor \frac{s_t}{2} \rfloor$. $\square$

Lemma 2.7 ([2]). *If s is a positive integer, then $\phi(s, s+1, \ldots, 2s-1) = s$.*

By Theorem 2.6 and Lemmas 2.4 and 2.7, we have

Corollary 2.8. *Let $D \in P_n$. (1) Let $R = \{s_1, \ldots, s_t\}$ $(t \geq 2)$ be a set of distinct lengths of cycles $C_{s_1}, \ldots, C_{s_t}$ such that $s_1 > \cdots > s_t \geq 1$, $gcd(s_1, \ldots, s_t) = 1$, and $|\bigcap_{i=1}^{t} V(C_{s_i})| = p$ $(\leq s_t)$. Then*

$$k(D) \leq n - p + (s_1 - 1)(s_t - 1) + \left\lfloor \frac{s_t}{2} \right\rfloor.$$

(2) Let $R = \{2t-1, 2t-2, \ldots, t\}$ $(t \geq 2)$ be a set of distinct lengths of cycles $C_t, C_{t+1}, \ldots, C_{2t-1}$ in D such that $|\bigcap_{i=1}^{t} V(C_{t+i-1})| = p$ $(\leq t)$. Then

$$k(D) \leq n - p + t + \left\lfloor \frac{t}{2} \right\rfloor.$$

3. The scrambling index of primitive digraphs

3.1. Primitive digraphs with d loops

A primitive digraph D is called a primitive digraph with d loops if and only if D contains exactly d loops. Let n, d be integer with $n \geq 2$ and $1 \leq d \leq n$. Let $P_n(d)$ denote the set of all primitive digraphs with n vertices and d loops. For convenience, let $R_t(X)$ denote the set of all those vertices which can be reached by a walk of length t in D starting from some vertex in X, where $\emptyset \neq X \subseteq V(D)$ and t is a nonnegative integer. If $t = 0$, then $R_0(X) = X$.

Lemma 3.1 ([12]). *Let D be a primitive digraph and $\emptyset \neq X \subseteq V(D)$. Then for nonnegative integers i, j, l, k, we have*

$$R_i(X) = R_{i-j}(R_j(X)) \quad (i \geq j) \quad \text{and} \quad \left|\bigcup_{l=0}^{k} R_l(X)\right| \geq \min\{|X| + k, n\}.$$

Let $L_{n,d}$ be the digraph with the set $V = \{1, 2, \ldots, n\}$ of vertices and the set $E = \{(i, i+1) \mid 1 \leq i \leq n-1\} \cup \{(n, 1)\} \cup \{(i, i) \mid n-d+1 \leq i \leq n\}$ of arcs. Obviously, $L_{n,d} \in P_n(d)$.

Lemma 3.2. *Let n, d be integers with $n \geq 2$ and $1 \leq d \leq n$. Then*

$$k(L_{n,d}) = n - \left\lceil \frac{d}{2} \right\rceil.$$

Proof. It is easy to check that

$$\bigcup_{l=0}^{n-\lceil \frac{d}{2} \rceil} R_l(\{i\}) = \left\{ i, i+1, \ldots, i+n-\left\lceil \frac{d}{2} \right\rceil \pmod n \right\}.$$

Let u, v be any pair of vertices of $L_{n,d}$. Note that $n-d+1, \ldots, n-1, n$ are loop vertices (namely, a vertex with a loop) of $L_{n,d}$. It follows that there exists a loop vertex $w \in [\bigcup_{l=0}^{n-\lceil \frac{d}{2} \rceil} R_l(\{u\})] \cap [\bigcup_{l=0}^{n-\lceil \frac{d}{2} \rceil} R_l(\{v\})]$. Therefore,

$$u \xrightarrow{n-\lceil \frac{d}{2} \rceil} w \quad \text{and} \quad v \xrightarrow{n-\lceil \frac{d}{2} \rceil} w.$$

Then $k_{u,v}(L_{n,d}) \leq n - \lceil \frac{d}{2} \rceil$, and $k(L_{n,d}) = \max_{u,v \in V(L_{n,d})}\{k_{u,v}(L_{n,d})\} \leq n - \lceil \frac{d}{2} \rceil$.

On the other hand, we have $k(L_{n,d}) \geq k_{1,n-\lceil \frac{d}{2} \rceil+1}(L_{n,d}) = n - \lceil \frac{d}{2} \rceil$.

Combining the above two relations, we obtain the desired result. $\square$

Theorem 3.3. *Let n, d be integers with $n \geq 2$ and $1 \leq d \leq n$. Let $D \in P_n(d)$. Then*

$$k(D) \leq n - \left\lceil \frac{d}{2} \right\rceil,$$

and this bound can be attained, $L_{n,d}$ is one of its extremal digraphs.

Proof. For any $u, v \in V(D)$, since $D \in P_n(d)$ is primitive, by Lemma 3.1,

$$\left|\bigcup_{l=0}^{n-\lceil \frac{d}{2}\rceil} R_l(\{u\})\right| \geq n - \left\lceil \frac{d}{2} \right\rceil + 1 \quad \text{and} \quad \left|\bigcup_{l=0}^{n-\lceil \frac{d}{2}\rceil} R_l(\{v\})\right| \geq n - \left\lceil \frac{d}{2} \right\rceil + 1.$$

It follows that $|[\bigcup_{l=0}^{n-\lceil \frac{d}{2}\rceil} R_l(\{u\})] \cap [\bigcup_{l=0}^{n-\lceil \frac{d}{2}\rceil} R_l(\{v\})]| \geq n - 2\lceil \frac{d}{2}\rceil + 2$.

Note that $D \in P_n(d)$ contains exactly d loops. We conclude that there exists a loop vertex $w \in [\bigcup_{l=0}^{n-\lceil \frac{d}{2}\rceil} R_l(\{u\})] \cap [\bigcup_{l=0}^{n-\lceil \frac{d}{2}\rceil} R_l(\{v\})]$. Moreover, since w is a loop vertex, it follows that

$$u \xrightarrow{n-\lceil \frac{d}{2}\rceil} w \quad \text{and} \quad v \xrightarrow{n-\lceil \frac{d}{2}\rceil} w.$$

Therefore, $k_{u,v}(D) \leq n - \lceil \frac{d}{2}\rceil$, and $k(D) = \max_{u,v\in V(D)}\{k_{u,v}(D)\} \leq n - \lceil \frac{d}{2}\rceil$.

Since $k(L_{n,d}) = n - \lceil \frac{d}{2}\rceil$ by Lemma 3.2, then the bound $k(D) \leq n - \lceil \frac{d}{2}\rceil$ can be attained, and $L_{n,d}$ is one of its extremal digraphs. □

Corollary 3.4. *Let D be a primitive digraph of order n with nonzero trace. Then*

$$k(D) \leq n - 1,$$

and this bound is best possible as shown by the digraph $L_{n,1}$.

Proof. Since D is a primitive digraph with nonzero trace, D contains at least one loop (namely, $d \geq 1$). By Theorem 3.3, $k(D) \leq n - \lceil \frac{d}{2}\rceil \leq n - \lceil \frac{1}{2}\rceil = n - 1$. Moreover, $k(L_{n,1}) = n - 1$ by Lemma 3.2. The results are obtained as desired. □

Let n, d, k_1, k_2 be integers with $1 \leq d < n$, $0 \leq k_1 \leq n - d - 1$ and $1 \leq k_2 \leq \lfloor \frac{d}{2}\rfloor - 1$. Let $L_{n,d}^{(n-\lceil \frac{d}{2}\rceil - k_1)}$ be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(n, i), (i, k_1 + 2) \mid 1 \leq i \leq k_1 + 1\} \cup \{(i, i + 1) \mid k_1 + 2 \leq i \leq n - 1\} \cup \{(i, i) \mid n - d + 1 \leq i \leq n\}$. Let $L_{n,d}^{(d-\lceil \frac{d}{2}\rceil + 1 - k_2)}$ be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(n, i), (i, j), (j, n - d + 2 + 2k_2) \mid 1 \leq i \leq n - d, n - d + 1 \leq j \leq n - d + 1 + 2k_2\} \cup \{(i, i + 1) \mid n - d + 2 + 2k_2 \leq i \leq n - 1\} \cup \{(i, i) \mid n - d + 1 \leq i \leq n\}$. Let $L_{n,d}^{(1)}$ be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(i, j), (j, i) \mid i \neq j\} \cup \{(i, i) \mid n - d + 1 \leq i \leq n\}$. Let $L_{n,n}^{(n-\lceil \frac{n}{2}\rceil - k_3)}$ be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(i, i + 1) \mid 1 \leq i \leq n - 2k_3 - 2\} \cup \{(n - 2k_3 - 1, i), (i, 1) \mid n - 2k_3 \leq i \leq n\} \cup \{(i, i) \mid 1 \leq i \leq n\}$, where $k_3 = 0, 1, \ldots, \lceil \frac{n-3}{2}\rceil$. Clearly, $L_{n,d}^{(j)} \in P_n(d)(1 \leq d \leq n, 1 \leq j \leq n - \lceil \frac{d}{2}\rceil)$.

Lemma 3.5. *Let n, d be integers with $n \geq 2$ and $1 \leq d \leq n$. Then*

$$k(L_{n,d}^{(j)}) = j, \quad \text{where } j = 1, 2, \ldots, n - \left\lceil \frac{d}{2} \right\rceil.$$

Proof. Let u, v be any pair of vertices of $L_{n,d}^{(j)}$. Note that $n - d + 1, \ldots, n - 1, n$ are the loop vertices of $L_{n,d}^{(j)}$. Similarly as the proof of Lemma 3.2, we conclude that there exists a loop vertex $w \in [\bigcup_{l=0}^{j} R_l(\{u\})] \cap [\bigcup_{l=0}^{j} R_l(\{v\})]$. It follows that

$$u \xrightarrow{j} w \quad \text{and} \quad v \xrightarrow{j} w.$$

Therefore, $k_{u,v}(L_{n,d}^{(j)}) \leq j$, and $k(L_{n,d}^{(j)}) = \max_{u,v\in V(L_{n,d}^{(j)})}\{k_{u,v}(L_{n,d}^{(j)})\} \leq j$.

On the other hand, for $d = 1, 2, \ldots, n - 1$, we have

Case 1. $j = 1$. It is obvious that $k(L_{n,d}^{(1)}) \geq 1$;

Case 2. $2 \leq j \leq d - \lceil \frac{d}{2}\rceil$. We have $k(L_{n,d}^{(j)}) \geq k_{1,n+k_2-\lfloor \frac{d-1}{2}\rfloor}(L_{n,d}^{(j)}) = j$;

Case 3. $d - \lceil \frac{d}{2}\rceil + 1 \leq j \leq n - \lceil \frac{d}{2}\rceil$. Note that $k(L_{n,d}^{(j)}) \geq k_{1,n-\lceil \frac{d}{2}\rceil+1}(L_{n,d}^{(j)}) = j$.

For $d = n$, we have $k(L_{n,n}^{(j)}) \geq k_{1,\lceil \frac{n}{2}\rceil - k_3}(L_{n,n}^{(j)}) = j$.

All in all, it can be seen that $k(L_{n,d}^{(j)}) = j$ for $j = 1, 2, \ldots, n - \lceil \frac{d}{2}\rceil$. □

Denote by $K(\Omega_n) = \{k(D) \mid D \in \Omega_n\}$ the scrambling index set for certain class Ω_n of primitive digraphs with n vertices. Now take an integer j with $1 \le j \le n - \lceil \frac{d}{2} \rceil$, and let $D = L_{n,d}^{(j)}$. Then by Lemma 3.5, we have $k(D) = j$. Since $D = L_{n,d}^{(j)} \in P_n(d)$, then

$$\left\{1, 2, \ldots, n - \left\lceil \frac{d}{2} \right\rceil \right\} \subseteq K(P_n(d)).$$

On the other hand, by Theorem 3.3 and the fact that $k(D) \ge 1$,

$$K(P_n(d)) \subseteq \left\{1, 2, \ldots, n - \left\lceil \frac{d}{2} \right\rceil \right\}.$$

Hence the ISP for $k(D)$ of $D \in P_n(d)$ is settled.

Theorem 3.6. *$K(P_n(d)) = \{1, 2, \ldots, n - \lceil \frac{d}{2} \rceil\}$, where $1 \le d \le n$.*

3.2. Primitive minimally strong digraphs

A digraph D is called strongly connected (strong for short) if for each ordered pair of distinct vertices x, y there is a path from x to y. A strong digraph D is called minimally strong provided each digraph obtained from D by the removal of an arc is not strong. As is well known, a matrix A is nearly reducible if and only if $D(A)$ is a minimally strong digraph [13,14]. In a strong digraph D, each vertex x has indegree $d_D^-(x)$ and outdegree $d_D^+(x)$ at least one. Besides, a vertex x is called critical if $d_D^-(x) = d_D^+(x) = 1$.

Let NR_n denote the set of all primitive, minimally strong digraphs of order n. It is easy to verify that when $n \le 3$, $NR_n = \emptyset$. The purpose of this subsection is to obtain the sharp lower and upper bounds for $k(D)$ of $D \in NR_n$ $(n \ge 4)$.

Lemma 3.7 ([9]). *Let $D = (V, E)$ be a minimally strong digraph and let $X \subseteq V$. Then D has the following properties:*

(1) *D has no arcs joining a vertex to itself (that is, D contains no loops).*
(2) *A circuit has no chords (an arc of D which is not an arc of a circuit π is called a chord of π if it joins two vertices of π).*
(3) *If $(x, y) \in E$, then (x, y) is an arc of every path from x to y.*
(4) *If the digraph D_X induced on X is strong, then D_X is minimally strong.*
(5) *If D has at least two vertices, then D contains a critical vertex.*

Let U_n $(n \ge 5)$ be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(1, 2), (2, 3), (3, 1), (1, 4), (4, 1)\} \cup \{(2, i), (i, 2) \mid 5 \le i \le n\}$. Let U_4 be the digraph with vertex set $V = \{1, 2, 3, 4\}$ and arc set $E = \{(1, 2), (2, 3), (3, 1), (1, 4), (4, 1)\}$. Clearly, $U_n \in NR_n$, where $n \ge 4$.

Lemma 3.8. $k(U_n) = \begin{cases} 4, & \text{if } n = 4, \\ 3, & \text{if } n \ge 5. \end{cases}$

Proof. *Case* 1. $n = 4$. It is easy to check that $i \xrightarrow{4} 1$ for $i = 1, 2, 3, 4$. Hence $k(U_4) \le 4$. Note that $R_3(\{2\}) = \{2, 4\}$ and $R_3(\{3\}) = \{1, 3\}$. Thus $R_3(\{2\}) \cap R_3(\{3\}) = \emptyset$. Then $k(U_4) \ge k_{2,3}(U_4) > 3$, and it follows that $k(U_4) = 4$.

Case 2. $n \ge 5$. Since $R_2(\{4\}) = \{2, 4\}$ and $R_2(\{5\}) = \{3, 5, \ldots, n\}$, we have $R_2(\{4\}) \cap R_2(\{5\}) = \emptyset$. Therefore, $k(U_n) \ge k_{4,5}(U_n) > 2$ $(n \ge 5)$.

On the other hand, it is not difficult to verify that

$$R_3(\{1\}) = \{1, 2, 4\}, \qquad R_3(\{2\}) = \{2, 3, \ldots, n\},$$
$$R_3(\{3\}) = R_3(\{4\}) = \{1, 3, 5, 6, \ldots, n\}, \qquad R_3(\{5\}) = \cdots = R_3(\{n\}) = \{1, 2\}.$$

Hence $R_3(\{i\}) \cap R_3(\{j\}) \ne \emptyset$ for $i \ne j \in V(U_n)$. Therefore, $k(U_n) = 3$ $(n \ge 5)$. $\square$

Theorem 3.9. *Let $D \in NR_n$ $(n \ge 4)$. Then*

$$k(D) = 4 \quad \text{if } n = 4, \quad \text{and} \quad k(D) \ge 3, \quad \text{if } n \ge 5.$$

Moreover, the bounds are best possible as shown by the digraph U_n $(n \ge 4)$.

Proof. *Case* 1. $n = 4$. Since $D \in NR_4$, D is isomorphic to U_4. By Lemma 3.8, we obtain that $k(D) = k(U_4) = 4$ immediately.

Case 2. $n \ge 5$. We need to show that $k(D) \ge 3$ by contradiction.

Subcase 2.1. $k(D) = 1$. Then for any two vertices $u, v \in V(D)$, there exists a vertex w such that $u \xrightarrow{1} w$ and $v \xrightarrow{1} w$. By Lemma 3.7(1), D contains no loops, which implies $w \ne u, v$. Considering the vertices v and w, there exists a vertex p such that $v \xrightarrow{1} p$ and $w \xrightarrow{1} p$. The vertex p may be u, but $p \ne v, w$ by Lemma 3.7(1). Note that $(v, p) \in E$, and $v \xrightarrow{1} w \xrightarrow{1} p$ is a path from v to p not containing the arc (v, p). It is a contradiction to Lemma 3.7(3).

Subcase 2.2. $k(D) = 2$. By Lemma 3.7(5), D contains a critical vertex, denoted by x. Suppose $R_1(\{x\}) = \{y\}$. Since $y \notin R_1(\{y\})$, then $k_{x,y}(D) = 2$.

We may assume that $R_1(\{y\}) = \{z_1, \dots, z_t\}$, where $z_j \neq y$ for $j = 1, \dots, t$. Thus $R_2(\{x\}) = \{z_1, \dots, z_t\}$. Since $k_{x,y}(D) = 2$, there is some vertex z_j ($j \in \{1, \dots, t\}$) such that $y \xrightarrow{2} z_j$. However, $z_j \in R_1(\{y\})$, which implies that either z_j is a loop vertex or there is a path from y to z_j of length 2 not containing the arc (y, z_j). It is a contradiction to Lemma 3.7(1) or (3).

Combining the above two subcases, we conclude that $k(D) \geq 3$. Moreover, by Lemma 3.8, $k(U_n) = 3$ when $n \geq 5$. The proof is finished. $\square$

Let H_n ($n \geq 4$) be the digraph with vertex set $V = \{1, 2, \dots, n\}$ and arc set $E = \{(i, i+1) \mid 1 \leq i \leq n-3\} \cup \{(n-2, 1), (n-3, n-1), (n-1, n), (n, 1)\}$.

Lemma 3.10. *Let n be an integer, $n \geq 4$. Then* $k(H_n) = \begin{cases} \dfrac{n^2-4n+7}{2}, & \text{if } n \text{ is odd}, \\ \dfrac{n^2-4n+8}{2}, & \text{if } n \text{ is even}. \end{cases}$

Proof. *Case* 1. n is odd. Let $l = \frac{n^2-4n+7}{2}$. It is not difficult to verify that

$$R_l(\{n\}) = R_l(\{n-2\}) = \begin{cases} \{2, 3, 4\}, & \text{if } n = 5, \\ \left\{2, 3, \dots, \dfrac{n+1}{2}\right\}, & \text{if } n \geq 7; \end{cases}$$

$$R_l(\{n-1\}) = \left\{1, 2, \dots, \frac{n-1}{2}\right\}; \qquad R_l(\{n-3\}) = \left\{n, 1, 2, \dots, \frac{n-1}{2}\right\};$$

$$R_l(\{n-i\}) = \left\{n-i+2, \dots, n-1, n, 1, \dots, \frac{n-2i+5}{2}\right\} \quad \left(i = 4, 5, \dots, \frac{n+3}{2}\right);$$

$$R_l\left(\left\{\frac{n-5}{2}\right\}\right) = \left\{\frac{n-1}{2}, \frac{n+1}{2}, \dots, n-2, n-1\right\} \quad (n \geq 7);$$

$$R_l(\{n-i\}) = \left\{n+2-i, \dots, \frac{3n+1}{2} - i\right\} \quad \left(i = \frac{n+7}{2}, \frac{n+9}{2}, \dots, n-1\right) \ (n \geq 9).$$

Thus $R_l(\{i\}) \cap R_l(\{j\}) \neq \emptyset$ for $i \neq j$, and it follows that $k(H_n) \leq l$.

On the other hand, $R_{l-1}(\{n-1\}) \cap R_{l-1}(\{\frac{n-3}{2}\}) = \emptyset$ since

$$R_{l-1}(\{n-1\}) = \left\{n, 1, 2, \dots, \frac{n-3}{2}\right\}; \qquad R_{l-1}\left(\left\{\frac{n-3}{2}\right\}\right) = \left\{\frac{n-1}{2}, \frac{n+1}{2}, \dots, n-1\right\}.$$

Consequently, $k(H_n) > l - 1$, and then $k(H_n) = l$.

Case 2. n is even. Let $m = \frac{n^2-4n+8}{2}$. It is not difficult to verify that

$$R_m(\{n\}) = R_m(\{n-2\}) = \begin{cases} \{1, 2, 3\}, & \text{if } n = 4, \\ \{n, 1, 2\}, & \text{if } n = 6, \\ \left\{\dfrac{n+4}{2}, \dots, n, 1, 2\right\}, & \text{if } n \geq 8; \end{cases}$$

$$R_m(\{n-1\}) = \begin{cases} \{4, 1\}, & \text{if } n = 4, \\ \left\{\dfrac{n+2}{2}, \dots, n, 1\right\}, & \text{if } n \geq 6; \end{cases} \qquad R_m(\{n-3\}) = \left\{\frac{n}{2}, \frac{n+2}{2}, \dots, n, 1\right\};$$

$$R_m(\{n-4\}) = \left\{\frac{n-2}{2}, \frac{n}{2}, \dots, n-2, n-1\right\} \quad (n \geq 6);$$

$$R_m(\{n-i\}) = \left\{\frac{n+6}{2} - i, \dots, n-i+2\right\} \quad \left(i = 5, 6, \dots, \frac{n+4}{2}\right) \ (n \geq 6);$$

$$R_m\left(\left\{\frac{n-6}{2}\right\}\right) = \left\{n, 1, 2, \dots, \frac{n-2}{2}\right\} \quad (n \geq 8);$$

$$R_m(\{n-i\}) = \left\{\frac{3n+4}{2} - i, \dots, n-1, n, 1, \dots, n+2-i\right\},$$

where $i = \frac{n+8}{2}, \frac{n+10}{2}, \ldots, n-1$ and $n \geq 10$. Thus $R_m(\{i\}) \cap R_m(\{j\}) \neq \emptyset$ for $i \neq j$, and it follows that $k(H_n) \leq m$. On the other hand, it is easy to see that

$$R_{m-1}(\{n-1\}) = \left\{\frac{n}{2}, \frac{n+2}{2}, \ldots, n-2, n-1\right\};$$

$$R_{m-1}\left(\left\{\frac{n-4}{2} \pmod n\right\}\right) = \left\{n, 1, 2, \ldots, \frac{n-4}{2}, \frac{n-2}{2}\right\}.$$

Hence $R_{m-1}(\{n-1\}) \cap R_{m-1}(\{\frac{n-4}{2} \pmod n\}) = \emptyset$. Thus $k(H_n) > m-1$, and then $k(H_n) = m$. □

Let Q_n $(n \geq 6)$ be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(i, i+1) \mid 1 \leq i \leq n-5\} \cup \{(n-4, n-2), (n-2, 1), (n-5, n-3), (n-3, n-1), (n-1, n), (n, 1)\}$.

Lemma 3.11. *Let n be an integer, $n \geq 6$. Then* $k(Q_n) = \begin{cases} \frac{n^2-6n+15}{2}, & \text{if } n \text{ is odd}, \\ \frac{n^2-6n+14}{2}, & \text{if } n \text{ is even}. \end{cases}$

Proof. *Case* 1. n is odd. Let $l = \frac{n^2-6n+15}{2}$. It is not difficult to verify that

$$R_l(\{n\}) = R_l(\{n-2\}) = \begin{cases} \{1, 2, 3, 4\}, & \text{if } n = 7; \\ \{n, 1, 2, 3\}, & \text{if } n = 9; \\ \{n-2, n-1, n, 1, 2, 3\}, & \text{if } n = 11; \\ \left\{\frac{n+5}{2}, \frac{n+7}{2}, \ldots, n-1, n, 1, 2, 3\right\}, & \text{if } n \geq 13; \end{cases}$$

$$R_l(\{n-1\}) = R_l(\{n-4\}) = \begin{cases} \{n, 1, 2\}, & \text{if } n = 7; \\ \{n-1, n-2, n, 1, 2\}, & \text{if } n = 9; \\ \left\{\frac{n+3}{2}, \frac{n+5}{2}, \ldots, n-1, n, 1, 2\right\}, & \text{if } n \geq 11; \end{cases}$$

$$R_l(\{n-3\}) = \begin{cases} \{n-2, n-1, n, 1\}, & \text{if } n = 7; \\ \left\{\frac{n+1}{2}, \frac{n+3}{2}, \ldots, n-1, n, 1\right\}, & \text{if } n \geq 9; \end{cases}$$

$$R_l(\{n-5\}) = \left\{\frac{n-1}{2}, \frac{n+1}{2}, \ldots, n, 1\right\}; \qquad R_l(\{n-6\}) = \left\{\frac{n-3}{2}, \frac{n-1}{2}, \ldots, n-1\right\};$$

$$R_l(\{n-7\}) = \left\{\frac{n-5}{2}, \frac{n-3}{2}, \ldots, n-6, n-5, n-4, n-3\right\} \quad (n \geq 9);$$

$$R_l(\{n-i\}) = \left\{\frac{n+9}{2} - i, \frac{n+11}{2} - i, \ldots, n+3-i\right\} \quad \left(i = 8, 9, \ldots, \frac{n+7}{2}\right) \ (n \geq 9);$$

$$R_l\left(\left\{\frac{n-9}{2}\right\}\right) = \left\{n, 1, 2, \ldots, \frac{n-5}{2}, \frac{n-3}{2}\right\} \quad (n \geq 11);$$

$$R_l\left(\left\{\frac{n-11}{2}\right\}\right) = \left\{n-2, n-1, n, 1, 2, \ldots, \frac{n-7}{2}, \frac{n-5}{2}\right\} \quad (n \geq 13);$$

$$R_l(\{n-i\}) = \left\{\frac{3n+5}{2} - i, \frac{3n+7}{2} - i, \ldots, n-1, n, 1, 2, \ldots, n+2-i, n+3-i\right\},$$

where $i = \frac{n+13}{2}, \frac{n+15}{2}, \ldots, n-2, n-1$ and $n \geq 15$. Hence $R_l(\{i\}) \cap R_l(\{j\}) \neq \emptyset$ for $i \neq j$, and it follows that $k(Q_n) \leq l$.

On the other hand, $R_{l-1}(\{n-3\}) \cap R_{l-1}(\{\frac{n-7}{2} \pmod n\}) = \emptyset$ since

$$R_{l-1}(\{n-3\}) = \left\{\frac{n-1}{2}, \frac{n+1}{2}, \ldots, n-2, n-1\right\};$$

$$R_{l-1}\left(\left\{\frac{n-7}{2} \pmod n\right\}\right) = \left\{n, 1, 2, \ldots, \frac{n-5}{2}, \frac{n-3}{2}\right\}.$$

Consequently, $k(Q_n) > l-1$, and then $k(Q_n) = l$.

Case 2. n is even. Let $m = \frac{n^2-6n+14}{2}$. It is not difficult to check that

$$R_m(\{n\}) = R_m(\{n-2\}) = \begin{cases} \{4, 5, 6, 1\}, & \text{if } n = 6, \\ \{3, 4, 5, 6, 7\}, & \text{if } n = 8, 10, \\ \left\{3, 4, \dots, \dfrac{n}{2}, \dfrac{n+2}{2}\right\}, & \text{if } n \geq 12; \end{cases}$$

$$R_m(\{n-1\}) = R_m(\{n-4\}) = \begin{cases} \{2, 3, 4, 5\}, & \text{if } n = 6, 8, \\ \left\{2, 3, \dots, \dfrac{n-2}{2}, \dfrac{n}{2}\right\}, & \text{if } n \geq 10; \end{cases}$$

$$R_m(\{n-3\}) = \begin{cases} \{1, 2, 3\}, & \text{if } n = 6, \\ \left\{1, 2, \cdots, \dfrac{n-4}{2}, \dfrac{n-2}{2}\right\}, & \text{if } n \geq 8; \end{cases}$$

$$R_m(\{n-5\}) = \begin{cases} \{n, 1, 2, 3\}, & \text{if } n = 6, \\ \left\{n, 1, 2, \dots, \dfrac{n-4}{2}, \dfrac{n-2}{2}\right\}, & \text{if } n \geq 8; \end{cases}$$

$$R_m(\{n-6\}) = \left\{n-2, n-1, n, 1, 2, \dots, \frac{n-6}{2}, \frac{n-4}{2}\right\} \quad (n \geq 8);$$

$$R_m(\{n-i\}) = \left\{n-i+3, n-i+4, \dots, n-1, n, 1, 2, \dots, \frac{n+6}{2} - i, \frac{n+8}{2} - i\right\},$$

where $i = 7, 8, \dots, \frac{n+6}{2}$ and $n \geq 8$;

$$R_m\left(\left\{\frac{n-8}{2}\right\}\right) = \left\{\frac{n-2}{2}, \frac{n}{2}, \dots, n-4, n-3, n-2, n-1\right\} \quad (n \geq 10);$$

$$R_m\left(\left\{\frac{n-10}{2}\right\}\right) = \left\{\frac{n-4}{2}, \frac{n-2}{2}, \dots, n-4, n-3\right\} \quad (n \geq 12);$$

$$R_m(\{n-i\}) = \left\{n+3-i, n+4-i, \dots, \frac{3n+2}{2} - i\right\},$$

where $i = \frac{n+12}{2}, \frac{n+14}{2}, \dots, n-1$ and $n \geq 14$. Then $R_m(\{i\}) \cap R_m(\{j\}) \neq \emptyset$ for $i \neq j$, and it follows that $k(Q_n) \leq m$. On the other hand, we have

$$R_{m-1}(\{n-3\}) = \left\{n, 1, 2, \dots, \frac{n-6}{2}, \frac{n-4}{2}\right\};$$

$$R_{m-1}\left(\left\{\frac{n-6}{2} \pmod n\right\}\right) = \left\{\frac{n-2}{2}, \frac{n}{2}, \dots, n-4, n-3, n-2, n-1\right\}.$$

Hence $R_{m-1}(\{n-3\}) \cap R_{m-1}(\{\frac{n-6}{2} \pmod n\}) = \emptyset$. Therefore, $k(Q_n) > m-1$, and then $k(Q_n) = m$. This completes the proof. □

Let W_n ($n \geq 6$) be the digraph with vertex set $V = \{1, 2, \dots, n\}$ and arc set $E = \{(i, i+1) \mid 1 \leq i \leq n-4\} \cup \{(n-3, 1), (n-4, n-2), (n-2, n-1), (n-1, n), (n, 1)\}$.

Lemma 3.12. *Let n be an even integer with $n \geq 6$. Then $k(W_n) = \frac{n^2-5n+10}{2}$.*

Proof. Let $m = \frac{n^2-5n+10}{2}$. Considering the digraph W_n, we have

$$R_m(\{n\}) = R_m(\{n-3\}) = \{2, 3, 5, \dots, n-3, n-2, n\};$$
$$R_m(\{n-1\}) = \{1, 2, 4, \dots, n-4, n-1\}; \qquad R_m(\{n-2\}) = \{1, 3, 5, \dots, n-3, n-2\};$$
$$R_m(\{n-4\}) = \{n-1, n, 1, 2, 4, \dots, n-4\};$$
$$R_m(\{n-5\}) = \{n-3, n-2, n-1, 1, 3, \dots, n-5\};$$
$$R_m(\{i\}) = \begin{cases} \{1, 3, \dots, i, i+2, i+3, i+5, \dots, n-4, n-1\} & (i \text{ is odd}), \\ \{2, 4, \dots, i, i+2, i+3, i+5, \dots, n-3, n-2, n\} & (i \text{ is even}), \end{cases}$$

where $n \geq 8$ and $1 \leq i \leq n-6$. Hence $R_m(\{i\}) \cap R_m(\{j\}) \neq \emptyset$ for $i \neq j$, and then $k(W_n) \leq m$. On the other hand, it is not difficult to obtain that

$$R_{m-1}(\{n-1\}) = \{1, 3, 5, \ldots, n-3, n-2\}; \qquad R_{m-1}(\{n-2\}) = \{2, 4, \ldots, n-4, n\}.$$

Hence $R_{m-1}(\{n-1\}) \cap R_{m-1}(\{n-2\}) = \emptyset$. Therefore, $k(W_n) > m-1$, and then $k(W_n) = m$. □

The result in the following lemma is due to Akelbek and Kirkland [3].

Lemma 3.13 ([3]). *Let D be a primitive digraph with n vertices and girth s. Then*

$$k(D) \leq n - s + \begin{cases} \left(\dfrac{s-1}{2}\right) n, & \text{when } s \text{ is odd}, \\ \left(\dfrac{n-1}{2}\right) s, & \text{when } s \text{ is even}. \end{cases}$$

Theorem 3.14. *Let $D \in NR_n$ $(n \geq 4)$. Then*

$$k(D) \leq \begin{cases} \dfrac{n^2-4n+7}{2}, & \text{if } n \text{ is odd}, \\ \dfrac{n^2-4n+8}{2}, & \text{if } n \text{ is even}, \end{cases}$$

with equality if and only if D is isomorphic to the digraph H_n.

Proof. Suppose $D \in NR_n$ $(n \geq 4)$ has an elementary circuit of length n. Then it follows from Lemma 3.7(2) that D has no arcs other than those of the circuit. This contradicts the hypothesis that D is primitive. Hence D has no elementary circuit of length n. Let s be the girth of D. Then $s < n$. If $s = n-1$, since D is primitive, D has an elementary circuit of length n, which is a contradiction. Thus $s \leq n-2$. Now we consider the following three cases.

Case 1. $s \leq n-4$. It follows from Lemma 3.13 that

$$\begin{aligned} k(D) &\leq \begin{cases} \dfrac{n}{2} + \left(\dfrac{n-2}{2}\right) s, & \text{if } s \text{ is odd}, \\ n + \left(\dfrac{n-3}{2}\right) s, & \text{if } s \text{ is even}. \end{cases} \\ &\leq \begin{cases} \dfrac{n}{2} + \left(\dfrac{n-2}{2}\right)(n-4) = \dfrac{n^2-5n+8}{2}, & \text{if } n \text{ is odd}, \\ n + \left(\dfrac{n-3}{2}\right)(n-4) = \dfrac{n^2-5n+12}{2}, & \text{if } n \text{ is even}. \end{cases} \\ &< \begin{cases} \dfrac{n^2-4n+7}{2}, & \text{if } n \text{ is odd}, \\ \dfrac{n^2-4n+8}{2}, & \text{if } n \text{ is even}. \end{cases} \end{aligned}$$

The strict inequality holds since $n \geq 6$ (because $s \leq n-4$, and $D \in NR_n$ contains no loops by Lemma 3.7(1)).

Case 2. $s = n-3$. Suppose α is an elementary circuit of length $n-3$. Since D is primitive and minimally strong, D has an elementary circuit β of length t, where $t = n-2$ or $t = n-1$.

Subcase 2.1. $t = n-2$. It follows from (2) and (4) of Lemma 3.7 that either 2 or 3 vertices of β are not the vertices of α. Hence the digraph D^* induced on the vertices of α and β is isomorphic to either H_{n-1} or Q_n.

If D^* is isomorphic to Q_n, since Q_n has n vertices, then by Lemma 3.11,

$$k(D) \leq k(Q_n) = \begin{cases} \dfrac{n^2-6n+15}{2} < \dfrac{n^2-4n+7}{2}, & \text{if } n \text{ is odd}, \\ \dfrac{n^2-6n+14}{2} < \dfrac{n^2-4n+8}{2}, & \text{if } n \text{ is even}. \end{cases}$$

The strict inequality holds since $n \geq 4$.

If D^* is isomorphic to H_{n-1}, without loss of generality, suppose $V(H_{n-1}) = \{1, 2, \ldots, n-1\}$ and the vertex $n \in V(D)$. Thus there is an arc (n, i) with $1 \leq i \leq n-1$ since D is strong. By Lemma 3.10, we have

$$k(D) \leq k(H_{n-1}) + 1 = \begin{cases} \dfrac{(n-1)^2-4(n-1)+10}{2} < \dfrac{n^2-4n+7}{2}, & \text{if } n \text{ is odd}, \\ \dfrac{(n-1)^2-4(n-1)+9}{2} < \dfrac{n^2-4n+8}{2}, & \text{if } n \text{ is even}. \end{cases}$$

The strict inequality holds since $n \geq 4$.

Subcase 2.2. $t = n-1$. It follows from (2) and (4) of Lemma 3.7 that we may assume D is isomorphic to the digraph W_n $(n \geq 6)$. Moreover, n shall be an even integer since $D \cong W_n$ is primitive. It follows from Lemma 3.12 that

$$k(D) \leq k(W_n) = \frac{n^2-5n+10}{2} < \frac{n^2-4n+8}{2}.$$

The strict inequality holds since $n \geq 4$.

Case 3. $s = n-2$. Since $D \in NR_n$, D has an elementary circuit of length $n-1$. Then by (2) and (4) of Lemma 3.7, D is isomorphic to H_n. The proof is finished by Lemma 3.10. □

Theorem 3.15. *For each integer k with $\frac{n^2-5n+8}{2} < k < \frac{n^2-4n+7}{2}$ if n is odd; $\frac{n^2-5n+12}{2} < k < \frac{n^2-4n+8}{2}$ if n is even, there is no digraph $D \in NR_n$ $(n \geq 6)$ with $k(D) = k$.*

Proof. From the proof of Theorem 3.14, it is easy to see that if $D \in NR_n$ is not isomorphic to H_n when $n \geq 6$. Then

$$k(D) \leq \begin{cases} \dfrac{n^2-5n+8}{2}, & \text{if } n \text{ is odd,} \\ \dfrac{n^2-5n+12}{2}, & \text{if } n \text{ is even.} \end{cases}$$

On the other hand, by Lemma 3.10, $k(H_n) = \begin{cases} \dfrac{n^2-4n+7}{2}, & \text{if } n \text{ is odd,} \\ \dfrac{n^2-4n+8}{2}, & \text{if } n \text{ is even.} \end{cases}$

Combining the above two relations, we obtain the results as desired. □

Theorem 3.15 actually means that there exist "gaps" in the scrambling index set of primitive minimally strong digraphs of order n.

3.3. Nearly decomposable digraphs

In this subsection, the scrambling index is studied for the following classes: primitive doubly symmetric digraphs, r-indecomposable digraphs $(r \geq 1)$, fully indecomposable digraphs, nearly decomposable digraphs, Cayley digraphs and primitive circulant digraphs, etc.

Lemma 3.16 ([7]). *Let n be an integer with $n \geq 2$. Let $D \in S_n$. Then*

$$k(D) = \left\lceil \frac{\exp(D)}{2} \right\rceil.$$

A symmetric digraph D is said to be doubly symmetric if (v_i, v_j) is an edge if and only if (v_{n+1-i}, v_{n+1-j}) is an edge $(i, j = 1, 2, \ldots, n)$. Let DS_n denote the set of all primitive doubly symmetric digraphs of order n. It is obvious that $DS_n \subseteq S_n$. Let $D_n^{(j)}$ $(n \geq 3, j = 1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1)$ be the digraph with vertex set $V = \{1, \ldots, n\}$ and arc set $E = \{(i, i+1), (i+1, i) \mid 2j+1 \leq i < n\} \cup \{(2j+1, i), (i, 2j+1) \mid 1 \leq i \leq j\} \cup \{(n, i), (i, n) \mid j < i \leq 2j\} \cup \{(i, i) \mid 1 \leq i \leq 2j\}$. Clearly, $D_n^{(j)} \in DS_n$ and $\exp(D_n^{(j)}) = n - 2j + 1$ [15,16]. By Lemma 3.16,

Lemma 3.17. *For $j = 1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1$, $k(D_n^{(j)}) = \begin{cases} \frac{n}{2} - j + 1, & \text{if } n \text{ is even,} \\ \frac{n-1}{2} - j + 1, & \text{if } n \text{ is odd.} \end{cases}$*

Lemma 3.18 ([15]). *Let n be an integer with $n \geq 2$ and let $D \in DS_n$. Then*

$$\exp(D) \leq n - 1.$$

Let $\overline{K_n}$ $(n \geq 2)$ be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(i, j), (j, i), (i, i) \mid i \neq j\}$. Note that $\overline{K_n} \in DS_n$, and $k(\overline{K_n}) = 1$. Combining Lemmas 3.16–3.18, the following theorem is obtained.

Theorem 3.19. *Let n be an integer with $n \geq 2$ and let $D \in DS_n$. Then*

$$k(D) \leq \begin{cases} \dfrac{n}{2}, & \text{if } n \text{ is even,} \\ \dfrac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

This bound can be attained as shown by the digraphs $D_n^{(1)}$ $(n \geq 3)$ or $\overline{K_n}$ $(n = 2)$. Moreover, $K(DS_n) = \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$.

Let n be an integer with $n \geq 2$. An $n \times n$ Boolean matrix A is called fully indecomposable if it contains no $k \times l$ zero submatrix with $k + l = n$. If A is fully indecomposable, $D(A)$ is said to be fully indecomposable. Let F_n be the set of all fully indecomposable digraphs of order n [17,1]. A nearly decomposable digraph D is a digraph such that $D \in F_n$, and $D - e \notin F_n$ for any $e \in E(D)$ [2,17]. Let NF_n be the set of all nearly decomposable digraphs of order n. It is known that every $D \in F_n \supseteq NF_n$ is primitive (see [2]).

Let r, n be integers with $1 \leq r < n$. An $n \times n$ Boolean matrix A is called r-indecomposable (shortly, r-inde) if it contains no $k \times l$ zero submatrix with $k + l = n - r + 1$ [17,18,11]. If A is r-inde, then $D(A)$ is said to be r-inde. Let $B_{n,r}$ be the set of all r-inde digraphs of order n. It is known that $B_{n,r} \supseteq B_{n,r+1}$ $(1 \leq r < n - 1)$, $B_{n,1} = F_n$, and every r-inde digraph $(r \geq 1)$ is primitive [17].

Lemma 3.20 ([18,11]). *Let r be an integer with $1 \leq r < n$. Then $D \in B_{n,r}$ if and only if $|R_1(X)| \geq |X| + r$ for all $X \subseteq V(D)$ with $1 \leq |X| \leq n - 1$.*

Theorem 3.21. *Let r, n be integers with $1 \leq r < n$. Let $D \in B_{n,r}$. Then*

$$k(D) \leq \left\lceil \frac{1}{r} \cdot \left\lfloor \frac{n}{2} \right\rfloor \right\rceil.$$

Proof. For any $u, v \in V(D)$, since $D \in B_{n,r}$, by Lemmas 3.1 and 3.20,

$$\begin{aligned} |R_{\lceil \lfloor \frac{n}{2} \rfloor \cdot \frac{1}{r} \rceil}(\{u\})| &= |R_1[R_{\lceil \lfloor \frac{n}{2} \rfloor \cdot \frac{1}{r} \rceil - 1}(\{u\})]| \geq |R_{\lceil \lfloor \frac{n}{2} \rfloor \cdot \frac{1}{r} \rceil - 1}(\{u\})| + r \\ &\geq \cdots \geq \left\lceil \left\lfloor \frac{n}{2} \right\rfloor \cdot \frac{1}{r} \right\rceil \cdot r + |\{u\}| \geq \left\lfloor \frac{n}{2} \right\rfloor \cdot \frac{r}{r} + 1 \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 \end{aligned}$$

and $|R_{\lceil \lfloor \frac{n}{2} \rfloor \cdot \frac{1}{r} \rceil}(\{v\})| \geq \lfloor \frac{n}{2} \rfloor + 1$. It follows that $|[R_{\lceil \lfloor \frac{n}{r} \rfloor \cdot \frac{1}{r} \rceil}(\{u\})] \cap [R_{\lceil \lfloor \frac{n}{r} \rfloor \cdot \frac{1}{r} \rceil}(\{v\})]| \geq 1$.

Therefore, there exists a vertex $w \in [R_{\lceil \lfloor \frac{n}{r} \rfloor \cdot \frac{1}{r} \rceil}(\{u\})] \cap [R_{\lceil \lfloor \frac{n}{r} \rfloor \cdot \frac{1}{r} \rceil}(\{v\})]$, namely,

$$u \xrightarrow{\lceil \lfloor \frac{n}{r} \rfloor \cdot \frac{1}{r} \rceil} w \quad \text{and} \quad v \xrightarrow{\lceil \lfloor \frac{n}{r} \rfloor \cdot \frac{1}{r} \rceil} w.$$

Hence $k_{u,v}(D) \leq \lceil \lfloor \frac{n}{2} \rfloor \cdot \frac{1}{r} \rceil$, and $k(D) = \max_{u,v \in V(D)} \{k_{u,v}(D)\} \leq \lceil \frac{1}{r} \cdot \lfloor \frac{n}{2} \rfloor \rceil$. $\square$

Corollary 3.22. *If $D \in B_{n,r}$ with $r \geq \lfloor \frac{n}{2} \rfloor$, then $k(D) = 1$.*

Let D_n^* $(n \geq 3)$ be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(1, i), (i, 1), (i, i) \mid 2 \leq i \leq n\}$. Let D_2^* be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(1, 2), (2, 1), (1, 1), (2, 2)\}$. Obviously, $D_n^* \in NF_n \subseteq F_n$, and $k(D_n^*) = 1$ $(n \geq 2)$. Now the scrambling index of $D \in F_n$ (resp. $D \in NF_n$) are investigated.

Theorem 3.23. *Let n be an integer with $n \geq 2$. Let $D \in F_n$ (or $D \in NF_n$). Then*

$$k(D) \leq \begin{cases} \dfrac{n}{2}, & \text{if } n \text{ is even}, \\ \dfrac{n-1}{2}, & \text{if } n \text{ is odd}. \end{cases}$$

This bound can be attained, $D_n^{(1)}$ $(n \geq 3)$ or D_2^* is one of its extremal digraphs.

Proof. By Theorem 3.21, since $D \in F_n = B_{n,1}$ (or $D \in NF_n \subseteq F_n$), we have

$$k(D) \leq \begin{cases} \dfrac{n}{2}, & \text{if } n \text{ is even}, \\ \dfrac{n-1}{2}, & \text{if } n \text{ is odd}. \end{cases}$$

On the other hand, when $n = 2$, $k(D_2^*) = 1$; when $n \geq 3$, by Lemma 3.17,

$$k(D_n^{(1)}) = \begin{cases} \dfrac{n}{2}, & \text{if } n \text{ is even}, \\ \dfrac{n-1}{2}, & \text{if } n \text{ is odd}. \end{cases}$$

Notice that $D_n^{(1)}$ $(n \geq 3)$ and $D_2^* \in NF_n \subseteq F_n$. This completes the proof. $\square$

Theorem 3.24. $K(F_n) = K(NF_n) = \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$ $(n \geq 2)$.

Proof. Note that $D_n^* \in NF_n \subseteq F_n$ and $k(D_n^*) = 1$ $(n \geq 2)$. Moreover, $D_n^{(j)} \in NF_n \subseteq F_n$ $(n \geq 3, j = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 1)$, and it follows from Lemma 3.17 that

$$\left\{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\} \subseteq K(F_n) \quad \text{and} \quad \left\{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\} \subseteq K(NF_n).$$

On the other hand, it follows from Theorem 3.23 that

$$K(F_n) \subseteq \left\{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\} \quad \text{and} \quad K(NF_n) \subseteq \left\{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}.$$

Hence the scrambling indices sets for F_n (resp. NF_n) are obtained. □

Let G be a multiplicative group of order n with identity element e, and let $A = \{a_1, a_2, \ldots, a_p\}$ be a subset of G. The (right) Cayley digraph [17] is the digraph $Cay(G, A) = (V, E)$, where $V = G$ and $E = \{(x, y) | x^{-1}y \in A\}$.

Lemma 3.25 *([17]). Let $A = \{a_1, a_2, \ldots, a_p\}$ be a subset of an Abelian group G. If $Cay(G, A)$ is primitive, then $Cay(G, A)$ is $\lceil \frac{p}{2} \rceil$-indecomposable.*

By Theorem 3.21 and Lemma 3.25, we immediately get that

Corollary 3.26. *Let $A = \{a_1, a_2, \ldots, a_p\}$ $(1 \leq p \leq n)$ be a subset of an Abelian group G. If $Cay(G, A)$ is primitive, then*

$$k(Cay(G, A)) \leq \left\lceil \left\lfloor \frac{n}{2} \right\rfloor \Big/ \left\lceil \frac{p}{2} \right\rceil \right\rceil.$$

Let $P = (p_{ij}) \in B_n$ denote the permutation matrix with $p_{i,i+1} = p_{n1} = 1$ for $i = 1, 2, \ldots, n-1$. A primitive circulant matrix $C = C\langle a_1, a_2, \ldots, a_p;\ n\rangle$ is a matrix of the form $C = P^{a_1} + P^{a_2} + \cdots + P^{a_p}$ $(0 \leq a_1 < \cdots < a_p < n)$. It is known that $D(C)$ is $Cay(Z_n, \{a_1, \ldots, a_p\})$ [17,2]. Then by Corollary 3.26, we have

Corollary 3.27. *Let $C = C\langle a_1, \ldots, a_p;\ n\rangle$ be a primitive circulant matrix. Then*

$$k(D(C)) \leq \left\lceil \left\lfloor \frac{n}{2} \right\rfloor \Big/ \left\lceil \frac{p}{2} \right\rceil \right\rceil.$$

3.4. Primitive micro-symmetric digraphs

A digraph D is called a micro-symmetric digraph if there is a pair i, j with $i \neq j$, such that (i, j) and (j, i) are both arcs. Let n be an integer with $n \geq 2$. Let MS_n denote the set of all primitive micro-symmetric digraphs of order n [12].

When $n = 2$, it is obvious that $k(D) = 1$ for $D \in MS_2$. Now the scrambling index $k(D)$ of $D \in MS_n$ for $n \geq 3$ is investigated.

Lemma 3.28. *Let n be an odd integer with $n \geq 3$ and let $D \in MS_n$. Then*

$$k(D) \leq 2n - 3.$$

Proof. If the girth of D is 1, then by Corollary 3.4, $k(D) \leq n - 1 \leq 2n - 3$. If $D \in MS_n$ contains no cycle of length 1, then the girth of D is 2. Then by Lemma 3.13, $k(D) \leq 2n - 3$. □

Lemma 3.29. *Let n be an even integer with $n \geq 4$ and let $D \in MS_n$. Then*

$$k(D) \leq 2n - 4.$$

Proof. Since $D \in MS_n$, D contains an odd cycle of length r and a cycle of length 2. Let C_2 (resp. C_r) denote a cycle of length 2 (resp. length r).

If $r = 1$, by Corollary 3.4, $k(D) \leq n - 1 \leq 2n - 4$. Now suppose $r \geq 3$.

Case 1. $|V(C_2) \cap V(C_r)| = 2$.

For any $u, v \in V(D)$, there exist $u_1, v_1 \in V(C_2) \subseteq V(C_r)$ such that

$$u \xrightarrow{n-2} u_1 \quad \text{and} \quad v \xrightarrow{n-2} v_1.$$

If $u_1 = v_1$, then $k_{u,v}(D) \leq n - 2 \leq 2n - 4$. Otherwise, $u_1 \neq v_1$, suppose u_1, v_1 divide C_r into two parts C^*, C^{**}. Note that r is odd, without loss of generality, we may assume $C^* : u_1 \xrightarrow{l_1} v_1$ and $C^{**} : v_1 \xrightarrow{l_2} u_1$ such that l_1 is even and l_2 is odd. Since l_1 is even and $v_1 \in V(C_2)$, we have

$$u \xrightarrow{n-2} u_1 \xrightarrow{l_1} v_1 \quad \text{and} \quad v \xrightarrow{n-2} v_1 \xrightarrow{l_1} v_1, \quad \text{where } l_1 \leq r - 1 \leq n - 2.$$

Hence $k_{u,v}(D) \le (n-2) + l_1 \le (n-2) + (n-2) \le 2n-4$.

Case 2. $|V(C_2) \cap V(C_r)| = 1$. Suppose $w_1 \in V(C_2) \cap V(C_r)$, $w_2 \in V(C_2) \setminus V(C_r)$.

For any $u, v \in V(D)$, there exist $u_1, v_1 \in V(C_2) \cup V(C_r)$ such that

$$u \xrightarrow{n-1-r} u_1 \quad \text{and} \quad v \xrightarrow{n-1-r} v_1.$$

If $u_1 = v_1$, then $k_{u,v}(D) \le n-1-r \le n-4 \le 2n-4$. Otherwise, $u_1 \ne v_1$, then we consider the following three cases:

Subcase 2.1. $u_1 = w_2$, $v_1 = w_1$. Note that $w_1 \in V(C_2) \cap V(C_r)$ and r is odd,

$$u \xrightarrow{n-1-r} u_1 = w_2 \xrightarrow{1} w_1 \xrightarrow{r-1} w_1 \quad \text{and} \quad v \xrightarrow{n-1-r} v_1 = w_1 \xrightarrow{r} w_1.$$

Hence $k_{u,v}(D) \le (n-1-r) + r \le n-1 \le 2n-4$.

Subcase 2.2. $u_1 = w_2,\ v_1 \in V(C_r) \setminus V(C_2)$. Suppose w_1, v_1 divide C_r into two parts C^*, C^{**}. Assume that $C^* : v_1 \xrightarrow{l_1} w_1$ and $C^{**} : w_1 \xrightarrow{l_2} v_1$.

If l_1 is odd, then $l_1 \le r-2$. It follows that

$$u \xrightarrow{n-1-r} u_1 = w_2 \xrightarrow{1} w_1 \xrightarrow{l_1-1} w_1 \quad \text{and} \quad v \xrightarrow{n-1-r} v_1 \xrightarrow{l_1} w_1.$$

Thus we have $k_{u,v}(D) \le (n-1-r) + l_1 \le n-3 \le 2n-4$.

If l_1 is even, then $l_1 \le r-1$ and $r - l_1 + 1$ is even. Thus

$$u \xrightarrow{n-1-r} u_1 = w_2 \xrightarrow{1} w_1 \xrightarrow{r} w_1 \quad \text{and} \quad v \xrightarrow{n-1-r} v_1 \xrightarrow{l_1} w_1 \xrightarrow{r-l_1+1} w_1.$$

Therefore, $k_{u,v}(D) \le (n-1-r) + (1+r) \le n \le 2n-4$.

Subcase 2.3. $u_1, v_1 \in V(C_r)$. Suppose u_1, v_1 divide C_r into two parts C^*, C^{**}. Without loss of generality, assume that $C^* : u_1 \xrightarrow{l_1} v_1$ and $C^{**} : v_1 \xrightarrow{l_2} u_1$ such that l_1 is even and l_2 is odd. Since there exists $w \in V(C_2)$ such that $v_1 \xrightarrow{n-2} w$,

$$u \xrightarrow{n-1-r} u_1 \xrightarrow{1_1} v_1 \xrightarrow{n-2} w \quad \text{and} \quad v \xrightarrow{n-1-r} v_1 \xrightarrow{n-2} w \xrightarrow{l_1} w, \quad \text{and hence}$$
$$k_{u,v}(D) \le (n-1-r) + (n-2) + l_1 \le (n-1-r) + (n-2) + (r-1) \le 2n-4.$$

Case 3. $|V(C_2) \cap V(C_r)| = 0$.

For any $u, v \in V(D)$, there exist $u_1, v_1 \in V(C_2) \cup V(C_r)$ such that

$$u \xrightarrow{n-2-r} u_1 \quad \text{and} \quad v \xrightarrow{n-2-r} v_1.$$

If $u_1 = v_1$, then $k_{u,v}(D) \le n-2-r \le n-5 \le 2n-4$. Otherwise, $u_1 \ne v_1$, then we consider the following three cases:

Subcase 3.1. $u_1, v_1 \in C_2$. Then there exist $u_2, v_2 \in V(C_r)$ such that

$$u_1 \xrightarrow{n-r} u_2 \quad \text{and} \quad v_1 \xrightarrow{n-r} v_2.$$

If $u_2 = v_2$, then $k_{u,v}(D) \le (n-2-r) + (n-r) \le 2n-8 \le 2n-4$. Otherwise, $u_2 \ne v_2$. Suppose u_2, v_2 divide C_r into two parts C^*, C^{**}. Without loss of generality, assume that $C^* : u_2 \xrightarrow{l_1} v_2$ and $C^{**} : v_2 \xrightarrow{l_2} u_2$ such that l_1 is even and l_2 is odd. Note that $v_1 \in C_2$, then we obtain that

$$u \xrightarrow{n-2-r} u_1 \xrightarrow{n-r} u_2 \xrightarrow{l_1} v_2 \quad \text{and} \quad v \xrightarrow{n-2-r} v_1 \xrightarrow{l_1} v_1 \xrightarrow{n-r} v_2.$$

Hence $k_{u,v}(D) \le (n-2-r) + (n-r) + l_1 \le (2n-2-2r) + (r-1) \le 2n-4$.

Subcase 3.2. $u_1, v_1 \in V(C_r)$. Similarly as Subcase 2.3, we have

$$k_{u,v}(D) \le 2n-4.$$

Subcase 3.3. $u_1 \in V(C_2)$, $v_1 \in V(C_r)$. Note that there exists $w \in V(C_2)$ such that $v_1 \xrightarrow{n-2} w$. If $w = u_1 \in V(C_2)$, since n is even, then

$$u \xrightarrow{n-2-r} u_1 \xrightarrow{n-2} u_1 \quad \text{and} \quad v \xrightarrow{n-2-r} v_1 \xrightarrow{n-2} w = u_1.$$

And we have $k_{u,v}(D) \le (n-2-r) + (n-2) \le 2n-7 \le 2n-4$.

If $w \ne u_1$, since $n-3+r$ is even, then

$$u \xrightarrow{n-2-r} u_1 \xrightarrow{1} w \xrightarrow{n-3+r} w \quad \text{and} \quad v \xrightarrow{n-2-r} v_1 \xrightarrow{r} v_1 \xrightarrow{n-2} w.$$

Therefore, $k_{u,v}(D) \le (n-2-r) + (n-2) + r \le 2n-4$.

Combining the above cases, $k(D) = \max_{u,v \in V(D)}\{k_{u,v}(D)\} \le 2n-4$. $\square$

When n (≥ 3) is odd, let M_n be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(i, i+1) \mid 1 \leq i \leq n-1\} \cup \{(n, 1), (2, 1)\}$; when n (≥ 4) is even, let M_n be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(i, i+1) \mid 1 \leq i \leq n-1\} \cup \{(n, n-1), (n-1, 1)\}$. Obviously, $M_n \in MS_n$ ($n \geq 3$).

Lemma 3.30. *Let n be an integer with $n \geq 3$. Then*

$$k(M_n) = \begin{cases} 2n-3, & \text{if } n \text{ is odd}, \\ 2n-4, & \text{if } n \text{ is even}. \end{cases}$$

Proof. When $n = 3$, we have $k(M_3) \geq k_{3,1}(M_3) = 3 = 2n - 3$.

When n (≥ 5) is odd, $k(M_n) \geq k_{3,4}(M_3) = 2n - 3$.

When n (≥ 4) is even, $k(M_n) \geq k_{1,2}(M_3) = 2n - 4$.

On the other hand, since $M_n \in MS_n$, by Lemmas 3.28 and 3.29,

$$k(M_n) \leq \begin{cases} 2n-3, & \text{if } n \text{ is odd}, \\ 2n-4, & \text{if } n \text{ is even}. \end{cases}$$

Consequently, the results are obtained as desired. $\square$

Combining the above three lemmas, we have the following theorem.

Theorem 3.31. *Let n be an integer with $n \geq 3$ and let $D \in MS_n$. Then*

$$k(D) \leq \begin{cases} 2n-3, & \text{if } n \text{ is odd}, \\ 2n-4, & \text{if } n \text{ is even}, \end{cases}$$

and this bound can be attained, M_n is one of its extremal digraphs.

When n (≥ 3) is odd, let $M_n^{(2n-3-2j)}$ ($j = 0, 1, \ldots, \frac{n-3}{2}$) be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(i, i+1) \mid 1 \leq i \leq n-1\} \cup \{(n, 1), (2, 1), (2, 2j+3)\}$; let $M_n^{(2n-6-2j)}$ ($j = 0, 1, \ldots, \frac{n-5}{2}$) be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(i, i+1) \mid 1 \leq i \leq n-2\} \cup \{(n-2, i) \mid i = 1, n, 2j+1\} \cup \{(n-1, n-2), (n, 2)\}$; let $M_n^{(2n-4)}$ be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(i, i+1) \mid 1 \leq i \leq n-1\} \cup \{(n, 1), (2, 1), (3, 2)\}$. When n (≥ 4) is even, let $M_n^{(2n-4-2j)}$ ($j = 0, 1, \ldots, \frac{n-4}{2}$) be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(i, i+1) \mid 1 \leq i \leq n-1\} \cup \{(n-1, 1), (n-1, 2j+1), (n, n-1)\}$; let $M_n^{(2n-5-2j)}$ ($j = 0, 1, \ldots, \frac{n-4}{2}$) be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(i, i+1) \mid 1 \leq i \leq n-2\} \cup \{(n-1, 1), (2, 1), (2, 2j+3)\}$.

For $n \geq 3$, let $\lambda_n = k(M_n)$. Then $M_n^{(j)} \in MS_n$ ($j = n-1, n, \ldots, \lambda_n - 1, \lambda_n$). Now the scrambling index set for MS_n is characterized.

Lemma 3.32. *Let n be an integer with $n \geq 3$. Then*

$$k(M_n^{(j)}) = j \quad \text{for } j = n-1, n, \ldots, \lambda_n - 1, \lambda_n.$$

Proof. Let u, v be any pair of vertices of $M_n^{(j)}$.

Case 1. n is odd. Then $\lambda_n = 2n - 3$.

Subcase 1.1. $j = n, n+2, \ldots, 2n-3$.

If $u, v \in \{1, 3, \ldots, n\}$, then $u \xrightarrow{n} 1$ and $v \xrightarrow{n} 1$. Hence $k_{u,v}(M_n^{(j)}) \leq n$.

If $u, v \in \{2, 4, \ldots, n-1\}$, then $u \xrightarrow{n-1} 1$, $v \xrightarrow{n-1} 1$. Thus $k_{u,v}(M_n^{(j)}) \leq n - 1$.

If $u \in \{1, 3, \ldots, n\}$ and $v \in \{2, 4, \ldots, n-1\}$, then $u \xrightarrow{j} 1$, $v \xrightarrow{j} 1$. Therefore, $k_{u,v}(M_n^{(j)}) \leq j$, and $k(M_n^{(j)}) = \max_{u,v \in V(M_n^{(j)})}\{k_{u,v}(M_n^{(j)})\} \leq j$.

On the other hand, $k(M_n^{(j)}) \geq k_{3,4}(M_n^{(j)}) = j$. Hence $k(M_n^{(j)}) = j$.

Subcase 1.2. $j = n-1, n+1, \ldots, 2n-6$. Similarly as Subcase 1.1, we have

$$k_{u,v}(M_n^{(j)}) \leq k_{1,2}(M_n^{(j)}) = j, \quad \text{and} \quad k(M_n^{(j)}) = j.$$

Subcase 1.3. $j = 2n - 4$. Similarly as Subcase 1.1, we have

$$k_{u,v}(M_n^{(2n-4)}) \leq k_{4,5}(M_n^{(2n-4)}) = 2n-4, \quad \text{and then } k(M_n^{(2n-4)}) = 2n-4.$$

Case 2. n is even. Then $\lambda_n = 2n - 4$.

Subcase 2.1. $j = n, n+2, \ldots, 2n-4$. Analogously as Subcase 1.1,

$$k_{u,v}(M_n^{(j)}) \leq k_{1,2}(M_n^{(j)}) = j, \quad \text{and then } k(M_n^{(j)}) = j.$$

Subcase 2.2. $j = n-1, n+1, \ldots, 2n-5$. Similarly as Subcase 1.1,

$$k_{u,v}(M_n^{(j)}) \leq k_{3,4}(M_n^{(j)}) = j, \quad \text{and hence } k(M_n^{(j)}) = j.$$

This completes the proof of Lemma 3.32. □

Theorem 3.33. $K(MS_n) = \{1, 2, \ldots, \lambda_n - 1, \lambda_n\}$ $(n \geq 3)$.

Proof. Since $S_n \subseteq MS_n$, by the results in [7], we have

$$\{1, 2, \ldots, n-1\} \subseteq K(MS_n).$$

Moreover, it follows from Lemma 3.32 that

$$\{n, n+1, \ldots, \lambda_n - 1, \lambda_n\} \subseteq K(MS_n).$$

On the other hand, by Theorem 3.21, when $n \geq 3$,

$$K(MS_n) \subseteq \{1, 2, \ldots, \lambda_n - 1, \lambda_n\}.$$

Therefore, the result is obtained as desired. □

4. Concluding remarks

The MIP and ISP for the scrambling index $k(D)$ of $D \in P_n(d), DS_n, F_n, NF_n, MS_n$ are settled respectively in this paper. However, the digraphs attained the sharp upper bounds (EMP) are not determined completely. It would be nice to settle the EMP in the further research. Moreover, the ISP for $k(D)$ of $D \in NR_n$ requires further discussion.

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