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Series parallel composition of greedy linear programming problems

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This paper is dedicated to Phil Wolfe on the occasion of his 65th birthday.

We study the concept of series and parallel composition of linear programming problems and show that greedy properties are inherited by such compositions. Our results are inspired by earlier work on compositions of flow problems. We make use of certain Monge properties as well as convexity properties which support the greedy method in other contexts.

Key words: Greedy algorithm, Monge arrays, series parallel graphs, linear programming, network flow, transportation problem, integrality, convexity.

1. Introduction

Hoffmann [7] showed that the transportation problem is solved by a greedy algorithm if the underlying cost array is a Monge array (so named after the mathematician G. Monge, who first considered such properties [11]). Meanwhile many new results concerning the question when greedy algorithms solve linear programming problems have been obtained (see [8] for a survey), but at the same time many aspects are still not fully understood.

In [2, 4, 3] Bein, Brucker and Tamir explore the concept of series parallel compositions of network flow problems. They consider linear programming descriptions of cost network flow problems and study the programming description of the flow problems that result when two networks are combined by a series or parallel composition. They show that the greedy algorithm solves the combined problem if it solves the original problems. Based on [2] and ideas presented in [1] Hoffman [9] generalized this work further. He shows that the

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compositions preserve the greedy property not only if path costs are obtained from edge costs by summation but also if they are obtained from more general operations, if they have certain monotonicity and Monge properties.

This paper is inspired by the earlier work in [3] and [9]. In this paper we show that under certain conditions the assumption that the underlying linear programs are specific descriptions of flow problems can, in fact, be dropped entirely.

We will state our main results now and prove them in Section 2. In Section 3 we discuss how earlier work can be reinterpreted in the framework of series parallel composition of linear programs. Section 3 also contains a lemma that links a certain convexity with Monge arrays. We close with a number of technical remarks.

In what follows we will assume that all matrices are nonnegative real matrices without zero columns. Consider then the two linear programming problems I and II:

$$\begin{aligned} \text{I:} \quad & \max \quad \sum_i c_i x_i \\ & \text{s.t.} \quad \sum_i x_i A_i \leqslant a, \\ & \qquad\quad x_i \geqslant 0, \end{aligned} \tag{1.1}$$

$$\begin{aligned} \text{II:} \quad & \max \quad \sum_j d_j y_j \\ & \text{s.t.} \quad \sum_j y_j B_j \leqslant b, \\ & \qquad\quad y_j \geqslant 0, \end{aligned} \tag{1.2}$$

where A_i and B_j denote the columns of A and B and along with those matrices all other constants a, b, c, d are nonnegative and real. Without loss of generality we will assume that

$$c_1 \geqslant c_2 \geqslant \cdots \quad \text{and} \quad d_1 \geqslant d_2 \geqslant \cdots. \tag{1.3}$$

and introduce intervals $K := [0, c_1]$ and $L := [0, d_1]$, which contain all c_i and d_j.

Furthermore we will consider the parametric programs I′ and II′ where the constraints $\sum_i x_i = v_{\mathrm{I}}$ and $\sum_j y_j = v_{\mathrm{II}}$ are added.

The *parallel composition* of I and II is then defined as

$$\begin{aligned} \text{III:} \quad & \max \quad \sum_i c_i x_i + \sum_j d_j y_j \\ & \text{s.t.} \quad \sum_i x_i A_i \leqslant a, \\ & \qquad\quad \sum_j y_j B_j \leqslant b, \\ & \qquad\quad x_i, y_j \geqslant 0, \end{aligned} \tag{1.4}$$

with $\sum_i x_i + \sum_j y_j = v_{\mathrm{III}}$ added for the parametric problem III′.

We now define the *series composition* of I and II. For a given function $F: K\times L\to\mathbb{R}^+$ it is defined as

$$\text{IV:}\quad \max \sum_{i,j} F(c_i, d_j) z_{ij}$$

$$\text{s.t.}\quad \sum_{ij} z_{ij}\binom{A_i}{B_j}\leqslant\binom{a}{b}, \tag{1.5}$$

$$z_{ij}\geqslant 0,$$

(the columns of IV are all possible combinations of I and II) with $\Sigma_{ij}z_{ij}=v_{\text{IV}}$ added for the parametric problem IV′.

In the following we will obtain results about the inheritance of certain properties under series and parallel compositions. A linear program such as I (or II) is called a *greedy* linear program if the vector $\bar{x}=(\bar{x}_1,\bar{x}_2,\ldots)$ found by successively maximizing x_1 then $x_2,\ldots$ satisfies

$$\sum_i c_i\bar{x}_i=\max\left\{\sum_i c_i x_i\colon \sum_i x_i A_i\leqslant a,\ x_i\geqslant 0\right\}. \tag{1.6}$$

For our context we introduce a somewhat stronger greedy property:
Let

$$v_{\text{I}}^*=\max\left\{\sum_i x_i\colon \sum_i x_i A_i\leqslant a,\ x_i\geqslant 0\right\}. \tag{1.7}$$

For any $0\leqslant v_{\text{I}}\leqslant v_{\text{I}}^*$ consider $\bar{x}^{v_{\text{I}}}$ the vector $\bar{x}$ truncated at v_{I}.[1] We then call I (or II respectively) a *strongly constrained parametrically greedy* (*s.c.p.g.*) linear program if

$$\sum_i c_i\bar{x}_i^{v_{\text{I}}}\ \text{maximizes I}'\ \text{for all}\ 0\leqslant v_{\text{I}}\leqslant v_{\text{I}}^*. \tag{1.8}$$

The notion of strongly greedy linear programs is quite natural. In fact, many programs that are greedy linear programs are also strongly greedy. Examples are polymatroids or the flow problems considered in [2, 4, 3]. But we do not know of any problems where the new aspects of "greediness" described in this paper illuminate any cases where a greedy algorithm was previously sought or is now joyously welcomed.

We are now ready to state two central results, which we will prove in the next section:

Theorem 1.1. *If linear program* I (1.1) *and* II (1.2) *are s.c.p.g., so is their parallel composition* III (1.4).

Theorem 1.2. *If linear program* I (1.1) *and* II (1.2) *are s.c.p.g., so is their series composition* IV (1.5) *if F has the following properties*:

$$F(\cdot, v)\ \textit{and}\ F(u,\cdot)\ \textit{are nondecreasing}, \tag{1.9}$$

[1] Vector z truncated by w is defined inductively by $z_1:=\min(w, z_1)$; $z_i:=\min(w-\Sigma_{k=1}^{i-1}z_k,\ z_i)$ for $i>1$.

$$\begin{aligned} &\textit{for each } v \in L, \quad F(\cdot, v) \textit{ is convex,} \\ &\textit{for each } u \in K, \quad F(u, \cdot) \textit{ is convex,} \end{aligned} \tag{1.10}$$

$$u_1 \geqslant u_2,\ v_1 \geqslant v_2 \quad \textit{imply} \quad F(u_1, v_1) + F(u_2, v_2) \geqslant F(u_1, v_2) + F(u_2, v_1). \tag{1.11}$$

Property (1.11) is known as the *Monge property*, which we mentioned at the beginning of this section. Note that if F is differentiable (1.9) becomes the condition that the first partials are nonnegative, (1.10) and (1.11) says that all second partials are nonnegative.

Based on compositions (1.4) and (1.5) one can introduce the notion of a *series parallel linear program*. A two terminal directed graph $G=(V, E)$ is called a *series parallel graph* if it fits the following recursive definition (see [13] for a detailed treatment of series parallel graphs): A single edge from one terminal s (usually called the *source*) to the other terminal t (usually called the *sink*) is a series parallel graph. If G_1 and G_2 are series parallel graphs with respective source sink pairs s_1, t_1 and s_2, t_2, their parallel composition is the graph obtained by identifying s_1 and s_2, and also identifying t_1 and t_2. Their series composition (G_1 followed by G_2) is the graph obtained by identifying t_1 and s_2.

Given a graph G, we associate with each $e \in G$ a linear program (e) of the form of I. Denote the data of the individual problem (e) by $A^{(e)}$, $a^{(e)}$, $c^{(e)}$, where the number of columns is $n^{(e)}$ and the number of rows is $m^{(e)}$. Assume that all $c_i^{(e)}$ are contained in some convex subset C of $\mathbb{R}$ such that $F: C \times C \to C$ is associative. If we let $F(u, v)$ be written as $u \circ v$ then we can define the *G-composition* of all the $(e)_{e \in E(G)}$ problems. The number of columns of the combined problem is

$$\sum_{p \in P} \prod_{e \in p} n^{(e)}$$

where P is the set of directed s–t paths in G. The variables are

$$z_{p:i_1,i_2,\ldots,i_k} \geqslant 0,$$

where $e_1, e_2, \ldots, e_k$ are the edges in p and $1 \leqslant i_1 \leqslant n^{(e_1)}, \ldots, 1 \leqslant i_k \leqslant n^{(e_k)}$. The corresponding cost coefficient is

$$c_{i_1} \circ c_{i_2} \circ \cdots \circ c_{i_k}.$$

For edge $e \in p$ let p have the form $e_1 e_2, \ldots, e_{r-1}, e, e_{r+1}, \ldots, e_k$; i.e. r is the position of e in p. Then the inequalities of the G-composition are

$$\sum_{p \ni e} \sum_{\substack{1 \leqslant i_1 \leqslant n^{(e_1)} \\ \vdots \\ 1 \leqslant i_k \leqslant n^{(e_k)}}} z_{p:i_1,i_2,\ldots,i_k} A_{i_r}^{(e)} \leqslant a^{(e)} \qquad \forall e \in E(G).$$

Thus iterating Theorems 1.1 and 1.2 we have the following result:

Theorem 1.3. *If G is a series parallel graph, and if $F: C \times C \to C$ is associative and satisfies* *properties* (1.9)–(1.11), *then the G-composition of s.c.p.g. linear programs is a s.c.p.g. linear program.* □

2. Proof of theorems

We will now discuss the validity of the Theorems 1.1 and 1.2. It is clear that the series composition (Theorem 1.2) is the interesting one, whereas the parallel case (Theorem 1.1) is straightforward. All one has to do for the parallel case is to convince oneself that an optimal solution for the composed problem can be obtained by merging the two original optimal greedy solutions.

We therefore concentrate on the series composition. For the proof of Theorem 1.2 we need the following majorization lemma:

Lemma 2.1. *Assume*

$$a_1 \geqslant \cdots \geqslant a_n, \qquad b_1 \geqslant \cdots \geqslant b_n; \tag{2.1}$$

$f_1, f_2, \ldots, f_n$ are nondecreasing convex functions on a real interval C containing all a_i and all b_i; (2.2)

$$i<j \text{ and } u>v \quad \text{imply} \quad f_i(u)+f_j(v) \geqslant f_i(v)+f_j(u). \tag{2.3}$$

Then if

$z=(z_1, \ldots, z_n)$ is a nonnegative vector such that

$$\sum_1^k z_i a_i \leqslant \sum_1^k z_i b_i, \quad k=1, \ldots, n, \tag{2.4}$$

we have

$$\sum_1^n z_i f_i(a_i) \leqslant \sum_1^n z_i f_i(b_i). \tag{2.5}$$

Proof. It is clear that it is sufficient to assume all $z_i>0$, which we do. We shall prove the lemma by induction on n. It clearly holds for $n=1$, where the only property of f_1 used is that it is nondecreasing.

For the inductive step we first consider the case where in addition to (2.1)–(2.4) we assume

$$\sum_1^n z_i a_i = \sum_1^n z_i b_i \tag{2.6}$$

and

every z_i is rational. (2.7)

From (2.7), there is a $\delta > 0$ such that

$$z_i = n_i \delta, \quad n_i \in \mathbb{N}^+, \ i = 1,\dots,n. \tag{2.8}$$

Let $N = \sum n_i$. Consider the sequences

$$a'_1 \geqslant a'_2 \geqslant \cdots \geqslant a'_N \quad \text{and} \quad b'_1 \geqslant b'_2 \geqslant \cdots \geqslant b'_N, \tag{2.9}$$

where the sequence $a'_1, a'_2,\dots,a'_N$ consists of n_1 a_1's, n_2 a_2's,..., in descending order, and similarly for the sequence $b'_1,b'_2,\dots,b'_N$. From (2.1), (2.4) and (2.8), we have

$$\begin{aligned} &a'_1 \leqslant b'_2, \\ &a'_1 + a'_2 \leqslant b'_1 + b'_2, \\ &\vdots \\ &a'_1 + \cdots + a'_N = b'_1 + \cdots + b'_N. \end{aligned} \tag{2.10}$$

It is well known (see [6]) that (2.9) and (2.10) imply that the vector $a' = (a'_1,\dots,a'_N)$ is in the polytope of all convex combinations of the vector $b' = (b'_1,\dots,b'_N)$ and its permutations. Since each f_i is convex, the function

$$f_1(a'_1) + \cdots + f_1(a'_{n_1}) + f_2(a'_{n_1+1}) + \cdots + f_2(a'_{n_1+n_2}) + \cdots + f_n(a'_N)$$

is a convex function on this polytope, so its maximum occurs at a vertex of the polytope, namely at one of the permutations of b'. But (2.3) implies that a maximizing vertex is b' itself. So

$$\sum n_i f_i(a_i) \leqslant \sum n_i f_i(b_i).$$

Multiply both sides by δ, use (2.8) and infer (2.5).

Now we must prove (2.5) without assuming (2.6) and (2.7). When $a_1 = b_1$ it is easy to see that the lemma follows from the induction hypothesis. So assume $a_1 < b_1$. It follows that, for any $\varepsilon > 0$, there exists $z'_1,\dots,z'_n > 0$,

$$z'_i \text{ is rational}, \quad i = 1,\dots,n, \tag{2.11}$$

$$|z'_i - z_i| < \varepsilon, \quad i = 1,\dots,n, \tag{2.12}$$

and

$$\sum_1^k z'_i a_i < \sum_1^k z'_i b_i, \quad k = 1,\dots,n. \tag{2.13}$$

Define $\alpha > 0$ using (2.13) by

$$\alpha z'_1 = \min_k \sum_1^k z'_i (b_i - a_i).$$

Then letting $a^* = (a^*_1,\dots,a^*_n)$ with

$$a^*_1 = a_1 + \alpha, \ a^*_2 = a_2,\dots,a^*_n = a_n$$

we have $a_1^* \geqslant \cdots \geqslant a_n^*$, and

$$\sum_1^k z_i' a_i^* \leqslant \sum_1^k z_i' b_i, \quad k = 1,\ldots,n,$$

with equality for some $k = k^*$.

If $k^* = n$, then, from our discussion of (2.6) and (2.7), we have, from (2.11),

$$\sum_1^n z_i' f(a_i^*) \leqslant \sum_1^n z_i' f(b_i). \tag{2.14}$$

If $k^* < n$, we have for the same reason

$$\sum_1^{k^*} z_i' f(a_i^*) \leqslant \sum_1^{k^*} z_i' f(b_i). \tag{2.15}$$

On the other hand, the induction hypothesis gives

$$\sum_{k^*+1}^{n} z_i f(a_i^*) \leqslant \sum_{k^*+1}^{n} z_i f(b_i). \tag{2.16}$$

But (2.12) and (2.14), or (2.12), (2.15) and (2.16) imply

$$\sum_1^n z_i f(a_i^*) \leqslant \sum_1^n z_i f(b_i). \tag{2.17}$$

But the definition of a^*, together with the fact that f_1 is nondecreasing (2.2), shows that (2.17) implies (2.5). □

Before we prove Theorem 1.2 we will first rewrite problem IV′.

The parametric problem

$$\text{IV}': \quad \max \quad \sum_{i,j} F(c_i, d_j) z_{ij}$$

$$\begin{aligned} \text{s.t.} \quad & \sum_{ij} z_{ij} \begin{pmatrix} A_i \\ B_j \end{pmatrix} \leqslant \begin{pmatrix} a \\ b \end{pmatrix}, \\ & \sum_{ij} z_{ij} = v, \\ & z_{ij} \geqslant 0, \end{aligned} \tag{2.18}$$

can be rewritten as

$$\left.\begin{array}{lll}\text{IV}'': & \max & \sum_{ij} F(c_i, d_j) z_{ij} \\ & \text{s.t.} & \sum_j z_{ij} = x_i, \\ & & \sum_i z_{ij} = y_j, \\ & & z_{ij} \geqslant 0,\end{array}\right\} \mathrm{T}(x, y)$$

$$\left.\begin{array}{l}\sum_i x_i A_i \leqslant a, \\ \sum_j y_j B_j \leqslant b, \\ \sum_i x_i = \sum_j y_j = v, \\ x_i, y_j \geqslant 0.\end{array}\right\} \mathrm{R} \tag{2.19}$$

As indicated we call the top part of problem (2.19), including the objective function $\mathrm{T}(x, y)$, and the remaining constraints R. Notice that $\mathrm{T}(x, y)$ is a transportation problem with right side x and y.

Hoffman [7] has shown that an optimal solution for $\mathrm{T}(x, y)$ is given by the northwest corner rule. Formally this solution can be represented in the following way[2]: On the real axis starting at 0, plot successive closed intervals $I_1, I_2, \ldots$ where

$$|I_i| \equiv \text{length of } I_i \text{ is } x_i. \tag{2.20}$$

The intervals I_i are referred to as *x-intervals*. Proceed in the same way with y_j to obtain *y*-intervals J_j. Then we have:

Remark 2.1. *An optimal solution to* $\mathrm{T}(x, y)$ *is given by*

$$z_{ij} = |I_i \cap J_j|. \qquad \square \tag{2.21}$$

Remark 2.2. *Let x^* and y^* be greedy solutions to* I′ *and* II′ *with parameter value v. Then z defined by* (2.21) *with respect to* $\mathrm{T}(x^*, y^*)$ *is a greedy solution of* IV′ (2.18).

Proof. The correctness of the remark follows from monotonicity of F and the monotonicity of the coefficients (1.3). We leave the verification to the reader.[3] □

Using Remark 2.1, let $G(x, y)$ be the value of an optimal solution z defined by (2.21) with respect to $\mathrm{T}(x, y)$. Then problem IV″ (2.19) can again be rewritten as

[2] We are unable to recall when we encountered this representation.

[3] Notice that ties are resolved in accordance with the northwest corner rule, not arbitrarily, cf. [9].

$$\text{IV}''' \quad \max \quad G(x, y)$$

$$\text{s.t.} \quad \sum_i x_i A_i \leqslant a,$$

$$\sum_j y_j B_j \leqslant b, \tag{2.22}$$

$$\sum_i x_i = \sum_j y_j = v,$$

$$x_i, y_j \geqslant 0.$$

We are now ready to prove the following lemma, from which Theorem 1.2 follows directly:

Lemma 2.2. *Let x, y be a feasible solution to* R, *and x^* be a greedy solutionn of* Γ'. *Then*

$$G(x^*, y) \geqslant G(x, y).$$

Proof. Consider (in the sense of (2.20)) the x-intervals I_i of x and y-intervals J_j of y and furthermore x^*-intervals I^*_{i*}. Now consider the common refinement of all these intervals K_l, numbered successively from the left. We want to invoke Lemma 2.1. To that end, we define numbers z_l as $|K_l|$. Numbers a_l and b_l are defined as follows: Set $a_l = c_i$ if $K_l \subset I_i$ and $b_l = c_{i*}$ if $K_l \subset I^*_{i*}$. Functions f_l are given by the rule: If $K_l \subset J_j$ then $f_l(u) = F(u, d_j)$.

Since problem I is greedy we have verified (2.1). Further since problem I is s.c.p.g., we obtain (2.4) by setting the parameter v_l successively to $\sum_1^k z_l$ for $k = 1,...,n$. As for f properties (1.9) and (1.10) of F imply (2.2) and (1.11) implies (2.3). So Lemma 2.1 implies (2.5), which is $G(x^*, y) \geqslant G(x, y)$. $\square$

In a similar way one shows $G(x, y^*) \geqslant G(x, y)$ and thus $G(x^*, y^*) \geqslant G(x, y)$. Therefore, by Remark 2.2, Theorem 1.2 is proved. $\square$

3. Earlier results

We shall begin with netflows and the results in [2, 4, 3] and [9]. They consider cost flow problems over a series parallel graph G. Associated with each edge $e \in E(G)$ are a nonnegative (usually integer) capacity $b(e)$ as well as a nonnegative cost $c(e)$. Then the program

$$\max \quad \sum_{p \in P} c(p)x(p)$$

$$\text{s.t.} \quad \sum_{p \ni e} x(p) \leqslant b(e) \quad \text{for all } e \in E(G), \tag{3.1}$$

$$x(p) \geqslant 0 \quad \text{for all } p \in P,$$

with $c(p)=c(e_1)+\cdots+c(e_k)$ and path decision variables $x(p)$, is the path-arc description of the cost flow problem on G (see [12] for a more detailed introduction to this formulation of flow problems).

Now define for each edge the trivial program

$$\begin{aligned} \max \quad & c(e)x(e) \\ \text{s.t.} \quad & 0 \leqslant x(e) \leqslant b(e), \end{aligned}$$

which is s.c.p.g.; then (3.1) is the G-composition of these programs, where $F(u, v)=u+v$. Therefore it follows that the cost flow problem on series parallel graphs is indeed s.c.p.g. This gives the main result of [2, 4, 3]. More generally this result holds for associative operations $F(u, v)=u \circ v$, when they satisfy (1.9)–(1.11). This implies some of the results in [9] and [1].

We now turn to the transportation problem. In [7] Hoffman has shown that a greedy algorithm solves the transportation in certain cases.

Given the problem

$$\begin{aligned} \text{TP:} \quad \max \quad & \sum_i \sum_j c_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_j x_{ij} \leqslant a_i, \\ & \sum_i x_{ij} \leqslant b_j, \\ & x_{ij} \geqslant 0, \end{aligned} \tag{3.2}$$

with $\sum_i a_i = \sum_j b_j$ and c_{ij}, a_i, $b_j \geqslant 0$. Then the parametric problem with parameter value $v = \sum_i a_i$ is the transportation problem (TP). If the array $(c_{ij})_{n \times m}$ has the properties

$$c_{i\cdot} \text{ and } c_{\cdot j} \text{ are nondecreasing in } i \text{ and } j; \tag{3.3}$$

$$\begin{aligned} & (c_{ij}) \text{ is a Monge array, i.e.} \\ & c_{i_1 j_1} + c_{i_2 j_2} \geqslant c_{i_1 j_2} + c_{i_2 j_1} \quad \text{for all } i_1 < i_2,\ j_1 < j_2; \end{aligned} \tag{3.4}$$

the problem (3.2) is greedy (and in fact s.c.p.g.).

We will now derive this result in the framework of Theorem 1.2. Although this result is used in proving Theorem 1.2, it is amusing to derive it as a corollary of Theorem 1.2. To this end, the following lemma on Monge arrays is needed; we postpone the proof of the lemma to the end of this section.

Lemma 3.1. *Given an $n \times m$ array (c_{ij}) satisfying (3.3) and (3.4) then there exist*

$$c_1 < c_2 < \cdots < c_n, \qquad d_1 < d_2 < \cdots < d_m,$$

and a function

$$F : [c_1, c_n] \times [d_1, d_n] \to \mathbb{R}^+$$

satisfying (1.9), (1.10) *and* (1.11) *such that*

$$F(c_i, d_j) = c_{ij} \quad \textit{for } i = 1,\dots,n \textit{ and } j = 1,\dots,m.$$

Consider then the linear programs

$$\begin{aligned} \max \quad & \sum_i c_i x_i \\ \text{s.t.} \quad & 0 \leqslant x_i \leqslant a_i \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \max \quad & \sum_j d_j y_j \\ \text{s.t.} \quad & 0 \leqslant y_j \leqslant b_j \end{aligned} \tag{3.6}$$

which are cleary s.c.p.g. as the parallel composition of trivial linear programs, where the c_i and d_j are as in Lemma 3.1. Now the transportation problem (3.2) is the series composition of (3.5) and (3.6), using the F of Lemma 3.1, which shows that (3.2) is indeed s.c.p.g.

In fact, Hoffman [7] did not make any monotonicity assumptions on c_{ij} and showed that the given Monge property, the northwest corner algorithm solves the transportation problem optimally. Hoffman's result however can also be put into our framework by observing that (c_{ij}) can be transformed into a monotone array (c'_{ij}) in such a way that $\sum c_{ij} z_{ij} - \sum c'_{ij} z_{ij}$ is a constant. To derive (c'_{ij}) from (c_{ij}) we subtract the first column from all columns and then subtract the first row from all rows. The validity of this transformation is easily verified.

Finally, we turn to the proof of Lemma 3.1:

Proof of Lemma 3.1. Given $c_1 < c_2 < \cdots < c_n$, $d_1 < d_2 < \cdots < d_m$, we define function $F: [c_1, c_n] \times [d_1, d_n] \to \mathbb{R}^+$ by

$$F(x,y) = \sum_{\mu, \nu = 0}^{1} \alpha_\nu \beta_\mu c_{i+\nu, j+\mu} \tag{3.7}$$

where $\alpha_0, \alpha_1, \beta_0, \beta_1$ are the coefficients of the unique representation of $(x, y) \in [c_i, c_{i+1}] \times [d_j, d_{j+1}]$ of the form

$$\begin{aligned} x &= \alpha_0 c_i + \alpha_1 c_{i+1}, \quad \alpha_0 + \alpha_1 = 1, \ \alpha_0, \alpha_1 \geqslant 0, \\ y &= \beta_0 d_j + \beta_1 d_{j+1}, \quad \beta_0 + \beta_1 = 1, \ \beta_0, \beta_1 \geqslant 0. \end{aligned} \tag{3.8}$$

Due to (3.3) the functions $F(c_i, \cdot)$ and $F(\cdot, d_j)$ are nondecreasing. Furthermore $c_1 < c_2 < \cdots < c_n$, $d_1 < d_2 < \cdots < d_m$, can be chosen in such a way that all functions $F(c_i, \cdot)$ and $F(\cdot, d_j)$ are also convex. We have to show that F satisfies properties (1.9), (1.10) and (1.11).

First (1.9), (1.10): Let $x \in [c_i, c_{i+1}]$. Then there exists an $0 \leqslant \alpha \leqslant 1$ such that

$$F(x, \cdot) = \alpha F(c_i, \cdot) + (1 - \alpha) F(c_{i+1}, \cdot).$$

As a convex combination of nondecreasing and convex functions $F(c_i,\cdot)$ and $F(c_{i+1},\cdot)$ the function $F(x,\cdot)$ is nondecreasing and convex. Repeat the arguments for $F(\cdot, y)$.

Now for (1.11): We first prove that

$$F(x_1, y_1)+F(x_2, y_2) \geqslant F(x_1, y_2)+F(x_2, y_1) \tag{3.9}$$

holds for points $P_1=(x_1, y_1)$, $P_2=(x_1, y_2)$, $P_3=(x_2, y_2)$, $P_4=(x_2, y_1)$ with

$$\begin{array}{lllll} x_1=\alpha_0 c_i+\alpha_1 c_{i+1}, & \alpha_0+\alpha_1=1,\ \alpha_0,\ \alpha_1 \geqslant 0, & \multirow{2}{*}{} & y_1=d_j, \\ x_2=\beta_0 c_i+\beta_1 c_{i+1}, & \beta_0+\beta_1=1,\ \beta_0,\ \beta_1 \geqslant 0, & \alpha_0>\beta_0, & y_2=d_{j+1}, \end{array} \tag{3.10}$$

(see Figure 1). Now (3.9) may be written in the form

$$\begin{aligned} & \alpha_0 c_{i,j+1}+\alpha_1 c_{i+1,j+1}+\beta_0 c_{i,j}+\beta_1 c_{i+1,j} \\ & \quad \leqslant \alpha_0 c_{i,j}+\alpha_1 c_{i+1,j}+\beta_0 c_{i,j+1}+\beta_1 c_{i+1,j+1} \end{aligned}$$

which is equivalent to

$$\begin{aligned} & \alpha_0(c_{i,j+1}+c_{i+1,j}-c_{i+1,j+1}-c_{i,j}) \\ & \quad \leqslant \beta_0(c_{i,j+1}+c_{i+1,j}-c_{i+1,j+1}-c_{i,j}). \end{aligned}$$

However the last inequality holds because $\alpha_0>\beta_0$ and

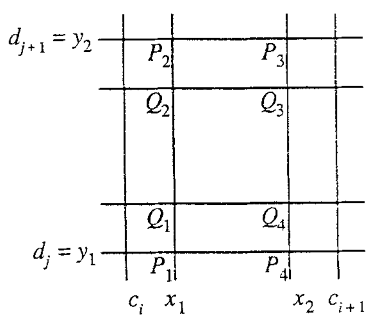

Fig. 1.

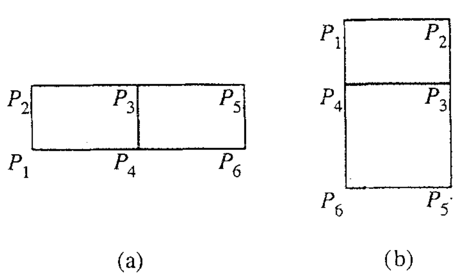

Fig. 2.

$$c_{i,j+1}+c_{i+1,j}-c_{i+1,j+1}-c_{i,j}\leqslant 0$$

due to (3.4).

Using the fact that (3.9) holds for the rectangle P_1,P_2,P_3,P_4 we derive in a similar way that (3.9) holds for the inner rectangle Q_1,Q_2,Q_3,Q_4. Next it is easy to show (see e.g. [5]) that property (3.9) is transitive in the sense that if (3.9) holds for P_1,P_2,P_3,P_4 and P_4,P_3,P_5,P_6 it also holds for P_1,P_2,P_5,P_6; see Figure 2(a) and (b). Using this transitivity and the previous result the argument can be repeated to show that (3.9) holds for arbitrary rectangles. □

4. Remarks

We first note that the results of Section 1 can also be formulated for minimization problems. The corresponding results require F to be concave rather than convex and in the Monge property "$\geqslant$" has to be replaced by "$\leqslant$".

The convexity assumption for Theorem 1.2 is indeed necessary. To see this, consider as program I,

$$\begin{array}{ll} \max & 12x+2y \\ \text{s.t.} & x+y+z\leqslant 1, \\ & 2x+y\leqslant 1, \\ & x,y,z\geqslant 0, \end{array}$$

and as program II the *dummy program*

$$\begin{array}{ll} \max & 3x \\ \text{s.t.} & x\leqslant 1, \\ & x\geqslant 0. \end{array}$$

Both programs I and II are s.c.p.g., but for $F(\cdot,\cdot)=\min(\cdot,\cdot)$ the series composition is not.[4]

If we weaken the notion of greedy by replacing $\Sigma_i x_i=v$ by $\Sigma_i x_i\leqslant v$ (now called *weakly greedy*) the result on parallel compositions is still true but the result for series compositions does not hold any longer. The series composition result can be carried over if in Theorem 1.2 we require F to have the additional property that

$$F(u,0)=F(0,v)=0 \quad \text{for all } u\in K,\ v\in L. \tag{4.1}$$

As an example, a function satisfying those properties if $F(u,v)=u\cdot v$.

It would be interesting to characterize those F that satisfy (1.9)–(1.11) (and property

[4]Dummy programs are not only useful for counterexamples: If the objective function $\Sigma_i c_i x_i$ of a s.c.p.g. is changed to $\Sigma_i f(c_i)x_i$ for a monotone and convex function f, the program remains s.c.p.g. To see this, all we have to do is consider the series composition with a dummy program and $F(\cdot,1):=f(\cdot)$.

(4.1) for the weak case). If F is assumed associative then there are severe restrictions. Several years ago, Jeremy Kahn [10] made significant inroads into the case where F is defined over the reals and has a fixed point $F(u, u) = u$.

Finally, what does it mean algebraically for a linear program to be s.c.p.g. Is it possible to characterize those triples (A, a, c) such that the corresponding linear program I is s.c.p.g.?

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References

[1] Y.P. Aneja, R. Chandrasekaran and K.P.K. Nair, "Classes of linear programs with integral optimal solutions," *Mathematical Programming Study* 25 (1985) 225–237.

[2] W.W. Bein, "Netflows, polymatroids, and greedy structures," Ph.D. Thesis, Universität Osnabrück (Osnabrück, Germany, 1986).

[3] W.W. Bein and P. Brucker, "Greedy concepts for network flow problems," *Discrete Applied Mathematics* 15 (1986) 135–144.

[4] W.W. Bein, P. Brucker and A. Tamir, "Minimum cost flow algorithms for series parallel networks," *Discrete Applied Mathematics* 10 (1985) 117–124.

[5] P.C. Gilmore, E.L. Lawler and D.B. Shmoys, "Well-solved special cases," in: E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan and D.B. Shmoys, eds., *The Travelling Salesman Problem – A Guided Tour of Combinatorial Optimization* (Wiley, New York, 1985) pp. 87–143.

[6] G.H. Hardy, J.E. Littlewood and G. Polya, *Inequalities* (Cambridge University Press, Cambridge, England, 1934).

[7] A.J. Hoffman, "On simple linear programming problems," in: V.L. Klee, ed., *Convexity, Proceedings of Symposia in Pure Mathematics, Vol. 7* (American Mathematical Society, Providence, RI, 1963) pp. 317–327.

[8] A.J. Hoffman, "On greedy algorithms that succeed," in: I. Anderson, ed., *Surveys in Combinatorics 1985. London Mathematical Society Lecture Note Series No. 103* (Cambridge University Press, Cambridge, England, 1985) pp. 97–112.

[9] A.J. Hoffman, "On greedy algorithms for series parallel graphs," *Mathematical Programming* 40 (1988) 197–204.

[10] J. Kahn, oral communication (1988).

[11] G. Monge, "Mémoire sur la théorie des déblai et des remblai," *Histoire de l'Academie Royale des Sciences (année 1781)* (Paris, 1784) pp. 666–704.

[12] C.H. Papadimitriou and K. Steiglitz, *Combinatorial Optimization* (Prentice-Hall, Englewood Cliffs, NJ, 1982).

[13] J. Valdes, "Parsing flowcharts and series-parallel graphs," Ph.D. Thesis, Stanford University (Stanford, CA, 1978).