A General Theory for the Equilibrium and Stability of Discrete Conservative Systems

By JOHN M. T. THOMPSON, Dept. of Civil Engineering, University College, London, Great Britain

1. Introduction

1. Historical Notes

Problems of instability play a key role in structural engineering, and it can be argued that almost all structural failures contain an element of instability. Indeed, even the necking of a tensile specimen is strictly a form of instability. This central role will inevitably increase as structural designs become more efficient, since 'optimum' designs, possessing often high degrees of symmetry, are by their very nature liable to fail by catastrophic instabilities.

The wide field of structural stability contains a substantial core of wholly elastic problems, generated in particular by the stringent demands of aeronautics and astronautics, and it is a general theory of elastic stability for conservative systems that is presented in this article. The response of these systems has an essential nonlinear character that was first examined in general terms by KOITER [1] in his classic dissertation at Delft in 1945. This general study was presented in the context of continuum elasticity, and it has been further developed by KOITER [2–13] and also by BUDIANSKY and HUTCHINSON and their associates [14–29].

A second line of study in the general nonlinear theory of elastic stability was initiated by the present author in 1963 in terms of generalized coordinates [30], following the linear studies of ZIEGLER [31, 32]. This approach has been extensively developed at University College London under Professor A. H. CHILVER [33–71], the work of ROORDA being worthy of particular note, and further significant developments have been made by SEWELL [72–75]. In this work four discrete critical points which are central to the field of elastic stability have been studied in great detail [THOMPSON [30, 40], etc.), and they form the subject of this article, a brief guide to the theoretical developments being given in the last section of this introduction.

2. Structural System

A conservative *structural* system described by the total potential energy function $V(Q_i, \Lambda)$ is here considered, where Q_i represents a set of n generalized coordinates and Λ is a 'loading' parameter. The work thus represents a contribution to the classical mechanics of discrete finite systems, although the most significant applications lie in conservative problems of nonlinear continuum elasticity. In this continuum field the theory is immediately valid for the discrete systems generated in many practical studies

by a finite and therefore approximate *modal* analysis, whether the mode-forms be continuous as in early analyses or localized and discontinuous as in the finite-element Rayleigh-Ritz procedure (FRAEIJS DE VEUBEKE [76], THOMPSON [50]). It is moreover tacitly assumed that the theory remains valid for the discrete systems generated in many classical continuum studies by an infinite modal analysis associated with a complete set of functions.

With these two fields of application in mind, the essential features of the work are developed in two complementary forms which together yield a deep insight into the structure of the general theory. The theory is thus presented first using the basic generalized coordinates without resort to a scheme of diagonalization, this development possessing a high degree of invariance and being readily applicable to the approximate finite-element analyses. Use is then made of a set of principal coordinates generated by a scheme of diagonalization, this development of the general theory yielding explicit solutions which can be directly employed in many classical studies. Examples in these two fields of application can be found in the work of THOMPSON and WALKER [57, 68].

In the general theory the total potential energy is taken to be a function both of the generalized coordinates and of the distinct algebraic variable Λ, the equilibrium and stability of the system being discussed at different but constant values of this variable. In specific applications this 'loading' parameter Λ can thus be equated to any controlled variable appearing in the energy function whose influence on the system we wish to discuss. Often Λ will be associated with the magnitude of prescribed forces acting on a system, but it might equally be identified as a dimension or an elastic modulus of the system, or as the magnitude of a prescribed displacement independent of the generalized coordinates (THOMPSON [34, 77]).

The class of system for which the total potential energy is linear in the loading parameter Λ exhibits some distinct properties and is treated in some detail as a specialization of the general system. In this treatment we shall, for purely semantic reasons, replace Λ by P and then write $V(Q_i, P) = U(Q_i) - P\,\mathcal{E}(Q_i)$. Here the $U(Q_i)$ can be regarded as a *generalized strain energy*, while P can be regarded as the magnitude of a *generalized force* acting through the *generalized displacement* $\mathcal{E}(Q_i)$. In an example of the stability of a boat (THOMPSON [40]) the height of the centre of gravity appears as a generalized force while the masses of the boat and ocean generate a generalized strain energy.

Returning to the general structural system described by the total potential energy function $V(Q_i, \Lambda)$ we shall be concerned with its statical equilibrium and its stability in the sense of LIAPOUNOFF [78], and to make the work self-contained we start by introducing two basic axioms. The first states that a stationary value of the total potential energy with respect to the generalized coordinates is necessary and sufficient for the equilibrium of the system. The second axiom states that a complete relative minimum of the total potential energy with respect to the generalized coordinates is necessary and sufficient for the stability of an equilibrium state of the system.

Associating the basic variables Q_i and Λ with rectangular axes in an $n+1$ dimensional EUCLIDEAN space we see that the n equilibrium equations specifying a stationary value of V with respect to the Q_i will define a series of equilibrium paths in this space, and it is the form and stability of these paths that will concern us in the

present work. In particular we shall be interested in the initial loss of stability of a structural system as it is 'loaded' with increasing Λ from an initial stable equilibrium state which we can without any loss of generality take to be the origin $Q_i = \Lambda = 0$.

3. *Four Critical Points*

In the light of the two basic axioms it is clear that there will be a close relationship between the equilibrium configurations and the stability of the system, and four basic transformations are shown *schematically* in figures 1 to 5. Here heavy lines represent equilibrium paths, continuous lines representing stable paths and broken lines representing unstable paths; plots of the total potential energy at various prescribed values of Λ are shown in lighter lines.

In the *limit point* of figure 1 the initially stable equilibrium path from the origin loses its stability on reaching the locally maximum value of the loading parameter Λ^C. This point arises in the mathematical treatment as the 'most general' stability phenomenon, and it is well-known to structural engineers in the response of shallow arches and domes.

At a prescribed value of $\Lambda = \Lambda^1$ less than Λ^C the total potential energy $V[Q_i, \Lambda^1]$ has a minimum with respect to Q_i on the stable rising region of the path and a maximum on the unstable falling region. As the prescribed value of Λ is increased the maximum and minimum coalesce so that at $\Lambda = \Lambda^C$ the total potential nergey $V[Q_i, \Lambda^C]$ has a horizontal point of inflexion at the critical equilibrium state. At the higher value of $\Lambda = \Lambda^2$ there are no local equilibrium states and the total potential energy $V[Q_i, \Lambda^2]$ has no local stationary point as shown. The critical equilibrium state is seen to be itself unstable, and the absence of local equilibrium states at values of Λ greater than Λ^C implies that a physical system under slowly increasing Λ must snap dynamically from the critical equilibrium state as indicated by the heavy arrow.

The phenomenon of the limit point has been drawn schematically in two dimensions in figure 1 on a plot of Λ against one of the Q_i and a more general schematic diagram is shown in three dimensions in figure 2 on a plot of Λ against two of the Q_i.

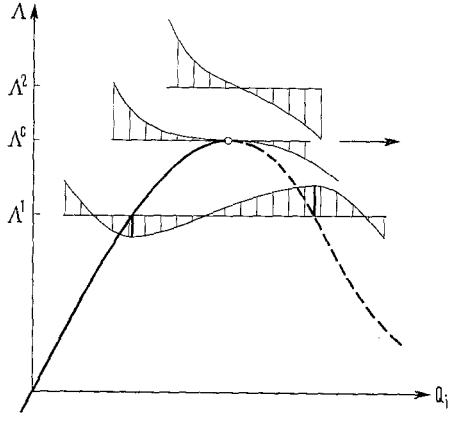

Figure 1
Energy curves in the limit point.

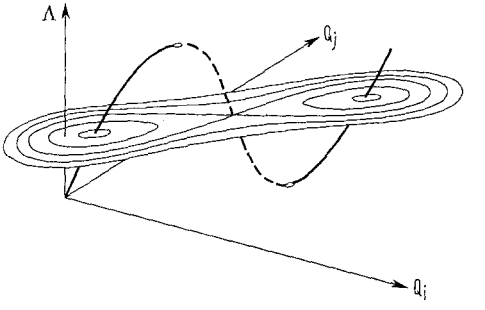

Figure 2
Energy surface in the limit point.

Here a remote rising region of the equilibrium path has been added and contours of total potential energy are drawn for a prescribed value of Λ less than the critical value.

Figures 1 and 2 yield considerable insight into the limiting phenomenon, but it must be remembered that we are dealing with a system with n degrees of freedom and care must be taken in drawing conclusions from these *schematic* figures. On an *actual* plot of Λ against one of the Q_i the limit point is for example *normally* seen as a smooth maximum, but it must be realized that for a *certain* choice of the coordinate Q_i the point *might* appear as a sharp cusp. The smooth maximum of a path in three-dimensional space can for example be seen as a cusp if the eye is directed along the horizontal tangent to the path.

For the specialized system in which V is linear in the loading parameter the limit point is seen as a smooth maximum on a plot of the generalized force P against its corresponding deflection $\mathcal{E}(Q_i)$ as shown in figure 6.

The second configuration of interest is the *asymmetric point of bifurcation* shown schematically in figure 3. Here the initially stable *fundamental* equilibrium path from the origin loses its stability on intersecting a distinct and continuous *post-buckling* equilibrium path. Both paths have a non-zero slope at the critical point, and with varying Λ the paths exhibit the *exchange of stability* discussed by Poincaré in 1885. At a prescribed value of $\Lambda = \Lambda^1$ less than the critical value Λ^C the total potential

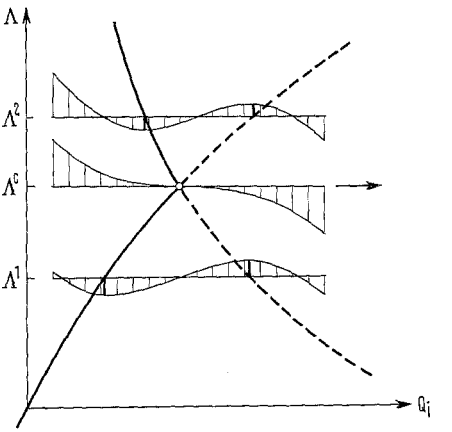

Figure 3
Energy curves in the asymmetric point of bifurcation.

energy $V[Q_i, \Lambda^1]$ has a minimum with respect to Q_i on the stable region of the fundamental path and a maximum with respect to Q_i on the unstable region of the post-buckling path. As the prescribed value of Λ is increased the maximum and minimum coalese so that at $\Lambda = \Lambda^C$ the total potential energy $V[Q_i, \Lambda^C]$ has a horizontal point of inflexion at the critical equilibrium state, after which a maximum and a minimum reappear as shown. The critical equilibrium state is itself unstable, so in the presence of small disturbances a physical system under slowly increasing Λ would snap dynamically from this critical state as indicated by the heavy arrow, despite the existance of stable equilibrium states at higher values of Λ.

As before we must remember that the system under consideration has n degrees of freedom, and care must be exercised in drawing conclusions from this schematic figure. Thus on an actual plot of Λ against one of the Q_i the asymmetric point of bifurcation is normally seen as the *direct* intersection of two equilibrium paths, but for a certain choice of the coordinate Q_i the two equilibrium paths might *touch* each other tangentially. The direct intersection of two paths in three-dimensional space can for example be seen as the touching of two paths if the eye is directed along the limiting horizontal segment.

Such a situation does indeed arise for the specialized system in which V is linear in the loading parameter, the two paths of the asymmetric point of bifurcation being seen to touch tangentially on a plot of the generalized force P against its corresponding deflection $\mathcal{E}(Q_i)$ as shown in figure 6.

The third configuration of interest is the *stable-symmetric point of bifurcation* shown schematically in figure 4. Here the initially-stable fundamental equilibrium path from the origin loses its stability on intersecting a distinct and continuous post-buckling path. The fundamental path has a non-zero slope at the critical point, while the stable rising post-buckling path has a zero slope at the point of intersection. At a prescribed value of $\Lambda = \Lambda^1$ less than the critical value Λ^C the total potential energy $V[Q_i, \Lambda^1]$ has a single stationary value with respect to Q_i, namely a minimum on the

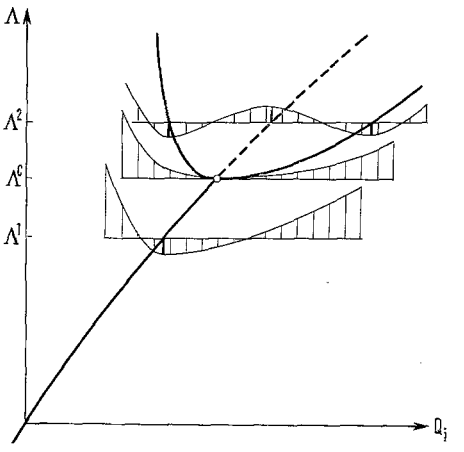

Figure 4
Energy curves in the stable-symmetric point of bifurcation.

stable region of the fundamental equilibrium path, and as the value of Λ is increased this minimum is transformed into two minima and a maximum as shown. The critical equilibrium state is itself stable, so a physical system under slowly increasing Λ would exhibit no dynamic snap but would follow the stable rising post-buckling path, the direction taken depending on the small disturbances assumed to be present.

An actual plot of Λ against one of the Q_i would normally have the appearance of the figure, but for a certain choice of the coordinate Q_i the post-buckling path might appear as a cusp striking the fundamental equilibrium path. Such a cusp does in fact arise on a plot of P against $\mathcal{E}(Q_i)$ for the specialized system as shown in figure 6.

The fourth configuration of interest is the *unstable-symmetric point of bifurcation* shown schematically in figure 5. Here the initially-stable fundamental equilibrium path from the origin again loses its stability on intersecting a distinct and continuous post-buckling path. The fundamental path has a non-zero slope at the critical point while the unstable falling post-buckling path has a zero slope at the point of intersection. At a prescribed value of $\Lambda = \Lambda^1$ less than the critical value Λ^C the total potential energy $V[Q_i, \Lambda^1]$ has three stationary values with respect to Q_i, namely two maxima on the unstable post-buckling path and a minimum on the stable region of the fundamental path. These three stationary points transform into a single

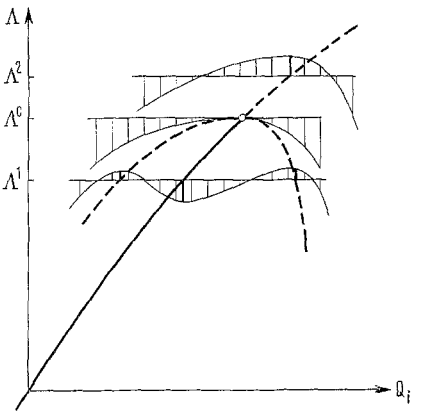

Figure 5
Energy curves in the unstable-symmetric point of bifurcation.

maximum with increasing Λ as shown. The critical equilibrium state is itself unstable, so a physical system under slowly increasing Λ would snap dynamically from the critical equilibrium state, the direction taken depending on the small disturbances assumed to be present.

An actual plot of Λ against one of the Q_i would normally have the appearance of the figure, but for a certain choice of the coordinate Q_i the post-buckling path might again appear as a cusp striking the fundamental equilibrium path. Such a cusp does in fact arise on a plot of P against $\mathcal{E}(Q_i)$ for the specialized system as shown in figure 6.

A word of explanation must be added here concerning the definition of the two symmetric points of bifurcation. For generality of treatment the terms stable-symmetric and unstable-symmetric are applied when the post-buckling equilibrium path has a zero slope, that is a zero first derivative, at the point of intersection, the former being used when the second derivative of this path is positive and the latter when it is negative. There is thus *no overall symmetry requirement* and the third derivative will in general be non-zero at the point of intersection. In many practical problems these symmetric points of bifurcation do however arise under conditions of overall symmetry and the general theory is then of course still applicable, certain terms simply vanishing identically. A common consequence of some overall symmetry arises for example for either of these two symmetric points of bifurcation in the $P - \mathcal{E}$

plot of figure 6, the two limbs of the post-buckling path being often completely superimposed so that they appear as one curve.

As we have mentioned, the limit point is well-known in the response of laterally-loaded shallow arches and domes. Limit points are moreover generated by imperfections in structures which nominally fail at a point of bifurcation, and an experimental study of this is reported by THOMPSON [79].

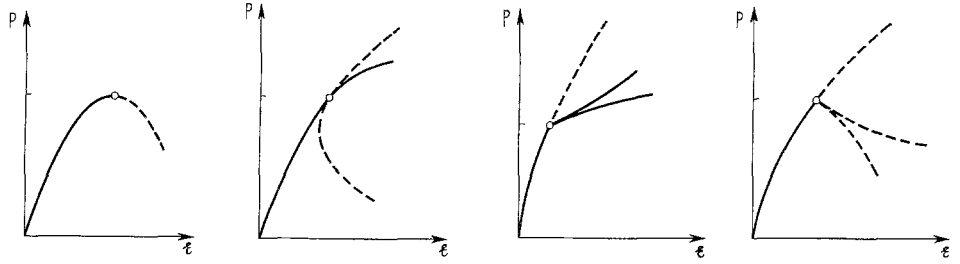

Figure 6

Load versus corresponding deflection for the specialized system in the limit point, the asymmetric and the two symmetric points of bifurcation.

In comparison with the symmetric points of bifurcation the asymmetric point is relatively rare, and the reason for this is not hard to find: bifurcations are usually generated by some degree of physical symmetry, and this symmetry is frequently sufficient to guarantee a symmetric point of bifurcation. The asymmetric point of bifurcation does however arise in the *rotationally-symmetric* buckling of a complete spherical shell under external pressure (THOMPSON [35]) and in the buckling of oblate spheroidal shells under external pressure (DANIELSON [24]). It also arises in the buckling of braced (BRITVEC and CHILVER [33]) and unbraced (GODLEY and CHILVER [46]) rigid-jointed frames, and some experiments on the former will shortly be presented.

The symmetric points of bifurcation, whether stable or unstable, arise throughout the spectrum of structural stability problems.

Simple struts and rings (CARRIER [80]) normally yield stable-symmetric points of bifurcation, while a strut on an elastic foundation (LEKKERKERKER [81], SLITER and BORESI [82]) can yield either a stable or an unstable symmetric point of bifurcation. Laterally-loaded arches frequently yield an unstable-symmetric point of bifurcation before the limiting maximum of a nonlinear fundamental equilibrium path (FUNG and KAPLAN [83]).

Simply-stiff pin-jointed frames yield stable and unstable symmetric points of bifurcation (BRITVEC [84, 85]).

Plates loaded in their plane normally yield stable-symmetric points of bifurcation while axially compressed cylindrical panels (KOITER [2], POPE [86]) exhibit both stable and unstable points of bifurcation, approaching the stable behaviour of plates and the unstable behaviour of cylindrical shells as the curvature parameter is varied.

The initial post-buckling behaviour of a wide variety of thin elastic shells has recently been studied by BUDIANSKY and HUTCHINSON and their associates at Harvard [14–29], and by virtue of their physical symmetry most of the shells considered were observed to exhibit symmetric points of bifurcation. Thus cylindrical

shells under torsion (BUDIANSKY [19]) and external pressure (BUDIANSKY and AMAZIGO [23]), toroidal shell segments (HUTCHINSON [21]), eccentrically-stiffened cylindrical shells (HUTCHINSON and AMAZIGO [22]) and oval cylindrical shells under axial compression (HUTCHINSON [27]) all exhibit discrete symmetric points of bifurcation, both stable and unstable depending on the parameters of the shell.

The study of the oval cylindrical shell is particularly worthy of note in demonstrating the value of bifurcation studies: previous large deflection analyses were seen to have completely missed significant local behaviour close to the point of bifurcation.

The shallow spherical dome under a point load (FITCH [25]) yields a discrete stable-symmetric point of bifurcation from a nonlinear fundamental path, while the same dome under uniform external pressure (FITCH and BUDIANSKY [26]) yields a discrete unstable-symmetric point of bifurcation.

Concluding our discussion of shell buckling we might finally note that the two notorious problems of shell buckling, namely the complete spherical shell under uniform external pressure and the long axially-compressed cylindrical shell, are both associated with simultaneous as opposed to discrete critical points (KOITER [5, 13], HUTCHINSON [20]).

The fine experimental work of ALMROTH, HOLMES and BRUSH [87] on the buckling of cylindrical shells under axial compression is however of interest, these workers employing accurate shell specimens fabricated by electrodeposition, following THOMPSON [88]. They used a mandrel inside the specimens to minimize plasticity and obtained different but repeatable buckling loads for the various specimens, the loads lying roughly between one half and three-quarters of the bifurcation loads of the corresponding perfect cylinders. They concluded that initial imperfections in the specimens rather than dynamic disturbances from the (laboratory) environment were the cause of the discrepancy between the experimental results and the bifurcation loads. They thus confirm the viewpoints of KOITER, BUDIANSKY and HUTCHINSON and the present work that the notorious discrepances in the buckling of thin shells are largely attributable to initial imperfections in the test specimens. The high-speed photographs taken during the buckling process (without a mandrel) are of further interest in showing that the initiation of buckling is associated with short-wavelength ripples which transform into the well-known large-amplitude diamond pattern during the dynamic snap.

4. Initial Imperfections

Structural systems can never be built precisely as planned and physical systems inevitably contain small *imperfections* associated with such things as geometrical errors and material defects. These imperfections can, as we have seen, drastically change the response of the system and the final section of the general theory is concerned with the loss of stability of an imperfect system derived as a perturbation of the 'perfect' system of the previous sections.

In this study a family of systems is generated by introducing the perturbation or *imperfection parameter* ε into the total potential energy function to give us the algebraic function $V(Q_i, \Lambda, \varepsilon)$. The system corresponding to $\varepsilon = 0$ is described as the *perfect system*, it being understood that this was the system of our earlier discussion, and systems corresponding to non-zero values of ε are described as *imperfect systems*.

Considering then the four discrete critical points in turn, *typical* equilibrium path configurations are shown on a plot of Λ against one of the Q_i in figures 7 to 10. Here heavy lines represent the equilibrium paths of the perfect system while light lines represent the equilibrium paths of the imperfect systems, continuous lines representing stable equilibrium paths and broken lines representing unstable equilibrium paths.

Figure 7 relates to the limit point, and we see that the response of an imperfect system is here not dissimilar from that of the corresponding perfect system. The peak or *failure load* Λ^M naturally varies with the imperfection parameter ε, and this variation is shown in the right-hand diagram of the figure, the function $\Lambda^M(\varepsilon)$ having normally a finite and non-zero slope and no singularity at the critical point.

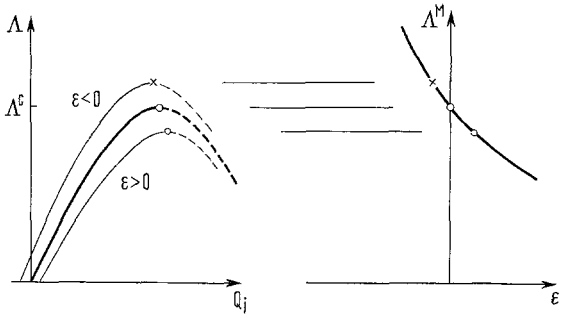

Figure 7
Imperfections in the limit point.

Figure 8 shows typical pictures for the asymmetric point of bifurcation, and we see that imperfections now play a significant role in changing the response of the system. Thus we see that for a small positive value of ε the system loses its stability at a limit point at a considerably reduced value of the loading parameter, while a system with a small negative value of ε exhibits no instability in the vicinity of the critical point and follows a rising stable path. The variation of the failure load Λ^M with the imperfection parameter ε is now of considerable interest and is shown in the right-hand diagram. For positive imperfections the function $\Lambda^M(\varepsilon)$ is locally parabolic, having no singularity at the critical point but having an infinite slope as shown; there is thus an *extreme sensitivity* to initial positive imperfections. Systems with negative

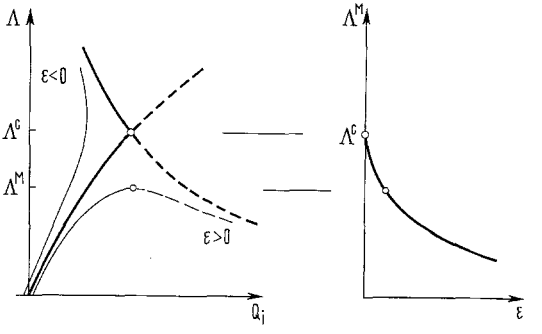

Figure 8
Imperfections in the asymmetric point of bifurcation.

imperfections having no local failure loads, their 'buckling' is simply characterized by a more-rapid growth of the deflections as the critical load of the perfect system is approached. There is thus no local branch of the $\Lambda^M(\varepsilon)$ curve for negative values of ε.

Figure 9 shows a typical picture for the stable-symmetric point of bifurcation, and we see that imperfections here play a less significant role in changing the response of the system. Small positive and small negative imperfections play similar roles, each yielding a continuously stable and rising equilibrium path as shown. Thus imperfect systems have no local failure load, 'buckling' being simply characterized by a more-rapid growth of the deflections as the critical load of the perfect system is approached.

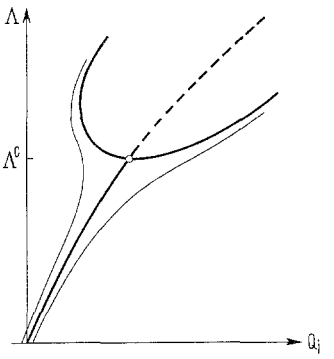

Figure 9
Imperfections in the stable-symmetric point of bifurcation.

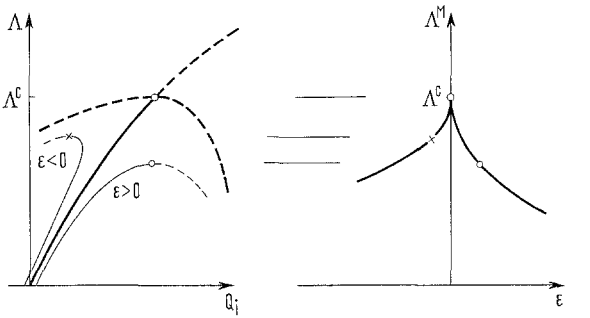

Figure 10
Imperfections in the unstable-symmetric point of bifurcation.

Figure 10 shows typical pictures for the unstable symmetric point of bifurcation, and we see that imperfections are here playing a significant role in changing the response of the system. Small positive and small negative imperfections again play similar roles, each now inducing failure at a limit point at a considerably reduced value of Λ. The variation of the failure load Λ^M with the imperfection parameter ε is now of considerable interest and is shown in the right-hand diagram. For both positive and negative imperfections the function $\Lambda^M(\varepsilon)$ now follows locally a two-thirds power law which yields a sharp cusp as shown, there being thus an *extreme sensitivity* to both positive and negative initial imperfections.

Many specific analyses of imperfection-sensitivity can be found in the work associated with KOITER [1–13] and BUDIANSKY and HUTCHINSON [14–29].

5. Experimental Study

The three discrete *branching* points were first studied experimentally by ROORDA [38] at University College, and some of his fine experimental results are reproduced in figures 11 and 12. A fairly 'rigid' loading system (THOMPSON [77]) was employed throughout the experiments, so that equilibrium states which would have been unstable under dead loading were easily studied.

Figure 11 shows the asymmetric branching behaviour of a simple rigid-jointed frame made from high-tensile steel strip to ensure elastic behaviour throughout. The

load P is applied nominally at the rigid joint, and a family of imperfect systems is generated by displacing its point of application a small distance f along the horizontal member as shown.

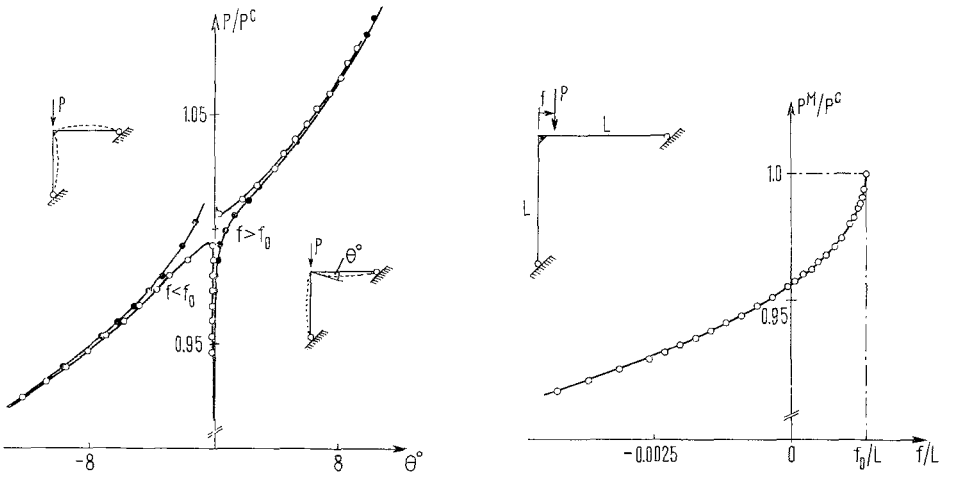

Figure 11

Experimental study of the asymmetric point of bifurcation.

It was found that a small positive value of $f = f_0$ was needed to cancel the effect of unavoidable geometrical and other imperfections in the frame, so we can identify $\varepsilon = f - f_0$ as an appropriate imperfection parameter when viewing the results within the framework of the present general theory. The equilibrium paths, shown on a plot of P/P^C against the rotation θ, represent the response of two imperfect systems, the solid circles corresponding to an imperfect system with $f > f_0$ and the open circles corresponding to an imperfect system with $f < f_0$. For each of these two imperfect systems the *natural* equilibrium path rising from the origin of coordinates is shown, together with the *complementary* equilibrium path which would not be encountered in a natural loading sequence but which can be readily located experimentally by manually reversing the deformation mode at high values of the load P. The curves of these two imperfect systems can be seen to band the response of the idealized perfect system for which f is precisely equal to f_0. The imperfection-sensitivity of the system was studied experimentally and is shown in the plot of P^M/P^C against f/L. The predicted initial parabolic variation is clearly seen, and a theoretical analysis of the imperfection-sensitivity of this simple frame by KOITER [10] has been shown to be in excellent agreement with these experimental results. Further pictures of an asymmetric point of bifurcation are shown in figure 6 of [38] for a similar rigid-jointed frame, the plot of P^M/P^C against f/L being here extended above the critical value of unity by generalizing P^M to include the minima of the complementary equilibrium paths.

Figure 12 shows the unstable-symmetric branching behaviour of a simple rigid-jointed arch made of high-tensile steel strip. Once again the load P is applied nominally at the rigid joint, and a family of imperfect systems is generated by displacing its point of application a small distance f as shown.

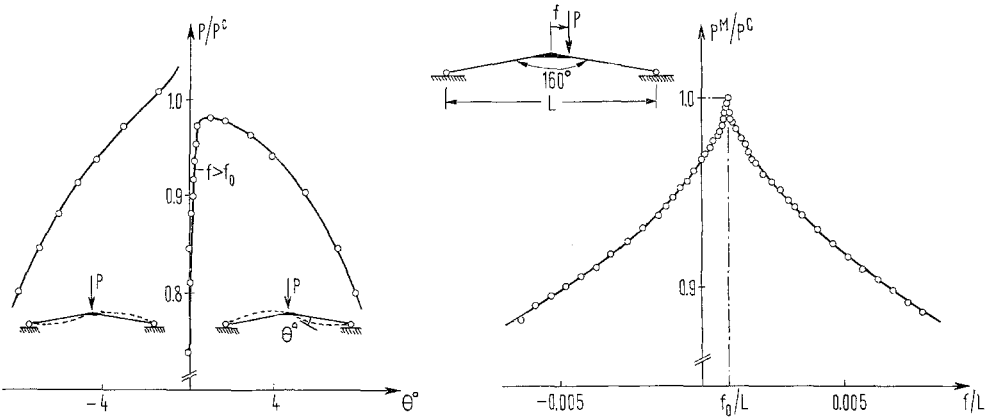

Figure 12
Experimental study of the unstable-symmetric point of bifurcation.

A small positive value of $f = f_0$ was needed to cancel the effect of unavoidable imperfections in the frame, and we can again identify $\varepsilon = f - f_0$ as an appropriate imperfection parameter. The equilibrium paths on a plot of P/P^C against θ here represent the response of a single imperfect system with $f > f_0$, the natural and the complementary paths being both located. The imperfection-sensitivity of the system is shown in the plot of P^M/P^C against f/L, and the expected two-thirds-power-law cusp is clearly seen.

6. *Outline of Theoretical Developments*

We conclude this introduction by outlining briefly the theoretical developments of the article.

Section 2 is concerned with the stability of an equilibrium state and the appropriate generalization, namely the stability of an equilibrium path, and is an extension of the author's initial paper [30].

Section 3 is concerned with the *normal* response of a system in the vicinity of a non-critical equilibrium state. The unique equilibrium path passing through such a state is studied using the form of perturbation analysis developed by SEWELL and THOMPSON ([72, 40] et seq) which has been used advantageously in the nonlinear analysis of continuous structures using the finite element Rayleigh-Ritz procedure (THOMPSON and WALKER [57], WALKER and HALL [59]). A number of new theorems are presented.

In Section 4 the possible responses of the system in the vicinity of a discrete critical point are examined and classified following [30], and the spatial forms of the four basic configurations are studied.

Section 5 represents a more-detailed study of the branching points, firstly without a scheme of diagonalization following [64] and [68] and secondly with such a scheme following the earlier work of the author [40]. The general analysis without a scheme of diagonalization is invaluable in studying the post-buckling of continuous structures using the finite element Rayleigh-Ritz procedure, and two applications have been presented by WALKER [58, 70].

In Section 6 the response of imperfect systems is discussed and the modest degree of imperfection-sensitivity associated with a limit point is demonstrated for the first time. The response of imperfect systems in the vicinity of the discrete branching points of section 5 is then studied and the asymptotic equations of imperfection-sensitivity are derived in terms of the post-buckling derivatives of the corresponding perfect systems both with and without the aid of a scheme of diagonalization following the authors earlier analyses [40, 64].

2. Stability of Equilibrium

1. Structural System

We consider a conservative *structural* system, that is to say a simple scleronomic and holonomic mechanical system (SYNGE and GRIFFITH [89]) described by the single-valued potential energy function $V(Q_i, \Lambda)$ which is assumed to be continuous and well-behaved. Here Q_i represents a set of n generalized coordinates and Λ represents a loading parameter. Such an energy function might arise directly from a discrete mechanical system, or indirectly from a continuous structural system rendered discrete by a modal analysis. The parameter Λ, although referred to as a loading parameter, is essentially any parameter of the system whose influence we wish to study.

To make the work self-contained, we now introduce two basic axioms for the system concerning its statical equilibrium and its stability in the sense of LIAPOUNOFF [78].

Axiom I

A stationary value of the total potential energy with respect to the generalized coordinates is necessary and sufficient for the equilibrium of the system.

This axiom can of course be proved from Newton's laws of motion for a simple conservative mechanical system.

Axiom II

A complete relative minimum of the total potential energy with respect to the generalized coordinates is necessary and sufficient for the stability of an equilibrium state of the system.

This axiom cannot be proved in *complete* generality for a simple conservative mechanical system, and a recent appraisal of the situation is presented by KOITER [7]. It seems clear however that a complete proof will eventually be found, and in the meantime we follow Koiter by expressing our complete faith in the axiom.

Associating the basic variables Q_i and Λ with rectangular axes in an $n+1$-dimensional Euclidean space we see that the n equilibrium equations specifying a stationary value of V with respect to the Q_i will define a series of equilibrium paths in this space, and it is the form and stability of these paths that will concern us in the present work.

We consider finally an important specialization of the preceding general system which can be made when the original total potential energy function $V(Q_i, \Lambda)$ is linear in the loading parameter Λ. When this linearity holds we shall, for purely

semantic reasons, replace Λ by P and then write

$$V(Q_i, P) = U(Q_i) - P\,\mathcal{E}(Q_i)\,.$$

Here the function $U(Q_i)$ can be regarded as a *generalized strain energy*, while P can be regarded as the magnitude of a *generalized force* acting through the *generalized displacement* $\mathcal{E}(Q_i)$.

2. Stability of an Equilibrium State

We consider now the stability of an equilibrium state

$$Q_i = Q_i^E\,,\quad \Lambda = \Lambda^E\,,$$

satisfying the n equilibrium equations

$$V_i[Q_j^E, \Lambda^E] = 0\,,$$

a subscript on V denoting partial differentiation with respect to the corresponding generalized coordinate.

Considering the increments from this state defined by

$$Q_i = Q_i^E + q_i\,,\quad V = V^E + v\,,$$

where

$$V^E = V[Q_i^E, \Lambda^E]\,,$$

we first write the Taylor expansion with respect to the generalized coordinates

$$v = \frac{1}{2}\,V_{ij}^E\,q_i\,q_j + \frac{1}{6}\,V_{ijk}^E\,q_i\,q_j\,q_k + \cdots$$

where

$$V_{ij}^E \equiv V_{ij}[Q_k^E, \Lambda^E]\,,$$

etc. The basic state being one of equilibrium, there is no linear term in this expansion. The dummy-suffix summation convention is employed, with all summations ranging from one to n.

The stability of the equilibrium state will depend in the first instance on the quadratic form

$$v^2 = \frac{1}{2}\,V_{ij}^E\,q_i\,q_j$$

and we start by reducing this to a diagonal form by means of a nonsingular linear transformation

$$q_i = \alpha_{ij}\,u_j \quad |\alpha_{ij}| \neq 0$$

with the inverse

$$u_i = \beta_{ij}\,q_j \quad |\beta_{ij}| \neq 0\,.$$

In the new coordinates we can write

$$v^2 = \frac{1}{2}\,C_i\,u_i^2$$

where the C_i are constants.

We now define a new energy function

$$D(u_i, \Lambda) \equiv V\,[Q_i^E + \alpha_{ij}\, u_j, \Lambda]$$

and writing

$$D = D^E + d$$

where

$$D^E \equiv D[0, \Lambda^E]$$

we expand D about the equilibrium state to give

$$d = \frac{1}{2}\, D_{ii}^E\, u_i^2 + \frac{1}{6}\, D_{ijk}^E\, u_i\, u_j\, u_k + \cdots .$$

Here subscripts on D denote partial differentiation with respect to the corresponding u_i, and the superscript E denotes evaluation at the equilibrium state. We see that

$$C_i = D_{ii}^E = D_{ii}[0, \Lambda^E]\,,$$

and

$$|\alpha_{ij}|^2 \cdot |V_{ij}^E| = |D_{ij}^E| = C_1\, C_2\, C_3\, C_4 \ldots C_n\,.$$

The quadratic form v^2 can be diagonalized in an infinite number of ways, and we shall suppose that one such way has been chosen, noting that the invariant properties of v^2 manifest themselves as invariant properties of the C_i. The number of positive C_i, the number of negative C_i and the number of zero C_i are independent of the choice of the diagonalizing transformation.

The coordinates u_i and the coefficients C_i play an important role in the following theory, and we shall call the u_i a set of *principal coordinates* and the C_i a set of *stability coefficients*.

We use these to make *qualified* statements about the stability of the equilibrium state. Thus if a particular stability coefficient C_r is positive we shall say that the equilibrium state is stable *with respect to* the corresponding principal coordinate u_r. If a particular stability coefficient C_s is negative we shall say that the equilibrium state is unstable *with respect to* the corresponding principal coordinate u_s. If a particular stability coefficient C_t is zero we shall say that the equilibrium state is *critical with respect to* the corresponding principal coordinate u_t.

The number of negative stability coefficients is called the *degree of instability*.

An equilibrium state is called *critical* without qualification if it is critical with respect to one or more of the principal coordinates, and we see that the two determinants $|V_{ij}|$ and $|D_{ij}|$ will vanish in a critical equilibrium state.

If the lowest stability coefficient is positive the equilibrium state is stable with respect to all of the principal coordinates and v^2 is *positive definite*. The equilibrium state is then clearly stable, and in contrast to critically stable equilibrium states whose stability depends on higher variations of V, we might say that the state is *thoroughly stable*.

If the lowest stability coefficient is negative the equilibrium state is unstable with respect to one or more of the principal coordinates and v^2 *admits negative values*. The equilibrium state is then clearly unstable, and in contrast to the critically unstable

equilibrium states whose instability depends on higher variations of V, we might say that the state is *thoroughly unstable.*

If the lowest stability coefficient is zero, the equilibrium state is critical with respect to one or more of the principal coordinates and stable with respect to the rest. The quadratic form v^2 is then *positive semi-definite* and supplies no decision about the stability of the equilibrium state. Higher-order terms in the expansion of v must then be examined, and we shall pursue this for a *discrete* critical point denoted by C at which a single stability coefficient is zero.

We consider then the case in which $C_1 = 0$ while $C_s > 0$ for $s \neq 1$. The quadratic form v^2 is thus positive semi-definite, being zero in the *critical direction* defined by

$$u_s = \beta_{sj}\, q_j = 0 \quad (\text{all } s \neq 1) .$$

We employ first the basic coordinates Q_i, and keeping Λ fixed at the value Λ^C we shall examine the variation of V along all coordinate paths leading from the equilibrium state. To do this we write a general path from the equilibrium state in the parametric form

$$Q_i = Q_i(s)$$

where s is some suitable parameter, vanishing at C, which we need not specify.

The variation of V along this path is then given by the defining equality

$$v(s) \equiv V[Q_i(s), \Lambda^C] - V^C$$

and we expand this variation as a function of s as follows

$$v(s) = \dot{v}^C s + \frac{1}{2}\ddot{v}^C s^2 + \frac{1}{6}\dddot{v}^C s^3 + \cdots$$

where a dot denotes differentiation with respect to s.

The coefficients of this series are readily obtained by the differentiation of the defining equality with respect to s. Thus

$$\dot{v} \equiv \frac{dv}{ds} = V_i\, \dot{Q}_i$$

and evaluating at the critical equilibrium state C we have the expected result

$$\dot{v}^C = 0 .$$

Differentiating a second time and evaluating we have

$$\ddot{v}^C = V_{ij}^C\, \dot{Q}_i^C\, \dot{Q}_j^C .$$

We see that $\ddot{v}^C$ is a positive semi-definite function of the n rates $\dot{Q}_i^C$, taking a (non-trivial) minimum value of zero for the set of rates associated with the critical direction, which is given by the equations

$$\frac{\partial \ddot{v}^C}{\partial \dot{Q}_i^C} = 2\, V_{ij}^C\, \dot{Q}_j^C = 0 .$$

Paths not starting in the critical direction yield positive values for $\ddot{v}^C$, and it remains to study higher variations of $v(s)$ along paths starting in the critical direction.

The third differentiation of the defining equality yields

$$\dddot{v} = V_{ijk}\,\dot{Q}_i\,\dot{Q}_j\,\dot{Q}_k + 3\,V_{ij}\,\ddot{Q}_i\,\dot{Q}_j + V_i\,\dddot{Q}_i$$

giving on evaluation for the *critical set of rates*

$$\dddot{v}^C = V^C_{ijk}\,\dot{Q}^C_i\,\dot{Q}^C_j\,\dot{Q}^C_k\,.$$

If this is non-zero, the critical equilibrium state is clearly unstable. If however this expression is zero we have no decision about the stability of the critical equilibrium state, and higher variations of $v(s)$ must be considered. We shall not pursue this any further in the basic coordinates Q_i, but we now consider the problem afresh in the principal coordinates u_i.

Proceeding as before we write a general coordinate path from the critical equilibrium state C in parametric form, choosing now u_1 as the independent variable as follows

$$u_i = u_i(u_1)\,.$$

Here it is to be understood that $u_1(u_1) = u_1$, and we write

$$u_{i1} \equiv \frac{du_i}{du_1}\,,$$

etc., it again being understood that

$$u_{11} \equiv \frac{du_1}{du_1} = 1\,, \quad u_{111} \equiv \frac{d^2u_1}{du_1^2} = 0\,,$$

etc.

The variation of the total potential energy along this path is given by the defining equality

$$d(u_1) \equiv D[u_i(u_1), \Lambda^C] - D^C$$

and we expand this variation as a function of u_1 as follows

$$d(u_1) = d^C_1 u_1 + \frac{1}{2}\,d^C_{11}u_1^2 + \frac{1}{6}\,d^C_{111}u_1^3 + \cdots$$

where

$$d^C_1 \equiv \left.\frac{dd}{du_1}\right|^C\,,$$

etc.

The coefficients of this series are readily obtained by the differentiation of the defining equality with respect to u_1. Thus

$$d_1 = D_i\,u_{i1}$$

giving at the critical equilibrium state

$$d^C_1 = 0\,.$$

Differentiating a second time we have

$$d_{11} = D_{ij}\,u_{i1}\,u_{j1} + D_i\,u_{i11}$$

giving on evaluation, since D^C_{ij} is a diagonal form with $D^C_{11} = 0$,

$$d^C_{11} = D^C_{ss}(u^C_{s1})^2$$

with a summation over s from two to n.

We see that d_{11}^C is a positive semi-definite function of the rates, being zero for

$$u_{s1}^C = 0 \quad (\text{for all } s \neq 1) ,$$

and we proceed to study higher variations of $d(u_1)$ *under this condition*. Differentiating a third time we find

$$d_{111}^C = D_{111}^C .$$

If this is non-zero the critical equilibrium state C is clearly unstable, while if this is zero we have no decision. We thus proceed to examine the fourth variation of $d(u_1)$ under the condition that $D_{111}^C = 0$.

Differentiating the defining equality a fourth time and evaluating we have

$$d_{1111}^C = D_{1111}^C + 6\, D_{s11}^C\, u_{s11}^C + 3\, D_{ss}^C (u_{s11}^C)^2 ,$$

summations on s ranging from two to n. This can be written as

$$d_{1111}^C = D_{1111}^C - 3\,\frac{(D_{s11}^C)^2}{D_{ss}^C} + 3\, D_{ss}^C \left[u_{s11}^C + \frac{D_{s11}^C}{D_{ss}^C} \right]^2 ,$$

and we see that we now have d_{1111}^C as a function of the $n-1$ second derivatives u_{s11}^C $(s \neq 1)$. This function clearly takes a *minimum* value of

$$d_{1111}^C = D_{1111}^C - 3 \sum_{s=2}^{s=n} \frac{(D_{s11}^C)^2}{D_{ss}^C} , \qquad d_{1111}^C \equiv \tilde{D}_{1111}^C$$

say, when

$$u_{s11}^C = -\frac{D_{s11}^C}{D_{ss}^C} \quad (s \neq 1) .$$

We see that if $\tilde{D}_{1111}^C$ is positive the critical equilibrium state C is stable, while if $\tilde{D}_{1111}^C$ is negative the equilibrium state is unstable. If this coefficient is zero, we have no decision and we must examine higher variations of $d(u_1)$, but we shall not pursue this problem further.

In the following sections we shall see that the stability of a critical equilibrium state, dependent as we have seen on D_{111}^C and $\tilde{D}_{1111}^C$, is closely related to the local form of the equilibrium paths.

3. Stability of an Equilibrium Path

Having studied the stability of an equilibrium state we shall now consider the appropriate generalization of the preceding analysis that will allow us to study the stability of an equilibrium path.

We start by writing an equilibrium path of interest in the single-valued parametric form

$$Q_i = Q_i^P(s) , \quad \Lambda = \Lambda^P(s)$$

where s is any suitable parameter specifying uniquely the position on the path: one suitable choice of s would of course be the distance along the path. We now introduce a 'sliding' set of axes q_i by means of the transformation

$$Q_i = Q_i^P(s) + q_i .$$

We further change coordinates by means of the transformation

$$q_i = \alpha_{ij}(s)\, u_j\,, \quad |\alpha_{ij}(s)| \neq 0\,,$$

chosen to diagonalize the quadratic coordinate form of V at every point on the path.

As we have seen before, this diagonalization can be accomplished in an infinite number of ways and we shall suppose that one such way has been chosen, merely insisting that α_{ij} is a continuous and single-valued function of s. In the $\Lambda - Q_i$ space the u_i-axes will thus slide along the path and in general rotate both as a set and relative to each other to give the axes varying orientation and obliquity.

Employing these principal axes we introduce the transformed energy function

$$B(u_i, \Lambda, s) \equiv V\,[Q_i^P(s) + \alpha_{ij}(s)\, u_j\,,\ \Lambda]$$

with the properties arising from its derivation

$$B_i[0, \Lambda^P(s), s] = 0\,, \quad B_{ij}[0, \Lambda^P(s), s] = 0 \quad (i \neq j)\,.$$

If the path is single-valued with respect to Λ in the region of interest, we can set $s = \Lambda$ to obtain the A-transformation. This transformation has proved invaluable in the discussion of branching points (THOMPSON [40]) and will be employed in the later sections. It is defined with reference to a fundamental path F by the equation

$$A(u_i, \Lambda) \equiv V\,[Q_i^F(\Lambda) + \alpha_{ij}(\Lambda)\, u_j, \Lambda]$$

and has analogous properties arising from its derivation.

The stability of any point on the path is now determined by the function

$$B[u_i, \Lambda^P(s), s]$$

and we can write

$$B_{ii}[0, \Lambda^P(s), s] \equiv C_i(s)$$

to obtain a set of stability coefficients which vary continuously along the path. We see that the degree of instability of a path can only change at a critical equilibrium state for which at least one stability coefficient is equal to zero. A change in the degree of instability, and in particular a loss of initial stability, will thus be characterized by the vanishing of the stability determinants $|V_{ij}|$ and $|B_{ij}|$.

We finally apply a B-transformation to each equilibrium path, matching the transformations at any point of intersection (bifurcation) so that at the intersection of two or more paths there will be a unique set of u_i-axes. Difficulties may conceivably arise in this scheme, particularly if the equilibrium paths contain closed loops, and we shall simply assume that the scheme can be accomplished in the immediate neighborhood of any equilibrium state of interest, including those associated with a point of bifurcation.

We observe finally that the B-transformation is indeed a generalization of the D-transformation by observing that it degenerates to the latter when we set $s = s^E$. Thus

$$B[u_i, \Lambda, s^E] = V\,[Q_i^P(s^E) + \alpha_{ij}(s^E)\, u_j, \Lambda] = D(u_i, \Lambda)\,.$$

The concepts developed here will be used in the later sections to discuss the stability changes that occur in the equilibrium paths passing through a critical equilibrium state.

3. Normal Behaviour

1. The General System

We consider now equilibrium solutions in the neighbourhood of a normal, that is to say a non-critical, equilibrium state (Q_j^E, Λ^E).

The n equilibrium equations are

$$V_i[Q_j, \Lambda] = 0 ,$$

and anticipating the existance of an equilibrium path passing through the equilibrium state we write

$$Q_j = Q_j(s) , \quad \Lambda = \Lambda(s) ,$$

with a view to constructing the parametric series solution

$$q_j(s) \equiv Q_j(s) - Q_j^E = \dot{Q}_j(0)\, s + \frac{1}{2}\, \ddot{Q}_j(0)\, s^2 + \cdots$$

$$\lambda(s) \equiv \Lambda(s) - \Lambda^E = \dot{\Lambda}(0)\, s + \frac{1}{2}\, \ddot{\Lambda}(0)\, s^2 + \cdots .$$

Here s might represent any suitable parameter defining progress along the path, but we shall in fact assume that s is to be equated to one of the basic variables $q_j = Q_j - Q_j^E$ or $\lambda = \Lambda - \Lambda^E$, the formulation naturally taking a particularly simple and symmetric form if s is equated to the loading parameter λ. A dot is used to denote differentiation with respect to s.

The equilibrium equations must be satisfied at every point on the equilibrium path, so we can write the characterizing equilibrium identity

$$V_i[Q_j(s), \Lambda(s)] \equiv 0 .$$

The left-hand side of each equation is now an implicit function of s, so we can differentiate each equation with respect to s as many times as we please. Differentiating once we have

$$V_{ij}\, \dot{Q}_j + V_i'\, \dot{\Lambda} = 0 ,$$

where a prime denotes differentiation with respect to Λ and the dummy-suffix summation convention is employed with all summations ranging from one to n. Differentiating a second time we have

$$(V_{ijk}\, \dot{Q}_k + V_{ij}'\, \dot{\Lambda})\, \dot{Q}_j + V_{ij}\, \ddot{Q}_j + (V_{ij}'\, \dot{Q}_j + V_i''\, \dot{\Lambda})\, \dot{\Lambda} + V_i'\, \ddot{\Lambda} = 0 ,$$

etc.

These equations can be evaluated at the equilibrium state (Q_j^E, Λ^E), and we shall refer to the *evaluated* equation obtained from the mth differentiation as the mth-order equilibrium equation. In this equation all path derivatives are evaluated at the equilibrium state E.

Allowing one of the basic variables, Q_j or Λ, to be independent by equating its increment, q_j or λ, to s, and correspondingly replacing its first derivative with respect to s by unity and its higher derivatives with respect to s by zero, we see that we have generated a series of linear problems which can in principle be solved for the remaining path derivatives.

Thus the first-order equilibrium equations represent a set of linear equations in the remaining first derivatives which can thus be found. These first derivatives can be substituted into the second-order equilibrium equations which now represent a set of linear equations in the remaining second derivatives which are thus readily obtained. The known first and second derivatives can be substituted into the third-order equilibrium equations which now represent a set of linear equations in the remaining third derivatives which are likewise readily obtained. In principle all the path derivatives can be obtained sequentially in this manner, and we note that each set of linear equations has the same basic matrix.

If we equate s to the loading parameter λ we obtain the more symmetric equations

$$V_{ij}\, Q_j' + V_i' = 0\,,$$

etc., a prime on the Q_j reminding us that s is now replaced by λ. The determinant of each set of the evaluated equations is now the non-zero stability determinant $|V_{ij}^E|$, so we see that there will be a unique solution for all the path derivatives $Q_j'(0)$, $Q_j''(0)$, $Q_j'''(0), \ldots$. There will thus be a *unique* equilibrium path through the non-critical equilibrium state.

The variation of the stability determinant $\Delta(Q_k, \Lambda) \equiv |V_{ij}(Q_k, \Lambda)|$ along the equilibrium path can be written as

$$\mathcal{D}(s) \equiv \Delta[Q_i(s), \Lambda(s)]$$

so that we have

$$\dot{\mathcal{D}}(s) = \Delta_i \dot{Q}_i + \Delta' \dot{\Lambda}\,,$$

etc., and this variation can be used to estimate the lowest critical load of a system (MASUR and SCHREYER [90], THOMPSON [51]).

The normal behaviour of the system is more readily examined if we employ a diagonalizing transformation, and we shall now repeat the preceding analysis using the D-transformation of the stability analysis.

A new energy function is defined as before by the equation

$$D[u_i, \Lambda] \equiv V\,[Q_i^E + \alpha_{ij}\, u_j\,, \Lambda]$$

the matrix α_{ij} diagonalizing the quadratic coordinate form of the potential energy at the equilibrium state E. The n equilibrium equations are now

$$D_i[u_j, \Lambda] = 0$$

and we write

$$u_j = u_j(s)\,, \quad \Lambda = \Lambda(s)$$

with a view to constructing a parametric series solution as before. Substituting into the equilibrium equations we have the characterizing equilibrium identity

$$D_i[u_j(s), \Lambda(s)] \equiv 0\,,$$

and differentiating repeatedly with respect to s we have

$$D_{ij}\,\dot{u}_j + D_i'\,\dot{\Lambda} = 0\,,$$

etc., as before. Evaluating at the equilibrium state E, the equations now take a particularly simple form since D_{ij}^E is diagonal. Thus the first-order equilibrium equations are simply

$$D_{ii}\,\dot{u}_i + D_i'\,\dot{\Lambda}\,|^E = 0 \quad (i = 1, 2, \ldots, n)\,,$$

there being now no summations involved, while the second-order equilibrium equations can be written as

$$D_{ijk}\,\dot{u}_j\,\dot{u}_k + 2\,D_{ij}'\,\dot{u}_j\,\dot{\Lambda} + D_{ii}\,\ddot{u}_i + D_i''(\dot{\Lambda})^2 + D_i'\,\ddot{\Lambda}\,|^E = 0 \quad (i = 1, 2, \ldots, n)\,.$$

These equations are readily solved once s is prescribed, and setting $s = u_1$ for example, we can immediately write down the corresponding rates. Thus setting $i = 1$ in the first-order equilibrium equation we have

$$\Lambda_1^E \equiv \frac{d\Lambda}{du_1}\Big|^E = -\frac{D_{11}}{D_1'}\Big|^E\,,$$

while setting $i = s \neq 1$ and using this first result we have

$$u_{s1}^E \equiv \frac{du_s}{du_1}\Big|^E = \frac{D_{11}}{D_{ss}}\frac{D_s'}{D_1'}\Big|^E \quad (s = 2, 3, \ldots, n)\,.$$

The second derivatives are readily written down, and we see that the unique equilibrium path passes through the non-critical equilibrium state in a simple fashion.

2. *The Specialized System*

The normal response of the specialized system, for which V is linear in the loading parameter and is written as

$$V[Q_i, P] \equiv U(Q_i) - P\,\mathcal{E}(Q_i)\,,$$

exhibits a number of significant features and will now be examined. The preceeding general developments are of course immediately applicable, and we shall employ the original Q_i coordinate system taking the loading parameter, denoted now by P, as the independent variable.

Thus equating to zero all energy derivatives involving more than one differentiation with respect to the loading parameter (such as V_i''), we have from the earlier work the ordered equilibrium equations

$$V_{ij}\,Q_j' + V_i'\,|^E = 0\,, \quad V_{ijk}\,Q_j'\,Q_k' + 2\,V_{ij}'\,Q_j' + V_{ij}\,Q_j''\,|^E = 0\,,$$

etc. In these equations a prime now denotes differentiation with respect to P.

These equations yield three equalities which will be required in the subsequent analysis. Multiplying each equation of the first set by the corresponding rate $Q_i'^E$ and summing the equations of this set we have the *first equality*

$$V_{ij}\,Q_i'\,Q_j' + V_i'\,Q_i'\,|^E = 0\,.$$

Repeating this operation, again on the first set of equations, but now multiplying by the corresponding second derivatives $Q_i''^E$ we have the *second equality*

$$V_{ij}\, Q_i''\, Q_j' + V_i'\, Q_i''\,|^E = 0\,.$$

Multiplying each equation of the second set by the corresponding rate $Q_i'^E$ and summing the equations of this set we have the *third equality*

$$V_{ijk}\, Q_i'\, Q_j'\, Q_k' + 2\, V_{ij}'\, Q_i'\, Q_j' + V_{ij}\, Q_i'\, Q_j''\,|^E = 0\,.$$

We consider now the variation of the corresponding deflection, $\mathcal{E}(Q_i)$, along the equilibrium path, writing

$$e(P) \equiv \mathcal{E}[Q_i(P)] - \mathcal{E}^E\,.$$

The first derivative of e is given by

$$e' = \mathcal{E}_i\, Q_i' = -V_i'\, Q_i'\,,$$

while the second derivative is given by

$$e'' = \mathcal{E}_{ij}\, Q_i'\, Q_j' + \mathcal{E}_i\, Q_i'' = -V_{ij}'\, Q_i'\, Q_j' - V_i'\, Q_i''\,.$$

Evaluating at the equilibrium state E and using the first equality we find that

$$e'^E \equiv \frac{de}{dP}\Big|^E = V_{ij}\, Q_i'\, Q_j'\,|^E\,,$$

the first result of interest.

We see that if the non-critical equilibrium state E is stable, so that V_{ij}^E is positive definite, the slope e'^E cannot be negative. We thus have the well-known result for the specialized system:

The unique equilibrium path passing through a thoroughly stable equilibrium state cannot have a negative slope on a plot of the generalized force against its corresponding deflection.

Evaluating the second derivative of e at the equilibrium state E and using the second and third equalities we have

$$e''^E \equiv \frac{d^2e}{dP^2}\Big|^E = -V_{ijk}\, Q_i'\, Q_j'\, Q_k' - 3\, V_{ij}'\, Q_i'\, Q_j'\,|^E\,,$$

the second result of interest. We see that we have obtained the second derivative e''^E in terms of the rates $Q_i'^E$, the second derivatives $Q_i''^E$ having been eliminated. We can thus write for the specialized system:

The curvature of the unique equilibrium path passing through a non-critical equilibrium state on a plot of the generalized force against its corresponding deflection is not dependent on the solution of the second-order equilibrium equations.

A consequence of this result, which will now be proved, is that a one-degree-of-freedom nonlinear Rayleigh-Ritz analysis employing the *current incremental displacement pattern*, will yield the correct path curvature on a plot of P against $\mathcal{E}$.

Considering then such an analysis, the assumed displacement pattern will be given by

$$Q_i = Q_i^E + A\, K_i$$

where the K_i are constants related to the true *current rates* by the equations

$$K_i = K\, Q_i'^{E} .$$

Here A is the single free amplitude parameter, and K is an unknown factor. The Rayleigh-Ritz energy function can be written as

$$V[A, P] \equiv V\,[Q_i^E + A\, K_i\,, P]\,,$$

$$U(A) - P\,\mathcal{E}(A) \equiv U\,[Q_i^E + A\, K_i] - P\,\mathcal{E}\,[Q_i^E + A\, K_i]\,,$$

and we have the relations

$$V_A = V_i K_i\,,\quad V_{AA} = V_{ij} K_i K_j\,,\quad V_{AAA} = V_{ijk} K_i K_j K_k\,,\quad V_A' = V_i' K_i\,,\quad V_{AA}' = V_{ij}' K_i K$$

a subscript A denoting differentiation with respect to A.

Now the one-degree-of-freedom *analytical model* generated by the nonlinear Rayleigh-Ritz procedure is described by the energy function

$$V[A, P] = U(A) - P\,\mathcal{E}(A)\,,$$

and we see that this model is itself a member of the class of specialized systems under consideration. The previous results of this section can thus be used to obtain the required Rayleigh-Ritz solutions.

Thus the first-order equilibrium equation of the analytical model is

$$V_{AA}\, A' + V_A' \,|^E = 0$$

giving

$$V_{ij}\, K_i\, K_j\, A' + V_i'\, K_i \,|^E = 0\,.$$

Remembering that $K_i = K\, Q_i'^{E}$, a comparison with the first equality shows that

$$A'^{E} = \frac{1}{K}$$

as we would expect. Employing again the preceeding results, the Rayleigh-Ritz prediction for e'^{E} is

$$e'^{E} = -V_A'\, A' \,|^E = -\frac{V_i'\, K_i}{K}\Big|^E = -V_i'\; Q_i' \,|^E\,,$$

which agrees with the exact solution, while the Rayleigh-Ritz prediction for the second derivative is

$$e''^{E} = -V_{AAA}(A')^3 - 3\; V_{AA}'(A')^2 \,|^E = -V_{ijk}\; Q_i'\; Q_j'\; Q_k' - 3\; V_{ij}'\; Q_i'\; Q_j' \,|^E\,,$$

which likewise agrees with the exact solution. We thus have the required result:

A one-degree-of-freedom nonlinear Rayleigh-Ritz analysis about a non-critical equilibrium state employing the correct incremental displacement pattern will yield the correct current slope and the correct current curvature at the equilibrium state on a plot of the generalized force against its corresponding deflection.

A number of similar theorems can be proved for systems exhibiting various forms of symmetry, but these systems will not be considered here.

4. Critical Behaviour

1. Perturbation Equations

We shall examine now the equilibrium solutions in the vicinity of a discrete critical equilibrium state (Q_i^C, Λ^C) at which a single stability coefficient is zero. We shall employ the diagonalized energy function D defined as before by the equation

$$D(u_i, \Lambda) \equiv V\,[Q_i^C + \alpha_{ij}\, u_j, \Lambda]$$

and we shall suppose that $D_{11}^C = 0$ while $D_{s\,s}^C \neq 0$ for $s \neq 1$.

We seek to express any equilibrium paths emanating from the critical point C in parametric form, and we shall take the critical principal coordinate u_1 as the independent variable, writing the paths as

$$u_j = u_j(u_1)\,, \quad \Lambda = \Lambda(u_1)\,.$$

Substituting these into the equilibrium equation $D_i = 0$ and differentiating repeatedly with respect to u_1 as before, we have the ordered equilibrium equations

$$D_{i\,i}\, u_{i\,1} + D_i'\, \Lambda_1\,|^C = 0\,,$$

$$D_{ijk}\, u_{j1}\, u_{k1} + 2\, D_{ij}'\, u_{j1}\, \Lambda_1 + D_{ii}\, u_{i11} + D_i''(\Lambda_1)^2 + D_i'\, \Lambda_{11}\,|^C = 0\,,$$

etc., where $i = 1, 2, \ldots, n$. Here, as earlier,

$$u_{j1} \equiv \frac{du_j}{du_1}\,, \quad u_{j11} \equiv \frac{d^2u_j}{du_1^2}\,,$$

so that

$$u_{11} = 1\,, \quad u_{111} = 0\,.$$

The stability determinant

$$\Delta(u_i, \Lambda) \equiv |\, D_{ij}(u_k, \Lambda)\,|$$

will vary along any equilibrium path, and we write

$$\mathcal{D}(u_1) \equiv \Delta[u_i(u_1), \Lambda(u_1)]$$

the first variation of $\mathcal{D}$ being thus given by

$$\mathcal{D}_1 \equiv \frac{d\mathcal{D}}{du_1} = \Delta_i\, u_{i\,1} + \Delta'\, \Lambda_1\,.$$

Now D_{ij} is diagonal at C with $D_{11}^C = 0$, so differentiating Δ by columns with respect to either u_i or Λ and evaluating at C only the leading determinant remains so we have

$$\Delta_i^C = D_{i\,11}\, D_{22}\, D_{33} \ldots D_{nn}\,|^C\,, \quad \Delta'^{\,C} = D_{11}'\, D_{22}\, D_{33} \ldots D_{nn}\,|^C$$

giving

$$\mathcal{D}_1^C = (D_{i\,11}\, u_{i\,1} + D_{11}'\, \Lambda_1)\, D_{22}\, D_{33} \ldots D_{nn}\,|^C\,.$$

This expression will subsequently allow us to examine the stability of the corresponding equilibrium path.

For the specialized P-system the D-transformation can be written out in full as

$$D(u_i, P) \equiv V[Q_i^C + \alpha_{ij} u_j, P] \equiv U[Q_i^C + \alpha_{ij} u_j] - P \mathcal{E}[Q_i^C + \alpha_{ij} u_j] \equiv \overline{U}(u_i) - P \overline{\mathcal{E}}(u_i)$$

and we write the path variation of the corresponding deflection in the form

$$e(u_1) \equiv \overline{\mathcal{E}}[u_i(u_1)] - \mathcal{E}^C .$$

The first derivative of e can thus be written down as follows

$$e_1 \equiv \frac{de}{du_1} = \overline{\mathcal{E}}_i u_{i1} = -D_i' u_{i1} .$$

2. *The Limit Point*

The equations of normal behaviour of section 3 indicate that a key coefficient in the critical behaviour will be $D_1'^C$, and we shall see that this coefficient differentiates the limiting and branching phenomena, the former arising when this coefficient is non-zero (figure 13).

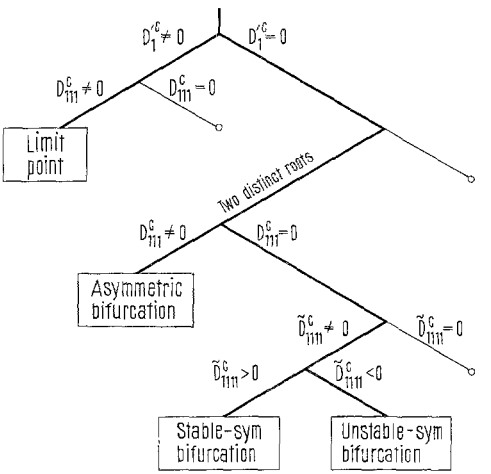

Figure 13
Classification of discrete critical points.

Specifying that $D_1'^C$ is non-zero, we proceed then to examine the equilibrium solutions in the vicinity of the critical equilibrium state C for which $D_{11}^C = 0$ while $D_{ss}^C \neq 0$ for $s \neq 1$.

With these conditions the first-order equilibrium equations yield

$$\Lambda_1^C = 0 , \quad u_{s1}^C = 0 \quad (s \neq 1)$$

and using these results the second-order equations yield

$$\Lambda_{11}^C = -\frac{D_{111}}{D_1'}^C$$

and

$$u_{s11}^C = \frac{D_{111} D_s' - D_{s11} D_1'}{D_{ss} D_1'}^C \quad (s \neq 1) .$$

The first derivative of $\mathcal{D}(u_1)$ becomes

$$\mathcal{D}_1^C = D_{111}\, D_{22}\, D_{33} \ldots D_{nn}\,|^C$$

and for the specialized system we have

$$e_1^C = -D_1'^C .$$

When $D_{111}^C \neq 0$ these solutions correspond to the well-known *limit* or snapping point. A single equilibrium path passes through the critical equilibrium state yielding a locally maximum or minimum value of Λ. The tangent to the path at point C is parallel to the u_1 axis, as shown in figure 14. On a plot of the loading parameter Λ against the critical principal coordinate u_1 or against the corresponding deflection of the specialized system $\mathcal{E}$, the path exhibits a smooth maximum or minimum, while on a plot of Λ against any non-critical principal coordinate u_s ($s \neq 1$) the path exhibits a sharp cusp (THOMPSON [30]).

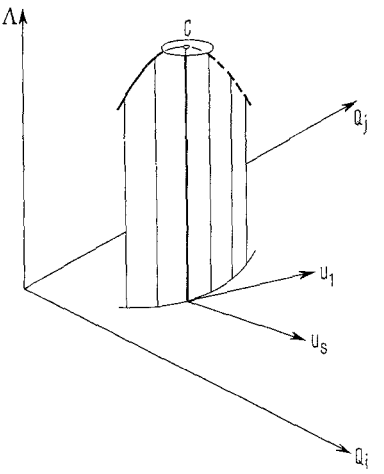

Figure 14
The spatial form of the limit point.

To examine the stability of the equilibrium path we now consider a set of sliding and rotating principal axes as outlined in section 2, taking the *fixed* u_1 as the independent variable. These new axes are chosen to match the fixed principal axes when the fixed u_1 equals zero, so we now have a set of stability coefficients $C_i(u_1)$ with the properties

$$C_1(0) = D_{11}^C = 0\,, \quad C_s(0) = D_{ss}^C \neq 0 \quad \text{for} \quad s \neq 1\,.$$

Now we see from the perturbation analysis that the stability determinant $\mathcal{D}(u_1)$ passes *through* zero at the critical point C, and it follows that the single stability coefficient $C_1(u_1)$ must pass *through* zero at this point. The degree of instability of the path thus changes by one at point C. This is shown in the figure in which a solid line denotes equilibrium states that are stable with respect to the sliding and rotating principal coordinate u_1, while a broken line denotes equilibrium states that are unstable with respect to this coordinate.

We can finally note that the critical equilibrium state is itself always unstable due to the non-zero cubic coefficient D^C_{111}.

3. Branching Points

Still restricting attention to a discrete critical point C at which $D^C_{11} = 0$ and $D^C_{ss} \neq 0$ for $s \neq 1$, let us now consider the case in which $D_1'^C = 0$.

For $i = 1$ the first-order equilibrium equation now yields no information, and setting $i = s \neq 1$ we have

$$u^C_{s1} = -\frac{D'_s}{D_{ss}} \Lambda_1 \Big|^C .$$

For $i = 1$ the second-order equilibrium equation yields

$$D_{111} + 2\sum_{s=2}^{s=n} D_{11s}\, u_{s1} + \sum_{s=2}^{s=n}\sum_{k=2}^{k=n} D_{1ks}\, u_{k1}\, u_{s1} + 2\, D'_{11}\, \Lambda_1 + 2\, \Lambda_1 \sum_{s=2}^{s=n} D'_{1s}\, u_{s1} + D''_1 (\Lambda_1)^2 \Big|^C = 0$$

and substituting for u^C_{s1} we have finally

$$A\, \Lambda_1^2 + B\, \Lambda_1 + C \,|^C = 0$$

where

$$A \equiv D''_1 - 2\sum_{s=2}^{s=n} \frac{D'_{1s}\, D'_s}{D_{ss}} + \sum_{s=2}^{s=n}\sum_{k=2}^{k=n} \frac{D_{1ks}\, D'_s\, D'_k}{D_{ss}\, D_{kk}} \Big|^C , \qquad B \equiv 2\, D'_{11} - 2\sum_{s=2}^{s=n} \frac{D_{11s}\, D'_s}{D_{ss}} \Big|^C ,$$

$$C \equiv D^C_{111} .$$

The variation of the corresponding deflection of the specialized system is given by

$$e^C_1 = P_1 \sum_{2}^{n} \frac{(D'_s)^2}{D_{ss}} \Big|^C .$$

Restricting attention to the case in which the quadratic equation for Λ^C_1 yields *two real and distinct roots* (including the special circumstance in which A vanishes so that we have the degenerate solutions $\Lambda^C_1 = \infty$ or $-C/B$) we see that we now have a point of bifurcation involving the intersection of two distinct equilibrium paths. Restricting our attention further to the initial loss of stability of a system under increasing Λ, such points of bifurcation can be classified as indicated in the diagram of figure 13.

If $C = D^C_{111}$ is not identically zero the quadratic equation is not satisfied by $\Lambda^C_1 = 0$, so the two equilibrium paths have distinct and non-zero slopes. We shall call such an intersection an *asymmetric* point of bifurcation or branching point.

Considering the variation of any u_s $(s \neq 1)$ with Λ we can write

$$u_s(u_1) \equiv u_s[\Lambda(u_1)]$$

giving on differentiation with respect to u_1

$$u_{s1} = u'_s\, \Lambda_1 .$$

Now $\Lambda_1^C \neq 0$ for either path, so we can write

$$u_s'^C = \frac{u_{s1}}{\Lambda_1}\Big|^C$$

and using the earlier result we have

$$u_s'^C = -\frac{D_s'}{D_{ss}}\Big|^C .$$

This holds for *either* path, so we see that the point of bifurcation has the characteristics shown in figure 15: a segment drawn between the two equilibrium paths at a constant

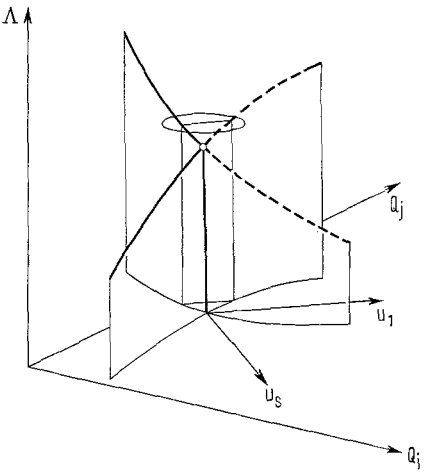

Figure 15
The spatial form of the asymmetric point of bifurcation.

value of Λ has the direction of u_1 as this value of Λ tends to Λ^C. Thus on a plot of the loading parameter Λ against the critical principal coordinate u_1 the two equilibrium paths have a direct intersection, while the two equilibrium paths *touch* each other on a plot of Λ against any non-critical principal coordinate u_s (THOMPSON [30]).

Considering the variation of the corresponding deflection of the specialized system with the magnitude of the generalized force P we can write

$$e(u_1) \equiv e[P(u_1)]$$

giving on differentiation with respect to u_1

$$e_1 = e' P_1 .$$

Now $P_1^C \neq 0$ for either path, so we can write

$$e'^C = \frac{e_1}{P_1}\Big|^C ,$$

and using the earlier result we have

$$e'^C = \sum_2^n \frac{(D_s')^2}{D_{ss}}\Big|^C .$$

This holds for *either* path, so the two paths of the asymmetric point of bifurcation are seen to *touch* on a plot of the load P against its corresponding deflection $\mathcal{E}$ (THOMPSON [30]) as shown in figure 6.

The stability of the two intersecting equilibrium paths of the asymmetric point of bifurcation is discussed later and we shall see that we have the *exchange of stabilities* described by POINCARÉ [91]. Thus considering two matching sets of sliding and rotating principal axes we see that only one stability coefficient, namely that associated with the critical principal coordinate u_1, is involved. Moving along one path with increasing Λ this stability coefficient changes in sign from positive to negative on passing through the critical equilibrium state, while moving along the other path with increasing Λ the corresponding stability coefficient changes in sign from negative to positive. This is shown in the figure in which a solid curve denotes equilibrium states which are stable with respect to the sliding and rotating critical principal coordinate u_1 while a broken curve denotes equilibrium states which are unstable with respect to this coordinate. Since we have earlier restricted attention to systems losing their initial stability it follows that the two paths are both everywhere stable with respect to the other sliding principal coordinates, so the solid curves represent equilibrium states that are stable without qualification.

The critical point of an asymmetric point of bifurcation is itself unstable due to the non-zero cubic coefficient D^C_{111}.

Finally if $C \equiv D^C_{111}$ is identically zero we see that one solution of the quadratic equation is given by $\Lambda^C_1 = 0$, the other solution being non-zero since we have earlier assumed the solutions to be distinct. Thus one of the intersecting equilibrium paths, path A say, has a non-zero initial slope while the other, path B say, has a zero initial slope. We shall call such a point of intersection a *symmetric* point of bifurcation or branching point.

Since u^C_{s1} is still given for $s \neq 1$ by the equation

$$u^C_{s1} = -\frac{D'_s}{D_{ss}} \Lambda_1 \Big|^C$$

we see that the rate of change with u_1 of any non-critical principal coordinate u_s is now zero for path B which is associated with $\Lambda^C_1 = 0$. The tangent at the critical point to path B is thus parallel to the u_1 axis.

Assuming that the initial loss of stability of the system under increasing Λ is associated with path A we shall see in section 5 that the precise path configurations depend on the sign of the previously defined coefficient $\tilde{D}^C_{1111}$ which determines the stability of the critical equilibrium state itself.

If $\tilde{D}^C_{1111}$ is positive the critical equilibrium state and the path B are both stable and the path B curves upwards as shown in figure 16. The intersection is then called a *stable-symmetric* point of bifurcation. Proceeding with increasing Λ along the path A the stability coefficient associated with a sliding and rotating critical principal coordinate u_1 changes from positive to negative at the critical equilibrium state, while along the initially-horizontal path B the corresponding stability coefficient is never negative but touches zero at the critical equilibrium state.

If $\tilde{D}^C_{1111}$ is negative the critical equilibrium state and the path B are both unstable and the path B curves downwards as shown in figure 17. The intersection is then called

an *unstable-symmetric* point of bifurcation. Proceeding with increasing Λ along the path A the stability coefficient associated with a sliding and rotating critical principal coordinate u_1 changes from positive to negative at the critical equilibrium state, while along the initially-horizontal path B the corresponding stability coefficient is never positive but touches zero at the critical equilibrium state.

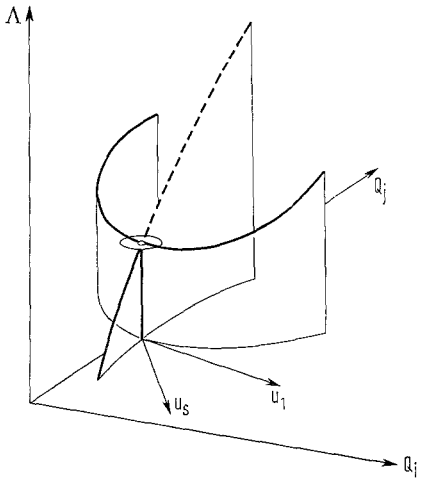

Figure 16
The spatial form of the stable-symmetric point of bifurcation.

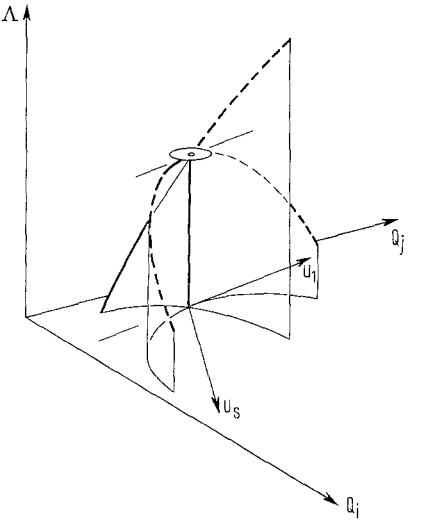

Figure 17
The spatial form of the unstable-symmetric point of bifurcation.

If $\tilde{D}^C_{1111}$ is zero more complex points of bifurcation are generated as indicated in the diagram of figure 13.

For both the symmetric points of bifurcation a plot of Λ against the fixed u_1 will be characterized by the direct intersection of two equilibrium paths, one path exhibiting a local maximum or minimum at the critical equilibrium state. On a plot of the loading parameter against any non-critical principal coordinate u_s or against the corresponding deflection of the specialized system $\mathcal{E}$ (figure 6) the degeneration of the equations indicates that this local maximum or minimum will appear as a cusp striking the other equilibrium path.

5. Branching Points

1. General Analysis

We shall now examine the branching points in more detail by employing a system of 'sliding' coordinates which represent changes in the generalized coordinates from a presumeably known 'fundamental' equilibrium path. We present first a general analysis which makes no resort to a scheme of diagonalization and which can therefore be directly employed in numerical analysis. The branching phenomena are then examined in greater detail in a special analysis which involves a scheme of continuous diagonalization.

Considering then the general system described by the total potential energy function $V(Q_i, \Lambda)$ we start the general analysis by supposing that in the region of interest the n equilibrium equations $V_i = 0$ yield a single-valued *fundamental* solution $Q_i = Q_i^F(\Lambda)$. A *sliding* set of incremental coordinates is defined by the n equations

$$Q_i = Q_i^F(\Lambda) + q_i$$

and we introduce the new energy function

$$W(q_i, \Lambda) \equiv V\,[Q_i^F(\Lambda) + q_i, \Lambda]\,.$$

The normal equilibrium and stability conditions hold good for this transformed energy function which has the properties

$$W_i[0, \Lambda] = 0\,, \quad W_i'[0, \Lambda] = 0\,,$$

etc. We have a valid mapping from the original $\Lambda - Q_i$ space to the new $\Lambda - q_i$ space in which the fundamental equilibrium path is given by $q_i = 0$.

Supposing now that a discrete critical point C lies on the fundamental equilibrium path at $\Lambda = \Lambda^C$, the determinant of W_{ij} will vanish at this point so we can write

$$|\,W_{ij}(0, \Lambda^C)\,| = 0\,.$$

We now seek to express any *post-buckling* equilibrium path emerging from this critical equilibrium state in the parametric form

$$q_j = q_j(q_1)\,, \quad \Lambda = \Lambda(q_1)$$

assuming without any essential loss of generality that the *first* generalized coordinate is a suitable independent variable. Substituting these parametric equations into the equilibrium equation $W_i = 0$ we have the characterizing equilibrium identity

$$W_i[q_j(q_1)\,, \quad \Lambda(q_1)] \equiv 0$$

where for convenience we have written q_1 as $q_1(q_1)$.

Proceeding as before we can now differentiate this equilibrium equation repeatedly with respect to q_1 to obtain the ordered equilibrium equations

$$W_{ij}\,q_{j1} + W_i'\,\Lambda_1 = 0\,,$$

$$(W_{ijk}\,q_{k1} + W_{ij}'\,\Lambda_1)\,q_{j1} + W_{ij}\,q_{j11} + (W_{ij}'\,q_{j1} + W_i''\,\Lambda_1)\,\Lambda_1 + W_i'\,\Lambda_{11} = 0\,,$$

etc., a subscript one denoting differentiation with respect to q_1. As before the notation is chosen for convenience, and we must remember that

$$q_{11} \equiv \frac{dq_1}{dq_1} = 1\,, \quad q_{111} \equiv \frac{d^2q_1}{dq_1^2} = 0\,,$$

etc.

Evaluating these equations at the critical point C, for which $q_i = 0$, $\Lambda = \Lambda^C$, and remembering that $W_i'\,|^C = W_i''\,|^C = 0$ we have

$$W_{ij}\,q_{j1}\,|^C = 0\,, \tag{5.1}$$

$$W_{ijk}\,q_{j1}\,q_{k1} + 2\,W_{ij}'\,q_{j1}\,\Lambda_1 + W_{ij}\,q_{j11}\,|^C = 0\,, \tag{5.2}$$

etc. Since $q_{11} = 1$ and assuming as before that q_1 is a suitable independent variable, we can solve the first equation for the rates q^C_{s1} where $s \neq 1$.

Multiplying the ith equation of the second set by q^C_{i1} and adding the n equations we have

$$W_{ijk}\, q_{i1}\, q_{j1}\, q_{k1} + 2\, W'_{ij}\, q_{i1}\, q_{j1}\, \Lambda_1 + W_{ij}\, q_{i1}\, q_{j11}\,|^C = 0$$

and observing that the last term vanishes by virtue of the first-order equilibrium equation we find

$$\Lambda^C_1 = -\left.\frac{W_{ijk}\, q_{i1}\, q_{j1}\, q_{k1}}{2\, W'_{ij}\, q_{i1}\, q_{j1}}\right|^C. \tag{5.3}$$

Here we have assumed that the critical equilibrium state is *simple* in the sense that the expression $W'_{ij}\, q_{i1}\, q_{j1}|^C$, which clearly plays a key role in the analysis, is non-zero.

We see that by this manipulation we have obtained the slope Λ^C_1 in terms of the rates q^C_{j1}, the second derivatives q^C_{j11} having been eliminated. We can thus write the general result:

The slope of the post-buckling path passing through a simple discrete critical point on a plot of the loading parameter against a (suitable) generalized coordinate is not dependent on the solution of the second-order equilibrium equations.

A consequence of this result is that a one-degree-of-freedom nonlinear Rayleigh-Ritz analysis employing the linear buckling mode will yield the correct value for this slope [36]. A formal proof of this is readily written down following the corresponding proof in section 3.

Returning to the analysis, knowing Λ^C_1 and remembering that $q_{111} = 0$, we can now solve the second-order equilibrium equation for the second derivatives q^C_{s11} $(s \neq 1)$ if more information is required.

This sequence can now be repeated as many times as we please. Thus multiplying the third-order equilibrium equation by q^C_{i1} and adding we can now find Λ^C_{11} without having to solve these equations for q^C_{j111}. We can thus write the general result:

For the post-buckling path passing through a simple discrete critical point the mth *derivative of the loading parameter with respect to a (suitable) generalized coordinate is not dependent on the solution of the* $(m+1)$th*-order equilibrium equations.*

The work so far is applicable whether or not the slope Λ^C_1 is zero. When Λ^C_1 is non-zero we have the previously discussed asymmetric point of bifurcation, and we shall now set $\Lambda^C_1 = 0$ to study the symmetric points of bifurcation in more detail.

Thus with $\Lambda^C_1 = 0$, the third-order equilibrium equation evaluated at the critical point can be written down and after multiplying by q^C_{i1} and summing the last term vanishes and we obtain the path curvature as

$$\Lambda^C_{11} = -\left.\frac{\tilde{W}_4}{3\, W'_{ij}\, q_{i1}\, q_{j1}}\right|^C,$$

where

$$\tilde{W}^C_4 \equiv W_{ijkl}\, q_{i1}\, q_{j1}\, q_{k1}\, q_{l1} + 3\, W_{ijk}\, q_{i1}\, q_{j1}\, q_{k11}\,|^C.$$

Let us consider now the post-buckling response of the specialized system on a plot of the load P against its corresponding deflection $\mathcal{E}$. The energy transformation of the general theory can first be written out in full as follows

$$W(q_i, P) \equiv V\,[Q^F_i(P) + q_i, P] \equiv U\,[Q^F_i(P) + q_i] - P\,\mathcal{E}\,[Q^F_i(P) + q_i]$$

and some required derivatives of W can be written down

$$W_i = U_i - P\,\mathcal{E}_i\,, \quad W' = U_i\,Q_i^{F\prime} - \mathcal{E} - P\,\mathcal{E}_i\,Q_i^{F\prime}\,, \quad W_j' = U_{ij}\,Q_i^{F\prime} - \mathcal{E}_j - P\,\mathcal{E}_{ij}\,Q_i^{F\prime}\,,$$

etc. Now $W_j'^{\,C} = 0$, so we have

$$\mathcal{E}_j^C = (U_{ij} - P\,\mathcal{E}_{ij})\,Q_i^{F\prime}\,|^C = W_{ij}\,Q_i^{F\prime}\,|^C$$

and multiplying by q_{j1}^C and adding we find

$$\mathcal{E}_j\,q_{j1}\,|^C = W_{ij}\,Q_i^{F\prime}\,q_{j1}\,|^C\,.$$

But the right-hand side of this last equation is zero by virtue of the first-order equilibrium equation, so we have the important result that

$$\mathcal{E}_j\,q_{j1}\,|^C = 0\,.$$

We consider now the change in the corresponding deflection defined by the equation

$$e(q_i,\,P) \equiv \mathcal{E}\,[Q_i^F(P) + q_i] - \mathcal{E}[Q_i^F(P)]$$

and we write down some required derivatives as follows,

$$e_i = \mathcal{E}_i\,, \quad e' = \mathcal{E}_i\,Q_i^{F\prime} - \mathcal{E}_i\,Q_i^{F\prime}\,, \quad e_j' = \mathcal{E}_{jk}\,Q_k^{F\prime}\,,$$

etc. The total variation of e with q_1 along the post-buckling equilibrium path is

$$e(q_1) \equiv e[q_j(q_1),\,P(q_1)]$$

giving on differentiation

$$\frac{de}{dq_1} = e_j\,q_{j1} + e'\,P_1\,,$$

etc., so using the earlier results we have

$$\frac{de}{dq_1}\,|^C = 0\,, \quad \frac{d^2e}{dq_1^2}\,|^C = \mathcal{E}_{jk}\,q_{j1}\,q_{k1} + 2\,P_1\,\mathcal{E}_{jk}\,Q_k^{F\prime}\,q_{j1} + \mathcal{E}_j\,q_{j11}\,|^C\,.$$

For the asymmetric point of bifurcation in which $P_1^C \neq 0$ the total post-buckling variation of e with P is well-behaved and we can write

$$e(q_1) \equiv e[P(q_1)]$$

so that

$$\frac{de}{dq_1} = \frac{de}{dP}\,P_1\,,$$

etc. We thus obtain the required results

$$\frac{de}{dP}\,|^C = 0\,, \quad \frac{d^2e}{dP^2}\,|^C = \frac{\mathcal{E}_{jk}\,q_{j1}\,q_{k1} + 2\,P_1\,\mathcal{E}_{jk}\,Q_k^{F\prime}\,q_{j1} + \mathcal{E}_j\,q_{j11}}{P_1^2}\,|^C\,.$$

Thus for the asymmetric point of bifurcation we see that on a plot of P against $\mathcal{E}$ the fundamental and post-buckling equilibrium paths touch each other at the critical equilibrium point C as discussed earlier and as shown in figure 6.

For the symmetric points of bifurcation in which $P_1^C = 0$ we see that the total post-buckling variation of e with P is associated with a singularity, but we can write

$$\frac{\Delta e}{\Delta P} = \frac{1/2\ (d^2e/dq_1^2)\ q_1^2 + \cdots}{1/2\ P_{11}\ q_1^2 + \cdots}$$

to obtain the limiting slope

$$\left.\frac{\Delta e}{\Delta P}\right|^C = \left.\frac{d^2e/dq_1^2}{P_{11}}\right|^C = \left.\frac{\mathcal{E}_{jk}\ q_{j1}\ q_{k1} + \mathcal{E}_j\ q_{j11}}{P_{11}}\right|^C .$$

Thus for the symmetric points of bifurcation we see that on a plot of P against $\mathcal{E}$ the post-buckling equilibrium path exhibits a cusp striking the fundamental equilibrium path at the critical point C as discussed earlier and as shown in figure 6.

General studies of the $P - \mathcal{E}$ response of specialized systems have been made by MANSFIELD [92] and POPE [93].

2. *Analysis with Diagonalization*

We shall now employ a scheme of continuous diagonalization to make a more-detailed study of these branching points.

Thus, having introduced the transformed energy function

$$W(q_i, \Lambda) \equiv V\ [Q_i^F(\Lambda) + q_i, \Lambda]$$

we now make a further change of coordinates by means of a non-singular linear transformation of the form $q_i = \alpha_{ij}(\Lambda)\ u_j$ to diagonalize the quadratic form corresponding to $W_{ij}(0, \Lambda)$. This diagonalization can be accomplished in an infinite number of ways and we shall suppose that one such way has been chosen, merely insisting that α_{ij} is a continuous and single-value function of Λ.

We now introduce the transformed energy function

$$A(u_i, \Lambda) \equiv V\ [Q_i^F(\Lambda) + \alpha_{ij}(\Lambda)\ u_j, \Lambda]$$

with the properties arising from its derivation

$$A_i(0, \Lambda) = A_i'(0, \Lambda) = A_i''(0, \Lambda) = \cdots = 0\ ,$$

$$A_{ij}(0, \Lambda) = A_{ij}'(0, \Lambda) = A_{ij}''(0, \Lambda) = \cdots = 0 \quad \text{for} \quad i \neq j\ .$$

Here a prime again denotes differentiation with respect to Λ.

We see that in the original $\Lambda - Q_i$ space the u_i axes slide along the fundamental path and in general rotate both as a set and relative to each other to give the axes varying orientation and obliquity. Since $Q_i^F(\Lambda)$ and $\alpha_{ij}(\Lambda)$ are continuous and single-valued, we see that we have a valid mapping from the original $\Lambda - Q_i$ space to the final $\Lambda - u_i$ space in which the fundamental path is given by $u_i = 0$.

Two special cases of the above coordinate transformation are worth noting. Firstly, we might insist that $\alpha_{ij}(\Lambda)$ should be orthogonal, to give us the T-transformation used by ROORDA [39]: the u_i axes will then slide and in general rotate, but will remain rectangular in $\Lambda - Q_i$ space. Secondly, it frequently happens in structural problems that $W(q_i, \Lambda) \equiv V\ [Q_i^F(\Lambda) + q_i, \Lambda]$ can be linearized with respect to Λ.

We can then write

$$W(q_i, \Lambda) = W^0(q_i) + \Lambda\, W^1(q_i)$$

and, if $W^0_{ij}(0)$ is positive definite, we can simultaneously diagonalize $W^0_{ij}(0)$ and $W^1_{ij}(0)$ by means of a linear transformation $q_i = \alpha_{ij}\, u_j$. We see that this gives rise to a special case of the A-transformation in which α_{ij} is not a function of Λ. The u_i axes now slide, are in general oblique, but do not rotate with Λ.

We can now start the analysis with the transformed energy function $A(u_i, \Lambda)$, the only necessary properties of which have been written down.

The energy function being diagonalized, the derivatives $A_{ii}(0, \Lambda)$ represent a set of stability coefficients, and we again focus attention on a *discrete* critical point C lying on the fundamental path, at which a *single* stability coefficient is zero. Thus we can write

$$A^C_{11} \equiv A_{11}(0, \Lambda^C) = 0\,, \quad A^C_{ss} \equiv A_{ss}(0, \Lambda^C) \neq 0 \quad \text{for} \quad s \neq 1\,,$$

where Λ^C is the critical value of Λ. Having specified a single-valued fundamental equilibrium path, such a critical point will in general correspond to a point of bifurcation.

Observing that $A'_{11}(0, \Lambda)$ is the rate of change of the critical stability coefficient along the fundamental path, we assume that the critical point is simple in the sense that

$$A'^C_{11} \neq 0\,.$$

The case in which this coefficient is equal to zero has been discussed briefly by POINCARÉ [91].

For future reference we now evaluate some derivatives of the stability determinant

$$\Delta(u_k, \Lambda) \equiv \left| A_{ij}(u_k, \Lambda) \right|.$$

Differentiating by columns with respect to either u_i or Λ, and evaluating at the critical point, only the leading determinant remains and we have finally

$$\Delta^C_i = A_{i11}\, A_{22}\, A_{33} \ldots A_{nn} \Big|^C\,, \quad \Delta'^C = A'_{11}\, A_{22}\, A_{33} \ldots A_{nn} \Big|^C\,,$$

all derivatives being evaluated at the critical point. We observe that Δ'^C is, by previous assumption, non-zero.

We evaluate, further, the second derivative Δ_{11} for the special case in which $A^C_{111} = 0$. Differentiating twice by columns and evaluating at the critical point, only determinants with a differentiated first column remain, and we have after some algebra

$$\Delta^C_{11} = \left\{ A_{1111} - 2 \sum_{s=2}^{s=n} \frac{(A_{s11})^2}{A_{ss}} \right\} A_{22}\, A_{33} \ldots A_{nn} \Big|^C\,.$$

Following the established scheme, we now write the post-buckling equilibrium path of the system in the form

$$u_j = u_j(u_1)\,, \quad \Lambda = \Lambda(u_1)$$

and substituting back into the equilibrium equation $A_i = 0$ we have the characterizing equilibrium identity

$$A_i[u_j(u_1), \Lambda(u_1)] \equiv 0 .$$

It should be noted carefully at this point that throughout this analysis the u_i axes are free to rotate and slide along the fundamental path. They thus always refer to the current value of Λ, and are not fixed at the critical point C.

Differentiating the characterizing equilibrium identity with respect to u_1 gives now

$$A_{ij}\, u_{j1} + A_i' \Lambda_1 = 0 ,$$

the subscript one denoting differentiation of the parametric post-buckling functions with respect to u_1. Evaluating this first-order equilibrium equation at the critical point C we see that for $i = 1$ the equation is identically satisfied, while for $i = s \neq 1$ the equation yields

$$u_{s1}^C = 0 \quad (s \neq 1)$$

the first derivative of interest.

Differentiating now the characterizing equilibrium identity a second time with respect to u_1 and evaluating at the critical point we have

$$\Lambda_1^C = -\frac{A_{111}}{2\,A_{11}'}\Bigg|^C$$

and

$$u_{s11}^C = -\frac{A_{s11}}{A_{ss}}\Bigg|^C \quad (s \neq 1) ,$$

the other two derivatives of immediate interest.

We see that when $A_{111}^C \neq 0$ the post-buckling slope Λ_1^C is non-zero and we have the asymmetric point of bifurcation.

We proceed now to determine higher derivatives for the special case in which A_{111}^C and (consequently) Λ_1^C are zero. Differentiating the characterizing equilibrium identity a third time with respect to u_1 and evaluating at the critical point we have after some algebra

$$\Lambda_{11}^C = -\frac{\tilde{A}_{1111}}{3\,A_{11}'}\Bigg|^C$$

where

$$\tilde{A}_{1111}^C \equiv A_{1111} - 3\sum_{s=2}^{s=n}\frac{(A_{s11})^2}{A_{ss}}\Bigg|^C .$$

Clearly this special case gives us the symmetric points of bifurcation, the stable-symmetric point when Λ_{11}^C is positive and the unstable-symmetric point when Λ_{11}^C is negative. More complex points of bifurcation are of course generated if $\tilde{A}_{1111}^C$ and (consequently) Λ_{11}^C are zero.

It can be seen that we have now determined the derivatives of primary interest for the post-buckling equilibrium paths of the three branching points under consideration, and it remains to examine the stability of the intersecting paths, and finally the stability of the critical equilibrium states themselves.

We shall assume, as before, that we are considering the initial loss of stability of the fundamental equilibrium path of a system under increasing Λ, so that all the non-

critical stability coefficients are positive ($A^C_{ss} > 0$ for $s \neq 1$) while the non-zero coefficient A'^C_{11} is negative. The fundamental path thus loses its initial stability as Λ increases through Λ^C.

Now if we follow the approach of section 2 and consider a new set of sliding and rotating principal coordinates for the *post-buckling* equilibrium path it is clear that in the vicinity of the critical point C the stability of the post-buckling equilibrium path will be dependent on the sign of the stability determinant $\Delta(u_k, \Lambda) = |A_{ij}|$. We thus proceed to examine the variation of this determinant along the post-buckling path, writing

$$\mathcal{D}(u_1) \equiv \Delta[u_j(u_1), \Lambda(u_1)]$$

so that

$$\mathcal{D}_1 \equiv \frac{d\mathcal{D}}{du_1} = \Delta_j u_{j1} + \Delta' \Lambda_1 .$$

Now $u^C_{s1} = 0$ for $s \neq 1$, and using the previously determined derivatives of Δ we have

$$\mathcal{D}^C_1 = (A_{111} + A'_{11} \Lambda_1) A_{22} A_{33} \dots A_{nn} |^C$$

and substituting now for Λ^C_1 we find that

$$\mathcal{D}^C_1 = \frac{1}{2} A_{111} A_{22} A_{33} \dots A_{nn} |^C$$

which clearly has the same sign as Λ^C_1.

Thus for the asymmetric point of bifurcation in which $A^C_{111} \neq 0$ we see that we have the exchange of stability discussed earlier and shown in figures 3 and 15.

For the symmetric points of bifurcation A^C_{111}, Λ^C_1 and $\mathcal{D}^C_1$ vanish and the second variation of $\mathcal{D}$ yields

$$\mathcal{D}^C_{11} = \Delta_{11} + \sum_{s=2}^{s=n} \Delta_s u_{s11} + \Delta' \Lambda_{11} = \frac{2}{3} \tilde{A}_{1111} A_{22} A_{33} \dots A_{nn} \Big|^C .$$

Thus when $\tilde{A}^C_{1111}$ and Λ^C_{11} are positive $\mathcal{D}^C_{11}$ is positive so the rising post-buckling path of the stable-symmetric point of bifurcation is stable (figures 4 and 16), while when $\tilde{A}^C_{1111}$ and Λ^C_{11} are negative $\mathcal{D}^C_{11}$ is also negative so the falling post-buckling path of the unstable-symmetric point of bifurcation is unstable (figures 5 and 17).

Since coordinate derivatives of A are equivalent to the coordinate derivatives of D in section 2, the stability of the critical equilibrium state itself is readily tested. Thus for the asymmetric point of bifurcation the critical equilibrium state C is unstable by virtue of the non-zero cubic coefficient A^C_{111}. For the stable-symmetric point of bifurcation the positive coefficient $\tilde{A}^C_{1111}$ shows that state C is stable, while for the unstable-symmetric point of bifurcation the negative coefficient $\tilde{A}^C_{1111}$ shows that the critical equilibrium state is then unstable.

As Koiter has observed, we see that the post-buckling behaviour of the system is closely linked to the stability of the critical equilibrium state itself.

6. Imperfect Systems

1. Formulation

In this final section we shall consider the response and loss of stability of an imperfect system derived as a perturbation of the 'perfect' system of the preceding analyses, paying particular attention to the important topic of imperfection-sensitivity. The limit point will be considered first, and then two branching analyses will be presented employing the general W-function and the diagonalized A-function respectively.

We start by generating a family of systems by introducing the perturbation or *imperfection parameter* ε into the energy function, giving us the single-valued function $V(Q_i, \Lambda, \varepsilon)$. The system corresponding to $\varepsilon = 0$ will be described as the *perfect system*, and it is to be understood that this was the system considered in the earlier sections. We likewise describe systems corresponding to non-zero values of ε as *imperfect systems*.

2. The Limit Point

We examine here the response of imperfect systems in the vicinity of the discrete limit point of section 4. Deflections of all systems being described by the same set of coordinates, we can still use the fixed principal axes of the earlier work by writing

$$D(u_i, \Lambda, \varepsilon) \equiv V\,[Q_i^C + \alpha_{ij}\, u_j, \Lambda, \varepsilon]\ .$$

The analysis thus starts with this transformed energy function, the only necessary properties of which refer to the perfect system and can be written as

$$D_{ij}^C = 0 \quad \text{for} \quad i \neq j\ , \quad D_{11}^C = 0\ ,$$

$$D_{ss}^C \neq 0 \quad \text{for} \quad s \neq 1\ , \quad D_1'^C \neq 0\ , \quad D_{111}^C \neq 0\ .$$

We anticipate equilibrium paths with the form of those of figure 7, in which a heavy line represents the equilibrium path of the perfect system while light lines represent the equilibrium paths of imperfect systems. We thus focus attention on the critical points of the imperfect systems which can be expressed by the relationships $u_j = u_j^M(\varepsilon)$, $\Lambda = \Lambda^M(\varepsilon)$, a point in load-coordinate space (u_j^M, Λ^M) corresponding to the critical point of a given imperfect system.

The critical equilibrium states are characterized by the conditions $D_i = \Delta = 0$ where $\Delta(u_k, \Lambda, \varepsilon) = |D_{ij}|$, so we can write the *characterizing identities*

$$D_i[u_j^M(\varepsilon), \Lambda^M(\varepsilon), \varepsilon] \equiv 0\ , \quad \Delta[u_j^M(\varepsilon), \Lambda^M(\varepsilon), \varepsilon] \equiv 0\ .$$

We can now perform an intrinsic analysis, similar to those employed for the perfect systems, to determine initial derivatives of the functions $u_j^M(\varepsilon)$, $\Lambda^M(\varepsilon)$, the latter being the required function of imperfection-sensitivity.

Thus differentiating the characterizing identities with respect to ε we have

$$D_{ij}\, \dot{u}_j^M + D_i'\, \dot{\Lambda}^M + \dot{D}_i = 0\ , \quad \Delta_i\, \dot{u}_i^M + \Delta'\, \dot{\Lambda}^M + \dot{\Delta} = 0\ ,$$

where a dot denotes differentiation with respect to ε. Evaluating these two equations

at the critical point of the perfect system, $u_i^M = 0$, $\Lambda^M = \Lambda^C$, $\varepsilon = 0$, gives the first derivatives $\dot{\Lambda}^{MC}$ and $\dot{u}_i^{MC}$. The former is of primary interest and arises directly from the first of these equations if we are prepared to assume what is readily proved by the second equation, namely that $\dot{u}_i^{MC} \neq \infty$.

Thus setting $i = 1$ in the first equation we have

$$D_1' \dot{\Lambda}^M + \dot{D}_1 \big|^C = 0$$

and since $D_1'^C \neq 0$ we can write

$$\dot{\Lambda}^{MC} = -\frac{\dot{D}_1}{D_1'}\Big|^C .$$

Thus, assuming that the imperfection has a direct action on the critical principal coordinate in the sense that $\dot{D}_1^C \neq 0$, we see that the peak or failure load Λ^M varies initially linearly with the magnitude of the imperfection parameter as shown in figure 7.

3. Branching Points

We examine now the response of imperfect systems in the vicinity of the discrete branching point of section 5 with a view to determining the equations of imperfection-sensitivity. Deflections of all systems are described by the same set of coordinates, so we can still use either the W-function or the A-function of section 5, and we consider first the W-function which can now be defined by the equation

$$W(q_i, \Lambda, \varepsilon) \equiv V\,[Q_i^F(\Lambda) + q_i, \Lambda, \varepsilon] .$$

The first analysis thus starts with this transformed energy function, the only necessary properties of which refer to the perfect system and are given in section 5.

One would naturally think of the critical points of the imperfect systems in the form $q_j = q_j^M(\varepsilon)$, $\Lambda = \Lambda^M(\varepsilon)$, a point (q_j^M, Λ^M) in load-coordinate space corresponding to a critical equilibrium state of a given imperfect system. For our purposes, however, it will be necessary to write these functional relationships in the inverted form

$$q_j^M = q_j^M(q_1^M) , \quad \Lambda^M = \Lambda^M(q_1^M) , \quad \varepsilon = \varepsilon^M(q_1^M) . \tag{6.1}$$

As ε^M tends to zero, the behaviour of the corresponding imperfect system will approach that of the perfect system, so these relationships will clearly be satisfied by

$$q_j^M = 0 , \quad \Lambda^M = \Lambda^C , \quad \varepsilon^M = 0 . \tag{6.2}$$

The critical equilibrium states satisfy the conditions $W_i = \Delta = 0$ where Δ is the stability determinant and a direct analysis along the lines of the following A-analysis is now possible using these conditions. The results of such a direct approach will however contain various derivatives of the determinant Δ and in a W-analysis vital comparisons with the post-buckling characteristics of the perfect system are not easily made. For this reason we here take a less-direct approach as follows.

The critical equilibrium states of the imperfect systems are characterized by the conditions $W_i = W_{ij}\, x_j = 0$ where $x_j(q_1^M)$ is a spiralling eigenvector associated with the limit points of the imperfect systems. This spiralling eigenvector plays an important but transitory role in the following analysis and makes no appearance in the final

results. The numerical analyst can thus use the results giving no thought to this vector employed in their derivation.

Writing the *characterizing identities* in full as

$$W_i[q_j^M(q_1^M), \Lambda^M(q_1^M), \varepsilon^M(q_1^M)] \equiv 0\,, \quad W_{ij}[q_j^M(q_1^M), \Lambda^M(q_1^M), \varepsilon^M(q_1^M)]\, x_j(q_1^M) \equiv 0$$

and introducing the normalizing condition that $x_1 = 1$ for all q_1^M, we can now perform an intrinsic perturbation analysis to determine the initial derivatives of the functions $q_j^M(q_1^M)$, $\Lambda^M(q_1^M)$, $\varepsilon^M(q_1^M)$.

Thus differentiating the characterizing identities once with respect to q_1^M we have

$$W_{ij}\, q_{j1}^M + W_i'\, \Lambda_1^M + \dot{W}_i\, \varepsilon_1^M = 0\,, \quad (W_{ijk}\, q_{k1}^M + W_{ij}'\, \Lambda_1^M + \dot{W}_{ij}\, \varepsilon_1^M)\, x_j + W_{ij}\, x_{j1} = 0\,, \tag{6.3}$$

and differentiating again with respect to q_1^M we have

$$\left.\begin{aligned}
&(W_{ijk}\, q_{k1}^M + W_{ij}'\, \Lambda_1^M + \dot{W}_{ij}\, \varepsilon_1^M)\, q_{j1}^M + W_{ij}\, q_{j11}^M + (W_{ij}'\, q_{j1}^M + W_i''\, \Lambda_1^M + \dot{W}_i'\, \varepsilon_1^M)\, \Lambda_1^M \\
&\qquad + W_i'\, \Lambda_{11}^M + (\dot{W}_{ij}\, q_{j1}^M + \dot{W}_i'\, \Lambda_1^M + \ddot{W}_i\, \varepsilon_1^M)\, \varepsilon_1^M + \dot{W}_i\, \varepsilon_{11}^M = 0\,, \\
&\{(W_{ijke}\, q_{e1}^M + W_{ijk}'\, \Lambda_1^M + \dot{W}_{ijk}\, \varepsilon_1^M)\, q_{k1}^M + W_{ijk}\, q_{k11}^M \\
&\qquad + (W_{ijk}'\, q_{k1}^M + W_{ij}''\, \Lambda_1^M + \dot{W}_{ij}'\, \varepsilon_1^M)\, \Lambda_1^M + W_{ij}'\, \Lambda_{11}^M \\
&\qquad + (\dot{W}_{ijk}\, q_{k1}^M + \dot{W}_{ij}'\, \Lambda_1^M + \ddot{W}_{ij}\, \varepsilon_1^M)\, \varepsilon_1^M + \dot{W}_{ij}\, \varepsilon_{11}^M\}\, x_j \\
&\qquad + (W_{ijk}\, q_{k1}^M + W_{ij}'\, \Lambda_1^M + \dot{W}_{ij}\, \varepsilon_1^M)\, x_{j1} \\
&\qquad + (W_{ijk}\, q_{k1}^M + W_{ij}'\, \Lambda_1^M + \dot{W}_{ij}\, \varepsilon_1^M)\, x_{j1} + W_{ij}\, x_{j11} = 0\,,
\end{aligned}\right\} \tag{6.4}$$

where a dot denotes differentiation with respect to ε. It will be seen that in analogy with the previous work, a subscript one is used to denote the differentiation of the functions (6.1) with respect to q_1^M, and that q_1^M is written as $q_1^M(q_1^M)$, $dq_1^M/dq_1^M = 1$ is written as q_{11}^M, etc. The normalizing condition on x_j gives us

$$x_1 = 1\,, \quad x_{11} = x_{111} = \cdots = 0\,,$$

and since in the limit as ε^M tends to zero the spiralling eigenvector x_j must be coincident with the eigenvector of the perfect system, we can write

$$x_j^C \equiv x_j(0) = q_{j1}^C\,.$$

Multiplying the first equation of (6.3) by the post-buckling rate of the perfect system, q_{i1}^C, adding the n equations and evaluating at the point represented by (6.2) we find

$$\dot{W}_i\, q_{i1}\, \varepsilon_1^M \,|^C = 0\,.$$

Assuming now that the imperfection parameter has a direct action on the buckling mode in the sense that

$$\dot{W}_i\, q_{i1} \,|^C \neq 0$$

we have the first result of interest

$$\varepsilon_1^{MC} = 0\,.$$

Evaluating the first equation of (6.3) now yields

$$W_{ij}\, q_{j1}^{M}\,|^{C} = 0$$

and a comparison with equation (5.1) of section 5 shows us that

$$q_{j1}^{MC} = q_{j1}^{C} \quad (= x_j^C)\,.$$

Multiplying the second equation of (6.3) by q_{i1}^{C} and adding the n equations, and using our assumption of section 5 that $W'_{ij}\, q_{i1}\, q_{j1}\,|^{C} \neq 0$ we obtain the second result of interest

$$\Lambda_1^{MC} = -\left.\frac{W_{ijk}\, q_{i1}\, q_{j1}\, q_{k1}}{W'_{ij}\, q_{i1}\, q_{j1}}\right|^{C}. \tag{6.5}$$

A comparison with equation (5.3) of section 5 shows us that

$$\Lambda_1^{MC} = 2\,\Lambda_1^{C}\,.$$

Multiplying the first equation of (6.4) by q_{i1}^{C} and adding now yields the third result of interest

$$\varepsilon_{11}^{MC} = \left.\frac{W_{ijk}\, q_{i1}\, q_{j1}\, q_{k1}}{\dot{W}_i\, q_{i1}}\right|^{C}. \tag{6.6}$$

The analysis is so far applicable whether or not $W_{ijk}\, q_{i1}\, q_{j1}\, q_{k1}\,|^{C}$ is zero. When this expression is non-zero we have the asymmetric point of bifurcation, and the asymptotic variation of the maximum load Λ^{M} with the imperfection parameter is then obtained from equations (6.5) and (6.6) as

$$\Lambda^{M} = \Lambda^{C} \pm (W_{ijk}^{C}\, q_{i1}^{C}\, q_{j1}^{C}\, q_{k1}^{C})^{1/2}\, \frac{[2\,\dot{W}_i^{C}\, q_{i1}^{C}\, \varepsilon]^{1/2}}{W_{ij}^{\prime C}\, q_{i1}^{C}\, q_{j1}^{C}}\,.$$

We see that we have here an initial parabolic variation as shown in figure 8, and that this variation is obtained entirely in terms of the *post-buckling rates of the perfect system*.

We shall now set $W_{ijk}\, q_{i1}\, q_{j1}\, q_{k1}\,|^{C} = 0$ to study the symmetric points of bifurcation, and from the above results we have immediately

$$\Lambda_1^{MC} = 0\,, \quad \varepsilon_{11}^{MC} = 0\,.$$

The second equation of (6.3) now yields

$$W_{ijk}\, q_{j1}\, q_{k1} + W_{ij}\, x_{j1}\,|^{C} = 0$$

while the first equation of (6.4) yields

$$W_{ijk}\, q_{j1}\, q_{k1} + W_{ij}\, q_{j11}^{M}\,|^{C} = 0\,.$$

A comparison with equation (5.2) of section 5 thus shows that

$$x_{j1}^{C} = q_{j11}^{MC} = q_{j11}^{C}\,.$$

The second equation of (6.4) now yields, after multiplying by q_{i1}^{C} and summing

$$\Lambda_{11}^{MC} = -\left.\frac{\tilde{W}_4}{W'_{ij}\, q_{i1}\, q_{j1}}\right|^{C}, \tag{6.7}$$

where as before

$$\tilde{W}_4^C \equiv W_{ijke}\, q_{i1}\, q_{j1}\, q_{k1}\, q_{e1} + 3\, W_{ijk}\, q_{i1}\, q_{j1}\, q_{k11}\,|^C .$$

We see that

$$\Lambda_{11}^{MC} = 3\, \Lambda_{11}^C .$$

Finally, the third differentiation of the first characterizing identity yields, after multiplying by q_{i1}^C and adding,

$$\varepsilon_{111}^{MC} = \frac{2\,\tilde{W}_4}{\dot{W}_i\, q_{i1}}\Big|^C . \tag{6.8}$$

When the numerator of this expression is non-zero we have a symmetric point of bifurcation. The asymptotic relationship between the maximum load Λ^M and the imperfection parameter is then obtained from equations (6.7) and (6.8) as

$$\Lambda^M = \Lambda^C - \frac{1}{2}\,(\tilde{W}_4^C)^{1/3}\,\frac{[3\,\dot{W}_i^C\, q_{i1}^C\, \varepsilon]^{2/3}}{W_{ij}'^C\, q_{i1}^C\, q_{j1}^C} .$$

We see that we have here an initial variation associated with a two-thirds power law as shown in figure 10, and that this variation is obtained entirely in terms of the *post-buckling derivatives of the perfect system.*

We shall now re-derive the equations of imperfection-sensitivity in terms of the A-function of section 5 which can now be defined by the equation

$$A(u_i, \Lambda, \varepsilon) \equiv V\,[Q_i^F(\Lambda) + \alpha_{ij}(\Lambda)\, u_j, \Lambda, \varepsilon] .$$

This second analysis thus starts with this transformed energy function, the only necessary properties of which refer to the perfect system and are given in section 5.

Corresponding to the assumption of the W-analysis, we assume throughout that the imperfection parameter has a direct action on the critical principal coordinate in the sense that

$$\dot{A}_1^C \equiv \dot{A}_1(0, \Lambda^C, 0) \neq 0 ,$$

where a dot again denotes differentiation with respect to ε. The imperfection is then a major imperfection in the terminology of ROORDA [39], who considers the action of both 'major' and 'minor' imperfections in an asymmetric point of bifurcation.

Introducing now the stability determinant

$$\Delta(u_k, \Lambda, \varepsilon) \equiv |\, A_{ij}(u_k, \Lambda, \varepsilon)\,| ,$$

we can readily establish the result, analogous to those of section 5, that

$$\dot{\Delta}^C = \dot{A}_{11}\, A_{22}\, A_{33} \ldots A_{nn}\,|^C .$$

One would naturally think of the critical points of the imperfect system in the form $u_j = u_j^M(\varepsilon)$, $\Lambda = \Lambda^M(\varepsilon)$, a point (u_j^M, Λ^M) in load-coordinate space corresponding to a critical equilibrium state of a given imperfect system. For our purposes, however, it will be necessary to write these functional relationships in the inverted form

$$u_j^M = u_j^M(u_1^M) , \quad \Lambda^M = \Lambda^M(u_1^M) , \quad \varepsilon = \varepsilon^M(u_1^M) . \tag{6.9}$$

As ε^M tends to zero, the behaviour of the corresponding imperfect system will approach that of the perfect system, so these relationships will clearly be satisfied by

$$u_j^M = 0\,, \quad \Lambda^M = \Lambda^C, \quad \varepsilon^M = 0\,. \tag{6.10}$$

The critical equilibrium states satisfying the conditions $A_i = \Delta = 0$, we can now perform an intrinsic analysis to determine the initial derivatives of $u_j^M(u_1^M)$, $\Lambda^M(u_1^M)$, $\varepsilon^M(u_1^M)$. Thus, substituting these functions, we can write the characterizing identities

$$A_i[u_j^M(u_1^M), \Lambda^M(u_1^M), \varepsilon^M(u_1^M)] \equiv 0\,, \quad \Delta[u_j^M(u_1^M), \Lambda^M(u_1^M), \varepsilon^M(u_1^M)] \equiv 0$$

and differentiating these functions of u_1^M with respect to u_1^M we have

$$A_{ij}\, u_{j1}^M + A_i'\, \Lambda_1^M + \dot{A}_i\, \varepsilon_1^M = 0\,, \quad \Delta_j\, u_{j1}^M + \Delta'\, \Lambda_1^M + \dot{\Delta}\, \varepsilon_1^M = 0\,.$$

It will be seen that following the earlier work a subscript one is used to denote the differentiation of the functions of (6.9) with respect to u_1^M and that u_1^M is written as $u_1^M(u_1^M)$, $du_1^M/du_1^M = 1$ is written as u_{11}^M, etc.

Evaluating these equations at the point represented by (6.10) we find

$$\varepsilon_1^{MC} = 0\,, \quad u_{s1}^{MC} = 0 \quad (s \neq 1)\,, \quad \Lambda_1^{MC} = -\frac{\Delta_1}{\Delta'}\Big|^C = -\frac{A_{111}}{A_{11}'}\Big|^C,$$

the first derivatives of interest.

Differentiating the characterizing identities a second time with respect to u_1^M we obtain

$$\varepsilon_{11}^{MC} = \frac{A_{111}}{\dot{A}_1}\Big|^C, \quad u_{s11}^{MC} = -\frac{A_{111}\dot{A}_s + A_{s11}\dot{A}_1}{A_{ss}\dot{A}_1}\Big|^C \quad (s \neq 1)$$

and

$$\Lambda_{11}^{MC} = -\frac{1}{\Delta'}\left\{\Delta_{11} - 2\Delta_1'\frac{A_{111}}{A_{11}'} - \sum_{s=2}^{s=n}\Delta_s\frac{A_{111}\dot{A}_s + A_{s11}\dot{A}_1}{A_{ss}\dot{A}_1} + \Delta''\left(\frac{A_{111}}{A_{11}'}\right)^2 + \dot{\Delta}\frac{A_{111}}{\dot{A}_1}\right\}\Big|^C.$$

We now have the required derivatives for the asymmetric point of bifurcation. As before, the relationship of greatest interest is of course $\Lambda^M(\varepsilon)$ giving the peak or failure load for a given imperfection, and specifying $A_{111}^C \neq 0$ we can write the asymptotic result

$$\Lambda^M = \Lambda^C \pm (A_{111}^C)^{1/2}\frac{[2\dot{A}_1^C\varepsilon]^{1/2}}{A_{11}'^C}.$$

This is, then, the parabolic relationship of figure 18 which is drawn for $A_{111}^C > 0$, $A_{11}'^C < 0$ and $\dot{A}_1^C > 0$.

For the special case in which $A_{111}^C = 0$ we have from the above equations

$$\Lambda_1^{MC} = 0\,, \quad \varepsilon_{11}^{MC} = 0\,, \quad u_{s11}^{MC} = -\frac{A_{s11}}{A_{ss}}\Big|^C \quad (s \neq 1)\,,$$

and, using the previously determined derivatives of Δ,

$$\Lambda_{11}^{MC} = -\frac{\tilde{A}_{1111}}{A_{11}'}\Big|^C,$$

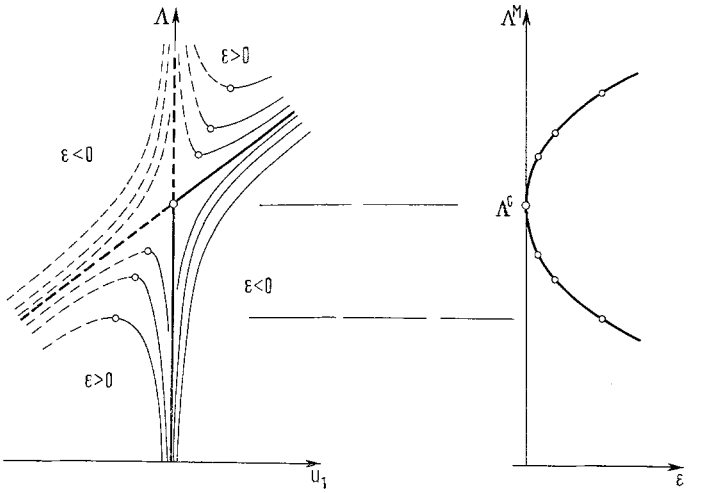

Figure 18
The asymmetric point of bifurcation.

where, as in section 5,

$$\tilde{A}^C_{1111} \equiv A_{1111} - 3\sum_{s=2}^{s=n} \frac{(A_{s11})^2}{A_{ss}}\bigg|^C .$$

We proceed to evaluate the higher derivatives for this special case. Differentiating the characterizing identities a third time with respect to u_1^M, and evaluating, we have after some algebra

$$\varepsilon^{MC}_{111} = -\frac{2\tilde{A}_{1111}}{\dot{A}_1}\bigg|^C .$$

We now have the required derivatives for the symmetric points of bifurcation, and specifying that $\tilde{A}^C_{1111} \neq 0$ we have the asymptotic relationship between the peak or failure load Λ^M and the imperfection parameter

$$\Lambda^M = \Lambda^C - \frac{1}{2}\,(\tilde{A}^C_{1111})^{1/3}\,\frac{[3\,\dot{A}^C_1\,\varepsilon]^{2/3}}{A'^C_{11}} .$$

This, then, is the initial two-thirds power law shown in figures 19 and 20, which are drawn for $A'^C_{11} < 0$ and $\dot{A}^C_1 > 0$.

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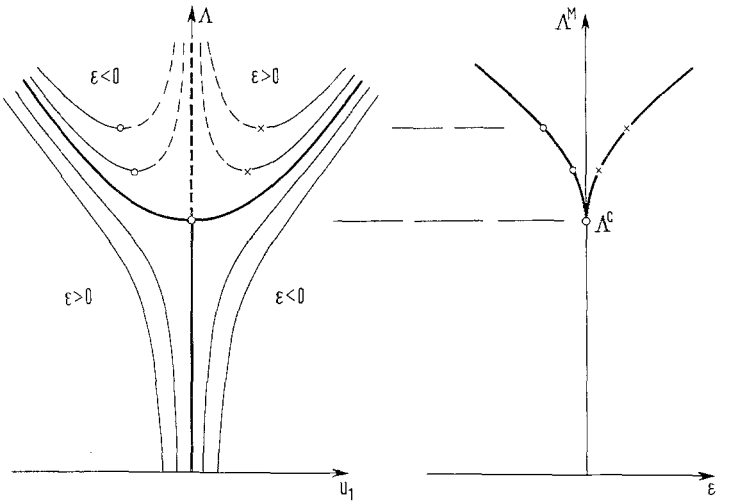

Figure 19
The stable-symmetric point of bifurcation.

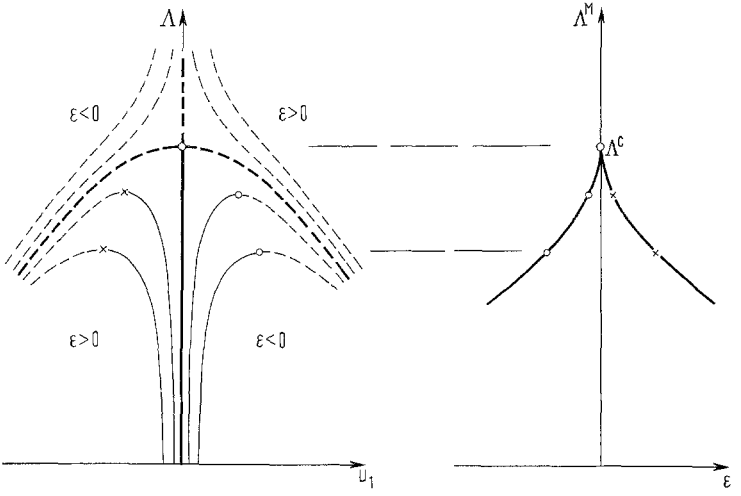

Figure 20
The unstable-symmetric point of bifurcation.

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Zusammenfassung

Eine vom Verfasser kürzlich entwickelte allgemeine Theorie der elastischen Stabilität wird in einheitlicher und moderner Form dargestellt. Die Theorie behandelt Gleichgewicht und Stabilität diskreter kontinuierlicher Systeme und ist somit ein Beitrag zur klassischen Mechanik, mit Anwendungen in der numerischen Untersuchung von in der Praxis auftretenden Körpern und Tragwerken.

Normales und kritisches Verhalten eines allgemeinen Systems werden durch einen natürlichen Störungsansatz klassifiziert und berechnet; beides mit oder ohne Diagonalisierungsschema. Das anfängliche überkritische Verhalten eines Systems bei einem diskreten kritischen Punkt wird ausführlich untersucht; besondere Aufmerksamkeit wird der wichtigen Frage der Imperfektionenanfälligkeit gewidmet.

Einige neue Theoreme werden aufgestellt.

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