First-Order Lagrangian Field Theories of Gravitation without Gauge Invariance.

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Summary. — The theory of gravitation is considered in the field-theoretical approach starting in the pseudo-Euclidean space-time. Gravitation is represented by a symmetric tensor potential $\psi_{\alpha\beta}$ containing one spin-2 part, one spin-1 part ($\psi_{\alpha\beta}{}^{;\beta}$) and two spin-0 parts ($\psi$ and $\psi_{\alpha\beta}{}^{;\alpha\beta}$). The first order of the theory is here considered, while dealing in particular with the case in which the field equations are not gauge invariant. It is analised in which sense it is possible to describe the field by a tensor not contributing the spin-1 part, owing to the zero divergence of the energy-momentum tensor.

1. – Introduction.

The flat–space-time approach to gravity has been widely discussed (1-5) (particularly in the first-order approximation (2)). It is known that, in the flat–space-time field-theoretical approach, gravity is represented by a second-rank, symmetric tensor potential $\psi_{\alpha\beta}$. This is the description that would be given by an ideal observer using ideal rods and clocks unaffected by gravity. It has been shown (3,5) that the consistency of the equations of motion and of the field equations implies an iterative procedure leading to a theory which coincides with general relativity when it is described by an observer using real rods and clocks (affected by gravity).

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(2) R. U. SEXL: *Forts. Phys.*, **15**, 269 (1967).
(3) S. DESER: *Gen. Rel. Grav.*, **1**, 9 (1970).
(4) G. CAVALLERI and G. SPINELLI: *Phys. Rev. D*, **12**, 2200, 2203 (1975).
(5) G. SPINELLI: *Rend. Accad. Lincei*, **63**, 71 (1977).

Here the first-order approximation only will be considered. In particular, one topic will be discussed which is interesting when comparing the gravitational with the electromagnetic field.

A second-rank tensor is decomposable into four spin parts. In the notation used by SEXL (2) it is (see also the appendix)

$$[\psi_{\alpha\beta}] = 0 \oplus 0 \oplus 1 \oplus 2 ,$$

where the two spin-0 parts are given by the scalars ψ and $\psi_{\sigma\lambda}^{;\sigma\lambda}$ and the spin-1 part by the vector $\psi_{\sigma\lambda}^{;\sigma}$ (which contains also the second of the above-mentioned scalar parts).

When constructing the first step of the iterative procedure one usually makes the hypothesis that gravity will be represented by a second-rank, symmetric tensor $\psi_{\alpha\beta}$ containing, outside the matter, a spin-2 part only.

In particular, the vector component is excluded because of theoretical considerations and experimental results. Actually, if the gravitational potential were a vector, the theory would be similar to electromagnetism. However, since gravitation is attractive, the gravitational energy for free fields would be negative and a radiating system would increase its internal energy (because of total-energy conservation). In other terms, the radiation reaction would cause a catastrophic behaviour instead of damping. From the point of view of quantum physics, if the energy for free fields were negative, a continuous production of elementary-particle pairs would occur. Even some fundamental experiments exclude a vector theory for gravitation. Indeed a vector theory predicts no deflection for light and a perihelion precession equal to 1/6 of the observed value. Moreover, it violates the equivalence principle even at first order in the gravitational constant G and, therefore, the results of the experiments rule it out. However, one could think of the presence of a vector component together with the pure tensor component. The above-mentioned drawbacks put an upper limit to this presence of the vector component. This upper limit being extremely low, it is reasonable to make the hypothesis that such a component does not exist at all.

The situation is qualitatively similar for what concerns the scalar components. The experimental results rule out the possibility of a purely scalar theory. As to the presence of the scalar component, it would imply observable effects. The experimental results nowadays can only give an upper limit to the presence of this component, but cannot completely exclude its existence. This upper limit is not so severe as in the case of the vector component. Recent experiments (6) allow for 1% of effects due to a possible scalar component.

(6) I. I. SHAPIRO, R. D. REASENBERG, P. E. MAC NEIL, R. B. GOLDSTEIN, J. P. BRENKLE, D. L. CAIN, T. KOMAREK, A. I. ZYGIELBAUM, W. F. CUDDIHY and W. H. MICHAEL jr.: *Journ. Geophys. Res.*, **82**, 4329 (1977).

Due to this low value it is justified the above-mentioned usual assumption that the potential $\psi_{\alpha\beta}$ is, outside the matter, a purely spin-2 tensor (*i.e.* without the vector and the scalar components). However, it is not so low to make completely unjustified the study of the case in which the scalar component is present.

It has been shown (²) that, in order to have a pure spin-2 theory, one has to impose the gauge invariance for the equations of motion and for the field equations. Here we will consider only the problem of the elimination of the vector component in the general case in which gauge invariance is not assumed.

2. – First-order Lagrangian densities, field equations and equations of motion.

In order to uniquely obtain the Lagrangian densities, let us make the following hypotheses:

1) The field equations and the equations of motion must be obtained by the variational principle

$$\delta\int d^4\Omega\, L = 0\,, \tag{1}$$

where $d^4\Omega$ is the invariant four-dimensional elementary volume and L is the Lagrangian density.

2) The gravitational potential is represented by a symmetric tensor $\psi_{\alpha\beta}$ which, outside the matter, does not contain the vector component.

3) The field equations contain derivatives of $\psi_{\alpha\beta}$ whose maximum order is the second one.

4) In the first step of the iterative procedure, the theory is linear with respect to the gravitational potential and (or) its derivatives.

5) The gravitational interaction has a long range.

6) The equivalence principle is satisfied, at least in its weak form (⁷).

The Lagrangian density can be written as

$$L = L^{(\mathrm{f})} + L^{(\mathrm{m})} + L^{(\mathrm{int})}\,, \tag{2}$$

where $L^{(\mathrm{f})}$ is the Lagrangian density for the field, $L^{(\mathrm{m})}$ is the Lagrangian density for the matter and $L^{(\mathrm{int})}$ the Lagrangian density for the interaction. $L^{(\mathrm{m})}$ and $L^{(\mathrm{int})}$ depend on the kind of matter we are considering. As to $L^{(\mathrm{f})}$ one can see

(⁷) K. S. THORNE, D. L. LEE and A. P. LIGHTMAN: *Phys. Rev. D*, **7**, 3563 (1973).

that the most general expression which satisfies hypotheses 2) and 3) is

$$(3)\qquad L^{(f)} = C_{(1)}\psi\psi + C_{(2)}\psi_{\mu\nu}\psi^{\mu\nu} + C_{(3)}\psi_{\mu\nu;\sigma}\psi^{\mu\nu;\sigma} + \\ + C_{(4)}\psi_{\mu\nu;\sigma}\psi^{\mu\sigma;\nu} + C_{(5)}\psi_{;\sigma}\psi^{;\sigma} + C_{(6)}\psi_{;\sigma}\psi_{\nu}^{;\nu\sigma},$$

where semi-colons denote the covariant differentiation, ψ is the trace $\psi_\alpha{}^\alpha$ and $C_{(i)}$ are 6 arbitrary constants.

For the sake of simplicity we will refer here to the case of incoherent neutral pointlike particles, but this, as will be seen, does not reduce the generality of the following considerations on the spin-1 component of the gravitational field. In this case it is

$$(4)\qquad L^{(m)} = -\,T^{\alpha\beta}a_{\alpha\beta}$$

and

$$(5)\qquad L^{(\text{int})} = fT^{\alpha\beta}\psi_{\alpha\beta}\,,$$

where $a_{\alpha\beta}$ is the fundamental metric tensor of the pseudo-Euclidean space-time, reducing in Cartesian co-ordinates to $\eta_{\alpha\beta} = \text{diag}\,(+1, -1, -1-1)$ and f is the coupling constant (related to the gravitational constant G by $G = f^2/8\pi$). $T^{\alpha\beta}$ is the energy-momentum tensor of the matter which, in the above-mentioned case of incoherent, neutral pointlike particles, is given by

$$(6)\qquad T^{\alpha\beta} = (-a)^{-\frac{1}{2}}\,\Sigma_{(n)}\,m_{(n)}\int \mathrm{d}s\,\dot{x}^\alpha\dot{x}^\beta\,\delta^4(x - z_{(n)})\,,$$

where a is the determinant of the matrix $\|a_{\alpha\beta}\|$, $m_{(n)}$ is the rest mass of the n-th particle, x^α are the general co-ordinates of the points of the space-time, while $z^\alpha_{(n)}$ are the co-ordinates of the n-th particle, $\dot{x}^\alpha = \mathrm{d}x^\alpha/\mathrm{d}s$ and $\mathrm{d}s = (a_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta)^{\frac{1}{2}}$. The integration in eq. (6) is made over a region containing all the particles.

The first-order field equations are obtained by varying the gravitational potential in eq. (1) in which eqs. (2)-(5) have been substituted. One obtains

$$(7)\qquad -2C_{(1)}a^{\alpha\beta}\psi - 2C_{(2)}\psi^{\alpha\beta} + 2C_{(3)}\Box\,\psi^{\alpha\beta} + C_{(4)}\psi^{\sigma(\alpha;\beta)}{}_{\sigma} + \\ + 2C_{(5)}a^{\alpha\beta}\Box\,\psi + C_{(6)}(\psi^{;\alpha\beta} + a^{\alpha\beta}\psi^{\nu\sigma}{}_{;\nu\sigma}) = fT^{\alpha\beta}\,.$$

It can be noticed that in the static case the first two terms give rise to solutions of the Yukawa form (that is exponentially decreasing from a pointlike source) instead of the Newtonian form. This corresponds, from a quantum-mechanical point of view, to having gravitons with nonzero rest mass. If we assume hypothesis 5) to be valid, that is that gravitational interaction has a long range, we have to exclude such terms. We will, therefore, assume

$$C_{(1)} = C_{(2)} = 0 \,. \tag{8}$$

The first-order equations of motion of the n-th particle are obtained by varying the co-ordinates of the n-th particle in the action integral in which eqs. (4) and (5) have been substituted. It is

$$[(1 + f\psi_{\mu\nu}\dot{z}^\mu\dot{z}^\nu)\dot{z}_\alpha - 2f\psi_{\alpha\lambda}\dot{z}^\lambda]_{;\sigma}\dot{z}^\sigma = -f\psi_{\mu\nu;\alpha}\dot{z}^\mu\dot{z}^\nu \,, \tag{9}$$

where the index n has been omitted for simplicity.

In cases of kinds of matter (for example, a continuum [8]) different from the one considered, eqs. (4) and (5) change, but it can be shown [8] that the form of eqs. (7) remains the same in each case. $T^{\alpha\beta}$ is the energy-momentum tensor in the absence of gravity and relevant to the considered kind of matter.

In any case the energy-momentum tensor is divergenceless. The fact that this would imply equations of motion inconsistent with eqs. (9) is what triggers the iterative procedure. The presence of the energy-momentum tensor as the source of the field equations (7) is required by the equivalence principle assumed in hypothesis 6). By definition the energy-momentum tensor is the one whose divergence equated to zero gives the equations of motion. The energy-momentum tensor which implies eqs. (9) is not given by (6), but can be easily obtained [4]. Substituting the new tensor on the r.h.s. of eqs. (7) will change such equations and, therefore, also the Lagrangian densities. This, in turn, will imply a change of the equations of motion. That is why the iteration begins. Here we are not primarily interested in the iteration, but we have mentioned it just to observe that the r.h.s. of eqs. (7) must be divergenceless, *i.e.*

$$T_{\alpha\beta}{}^{;\beta} = 0 \,, \tag{10}$$

at least up to terms whose order is higher than the one considered (here the first one). Moreover, it can be noticed that eqs. (10) are verified everywhere: inside and outside the matter.

3. – Elimination of the vector component of the gravitational tensor potential.

So far we have not used hypothesis 2) nor shown if it is satsfied.

Let us first show that the tensor potential $\psi_{\alpha\beta}$, solution of eqs. (7), generally contains the spin-1 part. After that, we will obtain that this vector component does not influence the observable effects, since the theory can be

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[8] G. SPINELLI: *Rend. Accad. Lincei* (in press, 1978).

described as well by a transformed potential which does not contain the spin-1 component.

Taking the divergence of both sides of eqs. (7) gives

$$(11)\qquad (2C_{(3)}+C_{(4)})\Box\,\psi^{\alpha\beta}{}_{;\beta}+(C_{(4)}+C_{(6)})\,\psi_{\beta\sigma}{}^{;\beta\sigma\alpha}+(2C_{(5)}+C_{(6)})\Box\,\psi^{;\alpha}=0\,.$$

Taking another divergence, one gets

$$(12)\qquad (2C_{(3)}+2C_{(4)}+C_{(6)})\Box\,\psi^{\alpha\beta}{}_{;\alpha\beta}+(2C_{(5)}+C_{(6)})\Box\Box\,\psi=0\,.$$

Because of eqs. (10), eqs. (11) and (12) are valid everywhere: outside and inside the matter. By setting, as usual, $\psi_{\alpha\beta}=0$ on the improper hypersurface which is the boundary of the four-dimensional pseudo-Euclidean space-time, an integration of eq. (12) can be made, eliminating a d'Alembertian and obtaining

$$(13)\qquad (2C_{(3)}+2C_{(4)}+C_{(6)})\,\psi^{\alpha\beta}{}_{;\alpha\beta}+(2C_{(5)}+C_{(6)})\Box\,\psi=0\,.$$

Another scalar equation can be obtained by taking the trace of both sides of eqs. (7). It is

$$(14)\qquad (2C_{(4)}+4C_{(6)})\,\psi^{\alpha\beta}{}_{;\alpha\beta}+(2C_{(3)}+8C_{(5)}+C_{(6)})\Box\,\psi=fT\,.$$

It can be noticed that the trace of the energy-momentum tensor on the r.h.s. of eq. (14) is zero outside the matter only. Outside the matter the system of eqs. (13) and (14) is homogeneous and gives

$$(15)\qquad \psi_{\alpha\beta}{}^{;\alpha\beta}=0$$

and

$$(16)\qquad \Box\,\psi=0\,,$$

if the coefficients are such that $(2C_{(3)}+2C_{(4)}+C_{(6)})(2C_{(3)}+8C_{(5)}+C_{(6)})\neq$ $\neq(2C_{(5)}+C_{(6)})(2C_{(4)}+4C_{(6)})$. Otherwise $\psi_{\alpha\beta}{}^{;\alpha\beta}$ is proportional to $\Box\,\psi$, but it is not necessarily zero. If the above-mentioned condition for the coefficients is satisfied, eqs. (15) and (16) hold outside the matter and can be substituted into eq. (11) which gives

$$(17)\qquad \Box\,\psi_{\alpha\beta}{}^{;\beta}=0\,,$$

outside the matter and under the condition $2C_{(3)}+C_{(4)}\neq 0$ to be added to the other condition.

In any case eqs. (16) and (17) are *not* sufficient to conclude that the vector component $\psi_{\alpha\beta}{}^{;\beta}$ and the scalar component ψ cannot be radiated or absorbed.

Indeed eqs. (16) and (17) hold outside the matter only. Therefore, eqs. (7) imply the presence in $\psi_{\alpha\beta}$ of a vector component notwithstanding the fact that $T_{\alpha\beta}{}^{;\beta} = 0$ everywhere.

However, it can be noticed that what is observable is not the tensor potential but the motion of the matter. The potential is a quantity which is convenient in order to calculate the motion of the matter. It is, therefore, possible to transform the potential $\psi_{\alpha\beta}$ if the field equations (7) and the equations of motion (9) are consequently transformed.

A general transformation of the potential should imply a form of the equations of motion difficult to be handled in practical cases. Therefore, it is here required that the transformations leave at least the equations of motion unchanged. This requirement presents many advantages.

First of all one can notice that the equations of motion (9) are the same that should be obtained by taking the geodesic motion in a proper Riemannian space-time and translating it into the pseudo-Euclidean space-time by the Rosen procedure [9,10]. Indeed the geodesic motion of a particle of co-ordinates $z^{*\alpha}$ in a Riemannian space-time with fundamental metric tensor $g_{\alpha\beta}$ are given by

$$\mathrm{D}\dot{z}^{*\alpha}/\mathrm{d}s^* = 0\,, \tag{18}$$

where $\mathrm{D}/\mathrm{d}s$ stands for the total covariant differentiation. If one translates eqs. (18) into the pseudo-Euclidean space-time by the Rosen procedure [9,10], *i.e.* by the rules

$$z^{*\alpha} = z^\alpha \tag{19}$$

and

$$g_{\alpha\beta} = a_{\alpha\beta} - 2f\psi_{\alpha\beta}\,, \tag{20}$$

one obtains the equations of motion (9) to first order. This fact permits convenient interpretations of the observable effects.

Moreover, and this is the main reason why we leave eqs. (9) unchanged, the calculations of the effects have already been done by the use of this formula and, therefore, it is much more convenient to change only the potential to be put into eqs. (9) instead of changing also the form of the equations of motion.

We need, therefore, the most general transformation which leaves eqs. (9) unchanged. Differently from what happens in the electromagnetic case, here we have to consider a co-ordinate transformation to be performed together with the potential transformation. The most general set of first-order transfor-

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(9) N. Rosen: *Phys. Rev.*, **57**, 147 (1940).
(10) G. Cavalleri and G. Spinelli: *Nuovo Cimento*, **21** B, 27 (1974).

mations under which eqs. (9) are invariant is given by

$$\bar{z}^\alpha = z^\alpha + 2f\Lambda^\alpha\,, \tag{21}$$

$$\bar{\psi}_{\alpha\beta} = \psi_{\alpha\beta} + \Lambda_{\alpha;\beta} + \Lambda_{\beta;\alpha}\,, \tag{22}$$

where Λ^α is an arbitrary four-vector.

The theory is said to be gauge invariant if also the field equations are invariant under the set of transformations (21) and (22). Even if this case is not the purpose of this work, let us briefly recall the results in order to show the differences with the general case. The assumption of the hypothesis of gauge invariance uniquely determines ([2]) the values of the coefficients $C_{(i)}$. The gauge-invariant, first-order field equations are

$$\Box\,\psi^{\alpha\beta} - \psi^{\sigma(\alpha;\beta)}{}_{\sigma} + \psi^{;\alpha\beta} + a^{\alpha\beta}(\psi^{\sigma\lambda}{}_{;\sigma\lambda} - \Box\,\psi) = fT^{\alpha\beta} \tag{23}$$

(one could as well multiply all the coefficients $C_{(i)}$ by an arbitrary constant, but this would be only a redefinition of the constant f). In this case, because of gauge invariance one can operate the transformations (21) and (22), moreover, requiring that the vector component and the scalar components vanish, *i.e.*

$$\bar{\psi}_{\mu\nu}{}^{;\mu} = 0\,, \tag{24}$$

$$\bar{\psi}_{\mu}^{\ \mu} = 0\,, \tag{25}$$

and, as a consequence of eqs. (24), also $\bar{\psi}_{\mu\nu}{}^{;\mu\nu} = 0$. The vector Λ^α has four arbitrary components, while the conditions (24) and (25) correspond to five scalar conditions. It can be easily shown ([11]) that eqs. (24) and (25) are compatible with the field equations (23) outside the matter. In other words one can conclude that, if the coefficients are the ones of eqs. (23), *i.e.* if the field equations and the equations of motion are gauge invariant, the field can be described *outside the matter* by a tensor not containing the spin-1 and the two spin-0 components.

Before passing to the general case, let us consider the intermediate particular case in which the coefficients satisfy the relationships

$$C_{(1)} = C_{(2)} = 0\,, \tag{8}$$

$$2C_{(5)} + C_{(6)} = 0\,, \tag{26}$$

$$2C_{(3)} + 2C_{(4)} + C_{(6)} = 0\,. \tag{27}$$

([11]) W. Wiss: *Helv. Phys. Acta*, **38**, 469 (1965).

Equations (8) are the usual ones which we assume throughout the paper. As to eqs. (26) and (27), they show that, even if we are not in the maximum generality, we are considering a case more general than the one gauge invariant under eqs. (21) and (22). Indeed they leave one arbitrary parameter (besides the one linked to the choice of the constant f). In this case the field equations are invariant under the set of gauge transformations

$$\bar{z}^\alpha = z^\alpha + 2f\Lambda^{;\alpha}, \tag{28}$$

$$\bar{\psi}_{\alpha\beta} = \psi_{\alpha\beta} + 2\Lambda_{;\alpha\beta}, \tag{29}$$

which is less wide than the set of eqs. (21) and (22). By the arbitrary scalar function Λ we can require that

$$\bar{\psi}_{\alpha\beta}^{\;;\alpha\beta} = 0, \tag{30}$$

thus eliminating one scalar component.

Now the requirement $T_{\alpha\beta}^{\;;\beta} = 0$ eliminates the vector component of $\bar{\psi}_{\alpha\beta}$. Indeed eqs. (11), because of eq. (26), imply

$$(2C_{(3)} + C_{(4)})\Box\,\bar{\psi}^{\alpha\beta}_{\;\;;\beta} = 0. \tag{31}$$

If $2C_{(3)} + C_{(4)} \neq 0$, this implies that the vector component can neither be radiated nor absorbed. Since eqs. (31) hold everywhere, choosing $\bar{\psi}_{\alpha\beta}^{\;;\beta} = 0$ on the boundary of the four-dimensional region leads to

$$\bar{\psi}_{\alpha\beta}^{\;;\beta} = 0, \tag{32}$$

everywhere. Moreover, even the other scalar component $\bar{\psi}$ is not present outside the matter. Indeed the requirement (30) implies

$$\Box\Box\Lambda = -\tfrac{1}{2}\psi_{\alpha\beta}^{\;;\alpha\beta}. \tag{33}$$

By this equation we have already used the arbitrariness of the scalar Λ. Therefore, it is

$$\bar{\psi} = 0, \tag{34}$$

only if

$$\Box\Lambda = -\tfrac{1}{2}\psi \tag{35}$$

is already a consequence of eq. (33) and of the field equations. This is just the case since eqs. (33) and (35) imply

$$\Box\psi = \psi_{\alpha\beta}^{\;;\alpha\beta}. \tag{36}$$

Equations (14), (26) and (27) imply

$$(2C_{(4)}+4C_{(6)})(\psi_{\alpha\beta}{}^{;\alpha\beta}-\Box\,\psi)=fT\,,\tag{37}$$

which is equal to eq. (36) outside the matter and if $2C_{(4)}+4C_{(6)}\neq 0$.

Concluding, in this case (coefficients satisfying eqs. (8), (26) and (27)) we have that $T_{\alpha\beta}{}^{;\beta}=0$ implies, if $2C_{(3)}+C_{(4)}\neq 0$, the vanishing of the vector component (and of the related scalar component) everywhere and the vanishing of the remaining scalar component outside the matter if $2C_{(4)}+4C_{(6)}\neq 0$, as in the preceding case of gauge invariance under eqs. (21) and (22).

Let us now come to the general case in which the coefficients are completely generic (except for $C_{(1)}=C_{(2)}=0$ as already said).

We have already seen that, if $\psi_{\alpha\beta}$ is not transformed, eqs. (7) imply the presence of the vector component. One, therefore, asks if it is possible to eliminate this component by a proper transformation of the potentials which does not change the field equations. Equations (7) (always with $C_{(1)}=C_{(2)}=0$) are not gauge invariant under the set of eqs. (21) and (22) which are the most general first-order transformations under which the equations of motion are invariant. However, this does not exclude the possibility that one particular transformation leaves the field equations invariant and eliminates the vector component by taking into account the fact that $T_{\alpha\beta}{}^{;\beta}=0$. In order to search for this transformation, one should verify that the equations

$$\psi^{\alpha\beta}{}_{;\beta}+\Box\,\Lambda^{\alpha}+\Lambda_{\beta}^{;\alpha\beta}=0\,,\tag{38}$$

by which one imposes that the vector components of $\bar{\psi}_{\alpha\beta}$ vanish, are compatible with the equations

$$(2C_{(3)}+C_{(4)})\Box\,\Lambda^{(\alpha;\beta)}+(2C_{(4)}+2C_{(6)})\Lambda_{\sigma}^{;\sigma\alpha\beta}+(4C_{(5)}+2C_{(6)})\,a^{\alpha\beta}\Box\,\Lambda_{\sigma}^{;\sigma}=0\,,\tag{39}$$

which require that the transformations leave eqs. (7) unchanged. The fact is that eqs. (38) and (39) are 14 scalar equations in four unknown scalar components of Λ^{α}. One could not even hope that the compatibility is ensured by eqs. (11), since these amount to four scalar equations only. Thus it seems that this way is a blind alley.

Let us, therefore, abandon the requirement that the field equations be invariant under the performed transformations. In other words we will only ask that the motion be described by eqs. (9), where $\psi_{\alpha\beta}$ can now be obtained by equations different from eqs. (7).

It is useful to assume the transformations

$$\psi_{\alpha\beta}{}^{;\beta}=h\psi_{;\alpha}\,,\tag{40}$$

where h is an arbitrary constant.

Because of the gauge noninvariance of eqs. (7), the use of eq. (40) changes eqs. (7) and, therefore, the solutions of such equations. On the other hand, eqs. (40) are compatible with eqs. (22) and, therefore, the solutions of the new field equations, put in the equations of motion, will give the same motion for the particle (properly trasforming the co-ordinates).

Now some passages will be done, whose main aim is to make appear on the l.h.s. of the field equations the l.h.s. of eqs. (23), which, we know, give a pure spin-2 field. Obviously some other terms will appear and they will have to be interpreted. This is a procedure inverse with respect to the one used in the fundamental paper by SEXL (2).

We start from eqs. (7) in which $C_{(1)} = C_{(2)} = 0$ for having long-range fields and put $C_{(3)} = \frac{1}{2}$ in order to have the usual coupling constant f. By using eqs. (7) and (40), we get

$$\Box\,\psi^{\alpha\beta} + (2C_{(4)}h + C_{(6)})\,\psi^{;\alpha\beta} + (2C_{(5)} + C_{(6)}h)\,a^{\alpha\beta}\Box\,\psi = fT^{\alpha\beta}\,. \tag{41}$$

Always by using eqs. (40) one gets

$$\begin{aligned}\Box\,\psi^{\alpha\beta} + \psi^{;\alpha\beta} - \psi_\sigma{}^{(\alpha;\beta)\sigma} + a^{\alpha\beta}(\psi_{\sigma\lambda}{}^{;\sigma\lambda} - \Box\,\psi) +&\\ + (2C_{(4)}h + C_{(6)} + 2h - 1)\,\psi^{;\alpha\beta} + (2C_{(5)} + C_{(6)}h - h + 1)\,a^{\alpha\beta}\Box\,\psi = fT^{\alpha\beta}\,.&\end{aligned} \tag{42}$$

The constant h is still at our disposal. We can use it in order to make the coefficients of the last two terms of the l.h.s. equal and opposite. If one chooses

$$h = -\frac{C_{(6)} + 2C_{(5)}}{2C_{(4)} + C_{(6)} + 1}\,, \tag{43}$$

it is

$$\Box\,\psi^{\alpha\beta} + \psi^{;\alpha\beta} - \psi_\sigma{}^{(\alpha;\beta)\sigma} + a^{\alpha\beta}(\psi_{\sigma\lambda}{}^{;\sigma\lambda} - \Box\,\psi) + b\psi^{;\alpha\beta} - ba^{\alpha\beta}\Box\,\psi = fT^{\alpha\beta}\,, \tag{44}$$

where

$$b = 2C_{(4)}h + C_{(6)} + 2h - 1\,. \tag{45}$$

Taking the trace of both sides of eqs. (44) and using eqs. (40) gives

$$(3h - 3b - 2)\Box\,\psi = fT\,. \tag{46}$$

From eqs. (44) and (46) one can see that the original field equations can be represented by the two equations

$$\Box\,\chi^{\alpha\beta} + \chi^{;\alpha\beta} - \chi_\sigma{}^{(\alpha;\beta)\sigma} + a^{\alpha\beta}(\chi_{\sigma\lambda}{}^{;\sigma\lambda} - \Box\,\chi) = fT^{\alpha\beta} \tag{47}$$

and

$$\Box\,\xi = f_2\,T\,, \tag{48}$$

where

$$\psi_{\alpha\beta} = \chi_{\alpha\beta} - \frac{f_2}{f} a_{\alpha\beta}\xi . \tag{49}$$

Indeed eqs. (46) and (48), valid all over the space-time, imply

$$\xi = \frac{f_2}{f}(3h - 3b - 2)\psi . \tag{50}$$

Now substituting eqs. (49) in (47) and taking into account (50) gives eqs. (44) if the relationship

$$\frac{2f_2^2}{f^2}(3h - 2 - 3b) = b , \tag{51}$$

which determins f_2, is verified.

In this way the motion is described by eqs. (9) containing the potential $\psi_{\alpha\beta}$. The latter is obtained by eqs. (49) and, therefore, contains one spin-2 and one spin-0 component. Indeed $\chi_{\alpha\beta}$ and ξ are determined by eqs. (47) and (48). Because of the form of the eqs. (47), $\chi_{\alpha\beta}$ is a pure spin-2 tensor.

The potential $\psi_{\alpha\beta}$ is a pure spin-2 tensor when $f_2 = 0$, which corresponds to $b = 0$ because of eq. (51).

The case considered before, in which eqs. (8), (26) and (27) are verfied (always by putting $C_{(3)} = \frac{1}{2}$), is the case in which eq. (43) is meaningless. However, the coefficients of the last two terms of the l.h.s. of eqs. (42) are already equal and opposite. In this particular case, one has h still disposable and can, therefore, choose it to make those two above-mentioned last terms of eqs. (42) vanish. In this way one obtains again the preceding result; that is in this particular case the potential can be described by a pure spin-2 tensor.

4. – Conclusions.

The first-order linear field equations have been considered for a gravitational theory. In particular, it has been examined the problem of the influence of the zero divergence for the energy-momentum tensor $T_{\alpha\beta}$. Generally, even by a transformation of the potential, if one requires not to change the field equations and the equations of motion, it seems that $T_{\alpha\beta}^{\ ;\beta} = 0$ does not imply the vanishing of the vector component of $\psi_{\alpha\beta}$. Exceptions are the cases in which the coefficients satisfy particular relationships. If the theory becomes gauge invariant, not only the vector component but also the scalar components vanish ouside the matter. The same happens where the constant coefficients satisfy eqs. (8), (26) and (27).

In the general case one can describe the theory by means of a potential containing only one spin-2 and one spin-0 component, but he has to change the field equations. Without changing the equations of motion it is possible to obtain the tensor $\psi_{\alpha\beta}$ which appears in such equations by a proper function (49) of a tensor $\chi_{\alpha\beta}$ and of a scalar ξ. The tensor $\chi_{\alpha\beta}$ obeys the usual gauge-invariant equations and can, therefore, be represented, outside the matter, by a pure spin 2.

* * *

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APPENDIX

Decomposition of a symmetric tensor.

Let us show that a symmetric tensor $\psi_{\alpha\beta}$ can be decomposed in the following way:

(A.1) $$\psi_{\alpha\beta} = \phi_{\alpha\beta} + \varrho_{(\alpha;\beta)} + \sigma_{;\alpha\beta} + \lambda a_{\alpha\beta}\,,$$

where $\phi_{\alpha\beta}$ is a symmetric tensor having

(A.2) $$\phi_{\alpha\beta}^{\;\;;\beta} = 0$$

and

(A.3) $$\phi_{\alpha}^{\;\alpha} = 0\,,$$

where ϱ_{α} is a vector such that

(A.4) $$\varrho_{\alpha}^{\;;\alpha} = 0\,,$$

and where σ and λ are two scalar functions. In order uniquely obtain the terms of the decomposition (A.1), it is convenient to take the divergence, the double divergence and the trace of both sides of eqs. (A.1). One gets, taking into account eqs. (A.2)-(A.4),

(A.5) $$\psi_{\alpha\beta}^{\;\;;\beta} = \Box\,\varrho_{\alpha} + \Box\,\sigma_{;\alpha} + \lambda_{;\alpha}\,,$$

(A.6) $$\psi_{\alpha\beta}^{\;\;;\alpha\beta} = \Box\Box\,\sigma + \Box\,\lambda\,,$$

(A.7) $$\psi = \Box\,\sigma + 4\lambda\,.$$

Let us suppose, as usual, that $\psi_{\alpha\beta}$, $\phi_{\alpha\beta}$, ϱ_{α}, λ and σ all have zero values and zero derivatives on the improper hypersurface which is the boundary of the four-dimensional pseudo-Euclidean space-time.

By setting

$$\varphi = \Box\sigma + \lambda\,, \tag{A.8}$$

eq. (A.6) becomes

$$\Box\varphi = \psi_{\alpha\beta}{}^{;\alpha\beta}\,. \tag{A.9}$$

The r.h.s. of eq. (A.9) is known and, therefore, (A.9) univocally determines φ once given the above-mentioned boundary conditions. Equations (A.7) and (A.8) are a linear system in the unknown functions λ and $\Box\sigma$ and it can be solved with respect to them. Also σ will be univocally obtained (owing to the boundary conditions). The same happens for eqs. (A.5); indeed, at this point, the last two terms of their r.h.s. are known terms and the differential equations (with the boundary conditions) can be univocally solved with respect to ϱ_α. Eventually, $\phi_{\alpha\beta}$ is given by eqs. (A.1) as a difference.

One can also immediatly verify that, if $\psi = 0$ and $\psi_{\alpha\beta}{}^{;\beta} = 0$, the tensor $\psi_{\alpha\beta}$ is equal to $\phi_{\alpha\beta}$ only. Indeed in this case eq. (A.6) gives $\Box\varphi = 0$, which leads to $\varphi = 0$ (always owing to the boundary conditions). Then the system (A.6) and (A.7) gives $\lambda = 0$ and $\Box\sigma = 0$, which, in turn, leads to $\sigma = 0$. Equations (A.5) become $\Box\varrho_\alpha = 0$, from which one gets $\varrho_\alpha = 0$.

● RIASSUNTO

La teoria della gravitazione è qui trattata nella formulazione in teoria dei campi nello spazio-tempo pseudoeuclideo. La gravitazione è rappresentata per mezzo di un potenziale tensoriale $\psi_{\alpha\beta}$ simmetrico che contiene dunque una parte a spin 2, una parte a spin 1 ($\psi_{\alpha\beta}{}^{;\beta}$) e due a spin 0 ($\psi$ e $\psi_{\alpha\beta}{}^{;\alpha\beta}$). Si tratta il primo ordine della teoria occupandosi in particolare del caso nel quale le equazioni di campo non sono varianti di gauge. Si discute in quale senso sia possibile descrivere il campo con un tensore che non contenga la parte a spin 1 in virtù del fatto che il tensore energia-impulso è solenoidale.

Лагранжианные полевые теории гравитации первого порядка без калибровочной инвариантности.

Резюме (*). — Рассматривается теория гравитации, исходя из псевдо-эвклидивого пространства-временеи. Гравитация представляется с помощью симметричного тензорного потенциала $\psi_{\alpha\beta}$, содержащего одну часть со спином 2, одну часть со спином 1 ($\psi_{\alpha\beta}{}^{;\beta}$) и две части со спином 0 (ψ и $\psi_{\alpha\beta}{}^{;\alpha\beta}$). В этой работе рассматривается теория первого порядка. В частности, рассматривается случай, в котором уравнения поля не являются калибровочно инвариантными. Анализируется возможность описания поля с помощью тензора, не содержащего часть со спином 1, вследствие нулевой расходимости тензора энергии-импульса.

(*) *Переведено редакцией.*