ABSOLUTELY CONVERGENT SERIES IN THE CANONICAL INDUCTIVE LIMITS

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Introduction. The canonical inductive limit (c.i.l.) of a sequence of locally convex spaces (l.c.s.) E_n (n = 1, 2,...) is defined as the inner inductive limit of the sequence of l.c.s. E_n, such that E_n is continuously embedded into E_{n+1}, n = 1, 2,... . The c.i.l. $E = \operatorname{ind}_{n\to} E_n$ is called regular [1] if any bounded in E set is contained and bounded in one of the l.c.s. E_n. In a number of papers (see, e.g., [1-7]) sufficient conditions have been obtained under which the c.i.l. of a sequence of l.c.s. is regular. In particular, in [4, 5] the regularity criterion has been found for the c.i.l. of a sequence of normed spaces.

According to [8], the c.i.l. $E = \operatorname{ind}_{n\to} E_n$ of l.c.s. E_n is α-regular (β-regular, respectively) if every set which is bounded in E is contained in one of the spaces E_n (respectively, if each bounded in E set contained in one of the spaces E_n is also bounded in one of the spaces E_m). The c.i.l. $E = \operatorname{ind}_{n\to} E_n$ of l.c.s. E_n is regular if and only if the space E is α-regular and β-regular simultaneously.

Following Korobeinik [9] we say that the c.i.l. $E = \operatorname{ind}_{n\to} E_n$ of l.c.s. E_n has the property (Y_0) if for any sequence $(x_k)_{k=1}^{\infty}$ of elements of E, such that the series $\sum_{k=1}^{\infty} x_k$ is absolutely convergent in E, there exists a number n for which the set $\{x_k\}_{k=1}^{\infty}$ is contained in E_n, and the series $\sum_{k=1}^{\infty} x_k$ absolutely converges in the space E_n. We shall say that the c.i.l. $E = \operatorname{ind}_{n\to} E_n$ of the sequence of l.c.s. E_n has the property α-(Y_0) [respectively, β-(Y_0)], if for any sequence $(x_k)_{k=1}^{\infty}$ of elements of E, such that the series $\sum_{k=1}^{\infty} x_k$ is absolutely convergent in E, there exists a number n for which the set $\{x_k\}_{k=1}^{\infty}$ is contained in E_n [respectively, if for any sequence $(x_k)_{k=1}$ of elements of E, which is contained in some space E_n, and such that the series $\sum_{k=1}^{\infty} x_k$ absolutely converges in E, there exists a number m, for which the series $\sum_{k=1}^{\infty} x_k$ is absolutely convergent in E_m]. Obviously, a c.i.l. E has the property (Y_0) if and only if E has the properties α-(Y_0) and β-(Y_0) simultaneously. Let us notice also that the property (Y_0) plays an essential role in the studying of the absolutely representing l.c.s. (see for this, e.g., [9, 10]).

Korobeinik [9] has shown that every c.i.l. of a sequence of nuclear Frechet spaces, each of which is compactly embedded into the following one, has the property (Y_0). Abanin [11] has proved a stronger result, showing that any nuclear LN*-space (i.e., the nuclear c.i.l. of a sequence of normed spaces with compact embeddings) has the property (Y_0). Moreover, sufficient conditions under which the c.i.l. of a sequence of l.c. sequence spaces has the property (Y_0) have been presented in [12].

In the present paper we have obtained criteria for each of the properties α-(Y_0) and β-(Y_0), and, as a corollary, the criterion for the property (Y_0) of the c.i.l. of a sequence of normed spaces. In particular, we have proved that the regularity of a c.i.l. of normed spaces is equivalent to the property (Y_0) of this c.i.l.

1. Some Auxiliary Results. Following Makarov [4, 5], we shall say that the c.i.l. E of a sequence of normed spaces E_n with closed (in E_n) unit balls $S_n = \{x \in E_n \mid \|x\|_n \leqslant 1\}$ has the property (F_1) if each ball S_n is closed in E. We shall say also that the c.i.l. E of a sequence of normed spaces E_n with closed (in E_n) unit balls S_n has the property α-(F_1) [β-(F_1), respectively], if the closure of every ball S_n in E is contained in a certain space

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$E_{m(n)}$ (respectively, if the closure of every ball S_n in E_n in the topology of E is bounded in some space $E_{p(n)}$).

If a c.i.l. of a sequence of normal spaces has the property (F_1), then, obviously, it has the properties α-(F_1) and β-(F_1) simultaneously, with $m(n) = p(n) = n$, $\forall$ n = 1, 2,... .

Let Q be an absolutely convex absorbing set in some l.c.s. F. The symbol p_Q will denote the Minkovskii functional of the set Q

$$p_Q(x) \overset{\text{def}}{=} \inf\{\alpha > 0 \mid x \in \alpha \cdot Q\}, \quad \forall x \in F.$$

Let $\rho = (\rho_k)_{k=1}^{\infty}$ be a sequence of real numbers. We will write $\rho > 0$ if $\rho_k > 0$, $\forall$ k = 1, 2,... . For the c.i.l. E of a sequence of normed spaces E_n with closed (in E_n) unit balls S_n the symbols $U_{\rho,n}$ will denote the set $\sum_{k=1}^{n} \rho_k S_k$, that is, the arithmetic sum of the sets $\rho_k \cdot S_k$, k = 1,...,n. Put $U_\rho \overset{\text{def}}{=} \bigcup_{n=1}^{\infty} U_{\rho,n}$. Obviously, for any sequence $\rho = (\rho_k)_{k=1}^{\infty}$ the relations $U_{\rho,n} \subseteq U_{\rho,n+1}$, $\forall$ n = 1, 2,... hold, and U_ρ is an absolutely convex set.

LEMMA 1. Let E be a c.i.l. of a sequence of normed spaces E_n with closed in (E_n) unit balls S_n. Then the collection $\{U_\rho\}_{\rho>0}$ forms a basis of neighborhoods (of the origin) in E.

Proof. It is easy to see that U_ρ is an environment in E, $\forall \rho > 0$. Let U be any neighborhood (of the origin) in E. Then the set $\frac{1}{2^k}U$, $\forall$ k = 1, 2,... is also a neighborhood in E, and therefore there exists a number $\rho_k > 0$ such that $\frac{1}{2^k}U \supseteq \rho_k \cdot S_k$. Consequently, $U \supseteq \sum_{k=1}^{n} \frac{1}{2^k}U \supseteq \sum_{k=1}^{n} \rho_k S_k = U_{\rho,n}$, $\forall$ n = 1, 2,..., from which $U \supseteq U_\rho$. We can also assume that $\rho_k < 1/2^k$, $\forall$ k = 1, 2,... .

Remark. Notice that a result corresponding to Lemma 1 in the case of a sequence of arbitrary l.c.s. is also true. Namely, let $E = \operatorname{ind}_{n\to} E_n$ be the c.i.l. of l.c.s. E_n with the basis V_n of absolutely convex neighborhoods, n = 1, 2,... . Then the collection of all possible sets $u_v \overset{\text{def}}{=} \bigcup_{k=1}^{\infty} u_{v,k}$, where $u_{v,k} \overset{\text{def}}{=} \sum_{i=1}^{k} v_i$, $v_i \in V_i$, i = 1,...,k, k = 1, 2,..., forms a basis of neighborhoods (of the origin) in E.

Further, we shall assume that if E is a c.i.l. of a sequence of normed spaces E_n and $x \in E \setminus E_n$, then $p_{S_n}(x) = \|x\|_n = +\infty$. Moreover, henceforth, we shall assume (without loss of generality) that for the c.i.l. E of a sequence of normed spaces E_n with unit balls S_n the relations $S_n \subseteq S_{n+1}$, $\forall$ n = 1, 2,... hold. Thus, if $\|\cdot\|_n$ is the norm in the space E_n, then $\|x\|_{n+1} \leqslant \|x\|_n$, $\forall x \in E_n$, $\forall n = 1, 2, \ldots$

LEMMA 2. Let E be the c.i.l. of a sequence of normed spaces E_n, which has the property α-(F_1). Let $\forall$ n = 1, 2,...,m(n)($\geqslant$n) be such a number that the closure $\bar{S}_n$ of the unit ball S_n in E is contained in $E_{m(n)}$. Then $\forall n = 1, 2, \ldots, \forall x \in E \setminus E_{m(n)}, \forall C > 0, \exists \rho = \rho(n, x, C) > 0$: $1/2 < \rho_n < 1$, $\rho_k < 1/2^{k-1}$, $\forall k \neq n$, and $p_{U_\rho}(x) \geqslant C$.

Proof. Let us fix arbitrary $n \geqslant 1$, $x \in E \setminus E_{m(n)}$, $C > 0$. Then $\frac{x}{C} \notin \frac{1}{2}\bar{S}_n$. Consequently, by virtue of Lemma 1 there exists $\tilde{\rho} > 0$, $\tilde{\rho}_k < 1/2^k$, $\forall$ k = 1, 2,..., such that $\frac{x}{C} \notin \frac{1}{2}S_n + U_{\tilde{\rho}}$. Put

$$\rho_k = \begin{cases} \tilde{\rho}_k, & k \neq n, \\ \frac{1}{2} + \tilde{\rho}_n, & k = n. \end{cases}$$

Then $\frac{1}{2} < \rho_n < 1$, $0 < \rho_k < \frac{1}{2^{k-1}}$, $\forall k \neq n$, and $\frac{x}{C} \notin U_\rho$, i.e., $p_{U_\rho}(x) \geqslant C$.

LEMMA 3. Let E be the c.i.l. of a sequence of normed spaces E_n, with the property β-(F_1). Let $\forall$ n = 1, 2,...,p(n)($\geqslant$n) be a number such that the closure $\bar{S}_n \cap E_n$ of the unit ball S_n in E_n in the topology of E is bounded in $E_{p(n)}$. Then $\forall n = 1, 2, \ldots, \exists \alpha_n > 0$, $\alpha_n \leqslant 1$: $\forall x \in E_n, \exists \rho = \rho(n, x) > 0$: $1/2 < \rho_n < 1$, $\rho_k < 1/2^{k-1}$, $\forall k \neq n$, and $p_{U_\rho}(x) \geqslant \alpha_n \cdot \|x\|_{p(n)}$.

Proof. Let us fix any $n \geqslant 1$. Let $\bar{S}_n \cap E_n \subseteq \beta_n \cdot S_{p(n)}$, $1 \leqslant \beta_n < +\infty$. Put $\alpha_n = 1/\beta_n$. If x = 0, then for an arbitrary ρ, $p_{U_\rho}(x) \geqslant \alpha_n \cdot \|x\|_{p(n)}$. Therefore, as ρ for x = 0 we can take any

$\rho > 0$, such that $1/2 < \rho_n < 1$ and $\rho_k < 1/2^{k-1}$, $\forall k \neq n$. Let $x \in E_n \setminus \{0\}$ and $\|x\|_{p(n)} = c_{p(n)}$. Then $\frac{x}{c_{p(n)}} \notin \frac{1}{2} S_{p(n)}$ and $\frac{\beta_n}{c_{p(n)}} x \notin \frac{1}{2}(\bar{S}_n \cap E_n)$; thus $\frac{\beta_n}{c_{p(n)}} x \notin \frac{1}{2} \bar{S}_n$, where $\bar{S}_n$ is the closure in E of the closed (in E_n) unit ball S_n of the space E_n. By Lemma 1 $\exists \tilde{\rho} > 0$: $\tilde{\rho}_k < 1/2^k$, k = 1, 2,... and $\frac{\beta_n}{c_{p(n)}} x \notin \frac{1}{2} S_n + U_{\tilde{\rho}}$. Put

$$\rho_k = \begin{cases} \tilde{\rho}_k, & k \neq n, \\ {}^1/_2 + \tilde{\rho}_k, & k = n. \end{cases}$$

Then $\frac{\beta_n}{c_{p(n)}} x \notin U_\rho$ and $p_{U_\rho}(x) \geqslant c_{p(n)}/\beta_n = \alpha_n \cdot \|x\|_{p(n)}$.

COROLLARY. Let E be a c.i.l. of a sequence of normed spaces E_n, which has the property (F_1). Then $\forall n = 1, 2, \ldots, \forall x \in E, \forall C > 0, \exists \rho' = \rho(n, x, C) > 0$: $1/2 < \rho_n < 1, \rho_k < \frac{1}{2^{k-1}}$, $\forall k \neq n$, and $p_{U_\rho}(x) \geqslant \min\{\|x\|_n, C\}$.

The above corollary follows immediately from Lemmas 2 and 3 by virtue of the fact that if $E = \operatorname{ind}_{n\to} E_n$ has the property (F_1) then m(n) = p(n) = n and $\beta_n = \alpha_n = 1$, $\forall$ n = 1, 2,... .

LEMMA 4. Let E be a c.i.l. of a sequence of normed spaces E_n. Then the following equality holds: $\forall \rho > 0, \forall x \in E$, $p_{U_\rho}(x) = \inf_{n \in \mathbf{N}} p_{U_{\rho,n}}(x)$.

Proof. Let us fix arbitrary $\rho > 0$ and $x \in E$. Since $U_{\rho,n} \subseteq U_\rho$, $\forall$n = 1, 2,..., then $p_{U_\rho}(x) \leqslant \inf_{n \in \mathbf{N}} p_{U_{\rho,n}}(x)$. We shall prove the opposite inequality. Let $\inf_{n \in \mathbf{N}} p_{U_{\rho,n}}(x) = 0$. Then $p_{U_\rho}(x) \geqslant \inf_{n \in \mathbf{N}} p_{U_{\rho,n}}(x)$. Let now $\inf_{n \in \mathbf{N}} p_{U_{\rho,n}}(x) = \alpha > 0$. Suppose that $p_{U_\rho}(x) = \beta < \alpha$. Then $\exists c$: $\beta < c < \alpha$ and $x \in c \cdot U_\rho = \bigcup_{n=1}^\infty c \cdot U_{\rho,n}$, from which $\exists n$: $x \in c \cdot U_{\rho,n}$. But in this case $p_{U_{\rho,n}}(x) \leqslant c$ and $\inf_{n \in N} p_{U_{\rho,n}}(x) \leqslant c < \alpha$. We have obtained a contradiction. Consequently, $\forall \rho > 0\ \forall x \in E\ p_{U_\rho}(x) = \inf_{n \in \mathbf{N}} p_{U_{\rho,n}}(x)$.

2. Criterion for the Property α-(Y_0). THEOREM 1. Let E be a c.i.l. of a sequence of normed spaces E_n. The following conditions are equivalent:

1) E has the property α-(Y_0);

2) E is an α-regular c.i.l.;

3) E has the property α-(F_1).

Proof. The implication 2) $\Rightarrow$ 1) is obvious. We shall prove the implication 3) $\Rightarrow$ 2). Let the space $E = \operatorname{ind}_{n\to} E_n$ have the property α-(F_1). We shall show that the space E is α-regular. Suppose the contrary, that is, that the space E is not α-regular. Then there exist a bounded in E set $\{x_k\}_{k=1}^\infty$ and an increasing sequence of natural numbers $(n_k)_{k=1}^\infty$, such that $x_k \in E_{n_{k+1}} \setminus E_{n_k}$, $\forall$ k = 1, 2,... . Let $\bar{S}_k$ be the closure in E of the closed (in E_k) unit ball S_k of the space E_k, k = 1, 2,... . Since E has the property α-(F_1), the ball $\bar{S}_k$ is contained in some space $E_{m(k)}$. We can always assume that $n_k \geqslant m(k)$, $\forall$ k = 1, 2,... . Then by Lemma 2 $\forall$ k = 1, 2,... there exists a sequence $\rho^{(k)} = (\rho_j^{(k)})_{j=1}^\infty > 0$ such that $1/2 < \rho_k^{(k)} < 1$, $\rho_j^{(k)} < 1/2^{j-1}$, $\forall$ j $\neq$ k and $p_{U_{\rho^{(k)}}}(x_k) \geqslant k$. Put $\rho_k = \min\{\rho_k^{(m)} \mid 1 \leqslant m \leqslant k\}$, k = 1, 2,... . Then $\forall$ k = 1, 2,..., if n $\leqslant$ k, we have

$$p_{U_{\rho,n}}(x_k) = p_{\sum_{j=1}^n \rho_j S_j}(x_k) \geqslant p_{2S_k}(x_k) = \frac{1}{2} p_{S_k}(x_k) = +\infty.$$

However, if n > k, then

$$p_{U_{\rho,n}}(x_k) \geqslant p_{2S_k + \sum_{j=k+1}^n \rho_j S_j}(x_k) = \frac{1}{4} p_{S_k/2 + \sum_{j=k+1}^n \rho_j S_j/4}(x_k) \geqslant$$

$$\geqslant \frac{1}{4} p_{\rho_k^{(k)} S_k + \sum_{j=k+1}^n \rho_j^{(k)} S_j/4}(x_k) \geqslant \frac{1}{4} p_{\sum_{j=k}^n \rho_j^{(k)} S_j}(x_k) \geqslant \frac{1}{4} p_{U_{\rho^{(k)},n}}(x_k) \geqslant \frac{1}{4} p_{U_{\rho^{(k)}}}(x_k) \geqslant \frac{k}{4}.$$

Consequently, by virtue of Lemma 4, $\sup_{k\in\mathbf{N}} p_{U_\rho}(x_k) \geqslant 1/4 \sup_{k\in\mathbf{N}} k = +\infty$. This contradicts the boundedness of the sequence $\{x_k\}_{k=1}^{\infty}$ in E. Thus the space E is α-regular.

Suppose now that E has the property α-(Y_0), but it is not α-regular. Then there exists a bounded in E set Q, which is not contained in any of the spaces E_n. Therefore ∀ n = 1, 2,... $\exists x_n \in Q : x_n \notin E_n$. The series $\sum_{k=1}^{\infty} x_k/2^k$ absolutely converges in E, but the set $\{x_k/2^k\}_{k=1}^{\infty}$ is not contained in any of the spaces E_n. Consequently, E does not have the property α-(Y_0). We have obtained a contradiction; thus 1) ⇒ 2).

Finally, because the closure in E of a bounded in E set is again bounded in E, the implication 2) ⇒ 3) follows.

COROLLARY. Let E be a c.i.l. of a sequence of normed spaces E_n, such that E_n is closed in E, ∀n = 1, 2,... . Then the space E is α-regular.

The above corollary has been obtained in [13] in a different way (for a wider class of l.c.s.).

3. Criterion for the Property β-(Y_0). THEOREM 2. Let E be a c.i.l. of a sequence of normed spaces E_n, which has the property β-(F_1). Then the space E has the property β-(Y_0).

Proof. Suppose that the space E has the property β-(F_1) but does not have the property β-(Y_0). Then there exists a sequence $(x_k)_{k=1}^{\infty}$ of elements of E, contained in some space E_q, such that the series $\sum_{k=1}^{\infty} x_k$ is absolutely convergent in E, but does not absolutely converge in any of E_m, m ⩾ q. Consequently, there exists an increasing sequence of entire nonnegative numbers $(n_k)_{k=q}^{\infty}$, such that $n_q = 0$ and

$$\sum_{l=n_k+1}^{n_{k+1}} \| x_l \|_{p(k)} \geqslant \frac{1}{\alpha_k}, \quad k = q, q+1, \ldots . \tag{1}$$

Here the number α_k, $\forall k \geqslant q$, is chosen accordingly to Lemma 3, and $p_k(k)$ (⩾k), as in Lemma 3, denotes the number for which the closure $\bar{S}_k \cap E_k$ of the closed (in E_k) unit ball S_k of the space E_k in E_k in the topology of E is bounded in $E_{p(k)}$. Applying Lemma 3 we obtain that $\forall k \geqslant q, \exists \rho^{(k)} = (\rho_n^{(k)})_{n=1}^{\infty} > 0$: $1/2 < \rho_k^{(k)} < 1$, $\rho_n^{(k)} < 1/2^{n-1}$, $\forall n \neq k$, and $p_{U_{\rho^{(k)}}}(x_l) \geqslant \alpha_k \cdot \| x_l \|_{p(k)}$, $\forall l = n_k + 1, \ldots, n_{k+1}$. Put $\rho_k = \min\{\rho_k^{(m)} \mid q \leqslant m \leqslant k\}$, $k \geqslant q$. Then $(\rho_k)_{k=q}^{\infty} > 0$ and $\rho_k < \frac{1}{2^{k-1}}$, $\forall k \geqslant q$. Let $\tilde{U}_{\rho,k} = \sum_{j=q}^{k} \rho_j S_j$, $k \geqslant q$, and $\tilde{U}_\rho = \bigcup_{k=q}^{\infty} \tilde{U}_{\rho,k}$. The set $\tilde{U}_\rho$ is an absolutely convex environment in E. Hence, for each k ⩾ q, if q ⩽ n ⩽ k, then $\forall l = n_k + 1, \ldots, n_{k+1}$

$$p_{\tilde{U}_{\rho,n}}(x_l) = p_{\sum_{j=q}^{n} \rho_j S_j}(x_l) \geqslant 1/2 \| x_l \|_k \geqslant 1/2 \| x_l \|_{p(k)}.$$

If, however, n > k, then $\forall l = n_k + 1, \ldots, n_{k+1}$

$$p_{\tilde{U}_{\rho,n}}(x_l) = p_{\sum_{j=q}^{n}\rho_j S_j}(x_l) \geqslant p_{2S_k + \sum_{j=k+1}^{n}\rho_j S_j}(x_l) \geqslant \frac{1}{4} p_{\sum_{j=1}^{n}\rho_j^{(k)} S_j}(x_l) \geqslant \frac{1}{4} p_{U_{\rho(k)}}(x_l) \geqslant \frac{\alpha_k}{4} \| x_l \|_{p(k)}.$$

Then, by Lemma 4 applied to the sequence of normed spaces $(E_n)_{n=q}^{\infty}$, $\forall k \geqslant q, \forall l = n_k + 1, \ldots, n_{k+1}$ × $p_{\tilde{U}_\rho}(x_l) \geqslant \frac{\alpha_k}{4} \| x_l \|_{p(k)}$. Therefore $\sum_{l=1}^{\infty} p_{\tilde{U}_\rho}(x_l) = \sum_{k=q}^{\infty} \sum_{l=n_k+1}^{n_{k+1}} p_{\tilde{U}_\rho}(x_l) \geqslant \sum_{k=q}^{\infty} \frac{\alpha_k}{4} \sum_{l=n_k+1}^{n_{k+1}} \| x_l \|_{p(k)} = +\infty$ by virtue of relations (1). This contradicts the absolute convergence of the series $\sum_{k=1}^{\infty} x_k$ in E.

THEOREM 1. Let E be a c.i.l. of a sequence of normed spaces E_n, which has the property β-(Y_0). Then the space E is β-regular.

Proof. Suppose that the space E is not β-regular. Then there exist a number n ⩾ 1 and a sequence $(x_k)_{k=n}^{\infty}$ of elements of E_n, bounded in E, but such that $\| x_k \|_k \geqslant 2^k$, $\forall k \geqslant n$. The series $\sum_{k=n}^{\infty} \frac{x_k}{2^k}$ absolutely converges in E; however, $\forall m \geqslant n$

$$\sum_{k=n}^{\infty} \frac{1}{2^k} \| x_k \|_m \geqslant \sum_{k=m}^{\infty} \frac{\| x_k \|_k}{2^k} = +\infty,$$

which contradicts the assumed property β-(Y_0) of E. Consequently, the space E is β-regular.

The following theorem directly follows from Theorems 2 and 3, and from the fact that the β-regularity of the c.i.l. E of a sequence of normed spaces implies the property β-(F_1) of E.

THEOREM 4. Let E be the c.i.l. of a sequence of normed spaces E_n. The following conditions are equivalent:

1) E has the property β-(Y_0);

2) E is a β-regular c.i.l.;

3) E has the property β-(F_1).

Remark. There exist c.i.l. of Banach spaces possessing the property α-(Y_0), but not the property β-(Y_0); finally, there are ones which have neither the property α-(Y_0) nor β-(Y_0). By virtue of Theorems 1 and 4 the examples corresponding to this situation are the examples presented in [8].

4. Criterion for the Property (Y_0). From Theorems 1 and 4 directly follows:

THEOREM 5. Let E be a c.i.l. of a sequence of normed spaces E_n. The following conditions are equivalent:

1) E has the property (Y_0);

2) E is a regular c.i.l.

We will show that each of conditions 1) and 2) of Theorem 5 is equivalent to the following one:

3) there exists an equivalent to $(E_n)_{n=1}^{\infty}$ sequence of normed spaces $(\tilde{E}_n)_{n=1}^{\infty}$, such that the space $E = \operatorname{ind}_{n\to} \tilde{E}_n$ has the property (F_1).

Here two sequences of l.c.s. $(G_n)_{n=1}^{\infty}$ and $(\tilde{G}_n)_{n=1}^{\infty}$ are called equivalent if $\forall n \exists m$: $G_n \hookrightarrow \tilde{G}_m$, $\forall p \exists l$: $\tilde{G}_p \hookrightarrow G_l$, where $\hookrightarrow$ is the symbol of a continuous embedding.

Indeed, the implication 3) $\Rightarrow$ 2) follows from Theorem 1 in [5]. Moreover, it is easy to show the implication 3) $\Rightarrow$ 2) with the help of Theorems 1 and 4 of the present paper. The validity of the implication 2) $\Rightarrow$ 3) follows immediately from Theorem 3 in [4]. It follows also from the simple fact that for the regular c.i.l. $E = \operatorname{ind}_{n\to} E_n$ the sequence $(\tilde{E}_n)_{n=1}^{\infty}$ of the normed spaces $\tilde{E}_n \stackrel{\text{def}}{=} \bigcup_{\alpha>0} \alpha\cdot\bar{S}_n$ with closed unit balls $\bar{S}_n$ ($\bar{S}_n$ is the closure of the unit ball S_n of the space E_n in E) is equivalent to $(E_n)_{n=1}^{\infty}$, and the space $E = \operatorname{ind}_{n\to} \tilde{E}_n$ has the property (F_1).

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LITERATURE CITED

1. L. V. Kantorovich and G. P. Akilov, Functional Analysis in Normed Spaces [in Russian], Fizmatgiz, Moscow (1959).
2. J. Sebastião e Silva, On Some Classes of Locally Convex Spaces Important in Applications [Russian translation], Vol. 1, Matematika (1957), pp. 60-77.
3. J. Dieudonne and L. Schwartz, Duality in the Spaces $(\mathcal{F})$ and $(L\mathcal{F})$ [Russian translation], Vol. 2, 2nd ed., Matematika (1958), pp. 77-107.
4. B. M. Makarov, "Inductive limits of normed spaces," Dokl. Akad. Nauk SSSR, 119, No. 6, 1092-1094 (1958).
5. B. M. Makarov, "Inductive limits of normed spaces," Vestn. Leningr. Gos. Univ., No. 13, Issue 3, 50-58 (1965).
6. K. Floret, "On bounded sets in inductive limits of normed spaces," Proc. AMS, 75, No. 2, 221-225 (1979).
7. K. Floret, "Some aspects of the theory of locally convex inductive limits," Funct. Anal. Surv. and Recent Results 2, Proc. 2nd Conf., Paderborn (1979).
8. B. M. Makarov, "Some pathological properties of inductive limits of Banach spaces," Usp. Mat. Nauk, 18, No. 3, 171-178 (1963).
9. Yu. F. Korobeinik, "A dual problem. 1. General results. Applications to Frechet spaces," Mat. Sb., 97, No. 2, 193-229 (1975).
10. Yu. F. Korobeinik, "Representing systems," Usp. Mat. Nauk, 36, No. 1, 73-126 (1981).
11. A. V. Abanin, "The property (Y_0)," Izv. SKNTs VSh. Est. Nauki, No. 4, 31-32 (1981).

12. Yu. F. Korobeinik, "A class of spaces with the property (Y_0)," Izv. SKNTs VSh. Est. Nauki, No. 4, 64 (1977).
13. I. Kucera and K. McKennon, "Bounded sets in inductive limits," Proc. AMS, 69, No. 1, 62-64 (1978).

LINDELÖF SPACES OF CONTINUOUS FUNCTIONS

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By "spaces" below we mean completely regular topological spaces. The set of all continuous maps of the space X into the space E, endowed with the topology of pointwise convergence, is denoted by $C_p(X, E)$ [or by $C_p(X)$ if E = R].

The goal of this note is the proof of a general assertion about mapping spaces being Lindelöf, which in particular combines the following familiar results into one: 1) the space $C_p(X)$ is Lindelöf if X is a closed (more generally almost $\aleph_0$-invariant, cf. [4] and Definition 3 below) subspace of a Σ-product of separable metric spaces [4], 2) the space $C_p(X)$ is hereditarily Lindelöf if X^n is hereditarily separable for each $n \in N$ [3, 11]. Moreover, we show that for a Corson compactum X, all spaces of the series: $C_p(X)$, $C_pC_p(X)$,... are Lindelöf. Previously Gul'ko [5, 6] established that the odd terms of this sequence are Lindelöf and the even terms are normal.

We follow [9] in terminology and notation. In particular, w(X) is the weight, nw(X) is the net weight, d(X) is the density, and $l(X)$ is the Lindelöf number of the space X. If φ is a cardinal-valued invariant, then let

$$\varphi_\infty(X) = \sup\{\varphi(X^n);\ n \in N\},$$
$$h\varphi(X) = \sup\{\varphi(Y);\ Y \subset X\}.$$

We denote by τ, λ certain infinite cardinals.

Definition 1. Let φ and φ* be cardinal invariants. We shall say that φ* is dual to φ if from $w(E) \leqslant \aleph_0$ and $\varphi_\infty(X) \leqslant \tau$ it follows that $\varphi^* C_p(X, E) \leqslant \tau$.

Instead of φ* one can also write φ^*_∞, but this is not an essential change because E^n is also a space of countable weight and $[C_p(X, E)]^n$ is homeomorphic to $C_p(X, E^n)$. Historically, Arkhangel'skii [1, 2] gave the first pairs of dual invariants by showing that nw* = nw and $l^* = t$. Velichko [3] and Zenor [11] established the equalities hd* = hl and hl^* = hd.

Definition 2. Let φ be a cardinal invariant. The class $\mathscr{P}(\varphi, \tau)$ consists of precisely those spaces X for which the following condition holds: for any sequence of sets $Y_n \subset X^n$ there exists a continuous map $\pi\colon X \to X$ such that $\varphi_\infty(\pi(X)) \leqslant \tau$ and $\pi^n(Y_n) \subset \overline{Y}_n$, $n \in N$. Here the line above denotes closure and $\pi^n\colon X^n \to X^n$ is the natural map which coincides with π coordinatewise.

Remark 1. If in each power X^n one chooses a set, not one at a time, and a countable number of them: $Y_{nm} \subset X^n$, then in this case too there exists a map π: X → X such that $\pi(Y_{nm}) \subset \overline{Y}_{nm}$ for all natural numbers n and m. In fact, let θ: N × N → N be a bijection such that θ(n, m) ⩾ n. In an obvious way each X^n can be arranged as a face in $X^{\theta(n,m)}$ and hence Y_{nm} can also be considered as situated in this space. But then we fall under the conditions formulated in Definition 2.

Remark 2. Without loss of generality one can consider the sequence Y_n to consist of closed sets and the map π to be such that $\pi^n(Y_n) \subset Y_n$, $n \in N$.

If the space X is "small" relative to φ, that is, if $\varphi_\infty(X) \leqslant \tau$, then it is clear that $X \in \mathscr{P}(\varphi, \tau)$, because as π one can take the identity map.

We get nontrivial examples by considering certain subspaces of Σ_λ- and σ-products, and also spaces of functions on them. We recall [2, 6] that by a Σ_λ-product (σ-product) of a family of spaces $\{X_i;\ i \in J\}$ we mean a subspace of the topological product $\Pi\{X_i;\ i \in J\}$,

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