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THE CONTACT PROBLEM OF ELECTROELASTICITY FOR A PLANE ELLIPTIC DIE

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We obtain an exact solution of the problem of the stress-strain state of an elastic piezoelectric half-space acted on by a rigid elliptic die with a flat base. The axis of symmetry of the body coincides with the direction of the field of preliminary polarization of the body. The solution is confined to the case of translational displacement of the die. We determine the quantities that characterize the mechanical and electric fields that arise in the region of contact of the die with the half-space.
Bibliography: 7 titles.

Contact problems for electroelastic (in particular, piezoceramic) materials are of both theoretical and practical interest. They belong to the class of problems with mixed boundary conditions, reflecting the contact condition of a deformable (piezoelectric) medium with an absolutely rigid body (the die). The formulation of these problems is based on the classical formulation of mixed boundary conditions for mechanical variables supplemented by conditions (basic or mixed) for the electrical components [1]. A number of authors [2–4] have studied contact problems for a piezoelectric half-plane. However, in practice piezoelectric elements of different shapes and sizes work in conditions of rather complicated loading. In this connection it becomes necessary to formulate and solve three-dimensional contact problems of electroelasticity. In the present paper we study the stress-strain state of a piezoceramic half-space that has been stamped by a rigid elliptic die with a flat base.

Analysis of the structure of the equations of state for a preliminarily polarized ceramic medium and experimental data show that in their mechanical and electrical properties piezoceramics behave like transversally-isotropic bodies. Here the axis of symmetry coincides with the direction of the field of preliminary polarization. We take the plane bounding the half-space to be the Oxy-plane. Let the Oz-axis be directed normally to the plane of isotropy toward the interior of the half-space, and let it coincide with the direction of the lines of force of the electric field of preliminary polarization. The Ox- and Oy-axes are arbitrarily oriented in the plane of isotropy. The base of the die is absolutely smooth. For the elliptic die in the scheme, the area of contact (S) is an elliptic region in the plane $z=0$

$$x^2+\frac{y^2}{1-e^2}\le a^2 \quad (1>e\ge 0,\ a>0). \tag{1}$$

We refer the boundary conditions to the unstrained surface of the electroelastic medium, that is, to the surface $z=0$. The boundary conditions for mechanical variables can be stated similarly to the conditions in problems of elasticity theory [5; 6]. We assume that in the absence of frictional forces the following forces are prescribed on the boundary surfaces:

$$\begin{gathered}\sigma_z=0,\ \tau_{xz}=\tau_{yz}=0 \quad \text{outside } S;\\ \sigma_z=-p(x,y),\ \tau_{xz}=\tau_{yz}=0 \text{ on } S.\end{gathered} \tag{2}$$

Here $p(x, y)$ are the surface forces acting on the boundary of the half-space on the area of contact. The law of distribution of this load is not known in advance and is determined while solving the problem. The equilibrium of the die under conditions (2) is possible only when a compressive force and moments are acting on it whose balancing forces satisfy the equations of equilibrium of the die:

$$P=\iint_S p(x,y)\,dx\,dy;\quad M_x=\iint_S yp(x,y)\,dx\,dy;\ M_y=\iint_S xp(x,y)\,dx\,dy. \tag{3}$$

The boundary condition for the displacement w of points of the region of contact can be expressed in terms of the quantities that determine the displacement of the die

$$w=\delta-\beta_y x+\beta_x y, \tag{4}$$

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where δ is the translational displacement of the die parallel to the Oz-axis, and β_x, β_y are the projections of the rotation vectors on the Ox- and Oy-axes.

For the electric variables we shall consider below two cases of physically realizable conditions. One is the prescription of the value of a required potential on the surface of contact S

$$\text{a)} \qquad \psi(x, y) = V_0 \tag{5}$$

and the other is requiring that the normal component of the electric induction vector $\vec{D}$ be zero:

$$\text{b)} \qquad \vec{n} \cdot \vec{D} = D_z = 0. \tag{6}$$

Starting from working conditions for piezoceramic elements that are practically realizable in the majority of cases, condition (6) for the electrical variables also holds on the rest of the boundary of the half-space (outside the area of contact) in the case of both boundary conditions (5) and boundary conditions (6). To be specific:

$$\vec{n} \cdot \vec{D} = D_z = 0 \qquad \text{outside } S. \tag{7}$$

The complete system of equations of statics of transversally isotropic piezoelectric bodies (the Oz-axis is directed along the axis of anisotropy) contains the following [1]:

the equations of equilibrium

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0, \quad \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0, \quad \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0; \tag{8}$$

the Cauchy relations

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z}, \quad \varepsilon_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \quad \varepsilon_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \quad \varepsilon_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}; \tag{9}$$

the equations of forced electrostatics

$$\operatorname{div} \vec{D} = 0, \quad \vec{E} = -\operatorname{grad} \psi \quad \left(\operatorname{curl} \vec{E} = 0\right) \tag{10}$$

and the linear equations of the piezoeffect in the ceramic or equations of state

$$\begin{gathered}
\sigma_x = c_{11}^E \varepsilon_x + c_{12}^E \varepsilon_y + c_{13}^E \varepsilon_z - e_{31} E_z, \\
\sigma_y = c_{12}^E \varepsilon_x + c_{11}^E \varepsilon_y + c_{13}^E \varepsilon_z - e_{31} E_z, \\
\sigma_z = c_{13}^E \left(\varepsilon_x + \varepsilon_y\right) + c_{33}^E \varepsilon_z - e_{33} E_z, \\
\tau_{yz} = c_{44}^E \varepsilon_{yz} - e_{15} E_y, \quad \tau_{xz} = c_{44}^E \varepsilon_{xz} - e_{15} E_x, \\
\tau_{xy} = c_{66}^E \varepsilon_{xy} = \frac{1}{2}\left(c_{11}^E - c_{12}^E\right) \varepsilon_{xy}, \quad D_x = \varepsilon_{11}^S E_x + e_{15} \varepsilon_{xz}, \\
D_y = \varepsilon_{11}^S E_y + e_{15} \varepsilon_{yz}, \quad D_z = \varepsilon_{33}^S E_z + e_{31}\left(\varepsilon_x + \varepsilon_y\right) + e_{33} \varepsilon_z.
\end{gathered} \tag{11}$$

Here $\vec{D}$ is the electroelastic displacement (induction) vector; $\vec{E}$ is the electric field intensity; c_{ij}^E are the moduli of elasticity measured at a constant electric field; e_{ij} are the piezomoduli; ε_{ij}^S are the dielectric permittivities measured at constant strain.

Substituting expressions (9) and (11) into relations (8) and (10), we obtain a system of equilibrium equations written for the variables u, v, w and the electric potential ψ. This is a complicated system of four coupled differential equations whose solution is representable in the form [7]

$$u = \sum_{j=1,2,3} \frac{\partial \Phi_j}{\partial x} + \frac{\partial \Phi_4}{\partial y}; \quad v = \sum_{j=1,2,3} \frac{\partial \Phi_j}{\partial y} - \frac{\partial \Phi_4}{\partial x}; \quad w = \sum_{j=1,2,3} k_j \frac{\partial \Phi_j}{\partial z}; \quad \psi = \sum_{j=1,2,3} l_j \frac{\partial \Phi_j}{\partial z}. \tag{12}$$

The functions $\Phi_j \left(j = \overline{1, 4}\right)$ must satisfy the equations

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \nu_j \frac{\partial^2}{\partial z^2}\right) \Phi_j = 0 \quad \left(j = \overline{1, 4}\right), \tag{13}$$

where $\nu_4 = 2c_{44}^E / \left(c_{11}^E - c_{12}^E\right)$, and ν_1, ν_2, ν_3 are the roots of the characteristic equation

$$\nu^3(A_1B_2 - C_1D_2) + \nu^2(A_1B_3 + A_2B_2 - C_1D_3 - C_2D_2) + \nu(A_2B_3 + A_3B_2 - C_2D_3 - C_3D_2) + A_3B_3 - C_3D_3 = 0; \quad (14)$$

$$A_1 = c_{11}^E e_{15},\ A_2 = \left(c_{44}^E + c_{13}^E\right)(e_{31} + e_{15}) - c_{11}^E e_{33} - c_{44}^E e_{15},\ A_3 = c_{44}^E e_{33},$$

$$B_2 = -\left[\varepsilon_{11}^S\left(c_{13}^E + c_{44}^E\right) + e_{15}(e_{31} + e_{15})\right],\ B_3 = \varepsilon_{33}^S\left(c_{13}^E + c_{44}^E\right) + e_{33}(e_{31} + e_{15}), \quad (15)$$

$$C_1 = -c_{11}^E\varepsilon_{11}^S,\ C_2 = (e_{15} + e_{31})^2 + c_{11}^E\varepsilon_{33}^S + c_{44}^E\varepsilon_{11}^S,\ C_3 = -c_{44}^E\varepsilon_{33}^S,$$

$$D_2 = e_{15}\left(c_{13}^E + c_{44}^E\right) - c_{44}^E(e_{31} + e_{15}),\ D_3 = c_{33}^E(e_{31} + e_{15}) - e_{33}\left(c_{13}^E + c_{44}^E\right);$$

k_j and l_j are parameters defined in terms of the roots $\nu_j (j = 1, 2, 3)$ of the characteristic equation (14) by the formulas

$$k_j = \left[\left(\nu_j c_{11}^E - c_{44}^E\right)(e_{15}\nu_j - e_{33}) + \nu_j\left(c_{44}^E + c_{13}^E\right)(e_{31} + e_{15})\right]\left[\left(c_{13}^E + c_{44}^E\right)(e_{15}\nu_j - e_{33}) - \left(c_{44}^E\nu_j - c_{33}^E\right)(e_{31} + e_{15})\right]^{-1},$$

$$l_j = \left[\left(\nu_j c_{11}^E - c_{44}^E\right)\left(c_{44}^E\nu_j - c_{33}^E\right) + \nu_j\left(c_{44}^E + c_{13}^E\right)^2\right]\left[(e_{31} + e_{15})\left(c_{44}^E\nu_j - c_{33}^E\right) - (e_{15}\nu_j - e_{33})\left(c_{13}^E + c_{44}^E\right)\right]^{-1}. \quad (16)$$

Let us introduce the notation $z_j = z/\sqrt{\nu_j}\left(j = \overline{1, 4}\right)$. Then the functions $\Phi_j(x, y, z_j)$ will be harmonic in suitable coordinate systems. We introduce the ellipsoidal coordinate systems

$$x^2 = \frac{a^2}{e^2}\rho_j^2\mu_j^2\lambda_j^2;\ y^2 = \frac{a^2}{e^2\left(1 - e^2\right)}\left(\rho_j^2 - e^2\right)\left(\mu_j^2 - e^2\right)\left(e^2 - \lambda_j^2\right);$$

$$z^2 = \nu_j z_j^2 = \frac{a^2\nu_j}{1 - e^2}\left(\rho_j^2 - 1\right)\left(1 - \mu_j^2\right)\left(1 - \lambda_j^2\right) \quad (17)$$

$$\left(0 \le \lambda_j^2 \le e^2;\ e^2 \le \mu_j^2 \le 1;\ 1 \le \rho_j^2 \le \infty;\ j = \overline{1, 4}\right).$$

The coordinate surfaces of these systems are ellipsoids ($\rho_j = \text{const}$), hyperboloids of one sheet ($\mu_j = \text{const}$) and hyperboloids of two sheets ($\lambda_j = \text{const}$). When $\rho_j = 1$, the ellipsoids $\rho_j = \text{const}$ degenerate into an ellipsoidal region in the plane $z = 0$ bounded by the curve on which $\mu_j = 1$. The surface $\mu_j = 1$ is the part of the plane $z = 0$ inside the ellipse on which $\mu_j = 1$. On the region of contact $S\left(\rho_j = 1\right)$ the elliptic coordinates assume the values

$$\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu;\ \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda \quad \text{when } \rho_j = 1. \quad (18)$$

On the rest of the boundary (outside the contact region) we have the equalities

$$\rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho;\ \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda;\ \mu_1 = \mu_2 = \mu_3 = \mu_4 = 1. \quad (19)$$

Taking account of (18) and (19), we can write the boundary conditions (2) and (4)–(7) as

$$\sigma_z = 0;\ \tau_{xz} = \tau_{yz} = 0;\ \vec{n}\cdot\vec{D} = D_z = 0 \text{ when } \mu_j = 1; \quad (20)$$

$$\text{a) } w = \delta - \beta_y x + \beta_x y;\ \tau_{xz} = \tau_{yz} = 0;\ \psi = V_0 \text{ when } \rho_j = 1; \quad (21)$$

$$\text{b) } w = \delta - \beta_y x + \beta_x y;\ \tau_{xz} = \tau_{yz} = 0;\ \vec{n}\cdot\vec{D} = D_z = 0 \text{ when } \rho_j = 1. \quad (22)$$

We determine the potential functions $\Phi_j\left(j = \overline{1, 4}\right)$ in the representation (12) as follows:

$$\Phi_j(x, y, z_j) = a_j P(x, y, z_j)\ (j = 1, 2, 3);\ \Phi_4(x, y, z_4) = 0, \quad (23)$$

where

$$P(x, y, z_j) = -\int\frac{d\tau}{\Delta(\tau)}\left[z_j - z(\tau)\right]\Bigg|_{\tau=\rho_j};\ \Delta(\tau) = \sqrt{\left(\tau^2 - 1\right)\left(\tau^2 - e^2\right)};\ z(\tau) = a^2\sqrt{\tau^2 - 1}\sqrt{1 - \frac{x^2}{a^2\tau^2} - \frac{y^2}{a^2\left(\tau^2 - e^2\right)}}; \quad (24)$$

and a_j are unknown constants. We then obtain the following relations for the components of the stresses and the projections of the induction vector:

$$\sigma_x = \sum_{j=1,2,3}\left[c_{11}^E\frac{\partial^2\Phi_j}{\partial x^2}+c_{12}^E\frac{\partial^2\Phi_j}{\partial y^2}+\left(c_{13}^E k_j+e_{31}l_j\right)\frac{\partial^2\Phi_j}{\partial z^2}\right];$$
$$\sigma_y = \sum_{j=1,2,3}\left[c_{12}^E\frac{\partial^2\Phi_j}{\partial x^2}+c_{11}^E\frac{\partial^2\Phi_j}{\partial y^2}+\left(c_{13}^E k_j+e_{31}l_j\right)\frac{\partial^2\Phi_j}{\partial z^2}\right];$$
$$\sigma_z = \sum_{j=1,2,3}\left(c_{33}^E k_j+e_{33}l_j-c_{13}^E\nu_j\right)\frac{\partial^2\Phi_j}{\partial z^2};$$
$$\tau_{xy}=\left(c_{11}^E-c_{12}^E\right)\sum_{j=1,2,3}\frac{\partial^2\Phi_j}{\partial x\partial y};\quad \tau_{xz}=\sum_{j=1,2,3}\left[c_{44}^E\left(1+k_j\right)+e_{15}l_j\right]\frac{\partial^2\Phi_j}{\partial x\partial z};\quad \tau_{yz}=\sum_{j=1,2,3}\left[c_{44}^E\left(1+k_j\right)+e_{15}l_j\right]\frac{\partial^2\Phi_j}{\partial y\partial z};\quad (25)$$
$$D_x = \sum_{j=1,2,3}\left[e_{15}\left(1+k_j\right)-\varepsilon_{11}^S l_j\right]\frac{\partial^2\Phi_j}{\partial x\partial z};$$
$$D_y = \sum_{j=1,2,3}\left[e_{15}\left(1+k_j\right)-\varepsilon_{11}^S l_j\right]\frac{\partial^2\Phi_j}{\partial y\partial z};$$
$$D_z = \sum_{j=1,2,3}\left(e_{33}k_j-\varepsilon_{33}^S l_j-\nu_j e_{31}\right)\frac{\partial^2\Phi_j}{\partial z^2}.$$

In what follows we shall assume that

$$\nu_j\left[e_{15}\left(1+k_j\right)-\varepsilon_{11}^S l_j\right]=e_{33}k_j-\nu_j e_{31}-\varepsilon_{33}^S l_j;\quad \nu_j\left[c_{44}^E\left(1+k_j\right)+e_{15}l_j\right]=e_{33}l_j-\nu_j c_{13}^E+c_{33}^E k_j. \quad (26)$$

Substituting the functions (23) into expressions (25), we find

$$\psi=\sum_{j=1,2,3}\frac{l_j}{\sqrt{\nu_j}}a_j\Psi_0(\rho_j);\quad w=\sum_{j=1,2,3}\frac{k_j}{\sqrt{\nu_j}}a_j\Psi_0(\rho_j);$$
$$\sigma_z=-\sum_{j=1,2,3}\left[c_{44}^E\left(1+k_j\right)+e_{15}l_j\right]\frac{a_j z_j \rho_j\left(\rho_j^2-e^2\right)}{a^2\Delta(\rho_j)\left(\rho_j^2-\mu_j^2\right)\left(\rho_j^2-\lambda_j^2\right)};$$
$$\tau_{xz}=-\sum_{j=1,2,3}\frac{1}{\sqrt{\nu_j}}\left[c_{44}^E\left(1+k_j\right)+e_{15}l_j\right]\frac{a_j x\Delta(\rho_j)}{a^2\rho_j\left(\rho_j^2-\mu_j^2\right)\left(\rho_j^2-\lambda_j^2\right)};\quad (27)$$
$$\tau_{yz}=-\sum_{j=1,2,3}\frac{1}{\sqrt{\nu_j}}\left[c_{44}^E\left(1+k_j\right)+e_{15}l_j\right]\frac{a_j y\rho_j\Delta(\rho_j)}{a^2\left(\rho_j^2-e^2\right)\left(\rho_j^2-\mu_j^2\right)\left(\rho_j^2-\lambda_j^2\right)};$$
$$D_z=-\sum_{j=1,2,3}\left[e_{15}\left(1+k_j\right)-\varepsilon_{11}^S l_j\right]\frac{a_j z_j \rho_j\left(\rho_j^2-e^2\right)}{a^2\Delta(\rho_j)\left(\rho_j^2-\mu_j^2\right)\left(\rho_j^2-\lambda_j^2\right)};$$
$$E_z=\sum_{j=1,2,3}\frac{l_j}{\nu_j}\frac{a_j z_j \rho_j\left(\rho_j^2-e^2\right)}{a^2\Delta(\rho_j)\left(\rho_j^2-\mu_j^2\right)\left(\rho_j^2-\lambda_j^2\right)},$$

where $\Psi_0(\rho)=F(\varphi, e)=\int_\rho^\infty\frac{d\tau}{\Delta(\tau)}$.

On the contact area ($\rho_j=1$) and outside it ($\mu_j=1$) we have the following expressions for these quantities.

$$\psi=\Psi_0(1)\sum_{j=1,2,3}\frac{l_j}{\sqrt{\nu_j}}a_j;\quad w=\Psi_0(1)\sum_{j=1,2,3}\frac{k_j}{\sqrt{\nu_j}}a_j;$$
$$\sigma_z=-\frac{1}{a}\frac{1}{\sqrt{\left(1-\mu^2\right)\left(1-\lambda^2\right)}}\sum_{j=1,2,3}\left[c_{44}^E\left(1+k_j\right)+e_{15}l_j\right]a_j;$$
$$\tau_{xz}=\tau_{yz}=0; \quad (28)$$

$$D_z = -\frac{1}{a}\frac{1}{\sqrt{(1-\mu^2)(1-\lambda^2)}}\sum_{j=1,2,3}\left[e_{15}(1+k_j)-\varepsilon_{11}^S l_j\right]a_j;$$

$$E_z = \frac{1}{a}\frac{1}{\sqrt{(1-\mu^2)(1-\lambda^2)}}\sum_{j=1,2,3}\frac{l_j}{\nu_j}a_j.$$

$$\psi = \Psi_0(\rho)\sum_{j=1,2,3}\frac{l_j}{\sqrt{\nu_j}}a_j;\quad w = \Psi_0(\rho)\sum_{j=1,2,3}\frac{k_j}{\sqrt{\nu_j}}a_j;\quad \sigma_z = 0;$$

$$\tau_{xz} = -\frac{1}{a^2}\frac{x\Delta(\rho)}{\rho(\rho^2-1)(\rho^2-\lambda^2)}\sum_{j=1,2,3}\frac{1}{\sqrt{\nu_j}}\left[c_{44}^E(1+k_j)+e_{15}l_j\right]a_j; \tag{29}$$

$$\tau_{yz} = -\frac{1}{a^2}\frac{y\rho\Delta(\rho)}{\Delta(\rho)(\rho^2-\lambda^2)}\sum_{j=1,2,3}\frac{1}{\sqrt{\nu_j}}\left[c_{44}^E(1+k_j)+e_{15}l_j\right]a_j;$$

$$D_z = 0;\quad E_z = 0.$$

For the projections of the balancing force vectors and moments on the contact area ($\rho_j = 1$) we find

$$P_x = \iint_S \tau_{xz}\,dx\,dy;\quad P_y = \iint_S \tau_{yz}\,dx\,dy;\quad P_z = \iint_S \sigma_z\,dx\,dy;$$
$$M_x = \iint_S y\sigma_z\,dx\,dy;\quad M_y = -\iint_S x\sigma_z\,dx\,dy;\quad M_z = \iint_S (x\tau_{yz}-y\tau_{xz})\,dx\,dy. \tag{30}$$

It follows from Eqs. (29) and (30) that

$$P_x = P_y = 0;\quad M_z = 0. \tag{31}$$

We determine the remaining quantities by computing the integrals that occur in (30)

$$P_z = -\frac{1}{a}\sum_{j=1,2,3}\left[c_{44}^E(1+k_j)+e_{15}l_j\right]a_j\iint_S\frac{dx\,dy}{\sqrt{(1-\mu^2)(1-\lambda^2)}} = -2\pi a\sum_{j=1,2,3}\left[c_{44}^E(1+k_j)+e_{15}l_j\right]a_j;$$

$$M_x = -\frac{1}{a}\sum_{j=1,2,3}\left[c_{44}^E(1+k_j)+e_{15}l_j\right]a_j\iint_S\frac{y\,dx\,dy}{\sqrt{(1-\mu^2)(1-\lambda^2)}} = 0; \tag{32}$$

$$M_y = \frac{1}{a}\sum_{j=1,2,3}\left[c_{44}^E(1+k_j)+e_{15}l_j\right]a_j\iint_S\frac{x\,dx\,dy}{\sqrt{(1-\mu^2)(1-\lambda^2)}} = 0.$$

It follows from relations (28) and (29) that the conditions on the boundary of the half-space hold if

$$\text{a)}\ \sum_{j=1,2,3}\frac{l_j}{\sqrt{\nu_j}}a_j = \frac{V_0}{\Psi_0(1)},\quad \sum_{j=1,2,3}\frac{m_j}{\sqrt{\nu_j}}a_j = 0; \tag{33}$$

$$\text{b)}\ \sum_{j=1,2,3}n_j a_j = 0,\quad \sum_{j=1,2,3}\frac{m_j}{\sqrt{\nu_j}}a_j = 0, \tag{34}$$

where

$$m_j = c_{44}^E(1+k_j)+e_{15}l_j;\quad n_j = e_{15}^E(1+k_j)-\varepsilon_{11}^S l_j\quad (j=1,2,3). \tag{35}$$

We obtain another equation for determining the unknowns from the first condition of (32)

$$\sum_{j=1,2,3}m_j a_j = -P_z/2\pi a. \tag{36}$$

Thus the stressed state of a piezoceramic half-space subject to the action of a flat elliptic die to which a normal force acting along the Oz-axis is applied is determined by the potential functions (23). The unknown coefficients $a_j\,(j=1,2,3)$ are found from Eqs. (33) and (36) in the case of the boundary conditions (20) and (21) and from equations (34) and (36) in the case of boundary conditions (20) and (22). They assume the respective values

$$\text{a)}\quad a_1 = -\frac{\sqrt{\nu_1}}{l_1c_1 - l_2c_2 + l_3c_3}\left(\frac{P_z}{2\pi a}b_1 - \frac{V_0}{\Psi_0(1)}c_1\right),$$

$$a_2 = \frac{\sqrt{\nu_2}}{l_1c_1 - l_2c_2 + l_3c_3}\left(\frac{P_z}{2\pi a}b_2 - \frac{V_0}{\Psi_0(1)}c_2\right), \tag{37}$$

$$a_3 = -\frac{\sqrt{\nu_3}}{l_1c_1 - l_2c_2 + l_3c_3}\left(\frac{P_z}{2\pi a}b_3 - \frac{V_0}{\Psi_0(1)}c_3\right),$$

where

$$b_1 = l_2m_3 - l_3m_2,\quad b_2 = l_1m_3 - l_3m_1,\quad b_3 = l_1m_2 - l_2m_1,$$
$$c_1 = m_2m_3\left(\sqrt{\nu_3} - \sqrt{\nu_2}\right),\quad c_2 = m_1m_3\left(\sqrt{\nu_3} - \sqrt{\nu_1}\right),$$
$$c_3 = m_1m_2\left(\sqrt{\nu_2} - \sqrt{\nu_1}\right);$$

$$\text{b)}\quad a_1 = -\frac{P_z}{2\pi a}\frac{\sqrt{\nu_1}\,d_1}{\sqrt{\nu_1}\,n_1c_1 - \sqrt{\nu_2}\,n_2c_2 + \sqrt{\nu_3}\,n_3c_3},$$

$$a_2 = \frac{P_z}{2\pi a}\frac{\sqrt{\nu_2}\,d_2}{\sqrt{\nu_1}\,n_1c_1 - \sqrt{\nu_2}\,n_2c_2 + \sqrt{\nu_3}\,n_3c_3}, \tag{38}$$

$$a_3 = -\frac{P_z}{2\pi a}\frac{\sqrt{\nu_3}\,d_3}{\sqrt{\nu_1}\,n_1c_1 - \sqrt{\nu_2}\,n_2c_2 + \sqrt{\nu_3}\,n_3c_3},$$

where

$$d_1 = \sqrt{\nu_2}n_2m_3 - \sqrt{\nu_3}n_3m_2,\quad d_2 = \sqrt{\nu_1}n_1m_3 - \sqrt{\nu_3}n_3m_1,$$
$$d_3 = \sqrt{\nu_1}n_1m_2 - \sqrt{\nu_2}n_2m_1.$$

Using the boundary condition for the displacement w, we compute the translational displacement of points of the base of the die

$$\text{a)}\quad \delta = -\frac{1}{l_1c_1 - l_2c_2 + l_3c_3}\left[\frac{P_z}{2\pi a}\Psi_0(1)(k_1b_1 - k_2b_2 + k_3b_3) - V_0(k_1c_1 - k_2c_2 + k_3c_3)\right]; \tag{39}$$

$$\text{b)}\quad \delta = -\frac{P_z}{2\pi a}\Psi_0(1)\frac{k_1d_1 - k_2d_2 + k_3d_3}{\sqrt{\nu_1}n_1c_1 - \sqrt{\nu_2}n_2c_2 + \sqrt{\nu_3}n_3c_3}. \tag{40}$$

The normal stress acting on the contact area $\left(\rho_j = 1\right)$ is

$$\sigma_z = \frac{1}{\sqrt{(1-\mu^2)(1-\lambda^2)}}\frac{P_z}{2\pi a^2}, \tag{41}$$

and we determine the normal components of the vectors of electric field induction and intensity and the electric potential on the contact area in the case of an electroded contact surface (20), (21) from the formulas

$$\psi^{(S)} = \iint_S \psi\,dx\,dy = a^2\sqrt{1-e^2}\,\pi\Psi_0(1)\sum_{j=1,2,3}\frac{l_j}{\sqrt{\nu_j}}a_j = a^2\sqrt{1-e^2}\,\pi V_0,$$

$$D_z^{(S)} = \iint_S D_z\,dx\,dy = -2\pi a\sum_{j=1,2,3}n_ja_j = \frac{1}{l_1c_1 - l_2c_2 + l_3c_3}\Big[P_z\left(\sqrt{\nu_1}n_1b_1 - \sqrt{\nu_2}n_2b_2 + \sqrt{\nu_3}n_3b_3\right) -$$

$$-2\pi a\frac{V_0}{\Psi_0(1)}\left(\sqrt{\nu_1}n_1c_1 - \sqrt{\nu_2}n_2c_2 + \sqrt{\nu_3}n_3c_3\right)\Big]; \tag{42}$$

$$E_z^{(S)} = \iint_S E_z\,dx\,dy = 2\pi a\sum_{j=1,2,3}\frac{l_j}{\nu_j}a_j = -\frac{1}{\sqrt{\nu_1\nu_2\nu_3}}\frac{1}{l_1c_1 - l_2c_2 + l_3c_3}\Big[P_z\left(\sqrt{\nu_2\nu_3}\,l_1b_1 - \sqrt{\nu_1\nu_3}\,l_2b_2 + \sqrt{\nu_1\nu_2}\,l_3b_3\right) -$$

$$-2\pi a\frac{V_0}{\Psi_0(1)}\left(\sqrt{\nu_2\nu_3}l_1c_1-\sqrt{\nu_1\nu_3}l_2c_2+\sqrt{\nu_1\nu_2}l_3c_3\right)\Bigg],$$

and in the case of a nonelectroded contact surface (20), (22) we have

$$D_z^{(S)}=0;$$
$$\psi^{(S)}=-\frac{P_z}{2}a\sqrt{1-e^2}\,\Psi_0(1)\frac{l_1d_1-l_2d_2+l_3d_3}{\sqrt{\nu_1}n_1c_1-\sqrt{\nu_2}n_2c_2+\sqrt{\nu_3}n_3c_3}; \tag{43}$$
$$E_z^{(S)}=-\frac{P_z}{\sqrt{\nu_1\nu_2\nu_3}}\frac{\sqrt{\nu_2\nu_3}l_1d_1-\sqrt{\nu_1\nu_3}l_2d_2+\sqrt{\nu_1\nu_2}l_3d_3}{\sqrt{\nu_1}n_1c_1-\sqrt{\nu_2}n_2c_2+\sqrt{\nu_3}n_3c_3}.$$

It follows from this solution that in the contact area between an electroelastic half-space and a rigid elliptic die with a flat base significant fields arise, both mechanical and electrical, and in general they increase without bound on the boundary of the region of contact. The formulas for computing the components of the electroelastic field have a very simple form.

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