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Convergence analysis of a parabolic nonlinear system arising in biology

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Abstract We investigate in this paper, the approximation of a semi-linear system of reaction diffusion equations in a bounded domain. The system is discretized using a $\mathbb{P}_r$ Lagrange finite method in space, and an implicit finite difference scheme in time. We analyze the scheme stability in the space $L^\infty((0,T), L^2(\Omega)) \cap L^2((0,T), H^1(\Omega))$. The error norm for approximate solutions is of order $O(h^r + \delta t)$.

Keywords Convergence · Evolution equations · Error estimates · Finite element · Parabolic system · Positive solution · Maximum principle · Stability

Mathematics Subject Classification (2000) Primary 65N12 · 65N30; Secondary 35K40 · 35B09

1 Introduction

We consider a model of reaction–diffusion equations of the form

$$
\begin{aligned}
\frac{\partial u_i}{\partial t} - d_i \Delta u_i + q_i(u) u_i &= f_i(u) && \text{in } \Omega\times]0,T[,\\
\frac{\partial u_i}{\partial n} &= 0 && \text{on } \partial\Omega\times]0,T[,\\
u_i(0,x) = u_{i0}(x) &\geq 0 && \text{in } \Omega,
\end{aligned}
\tag{1}
$$

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for all $i = 1, 2, \ldots, 6$. We have denoted $u = (u_1, u_2, \ldots, u_6)$; Ω is a bounded domain in $\mathbb{R}^3$; If we use a $\mathbb{P}_r$ Lagrange finite element with $r \geq 1$, the boundary $\partial\Omega$ needs to be of class $\mathcal{C}^{r+1}$; $d_i > 0$; $T > 0$ is a given time; the functions $q_i(u)$ and $f_i(u)$ are of the form

$$\begin{aligned}
&q_1(u) = -\frac{ku_2}{K_s + u_2} + \mu + r + \nu, \quad q_2(u) = \frac{ku_1}{K_s + u_2},\\
&q_3(u) = \frac{c_1 u_5}{K_m + u_5}, \quad q_4(u) = \frac{c_2 u_5}{K_m + u_5}, \quad q_5(u) = \zeta, \quad q_6(u) = 0,\\
&f_1(u) = 0, \quad f_2(u) = \frac{(c_1 u_3 + c_2 u_4)u_5}{K_m + u_5} + \frac{\zeta u_5 + \mu u_1}{2},\\
&f_3(u) = \frac{\zeta u_5 + \mu u_1}{2}, \quad f_4(u) = 0, \quad f_5(u) = \nu u_1, \quad f_6(u) = \theta u_1;
\end{aligned} \tag{2}$$

the parameters $k, K_s, K_m, \mu, \nu, \zeta, c_1, c_2$ et θ are all nonnegatives; The initial conditions u_{i0} are taken in the functional space $H^{r+1}(\Omega)$.

This model was first proposed in [7]. It describes soil organic matter decomposition by microorganisms. The questions of existence, unicity and positivity of the solution verifying $u_i \in L^2((0,T), H^1(\Omega)) \cap \mathcal{C}^0([0,T], L^2(\Omega))$ and $\dfrac{\partial u_i}{\partial t} \in L^2((0,T), H^{-1}(\Omega))$ was studied in [5] using variational and semi-groups methods.

Parabolic equations arise in many fields notably in soil sciences, where biological activity can be modelled by means of reaction–diffusion systems, see [4,8,11]. The solution must usually be computed with a numerical scheme. The implementation and analysis of adequate numerical scheme for nonlinear systems constitute a wide research subject. And it is often difficult to show that the approximate solutions converge to the weak one.

In this paper, we study the approximation of system (1) using $\mathbb{P}_r$ Lagrange finite element discretization in space, and an implicit Euler finite difference discretization in time. We show that the scheme maintains the solution's positivity. We obtain the stability in $L^\infty((0,T), L^2(\Omega)) \cap L^2((0,T), H^1(\Omega))$. We also deal with the error estimate which is of order $O(h^r + \delta t)$. For more details concerning the approximation of nonlinear reaction diffusion systems, we refer readers to the following papers [1–3,6].

We first present in Sect. 2 the discretization and its positivity preservation property. Then we study in Sect. 3 the stability. And finally Sect. 4 presents the study of the approximation error.

2 The discretization

2.1 The weak formulation

To obtain the weak formulation of (1), we make the change of variable $w_i(t) = e^{-\beta t} u_i(t)$, where $\beta > 0$. We decompose β as follows $\beta = \mu + \lambda$ with $\mu > 0$ and $\lambda > 0$. Then we multiply the first equation of System (1) by a test function $v_i \in H^1(\Omega)$ and integrate it by parts. It arises the formulation: for all $t \in]0, T[$, find $w = (w_i)_{i \in \{1,2,\ldots,6\}}$ with $w_i \in L^2((0,T), H^1(\Omega)) \cap \mathcal{C}^0([0,T], L^2(\Omega))$ and $\dfrac{\partial w_i}{\partial t} \in L^2((0,T); H^{-1}(\Omega))$ such that

$$\begin{cases} \left(\dfrac{\partial w_i}{\partial t}, v_i\right) + b_i(w_i, v_i) + (\Phi_i(w) w_i, v_i) = (\tilde{f}_i(w), v_i) \quad \forall\, v_i \in H^1(\Omega), \\ w_i(t = 0) = u_{i0}, \end{cases} \tag{3}$$

where the notation (., .) stands for the scalar product in $L^2(\Omega)$. We have defined

$$b_i(w_i, v_i) = d_i \int_\Omega \nabla w_i \nabla v_i dx + \mu \int_\Omega w_i v_i dx, \tag{4}$$

$$\Phi_i(w) = \lambda + q_i(e^{\beta t} w) \tag{5}$$

and

$$\tilde{f}_i(w) = e^{-\beta t} f_i(e^{\beta t} w). \tag{6}$$

The bilinear form $b_i(.,.)$ is coercive in $H^1(\Omega)$ with $\alpha_i = \min(d_i, \mu)$ as coercivity constant. Let us define $\alpha = \min_{i=1,\dots,6}(\alpha_i) > 0$.

Remark Existence of solution has been studied in [5] by using variational method and the semigroup theory and is given by the following theorem:

Theorem 1 *If the initial condition u_0 is nonnegative and*

$$u_{10} \in L^\infty(\Omega) \quad and \quad u_{i0} \in L^2(\Omega) \tag{7}$$

for $i \in \{2, ldots, 6\}$, the system (1) admits an unique nonnegative solution

$$u_i \in L^2((0,T), H^1(\Omega)) \cap \mathcal{C}^0([0,T], L^2(\Omega)) \, and \, \frac{\partial u_i}{\partial t} \in L^2((0,T), H^{-1}(\Omega)) \tag{8}$$

for $N = 3$.

Proof See [5]. □

According to the Theorem 1, the property $u_0 \geq 0$ implies that $w \geq 0$. As a result, we verify that there exists a constant $\lambda > 0$ such that

$$\Phi_i(w) = q_i(w) + \lambda \geq 0 \quad \text{for all } i = 1, 2, \dots, 6. \tag{9}$$

This last inequality which is satisfied for all $w \geq 0$, is necessary for proving the forthcoming results.

2.2 Discret space and approximation properties

In time, we discretize the interval $[0, T]$ by defining $0 = t_0 < t_1 < \cdots < t_N = T$ with an uniform time step $\delta t = \dfrac{T}{N}$ and $N \in \mathbb{N}^*$. We also discretize the domain Ω with a regular mesh $\mathcal{T}_h$, where h is the diameter of the biggest mesh element.

We seek the approximate solution in the discret space V_h defined on the mesh $\mathcal{T}_h$. The space V_h is defined as follows

$$V_h = \{v \in \mathcal{C}(\overline{\Omega}) \,:\, v_{|K} \in \mathbb{P}_r, \, \forall K \in \mathcal{T}_h\}, \tag{10}$$

where $\mathbb{P}_r$ is the space of polynomials whose degree is less than $r \in \mathbb{N}^*$. We can define the following interpolation operator

$$\prod_h^r : H^1(\Omega) \to V_h \tag{11}$$

satisfying the estimate

$$\left| v - \prod_h^r v \right|_{H^m(\Omega)} \leq C h^{r+1-m} |v|_{H^{r+1}(\Omega)} \quad \forall\, v \in H^{r+1}(\Omega), \quad m = 0, 1, \tag{12}$$

with $C > 0$, see Theorem 4.5 at p. 93 in [9]. We use the notations

$$|v|_{H^r(\Omega)} = \left(\int_\Omega |D^r v|^2 dx \right)^{\frac{1}{2}}, \quad \|v\|_{H^r(\Omega)} = \left(\sum_{|\alpha|=0}^r \int_\Omega |D^\alpha v|^2 dx \right)^{\frac{1}{2}} \tag{13}$$

for the norms. We also define the orthogonal projection operator

$$\prod_{i,h}^r : H^1(\Omega) \to V_h \tag{14}$$

associated to bilinear form $b_i(.,.) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$, verifying

$$b_i(v, v_h) = b_i\left(\prod_{i,h}^r v, v_h \right) \quad \forall\, v_h \in V_h. \tag{15}$$

For all $v \in H^{r+1}(\Omega)$, we have

$$\left\| \prod_{i,h}^r v - v \right\|_{H^1(\Omega)} + h^{-1} \left\| \prod_{i,h}^r v - v \right\|_{L^2(\Omega)} \leq C h^p |v|_{H^{p+1}(\Omega)}, \quad 0 \leq p \leq r, \tag{16}$$

where $C > 0$ is a constant, see [10]. We finally remember the following space inclusion result

$$H^k(\Omega) \subset \mathcal{C}^m(\overline{\Omega}) \quad \text{if } k > m + \frac{n}{2}, \tag{17}$$

where $k, m \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$, see property 2.3 at p. 21 in [10].

2.3 The finite element method

We consider a numerical scheme obtained from a $\mathbb{P}_r$, $r \geq 1$ Lagrange finite element method in space and an implicit Euler finite difference discretization in time: find $w_{i,h}^n \in V_h$ such that

$$w_{i,h}^0 = \prod_h^r u_{i0} \tag{18}$$

and

$$\begin{aligned} &\left\langle \frac{w_{i,h}^n - w_{i,h}^{n-1}}{\delta t}, v_{i,h} \right\rangle + b_i(w_{i,h}^n, v_{i,h}) + \left\langle \Phi_i(w_h^{n-1}) w_{i,h}^n, v_{i,h} \right\rangle \\ &\quad = \langle \tilde{f}_i(w_h^{n-1}), v_{i,h} \rangle \text{ for all } v_{i,h} \in V_h, \end{aligned} \tag{19}$$

for $n = 1, \ldots, N$. We have noted $w_h^n = (w_{i,h}^n)_{i=1,\ldots,6}$.

Remark According to the property (17), we verify that, for all $r \in \mathbb{N}^*$ we have

$$u_{i0} \in H^{r+1}(\Omega) \subset \mathcal{C}(\overline{\Omega}). \tag{20}$$

It comes that $\prod_h^r u_{i0}$ is well defined.

2.4 The approximation property

We start by establishing the positivity of the discret solution.

Lemma 1 *The hypothesis $w_{i,h}^0 \geq 0$ implies that $w_{i,h}^n \geq 0$ for all $n \in \mathbb{N}$.*

Proof We note that the functions

$$w_{i,h}^{n+} = \max(0, w_{i,h}^n), \qquad w_{i,h}^{n-} = \max(0, -w_{i,h}^n) \tag{21}$$

satisfy the following properties

$$w_{i,h}^n = w_{i,h}^{n+} - w_{i,h}^{n-}, \qquad w_{i,h}^{n-} w_{i,h}^{n+} = 0. \tag{22}$$

We prove the lemma by recurrence.

The initial condition implies $w_{i,h}^0 \geq 0$.

Let us suppose that $w_{i,h}^{n-1} \geq 0$ for $n \in \mathbb{N}^*$.

We show that $w_{i,h}^n \geq 0$.

By choosing $v_{i,h} = w_{i,h}^{n-}$ in the scheme (19) and observing that

$$\langle w_{i,h}^n - w_{i,h}^{n-1}, w_{i,h}^{n-} \rangle = -\|w_{i,h}^{n-}\|_{L^2(\Omega)}^2 - \langle w_{i,h}^{n-1}, w_{i,h}^{n-} \rangle, \tag{23}$$

$$b_i(w_{i,h}^n, w_{i,h}^{n-}) = -b_i(w_{i,h}^{n-}, w_{i,h}^{n-}) \tag{24}$$

and

$$\langle \Phi_i(w_h^{n-1}) w_{i,h}^n, w_{i,h}^{n-} \rangle = -\langle \Phi_i(w_h^{n-1}) w_{i,h}^{n-}, w_{i,h}^{n-} \rangle, \tag{25}$$

we obtain from (19)

$$\begin{aligned} \|w_{i,h}^{n-}\|_{L^2(\Omega)}^2 + \langle w_{i,h}^{n-1}, w_{i,h}^{n-} \rangle = -\delta t \Big(& b_i(w_{i,h}^{n-}, w_{i,h}^{n-}) + \langle \Phi_i(w_h^{n-1}) w_{i,h}^{n-}, w_{i,h}^{n-} \rangle \\ & + \langle \tilde{f}_i(w_h^{n-1}), w_{i,h}^{n-} \rangle \Big). \end{aligned} \tag{26}$$

We remark that $w_h^{n-1} \geq 0$ implies $\Phi_i(w_h^{n-1}) \geq 0$ and $\tilde{f}_i(w_h^{n-1}) \geq 0$. By using the inequalities

$$b_i(w_{i,h}^{n-}, w_{i,h}^{n-}) = \int_\Omega d_i |\nabla w_{i,h}^{n-}|^2 + \mu |w_{i,h}^{n-}|^2 dx \geq 0, \tag{27}$$

$$\langle \Phi_i(w_h^{n-1}) w_{i,h}^{n-}, w_{i,h}^{n-} \rangle = \int_\Omega \Phi_i(w_h^{n-1}) |w_{i,h}^{n-}|^2 dx \geq 0 \tag{28}$$

and

$$\langle \tilde{f}_i(w_h^{n-1}), w_{i,h}^{n-} \rangle = \int_\Omega \tilde{f}_i(w_h^{n-1}) w_{i,h}^{n-}\, dx \geq 0, \tag{29}$$

the statement (26) implies

$$\langle w_{i,h}^{n-1}, w_{i,h}^{n-}\rangle \leq -\|w_{i,h}^{n-}\|_{L^2(\Omega)}^2. \tag{30}$$

The functions $w_{i,h}^{n-1}$ and $w_{i,h}^{n-}$ are nonnegatives, this implies that the scalar product $< w_{i,h}^{n-1}, w_{i,h}^{n-} >$ is also nonnegative. It follows that $-\|w_{i,h}^{n-}\|_{L^2(\Omega)}^2 \geq 0$ and consequently we have $w_{i,h}^{n-} = 0$. We finally conclude that $w_{i,h}^{n} \geq 0$. □

3 The stability

In this section, we demonstrate the scheme stability

Let us recall the Gronwall discret lemma.

Lemma 2 *Let us assume that k_n is a nonnegative sequence, and the sequence ϕ_n satisfies*

$$\phi_0 \leq g_0, \qquad \phi_n \leq g_0 + \sum_{m=0}^{n-1} p_m + \sum_{m=0}^{n-1} k_m \phi_m, \quad n \geq 1. \tag{31}$$

If $g_0 \geq 0$ and $p_m \geq 0$ for $m \geq 0$, we have

$$\phi_n \leq \left(g_0 + \sum_{m=0}^{n-1} p_m\right) exp\left(\sum_{m=0}^{n-1} k_m\right), \quad n \geq 1. \tag{32}$$

Proof See [10], chapter 1. □

For the stability, we get the following lemma.

Lemma 3 *The scheme (19) is stable in $L^2(\Omega)$ and $H^1(\Omega)$. It means that*

$$\|w_h^n\|_{L^2(\Omega)}^2 + \alpha\delta t \sum_{k=1}^{n} \|w_h^k\|_{H^1(\Omega)}^2 \leq \|w_h^0\|_{L^2(\Omega)}^2 exp\left(\frac{TM}{\alpha}\right), \quad n \geq 1 \tag{33}$$

where $M > 0$.

Proof By posing $v_{i,h} = w_{i,h}^n$ in the scheme (19), and observing that

$$\langle w_{i,h}^n - w_{i,h}^{n-1}, w_{i,h}^n\rangle = \|w_{i,h}^n\|_{L^2(\Omega)}^2 - \langle w_{i,h}^{n-1}, w_{i,h}^n\rangle, \tag{34}$$

$$b_i(w_{i,h}^n, w_{i,h}^n) \geq \alpha\|w_{i,h}^n\|_{H^1(\Omega)}^2 \tag{35}$$

and

$$\langle \Phi_i(w_h^{n-1}) w_{i,h}^n, w_{i,h}^n\rangle \geq 0, \tag{36}$$

we obtain

$$\|w_{i,h}^n\|_{L^2(\Omega)}^2 + \alpha\delta t\|w_{i,h}^n\|_{H^1(\Omega)}^2 \leq \langle w_{i,h}^{n-1}, w_{i,h}^n\rangle + \delta t\langle \tilde{f}_i(w_h^{n-1}), w_{i,h}^n\rangle. \tag{37}$$

By applying the Young inequality to the first term and the generalized Young inequality to the second term in the right hand side, we get

$$\|w_{i,h}^n\|_{L^2(\Omega)}^2 + (2\alpha - \epsilon)\delta t\|w_{i,h}^n\|_{H^1(\Omega)}^2 \leq \|w_{i,h}^{n-1}\|_{L^2(\Omega)}^2 + \frac{\delta t}{\epsilon}\|\tilde{f}_i(w_h^{n-1})\|_{L^2(\Omega)}^2. \tag{38}$$

By choosing $\epsilon = \alpha$, summing the inequalities for $k = 1, ..., n$ and using

$$\exists\, M_i > 0 \;:\; \|\tilde{f}_i(w_h^{n-1})\|_{L^2(\Omega)} \leq M_i \|w_h^{n-1}\|_{L^2(\Omega)}, \tag{39}$$

it comes

$$\|w_{i,h}^n\|_{L^2(\Omega)}^2 + \alpha\delta t \sum_{k=1}^{n} \|w_{i,h}^k\|_{H^1(\Omega)}^2 \;\leq\; \|w_{i,h}^0\|_{L^2(\Omega)}^2 + \frac{\delta t\, M_i}{\alpha} \sum_{k=1}^{n} \|w_h^{k-1}\|_{L^2(\Omega)}^2. \tag{40}$$

Then we sum over $i = 1, ..., 6$ to get

$$\|w_h^n\|_{L^2(\Omega)}^2 + \alpha\delta t \sum_{k=1}^{n} \|w_h^k\|_{H^1(\Omega)}^2 \;\leq\; \|w_h^0\|_{L^2(\Omega)}^2 + \frac{\delta t\, M}{\alpha} \sum_{k=1}^{n} \|w_h^{k-1}\|_{L^2(\Omega)}^2. \tag{41}$$

with $M > 0$.

According to the discret Gronwall lemma, we obtain the estimate

$$\|w_h^n\|_{L^2(\Omega)}^2 + \alpha\delta t \sum_{k=1}^{n} \|w_h^k\|_{H^1(\Omega)}^2 \;\leq\; \|w_h^0\|_{L^2(\Omega)}^2 \exp\Big(\frac{T\, M}{\alpha}\Big). \tag{42}$$

□

4 Approximation error estimate

Let us assume that $w_i(t) \in H^r(\Omega) \subset C(\overline{\Omega})$ for all $t \in [0, T]$, which is true for $r > \dfrac{n}{2}$. We establish here the approximation error estimate.

Let us define

$$\epsilon_{i,h}^n = w_{i,h}^n - \prod_{i,h}^{r} w_i(t_n), \tag{43}$$

and subtract from the scheme (19), the following term

$$\left\langle \frac{\prod_{i,h}^r \big(w_i(t_n) - w_i(t_{n-1})\big)}{\delta t}, v_{i,h} \right\rangle + b_i\left(\prod_{i,h}^{r} w_i(t_n), v_{i,h}\right) + \left\langle \Phi_i(w_h^{n-1}) \prod_{i,h}^{r} w_i(t_n), v_{i,h} \right\rangle. \tag{44}$$

It comes

$$\begin{aligned} &\left\langle \frac{\epsilon_{i,h}^n - \epsilon_{i,h}^{n-1}}{\delta t}, v_{i,h} \right\rangle + b_i(\epsilon_{i,h}^n, v_{i,h}) + \langle \Phi_i(w_h^{n-1})\epsilon_{i,h}^n, v_{i,h}\rangle \\ &= \langle \tilde{f}_i(w_h^{n-1}), v_{i,h}\rangle - \left\langle \frac{\prod_{i,h}^r \big(w_i(t_n) - w_i(t_{n-1})\big)}{\delta t}, v_{i,h} \right\rangle - b_i\Big(\prod_{i,h}^{r} w_i(t_n), v_{i,h}\Big) \\ &\quad - \left\langle \Phi_i(w_h^{n-1}) \prod_{i,h}^{r} w_i(t_n), v_{i,h} \right\rangle. \end{aligned} \tag{45}$$

According to the relations (3) and (15), we verify

$$b_i\left(\prod_{i,h}^{r} w_i(t_n), v_{i,h}\right) = b_i(w_i(t_n), v_{i,h}), \tag{46}$$

$$= \left\langle -\frac{\partial w_i(t_n)}{\partial t} - \Phi_i(w(t_n))w_i(t_n) + \tilde{f}_i(w(t_n)), v_{i,h}\right\rangle.$$

Then we get

$$\left\langle \frac{\epsilon_{i,h}^n - \epsilon_{i,h}^{n-1}}{\delta t}, v_{i,h}\right\rangle + b_i(\epsilon_{i,h}^n, v_{i,h}) + \langle \Phi_i(w_h^{n-1})\epsilon_{i,h}^n, v_{i,h}\rangle = \langle \delta_{i,h}^n, v_{i,h}\rangle \tag{47}$$

with

$$\delta_{i,h}^n = E_{1,i}^n + E_{2,i}^n + E_{3,i}^n, \tag{48}$$

where

$$E_{1,i}^n = \frac{\partial w_i(t_n)}{\partial t} - \frac{1}{\delta t}\prod_{i,h}^{r}\Big(w_i(t_n) - w_i(t_{n-1})\Big), \tag{49}$$

$$E_{2,i}^n = \tilde{f}_i(w_h^{n-1}) - \tilde{f}_i(w(t_n)), \tag{50}$$

$$E_{3,i}^n = \Phi_i(w(t_n))w_i(t_n) - \Phi_i(w_h^{n-1})\prod_{i,h}^{r} w_i(t_n). \tag{51}$$

The following lemma presents an estimate of $\epsilon_{i,h}^n$ in the space $L^2(\Omega)$ and $H^1(\Omega)$.

Lemma 4 $\epsilon_{i,h}^n$ *satisfies the following estimate*

$$\|\epsilon_{i,h}^n\|_{L^2(\Omega)}^2 + \alpha\delta t\sum_{m=1}^{n}\|\epsilon_{i,h}^m\|_{H^1(\Omega)}^2 \quad \le \|\epsilon_{i,h}^0\|_{L^2(\Omega)}^2 + \frac{\delta t}{\alpha}\sum_{m=1}^{n}\|\delta_{i,h}^m\|_{L^2(\Omega)}^2. \tag{52}$$

Proof We choose $v_{i,h} = \epsilon_{i,h}^n$ in Eq. (47). By using Cauchy–Schwarz inequality at the right, the bilinear form $b_i(.,.)$ coercivity, the positivity of $\Phi_i(w_h^{n-1})$ and the following equality:

$$\langle \epsilon_{i,h}^n - \epsilon_{i,h}^{n-1}, \epsilon_{i,h}^n\rangle = \|\epsilon_{i,h}^n\|_{L^2(\Omega)}^2 - \langle \epsilon_{i,h}^{n-1}, \epsilon_{i,h}^n\rangle, \tag{53}$$

we get

$$\|\epsilon_{i,h}^n\|_{L^2(\Omega)}^2 + \alpha\delta t\|\epsilon_{i,h}^n\|_{H^1(\Omega)}^2 \le \langle \epsilon_{i,h}^{n-1}, \epsilon_{i,h}^n\rangle + \delta t\|\delta_{i,h}^n\|_{L^2(\Omega)}\|\epsilon_{i,h}^n\|_{L^2(\Omega)}. \tag{54}$$

We apply Young inequality to the first term at the right side of inequality. Afterwards we use the generalized Young inequality for the second term. We obtain

$$\|\epsilon_{i,h}^n\|_{L^2(\Omega)}^2 + (2\alpha - \epsilon)\delta t\|\epsilon_{i,h}^n\|_{H^1(\Omega)}^2 \le \|\epsilon_{i,h}^{n-1}\|_{L^2(\Omega)}^2 + \frac{\delta t}{\epsilon}\|\delta_{i,h}^n\|_{L^2(\Omega)}^2. \tag{55}$$

By choosing $\epsilon = \alpha$ and adding this last inequality for $m = 1, \ldots, n$, it comes

$$\|\epsilon_{i,h}^n\|_{L^2(\Omega)}^2 + \alpha\delta t\sum_{m=1}^{n}\|\epsilon_{i,h}^m\|_{H^1(\Omega)}^2 \le \|\epsilon_{i,h}^0\|_{L^2(\Omega)}^2 + \frac{\delta t}{\alpha}\sum_{m=1}^{n}\|\delta_{i,h}^m\|_{L^2(\Omega)}^2. \tag{56}$$

□

The following lemma permits to overestimate $\epsilon_{i,h}^0$.

Lemma 5 *There is a constant $C > 0$ such that*

$$\|\epsilon_{i,h}^0\|_{L^2(\Omega)} \leq Ch^{r+1}|u_{i0}|_{H^{r+1}(\Omega)}. \tag{57}$$

Proof The initial condition is $w_{i,h}^0 = \prod_h^r u_{i0}$, then we get

$$\epsilon_{i,h}^0 = \prod_h^r u_{i0} - \prod_{i,h}^r u_{i0}. \tag{58}$$

This implies

$$\|\epsilon_{i,h}^0\|_{L^2(\Omega)} \leq \left\| \prod_h^r u_{i0} - u_{i0} \right\|_{L^2(\Omega)} + \left\| u_{i0} - \prod_{i,h}^r u_{i0} \right\|_{L^2(\Omega)}. \tag{59}$$

The first term at the right of the inequality satisfy

$$\left\| \prod_h^r u_{i0} - u_{i0} \right\|_{L^2(\Omega)} \leq C_1 h^{r+1}|u_{i0}|_{H^{r+1}(\Omega)}, \tag{60}$$

according to the inequality (12) on the interpolation operator. We have chosen $m = 0$ and $C_1 > 0$ is a constant.

The second term at the right of the inequality satisfy

$$\left\| u_{i0} - \prod_{i,h}^r u_{i0} \right\|_{L^2(\Omega)} \leq C_2 h^{r+1}|u_{i0}|_{H^{r+1}(\Omega)}, \tag{61}$$

in pursuance of the orthogonal projection inequality (16). Here we have set $p = r$ and $C_2 > 0$ is a constant.

We deduce that there is a constant $C > 0$ such that

$$\|\epsilon_{i,h}^0\|_{L^2(\Omega)} \leq Ch^{r+1}|u_{i0}|_{H^{r+1}(\Omega)}. \tag{62}$$

□

Lemma 6 *For all $i = 1, ..., 6$, f_i satisfies*

$$\|f_i(u) - f_i(v)\|_{L^2(\Omega)} \leq C\|u - v\|_{L^2(\Omega)} \tag{63}$$

for all nonnegatives $v = (v_i)_{i=1,...,6}$ and $u = (u_i)_{i=1,...,6}$ in $H^r(\Omega)$.

Proof The functions f_i for $i \neq 2$ are affines, so the property is verified. Concerning f_2, we note that the term $\dfrac{\zeta u_5 + \mu u_1}{2}$ is affine. We want to show that the function

$$g(u) = \frac{u_5}{K_m + u_5}(c_1 u_3 + c_2 u_4) \tag{64}$$

verifies

$$\|g(u) - g(v)\|_{L^2(\Omega)} \leq C\|u - v\|_{L^2(\Omega)}, \tag{65}$$

with $C > 0$. Let us define $\alpha(u_5) = \dfrac{u_5}{K_m + u_5}$. We verify

$$g(u) - g(v) = \alpha(u_5)\big(c_1(u_3 - v_3) + c_2(u_4 - v_4)\big) + (\alpha(u_5) - \alpha(v_5))\big(c_1 v_3 + c_2 v_4\big). \tag{66}$$

The derivative of $\alpha(u_5)$ is $\dfrac{K_m}{(K_m + u_5)^2}$. It is bounded because u_5 is nonnegative. It follows that

$$|g(u) - g(v)| \leq c_1|u_3 - v_3| + c_2|u_4 - v_4| + \frac{c_1|v_3| + c_2|v_4|}{K_m}|u_5 - v_5|. \tag{67}$$

We recall that the functions v_3 and v_4 are taken in the space $H^r(\Omega) \subset C(\overline{\Omega})$, then we deduce that there exists $C > 0$ such that

$$\|g(u) - g(v)\|_{L^2(\Omega)} \leq C\|u - v\|_{L^2(\Omega)}. \tag{68}$$

□

We present on the three next lemmas the estimates on $E^m_{1,i}$, $E^m_{2,i}$ and $E^m_{3,i}$ in $L^2(\Omega)$.

Lemma 7 *There is a constant $C > 0$ such that*

$$\|E^m_{1,i}\|_{L^2(\Omega)} \leq C\left(\int_{t_{m-1}}^{t_m} \left\|\frac{\partial^2 w_i(s)}{\partial t^2}\right\|_{L^2(\Omega)} ds + \frac{h^r}{\delta t}\int_{t_{m-1}}^{t_m} \left|\frac{\partial w_i(s)}{\partial t}\right|_{H^r(\Omega)} ds\right). \tag{69}$$

Proof The triangular inequality implies

$$\begin{aligned}\|E^m_{1,i}\|_{L^2(\Omega)} \leq &\left\|\frac{\partial w_i(t_m)}{\partial t} - \frac{w_i(t_m) - w_i(t_{m-1})}{\delta t}\right\|_{L^2(\Omega)} \\ &+\left\|\frac{1}{\delta t}\left(I - \prod_{i,h}^{r}\right)\Big(w_i(t_m) - w_i(t_{m-1})\Big)\right\|_{L^2(\Omega)}.\end{aligned} \tag{70}$$

The development of Taylor with an integral residu involves

$$\begin{aligned}\|E^m_{1,i}\|_{L^2(\Omega)} \leq &\left\|\frac{1}{\delta t}\int_{t_{m-1}}^{t_m} (s - t_{m-1})\frac{\partial^2 w_i(s)}{\partial t^2} ds\right\|_{L^2(\Omega)} \\ &+\left\|\frac{1}{\delta t}\int_{t_{m-1}}^{t_m}\left(I - \prod_{i,h}^{r}\right)\frac{\partial w_i(s)}{\partial t} ds\right\|_{L^2(\Omega)} \\ \leq &\int_{t_{m-1}}^{t_m}\left\|\frac{\partial^2 w_i(s)}{\partial t^2}\right\|_{L^2(\Omega)} ds + \frac{1}{\delta t}\int_{t_{m-1}}^{t_m}\left\|\left(I - \prod_{i,h}^{r}\right)\frac{\partial w_i(s)}{\partial t}\right\|_{L^2(\Omega)} ds \\ \leq &\, C\left(\int_{t_{m-1}}^{t_m}\left\|\frac{\partial^2 w_i(s)}{\partial t^2}\right\|_{L^2(\Omega)} ds + \frac{h^r}{\delta t}\int_{t_{m-1}}^{t_m}\left|\frac{\partial w_i(s)}{\partial t}\right|_{H^r(\Omega)} ds,\right),\end{aligned} \tag{71}$$

where $C > 0$ is a constant. For the last inequality, we have used the property (16) with $p = r - 1$. □

Lemma 8 *There exists constants $C^i > 0$ such that*

$$\|E^m_{2,i}\|_{L^2(\Omega)} \leq C^i \left(\|\epsilon_h^{m-1}\|_{L^2(\Omega)} + h^r max_{t\in[0,T]}|w(t)|_{H^r(\Omega)} + \int_{t_{m-1}}^{t_m} \left\| \frac{\partial w(s)}{\partial t} \right\|_{L^2(\Omega)} ds \right), \tag{72}$$

for $i = 1, .., 6$.

Proof The Lemma 6 implies

$$\begin{aligned} \|E^m_{2,i}\|_{L^2(\Omega)} &\leq C^i \|w_h^{m-1} - w(t_m)\|_{L^2(\Omega)} \\ &\leq C^i \left(\|w_h^{m-1} - w(t_{m-1})\|_{L^2(\Omega)} + \|w(t_{m-1}) - w(t_m)\|_{L^2(\Omega)} \right), \end{aligned} \tag{73}$$

where $C^i > 0$ is a constant. The property (16) permits to overvalue the first term at the right of the inequality

$$\begin{aligned} \left\| w_h^{m-1} - w(t_{m-1}) \right\|_{L^2(\Omega)} &\leq \left\| w_h^{m-1} - \prod_{i,h}^{r} w(t_{m-1}) \right\|_{L^2(\Omega)} + \left\| \left(I - \prod_{i,h}^{r} \right) w(t_{m-1}) \right\|_{L^2(\Omega)}, \\ &\leq \|\epsilon_h^{m-1}\|_{L^2(\Omega)} + Ch^r |w(t_{m-1})|_{H^r(\Omega)}, \end{aligned} \tag{74}$$

where $C > 0$ is a constant.

By using the Taylor development, we have

$$\left\| w(t_{m-1}) - w(t_m) \right\|_{L^2(\Omega)} \leq \left\| \int_{t_{m-1}}^{t_m} \frac{\partial w(s)}{\partial t} ds \right\|_{L^2(\Omega)} \leq \int_{t_{m-1}}^{t_m} \left\| \frac{\partial w(s)}{\partial t} \right\|_{L^2(\Omega)} ds. \tag{75}$$

From these two last inequalities, we deduce that there exists a constant denoted $C^i > 0$ such that

$$\|E^m_{2,i}\|_{L^2(\Omega)} \leq C^i \left(\|\epsilon_h^{m-1}\|_{L^2(\Omega)} + h^r \max_{t\in[0,T]}|w(t)|_{H^r(\Omega)} + \int_{t_{m-1}}^{t_m} \left\| \frac{\partial w(s)}{\partial t} \right\|_{L^2(\Omega)} ds \right). \tag{76}$$

□

Lemma 9 *There exists a constant $C > 0$ such that*

$$\|E^m_{3,i}\|_{L^2(\Omega)} \leq C \left(\left\|\epsilon_h^{m-1}\right\|_{L^2(\Omega)} + h^r max_{t\in[0,T]}|w(t)|_{H^r(\Omega)} + \int_{t_{m-1}}^{t_m} \left\| \frac{\partial w(s)}{\partial t} \right\|_{L^2(\Omega)} \right). \tag{77}$$

Proof The triangular inequality implies

$$\begin{aligned} \|E^m_{3,i}\|_{L^2(\Omega)} \leq & \left\| (\Phi_i(w(t_m)) - \Phi_i(w_h^{m-1}))w_i(t_m) \right\|_{L^2(\Omega)} \\ & + \left\| \Phi_i(w_h^{m-1}) \left(I - \prod_{i,h}^{r} \right) w_i(t_m) \right\|_{L^2(\Omega)}. \end{aligned} \tag{78}$$

As $w_i(t_m) \in H^r(\Omega) \subset \mathcal{C}^0(\overline{\Omega})$, there exists a constant $C_i > 0$ such that

$$\sup_{x\in\overline{\Omega}} |w_i(t_m)| < C_i. \tag{79}$$

We recall that $\|w_{i,h}^m\|_{L^2(\Omega)}$ is bounded according to the Lemma 3. V_h is a space with a finite dimension, so the norms L^2 and L^∞ are equivalents.Then we deduce that there exists a constant $D_i > 0$ such that

$$\|\Phi_i(w_h^{m-1})\|_{L^\infty(\Omega)} < D_i. \tag{80}$$

It gives

$$\|E_{3,i}^m\|_{L^2(\Omega)} \leq C\left(\left\|\Phi_i(w(t_m)) - \Phi_i(w_h^{m-1})\right\|_{L^2(\Omega)} + \left\|\left(I - \prod_{i,h}^r\right) w_i(t_m)\right\|_{L^2(\Omega)}\right), \tag{81}$$

where $C > 0$ is a constant. Moreover inequality (16) implies

$$\|E_{3,i}^m\|_{L^2(\Omega)} \leq C\Big(\left\|\Phi_i(w(t_m)) - \Phi_i(w_h^{m-1})\right\|_{L^2(\Omega)} + h^r\left|w_i(t_m)\right|_{H^r(\Omega)}\Big). \tag{82}$$

The Taylor expansion of Φ_i about w_h^{m-1} reads

$$\Phi_i(w(t_m)) - \Phi_i(w_h^{m-1}) = \int_{w_h^{m-1}}^{w(t_m)} \nabla\Phi_i(\zeta)d\zeta. \tag{83}$$

The functions w_h^{m-1} and $w(t_m)$ are bounded in $L^\infty(\Omega)$ and nonnegatives. Consequently there is a constant $C > 0$ such that

$$|\nabla\Phi_i(\zeta)| \leq C. \tag{84}$$

The relations (82), (83) and (84) imply

$$\|E_{3,i}^m\|_{L^2(\Omega)} \leq C\Big(\left\|w(t_m) - w_h^{m-1}\right\|_{L^2(\Omega)} + h^r\left|w_i(t_m)\right|_{H^r(\Omega)}\Big). \tag{85}$$

By a proof similar to that of Lemma 8, we obtain

$$\|E_{3,i}^m\|_{L^2(\Omega)} \leq C\left(\|\epsilon_h^{m-1}\|_{L^2(\Omega)} + h^r \max_{t\in[0,T]}|w(t)|_{H^r(\Omega)} + \int_{t_{m-1}}^{t_m} \left\|\frac{\partial w(s)}{\partial t}\right\|_{L^2(\Omega)}\right). \tag{86}$$

□

We use the previous results to demonstrate the following Lemma.

Lemma 10 *There is a constant C_T which depends on T such that*

$$\begin{aligned}\delta t\sum_{m=1}^{n}\|\delta_{i,h}^m\|_{L^2(\Omega)}^2 \leq C_T\Bigg(\delta t^2\left(\int_0^T\left\|\frac{\partial^2 w(s)}{\partial t^2}\right\|_{L^2(\Omega)}^2 ds + \int_0^T\left\|\frac{\partial w(s)}{\partial t}\right\|_{L^2(\Omega)}^2 ds\right)\\ +h^{2r}\left(\int_0^T\left|\frac{\partial w(s)}{\partial t}\right|_{H^r(\Omega)}^2 ds + max_{t\in[0,T]}|w(t)|_{H^r(\Omega)}^2\right), +\delta t\sum_{m=0}^{n-1}\|\epsilon_h^m\|_{L^2(\Omega)}^2\Bigg),\end{aligned} \tag{87}$$

for all $i = 1, 2, ..., 6$.

Proof We compute the square of the Lemmas 7, 8, 9 inequalities and apply Cauchy–Schwarz to the right term inequality to get

$$\|E_{1,i}^m\|_{L^2(\Omega)}^2 \le C\left(\delta t\int_{t_{m-1}}^{t_m}\left\|\frac{\partial^2 w(s)}{\partial t^2}\right\|_{L^2(\Omega)}^2 ds + \frac{h^{2r}}{\delta t}\int_{t_{m-1}}^{t_m}\left|\frac{\partial w(s)}{\partial t}\right|_{H^r(\Omega)}^2 ds\right), \quad (88)$$

$$\|E_{2,i}^m\|_{L^2(\Omega)}^2 \le C\left(\|\epsilon_h^{m-1}\|_{L^2(\Omega)}^2 + h^{2r}\max_{t\in[0,T]}|w(t)|_{H^r(\Omega)}^2 + \delta t\int_{t_{m-1}}^{t_m}\left\|\frac{\partial w(s)}{\partial t}\right\|_{L^2(\Omega)}^2 ds\right) \quad (89)$$

and

$$\|E_{3,i}^m\|_{L^2(\Omega)}^2 \le C\left(\|\epsilon_h^{m-1}\|_{L^2(\Omega)}^2 + h^{2r}\max_{t\in[0,T]}|w(t)|_{H^r(\Omega)}^2 + \delta t\int_{t_{m-1}}^{t_m}\left\|\frac{\partial w(s)}{\partial t}\right\|_{L^2(\Omega)}^2 ds\right). \quad (90)$$

We deduce

$$\begin{aligned}\delta t\sum_{m=1}^{n}\|\delta_{i,h}^m\|_{L^2(\Omega)}^2 \le C_T\Bigg(&\delta t^2\left(\int_0^T\left\|\frac{\partial^2 w(s)}{\partial t^2}\right\|_{L^2(\Omega)}^2 ds + \int_0^T\left\|\frac{\partial w(s)}{\partial t}\right\|_{L^2(\Omega)}^2 ds\right)\\ &+h^{2r}\left(\int_0^T\left|\frac{\partial w(s)}{\partial t}\right|_{H^r(\Omega)}^2 ds + \max_{t\in[0,T]}|w(t)|_{H^r(\Omega)}^2\right)\\ &+\delta t\sum_{m=0}^{n-1}\|\epsilon_h^m\|_{L^2(\Omega)}^2\Bigg),\end{aligned} \quad (91)$$

where C_T is a constant depending on T. □

We present now the main theorem on the error estimate.

Theorem 2 *If the initial condition $u_0 = (\prod_h^r u_{i0})_{i=1,\ldots,6}$ and the weak solution w are enough regulars, we get the following estimate*

$$\begin{aligned}\|w_h^n - w(t_n)\|_{L^2(\Omega)}^2 + \alpha\delta t\sum_{m=1}^{n}\|w_h^m - w(t_m)\|_{H^1(\Omega)}^2 \le C_T\Bigg(&h^{2(r+1)}|w_0|_{H^{r+1}(\Omega)}^2\\ &+\delta t^2\left(\int_0^T\left\|\frac{\partial^2 w(s)}{\partial t^2}\right\|_{L^2(\Omega)}^2 ds + \int_0^T\left\|\frac{\partial w(s)}{\partial t}\right\|_{L^2(\Omega)}^2 ds\right)\\ &+h^{2r}\left(\int_0^T\left|\frac{\partial w(s)}{\partial t}\right|_{H^r(\Omega)}^2 ds + max_{t\in[0,T]}|w(t)|_{H^{r+1}(\Omega)}^2\right)\Bigg),\end{aligned} \quad (92)$$

where C_T is a constant depending on T.

Proof We deduce from the Lemmas 4, 5 and 10

$$
\begin{aligned}
&\|\epsilon_{i,h}^n\|_{L^2(\Omega)}^2 + \alpha\delta t \sum_{m=1}^{n} \|\epsilon_{i,h}^m\|_{H^1(\Omega)}^2 \leq C_T \Bigg(\delta t \sum_{m=0}^{n-1} \|\epsilon_h^m\|_{L^2(\Omega)}^2 + h^{2(r+1)} |w_{0i}|_{H^{r+1}(\Omega)}^2 \\
&+\delta t^2 \left(\int_0^T \left\| \frac{\partial^2 w(s)}{\partial t^2} \right\|_{L^2(\Omega)}^2 ds + \int_0^T \left\| \frac{\partial w(s)}{\partial t} \right\|_{L^2(\Omega)}^2 ds \right) \\
&+h^{2r} \left(\int_0^T \left| \frac{\partial w(s)}{\partial t} \right|_{H^r(\Omega)}^2 ds + \max_{t\in[0,T]} |w(t)|_{H^r(\Omega)}^2 \right) \Bigg).
\end{aligned}
\tag{93}
$$

By summing the previous inequalities for $i = 1, \ldots, 6$, it comes

$$
\begin{aligned}
&\|\epsilon_{h}^n\|_{L^2(\Omega)}^2 + \alpha\delta t \sum_{m=1}^{n} \|\epsilon_{h}^m\|_{H^1(\Omega)}^2 \leq C_T \Bigg(\delta t \sum_{m=0}^{n-1} \|\epsilon_h^m\|_{L^2(\Omega)}^2 + h^{2(r+1)} |w_{0}|_{H^{r+1}(\Omega)}^2 \\
&+\delta t^2 \left(\int_0^T \left\| \frac{\partial^2 w(s)}{\partial t^2} \right\|_{L^2(\Omega)}^2 ds + \int_0^T \left\| \frac{\partial w(s)}{\partial t} \right\|_{L^2(\Omega)}^2 ds \right) \\
&+h^{2r} \left(\int_0^T \left| \frac{\partial w(s)}{\partial t} \right|_{H^r(\Omega)}^2 ds + \max_{t\in[0,T]} |w(t)|_{H^r(\Omega)}^2 \right) \Bigg).
\end{aligned}
\tag{94}
$$

The discret Gronwall lemma implies

$$
\begin{aligned}
&\|\epsilon_{h}^n\|_{L^2(\Omega)}^2 + \alpha\delta t \sum_{m=1}^{n} \|\epsilon_{h}^m\|_{H^1(\Omega)}^2 \\
&\leq C_T \Bigg(h^{2(r+1)} |w_{0}|_{H^{r+1}(\Omega)}^2 + \delta t^2 \left(\int_0^T \left\| \frac{\partial^2 w(s)}{\partial t^2} \right\|_{L^2(\Omega)}^2 ds + \int_0^T \left\| \frac{\partial w(s)}{\partial t} \right\|_{L^2(\Omega)}^2 ds \right) \\
&+h^{2r} \left(\int_0^T \left| \frac{\partial w(s)}{\partial t} \right|_{H^r(\Omega)}^2 ds + \max_{t\in[0,T]} |w(t)|_{H^r(\Omega)}^2 \right) \Bigg).
\end{aligned}
\tag{95}
$$

The interpolation error inequality (12) implies

$$
\left\| \left(I - \prod_{i,h}^{r} \right) w(t_m) \right\|_{L^2(\Omega)} \leq C h^r |w(t_m)|_{H^r(\Omega)}
\tag{96}
$$

and

$$
\left\| \left(I - \prod_{i,h}^{r} \right) w(t_m) \right\|_{H^1(\Omega)} \leq C h^r |w(t_m)|_{H^{r+1}(\Omega)}.
\tag{97}
$$

We apply the triangular inequality and introduce the term $\prod_{i,h}^{r} w(t_n)$. From the inequalities (95), (96) and (97), we have

$$\begin{aligned}\|w_h^n - w(t_n)\|^2_{L^2(\Omega)} + \alpha\delta t \sum_{m=1}^{n} \|w_h^m - w(t_m)\|^2_{H^1(\Omega)} &\leq C_T \Bigg(h^{2(r+1)}|w_0|^2_{H^{r+1}(\Omega)} \\ &+ \delta t^2 \left(\int_0^T \left\|\frac{\partial^2 w(s)}{\partial t^2}\right\|^2_{L^2(\Omega)} ds + \int_0^T \left\|\frac{\partial w(s)}{\partial t}\right\|^2_{L^2(\Omega)} ds\right) \\ &+ h^{2r}\left(\int_0^T \left|\frac{\partial w(s)}{\partial t}\right|^2_{H^r(\Omega)} ds + \max_{t\in[0,T]}|w(t)|^2_{H^{r+1}(\Omega)}\right)\Bigg). \quad \square\end{aligned} \tag{98}$$

5 Numerical example

Numerical example is given for the corresponding discrete problem (19) of the problem (1). The solution is computed using the finite element solver Freefem3d. The triangulation is obtained from the domain $\Omega =]0, 1[\times]0, 1[\times]0, 1[$. The number of nodes and elements are respectively about 8,000 and 6,859. The time step is $dt = 1$.

The functions used to initialyze the solution are

$$u_1(x, y) = 0.1 * \sin(2 * \pi * x/10), \qquad u_4(x, y) = \cos(2 * \pi * x/10), \tag{99}$$

$$u_i(x, y) = 0 \text{ for all } i = 2, 3, 5, 6. \tag{100}$$

The parameter's values are

$$k = 0.7, \quad K_b = 0.264, \quad K_m = 0.001, \quad \mu = 0.001, \quad \theta = 0.02, \tag{101}$$

$$\nu = 0.001, \quad \zeta = 0.03, \quad c_1 = 0.008 \quad c_2 = 0.1. \tag{102}$$

The figure 1 shows the evolution masses of u_1 and u_4 noted BM and FOM. The figure 2 presents the cumulates masses, which theorically must be constant.

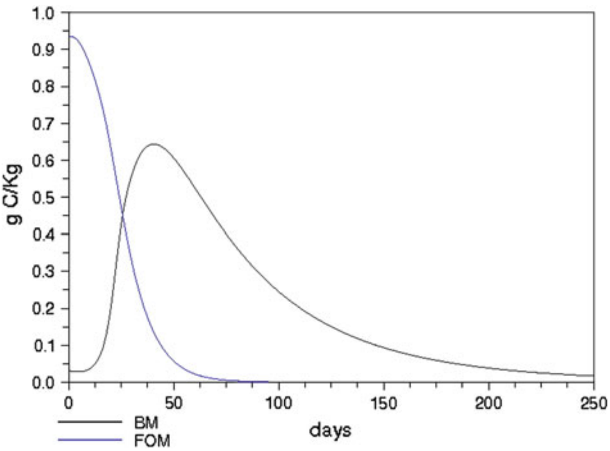

Fig. 1 Masse of u_1 and u_4

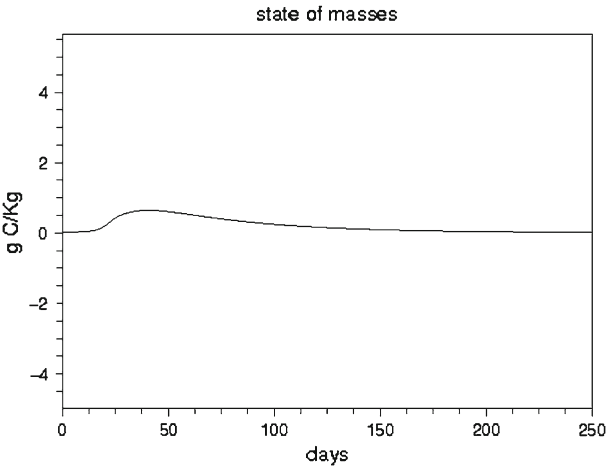

Fig. 2 Cumulates masses

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