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# DYNAMIC ANALYSIS OF 3-D FLEXIBLE EMBEDDED FOUNDATIONS BY A FREQUENCY DOMAIN BEM–FEM

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## SUMMARY

A study on the dynamic response of three-dimensional flexible foundations of arbitrary shape, embedded in a homogenous, isotropic and linear elastic half-space is presented. Both massive and massless foundations are considered. The soil–foundation system is subjected to externally applied forces, and/or to obliquely incident seismic waves. The numerical method employed is a combination of the frequency domain Boundary Element Method, which is used to simulate the elastic soil medium, and the Finite Element Method, on the basis of which the stiffness matrix of the foundation is obtained. The foundation and soil media are combined by enforcing compatibility and equilibrium conditions at their common interface. Both relaxed and completely bonded boundary conditions are considered. The accuracy of the proposed methodology is partially verified through comparison studies with results reported in the literature for rigid embedded foundations.

## 1. INTRODUCTION

The assumption of the foundation being rigid, even though it appears to be reasonable for massive structures, is not always valid. Indeed, out of plane deformations of foundation plates have been reported from field studies on actual buildings, e.g. References 18, 21, 61 and 63. However, only a few works from those available in the literature deal with the dynamic analysis of flexible foundations. In the following literature review only those works relative to the subject of flexible embedded foundations are mentioned. More comprehensive reviews on the general SSI problem can be found elsewhere, see e.g. Richart *et al.*[45] Gazetas,[16] Karabalis and Beskos.[29]

Analytical methods of solution of the above problem have been reported by Oien[40] for the case of a strip footing subjected to plane wave excitation, by Iguchi[22] for a long rectangular mat, flexible in its long dimension, subjected to obliquely incident seismic waves, by Lin[35] and Krenk and Schmidt[33] for a flexible circular plate under the influence of externally applied loads and by Iguchi and Luco[24] for flexible circular plates with a rigid core on a viscoelastic uniform or layered half-space.

A number of numerical solutions have also been reported. Savidis and Richter,[48] Ray,[43,44] Iguchi and Luco,[23] Whittaker and Christiano[53,54] and Avanessian *et al.*[2] combined successfully the FEM and analytical results from Lamb's problem[34] in their three-dimensional studies of flexible surface foundations. Tassoulas and Kausel[50] studied the effect of the rigid sidewall on the dynamic stiffness of rigid embedded circular footings, while Riggs and Waas[46] studied the influence of the foundation flexibility on the dynamic response of surface flexible foundations on a viscoelastic stratum. The Finite Element Method (FEM) has been used for the dynamic analysis of two-dimensional surface or embedded flexible foundations in conjunction with several specialized techniques e.g. substructuring,[20,52] infinite elements[6,31,37] and substructure deletion and cloning.[9,10] Three-dimensional analyses have also been reported by Dasgupta and Rao[11] and Gupta *et al.*[19]

Several simplified models have also been proposed to study approximately the dynamic stiffness of rigid foundations of relatively simple geometric shapes. The most recent of those models can be found in the work of

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Gazetas, Dobry and Tassoulas[17] and Wolf and Somaini.[60] An approximate analytical model for rigid circular embedded foundations was first reported by Beredugo and Novak[3] and Novak and Beredugo.[39]

The Boundary Element Method (BEM) has been applied to the general SSI area of research through either a frequency domain formulation, e.g. References 8, 12 and 13, or a direct step-by-step time domain formulation, e.g. References 27, 28, 58 and 59. However, the BEM has found a limited application so far in problems involving flexible foundations, and then only surface foundations of simple geometries have been treated. Ottenstreuer,[41] Karabalis[26] and Karabalis and Beskos[27] used the frequency and time domain BEM formulations, respectively, to study the dynamic response of three-dimensional flexible surface foundations. Wolf and Darbre[55,56] and Abascal and Dominguez[1] obtained the dynamic stiffnesses of a soil medium by the BEM for embedded foundations, while Spyrakos and Beskos[49] analysed the corresponding two-dimensional rigid foundation problem using the time domain BEM. Three-dimensional rigid embedded foundations have also been analysed by the BEM using infinite domain solutions, e.g. Dominguez[12,13] in the frequency domain and Karabalis and Beskos[28] in the time domain. In the above studies the flexibility of the foundation plate is accounted for wherever it is required, by means of a FEM discretization using the thin plate theory which has been found to produce satisfactory results for static analyses.[5,14]

Several authors have studied the problem of obliquely incident waves impinging on a rigid or flexible foundation. Wong and Luco[62] were the first to study the dynamic response of a rigid massless rectangular foundation while Whittaker and Christiano[54] studied the case of flexible surface foundations. The frequency domain BEM has been applied by Dominguez[13] for the study of rigid massless foundations and by Rizzo *et al.*[47] for the study of the radiation and scattering of elastic waves. Karabalis and Beskos[26-29] used the time domain BEM to study the influence of obliquely incident seismic waves of arbitrary time variation on rigid surface or embedded, and flexible surface foundations.

This work addresses the problem of the interaction between a 3-D flexible embedded foundation of arbitrary shape and a homogenous, isotropic, linear elastic half-space where no uplift or lateral separation occur at the contact surface between the two media. Both massive and massless foundations are considered, undergoing vertical, horizontal, torsional and rocking motion or combinations of the above. The forcing functions of the problem under consideration are externally applied harmonic loads and/or obliquely incident seismic waves. The foundation–soil system is analysed numerically in the frequency domain by a FEM–BEM combination. The BEM is used, along with infinite space solutions, to simulate the soil medium which is modelled as a homogenous, isotropic, linear elastic half-space, while the FEM is used for the flexible, linear elastic foundation structure.

## 2. FORMULATION AND NUMERICAL IMPLEMENTATION

### 2.1. *General formulation*

The geometry of the problem as well as the coordinate system employed and the related nomenclature are shown in Figure 1. The soil foundation system can be viewed as the assemblange of two substructures, namely the soil and the foundation media. The pertinent governing equations for each medium are developed separately and are finally combined by means of the equilibrium and compatibility conditions written for the contact surface between the soil and the foundation. The following general frequency domain BEM formulation is based on the work of Cruse and Rizzo,[7], while details on the substructure formulation used in this work, can be found in various sources, e.g. References 20, 28 and 30.

(i) *Soil substructure.* Let $R$ represent a homogeneous isotropic and linear elastic region bounded by a surface $B$. Assuming harmonic motion, i.e. the displacement of point $\mathbf{x}$ is given by $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x})e^{i\omega t}$, the equations of motion for the region $R$ are given by

$$(c_1^2 - c_2^2)u_{i,ij} + c_2^2 u_{j,ii} + f_i = -\omega^2 \bar{u}_j \tag{1}$$

where $c_1$ and $c_2$ are the dilatational and shear wave velocities, respectively, given by

$$c_1^2 = (\lambda + 2\mu)/\rho_s \quad c_2^2 = (\mu/\rho_s) \tag{2}$$

$\rho_s$ being the mass density of the soil medium, $\lambda$ and $\mu$ the Lamé constants and $f_i$ the body force per unit mass. Assuming zero body forces and initial conditions, the displacements $u_k$ at any point $\xi$ of the surface $B$ can be expressed as

$$\beta(\xi)u_k(\xi,\omega) = \int_B [u_{ik}(\mathbf{x},\xi,\omega)t_{(\mathbf{n})i}(\mathbf{x},\omega) - t_{(\mathbf{n})ik}(\mathbf{x},\xi,\omega)u_i(\mathbf{x},\omega)]\mathrm{d}B(\mathbf{x}) \tag{3}$$

where

$$t_{(\mathbf{n})i}(\mathbf{x},t) = t_{ij}(\mathbf{x},t)n_j \tag{4}$$

are surface tractions, $\mathbf{n}$ is the unit outward normal vector on surface $B$ and $\omega$ is the frequency variable. The $\beta(\xi)$ parameter is a function of the geometry of the boundary $B$ in the vicinity of $\xi$, and for the purposes of this work, where only smooth boundaries are considered, $\beta(\xi)$ assumes the constant value of 0·5. The Latin subscripts take the values 1, 2 and 3 and the summation convention is assumed over repeated indices. The pair of fundamental solutions $u_{ik}(\mathbf{x},\xi,\omega)$ and $t_{(\mathbf{n})ik}(\mathbf{x},\xi,\omega)$ correspond, respectively, to the displacements and tractions in an infinite solid medium due to a unit harmonic force $f_k(\mathbf{x},\omega)$ concentrated at a point $\mathbf{x}$, i.e.

$$\rho_s f_k(\mathbf{x},\omega) = f(\omega)\delta(\mathbf{x}-\xi)e_k \tag{5}$$

where $\mathbf{e}$ denotes the directionality of the force vector. The tensors $u_{ik}$ and $t_{(\mathbf{n})ik}$ depend on the material properties of the soil medium, the frequency variable $\omega$, as well as the distance

$$r = \sqrt{[(x_i-\xi_i)(x_i-\xi_i)]} \tag{6}$$

between the source point $\mathbf{x}$ and the receiver point $\xi$. Explicit formulae for the fundamental solutions $u_{ik}(\mathbf{x},\xi,\omega)$ and $t_{(\mathbf{n})ik}(\mathbf{x},\xi,\omega)$ can be found in various sources, e.g. Karabalis and Beskos.[29]

For the purposes of this work the BIE (3) can be discretized by dividing the surface $B$ into an $M$ number of rectangular elements, as shown in Figure 1. Subsequently, an assumption has to be made with regard to the spatial variation of all the boundary quantities over each one of these boundary elements. Based on the suggestion of several authors, see e.g. References 12, 30, 41 and 42, a uniform distribution of displacements and contact tractions over each element has been assumed in this work. A direct consequence of this approximation is a collocation of the element tractions and displacements at the centre of the corresponding element, and thus the BIE (3) can be written in the following matrix summation form:

$$[0{\cdot}5]\{u^R\} = \sum_{s=1}^{M} \{[G^{R,s}]\{t^s\} - [T^{R,s}]\{u^s\}\}, \qquad R = 1,2,\ldots,M \tag{7}$$

where $\{u^R\}$ represents the $(3\times 1)$ displacement vector of the element $R$ (receiver), and $(t^s)$ and $\{u^s\}$ represent, respectively, the $(3\times 1)$ traction and displacement vectors of the element $s$ (source). The $(3\times 3)$ influence

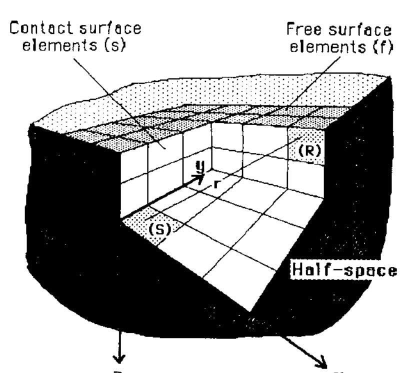


Figure 1. Geometry and nomenclature

matrices $[G^{R,s}]$ and $[T^{R,s}]$ are given by

$$G_{ij}^{R,s} = \int_{(s)} u_{ij}(\mathbf{x}, \xi, \omega)\mathrm{d}s(\mathbf{x})$$

$$T_{ij}^{R,s} = \int_{(s)} t_{(\mathrm{n})ij}(\mathbf{x}, \xi, \omega)\mathrm{d}s(\mathbf{x}) \tag{8}$$

where $(s)$ represents the area of the source element. Since a summation over a finite number of boundary elements $M$ is indicated at the right-hand side of equation (7) a truncation of the infinite free surface boundary is implied. This is necessary in view of the infinite space fundamental solutions used in this work. However, as has been previously reported by Karabalis and Mohammadi[30] and Mohammadi and Karabalis[38] for surface foundations, this approximation is not expected to compromise the accuracy of the solution. Indeed, it is shown in the numerical results section of this work that one, two or three zones of free-field elements, of size comparable to that of the elements used on the contact surface, produce essentially identical results.

Owing to the form of the fundamental solutions $G_{ij}$ and $T_{ij}$ a potential problem might be expected at the so-called singular elements, i.e. $s = R$. However, it can be shown, see e.g. References 12 and 38, that this difficulty can be circumvented by means of a series expansion. It can also be shown that a straightforward Gauss integration of these 'singular' expressions produces as accurate results.[15] Equation (7) can be written for every boundary element $R = 1, 2 \ldots, M$, in which case the resulting system of $3 \times M$ linear algebraic equations can be cast in a matrix form as

$$(0{\cdot}5)\{u\} = [G]\{t\} - [T]\{u\} \tag{9}$$

The solution of this system of equations requires the specification of $3M$ boundary conditions prior to its solution for some particular value of the frequency parameter $\omega$. If the subscript f is used to designate free-surface elements and the subscript s is used for those elements on the soil–foundation contact surface, as shown in Figure 1, then equation (9) can be rewritten in a partitioned form as

$$0{\cdot}5\begin{Bmatrix} u_\mathrm{f} \\ u_\mathrm{s} \end{Bmatrix} = \begin{bmatrix} G_\mathrm{ff} & G_\mathrm{fs} \\ G_\mathrm{sf} & G_\mathrm{ss} \end{bmatrix}\begin{Bmatrix} t_\mathrm{f} \\ t_\mathrm{s} \end{Bmatrix} - \begin{bmatrix} T_\mathrm{ff} & T_\mathrm{fs} \\ T_\mathrm{sf} & T_\mathrm{ss} \end{bmatrix}\begin{Bmatrix} u_\mathrm{f} \\ u_\mathrm{s} \end{Bmatrix} \tag{10}$$

Applying the boundary condition $\{t_\mathrm{f}\} = 0$, i.e. the free-surface tractions are zero, in equation (10), one can easily eliminate the free-surface displacements and obtain a relationship of the form

$$\{u_\mathrm{s}\} = [R(\omega)]\{t_\mathrm{s}\} \tag{11}$$

which represents a system of $3M_\mathrm{s}$ equations, $M_\mathrm{s}$ being the number of the boundary elements on the contact surface, and

$$[R(\omega)] = [[I] - 4[T_\mathrm{sf}]\{[I] + 2[T_\mathrm{ff}]\}^{-1}[T_\mathrm{fs}] + 2[T_\mathrm{ss}]]^{-1}$$

$$\times [2[G_\mathrm{ss}] - 4[T_\mathrm{sf}]\{[I] + 2[T_\mathrm{ff}]\}^{-1}[G_\mathrm{sf}]] \tag{12}$$

Furthermore, equation (12) can be modified into a relationship between displacements and forces applied at the centres of the boundary elements of the contact surface, i.e.

$$\{u_\mathrm{s}\} = [R(\omega)][A]^{-1}\{P_\mathrm{s}\} \tag{13}$$

where $[A]$ is a $(3M_\mathrm{s} \times 3M_\mathrm{s})$ diagonal matrix, $A_{mm}$ being equal to the area of the element $m(m = 1, \ldots, M_\mathrm{s})$ and $\{P_\mathrm{s}\}$ being the $(3M_\mathrm{s} \times 1)$ contact force vector.

(ii) *Foundation substructure.* Following standard Finite Element procedures, see e.g. Zienkiewicz and Cheung,[65] the foundation medium is subdivided into a number of rectangular elements. For the purposes of this work each rectangular element is defined by four nodal points and a total of twenty degrees of freedom, i.e.

two in-plane displacements, one transverse displacement and two rotations per node (Figure 2). Finally, a stiffness equation, relating nodal forces $\{P_{\text{elast}}\}$ to nodal displacements $\{u_{\text{f}}\}$ can be written in the form

$$[\bar{K}_{\text{c}}]\{u_{\text{f}}\} = \{P_{\text{elast}}\} \tag{14}$$

where

$$[\bar{K}_{\text{c}}] = [[K_{\text{c}}] - [M_{\text{c}}]\omega^2] \tag{15}$$

is the condensed impedance matrix of the foundation substructure, after the rotational degrees of freedom have been eliminated, and $[K_{\text{c}}]$ and $[M_{\text{c}}]$ are, respectively, the condensed stiffness and mass matrices of the foundation substructure. The elimination of the undesired degrees of freedom can be achieved by standard static condensation operations on the complete stiffness matrix $[K]$ and mass matrix $[M]$, see e.g. Reference 64.

(iii) *Soil–foundation system.* On the assumption that no slippage or uplift occurs at the soil–foundation interface, the conditions of nodal forces equilibrium and nodal displacements compatibility can be used to match the two independently modelled substructures. In doing so, a difficulty arises from the fact that the discretized force–displacement relationship for the soil medium, expressed in equation (13), is written in terms of the resultant force and the average displacement related to the centre of each boundary element, while the corresponding force–displacement relationship of the foundation medium is given in terms of nodal forces and displacements. This difficulty can be circumvented by the use of an averaging procedure introduced by Iguchi and Luco.[23] The displacement vector $\{u_{\text{s}}\}_m$ at the centre of the boundary element $m$ can be expressed in terms of the displacement vectors $\{u_{\text{sc}}\}_{m1}$, $\{u_{\text{sc}}\}_{m2}$, $\{u_{\text{sc}}\}_{m3}$ and $\{u_{\text{sc}}\}_{m4}$ at its corner points as

$$\{u_{\text{s}}\}_m = 0{\cdot}25[\{u_{\text{sc}}\}_{m1} + \{u_{\text{sc}}\}_{m2} + \{u_{\text{sc}}\}_{m3} + \{u_{\text{sc}}\}_{m4}] \tag{16}$$

For the total number of $M_{\text{s}}$ boundary elements used in the problem under consideration an $M_{\text{s}}$ number of equations (16) can be written in the following compact form:

$$\{u_{\text{s}}\} = [T]\{u_{\text{sc}}\} \tag{17}$$

where $\{u_{\text{sc}}\}$ is the displacement vector of the soil medium collocated at the nodal points of the finite element discretization scheme used for the foundation substructure and $[T]$ is the transformation matrix implied by equation (16). After this transformation has been established the compatibility condition between the displacements of the foundation and the soil substructures can be simply stated as

$$\{u_{\text{sc}}\} = \{u_{\text{f}}\} = \{u\} \tag{18}$$

A transformation of the resultant forces from the centre of each rectangular boundary element to its four corners can be accomplished through a procedure similar to that described for the displacements. The final result of this transformation can be written as

$$\{P_{\text{sc}}\} = [T]^{\text{T}}\{P_{\text{s}}\} \tag{19}$$

where $\{P_{\text{sc}}\}$ is the force vector on the soil medium collocated at the nodal points of the finite element

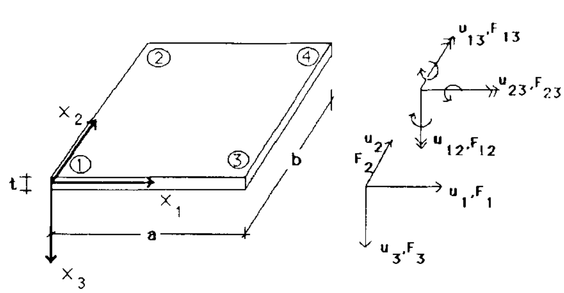


Figure 2. A typical finite element (after Zienkiewicz and Cheung [65])

discretization used for the foundation substructure. Considering next the internal (elastic) and the externally applied nodal forces on the foundation, as shown in Figure 3 for the node $i$, equilibrium requirements dictate that

$$\{P_{\text{elast}}\} = \{P_{\text{ext}}\} - \{P_{\text{sc}}\} \tag{20}$$

where $\{P_{\text{sc}}\}$ is the nodal force vector due to the subgrade reaction and $\{P_{\text{ext}}\}$ is the externally applied nodal force vector.

Direct substitution of equations (13), (14) and (17–19) into equation (20) yields the required relationship between externally applied loads and nodal displacements. This relationship has the form

$$\{u\} = [[\bar{K}_c] + [S(\omega)]]^{-1}\{P_{\text{ext}}\} \tag{21}$$

where

$$[S(\omega)] = [T][[R(\omega)][A]^{-1}]^{-1}[T]^{\mathrm{T}} \tag{22}$$

The matrix $S(\omega)$ is in general non-symmetric due to the approximations involved in the BEM representation of the soil medium. However, the symmetry requirement of the overall compliance matrix of equation (21) can be satisfied easily by a procedure that minimizes the square of the errors in the non-symmetric off-diagonal terms of the matrix $[S(\omega)]$, see Reference 4. Therefore, the new symmetric matrix $[S'(\omega)]$ can be expressed as

$$[S'(\omega)] = 0{\cdot}5[[S(\omega)] + [S(\omega)]^{\mathrm{T}}] \tag{23}$$

### 2.2. *Relaxed boundary conditions*

The 'relaxed boundary conditions' approximation is based on the observation that only a weak coupling exists among certain modes of vibration of a rigid foundation.[26,29,38] The alternative assumption of 'complete bond' between the soil and the foundation, on the basis of which the general formulation of the previous section has been developed, has also been discussed for the case of rigid foundations by several authors, see e.g. References 12, 27 and 30. The complete bond approximation, while it is believed to be more realistic, requires a substantially increased computational effort, see e.g. Reference 38. This is due, primarily, to the fact that the infinite rather than the half-space solutions are used in conjunction with the BEM in order to simulate the semi-infinite soil medium. Apparently, use of infinite space fundamental solutions requires a boundary element discretization of the entire free-surface of the half-space which increases the required numerical effort as compared to the assumption of the 'relaxed' boundary conditions for which only the contact surface needs to be discretized. Complete bond between the soil and the foundation at a minimum amount of discretization, limited to just the contact surface, is possible only if half-space fundamental solutions are used.[25,32]

The non-relaxed boundary condition approximation is based on the observation that the off-diagonal terms of equation (8) are very small, and therefore their influence on the final solution of the problem can be neglected. As an immediate consequence of the above simplification the influence tensor $[T_{ij}]$ vanishes among boundary elements lying on the same plane. Furthermore, for a singular plane element all the terms of the traction tensor and all the off-diagonal terms of the displacement tensor vanish. Thus, a direct substitution in

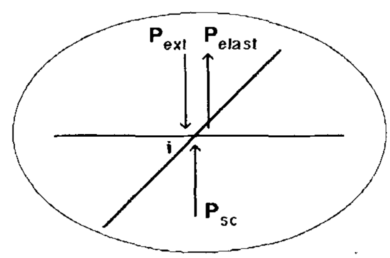


Figure 3. Nodal forces applied on the foundation substructure

equation (12) of $[T_{\mathrm{ff}}] = 0$ yields the following expression for $[R(\omega)]$:

$$[R(\omega)] = [[I] - 4[T_{\mathrm{sf}}][T_{\mathrm{fs}}] + 2[T_{\mathrm{ss}}]]^{-1}[2[G_{\mathrm{ss}}] - 4[T_{\mathrm{sf}}][G_{\mathrm{sf}}]] \tag{24}$$

The formulation presented previously for non-relaxed boundary conditions can also be used for relaxed boundary conditions if the expression for $R(\omega)$ given by equation (12) is substituted by the one given by equation (24) in equation (22).

### 2.3. *Obliquely incident seismic waves*

In this section the dynamic response of a flexible embedded foundation perfectly bonded to an elastic half-space and subjected to the action of both external forces and obliquely incident seismic waves is examined. The starting point of this formulation is Thau's suggestion[51] to split the total displacement and traction fields into two parts; the free field due to the impinging seismic waves, and the scattered field generated by the motion of the foundation in the absence of any seismic excitation. This is expressed mathematically for the purposes of the methodology proposed in this work, by

$$\begin{aligned} \{u_{\tau c}\} &= \{u_{gc}\} + \{u_{sc}\} \\ \{t_{\tau c}\} &= \{t_{gc}\} + \{t_{sc}\} \end{aligned} \tag{25}$$

where $\{u_{\tau c}\}$ and $\{t_{\tau c}\}$ are the total displacement and traction vectors described at the nodes of the finite element discretization scheme, while the subscripts g and s are used to indicate their free and scattered components, respectively, and the subscript c indicates corners of the boundary element discretization scheme or nodes of the finite element discretization mesh.

Equation (25b) can also be expressed in terms of nodal forces as

$$\{P_{\tau f}\} = \{P_{gc}\} + \{P_{sc}\} \tag{26}$$

Recalling the relationship between external forces and their corresponding displacements for the subgrade derived in Section 2.1

$$\{P_{sc}\} = [S(\omega)]\{u_{sc}\} \tag{27}$$

and combining with equation (25a), one obtains

$$\{P_{sc}\} = [S(\omega)][\{u_{\tau c}\} - \{u_{gc}\}] \tag{28}$$

Equilibrium of nodal forces yields

$$\{P_{\mathrm{ext}}\} = \{P_{gc}\} + \{P_{sc}\} + \{P_{\mathrm{elast}}\} \tag{29}$$

where $\{P_{\mathrm{ext}}\}$ is the total external force vector applied on the foundation and $\{P_{\mathrm{elast}}\}$ is the internal elastic force vector of the foundation substructure. A direct substitution of equation (28) and (14) into equation (29) yields, after setting $\{P_{\mathrm{ext}}\} = 0$ for simplicity,

$$\{u_{\tau c}\} = [[S(\omega)] + [\bar{K}_c]]^{-1}[[S(\omega)]\{u_{gc}\} - \{P_{gc}\}] \tag{30}$$

Using the averaging scheme described in Section 2.1, one can express the force vector $\{P_{gc}\}$ applied at the corners of the boundary elements as

$$\{P_{gc}\} = [T]^{\mathrm{T}}\{P_g\} = [T]^{\mathrm{T}}[A]\{t_g\} \tag{31}$$

where the $\{P_g\}$ and $\{t_g\}$ are the force and traction vectors, respectively, applied at the centres of the boundary elements. Finally, by direct substitution of equation (31) into equation (30) a relationship between the unknown displacements at the soil–foundation interface and the known seismic excitation can be obtained in the form

$$\{u_{\tau c}\} = [[S(\omega)] + [\bar{K}_c]]^{-1}[[S(\omega)]\{u_{gc}\} - [T]^{\mathrm{T}}[A]\{t_g\}] \tag{32}$$

It should be noted that the displacement vector $\{u_{gc}\}$ is described at the nodal points of the finite element discretization while the traction vector $\{t_g\}$ is defined at the centres of the corresponding boundary elements. This inconsistency is due to the fact that a description of the tractions at the nodal points of the finite element mesh becomes an unnecessarily complicated matter, particularly at points belonging to more than one plane. This problem is circumvented if tractions are defined at the centres of the boundary elements where no ambiguity exists as to the orientation of their normal plane of action.

## 3. NUMERICAL RESULTS

The general frequency domain BEM–FEM methodology described in the previous sections is applied here to the study of the response of flexible embedded foundations subjected to externally applied harmonic loads or seismic wave excitations. The case of embedded foundations with rigid walls and flexible floor, as well as results pertaining to flexible walls and the floor of the foundation are presented. The geometry of the soil–foundation system under consideration is depicted in Figure 4. The soil elastic properties are taken as modulus of elasticity $E_s = 8{\cdot}84 \times 10^6$ psi ($6{\cdot}1 \times 10^4$ N/mm$^2$), Poisson's ratio $\nu_s = 0{\cdot}3$ and mass density $\rho_s = 2{\cdot}82 \times 10^{-4}$ lb.s$^2$/in$^4$ ($3{\cdot}01 \times 10^{-3}$ kg/cm$^3$). The Poisson's ratio and the thickness of the foundation have been kept constant throughout this work and are $\nu_f = 0{\cdot}3$ and h = 3·0 in (7·62 cm). However, in order to study the effects of the varying flexibility ratio between the rigid vertical walls and the horizontal floor, the modulus of elasticity of the wall has been kept constant at $E_w = 50 \times 10^9$ psi ($3{\cdot}45 \times 10^8$ N/mm$^2$), while the modulus of elasticity of the floor varies from $E_f = 50 \times 10^9$ to $50 \times 10^6$ psi ($3{\cdot}45 \times 10^5$ N/mm$^2$). For all practical purposes, when $E_f$ reaches its upper limit the entire foundation behaves as a rigid body.

### 3.1. *Relaxed boundary conditions*

As a first example the vertical and rocking motion of a rigid embedded foundation are considered. This is done in an effort to verify the validity and accuracy of the formulation developed in this work, through comparisons with results obtained by Dominguez[12] and Karabalis and Beskos,[27] who used the simplification of perfectly rigid embedded foundations, i.e. only six degrees of freedom, in their studies. A direct comparison with results pertaining to the flexible embedded foundation problem is not possible since to the authors' best knowledge no such results have been reported in the literature, as yet. In Figure 5(a) the vertical response of the rigid embedded foundation defined previously is plotted versus the dimensionless frequency $a_0$ which is described as

$$a_0 = \omega b / c_2 \tag{33}$$

$2b$ being the largest horizontal dimension of the foundation and $c_2$ the shear wave velocity. The amplitude of the harmonic-vertical force is $P_{03}$ = 200 kip (889·6 kN). Two sets of results obtained in this work are shown in Figure 5(a) along with results reported by Dominguez.[12] The first set corresponds to the discretization shown

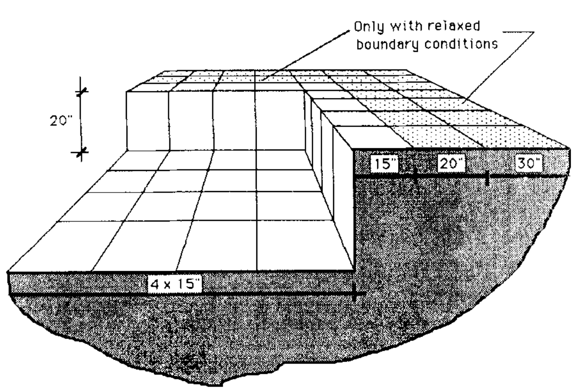


Figure 4. Foundation geometry and material constants

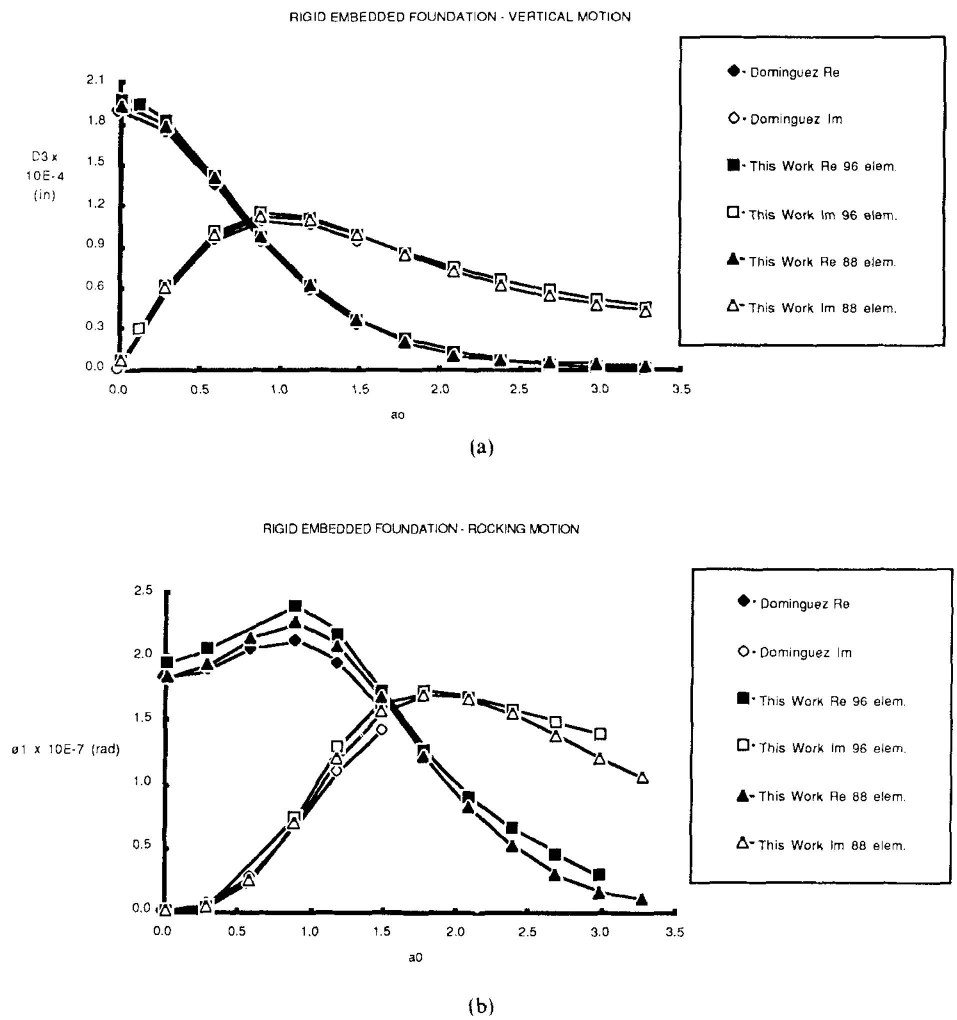


Figure 5. Harmonic response versus dimensionless frequency for a rigid embedded foundation

in Figure 4 with a total number of 96 boundary elements, while the other has been obtained using a $6 \times 6$ element discretization on the floor, one row of elements on the wall and one row of elements on the free surface. A $4 \times 4$ discretization scheme is employed for the integrations indicated in equation (8). Similarly in Figure 5(b) the rocking motion of the rigid embedded foundation due to a harmonic moment of amplitude $M_{02} = 200$ in-kip (22·6 kN-m) is plotted versus the dimensionless frequency $a_0$. The agreement of the results obtained in this work with those reported in Reference 12 is excellent. Furthermore, the close agreement of the results obtained by the two discretization schemes utilized in this work indicates the very small influence of the free-field discretization upon the final solution of the problem.

As a second example the effect of the wall-to-floor flexibility ratio on the dynamic response of the embedded foundation is studied. Thus, in Figure 6 a set of three-dimensional plots is shown where the amplitude of the vertical motion (magnified by a factor of $10^5$), due to a vertical force of amplitude $P_{03} = 200$ kip (889·6 kN) concentrated at the centre of the foundation, is given for a dimensionless frequency $a_0 = 0{\cdot}3$ and a range of wall-to-floor flexibility ratios varying from $E_w/E_f = 1$ (rigid foundation) to $E_w/E_f = 1000$ (very flexible floor, rigid walls). From these results it is evident, as one might have expected, that although the wall-to-floor flexibility ratio does affect substantially the deformation patterns of the floor, it does not seem to have a significant influence on the overall rigid body motion of the wall.

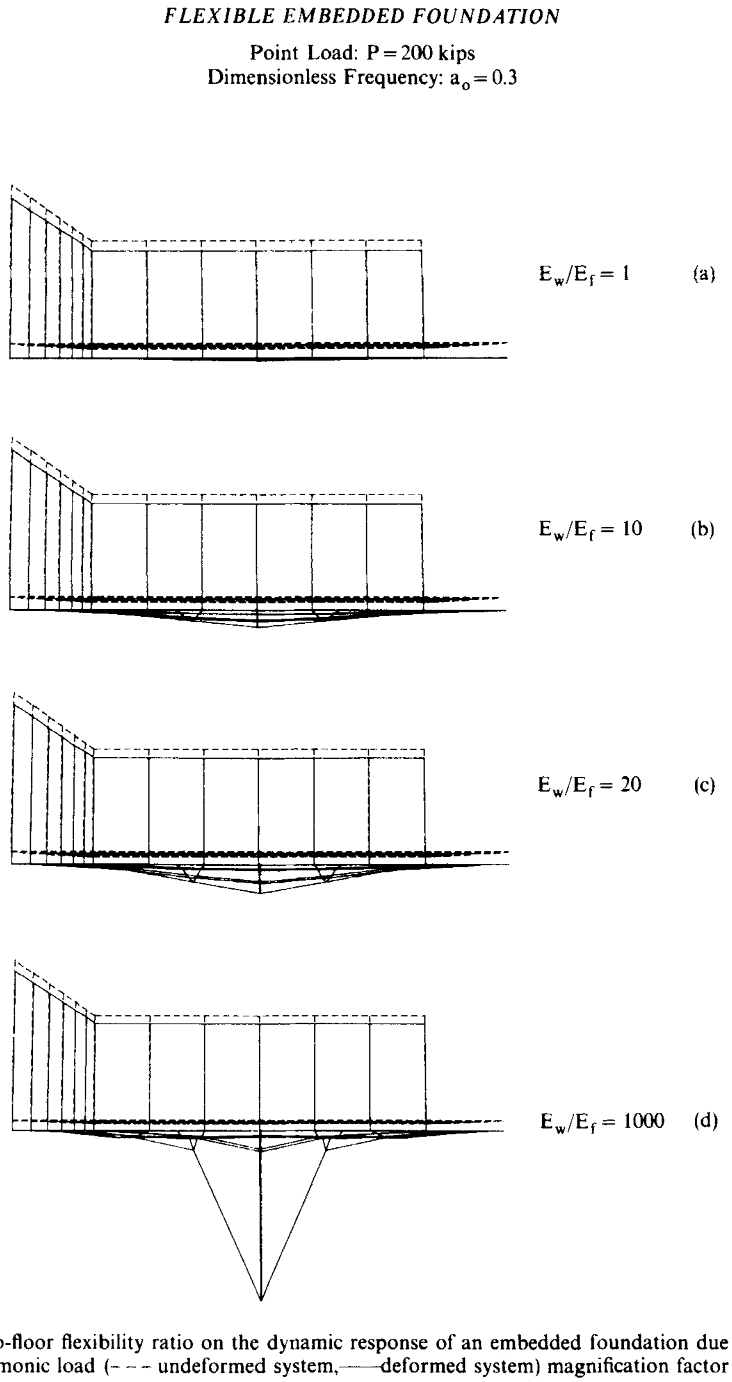


Figure 6. Effect of wall-to-floor flexibility ratio on the dynamic response of an embedded foundation due to a concentrated vertical harmonic load (- - - undeformed system,—— deformed system) magnification factor $10^5$

Subsequently, in Figure 7 another set of three-dimensional plots is shown where the ratio $E_w/E_f$ is kept constant at 1000 (very flexible floor, rigid walls), while the dimensionless frequency is varying from $a_o = 0{\cdot}3$ to 3·0. The results shown in Figure 7 are due to a harmonic vertical distributed load of magnitude $q_{03} = 178$ psi (122·7 N/cm$^2$). Again the results are in accordance with physical intuition, i.e. at low frequencies the overall rigid motion of the wall is large and the real part of the motion appears to be predominant with a weak rigid-

*FLEXIBLE EMBEDDED FOUNDATION*
Distributed Load: $q = 178$ lb/in$^2$
$E_w/E_f = 1000$

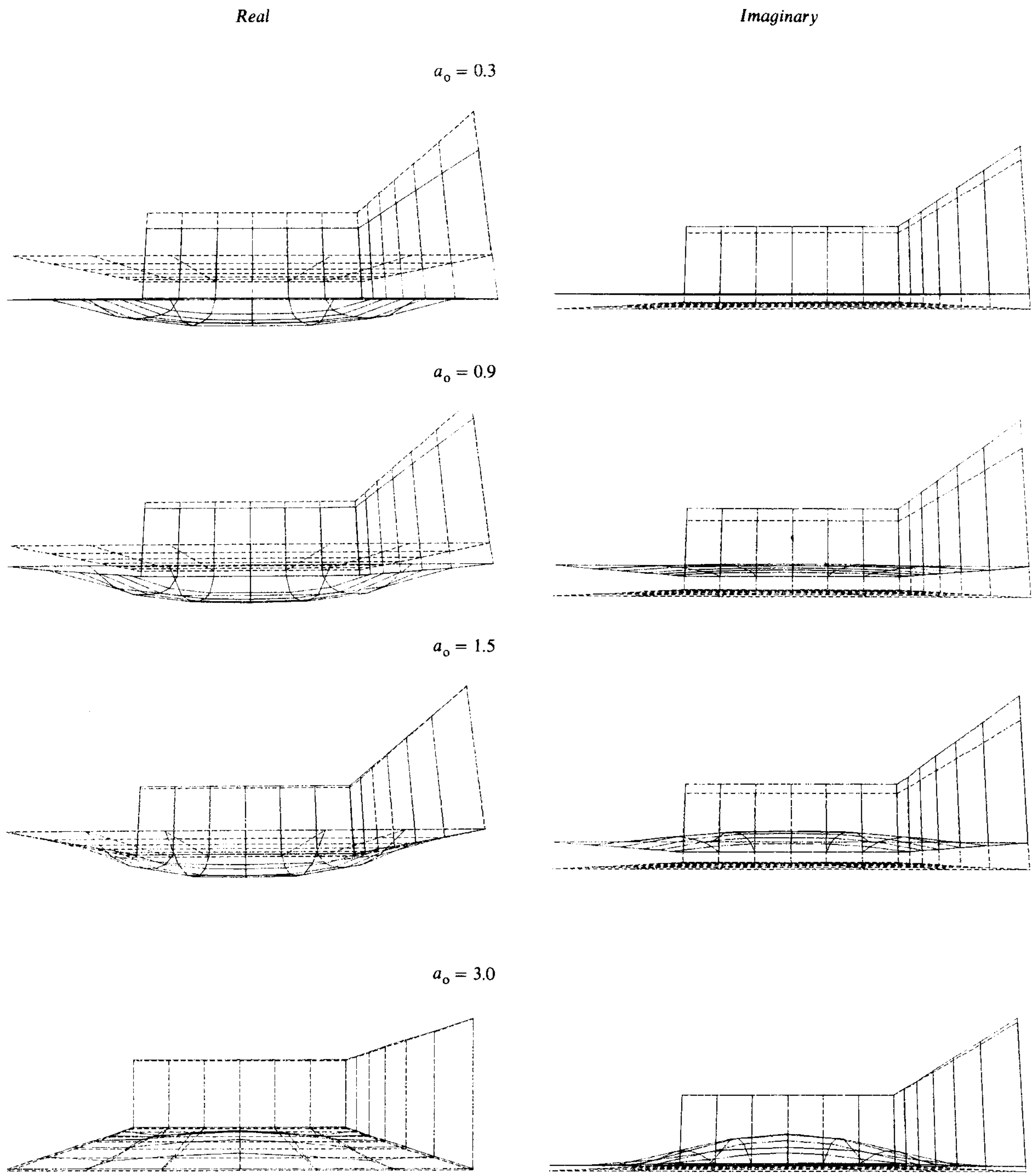


Figure 7. Dynamic response versus dimensionless frequency of an embedded foundation with rigid walls and flexible floor due to a vertical harmonic distributed load (- - - undeformed system,——deformed system) magnification factor $10^5$

body-like imaginary part, while at higher frequencies the roles of the real and the imaginary parts of the motion are reversed and the overall rigid body motion is diminished.

Similar results for a variety of forcing functions and flexibility ratios and a wide range of frequencies can be found in Gaitanaros.[15]

### 3.2. *Non-relaxed boundary conditions*

First the effect of different discretization schemes on the response of a flexible embedded foundation is examined. The elastic constants of the soil medium are kept the same as in Section 3.1. The same integration scheme as before, i.e. a $4 \times 4$ Gaussian quadrature point, is used in the evaluation of the integrals of equation (8). In order to compare results obtained in this section with those obtained using the relaxed boundary condition formulation, the vertical and rocking responses of a rigid embedded foundation are first examined. Thus, as a first example, for a wall-to-floor flexibility ratio $E_w/E_f = 1$ (rigid foundation), the response of the foundation to a vertical harmonic force of 200 kip (889·6 kN) is plotted versus the dimensionless frequency $a_0$. Three sets of results are obtained here, using relaxed boundary conditions, each corresponding to the discretization scheme of Figure 4 with $4 \times 4$ elements on the floor, one row of elements on each wall and one (1), two (2) or three (3) zones of free-field elements. The results are plotted in Figure 8, where it is apparent that the influence of the free-surface elements on the overall response of the foundation is negligible. Thus, for all the following examples only one zone of free-surface elements is used, in an effort to reduce the required amount of CPU time and computer storage. A comparison of the results obtained using the non-relaxed boundary condition assumption with one zone of free-field elements to those obtained using relaxed boundary conditions, for the cases of the vertical and the rocking motions, reveals the minor differences of the two approximations.

As a second example, a series of plots involving the rocking motion of the flexible embedded foundation is shown in Figure 9. In this case a pair of equal and opposite forces of magnitude 6·66 kip (29·6 kN) is applied at a distance $e = 15$ in (38·1 cm) on each side of the centre of the foundation and the wall-to-floor flexibility ratio $E_w/E_f$ varies from 1 to 1000. The results obtained are similar to those from relaxed boundary conditions,[15] the only difference being the relative movement of the walls of the foundation not only in the $z$-direction but also in the $x$- and $y$-directions. Obviously, these deformation patterns cannot be monitored if relaxed boundary conditions are used. A variety of similar results, demonstrating the versatility of the proposed methodology in handling any kind of loading patterns, can be found in Gaitanaros.[15]

### 3.3. *Massive foundations*

The methodology developed previously can be used to study the response of a massive flexible embedded foundations. The modified dimensionless mass ratio

$$B_z = \frac{1 - \nu}{4} \frac{m_f}{\rho_s r_0^3} \tag{34}$$

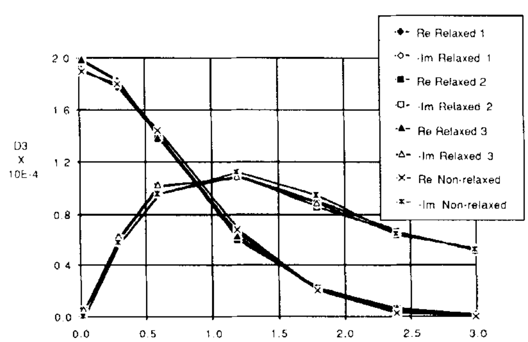


Figure 8. Comparison of relaxed versus non-relaxed boundary conditions for the harmonic response of a rigid embedded foundation

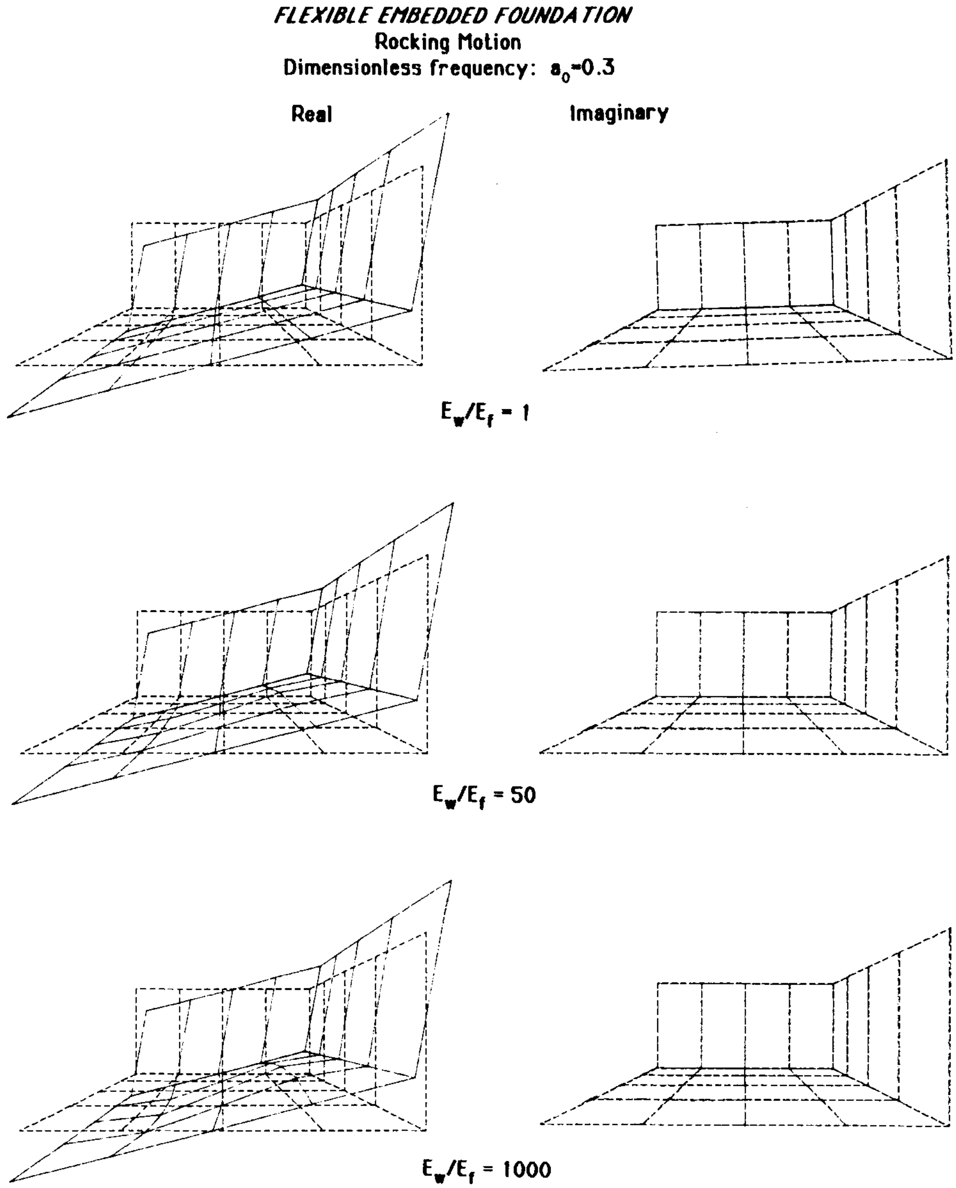


Figure 9. Effect of wall-to-floor flexibility ratio on the dynamic response of an embedded foundation due to a pair of equal and opposite vertical harmonic concentrated forces (- - -undeformed system,——deformed system) magnification factor $10^5$

originally introduced by Lysmer[36] for the vertical vibration of a surface rigid circular footing is used in this work, where $m_f$ is the total mass of the vibrating foundation, $r_0$ is the radius of an equivalent circular contact area, and $\rho_s$ and $\nu$ are the mass density and Poisson's constant of the soil. In this study an equivalent radius $r_0$ is calculated as

$$r_0 = \sqrt{\left(\frac{b_x b_y}{\pi}\right)} \tag{35}$$

$b_x$ and $b_y$ being the horizontal dimensions of the foundation.

In Figure 10 the vertical response of the massive flexible foundation is plotted versus dimensionless frequency for values of $B_z$ in the range 0–5. The results obtained in this section are in qualitative agreement with those reported in Reference 45 for a surface circular machine foundation. A direct comparison of the results obtained in this work for massive embedded foundations is not possible because no results of similar problems have appeared in the literature as yet.

The results shown in Figure 10 have been obtained using relaxed boundary conditions. However, in view of the results obtained in section 3.2 for massless foundations the non-relaxed boundary condition approximation is not expected to produce substantial differences.

### 3.4. *Seismic waves*

The present section studies the response of massless and massive flexible embedded foundations due to the influence of harmonic seismic wave excitations.

A plane P-SV harmonic wave (see Figure 11), propagating in a direction parallel to one of the vertical sides of the flexible embedded foundation has been chosen to represent the seismic forcing function. However, any other arbitrary spatial variation of the incoming seismic waves can be handled with equal ease. Based on the work of several authors, e.g. Wolf and Obernhuber[57] and Dominguez,[13] the amplitudes of the P- and SV-waves are determined by (i) the ratio of the horizontal to the vertical free-field motion at the control point, which is taken to be similar to that in Reference [13], and (ii) by the angle of incidence $\psi_{sv}$ of the SV-waves which is set equal to 70,80 or 90 degrees, see Figure 11. The ratio of the horizontal to the vertical free-field motion at the control point is given in Table I for a set of fractions and multiples of the fundamental circular frequency $\omega_0$ of a layer of soil corresponding to the embedment, defined as

$$\omega_0 = 2\pi \frac{c_2}{4E} \tag{36}$$

where $E$ is the embedment of the foundation.

In Figures 12(a, b, c) the dynamic response of a flexible embedded foundation is plotted for angles of incidence $\psi_{SV} = 70°$, $80°$ and $90°$ and for $B_z$ varying from 0 to 5. The frequency is kept constant at $\omega_0 = 2.1$. In

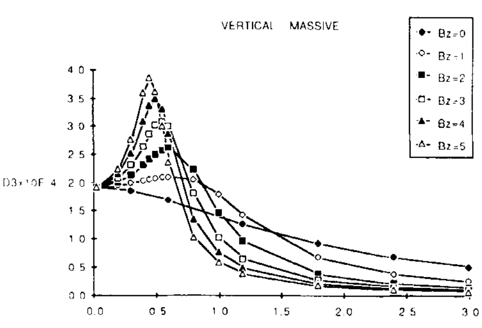


Figure 10. Vertical response of a massive rigid embedded foundation versus mass and dimensionless frequency due to a vertical harmonic force

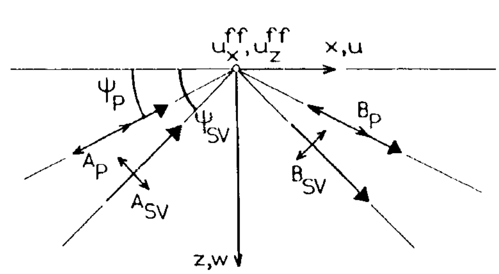


Figure 11. Nomenclature for a P-SV Wave (after Karabalis and Beskos [28])

*MASSIVE FOUNDATION*
Dimensionless frequency $a_c = 2.1$

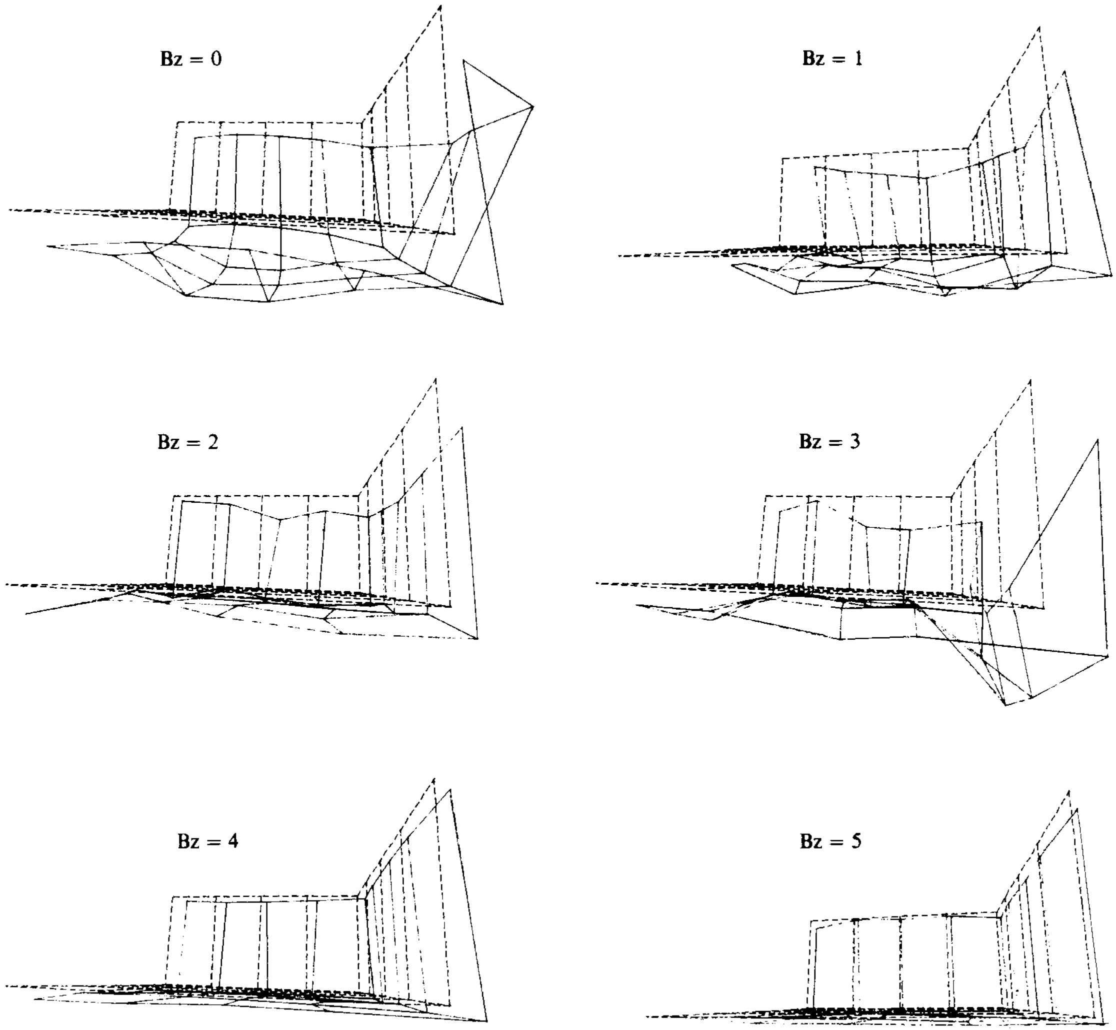


Figure 12(a). Dynamic response of a flexible embedded foundation versus mass due to a P-SV seismic wave (angle of incidence 70°) (- - -undeformed system,——deformed system)

Table I

| Frequency | $(u_z^{\mathrm{ff}}/u_x^{\mathrm{ff}})z = 0$ |
|---|---|
| $\omega_0/5$ | 0·676 |
| $2\omega_0/5$ | 0·723 |
| $3\omega_0/5$ | 0·838 |
| $4\omega_0/5$ | 0·939 |
| $\omega_0$ | 1·035 |
| $6\omega_0/5$ | 1·128 |
| $7\omega_0/5$ | 1·158 |
| $8\omega_0/5$ | 1·199 |
| $9\omega_0/5$ | 1·250 |

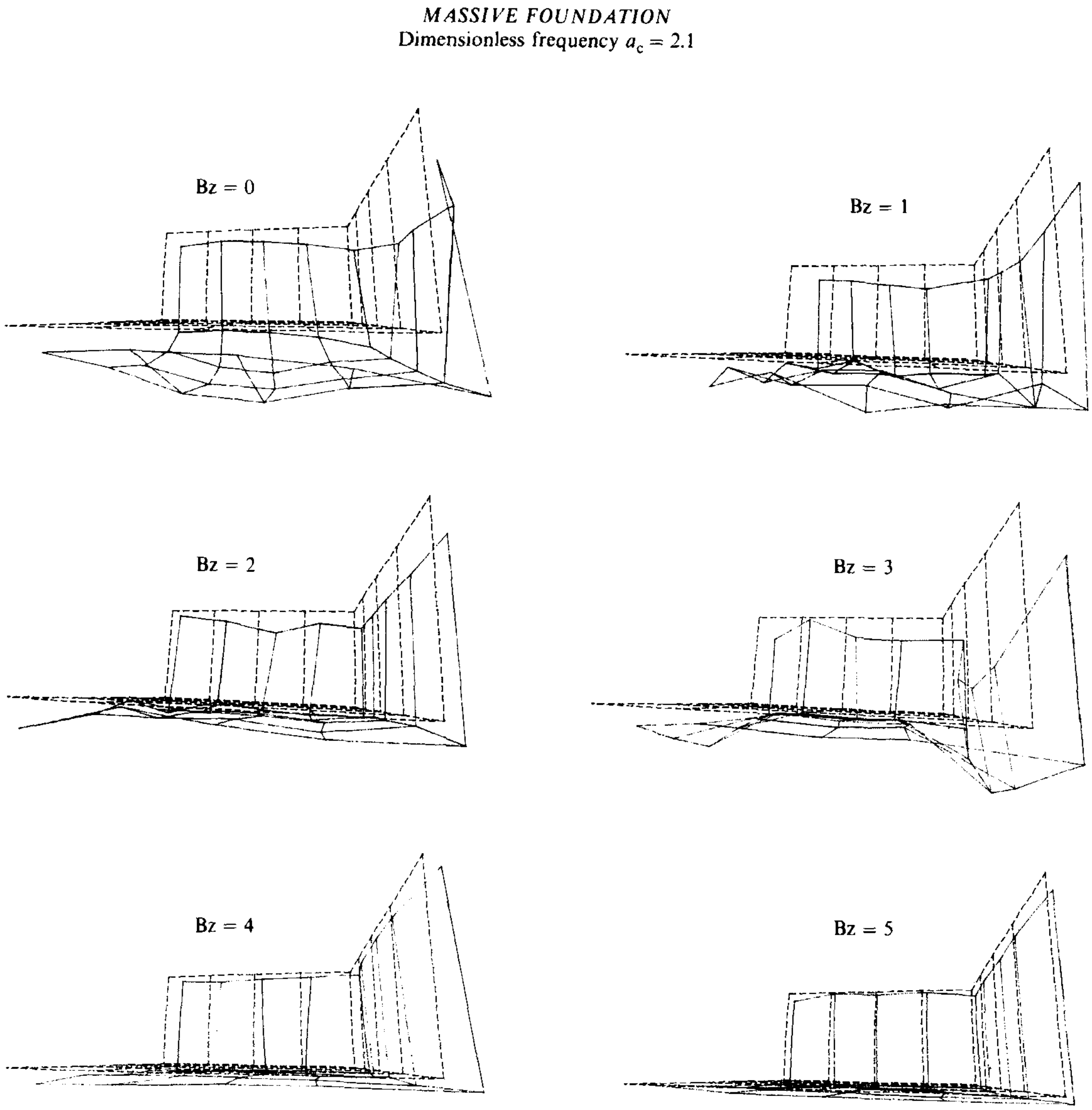


Figure 12(b). Dynamic response of a flexible embedded foundation versus mass due to a P-SV seismic wave (angle of incidence 80°) (- - -undeformed system,——deformed system)

addition Figures 13(a, b, c) show the vertical, horizontal and rocking responses of a rigid, massive, embedded foundation for the same variation of $\psi_{SV}$ and $\omega_0$. Direct comparison studies based on the above results are not possible since no similar works are available in the literature, as yet.

## 4. CONCLUSIONS

A combination of the BEM and FEM is used for the study of the dynamic response of 3-D flexible embedded foundations. The proposed methodology combines the advantages of both worlds since (i) the BEM is used for the elastic half-space, requiring a minimum discretization of only the soil–foundation interface and a small portion of the surrounding free surface, while at the same time the radiation condition is taken into account automatically, and (ii) the efficiency and convenience of the FEM in the analysis of a three-dimensional finite structure are fully utilized.

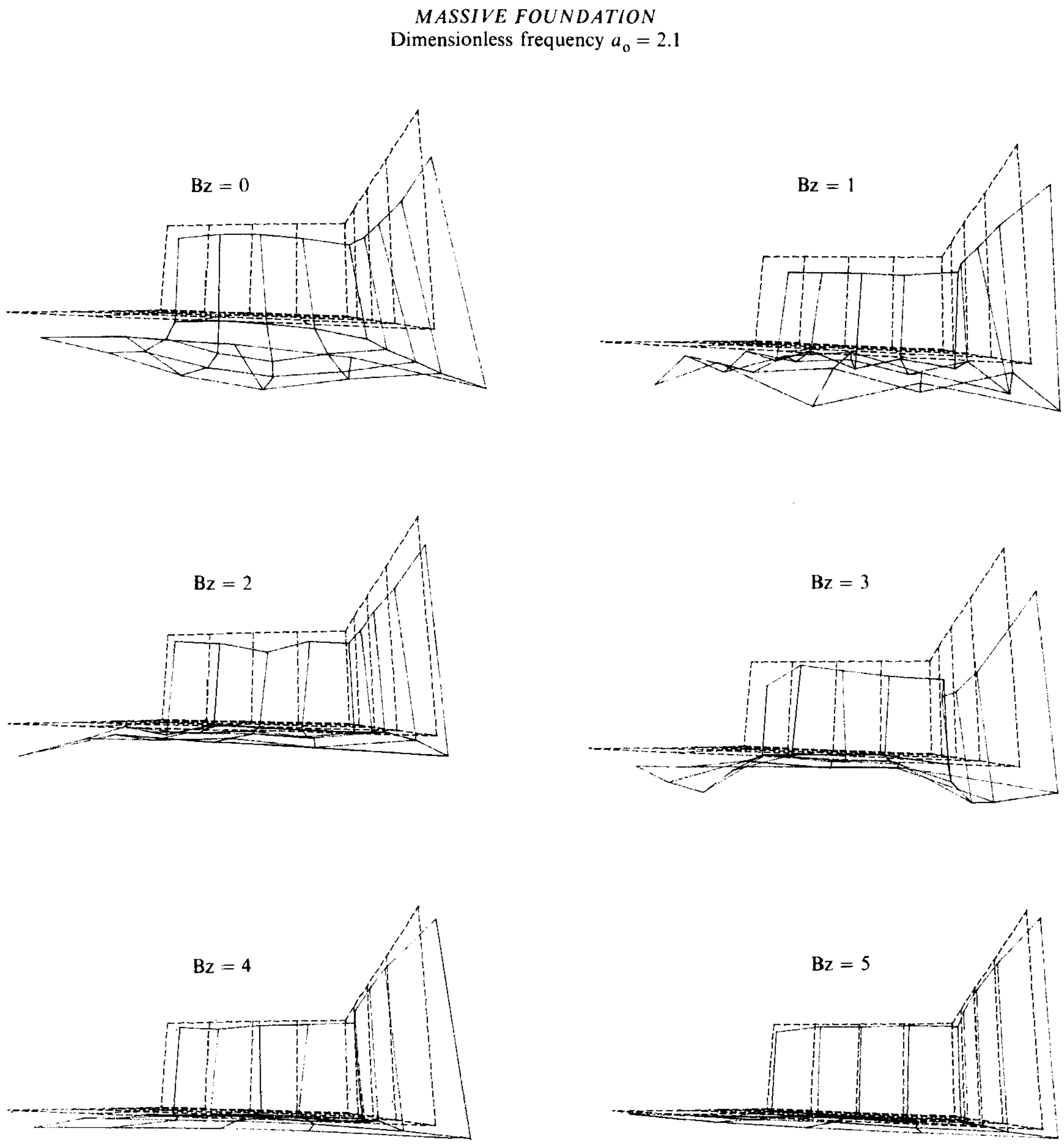


Figure 12(c). Dynamic response of a flexible embedded foundation versus mass due to a P-SV seismic wave (angle of incidence 90°) (- - -undeformed system,——deformed system)

The benefits of the simplifying assumption of relaxed boundary conditions, in order to minimize the required computer effort, have been demonstrated. The results obtained under this assumption are compared to those involving complete bond between soil and foundation. These comparison studies reveal a close agreement between the above two boundary approximations.

Several discretization schemes have been employed in the above analysis, proving the convergence of the proposed methodology. It has also been shown that the influence of the free-surface elements to the overall response of the foundation is small. The flexibility of the foundation has been a varying factor and, for purposes of comparison with previous works, results for rigid embedded foundations are also presented in this work.

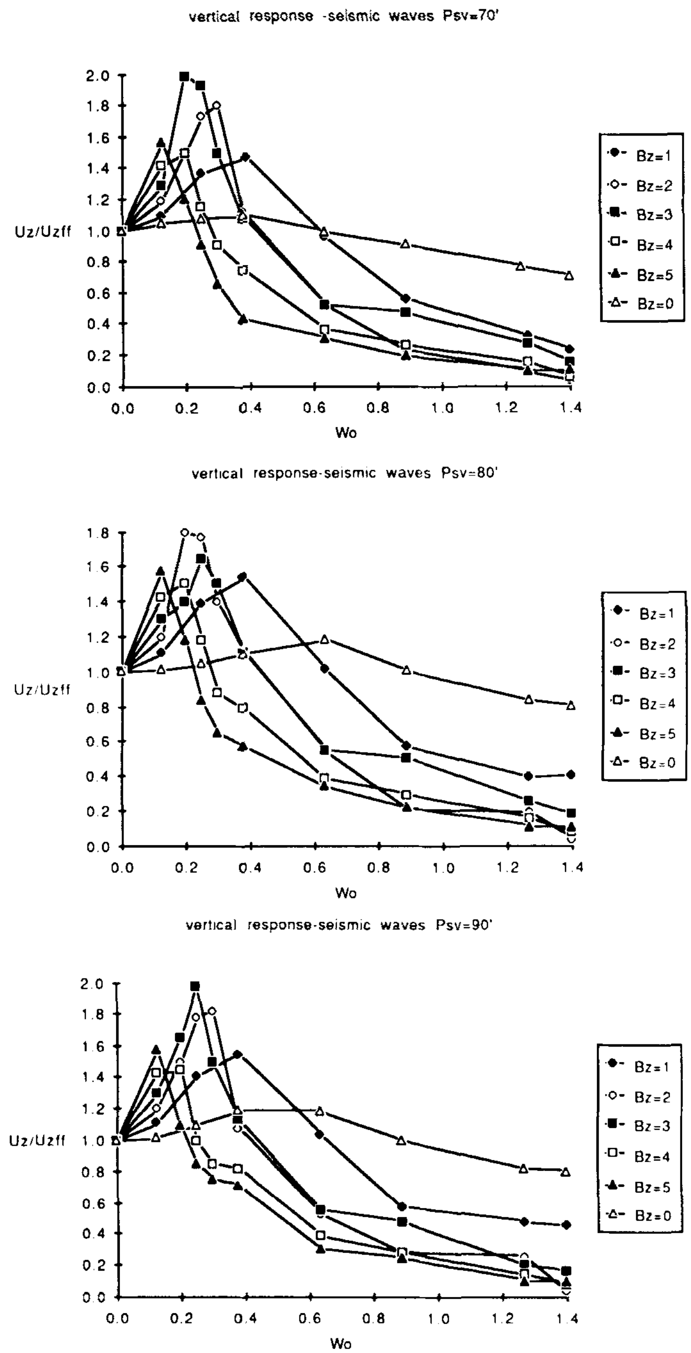


Figure 13(a). Vertical response of a rigid embedded foundation versus dimensionless frequency, mass and angle of incidence due to a P-SV seismic wave

It has also been shown through parametric studies that the flexibility of the foundation is an important factor in assessing the dynamic behaviour of an embedded foundation. Local deformations attributed to the flexibility of the foundation can exceed by orders of magnitude the overall motion of the foundation which resembles a rigid-like motion. Additional parametric studies on this subject are under way, the results of which will be published soon by the authors.

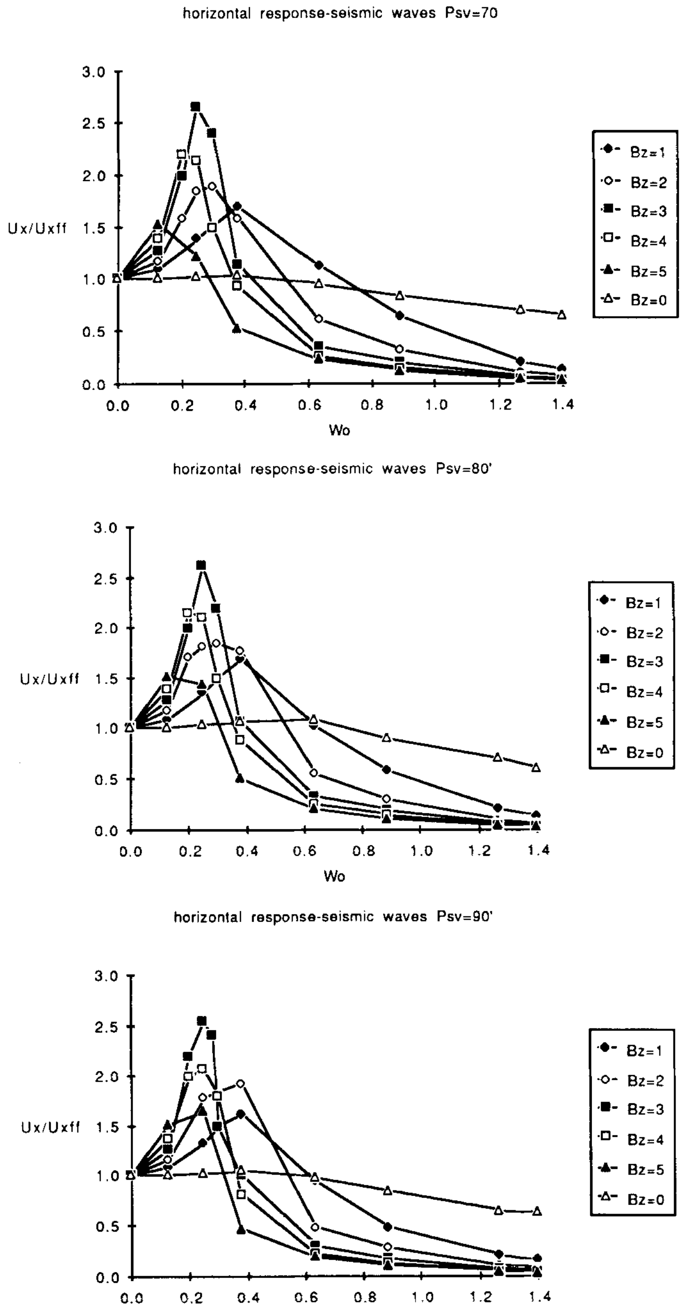


Figure 13(b). Horizontal response of a rigid embedded foundation versus dimensionless frequency mass, and angle of incidence due to a P-SV seismic wave

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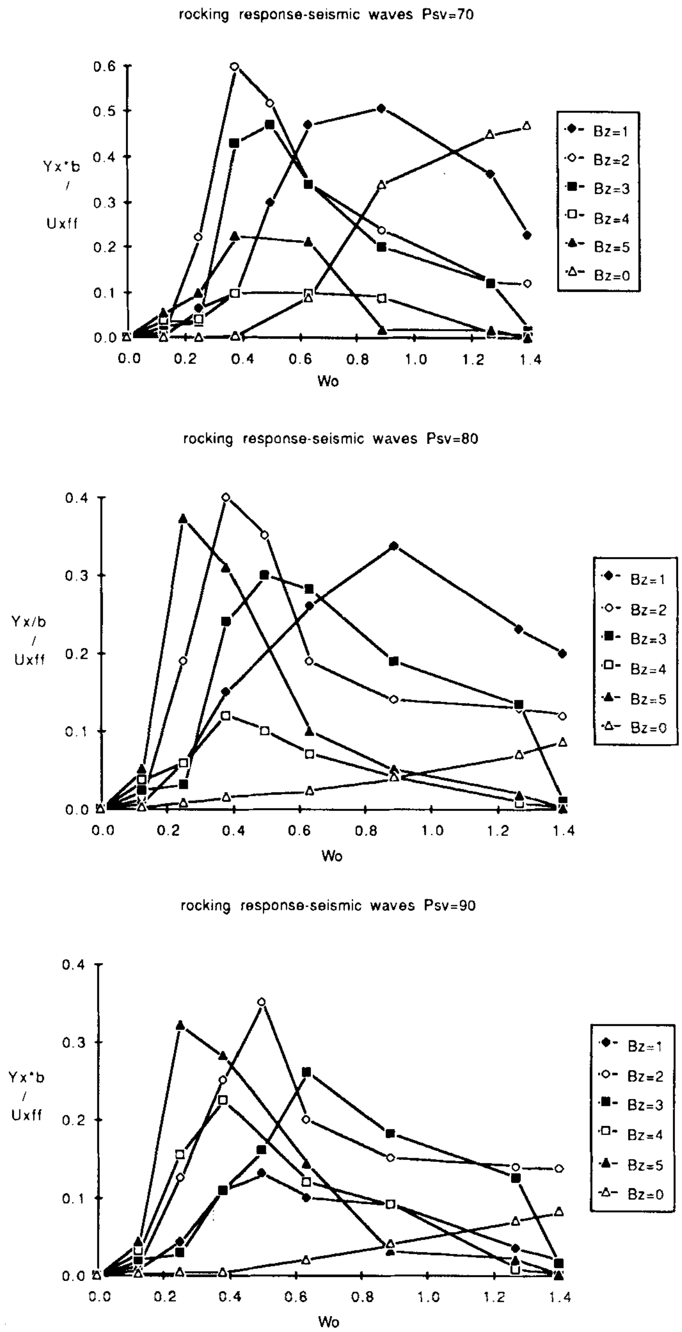


Figure 13(c). Rocking response of a rigid embedded foundation versus dimensionless frequency, mass and angle of incidence due to a P-SV seismic wave

## REFERENCES

1. R. Abascal and J. Dominguez, 'Vibrations of footings on zoned viscoelastic soils', *J. eng. mech. div. ASCE* **112**, 433–447 (1986).
2. V. Avanessian, R. Muki and S. B. Dong, 'Axisymmetric soil–structure interaction by global–local finite elements', *Earthquake eng. struct. dyn.* **14**, 355–367 (1986).
3. Y. O. Beredugo and M. Novak, 'Coupled horizontal and rocking vibration of embedded footings', *Canadian geotech. j.* **9**, 477–497 (1972).

4. C. A. Brebbia and S. Walker, *Boundary Element Techniques in Engineering*, Butterworths, London, 1980.
5. Y. K. Cheung and O. C. Zienkiewicz, 'Plates and tanks on elastic foundations—an application of finite element method', *Int. j. solids struct.* **1**, 451–461 (1965).
6. Y. K. Chow and I. M. Smith, 'Infinite elements for dynamic foundation analysis', in *Numerical Methods in Geomechanics, Edmonton* (Ed. Z. Einsenstein), A. A. Balkema, Rotterdam, 1982, pp. 15–22.
7. T. A. Cruse and F. J. Rizzo, 'A direct formulation and numerical solution of the general transient elastodynamic problem. I', *J. math. anal. appl.* **22**, 244–259 (1968).
8. T. A. Cruse, 'A direct formulation and numerical solution of the general transient elastodynamic problem. II', *J. math. anal. appl.* **22**, 341–355 (1968).
9. G. Dasgupta, 'Foundation impedance matrices by substructure deletion', *J. eng. mech. div. ASCE* **106**, 517–523 (1980).
10. G. Dasgupta, 'A finite element formulation for unbounded homogeneous continuum', *J. appl. mech. ASME* **49**, 136–140 (1982).
11. S. P. Dasgupta and N. S. V. K. Rao, 'Dynamics of rectangular footings by finite elements', *J. geotech. eng. div. ASCE* **104**, 621–637 (1978).
12. J. Dominguez, 'Dynamic stiffness of rectangular foundations', *Publication No. R*78-20, Department of Civil Engineering, M. I. T., Cambridge, MA, 1978.
13. J. Dominguez, 'Response of embedded foundations to travelling waves', *Publication No. R*78–24, Department of Civil Engineering, M.I.T., Cambridge, MA, 1978.
14. B. Frederick, 'Thick rectangular plates on an elastic foundation', *Trans. ASCE* **122**, 1069–1085 (1957).
15. A. P. Gaitanaros, 'Dynamic analysis of three dimensional flexible embedded foundations', *Master Thesis*, University of South Carolina, Columbia, SC 29208, 1986.
16. G. Gazetas, 'Analysis of machine foundation vibrations :state of the art', *Soil dyn. earthquake eng.* **2**, 1–42 (1983).
17. G. Gazetas, R. Dorby, and J. L. Tassoulas, 'vertical, response of arbitrarily shaped embedded foundations, *J. geotech. eng. div. ASCE* **111**, 750–771 (1985).
18. B. Goschy, 'Soil–foundation–structure interaction', *J. struct. div. ASCE* **104**, 749–761 (1978).
19. S. Gupta, J. Penzien, T. W. Lin and C. S. Yeh, 'Three-dimensional hybrid modelling of soil–structure interaction', *Earthquake eng. struct. dyn.* **10**, 69–87 (1982).
20. J. A. Gutierrez and A. K. Chopra, 'A substructure method for earthquake analysis of structures including soil–structure interaction', *Earthquake eng. struct. dyn.* **6**, 51–69 (1978).
21. D. E. Hudson, 'Dynamic tests of full-scale structures', *J. eng. mech. div. ASCE* **103**, 1141–1157 (1977).
22. M. Iguchi, 'A basic study on the behavior of long dimensional size buildings during earthquakes', *Proc. 6th world conf. earthquake eng.* New Delhi, India 1490–1495 (1977).
23. M. Iguchi and J. E. Luco, 'Dynamic response of flexible rectangular foundations on an elastic half-space', *Earthquake eng. struct. dyn.* **9**, 239–249 (1981).
24. M. Iguchi and J. E. Luco, 'Vibration of flexible plate on viscoelastic medium', *J. eng. mech. div. ASCE* **108**, 1103–1120 (1982).
25. L. R. Johnson, 'Green's function for Lamb's problem, *Geophys. j. astronom. soc.* **37**, 99–131 (1974).
26. D. L. Karabalis, 'Dynamic response of three dimensional foundations', *Ph.D. Thesis*, University of Minessota, Mineapolis, MN, 1984.
27. D. L. Karabalis, and D. E. Beskos, 'Dynamic response of 3-D flexible foundations by time domain BEM and FEM', *Soil dyn. earthquake eng.* **4**, 91–101 (1985).
28. D. L. Karabalis and D. E. Beskos, 'Dynamic response of 3-D embedded foundations by the boundary element method', *Comput. methods appl. mech. eng.* **56**, 91–119 (1986).
29. D. L. Karabalis and D. E. Beskos, 'Dynamic soil–structure interaction', in *Boundary Element Methods in Mechanics* (Ed. D. E. Beskos), North-Holland, Amsterdam, 1987, Chapter 11.
30. D. L. Karabalis and M. Mohammadi, 'The application of the boundary element method to dynamic soil–structure interaction problems: computational aspects', *Proc. 13th southeastern conf. theoret. appl. mech.* University of South Carolina, Columbia, SC, 321–328 (1986).
31. E. Kausel, J. M. Roesset and G. Waas, 'Dynamic analysis of footings on layered media', *J. eng. mech. div. ASCE* **101**, 679–693 (1975).
32. S. Kobayashi and N. Nishimura, 'Green's tensors for elastic half-space—an application of boundary integral equation method', *Memoirs fac. eng. Kyoto Univ.* **42**, Part 2, 228–241 (1980).
33. S. Krenk and H. Schmidt, 'Vibration of an elastic circular plate on an elastic half space—a direct approach', *J. appl. mech. ASME* **48**, 161–168 (1981).
34. H. Lamb, 'On the propagation of tremors over the surface of an elastic solid', *Philos. trans. roy. soc. London* **A203**, 1–42 (1904).
35. Y. J. Lin, 'Dynamic response of circular plates resting on viscoelastic half-space', *J. appl. mech. ASME* **45**, 379–384 (1978).
36. J. Lysmer, 'Vertical motions of rigid footings', *Ph.D. Thesis*, University of Michigan, Ann Arbor, 1965.
37. F. Medina and J. Penzien, 'Infinite elements for elastodynamics', *Earthquake eng. struct. dyn.* **10**, 699–709 (1982).
38. M. Mohammadi and D. L. Karabalis, '3-D soil–structure interaction analysis by BEM: comparison studies and computational aspects', *Int. j. numer. methods eng.* (submitted).
39. M. Novak and Y. O. Beredugo, 'Vertical vibration of embedded footings', *J. soil mech. found. div. ASCE* **98**, 1291–1310 (1972).
40. M. A. Oien, 'Steady motion of a plate and an elastic half-space', *J. appl. mech. ASME* **40**, 478–484 (1973).
41. M. Ottenstreuer, 'Frequency dependent dynamic response of footings', *Proc. soil dyn. earthquake eng. conf.* Southampton, England 799–809 (1989).
42. M. Ottenstreuer and G. Schmid, 'Boundary elements applied to soil–foundation interaction', *Proc. 3rd int. sem. recent adv. boundary element methods* Irvine Calif 293–309 (1981).
43. D. Ray, 'Three-dimensional continuum-finite element formulation for dynamic impedance evaluation of arbitrarily shaped foundations', in *Numerical Methods in Geomechanics, Aachen*, 1979 (Ed. W. Witke), A. A. Balkema, Rotterdam, 1979, pp. 77–78.
44. D. Ray, '3-D half-space solution for flexible foundations', *3rd EMD/ASCE specialty conf.* Austin, Texas 738–741 (1979).
45. F. E. Richart, R. D. Woods and J. R. Hall, '*Vibration of Soils and Foundations*', Prentice-Hall, Englewood Cliffs, N. J., 1970.
46. H. R. Riggs and G. Waas, 'Influence of foundation flexibility on soil–structure interaction', *Earthquake eng. struct. dyn.* **13**, 597–615 (1985).

47. F. J. Rizzo, D. J. Shippy and M. Rezayat, 'Boundary integral equation analysis for a class of earth–structure interaction problems', Final Report, Department of Engineering Mechanics, College of Engineering, University of Kentucky, 1985.
48. S. A. Savidis and T. Richter, 'Dynamic response of elastic plates on the surface of the half-space', *Int. j. numer. anal. methods geomech.* **3**, 245–254 (1979).
49. C. C. Spyrakos and D. E. Beskos, 'Dynamic response of flexible strip foundations by boundary and finite elements', *Soil dyn. earthquake eng.* **5**, 84–96 (1986).
50. J. L. Tassoulas and E. Kausel, 'On the effect of the rigid sidewall on the dynamic stiffness of embedded circular footings', *Earthquake eng. struct. dyn.* **11**, 403–413 (1983).
51. S. A. Thau, 'Radiation and scattering from a rigid inclusion in an elastic medium, *J. appl. mech. ASME* **34**, 509–511 (1967).
52. A. K. Vaish and A. K. Chopra, 'Earthquake finite element analysis of structure–foundation systems', *J. eng. mech. div. ASCE* **100**, 1101-1116 (1974).
53. W. L. Whittaker and P. Christiano, 'Dynamic response of plate on elastic half-space', *J. eng. mech. div. ASCE* **108**, 133–154 (1982).
54. W. L. Whittaker and P. Christiano, 'Response of a plate and elastic half-space to harmonic waves', *Earthquake eng. struct dyn.* **10**, 255–266 (1982).
55. J. P. Wolf and G. R. Darbre, 'Dynamic-stiffness matrix of soil by the boundary element method: conceptual aspect, *Earthquake eng. struct. dyn.* **12**, 385–400.(1984).
56. J. P. Wolf and G. R. Darbre, 'Dynamic-stiffness matrix of soil by the boundary element method: embedded foundation', *Earthquake eng. struct. dyn.* **12**, 401–416 (1984).
57. J. P. Wolf and P. Obernhuber, 'Free-field response from inclined SV- and P-waves and Rayleigh-waves', *Earthquake eng. struct. dyn.* **10**, 847–869 (1982).
58. J. P. Wolf and P. Obernhuber, 'Non-linear soil structure interaction analysis using dynamic stiffness or flexibility of soil in the time domain', *Earthquake eng. struct. dyn.* **13**, 195–212 (1985).
59. J. P. Wolf and P. Obernhuber, 'Non-linear soil structure interaction analysis using Green's function of soil in the time domain', *Earthquake eng. struct. dyn.* **13**, 213–223 (1985).
60. J. P. Wolf and D. R. Somaini, 'Approximate dynamic model of embedded foundation in time domain', *Earthquake eng. struct. dyn.* **14**, 683–703 (1986).
61. F. S. Wong, 'Uncertainties in dynamic SSI', *J. eng. mech. div. ASCE* **110**, 308–324 (1984).
62. H. L. Wong and J. E. Luco, 'Dynamic response of rectangular foundations in obliquely incident seismic waves', *Earthquake eng. struct. dyn.* **6**, 3–16 (1978).
63. H. L. Wong, J. E. Luco and M. D. Trifunac, 'Contact stresses and ground motion generated by soil-structure interaction', *Earthquake eng. struct. dyn.* **5**, 67–69 (1977).
64. O. C. Zienkiewicz, *The Finite Element Method*, 3rd edn, McGraw-Hill, New York, 1977.
65. O. C. Zienkiewiz and Y. K. Cheung, 'Finite element method of analysis for arch dam shells and comparison with finite difference procedures', in *Theory of Arch Dams* (Ed. J. R. Rydzewski), Pergamon, Oxford, 1965.