Between Generalized Clifford Algebras and $gl(\infty, \mathbf{C})$

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Abstract. In this paper we will discuss the correspondence between generalized Clifford algebras and the $gl(N, \mathbf{C})$. We will also examine the limit $N \to \infty$. Indeed, in this situation the generators of $\mathbf{C}^{\mathbf{r}}_{\infty}$ are identified with the infinite matrix $\Gamma_1 = \sum_{i \in z} E_{i,i+1}$ where $E_{i,j}$ denotes the basis of the $gl(\infty, \mathbf{C})$ algebra.

1. Introduction

The aim of this note is to establish a correspondence between the generalized Clifford algebras $\mathbf{C}^{\mathbf{r}}_{\mathbf{N}}$ and the $gl(N, \mathbf{C})$ algebra. More precisely we will see that for any pair (i, j), the generators γ_i and γ_j of $\mathbf{C}^{\mathbf{r}}_{\mathbf{N}}$ with $(i < j)$ are in correspondence with the matrices elements of $gl(N, \mathbf{C})$. An interesting case is obtained when $N \to \infty$, in this case the algebras $\mathbf{C}^{\mathbf{r}}_{\infty}$ is embedded in the $gl(\infty, \mathbf{C})$. Firstly recall that the real standard Clifford algebra $\mathbf{C}^{\mathbf{r}}_{\mathbf{2}}$ is generated by a set of basis vectors $\{\gamma_1, \gamma_2, ...\gamma_r\}$ and defining relations:

$$\{\gamma_i, \gamma_j\} = 2g_{ij} \tag{1}$$

where g_{ij} are the real coefficients of a non degenerate symmetric bilinear form. In the set of orthonormal vectors the defining relations become

$$\gamma_i\gamma_j = -\gamma_j\gamma_i \qquad \gamma_i^2 = \pm 1 \tag{2}$$

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Hence these algebras admit a $\mathbf{Z_2}$-graded structure. However mathematicians have obtained new algebras defined from a n-linear relations and leading to an underlying $\mathbf{Z_n}$-graded structures, the so-called generalized Clifford algebras and its generalized Grassmann algebras, which emerge naturally from various contexts: in the natural extension of complex numbers e.g multicomplex numbers [1], in generalized statistics [2] and even in 2D fractional supersymmetries theories [3] and this is one of the reasons for extensive attention paid to divers aspects of these structures. The aim of this work is to establish the connection between these structures and the $gl(N, \mathbf{C})$. The limit $N \to \infty$ is also in order. The work is organized as follows, in Sec. 2, we sketch the basic notion and useful properties of the generalized Clifford algebras and its Grassmann algebras. In Sec. 3, we will study our correspondence, we will see that this correspondence can be forced to morphism structure. At the end we will examine the limit $N \to \infty$.

2. The Generalized Clifford and Grassmann Algebras

In brief, recall that the generalized associative Clifford algebras and its associated Grassmann algebras have been investigated by many authors in many occasions. Here we will sketch only the basic notions connected with these structures (see e.g 4,5,6 for more details). Then the generalized Clifford algebras $\mathbf{C_N^r}$ are generated by the set of r canonical generators $\gamma_1, \gamma_2, ...\gamma_r$ fulfilling the following relations, defining them as a set of orthonormal vectors

$$\gamma_i\gamma_j = w\gamma_j\gamma_i \quad (i < j), \quad \gamma_i^N = \pm 1 \tag{3}$$

where the parameter w is chosen to be a $N-$th primitive root of the unity i.e $w = exp(2i\pi/N)$ without loss of the generality. The associated generalized Grassmann algebras $\mathbf{G_N^r}$ are defined by the same above relations where we substitute the relation $\gamma_i^N = \pm 1$ by $\gamma_i^N = 0$, to end this recall note that this latter, i.e $\mathbf{G_N^r}$, is the fundamental tool in the concept of fractional supersymmetries theories.

3. Correspondence Between $\mathbf{C}_{\mathbf{N}}^{\mathbf{r}}$ and $gl(N, \mathbf{C})$

In this section we will give the correspondence of generalized Clifford algebras and $gl(N, \mathbf{C})$. To begin with, let us consider the following matrices in $gl(N, \mathbf{C})$

$$G_1 = \begin{pmatrix} 1 & 0 & .. & .. & 0 \\ 0 & w & . & . & . \\ . & . & . & \ddots & . \\ . & . & . & . & 0 \\ 0 & . & . & 0 & w^{N-1} \end{pmatrix} \quad G_2 = \begin{pmatrix} 0 & a_1 & .. & .. & 0 \\ 0 & 0 & a_2 & . & . \\ . & . & . & \ddots & . \\ . & . & . & . & a_{N-1} \\ a_N & . & . & 0 & 0 \end{pmatrix} \tag{4}$$

It is easy to verify that:

$$G_1 G_2 = w G_2 G_1 \quad (G_1)^N = \mathbf{1}_{N\times N} \quad (G_2)^N = \prod_{i=0}^{i=N-1} a_i \mathbf{1}_{N\times N} \tag{5}$$

Let us now consider a subspace spanned by these matrices, the matrix $(G_2)^N$ satisfies an additive constraint such as: $(G_2)^N = \prod_{i=0}^{i=N-1} a_i \mathbf{1}_{N\times N}$ $=\mathbf{1}_{N\times N}$. As an immediate consequence of this contract the parameters a_i satisfy the following constraint: $\prod_{i=0}^{i=N-1} a_i = 1$. Now we are in position to give the following assertion:

For any pair (i, j) with $i < j$, the mapping $\gamma_i \to G_1$ and $\gamma_j \to G_2$ gives the realization of $\mathbf{C}_{\mathbf{N}}^{\mathbf{r}}$ in $gl(N, \mathbf{C})$. The proof follows from the relations (3) and (5).

This gives the correspondence between $\mathbf{C}_{\mathbf{r}}^{\mathbf{N}}$ and $gl(N, \mathbf{C})$. This correspondence is not a morphism structure due to the relation (3), namely $\gamma_i^2 = \pm 1$. However this is true on just a subspace of $\mathbf{C}_{\mathbf{r}}^{\mathbf{N}}$ where $\gamma_i^N = 1$.

An interesting case is obtained in the limit $N \to \infty$. Indeed in this situation the $\mathbf{C}_{\mathbf{r}}^{\infty}$ is embedded in the $gl(\infty, \mathbf{C})$ algebra. Before going ahead, let us recall some basic notions connected with $gl(\infty, \mathbf{C})$; the Lie algebra of infinite matrices which is defined by:

$$gl(\infty, \mathbf{C}) = \{(A_{ij})_{ij\in Z} / A_{ij} = 0 \quad \text{for } |i - j| >> 0\} \tag{6}$$

These are all the infinite dimensional matrices with only a finite number of nonzero diagonals. This infinite dimensional Lie algebra has as basis the matrices $E_{i,j}$ with 1 in the (i, j) entry and 0 anywhere else. If we denote by

ν_i the infinite column vectors with 1 in the i-th row and zero everywhere else, then

$$E_{i,j}\nu_k = \delta_{jk}\nu_i \tag{7}$$

the basis $E_{i,j}$ has the property

$$E_{i,j}E_{k,l} = \delta_{jk}E_{i,l} \tag{8}$$

and the commutation relation between two matrices is $[E_{i,j}, E_{k,l}] = \delta_{jk}E_{i,l} - \delta_{il}E_{k,j}$

Now, we turn to our embedding. In fact, in this situation the generators of $\mathbf{C}^{\mathbf{r}}_{\infty}$ are identified with the infinite matrix $\Gamma_1 = \sum_{i\in z} E_{i,i+1}$. Finally, we would like to note that a connection between generalized Clifford algebra and some infinite dimensional Lie algebras was obtained in [7,8].

4. Conclusion

In this note we have proposed the correspondence between generalized Clifford algebras and $gl(N, \mathbf{C})$, we have also examined the limit $N \to \infty$. In fact, in this situation we have the embedding of $\mathbf{C}^{\mathbf{r}}_{\infty}$ in $gl(\infty, \mathbf{C})$.

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