Proof. Since $\psi(\xi)$ attains a minimum on U_n we can assume that $\psi(\xi)>1$. By Theorem 4, applies to the function $F(\xi)=\log\psi(\xi)$, there exist a singular measure $\lambda\geqslant 0$ and a function $K(z)\in H(D_n)$ such that $\operatorname{Re}K=P[\log\psi-d\lambda]$. We set $G(z)=e^{K(z)}$. Then $\log|G|\leqslant P[\log\psi]$ and, by virtue of the boundedness of the function $\psi G\in H^\infty(D_n)$. Consequently, $\log|G|=\operatorname{Re}K$ has an admissible limit almost everywhere on U_n, equal to $\log\psi$.

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NUMERICAL RANGE OF A LINEAR RELATION AND MAXIMAL RELATIONS

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UDC 513.8+517.4

1. Let θ be a linear binary relation in a Hilbert space H, i.e.,

$$x\theta x'\leftrightarrow\{x,\ x'\}\in G_\theta\subset H\oplus H;\ x,\ x'\in H,$$

where G_θ is some linear manifold in $H\oplus H$ (the graph of the relation θ). By the domain of definition D_θ and the range R_θ of the relation θ we mean, respectively, the set of the first and the set of the second components of the elements $\{x,\ x'\}\in G_\theta$. We set $\theta(x)=\{x'\in H:x\theta x'\}$. The conditions $\theta(0)=\{0\}$ is necessary and sufficient that the relation be a (single-valued) operator $T:D_\theta\to H$, i.e., that $x\theta x'\leftrightarrow x'=Tx$, $D_T=D_\theta$.

Definition 1. By the numerical range of a linear relation θ in H we mean the set $W(\theta)\subset \boldsymbol{C}$ of the values of the inner product $(x',\ x)$ for all possible $x\theta x'$, $\|x\|=1$.

By Hausdorff's theorem, the numerical range of an arbitrary linear operator T in H is a convex set in $\boldsymbol{C}$ (closed if $\dim H<\infty$).

THEOREM 1. The numerical range $W(\theta)$ of an arbitrary linear relation θ in H is a convex set in the complex plane for $\dim D_\theta<\infty$ it is closed and bounded or $=\boldsymbol{C}$).

Proof. Modifying the arguments from [1, p. 120] we assume that $\lambda_1,\ \lambda_2\in W(\theta)$, i.e., there exist $x_1\theta x_1'$, $x_2\theta x_2'$ such that $\lambda_1=(x_1',\ x_1)$ $\lambda_2=(x_2',\ x_2)$ $\|x_1\|=\|x_2\|=1$. Assume that x_1, x_2 are linearly independent and let P be the orthoprojection onto their linear span E. We define in E a linear operator T by the conditions $Tx_1=Px_1'$, $Tx_2=Px_2'$. Then $\lambda_1=(Tx_1,x_1)$, $\lambda_2=(Tx_2,\ x_2)$ and, by Hausdorff's theorem for the two-dimensional case, for any λ from the segment joining the points λ_1 and λ_2, there exists a normalized vector $x=\alpha_1x_1+\alpha_2x_2$, such $\lambda=(Tx,\ x)$, i.e., $\lambda=(P(\alpha_1x_1'+\alpha_2x_2'),\ \alpha_1x_1+\alpha_2x_2)=(\alpha_1x_1'+\alpha_2x_2',\ \alpha_1x_1+\alpha_2x_2)$, and, moreover, $(\alpha_1x_1+\alpha_2x_2)\,\theta\,(\alpha_1x_1'+$

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$\alpha_2 x_2')$ by virtue of the linearity of the relation θ and, thus, $\lambda \in W(\theta)$, and in the case of the linear independence of x_1, x_2 the convexity of $W(\theta)$ is proved.

Assume now that x_1, x_2 are colinear but $\lambda_1 \neq \lambda_2$. This case is possible only for relations, while for operators it does not occur. We have $x_2 = \alpha x_1$ for some $\alpha \in \boldsymbol{C}$, $|\alpha| = 1$. Therefore, $(\alpha x_1)\theta x_2'$ and $(\alpha x_1)\theta(\alpha x_1')$, from where there follows that $0\theta(x_2' - \alpha x_1')$, and, moreover, $(x_2' - \alpha x_1', x_1) \neq 0$, since otherwise we would have $\lambda_1 = (\alpha x_1', \alpha x_1) = (x_2', \alpha x_1) = \lambda_2$. We denote $h = x_2' - \alpha x_1'$. We have $0\theta h$, $(h, x_1) \neq 0$, $x_1 \in D_\theta$. From here we obtain $W(\theta) = \boldsymbol{C}$ by the following lemma.

<u>LEMMA 1.</u> If $0 \theta h$ and $h \notin D_\theta^\perp = H \ominus \bar{D}_\theta$, then $W(\theta) = \boldsymbol{C}$. If $W(\theta) \neq \boldsymbol{C}$, then $\theta(0) \perp D_\theta$.

<u>Proof of the Lemma</u>. If $0\theta h$ and $h \notin D_\theta^\perp$, then there exists $x \in D_\theta$, $\|x\| = 1$, such that $(h, x) \neq 0$. But then $x\theta x'$ for some $x' \in H$ and, therefore, for any $\zeta \in \boldsymbol{C}$ we have $x\theta(x' + \zeta h)$ and $W(\theta) \supset \{(x' + \zeta h, x), \forall \zeta \in \boldsymbol{C}\} = \boldsymbol{C}$.

The lemma is proved. In order to conclude the proof of the theorem we note that if $\dim D_\theta < \infty$, then either $\theta(0) \not\subset D_\theta^\perp$, and then $W(\theta) = \boldsymbol{C}$ by lemma, or $\theta(0) \perp D_\theta$, and then $W(\theta)$ is closed and bounded in $\boldsymbol{C}$ as the continuous image of a compactum, namely of the unit sphere in a finite-dimensional space under the mapping $x \to (Tx, x)$ (here by the condition $Tx = Px'$, where $x\theta x'$ and P is the orthoprojection onto $\bar{D}_\theta$, one has defined an operator $T: D_\theta \to D_\theta$, since $\theta(0) \perp D_\theta$). The theorem is completely proved.

A relation θ_1 is said to be an extension of the relation θ, if $x\theta x' \Rightarrow x\theta_1 x'$, and a proper extension if $G_{\theta_1} \neq G_\theta$. To the closure of a relation there corresponds the closure of its graph.

A relation θ is said to be Hermitian (selfadjoint) [2] if from $x\theta x'$, $y\theta y'$ there follows $(x', y) - (x, y') = 0$ (1) and if from the validity of (1) for some pair $x, x' \in H$ for all possible $y\theta y'$ there follows $x\theta x'$. Every Hermitian relation is equivalent to the equation $\cos A \cdot x' - \sin A \cdot x = 0$ (2) with some operator $A = A^*$, $\|A\| \leq \pi/2$, and also to the equation $(U - I)x' + i(U + I)x = 0$ with a unitary operator $U = U_\theta$ (called the Cayley transform of the relation θ). The operator $U = -e^{2iA}$ is defined uniquely by the relation θ. Every relation (2) or (3) is Hermitian [2] for every $A = A^*$. Obviously, for a Hermitian relation we have $W(\theta) \subset \boldsymbol{R}$.

A relation θ is said to be dissipative [3] if $W(\theta) \subset \boldsymbol{C}^+ = \{\zeta \in \boldsymbol{C} : \operatorname{Im} \zeta \geqslant 0\}$, accumulative if $W(\theta) \subset \boldsymbol{C}^-$, $(\operatorname{Im} \zeta \leqslant 0)$, accretive if $W(\theta) \subset \{\zeta \in \boldsymbol{C} : \operatorname{Re} \zeta \geqslant 0\}$, symmetric under the condition (1), i.e., for $W(\theta) \subset \boldsymbol{R}$. If a dissipative relation does not have proper dissipative extension, then it is said to be maximal-dissipative (m-dissipative). In a similar manner one defines an m-accretive relation, etc.

As shown in [3], if $U \in B(H)$ is a contraction $(\|U\| \leqslant 1)$, equation (3) defines an m-dissipative relation and each m-dissipative relation can be represented in this form. The relation (3) is maximally symmetric if the operator U is isometric.

By analogy with operators [4], a relation θ is said to be sectorial [5] if $W(\theta)$ is contained in a sector of the complex ζ -plane $|\arg(\zeta - \gamma)| \leqslant \varphi < \pi/2$, $\gamma \in \boldsymbol{R}$ (4), and m-sectorial if it does not have proper extensions with numerical range in the semiplane $\operatorname{Re} \zeta \geqslant \alpha$ for any

$\alpha \in \boldsymbol{R}$. Then γ and φ are called the vertex and the semiangle of the sectorial relation (they are not defined uniquely).

Definition 2. Assume that there is given some closed set $K \subset \boldsymbol{C}$, $K \neq \boldsymbol{C}$. An operator $T: D_T \to \bar{D}_T \subset H$ is said to be K-maximal in $\bar{D}_T$, if $W(T) \subset K$ and T does not have proper extension T_1 and $\bar{D}_T$ such that $W(T_1) \subset K$. Similarly, a linear relation θ in H is said to be K-maximal if $W(\theta) \subset K$ and the relaton θ does not have proper extensions θ_1 such that $W(\theta_1) \subset K \neq \boldsymbol{C}$.

Example 1. The operator $\Lambda y = -iy'$, $y(0) = 0$, $0 \leqslant t < \infty$ is K-maximal in $L^2(0, \infty)$ for $K = \boldsymbol{R}$ and for $K = \boldsymbol{C}^-$ (a maximal symmetric, resp. and m-accumulative operator) but it is not K-maximal for $K = \boldsymbol{C}^+$ (m-dissipative) since it admits a proper dissipative (maximal) extension $\Lambda_1 = \Lambda^*$ in $L^2(0, \infty): \Lambda \subset \Lambda_1 \neq \Lambda$, $\Lambda_1 y = -iy'$, $\operatorname{Im}(\Lambda_1 y, y) = |y(0)|^2/2 \geqslant 0$.

THEOREM 2. In order that a linear relation θ in H be K-maximal, $W(\theta) \subset K = \bar{K} \neq \boldsymbol{C}$, it is necessary and sufficient that there exist an operator $T: D_T = D_\theta \to \bar{D}_\theta$ K-maximal in $\bar{D}_\theta$ such that

$$x\theta x' \leftrightarrow x' - Tx \in D_\theta^\perp,\ R_T \subset \bar{D}_\theta. \tag{5}$$

By virtue of (5), the correspondence θNT is one-to-one. The operator T and the relation are closed. This theorem generalizes the representation of Hermitian relations, established by the corollary to Theorem 1 of [2], and of m-sectorial relations [5]).

LEMMA 2. If the relation θ is K-maximal for some $K \supset W(\theta)$, then from $y \perp D_\theta$ there follows $0\theta y$, i.e., $D_\theta^\perp \subset \theta(0)$.

Proof of the Lemma. If $0\theta y$ does not hold for some $y \perp D_\theta$, then the relation θ_1, defined by the condition

$$x\theta_1 x' \leftrightarrow \exists \zeta \in \boldsymbol{C}: x\theta(x' - \zeta y), \tag{6}$$

would turn out to be a proper extension of the relation and, moreover, $W(\theta_1) = W(\theta) \subset K$ by virtue of (6), since $(x', x) = (x' - \zeta y, x)$, due to $y \perp D_\theta$, $x \in D_\theta$. We have obtained a contradiction with the conditions of the K-maximality of the relation θ. The lemma is proved.

Proof of the Theorem. From Lemma 1 and Lemma 2 we have for a K-maximal relation θ in H, that $0\theta y \leftrightarrow y \perp D_\theta$; $\theta(0) = D_\theta^\perp$ (7). We denote by P the orthoprojection from H onto $\bar{D}_\theta$. Then by the condition $x\theta x' \leftrightarrow Tx = Px'$ (8) by virtue of (7), on $T: D_T = D_\theta \to \bar{D}_\theta \subset H$, there is defined the operator $T: D_T = D_\theta \to \bar{D}_\theta \subset H$, satisfying (5) and, therefore, also $W(T) = W(\theta)$. If T would have a proper extension T_1 and $\bar{D}_\theta$, for which $W(\theta_1) \subset K$, then this extension by formula (5) would generate a proper extension θ_1 of the relation θ, for which we would have $W(T_1) \subset K$; this is not allowed by the conditions of the theorem. Thus, the operator T is K-maximal in $H_1 = \bar{D}_\theta \subset H$. Since it is defined on a set that is dense in H_1 and $W(T) \neq \boldsymbol{C}$, it follows that it is closable by virtue of Theorem V.3.4 of [4]; therefore, it is closed since otherwise its closure $T^\sim$ would be its proper extension in H_1 with $W(T^\sim) \subset \overline{W(T)} \subset \bar{K} = K$.

*If θ (5) is K-maximal, then $R_\lambda(\theta) = (T - \lambda)^{-1} P_\theta$, P_θ being the orthoprojection onto $\bar{D}_\theta$.

In a similar way, also the K-maximal relation θ is closed by virtue of $\overline{W(\theta)} \subset K \neq \boldsymbol{C}$. Equality $W(T) = W(\theta)$ is obvious by virtue of [5]. The necessity of the conditions of the theorem is proved. We prove their sufficiency, i.e., the K-maximality of the relation θ in H, generated according to formula (5) by the operator T, K-maximal in $\bar{D}_T$. We have $W(\theta) = W(T) \subset K$. We assume that there exists $\theta_1 \supset \theta$ a proper extension with $W(\theta_1) \subset K$ and assume that $x_1\theta_1 x_1'$, but not $x_1\theta x_1'$. Then $\bar{D}_{\theta_1} = \bar{D}_\theta$, since otherwise we would not have $D_{\theta_1} \perp \theta_1(0) \supset \theta(0)$ and by Lemma 1, we should have $W(\theta_1) = \boldsymbol{C}$, in spite of the assumption. Therefore $(x_1', x_1) = (Px_1', x_1) \in K$, where P is the orthoprojection onto $\bar{D}_\theta$. Moreover, $x_1 \notin D_T = D_\theta$, otherwise we would have $x_1\theta(Tx_1)$ and $Px_1' \neq Tx_1$ (since $\{x_1, x_1'\} \notin G_\theta$), and from here $x_1\theta_1(Tx_1)$, $x_1\theta_1(Px_1')$ and $0\theta_1(Tx_1 - Px_1')$, $0 \neq Tx_1 - Px_1' \in \bar{D}_{\theta_1}$, which, by Lemma 1, would lead to $W(\theta_1) = \boldsymbol{C}$, in spite of the assumption. Consequently, setting $T_1 \supset T$, $T_1 x_1 = Px_1'$, we would obtain a proper extension T_1 of the operator T in $\bar{D}_T$, and $W(T_1) \subset K$, in spite of the K-maximality of T in D_θ. The theorem is proved.

COROLLARY 1 [5]. The relation θ is m-sectorial in H if and only if there exists an operator $T: D_T = D_\theta \to \bar{D}_\theta$ m-sectorial in $\bar{D}_\theta$ such that $W(T) = W(\theta)$ and (5) holds.

2. Based on the previous considerations, we extend the concept of the Friedrichs extension of sectorial, densely defined operators [4], to sectorial relations and, in particular, to sectorial operators with a domain of definition that is not dense in H. (We mention that the symmetric semibounded relations are special cases of the sectorial relations. Regarding their Friedrichs extensions, see [5, 6] and the earlier investigations of A. V. Shtraus, based on M. G. Krein's theory on the extension of semibounded operators).

As shown in Lemma 1, if $W(\theta) \neq \boldsymbol{C}$, then $\theta(0) \perp D_\theta$. . Therefore, each linear relation θ with $W(\theta) \neq \boldsymbol{C}$ generates by formula (8) an operator T which, in turn, by formula $x\theta_1 x' \leftrightarrow x' - Tx \in D_\theta^\perp$, $R_T \subset \bar{D}_\theta$, (9) generates a relation $\theta_1 \supset \theta$, $D_{\theta_1} = D_\theta$ and $\theta_1(0) = D_\theta^\perp \supset \theta(0)$, $W(\theta_1) = W(\theta) = W(T)$. If θ is a sectorial relation in H, then the operator T is sectorial and densely defined in $H_1 = \bar{D}_\theta$, and, therefore, admits in H_1 an m-sectorial Friedrichs extension T^{F_1} [4, Chap. VI. Sec. 2], which is constructed by means of the closure in H_1 of the quadratic form (Tx, x), $x \in D_\theta$, and, therefore, $W(T^{F_1}) \subset \overline{W(T)} = \overline{W(\theta)}$, $R_{T^{F_1}} \subset H_1 = \bar{D}_\theta$.

By formula (5), with T^{F_1} instead of T, we define an m-sectorial relation $\theta^F \supset \theta$, $W(\theta^F) \subset \overline{W(\theta)}$. θ^F is called the Friedrichs extension (in H) of the sectorial relation θ.

To each sectorial relation θ in H there corresponds a quadratic closed sectorial form; $a_\theta(x) = (x', x)$, $x\theta x'$. This form does not depend on the choice of the vector x', since $\theta(0) \perp D_\theta$ and $a_\theta(x) = (Tx, x)$, $x \in D_\theta$.

THEOREM 3. Among all possible m-sectorial extensions of the sectorial relation θ in H, there exists a unique one having domain of definition of the closure of the quadratic form $a_\theta(x)$. It is also the unique extension for which the closure of the quadratic form has the smallest domain of definition, i.e., contained in the domain of definition of the closure of the form corresponding to any other m-sectorial extension of the relation θ. This relation is the above constructed Friedrichs extension θ^F of the relation θ.

The proof of the theorem follows from the previous constructions and from Theorems VI.2.10 and VI.2.11 of [4].

COROLLARY 2. If $T:D_T\to H$ is a sectorial operator with a domain of definition that is not dense in H (the inclusion $R_T\subset \bar{D}_T$ is not assumed), then its Friedrichs extension in H is an m-sectorial relation T^F, generated by the formula $xT^Fx'\leftrightarrow x'-(PT)^{F_1}x\in D_T^{\perp}$, where P is the orthoprojection onto $\bar{D}_T$, F_1 denotes the Friedrichs extension of the operator in the subspace $H_1=\bar{D}_T$, F is the Friedrichs extension in H. For the resolvent of the Friedrichs extension T^F for $\lambda\in C\setminus\overline{W(T)}$ we have the formula $R_\lambda(T^F)=R_\lambda^{(1)}((PT)^{F_1})P$, where $R_\lambda^{(1)}(A)$ is the resolvent of the operator A in H_1, which is a bounded operator for $\lambda\notin\overline{W(A)}$ (T^F reduces to an operator only for $H_1=H$. The boundedness of the resolvent follows from Theorem 42 [1, p. 119 of the Russian edition]).

Remark 1. Making use of the known fact of the existence of m-dissipative extensions of a dissipative operator, of Theorem 2, and of formula (9), it is easy to show the existence of m-dissipative extensions for any dissipative relation (which, however, is also known).

3. Now we investigate when is the relation θ, generated by the equation $x\theta x'\leftrightarrow Cx'-Bx=0$ (10) with unbounded operators B and C, Hermitian.

In this case it is natural to consider also the question of the Hermitian property of the closure $\theta^{\sim}$ of the relation θ (10). The simplest examples are: 1) $x'-Bx=0$, where $B=B^*$ is an unbounded operator. Here the relation $x\theta x'$ is Hermitian. 2) $Bx'-Bx=0$, where B is an injective operator, densely defined in H, with $D_B\neq H$. In this case the relation $x\theta x'$ is not Hermitian but its closure $\theta^{\sim}$ is Hermitian, equivalent namely to the equation $x'=x$ (in this case, θ may be said to be "essentially Hermitian").

Parallel with the relation θ (10) we introduce its restriction $\theta_1\subset\theta$: $x\theta_1x'\leftrightarrow C_1x'-B_1x=0$ (11) where $B_1\subset B$, $D_{B_1}=D_B\cap(D_C\oplus D_C^{\perp})$, $C_1\subset C$, $D_{C_1}=D_C\cap(D_B\oplus D_B^{\perp})$ (12).

THEOREM 4. Assume that B and C in (10) are arbitrarily given linear operators, possibly unbounded, not closed* and with domains of definitions D_B and D_C, not dense in H, acting from H into an arbitrary linear space E.

The following conditions 1-4 are necessary in order that the relation θ, defined by the equation (10), be Hermitian, and sufficient in order that it be the Hermitian closure of the relation $\theta_1\subset\theta$, defined in (11), (12) (obviously, $\theta_1^{\sim}\subset\theta^{\sim}$):

$$1.\ D_B^{\perp}\subset \operatorname{Ker} C,\ D_C^{\perp}\subset\operatorname{Ker} B.\ 2.\ \overline{D_B\cap D_C}=\bar{D}_B\cap\bar{D}_C.$$

3. The operators $B\pm iC$ are invertible on their ranges (injective).

4. The closure $U_1^{\sim}$ of the operator U_1,

$$U_1=(B+iC)^{-1}(B-iC) \tag{13}$$

in the space $H_1=\overline{D_B\cap D_C}$ exists and it is unitary.

*If E is such that these concepts make sense; for example, if it is a topological space.

Under these conditions the operator

$$U = U_{\tilde{1}} \oplus I_{D_B^\perp} \oplus (-I_{D_C^\perp}) \tag{14}$$

is unitary and it is the Cayley transform of the relation $\theta_{\tilde{1}}$, which can be represented in terms of U (14) in the form (3).

Proof. Necessity. Assume that the relation (10) is Hermitian. Then it is $\boldsymbol{R}$ -maximal and by Lemma 2 we have $D_\theta^\perp \subset \theta(0)$, whence $D_B^\perp \subset D_\theta^\perp \subset \theta(0) = \operatorname{Ker} C$. Since θ (10) is Hermitian, then the inverse relation $Bx' - Cx = 0$, is also Hermitian; from here $D_C^\perp \subset \operatorname{Ker} B$ and 1 is proved.

We prove condition 2. By virtue of (2), where $A = A^*$, $\|A\| < \pi/2$, the Hermitian relation θ can be represented in the following parametric form: $x = \cos A \cdot h, x' = \sin A \cdot h, \forall h \in H$, from where it is clear that $D_B \supset R_{\cos A}$, $D_C \supset R_{\sin A}$.

Let $E(t)$ be the partition of the identity of the operator A. We set

$$H_c = E([-\pi/4, \pi/4])H,$$
$$H_s = E([-\pi/2, -\pi/4) \cup (\pi/4, \pi/2])H.$$

It is easy to see that

$$H_c \subset R_{\cos A} \subset D_B; \; H_s \subset R_{\sin A} \subset D_C, \tag{15}$$
$$H_c \oplus H_s = H, \; H_s = H_c^\perp. \tag{16}$$

Indeed, if, for example $h \in H_c$, then $h = \cos A \cdot g$, where $g = \int_{-\pi/4-0}^{\pi/4+0} (1/\cos t)\, dE(t)\, h$ and thus $h \in R_{\cos A}$, $H_c \subset R_{\cos A}$.

We denote by P_c and $P_s = P_c^\perp$ the orthoprojections onto H_c and onto H_s respectively.

By virtue of (15) we have

$$D_B = H_c \oplus P_s D_B = H_c \oplus L_s, \; D_C = H_s \oplus P_c D_C = H_s \oplus L_c \tag{17}$$

where H_c, H_s are the subspaces [16], they are closed, while $L_s \subset H_s$, $L_c \subset H_c$ are some linear manifolds. From (17) we have $D_B \cap D_C = L_s \oplus L_c$, $\overline{D_B \cap D_C} = \overline{L_s \oplus L_c} = \bar{L}_s \oplus \bar{L}_c$. On the other hand,

$$\bar{D}_B = \overline{H_c \oplus L_s} = H_c \oplus \bar{L}_s, \; \bar{D}_C = \overline{H_s \oplus L_c} = H_s \oplus \bar{L}_c,$$

from where

$$\bar{D}_B \cap \bar{D}_C = \bar{L}_s \oplus \bar{L}_c.$$

Thus

$$\overline{D_B \cap D_C} = \bar{D}_B \cap \bar{D}_C,$$

which is what we intended to prove.

We prove condition 3. If, for example, $(B + iC)x = 0$, then $Cx - B(ix) = 0$, i.e., $(ix)\theta x$, from where $x = 0$, since $W(\theta) \subset \boldsymbol{R}$. Thus, the operator $B + iC$ is injective. The injectivity of $B - iC$ is established in a similar manner.

We prove condition 4. By virtue of the injectivity of the operators $B+iC$ there exists an (injective) operator $U_1=(B+iC)^{-1}(B-iC)$. Obviously, $D_{U_1}\subset D_B\cap D_C$, $R_{U_1}\subset D_B\cap D_C$.

We prove that here equalities rather than inclusions prevail and that U_1 is isometric. If $f=U_1h$ for some $h\in D_B\cap D_C$, then

$$(B+iC)f=(B-iC)h \tag{18}$$

or

$$C(i(f+h))-B(h-f)=0,\ \text{т. е.}\ (h-f)\theta(i(h+f)). \tag{19}$$

Setting in (3) $U=U_\theta$,

$$x=h-f,\ x'=i(h+f) \tag{20}$$

we obtain $f=U_\theta h$, i.e., $U_1\subset U_\theta|(D_B\cap D_C)$. Therefore U_1 is isometric and has a closure $\tilde{U_1}$, whose unitarity in the subspace $H_1=\overline{D_B\cap D_C}$ will be established if we show that

$$U_\theta(D_B\cap D_C)=D_B\cap D_C. \tag{21}$$

Indeed, then for $h\in D_B\cap D_C$ we shall have $U_\theta h=f\in D_B\cap D_C$ and there exists a pair $x,\ x'\in H$, $x\theta x'$, such that (20) holds; therefore, also (18) holds, i.e., $U_\theta\restriction(D_B\cap D_C)\subset U_1$, and thus $U_1=U_\theta\restriction(D_B\cap D_C)$, from where $\tilde{U_1}=U_\theta\restriction H_1$, $D_{\tilde{U_1}}=R_{\tilde{U_1}}=H_1=\overline{D_B\cap D_C}$, i.e., $\tilde{U_1}$ is unitary in H_1.

Thus, it remains to prove (21). From (2), (3) we have that $U_\theta=-e^{2iA}$. Therefore, for any $h\in H$ we have $(U_\theta-I)h=-2\cos A\cdot e^{iA}h\in R_{\cos A}\subset D_B$, $(U_\theta+I)h=-2i\sin A\cdot e^{iA}h\in R_{\sin A}\subset D_C$. Thus, if $h\in D_B$, then also $U_\theta h=(U_\theta-I)h+h\in D_B$, i.e., $U_\theta D_B\subset D_B$. Similarly, $U_\theta^{-1}D_B\subset D_B$, and therefore, $U_\theta D_B=D_B$. In exactly the same way $U_\theta D_C=D_C$, and therefore, $U_\theta(D_B\cap D_C)=(U_\theta D_B)\cap(U_\theta D_C)=D_B\cap D_C$, and the necessity of all the conditions of the theorem is proved.

<u>Sufficiency</u>. We set

$$U'=U_1\oplus I_{D_B^\perp}\oplus(-I_{D_C^\perp}), \tag{22}$$

where U_1 is defined by the formula (13), and we consider the linear relation $x\rho x'\leftrightarrow(U'-I)x'+i(U'+I)x=0$ (23). We have

$$U'-I=(U_1-I)_{D_B\cap D_C}\oplus 0_{D_B^\perp}\oplus(-2I_{D_C^\perp});$$
$$U'+I=(U_1+I)_{D_B\cap D_C}\oplus 2I_{D_B^\perp}\oplus 0_{D_C^\perp}.$$

Therefore, if $x\rho x'$, then

$$P_{D_B^\perp}x=0,\ P_{D_C^\perp}x'=0,\ P_{H_1}x\in D_B\cap D_C,\ P_{H_1}x'\in D_B\cap D_C,$$
$$(U'-I)x'\in D_B\cap D_C,\ (U'+I)x\in D_B\cap D_C.$$

Thus, multiplying (23) by the injective operator $B+iC$ with domain of definition $D_B\cap D_C$, we obtain the equivalent equation

$$(\{(B-iC)-(B+iC)\}_{D_B\cap D_C}\oplus 0_{D_B^\perp})x'+i(\{(B-iC)+(B+iC)\}_{D_B\cap D_C}\oplus 0_{D_C^\perp})x=0,$$

generating a relation ρ in H. This equation can be written in the form $x\rho x'\leftrightarrow(C\restriction(D_B\cap D_C)\oplus 0_{D_B^\perp})x'-(B\restriction(D_B\cap D_C)\oplus 0_{D_C^\perp})x=0.$

Since by condition 1 we have $D_B^{\perp} \subset \operatorname{Ker} C$, $D_C^{\perp} \subset \operatorname{Ker} B$, it follows that

$$C \upharpoonright (D_B \cap D_C) \oplus 0_{D_B^{\perp}} = C \upharpoonright (D_B \cap D_C \oplus D_B^{\perp})$$
$$= C \upharpoonright \{D_C \cap (D_B \oplus D_B^{\perp})\} = C_1, \; B \upharpoonright (D_B \cap D_C) \oplus 0_{D_C^{\perp}} = B_1,$$

(see (12)) and, therefore $\rho = \theta_1 \subset \theta$, $\rho^{\sim} = \theta_1^{\sim} \subset \theta^{\sim}$.

We note that ρ (23) can be represented in the parametric form

$$x' = -i(U' + I)h, \; x = (U' - I)h,$$
$$\forall h \in D_B \cap D_C \oplus D_B^{\perp} \oplus D_C^{\perp} = D_{U'}$$

and, moreover, $h = (ix' - x)/2$, while the operator U' is bounded. Therefore, the closure $\rho^{\sim}$ is obtained by passing from U' to $U'^{\sim}$ in the parametric representation of the relation ρ, and thus also in (23):

$$x\rho^{\sim}x' \leftrightarrow (U'^{\sim} - I)x' + i(U'^{\sim} + I)x = 0.$$

It is easy to see that the operator $U'^{\sim}$ (22) is unitary in H since by assumption $U_1^{\sim}$ is unitary in

$$H_1 = \overline{D_B \cap D_C} = \bar{D}_B \cap \bar{D}_C,$$

and since

$$D_B \cap \bar{D}_C \oplus D_B^{\perp} \oplus D_C^{\perp} = H.$$

Thus, the relation $\rho^{\sim}$, i.e., $\theta_1^{\sim}$, is Hermitian and its Cayley transform is

$$U'^{\sim} = U_1^{\sim} \oplus I_{D_B^{\perp}} \oplus (-I_{D_C^{\perp}}) = U,$$

this proves the properties of the operator U (14), the sufficiency of the conditions of the theorem, and the entire Theorem 4.

(We mention that this theorem has been communicated in [2, Theorem 3] in a not entirely precise form.)

In conclusion we give an example that shows that the closure of the relation θ (10), and not its restriction θ_1 (11), (12), may turn out, under the assumptions of Theorem 4, to be broader than a Hermitian relation, i.e., it is not Hermitian but contains it as its restriction. Indeed, let B be and invertible operator, densely defined in $D_B \neq H$, $C \supset B$ is its proper extension such that D_C is the linear hull of D_B and $h \notin D_B$, and let $Ch = 0$. Then all the conditions of Theorem 4 are satisfied, but $0\theta h$ by virtue of (10), while $D_\theta = D_B$ is dense in H and, by Lemma 1, $W(\theta) = \mathbf{C}$, so that θ is not even a symmetric relation. At the same time, $x\theta_1^{\sim}x' \leftrightarrow x' = x$ is a Hermitian relation.

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