Note

An empty interval in the spectrum of small weight codewords in the code from points and k-spaces of PG(n, q)

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A R T I C L E   I N F O

A B S T R A C T

Let $C_k(n, q)$ be the p-ary linear code defined by the incidence matrix of points and k-spaces in PG(n, q), $q = p^h$, p prime, $h \geqslant 1$. In this paper, we show that there are no codewords of weight in the open interval $]\frac{q^{k+1}-1}{q-1}, 2q^k[$ in $C_k(n, q) \setminus C_{n-k}(n, q)^\perp$ which implies that there are no codewords with this weight in $C_k(n, q) \setminus C_k(n, q)^\perp$ if $k \geqslant n/2$. In particular, for the code $C_{n-1}(n, q)$ of points and hyperplanes of PG(n, q), we exclude all codewords in $C_{n-1}(n, q)$ with weight in the open interval $]\frac{q^{n}-1}{q-1}, 2q^{n-1}[$. This latter result implies a sharp bound on the weight of small weight codewords of $C_{n-1}(n, q)$, a result which was previously only known for general dimension for q prime and $q = p^2$, with p prime, $p > 11$, and in the case $n = 2$, for $q = p^3$, $p \geqslant 7$ [K. Chouinard, On weight distributions of codes of planes of order 9, Ars Combin. 63 (2002) 3–13; V. Fack, Sz.L. Fancsali, L. Storme, G. Van de Voorde, J. Winne, Small weight codewords in the codes arising from Desarguesian projective planes, Des. Codes Cryptogr. 46 (2008) 25–43; M. Lavrauw, L. Storme, G. Van de Voorde, On the code generated by the incidence matrix of points and hyperplanes in PG(n, q) and its dual, Des. Codes Cryptogr. 48 (2008) 231–245; M. Lavrauw, L. Storme, G. Van de Voorde, On the code generated

by the incidence matrix of points and k-spaces in PG(n, q) and its dual, Finite Fields Appl. 14 (2008) 1020–1038].

1. Definitions

Let PG(n, q) denote the n-dimensional projective space over the finite field $\mathbb{F}_q$ with q elements, where $q = p^h$, p prime, $h \geqslant 1$, and let V$(n + 1, q)$ denote the underlying vector space. Let θ_n denote the number of points in PG(n, q), i.e., $\theta_n = (q^{n+1} - 1)/(q - 1)$.

We define the incidence matrix $A = (a_{ij})$ of points and k-spaces in the projective space PG(n, q), $q = p^h$, p prime, $h \geqslant 1$, as the matrix whose rows are indexed by the k-spaces of PG(n, q) and whose columns are indexed by the points of PG(n, q), and with entry

$$a_{ij} = \begin{cases} 1 & \text{if point } j \text{ belongs to } k\text{-space } i, \\ 0 & \text{otherwise.} \end{cases}$$

The p-ary linear code of points and k-spaces of PG(n, q), $q = p^h$, p prime, $h \geqslant 1$, is the $\mathbb{F}_p$-span of the rows of the incidence matrix A. We denote this code by $C_k(n, q)$. The *support* of a codeword c, denoted by supp(c), is the set of all non-zero positions of c. The *weight* of c is the number of non-zero positions of c and is denoted by wt(c). Often we identify the support of a codeword with the corresponding set of points of PG(n, q). We let (c_1, c_2) denote the scalar product in $\mathbb{F}_p$ of two codewords c_1, c_2 of $C_k(n, q)$. Furthermore, if T is a set of points of PG(n, q), then the incidence vector of this set is also denoted by T. The dual code $C_k(n, q)^\perp$ is the set of all vectors orthogonal to all codewords of $C_k(n, q)$, hence

$$C_k(n, q)^\perp = \left\{ v \in V(\theta_n, p) \;\middle\|\; (v, c) = 0, \;\; \forall c \in C_k(n, q) \right\}.$$

It is easy to see that $c \in C_k(n, q)^\perp$ if and only if $(c, K) = 0$ for all k-spaces K of PG(n, q).

2. Previous results

The p-ary linear code of points and lines of PG$(2, q)$, $q = p^h$, p prime, $h \geqslant 1$, is studied in [1, Chapter 6]. In [1, Proposition 5.7.3], the codewords of minimum weight of the code of points and hyperplanes of PG(n, q), $q = p^h$, p prime, $h \geqslant 1$, are determined. The first results on codewords of small weight in the p-ary linear code of points and lines in PG$(2, p)$, p prime, were proved by McGuire and Ward [10], where they proved that there are no codewords of $C_1(2, p)$, p an odd prime, in the interval $[p + 2, 3(p + 1)/2]$. This result was extended by Chouinard (see [3,4]) where he proves the following result.

Result 1. *(See [3,4].) In the p-ary linear code arising from* PG$(2, p)$, p *prime, there are no codewords with weight in the closed interval* $[p + 2, 2p - 1]$.

This result shows that there is a gap in the weight enumerator of the code $C_1(2, p)$ of points and lines in PG$(2, p)$, p prime. In Corollary 21, Result 1 is extended to the code of points and k-spaces in PG(n, p), p prime, $p > 5$.

In the case where q is not a prime, we improve on results of [5,7,8], where the authors exclude codewords of small weight in $C_{n-1}(n, q)$, $q = p^h$, p prime, $h \geqslant 1$, respectively $C_k(n, q) \setminus C_k(n, q)^\perp$, $q = p^h$, p prime, $h \geqslant 1$, corresponding to linear small minimal blocking sets, which implied Results 2 and 3. For the definition of a blocking set, see the next section.

Result 2. *(See [7, Corollary 3].) The only possible codewords c of* $C_{n-1}(n, q)$, $q = p^h$, p *prime*, $h \geqslant 1$, *of weight in the open interval* $]\theta_{n-1}, 2q^{n-1}[$ *are the scalar multiples of non-linear minimal blocking sets, intersecting every line in* 1 (mod p) *points.*

Result 3. *(See [8, Corollary 2].) For $k \geqslant n/2$, the only possible codewords c of $C_k(n,q) \setminus C_k(n,q)^{\perp}$, $q = p^h$, p prime, $h \geqslant 1$, of weight in the open interval $]\theta_k, 2q^k[$ are scalar multiples of non-linear minimal k-blocking sets of $PG(n,q)$, intersecting every line in $1 \pmod{p}$ or zero points.*

Remark 4. It is believed (and conjectured, see [11, Conjecture 3.1]) that all small minimal blocking sets are linear. If that conjecture is true, then Result 2 eliminates all possible codewords of $C_{n-1}(n,q)$, $q = p^h$, p prime, $h \geqslant 1$, of weight in the open interval $]\theta_{n-1}, 2q^{n-1}[$, and Result 3 eliminates all codewords of $C_k(n,q) \setminus C_k(n,q)^{\perp}$, $q = p^h$, p prime, $h \geqslant 1$, of weight in the open interval $]\theta_k, 2q^k[$ if $k \geqslant n/2$.

In this article, we avoid the obstacle of this non-solved conjecture and improve on Results 2 and 3 by showing that there are no codewords in $C_k(n,q) \setminus C_{n-k}(n,q)^{\perp}$, $q = p^h$, p prime, $p > 5$, $h \geqslant 1$, in the open interval $]\theta_k, 2q^k[$, which implies that there are no codewords in the open interval $]\theta_k, 2q^k[$ in $C_k(n,q) \setminus C_k(n,q)^{\perp}$ if $k \geqslant n/2$. Using the results of [8], we show that there are no codewords in $C_k(n,q)$, $q = p^h$, p prime, $h \geqslant 1$, $p > 7$, with weight in the open interval $]\theta_k, (12\theta_k + 6)/7[$.

In the case that $k = n - 1$, we show that there are no codewords in $C_{n-1}(n,q)$, $q = p^h$, p prime, $h \geqslant 1$, in the open interval $]\theta_{n-1}, 2q^{n-1}[$. These bounds are sharp: codewords of minimum weight in $C_{n-1}(n,q)$ have been characterized as scalar multiples of incidence vectors of hyperplanes (see [1, Proposition 5.7.3]), and codewords of weight $2q^{n-1}$ can be obtained by taking the difference of the incidence vectors of two hyperplanes.

3. Blocking sets

A *blocking set* of $PG(n,q)$ is a set K of points such that each hyperplane of $PG(n,q)$ contains at least one point of K. A blocking set K is called *trivial* if it contains a line of $PG(n,q)$. These blocking sets are also called 1-*blocking sets* in [2]. In general, a *k-blocking set* K in $PG(n,q)$ is a set of points such that any $(n-k)$-dimensional subspace intersects K. A k-blocking set K is called *trivial* if there is a k-dimensional subspace contained in K. If an $(n-k)$-dimensional space contains exactly one point of a k-blocking set K in $PG(n,q)$, it is called a *tangent $(n-k)$-space* to K, and a point P of K is called *essential* when it belongs to a tangent $(n-k)$-space of K. A k-blocking set K is called *minimal* when no proper subset of K is also a k-blocking set, i.e., when each point of K is essential. A k-blocking set is called *small* if it contains less than $3(q^k + 1)/2$ points.

In order to define a *linear k-blocking set*, we introduce the notion of a Desarguesian spread.

By field reduction, the points of $PG(n,q)$, $q = p^h$, p prime, $h \geqslant 1$, correspond to $(h-1)$-dimensional subspaces of $PG((n+1)h - 1, p)$, since a point of $PG(n,q)$ is a 1-dimensional vector space over $\mathbb{F}_q$, and so an h-dimensional vector space over $\mathbb{F}_p$. In this way, we obtain a partition $\mathcal{D}$ of the point set of $PG((n+1)h - 1, p)$ by $(h-1)$-dimensional subspaces. In general, a partition of the point set of a projective space by subspaces of a given dimension k is called a *spread*, or a *k-spread* if we want to specify the dimension. The spread we have obtained here is called a *Desarguesian spread*. Note that the Desarguesian spread satisfies the property that each subspace spanned by two spread elements is again partitioned by spread elements.

Definition 5. Let $\mathcal{D}$ be a Desarguesian $(h-1)$-spread of $PG((n+1)h - 1, p)$ as defined above. If U is a subset of $PG((n+1)h - 1, p)$, then we write $\mathcal{B}(U) = \{R \in \mathcal{D} \,\|\, U \cap R \neq \emptyset\}$.

In analogy with the correspondence between the points of $PG(n,q)$ and the elements of a Desarguesian spread $\mathcal{D}$ in $PG((n+1)h - 1, p)$, we obtain the correspondence between the lines of $PG(n,q)$ and the $(2h - 1)$-dimensional subspaces of $PG((n+1)h - 1, p)$ spanned by two elements of $\mathcal{D}$, and in general, we obtain the correspondence between the $(n-k)$-spaces of $PG(n,q)$ and the $((n-k+1)h - 1)$-dimensional subspaces of $PG((n+1)h - 1, p)$ spanned by $n - k + 1$ elements of $\mathcal{D}$. With this in mind, it is clear that any hk-dimensional subspace U of $PG(h(n+1) - 1, p)$ defines a k-blocking set $\mathcal{B}(U)$ in $PG(n,q)$. A blocking set constructed in this way is called a *linear k-blocking set*. Linear k-blocking sets were first introduced by Lunardon [9, Section 5], although there a different approach is used. For more on the approach explained here, we refer to [6, Chapter 1].

4. Results

In [12], Szőnyi and Weiner proved the following result on small blocking sets.

Result 6. *(See [12, Theorem 2.7].) Let B be a minimal blocking set of $\mathrm{PG}(n, q)$ with respect to k-dimensional subspaces, $q = p^h$, $p > 2$ prime, $h \geqslant 1$, and assume that $|B| < 3(q^{n-k} + 1)/2$. Then any subspace that intersects B, intersects it in $1 \pmod{p}$ points.*

In [8], Lavrauw et al. proved the following results.

Result 7. *The support of a codeword $c \in \mathsf{C}_k(n, q)$, $q = p^h$, p prime, $h \geqslant 1$, with weight smaller than $2q^k$, for which $(c, S) \neq 0$ for some $(n - k)$-space S, is a minimal k-blocking set in $\mathrm{PG}(n, q)$. Moreover, c is a scalar multiple of a certain incidence vector, and $\mathrm{supp}(c)$ intersects every $(n - k)$-dimensional space in $1 \pmod{p}$ points.*

Lemma 8. *Let $c \in \mathsf{C}_k(n, q)$, $q = p^h$, p prime, $h \geqslant 1$, then there exists a constant $a \in \mathbb{F}_p$ such that $(c, U) = a$, for all subspaces U of dimension at least $n - k$.*

In the same way as is done by the authors in [8, Theorem 19], one can prove Lemma 9, which shows that all minimal k-blocking sets of size less than $2q^k$ and intersecting every $(n - k)$-space in $1 \pmod{p}$ points, are small.

Lemma 9. *Let B be a minimal k-blocking set in $\mathrm{PG}(n, q)$, $n \geqslant 2$, $q = p^h$, p prime, $p > 5$, $h \geqslant 1$, intersecting every $(n - k)$-dimensional space in $1 \pmod{p}$ points. If $|B| \in \,]\theta_k, 2q^k[$, then*

$$|B| < \frac{3(q^k - q^k/p)}{2}.$$

Lemma 10. *Let B_1 and B_2 be small minimal $(n - k)$-blocking sets in $\mathrm{PG}(n, q)$, $q = p^h$, p prime, $h \geqslant 1$. Then $B_1 - B_2 \in \mathsf{C}_k(n, q)^{\perp}$.*

Proof. It follows from Result 6 that $(B_i, \pi_k) = 1$ for all k-spaces π_k, $i = 1, 2$. Hence $(B_1 - B_2, \pi_k) = 0$ for all k-spaces π_k. This implies that $B_1 - B_2 \in \mathsf{C}_k(n, q)^{\perp}$. $\quad\square$

Lemma 11. *Let c be a codeword of $\mathsf{C}_k(n, q)$, $q = p^h$, p prime, $h \geqslant 1$, with weight smaller than $2q^k$, for which $(c, S) \neq 0$ for some $(n - k)$-space S, and let B be a small minimal $(n - k)$-blocking set. Then $\mathrm{supp}(c)$ intersects B in $1 \pmod{p}$ points.*

Proof. Let c be a codeword of $\mathsf{C}_k(n, q)$ with weight smaller than $2q^k$, for which $(c, S) \neq 0$ for some $(n - k)$-space S. Lemma 10 shows that $(c, B_1 - B_2) = 0 = (c, B_1) - (c, B_2)$ for all small minimal $(n - k)$-blocking sets B_1 and B_2. Hence (c, B), with B a small minimal $(n - k)$-blocking set, is a constant. Result 7 shows that c is a codeword only taking values from $\{0, a\}$, so $(c, B) = a(\mathrm{supp}(c), B)$, hence $(\mathrm{supp}(c), B)$ is a constant too. Let B_1 be an $(n - k)$-space, then Result 7 shows that $(\mathrm{supp}(c), B_1) = 1$. Since B_1 is a small minimal $(n - k)$-blocking set, the number of intersection points of $\mathrm{supp}(c)$ and B is equal to $1 \pmod{p}$ for any small minimal blocking set B. $\quad\square$

It follows from Lemma 8 that, for $c \in \mathsf{C}_k(n, q)$ and S an $(n - k)$-space, (c, S) is a constant. Hence, either $(c, S) \neq 0$ for all $(n - k)$-spaces S, or $(c, S) = 0$ for all $(n - k)$-spaces S. In this latter case, $c \in \mathsf{C}_{n-k}(n, q)^{\perp}$.

Theorem 12. *There are no codewords in $\mathsf{C}_k(n, q) \setminus \mathsf{C}_{n-k}(n, q)^{\perp}$, $1 \leqslant k \leqslant n - 1$, $2 \leqslant n$, with weight in the open interval $]\theta_k, 2q^k[$, $q = p^h$, p prime, $p > 5$, $h \geqslant 1$.*

Proof. Let Y be a linear small minimal $(n-k)$-blocking set in $\mathrm{PG}(n,q)$. As explained in Section 3, Y corresponds to a set $\bar{Y} = \mathcal{B}(\pi)$ of $(h-1)$-dimensional spread elements intersecting a certain $(h(n-k))$-space π in $\mathrm{PG}(h(n+1)-1,p)$. Let c be a codeword of $C_k(n,q) \setminus C_{n-k}(n,q)^{\perp}$ with weight at most $2q^k - 1$. Result 7 and Lemma 9 show that $\mathrm{supp}(c)$ is a small minimal k-blocking set B. This blocking set B corresponds to a set $\bar{B}$ of $|B|$ spread elements in $\mathrm{PG}(h(n+1)-1,p)$. Since $\mathrm{supp}(c)$ and Y intersect in 1 (mod p) points (see Lemma 11), $\bar{B}$ and $\bar{Y}$ intersect in 1 (mod p) spread elements. Since all spread elements of $\bar{Y}$ intersect π, there are 1 (mod p) spread elements of $\bar{B}$ that intersect π.

But this holds for any $(h(n-k))$-space π' in $\mathrm{PG}(h(n+1)-1,p)$, since any $(h(n-k))$-space π' corresponds to a linear small minimal $(n-k)$-blocking set Y' in $\mathrm{PG}(n,q)$.

Let $\tilde{B}$ be the set of points contained in the spread elements of the set $\bar{B}$. Since a spread element that intersects a subspace of $\mathrm{PG}(h(n+1)-1,p)$ intersects it in 1 (mod p) points, $\tilde{B}$ intersects any $(h(n-k))$-space in 1 (mod p) points. Moreover, $|\tilde{B}| = |B| \cdot (p^h - 1)/(p-1) \leqslant 3(p^{hk} - p^{hk-1}) \cdot (p^h - 1)/(2(p-1)) < 3(p^{h(k+1)-1} + 1)/2$ (see Lemma 9). This implies that $\tilde{B}$ is a small $(h(k+1)-1)$-blocking set in $\mathrm{PG}(h(n+1)-1,p)$.

Moreover, $\tilde{B}$ is minimal. This can be proved in the following way. Let R be a point of $\tilde{B}$. Since B is a minimal k-blocking set in $\mathrm{PG}(n,q)$, there is a tangent $(n-k)$-space S through the point R' of $\mathrm{PG}(n,q)$ corresponding to the spread element $\mathcal{B}(R)$. Now S corresponds to an $(h(n-k+1)-1)$-space π' in $\mathrm{PG}(h(n+1)-1,p)$, such that $\mathcal{B}(R)$ is the only element of $\bar{B}$ in π'. This implies that through R, there is an $(h(n-k))$-space in π' containing only the point R of $\tilde{B}$. This shows that through every point of $\tilde{B}$, there is a tangent $(h(n-k))$-space, hence that $\tilde{B}$ is a minimal $(h(k+1)-1)$-blocking set.

Result 6 implies that $\tilde{B}$ intersects any subspace of $\mathrm{PG}(h(n+1)-1,p)$ in 1 (mod p) or zero points. This implies that a line is skew, tangent or entirely contained in $\tilde{B}$, hence $\tilde{B}$ is a subspace of $\mathrm{PG}(h(n+1)-1,p)$, with at most $3(p^{h(k+1)-1}+1)/2$ points, intersecting every $(h(n-k))$-space. Moreover, it is the point set of a set of $|B|$ spread elements. Hence, $\bar{B}$ is the set of spread elements corresponding to a k-space in $\mathrm{PG}(n,q)$, so $\mathrm{supp}(c)$ has size θ_k. $\quad\square$

In [8], Lavrauw et al. determined a lower bound on the weight of the code $C_k(n,q)^{\perp}$.

Result 13. *The minimum weight of $C_k(n,q)^{\perp}$, $q = p^h$, p prime, $h \geqslant 1$, $2 \leqslant n$, $1 \leqslant k \leqslant n-1$, is at least $(12\theta_{n-k}+2)/7$ if $p = 7$, and at least $(12\theta_{n-k}+6)/7$ if $p > 7$.*

Theorem 14. *For $q = p^h$, p prime, $h \geqslant 1$, $2 \leqslant n$, $1 \leqslant k \leqslant n-1$, there are no codewords in $C_k(n,q)$ with weight in the open interval $]\theta_k, (12\theta_k+2)/7[$ if $p = 7$ and there are no codewords in $C_k(n,q)$ with weight in the open interval $]\theta_k, (12\theta_k+6)/7[$ if $p > 7$.*

Proof. This follows immediately from Theorem 12 and Result 13. $\quad\square$

In [8], the authors proved the following result.

Result 15. *(See [8, Lemma 3].) Assume that $k \geqslant n/2$. A codeword c of $C_k(n,q)$ is in $C_k(n,q) \cap C_k(n,q)^{\perp}$ if and only if $(c, U) = 0$ for all subspaces U with $\dim(U) \geqslant n-k$.*

Corollary 16. *If $k \geqslant n/2$, $C_k(n,q) \setminus C_{n-k}(n,q)^{\perp} = C_k(n,q) \setminus C_k(n,q)^{\perp}$, $q = p^h$, p prime, $h \geqslant 1$.*

Proof. It follows from Result 15 that $C_k(n,q) \cap C_{n-k}(n,q)^{\perp} = C_k(n,q) \cap C_k(n,q)^{\perp}$ if $k \geqslant n/2$. $\quad\square$

In [7], the authors proved the following result.

Result 17. *(See [7, Theorem 5].) The minimum weight of $C_{n-1}(n,q) \cap C_{n-1}(n,q)^{\perp}$ is equal to $2q^{n-1}$.*

Result 18. *(See [8, Theorem 12].) The minimum weight of* $C_k(n, p)^{\perp}$, *where p is a prime, is equal to* $2p^{n-k}$, *and the codewords of weight* $2p^{n-k}$ *are the scalar multiples of the difference of two $(n-k)$-spaces intersecting in an $(n-k-1)$-space.*

Theorem 12, Corollary 16, and Result 17 yield the following corollary, which gives a sharp empty interval on the size of small weight codewords of $C_{n-1}(n, q)$, since θ_{n-1} is the weight of a codeword arising from the incidence vector of an $(n-1)$-space and $2q^{n-1}$ is the weight of a codeword arising from the difference of the incidence vectors of two $(n-1)$-spaces.

Corollary 19. *There are no codewords with weight in the open interval* $]\theta_{n-1}, 2q^{n-1}[$ *in the code* $C_{n-1}(n, q)$, $q = p^h$, *p prime, $h \geqslant 1$, $p > 5$.*

In the planar case, this yields the following corollary, which improves on the result of Chouinard mentioned in Result 1.

Corollary 20. *There are no codewords with weight in the open interval* $]q+1, 2q[$ *in the p-ary linear code of points and lines of* $PG(2, q)$, $q = p^h$, *p prime, $h \geqslant 1$, $p > 5$.*

In this case, the weight $q+1$ corresponds to the incidence vector of a line, and the weight $2q$ can be obtained by taking the difference of the incidence vectors of two different lines.

Theorem 12 and Result 18 yield the following corollary, extending the result of Chouinard mentioned in Result 1 to general dimension.

Corollary 21. *There are no codewords with weight in the open interval* $]\theta_k, 2q^k[$ *in the code* $C_k(n, q)$, $n \geqslant 2$, $1 \leqslant k \leqslant n-1$, *$q$ prime, $q > 5$.*

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