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Stability of characterizations of normal distributions based on the conditional expected values of the sample skewness and the sample kurtosis

Truc T. Nguyen[1], **Khoan T. Dinh**[2]

[1] Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, OH 43403-0221, USA (e-mail: tnguyen@bgnet.bgsu.edu)
[2] US Environmental Protection Agency, Washington, DC 20460, USA (e-mail: dinh.khoan@epamail.epa.gov)

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Abstract. Characterizations of normal distributions given by Nguyen and Dinh (1998) based on conditional expected values of the sample skewness and the sample kurtosis, given the sample mean and the sample variance, are shown to be stable.

Key words: Conditional density function, characteristic function, conditional moment, small parameter

1. Introduction

Nguyen and Dinh (1998) gave characterizations of normal distributions based on conditional expected values of the sample skewness and the sample kurtosis. They also showed that the goodness-of-fit tests for normality based on these characterizations are exactly the tests of normality using the sample skewness and the sample kurtosis. In this note we go to show that all these characterizations are stable. The stability of these characterizations is equivalent to the robustness of the skewness test and the kurtosis test of normality limited to the set of distributions in the results.

Let a distribution F be characterized by a property S of a random sample $X_1, \dots, X_n$. This characterization of F based on S is stable if the property S is replaced by a weaker property S_ε, where ε is a "small parameter", and the distribution F_ε is close to F in terms of ε (Kagan, Linnik, and Rao (1973)).

Definition 1.1. *A distribution F is ε-normal if there exist $\mu \in R$ and $\sigma > 0$ such that* $\mathrm{Sup}_x |F(x) - \Phi((x-\mu)/\sigma)| \le \varepsilon$, *where Φ is the distribution function of a standard normal distribution.*

The problem of stability of a characterization was studied by several authors such as Sapogov (1951, 1959), Zolotarev (1968), Machis (1969), Hoang

(1968), Maloshevskii (1968), Shalaeskii (1959), Mashalkin (1968), Kagan (1970), Klebanov and Melamed (1980, 1981), Klebanov and Yanushkevichene (1982), Klebanov, Dimitrov, and Rachev (1982). The following result of Esseen (1945) is used (Kagan, Linnik, and Rao (1973)). It gives the estimate of the closeness of two distribution functions F and G in terms of the closeness of their characteristic functions φ and ψ on an arbitrary interval $[-T, T]$.

Theorem 1.1. *Let F and G be distribution functions and φ and ψ their respective characteristic functions. If G has the probability density function G', then*

$$\mathrm{Sup}_x|F(x) - G(x)| \leq \frac{1}{\Pi}\int_{-T}^{T}\left|\frac{\varphi(t) - \psi(t)}{t}\right| dt + \frac{24}{\Pi T}\mathrm{Sup}_x|G'(x)|. \tag{1.1}$$

The following inequality will be used several times in the proofs of the theorem in Section 2 and Section 3 below.

For any complex number z,

$$|e^z - 1| \leq |z|e^{|z|}. \tag{1.2}$$

2. Stability of the characterization based on the sample skewness

We show that the characterization results given by Theorem 2.1, Corollary 2.1, and Corollary 2.2 of Nguyen and Dinh (1998) are stable.

Theorem 2.1. *Let $X_1, \ldots, X_n$, $n > 2$, be a random sample from a non-degenerate distribution F, with a finite third moment. Assume that $E[(X_1 - Z_1)^3 | Z_1, Z_2] = b_n(z_1, z_2)$, with $\left|\frac{n^2}{(n-1)(n-2)} b_n(Z_1, Z_2)\right| < \varepsilon^n$ for all $Z_1, Z_2, \varepsilon < e^{-(\sigma^2+1)}, \mu$ and σ^2 are the mean and the variance of F, respectively, $Z_1 = \sum_{j=1}^{n} X_j/n$, $Z_2 = \sum_{j=1}^{n}(X_j - Z_1)^2$. Then $\mathrm{Sup}_x|F(x) - \Phi((x-\mu)/\sigma)| < C\left(\ln\frac{1}{\varepsilon}\right)^{-1/3}$, that is, F is $C\left[\ln\left(\frac{1}{\varepsilon}\right)\right]^{-1/3}$ – normal, where C is a constant depending only on σ^2.*

Proof: The proof of this theorem is given in Appendix B. In this proof, we show that $\mathrm{Sup}_x|F(x) - \Phi((x-\mu)/\sigma)| \leq C[\ln(1/\varepsilon)]^{-1/3}$, where Φ is the distribution function of a standard normal distribution, and $C = \frac{e}{9(\sigma^2+1)\Pi} + \frac{12\sqrt{2}(\sigma^2+1)^{1/3}}{\sigma\Pi^{3/2}}$.

Corollary 2.1. *Let $X_1, \ldots, X_n$, $n > 2$, be a random sample from a non-degenerate distribution F with a finite third moment. Assume that $E[\sqrt{b_1}|z_1, z_2] = c_n(z_1, z_2)$, with $\left|\frac{n^2}{(n-1)(n-2)}(z_2/n)^{3/2} c_n(z_1, z_2)\right| \leq \varepsilon^n$ for all possible values of z_1 and $z_2, \varepsilon < e^{-(\sigma^2+1)}$, σ^2 is the variance of F and b_1 is the sample skewness. Then F is $C[\ln(1/\varepsilon)]^{-1/3}$ – normal, where C is a constant depending only on σ^2.*

Proof: From $E[\sqrt{b_1}|z_1, z_2] = c_n(z_1, z_2)$ we have $E[(X_1 - Z_1)^3|z_1, z_2] = (z_2/n)^{3/2} c_n(z_1, z_2)$. Hence, by Theorem 2.1, F is $C[\ln(1/\varepsilon)]^{-1/3}$ – normal.

Corollary 2.2. *Let* $X_1, \ldots, X_n$, $n > 2$, *be a random sample from a non-degenerate distribution F with a finite third moment. Assume that* $f_{X_1}(x|z_1, z_2)$

$$= K(n)\left[1 - \frac{n-1}{n}(x - z_1)^2/z_2\right]^{(n-4)/2} \Big/ \sqrt{z_2} + d_n(x, z_1, z_2)$$

$$\textit{for } (x - z_1)^2 < \frac{n}{n-1} z_2, \quad \textit{and } 0 \textit{ otherwise}, \tag{2.1}$$

where

$$K(n) = \frac{\sqrt{n}\Gamma((n-1)/2)}{\sqrt{(n-1)\Pi}\Gamma((n-2)/2)},$$

$$\textit{with } \frac{n^2}{(n-1)(n-2)}\left|\int_B (x - z_1)^3\, d_n(x, z_1, z_2)\, dx\right| \le \varepsilon^n,$$

for all possible values of $z_1, z_2, B = \left\{x : (x - z_1)^2 < \frac{n}{n-1} z_2\right\}$, $\varepsilon < e^{-\sigma^2+1}$, σ^2 *is the variance of F. Then F is* $C[\ln(1/\varepsilon)]^{-1/3}$ *– normal, where C is a constant depending only on* σ^2.

Proof: From (2.1), by letting

$$a = z_1 - \sqrt{((n-1)/n)z_2} \quad \text{and} \quad b = z_1 + \sqrt{((n-1)/n)z_2},$$

$$\begin{aligned}
&|E[(X_1 - Z_1)^3|z_1, z_2]| \\
&\quad = \left|\int_a^b (x - z_1)^3 K(n)[1 - (n/(n-1)(x - z_1)^2/z_2]^{(n-4)/2}/\sqrt{z_2}\, dx \right. \\
&\qquad \left. + \int_a^b (x - z_1)^3\, d_n(x, z_1, z_2)\, dx\right| \\
&\quad = \left|\int_a^b (x - z_2)^3\, d_n(x, z_1, z_2)\, dx\right| \\
&\quad \le \frac{(n-1)(n-2)}{n^2}\varepsilon^n.
\end{aligned}$$

The result then is obtained from Theorem 2.1.

3. Stability of the characterization based on the sample kurtosis

In this section we show that the characterizations given by Theorem 2.2 and Corollary 2.4 of Nguyen and Dinh (1998) are stable.

Theorem 3.1. *Let $X_1, \ldots, X_n$, $n > 2$, be a random sample from a non-degenerate distribution F with a finite fourth moment. Assume that*

$$E[(X_1 - Z_1)^4 | z_1, z_2] = \frac{3(n-1)}{n^2(n+1)} z_2^2 + b_n(z_1, z_2), \tag{3.1}$$

where $\dfrac{n^2(n+1)}{(n-1)(n-2)} |b_n(z_1, z_2)| \leq \varepsilon^n$ *for all possible values of* z_1, z_2, *and* $\varepsilon < e^{-(\sigma^2+1)}$, $\mu_3 = 3\mu\sigma^2 + \mu^3$, *where* μ, σ^2, *and* μ_3 *are the mean, the variance, and the third moment of F, respectively. Then F is* $C[\ln(1/\varepsilon)]^{-1/3}$ *– normal, where C is a constant depending only on* σ^2.

Proof: The technique used in the proof of this theorem is similar to the technique used in the proof of Theorem 2.1, then the proof of this theorem is omitted. In this proof, we show that

$$\mathrm{Sup}_x |F(x) - \Phi((x-\mu)/\sigma)| \leq C[\ln(1/\varepsilon)]^{-1/3},$$

where $C = e/48(\sigma^2 + 1)\Pi + 12\sqrt{2}(\sigma^2 + 1)^{1/4}/\sigma\Pi^{3/2}$.

Corollary 3.1. *Let $X_1, \ldots, X_n$, $n > 2$, be a random sample from a non-degenerate distribution F with a finite fourth moment. Assume that*

$$E[b_2 | z_1, z_2] = \frac{3(n-1)}{n+1} + c_n(z_1, z_2),$$

$$\text{with } \left| \frac{n(n+1)z_2^2}{(n-1)(n-2)(n-3)} c_n(z_1, z_2) \right| < \varepsilon^n$$

for all possible values of $z_1, z_2, \varepsilon < e^{-(\sigma^2+1)}$, $\mu_3 = \mu^3 + 3\mu\sigma^2$, *where* μ, σ^2, *and* μ_3 *are the mean, the variance, and the third moment of F, respectively,* b_2 *is the sample kurtosis. Then F is* $C[\ln(1/\varepsilon)]^{-1/3}$ *– normal, where C is a constant depending only on* σ^2.

4. Example

The results of stability given in Section 2 and Section 3 have meaning if any set of distributions given in these sections does not contain only the normal distributions. This fact is given by the following example.

Let $G(x)$ be a distribution with density function

$$g(x) = (1 - \varepsilon) \frac{1}{\sqrt{2\Pi}} e^{-x^2/2} + \varepsilon f(x), \tag{4.1}$$

where ε is a small positive number, and $f(x)$ is the density function of a given non normal distribution function $F(x)$ of support the set of all real numbers, with a zero mean, a unit variance, a zero third moment, and $f(x)$ is given by

$$f(x) = \frac{1}{\sqrt{2\Pi\sigma^2}} e^{-x^2/2\sigma^2} \quad \text{for } |x| > \sqrt{\frac{-\ln(\sigma^2)}{1/\sigma^2 - 1}}, \tag{4.2}$$

and

$$f(x) \geq \frac{1}{\sqrt{2\Pi^2}} e^{-x^2/2} \quad \text{otherwise},$$

where $\sigma^2 < 1$.

The distribution with density given by (4.1) satisfies all the properties of the sets of distributions given in Section 2 and Section 3. The proof of these properties is given in Appendix A.

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Appendix A: Proof of Example in Section 4

In this appendix, we prove that the distribution given in (4.1) satisfies all the properties of the sets of distributions given in Section 2 and Section 3.

Let $L = \mathrm{Sup}_x \dfrac{f(x)}{\frac{1}{\sqrt{2\Pi}} e^{-x^2/2}}$. Hence, $L > 1$, and $f(x) \leq L \frac{1}{\sqrt{2\Pi}} e^{-x^2/2}$ for all x. Let $X_1, \ldots, X_n, n > 2$, be a random sample from the density $g(x)$. The joint density of $X_1, \ldots, X_{n-2}, Z_1$, and Z_2 is

$$g(x_1, \ldots, x_{n-2}, z_1, z_2) = \prod_{j=1}^{n} \left[\frac{1-\varepsilon}{\sqrt{2\Pi}} e^{-x_j^2/2} + \varepsilon f(x_j) \right] \quad A(x_1, \ldots, x_{n-2}, z_1, z_2), \tag{A.1}$$

where $A = \sum |J|$ is the sum of all the Jacobians of the transformations from $(x_1, \ldots, x_n)$ to $(x_1, \ldots, x_{n-2}, z_1, z_2)$, defined by

$$x_{n-1} + x_n = nz_1 - \sum_{j=1}^{n-2} x_j, \quad \text{and} \quad x_{n-1}^2 + x_n^2 = z_2 + nz_1^2 - \sum_{j=1}^{n-2} x_j^2.$$

Let $I = \{1, \ldots, n\}$ and let J_m be a proper subset of size m of I, $m = 0, \ldots, n-1$. The density given in (A.1) can be written as

$$\begin{aligned} g(x_1, \ldots, x_{n-2}, z_1, z_2) = {} & (2\Pi)^{-n/2} e^{-\sum_{j=1}^{n} x_j^2/2} A(x_1, \ldots, x_{n-2}, z_1, z_2) \\ & + B_n(x_1, \ldots, x_{n-2}, z_1, z_2), \end{aligned} \tag{A.2}$$

where

$$B_n(x_1,\dots,x_{n-2},z_1,z_2)=\sum_{m=0}^{n-1}\sum_{J_m}(2\Pi)^{-m/2}e^{-\sum\limits_{j\in J_m}x_j^2/2}x$$
$$\times\left[\prod_{j\in I-J_m}(f(x_j)-(2\Pi)^{-1/2}e^{-x_j^2/2})\varepsilon^{n-m}\right]$$
$$\times A(x_1,\dots,x_{n-1},z_1,z_2). \tag{A.3}$$

From (A.3) and $f(x)\le L(2\Pi)^{-1/2}e^{-x^2/2}$, it is trivial that

$$|B_n(x_1,\dots,x_{n-2},z_1,z_2)|<n(1+L)^n\varepsilon(2\Pi)^{-n/2}e^{-\sum\limits_{j=1}^{n}x_j^2/2}$$
$$\times A(x_1,\dots,x_{n-2},z_1,z_2). \tag{A.4}$$

Hence, the joint density of Z_1 and Z_2 is

$$g_{Z_1Z_2}(z_1,z_2)=f_1(z_1,z_2)+H_n(z_1,z_2), \tag{A.5}$$

where

$$f_1(z_1,z_2)=\frac{\sqrt{n}}{\sqrt{2\Pi}}e^{-nz_1^2/2}x\frac{1}{\Gamma((n-1)/2)2^{(n-1)/2}}z_2^{(n-1)/2-1}e^{-z_2/2}, \tag{A.6}$$

and

$$H_n(z_1,z_2)=\int\dots\int B_n(x_1,\dots,x_{n-2},z_1,z_2)\prod_{j=1}^{n-2}dx_j. \tag{A.7}$$

For all possible values of z_1, z_2

$$|H_n(z_1,z_2)|=\left|\int\dots\int B_n(x_1,\dots,x_{n-2},z_1,z_2)\prod_{j=1}^{n-2}dx_j\right|$$
$$<[(1+L\varepsilon)^n-1]\frac{\sqrt{n}}{\sqrt{2\Pi}}e^{-nz_1^2/2}$$
$$\times\frac{1}{\Gamma((n-1)/2)2^{(n-1)/2}}z_2^{(n-1)/2-1}e^{-z_2/2}$$
$$<n(1+L)^n\varepsilon\frac{\sqrt{n}}{\sqrt{2\Pi}}e^{-nz_1^2/2}$$
$$\times\frac{1}{\Gamma((n-1)/2)2^{(n-1)/2}}z_2^{(n-1)/2-1}e^{-z_2/2}$$
$$=n(1+L)^n\varepsilon f_1(z_1,z_2) \tag{A.8}$$

Assume that ε is chosen such that $n(1+L)^n\varepsilon < 1/2$, that is, $\varepsilon < n^{-1}(1+L)^{-n}/2$, then $|H_n(z_1, z_2)| < f_1(z_1, z_2)/2$ for all possible values of z_1, z_2.

Moreover, from (A.1) and (A.3),

$$|B_n(x_1, \ldots, x_{n-2}, z_1, z_2)| < n2^n\varepsilon(2\Pi\sigma^2)^{-n/2}e^{-\sum_{j=1}^{n} x_j^2/2\sigma^2} \times A(x_1, \ldots, x_{n-1}, z_1, z_2) \tag{A.9}$$

and then

$$|H_n(z_1, z_2)| < n2^n\varepsilon f_\sigma 2(z_1, z_2), \tag{A.10}$$

where

$$f_\sigma 2(z_1, z_2) = \frac{\sqrt{n}}{\sqrt{2\Pi\sigma^2}}e^{-nz_1^2/2\sigma^2}\frac{1}{\Gamma((n-1)/2)(2\sigma^2)^{(n-1)/2}}e^{-z_2/2\sigma^2}. \tag{A.11}$$

From (A.2) and (A.5), the conditional density of X_1 given Z_1 and Z_2 is

$$fX_1(x|z_1, z_2) = g_0(x|z_1, z_2) + d_n(x, z_1, z_2), \tag{A.12}$$

where

$$g_0(x|z_1, z_2) = K(n)\left[1 - \frac{n-1}{n}(x-z_1)^2/z_2\right]^{(n-4)/2} \Big/ \sqrt{z_2}, \tag{A.13}$$

for $(x-z_1)^2 < \frac{n}{n-1}z_2$ and

$$d_n(x, z_1, z_2) = \frac{-H_n(z_1, z_2)g_0(x|z_1, z_2)}{f_0(z_1, z_2) + H_n(z_1, z_2)} + \frac{\int \ldots \int B_n(x_1, \ldots, x_{n-2}, z_1, z_2)\prod_{j=2}^{n-2} dx_j}{f_0(z_1, z_2) + H_n(z_1, z_2)}. \tag{A.14}$$

From (A.8), (A.10), and (A.14)

$$\begin{aligned} |d_n(x, z_1, z_2)| &< \frac{[|H_n(z_1, z_2)| + n2^n\varepsilon f_\sigma 2(z_1, z_2)]g_0(x|z_1, z_2)}{f_0(z_1, z_2) + H_n(z_1, z_2)} \\ &< \frac{n2^{n+1}\varepsilon f_\sigma 2(z_1, z_2)g_0(x|z_1, z_2)}{f_0(z_1, z_2)/2} \\ &= n2^{n+2}\varepsilon g_0(x|z_1, z_2)e^{-(nz_1^2+z_2)(1/\sigma^2-1)/2}/(\sigma^2)^{(n-2)/2}. \end{aligned} \tag{A.15}$$

Hence,

$$\begin{aligned}
&\left|\int (x-z_1)^3\, d_n(x,z_1,z_2)\, dx\right| \\
&\quad\le \int |(x-z_1)^3|\,|d_n(x,z_1,z_2)|\, dx \\
&\quad\le [n2^{n+2}\varepsilon e^{-(nz_1^2+z_2)(1/\sigma^2-1)/2}/(\sigma^2)^{(n-2)/2}] \\
&\qquad\times \int |(x-z_1)^3| g_0(x|z_1,z_2)\, dx \\
&\quad= [nz^{n+2}\varepsilon e^{-(nz_1^2+z_2)(1/\sigma^2-1)/2}/(\sigma^2)^{(n-2)/2}] \\
&\qquad\times [4nK(n)z_2^{3/2}/(n-1)^2(n-2)] \\
&\quad\le [n2^{n+2}\varepsilon e^{-z_2(1/\sigma^2-1)/2}/(\sigma^2)^{(n-2)/2}] \\
&\qquad\times [4nK(n)z_2^{3/2}/(n-1)^2(n-2)] \\
&\quad\le \frac{(3)^{3/2} z^{n+4} n^{5/2}\varepsilon\Gamma((n-1)/2)}{e^{3/2}(\sigma^2)^{(n-5)/2}(1-\sigma^2)^{3/2}(n-1)^{5/2}(n-2)\sqrt{\Pi}\,\Gamma((n-2)/2)},
\end{aligned} \tag{A.16}$$

and

$$\left|\frac{n^2}{(n-1)(n-2)}\int (x-z_1)^3\, d_n(x,z_1,z_2)\, dx\right| \le \frac{2187(2)^{n-2}e^{3/2}\varepsilon}{(\sigma^2)^{(n-5)/2}(1-\sigma^2)^{3/2}\sqrt{\Pi}} \tag{A.17}$$

If we replace ε by $\dfrac{(\sigma^2)^{(n-5)/2}(1-\sigma^2)^{3/2}\sqrt{\Pi}\varepsilon^n}{2187(2)^{n-2}e^{3/2}}$, all the conditions in Corollary 2.2, then Theorem 2.1 and Corollary 2.1 are satisfied. Similarly,

$$\frac{n^2(n+1)}{(n-1)(n-2)}\left|\int (x-z_1)^4\, d_n(x,z_1,z_2)\, dx\right| \le \frac{6561(2)^{n-1}e^2\varepsilon}{(\sigma^2)^{(n-6)/2}(1-\sigma^2)^2} \tag{A.18}$$

If we replace ε by $\dfrac{(\sigma^2)^{(n-6)/2}(1-\sigma^2)^2\varepsilon^n}{6561(2)^{n-1}e^2}$, all the conditions in Theorem 3.1, then Corollary 3.1 are satisfied.

Appendix B: Proof of Theorem 2.1

Without loss of generality, asume that $E[b_n(Z_1,Z_2)] = 0$. To simplify the notations in this proof, assume that $\left|\dfrac{n^2}{(n-1)(n-2)} b_n(Z_1,Z_2)\right| \le \varepsilon$, for all possible values of Z_1, and Z_2, and $\varepsilon < 1$. From $E[(X_1-Z_1)^3|Z_1,Z_2] = b_n(Z_1,Z_2)$, $E[(X_1-Z_1)^3|Z_1] = E_{Z_2|Z_1}[[E(X_1-Z_1)^3|Z_1,Z_2]] = E[b_n(Z_1,Z_2)|Z_1]$.

Set $S_n(Z_1) = E[b_n(Z_1, Z_2)|Z_1]$, then

$$E[(X_1 - Z_1/n)^3|Z_1] = S_n(Z_1), \tag{B.1}$$

where $|S_n(Z_1)| = |E[b_n(Z_1, Z_2)|Z_1)]| \leq \dfrac{(n-1)(n-2)}{n^2}\varepsilon$.

Multiplying both sides of (B.1) by $\exp(itZ_1)$, taking the expectation, and simplifying, it leads to the following equation for the characteristic function $\varphi(t)$ of F

$$\begin{aligned}
&\varphi'''(t)\varphi^{n-1}(t) - 3\varphi''(t)\varphi'(t)\varphi^{n-2}(t) + 2\varphi'3(t)\varphi^{n-3}(t) \\
&\quad = \frac{n^2}{(n-1)(n-2)} E[\exp(itZ_1)S_n(Z_1)].
\end{aligned} \tag{B.2}$$

Set $R_n(t) = \dfrac{n^2}{(n-1)(n-2)} E[\exp(itZ_1)S_n(Z_1)]$, then $|R_n(t)| < \varepsilon$ for all real values t.

Let $\delta(\varepsilon) = \mathrm{Sup}\{\tau : |\varphi(t)| \geq \varepsilon^{1/2n}$ for $|t| \leq \tau\}$. For $|t| < \delta(\varepsilon)$, devide both sides of (B.2) by $\varphi^n(t)$, it is equivalent to

$$(\ln \varphi(t))''' = \frac{R_n(t)}{\varphi^n(t)}. \tag{B.3}$$

From the given conditions of the theorem, the equation (B.3) has a solution satisfying the following relation

$$\varphi(t) = \exp\left\{-\frac{1}{2}\sigma^2 t^2 + i\mu t + \int_0^t \int_0^w \int_0^v \frac{R_n(u)}{\varphi^n(u)}\, du\, dv\, dw\right\}, \tag{B.4}$$

where μ and σ^2 are the mean and the variance of F, respectively. For any $|t| < \delta(\varepsilon)$,

$$\begin{aligned}
\left| Re \int_0^t \int_0^w \int_0^t \frac{R_n(u)}{\varphi^n(u)}\, du\, dv\, dw \right| &\leq \left| \int_0^t \int_0^w \int_0^v \frac{R_n(u)}{\varphi^n(u)}\, du\, dv\, dw \right| \\
&\leq \left| \int_0^t \int_0^w \int_0^v \frac{|R_n(u)|}{|\varphi^n(u)|}\, du\, dv\, dw \right| \\
&\leq \left| \int_0^t \int_0^w \int_0^v \frac{\varepsilon}{\varepsilon^{1/2}}\, du\, dv\, dw \right| \\
&= \frac{\sqrt{\varepsilon}|t|^3}{6},
\end{aligned} \tag{B.5}$$

and then

$$\begin{aligned}
|\varphi(t)| &= \exp\left\{-\frac{1}{2}\sigma^2 t^2 + Re \int_0^t \int_0^w \int_0^v \frac{R_n(u)}{\varphi^n(u)}\, du\, dv\, dw\right\} \\
&\geq \exp\left\{-\frac{1}{2}\sigma^2 t^2 - \frac{\sqrt{\varepsilon}|t|^3}{6}\right\}.
\end{aligned} \tag{B.6}$$

Since the even function in the last term of (B.6) is strictly decreasing in $|t|$, and $|\varphi(t)| \geq \varepsilon^{1/2n}$ for $|t| \leq \delta(\varepsilon)$, $\delta(\varepsilon)$ is not smaller than the positive solution of the equation $\exp\left\{-\frac{1}{2}\sigma^2 t^2 - \frac{\sqrt{\varepsilon}|t|^3}{6}\right\} = \varepsilon^{1/2n}$, or equivalently, of the equation

$$\sigma^2 t^2 + \frac{\sqrt{\varepsilon}|t|^3}{3} = \frac{1}{n}\ln\left(\frac{1}{\varepsilon}\right). \tag{B.7}$$

The function on the left side of (B.7) is an even function, then (B.7) has two opposite real solutions. Let t_0 be the positive solution of (B.7). Then $t_0 \leq \delta(\varepsilon)$.

For $t > 0$, (A.7) becomes

$$\sigma^2 t^2 + \frac{\sqrt{\varepsilon} t^3}{3} = -\frac{1}{n}\ln\varepsilon. \tag{B.8}$$

Depending on the values of t we have, for $t > 1$, $\left(\sigma^2 + \frac{\sqrt{\varepsilon}}{3}\right)t^2 \leq \sigma^2 t^2 + \frac{\sqrt{\varepsilon}t^3}{3} \leq \left(\sigma^2 + \frac{\sqrt{\varepsilon}}{3}\right)t^3$, and for $t \leq 1$, $\left(\sigma^2 + \frac{\sqrt{\varepsilon}}{3}\right)t^3 \leq \sigma^2 t^2 + \frac{\sqrt{\varepsilon}t^3}{3} \leq \left(\sigma^2 + \frac{\sqrt{\varepsilon}}{3}\right)t^2$.
Hence, the solution t_0 of the equation (2.7) satisfies

$$t_0 \geq \min\left\{\left[\ln(1/\varepsilon)\Big/ n\left(\sigma^2 + \frac{\sqrt{\varepsilon}}{3}\right)\right]^{1/2}, \quad \left[\ln(1/\varepsilon)\Big/ n\left(\sigma^2 + \frac{\sqrt{\varepsilon}}{3}\right)\right]^{1/3}\right\}.$$

This implies

$$t_0 \geq \min\{[\ln(1/\varepsilon)/n(\sigma^2+1)]^{1/2}, [\ln(1/\varepsilon)/n(\sigma^2+1)]^{1/3}\}.$$

Assume that $\ln(1/\varepsilon) \geq n(\sigma^2+1)$, or equivalently, $\varepsilon \leq e^{-n(\sigma^2+1)}$, then

$$\delta(\varepsilon) \geq t_0 \geq \delta_0(\varepsilon) = (\ln(1/\varepsilon)/n(\sigma^2+1))^{1/3}. \tag{B.9}$$

For $|t| \leq \delta_0(\varepsilon)$, using the inequality (1.2),

$$\begin{aligned}
&\left|\varphi(t) - \exp\left\{-\frac{1}{2}\sigma^2 t^2 + i\mu t\right\}\right| \\
&\quad = \exp\left\{-\frac{1}{2}\sigma^2 t^2\right\}\left|\exp\left\{\int_0^t\int_0^w\int_0^v \frac{R_n(u)}{\varphi^n(u)}\,du\,dv\,dw\right\} - 1\right| \\
&\quad \leq \exp\left\{-\frac{1}{2}\sigma^2 t^2\right\}\left|\int_0^t\int_0^w\int_0^v \frac{R_n(u)}{\varphi^n(u)}\,du\,dv\,dw\right| \\
&\qquad \times \exp\left\{\left|\int_0^t\int_0^w\int_0^v \frac{R_n(u)}{\varphi^n(u)}\,du\,dv\,dw\right|\right\} \\
&\quad \leq \frac{\sqrt{\varepsilon}|t|^3}{6}\exp\left\{-\frac{1}{2}\sigma^2 t^2 + \frac{\sqrt{\varepsilon}|t|^3}{6}\right\}
\end{aligned}$$

$$\le \frac{\sqrt{\varepsilon}|t|^3}{6}\exp\left\{\frac{\sqrt{\varepsilon}|t|^3}{6}\right\}$$

$$\le \frac{\sqrt{\varepsilon}|t|^3}{6}\exp\{\sqrt{\varepsilon}\ln(1/\varepsilon)/6n(\sigma^2+1)\} \le \frac{\sqrt{\varepsilon}|t|^3}{6}\,e,$$

since $|t|^3$ is increasing in $|t|$, and for $\varepsilon > 0$, we have $\varepsilon^{1/2}\ln(1/\varepsilon) < 1$.

By Theorem 1.1, for $|T| \le \delta_0(\varepsilon)$,

$$\mathrm{Sup}_x|F(x) - \Phi((x-\mu)/\sigma)| \le \frac{1}{\Pi}\int_{-T}^{T}\left|\frac{\varphi(t) - \exp\left\{-\frac{1}{2}\sigma^2t^2 + i\mu t\right\}}{t}\right| dt$$

$$+\frac{24}{\Pi T}\,\mathrm{Sup}_x|\Phi'((x-\mu)/\sigma)|$$

$$\le \frac{1}{\Pi}\int_{-T}^{T}\frac{e\sqrt{\varepsilon|t|^2}}{6}\,dt + \frac{12\sqrt{2}}{T\sigma\Pi^{3/2}}$$

$$= \frac{e\sqrt{\varepsilon}T^3}{9\Pi} + \frac{12\sqrt{2}}{T\sigma\Pi^{3/2}}. \tag{B.10}$$

Take $T = \delta_0(\varepsilon) = [\ln(1/\varepsilon)/n(\sigma^2+1)]^{1/3}$ and substitute in (B.10),

$$\mathrm{Sup}_x|F(x) - \Phi((x-\mu)/\sigma)|$$

$$\le e\sqrt{\varepsilon}\ln(1/\varepsilon)/9n(\sigma^2+1)\Pi$$

$$+ 12\sqrt{2}[n(\sigma^2+1)]^{1/3}[\ln(1/\varepsilon)]^{-1/3}/\sigma\Pi^{3/2}. \tag{B.11}$$

For any $\varepsilon > 0$, $\sqrt{\varepsilon}\ln\left(\frac{1}{\varepsilon}\right)$ is smaller than $\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{-1/3}$, and $n^{1/3} > n^{-1}$, then from (B.11),

$$\mathrm{Sup}_x|F(x) - \Phi((x-\mu)/\sigma)$$

$$\le \left[\frac{en^{1/3}}{9(\sigma^2+1)\Pi} + \frac{12\sqrt{2}(n(\sigma^2+1))^{1/3}}{\sigma\Pi^{3/2}}\right]\left(\ln\frac{1}{\varepsilon}\right)^{-1/3}. \tag{B.12}$$

In order for the right side of the inequality (B.12) to not depend on n, we can choose ε^n instead of ε, that is, if $\left|\frac{n^2}{(n-1)(n-2)}b_n(z_1,z_2)\right| < \varepsilon^n$, then

$$\mathrm{Sup}_x|F(x)-\Phi((x-\mu)/\sigma)| \le \left[\frac{e}{9(\sigma^2+1)\Pi} + \frac{12\sqrt{2}(\sigma^2+1)^{1/3}}{\sigma\Pi^{3/2}}\right](\ln(1/\varepsilon))^{-1/3}$$

$$= C[\ln(1/\varepsilon)]^{-1/3}, \tag{B.13}$$

where $C = \frac{e}{9(\sigma^2+1)\Pi} + \frac{12\sqrt{2}(\sigma^2+1)^{1/3}}{\sigma\Pi^{3/2}}$. Then F is $C\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{-1/3}$ – normal, where $\varepsilon^n < e^{-n(\sigma^2+1)}$, or equivalently, $\varepsilon < e^{-(\sigma^2+1)}$. The constant C depends only on σ^2, and this theorem is proved.

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