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Minus total k-subdomination in graphs

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Abstract Let $G = (V, E)$ be a simple graph without isolated vertices. For positive integer k, a 3-valued function $f : V \to \{-1, 0, 1\}$ is said to be a minus total k-subdominating function (MTkSF) if $\sum\limits_{u\in N(v)} f(u) \geqslant 1$ for at least k vertices v in G, where $N(v)$ is the open neighborhood of v. The minus total k-subdomination number $\gamma_{kt}^-(G)$ equals the minimum weight of an MTkSF on G. In this paper, the values on the minus total k-subdomination number of some special graphs are investigated. Several lower bounds on γ_{kt}^- of general graphs and trees are obtained.

Keywords minus total k-subdomination, path, complete graph, complete bipartite graph, bound
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Introduction

All graphs considered here are finite, undirected, simple, and without isolated vertices. For standard graph theory terminology not given here, one can refer to [1–2]. Specifically, let $G = (V, E)$ be a graph with the vertex set V and edge set E. The open neighborhood of v is $N(v) = \{u \in V | \ uv \in E\}$ and the closed neighborhood of v is $N[v] = \{v\} \cup N(v)$. The degree of a vertex v in G is $d(v) = |N(v)|$. Let f be a real valued function on V. For a non-empty subset S of V, we define

$$f(S) = \sum_{v\in S} f(v),$$

and the weight of a minus total k-subdominating function (MTkSF) f on G is $f(V)$. $G[S]$ denotes the subgraph of G induced by S. $\Delta(G)$ and $\delta(G)$ denote the maximum degree and the minimum degree of vertices of G, respectively. When no ambiguity can occur, we often simply write Δ and δ instead. Next we give some basic definitions.

A signed total dominating function (STDF) of G is defined in [3–4] as a function $f : V \to \{-1, 1\}$ such that $f(N(v)) \geqslant 1$ for every $v \in V$. The signed total domination number (STDN) of G, denoted by $\gamma_t^s(G)$, is the minimum weight of an STDF.

Let $G = (V, E)$ be a graph. For $k \in \mathbf{Z}^+$, $1 \leqslant k \leqslant |V|$, a function $f : V \to \{-1, 0, 1\}$ is said to be an MTkSF on G in [5] if $f(N(v)) \geqslant 1$ for at least k vertices v in G. The minus total k-subdomination number (MTkSN) of G, denoted by $\gamma_{kt}^-(G)$, is equal to $\min\{f(V) \mid f$ is an MTkSF on $G\}$. Especially, if $k = |V|$, then the minus total k-subdomination number is the minus total domination number (MTDN) $\gamma_t^-(G)$ of G. Minus total domination has been studied in, for example, [6–9]. When we simply change "open" neighborhood $N(v)$ in the definition of minus total k-subdomination to "closed" neighborhood $N[v]$, we can define the minus k-subdomination number of a graph. Minus k-subdomination has been studied in, for example, [10–13].

Harris, *et al.*[6] showed that the decision problems for the minus total k-subdomination number of a graph are NP-complete respectively, even when the graph is restricted to a bipartite graph or a chordal graph. Hence it is of interest to determine values and bounds on the minus total k-subdomination number of a graph. In this paper, we obtain the values of γ_{kt}^- of some special graphs and establish several lower bounds on γ_{kt}^- of general graphs and trees.

1 Minus total k-subdomination numbers of some graphs

Let f be an MTkSF of $G = (V, E)$. We say that $v \in V$ is covered by f if $f(N(v)) \geqslant 1$, and denote by C_f the

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set of vertices covered by f. Let $P_f = \{v \in V | f(v) = 1\}$, $M_f = \{v \in V | f(v) = -1\}$, $Q_f = \{v \in V | f(v) = 0\}$, and $B_f = \{v \in V | f(N(v)) = 1\}$.

Theorem 1.1 For any path $P_n (n \geqslant 2)$, $1 \leqslant k \leqslant n-1$,

$$\gamma_{kt}^-(P_n) = \begin{cases} -\left\lfloor \dfrac{k}{2} \right\rfloor, & \text{if } n \text{ is odd and } k = \frac{n+1}{2}, \\ \dfrac{3k}{2} + 1 - n, & \text{if } n \text{ is odd, } k \text{ is even,} \\ & 2k \geqslant n+3 \text{ and } 2k - n \equiv 3 \pmod 4, \\ \left\lceil \dfrac{3k}{2} \right\rceil - n, & \text{otherwise.} \end{cases}$$

Proof Let $P_n : v_1 v_2 v_3 \cdots v_n$ be a path on n vertices, and f be a minimum MTkSF assigning the value -1 to the vertices of $P_n = (V, E)$ as many as possible. Thus $\gamma_{kt}^-(P_n) = |P_f| - |M_f|$. Let I denote the set of all isolated vertices in $P_n[C_f]$. We first prove that $f(v) = -1$ for each $v \in I$. If not, there exists at least a vertex $v' \in I$ such that $f(N(v')) \geqslant 1$, but $f(v') \neq -1$. We define another function f' by $f'(v') = -1$ and $f'(v) = f(v)$ for all $v \in V/\{v'\}$. Obviously, f' is also an MTkSF, but $f'(V) < f(V)$ is a contradiction.

Case 1.1 $I = C_f$.

This is to say, all vertices in $P_n[C_f]$ are isolated vertices, and the vertices not adjacent to the vertices of I are assigned the value of -1. If n is odd, $k \leqslant |C_f| \leqslant \frac{n+1}{2}$; if n is even, $k \leqslant |C_f| \leqslant \frac{n}{2}$. Clearly, for any $v \in C_f, N(v) \subseteq P_f \cup Q_f$ and $|N(v)| = 1$ or 2.

Subcase 1.1.1 n is even. Obviously, $|P_f| + |Q_f| \geqslant |C_f| \geqslant k$, and $|P_f| \geqslant |Q_f|$. Then $|M_f| \leqslant n - k$. Thus $\gamma_{kt}^-(P_n) \geqslant \frac{k}{2} - (n-k) = \frac{3k}{2} - n$, and then $\gamma_{kt}^-(P_n) \geqslant \lceil \frac{3k}{2} \rceil - n$.

Subcase 1.1.2 n is odd. If $k = \frac{n+1}{2}$, then $|P_f| + |Q_f| = |C_f| - 1 = k - 1$, and $|P_f| \geqslant |Q_f| + 1$. In this case, the number of isolated vertices in $P_n[C_f]$ is $\frac{n+1}{2}$, and $I = C_f$. So $|M_f| = \frac{n+1}{2} = k$. Thus $\gamma_{kt}^-(P_n) \geqslant \frac{k}{2} - k = -\frac{k}{2}$, by the integrity of $\gamma_{kt}^-(P_n)$, we have $\gamma_{kt}^-(P_n) \geqslant -\lfloor \frac{k}{2} \rfloor$. If $k < \frac{n+1}{2}$, similar to Subcase 1.1.1, we can obtain $\gamma_{kt}^-(P_n) \geqslant \lceil \frac{3k}{2} \rceil - n$.

Case 1.2 $I \subset C_f$.

In this case, we have

$$C_f - I \subset P_f \cup Q_f. \tag{1}$$

Furthermore, for every vertex $v \in I$, we have

$$N(v) \subseteq P_f \cup Q_f \text{ and } |N(v)| \in \{1, 2\}. \tag{2}$$

It follows from (1) and (2) that $|P_f| + |Q_f| \geqslant |C_f| \geqslant k$. Obviously, $|P_f| \geqslant |Q_f|$, otherwise we can find another minimum MTkSF f' with more vertices of -1. Then $|M_f| \leqslant n - k$. In a way similar to that in Subcase 1.1.1, we can obtain $\gamma_{kt}^-(P_n) \geqslant \lceil \frac{3k}{2} \rceil - n$. Consequently,

$$\gamma_{kt}^-(P_n) \geqslant \begin{cases} -\left\lfloor \dfrac{k}{2} \right\rfloor, & \text{if } n \text{ is odd and } k = \frac{n+1}{2}, \\ \left\lceil \dfrac{3k}{2} \right\rceil - n, & \text{otherwise.} \end{cases}$$

On the other hand, define a function $g : V \to \{-1, 0, 1\}$ in different cases as follows:

(i) $k \leqslant \frac{n}{2}$. Define

$$g(v_i) = \begin{cases} 1, & \text{if } 2 \leqslant i \leqslant 2k, \text{ and } i = 4j-2,\ j \in \mathbf{Z}^+ \\ & \text{and } 1 \leqslant j \leqslant \frac{k+1}{2}, \\ 0, & \text{if } 2 \leqslant i \leqslant 2k, \text{ and } i = 4j,\ j \in \mathbf{Z}^+ \\ & \text{and } 1 \leqslant j \leqslant \frac{k}{2}, \\ -1, & \text{otherwise.} \end{cases}$$

Then g is an MTkSF of P_n with weight $g(V) = \lceil \frac{k}{2} \rceil - (n - \lceil \frac{k}{2} \rceil - \lfloor \frac{k}{2} \rfloor) = \lceil \frac{3k}{2} \rceil - n$.

(ii) n is odd and $k = \frac{n+1}{2}$. Define

$$g(v_i) = \begin{cases} 1, & \text{if } i = 4j-2, \text{ and the vertex } v_{n-1}, \\ & j \in \mathbf{Z}^+ \text{ and } 1 \leqslant j \leqslant \frac{n-1}{4}, \\ 0, & \text{if } i = 4j,\ j \in \mathbf{Z}^+ \text{ and } 1 \leqslant j \leqslant \frac{n-3}{4}, \\ -1, & \text{otherwise.} \end{cases}$$

Then g is an MTkSF of P_n with weight $g(V) = \lceil \frac{k}{2} \rceil - k = -\lfloor \frac{k}{2} \rfloor$.

(iii) n is even and $\frac{n}{2} < k \leqslant n-1$. In the following subcases, we define an MTkSF g with the number of -1 being $n-k$, and the number of vertices in C_f being $n - k + n - 2(n-k) = k$.

(a) k is odd, and then $n-k$ is odd. If $2k - n \equiv 0$(mod 4), then defining g by

$$(g(v_1), \cdots, g(v_n)) = (\underbrace{\overbrace{-1, 1, -1, 0}, \cdots, -1, 1, -1, 0}, \underbrace{\overbrace{1, 1, 0, 0}, \cdots, 1, 1, 0, 0}, 1, -1),$$

we can check that g is an MTkSF of weight $g(V) = \frac{n-k+1}{2} + \frac{2k-n}{2} - (n-k) = \frac{3k+1}{2} - n$. If $2k - n \equiv 2$ (mod 4), then defining g by

$$(g(v_1), \cdots, g(v_n)) = (\underbrace{\overbrace{-1, 1, -1, 0}, \cdots, -1, 1, -1, 0}, \underbrace{\overbrace{0, 1, 1, 0}, \cdots, 0, 1, 1, 0}, 0, 1, 1, -1),$$

we can check that g is an MTkSF of weight $g(V) = \frac{n-k+1}{2} + \frac{2k-n-2}{2} + 1 - (n-k) = \frac{3k+1}{2} - n$.

(b) k is even, and then $n-k$ is even. If $2k-n\equiv 0 \pmod 4$, then defining g by

$$(g(v_1),\cdots,g(v_n))=(\underbrace{\overbrace{-1,1,-1,0},\cdots,-1,1,-1,0},$$

$$\underbrace{\overbrace{0,1,1,0},\cdots,0,1,1,0}),$$

then the weight $g(V)=\frac{n-k}{2}+\frac{2k-n}{2}-(n-k)=\frac{3k}{2}-n$. If $2k-n\equiv 2 \pmod 4$, then defining g by

$$(g(v_1),\cdots,g(v_n))=(\underbrace{\overbrace{-1,1,-1,0},\cdots,-1,1,-1,0},$$

$$-1,1,\underbrace{\overbrace{0,0,1,1},\cdots,0,0,1,1},0,0,1,-1),$$

we can check that g is an MTkSF with weight $g(V)=\frac{n-k-2}{2}+2+\frac{2k-n-2}{2}-(n-k)=\frac{3k}{2}-n$.

(iv) n is odd and $\frac{n+1}{2}<k\leqslant n-1$. In the following subcases, we also define an MTkSF g with the number of -1 being $n-k$, and the number of vertices in C_f is $n-k+n-2(n-k)=k$.

(a) k is odd. Then $n-k$ is even. If $2k-n\equiv 1 \pmod 4$, then defining g by

$$(g(v_1),\cdots,g(v_n))=(\underbrace{\overbrace{-1,1,-1,0},\cdots,-1,1,-1,0},$$

$$-1,1,-1,1,\underbrace{\overbrace{0,0,1,1},\cdots,0,0,1,1},0),$$

we can check that g is an MTkSF of weight $g(V)=\frac{n-k}{2}+1+\frac{2k-n-1}{2}-(n-k)=\frac{3k+1}{2}-n$. If $2k-n\equiv 3 \pmod 4$, then defining g by

$$(g(v_1),\cdots,g(v_n))=(\underbrace{\overbrace{-1,1,-1,0},\cdots,-1,1,-1,0},$$

$$\underbrace{\overbrace{0,1,1,0},\cdots,0,1,1,0},0,1,1),$$

we can check that g is an MTkSF with weight $g(V)=\frac{n-k}{2}+\frac{2k-n+1}{2}-(n-k)=\frac{3k+1}{2}-n$.

(b) k is even. Then $n-k$ is odd. If $2k-n\equiv 1 \pmod 4$, then defining g by

$$(g(v_1),\cdots,g(v_n))=(\underbrace{\overbrace{-1,1,-1,0},\cdots,-1,1,-1,0},$$

$$-1,1,\underbrace{\overbrace{0,0,1,1},\cdots,0,0,1,1},0),$$

we can check that g is an MTkSF of weight $g(V)=\frac{n-k+1}{2}+\frac{2k-n-1}{2}-(n-k)=\frac{3k}{2}-n$. If $2k-n\equiv 3 \pmod 4$, then defining g by

$$(g(v_1),\cdots,g(v_n))=(\underbrace{\overbrace{-1,1,-1,0},\cdots,-1,1,-1,0},$$

$$-1,1,\underbrace{\overbrace{0,0,1,1},\cdots,0,0,1,1},0,1,1).$$

It is easy to check that g is an MTkSF of weight $g(V)=\frac{n-k+1}{2}+\frac{2k-n-3}{2}+2-(n-k)=\frac{3k}{2}+1-n$. Next, we will prove that g is a minimum MTkSF in this subcase. Denote S as one component in $P_n[C_f]-I$. Supposing there exist components with consecutive -1 in P_n, we can put them together as one component T. Let us consider the component $R=S\bigcup\{v_i\}\bigcup T$ in P_n, where v_i is the vertex linking S and T. Then $f(v_i)\geqslant 0$, so $f(R)$ is at least $\frac{|S|}{2}-|T|$. On the other hand, we denote $|R|=n'=|S|+|T|+1$ and $k'=|S|$. Through the above analysis, we can find an MTkSF g' with the number of -1 being $|T|+2$ or $n'-k'=|T|+1\geqslant|T|$ and the weight at most $\frac{3k'}{2}+1-n'=\frac{|S|}{2}-|T|\leqslant\frac{|S|}{2}-|T|$, a contradiction. Using the same method, we can also deduce that there do not exist vertices in I between S and T. So there is not a component with consecutive -1 in P_n. Consequently, we can check that g is a minimum MTkSF with the number of -1 large enough, and then $\gamma_{kt}^{-}(P_n)=\frac{3k}{2}+1-n$ in this subcase.

Finally, according to $\gamma_{kt}^{-}(P_n)\leqslant g(V)$, consequently,

$$\gamma_{kt}^{-}(P_n)=\begin{cases}-\left\lfloor\dfrac{k}{2}\right\rfloor, & \text{if } n \text{ is odd and } k=\frac{n+1}{2},\\ \dfrac{3k}{2}+1-n, & \text{if } n \text{ is odd, } k \text{ is even, and}\\ \quad 2k-n\equiv 3 \pmod 4, & \\ \left\lceil\dfrac{3k}{2}\right\rceil-n, & \text{otherwise.}\end{cases}$$

Next we will discuss the value of $\gamma_t^{-}(P_n)$, *i.e.*, the case of $\gamma_{kt}^{-}(P_n)$ with $k=n$. When $k=n$, no vertices in P_n are assigned the values of -1. Otherwise, at least one vertex v of P_n adjacent to the vertex with the value of -1 does not satisfy $f(N(v))\geqslant 1$. So we can easily obtain the following proposition.

Proposition 1.1 For any path P_n $(n\geqslant 2)$,

$$\gamma_t^{-}(P_n)=\begin{cases}\dfrac{n}{2} & \text{for } n\equiv 0 \pmod 4,\\ \dfrac{n+1}{2} & \text{for } n\equiv 1 \pmod 4,\\ \dfrac{n+2}{2} & \text{for } n\equiv 2 \pmod 4,\\ \dfrac{n+1}{2} & \text{for } n\equiv 3 \pmod 4.\end{cases}$$

Theorem 1.2 For any complete graph $K_n (n\geqslant 2)$,

$$\gamma_{kt}^{-}(K_n)=\begin{cases}0, & 1\leqslant k\leqslant \frac{n}{2},\\ 1, & \frac{n}{2}<k\leqslant n-1,\\ 2, & k=n.\end{cases}$$

Proof Let f be a minimum MTkSF on K_n.

Subcase 1.2.1 $1 \leqslant k \leqslant \frac{n}{2}$

Since there exists at least one vertex $v \in V$ with $f(N(v)) = f(V) - f(v) \geqslant 1$, it follows that $f(V) \geqslant f(v) + 1 \geqslant 0$. If n is even, we define $g_1 : V \to \{-1, 0, 1\}$ by

$$g_1(x) = \begin{cases} 1, & \dfrac{n}{2} \text{ vertices in } V, \\ -1, & \text{otherwise.} \end{cases}$$

And if n is odd, we define $g_2 : V \to \{-1, 0, 1\}$ by

$$g_2(x) = \begin{cases} 1, & \frac{n-1}{2} \text{ vertices in } V, \\ 0, & \text{one vertex in } V, \\ -1, & \text{otherwise.} \end{cases}$$

Then g_1 and g_2 are MTkSFs of K_n of weight $g_1(V) = g_2(V) = 0$, so $\gamma^-_{kt}(K_n) \leqslant g_1(V)$ or $g_2(V)$. Consequently, in this case $\gamma^-_{kt}(K_n) = 0$.

Subcase 1.2.2 $\frac{n}{2} < k \leqslant n$

Similar to Subcase 1.2.1, $|P_f| - |M_f| = f(V) \geqslant 0$. Since $|P_f| + |M_f| + |Q_f| = n$, $|P_f| + |Q_f| \geqslant (n + |Q_f|)/2 \geqslant n/2$. Since there exist at least k vertices $v \in V$ such that $f(N(v)) \geqslant 1$, $|C_f| \geqslant k > \frac{n}{2}$. Then there exists at least one vertex $u \in Q_f \cup P_f$ such that $f(N(u)) = f(V) - f(u) \geqslant 1$, so $f(V) \geqslant 1$ or 2. Especially, when $k = n$, then $f(V) \geqslant 2$. Supposing $n/2 < k \leqslant n - 1$, we define $g_1 : V \to \{-1, 0, 1\}$ by

$$g_1(x) = \begin{cases} 1, & \text{any one vertex in } V, \\ 0, & \text{otherwise.} \end{cases}$$

Then g_1 is an MTkSF of K_n of weight $g_1(V) = 1$, so $\gamma^-_{kt}(K_n) \leqslant g_1(V) = 1$. Therefore, in this subcase, $\gamma^-_{kt}(K_n) = 1$. Supposing $k = n$, we define $g_2 : V \to \{-1, 0, 1\}$ by

$$g_2(x) = \begin{cases} 1, & \text{any two vertices in } V, \\ 0, & \text{otherwise.} \end{cases}$$

Then g_2 is an MTkSF of K_n of weight $g_2(V) = 2$, so $\gamma^-_{kt}(K_n) \leqslant g_2(V) = 2$. Therefore, in this subcase, $\gamma^-_{kt}(K_n) = 2$.

Theorem 1.3 For any complete bipartite graph $K_{m,n} (n \geqslant m \geqslant 1)$,

$$\gamma^-_{kt}(K_{m,n}) = \begin{cases} 1 - n, & \text{if } 1 \leqslant k \leqslant n, \\ 2, & \text{if } n < k \leqslant m + n. \end{cases}$$

Proof Let $K_{m,n} = (V, E)$, X and Y be the bipartite sets of $K_{m,n}$ with $|X| = m$ and $|Y| = n$. Among all the minimum MTkSFs on $K_{m,n}$, let f be one that assigns the value -1 to vertices of Y as many as possible. Denote $X^+ = \{v \in X \mid f(v) = 1\}$, $X^- = \{v \in X \mid f(v) = -1\}$, and $X^0 = \{v \in X \mid f(v) = 0\}$. Denote $Y^+ = \{v \in Y \mid f(v) = 1\}$, $Y^- = \{v \in Y \mid f(v) = -1\}$, and $Y^0 = \{v \in Y \mid f(v) = 0\}$. Then $\gamma^-_{kt}(K_{m,n}) = f(V) = f(X) + f(Y) = |X^+| - |X^-| + |Y^+| - |Y^-|$.

Case 1.3 $1 \leqslant k \leqslant n$

We show that in this case $Y = Y^-$, *i.e.*, each vertex of Y is assigned the value -1 under f. Assume to the contrary that $Y^0 \cup Y^+ \neq \emptyset$.

If $f(X) \geqslant 1$, then let $f_1 : V \to \{-1, 0, 1\}$ be defined as follows: $f_1(v) = -1$ if $v \in Y^0 \cup Y^+$ and $f_1(v) = f(v)$ if $v \notin Y^0 \cup Y^+$. Since $f_1(N(\omega)) = f(X) \geqslant 1$ for each $\omega \in Y$, it follows that f_1 is an MTkSF on $K_{m,n}$ of weight less than that of f, which is a contradiction.

If $f(X) \leqslant 0$, then $|X^+| \leqslant |X^-|$. Since there exist k vertices v of V such that $f(N(v)) \geqslant 1$, it follows that $f(Y) \geqslant 1$, *i.e.*, $|Y^+| > |Y^-|$. Then $|Y^+| > \frac{1}{2}(|Y| - |Y^0|)$, so $|Y^+| + |Y^0| > \frac{1}{2}(|Y| + |Y^0|) \geqslant \frac{1}{2}|Y| \geqslant \frac{1}{2}|X|$. Let $f_2 : V \to \{-1, 0, 1\}$ be defined as follows: $f_2(v) = -1$ for $\lceil (|X| + 1)/2 \rceil$ vertices v of Y^+ or Y^0, $f_2(v) = 1$ for $\lceil (|X| - |X^0| + 1)/2 \rceil$ vertices u of X, and $f_2(v) = f(v)$ for all remaining vertices v of V. Since $f_2(N(y)) = f_2(X) \geqslant 1$ for each $y \in Y$, it follows that f_2 is an MTkSF on $K_{m,n}$ of weight $f_2(V) \leqslant f(V)$. However, f_2 assigns the value -1 to more vertices of Y than f does, contrary to our choice of f. We deduce, therefore, that $Y = Y^-$.

Let y be a vertex in Y for which $f(N(y)) \geqslant 1$. Then $|X^+| - |X^-| = f(X) = f(N(y)) \geqslant 1$. Thus $\gamma^-_{kt}(K_{m,n}) = |X^+| - |X^-| + |Y^+| - |Y^-| \geqslant 1 - n$.

Next we define an MTkSF $g : V \to \{-1, 0, 1\}$ by

$$g(x) = \begin{cases} 1, & \text{one vertex in } X, \\ 0, & \text{other vertices in } X, \\ -1, & \text{otherwise.} \end{cases}$$

Then g is an MTkSF of $K_{m,n}$ with weight $g(V) = 1 - n$, and $\gamma^-_{kt}(K_{m,n}) \leqslant g(V) = 1 - n$. Consequently, if $k \leqslant n$, $\gamma^-_{kt}(K_{m,n}) = 1 - n$.

Case 1.4 $n < k \leqslant m + n$

In this case, there exist $y \in Y$ and $x \in X$ such that $f(N(y)) \geqslant 1$ and $f(N(x)) \geqslant 1$. Then $f(X) = f(N(y)) \geqslant 1$ and $f(Y) = f(N(x)) \geqslant 1$. Thus $\gamma^-_{kt}(K_{m,n}) = f(X) + f(Y) \geqslant 2$. We now define an MT$k$SF $g : V \to \{-1, 0, 1\}$ as follows:

$$g(x) = \begin{cases} 1, & \text{one vertex in } X \text{ and one vertex in } Y, \\ 0, & \text{otherwise.} \end{cases}$$

Then g is an MTkSF of $K_{m,n}$ with weight 2, so $\gamma^-_{kt}(K_{m,n}) \leqslant g(V) = 2$. Consequently, if $n < k \leqslant m + n$, $\gamma^-_{kt}(K_{m,n}) = 2$.

Corollary 1.1 For any star $K_{1,n-1} (n \geqslant 2)$,

$$\gamma^-_{kt}(K_{1,n-1}) = \begin{cases} 2 - n, & \text{if } 1 \leqslant k \leqslant n - 1, \\ 2, & \text{if } k = n. \end{cases}$$

2 Lower bounds on γ_{kt}^-

Lemma 2.1 For any tree $T=(V,E)$ on n vertices ($n\geqslant 2$), $\gamma_t^-(T)\geqslant 2$, and the equality holds if only if each vertex v of T is an odd vertex and v is at least adjacent to $\frac{d_T(v)-1}{2}$ leaves of T.

Proof Let f be any minimum minus total daminating function (MTDF) of T. If $M_f=\emptyset$, obviously, $\gamma_t^-(G)\geqslant 2$. So we may assume there exists a vertex $v\in M_f$. Let T be rooted at v. Since $f(N(v))\geqslant 1$, at least one adjacent vertex x of v is assigned $+1$ under f. On the other hand, $f(N(x))\geqslant 1$ and $f(v)=-1$, so at least two adjacent vertices x_1, x_2 of x are assigned $+1$ under f. If $M_f=\{v\}$, we have $\gamma_t^-(T)=|P_f|-|M_f|\geqslant 3-1=2$. If $M_f-\{v\}\neq\emptyset$, let $y_1\in M_f-\{v\}$, and y_1 be a child of vertex y. Since $f(N(y))\geqslant 1$, there exists at least one brother y_2 of y_1 that belongs to the set P_f. Consequently, we have $|P_f|\geqslant |M_f|+2$. Thus $\gamma_t^-(T)=|P_f|-|M_f|\geqslant 2$.

From the definitions of STDN and MTDN, it is easily seen that $\gamma_t^s(G)\geqslant\gamma_t^-(G)$ for a graph G. Also, we know from [3] that $\gamma_t^s(T)=2$ if and only if each vertex v of T is an odd vertex and v is at least adjacent to $(d_T(v)-1)/2$ leaves of T. So, in this case, we obtain $\gamma_t^-(T)\leqslant\gamma_t^s(G)=2$, and then $\gamma_t^-(G)=2$. Consequently, the bound is sharp.

Theorem 2.1 For any tree T on n vertices ($n\geqslant 2$),

$$\gamma_{kt}^-(T)\geqslant\begin{cases}2-n, & \text{if } 1\leqslant k\leqslant n-1,\\ 2, & \text{if } k=n,\end{cases}$$

with equality for $T=K_{1,n-1}$.

Proof Let f be any minimum MTkSF on T. By the definition of MTkSF, we know that $f(N(v))\geqslant 1$ for at least one vertex v of T. Then at least one adjacent vertex x of v is assigned $+1$ under f, so we have $\gamma_{kt}^-(T)\geqslant 1-(n-1)=2-n$. Thus, by Lemma 2.1 and Corollary 1.1, the result is true.

Theorem 2.2 For any graph G of order n with maximum degree $\varDelta$ and minimum degree $\delta\geqslant 1$,

$$\gamma_{kt}^-(G)\geqslant\frac{(\delta-3\varDelta)n+2(\varDelta+1)k}{\varDelta+\delta}n.$$

Proof Let f be a minimum MTkSF on $G=(V,E)$. Then $P_f=P_\varDelta\cup P_\delta\cup P_\varTheta$, where $P_\varDelta$ and P_δ are sets of all vertices of P_f with degree equal to $\varDelta$ and δ, respectively, and $P_\varTheta$ contains all other vertices in P_f. We define $M_f=M_\varDelta\cup M_\delta\cup M_\varTheta$ and $Q_f=Q_\varDelta\cup Q_\delta\cup Q_\varTheta$ similarly to P_f. Further, for $i\in\{\varDelta,\delta,\varTheta\}$, let V_i be defined by $V_i=P_i\cup M_i\cup Q_i$. Then $n=|V_\varDelta|+|V_\delta|+|V_\varTheta|$. Since for at least k vertices $v\in V$, $f(N(v))\geqslant 1$, we have

$$\sum_{v\in V}f(N(v))\geqslant k-\varDelta(n-k)=(\varDelta+1)k-\varDelta n.$$

The sum $\sum\limits_{v\in V}f(N(v))$ counts the value $f(v)$ exactly $d(v)$ times for each vertex $v\in V$, *i.e.*, $\sum\limits_{v\in V}f(N(v))=\sum\limits_{v\in V}f(v)d(v)$. Thus

$$\sum_{v\in V}f(v)d(v)\geqslant(\varDelta+1)k-\varDelta n.$$

Dividing the sum up into the six summations and replacing $f(v)$ with the corresponding value of 1, -1 and 0 yield

$$\sum_{v\in P_\varDelta}d(v)+\sum_{v\in P_\delta}d(v)+\sum_{v\in P_\varTheta}d(v)-\sum_{v\in M_\varDelta}d(v)-\sum_{v\in M_\delta}d(v)-\sum_{v\in M_\varTheta}d(v)\geqslant(\varDelta+1)k-\varDelta n.$$

Thus we have

$$\varDelta|P_\varDelta|+\delta|P_\delta|+(\varDelta-1)|P_\varTheta|-\varDelta|M_\varDelta|-\delta|M_\delta|-(\delta+1)|M_\varTheta|\geqslant(\varDelta+1)k-\varDelta n.$$

For $i\in\{\varDelta,\delta,\varTheta\}$, we replace $|P_i|$ with $|V_i|-|M_i|-|Q_i|$ in the above inequality.
Therefore,

$$\begin{aligned}&\varDelta|V_\varDelta|+\delta|V_\delta|+(\varDelta-1)|V_\varTheta|\\&\quad\geqslant(\varDelta+1)k-\varDelta n+2\varDelta|M_\varDelta|+2\delta|M_\delta|+(\varDelta+\delta)|M_\varTheta|+\varDelta|Q_\varDelta|+\delta|Q_\delta|+(\varDelta-1)|Q_\varTheta|.\end{aligned}$$

It follows that

$$\begin{aligned}&2\varDelta n-(\varDelta+1)k\\&\quad\geqslant 2\varDelta|M_\varDelta|+2\delta|M_\delta|+(\varDelta+\delta)|M_\varTheta|+(\varDelta-\delta)|V_\delta|+|V_\varTheta|+\varDelta|Q_f|-(\varDelta-\delta)|Q_\delta|-|Q_\varTheta|\\&\quad=2\varDelta|M_\varDelta|+2\delta|M_\delta|+(\varDelta+\delta)|M_\varTheta|+(\varDelta-\delta)(|P_\delta|+|M_\delta|)+(|P_\varTheta|+|M_\varTheta|)+\varDelta|Q_f|.\\&\quad=2\varDelta|M_\varDelta|+(\varDelta+\delta)|M_\delta|+(\varDelta+\delta+1)|M_\varTheta|+(\varDelta-\delta)|P_\delta|+|P_\varTheta|+\varDelta|Q_f|\\&\quad\geqslant(\varDelta+\delta)|M_\varDelta|+(\varDelta+\delta)|M_\delta|+(\varDelta+\delta)|M_\varTheta|+\varDelta|Q_f|\\&\quad=(\varDelta+\delta)|M_f|+\varDelta|Q_f|.\end{aligned}$$

Therefore,

$$|M_f|\leqslant\frac{2\varDelta n-(\varDelta+1)k-\varDelta|Q_f|}{\varDelta+\delta}.$$

So

$$\begin{aligned}\gamma_{kt}^-(G)&=|P_f|-|M_f|=n-2|M_f|-|Q_f|\\&\geqslant n-2\frac{2\varDelta n-(\varDelta+1)k-\varDelta|Q_f|}{\varDelta+\delta}-|Q_f|\\&\geqslant\frac{(\delta-3\varDelta)n+2(\varDelta+1)k}{\varDelta+\delta}.\end{aligned}$$

By Theorem 2.2, we immediately obtain the following result:

Theorem 2.3 For every r-regular graph G of order n,

$$\gamma_{kt}^{-}(G) \geqslant \frac{(r+1)k - rn}{r}.$$

In particular, if $k = n$, then we have

Corollary 2.1 For every r-regular graph $G = (V, E)$ of order n,

$$\gamma_{t}^{-}(G) \geqslant \frac{n}{r}$$

and this bound is sharp.

Note The lower bound is sharp by considering a complete bipartite graph $K_{r,r}$. According to Theorem 1.3, we know $\gamma_{kt}^{-}(K_{r,r}) = 2 = (r+r)/r = n/r$.

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