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On σ-subnormal and σ-permutable subgroups of finite groups

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ABSTRACT

Let $\sigma = \{\sigma_i | i \in I\}$ be some partition of the set $\mathbb{P}$ of all primes, that is, $\mathbb{P} = \cup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. Let G be a finite group. We say that G is: *σ-primary* if G is a σ_i-group for some $i \in I$; *σ-soluble* if every chief factor of G is σ-primary. We say that a set $\mathcal{H} = \{H_1, \dots, H_t\}$ of Hall subgroups of G, where H_i is σ-primary ($i = 1, \dots, t$), is a *complete Hall set of type σ* of G if $(|H_i|, |H_j|) = 1$ for all $i \neq j$ and $\pi(G) = \pi(H_1) \cup \dots \cup \pi(H_t)$. We say that a subgroup A of G is: *σ-subnormal* in G if there is a subgroup chain $A = A_0 \leq A_1 \leq \dots \leq A_n = G$ such that either A_{i-1} is normal in A_i or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \dots, t$; *σ-permutable* in G if G has a complete Hall set $\mathcal{H}$ of type σ such that A is $\mathcal{H}^G$-permutable in G, that is, $AH^x = H^xA$ for all $x \in G$ and all $H \in \mathcal{H}$. We study the relationship between the σ-subnormal and σ-permutable subgroups of G. In particular, we prove that every σ-permutable subgroup of G is σ-subnormal, and we classify finite σ-soluble groups in which every σ-subnormal subgroup is σ-permutable.

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1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. If n is an integer, then the symbol $\pi(n)$ denotes the set of all primes dividing $|n|$; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G.

In what follows, $\sigma = \{\sigma_i | i \in I\}$ is some partition of $\mathbb{P}$, that is, $\mathbb{P} = \cup_{i\in I}\sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$.

We put $\sigma(n) = \{\sigma_i \cap \pi(n) \mid i \in I, \sigma_i \cap \pi(n) \neq \emptyset\}$, $\sigma(G) = \sigma(|G|)$ and we say that G is *σ-primary* if either $G = 1$ or $|\sigma(G)| = 1$.

A set $\mathcal{S}$ of Sylow subgroups of G is called a *complete set of Sylow subgroups* of G if $\mathcal{S}$ contains exact one Sylow p-subgroup of G for every prime p dividing $|G|$. By analogy with it, we say that a set $\mathcal{H} = \{H_1, \dots, H_t\}$ of Hall subgroups of G, where H_i is σ-primary ($i = 1, \dots, t$), is a *complete Hall set of G of type σ* if $(|H_i|, |H_j|) = 1$ for all $i \neq j$ and $\pi(G) = \pi(H_1) \cup \dots \cup \pi(H_t)$. Following [1], the group G is a *σ-group* if it possesses a complete Hall set of type σ.

Let $\mathcal{L}$ be some non-empty set of subgroups of G. Then a subgroup A of G is called *$\mathcal{L}$-permutable* if $AH = HA$ for all $H \in \mathcal{L}$; *$\mathcal{L}^G$-permutable* if $AH^x = H^xA$ for all $H \in \mathcal{L}$ and all $x \in G$. If A is $\mathcal{L}^G$-permutable, where $\mathcal{L}$ is a complete set of Sylow subgroups of G, then A is called *S-quasinormal* or *S-permutable* in G. By analogy with it, we say that a subgroup A of G is *σ-permutable* in G if G has a complete Hall set $\mathcal{H}$ of type σ such that A is $\mathcal{H}^G$-permutable.

The S-permutable subgroups possess a series of interesting properties and they are closely related to subnormal subgroups. For instance, if H is an S-permutable subgroup of G, then H is subnormal in G (Kegel [2]) and the quotient H/H_G is nilpotent (Deskins [3]) and so, in fact, H^G/H_G is nilpotent. Moreover, H/H_G is nilpotent also in the case when H is subnormal and permutes with all members of some complete set of Sylow subgroups of G [4]. The S-permutable subgroups of G form a sublattice of the lattice of all subnormal subgroups of G (Kegel [2]). The description of PST-groups, that are groups, in which every subnormal subgroup is S-permutable, was first obtained by Agrawal [5], for the soluble case, and by Robinson in [6], for the general case. In the further publications, authors (see, for example, the recent papers [7–16]) have found out and described many other interesting characterizations of soluble PST-groups.

In fact, the following concept is a modification of the main concept in [17].

Definition 1.1. We say that a subgroup A of G is *σ-subnormal* in G if there is a subgroup chain $A = A_0 \leq A_1 \leq \dots \leq A_n = G$ such that either A_{i-1} is normal in A_i or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \dots, n$.

Before continuing, let's consider the following elementary example.

Example 1.2. Let p, q, r, t be different primes, where q and r divide $p - 1$. Let A be a non-abelian group of order pr, T a group of order t and $P \rtimes Q$ a non-abelian group of order pq. Let $V = (P \rtimes Q) \wr T = K \rtimes T$, where K is the base group of the regular wreath product V, and let $H = Q^\natural$ and $L = P^\natural$ (we use here the terminology in [18, A]). Let $G = V \wr A = K_0 \rtimes A$, where K_0 is the base group of the regular wreath product G. If $\sigma = \{\{p, q\}, \{p, q\}'\}$, then the subgroup H is σ-subnormal in G (see Lemma 2.6(6) below) but it is not subnormal in G.

Now, let $\sigma = \{\{p, r\}, \{p, r\}'\}$. It is clear that L is normal in K_0, so L is σ-permutable in G (see Lemma 3.1 below). Finally, suppose that L is S-permutable in G and let R be a Sylow r-subgroup of A. Then for every $x \in G$ we have $LR^x = R^x L$, so $L = LR^x \cap K_0$ is normal in LR^x. Therefore $G = K_0 R^G \leq N_G(L)$, contrary to [18, A, 18.5(a)]. Hence L is not S-permutable in G.

The results in [5–16] are the motivations for the following

Question. Let G be a σ-group. What is the structure of G provided that every σ-subnormal subgroup of G is σ-permutable?

We do not know the answer to this question in the case of an arbitrary σ-group G. But we give a complete classification of such groups in the case of σ-soluble groups.

Definition 1.3. We say that G is: *σ-soluble* if every chief factor of G is σ-primary; *σ-nilpotent* if $(H/K) \rtimes (G/C_G(H/K))$ is σ-primary for every chief factor H/K of G.

Note that every σ-nilpotent group is also σ-soluble, and G is σ-soluble if and only if it is σ_i-separable for all $i \in I$; G is soluble (respectively nilpotent) if and only if it is σ-soluble (respectively σ-nilpotent), where σ is the smallest partition of $\mathbb{P}$, that is, for any $i \in I$, σ_i is a one-element set. Finally note that G is π-separable if and only if it is σ-soluble, where $\sigma = \{\pi, \pi'\}$.

We use $\mathfrak{N}_\sigma$ to denote the class of all σ-nilpotent groups. The symbol $\mathrm{G}^{\mathfrak{N}_\sigma}$ denotes the σ-nilpotent residual of G, that is, the intersection of all normal subgroups N of G with σ-nilpotent quotient G/N.

We say that a Hall subgroup H of G is a *σ-Hall* subgroup of G provided $\sigma(H) \subseteq \sigma(G)$.

Theorem A. *Let G be a σ-soluble group. Then every σ-subnormal subgroup of G is σ-permutable if and only if $G = D \rtimes M$, where $D = G^{\mathfrak{N}_\sigma}$ is an abelian σ-Hall subgroup of odd order of G such that every element of M induces a power automorphism of D.*

The proof of Theorem A is based on many observations about σ-subnormal and σ-permutable subgroups obtained in this paper. In particular, we use the following result, which generalizes the above-mentioned results in [2,3].

Theorem B. *Let G be a σ-group and $\mathcal{H} = \{H_1, \ldots, H_t\}$ be a complete Hall set of type σ of G. Let $D = G^{\mathfrak{N}_\sigma}$ and H be a subgroup of G.*

(i) *If H is $\mathcal{H}^D$-permutable, then H is σ-subnormal in G.*

(ii) *If H is $\mathcal{H}^G$-permutable, then H^G/H_G is σ-nilpotent and $D \leq N_G(H)$.*

We say that G is a *σ-group of Sylow type* if every subgroup of G is a D_{σ_i}-group for all $i \in I$. Note that by the Feit–Thompson Odd Theorem and Theorem 3.6 in [19, 6], every σ-soluble group is a σ-group of Sylow type. Note also that the group $PSL(2,11)$ and the Mathieu group M_{11} are σ-groups of Sylow type, where $\sigma = \{\sigma_i \mid i \in I\}$ such that $\sigma_1 = \{5, 11\}$ and σ_i is a one-element set for all $i \neq 1$; the Lyons group Ly is a σ-groups of Sylow type, where $\sigma = \{\sigma_i \mid i \in I\}$ such that $\sigma_1 = \{11, 67\}$ and σ_i is a one-element set for all $i \neq 1$.

As another application of Theorem B, we prove also the following result.

Theorem C. *Let G be a σ-group of Sylow type. Then the set of all σ-permutable subgroups of G forms a sublattice of the lattice of all σ-subnormal subgroups of G.*

Note that Theorem C not only generalizes the above mentioned Kegel result on the lattice of the S-permutable subgroups but also gives a shorter proof of it.

Finally, note that all results of this paper remain to be new in the universe of all soluble groups.

2. Basic lemmas and propositions

We use $\mathfrak{S}_\sigma$ to denote the class of all σ-soluble groups.

Direct calculations show that the following lemma is true.

Lemma 2.1. (i) *The class $\mathfrak{S}_\sigma$ is closed under taking direct products, homomorphic images and subgroups. Moreover, any extension of the σ-soluble group by a σ-soluble group is a σ-soluble group as well.*

(ii) *$\mathfrak{S}_\sigma \subseteq \mathfrak{S}_{\sigma^*}$ for any partition $\sigma^* = \{\sigma_j^* \mid j \in J\}$ of $\mathbb{P}$ such that $J \subseteq I$ and $\sigma_j \subseteq \sigma_j^*$ for all $j \in J$.*

Let A, B and R be subgroups of G. Then A is said to *R-permute* with B [20] if for some $x \in R$ we have $AB^x = B^x A$.

In what follows, Π is always supposed to be a subset of the set σ and $\Pi' = \sigma \setminus \Pi$.

We say that: *n is a Π-number* if $\pi(n) \subseteq \cup_{\sigma_i \in \Pi} \sigma_i$; a subgroup H of G is a *Π-subgroup of G* if $|H|$ is a Π-number; a σ-Hall subgroup H of G is a *Hall Π-subgroup* of G if H is a Π-subgroup of G and $|G : H|$ is a Π'-number.

If G has a complete Hall set $\mathcal{H} = \{H_1, \ldots, H_t\}$ of type σ such that $H_iH_j = H_jH_i$ for all i, j, then we say that $\mathcal{H}$ is a *σ-basis* of G.

By the classical Hall theorem, G is soluble if and only if it has a Sylow basis. The direct analogue of this result for σ-soluble groups is not true in general. Indeed, let $\sigma = \{\{2, 3\}, \{2, 3\}'\}$. Then the alternating group A_5 of degree 5 has a σ-basis and it is not σ-soluble. Nevertheless, the following generalization of the Hall result is true.

Proposition 2.2. *(See [21,22].) Let R be the product of all normal σ-soluble subgroups of G. Then any two of the following conditions are equivalent:*

(i) *G is σ-soluble.*

(ii) *For any Π, G has a Hall Π-subgroup E and every Π-subgroup of G is contained in a conjugate of E. Moreover, every σ-Hall subgroup of G R-permutes with every Sylow subgroup of G.*

(iii) *G has a σ-basis $\{H_1, \dots, H_t\}$ such that for each $i \neq j$ every Sylow subgroup of H_i R-permutes with every Sylow subgroup of H_j.*

Let $\mathfrak{F}$ be a class of groups. A chief factor H/K of G is called *$\mathfrak{F}$-central* in G provided $(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{F}$.

Proposition 2.3. *(See [21,22].) Any two of the following conditions are equivalent:*

(i) *G is σ-nilpotent.*

(ii) *Every chief factor of G is $\mathfrak{N}_\sigma$-central.*

(iii) *G has a complete Hall set $\mathcal{H} = \{H_1, \dots, H_t\}$ of type σ such that $G = H_1 \times \dots \times H_t$.*

(iv) *G has a complete Hall set $\mathcal{H} = \{H_1, \dots, H_t\}$ of type σ such that every member of $\mathcal{H}$ is σ-subnormal in G.*

(v) *Every subgroup of G is σ-subnormal in G.*

(vi) *Every maximal subgroup of G is σ-subnormal in G.*

Corollary 2.4. *If A and B are normal σ-nilpotent subgroups of G, then AB is σ-nilpotent.*

Proof. Without loss of generality we can assume that $AB = G$. Then, by Lemma 2.1, G is σ-soluble. Hence by Proposition 2.2, G has a complete Hall set $\mathcal{H} = \{H_1, \dots, H_t\}$ of type σ. Without loss of generality we can assume that H_i is a σ_i-group for all $i = 1, \dots, t$. Then, by Proposition 2.3, $A = (A \cap H_1) \times \dots \times (A \cap H_t)$ and $B = (B \cap H_1) \times \dots \times (B \cap H_t)$, where $A \cap H_i$ and $B \cap H_i$ are Hall σ_i-subgroups of A and B, respectively. Then $(A \cap H_i)(B \cap H_i)$ is a normal Hall σ_i-subgroup of G, so G is σ-nilpotent. The corollary is proved. $\square$

We call the product of all normal σ-nilpotent subgroups of G the *σ-Fitting subgroup* of G and denote it by $F_\sigma(G)$.

Lemma 2.5. *The class $\mathfrak{N}_\sigma$ is closed under taking direct products, homomorphic images and subgroups. Moreover, if H is a normal subgroup of G and $H/H \cap \Phi(G)$ is σ-nilpotent, then E is σ-nilpotent.*

Proof. In view of Proposition 2.3 and Corollary 2.4, it is enough to prove that if $H/H \cap \Phi(G)$ is a σ_i-group, then H has a normal Hall σ_i-subgroup. Let $D = O_{\sigma_i{}'}(H)$. Then, since $H \cap \Phi(G)$ is nilpotent, D is a Hall $\sigma_i{}'$-subgroup of H. Hence by the Schur–Zassenhaus theorem, H has a Hall σ_i-subgroup, say E. It is clear that H is σ_i'-soluble, so any two Hall σ_i-subgroups of H are conjugated in H. Therefore by the Frattini Argument we have $G = HN_G(E) = (E(H \cap \Phi(G)))N_G(E) = N_G(E)$. Thus E is normal in G. The lemma is proved. $\square$

We use $O^{\Pi}(G)$ to denote the subgroup of G generated by all its Π'-subgroups. Instead of $O^{\{\sigma_i\}}(G)$ we write $O^{\sigma_i}(G)$.

Lemma 2.6. *Let A, K and N be subgroups of G. Suppose that A is σ-subnormal in G and N is normal in G.*

(1) *$A \cap K$ is σ-subnormal in K.*

(2) *If K is a σ-subnormal subgroup of A, then K is σ-subnormal in G.*

(3) *If K is σ-subnormal in G, then $A \cap K$ and $\langle A, K\rangle$ are σ-subnormal in G.*

(4) *AN/N is σ-subnormal in G/N.*

(5) *If $N \leq K$ and K/N is σ-subnormal in G/N, then K is σ-subnormal in G.*

(6) *If $K \leq A$ and A is σ-nilpotent, then K is σ-subnormal in G.*

(7) *If $H \neq 1$ is a Hall Π-subgroup of G and A is not a Π'-group, then $A \cap H \neq 1$ is a Hall Π-subgroup of A.*

(8) *If $|G : A|$ is a Π-number, then $O^{\Pi}(A) = O^{\Pi}(G)$.*

(9) *If N is a Π-group of G, then $N \leq N_G(O^{\Pi}(A))$.*

(10) *If A is a σ-*Hall subgroup *of G, then A is normal in G.*

(11) *If G is a σ-group and A is σ-nilpotent, then A is contained in $F_{\sigma}(G)$.*

Proof. Assume that this lemma is false and let G be a counterexample of minimal order. By hypothesis, there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_r = G$ such that either A_{i-1} is normal in A_i or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, r$. Let $M = A_{r-1}$. We can assume without loss of generality that $M \neq G$.

(1) Consider the chain $K_0 = K \cap A_0 \leq K \cap A_1 \leq \cdots \leq K \cap A_r = K$. If A_{i-1} is normal in A_i, then evidently $K \cap A_{i-1}$ is normal in $K \cap A_i$. Now suppose that $A_i/(A_{i-1})_{A_i}$ is a σ_i-group. Then $(A_i \cap K)(A_{i-1})_{A_i}/(A_{i-1})_{A_i} \simeq A_i \cap K/(A_{i-1})_{A_i} \cap K$ is a σ_i-group, where $(A_{i-1})_{A_i} \cap K$ is normal in $A_i \cap K$ and so $(A_{i-1})_{A_i} \cap K \leq (K \cap A_{i-1})_{K\cap A_i}$. Hence $(K \cap A_i)/(K \cap A_{i-1})_{K\cap A_i}$ is a σ_i-group. Hence $A \cap K$ is σ-subnormal in K.

Assertions (2) and (5) are evident.

(3) By Assertions (1) and (2), $A \cap K$ is σ-subnormal in G.

Now we show that $\langle A, K\rangle$ is σ-subnormal in G. A subgroup H of G is said to be $\mathfrak{F}$-*subnormal* in the sense of Kegel [17] or K-$\mathfrak{F}$-*subnormal* in G (see p. 236 in [23]) in G if there exists a chain of subgroups

$$H = H_0 \leq H_1 \leq \cdots \leq H_t = G$$

such that either H_{i-1} is normal in H_i or $H_i/(H_{i-1})_{H_i} \in \mathfrak{F}$ for all $i = 1, \ldots, t$. Kegel proved [17] that if $\mathfrak{F}$ is the class of all π-groups for some non-empty subset π of $\mathbb{P}$, then the set of all K-$\mathfrak{F}$-subnormal subgroups of G forms a sublattice of the lattice of all subgroups of G. Therefore from Proposition 2.3 and Theorem 6.3.9 and Lemma 6.3.11 in [23] it follows that the set of all K-$\mathfrak{N}_{\sigma}$-subnormal subgroups of G forms a sublattice of the lattice of all subgroups of G. Finally, note that in view of Proposition 2.3, a subgroup A of a σ-group G is σ-subnormal in G if and only if it is K-$\mathfrak{N}_{\sigma}$-subnormal in G. Therefore $\langle A, K\rangle$ is σ-subnormal in G.

(4) Consider the chain

$$AN/N = A_0N/N \leq A_1N/N \leq \cdots \leq A_rN/N = G/N.$$

Assume that $A_{i-1}N/N$ is not normal in A_iN/N. Then $L = A_{i-1}$ is not normal in $T = A_i$ and so T/L_T is a σ_i-group for some $i \in I$, by hypothesis. Then

$$(T/L_T)/(L_T(T \cap N)/L_T) = (T/L_T)/((T \cap NL_T)/L_T) \simeq T/(T \cap NL_T) \simeq$$
$$TN/L_TN \simeq (TN/N)/(L_TN/N)$$

is a σ_i-group. But $L_TN/N \leq (LN/N)_{TN/N}$. Hence $(TN/N)/(LN/N)_{TN/N}$ is σ-primary. Hence AN/N is σ-subnormal in G/N.

(6) Since A is σ-nilpotent, every subgroup of A is σ-subnormal in A by Proposition 2.3(v). Thus this assertion is a corollary of Assertion (2).

(7) First we show that $M \cap H \neq 1$ is a Hall Π-subgroup of M. If either $H \leq M$ or M is normal in G, it is evident. Assume that $K = M_G \neq M$ and $H \nleq M$. Then $|G : K|$ is a σ_i-number for some $i \in I$, where $\sigma_i \in \Pi$. Therefore, in the case when $K = 1$, G is a σ_i-group and so the assertion is true.

Now assume that $K \neq 1$. Then, by the choice of G, $(HK/K) \cap (M/K) = (H \cap M)K/K \neq 1$ is a Hall Π-subgroup of M/K. Hence $|M : K(H \cap M)|$ is a Π'-number. On the other hand, $H \cap K$ is a Hall Π-subgroup of K since K is normal in G. Therefore $|M : H \cap M| = |M : K(H \cap M)||K : K \cap H|$ is a Π'-number. Hence $M \cap H \neq 1$ is a Hall σ_i-subgroup of M.

Since A is σ-subnormal in M and $|M| < |G|$, the choice of G implies that $H \cap A = (H \cap M) \cap A$ is a Hall Π-subgroup of A.

(8) It is clear that $|M : A|$ and $|G : M|$ are Π-numbers. Moreover, A is σ-subnormal in M, so by the choice of G we have $O^{\Pi}(A) = O^{\Pi}(M)$.

Since $|G : M|$ is a Π-number, G/M_G is a Π-number. Therefore every Π'-subgroup of G is contained in M_G, so $O^{\Pi}(G) = O^{\Pi}(M) = O^{\Pi}(A)$.

(9) It is clear that $|AN : A|$ is a Π-number. On the other hand, A is σ-subnormal in AN by Assertion (1). Hence $N \leq N_G(O^{\Pi}(AN)) = N_G(O^{\Pi}(A))$ by Assertion (1).

(10) It is clear that A^x is a σ-subnormal σ-Hall subgroup of G for all $x \in G$. Therefore this assertion is a corollary of Assertion (7).

(11) In view of Proposition 2.3, it is enough to consider the case when A is a σ_i-group for some $i \in I$. By hypothesis, G has a Hall σ_i-subgroup, say H. Then, by Assertion (7), for any $x \in G$ we have $A \leq H^x$. Hence $A^G \leq H_G \leq F_\sigma(G)$.

The lemma is proved. $\square$

Lemma 2.7. *Let A and K be subgroups of G and H a Hall Π-subgroup of G. Suppose that $AH = HA$, A is σ-subnormal in AH and $H \cap K$ is a Hall Π-subgroup of K. Then $AH \cap K = (A \cap K)(H \cap K)$.*

Proof. Since $|AH : A| = |H : H \cap A|$ is a Π-number, $O^{\Pi}(A) = O^{\Pi}(AH)$ by Lemma 2.6(8). Hence every Π'-subgroup of AH is contained in A. Therefore $|(AH \cap K) : (A \cap K)|$ is a Π-number. But then $AH \cap K = (A \cap K)(H \cap K)$. The lemma is proved. $\square$

The following lemma also can be proved by direct calculations.

Lemma 2.8. *Let H, K and N be subgroups of a σ-group G. Let $\mathcal{H} = \{H_1, \dots, H_t\}$ be a complete Hall set of type σ of G and $\mathcal{L} = \mathcal{H}^K$. Suppose that H is $\mathcal{L}$-permutable and N is normal in G.*

(1) *If $H \leq E \leq G$, then H is $\mathcal{L}^*$-permutable, where $\mathcal{L}^* = \{H_1 \cap E, \dots, H_t \cap E\}^{K \cap E}$. In particular, if G is a σ-group of Sylow type and H is σ-permutable in G, then H is σ-permutable in E.*

(2) *The subgroup HN/N is $\mathcal{L}^{**}$-permutable, where $\mathcal{L}^{**} = \{H_1N/N, \dots, H_tN/N\}^{KN/N}$.*

(3) *If G is a σ-group of Sylow type and E/N is a σ-permutable subgroup of G/N, then E is σ-permutable.*

(4) *If K is $\mathcal{L}$-permutable, then $\langle H, K \rangle$ is $\mathcal{L}$-permutable* [18, A, 1.6(a)].

3. Proofs of Theorems A, B and C

Proof of Theorem B. Assume that this theorem is false and let (G, H) be a counterexample with $|G| + |G : H|$ as small as possible. Then $t > 1$ and $H^G \neq H$. Thus, one of the subgroups of $\mathcal{H}$ does not normalize H. We can assume that $H_1 \nleq N_G(H)$.

We start with two general claims.

Claim (1) $H_G = 1$.

Assume that $H_G \neq 1$. It is clear that $\{H_1H_G/H_G, \dots, H_tH_G/H_G\}$ is a complete Hall set of type σ of G/H_G and $DH_G/H_G = G^{\mathfrak{N}_\sigma}H_G/H_G = (G/H_G)^{\mathfrak{N}_\sigma}$. Therefore by Lemma 2.8(2) the hypothesis holds for G/H_G. If H is $\mathcal{H}^D$-permutable, then the choice of G implies that H/H_G is σ-subnormal in G/H_G, so H is σ-subnormal in G by Lemma 2.6(5). If H is $\mathcal{H}^G$-permutable, then H^G/H_G is σ-nilpotent and $DH_G/H_G \leq N_{G/H_G}(H/H_G) = N_G(H)/H_G$, so $D \leq N_G(H)$. Therefore $H_G = 1$.

Claim (2) *If H is $\mathcal{H}^G$-permutable, then $H_j^G \leq N_G(H)$ for all $j > 1$.*

We follow a similar argument to the one in the proof of Theorem A in [24] (see also the proof of Statement 2 of [25, 1.2.14]).

Assume that for some $x \in G$ and $j > 1$ we have $H_j^x \nleq N_G(H)$, and let $A = \langle H^{H_1} \rangle$ and $B = \langle H^{H_j^x} \rangle$. Then $A^G = H^G = B^G$ and $H < A \leq HH_1$ and $H < B \leq HH_j^x$, which implies that $H = A \cap B$ and so $1 = H_G = A_G \cap B_G$. Finally, A and B are $\mathcal{H}^G$-permutable by Lemma 2.8(4). Therefore the choice of G implies that A^G/A_G and B^G/B_G are σ-nilpotent, hence $H^G \simeq H^G/A_G \cap B_G$ is σ-nilpotent and so H is σ-subnormal in G by Lemma 2.6(6).

Proof of (i). We can assume without loss of generality that H_i is a σ_i-group for all $i = 1, \dots, t$. Let $E_i = H_i \cap HD$. Since $H_iH = HH_i$ by hypothesis, $H_i \cap H$ is a Hall σ_i-subgroup of H. Hence $E_i = (H_i \cap H)(H_i \cap D)$ and so $|HD : E_i|$ is a σ_i'-number. It follows that E_i is a Hall σ_i-subgroup of HD. Therefore $E_1, \dots, E_t$ is a complete Hall set of type σ of HD. Moreover, for any $x \in D$ we have

$$HH_i^x \cap HD = H(H_i^x \cap HD) = H(H_i \cap HD)^x = HE_i^x = E_i^x H$$

since $HH_i^x \cap HD$ is a subgroup of HD. Furthermore, since $HD/D \simeq H/H \cap D$ and $HD/D \leq G/D \in \mathfrak{N}_\sigma$, it follows that $(HD)^{\mathfrak{N}_\sigma} \leq D$ by Lemma 2.5. Therefore the hypothesis holds for HD.

If $HD < G$, H is σ-subnormal in HD, by minimality of $|G| + |G : H|$. By Lemmas 2.6(2)(5) and 2.5 we obtain a contradiction. Hence $G = HD$ and so, in fact, H is H^G-permutable. From Claim (2) we get that $H_j^G \leq N_G(H)$ for every $j > 1$. Let $E = H_2^G \cdots H_t^G$. Then $E \leq N_G(H)$ or, equivalently, H is normal in HE. On the other hand, $G/E \simeq H_1/(H_1 \cap E)$ and hence HE/E is σ-subnormal in G/E. By Lemma 2.6(5), HE is σ-subnormal in G. Thus H is σ-subnormal in G. This contradicts our assumption. Hence (i) holds.

Proof of (ii). By Part (i), H is σ-subnormal in G and then in HH_1 by Lemma 2.6(1). Hence $O^{\sigma_1}(H) = O^{\sigma_1}(HH_1)$ by Lemma 2.6(8). Therefore $H_1 \leq N_G(O^{\sigma_1}(H))$. By Claim (2), $H_j^G \leq N_G(H)$ and so $H_j^G \leq N_G(O^{\sigma_1}(H))$ for all $j > 1$. Then $O^{\sigma_1}(H)$ is normal in G. By Claim (1), $O^{\sigma_1}(H) = 1$, and hence H is a σ_1-group. Therefore H is σ-nilpotent. Since H is σ-subnormal in G, $H \leq F_\sigma(G)$ by Lemma 2.6(11). Hence H^G is σ-nilpotent.

Let $L_i = O^{\sigma_i'}(H)$, for all $i = 1, \dots, t$. Then $H = L_1 \cdots L_t$ and $N_G(H) = N_G(L_1) \cap \cdots \cap N_G(L_t)$. By Lemma 2.6(8), $H_i^x \leq N_G(O^{\sigma_i}(H))$ for all $x \in G$. This is to say that, $H_i^G \leq N_G(O^{\sigma_i}(H))$. Hence $H_i^G \leq N_G(L_j)$, for all $j \neq i$. Since $O^{\sigma_i}(G) = \Pi_{j \neq i} H_j^G$, we have $O^{\sigma_i}(G) \leq N_G(L_i)$, for all $i = 1, \dots, t$. But then $D \leq O^{\sigma_1}(G) \cap \cdots \cap O^{\sigma_t}(G) \leq N_G(L_1) \cap \cdots \cap N_G(L_t) = N_G(H)$.

The theorem is proved. □

Lemma 3.1. *Let H be a σ_1-subgroup of a σ-group G. Then H is σ-permutable in G if and only if $O^{\sigma_1}(G) \leq N_G(H)$.*

Proof. First assume that H is σ-permutable in G and let $\mathcal{H} = \{H_1, \dots, H_t\}$ be a complete Hall set of type σ of G such that H is $\mathcal{H}^G$-permutable. Then H is σ-subnormal in G by Theorem B(i). We can assume without loss of generality that H_1 is a σ_1-group. Let $i > 1$ and $x \in G$. Then $HH_i^x = H_i^x H$ and H is σ-subnormal in HH_i^x by Lemma 2.6(1). Hence $O^{\sigma_i}(HH_i^x) = O^{\sigma_i}(H) = H$ by Lemma 2.6(8). Hence $H_i^x \leq N_G(H)$. Therefore $O^{\sigma_1}(G) = \langle \{H_i^x | i \neq 1, x \in G\} \rangle \leq N_G(H)$.

Now assume that $O^{\sigma_1}(G) \leq N_G(H)$. Then H is $\{H_2, \dots, H_t\}^G$-permutable. On the other hand, since $G/O^{\sigma_1}(G)$ is a σ_1-group, $O^{\sigma_1}(G)H$ is σ-subnormal in G by Lemma 2.6(5). Therefore H is σ-subnormal in G by Lemma 2.6(2) and so $H \leq H_1^x$ for all $x \in G$ by Lemma 2.6(7). Hence $HH_1^x = H_1^x = H_1^x H$. Therefore H is σ-permutable in G. The lemma is proved. □

If H is a Hall $\sigma_i{}'$-subgroup of G, then we say that H is a *σ_i-complement* of G.

We will say that a group G is a *$P\Sigma T$-group* if $G = D \rtimes M$, where $D = G^{\mathfrak{N}_\sigma}$ is an abelian σ-Hall subgroup of odd order of G such that every element of M induces a power automorphism of D.

Proof of Theorem A. Let $\mathcal{H} = \{H_1, \ldots, H_t\}$ be a complete Hall set of type σ of G. We can assume without loss of generality that H_i is a σ_i-group for all $i = 1, \ldots, t$.

Necessity. Assume that this is false and let G be a counterexample of minimal order. Then $D \neq 1$, so $t > 1$. Since G is σ-soluble, every Hall σ_i-subgroup E_i of G is a conjugate of H_i and so $HE_i = E_iH$.

(a) *If R is a non-identity normal subgroup of G, then the hypothesis holds for G/R. Hence G/R is a $P\Sigma T$-group.*

Let H/R be a σ-subnormal subgroup of G/R. Then H is σ-subnormal in G by Lemma 2.6(5). Hence H is σ-permutable in G by hypothesis, so H/R is σ-permutable in G/R by Lemma 2.8(2). Therefore the hypothesis holds for G/R, so G/R is a $P\Sigma T$-group by the choice of G.

(b) *If E is a proper σ-subnormal subgroup of G, then $E^{\mathfrak{N}_\sigma} \leq D$ and E is a $P\Sigma T$-group.*

Indeed, since $E/E \cap D \simeq ED/D \in \mathfrak{N}_\sigma$ and $\mathfrak{N}_\sigma$ is a hereditary class by Lemma 2.5, $E/E \cap D \in \mathfrak{N}_\sigma$. Hence $E^{\mathfrak{N}_\sigma} \leq D$.

Since every σ-subnormal subgroup H of E is σ-subnormal in G by Lemma 2.6(2), H is σ-permutable in G by hypothesis and so H is σ-permutable in E by Lemma 2.8(1) since every σ-soluble group is a σ-group of Sylow type. Therefore E is a $P\Sigma T$-group by the choice of G.

(c) *D is soluble.*

In view of Claim (a), it is enough to show that a minimal normal subgroup R of G contained in D is abelian. Let V be a maximal subgroup of R. Since G is σ-soluble by hypothesis, R is σ-primary, say R is a σ_1-group. Hence $R \leq D \leq O^{\sigma_1}(G)$. On the other hand, V is σ-subnormal in G by Lemma 2.6(6), so V is σ-permutable in G. Hence $R \leq N_G(V)$ by Lemma 3.1. Therefore R is abelian.

(d) *D is a σ-Hall subgroup of G. Hence D has a complement in G, say M.*

Suppose that this is false and let U be a Hall σ_i-subgroup of D such that $1 < U < H_i$. Without loss of generality we can assume that $i = 1$.

(1) *If R is a minimal normal subgroup of G contained in D, then $R = U$ is a Sylow p-subgroup of D for some prime $p \in \sigma_1$ and a p-complement of D is a σ-Hall subgroup of G. Hence R is the unique minimal normal subgroup of G contained in D and $R = G_p \cap D$, where G_p is a Sylow p-subgroup of G contained in H_1.*

Since D is soluble by Claim (c), R is a p-group for some prime p. Moreover, $D/R = (G/R)^{\mathfrak{N}_\sigma}$ is a σ-Hall subgroup of G/R by Claim (a). Hence UR/R is a σ-Hall subgroup of G/R. It is clear that $UR \neq H_1$. Hence $R = U$ is a Sylow p-subgroup of D. It is also clear that a p-complement of D is a σ-Hall subgroup of G.

(2) $R \nleq \Phi(G)$.

Assume that $R \leq \Phi(G)$. Then $D \neq R$ by Lemma 2.5. On the other hand, D/R is a p'-group by Claim (1). Hence $O_{p'}(D) \neq 1$ by Lemma 2.5. But $O_{p'}(D)$ is characteristic in D and so it is normal G, which contradicts (1).

(3) *If G has a minimal normal subgroup $L \neq R$, then $G_p = R \times (L \cap G_p)$. Hence $O_{p'}(G) = 1$.*

Indeed, $L \nleq D$ by Claim (1). On the other hand, $DL/L \simeq D$ is a σ-Hall subgroup of G/L by Claim (a). Hence $G_pL/L = RL/L$, so $G_p = R \times (L \cap G_p)$. Thus $O_{p'}(G) = 1$.

(4) *$C_G(R) = R \times V$, where $V = C_G(R) \cap M \leq H_1$.*

In view of Claims (1) and (2), there is a maximal subgroup M of G such that $G = R \rtimes M$ and so $C_G(R) = R \times V$, where $V = C_G(R) \cap M$ is a normal subgroup of G. By Claim (1), $V \cap D = 1$ and hence $V \simeq DV/D$ is σ-nilpotent. Let W be a σ_1-complement of V. Then W is characteristic in V and so it is normal in G. Therefore we have (4) by Claim (3).

(5) $G_p \neq H_1$.

Assume that $G_p = H_1$. Let Z be a subgroup of order p in $Z(G_p) \cap R$. Then, since $R \leq D \leq O^{\sigma_1}(G) = O^p(G)$, Z is normal in G by Lemma 3.1. Hence $|R| = p$ and so $R < C_G(G)$. Then $C_G(R) = R \times L = G_p = C_G(G_p)$, where L is a minimal normal subgroup of G by Claims (3) and (4). Therefore every subgroup of G_p is normal in G by Lemma 3.1. Hence $|L| = p$. Let $R = \langle a \rangle$, $L = \langle b \rangle$ and $N = \langle ab \rangle$. Then $N \nleq D$, so in view of the G-isomorphisms $DN/D \simeq N \simeq NL/L \simeq G_p/L = RL/L \simeq R$ we get that $R \rtimes (G/C_G(R)) = R \times (G/G_p)$ is a p-group. This contradiction shows that we have (5).

Final contradiction for (d). By Proposition 2.2, G has a σ_1-complement E and a Sylow p-subgroup P such that $EP = PE$. Let $V = (EP)^{\mathfrak{N}_\sigma}$. By Claim (5), $EP \neq G$. On the other hand, $R \leq D \leq EP$ by Claim (1), so EP is σ-subnormal in G by Lemma 2.6(5). Therefore by Claim (b) we have $V \leq D$, V is a σ-Hall subgroup of EP and every subgroup of V is normal in EP. But then, since the Sylow p-subgroup P of G is contained in EP, V is a p'-group and so $V \leq C_G(R) \leq H_1$. Hence $V \leq D \cap H_1 = R$, so $V = 1$. Therefore EP is σ-nilpotent and so $E \leq C_G(R) \leq H_1$. Hence $E = 1$ and so $t = 1$, a contradiction. Hence we have (d).

(e) *At least one of the subgroups $H_1, \dots, H_t$ is normal in G and contained in D.*

Since G is σ-soluble, $D < G$. On the other hand, D is a σ-Hall subgroup of G by Claim (d), and D is a $P\Sigma T$-group by Claim (b). Therefore every σ-Hall subgroup of D is a σ-Hall subgroup of G, so for some i, $H_i \leq D$ and H_i is normal in G.

(f) *D is nilpotent.*

Assume that this is false and let R be a minimal normal subgroup of G. Then $RD/R = (G/R)^{\mathfrak{N}_\sigma}$ is abelian by Claim (a). Therefore R is the unique minimal normal subgroup of G and $R \nleq \Phi(G)$ by Lemma 2.5. Therefore $R = C_G(R)$ is a p-group for some prime p by [18, A, 15.2]. By Claim (e), one of the subgroups $H_1, \dots, H_t$, say H_1, is normal in G and contained in D. Therefore $R \leq H_1$. Let L be a subgroup of order p in R. Then $O^{\sigma_1}(G) \leq N_G(L)$ by Lemma 3.1. But $G/O^{\sigma_1}(G)$ is a σ_1-group and so from $H_1 \leq D \leq O^{\sigma_1}(G)$ we get that $O^{\sigma_1}(G) = G$. Therefore $R = L$, so $G/C_G(R) = G/R$ is a cyclic group. Hence G is supersoluble and therefore D is nilpotent.

(g) *Every subgroup H of D is normal in G.*

Since D is a σ-Hall nilpotent subgroup of G by Claims (d) and (f), it is enough to consider the case when H is a non-identity σ_i-group for some $i \in I$. In this case we have $H_i \leq D$ and so $O^{\sigma_1}(G) = G$. But then H is normal in G by Lemma 3.1.

(h) *If p is a prime such that $(p-1, |G|) = 1$, then p does not divide $|D|$. In particular, $|D|$ is odd. Hence D is abelian.*

Assume that this is false. Then, by Claim (g), D has a maximal subgroup E such that $|D : E| = p$ and E is normal in G. Then $C_G(D/E) = G$, so since D is a Hall subgroup of G, $G/E = (D/E) \times (ME/E)$, where $ME/E \simeq M \simeq G/D$ is σ-nilpotent. Therefore G/E is σ-nilpotent by Lemma 2.5. But then $D \leq E$, a contradiction. Hence p does not divide $|D|$. In particular, $|D|$ is odd. But in view of Claim (g), D is a Dedekind group. Hence D is abelian

From Claims (d)–(h) we get that the necessity conditions are true for G, which contradicts the choice of G.

Sufficiency. It is enough to show that if $G = D \rtimes M$, where D is a σ-Hall subgroup of G such that every subgroup of D is normal in G and M is σ-nilpotent, then every σ-subnormal subgroup of G is σ-permutable in G. Assume that this is false and let G be a counterexample of minimal order. Then $1 < D < G$ and for some σ-subnormal subgroup H of G and for some $x \in G$ and $k \in I$ we have $HH_k^x \neq H_k^x H$. Let $E = H_k^x$. Since G is σ-soluble and D is a σ-Hall subgroup of G, we can assume without loss of generality that $M = H_i \times \cdots \times H_t$ for some $i > 1$. Then, since every subgroup of D is normal in G, $i \leq k \leq t$.

Assume that $H_G \neq 1$. It is clear that the hypothesis holds for G/H_G, so the choice of G implies that $(H/H_G)(EH_G/H_G) = (EH_G/H_G)(H/H_G)$. Therefore $HE = HEH_G = EH_G H_G H = EH_G H = EH$, a contradiction. Hence $H_G = 1$, which implies that $H \cap D = 1$. It follows that $H \simeq DH/D$ is σ-nilpotent. Therefore we can assume without loss of generality that $H \leq M$. Hence $H = (H \cap H_i) \times \cdots \times (H \cap H_t)$, so for some j we have $(H \cap H_j)E \neq E(H \cap H_j)$. Let $V = H \cap H_j$. Then V is clearly σ-subnormal in G, so $j \neq k$ by Lemma 2.6(7).

Since ED/D is a Hall σ_k-subgroup of G/D, $(VD/D)(ED/D) = (ED/D)(VD/D)$. Hence VED is a subgroup of G. Assume that $VED < G$. It is clear that the hypothesis holds for VED. On the other hand, V is clearly σ-subnormal in VED. Hence $EV = VE$ by the choice of G, a contradiction. Therefore $VED = G$. It follows that $V = H_j$. But then V is normal in G by Lemma 2.6(10) and so $VE = EV$. This contradiction completes the proof of the result. □

Lemma 3.2. *Let H and K be subgroups of G. If G is a σ-group of Sylow type and H is σ-permutable in G, then $H \cap K$ is σ-permutable in K.*

Proof. Let K_i be a Hall σ_i-subgroup of K. Then for some Hall σ_i-subgroup E_i of G we have $K_i = E_i \cap K$. Moreover, since G is a σ-group of Sylow type, H permutes with E_i and H is σ-subnormal in HE_i by Theorem B(ii) and Lemma 2.6(1), so $HE_i \cap K = (H \cap K)(E_i \cap K) = (H \cap K)K_i = K_i(H \cap K)$ by Lemma 2.7. Hence $H \cap K$ is σ-permutable in K. The lemma is proved. □

Proof of Theorem C. In fact, in view of Lemma 2.8(4), we have only to show that if A and B are σ-permutable subgroups of G, then $A \cap B$ is σ-permutable in G. Assume that this is false and let G be a counterexample of minimal order. Then $|\sigma(G)| > 1$, and for some i and for some Hall σ_i-subgroup H of G we have $(A \cap B)H \neq H(A \cap B)$.

Let $D = AH \cap BH$. Then $A \cap D$ and $B \cap D$ are σ-permutable in D by Lemma 3.2. Assume that $D < G$. Then the choice of G implies that $A \cap B = (A \cap D) \cap (B \cap D)$ is σ-permutable in D by the choice of G. Hence $(A \cap B)H = H(A \cap B)$, a contradiction. Therefore $D = G$, so $G = AH = BH$. Hence $|G : A|$ and $|G : B|$ are σ_i-numbers. By Theorem B(i), A and B are σ-subnormal in G. Hence by Lemma 2.6(8) we have

$$O^{\sigma_i}(A) = O^{\sigma_i}(G) = O^{\sigma_i}(B).$$

Therefore, since $|\sigma(G)| > 1$, it follows that $V = A_G \cap B_G \neq 1$.

By Lemma 2.8(2), A/V and B/V are σ-permutable in G/V. Hence the choice of G implies that

$$(A \cap B/V)(HV/V) = ((A/V) \cap (B/V))(HV/V) =$$
$$(HV/V)((A/V) \cap (B/V)) = (HV/V)(A \cap B/V)$$

by the choice of G. But then $(A \cap B)H = (A \cap B)HV = HV(A \cap B) = H(A \cap B)$. This contradiction completes the proof of the result. □

Corollary 3.3. *(See Kegel [2] or Theorem 1.2.19 in [25].) The set of all S-permutable subgroups of G forms a sublattice of the lattice of all subnormal subgroups of G.*

4. Final remarks, some other applications and open questions

1. In view of Theorem B(i), the class of all σ-soluble groups G, in which σ-permutability is a transitive relation on G, that is, every σ-permutable subgroup of a σ-permutable subgroup of G is σ-permutable in G, coincides with the class of all $P\Sigma T$-groups.

2. As another application of Theorem B we prove the following fact.

Theorem 4.1. *Let G be a σ-group of Sylow type. Then a subgroup A of G is σ-permutable in G if and only if A is σ-subnormal in G and A is σ-permutable in $\langle A, x\rangle$ for all $x \in G$.*

Proof. In view of Theorem B(i) and Lemma 2.8(1), it is enough to prove that if A is σ-subnormal in G and σ-permutable in $\langle A, x\rangle$ for all $x \in G$, then A is σ-permutable in G. Assume that this is false and let G be a counterexample with $|G| + |A|$ minimal. Then G is not σ-primary, and for some i and for some Hall σ_i-subgroup H of G we have $AH \neq HA$.

Assume that $\langle A, H\rangle < G$. In view of Lemma 2.6(1), A is σ-subnormal in $\langle A, H\rangle$. Therefore the hypothesis holds for $\langle A, H\rangle$, so A is σ-permutable in $\langle A, H\rangle$ by the choice of G. But then, since G is a σ-group of Sylow type by hypothesis, A permutes with every Hall σ_i-subgroup of $\langle A, H\rangle$ and hence $AH = HA$. This contradiction shows that $\langle A, H\rangle = G$.

Now let $x \in H$ and $L = \langle A, x\rangle$. Then A is σ-permutable in L by hypothesis, so for some Hall σ_i-subgroup E of L we have $x \in E$ and $AE = EA$. It follows that $|AE : A|$ is a σ_i-number and so $O^{\sigma_i}(AE) = O^{\sigma_i}(A)$ by Lemma 2.6(8). Hence $x \in N_G(O^{\sigma_i}(A))$. Therefore $O^{\sigma_i}(A)$ is normal in G. But since $AH \neq HA$, $A \nleq H$ and so in view of Lemma 2.6(7) we have $O^{\sigma_i}(A) \neq 1$. Therefore, $A_G \neq 1$.

Note that $\langle A/A_G, xA_G\rangle = \langle A, x\rangle/A_G$, where A/A_G is σ-permutable in $\langle A, x\rangle/A_G$ by Lemma 2.8(2). Also, A/A_G is σ-subnormal in G/A_G by Lemma 2.6(4). Hence the hypothesis holds for G/A_G, so A/A_G is σ-permutable in G/A_G by the choice of G. Hence A is σ-permutable in G by Lemma 2.6(3). This contradiction completes the proof of the result. □

Recall that a subgroup A of G is subnormal in G if and only if A is subnormal in $\langle A, x\rangle$ for all $x \in G$ [18, A, 14.10]. Therefore from Theorem 4.1 we get the following known result.

Corollary 4.2. *(See Ballester-Bolinches and Esteban-Romero [26] or Theorem 1.2.13 in [25].) A subgroup A of G is S-permutable in G if and only if A is S-permutable in $\langle A, x\rangle$ for all $x \in G$.*

3. Let $\mathfrak{X}$ be a class of groups. A group G is called $\mathfrak{X}$-*critical* if G is not in $\mathfrak{X}$ but all proper subgroups of G are in $\mathfrak{X}$ [18, p. 517]. An $\mathfrak{N}$-critical group, where $\mathfrak{N}$ is the class of all nilpotent groups, is called a *Schmidt group*.

Example 4.3. Let $G = (C_5 \rtimes C_2) \times S_4$, where $C_5 \rtimes C_2$ is a non-abelian group of order 10 and S_4 is the symmetric group of degree 4. Let $\sigma = \{\{2, 3\}, \{2, 3\}'\}$. Then $F_\sigma(G) = C_5S_4$ and $G/F_\sigma(G) \simeq C_2$ is abelian. Now let H be any Schmidt subgroup of G. If 5 divides $|H|$, then $C_5 \leq H$ and so G/H_G is a $\{2, 3\}$-group. Hence H is σ-subnormal in G. If 5 does not divide $|H|$, then H is not $\mathfrak{N}_\sigma$-critical subgroup of G.

Example 4.3 was a motivation for the following result.

Theorem 4.4. *(See Skiba [21].) Suppose that G is a non-σ-nilpotent σ-group. If every $\mathfrak{N}_\sigma$-critical subgroup of G is σ-subnormal in G, then $G/F_\sigma(G)$ is abelian.*

Corollary 4.5. *(See Semenchuk [27].) Suppose that G is a non-nilpotent group. If every Schmidt subgroup of G is subnormal in G, then G is metanilpotent.*

Corollary 4.6. *(See Monakhov and Knyagina [28].) Suppose that G is a non-nilpotent group. If every Schmidt subgroup of G is subnormal in G, then $G/F(G)$ is abelian.*

4. A full description of groups with subnormal Schmidt subgroups was obtained by V.A. Vedernikov in [29].

Question 4.7. Describe σ-groups in which every $\mathfrak{N}_\sigma$-critical subgroup is σ-subnormal.

5. We say that a σ-group G is *σ-dispersive* provided G has a normal series $1 = G_1 < G_2 < \cdots < G_{t-1} < G_t = G$ and a complete Hall set $\mathcal{H} = \{H_1, \ldots, H_t\}$ of type σ such that $G_iH_i = G_{i+1}$ for all $i = 1, \ldots, t-1$.

In view of Theorem A, every σ-soluble group in which every σ-subnormal subgroup is σ-permutable is σ-dispersive.

Question 4.8. Let G be a σ-soluble group and $|\sigma(G)| = n$. Assume that every $(n+1)$-maximal subgroup of G is σ-subnormal. Is it true then that G is σ-dispersive?

In the case when σ is the smallest partition of $\mathbb{P}$ the answer to Question 4.8 is positive [30].

6. In view of [31, IV, 5.4] the following question seems to be natural.

Question 4.9. Is it true that every $\mathfrak{N}_\sigma$-critical group is σ-soluble?

7. Finally, we draw the readers' attention to the following two open problems.

Question 4.10. Suppose that for every $x \in G$, the subgroup H of G is σ-subnormal in $\langle H, x\rangle$. Is it true then that H is σ-subnormal in G?

Question 4.11. Let H and K be σ-subnormal subgroups of G such that $\pi(H/H^{\mathfrak{N}_\sigma}) \cap \pi(K/K^{\mathfrak{N}_\sigma})$ is empty. Is it true then that $HK = KH$?

In the case when σ is the smallest partition of $\mathbb{P}$ the answers to Questions 4.10 and 4.11 are positive (see Wielandt's theorems [18, A, 14.10] and [32, 4.1.2], respectively).

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References

[1] W. Guo, A.N. Skiba, Finite groups with permutable complete Wielandt sets of subgroups, J. Group Theory 18 (2) (2015) 191–200, http://dx.doi.org/10.1515/jgth-2014-0045.

[2] O.H. Kegel, Sylow-Gruppen und Subnormalteiler endlicher Gruppen, Math. Z. 78 (1962) 205–221.
[3] W.E. Deskins, On quasinormal subgroups of finite groups, Math. Z. 82 (1963) 125–132.
[4] A. Ballester-Bolinches, R. Esteban-Romero, A.A. Heliel, M.O. Almestadi, $\mathcal{Z}$-permutable subgroups of finite groups, preprint.
[5] R.K. Agrawal, Finite groups whose subnormal subgroups permute with all Sylow subgroups, Proc. Amer. Math. Soc. 47 (1975) 77–83.
[6] D.J.S. Robinson, The structure of finite groups in which permutability is a transitive relation, J. Aust. Math. Soc. 70 (2001) 143–159.
[7] R.A. Brice, J. Cossey, The Wielandt subgroup of a finite soluble groups, J. Lond. Math. Soc. 40 (1989) 244–256.
[8] J.C. Beidleman, B. Brewster, D.J.S. Robinson, Criteria for permutability to be transitive in finite groups, J. Algebra 222 (1999) 400–412.
[9] A. Ballester-Bolinches, R. Esteban-Romero, Sylow permutable subnormal subgroups, J. Algebra 251 (2002) 727–738.
[10] A. Ballester-Bolinches, J.C. Beidleman, H. Heineken, Groups in which Sylow subgroups and subnormal subgroups permute, Illinois J. Math. 47 (2003) 63–69.
[11] A. Ballester-Bolinches, J.C. Beidleman, H. Heineken, A local approach to certain classes of finite groups, Comm. Algebra 31 (2003) 5931–5942.
[12] M. Asaad, Finite groups in which normality or quasinormality is transitive, Arch. Math. 83 (4) (2004) 289–296.
[13] A. Ballester-Bolinches, J. Cossey, Totally permutable products of finite groups satisfying SC or PST, Monatsh. Math. 145 (2005) 89–93.
[14] K. Al-Sharo, J.C. Beidleman, H. Heineken, et al., Some characterizations of finite groups in which semipermutability is a transitive relation, Forum Math. 22 (2010) 855–862.
[15] J.C. Beidleman, M.F. Ragland, Subnormal, permutable, and embedded subgroups in finite groups, Cent. Eur. J. Math. 9 (4) (2011) 915–921.
[16] X. Yi, A.N. Skiba, Some new characterizations of PST-groups, J. Algebra 399 (2014) 39–54.
[17] O.H. Kegel, Untergruppenverbande endlicher Gruppen, die den subnormalteilerverband echt enthalten, Arch. Math. 30 (3) (1978) 225–228.
[18] K. Doerk, T. Hawkes, Finite Soluble Groups, Walter de Gruyter, Berlin, New York, 1992.
[19] D. Gorenstein, Finite Groups, Harper & Row Publishers, New York, Evanston, London, 1968.
[20] W. Guo, K.P. Shum, A.N. Skiba, X-semipermutable subgroups of finite groups, J. Algebra 315 (2007) 31–41.
[21] A.N. Skiba, On some properties of finite σ-soluble and σ-nilpotent groups, preprint.
[22] A.N. Skiba, On σ-properties of finite groups I, PFMT 4 (21) (2014) 89–96.
[23] A. Ballester-Bolinches, L.M. Ezquerro, Classes of Finite Groups, Springer-Verlag, Dordrecht, 2006.
[24] P. Schmid, Subgroups permutable with all Sylow subgroups, J. Algebra 207 (1998) 285–293.
[25] A. Ballester-Bolinches, R. Esteban-Romero, M. Asaad, Products of Finite Groups, Walter de Gruyter, Berlin, New York, 2010.
[26] A. Ballester-Bolinches, R. Esteban-Romero, On finite soluble groups in which Sylow permutability is a transitive relation, Acta Math. Hungar. 101 (2003) 193–202.
[27] V.N. Semenchuk, Finite groups with a system of minimal non-$\mathfrak{F}$-groups, in: Subgroup Structure of Finite Groups, Nauka i Tehnika, Minsk, 1981, pp. 138–139.
[28] V.S. Monakhov, V.N. Knyagina, On finite groups with some subnormal Schmidt subgroups, Sib. Math. J. 45 (6) (2004) 1316–1322.
[29] V.A. Vedernikov, Finite groups with subnormal Schmidt subgroups, Algebra Logika 46 (6) (2007) 669–687.
[30] A. Mann, Finite groups whose n-maximal subgroups are subnormal, Trans. Amer. Math. Soc. 132 (1968) 395–409.
[31] B. Huppert, Endliche Gruppen I, Springer-Verlag, Berlin, Heidelberg, New York, 1967.
[32] J.C. Lennox, S.E. Stonehewer, Subnormal Subgroups of Groups, Clarendon Press, Oxford, 1987.