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CONSTRAINED MOTION OF DEFORMABLE BODIES

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SUMMARY

In this paper, issues related to the dynamic modelling of constrained deformable bodies that undergo large rigid body displacements are discussed. Particular attention is focused on *finite element formulations*. It is shown that the use of nodal co-ordinates and shape functions to describe the finite rotation of some of the commonly used finite elements leads to a linearization of the kinematics and dynamic relationships. The structure of the non-linear dynamic equations that govern the motion of deformable bodies that undergo large displacements is examined. Comments on the finite element formulation of the *invariants of motion*, the definition of the *generalized forces* and *moments* in flexible body dynamics and the computational strategy used for the automatic generation of the equations of motion are made. The computer formulation of the *joint constraints* between deformable bodies as well as the numerical algorithms currently used in many of the general purpose computer programs that are based on the augmented formulation are discussed. A decoupled joint-elastic acceleration recursive formulation is also presented. This formulation leads to a small system of acceleration equations whose dimensions are independent of the number of the elastic degrees of freedom of the system. In this paper, the coupling between the displacements of the deformable bodies is classified as *kinematic*, *inertia* and *elastic*. In view of this classification, comments on the validity of using the *updated finite element Lagrangian formulation* and the *4 × 4 transformation matrix* in the dynamic analysis of flexible multibody systems are made. The coupling between the finite rotation and the wave motion in constrained deformable bodies is also discussed.

1. INTRODUCTION

Several issues related to the dynamics of constrained deformable bodies that undergo large rigid body displacements are discussed in this paper. The validity of using the nodal co-ordinates and the shape functions to describe the finite rotations of some of the finite elements commonly used in the analysis of flexible bodies is examined. This examination sheds light on some of the problems associated with the use of the *updated finite element Lagrangian formulation* in the large displacement analysis of deformable bodies. It also provides motivation for the definition of the co-ordinate systems used in the total Lagrangian formulation presented in previous publications.[1,2] The fact that most of the element shape functions can be used to describe large translational displacements is crucial in the development of the non-linear dynamic formulation presented in this paper. By using this fact and a set of co-ordinate systems that define the configuration of the finite element, the *principle of virtual work in dynamics* or *Lagrange's equation* can be used to derive the *non-linear generalized Newton–Euler equations*[2,3] for deformable bodies that undergo large rigid body displacements. These equations can be expressed in terms of a unique set of *invariants of motion* that depend on the assumed displacement field and can be evaluated in advance in a preprocessor computer program.

In developing the equations of motion of deformable bodies, special attention must be paid to the definition of forces and moments.[4] The concept of the equivalence of two systems of forces in

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rigid body dynamics is not applicable to deformable body dynamics. A force that acts at a point on a deformable body is equivalent to a system, defined at another point, that consists of the same force, a moment that depends on the relative displacement between the two points and a set of generalized elastic forces that depend on the finite rotation of the body.[4] This is a subject of particular interest in *control applications*, since in many cases the motion of the system is specified and the interest is focused on defining the *joint control forces* that produce the desired motion. Nonetheless, a close examination of the structure of the mass matrix and the forces in deformable body dynamics and the proper identification of the invariants leads to a systematic procedure for the automatic generation of the inertia and stiffness characteristics of deformable bodies in multibody systems.

Once the structure of the non-linear dynamic equations that govern the unconstrained motion of deformable bodies is defined, two approaches can be used to formulate the multibody equations of motion. These are the *augmented* and the *recursive formulations*. In the augmented formulation, the multibody equations of motion are formulated in terms of a set of variables that include both the dependent and independent co-ordinates. In this type of formulation, constraints between the variables are formulated using a set of linear and/or non-linear algebraic constraint equations that depend on the system co-ordinates and possibly on time. This leads to a mixed system of *algebraic* and *differential equations* that must be solved simultaneously using matrix and computer methods.[2] In the recursive formulations, the equations of motion are formulated in terms of the joint variables or the system degrees of freedom. This leads to a smaller system of strongly coupled equations.[3–5, 27] In this case, one obtains only a set of differential equations that can be integrated numerically in order to define the state of the system. In most existing recursive methods, decoupling the joint and elastic accelerations requires the inversion or the **LU** factorization of non-linear matrices whose dimensions depend on the number of the system elastic degrees of freedom. A recursive method that leads to a loosely coupled system of equations was proposed.[19–21] This method can be used to eliminate the coupling between the joint and elastic accelerations while maintaining the inertia coupling between the rigid body and the elastic displacements. The use of this method leads to a set of acceleration equations in which the dimensions of the coefficient matrices are independent of the number of elastic degrees of freedom.

Another topic of particular significance in the analysis of deformable bodies that undergo large displacements is the coupling between the displacements. The *coupling* between the finite rotations and the deformation displacements has a significant effect on the dynamics of deformable bodies. Significant changes in the *wave phenomenon* occur as the result of the finite rotation.[6,7] For example, elastic waves in a perfectly elastic non-rotating rods propagate with the same phase velocity. Consequently, the group velocity is constant and is independent of the wave number or the dimension of the rod. *Dispersion*, however, occurs as the result of the finite rotation and its coupling with the deformation displacements.[7] The *phase velocities* of harmonic waves are no longer equal and consequently the *group velocity* becomes dependent on the wave number.

2. FINITE ELEMENTS AND FINITE ROTATIONS

Several finite element formulations have been proposed for the large displacement analysis of deformable bodies. One of the commonly used approaches is the *updated Lagrangian formulation*. In the updated Lagrangian formulation, a *convected co-ordinate system* is attached to each finite element and hence it shares its rigid body motion. By using a relatively small step size in the numerical integration, the displacement of the element between two co-ordinate systems is described using the shape function and the nodal co-ordinates of the element. The current deformed state is used as the new reference state prior to the next incremental step in the transient dynamic solution. The updated Lagrangian formulation leads to a simple system of dynamic equations in which the element mass matrix defined in the convected co-ordinate system is constant. Furthermore, the use of the *lumped mass technique* leads to a constant element mass matrix in the global co-ordinate system.[28] Since several of the shape functions of *beams*, *plates* and *shells* can not be used to describe finite rotations, the use of the updated Lagrangian formulation leads to a subtle linearization of the resulting dynamic equations. The limitations on the use of these shape functions in the large displacement analysis of deformable bodies can be demonstrated. To this end, we use the shape function of the six degree of freedom, two node planar beam element. Each node is assumed to have three co-ordinates; two describe the translation and one describes the slope at this nodal point. The vector of nodal co-ordinates of the element j on the deformable body i can be written as

$$\mathbf{e}^{ij} = [e_1^{ij} \quad e_2^{ij} \quad e_3^{ij} \quad e_4^{ij} \quad e_5^{ij} \quad e_6^{ij}]^{\mathrm{T}} \tag{1}$$

where $e_1^{ij}, e_2^{ij}, e_4^{ij}$ and e_5^{ij} are the translational nodal co-ordinates, while e_3^{ij} and e_6^{ij} are the slopes at the two nodal points. An element shape function associated with this set of nodal co-ordinates is

$$\bar{\mathbf{S}}^{ij} = \begin{bmatrix} 1-\xi & 0 & 0 & \xi & 0 & 0 \\ 0 & 1-3\xi^2+2\xi^3 & l(\xi-2\xi^2+\xi^3) & 0 & 3\xi^2-2\xi^3 & l(\xi^3-\xi^2) \end{bmatrix}^{ij} \tag{2}$$

where $\xi = x/l$, and x is the spatial co-ordinate and l is the length of the element. Note that a general rigid body translation can be described by the vector

$$\mathbf{R} = [R_x \quad R_y]^{\mathrm{T}} \tag{3}$$

where R_x and R_y are the displacements of an arbitrary point on the element. As the result of this rigid body translation, the vector of nodal co-ordinates becomes

$$\begin{aligned} \mathbf{e}_t^{ij} &= [e_1^{ij}+R_x \quad e_2^{ij}+R_y \quad e_3^{ij} \quad e_4^{ij}+R_x \quad e_5^{ij}+R_y \quad e_6^{ij}]^{\mathrm{T}} \\ &= \mathbf{e}^{ij} + \mathbf{R}_e \end{aligned} \tag{4}$$

where $\mathbf{e}^{ij}$ as defined by equation (1) is the vector of nodal co-ordinates before the rigid body translation and $\mathbf{R}_e$ is the vector

$$\mathbf{R}_e = [R_x \quad R_y \quad 0 \quad R_x \quad R_y \quad 0]^{\mathrm{T}}$$

By using simple matrix multiplication, it can be shown that

$$\bar{\mathbf{S}}^{ij}\mathbf{R}_e = \mathbf{R}$$

That is

$$\bar{\mathbf{S}}^{ij}\mathbf{e}_t^{ij} = \bar{\mathbf{S}}^{ij}\mathbf{e}^{ij} + \bar{\mathbf{S}}^{ij}\mathbf{R}_e = \bar{\mathbf{S}}^{ij}\mathbf{e}^{ij} + \mathbf{R}$$

This implies that the element nodal co-ordinates can be used to describe an arbitrarily large rigid body translation. This is a basic assumption which is utilized in our formulation and its significance becomes apparent when the dynamic equations are formulated in terms of a minimum number of independent invariants of motion.[8]

Finite rotation

If the finite element undergoes a pure rotation defined by the angle θ, the position vector of an arbitrary point at a distance x from its end as the result of this rotation is

$$\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} x\cos\theta \\ x\sin\theta \end{bmatrix} \tag{5}$$

Using this equation and the definition of the slope, one has

$$e_3^{ij} = e_6^{ij} = \frac{\partial v}{\partial x} = \sin\theta \tag{6}$$

In this case, the vector of nodal co-ordinates becomes

$$\mathbf{e}_r^{ij} = [0 \quad 0 \quad \sin\theta \quad l\cos\theta \quad l\sin\theta \quad \sin\theta]^{\mathrm{T}}$$

where l is the length of the element. It can be shown, by direct matrix multiplication, that

$$\bar{\mathbf{S}}^{ij}\mathbf{e}_r^{ij} = \begin{bmatrix} x\cos\theta \\ x\sin\theta \end{bmatrix}$$

That is, the shape function of the beam element, as defined by equation (2), can be used to describe finite rotations provided that the slopes at the nodal points can be defined using *trigonometric functions* as in equation (6). In the finite element formulation such a definition can not be made since trigonometric functions lack any physical meaning. The use of trigonometric functions to define the nodal co-ordinates introduces technical difficulties in assembling the finite elements and in transforming the nodal co-ordinates from one co-ordinate system to another.[8,9] On the other hand, if the rotation is assumed to be small, one has

$$e_3^{ij} = e_6^{ij} \approx \theta$$

Infinitesimal rotations can be treated as vectors. Therefore, the rule of transforming vectors from one co-ordinate system to another can be applied to the transformation of the nodal co-ordinates. Furthermore, the slopes as defined by the preceding equation have physical meaning and consequently no technical problems arise when the elements are assembled.

Co-ordinate systems

Using a similar procedure as the one described in this section it can be shown that most of the commonly used shape functions can describe an arbitrary large rigid body translation. As demonstrated by the beam example presented in this section, some of the shape functions cannot be used to describe an arbitrary finite rotation of the element. Even though in the cases where the element nodal co-ordinates can be used to describe finite rigid body rotations as in the case of *triangular*, *rectangular*, *solid* and *tetrahedral elements*, the use of the nodal co-ordinates is not convenient in describing the relative finite rotations between the components of the multibody system.

Using the fact that the element shape function can be used to describe an arbitrary large rigid body translation, the location of an arbitrary point on the element can be defined in an *intermediate element co-ordinate system* $\bar{X}^{ij}\bar{Y}^{ij}\bar{Z}^{ij}$ whose axes, as shown in Figure 1, are parallel to the axes of the *element co-ordinate system* $X^{ij}Y^{ij}Z^{ij}$ as[1,2]

$$\bar{\mathbf{u}}_i^{ij} = \bar{\mathbf{S}}^{ij}(\mathbf{e}_o^{ij} + \mathbf{e}_f^{ij}) \tag{7}$$

where $\bar{\mathbf{u}}_i^{ij}$ is the position vector of the arbitrary point on the element defined in the intermediate element co-ordinate system, $\mathbf{e}_o^{ij}$ is the vector of nodal co-ordinates in the undeformed state and $\mathbf{e}_f^{ij}$ is the vector of nodal deformations. The origin of the intermediate element co-ordinate system $X^{ij}Y^{ij}Z^{ij}$ is assumed to be rigidly connected to the origin of the *body co-ordinate system* $X^iY^iZ^i$. In this case, the global position vector of an arbitrary point on the element j on the deformable

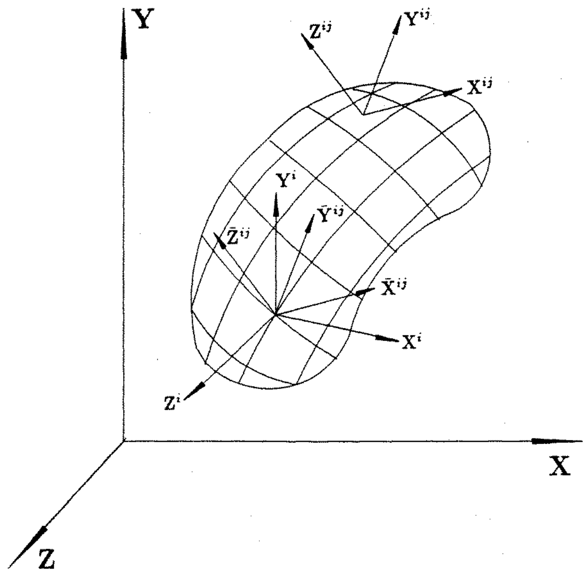

Figure 1. Co-ordinate systems

body i can be written as[1,2]

$$\mathbf{r}^{ij} = \mathbf{R}^i + \mathbf{A}^i \bar{\mathbf{u}}^{ij} \tag{8}$$

where $\mathbf{R}^i$ is the global position vector of the origin of the body co-ordinate system, $\mathbf{A}^i$ is the transformation matrix from the body to the global co-ordinate system and $\bar{\mathbf{u}}^{ij}$ is the local position vector of the arbitrary point defined as

$$\bar{\mathbf{u}}^{ij} = \mathbf{S}^{ij}\mathbf{B}^{ij}\mathbf{B}^i\mathbf{q}_f^i \tag{9}$$

in which $\mathbf{B}^{ij}$ is the *Boolean matrix* that describes the element connectivity, $\mathbf{B}^i$ is the matrix of the reference conditions,[2,10] $\mathbf{q}_f^i$ is the vector of elastic co-ordinates of the deformable body i and $\mathbf{S}^{ij}$ is the shape matrix of the element j defined in the body co-ordinate system.[10] Note that this description of motion does not imply any linearization of the kinematic relationships provided that the element rotations in the case of beams and plates with respect to the body co-ordinate system are assumed to be small. If the shape function of the element can be used to describe arbitrary finite rotations such as in the case of two dimensional triangular and rectangular elements and the case of three dimensional solid and tetrahedral elements, the kinematic equations presented in this section can be used without any assumption of linearization in the large deformation analysis of flexible multibody systems.

3. DEFINITION OF FORCES IN FLEXIBLE BODY DYNAMICS

Several techniques can be used to derive the dynamic equations of the deformable body i that undergoes large rigid body displacements. In the case of an *unconstrained deformable body*, the application of the *principle of virtual work in dynamics* leads to

$$\mathbf{Q}_{\mathrm{i}}^i = \mathbf{Q}_{\mathrm{e}}^i \tag{10}$$

where $\mathbf{Q}_{\mathrm{i}}^{i}$ is the vector of the *generalized inertia forces* and $\mathbf{Q}_{\mathrm{e}}^{i}$ is the vector of *applied external* and *elastic forces*. In *Lagrange's equation* the generalized inertia forces are expressed in terms of the kinetic energy, while in the *Gibbs–Appel equation* the generalized inertia forces are expressed in terms of the acceleration function. Both can be derived using the basic definition of the virtual work of the inertia forces defined as

$$\delta W_{\mathrm{i}} = \int_{V^i} \rho^i \ddot{\mathbf{r}}^{i\mathrm{T}} \delta \mathbf{r}^i \mathrm{d}V^i = \int_{V^i} \rho^i \ddot{\mathbf{r}}^{i\mathrm{T}} \frac{\partial \mathbf{r}^i}{\partial \mathbf{q}^i} \delta \mathbf{q}^i \mathrm{d}V^i$$
$$= \mathbf{Q}_{\mathrm{i}}^{i\mathrm{T}} \delta \mathbf{q}^i \tag{11}$$

where ρ^i and V^i are, respectively, the mass density and volume of the deformable body i, $\ddot{\mathbf{r}}^i$ is the acceleration vector of an arbitrary point on the deformable body and $\mathbf{q}^i$ is the vector of generalized co-ordinates of the body which can be defined using the *absolute reference* and the *elastic relative co-ordinates* as

$$\mathbf{q}^i = [\mathbf{R}^{i\mathrm{T}} \quad \boldsymbol{\theta}^{i\mathrm{T}} \quad \mathbf{q}_f^{i\mathrm{T}}]^{\mathrm{T}} \tag{12}$$

in which $\boldsymbol{\theta}^i$ is the set of rotational co-ordinates used to describe the orientation of the deformable body and $\mathbf{R}^i$ and $\mathbf{q}_f^i$ are as previously defined. It follows from equation (11) that the generalized inertia forces are

$$\mathbf{Q}_{\mathrm{i}}^{i\mathrm{T}} = \int_{V^i} \rho^i \ddot{\mathbf{r}}^{i\mathrm{T}} \frac{\partial \mathbf{r}^i}{\partial \mathbf{q}^i} \mathrm{d}V^i \tag{13a}$$

which is the same as

$$\mathbf{Q}_{\mathrm{i}}^{i\mathrm{T}} = \int_{V^i} \rho^i \ddot{\mathbf{r}}^{i\mathrm{T}} \frac{\partial \dot{\mathbf{r}}^i}{\partial \dot{\mathbf{q}}^i} \mathrm{d}V^i \tag{13b}$$

since

$$\frac{\partial \mathbf{r}^i}{\partial \mathbf{q}^i} = \frac{\partial \dot{\mathbf{r}}^i}{\partial \dot{\mathbf{q}}^i} \tag{14}$$

Using the kinematic relationships presented in the preceding section, the relationship between the angular acceleration $\boldsymbol{\alpha}^i$ of the co-ordinate system of the deformable body i and the time derivatives of the orientational co-ordinates, and equation (10), one obtains the *generalized Newton–Euler equations* for the deformable body as

$$\begin{bmatrix} \mathbf{m}_{RR}^i & \mathbf{m}_{R\theta}^i & \mathbf{m}_{Rf}^i \\ & \mathbf{m}_{\theta\theta}^i & \mathbf{m}_{\theta f}^i \\ \text{symmetric} & & \mathbf{m}_{ff}^i \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{R}}^i \\ \boldsymbol{\alpha}^i \\ \ddot{\mathbf{q}}_f^i \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_R^i \\ \mathbf{Q}_\alpha^i \\ \mathbf{Q}_f^i - \mathbf{K}_{ff}^i \mathbf{q}_f^i \end{bmatrix} + \begin{bmatrix} \mathbf{F}_R^i \\ \mathbf{F}_\alpha^i \\ \mathbf{F}_f^i \end{bmatrix} \tag{15}$$

where $\mathbf{m}_{RR}^i$, $\mathbf{m}_{R\theta}^i$, $\mathbf{m}_{Rf}^i$, $\mathbf{m}_{\theta\theta}^i$, $\mathbf{m}_{\theta f}^i$ and $\mathbf{m}_{ff}^i$ are the components of the mass matrix, $\mathbf{K}_{ff}^i$ is the stiffness matrix, $\mathbf{Q}^i = [\mathbf{Q}_R^{i\mathrm{T}} \quad \mathbf{Q}_\alpha^{i\mathrm{T}} \quad \mathbf{Q}_f^{i\mathrm{T}}]^{\mathrm{T}}$ is the vector of externally applied forces, and $\mathbf{F}^i = [\mathbf{F}_R^{i\mathrm{T}} \quad \mathbf{F}_\alpha^{i\mathrm{T}} \quad \mathbf{F}_f^{i\mathrm{T}}]^{\mathrm{T}}$ is a quadratic velocity vector that absorbs the *Coriolis* and the *centrifugal force components.*

Invariants of motion

As the result of the finite rotation, the mass matrix of equation (15) is a non-linear function of the co-ordinates while the Coriolis and centrifugal forces are non-linear functions of the co-ordinates and velocities. It can be shown, however, that the non-linear mass matrix and the non-linear Coriolis and centrifugal forces can be expressed in terms of a set of invariants that depend on the assumed displacement field. These invariants can be developed for each finite element j on the deformable body i. The invariants of the deformable body i can then be obtained by assembling the invariants of its finite elements using a standard finite element assembly procedure. If the shape function of the finite element can be used to describe large rigid body translations in three orthogonal directions, it can be shown that the invariants of the element j on the deformable body i are[1,2]

$$\mathbf{I}_1^{ij} = \int_{V^{ij}} \rho^{ij} \mathbf{S}^{ij} \, \mathrm{d}V^{ij} \tag{16a}$$

$$\mathbf{I}_{kl}^{ij} = \int_{V^{ij}} \rho^{ij} \mathbf{S}_k^{ij\mathrm{T}} \mathbf{S}_l^{ij} \, \mathrm{d}V^{ij}, \qquad k, l = 1, 2, 3 \tag{16b}$$

where ρ^{ij} and V^{ij} are, respectively, the mass density and volume of the element j and $\mathbf{S}_k^{ij}$ is the kth row of the element shape function. The invariants of the body i can simply be obtained as

$$\mathbf{I}_1^i = \sum_{j=1}^{n_e} \mathbf{I}_1^{ij} \tag{17a}$$

$$\mathbf{I}_{kl}^i = \sum_{j=1}^{n_e} \mathbf{I}_{kl}^{ij} \tag{17b}$$

where n_e is the total number of the finite elements used to discretize the deformable body i.

Note that the invariants of equations 16(a) and (b) are given in their *consistent mass* form. These invariants can also be expressed in a *lumped mass* form.[2,11,12] In this latter case, the structure of the mass matrix does not change and it remains a non-linear function of the co-ordinates. This is one of the basic differences between the approach presented in this paper and the updated Lagrangian formulation implemented in some of the existing finite element computer programs.

Forces in flexible body dynamics

In rigid body dynamics, a force that acts at a point on the body is *equivalent* or *equipollent* to a system of forces, defined at another point, that consists of the same force and a moment. Consequently, the force is *a sliding vector* defined by its magnitude, direction and its line of action. On the other hand, a moment in rigid body dynamics is defined only by its magnitude and direction. It is a *free vector* which is independent of a point of application since the angular velocity is a characteristic of the whole rigid body and not of a particular point on the body. In deformable body dynamics, however, a force that acts at a point is equivalent to the same force, a moment that depends on the deformation of the body, and a set of generalized elastic forces that depend on the finite rotation and the assumed displacement field.[4] Furthermore, a moment in flexible body dynamics is no longer a free vector, it is defined by its magnitude, its direction and its point of application.[4]

In many *control applications*, the desired motion of a system is specified and the interest is focused on determining the *joint control forces* that produce this desired motion. This *inverse dynamics problem* must be carefully handled in view of the definition of forces and moments in flexible body dynamics. In Reference 4, a procedure for determining the generalized joint forces is presented and its use is demonstrated using multibody examples. An important application which is currently being examined using the procedure outlined in Reference 4 is the multibody *space*

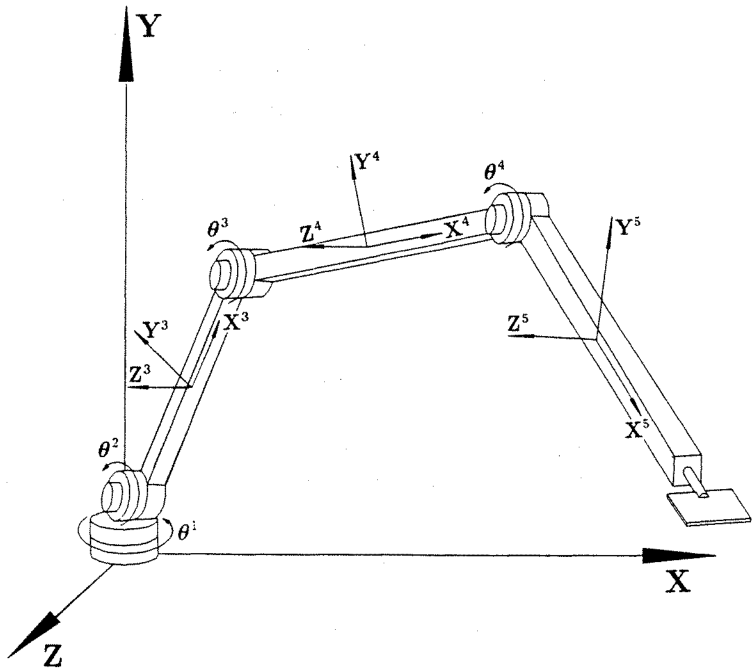

Figure 2. The space crane

crane shown in Figure 2. This multibody system will be used in the assembly and the construction of massive systems which are too large to be placed in orbit by a single vehicle.[13] Because of its large size, several difficulties are encountered in the analysis, design and control of the space crane. Among these difficulties are the structural flexibility, the location of the sensors and actuators that are required to produce the desired motion, and the precise definition of the joint forces that move the components of the crane. The effect of the structural flexibility on these forces must be taken into consideration.

Selection of the elastic co-ordinates

The generalized Newton–Euler equations as defined by equation (15) are formulated in terms of a coupled set of reference and elastic nodal variables. This is a *finite element formulation* which was obtained using the physical nodal co-ordinates of the finite elements used to discretize the deformable body *i*. This matrix equation can be solved using matrix and computer methods for the reference motion and the elastic nodal co-ordinates of the elements. This formulation, therefore, should not be viewed as a *component mode* type of formulation. In a component mode formulation, the deformable body can be treated as one element whose deformation is described using a set of assumed modes. The differences between the finite element formulation and the assumed mode technique and the difference between the obtained invariants of motion in both cases must be clear. Component modes, however, can be used in a finite element formulation in order to reduce the number of elastic co-ordinates and eliminate insignificant high frequency modes. To this end, a set of assumed modes that can be determined by solving an eigenvalue problem or can be determined using *experimental modal analysis* techniques[14] may be used. Let $\mathbf{B}^i_{\mathrm{m}}$ be the modal matrix that contains a set of assumed modes that are determined experimentally or by solving the *eigenvalue problem.* A change from the space of the physical nodal co-ordinates to the space of modal co-ordinates can be achieved by using the modal transformation $\mathbf{B}^i_{\mathrm{m}}$. In this case, one must realize that there is no change in the structure of equation (15); one only has to express the invariants of equation (17) in their modal form. These invariants can be transformed to their modal form according to

$$(\mathbf{I}^i_1)_{\mathrm{m}} = \mathbf{I}^i_1 \mathbf{B}^i_{\mathrm{m}} \tag{18a}$$

$$(\mathbf{I}^i_{kl})_{\mathrm{m}} = \mathbf{B}^{i\mathrm{T}}_{\mathrm{m}} \mathbf{I}^i_{kl} \mathbf{B}^i_{\mathrm{m}} \tag{18b}$$

That is, the formulation remains the same and any change in the basis of the elastic nodal co-ordinates can be achieved by transforming the invariants of motion.

Computational strategy

The fact that the non-linear dynamic equations of the deformable bodies that undergo large displacements can be expressed in terms of a set of invariants of motion suggests a two-stage computational strategy. In the first stage, the invariants of motion as well as the conventional stiffness matrix are evaluated in a *preprocessor computer program.* This program systematically constructs the invariants and stiffness matrices of the finite elements of each deformable body in the multibody system. These element matrices are then assembled in order to obtain the matrices of the deformable bodies in the system. If the modal co-ordinates are to be used to reduce the number of co-ordinates of some deformable bodies in the system, the invariants as well as the stiffness matrices of these bodies can be expressed in their modal form in the preprocessor computer program.[2] The output of the preprocessor is a set of data that remain constant throughout the motion of the bodies. These data are used as part of the input data to the *main processor* used for the dynamic simulation. The computational algorithm of the main processor can be based on either the *augmented* or the *recursive formulation.* The same preprocessor can be used in both cases since the invariants of motion are characteristics of the deformable body and they do not depend on the approach used for formulating the dynamic equations of the multibody system.

4. AUGMENTED FORMULATION

In the augmented formulation, the dynamic equations of the flexible multibody system are formulated in terms of a set of co-ordinates that are not totally independent. The relationships between these co-ordinates are formulated using a set of non-linear algebraic constraint equations that describe the mechanical joints and the specified motion trajectories in the multibody system. These kinematic constraint equations can be introduced to the dynamic formulation using the vector of algebraic constraint equations, which can be written compactly as

$$\mathbf{C}(\mathbf{q}, \mathbf{t}) = \mathbf{0} \tag{19}$$

where $\mathbf{C}$ is the vector of the kinematic constraint equations that can be a linear or non-linear function of the system generalized co-ordinates $\mathbf{q}$ and time t.

In the *augmented formulation*, the equations of motion can be written compactly as[2]

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}^{\mathrm{T}}_{\mathbf{q}}\boldsymbol{\lambda} = \mathbf{Q}_e + \mathbf{F} \tag{20}$$

where $\mathbf{M}$ is the system mass matrix, $\mathbf{C_q}$ is the Jacobian matrix of the kinematic constraints, $\boldsymbol{\lambda}$ is the vector of Lagrange multipliers, $\mathbf{Q}_e$ is the vector of externally applied and elastic forces and $\mathbf{F}$ is the vector of Coriolis and centrifugal forces.

Computer formulation of the joint constraints

Figure 3 shows examples of some of the mechanical joints that are often encountered in several industrial and technological applications. The *spherical joint* shown in Figure 3(a) has three degrees of freedom which allow three independent relative rotations between the two bodies connected by this joint. The *cylindrical joint* shown in Figure 3(b) has two degrees of freedom since it allows relative translation along, and relative rotation about the joint axis. The *revolute* and *prismatic* joints shown, respectively, in Figures 3(c) and (d) have only one degree of freedom. The mathematical formulation of these joints can be expressed in the form of equation (19). For example, in the case of the *spherical joint* we require that two points on body i and body j, which are connected by this joint, remain in contact throughout the motion of the two bodies. In terms of the absolute co-ordinates, this condition can be expressed in the form of equation (19) as

$$\mathbf{R}^i + \mathbf{A}^i\bar{\mathbf{u}}_p^i - \mathbf{R}^j - \mathbf{A}^j\bar{\mathbf{u}}_p^j = \mathbf{0} \tag{21}$$

where superscripts i and j refer, respectively, to bodies i and j and $\bar{\mathbf{u}}_p^i$ and $\bar{\mathbf{u}}_p^j$ are the local position vectors of the joint definition points on body i and body j, respectively. Note that $\bar{\mathbf{u}}_p^i$ and $\bar{\mathbf{u}}_p^j$, in flexible body dynamics, are implicit functions of time since they depend on the deformation of the bodies.

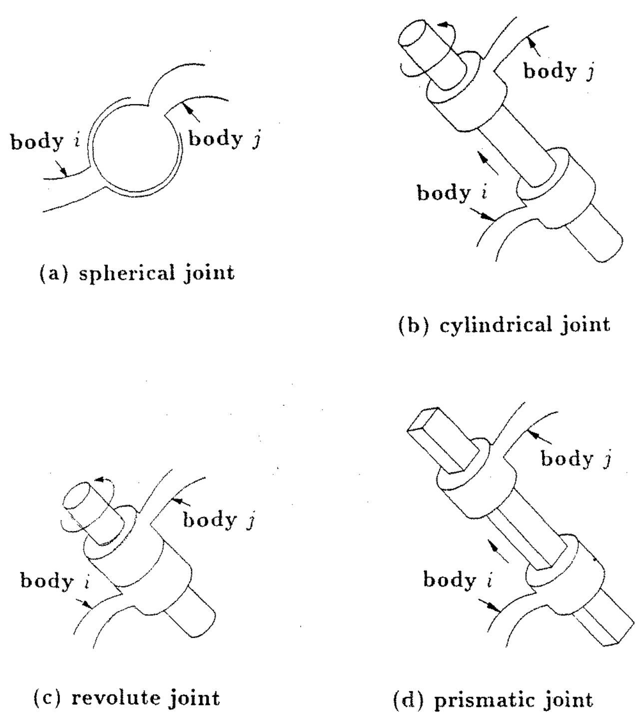

Figure 3. Examples of mechanical joints

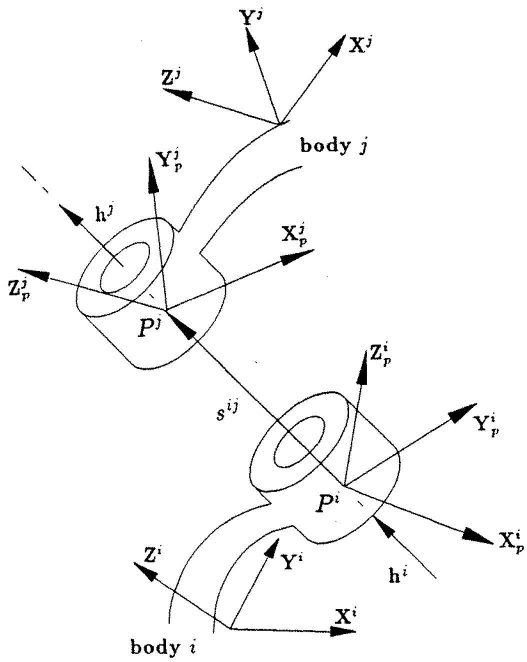

Figure 4. Intermediate joint co-ordinate systems

In order to be able to formulate the kinematic constraints that describe the cylindrical, revolute and prismatic joints in flexible body dynamics a set of *intermediate body fixed joint co-ordinate systems* must be introduced. Figure 4 shows body i and body j that are connected by a *cylindrical joint* that allows relative translation and rotation between the two bodies. Let $X^i Y^i Z^i$ and $X^j Y^j Z^j$ be the co-ordinate systems of body i and body j, respectively. For the convenience of describing the large relative displacements between the two bodies, the intermediate body fixed joint co-ordinate systems $X^i_p Y^i_p Z^i_p$ and $X^j_p Y^j_p Z^j_p$ are introduced. These co-ordinate systems are assumed to have zero mass and inertia and their origins are assumed, in the finite element formulation, to be rigidly, attached to nodal points on the two bodies. The relative motion between the two bodies is assumed to be along the joint axis. We make the assumption that the joint axis can be described by a rigid line. Let $\mathbf{h}^i$ be a vector drawn on body i along the joint axis. Similarly, let $\mathbf{h}^j$ be a vector drawn on body j along the joint axis, as shown in Figure 4. As shown in the figure, the vector $\mathbf{s}^{ij}$ has a variable magnitude since it connects points P^i and P^j on bodies i and j, respectively. The kinematic constraint equations for the cylindrical joint can be written as

$$\left.\begin{aligned} \mathbf{h}^i \times \mathbf{h}^j = \mathbf{0} \\ \mathbf{h}^i \times \mathbf{s}^{ij} = \mathbf{0} \end{aligned}\right\} \tag{22}$$

where

$$\mathbf{s}^{ij} = \mathbf{R}^i + \mathbf{A}^i \bar{\mathbf{u}}^i_p - \mathbf{R}^j - \mathbf{A}^j \bar{\mathbf{u}}^j_p$$

$$\mathbf{h}^i = \mathbf{A}^i \mathbf{A}^i_p \bar{\mathbf{h}}^i$$

$$\mathbf{h}^j = \mathbf{A}^j \mathbf{A}^j_p \bar{\mathbf{h}}^j$$

in which $\bar{\mathbf{h}}^i$ and $\bar{\mathbf{h}}^j$ are constant vectors defined in the intermediate body fixed joint co-ordinate systems $X_p^i Y_p^i Z_p^i$ and $X_p^j Y_p^j Z_p^j$, respectively, and $\mathbf{A}_p^i$ and $\mathbf{A}_p^j$ are the transformation matrices from the intermediate co-ordinate systems to the body co-ordinate systems. If the deformations of bodies i and j are assumed to be small, $\mathbf{A}_p^i$ and $\mathbf{A}_p^j$ are infinitesimal rotation matrices that can be expressed in terms of the slopes at the nodal points.[5] Note also that the constraint equations of equation (22) have only four independent algebraic equations which are non-linear in the reference and elastic co-ordinates of the two bodies.

The *revolute joint* can be considered as a special case of the cylindrical joint. In this case, the length of the vector $\mathbf{s}^{ij}$ is constant. Therefore, the kinematic constraint equations of the revolute joint are

$$\left.\begin{aligned} \mathbf{h}^i \times \mathbf{h}^j &= \mathbf{0} \\ \mathbf{h}^i \times \mathbf{s}^{ij} &= \mathbf{0} \\ \mathbf{s}^{ij\mathrm{T}} \mathbf{s}^{ij} &= c \end{aligned}\right\} \tag{23}$$

where c is a constant scalar.

Similarly, the constraint equations of the *prismatic joint* are

$$\left.\begin{aligned} \mathbf{h}^i \times \mathbf{h}^j &= \mathbf{0} \\ \mathbf{h}^i \times \mathbf{s}^{ij} &= \mathbf{0} \\ \mathbf{n}^{i\mathrm{T}} \mathbf{n}^j &= \mathbf{0} \end{aligned}\right\} \tag{24}$$

where $\mathbf{n}^i$ and $\mathbf{n}^j$ are two vectors drawn perpendicular to the joint axis on body i and body j, respectively. Note that $\mathbf{n}^i$ and $\mathbf{n}^j$ can be defined as

$$\mathbf{n}^i = \mathbf{A}^i \mathbf{A}_p^i \bar{\mathbf{n}}^i$$
$$\mathbf{n}^j = \mathbf{A}^j \mathbf{A}_p^j \bar{\mathbf{n}}^j$$

in which $\bar{\mathbf{n}}^i$ and $\bar{\mathbf{n}}^j$ are constant vectors defined in the intermediate joint co-ordinate systems $X_p^i Y_p^i Z_p^i$ and $X_p^j Y_p^j Z_p^j$. That is, the vectors $\mathbf{n}^i$ and $\mathbf{n}^j$ must be iteratively updated during the dynamic simulation.

Many of the kinematic constraints that describe mechanical joints in this section are formulated using the notation of the cross product between two vectors. Examples of these constraints are the equations of the cylindrical joint given by equation (22). Each of these equations provides three equations; only two of them are independent. A criterion for the selection of a proper set of independent equations[15] is to compare the absolute values of the components of one of the vectors that appear in the cross product equation and select the two scalar equations having the largest components. This approach, as demonstrated by Nikravesh,[15] can be used to avoid the singularities that occur at special configurations.

Solution for the accelerations

Equations (19) and (20) represent a system of algebraic and differential equations which can be solved using computer and numerical methods. In order to be able to numerically integrate this system, equations (19) and (20) must be solved for the vector of accelerations. To this end, equation (19) is differentiated twice with respect to time. This leads to

$$\mathbf{C}_{\mathbf{q}} \ddot{\mathbf{q}} = \mathbf{Q}_{\mathrm{c}} \tag{25}$$

where $\mathbf{Q}_c$ is a vector that absorbs terms that are quadratic in the velocity.[2] If equations (20) and (25) are combined one obtains a system of matrix equation that is linear in the vectors of accelerations and Lagrange multipliers. This matrix equation can be written as

$$\begin{bmatrix} \mathbf{M} & \mathbf{C}_{\mathbf{q}}^{T} \\ \mathbf{C}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_e + \mathbf{F} \\ \mathbf{Q}_c \end{bmatrix} \tag{26}$$

This system of equations can be solved for the generalized reference and elastic accelerations as well as the vector of Lagrange multipliers. Note that, in this case, the obtained solution contains both the dependent and independent accelerations. Lagrange multipliers can be used to determine the generalized forces of the joint constraints and specified motion trajectories. The actual reaction and driving forces and torques can be obtained using the procedure described in Reference 4.

Another alternate approach, but numerically different, is to use the generalized co-ordinate partitioning.[16] In this case, the vector of system generalized co-ordinates can be written as

$$\mathbf{q} = [\mathbf{q}_i^T \quad \mathbf{q}_d^T]^T \tag{27}$$

where $\mathbf{q}_i$ is the vector of system independent co-ordinates, and $\mathbf{q}_d$ is the vector of dependent co-ordinates. According to this co-ordinate partitioning, equation (25) can be written as

$$\mathbf{C}_{\mathbf{q}_i}\ddot{\mathbf{q}}_i + \mathbf{C}_{\mathbf{q}_d}\ddot{\mathbf{q}}_d = \mathbf{Q}_c \tag{28}$$

where $\mathbf{C}_{\mathbf{q}_i}$ and $\mathbf{C}_{\mathbf{q}_d}$ are the sub-Jacobians associated with the independent and dependent co-ordinates, respectively. The matrix $\mathbf{C}_{\mathbf{q}_d}$ is a square matrix and if the kinematic constraint equations are assumed to be linearly independent, the dependent co-ordinates can be selected such that the matrix $\mathbf{C}_{\mathbf{q}_d}$ is non-singular. Wehage[16] used the **LU** factorization method to identify the independent co-ordinates. Other techniques such as the *singular value decomposition*[17] and the **QR**[18] method that involves *Householder iterations* are also proposed.

Equation (28) can then be used to write the dependent co-ordinates in terms of the independent ones. In this case one has

$$\ddot{\mathbf{q}}_d = \mathbf{B}_{di}\ddot{\mathbf{q}}_i + \mathbf{C}_{\mathbf{q}_d}^{-1}\mathbf{Q}_c$$

in which

$$\mathbf{B}_{di} = -\mathbf{C}_{\mathbf{q}_d}^{-1}\mathbf{C}_{\mathbf{q}_i}$$

Therefore, the total vector of system accelerations can be written in terms of the independent accelerations as

$$\ddot{\mathbf{q}} = \begin{bmatrix} \ddot{\mathbf{q}}_i \\ \ddot{\mathbf{q}}_d \end{bmatrix} = \mathbf{C}_{di}\ddot{\mathbf{q}}_i + \bar{\mathbf{Q}}_c \tag{29}$$

where

$$\mathbf{C}_{di} = \begin{bmatrix} \mathbf{I} \\ \mathbf{B}_{di} \end{bmatrix}, \quad \bar{\mathbf{Q}}_c = \begin{bmatrix} \mathbf{0} \\ \mathbf{C}_{\mathbf{q}_d}^{-1}\mathbf{Q}_c \end{bmatrix}$$

Substituting equation (29) into equation (20), premultiplying by the transpose of the matrix $\mathbf{B}_d^i$, and using the fact that $\mathbf{B}_{di}^T\mathbf{C}_{\mathbf{q}}^T = \mathbf{0}^2$, the vector of Lagrange multipliers can be eliminated from equation (20). This leads to the reduced system of equations[2]

$$\mathbf{M}_{ii}\ddot{\mathbf{q}}_i = \mathbf{R}_i \tag{30}$$

where $\mathbf{M}_{ii}$ is the mass matrix associated with the independent co-ordinates and $\mathbf{R}_i$ is the vector of generalized forces associated with those co-ordinates.

As pointed out in Reference 2, the use of the *embedding technique* that leads to equation (30) is not computationally as efficient as the use of equation (26) to solve for the accelerations. For this reason, equation (26) is used with *sparse matrix techniques* in several commercially available multibody computer programs.

Numerical solution of the augmented equations

Once the vector of accelerations is determined several numerical integration procedures can be used for solving for the co-ordinates and velocities. Perhaps the simplest approach is to integrate both the dependent and independent accelerations assuming that the error in the numerical integration is small enough such that the kinematic constraint relationships of equation (19) are not significantly violated. Clearly in this case, one must use an accurate numerical integration method in order to avoid large errors in the numerical solution.

A second alternative is to use *Wehage's algorithm* in which the independent accelerations are identified and integrated forward in time using direct numerical integration methods. The numerical solution defines the independent co-ordinates and velocities. The dependent co-ordinates and velocities can then be determined using the kinematic constraint relationships.[2,16] Since the kinematic constraints equations can be non-linear functions of the system variables, in Wehage's algorithm a *Newton–Raphson algorithm* is used in order to solve for the dependent variables.

A third alternative is to numerically integrate all the accelerations, dependent and independent, in order to obtain the dependent and independent co-ordinates and velocities. The dependent co-ordinates and velocities can then be adjusted in order to satisfy the kinematic relationships. A Newton–Raphson algorithm must also be used in this case. Furthermore, this approach has an additional disadvantage since most accurate numerical integration routines store the time history of the variables. Any adjustment made in these variables outside the integration routine can be a source of numerical problems.

5. RECURSIVE AND PROJECTION METHODS

In the preceding section, the use of the augmented formulation in the computer aided analysis of flexible multibody systems is discussed. In this type of formulation, the kinematic and dynamic equations are formulated in terms of a mixed set of dependent and independent co-ordinates. In this case, one may introduce Lagrange multipliers, or use the embedding technique to reduce the number of dynamic equations to a minimum set. In this section other alternate approaches that can be used in the analysis of flexible multibody systems are discussed. In these approaches the system kinematic and dynamic equations are formulated in terms of the system joint degrees of freedom. If two bodies are connected by a joint, the co-ordinates of one body can be expressed in terms of the co-ordinates of the other body as well as the joint degrees of freedom.[3,5] Using these displacement relationships, the velocity and acceleration equations can be obtained by direct differentiation. For example, if two bodies are connected by cylindrical joint as shown in Figure 4, the relationship between the reference and elastic accelerations of body i and the reference and elastic accelerations of body j and the joint accelerations can be written as[19–21]

$$\ddot{\mathbf{q}}^i = \mathbf{G}^i\ddot{\mathbf{q}}^j + \mathbf{H}^i\ddot{\mathbf{P}}^i + \boldsymbol{\gamma}^i \tag{31}$$

where $\mathbf{G}^i$ and $\mathbf{H}^i$ are velocity influence coefficient matrices that depend on the co-ordinates of the two bodies, $\boldsymbol{\gamma}^i$ is a vector that absorbs terms that are quadratic in the velocities, $\ddot{\mathbf{q}}^i$ and $\ddot{\mathbf{q}}^j$ are the vectors of reference and elastic accelerations of bodies i and j, respectively, and $\ddot{\mathbf{P}}^i$ is the vector of the joint and elastic accelerations of body i. In the case of the constrained motion, the *generalized Newton–Euler equations* of equation (15) can be written for the deformable body i as

$$\mathbf{M}^i\ddot{\mathbf{q}}^i = \mathbf{Q}_e^i + \mathbf{Q}_v^i + \mathbf{Q}_R^i \tag{32}$$

where $\mathbf{M}^i$ is the mass matrix, $\mathbf{Q}_e^i$ is the vector of externally applied and elastic forces and moments, $\mathbf{Q}_v^i$ is the vector of Coriolis and centrifugal force components, and $\mathbf{Q}_R^i$ is the vector of reaction forces and moments. Note that equation (31) can be used to eliminate the reference and elastic accelerations of body i from equation (32). This leads to a set of dynamic equations of body i expressed in terms of the reference and elastic accelerations of body j and the joint accelerations. Furthermore, in these equations, the joint reaction forces between the two bodies are automatically eliminated. This procedure can be continued from one body to another until the base body is reached, leading to a system of dynamic equations expressed in terms of the degrees of freedom.

Most existing recursive methods lead to small systems of strongly coupled equations. The coefficient matrix of the acceleration equation is dense and non-linear as the result of the large relative displacement between the interconnected bodies. Decoupling the joint and elastic accelerations in these equations will require finding the inverse or the **LU** factorization of non-linear matrices whose dimension depends on the number of elastic degrees of freedom. Consequently, speaking of the order of an algorithm becomes meaningless since the number of elastic co-ordinates varies from one body to another. Recently, a recursive method[19–21] that systematically decouple the joint and elastic accelerations was proposed. In this method, the generalized Newton–Euler equations, the relationship between the absolute, elastic and joint accelerations, and the reaction force equations are combined in order to form a system of loosely coupled equations which has a sparse matrix structure. By using matrix partitioning, the coupling between the joint and elastic accelerations can be eliminated. This leads to smaller system of equations in the joint accelerations and joint reaction forces. The dimension of the coefficient matrix in this system is independent of the number of elastic co-ordinates. This procedure can be demonstrated by utilizing equation (31) to write the absolute and elastic accelerations in terms of the joint and elastic accelerations as

$$\begin{bmatrix} \ddot{\mathbf{q}}_r \\ \ddot{\mathbf{q}}_f \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{H}}_{PP} & \bar{\mathbf{H}}_{Pf} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{P}}_r \\ \ddot{\mathbf{q}}_f \end{bmatrix} + \begin{bmatrix} \boldsymbol{\gamma}_r \\ \mathbf{0} \end{bmatrix} \tag{33}$$

where subscript r and f refer, respectively, to the absolute reference and elastic co-ordinates, $\mathbf{P}_r$ is the vector of system joint co-ordinates, $\bar{\mathbf{H}}_{PP}$ and $\bar{\mathbf{H}}_{Pf}$ are velocity influence coefficient matrices, and $\boldsymbol{\gamma}_r$ is a vector that absorbs terms which are quadratic in the velocities.

The equations of motion of the flexible multibody system expressed in terms of the absolute co-ordinates can be written as

$$\mathbf{M}\ddot{\mathbf{q}} = \mathbf{Q} + \mathbf{F} \tag{34}$$

where $\mathbf{M}$ is the system mass matrix, $\ddot{\mathbf{q}}$ is the vector of absolute accelerations, $\mathbf{Q}$ is the vector of forces that absorbs applied, centrifugal, Coriolis and elastic forces, and $\mathbf{F}$ is the vector of joint reaction forces. Equation (34) can also be written as

$$\begin{bmatrix} \mathbf{M}_{rr} & \mathbf{M}_{rf} \\ \mathbf{M}_{fr} & \mathbf{M}_{ff} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_r \\ \ddot{\mathbf{q}}_f \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_r \\ \mathbf{Q}_f \end{bmatrix} + \begin{bmatrix} \mathbf{F}_r \\ \mathbf{F}_f \end{bmatrix} \tag{35}$$

where, as previously pointed out, subscripts r and f refer, respectively, to the rigid body and elastic co-ordinates.

The joint reaction forces must satisfy the identity

$$\begin{bmatrix} \bar{\mathbf{H}}_{\mathrm{PP}}^{\mathrm{T}} & \mathbf{0} \\ \bar{\mathbf{H}}_{\mathrm{P}f}^{\mathrm{T}} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{F}_r \\ \mathbf{F}_f \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{36}$$

and consequently

$$\bar{\mathbf{H}}_{\mathrm{PP}}^{\mathrm{T}} \mathbf{F}_r = \mathbf{0} \tag{37a}$$

$$\mathbf{F}_f = -\bar{\mathbf{H}}_{\mathrm{P}f}^{\mathrm{T}} \mathbf{F}_r \tag{37b}$$

These equations show that the joint forces associated with the elastic co-ordinates do not introduce new independent variables. These forces can be determined by using the joint forces associated with the reference co-ordinates.

Substituting equation (37b) into equation (35), one obtains

$$\mathbf{M}_{rr}\ddot{\mathbf{q}}_r + \mathbf{M}_{rf}\ddot{\mathbf{q}}_f = \mathbf{Q}_r + \mathbf{F}_r \tag{38a}$$

$$\mathbf{M}_{fr}\ddot{\mathbf{q}}_r + \mathbf{M}_{ff}\ddot{\mathbf{q}}_f = \mathbf{Q}_f - \bar{\mathbf{H}}_{\mathrm{P}f}^{\mathrm{T}} \mathbf{F}_r \tag{38b}$$

The first matrix equation in equation (33) can be written as

$$\ddot{\mathbf{q}}_r = \bar{\mathbf{H}}_{\mathrm{PP}}\ddot{\mathbf{P}}_r + \bar{\mathbf{H}}_{\mathrm{P}f}\ddot{\mathbf{q}}_f + \boldsymbol{\gamma}_r \tag{39}$$

Combining equations (37a), (38a), (38b) and (39), one obtains

$$\begin{bmatrix} \mathbf{M}_{rr} & \mathbf{M}_{rf} & \mathbf{I} & \mathbf{0} \\ \mathbf{M}_{fr} & \mathbf{M}_{ff} & -\bar{\mathbf{H}}_{\mathrm{P}f}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{I} & -\bar{\mathbf{H}}_{\mathrm{P}f} & \mathbf{0} & -\bar{\mathbf{H}}_{\mathrm{PP}} \\ \mathbf{0} & \mathbf{0} & -\bar{\mathbf{H}}_{\mathrm{PP}}^{\mathrm{T}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_r \\ \ddot{\mathbf{q}}_f \\ -\mathbf{F}_r \\ \ddot{\mathbf{P}}_r \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_r \\ \mathbf{Q}_f \\ \boldsymbol{\gamma}_r \\ \mathbf{0} \end{bmatrix} \tag{40}$$

This system of equations has a dimension equal to $12n + n_f + n_r$ where n is the total number of bodies, n_f is the total number of elastic degrees of freedom and n_r is the total number of joint co-ordinates. The coefficient matrix in equation (40) is symmetric and sparse. This system can be solved in order to obtain the absolute, joint and elastic accelerations as well as the joint reaction forces. Note that the joint and elastic accelerations are coupled in this equation. If the number of elastic degrees of freedom is large, the solution of equation (40) at every time step can be computationally expensive. The joint and elastic accelerations can, however, be decoupled leading to a smaller system of equation whose dimension is independent of the number of elastic degrees of freedom. To this end, one can utilize the fact that the matrix $\mathbf{M}_{ff}$ is a constant positive definite matrix. This is usually the case when a consistent mass formulation is used or when the modal co-ordinates are employed. Using $\mathbf{M}_{ff}$ as the pivot element in equation (40), one can use a simple Gauss–Jordan elimination procedure to obtain the following reduced system of equations

$$\begin{bmatrix} (\mathbf{M}_{rr} - \mathbf{M}_{rf}\mathbf{M}_{ff}^{-1}\mathbf{M}_{fr}) & (\mathbf{I} + \mathbf{M}_{rf}\mathbf{M}_{ff}^{-1}\bar{\mathbf{H}}_{\mathrm{P}f}^{\mathrm{T}}) & \mathbf{0} \\ (\mathbf{I} + \bar{\mathbf{H}}_{\mathrm{P}f}\mathbf{M}_{ff}^{-1}\mathbf{M}_{fr}) & -\bar{\mathbf{H}}_{\mathrm{P}f}\mathbf{M}_{ff}^{-1}\bar{\mathbf{H}}_{\mathrm{P}f}^{\mathrm{T}} & -\bar{\mathbf{H}}_{\mathrm{PP}} \\ \mathbf{0} & -\bar{\mathbf{H}}_{\mathrm{PP}}^{\mathrm{T}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_r \\ -\mathbf{F}_r \\ \ddot{\mathbf{P}}_r \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_r - \mathbf{M}_{rf}\mathbf{M}_{ff}^{-1}\mathbf{Q}_f \\ \boldsymbol{\gamma}_r + \bar{\mathbf{H}}_{\mathrm{P}f}\mathbf{M}_{ff}^{-1}\mathbf{Q}_f \\ \mathbf{0} \end{bmatrix} \tag{41}$$

Observe that the dimension of this system of equation is independent of the number of elastic co-ordinates of the system. Furthermore, the coefficient matrix remains symmetric. This system can be solved for the absolute reference and joint accelerations as well as the joint reaction forces. The elastic accelerations can then be obtained by solving equation (38b). Since $\mathbf{M}_{ff}$ is a constant matrix, the solution for the elastic acceleration is trivial, especially in the case of using the modal co-ordinates because $\mathbf{M}_{ff}$ is a diagonal matrix in this case. It can be shown that the matrices $(\mathbf{M}_{rr} - \mathbf{M}_{rf}\mathbf{M}_{ff}^{-1}\mathbf{M}_{fr})$ and $\bar{\mathbf{H}}_{Pf}\mathbf{M}_{ff}^{-1}\bar{\mathbf{H}}_{Pf}^{T}$ on the main diagonal of the coefficient matrix in equation (41) are block diagonal matrices. Consequently, the *recursive projection* procedure described in References 19–21 can be applied. Clearly, this procedure has a computational advantage over existing order n algorithms, because it is independent of the number of elastic degrees of freedom of the system.

6. COUPLING EFFECTS

Several investigations have been conducted recently in order to examine the effect of the coupling between various displacements in the analysis of flexible multibody systems. The coupling between the displacements of a deformable body that undergoes large displacements can be classified as *kinematic, inertia,* or *elastic coupling.* In this section, these three types of coupling will be discussed. Before this section is concluded a brief discussion on the effect of the finite rotation on the *wave propagation* is also presented.

Kinematic coupling

In Euler–Bernoulli theory for straight beams, it is assumed that there is no coupling between the longitudinal and the transverse displacements. Mode shapes are either bending or axial modes. Kinematic coupling between the transverse and axial displacements, however, results from considering the effect of the rotary inertia[22] and/or the initial curvature of the beam.[23,24] In this case, the mode shapes contain both the axial and bending effects. It is shown in several investigations[22–24] that these kinematic coupling effects can have significant effects on the dynamic response of the elastic members that undergo large rigid body displacements.

Inertia coupling

Coupling between the large rigid body displacement and the elastic deformation as well as coupling between the transverse and longitudinal displacements exists as the result of the finite rotation. Even in the cases in which the longitudinal and transverse displacements are not kinematically coupled, inertia coupling between these two types of motion does still exist as the result of the rigid body displacements. Furthermore, in many applications, the elastic deformation can have a significant effect on the large rigid body motion as the result of the inertia coupling. This inertia coupling is represented by non-linear terms in the mass matrix and the Coriolis and centrifugal force components. The inertia tensor of the deformable body, defined in the body co-ordinate system, is no longer a constant matrix since it depends on the body deformation.

Several investigators in their analysis of spatial mechanisms and robotics have used the 4×4 transformation matrix in order to describe the large relative displacements between two deformable bodies. The Denavit–Hartenberg 4×4 transformation matrix, widely used in the analysis of rigid body systems, depends on four parameters. These are the *link length, link twist, joint angle* and *link offset.* The joint angle and the link offset describe the large relative rotational and translational displacements between the two bodies. The link length and link twist are assumed to be constant parameters that depend on the shape of the links. Clearly, in the case of deformable bodies, the link length and link offset are no longer constant. Consequently, the elastic deformation has an effect on the 4×4 transformation matrix. This effect, however, has been neglected in most of the investigations that used the 4×4 Denavit–Hartenberg transformation matrix. Such an assumption, which implies that the elastic deformation has a negligible effect on the transformation between the co-ordinate systems, leads to a partial linearization of the kinematic relationships. The effect of this linearization on the terms that represent the inertia coupling between the rigid body motion and the elastic deformation needs to be examined. Similar comments can also be made regarding the use of the *finite element Lagrangian formulations* in the analysis of deformable bodies that undergo finite rotations. As discussed in Section 2, the use of the nodal co-ordinates of beam and plate elements to describe finite rotations leads to linearization of the terms that define the order of the element rotation.[8,9]

Elastic coupling

Elastic coupling between the longitudinal and transverse displacements of structural components results from considering the effect of geometric elastic non-linearities. For example, if non-linear strain–displacement relationships are used, one obtains, in addition to the conventional stiffness matrix, the geometric stiffness matrix which includes elastic coupling between the longitudinal and transverse displacements. The effect of this coupling on the dynamics of deformable components in multibody systems has been examined in several publications.[25,26]

Effect of the finite rotation on the wave motion

The significant effect of the coupling between the rigid body motion and the elastic deformation has been recognized and discussed in several publications. While the elastic deformation can be represented as a wave motion, the effect of the finite rotation on the wave phenomenon has been examined only recently. By understanding the relationship between the rigid body motion and the elastic deformation on the microscopic level, one will be able to better understand the dynamics of deformable bodies that undergo large rigid body displacements.

In perfectly elastic non-rotating rods, the wave motion can be described as the sum of an infinite number of harmonic waves travelling with the same *phase velocity*. This phase velocity is constant and is defined as

$$c_0 = \sqrt{\frac{E}{\rho}}$$

where E and ρ are, respectively, the modulus of elasticity and the mass density of the rod. Since all the harmonic waves have the same phase velocity, the *group velocity* is constant and is equal to the phase velocity. In this case, the elastic medium is said to be *non-dispersive*. Observe that in a non-dispersive medium, the phase and group velocities are independent of the wave number and the dimensions of the rod. Dispersion, however, occurs as the result of the finite rotation.[6,7] Harmonic waves no longer travel with the same phase velocity. Consequently, the group velocity becomes dependent on the wave number. Furthermore, both the group and phase velocities depend on the finite rotation and become dependent on the dimension of the rod. It was also shown[6,7] that the phase velocity of low frequency harmonic waves is more affected by the finite rotation as compared to the high frequency harmonics.

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