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Density gradient enhanced topology optimization of continuum structures

The formulation of topology optimization problems for continuum structures usually requires the introduction of the material density as design variable. In this context the design domain is discretized by finite elements and a constant density distribution within the elements is assumed. To prevent the occurrence of optimal designs characterized by extended regions with intermediate density values, special penalty methods are used. The introduction of those methods results in discontinuities in the global density distribution and often leads to designs containing unfavourable microstructures such as the well-known checkerboard patterns. To obtain designs free of microstructures, we propose a global density gradient based regularization and discuss the results of numerical studies of several maximum-stiffness-design problems based on the presented approach.

1. Maximum-stiffness-design problem

The general topology optimization problem usually results in the determination of the optimal material distribution in a defined design domain Ω. In this context an isotropic material is used and a discrete material-indicator function

$$\rho(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \Omega^s \\ 0, & \mathbf{x} \in \Omega^e \end{cases} \tag{1}$$

is introduced as design variable [1], which divides the design domain Ω into a solid region Ω^s and an empty region Ω^e. To obtain a continuous optimization problem the indicator function is identified as the material density and intermediate values between 0 and 1 are admitted. Furthermore, a linear relationship between density and Young's modulus E of the material is postulated. In the case of linear-elastic maximum-stiffness-design the corresponding so-called relaxed optimization problem can be formulated in the following form

$$\min_{\rho} \left\{ \int_{\Omega} \mathbf{f}_v^T \mathbf{u} \, d\Omega + \int_{\Gamma_t} \mathbf{f}_t^T \mathbf{u} \, d\Gamma_t \right\} \tag{2}$$

subject to :

$$\int_{\Omega} \boldsymbol{\varepsilon}^T(\mathbf{u}) \, \mathbf{C}(E) \, \boldsymbol{\varepsilon}(\delta \mathbf{u}) \, d\Omega = \int_{\Omega} \mathbf{f}_v^T \delta \mathbf{u} \, d\Omega + \int_{\Gamma_t} \mathbf{f}_t^T \delta \mathbf{u} \, d\Gamma_t \ , \quad E = \rho E_0 \ , \quad \int_{\Omega} \rho \, d\Omega \leq M_0 \ , \quad 0 \leq \rho \leq 1. \tag{3}$$

The displacement field $\mathbf{u}$ caused by body forces $\mathbf{f}_v$ and traction forces $\mathbf{f}_t$, acting on the boundary Γ_t of the design domain, is determined by the variational form of the equilibrium condition, which can be handled as a constraint to the optimization problem. In this context $\mathbf{C}$ denotes the material rigidity tensor and $\boldsymbol{\varepsilon}$ corresponds to linearized strains. To solve the equilibrium problem the design domain is usually discretized by finite elements and a constant density distribution within the elements is assumed. Furthermore, a mass constraint and the mentioned restriction on the admissible density values have to be introduced to obtain a well-posed problem.

Optimal designs, based on the above formulation, contain in consequence of the introduced relaxation so-called "gray" regions characterized by intermediate values of material density. This effect can be observed in the example stated below, showing the numerical solution of the maximum-stiffness-design problem for an exemplary design domain, boundary conditions and loading (figure 1).

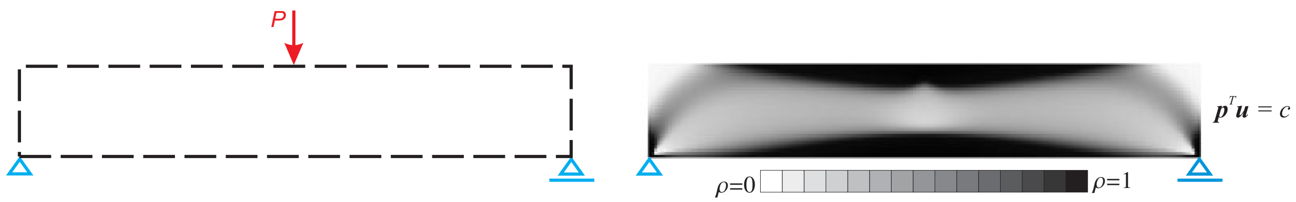

Figure 1: Design domain & optimized structure

Since the realization of inhomogeneous materials with intermediate material densities turns out to be difficult from the engineering point of view, special penalty methods are used to prevent the occurrence of extended regions with intermediate density values. In this context the SIMP-approach [1], based on a non-linear relationship between density and Young's modulus of the material in the form

$$E = \rho^p E_0 \quad , \quad p \geq 1 \ , \tag{4}$$

has achieved a widespread acceptance. The introduction of the SIMP-approach or equivalent penalty methods results in discontinuities in the global density distribution and often leads to designs containing unfavourable microstructures such as the well-known checkerboard-like patterns [2], where density values 0 and 1 alternate as shown in figure 2.

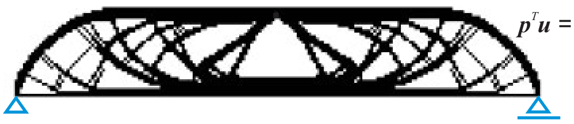

Figure 2: Optimized structure (SIMP-based)

2. Density gradient enhanced maximum-stiffness-design (Extended SIMP-approach)

To obtain optimal designs free of extended regions characterized by intermediate density values, as well as microstructures, we introduce a global density gradient based penalty-functional

$$\lambda(\nabla\rho) = \int_\Omega (\nabla\rho)^T \, \nabla\rho \, d\Omega \ , \tag{5}$$

which prevents the formation of extended regions with oscillating density values characterized by high density gradients. Furthermore, we modify the known SIMP-approach by including the above functional in the form

$$E = \rho^p E_o e^{-\gamma\lambda(\nabla\rho)} \quad , \quad p \geq 1 \ , \ \gamma \geq 0 \ , \tag{6}$$

where γ corresponds to an additional penalty parameter. In this context a density gradient has to be defined, whose existence requires a continuous density distribution within the whole design domain. Consequently, a finite element type with density as nodal variable is formulated, which allows a variable density within the element.
The solution of the above problem results, on the basis of the presented approach, in the following optimal design

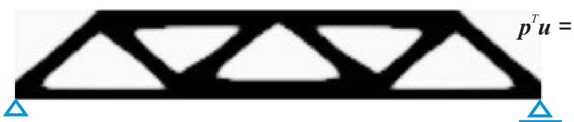

Figure 3: Optimized structure (X-SIMP-based)

which can be considered as an admissible solution from the engineering point of view. Numerical studies of different maximum-stiffness-design problems, based on the X-SIMP-approach, demonstrate that the optimal designs solely consist of extended "black" and "white" regions. Consequently, material distributions containing holes at the macroscopic level can be obtained, which are considerably easier to realize than complex microstructures. In this context a small decrease of the achieved structure-stiffness, in consequence of the additional density gradient constraint, is usually observed, whose size, however, is not significant. Furthermore, the determined designs are characterized by a smooth boundary of the structure as a result of the density gradient penalization. Finally, the introduced approach constrains the total length of the interface of "black" and "white" regions, so that mesh-independent results are obtained.

3. References

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