PREDICTION OF CONFIDENCE LIMITS FOR RELIABILITY INDEXES WITH LINEAR DEPENDENCE ON PARAMETERS

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We construct the confidence limits for a reliability index when the reliability-determining stochastic process is linear in random parameters.

1. Statement of the Problem

Reliability theory often uses a parametric model of failures, assuming that system availability depends on some determining parameter whose variation in time is described by the stochastic process $\xi(t)$, and the system availability condition in period $[0, T]$ is

$$A(T)=\left\{\xi(t)\in D,\ \forall t\in[0,T]\right\}. \tag{1}$$

The set D of parameter values for which the system is failfree is called the tolerance field.

In case of failure due to wear and tear, we naturally consider processes $\xi(t)$ with monotone sampling functions, e.g., monotone decreasing processes, identifying the tolerance field with the interval $[x_0, \infty)$ or $(-\infty, x_0]$. The availability condition A(T) in this case takes the form

$$A(T)=\left\{\xi(t)\gtrless x_0,\ \forall t\in[0,T]\right\}=\left\{\xi(T)\gtrless x_0\right\}.$$

With regard to the process $\xi(t)$ we assume that this is a so-called "semistochastic" process [2] with a given analytical expression of the function $\xi(t)$ which includes random parameters.

We assume that the process $\xi(t)$ is linear in the random parameters α_1, i.e.,

$$\xi(t)=\alpha_1\varphi_1(t)+\alpha_2\varphi_2(t)+\ldots+\alpha_k\varphi_k(t), \tag{2}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are independent and normally distributed with $(\mu_i, \sigma^2\theta_i^2)$; θ_i are known constants, $\varphi_i(t)$ are linearly independent functions for $i = 1, 2, \ldots, k$. In order to ensure that the sampling functions of $\xi(t)$ are monotone decreasing, we stipulate that the functions $\varphi_i(t)$ are nonincreasing and $P\{\alpha_i<0\}\ll 1$ for $i = 1, 2, \ldots, k$.

The construction of confidence limits starts with given measurements of n independent sampling functions $\xi_1(t),\ldots,\xi_n(t)$ at the time moments $t_1, t_2, \ldots, t_\ell$ $(\ell\geqslant k)$. The measurements are subject to random errors, normally distributed with $(0,\sigma^2\Delta^2)$, where Δ is known.

The observations are thus representable by random vectors

$$\vec{a}_i=(\xi_i(t_1)+\varepsilon_{i1};\ \xi_i(t_2)+\varepsilon_{i2};\ldots;\xi_i(t_\ell)+\varepsilon_{i\ell}),\ i=1,\ldots,n,$$

which are independent and identically normally distributed with mean $m_a=F\vec{\mu}$ and variance $D_a=\sigma^2(F\theta F^T+\Delta^2 I)$. Here

N. E. Bauman Moscow Technical College. Translated from Statisticheskie Metody, pp. 134-141, 1982.

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$$F=\begin{bmatrix}\varphi_1(t_1) & \varphi_2(t_1) & \dots & \varphi_k(t_1)\\ \varphi_1(t_2) & \varphi_2(t_2) & \dots & \varphi_k(t_2)\\ \dots & \dots & \dots & \dots\\ \varphi_1(t_\ell) & \varphi_2(t_\ell) & \dots & \varphi_k(t_\ell)\end{bmatrix},\quad \vec{\mu}=\begin{bmatrix}\mu_1\\ \mu_2\\ \dots\\ \mu_k\end{bmatrix},\quad \theta=\begin{bmatrix}\theta_1^2 & 0 & \dots & 0\\ 0 & \theta_2^2 & \dots & 0\\ \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & \theta_k^2\end{bmatrix},$$

I is the identity matrix of order ℓ.

In this paper, we consider the prediction of confidence limits for the probability $P(t) = P\{A(t)\}$ of failfree operation of the system during the time $[0, t]$, $t \geqslant t_\ell$, and construct the lower confidence limit t_γ on the γ-reserve of the system, given by

$$P\{\tau > t_\gamma\} = \gamma, \tag{3}$$

where the random variable τ is the time to failure.

2. Prediction of the Lower Confidence Limit (LCL) for P(t)

Consider the process $\xi(t)$ with decreasing sampling functions bounded from below, i.e.,

$$P(t) = P\{\xi(t) > x_0\}. \tag{4}$$

Since $\alpha_1, \alpha_2, \dots, \alpha_k$ are normally distributed, then $\xi(t)$ is also normally distributed with

$$\begin{aligned}\mu(t) &= \mu_1\varphi_1(t) + \mu_2\varphi_2(t) + \dots + \mu_k\varphi_k(t),\\ \sigma^2(t) &= \sigma^2\cdot\sum_{i=1}^{k}\theta_i^2\varphi_i^2(t) = \sigma^2 B(t),\end{aligned} \tag{5}$$

where

$$B(t) = \sum_{i=1}^{k}\theta_i^2\varphi_i^2(t).$$

Hence

$$P(t) = P\{\xi(t) > x_0\} = \Phi(H(t)), \tag{6}$$

where $H(t) = [\mu(t) - x_0]/\sigma\sqrt{B(t)}$, and $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int^{x} e^{-\frac{u^2}{2}}du$ is the Laplace function.

Note that since $\Phi(x)$ is monotone, it suffices to find the LCL for a given level γ for $H(t)$ and then apply the relationship

$$\underline{P}_\gamma(t) = \Phi(\underline{H}_\gamma(t)), \tag{7}$$

where $\underline{P}_\gamma(t)$ is the level-γ LCL for P(t),

$\underline{H}_\gamma(t)$ is the level-γ LCL for H(t).

In order to find the unknown parameters entering H(t), specifically, $\mu_1, \mu_2, \dots, \mu_k, \sigma^2$, we represent the data in the following form:

Since

$$\xi_i(t_j) + \varepsilon_{ij} = \sum_{m=1}^{k} \alpha_{im}\varphi_m(t_j) + \varepsilon_{ij}$$

$$= \sum_{m=1}^{k} \mu_m \varphi_m(t_j) + \sum_{m=1}^{k} (\alpha_{im} - \mu_m)\varphi_m(t_j) + \varepsilon_{ij},$$

the vectors $\vec{a}_i$ are representable in the form

$$\vec{a}_i = F\vec{\mu} + \vec{e}_i, \tag{8}$$

where $\vec{e}_i$ are unknown normally distributed vectors with mean $\vec{0}$ and variance matrix

$$D_e = D_a = \sigma^2(F\theta F^T + \Delta^2 I).$$

As the unbiased estimator of the mean vector $\vec{\mu}$ we take the OLS estimator [1]

$$\hat{\vec{\mu}} = (F^T W F)^{-1} F^T W \vec{\bar{a}}, \tag{9}$$

where $\vec{\bar{a}} = \frac{1}{n}\sum_{i=1}^{n} \vec{a}_i$ and $W = (F\theta F^T + \Delta^2 I)^{-1}$. As the unbiased estimator of variance $\hat{\sigma}^2$ we take [1]

$$\hat{\sigma}^2 = \frac{1}{n\ell - k} \sum_{i=1}^{n} (\vec{a}_i - F\hat{\vec{\mu}})^T W (\vec{a}_i - F\hat{\vec{\mu}}). \tag{10}$$

Here

$$V^2 = \frac{\hat{\sigma}^2(n\ell - k)}{\sigma^2}$$

is χ^2-distributed with $(n\ell - k)$ degrees of freedom [1], and $\hat{\mu}(t) = \sum_{i=1}^{k} \hat{\mu}_i \varphi_i(t)$ is normally distributed,

$$M(\hat{\mu}(t)) = \sum_{i=1}^{k} \mu_i \varphi_i(t),$$

$$D(\hat{\mu}(t)) = \frac{\sigma^2}{n} \vec{\varphi}^T(t)(F^T W F)^{-1}\vec{\varphi}(t),$$

where $\vec{\varphi}(t) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_k(t))^T$. Therefore, the statistic

$$\hat{G} = \frac{[\hat{\mu}(t) - x_0]\sqrt{n(n\ell - k)}}{\sigma\sqrt{V^2 \vec{\varphi}^T(t)(F^T W F)^{-1}\vec{\varphi}(t)}} = \frac{[\hat{\mu}(t) - x_0]\sqrt{n}}{\sqrt{\hat{\sigma}^2 \vec{\varphi}^T(t)(F^T W F)^{-1}\vec{\varphi}(t)}} \tag{11}$$

follows the noncentral t-distribution with $(n\ell - k)$ degrees of freedom and noncentrality parameter

$$G(t) = \frac{[\mu(t) - x_0]\sqrt{n}}{\sigma\sqrt{\vec{\varphi}^T(t)(F^T W F)^{-1}\vec{\varphi}(t)}} = \frac{H(t)}{C(t)}, \tag{12}$$

where $C(t) = \frac{\sqrt{\vec{\varphi}^T(t)(F^T W F)^{-1}\vec{\varphi}(t)}}{\sqrt{n}\cdot B(t)}$ is a known quantity.

Using the tables of [3, 4, 5], we can find the LCL $\underline{G}_\gamma(t)$ for G(t) given the level γ, the number of degrees of freedom $(n\ell - k)$, and the sample value $\hat{G}(t)$:

$$\underline{G}_\gamma(t) = \delta(n\ell - k;\ 1-\gamma;\ \hat{G}(t)). \tag{13}$$

Using the relationship (12) between G(t) and H(t), we obtain the level-γ LCL for H(t) in the form

$$\underline{H}_{\gamma}(t) = C(t)\cdot\delta\left(n\ell-k;\, 1-\gamma;\, \frac{\hat{H}(t)}{C(t)}\right), \tag{14}$$

where

$$\hat{H}(t) = \frac{\hat{\mu}(t)-x_0}{\hat{\sigma}\sqrt{B(t)}}. \tag{15}$$

Using the relationship between $\underline{P}_{\gamma}(t)$ and $\underline{H}_{\gamma}(t)$ we finally obtain

$$\underline{P}_{\gamma}(t) = \Phi\left(C(t)\cdot\delta\left(n\ell-k;\, 1-\gamma;\, \frac{\hat{H}(t)}{C(t)}\right)\right). \tag{16}$$

It can be shown that the linearized variance estimator of $\hat{G}(t)$ is

$$1+\frac{1}{2(n\ell-k)}\hat{G}^2(t).$$

Therefore, using the normal approximation to the noncentral t-distribution, we obtain an approximate LCL for G(t) in the form

$$\underline{G}^*_{\gamma}(t) = \hat{G}(t) - z_{\gamma}\sqrt{1+\frac{1}{2(n\ell-k)}\hat{G}^2(t)},$$

and so

$$\underline{H}_{\gamma}(t) = \hat{H}(t) - z_{\gamma}\sqrt{C^2(t)+\frac{1}{2(n\ell-k)}\hat{H}^2(t)}$$

and

$$\underline{P}_{\gamma}(t) = \Phi\left(\hat{H}(t) - z_{\gamma}\sqrt{C^2(t)+\frac{1}{2(n\ell-k)}\hat{H}^2(t)}\right). \tag{17}$$

Here z_{γ} is the standard normal quantile. Along the same lines we can obtain exact and approximate expressions for the upper confidence limit

$$\bar{P}_{\gamma}(t) = \Phi\left(C(t)\cdot\delta\left(n\ell-k;\, \gamma;\, \frac{\hat{H}(t)}{C(t)}\right)\right),$$

$$\bar{P}^*_{\gamma} = \Phi\left(\hat{H}(t) + z_{\gamma}\sqrt{C^2(t)+\frac{\hat{H}^2(t)}{2(n\ell-k)}}\right). \tag{18}$$

3. Predicting the LCL for the γ-Reserve of the System

Let us consider under the same conditions the problem of interval estimation of the γ-reserve of the system with the determining parameter $\xi(t)$. The time to failure is defined as $\tau = \inf(t, \xi(t) < x_0)$. If all the sampling functions of $\xi(t)$ are strictly monotone, then τ is defined as the value of t for which $\xi(t) = x_0$. Here

$$P\{\tau > t\} = P\{\xi(t) > x_0\} = P(t). \tag{19}$$

Define the LCL $\underline{t}_{\gamma q}$ of level q for the γ-reserve by the relation

$$P\{t_\gamma > \underline{t}_{\gamma q}\} = q. \tag{20}$$

<u>LEMMA.</u> The LCL $\underline{t}_{\gamma q}$ is the solution of the equation

$$\underline{P}_q(t) = \gamma, \tag{21}$$

where $\underline{P}_q(t)$ is the level-q LCL for P(t).

<u>Proof.</u> The relation (20) is equivalent to the following condition holding for some process $\xi(t)$ independent of $\underline{t}_{\gamma q}$ with the above characteristics:

$$P\{P_\xi(\tau > \underline{t}_{\gamma q}) > \gamma\} = q,$$

which by (19) is equivalent to

$$P\{P(\underline{t}_{\gamma q}) > \gamma\} = q.$$

Comparing the last expression with the definition of level-q LCL for P(t),

$$P\{P(t) > \underline{P}_q(t)\} = q,$$

we note that $\underline{t}_{\gamma q}$ is that value of t which satisfies (21). Q.E.D.

We can apply the lemma to write an equation for $\underline{t}_{\gamma q}$ using the exact expression (16). However, since using tables of the noncentral t-distribution is not a simple undertaking, it is better to use the approximate expression (17). Then the sought LCL $\underline{t}_{\gamma q}$ is the root t of the equation

$$\Phi\left(\hat{H}(t) - z_\gamma \sqrt{C^2(t) + \frac{1}{2(nl-k)} \hat{H}^2(t)}\right) = \gamma$$

or, equivalently, of the equation

$$\hat{H}(t) - z_\gamma \sqrt{C^2(t) + \frac{1}{2(nl-k)} \hat{H}^2(t)} = z_\gamma. \tag{22}$$

This equation is easily solved numerically, e.g., by Newton's method.

As the starting approximation, we may take the point estimator t_γ obtained from the relation $\Phi(\hat{H}(t)) = \gamma$, which may be rewritten in the following equivalent form:

$$\frac{\hat{\mu}_1 \varphi_1(t) + \ldots + \hat{\mu}_k \varphi_k(t) - x_0}{\hat{\sigma}\sqrt{B(t)}} = z_\gamma. \tag{23}$$

Uniqueness of the solution of (23) follows from our assumptions.

<u>4. Example</u>

Let us consider the linear model $\xi(t) = \alpha_1 - \alpha_2 t$ for $\theta_1 = 1$, $\theta_2 = 0{,}1$, $\Delta = 0{,}05$, with four sampling functions of the process $\xi(t)$ observed at the moments $t_1 = 0$, $t_2 = 3$, $t_3 = 6$ months.

We need to determine the LCL of level $\gamma = 0.9$ for $P(t)$ for $t = 10$ months and the LCL of level 0.8 for the 0.9-reserve of the system given that the critical level is $x_0 = 3$.

For our case, $n = 4$, $\ell = 3$, $k = 2$.

The data:

$$\vec{a}_1 = \begin{bmatrix} 9.12 \\ 7.85 \\ 6.67 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 7.11 \\ 6.00 \\ 4.80 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} 9.26 \\ 8.37 \\ 7.59 \end{bmatrix}, \vec{a}_4 = \begin{bmatrix} 6.86 \\ 5.98 \\ 5.16 \end{bmatrix};$$

the matrices:

$$F = \begin{bmatrix} 1 & 0 \\ 1 & 3 \\ 1 & 6 \end{bmatrix}, \qquad \theta = \begin{bmatrix} 1 & 0 \\ 0 & 0{,}01 \end{bmatrix}.$$

From (9) we calculate the estimator of the mean vector $\hat{\vec{\mu}}$, and from (10) the variance estimator σ^2:

$$\hat{\vec{\mu}} = \begin{bmatrix} 8.08 \\ 0.34 \end{bmatrix}, \qquad \sigma^2 = 1.2086 .$$

The functions $B(t)$, $C(t)$, $\hat{H}(t)$ for this case take the form

$$B(t) = 1 + 0.01t^2, \quad C(t) = \sqrt{\frac{0.825 - 0.019t + 0.01t^2}{4(1 + 0.01t^2)}},$$

$$\hat{H}(t) = \frac{5.08 - 0.34t}{\sqrt{1.2086(1 + 0{,}01t^2)}} .$$

In order to construct the LCL for $P(t)$ for $t = 10$, we evaluate

$$B(10) = 2, \quad C(10) = 0.478, \quad C^2(10) = 0.227, \quad \hat{H}(10) = 1.08$$

and

$$\underline{P}_{0.9}(10) = \Phi\left(1.08 - 1.28\sqrt{0.227 + (1.08)^2/20}\right) = \Phi(0{,}4) \approx 0{,}655.$$

In order to find the 0.8LCL for $t_{0.9}$, we rewrite (22) in the form

$$1.28\sqrt{1 + 0.01t^2} - 4.621 + 0.309t + \sqrt{0.0051t^2 - 0.0975t + 0.8683} = 0.$$

Solving this equation by Newton's method, we take the starting approximation from (23), which in this case gives $t_0 = 9.3$. For $\underline{t}_{0.9,0.8}$ we obtain $\underline{t}_{0.9,0.8} = 7.6$ in two steps by Newton's method to within 0.05.

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