ASYMPTOTIC BEHAVIOR OF EIGENVALUES OF NON-SELF-ADJOINT OPERATORS

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In this article we, with certain restrictions, establish the weak equivalence of distribution functions of the eigenvalues and s-values of a compact operator. We give applications to differential operators, whose spectra cannot be condensed to a ray.

1. Let σ_∞ denote the class of totally continuous operators, defined in the separable Hilbert space H. Denote by $\lambda_n(A)$, $s_n(A)$ ($n = 1, 2, 3, \ldots$), respectively, the eigenvalues and s-values of the operator $A \in \sigma_\infty$, numerated in the order of nonincrease of modules. We recall that $s_n^2(A) \leqslant \lambda_n(A^*A)$.

Denote by the number θ_T, for a linear operator T with region of definition D_T, the angle of the cone $W_T = \{Tf, f), f \in D_T\}$.

For the operator $A \in \sigma_\infty$, consider the functions

$$n(r, A) = \sum_{r\,|\,\lambda_n(A)\,|\geqslant 1} 1, \quad \widetilde{n}(r, A) = \sum_{r s_n(A) \geqslant 1} 1.$$

Moreover, for $p > 0$ set $\sigma_p(H) = \{A \in \sigma_\infty(H), \sum s_n^p(A) < +\infty\}$.

2. THEOREM 1. Let the dissipative operator $A \in \sigma_p$, where $p \leqslant \frac{\pi}{2}\theta_A^{-1}$, and let the following condition hold: $\tilde{n}(2r, A) = O(1)\tilde{n}(r, A)$. Then $\tilde{n}(r, A) \asymp n(r, A)$.

Remark. The example of the operator of fractional integration (see [1]) shows that the condition $A \in \sigma_p$, $p \leqslant \frac{\pi}{2}\theta_A^{-1}$, in Theorem 1 can never be replaced by the weaker statement $s_n(A) = O(n^{-1/2p})$, $p \leqslant \frac{\pi}{2}\theta_A^{-1}$.

3. Using the results of [2, §2], the proof of the theorem reduces to the case of integral p. Without loss of generality, we may assume that the values of the form $z = (Af, f)$ lie in the angle $0 \leqslant \arg z \leqslant \pi/2p$. Since $T = AP$ is a dissipative operator, it follows from Theorem 2 of [2] that $A = T^{1/p}$, in the sense of [2]. Applying once more the results of [2, §2], the proof reduces to the case $p = 1$, such that in this case the operator $B = A^{-1}$ exists, and $\operatorname{Re}(B + tE)^{-1} \geqslant 0$, $\operatorname{Im}(B + tE)^{-1} \geqslant 0$ for all $t > 0$. From the above inequalities we have

$$|(B + tE)^{-1}|_1 \leqslant 2\int (t + \lambda)^{-1} dn(\lambda, A).$$

Hence, and since the operator $(B + tE)(\sqrt{B^*B} + tE)^{-1}$ is uniformly bounded in $t \in R_1^+$, we obtain the estimate

$$\int (\lambda + t)^{-1} d\tilde{n}(\lambda, A) \asymp \int (\lambda + t)^{-1} dn(\lambda, A),$$

since from the condition $\tilde{n}(2r, A) = O(1)\tilde{n}(r, A)$ it follows from the results of [3] that $n(r, A) = O(\tilde{n}(r, A))$. Moreover, the Tauber theorems apply.

4. Let T and L be invertible closed operators with discrete spectra; $T^* = T \geqslant 0$. Set $N(\lambda, T) = n(\lambda, T^{-1})$, $N(\lambda, L) = n(\lambda, L^{-1})$, $k(\lambda, T) = \tilde{n}(\lambda, T^{-1})$. Suppose that $D_T = D_L$ and for all $u \in D_T$ we have the inequalities

$$c_1 \,|\, Tu \,| \leqslant |\, Lu \,| \leqslant c_2 |\, Tu \,| \tag{1}$$

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with some constants $c_1, c_2 > 0$. Since the function $k(\lambda, L)$ coincides with the maximal (in dimension of the linear manifolds M such that $|Lu| \leqslant \lambda |u|$ for all $u \in M$, from (1) we have the inequalities

$$N(c_2^{-1}\lambda, T) \leqslant k(\lambda, L) \leqslant N(c_1^{-1}\lambda, T).$$

Hence, if $N(2\lambda, T) = O(1)N(\lambda, T)$ then $N(\lambda, T) \asymp k(\lambda, L)$. If, moreover, the resolvent of the operator T belongs to the class σ_p, where $p \leqslant \frac{\pi}{2}\theta_L^{-1}$, and $\theta_L \leqslant \pi$, from Theorem 1 we obtain $N(\lambda, L) \asymp N(\lambda, T)$.

5. Let Ω be a bounded region in R_n. Denote by $\overset{0}{H}_m(\Omega)$, where m is an integer, the closure of $C_0^\infty(\Omega)$ in the norm $|u|_m = \sum_{|\alpha| \leqslant m} |D^\alpha u|_{L_2(\Omega)}$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, and $D^\alpha = \left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{1}{i}\frac{\partial}{\partial x_n}\right)^{\alpha_n}$. Let L denote the closure in $L_2(\Omega)$ of the operator

$$L_0 = \sum_{|\alpha|, |\beta| \leqslant m} D^\alpha (a_{\alpha\beta}(\cdot) D^\beta), \quad D_{L_0} = C^\infty(\Omega) \cap \overset{\circ}{H}_m(\Omega),$$

where $a_{\alpha\beta}(\cdot) \in C^\infty(\Omega)$, for all $|\alpha|, |\beta| \leqslant m$. Suppose that the values of the function $z(x, s) = \sum_{|\alpha| = |\beta| = m} a_{\alpha\beta}(x) s^\alpha s^\beta$ $(x \in \Omega, s \in R_n)$ form an angle of $\theta < \min\left(\frac{\pi m}{n}, \pi\right)$, where $z(x, s) \neq 0$ if $s \neq 0$.

THEOREM 2. For the conditions formulated above, the operator L has a discrete spectrum, and $N(\lambda, L) \asymp \lambda^{n/2m}$.

6. Consider the Sturm–Liouville operator $L = -\frac{d^2}{dt^2} + q(t)$ in $L_2(R_1^+)$, with boundary condition $y(0) = 0$. Suppose that $|\operatorname{Im} q(t)| \leqslant \operatorname{tg}\frac{\pi}{4p} \operatorname{Re} q(t)$ and $|q(t)|^{0.5-p} \in L_1(R_1^+)$, where $p > 1/2$. Let $\varphi(2\lambda) = O(\varphi(\lambda))$, where $\varphi(\lambda) = \operatorname{mes}\{t \,\|\, q(t)| \leqslant \lambda\}$ and $q'(t) = o(|q(t)|^{1.5})$.

THEOREM 3. For the conditions formulated above the operator L has discrete spectrum, and $N(\lambda, L) \asymp \sqrt{\lambda}\varphi(\lambda)$.

In connection with Theorem 3, we note [4-6], in which the asymptotics of the spectrum of a Sturm–Liouville operator with complex potential are also studied.

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