Journal of Theoretical Probability, Vol. 9, No. 4, 1996

Erdős–Rényi–Shepp Laws and Weighted Sums of Independent Identically Distributed Random Variables

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Received April 21, 1995; revised December 1, 1995

Let $X_0, X_1, X_2, \ldots$ be i.i.d. random variables with $E(X_0)=0$, $E(X_0^2)=1$, $E(\exp\{tX_0\})<\infty$ $(|t|<t_0)$ and partial sums S_n. Starting from Shepp's version of the well-known Erdős–Rényi–Shepp law

$$\limsup_{n\to\infty}\left([c\log n]^{-1}\right)(S_{n+[c\log n]}-S_n)=\alpha \qquad \text{a.s.}$$

where α is a number depending upon c and the distribution of X_0, we show that other weighted sums $V(n)=\sum a_j(n)\,X_j$ exhibit a similar lim sup behavior, if the weights satisfy certain regularity conditions. We also prove for such weighted sums certain versions of the classical Erdős–Rényi law.

KEY WORDS: Erdős–Rényi laws; weighted sums; i.i.d. random variables.

1. INTRODUCTION

Let (Ω, Σ, P) be a probability space and suppose that $X_0, X_1, X_2, \ldots$ are independent, identically distributed (i.i.d.) random variables defined on this space.

Studying almost sure convergence of partial sums and increments of partial sums one faces two typical kinds of behavior characterized as almost sure noninvariance principle (ASNIP) and almost sure invariance principle (ASIP), see Ref. 3. The classical example of the first type is the Erdős–Rényi–Shepp law as e.g. in Refs. 12, 13, and 23.

Theorem S. Let (X_n) be a sequence of i.i.d. random variables such that $E(X_0)=0$, $E(X_0^2)=1$ and $E(e^{tX_0})$ is finite in a neighborhood of 0.

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0894-9840/96/1000-0961$09.50/0

Define the cumulant generating function $C(t) := \log E(e^{tX_0})$ and its Fenchel- (convex-) conjugate

$$C^*(x) := \sup_t \{xt - C(t)\} \quad (x > 0)$$

If $c = c(\alpha)$ and $\alpha = \alpha(c)$ are linked by $C^*(\alpha) = 1/c$, we have

$$\limsup_{n \to \infty} \frac{1}{[c \log n]} \sum_{n \leqslant k < n + [c \log n]} X_k = \alpha(c) \qquad \text{a.s.}$$

Conversely the law of X_0 can be recovered from the limits by varying c.

There exist various generalizations and refinements of this theorem, see Refs. 10–14. The characteristic feature of all these theorems is the fact that the limit behavior of the increments of the partial sums S_n considered in these Theorems depends on the entire distribution of X_0 (ASNIP).

This is in contrary to the corresponding Kolmogorov-type of strong laws of large numbers where the limiting behavior depends on the distribution F only through its moments (ASIP), recall e.g., the classical strong law of large numbers.

To present a rather general result of the Kolmogorov-type for increments of partial sums and weighted means of i.i.d. random variables we need two notations (compare Ref. 6, Sections 2.11 and 2.1). We say that a measurable function $f: \mathbb{R} \to (0, \infty)$

(i) is *self-neglecting* if it is continuous, $o(x)$ as $(x \to \infty)$ and

$$\frac{f(x + uf(x))}{f(x)} \to 1 \ (x \to \infty) \qquad \text{for all} \quad u \in \mathbb{R}$$

(ii) has *bounded increase* if

$$\frac{f(\lambda x)}{f(x)} < \kappa \lambda^{\alpha} \ (1 \leqslant \lambda \leqslant A,\ x \geqslant x_0)$$

for suitable $A > 1$, κ, x_0, α.

Then the following result is basically contained in Theorem 3 in Ref. 5, see also Ref. 3, with (iii) respectively (iv) following from Refs. 20 and 24, respectively Ref. 18, where the weights considered arise from summability theory.

Theorem BG. If ϕ is self-neglecting and increasing with inverse function ψ having a derivative ψ' of bounded increase, the following statements are equivalent:

(i) $E(\psi(|X_0|)) < \infty$, $E(X_0) = \mu$;

(ii) $[1/u\phi(n)] \sum_{n \leqslant k < n + u\phi(n)} X_k \to \mu$ $\forall u > 0$, $(n \to \infty)$ a.s.; i.e., in the language of summability theory $X_n \to \mu$ a.s. (M_ϕ), where (M_ϕ) is the running mean with length function ϕ;

(iii) $[1/\sqrt{2\pi}\,\phi(n)] \sum_{k=0}^{\infty} \exp\{-[(k-n)^2/2\phi(n)^2]\}\, X_k \to \mu$ a.s. $(n \to \infty)$; i.e., $X_n \to \mu$ a.s. (V_ϕ), where (V_ϕ) is the generalized Valiron mean;

(iv) $\sum_{k=0}^{\infty} [\exp\{-g(k) + kg'(t)\}/p(e^{g'(t)})]\, X_k \to \mu$ a.s. $(t \to \infty)$, with $p(x) := \sum_{k=0}^{\infty} e^{-g(k)} x^k$ and g a suitable function (see Section 3) with $1/\sqrt{g''(n)} = \phi(n)$, hence $X_n \to \mu$ a.s. (P), where (P) is the power series method having weights $p_k = \exp\{-g(k)\}$.

Bingham,[3] and Bingham and Goldie[5] even point out that their condition "ψ' being of bounded increase" is important since otherwise one could choose $\phi(n) = c \cdot \log n$ (that means $\psi'(t) = (1/c)\, e^{t/c}$ which is not of bounded increase) and then the equivalence of (i) and (ii) in Theorem BG would contradict the Erdős–Rényi–Shepp laws.

In this paper, we want to investigate whether a similar behavior as in Theorem S occurs for a larger class of weighted means than just for running means. This investigation will in particular complement Theorem BG and the discussion of its boundary cases. Therefore we consider transformations of discrete or continuous type which may (but not necessarily have to) be summability methods. We only need a certain behavior of the sums of powers of the weights. Thinking of the trigonometric system (orthogonal random variables on $[0, 1]$) where weighted means with smoothly decreasing weights behave nicer than those with rectangular shaped weights (compare the convergence of the partial sums of Fourier series or smoothed periodograms in Time Series Analysis) it might be possible that smooth weights applied to i.i.d. random variables have a nicer limiting behavior as well. But this is not the case as we can see from our main result presented in the next section. Its proof together with the related exponential inequalities are given in Section 2, whereas the technical results are postponed to Section 4, where we prove e.g., the exponential inequalities. In Section 3, we state and prove the Erdős–Rényi version of our Theorem and apply our results to a wide variety of examples.

2. MAIN RESULTS

Let $X_0, X_1, X_2, \ldots$ be independent, identically distributed (i.i.d.) random variables with

$$\begin{cases} \text{(i)} & E(X_0)=0,\ E(X_0^2)=1 \\ \text{(ii)} & \text{there exists a } t_0>0 \text{ such that the } \textit{moment generating} \\ & \textit{function (mgf) } M(t):=E(\exp tX_0)<\infty \text{ if } |t|<t_0 \end{cases} \tag{2.1}$$

Denoted earlier by

$$C(t):=\log M(t)=\sum_{\nu=1}^{\infty}\frac{c_\nu}{\nu!}t^\nu$$

the *cumulant generating function* (*cgf*), which exists for $|t|<t_0$. Observe that $c_1=0$ and that $C(t)$ is a strictly convex function.

Let $a_j(\lambda)$, $j=0, 1, 2,...;$ $\lambda>0$, where λ is a continuous parameter or $\lambda\in\mathbb{N}$, be real numbers such that for each $\nu=1, 2,...$

$$\sum_{j=0}^{\infty}|a_j(\lambda)|^\nu<\infty \qquad \text{for all } \lambda$$

We consider the following weighted means of the sequence (X_n):

$$V(\lambda):=\sum_{j=0}^{\infty}a_j(\lambda)\,X_j$$

$V(\lambda)$ can be defined almost surely (a.s.), because $EX_0^2<\infty$ and $\sum_{j=0}^{\infty}a_j(\lambda)^2<\infty$.

In studying the almost sure convergence of this expression and especially calculating its cumulant generating function, see Lemma 2 and its proof, the following conditions turn out to be suitable.

Let ϕ: $[0, \infty)\to[0, \infty)$ be a nondecreasing function with $\phi(\lambda)\to\infty$ $(\lambda\to\infty)$ and (a_ν) be a sequence of real numbers with $a_\nu\neq 0$ such that $a:=\lim_{\nu\to\infty}\sqrt[\nu]{|a_\nu|}$ exists and

$$\sum_{j=0}^{\infty}a_j(\lambda)^\nu=\frac{a_\nu}{(\phi(\lambda))^{\nu-1}}R(\nu,\lambda) \qquad \text{for all } \lambda \text{ and } \nu \tag{2.2}$$

where for the error term $R(\nu, \lambda)$ the following condition holds

$$|R(\nu,\lambda)-1|\leqslant r_\nu\frac{(1+\delta(\lambda))^\nu}{\phi(\lambda)} \qquad \text{for all } \lambda \text{ and } \nu \tag{2.3}$$

with $\delta(\lambda)\to 0$ $(\lambda\to\infty)$ and $\limsup_{\nu\to\infty}\sqrt[\nu]{r_\nu}\leqslant 1$.

Define the following "asymptotic cumulant generating function" of $V(\lambda)$

$$\chi(t) := \sum_{\nu=2}^{\infty} \frac{a_\nu c_\nu}{\nu!} t^\nu \qquad \text{for} \quad |t| < t_0^* (= t_0/a) \tag{2.4}$$

Related to the function χ are the following quantities:

$$A := \lim_{t \to t_0^*} \chi'(t) \qquad \text{and} \qquad C_0 := 1 \Big/ \int_0^{t_0^*} u\chi''(u)\, du$$

These numbers are needed in the following to specify the range of the parameters α and c (compare Theorems S and 1). In Section 4, we show that $A > 0$ and $C_0 < \infty$, so that the range of these parameters is not empty.

We say that $V(\lambda)$ satisfies (S) if there exist nondecreasing functions $m(\cdot)$ and $\varepsilon(\cdot)$ such that

$$\sum_{|j - m(\lambda)| > \varepsilon(\lambda)\phi(\lambda)} |a_j(\lambda)| \log(j+2) \to 0 \ (\lambda \to \infty) \tag{S1}$$

and if there exists a $\gamma > 0$ such that for the sequence $n_k := k(\log k)^\gamma$, $k = 2, 3,\ldots$ we have

$$m(n_{k+1}) - m(n_k) - 2\varepsilon(n_{k+1})\,\phi(n_{k+1}) > 0 \tag{S2}$$

In the sequel this condition will be used to guarantee that the dependence between the $V(\lambda)$'s will be sufficiently weak for different λ's. In most special cases discussed later we can choose $m(\lambda) = \kappa\lambda$ with some constant $\kappa > 0$, $\varepsilon(\lambda) = \log\lambda$ and $n_k = k(\log k)^3$.

Before we state and prove our main result we give *exponential inequalities*, which are crucial for our proof, but also useful in their own right, e.g., in studying large deviations the weighted means. We postpone their proof to the last section.

Lemma 1. Let these assumptions hold true and define $\chi^*(x) := \sup_t\{tx - \chi(t)\}$ for $x > 0$. Then $\chi^*(x)$ is continuous and increasing on $[0, A)$ with values in $[0, 1/C_0)$ and there exist constants $\kappa_1, \kappa_2 > 0$ such that for all $x \in [0, A)$ and $\lambda > 0$

(i) $P(V(\lambda) \geqslant x) \leqslant \exp\{-\phi(\lambda)(\chi^*(x) - [\kappa_1/\phi(\lambda)])\}$ and

(ii) $P(V(\lambda) \geqslant x) \geqslant \exp\{-\phi(\lambda)(\chi^*(x) + [\kappa_2/\sqrt{\phi(\lambda)}])\}$.

Remark 1. Since $\chi^*(\cdot)$ is increasing and continuous and $0<\phi(\lambda)\to\infty$ $(\lambda\to\infty)$, we have for any $x\in(0,A)$ and every sufficiently small $\varepsilon>0$

$$P(V(\lambda)\geqslant x)\leqslant\exp\{-\phi(\lambda)\,\chi^*(x-\varepsilon)\}$$

and

$$P(V(\lambda)\geqslant x)\geqslant\exp\{-\phi(\lambda)\,\chi^*(x+\varepsilon)\}\qquad\text{for}\quad\lambda>\lambda_0(\varepsilon,x)$$

We can now state our main result in Theorem 1.

Theorem 1. Let (X_n) be a sequence of i.i.d. random variables satisfying Eq. (2.1) and $V(\lambda)$ be a transformation with weights satisfying Eqs. (2. 2) and (2.3) and for statements (ii) and (iii) next also (S). Then we have

(i) $\lim_{n\to\infty}V(n)=0$ a.s. provided that $\phi(n)/\log n\to\infty$ $(n\to\infty)$.

(ii) $\limsup_{n\to\infty}V(n)=\chi^{*-1}(1/c)$ a.s. provided that $\phi(n)/\log n\to c\in(C_0,\infty)$ $(n\to\infty)$ and the entire law of X_0 can be recovered from the limits by varying c in (C_0,∞) (Erdős–Rényi–Shepp-law).

(iii) If $A=\infty$ we have $\limsup_{n\to\infty}V(n)=\infty$ a.s. provided that $\phi(n)/\log n\to 0$ $(n\to\infty)$.

Proof.

(i) By Lemma 1 we have for small $\varepsilon>0$ and $n>n_0(\varepsilon)$

$$P(V(n)\geqslant\varepsilon)\leqslant\exp\left\{-\phi(n)\left(\chi^*(\varepsilon)-\frac{\kappa_1}{\phi(n)}\right)\right\}\leqslant\exp\left\{-\phi(n)\frac{\chi^*(\varepsilon)}{2}\right\}$$

Since $\phi(n)/\log n\to\infty$, we have for $n\geqslant n_1$

$$\frac{\phi(n)}{2\log n}\frac{\chi^*(\varepsilon)}{2}\geqslant 1$$

and so we get for $N:=\max\{n_0,n_1,2\}$

$$\sum_{n=N}^{\infty}P(V(n)\geqslant\varepsilon)\leqslant\sum_{n=N}^{\infty}\exp\left\{-2\log n\frac{\phi(n)}{2\log n}\frac{\chi^*(\varepsilon)}{2}\right\}\leqslant\sum_{n=N}^{\infty}\frac{1}{n^2}<\infty$$

Therefore it follows by the Borel–Cantelli Lemma that

$$V(n)\leqslant\varepsilon\qquad\text{a.s.}$$

Repeating the argument for the sequence $(-X_n)$ yields

$$V(n)\geqslant-\varepsilon\qquad\text{a.s.}$$

and hence (i) follows.

(ii) We start by showing that for $c \in (C_0, \infty)$

$$\limsup_{n \to \infty} V(n) \leqslant \chi^{*-1}(1/c) \qquad \text{a.s.} \tag{2.5}$$

Let $\varepsilon > 0$ be given such that $x := \chi^{*-1}(1/c) + 2\varepsilon < A$. By Lemma 1 we get for large n

$$P(V(n) \geqslant x) \leqslant \exp\left\{-\log n \frac{\phi(n)}{\log n} (\chi^*(\chi^{*-1}(1/c) + \varepsilon))\right\}$$

Since χ^* is increasing there exists a $\delta = \delta(\varepsilon) < 1$ such that

$$\chi^*(\chi^{*-1}(1/c) + \varepsilon)) \geqslant \frac{1+\delta}{c}$$

Since $\phi(n)/\log n \to c$ we have for large n

$$\left|\frac{\phi(n)}{\log n} - c\right| \leqslant \frac{\delta c}{2}$$

$\rangle$Hence

$$P(V(n) \geqslant x) \leqslant \exp\left\{-\log n \frac{1}{c}(1+\delta)\left(1-\frac{\delta}{2}\right)c\right\} = \exp\{-\log n(1+\tilde{\delta})\}$$

with $\tilde{\delta} = \frac{1}{2}(\delta - \delta^2) > 0$. Now Eq. (2.5) follows via the Borel–Cantelli Lemma.

We now show

$$\limsup_{n \to \infty} V(n) \geqslant \chi^{*-1}(1/c) \qquad \text{a.s.} \tag{2.6}$$

Let $\varepsilon > 0$ be given such that $x := \chi^{*-1}(1/c) - \varepsilon > 0$.

Since the mgf of X_0 exists in a neighborhood of 0, we have

$$|X_n| = \mathbf{O}(\log(n+2)) \qquad \text{a.s.}$$

Therefore we obtain by condition (S1) on the transformation

$$\left|\sum_{|j-m(n)| > \varepsilon(n)\,\phi(n)} X_j a_j(n)\right|$$

$$\leqslant \kappa \sum_{|j-m(n)| > \varepsilon(n)\,\phi(n)} |a_j(n)| \log(j+2) = \mathbf{o}(1) \tag{2.7}$$

with some constant $\kappa > 0$. Hence by Lemma 1 we have for large n

$$P\left(\sum_{|j-m(n)| \leqslant \varepsilon(n)\,\phi(n)} X_j a_j(n) \geqslant x - 2\varepsilon\right) \geqslant P(V(n) \geqslant x - \varepsilon)$$

$$\geqslant \exp\left\{-\log n \frac{\phi(n)}{\log n} \chi^*(x)\right\}$$

Using again that χ^* is increasing we find a $0 < \delta = \delta(\varepsilon) < 1$ such that

$$\chi^*(x) = \chi^*(\chi^{*-1}(1/c) - \varepsilon) \leqslant \frac{1-\delta}{c}$$

and therefore

$$P\left(\sum_{|j-m(n)| \leqslant \varepsilon(n)\,\phi(n)} X_j a_j(n) \geqslant x - 2\varepsilon\right) \geqslant \exp\left\{-\log n \frac{1}{c}(1-\delta)\left(1+\frac{\delta}{2}\right) c\right\}$$

$$= \exp\{-\log n(1-\tilde{\delta})\}$$

We set $\tilde{V}(n) := \sum_{|j-m(n)| \leqslant \varepsilon(n)\,\phi(n)} X_j a_j(n)$, and use the sequence (n_k) given by (S2) to obtain

$$P(\tilde{V}(n_k) \geqslant x - 2\varepsilon) \geqslant \exp\{-\log(k(\log k)^{\gamma})(1-\tilde{\delta})\} \geqslant \kappa \exp\{-\log k(1-\tilde{\delta}/2)\}$$

with some $\kappa > 0$. Hence

$$\sum_{k=2}^{\infty} P(\tilde{V}(n_k) \geqslant x - 2\varepsilon) \geqslant \kappa \sum_{k=2}^{\infty} \frac{1}{k^{1-\tilde{\delta}/2}} = \infty$$

Since the $\tilde{V}_{n_k}$ are independent by the choice of (n_k) given by (S2) we obtain by the Borel–Cantelli Lemma

$$\limsup_{n \to \infty} \tilde{V}(n) \geqslant x - 2\varepsilon = \chi^*(1/c) - 3\varepsilon \qquad \text{a.s.}$$

Using Eq. (2.7) again and the fact that ε was chosen arbitrarily we obtain Eq. (2.6). Combining Eq. (2.5) with Eq. (2.6) we obtain the first part of (ii).

Since $\chi(t)$ is strictly convex in the interval $[0, t_0^*)$, (compare Section 4, proof of Lemma 1), we can repeat the arguments of the classical proof as e.g., in Ref. 9, Theorem 2.4.5 or Ref. 13, Theorem 1 to see that knowledge of the limits uniquely determines $\chi(t)$. Then we use our knowledge of the sequence (a_ν) of positive numbers to calculate the cumulant generating function. This determines the characteristic function by analytic continuation, and hence the distribution.

(iii) Let $M>0$ be given. We show

$$\limsup_{n\to\infty} \tilde{V}(n) \geqslant M \qquad \text{a.s.}$$

with $\tilde{V}(n)$ as in (ii). Since $A=\infty$, we can use Lemma 1 and obtain for large n and some $\varepsilon>0$

$$P(\tilde{V}(n)\geqslant M)\geqslant P(V(n)\geqslant M+1)\geqslant \exp\{-\phi(n)\,\chi^*(M+1+\varepsilon)\}$$

Since $\phi(n)/\log n\to 0$ we get for large n

$$\frac{\phi(n)}{\log n}\chi^*(M+1+\varepsilon)<\frac{1}{2}$$

and therefore using the same sequence (n_k) as in (ii) we have with some $\kappa>0$

$$\sum_{k=1}^{\infty} P(\tilde{V}(n_k)\geqslant M)\geqslant \kappa \sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}=\infty$$

Hence (iii) follows from the Borel-Cantelli Lemma. □

3. RELATED RESULTS AND EXAMPLES

Using Theorem 1 we can (under further conditions on the weights) prove a version of the classical Erdős–Rényi law as it is stated for running means; e.g., in Ref. 9, Theorem 2.4.3.

We assume that we have a supply of i.i.d. r.v. X_n, $n\in\mathbb{Z}$ satisfying Eq. (2.1) and that we have weights $a_j(n)$ such that the function $\phi(\cdot)$ is of the form $\phi(n)=c\log(n)$ with some constant $c>C_0$. Using the center-function $m(n)$ from condition (S) we define for $1\leqslant l\leqslant n$ weights $a_{jl}(n)$ with shifted center by

$$a_{jl}(n):=\begin{cases} a_{m(n)-m(l)+j}(n) & \text{if} \quad -(m(n)-m(l))\leqslant j \\ 0 & \text{otherwise}\end{cases}$$

Then we define

$$V_n(l):=\sum_{j\in\mathbb{Z}} a_{jl}(n)\,X_j$$

Observe that $V_n(n)=V(n)$.

We say that the weighted mean V satisfies the condition (S3) if for any $\rho > 0$ and $n^\rho \leqslant \mu \leqslant (n+1)^\rho$

$$\max_{1 \leqslant l \leqslant \mu} \left\{ \sum_{|j - m(l)| \leqslant c \log \mu\varepsilon(\mu)} |a_{jl}(\mu) - a_{jl}((n+1)^\rho)| \log(|j|+2) \right\} = \mathbf{o}(1) \qquad \text{(S3)}$$

(To avoid complicated notation we use n^ρ meaning $[n^\rho]$.) Then we have the following

Theorem 2. Suppose that V satisfies Eqs. (2.2) and (2.3) with the function $\phi(n) = c \log n$, $c > C_0$ and the conditions (S1)–(S3) with the function $\varepsilon(n) = \log^\beta n$ for some $\beta \geqslant 0$, then for

$$T_n := \max_{1 \leqslant l \leqslant n} V_n(l)$$

we have

$$\lim_{n \to \infty} T_n = \chi^{*-1}(1/c) \qquad \text{a.s.}$$

Proof. We have

$$\begin{aligned} T_n &= \max_{1 \leqslant l \leqslant n} \left\{ \sum_{|j - m(l)| \leqslant c \log n\varepsilon(n)} a_{jl}(n) X_j + \sum_{|j - m(l)| > c \log n\varepsilon(n)} a_{jl}(n) X_j \right\} \\ &\leqslant \max_{1 \leqslant l \leqslant n} \left\{ \sum_{|j - m(l)| \leqslant c \log n\varepsilon(n)} a_{jl}(n) X_j \right\} + \max_{1 \leqslant l \leqslant n} \left\{ \sum_{|j - m(l)| > c \log n\varepsilon(n)} a_{jl}(n) X_j \right\} \\ &= \tilde{T}_n + R_n \end{aligned}$$

Define also

$$\tilde{T}_n^* = \max_{1 \leqslant l \leqslant n} \left\{ \sum_{|j - m(l)| \leqslant c \log n\varepsilon(n) - 1} a_{jl}(n) X_j \right\}$$

We start by estimating R_n. Using the monotonicity of the functions $m(\cdot)$ and $\log(\cdot)$ we obtain by Eq. (2.1) and condition (S1) (observe that since the weights are shifted $|a_{jl}(n)| \log(|j|+2)$ is maximal for l such that $|j|$ becomes maximal)

$$R_n \leqslant \sum_{|j - m(n)| > c \log n\varepsilon(n)} |a_{jn}(n)| \log(j+2) = \mathbf{o}(1) \qquad \text{a.s.}$$

Set $x_c := \chi^{*-1}(1/c)$ and let $\varepsilon > 0$ be sufficiently small. The exponential bounds yield

$$P(T_n \geqslant x_c + \varepsilon) \leqslant \sum_{l=1}^{n} P(V_n(l) \geqslant x_c + \varepsilon)$$
$$\leqslant n \exp\{-\log n(1+\delta)\} = \exp\{-\delta \log n\}$$

with some $\delta = \delta(\varepsilon)$. Now choose ρ such that $\rho\delta > 1$, then by the Borel–Cantelli Lemma

$$P(T_{n^\rho} \geqslant x_c + \varepsilon \ i.o.) = 0 \text{ and hence } \limsup_{n\to\infty} T_{n^\rho} \leqslant x_c \qquad \text{a.s.}$$

Neglecting the tails, possible since $R_n = \mathbf{o}(1)$, we get $\limsup_{n\to\infty} \tilde{T}_{n^\rho} \leqslant x_c$ a.s. and likewise $\limsup_{n\to\infty} \tilde{T}^*_{n^\rho} \leqslant x_c$ a.s. For $n^\rho \leqslant \mu < (n+1)^\rho$ we get

$$\tilde{T}_\mu = \max_{1\leqslant l\leqslant\mu} \left\{ \sum_{|j-m(l)| \leqslant c\log\mu\varepsilon(\mu)} a_{jl}(\mu)\, X_j \right\}$$
$$\leqslant \max_{1\leqslant l\leqslant\mu} \left\{ \sum_{|j-m(l)| \leqslant c\log\mu\varepsilon(\mu)} a_{jl}((n+1)^\rho)\, X_j \right\}$$
$$+ \max_{1\leqslant l\leqslant\mu} \left\{ \sum_{|j-m(l)| \leqslant c\log\mu\varepsilon(\mu)} \{a_{jl}(\mu) - a_{jl}((n+1)^\rho)\}\, X_j \right\}$$
$$\leqslant \max\{\tilde{T}_{(n+1)^\rho}, \tilde{T}^*_{(n+1)^\rho}\} + D_\mu$$

(Observe that $\log^{1+\beta} n^\rho - \log^{1+\beta}(n+1)^\rho \to 0$ $(n\to\infty)$). By conditions Eq. (2.1) and condition (S3) we obtain that $D_\mu = \mathbf{o}(1)$ a.s. Hence (using again the fact that we can neglect the tails) we get

$$\limsup_{\mu\to\infty} T_\mu \leqslant \limsup_{n\to\infty} T_{(n+1)^\rho} \leqslant x_c \qquad \text{a.s.}$$

Furthermore using Theorem 1

$$\liminf_{\mu\to\infty} T_\mu \geqslant \limsup_{\mu\to\infty} V_\mu(\mu) = \limsup_{\mu\to\infty} V(\mu) = x_c \qquad \text{a.s.}$$

Therefore Theorem 2 is proved. □

We now discuss some examples of weighted means for which our theorems can be applied. In the sequel (s_n) always denotes a sequence of complex numbers.

Running means (Moving averages): Let $m := (m_n)$ be a (strictly) increasing sequence of positive integers and $\phi(n)$ an nondecreasing function. Define the running mean

$$D_m(n) := \frac{1}{\phi(n)} \sum_{m_n \leqslant j < m_n + \phi(n)} s_j$$

Hence

$$a_j(n) = \begin{cases} 1/\phi(n) & m_n \leqslant j < m_n + \phi(n) \\ 0 & \text{otherwise} \end{cases}$$

and for $\nu = 1, 2,...$ follows

$$\sum_{j=0}^{\infty} a_j(n)^\nu = \frac{[\phi(n)]}{\phi(n)^\nu} = \frac{([\phi(n)])}{\phi(n)} \frac{1}{\phi(n)^{\nu-1}}$$

Therefore we have $a_\nu = 1$, $R(\nu, n) = 1 + \mathbf{O}(1/\phi(n))$ and see that Eqs. (2.2) and (2.3) are satisfied.

With centering function $m(n) := m_n$ and $\varepsilon(n) :\equiv 1$ condition (S1) is trivially satisfied, whereas (S2) (with $n_k = k(\log k)^\gamma$ some $\gamma > 0$) can be satisfied if $m_n/n \to \xi > 0$ ($\xi = \infty$ is possible) or $m_n/n \sim 1/\log^\alpha n$, $\alpha > 0$, but choices like $m_n = n^{1-\varepsilon}$, $\varepsilon > 0$ are not admissible, actually the lim sup will be smaller in this case.

Therefore Theorem 1 holds for such means and especially if we use $m_n = n + 1$, $\phi(n) = [c \log n]$, $c \in (C_0, \infty)$ (observe that in this case $\chi(t) \equiv C(t)$), we obtain the well-known Theorem of Shepp as stated in Refs. 12, 13, and 23.

In Theorem 2, we have in case $\phi(n) = c \log n$

$$V_n(l) = \frac{1}{c \log n} \sum_{l \leqslant j < l + c \log n} X_j$$

Since for $|j - l| \leqslant c \log \mu$ we have $a_{jl}(\mu) - a_{jl}((n+1)^\rho) = \mathbf{O}(1/n \log^2(n)^\rho)$ (S3) is fulfilled. Since $\log n/[\log n] \to 1$ we obtain from Theorem 2 the well-known Erdős–Rényi Theorem as e.g., in Ref. 9, Theorem 2.4.3. Further refinements and extensions of this Theorem have been discussed in Refs. 11 and 14.

Generalized Valiron-methods: Because of its relevance in probability theory (see Ref. 2) we focus on this method of summability.

Let $\phi(n)$ be a self-neglecting function with $\phi(n) \to \infty$ $(n \to \infty)$. Then we say (s_n) is summable to s by the generalized Valiron-method (V_ϕ) if

$$\frac{1}{\sqrt{2\pi}\,\phi(n)} \sum_{j=0}^{\infty} s_j \exp\left\{-\frac{1}{2}\left(\frac{j-n}{\phi(n)}\right)^2\right\} \to s \ (n \to \infty)$$

The classical Valiron method uses $\phi(n) = \sqrt{n}$. We have

$$a_j(n) = \frac{1}{\sqrt{2\pi}\,\phi(n)} \exp\left\{-\frac{1}{2}\left(\frac{j-n}{\phi(n)}\right)^2\right\}$$

We obtain using the functional equation of the ϑ-function, see Ref. 8 [p. 63], for $\nu \in \mathbb{N}$

$$\begin{aligned}
\sum_{j=0}^{\infty} a_j(n)^\nu &= \frac{1}{(\sqrt{2\pi}\,\phi(n))^\nu} \sum_{j=0}^{\infty} \exp\left\{-\frac{\nu}{2}\left(\frac{j-n}{\phi(n)}\right)^2\right\} \\
&= \frac{1}{(\sqrt{2\pi}\,\phi(n))^{\nu-1}} \frac{1}{\sqrt{\nu}} \left(1 + 2\sum_{j=1}^{\infty} \exp\left\{-2\pi^2 j^2 \frac{\phi(n)^2}{\nu}\right\}\right)
\end{aligned}$$

Now we set

$$a_\nu := \frac{1}{\sqrt{\nu}\,\sqrt{2\pi}^{\,\nu-1}} \qquad \text{and} \qquad R(n, \nu) := 1 + 2\sum_{j=1}^{\infty} \exp\left\{-2\pi^2 j^2 \frac{\phi(n)^2}{\nu}\right\}$$

Breaking the sum in two parts we obtain for the error term

$$\begin{aligned}
|R(n, \nu) - 1| &= 2\left(\sum_{j=1}^{\nu-1} \cdots + \sum_{j=\nu}^{\infty} \exp\left\{-2\pi^2 j^2 \frac{\phi(n)^2}{\nu}\right\}\right) \\
&\leqslant 2\int_0^{\nu-1} \exp\left\{-2\pi^2 t^2 \frac{\phi(n)^2}{\nu}\right\} dt + 2\sum_{j=\nu}^{\infty} \exp\{-2\pi^2 j \phi(n)^2\} \\
&\leqslant \frac{\sqrt{2\nu}}{\pi\phi(n)} \int_0^{\infty} \exp\{-t^2\}\, dt + 2\,\frac{\exp\{-2\pi^2 \nu \phi(n)^2\}}{1 - \exp\{-2\pi^2 \phi(n)^2\}} \\
&\leqslant \frac{\sqrt{\nu}}{\sqrt{2\pi}\,\phi(n)} + 3\exp\{-\phi(n)^2\} \leqslant \frac{c \cdot (\nu+1)}{\phi(n)}
\end{aligned}$$

Hence with $\delta(n) \equiv 0$ and $r_\nu = c \cdot (\nu + 1)$ we see that conditions in Eqs. (2.2) and (2.3) hold true. Using $m(n) := n$ as centering function and $\varepsilon(n) := \log n$ we obtain by techniques introduced by Valiron,[25] p. 269, (compare Ref. 17 and also Ref. 16, Ch. 9 esp. Sections 9.10–16 and notes) the validity of the conditions (S1) and (S2). Hence our Theorem 1 is applicable for Valiron-methods.

For Theorem 2 (Erdős–Rényi law) we have

$$a_{jl}(n) := \frac{1}{\sqrt{2\pi}\, c \log n} \exp\left\{ -\frac{1}{2} \left(\frac{j-l}{c \log n} \right)^2 \right\}$$

To see that (S3) holds true observe that

$$\left| \frac{a_{jl}(\mu)}{a_{jl}((n+1)^\rho)} - 1 \right| = \left| \exp\left\{ -\frac{1}{2} \left(\frac{j-l}{c} \right)^2 \left(\frac{1}{(\log \mu)^2} - \frac{1}{(\log (n+1)^\rho)^2} \right) \right\} - 1 \right|$$

For $n^\rho \leqslant \mu \leqslant (n+1)^\rho$ we get

$$\left| \frac{1}{(\log \mu)^2} - \frac{1}{(\log (n+1)^\rho)^2} \right| \leqslant \frac{1}{(\log n^\rho)^3\, n}$$

Now knowing that $|j-l| \leqslant c \log^2 (n+1)^\rho$ we obtain using the standard inequality $|e^{-x} - 1| \leqslant |x|$ for small x

$$\left| \frac{a_{jl}(\mu)}{a_{jl}((n+1)^\rho)} - 1 \right| \leqslant \frac{\log (n+1)^\rho}{n}$$

independent of l. Therefore

$$\max_{1 \leqslant l \leqslant \mu} \left\{ \sum_{|j-l| \leqslant c \log^2 \mu} \log(|j|+2)\, a_{jl}((n+1)^\rho) \left| \frac{a_{jl}(\mu)}{a_{jl}((n+1)^\rho)} - 1 \right| \right\}$$
$$= \mathbf{O}\left(\frac{\log^2 (n+1)^\rho}{n} \right)$$

This proves (S3) and hence we have Theorem 2 for Valiron methods.

Generalized Nörlund and Power series methods: We briefly discuss two further classes of summability methods related to the Valiron methods, see Refs. 19–21, and 24.

Let (p_n) be a sequence of positive numbers such that the power series

$$p(t) := \sum_{k=0}^{\infty} p_k t^k$$

has radius of convergence $R > 0$. Define the convolution

$$(p * p)_n := \sum_{k=0}^{n} p_{n-k} p_k$$

Then we say the sequence (s_n) is summable to s by

(i) the **generalized Nörlund mean** (N, p, p) if

$$\sigma_n := \frac{1}{(p * p)_n} \sum_{k=0}^{n} p_{n-k} p_k s_k \to s \ (n \to \infty)$$

(ii) the **power series method** (J_p) if $p_s(t) := \sum_{k=0}^{\infty} p_k s_k t^k$ has radius of convergence greater or equal than R and if

$$\sigma(t) := \frac{p_s(t)}{p(t)} \to s \ (t \to R-)$$

Suppose now that the real function g satisfies the following conditions

$$(\mathrm{C}) \quad \begin{cases} g \in C_2[0, \infty), & g''(t) \text{ is positive and decreasing} \\ & t^2 g''(t) \text{ is increasing on } [0, \infty) \end{cases}$$

We use in the following the sequences $p_n = e^{-g(n)}$. The weighted means corresponding to this summability methods are:

(i) in the generalized Nörlund case

$$V^N(n) = \sum_{j=0}^{\infty} a_j(n)\, X_j \qquad \text{with} \quad a_j(n) := \begin{cases} \dfrac{p_{n-j} p_j}{(p * p)_n} & 0 \leqslant j \leqslant n \\ 0 & n < j \end{cases}$$

(ii) in the (discrete) power series case ($R = \infty$ after the transformation $t = e^{g'(x)}$)

$$V^P(n) = \sum_{j=0}^{\infty} a_j(n)\, X_j \qquad \text{with} \quad a_j(n) := \frac{\exp\{-g(j) + jg'(n)\}}{p(e^{g'(n)})}, \qquad j \in \mathbb{N}$$

Using mostly the techniques introduced in Ref. 7, Section 4 esp. Lemma 2, see also Ref. 21, we can show that if we define

$$\phi(n) := \frac{1}{\sqrt{g''(n)}}$$

conditions in Eqs. (2.2) and (2.3) are fulfilled with

$$a_\nu = \frac{1}{\sqrt{2\pi}^{\,\nu-1} \sqrt{\nu}}$$

That conditions (S1) and (S2) hold true is also a consequence of Ref. 7. Therefore we can apply Theorem 1 for this methods (for part (i) compare Ref. 19).

For the generalized Nörlund method we have $m(n) = [n/2]$ and hence

$$V_n(l) = \sum_{j=-[(n-l)/2]}^{[(n+l)/2]} a_{jl}(n)\, X_j \qquad \text{with} \quad a_{jl}(n) = \frac{p_{[(n+l)/2]-j}\, p_{[(n-l)/2]+j}}{(p*p)_n}$$

If $g(\cdot)$ is such that $g''(n) \sim c \log^2 n$, $c > C_0^2$ then (S3) can be verified by techniques from Ref. 7. Hence we also get an Erdős–Rényi law for these methods.

To obtain an Erdős–Rényi law for the power series methods (J_p) it is more convenient to use the close relation of these methods to Valiron methods (see Refs. 20 and 24) instead of verifying (S3). The results of Refs. 7 and 24 enable us furthermore to obtain our results for the continuous power series method case, where the only remaining task is to obtain bounds for the gaps, i.e., to show that

$$\sup_{n \leqslant \lambda < n+1} |V^p(\lambda) - V^p(n)| = o(1) \qquad \text{a.s.}$$

Kernel-estimators: In nonparametric regression so-called kernel estimators $\hat{\mu}(x)$ are used for estimating a smooth regression function $\mu(x)$, $x \in [0, 1]$, see Ref. 22. For the simplest estimator the stochastic error-term can be written as

$$\hat{\mu}(x) - E(\hat{\mu}(x)) = \sum_{j=0}^{n} K\left(\frac{x - j/n}{b_n}\right) \frac{X_j}{nb_n}$$

where the bandwidth satisfies $b_n \to 0$, $nb_n \to \infty$, $x \in [0, 1]$. For the kernel function K we assume that $K \in L(\mathbb{R})$, has support $[-1, 1]$, is Lipschitz continuous and satisfies $\int_{-1}^{1} K(u)\, du = 1$. Under our assumptions on (X_n) we have e.g., for $x = \frac{1}{2}$

$$\hat{\mu}(x) - E(\hat{\mu}(x)) = \sum_{nb_n \leqslant j - n/2 \leqslant nb_n} K\left(\frac{1/2 - j/n}{b_n}\right) \frac{X_j}{nb_n}$$

If we define $\phi(n) := nb_n$, $m(n) := n/2$ we can apply our Theorem 1, since the weights $a_j(n) := K((n/2 - j)/\phi(n))(1/\phi(n))$ satisfiy our basic assumptions with $a_v = \int_{-1}^{1} K^v(u)\, du$.

4. AUXILIARY RESULTS

Lemma 2. The cumulant generating function

$$\psi_\lambda(t) := \log E(\exp\{V(\lambda)\, t\})$$

of $V(\lambda)$ is well defined for each λ and all t with $|t/\phi(\lambda)| < t_0^*$.

We have for $|t/\phi(\lambda)| \leqslant \tau < t_0^*$ as $\lambda \to \infty$

$$\psi_\lambda(t) = \phi(\lambda)\, \chi(t/\phi(\lambda)) + \mathbf{O}(1)$$

$$\psi''_\lambda(t) = \frac{1}{\phi(\lambda)}\, \chi''(t/\phi(\lambda)) + \mathbf{O}\left(\frac{1}{\phi(\lambda)^2}\right)$$

$$\psi^{(4)}_\lambda(t) = \frac{1}{\phi(\lambda)^3}\, \chi^{(4)}(t/\phi(\lambda)) + \mathbf{O}\left(\frac{1}{\phi(\lambda)^4}\right)$$

where the $\mathbf{O}$ constants depend on τ (but not on λ).

Proof. We have using the independence assumption and Eq. (2.2) that

$$\begin{aligned}
\psi_\lambda(t) &= \sum_{j=0}^{\infty} C(a_j(\lambda)\, t) = \sum_{j=0}^{\infty} \sum_{\nu=2}^{\infty} \frac{c_\nu}{\nu!} (a_j(\lambda)\, t)^\nu \\
&= \sum_{\nu=2}^{\infty} \frac{c_\nu}{\nu!} \left(\sum_{j=0}^{\infty} a_j(\lambda)^\nu \right) t^\nu = \phi(\lambda) \sum_{\nu=2}^{\infty} \frac{c_\nu}{\nu!} a_\nu R(\nu, \lambda) \left(\frac{t}{\phi(\lambda)} \right)^\nu \\
&= \phi(\lambda)\, \chi\left(\frac{t}{\phi(\lambda)} \right) + \phi(\lambda) \sum_{\nu=2}^{\infty} \frac{c_\nu}{\nu!} a_\nu (R(\nu, \lambda) - 1) \left(\frac{t}{\phi(\lambda)} \right)^\nu
\end{aligned}$$

with the first equality and the change of summation being justified by absolute convergence of the last series. This can be seen using Eq. (2.3)

$$\begin{aligned}
\left| \sum_{\nu=2}^{\infty} \frac{c_\nu}{\nu!} a_\nu (R(\nu, \lambda) - 1) \left(\frac{t}{\phi(\lambda)} \right)^\nu \right| &\leqslant \sum_{\nu=2}^{\infty} \frac{|c_\nu|}{\nu!} |a_\nu|\, |(R(\nu, \lambda) - 1)| \left(\frac{|t|}{\phi(\lambda)} \right)^\nu \\
&\leqslant \frac{1}{\phi(\lambda)} \sum_{\nu=2}^{\infty} \frac{|c_\nu a_\nu|}{\nu!} |r_\nu|\, (\tau(1 + \delta(\lambda)))^\nu \\
&= \mathbf{O}\left(\frac{1}{\phi(\lambda)} \right)
\end{aligned}$$

For the last step we used $\tau\,|1+\delta(\lambda)| < t_0^*$ for $\lambda \geqslant \lambda_0$, which implies the convergence of the last series. From

$$\psi_\lambda(t) = \phi(\lambda)\left\{\chi\left(\frac{t}{\phi(\lambda)}\right) + \sum_{\nu=2}^{\infty} \frac{c_\nu}{\nu!} a_\nu (R(\nu, \lambda) - 1)\left(\frac{t}{\phi(\lambda)}\right)^\nu\right\}$$

we obtain by componentwise differentiation

$$\psi_\lambda^{(\mu)}(t) = \phi(\lambda)^{(1-\mu)}\left\{\chi^{(\mu)}\left(\frac{t}{\phi(\lambda)}\right) + \sum_{\nu=\max\{2-\mu,\,0\}}^{\infty} \frac{c_{\nu+\mu} a_{\nu+\mu}}{\nu!}\right.$$
$$\left.\times (R(\nu+\mu, \lambda) - 1)\left(\frac{t}{\phi(\lambda)}\right)^\nu\right\}$$

Hence using as before this inequality in Eq. (2.3) we obtain for $\mu = 0, 1,\ldots$

$$\psi_\lambda^{(\mu)}(t) = \phi(\lambda)^{(1-\mu)}\chi^{(\mu)}\left(\frac{t}{\phi(\lambda)}\right) + \mathbf{O}\left(\frac{1}{\phi(\lambda)^\mu}\right) \qquad \square$$

Remark 2. Since (the random variable) $V(\lambda)$ satisfies $E(V(\lambda)) = 0$, $0 < E(V(\lambda)^2) = \sum a_j(\lambda)^2 < \infty$ and its mgf exists in a neighborhood of 0, $\psi_\lambda(t)$ is strictly convex in $[0, t_0^*\phi(\lambda))$ for each λ.

Proof of Lemma 1. Using Lemma 2 we see that $\chi(t)$ is convex in $[0, t_0^*)$. Since $\chi''(z)$ is holomorphic for $|z| < t_0^*$ and $\chi''(0) > 0$, $\chi''(t) \geqslant 0$ in $(0, t_0^*)$ $\chi'(t)$ has to be strictly increasing (since $\chi''(\cdot)$ can only have isolated roots) and therefore $\chi(t)$ is strictly convex in $[0, t_0^*)$. Following the lines of Ref. 13, proof of Theorem 1, we now can establish that $\chi^*(x)$ is continuous and increasing in $[0, A)$ and that there exists a $1-1$ correspondence between $\chi^{*-1}(1/c)$ and $\chi'(t_c)$ for $c \in (C_0, \infty)$ and $t_c \in (0, t_0^*)$. In particular we have $\chi^*(x) = t^*x - \chi(t^*)$ with $t^* = \chi'^{-1}(x)$.

Now define

$$\psi_\lambda^*(x) := \sup_t \{tx - \psi_\lambda(t)\}$$

Since $\psi_\lambda(\cdot)$ is strictly convex by Remark 2 we likewise get $\psi_\lambda^*(x) = t^*(\lambda)\,x - \psi_\lambda(t^*(\lambda))$ with $t^*(\lambda) = \psi_\lambda'^{-1}(x)$. Since $\psi_\lambda'(t^*(\lambda)) = x$ we get by Lemma 2 that

$$\chi'\left(\frac{t^*(\lambda)}{\phi(\lambda)}\right) + \mathbf{O}\left(\frac{1}{\phi(\lambda)}\right) = x$$

Now $\chi'(t^*) = x$, therefore we get by the continuity and strict convexity of χ

$$t^*(\lambda) = t^*\phi(\lambda) + \mathbf{O}(1) \tag{4.1}$$

Using Lemma 2 again and the continuity of $\chi^{(\mu)}$ we obtain by Eq. (4.1)

$$\psi_\lambda^{(\mu)}(t^*(\lambda)) = \phi(\lambda)^{(1-\mu)}\chi^{(\mu)}\left(\frac{t^*(\lambda)}{\phi(\lambda)}\right) + \mathbf{O}\left(\frac{1}{\phi(\lambda)^\mu}\right) = \mathbf{O}\left(\frac{1}{\phi(\lambda)^{(\mu-1)}}\right) \tag{4.2}$$

To prove (i) observe that for any $0 < t < t_0^*\phi(\lambda)$

$$\begin{aligned} P(V(\lambda) \geqslant x) &= P(\exp\{tV(\lambda)\} \geqslant \exp\{tx\}) \leqslant \exp\{-(tx - \psi_\lambda(t))\} \\ &\leqslant \exp\{-(tx - \phi(\lambda)\,\chi(t/\phi(\lambda)) - \kappa_1)\} \\ &= \exp\left\{-\phi(\lambda)\left(x\frac{t}{\phi(\lambda)} - \chi\left(\frac{t}{\phi(\lambda)}\right) - \frac{\kappa_1}{\phi(\lambda)}\right)\right\} \end{aligned}$$

with some constant $\kappa_1 > 0$. Therefore using $t = t^*\phi(\lambda)$

$$P(V(\lambda) \geqslant x) \leqslant \exp\left\{-\phi(\lambda)\left(\chi^*(x) - \frac{\kappa_1}{\phi(\lambda)}\right)\right\}$$

To prove (ii) we use the well-established technique of associated distributions, see Ref. 15, XVI.7. In our context we follow in particular the ideas of Ref. 1, p. 145. Set

$$Z(\lambda) := V(\lambda) - x$$

Let $T(\lambda)$ be a random variable with distribution (for each fixed λ)

$$H_\lambda(x) := \int_{-\infty}^{x} e^{t^*(\lambda)\omega}\,\frac{dF_{Z(\lambda)}(\omega)}{M_{Z(\lambda)}(t^*(\lambda))}$$

with $M_{Z(\lambda)}(v)$ being the mgf of $Z(\lambda)$, i.e.,

$$M_{Z(\lambda)}(v) = E(\exp\{vZ(\lambda)\}) = \exp\{-(vx - \psi_\lambda(v))\}$$

Therefore we have for the mgf of $T(\lambda)$ that

$$\begin{aligned} M_{T(\lambda)}(\xi) &= \int_{-\infty}^{\infty} e^{\xi\omega}\,dH_\lambda(\omega) = \int_{-\infty}^{\infty} e^{(\xi + t^*(\lambda))\omega} d\,\frac{F_{Z(\lambda)}(\omega)}{M_{Z(\lambda)}(t^*(\lambda))} \\ &= \frac{M_{Z(\lambda)}(\xi + t^*(\lambda))}{M_{Z(\lambda)}(t^*(\lambda))} \end{aligned}$$

Now we can compute the moments of $T(\lambda)$ using its mgf. Since $\psi'_\lambda(t^*(\lambda)) = x$ we get

$$E(T(\lambda)) = M'_{T(\lambda)}(\xi)\,|_{\xi=0} = \frac{M'_{Z(\lambda)}(t^*(\lambda))}{M_{Z(\lambda)}(t^*(\lambda))} = 0$$

Furthermore

$$\begin{aligned} s^2(\lambda) := E(T^2(\lambda)) &= \frac{M''_{Z(\lambda)}(t^*(\lambda))}{M_{Z(\lambda)}(t^*(\lambda))} \\ &= \{(x-\psi'_\lambda(t^*(\lambda)))^2 + \psi''_\lambda(t^*(\lambda))\} = \psi''_\lambda(t^*(\lambda)) \end{aligned}$$

and

$$E(T^4(\lambda)) = \frac{M^{(4)}_{Z(\lambda)}(t^*(\lambda))}{M_{Z(\lambda)}(t^*(\lambda))} = \psi^{(4)}_\lambda(t^*(\lambda)) + 3(\psi''_\lambda(t^*(\lambda)))^2$$

Define $\rho(\lambda) := P(T(\lambda) \geqslant 0) > 0$. It follows that

$$P(Z(\lambda) \geqslant 0) = \int_0^\infty e^{-t^*(\lambda)\,\omega} \frac{dH_\lambda(\omega)}{\rho(\lambda)} M_{Z(\lambda)}(t^*(\lambda))\,\rho(\lambda)$$

where $dH_\lambda(\cdot)/\rho(\lambda)$ defines a regular probability measure. Since

$$\exp\{-\theta(\lambda)\} := \frac{P(Z(\lambda) \geqslant 0)}{M_{Z(\lambda)}(t^*(\lambda))} \leqslant 1$$

we obtain taking logarithms and using Jensen's inequality that

$$\begin{aligned} -\theta(\lambda) &= \log\left(\int_0^\infty e^{-t^*(\lambda)\,\omega} \frac{dH_\lambda(\omega)}{\rho(\lambda)}\right) + \log \rho(\lambda) \\ &\geqslant \frac{-t^*(\lambda)}{\rho(\lambda)} \int_0^\infty \omega\, dH_\lambda(\omega) + \log \rho(\lambda) \\ &\geqslant \frac{-s(\lambda)\, t^*(\lambda)}{\rho(\lambda)} + \log \rho(\lambda) \end{aligned}$$

By Theorem 9.2 in Ref. 1 we have

$$\rho(\lambda) = P(T(\lambda) \geqslant 0) \geqslant \frac{s(\lambda)^4}{4E(T^4(\lambda))} = \frac{(\psi''_\lambda(t^*(\lambda)))^2)}{4(\psi^{(4)}_\lambda(t^*(\lambda)) + 3(\psi''_\lambda(t^*(\lambda)))^2)} \to \frac{1}{12}$$

for $\lambda \to \infty$ by the asymptotics in Lemma 1. This gives $\rho(\lambda) > \frac{1}{24}$ for $\lambda > \lambda_0$. By Eq. (4.1) and (4.2) we get

$$t^*(\lambda)\, s(\lambda) = \sqrt{\phi(\lambda)}\left(\mathbf{O}(1) + \mathbf{O}\left(\frac{1}{\sqrt{\phi(\lambda)}}\right)\right)$$

Combining these bounds we get for some constant $\alpha > 0$ and for $\lambda > \lambda_0$

$$\exp\{-\theta(\lambda)\} \geqslant \exp\{-\alpha\sqrt{\phi(\lambda)}\}$$

Hence it follows that

$$\begin{aligned} P(Z(\lambda) \geqslant 0) &= M_{Z(\lambda)}(t^*(\lambda)) \exp\{-\theta(\lambda)\} \\ &\geqslant \exp\{-(xt^*(\lambda) - \psi_\lambda(t^*(\lambda)))\} \exp\{-\alpha\sqrt{\phi(\lambda)}\} \end{aligned}$$

Now we use once more Eqs. (4.1) and (4.2) and the continuity of χ to conclude

$$P(Z(\lambda) \geqslant 0) \geqslant \exp\left\{-\phi(\lambda)\left(\chi^*(x) + \frac{\kappa_2}{\sqrt{\phi(\lambda)}}\right)\right\}$$

with a constant κ_2. □

REFERENCES

1. Billingsley, P. (1986). Probability and Measure, *Wiley Series in Probability and Mathematical Statistics*, J. Wiley, New York.
2. Bingham, N. H. (1984). On Valiron and circle convergence, *MZ*. **186**, 273–286.
3. Bingham, N. H. (1988). Moving averages, In Edgar G. G., and Sucheston, L. (eds.), *Almost Everywhere Convergence I* (*Proc. Conf. Columbus*, Ohio, 1988, pp. 131–145), New York, Academic Press.
4. Bingham, N. H., and Tenenbaum, G. (1986). Riesz and Valiron means and fractional moments, *Math. Proc. Camb. Phil. Soc.* **99**, 143–149.
5. Bingham, N. H., and Goldie, C. M. (1988). Riesz means and self-neglecting functions, *MZ*. **199**, 443–454.
6. Bingham, N. H., Goldie, C. M., and Teugels, J. L. (1987). Regular Variation, *Encyclopedia Math. and Appl.*, Cambridge University Press, Cambridge.
7. Borwein, D., and Kratz, W. (1994). An O-Tauberian theorem and a high indices theorem for power series methods of summability, *Math. Proc. Camb. Phil. Soc.* **115**, 365–375.
8. Courant, R., and Hilbert, D. (1968). *Methoden der Mathematischen Physik* (3. Auflage), Springer-Verlag, Berlin.
9. Csörgő, M., and Rèvèsz, P. (1981). *Strong Approximations in Probability and Statistics*, Academic Press, New York.
10. Csörgő, M., and Shao, Q.-M. (1994). A self-normalized Erdős–Rényi type strong law of large numbers, *Stoch. Proc. Appl.* **50**, 187–196.

11. Csörgő, M., and Steinebach, J. (1981). Improved Erdős–Rényi and strong approximation laws for increments of partial sums, *Ann. Prob.* **9**(6), 988–996.
12. de Acosta, A., and Kuelbs, J. (1983). Limit theorems for moving averages of independent random vectors, *Z.W. verw. Geb.* **64**, 67–123.
13. Deheuvels, P., Devroye, L., and Lynch, J. (1986). Exact convergence rates in the limit theorems of Erdős–Rényi and Shepp, *Ann. Prob.* **14**(1), 209–223.
14. Deheuvels, P., and Steinebach, J. (1987). Exact convergence rates in strong approximation laws for large increments of partial sums, *Prob. Th. Rel. Fields* **76**, 336–393.
15. Feller, W. (1970). *An Introduction to Probability Theory and Its Applications II*, 2nd Ed., John Wiley, New York.
16. Hardy, G. H. (1949). *Divergent Series*, Oxford Press, Oxford.
17. Kiesel, R. (1990). Taubersätze und Starke Gesetze für Potenzreihenverfahren, Universität Ulm, Dissertation.
18. Kiesel, R. (1993). Power series methods and almost sure convergence, *Math. Proc. Camb. Phil. Soc.* **113**, 195–204.
19. Kiesel, R. (1993). General Nörlund transforms and power series methods, *MZ.* **214**, 273–286.
20. Kiesel, R., and Stadtmüller, U. (1991). Tauberian theorems for general power series methods, *Math. Proc. Camb. Phil. Soc.* **110**, 483–490.
21. Kiesel, R., and Stadtmüller, U. (1994). Tauberian and convexity theorems for certain (N, p, q) means, *Can. J. Math.* **46**, 982–994.
22. Müller, H. G. (1988). Nonparametric regression analysis of longitudinal data. *Lecture Notes in Statistics*, Vol. 46, Springer-Verlag, New York.
23. Shepp, L. A. (1964). A limit law concerning moving averages, *Ann. Math. Stat.* **35**(1), 424–428.
24. Stadtmüller, U. (1996). On a family of summability methods and one-sided Tauberian conditions, *J. Math. Anal. Appl.* (to appear).
25. Valiron, G. (1917), Remarques sur la sommation des séries divergentes par les méthods de Borel, *Rend. Pal.* **42**, 267–284.