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Solitons with fission and fusion behaviors in a variable coefficient Broer–Kaup system

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Abstract

Using an extended homogeneous balance approach and a linear variable separation method, a general variable separation excitation of the (2 + 1)-dimensional variable coefficient Broer–Kaup (VCBK) system is derived. Based on the derived solution, we reveal soliton fission and fusion phenomena in the (2 + 1)-dimensional soliton system.

1. Introduction

Usually, the collisions between solitons of integrable models are regarded to be completely elastic. That is to say, the amplitude, velocity and wave shape of a soliton do not change after nonlinear interaction [1,2]. However, for some special solutions of certain (2 + 1)-dimensional models in our colleagues' and our recent study, the interactions among solitonic excitations like peakons and compactons are not completely elastic since their shapes or amplitudes are changed after their collisions [3,4]. Furthermore, for some (1 + 1)-dimensional models, two or more solitons may fuse into one soliton at a special time while for sometimes one soliton may fission into two or more solitons at other special time [5]. These phenomena are often called soliton fusion and soliton fission respectively. Actually, the soliton fusion and fission phenomena have been observed in many physical systems such as organic membrane and macromolecular material [6], and physical fields like plasma physics, nuclear physics and hydrodynamics [7]. Recently, Zhang [8] and Lin et al. [9] studied the evolutions of soliton solutions for two (1 + 1)-dimensional PDEs with time and revealed the solitons fission and soliton fusion phenomena. In a similar way, Wang et al. [10] further discussed two (1 + 1)-dimensional models, the Burgers equation and the Sharma–Tasso–Olver equation, via the Hirota's direct method, and also found the soliton fission and soliton fusion phenomena. Now an interesting and important problem is that are there soliton fission and fusion phenomena in higher dimensions? The main purpose of our present paper is searching for some possible soliton fission and soliton fusion phenomena in (2 + 1)-dimensions. As a concrete example, we consider following (2 + 1)-dimensional variable coefficient Broer–Kaup system (VCBK) [11]

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$$H_{yt} - B(t)[H_{xxy} - 2(HH_x)_y - 2G_{xx}] = 0, \tag{1}$$

$$G_t + B(t)\left[G_{xx} + 2(GH)_x\right] = 0, \tag{2}$$

where $B(t)$ is an arbitrary function of time t. It is evident that when $B(t)$ = constant, the VCBK system will be degenerated to the celebrated (2 + 1)-dimensional Broer–Kaup system (BK) [12], which may be derived from inner parameter dependent symmetry constraint of Kadomtsev–Petviashvili model [13]. When $y = x$, the (2 + 1)-dimensional BK system will be reduced further to the usual (1 + 1)-dimensional BK system, which can be used to describe the propagation of long waves in shallow water [14]. Using some suitable dependent and independent variable transformations, Chen and Li [15] have proved that the (2 + 1)-dimensional BK system can be transformed to the (2 + 1)-dimensional dispersive long wave equation (DLWE) [16] and (2 + 1)-dimensional Ablowitz–Kaup–Newell–Segur system (AKNS) [17]. Actually, the (2 + 1)-dimensional BK system has been widely investigated by many researchers [18–21]. However, to the best of our knowledge, the studies on soliton fission and soliton fusion phenomena for the (2 + 1)-dimensional VCBK system were not reported in the preceding literature.

2. General solution to the (2 + 1)-dimensional VCBK system

As is well known, for a given nonlinear partial differential model, one can utilize different approaches. One of the useful and powerful method is an extended homogeneous balance approach (EHBA). According to the EHBA principle (which can also be obtained through the standard truncated Painlevé expansion), let us begin with a Painlevé–Bäcklund transformation

$$H = (\ln f)_x + H_1, \quad G = (\ln f)_{xy} + G_2, \tag{3}$$

where $f = f(x, y, t)$ is an arbitrary function of variables $\{x, y, t\}$ to be determined and $\{H_1, G_2\}$ are two arbitrary seed solutions. For convenience of the following discussion, we choose the seed solutions $\{H_1, G_2\}$ to be $H_1 = H_1(x, t)$, $G_2 = 0$, where $H_1(x, t)$ is an arbitrary function of indicated arguments. Substituting Eq. (3) with the seed solutions into Eqs. (1) and (2) yields an identical trilinear equation

$$[f^2\partial_{xy} - f(f_x\partial_y + f_y\partial_x + f_{xy}) + 2f_xf_y][f_t + B(t)f_{xx} + 2B(t)H_1f_x] = 0. \tag{4}$$

It can be easily seen that if f satisfies

$$f_t + B(t)f_{xx} + 2B(t)H_1f_x = 0, \tag{5}$$

Eq. (4) is satisfied automatically. Since Eq. (5) is only a linear equation, one can certainly utilize the linear superposition theorem. For instance

$$f = Q_0(y) + \sum_{k=1}^{N} P_k(x, t)Q_k(y, t), \tag{6}$$

where the variable separated functions $P_k(x, t) \equiv P_k$ and $Q_k(y, t) \equiv Q_k$ ($k = 1, 2, \ldots, N$) are only the functions of $\{x, t\}$ and $\{y, t\}$, respectively, and $Q_0(y) \equiv Q_0$. Inserting the ansatz (6) into Eq. (5) yields following set of variable separation equations:

$$P_{kt} + B(t)(2H_1P_{kx} + P_{kxx}) + \sum_{l=1}^{M} C_{kl}(t)P_k = 0,$$

$$Q_{kt} - \sum_{l=1}^{M} C_{kl}(t)Q_k = 0 \quad (l = 1, 2, \ldots, M), \tag{7}$$

where $C_{kl}(t)$ ($k = 1, 2, \ldots, N$; $l = 1, 2, \ldots, M$) are arbitrary functions of time t. Then a general variable separation excitation for the VCBK system yields

$$H = \frac{\sum_{k=1}^{N} P_{kx}Q_k}{Q_0 + \sum_{k=1}^{N} P_kQ_k} + H_1, \tag{8}$$

$$G = \frac{\sum_{k=1}^{N} P_{kx}Q_{ky}}{Q_0 + \sum_{k=1}^{N} P_kQ_k} - \frac{\sum_{k=1}^{N} P_{kx}Q_k(Q_{0y} + \sum_{k=1}^{N} P_kQ_{ky})}{(Q_0 + \sum_{k=1}^{N} P_kQ_k)^2}, \tag{9}$$

where H_1, P_k and Q_k admit Eq. (7).

In order to discuss some interesting properties of the above general excitation (8) and (9), we should make some simplification further. Here we consider a simplest case: $N = M = 1$, $\{P_1, Q_1\} \equiv \{P, Q\}$, $C_{11}(t) \equiv c(t)$. Then the above Eqs. (6) and (7) become

$$f = Q_0 + PQ, \tag{10}$$

$$P_t + B(t)(P_{xx} + 2H_1P_x) + c(t)P = 0,$$

$$Q_t - c(t)Q = 0. \tag{11}$$

It is easy to obtain general solutions of Eq. (11). Since $H_1(x,t)$ is an arbitrary seed solution, we can view P as an arbitrary function of $\{x,t\}$, then the seed solution H_1 is fixed by the first equation of (11),

$$H_1 = -\frac{P_t + B(t)P_{xx} + c(t)P}{2B(t)P_x}. \tag{12}$$

As to the second equation of (11), its general solution has the form

$$Q(y,t) = \varphi(y)\exp\int^t c(t)\,\mathrm{d}t, \tag{13}$$

where $\varphi(y) = \varphi$ is an arbitrary function of y.

Finally, we derive a special variable separation excitation of the (2 + 1)-dimensional VCBK system

$$H = \frac{P_x\varphi\exp\int^t c(t)\,\mathrm{d}t}{Q_0 + P\varphi\exp\int^t c(t)\,\mathrm{d}t} - \frac{P_t + B(t)P_{xx} + c(t)P}{2B(t)P_x}, \tag{14}$$

$$G = \frac{P_x(\varphi_yQ_0 - \varphi Q_{0y})\exp\int^t c(t)\,\mathrm{d}t}{\left[Q_0 + P\varphi\exp\int^t c(t)\,\mathrm{d}t\right]^2} \tag{15}$$

with four arbitrary functions $P(x,t)$, $Q_0(y)$, $\varphi(y)$ and $c(t)$.

3. Some special solutions: fission and fusion solitons in the VCBK system

Here we do not study the general field (9), and only discuss the special field G expressed by (15). Actually, even in this special situation, one can still find abundant localized structures for the (2 + 1)-dimensional VCBK system. As the arbitrariness of characteristic functions P, Q_0, φ and $c(t)$ is included in the special field (15), G obviously possesses quite rich structures such as dromion, peakon, compacton and foldon when we select the arbitrary functions P, Q_0, φ if and $c(t)$ appropriately, which also implies that some exotic behaviors may propagate along with the above lines [22,23].

From the above brief discussions, one may deduce that the field G would exist some novel properties that have not been revealed. Recently, it is reported both theoretically and experimentally that fission and fusion phenomena can happen for (1 + 1)-dimensional solitons or solitary waves [10]. Now we focus our attention on these intriguing fusion and fission phenomena for the special field G in (2 + 1)-dimensions, which may exist in certain situations. For instance, when we select the arbitrary functions $P(x,t)$ and $\varphi(y)$ to be

$$P(x,t) = 1 + 2\exp(x - 2t) + \begin{cases} \exp(x+t), & x+t \leqslant 0, \\ -\exp((-x-t)+2), & x+t > 0, \end{cases}$$

$$\varphi(y) = 1 + \exp(2y), \tag{16}$$

and $Q_0 = 1$, $c(t) = 0$ in Eq. (15), then we can obtain a new kind of fission solitary wave solution for the field G. Fig. 1 shows an evolutional profile of the solitary wave solution for the corresponding field G (15), which depicts a fission phenomenon. From Fig. 1, one can clearly see that the one soliton fissions into two solitons. It is interesting to mention that the left travelling soliton along with the negative x-axis, i.e. one of the pairs of solitons that emerge after the fission, is stable and do not undergo additional fissions as running program for longer periods of time till to $t = 10^3$. However, the right travelling soliton along with the positive x-axis is unstable and will fission further into many oscillating solitons as time $t > 11$, their shapes and amplitudes are changed with time.

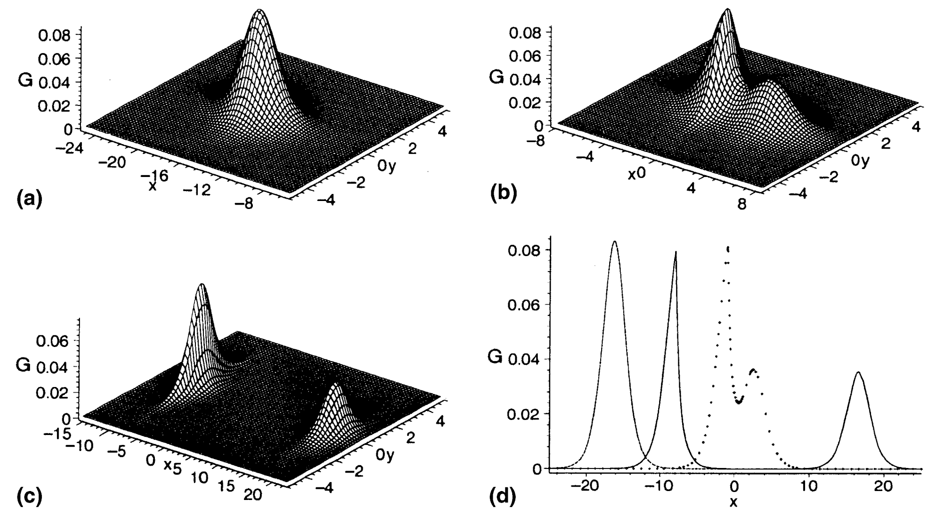

Fig. 1. The evolutional profile of one soliton fission into two solitons for the field G (15) with condition (16) at different times: (a) $t = -6$, (b) $t = 0$, (c) $t = 6$. (d) A sectional view related to (a)–(c) at $y = 0$: dashed line (a), dotted line (b) and solid line (c).

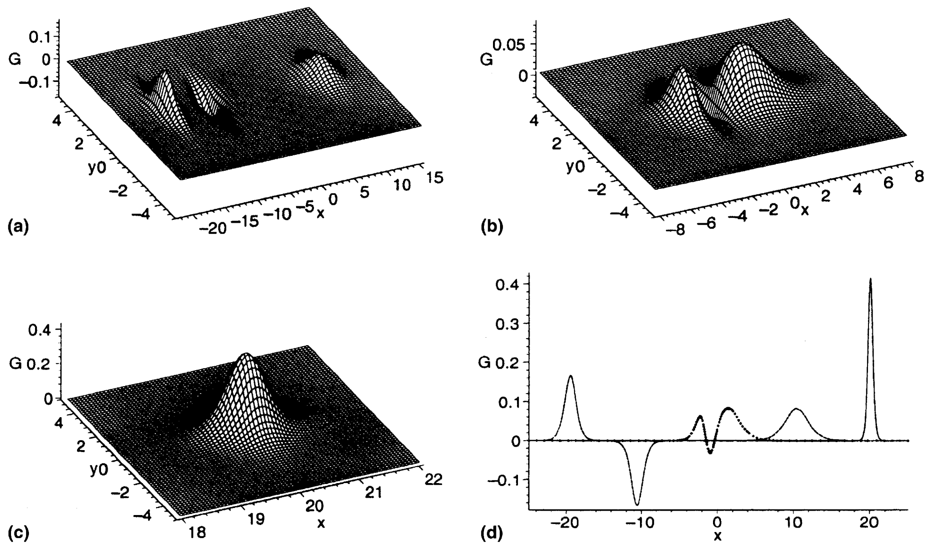

Fig. 2. Three solitons fuse into one soliton evolutional plot of the field G (15) with condition (17) at different times: (a) $t = -10$, (b) $t = -1$, (c) $t = 20$. (d) A sectional view related to (a)–(c) at $y = 0$: dashed line (a), dotted line (b) and solid line (c).

Along with the above line, when we consider $P(x,t)$ and $\varphi(y)$ to be

$$P(x,t) = 1 + \frac{\exp(5x-5t) + 0.8\exp(2x-3t) + \exp(2x-4t)}{(1+\exp(2x-3t))^2}, \qquad \varphi(y) = 1 + \exp(2y) \tag{17}$$

and $Q_0 = 1$, $c(t) = 0$ for the special field G (15), then we can obtain another new type of fusion solitary wave, which possesses apparently different property compared with Fig. 1. From Fig. 2, one can find that three solitons fuse into one soliton finally. The fused single soliton remains stable for subsequent times as we run program for rather long times ($t = 10^3$).

Certainly, if we consider another selections for the arbitrary functions P, Q_0, φ and $c(t)$ such as Jacobian functions or the solutions of the well-known Lorenz chaotic system, then we may derive some novel solitary wave solutions with double periodic properties or chaotic behaviors, which are neglected in our present paper. Actually, because there exist some arbitrary characteristic functions P, Q_0, φ and $c(t)$ in the special field G, any exotic behaviors may engender along with the above mentioned ideas.

4. Summary and discussion

In summary, by means of an extended homogeneous balance approach and a linear variable separation method, the (2 + 1)-dimensional VCBK system is successfully solved. Based on a special solution of the derived general solution with arbitrary functions, we list two simple examples, soliton fission and fusion solutions for the (2 + 1)-dimensional VCBK system. Usually, it is considered that the interactions among solitons are completely elastic. However, in some special cases, the soliton collisions may be nonelastic or completely nonelastic. From the brief analysis in our present paper, we can see that these intriguing phenomena like the soliton fission and fusion can occur in a higher dimensional soliton system if we choose appropriate initial conditions or boundary conditions, which are similar to some work in (1 + 1)-dimensions carried out by several authors [8–10]. Although we have given out some soliton fusion and fission phenomena in (2 + 1)-dimensions, it is obvious that there are still many significant and interesting problems waiting for further discussions. Just as the authors [8–10] have pointed out in (1 + 1)-dimensional cases, what is the necessary and sufficient condition for soliton fission and soliton fusion? What is the general equation for the distribution of the energy and momentum after soliton fission and soliton fusion? How can we use the soliton fission and soliton fusion of integrable models to investigate practically observed soliton fission and soliton fusion in the experiments? These are all the pending issues. Actually, our present short note is merely an initial work, due to widely potential applications of soliton theory, to learn more about the soliton fission and fusion properties and their applications in reality are worthy of further study.

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