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Spectral statistics in a quantum optical model

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Abstract. In this paper, we have investigated spectral statistics in a quantum optical model, the condition for the random matrix theory is analysed and illustrated with numerical results.

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Although an analytical work is hard, the chaotic behavior of quantum systems has received considerable interest forthe past few years [1–4]. The spectral analysis with respect to the random matrix theory [5] has been generally used to show chaotic behaviours of autonomus quantum systems. Since then, the correspondence between the character of the classical motion and the statistical properties of the spectrum of analogous quantum system has been well established [6], i.e., classically integrable systems lead to uncorrelated quantum levels (Poission distribution), while the eigenvalues of systems are supposed to have the same statistical properties as the eigenvalues of random matrices belonging to Gaussian Orthogonal Ensemble (GOE) (Wigner distribution) under the condition in which the classical systems are chaotic.

A physically interesting example which belongs to the quantum chaos field is the most basic quantum-optical model of a two-level atom interacting with a single mode of a quantized electromagnetic field [7]. For sufficiently strong coupling between the field and the atom, some non-classical dynamical features of the system were found to behave irregularly under chosen initial condition [8], and its semi-classical system can exhibit chaotic dynamics [9]. However, the numerical studies of its corresponding energy spectrum have shown that the nearest-level spacing was highly correlated and regular [8,10], which were inconsistent with the above judgement. Graham and Höhnerbach [11] have indicated that such results were closely related to the single two-level atom being considered, the model with a number of two-level atoms can give statistical behavior of the GOE. In this paper, we will analyse this problem further and wish to indicate the condition for random matrix theory and the classical-quantum correspondence in this system.

The system considered here describes several two-level atoms interacting with a single mode of a quantized electromagnetic field. Its Hamiltonian reads [11]

$$H = \Omega a^+ a + \omega S_z + 2G(a^+ + a)S_x \quad (\hbar = 1) \tag{1}$$

where $S_z = \frac{1}{2}\sum_{i=1}^{N_A} \sigma_{z,i}$, $S_x = \frac{1}{2}\sum_{i=1}^{N_A} \sigma_{x,i}$, $\sigma_{x,i}$, $\sigma_{y,i}$ and $\sigma_{z,i}$ are the Pauli spin matrices for the individual two-level atoms. N_A is the total number of two-level atoms, ω is the atomic transition frequency, a^+ and a are the creation and annihilation operators of the field with frequency Ω and the commutation relation $[a, a^+] = 1$, G is the atom-field coupling constant.

Usually, the strength of the field, i.e., mean photon number $\overline{N_F}$, is not explicitly mentioned in numerical calculations. Taking it into consideration, there are altogether five parameters in this system, i.e., Ω, ω, G, N_A and $\overline{N_F}$. Noticing that only their relative values are relevant to studies of spectral statistics, we have four parameters at our disposal. Correspondingly, we have different parameter conditions for the model.

In the usual discussion, the number of atoms N_A is fixed, and so is the mean photon number $\overline{N_F}$, whereupon, only two relative parameters remain, then the Hamiltonian (1) can be written as follows:

$$\bar{H} = \frac{H}{\sqrt{\Omega\omega}} = \sqrt{\frac{\Omega}{\omega}} a^+ a + \sqrt{\frac{\omega}{\Omega}} S_z + 2\frac{G}{\sqrt{\Omega\omega}} S_x(a^+ + a) \tag{2}$$

Clearly, there are some integrable limits and those around the integrable limits can be treated pertubatively from the corresponding ones.

(1) $\frac{G}{\sqrt{\Omega\omega}} \ll 1$ and $\frac{\Omega}{\omega} \approx 1$, we have

$$\overline{H^{(0)}} = \frac{H^{(0)}}{\sqrt{\Omega\omega}} = \sqrt{\frac{\Omega}{\omega}} a^+ a + \sqrt{\frac{\omega}{\Omega}} S_z + \frac{G}{\sqrt{\Omega\omega}}(aS_+ + a^+ S_-)$$

$$\overline{V} = \frac{V}{\sqrt{\Omega\omega}} = \frac{G}{\sqrt{\Omega\omega}}(aS_- + a^+ S_+) \tag{3}$$

the conserved quantity besides the energy H is $a^+a + S_z$. If the pertubative term V in H is neglected, we have just the the rotating wave approximation (RWA).

(2) $\dfrac{G}{\sqrt{\Omega\omega}} \gg 1$ and $\dfrac{\Omega}{\omega} \approx 1$, we have

$$\overline{H^{(0)}} = \frac{H^{(0)}}{\sqrt{\Omega\omega}} = 2\frac{G}{\sqrt{\Omega\omega}} S_x(a^+ + a)$$

$$\bar{V} = \frac{V}{\sqrt{\Omega\omega}} = \sqrt{\frac{\Omega}{\omega}}\, a^+a + \sqrt{\frac{\omega}{\Omega}}\, S_z \tag{4}$$

the conserved quantities are S_x and $a^+ + a$. When the coupling of atom-field interaction is very strong, we can consider the approximation that all the states of different photon numbers and different values of S_z are degenerated.

(3) $\dfrac{\Omega}{\omega} \ll 1$ and $\dfrac{\Omega}{\sqrt{\Omega\omega}} \approx 1$, we have

$$\overline{H^{(0)}} = \frac{H^{(0)}}{\sqrt{\Omega\omega}} = \sqrt{\frac{\omega}{\Omega}} S_z + 2\frac{G}{\sqrt{\Omega\omega}} S_x(a^+ + a) + \frac{1}{4}\sqrt{\frac{\Omega}{\omega}}(a^+ + a)^2$$

$$\bar{V} = \frac{V}{\sqrt{\Omega\omega}} = \sqrt{\frac{\Omega}{\omega}}\, a^+a \tag{5}$$

the conserved quantity besides the energy H is $a^+ + a$. If V is omitted, this case can be considered as the degenerate approximation of the states with different photon numbers.

(4) $\dfrac{\Omega}{\omega} \gg 1$ and $\dfrac{G}{\sqrt{\Omega\omega}} \approx 1$ we have

$$\overline{H_{(0)}} = \frac{H^{(0)}}{\sqrt{\Omega\omega}} = \sqrt{\frac{\Omega}{\omega}} a^+a + 2\frac{G}{\sqrt{\Omega\omega}} S_x(a^+ + a)$$

$$\bar{V} = \frac{V}{\sqrt{\Omega\omega}} = \sqrt{\frac{\omega}{\Omega}}\, S_z \tag{6}$$

S_x is the conserved quantity besides the energy H. If we omit V, this case can be taken as the degenerate approximation of the states with different values of S_z.

Graham and Höhnerbach [8] have discussed the integrable limits (1)(3)(4) given above in the model with a single two-level atom.

In order to study the effect of the mean photon number $\overline{N_F}$ and atom number N_A, we rewrite the Hamiltonian (1) as follows,

$$H = \overline{N_F}\Omega\left(\frac{a^+a}{\overline{N_F}}\right) + N_A\omega\left(\frac{S_z}{N_A}\right) + 2GN_A\sqrt{\overline{N_F}}\left(\frac{S_x}{N_A}\right)\left(\frac{a^+ + a}{\sqrt{\overline{N_F}}}\right) \tag{7}$$

where S_i/N_A, $i = x, y, z$, $(a^+ + a)/\sqrt{2\overline{N_F}}$ and $i(a^+ - a)/\sqrt{2\overline{N_F}}$ are the dynamical variables satisfying the following commutation relations:

$$\left[\left(\frac{S_i}{N_A}\right), \left(\frac{S_j}{N_A}\right)\right] = i\frac{1}{N_A}\sum_k \varepsilon_{ijk}\left(\frac{S_k}{N_A}\right) \tag{8}$$

$$\left[\frac{a^+ + a}{\sqrt{2\overline{N_F}}}, \frac{i(a^+ - a)}{\sqrt{2\overline{N_F}}}\right] = i\frac{1}{\overline{N_F}} \tag{9}$$

If $\overline{N_F}\Omega$, $N_A\omega$ and $2GN_A\sqrt{\overline{N_F}}$ are kept fixed for systems with different N_A and $\overline{N_F}$, the only difference among them lies in the quantal effect due to the difference in (8) and (9). If $1/N_A$, and $1/\overline{N_F}$ approach zero (i.e., N_A and $\overline{N_F}$ become infinitely large), (8) and (9) turn into classical Poisson brackets, they result in the same classical limit, moreover, the classical counterpart of the quantum system is obtained. However, for systems with same Ω, ω, G and $\overline{N_F}$ but different N_A, one can see from (7) that there should exist different classical limits corresponding to different N_A, furthermore, there exist different quantal effect, which can be seen in the following numerical results. Therefore, one should make sure what is the proper classical counterpart in considering the quantum-classical correspondence.

Before we examine the requirements of the analysis of spectral data and the random matrix theory, we introduce the effective Hamiltonian matrices. We consider such a quasi-stationary process, where the Hamiltonian is chosen as

$$H(\lambda(t)) = H^0 + \lambda(t)(H - H^0) \tag{10}$$

where λ is a time-dependent parameter satisfying $\lambda(0) = 0$, and $d\lambda(t)/dt = \varepsilon$ (ε is infinitely small). Suppose H^0 is commutable with a CSCO I_α^0 corresponding to a certain subgroup chain of the dynamical group such that

$$H^0|\phi_i^0\rangle = E_i^0|\phi_i^0\rangle \tag{11}$$

$$I_\alpha^0|\phi_i^0\rangle = \Lambda_m|\phi_i^0\rangle \tag{12}$$

Then after ignoring a phase factor, the solution of the Schrödinger equation with the initial condition $|\phi_i^0\rangle$ is just the eigenvector of the Hamiltonian $H(\lambda)$. In this sense, we can consider the stationary problem just as a time-dependent case.

In a partly integrable classical system, whether a trajectory remains on a KAM or not varies with the initial conditions. Now the same problem appears as to whether solutions of the Schrödinger equation can be still characterized by a set of quantum numbers when the initial conditions are within a certain energy region. Stimulated by this, we try to use the effective Hamiltonians related to different energy regions to investigate the local quantum signatures of classical nonintegrability of corresponding energy surfaces.

Assuming that there are N eigenstates of $H(\lambda)$ in the energy region μ. Eigenvectors $|\phi_i(\lambda)\rangle (i \in [\mu])$ are within this energy region while eigenvectors $|\phi_i(\lambda)\rangle (i \notin [\mu])$ are

outside this energy region. The procedure to get the effective Hamiltonians is as follows:

1. With the help of the projection operator

$$P_\mu = 1 - \sum_{i \notin [\mu]} |\phi_i(\lambda)\rangle\langle\phi_i(\lambda)| \tag{13}$$

we can obtain a set of linearly independent state vectors from $|\phi_i^0\rangle (i \in [\mu])$:

$$|\zeta_i\rangle = P_\mu |\phi_i^0\rangle, i \in [\mu] \tag{14}$$

2. By using the Schmidt method, we can obtain from $|\zeta_i\rangle$ a complete set of orthogonal basic vectors $|\eta_i\rangle (i \in [\mu])$ of H_μ for the subspace μ:

$$\langle\eta_i|\eta_j\rangle = \delta_{ij}, i, j \in [\mu] \tag{15}$$

$$\sum_{i \in [\mu]} |\eta_i\rangle\langle\eta_i| = P_\mu \tag{16}$$

3. Carrying out the same process for different energy regions, we have projection operators for different subspaces. Apparently

$$\sum_\mu P_\mu = \sum_i |\eta_i\rangle\langle\eta_i| = 1 \tag{17}$$

thus $|\eta_i\rangle$ obtained for all subspaces from an orthogonal basis set for the whole space. The relation between $|\phi_i^0\rangle$, $|\phi_i(\lambda)\rangle$ and $|\eta_i(\lambda)\rangle$ can be used to define orthogonal transformations $U(\lambda, 0)$, $U_c(\lambda, 0)$ and $U_l(\lambda)$ as follows:

$$|\phi_i(\lambda)\rangle = U(\lambda, 0)|\phi_i^0\rangle = \sum_j |\phi_j^0\rangle\langle\phi_j^0|U(\lambda, 0)|\phi_i^0\rangle \tag{18}$$

$$|\eta_i(\lambda)\rangle = U_c(\lambda, 0)|\phi_i^0\rangle = \sum_j |\phi_j^0\rangle\langle\phi_j^0|U_c(\lambda, 0)|\phi_i^0\rangle \tag{19}$$

$$|\phi_i(\lambda)\rangle = U_l(\lambda)|\eta_i(\lambda)\rangle = \sum_j |\eta_j\rangle\langle\eta_j|U_l(\lambda, 0)|\eta_i\rangle \tag{20}$$

obviously

$$\langle\eta_j(\lambda)|\phi_i(\lambda)\rangle = \langle\eta_j(\lambda)|U_l(\lambda)|\eta_i(\lambda)\rangle = 0 \tag{21}$$

if $i \in [\mu]$, $j \in [\nu]$ with $\mu \neq \nu$, we see that $\langle\eta_j(\lambda)|U_l(\lambda)|\eta_i(\lambda)\rangle$ is a block matrix and hence the Hamiltonian matrix $\langle\eta_j(\lambda)|H(\lambda)|\eta_i(\lambda)\rangle$ which is diagonalized by the block matrix $U_l(\lambda)$ should be also a block matrix. Therefore, we have

$$H = H_{\text{eff};1} \oplus H_{\text{eff};2} \oplus \cdots \tag{22}$$

where

$$H_{\text{eff};\mu} = P_\mu H P_\mu = \sum_{ij \in [\mu]} |\eta_i\rangle\langle\eta_i|H|\eta_j\rangle\langle\eta_j| \tag{23}$$

as the effective Hamiltonian matrix for the subspace μ.

In this way, the diagonalizaion for the Hamiltonian matrix is resolved into two steps. The first step for obtaining the effective Hamiltonian matrices is performed with the decoupling transformation $U_c(\lambda, 0)$ such that the interactions with states outside the relevant subspaces are well taken into account, while the second one is to diagonalize the effective Hamiltonain matrices for different energy regions with a further orthogonal transformation $U_l(\lambda)$. So the array of eigenvectors thus obtained as just the $U_l(\lambda)$ matrix given by (20) and the eigenenergies thus obtained is just the $U_l(\lambda)$ from $H(\lambda)$.

After obtaining the effective Hamiltonian, we begin to show the requirements of the analysis of spectral data and the random matrix theory. In the analysis of spectral data, we first scale the data in different energy regions to the same mean energy and the same mean level spacing, then lump them together and treat them as an ensemble of random entities. In order to have meaningful results, there should exist sufficiently many levels in each energy region (i.e., $\Delta N \gg 1$), and the unfolded data should be indeed uncorrelated due to dynamical reasons [4]. While in random matrix theory, we should have the corresponding requirement. The ensemble of random matrices should be the effective Hamiltonian matrices of the corresponding energy region; also, they should be unfolded in the same manner as for the numerical data. However, that the unfolded effective Hamiltonian matrices can be taken as random matrix ensemble must be subjected to definite dynamical conditions [4]. When eigenvectors of the quantum system H expressed into eigenvectors of $H^{(0)}$ have strong mixing in global regions much larger than the energy regions considered (i.e., $M \gg \Delta N$), the informations of the $M \times M$ submatrix of H should be brought into the $\Delta N \times \Delta N$ matrix of the effective Hamiltonian H_{eff}, thus, H_{eff} becomes very much complicated as if the matrix elements are completely uncorrelated. Then, after unfolding, the randomness of the Hamiltonian matrices can be be regarded as random matrices only if the condition

$$1 \ll \Delta N \ll M \tag{24}$$

is fulfilled.

Now, we indicate, in the two-level system, that the above condition (24) can not be satisfied when N_A is small, such as $N_A = 1$ [8, 10]. If $\Omega/\omega = 1$, when $G/\sqrt{\Omega\omega} = 0$, the states with same value of $a^+a + S_z$ are degenerate, but here $S_z = \pm\frac{1}{2}$, they are twofold degenerate states. When $G/\sqrt{\Omega\omega}$ increases a little, the atom-field interaction mainly splits the twofold degenerate states and can not cause strong mixing between the states with different values of $a^+a + S_z$. When $G/\sqrt{\Omega\omega}$ increase more, the states originally with different values of $a^+a + S_z$ can be mixed strongly when they come close to each other. But due to the twofold degeneracy, the strong mixing can occur only in rather small region, the condition (24) is still not satisfied. If $G/\sqrt{\Omega\omega} \gg 1$, it turns to the second integrable limit discussed before, the condition (24) is also not satisfied. Therefore, the condition (24) for the GOE statistics can not be fulfilled for any intensity of the atom-field interaction in this case.

In order to illustrate the previous discussions, we give in the following numerical results for cases under various conditions. We take $|m, n\rangle$ as basis, where $S_z|m, n\rangle = m|m, n\rangle$, $a^+a|m, n\rangle = n|m, n\rangle$ (m is in the range $-N_A/2 \sim N_A/2$, and $n \geq 0$, integer). Then, following the numerical method [8] of truncating an infinite matrix to finite order, we can find the energy eigenvalues. The diagonalization of the Hamiltonian matrix is performed in a 400-dimensional space spanned by basic states with even quantum numbers of $a^+a + S_z + \frac{1}{2}$ (or 0).

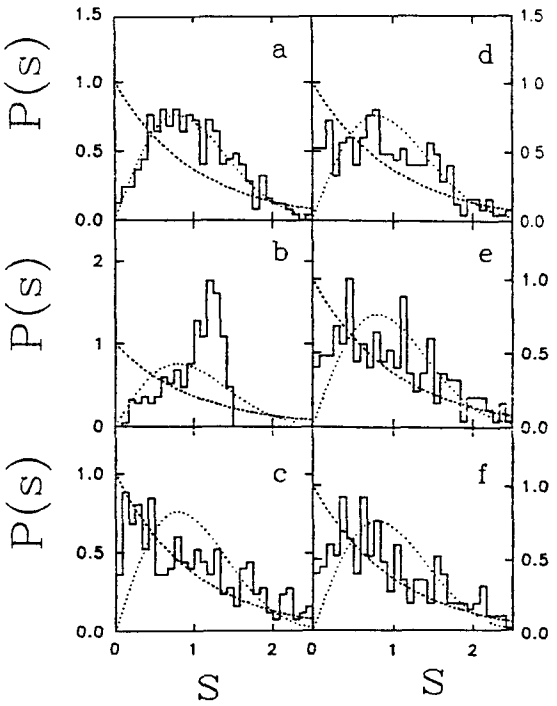

Fig. 1. Histogram of nearest-neighboring level spacings $P(S)$ **a** $\Omega/\omega=1$, $G/\sqrt{\Omega\omega}=0.2$, $N_A=21$, $N_F=0-40$ **b** $\Omega/\omega=1$, $G/\sqrt{\Omega\omega}=0.2$, $N_A=1$, $N_F=0-200$ **c** $\Omega/\omega=1$, $G/\sqrt{\Omega\omega}=0.09$, $N_A=21$, $N_F=0-40$ **d** $\Omega/\omega=1$, $G/\sqrt{\Omega\omega}=1$, $N_A=21$, $N_F=0-40$ **e** $\Omega/\omega=0.09$, $G/\sqrt{\Omega\omega}=0.3$, $N_A=21$, $N_F=0-110$ **f** $\Omega/\omega=9.0$, $G/\sqrt{\Omega\omega}=0.3$, $N_A=21$, $N_F=0-28$ dashed line for Poisson statistics and dotted line for GOE statistics

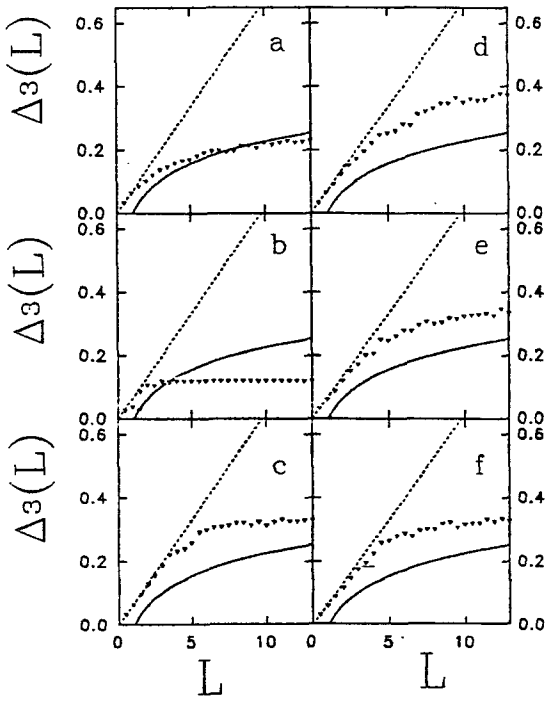

Fig. 2. $\Delta_3(L)$ statistics, dashed line for Poisson statistics, Solid line for GOE statistics and Triangle down for numerical results, others same as for Fig. 1

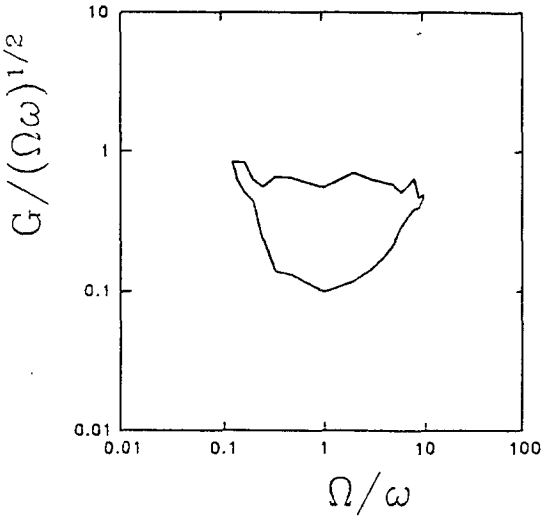

Fig. 3. GOE statistics for $N_A=21$ appear within the regions circuled by the solid line

The lowest 300 levels are used in statistical analyses. The ranges of photon numbers for different cases are then determined.

In Fig. 1, we plot the distributions of nearest-neighboring level spacings $P(S)$ and Fig. 2. is the $\Delta_3(L)$ statistics. Case (a) differs largely from the four integrable limits discussed before, the perturbing term viewed from any integrable limit and the number of atoms are both big enough to lead to strong mixing in a global region, the condition (24) is fulfilled, so GOE statistics are displayed as shown in Fig. 1a and Fig. 2a. In case (b), N_A is much smaller than the previous case, though other parameters are unchanged. As discussed before, the condition (24) cannot be fulfilled, so the GOE statistics cannot be exhibited in Fig. 1b and Fig. 2b. Noticing that case (a) and (b) have different classical limits, it is no wonder why they give different quantum behaviors. Cases (c)(d)(e)(f) are rather close to the integrable limits (1)(2)(3)(4), respectively. With respect to them, the perturbing terms are rather small and cannot lead to strong mixing in global regions to fulfill the condition (24). The numerical results for them are shown in Fig. 1c–f and Fig. 2c–f, GOE statistics are not exhibited. Furthermore, in order to reflect clearly the integrable limits discussed above, as an example, we have approximately plotted a parameter plot shown in Fig. 3 for $N=21$, where GOE statistics can be observed within bounds of the solid line.

The above discussions show that: (1) in the spectral statistics for quantum systems the condition (24) for GOE should be considered; (2) in considering the quantum-classical correspondence, the classical counterpart of the quantum system should be properly taken. Only in this way, the inference mentioned at the beginning of the paper can be valid.

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