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The stability of the equilibrium position of Hamiltonian systems with two degrees of freedom☆

Yu. N. Bibikov

St Petersburg, Russia

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ABSTRACT

The stability of the equilibrium position at the origin of coordinates of a Hamiltonian system with two degrees of freedom with a Hamiltonian, the unperturbed part of which generates oscillators with a cubic restoring force, is considered. It is proved that the equilibrium position is Lyapunov conditionally stable for initial values which do not belong to a certain surface of the Hamiltonian level. A reduction of the system onto this surface shows that, in the generic case, unconditional Lyapunov stability also occurs.

The case of two linear oscillators has been investigated in the classical papers of Arnold and Moser, while the case of one linear and one cubic oscillator (and, in general, with any odd exponent) was investigated by Sokolskii.[1]

1. Formulation of the problem and main results

We will consider a substantially analytical Hamiltonian system with two degrees of freedom in the neighbourhood of an equilibrium position at the origin of coordinates. Suppose the Hamiltonian of the system has the form $H = H^0 + H^1$ with an unperturbed part

$$H^0 = \frac{1}{4}\lambda_1(p_1^4 + 2q_1^2) - \frac{1}{4}\lambda_2(p_2^4 + 2q_2^2); \quad \lambda_i > 0, \quad i = 1, 2 \tag{1.1}$$

and an expansion of the perturbation H^1 in powers of p_i, q_i ($i = 1, 2$) does not contain terms of order lower than five, if, considering the variables p_i as quantities of the first measurement, we assign a measurement, equal to two, to the variables q_i.

Such a system arises, for example, when investigating conservative perturbations of two weakly connected oscillators with a cubic restoring force

$$\ddot{p}_i + p_i^3 = 0, \quad i = 1, 2$$

Note also that we can reduce any Hamiltonian H^0 to the form (1.1), the coefficients of which for q_1^2 and q_2^2 are arbitrary positive constants.

Theorem 1 The equilibrium position of a system with Hamiltonian H is conditionally stable for initial data satisfying the condition $H \neq 0$.

Consider the surface of level $H = 0$.

We will change to "action-angle" variables using the formulae

$$p_1 = (3x)^{1/3}\mathrm{Cs}\varphi_1, \quad q_1 = (3x)^{2/3}\mathrm{Sn}\varphi_1, \quad p_2 = (3y)^{1/3}\mathrm{Cs}\varphi_2, \quad q_2 = (3y)^{2/3}\mathrm{Sn}\varphi_2 \tag{1.2}$$

where $x \geq 0$, $y \geq 0$, and $\mathrm{Cs}\varphi$ and $\mathrm{Sn}\varphi$ are functions, introduced by Lyapunov,[2] defined by the relations

$$\mathrm{Cs}'\varphi = -\mathrm{Sn}\varphi, \quad \mathrm{Sn}'\varphi = \mathrm{Cs}^3\varphi, \quad \mathrm{Cs}0 = 1, \quad \mathrm{Sn}0 = 0; \quad \mathrm{Cs}^4\varphi + 2\mathrm{Sn}^2\varphi \equiv 1$$

These functions are periodic with a certain period T.

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E-mail address: bibicoff@yandex.ru

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Since $dp_1 \wedge dq_1 = dx \wedge d\varphi_1$ and $dp_2 \wedge dq_2 = dy \wedge d\varphi_2$, replacement (1.2) is canonical. As a result we obtain the Hamiltonian

$$F(x, y, \varphi_1, \varphi_2) = f(x, y) + f^*(x, y, \varphi_1, \varphi_2)$$

where $f(x, y)$ is the unperturbed part of the Hamiltonian, $f^*(x,y,\varphi_1,\varphi_2)$ is a perturbation, where

$$f = \frac{1}{4}\lambda_1(3x)^{4/3} - \frac{1}{4}\lambda_2(3y)^{4/3} \tag{1.3}$$

and f^* is a series in powers of $x^{1/3}, y^{1/3}$ with coefficients, T-periodic in φ_1 and φ_2, the expansion of which contains no terms of order lower than the fifth in $x^{1/3}, y^{1/3}$.

We will reduce the system on the surface of level $F = 0$. Expanding the equation $F = 0$ in terms of y, we obtain $y = G(x, \varphi_1, \varphi_2)$. In the system

$$\dot{x} = -F_{\varphi_1}, \quad \dot{y} = -F_{\varphi_2}, \quad \dot{\varphi}_1 = F_x, \quad \dot{\varphi}_2 = F_y$$

(here and henceforth a subscript letter denotes a partial derivative with respect to the corresponding variable) we put $y = G(x, \varphi_1, \varphi_2)$. Since $F(x, G, \varphi_1, \varphi_2) = 0$ we have

$$F_y G_{\varphi_1} + F_{\varphi_1} = 0, \quad F_x + F_y G_x = 0$$

Consequently

$$\frac{dx}{d\varphi_2} = -\frac{F_{\varphi_1}}{F_y} = G_{\varphi_1}, \quad \frac{d\varphi_1}{d\varphi_2} = \frac{F_x}{F_y} = -G_x$$

Thus, on the surface of level $F = 0$, the motions are described by a Hamiltonian system with Hamiltonian $G(x, \varphi_1, \varphi_2)$ and with an independent variable φ_2. This is a system with one degree of freedom, but is not autonomous, which depends periodically on the "time" φ_2. This system can be investigated by standard Kolmogorov-Arnold-Moser (KAM)-theory methods.

Since

$$F = \frac{1}{4}\lambda_1(3x)^{4/3} - \frac{1}{4}\lambda_2(3y)^{4/3} + f^*$$

we have

$$G = \kappa x + g(x^{1/3}, \varphi_1, \varphi_2); \quad g = O(x^{4/3}), \quad \kappa = \left(\frac{\lambda_1}{\lambda_2}\right)^{3/4}$$

Consequently, the system reduced on the surface of level $F = 0$ has the form

$$\frac{dx}{d\varphi_2} = g_{\varphi_1}, \quad \frac{d\varphi_1}{d\varphi_2} = -\kappa - g_x \tag{1.4}$$

If we add the equations $\dot{r} = -g_{\varphi_2}$ and $\dot{\varphi}_2 = 1$ to system (1.4), we obtain an autonomous Hamiltonian system with Hamiltonian

$$G_1 = \kappa x + g(x^{1/3}, \varphi_1, \varphi_2) + r$$

We will change to new canonical variables $z, \theta, \rho, \varphi_2$ using the generating function

$$S = x\theta + r\varphi_2 + s(\theta, \varphi_2)x^{4/3}$$

i.e., using the replacement

$$z = x + S_\theta, \quad \varphi_1 = \theta + S_x, \quad \rho = r + S_{\varphi_2} \tag{1.5}$$

We will choose a function $S(\theta, \varphi_2)$ so that the new Hamiltonian has the form

$$E = \kappa z + dz^{4/3} + O(z^{5/3}) + \rho$$

where d is a certain constant. Substituting expressions (1.5) into the representation for G_1 and equating coefficients of $z^{4/3}$ in the representations for G_1 and E, we obtain the equation

$$\kappa s_\theta + s_{\varphi_2} = g_1(\theta, \varphi_2) - d$$

where g_1 is the coefficient of $x^{4/3}$ of the expansion of the function g in powers of $x^{1/3}$. This equation has a periodic solution if the number κ is the Diophantine and d is the mean value of the coefficient $g_1(\theta, \varphi_2)$. If $d \neq 0$, the Poincaré representation is twisting,[3] and system (1.4), in any neighbourhood of the origin of coordinates, has invariant two-dimensional tori, which divide three-dimensional phase space. Consequently, each trajectory either belongs to a invariant torus, or is "trapped" between two such tori. Hence we obtain the following theorem from Theorem 1.

Theorem 2. If the number κ is Diophantine, and the mean value of the coefficient $g_1(\theta_1, \varphi_2)$ is non-zero, the equilibrium position of the system with Hamiltonian $H(p_1, q_1, p_2, q_2)$ is Lyapunov stable.

If the unperturbed part of the Hamiltonian has a special form, the condition for stability can be expressed directly in terms of its coefficients.

For example, the following holds.

Theorem 3. Suppose the unperturbed part of the Hamiltonian H has the form

$$H^0 = \frac{1}{4}\lambda_1 h_1 - \frac{1}{4}\lambda_2 h_2 + \beta_{11}h_1^2 + 2\beta_{12}h_1 h_2 + \beta_{22}h_2^2; \quad h_j = p_j^4 + 2q_j^2; \quad j = 1, 2 \tag{1.6}$$

while the perturbation H^1 contains no terms of order less than nine, if, as previously, we denote the first measurement by p_j, and the second measurement by the variable q_j. If

$$\det \begin{Vmatrix} \beta_{11} & \beta_{12} & \lambda_1 \\ \beta_{12} & \beta_{22} & -\lambda_2 \\ \lambda_1 & -\lambda_2 & 0 \end{Vmatrix} \neq 0 \tag{1.7}$$

the equilibrium position of the system is Lyapunov stable.

2. Geometrical interpretation of Theorem 1

Consider the Hamiltonian system of general form in the variables $x \geq 0$, $y \geq 0$ – the "action" and φ_1, $\varphi_2(\mathrm{mod}2\pi)$ – the "angle" with Hamiltonian

$$F = f(x, y) + f^*(x, y, \varphi_1, \varphi_2)$$

the unperturbed part f of which is independent of the angle variables, while the perturbation f^* is infinitesimally small with respect to f when the neighbourhood of the equilibrium position contracts to the origin of coordinates $x = 0$, $y = 0$.

We will assume that the Hamiltonian F is obtained from the actual analytic Hamiltonian by changing to "action-angle" variables.

For the unperturbed system, two-dimensional invariant tori correspond to the points (x, y). According to the KAM-theory for perturbations, those tori are preserved for which the following equality is satisfied

$$f_y = \Delta f_x \tag{2.1}$$

where the parameter Δ satisfies the Diophantine condition

$$|k_1 + k_2\Delta| > K(|k_1| + |k_2|)^{-2} \tag{2.2}$$

$$k_1, k_2 \in Z, \quad |k_1| + |k_2| \neq 0, \quad K > 0 \tag{2.3}$$

We will assume that the "isoenergetic nondegeneracy" condition

$$A = \det \begin{Vmatrix} f_{xx} & f_{xy} & f_x \\ f_{xy} & f_{yy} & f_y \\ f_x & f_y & 0 \end{Vmatrix} \neq 0 \tag{2.4}$$

is satisfied.

Then,[4] if the perturbation is sufficiently small, the invariant tori exist on each surface of the level $F = \mathrm{const}$ in any neighbourhood of the equilibrium position, and, when the neighbourhood is reduced, if in inequality (2.2) $K \to 0$ sufficiently rapidly, the measure of the union of the invariant tori on this surface of the level, like an infinitesimal quantity, is equivalent to the measure of relative neighbourhood. The geometrical meaning of isoenergetic non-degeneracy is as follows: for an unperturbed Hamiltonian the curves $f(x, y) = c$ in the first quadrant of the Oxy plane and the curves (2.1) intersect transversely.

Each two-dimensional torus divides the three-dimensional surface $F(x, y, \varphi_1, \varphi_2) = c$. The uniformity of the value of the neighbourhood, in which these propositions are satisfied, relative to the constant c, is ensured by the independence of the quantity K from c. Hence, Lyapunov stability of the equilibrium position occurs.

Example 1. In the Arnold-Moser case (Ref. 5, Appendix 8) we have

$$f = \lambda_1 x - \lambda_2 y + \frac{1}{2}\beta_{11}x^2 + \beta_{12}xy + \frac{1}{2}\beta_{22}y^2$$

Condition (2.3) reduces to inequality (1.7), which ensures Lyapunov stability of the equilibrium position.

Example 2. The Sokolskii case (Ref. 1, Theorem 4.1). Here

$$H^0 = \frac{1}{4}\lambda_1(p_1^4 + 2q_1^2) - \frac{1}{2}\lambda_2(p_2^2 + q_2^2)$$

In "action-angle" variables

$$f = \lambda_1 x^{4/3} - \lambda_2 y$$

the straight lines x = const correspond to equality (2.1), while the parabolas with exponent 4/3, which intersect with these straight lines transversely, correspond to the equality $f = c$, which ensures stability of the equilibrium position.

We will now consider the unperturbed Hamiltonian (1.3). We have

$$f_x = \lambda_1(3x)^{1/3}, \quad f_y = -\lambda_2(3y)^{1/3}, \quad f_{xx} = \lambda_1(3x)^{-2/3}, \quad f_{yy} = -\lambda_2(3y)^{-2/3}, \quad f_{xy} = 0$$

The curves $f = 0$ and $A = 0$ coincide and are given by the equation

$$y = \kappa x \tag{2.5}$$

The curves $f = c$ with $c \neq 0$ are hyperbolae with exponent 4/3 and asymptotes (2.5). Straight lines (2.1) pass through the origin of coordinates and intersect these hyperbolae transversely. Consequently, the existence of invariant tori is not guaranteed solely on the surface $F = 0$. This is the content of Theorem 1.

3. Proofs of Theorems 1 and 3

Consider the Hamiltonian

$$F = f(x, y) + f^*(x, y, \varphi_1, \varphi_2)$$

with unperturbed part (1.3), corresponding to expression (1.1). The system of differential equations has the form

$$\dot{x} = -f^*_{\varphi_1}, \quad \dot{y} = -f^*_{\varphi_2}$$
$$\dot{\varphi}_1 = \lambda_1(3x)^{1/3} + f^*_x, \quad \dot{\varphi}_2 = -\lambda_2(3y)^{1/3} + f^*_y \tag{3.1}$$

Taking into account the discussion in Section 2, it is necessary to convince ourselves that the orders of infinitesimals of the perturbations stated in Theorems 1 and 3 are sufficient to be able to use the results of the KAM-theory. We will formulate the statements of this theory in the form,[6,7] which is used below.

Consider the Hamiltonian or the inverse system of differential equations

$$\dot{z} = Z(z, \varphi, a), \quad \dot{\varphi} = a + \Phi(z, \varphi, a) \tag{3.2}$$

where z and φ are vector variables with dimensions n, and Z and Φ are actually analytic functions, 2π-periodic in the components of the vector φ when $||z|| < \delta_0$, $||m\varphi|| < r_0$. The parameter a varies in the p_0-neighbourhood ($p_0 = \gamma\delta_0/2$) of the set

$$M = \{\alpha = (\alpha_1, \ldots, \alpha_n) | k_1\alpha_1 + \ldots + k_n\alpha_n| > \gamma|k|^{-\tau}\} \tag{3.3}$$

where $\gamma > 0$, $\tau > 0$, k_i are integers $|k| = |k_1| + \ldots + |k_n| > 0$. Suppose the functions Z and Φ satisfy the inequalities

$$\|Z\| < r_0^{\tau+1}\gamma\delta_0^2, \quad \|\Phi\| \leq r_0^{\tau+2}\gamma\delta_0$$

Then a function $a_0(\alpha)$, $\alpha \in M$ exists such that system (3.2) when $a = a_0(\alpha)$ has quasi-periodic solutions

$$z = g(\alpha t + \varphi_0, \alpha), \quad \varphi = \alpha t + \varphi_0 + h(\alpha t + \varphi_0, \alpha) \tag{3.4}$$

where φ_0 is an arbitrary constant vector and α is a vector of the basis frequencies. Here

$$a_0(\alpha) = \alpha + \Gamma(\alpha) \tag{3.5}$$

where $\Gamma(\alpha)$ has the same order of infinitesimals as Φ.

Quasi-periodic solutions (3.4) fill n-dimensional invariant tori of the phase space of system (3.2).

Remark. If the period of the functions Z and Φ differs from 2π in the components φ, it must be normalized to 2π. The Hamiltonian here is multiplied by a constant. Precisely this situation arises in the present paper. The constant factor is assumed to be accumulated in the coefficients λ_1 and λ_2.

In system (3.1) we change to the variables z_1 and z_2 using the formulae

$$3x = \varepsilon^3\lambda_2^{3/4}\rho^3\cos^3\theta + \varepsilon^{7/2}\rho^3 z_1, \quad 3y = \varepsilon^3\lambda_1^{3/4}\rho^3\sin^3\theta + \varepsilon^{7/2}\rho^3 z_2$$
$$\rho > 0, \quad \delta < \theta < \pi/2 - \delta, \quad |z_i| < 1, \quad i = 1, 2 \text{ (}\varepsilon \text{ is a small positive perameter} \tag{3.6}$$

In the new variables, system (3.1) takes the form

$$\dot{z}_i = O(\varepsilon^{3/2}), \quad \dot{\varphi}_i = \varepsilon a_i(\rho, \theta) + O(\varepsilon^{3/2}), \quad i = 1, 2$$
$$a_1(\rho, \theta) = \lambda_1\lambda_2^{1/4}\rho\cos\theta, \quad a_2(\rho, \theta) = -\lambda_1^{1/4}\lambda_2\rho\sin\theta \tag{3.7}$$

Its right-hand side vanishes when $\rho = 0$.

We obtain a system of the form (3.2). Suppose the numbers α_1 and α_2 satisfy the inequality

$$|k_1\alpha_1 + k_2\alpha_2| > K(|k_1| + |k_2|)^{-2} \tag{3.8}$$

and conditions (2.3) are satisfied (we can take $K = \varepsilon^{1/4}$). We will assume that α_i varies in the neighbourhood of the number a_i. Note that $\gamma = K\varepsilon$, where γ is the constant from inequality (3.3). If the quantity K is sufficiently small, the numbers $\Delta = \alpha_2/\alpha_1$, which satisfy inequality (2.2), equivalent to (3.8), occupy a large part of any section of the numerical straight line.

We will further assume that $\theta \le \pi/4$. If $\theta > \pi/4$, we must put $\Delta = \alpha_1/\alpha_2$.

By virtue of equality (3.5), the parameters ρ and θ, which satisfy system of equations

$$\varepsilon a_i(\rho, \theta) = \varepsilon\alpha_i + \Gamma_i(\alpha_1, \alpha_2, \varepsilon), \quad i = 1, 2 \tag{3.9}$$

generate invariant two-dimensional tori of system (3.7) (and of course of system (3.1) also) with quasi-periodic solutions (3.4).

Substituting expressions (3.6) into Hamiltonian F and dividing by $\varepsilon^{7/3}\rho^3$, we obtain

$$F_1 = \frac{3}{4}\varepsilon^{1/2}\lambda_1\lambda_2\rho(\cos^4\theta - \sin^4\theta) + \rho\varepsilon Q(\varepsilon, \rho)$$

The function $Q(\varepsilon, \rho)$ is analytic in the neighbourhood of the point $\rho = 0$, $\varepsilon = 0$.

Using equality (3.9) we obtain

$$F_1 = \frac{3}{4}\varepsilon^{1/2}\alpha_1(\lambda_2^{3/4} + \Delta\lambda_1^{3/4})(\cos\theta + \sin\theta) + O(\varepsilon)$$

Fixing the value of Δ, which satisfies condition (2.2), we obtain that if $\Delta \neq -1/\kappa$, when α_1 changes (and there is a corresponding change in α_2) Hamiltonian F_1 takes all non-zero values. Invariant tori thereby exist on all the corresponding surfaces of the level of the Hamiltonian, apart, possibly, from the surface $F = 0$. Hence Theorem 1 follows.

We now turn to the Hamiltonian with unperturbed part (1.6), written in the variables x, y, φ_1, φ_2($h_1 = (3x)^{4/3}$, $h_2 = (3y)^{4/3}$). We will henceforth discuss the proof of Theorem 1 similarly. Taking into account the order of infinitesimals of the perturbation, in inequality (3.8) we can replace K by $K\varepsilon^7$. Then $\gamma = K\varepsilon^8$.

Taking Theorem 1 into account, it is sufficient to consider the case $\Delta = -1/\kappa$. Then $\theta = 1 + o(\varepsilon^{1/2})$. Substituting expressions (3.6) into the Hamiltonian and dividing by $\varepsilon^{7/3}\rho^3$, we obtain

$$F = \frac{1}{16}\varepsilon^{9/2}\rho^5(\beta_{11}\lambda_2^2 + 2\beta_{12}\lambda_1\lambda_2 + \beta_{22}\lambda_1^2)(1 + o(\varepsilon^{1/2}))$$

Since, by virtue of (3.9), ρ is proportional to α_1, when α_1 changes the Hamiltonian will also change, running through all values in the neighbourhood of zero, if the expression in brackets is non-zero, which is equivalent to condition (1.7). Taking into account the proof of Theorem 1, we conclude that all the surfaces of the level of the Hamiltonian, close to zero, including $F = 0$, contain invariant tori in any neighbourhood of the origin of coordinates. Hence Theorem 3 follows.

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