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Coexistence of unbounded solutions and periodic solutions of Liénard equations with asymmetric nonlinearities at resonance

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Abstract In this paper, we deal with the existence of unbounded orbits of the mapping

$$\begin{cases} \theta_1 = \theta + 2n\pi + \frac{1}{\rho}\mu(\theta) + o(\rho^{-1}), \\ \rho_1 = \rho + c - \mu'(\theta) + o(1), \quad \rho \to \infty, \end{cases}$$

where n is a positive integer, c is a constant and $\mu(\theta)$ is a 2π-periodic function. We prove that if $c > 0$ and $\mu(\theta) \neq 0$, $\theta \in [0, 2\pi]$, then every orbit of the given mapping goes to infinity in the future for ρ large enough; if $c < 0$ and $\mu(\theta) \neq 0$, $\theta \in [0, 2\pi]$, then every orbit of the given mapping goes to infinity in the past for ρ large enough. By using this result, we prove that the equation $x'' + f(x)x' + ax^+ - bx^- + \phi(x) = p(t)$ has unbounded solutions provided that a, b satisfy $1/\sqrt{a} + 1/\sqrt{b} = 2/n$ and $F(x)(= \int_0^x f(s)ds)$, and $\phi(x)$ satisfies some limit conditions. At the same time, we obtain the existence of 2π-periodic solutions of this equation.

Keywords: Liénard equations, unbounded solutions, periodic solutions

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1 Introduction

We are concerned with the unboundedness of solutions of the second-order differential Liénard equation

$$x'' + f(x)x' + ax^+ - bx^- + \phi(x) = p(t), \tag{1.1}$$

where a, b are positive constants, $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$, f, $\phi : \mathbb{R} \to \mathbb{R}$ are continuous, $p : \mathbb{R} \to \mathbb{R}$ is continuous and 2π-periodic.

When $f(x) \equiv 0$, (1.1) has been widely studied for a long time. Many authors studied the existence and multiplicity of periodic solutions by various methods, such as, critical point theory, phase-plane analysis combined with fixed point theorems and continuation methods[1−7]. Concerning the boundedness of solutions of (1.1), Ortega[8] first proved that all solutions of equation

$$x'' + ax^+ - bx^- = 1 + \varepsilon h(t)$$

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are bounded provided that $a \neq b$, h is sufficiently smooth and $|\varepsilon|$ is small enough. Lately, this result was extended to the more general case by several authors. We refer to [9–12] and the references therein. When a, b satisfy $1/\sqrt{a}+1/\sqrt{b}=2/n$, $n \in \mathbb{N}$, Wang[11] studied the boundedness of equation

$$x''+ax^+-bx^-+\phi(x)=p(t) \tag{1.2}$$

by assuming that $\phi(x)$, $p(t)$ are sufficiently smooth and the limits

$$\lim_{x\to+\infty}\phi(x)=\phi(+\infty), \quad \lim_{x\to-\infty}\phi(x)=\phi(-\infty)$$

exist and are finite; moreover,

$$\Sigma(\theta)=\frac{n}{\pi}\left[\frac{\phi(+\infty)}{a}-\frac{\phi(-\infty)}{b}\right]-\frac{1}{2\pi}\int_0^{2\pi}p(t)c\left(t+\frac{\theta}{n}\right)dt\neq 0, \quad \theta\in[0,2\pi],$$

where $c(t)$ is a 2π-periodic solution of equation $x''+ax^+-bx^-=0$ satisfying the initial value condition $c(0)=0$, $c'(0)=1$. In fact, $c(t)$ can be expressed in the form:

$$c(t)=\begin{cases}\dfrac{1}{\sqrt{a}}\sin\sqrt{a}t, & 0\leqslant t\leqslant\dfrac{\pi}{\sqrt{a}},\\ -\dfrac{1}{\sqrt{b}}\sin\sqrt{b}\left(t-\dfrac{\pi}{\sqrt{a}}\right), & \dfrac{\pi}{\sqrt{a}}\leqslant t\leqslant\frac{2\pi}{n}.\end{cases}$$

With respect to the unboundedness of solutions of (1.2), Alonso and Ortega[13] proved the existence of periodic function $p(t)$ such that all solutions of equation $x''+ax^+-bx^-=p(t)$ with large initial value conditions are unbounded provided that $1/\sqrt{a}+1/\sqrt{b}$ is a rational number. To deal with this problem, they studied the dynamics of a class of mapping defined on the plane, which has an asymptotic expression in polar coordinates,

$$\begin{cases}\theta_1=\theta+2n\pi+\dfrac{1}{\rho}\mu(\theta)+o(\rho^{-1}),\\ \rho_1=\rho+\mu'(\theta)+o(1), \quad \rho\to+\infty,\end{cases}$$

where $n>0$ is an integer and μ is continuous and 2π-periodic. Under the condition that all zeroes of μ are non-degenerate, they proved the existence of orbits that go to infinity in the future or in the past by constructing invariant fields near the zeroes of μ. On the basis of this conclusion, they obtained the unbounded solutions of equation $x''+ax^+-bx^-=p(t)$. It is well known that, for a linear equation ($a=b$), the existence of a periodic solution should imply the boundedness of all solutions. But, for a nonlinear equation ($a\neq b$), the result in [13] shows that the unbounded solutions and the periodic solutions can coexist. Fabry and Mawhin[14] generalized the result in [13]. They proved all solutions of (1.2) with large initial values are unbounded provided that the function Σ has non-degenerate zeroes. For other related works, one can check [15–17].

The aim of this paper is to study the unbounded solutions and the periodic solutions of (1.1). We will study the dynamics of a class of mapping defined on the plane, which has the asymptotic expression in polar coordinates as follows,

$$\begin{cases}\theta_1=\theta+2n\pi+\frac{1}{\rho}\mu(\theta)+o(\rho^{-1}),\\ \rho_1=\rho+c-\mu'(\theta)+o(1), \quad \rho\to+\infty,\end{cases}$$

with $\mu(\theta) \neq 0$, $\theta \in [0, 2\pi]$. Since μ has no zeroes, the methods in [13, 14] are invalid. We prove the existence of orbits which go to infinity in the future or in the past by taking the transformations to reduce the term $c - \mu'(\theta)$ to the case with a definite sign. On the basis of this conclusion, we deal with the unboundedness of solutions of (1.1). Meanwhile, we can still prove the existence of periodic solutions of (1.1).

Throughout this paper, we always use the notations $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Z}$ to denote the real number set, the nature number set and the integer number set, respectively.

2 Existence of unbounded orbits of a planar mapping

Given $\sigma > 0$, denote by B_σ the open ball centered at the origin and with radius σ. Set $E_\sigma = \mathbb{R}^2 \backslash B_\sigma$. Assume that the mapping $P : E_\sigma \to \mathbb{R}^2$ is a one-to-one and continuous mapping, whose lift (also denoted by P) can be expressed in the form

$$P : \quad \begin{cases} \theta_1 = \theta + 2n\pi + \dfrac{1}{\rho}\mu(\theta) + h_1(\rho, \theta), \\ \rho_1 = \rho + c - \mu'(\theta) + h_2(\rho, \theta), \end{cases} \tag{2.1}$$

where $n > 0$ is an integer, c is a constant, $\mu \in C^2(S^1)$ with $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and h_1, h_2 are continuous, 2π-periodic in θ and satisfy the condition as follows:

$$h_1(\rho, \theta) = o(\rho^{-1}), \quad h_2(\rho, \theta) = o(1), \quad \rho \to +\infty. \tag{2.2}$$

Given a point (ρ_0, θ_0), denote by $\{(\rho_j, \theta_j)\}$ the orbit of the mapping P through the point (ρ_0, θ_0). That is to say $P(\rho_j, \theta_j) = (\rho_{j+1}, \theta_{j+1})$.

Proposition 2.1. *Assume that the condition* (2.2) *holds. Then the following conclusion holds*:

(1) *if* $c > 0$ *and* $\mu(\theta) \neq 0$, $\theta \in [0, 2\pi]$, *then there exists* $R_0 > 0$ *such that for* $\rho_0 \geqslant R_0$, *the orbit* $\{(\rho_j, \theta_j)\}$ *exists in the future and satisfies* $\lim_{j\to+\infty} \rho_j = +\infty$;

(2) *if* $c < 0$ *and* $\mu(\theta) \neq 0$, $\theta \in [0, 2\pi]$, *then there exists* $R_0 > 0$ *such that for* $\rho_0 \geqslant R_0$, *the orbit* $\{(\rho_j, \theta_j)\}$ *exists in the past and satisfies* $\lim_{j\to-\infty} \rho_j = +\infty$.

To prove Proposition 2.1, we take a transformation as in [13]. Let $1/\rho = \delta r$, where $\delta > 0$ is a parameter to be determined later. Under this transformation, the mapping P becomes

$$\bar{P} : \begin{cases} \theta_1 = \theta + 2n\pi + \delta r \mu(\theta) + h_{11}(r, \theta; \delta), \\ r_1 = r + \delta r^2(-c + \mu'(\theta)) + \delta r^2 h_{21}(r, \theta; \delta), \end{cases}$$

where h_{11}, h_{21} are defined as follows:

$$h_{11}(r, \theta; \delta) = h_1(\delta^{-1}r^{-1}, \theta),$$

$$h_{21}(r, \theta; \delta) = -h_2(\delta^{-1}r^{-1}, \theta) + \frac{[c - \mu'(\theta) + h_2(\delta^{-1}r^{-1}, \theta)]^2}{\delta^{-1}r^{-1} + c - \mu'(\theta) + h_2(\delta^{-1}r^{-1}, \theta)}.$$

From (2.2) we have

$$\lim_{\delta\to0^+} \delta^{-1}r^{-1}h_{11}(r, \theta; \delta) = \lim_{\delta\to0^+} \delta^{-1}r^{-1}h_1(\delta^{-1}r^{-1}, \theta) = 0; \qquad \lim_{\delta\to0^+} h_{21}(r, \theta; \delta) = 0 \tag{2.3}$$

uniformly for $\theta \in [0, 2\pi]$ and sufficiently small r.

Next, we shall introduce another transformation which reduces the term $\delta r^2(-c + \mu'(\theta))$ in mapping $\bar{P}$ to the case with a definite sign. We only deal with the case $\mu(\theta) > 0$, $\theta \in [0, 2\pi]$. The case $\mu(\theta) < 0$, $\theta \in [0, 2\pi]$ can be handled similarly. Consider the system

$$\theta' = r\mu(\theta), \qquad r' = r^2\mu'(\theta), \quad (r > 0). \tag{2.4}$$

The first integral of (2.4) is $I(r,\theta) = \frac{\mu(\theta)}{r}$. Therefore, the orbits of (2.4) can be expressed in the form

$$\Gamma_h : I(r,\theta) = \frac{\mu(\theta)}{r} = h,$$

where h is an arbitrary constant. Let $(r(t), \theta(t))$ be the solution of (2.4) lying on the curve Γ_h. Of course, if $\mu(\theta) \neq 0$, then $(r(t), \theta(t))$ is a periodic solution. Denote by $T(h)$ the minimal period of $(r(t), \theta(t))$. From the first equation of (2.4) we have

$$T(h) = h\int_0^{2\pi} \frac{d\theta}{\mu^2(\theta)} = dh, \quad \Big(d = \int_0^{2\pi} \frac{d\theta}{\mu^2(\theta)}\Big).$$

Now, we can introduce the frequency function

$$\omega(h) = \frac{2\pi}{T(h)} = \frac{2\pi}{dh}.$$

Given (r, θ), let us define a function as in [18],

$$K(r,\theta) = \frac{\mu(\theta)}{r}\int_0^{\theta} \frac{ds}{\mu^2(s)}.$$

Obviously, the quantity $K(r, \theta)$ denotes the time for a solution $(r(t), \theta(t))$ to go from the vertical axis $\theta = 0$ to the point (r, θ). Applying the functions ω and K, we can define

$$\tau(\theta) = \omega(I(r,\theta))K(r,\theta) = \frac{2\pi}{d}\int_0^{\theta} \frac{ds}{\mu^2(s)}.$$

This function satisfies $\tau(\theta + 2\pi) = \tau(\theta) + 2\pi$. We can now define the mapping $\Psi : R^+ \times S^1 \to R^+ \times S^1$ as follows:

$$\Psi : (r, \theta) \to (I, \tau) = (I(r, \theta), \tau(\theta))$$

with

$$I = I(r,\theta) = \frac{\mu(\theta)}{r}, \qquad \tau = \tau(\theta) = \frac{2\pi}{d}\int_0^{\theta} \frac{ds}{\mu^2(s)}.$$

It is easy to check that the mapping Ψ is a bijective mapping. Let Ψ^{-1} be the inverse function of Ψ. Write $\Psi^{-1} : (I, \tau) \to (r, \theta) = (r(I, \tau), \theta(\tau))$ with

$$r(I,\tau) = \frac{\mu(\theta(\tau))}{I}, \qquad \frac{2\pi}{d}\int_0^{\theta(\tau)} \frac{ds}{\mu^2(s)} = \tau.$$

Finally, we define a mapping $\hat{P} = \Psi \circ \bar{P} \circ \Psi^{-1} : (I, \tau) \to (I_1, \tau_1) = \hat{P}(I, \tau)$.

Lemma 2.2. *Assume that* (2.2) *holds and* $\mu(\theta) > 0$, $\theta \in [0, 2\pi]$. *Then*

$$\hat{P} : \begin{cases} \tau_1 = \tau + 2n\pi + \delta\omega(I) + \delta h_{12}(I, \tau; \delta), \\ I_1 = I + \delta c\mu(\theta(\tau)) + \delta h_{22}(I, \tau; \delta), \end{cases}$$

where $h_{12}(I,\tau;\delta)$, $h_{22}(I,\tau;\delta)$ *satisfy* $\lim_{\delta\to 0^+} h_{12}(I,\tau;\delta)=0, \lim_{\delta\to 0^+} h_{22}(I,\tau;\delta)=0$ *uniformly for* $\tau\in R$ *and* I *large enough.*

Proof. From the definition of Ψ^{-1} we know that $\Psi^{-1}(I,\tau)=(r(I,\tau),\theta(\tau))$. Therefore, $(r_1,\theta_1)=\bar{P}\circ\Psi^{-1}(I,\tau)$ has the form

$$\begin{cases}\theta_1=\theta(\tau)+2n\pi+\delta r(I,\tau)\mu(\theta(\tau))+h_{11}(r(I,\tau),\theta(\tau);\delta),\\ r_1=r(I,\tau)+\delta r^2(I,\tau)(-c+\mu'(\theta(\tau)))+\delta r^2(I,\tau)h_{21}(r(I,\tau),\theta(\tau);\delta).\end{cases}$$

Recalling $r(I,\tau)=\mu(\theta(\tau))/I$, we have

$$\begin{cases}\theta_1=\theta(\tau)+2n\pi+\dfrac{\delta\mu^2(\theta(\tau))}{I}+h_{11}\Big(\dfrac{\mu(\theta(\tau))}{I},\theta(\tau);\delta\Big),\\ r_1=\dfrac{\mu(\theta(\tau))}{I}+\dfrac{\delta\mu^2(\theta(\tau))(-c+\mu'(\theta(\tau)))}{I^2}+\dfrac{\delta\mu^2(\theta(\tau))}{I^2}h_{21}\Big(\dfrac{\mu(\theta(\tau))}{I},\theta(\tau);\delta\Big).\end{cases}$$

Next, we shall give the asymptotic expression of (I_1,τ_1). We know

$$I_1=\frac{\mu(\theta_1)}{r_1},\qquad \tau_1=\frac{2\pi}{d}\int_0^{\theta_1}\frac{ds}{\mu^2(s)}.$$

Expanding $\mu(\theta_1)$ we get

$$\mu(\theta_1)=\mu(\theta(\tau))+\frac{\delta\mu'(\theta(\tau))\mu^2(\theta(\tau))}{I}+\bar{h}_{11},$$

where $\bar{h}_{11}=\bar{h}_{11}(I,\theta;\delta)$ is defined by

$$\begin{aligned}\bar{h}_{11}= {}&\mu'(\theta(\tau))h_{11}\Big(\frac{\mu(\theta(\tau))}{I},\theta(\tau);\delta\Big)\\ &+\int_0^1(1-s)\mu''\Big[\theta(\tau)+s\frac{\delta\mu^2(\theta(\tau))}{I}+sh_{11}\Big(\frac{\mu(\theta(\tau))}{I},\theta(\tau);\delta\Big)\Big]\\ &\quad\cdot\Big[\frac{\delta\mu^2(\theta(\tau))}{I}+h_{11}\Big(\frac{\mu(\theta(\tau))}{I},\theta(\tau);\delta\Big)\Big]^2ds.\end{aligned}$$

On the other hand,

$$\begin{aligned}\tfrac{1}{r_1}&=\frac{I}{\mu(\theta(\tau))\big(1+\frac{\delta\mu(\theta(\tau))(-c+\mu'(\theta(\tau)))}{I}+\frac{\delta\mu(\theta(\tau))}{I}h_{21}\big(\frac{\mu(\theta(\tau))}{I},\theta(\tau);\delta\big)\big)}\\ &=\tfrac{I}{\mu(\theta(\tau))}+\delta(c-\mu'(\theta(\tau)))+\delta\bar{h}_{21},\end{aligned}$$

with $\bar{h}_{21}=\bar{h}_{21}(I,\tau;\delta)$ defined by

$$\bar{h}_{21}=-h_{21}\Big(\frac{\mu(\theta(\tau))}{I},\theta(\tau);\delta\Big)+\frac{\delta\mu(\theta(\tau))}{I}\frac{\big[-c+\mu'(\theta(\tau))+h_{21}\big(\frac{\mu(\theta(\tau))}{I},\theta(\tau);\delta\big)\big]^2}{1+\frac{\delta\mu(\theta(\tau))}{I}\big[-c+\mu'(\theta(\tau))+h_{21}\big(\frac{\mu(\theta(\tau))}{I},\theta(\tau);\delta\big)\big]}.$$

Therefore,

$$\begin{aligned}I_1= {}&I+\delta c\mu(\theta(\tau))+\delta\mu(\theta(\tau))\bar{h}_{21}(I,\tau;\delta)+\frac{\delta^2(c-\mu'(\theta(\tau)))}{I}\mu'(\theta(\tau))\mu^2(\theta(\tau))\\ &+\frac{\delta^2\mu'(\theta(\tau))\mu^2(\theta(\tau))}{I}\bar{h}_{21}(I,\tau;\delta)+\frac{I}{\mu(\theta(\tau))}\bar{h}_{11}(I,\tau;\delta)+\delta(c-\mu'(\theta(\tau)))\bar{h}_{11}(I,\tau;\delta)\\ &+\delta\bar{h}_{11}(I,\tau;\delta)\bar{h}_{21}(I,\tau;\delta).\end{aligned}$$

Set

$$\begin{aligned}h_{22}(I,\tau;\delta)=&\ \mu(\theta(\tau))\bar{h}_{21}(I,\tau;\delta)+\frac{\delta(c-\mu'(\theta(\tau)))}{I}\mu'(\theta(\tau))\mu^2(\theta(\tau))\\&+\frac{\delta\mu'(\theta(\tau))\mu^2(\theta(\tau))}{I}\bar{h}_{21}(I,\tau;\delta)+\frac{I}{\delta\mu(\theta(\tau))}\bar{h}_{11}(I,\tau;\delta)\\&+(c-\mu'(\theta(\tau)))\bar{h}_{11}(I,\tau;\delta)+\bar{h}_{11}(I,\tau;\delta)\bar{h}_{21}(I,\tau;\delta).\end{aligned}$$

Consequently,

$$I_1=I+\delta c\mu(\theta(\tau))+\delta h_{22}(I,\tau;\delta).$$

In what follows, we shall prove that $\lim_{\delta\to0^+}h_{22}(I,\tau;\delta)=0$ uniformly for $\tau\in[0,2\pi]$ and I large enough. Since $\mu(\theta)\neq0$, for $\theta\in[0,2\pi]$, it follows from (2.3) that

$$\lim_{\delta\to0^+}\delta^{-1}Ih_{11}\Big(\frac{\mu(\theta(\tau))}{I},\theta(\tau);\delta\Big)=0;\qquad \lim_{\delta\to0^+}h_{21}\Big(\frac{\mu(\theta(\tau))}{I},\theta(\tau);\delta\Big)=0 \tag{2.5}$$

uniformly for $\tau\in[0,2\pi]$ and sufficiently large I. Furthermore, we have

$$\lim_{\delta\to0^+}\delta^{-1}I\bar{h}_{11}(I,\tau;\delta)=0;\qquad \lim_{\delta\to0^+}\bar{h}_{21}(I,\tau;\delta)=0 \tag{2.6}$$

uniformly for $\tau\in[0,2\pi]$ and I large enough. According to (2.6) we have

$$\lim_{\delta\to0^+}\bar{h}_{11}(I,\tau;\delta)=0;\quad \lim_{\delta\to0^+}\mu(\theta(\tau))\bar{h}_{21}(I,\tau;\delta)=0,$$

and

$$\lim_{\delta\to0^+}\frac{\delta\mu'(\theta(\tau))\mu^2(\theta(\tau))}{I}\bar{h}_{21}(I,\tau;\delta)=0;\quad \lim_{\delta\to0^+}\frac{I}{\delta\mu(\theta(\tau))}\bar{h}_{11}(I,\tau;\delta)=0$$

uniformly for $\tau\in[0,2\pi]$ and I large enough. On the other hand, it is easy to see that

$$\lim_{\delta\to0^+}\frac{\delta(c-\mu'(\theta(\tau)))}{I}\mu'(\theta(\tau))\mu^2(\theta(\tau))=0;\quad \lim_{\delta\to0^+}\bar{h}_{11}(I,\tau;\delta)\bar{h}_{21}(I,\tau;\delta)=0,$$

uniformly for $\tau\in[0,2\pi]$ and sufficiently large I. Combining the above limits, we obtain the conclusion that $\lim_{\delta\to0^+}h_{22}(I,\tau;\delta)=0$ uniformly for $\tau\in[0,2\pi]$ and I large enough.

Now we are in a position to prove $\tau_1=\tau+2n\pi+\delta\omega(I)+\delta h_{12}(I,\tau;\delta)$. From the definition of τ_1 we have that

$$\tau_1=\frac{2\pi}{d}\int_0^{\theta_1}\frac{ds}{\mu^2(s)}=\frac{2\pi}{d}\int_0^{2n\pi+\theta(\tau)+\frac{\delta\mu^2(\theta(\tau))}{I}+h_{11}(\frac{\mu(\theta(\tau))}{I},\theta(\tau);\delta)}\frac{ds}{\mu^2(s)}.$$

Hence,

$$\begin{aligned}\tau_1=&\frac{2\pi}{d}\int_0^{2n\pi}\frac{ds}{\mu^2(s)}+\frac{2\pi}{d}\int_{2n\pi}^{2n\pi+\theta(\tau)}\frac{ds}{\mu^2(s)}+\frac{2\pi}{d}\int_{2n\pi+\theta(\tau)}^{2n\pi+\theta(\tau)+\frac{\delta\mu^2(\theta(\tau))}{I}}\frac{ds}{\mu^2(s)}\\&+\frac{2\pi}{d}\int_{2n\pi+\theta(\tau)+\frac{\delta\mu^2(\theta(\tau))}{I}}^{2n\pi+\theta(\tau)+\frac{\delta\mu^2(\theta(\tau))}{I}+h_{11}(\frac{\mu(\theta(\tau))}{I},\theta(\tau);\delta)}\frac{ds}{\mu^2(s)}.\end{aligned}$$

In terms of the definition of d and the function $\theta(\tau)$, we get that

$$\frac{2\pi}{d}\int_0^{2n\pi}\frac{ds}{\mu^2(s)}=2n\pi,\quad \frac{2\pi}{d}\int_{2n\pi}^{2n\pi+\theta(\tau)}\frac{ds}{\mu^2(s)}=\tau. \tag{2.7}$$

Write

$$\int_{2n\pi+\theta(\tau)}^{2n\pi+\theta(\tau)+\frac{\delta\mu^2(\theta(\tau))}{I}} \frac{ds}{\mu^2(s)} = \int_{2n\pi+\theta(\tau)}^{2n\pi+\theta(\tau)+\frac{\delta\mu^2(\theta(\tau))}{I}} \frac{ds}{\mu^2(\theta(\tau))} + \int_{2n\pi+\theta(\tau)}^{2n\pi+\theta(\tau)+\frac{\delta\mu^2(\theta(\tau))}{I}} \frac{(\mu(\theta(\tau))+\mu(s))(\mu(\theta(\tau))-\mu(s))}{\mu^2(s)\mu^2(\theta(\tau))}ds.$$

Since $\mu(\theta) \neq 0$, $\theta \in [0, 2\pi]$, there exists a constant $\gamma > 0$ such that

$$\left| \int_{2n\pi+\theta(\tau)}^{2n\pi+\theta(\tau)+\frac{\delta\mu^2(\theta(\tau))}{I}} \frac{(\mu(\theta(\tau))+\mu(s))(\mu(\theta(\tau))-\mu(s))}{\mu^2(s)\mu^2(\theta(\tau))}ds \right| \leqslant \frac{\gamma\delta^2}{I^2}. \tag{2.8}$$

Therefore,

$$\frac{2\pi}{d} \int_{2n\pi+\theta(\tau)}^{2n\pi+\theta(\tau)+\frac{\delta\mu^2(\theta(\tau))}{I}} \frac{ds}{\mu^2(s)} = \frac{2\pi\delta}{dI} + \delta\bar{h}_{12}(I,\tau;\delta) \tag{2.9}$$

with

$$\bar{h}_{12}(I,\tau;\delta) = \frac{2\pi}{d\delta} \int_{2n\pi+\theta(\tau)}^{2n\pi+\theta(\tau)+\frac{\delta\mu^2(\theta(\tau))}{I}} \frac{(\mu(\theta(\tau))+\mu(s))(\mu(\theta(\tau))-\mu(s))}{\mu^2(s)\mu^2(\theta(\tau))}ds.$$

From (2.8) we know that

$$\lim_{\delta\to 0^+} \bar{h}_{12}(I,\tau;\delta) = 0, \tag{2.10}$$

uniformly for $\tau \in [0, 2\pi]$ and I large enough. Similarly, there exists a constant $\bar{\gamma} > 0$ such that

$$\left| \int_{2n\pi+\theta(\tau)+\frac{\delta\mu^2(\theta(\tau))}{I}}^{2n\pi+\theta(\tau)+\frac{\delta\mu^2(\theta(\tau))}{I}+h_{11}(\frac{\mu(\theta(\tau))}{I},\theta(\tau);\delta)} \frac{ds}{\mu^2(s)} \right| \leqslant \bar{\gamma} \left| h_{11}\left(\frac{\mu(\theta(\tau))}{I}, \theta(\tau); \delta\right) \right|. \tag{2.11}$$

Set

$$\hat{h}_{12}(I,\tau;\delta) = \frac{2\pi}{d\delta} \int_{2n\pi+\theta(\tau)+\frac{\delta\mu^2(\theta(\tau))}{I}}^{2n\pi+\theta(\tau)+\frac{\delta\mu^2(\theta(\tau))}{I}+h_{11}(\frac{\mu(\theta(\tau))}{I},\theta(\tau);\delta)} \frac{ds}{\mu^2(s)}. \tag{2.12}$$

According to (2.5) and (2.11), we have

$$\lim_{\delta\to 0^+} \hat{h}_{12}(I,\tau;\delta) = 0. \tag{2.13}$$

uniformly for $\tau \in [0, 2\pi]$ and sufficiently large I. It follows from (2.7), (2.9) and (2.12) that

$$\tau_1 = \tau + 2n\pi + \delta\omega(I) + \delta h_{12}(I,\tau;\delta)$$

with $h_{12}(I,\tau;\delta) = \bar{h}_{12}(I,\tau;\delta) + \hat{h}_{12}(I,\tau;\delta)$. From (2.10) and (2.13) we know

$$\lim_{\delta\to 0^+} h_{12}(I,\tau;\delta) = 0$$

uniformly for $\tau \in [0, 2\pi]$ and I large enough. Up to now, we have finished the proof of Lemma 2.2.

Proof of Proposition 2.1. Assume $\mu(\theta) > 0$, $\theta \in [0, 2\pi]$. Given a point (I_0, τ_0), denote by $\{(I_j, \tau_j)\}$ the orbit of the mapping $\hat{P}$ through the point (I_0, τ_0). We will prove that $I_j \to +\infty$, which implies $\rho_j \to +\infty$, as $j \to +\infty$ (or $-\infty$) according to the sign of c. We will proceed in two steps.

(1) $c>0$. Set $m_0=\min\{c\mu(\theta):\theta\in[0,2\pi]\}>0$. From Lemma 2.2 we know that there exist a sufficiently small $\delta_0>0$ and a sufficiently large $l_0>0$ such that $|h_{22}(I,\tau;\delta_0)|\leqslant m_0/2$, for $I\geqslant l_0$ and $\tau\in[0,2\pi]$. Therefore, for $I_0\geqslant l_0$,

$$I_1=I_0+\delta_0 c\mu(\theta(\tau_0))+\delta_0 h_{22}(I_0,\tau_0;\delta_0)\geqslant I_0+\delta_0 m_0/2,$$

which implies that $I_1\geqslant I_0\geqslant l_0$. Inductively, we have that, for $j=1,2,\ldots$,

$$I_{j+1}=I_j+\delta_0 c\mu(\theta(\tau_j))+\delta_0 h_{22}(I_j,\tau_j;\delta_0)\geqslant I_j+\delta_0 m_0/2\geqslant I_0+\delta_0 m_0(j+1)/2,$$

which yields

$$\lim_{j\to+\infty} I_j=+\infty. \tag{2.14}$$

Since $\mu(\theta)>0$, $\theta\in[0,2\pi]$ and $r_j=r(I_j,\theta_j)=\mu(\theta(\tau_j))/I_j$, it follows from (2.14) that

$$\lim_{j\to+\infty} r_j=0,$$

which, together with the transformation $1/\rho=\delta_0 r$, implies that $\lim_{j\to+\infty}\rho_j=+\infty$.

(2) $c<0$. Set $m_1=\min\{-c\mu(\theta):\theta\in[0,2\pi]\}>0$. According to Lemma 2.2, there exist the constants $\delta_1>0$ and $l^0>0$ such that $|h_{22}(I,\tau;\delta_1)|\leqslant m_1/2$, for $I\geqslant l^0$ and $\tau\in[0,2\pi]$. Let $S=\{(I,\tau):I\geqslant l^0,\tau\in\mathbb{R}\}$. From the expression of the mapping $\hat{P}$ we know that $\hat{P}(S)$ contains a neighborhood of infinity. Hence, there exists a constant $l_1>0$ such that if $I_0\geqslant l_1$ and $\hat{P}^{-1}(I_0,\tau_0)=(I_{-1},\tau_{-1})$, then $I_{-1}\geqslant l^0$. Since

$$\begin{cases}\tau_0=\tau_{-1}+2n\pi+\delta_1\omega(I_{-1})+\delta_1 h_{12}(I_{-1},\tau_{-1};\delta_1),\\ I_0=I_{-1}+\delta_1 c\mu(\theta(\tau_{-1}))+\delta_1 h_{22}(I_{-1},\tau_{-1};\delta_1),\end{cases}$$

we know that

$$\begin{cases}\tau_{-1}=\tau_0-2n\pi-\delta_1\omega(I_{-1})-\delta_1 h_{12}(I_{-1},\tau_{-1};\delta_1),\\ I_{-1}=I_0+\delta_1|c|\mu(\theta(\tau_{-1}))-\delta_1 h_{22}(I_{-1},\tau_{-1};\delta_1).\end{cases} \tag{2.15}$$

From the second equality of (2.15) we know that if $I_0\geqslant l_1$, then

$$I_{-1}=I_0+\delta_1|c|\mu(\theta(\tau_{-1}))-\delta_1 h_{22}(I_{-1},\tau_{-1};\delta_1)\geqslant I_0+\delta_1 m_1/2.$$

Inductively, we have that, for $j=-1,-2,\ldots$,

$$I_j=I_{j+1}+\delta_1|c|\mu(\theta(\tau_j))-\delta_1 h_{22}(I_j,\tau_j;\delta_1)\geqslant I_{j+1}+\delta_1 m_1/2\geqslant I_0-\delta_1 m_1 j/2,$$

which results in $\lim_{j\to-\infty} I_j=+\infty$. Therefore, we conclude that $\lim_{j\to-\infty}\rho_j=+\infty$.

3 Unbounded solutions and periodic solutions of Liénard equations

In this section, we will deal with the unboundedness of solutions and the existence of 2π-periodic solutions of (1.1) if $F(x)(=\int_0^x f(u)du)$ and $\phi(x)$ have finite limits. To this end, we will study the properties of the Poincaré mapping of the system $x'=y-F(x)$, $y'=-(ax^+-bx^-+\phi(x))+p(t)$. Therefore, throughout this section, we always assume that $\phi(x)$ is locally Lipschitz and the following conditions hold,

(H_1) $1/\sqrt{a}+1/\sqrt{b}=2/n,\ n\in\mathbb{N}$,

(H_2) $\exists F(\pm\infty)\in\mathbb{R}:\lim_{x\to+\infty}F(x)=F(+\infty),\ \lim_{x\to-\infty}F(x)=F(-\infty)$,

(H_3) $\exists \phi(\pm\infty)\in\mathbb{R}:\lim_{x\to+\infty}\phi(x)=\phi(+\infty),\ \lim_{x\to-\infty}\phi(x)=\phi(-\infty)$.

From sec. 1 we know that the function $c(t)$ is a 2π-periodic solution of equation $x''+ax^+-bx^-=0$ satisfying the initial condition $c(0)=0, c'(0)=1$. The derivative of $c(t)$ will be denoted by $s(t)=c'(t)$. Obviously, $c(t)$ and $s(t)$ satisfy the following properties:

(i) $c(t+2\pi/n)=c(t)$, $s(t+2\pi/n)=s(t)$;

(ii) $c(t)\in C^2(\mathbb{R})$, $s(t)\in C^1(\mathbb{R})$;

(iii) $c'(t)=s(t)$, $s'(t)=-(ac^+(t)-bc^-(t))$;

(iv) $s(t)^2+ac^+(t)^2+bc^-(t)^2=1,\quad \forall\, t\in\mathbb{R}$.

Define a mapping

$$\Phi:\ (\theta,I)\in S^1\times(0,+\infty)\to(x,y)\in\mathbb{R}^2\backslash\{0\}$$

with

$$x=\gamma I^{\frac12}c\Big(\frac{\theta}{n}\Big),\quad y=\gamma I^{\frac12}s\Big(\frac{\theta}{n}\Big),$$

where $\gamma=\sqrt{2n}$, $S^1=\mathbb{R}/2\pi\mathbb{Z}$.

Now, we consider the equivalent system of (1.1),

$$x'=y-F(x),\quad y'=-(ax^+-bx^-+\phi(x))+p(t). \tag{3.1}$$

Let $(x(t),y(t))=(x(t,x_0,y_0),y(t,x_0,y_0))$ be the solution of (3.1) through the initial point (x_0,y_0). Under the condition that $\phi(x)$ is locally Lipschitz and (H_2), (H_3) hold, we can prove by the standard methods that the solution is defined uniquely on whole t-axis. Thus we can define the Poincaré mapping $P_0:\mathbb{R}^2\to\mathbb{R}^2$ as follows:

$$P_0:(x_0,y_0)\to(x(2\pi,x_0,y_0),y(2\pi,x_0,y_0)).$$

Under the transformation Φ, (3.1) becomes

$$\begin{cases}\dfrac{d\theta}{dt}=n-\dfrac{\gamma}{2}I^{-\frac12}F\Big(\gamma I^{\frac12}c\Big(\dfrac{\theta}{n}\Big)\Big)s\Big(\dfrac{\theta}{n}\Big)+\dfrac{\gamma}{2}I^{-\frac12}\phi\Big(\gamma I^{\frac12}c\Big(\dfrac{\theta}{n}\Big)\Big)c\Big(\dfrac{\theta}{n}\Big)\\ \qquad\quad -\dfrac{\gamma}{2}I^{-\frac12}p(t)c\Big(\dfrac{\theta}{n}\Big),\\ \dfrac{dI}{dt}=-\dfrac{\gamma}{n}I^{\frac12}F(\gamma I^{\frac12}c\Big(\dfrac{\theta}{n}\Big))\Big[ac^+\Big(\dfrac{\theta}{n}\Big)-bc^-\Big(\dfrac{\theta}{n}\Big)\Big]-\dfrac{\gamma}{n}I^{\frac12}\phi\Big(\gamma I^{\frac12}c\Big(\dfrac{\theta}{n}\Big)\Big)s\Big(\dfrac{\theta}{n}\Big)\\ \qquad\quad +\dfrac{\gamma}{n}I^{\frac12}p(t)s\Big(\dfrac{\theta}{n}\Big).\end{cases} \tag{3.2}$$

Denote by $(\theta(t),I(t))=(\theta(t,\theta_0,I_0),I(t,\theta_0,I_0))$ the solution of (3.2) satisfying an initial condition $\theta(0)=\theta_0$, $I(0)=I_0$ with $x_0=\gamma I_0^{\frac12}c(\dfrac{\theta_0}{n}), y_0=\gamma I_0^{\frac12}s(\dfrac{\theta_0}{n})$. Then the Poincaré mapping P_0 becomes

$$P_1:(\theta_0,I_0)\to(\theta_1,I_1)=(\theta(2\pi,\theta_0,I_0),I(2\pi,\theta_0,I_0)).$$

From the second equality of (3.2) we get

$$\frac{dI^{\frac12}}{dt}=-\frac{\gamma}{2n}F\Big(\gamma I^{\frac12}c\Big(\frac{\theta}{n}\Big)\Big)\Big[ac^+\Big(\frac{\theta}{n}\Big)-bc^-\Big(\frac{\theta}{n}\Big)\Big]-\frac{\gamma}{2n}\phi\Big(\gamma I^{\frac12}c\Big(\frac{\theta}{n}\Big)\Big)s\Big(\frac{\theta}{n}\Big)+\frac{\gamma}{2n}p(t)s\Big(\frac{\theta}{n}\Big). \tag{3.3}$$

It follows from (3.3) that

$$I(t)^{\frac{1}{2}} = I_0^{\frac{1}{2}} + O(1), \quad t \in [0, 2\pi]. \tag{3.4}$$

Furthermore, we have that, for $I_0 \to +\infty$, $I(t)^{-\frac{1}{2}} = I_0^{-\frac{1}{2}} + O(I_0^{-1})$, $t \in [0, 2\pi]$, which, together with the first equality of (3.2), implies that $\frac{d\theta}{dt} = n + O(I_0^{-\frac{1}{2}})$. Consequently,

$$\theta(t) = \theta_0 + nt + O(I_0^{-\frac{1}{2}}), \quad t \in [0, 2\pi], \tag{3.5}$$

which, together with (3.3), yields

$$\begin{aligned}\frac{dI^{\frac{1}{2}}}{dt} &= -\frac{\gamma}{2n} F\Big(\gamma I_0^{\frac{1}{2}} c\Big(t + \frac{\theta_0}{n}\Big) + O(1)\Big)\Big[ac^+\Big(t + \frac{\theta_0}{n}\Big) - bc^-\Big(t + \frac{\theta_0}{n}\Big)\Big] \\ &\quad -\frac{\gamma}{2n}\phi\Big(\gamma I_0^{\frac{1}{2}} c\Big(t + \frac{\theta_0}{n}\Big) + O(1)\Big)s\Big(t + \frac{\theta_0}{n}\Big) + \frac{\gamma}{2n}p(t)s\Big(t + \frac{\theta_0}{n}\Big) + O(I_0^{-\frac{1}{2}}).\end{aligned} \tag{3.6}$$

Integrating both sides of (3.6) over interval $[0, 2\pi]$, we get

$$\begin{aligned}I_1^{\frac{1}{2}} = I_0^{\frac{1}{2}} &- \frac{\gamma}{2n}\int_0^{2\pi} F\Big(\gamma I_0^{\frac{1}{2}} c\Big(t+\frac{\theta_0}{n}\Big)+O(1)\Big)\Big[ac^+\Big(t+\frac{\theta_0}{n}\Big)-bc^-\Big(t+\frac{\theta_0}{n}\Big)\Big]dt \\ &-\frac{\gamma}{2n}\int_0^{2\pi}\phi\Big(\gamma I_0^{\frac{1}{2}} c\Big(t+\frac{\theta_0}{n}\Big)+O(1)\Big)s\Big(t+\frac{\theta_0}{n}\Big)dt+\frac{\gamma}{2n}\int_0^{2\pi} p(t)s\Big(t+\frac{\theta_0}{n}\Big)dt+O(I_0^{-\frac{1}{2}}).\end{aligned}$$

Similarly, substituting (3.5) in the first equality of (3.2), we obtain that for $t \in [0, 2\pi]$

$$\begin{aligned}\frac{d\theta}{dt} &= n - \frac{\gamma}{2}I_0^{-\frac{1}{2}}F\Big(\gamma I_0^{\frac{1}{2}} c\Big(t + \frac{\theta_0}{n}\Big) + O(1)\Big)s\Big(t + \frac{\theta_0}{n}\Big) \\ &\quad +\frac{\gamma}{2}I_0^{-\frac{1}{2}}\phi\Big(\gamma I_0^{\frac{1}{2}} c\Big(t + \frac{\theta_0}{n}\Big) + O(1)\Big)c\Big(t + \frac{\theta_0}{n}\Big) - \frac{\gamma}{2}I_0^{-\frac{1}{2}}p(t)c\Big(t + \frac{\theta_0}{n}\Big) + O(I_0^{-1}).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\theta_1 = \theta_0 + 2n\pi &- \frac{\gamma}{2}I_0^{-\frac{1}{2}}\int_0^{2\pi} F\Big(\gamma I_0^{\frac{1}{2}} c\Big(t + \frac{\theta_0}{n}\Big) + O(1)\Big)s\Big(t + \frac{\theta_0}{n}\Big)dt \\ &+\frac{\gamma}{2}I_0^{-\frac{1}{2}}\int_0^{2\pi}\phi\Big(\gamma I_0^{\frac{1}{2}} c\Big(t + \frac{\theta_0}{n}\Big) + O(1)\Big)c\Big(t + \frac{\theta_0}{n}\Big)dt \\ &-\frac{\gamma}{2}I_0^{-\frac{1}{2}}\int_0^{2\pi} p(t)c\Big(t + \frac{\theta_0}{n}\Big)dt + O(I_0^{-1}).\end{aligned}$$

Set $\rho = I^{1/2}$. Then we get

$$\begin{cases}\theta_1 = \theta_0 + 2n\pi - \frac{\gamma}{2}\rho_0^{-1}\int_0^{2\pi} F\Big(\gamma\rho_0 c\Big(t + \frac{\theta_0}{n}\Big) + O(1)\Big)s\Big(t + \frac{\theta_0}{n}\Big)dt \\ \qquad +\frac{\gamma}{2}\rho_0^{-1}\int_0^{2\pi}\phi(\gamma\rho_0 c\Big(t + \frac{\theta_0}{n}\Big) + O(1)\Big)c\Big(t + \frac{\theta_0}{n}\Big)dt \\ \qquad -\frac{\gamma}{2}\rho_0^{-1}\int_0^{2\pi} p(t)c\Big(t + \frac{\theta_0}{n}\Big)dt + O(\rho_0^{-2}), \\ \rho_1 = \rho_0 - \frac{\gamma}{2n}\int_0^{2\pi} F\Big(\gamma\rho_0 c\Big(t + \frac{\theta_0}{n}\Big) + O(1)\Big)\Big[ac^+\Big(t + \frac{\theta_0}{n}\Big) - bc^-\Big(t + \frac{\theta_0}{n}\Big)\Big]dt \\ \qquad -\frac{\gamma}{2n}\int_0^{2\pi}\phi\Big(\gamma\rho_0 c\Big(t + \frac{\theta_0}{n}\Big) + O(1)\Big)s\Big(t + \frac{\theta_0}{n}\Big)dt \\ \qquad +\frac{\gamma}{2n}\int_0^{2\pi} p(t)s\Big(t + \frac{\theta_0}{n}\Big)dt + O(\rho_0^{-1}).\end{cases} \tag{3.7}$$

Write

$$\psi_1(\rho_0,\theta_0)=\int_0^{2\pi}F\Big(\gamma\rho_0 c\Big(t+\frac{\theta_0}{n}\Big)+O(1)\Big)s\Big(t+\frac{\theta_0}{n}\Big)dt,$$
$$\psi_2(\rho_0,\theta_0)=\int_0^{2\pi}F\Big(\gamma\rho_0 c\Big(t+\frac{\theta_0}{n}\Big)+O(1)\Big)\Big[ac^+\Big(t+\frac{\theta_0}{n}\Big)-bc^-\Big(t+\frac{\theta_0}{n}\Big)\Big]dt,$$
$$\psi_3(\rho_0,\theta_0)=\int_0^{2\pi}\phi\Big(\gamma\rho_0 c\Big(t+\frac{\theta_0}{n}\Big)+O(1)\Big)s\Big(t+\frac{\theta_0}{n}\Big)dt,$$
$$\psi_4(\rho_0,\theta_0)=\int_0^{2\pi}\phi\Big(\gamma\rho_0 c\Big(t+\frac{\theta_0}{n}\Big)+O(1)\Big)c\Big(t+\frac{\theta_0}{n}\Big)dt.$$

By the Lebesgue dominated convergence theorem, it is not hard to prove

Lemma 3.1. *Assume that the conditions* (H_1)*,* (H_2) *and* (H_3) *hold. Then, for* $\rho_0\to+\infty$*, we have that*

$$\psi_1(\rho_0,\theta_0)=o(1),\qquad \psi_2(\rho_0,\theta_0)=2n(F(+\infty)-F(-\infty))+o(1),$$
$$\psi_3(\rho_0,\theta_0)=o(1),\qquad \psi_4(\rho_0,\theta_0)=2n(\phi(+\infty)/a-\phi(-\infty)/b)+o(1),$$

uniformly for $\theta_0\in[0,2\pi]$.

According to Lemma 3.1 and (3.7), we have

Lemma 3.2. *Under the same conditions of Lemma* 3.1*, the mapping* $P_2:(\theta_0,\rho_0)\to(\theta_1,\rho_1)$ *can be expressed in the form:*

$$\begin{cases}\theta_1=\theta_0+2n\pi+\dfrac{\gamma}{2}\rho_0^{-1}\Big[2n(\phi(+\infty)/a-\phi(-\infty)/b)-\displaystyle\int_0^{2\pi}p(t)c\Big(t+\frac{\theta_0}{n}\Big)dt\Big]+o(\rho_0^{-1}),\\ r_1=\rho_0-\dfrac{\gamma}{2n}\Big[2n(F(+\infty)-F(-\infty))-\displaystyle\int_0^{2\pi}p(t)s\Big(t+\frac{\theta_0}{n}\Big)dt\Big]+o(1),\quad \rho_0\to+\infty.\end{cases}$$

Set $\mu(\theta)=n\gamma(\phi(+\infty)/a-\phi(-\infty)/b)-\frac{\gamma}{2}\int_0^{2\pi}p(t)c(t+\frac{\theta}{n})dt$*. Clearly,* $\mu(\theta)\in C^2(S^1)$ *and* $\mu'(\theta)=-\frac{\gamma}{2n}\int_0^{2\pi}p(t)s(t+\frac{\theta}{n})dt$*. Hence, the mapping* P_2 *has the form:*

$$P_2:\begin{cases}\theta_1=\theta_0+2n\pi+\rho_0^{-1}\mu(\theta_0)+o(\rho_0^{-1}),\\ \rho_1=\rho_0-\gamma(F(+\infty)-F(-\infty))-\mu'(\theta_0)+o(1),\quad \rho_0\to+\infty.\end{cases}\tag{3.8}$$

Now, we are in a position to state the main theorem of this section.

Theorem 3.3. *Assume that* (H_1)*,* (H_2)*,* (H_3) *hold and*

$$\mu(\theta)=2n\Big(\frac{\phi(+\infty)}{a}-\frac{\phi(-\infty)}{b}\Big)-\int_0^{2\pi}p(t)c\Big(t+\frac{\theta}{n}\Big)dt\neq0,\quad \theta\in[0,2\pi].$$

Then (1.1) *has at least one* 2π*-periodic solution. Moreover, the following conclusions hold:*

(1) *if* $F(+\infty)<F(-\infty)$*, then there exists* $\Gamma>0$ *such that, for* $x_0^2+y_0^2\geqslant\Gamma^2$*, the solution* $x(t)$ *with* $x(0)=x_0$*,* $x'(0)=y_0$ *satisfies* $\lim\limits_{t\to+\infty}(|x(t)|+|x'(t)|)=+\infty$.

(2) *if* $F(+\infty)>F(-\infty)$*, then there exists* $\Gamma>0$ *such that, for* $x_0^2+y_0^2\geqslant\Gamma^2$*, the solution* $x(t)$ *with* $x(0)=x_0$*,* $x'(0)=y_0$ *satisfies* $\lim\limits_{t\to-\infty}(|x(t)|+|x'(t)|)=+\infty$.

Proof. We proceed in two steps.

(i) To deal with the existence of 2π-periodic solutions of (1.1). Since $\mu(\theta) \neq 0$, it is guaranteed that the image (ρ_1, θ_1) of (ρ_0, θ_0) under P_2 never lies on the ray $\theta = \theta_0$ when ρ_0 is large enough. According to the Poincaré-Bohl theorem[19], the mapping P_2 possesses at least one fixed point. Consequently, (1.1) has at least one 2π-periodic solution.

(ii) To deal with the unboundedness of solutions of (1.1). Assume $F(+\infty) < F(-\infty)$. From (3.8) and Proposition 2.1 we know that there exists $R_0 > 0$ such that, if $\rho_0 \geqslant R_0$, then the orbit $\{(\rho_j, \theta_j)\}$ exists in the future and satisfies $\lim_{j\to+\infty} \rho_j = +\infty$. Therefore, if $I_0 \geqslant R_0^2$, then the orbit $\{(I_j, \theta_j)\}$ with $(I_j = \rho_j^2)$ exists in the future and satisfies $\lim_{j\to+\infty} I_j = +\infty$. On the other hand, according to (3.4), we know that there exists the constant $d_0 > 0$ such that $|I(t)^{1/2} - I_0^{1/2}| \leqslant d_0$, $t \in [0, 2\pi]$. Hence, $\lim_{t\to+\infty} I(t) = +\infty$. By the expression of the transformation Φ we have that $ax^{+2}(t) + bx^{-2}(t) + y(t)^2 = 2nI(t)$, which implies that $\lim_{t\to+\infty}(|x(t)| + |y(t)|) = +\infty$. From the boundedness of $F(x)$ and $x'(t) = y(t) - F(x)$ we obtain that $\lim_{t\to+\infty}(|x(t)| + |x'(t)|) = +\infty$. The case $F(+\infty) > F(-\infty)$ can be handled similarly.

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References

[1] Fučik S. Solvability of Nonlinear Equations and Boundary Value Problems. Dordrecht: Reidel, 1980

[2] Dancer N. Boundary-value problems for weakly nonlinear ordinary differential equations. *Bull Aust Math Soc*, **15**: 321–328 (1976)

[3] Lazer A C, McKenna P J. Existence, uniqueness and stability of oscillations in differential equations with asymmetric nonlinearities. *Trans Am Math Soc*, **315**: 721–739 (1989)

[4] Fabry C, Fonda A. Nonlinear resonance in asymmetric oscillators. *J Differ Equ*, **147**: 58–78 (1998)

[5] Qian D. Infinity of subharmonics for asymmetric Duffing equations with the Lazer-Leach-Dancer condition. *J Differ Equ*, **171**: 233–250 (2001)

[6] Wang Z. Existence and multiplicity of periodic solutions of the second order differential equations with jumping nonlinearities. *Acta Math Sin English Ser*, **18**: 615–624 (2002)

[7] Capietto A, Wang Z. Periodic solutions of Liénard equations with asymmetric nonlinearities at resonance. *J London Math Soc*, **68**: 119–132 (2003)

[8] Ortega R. Asymmetric oscillators and twist mappings. *J London Math Soc*, **53**: 325–342 (1996)

[9] Liu B. Boundedness in asymmetric oscillations. *J Math Anal Appl*, **231**: 355–373 (1999)

[10] Liu B, Wang Y. Invariant tori in nonlinear oscillations. *Sci China Ser A-Math*, **42**: 1047–1058 (1999)

[11] Wang X. Invariant tori and boundedness of solutions for asymmetric oscillations. *Acta Math Sin English Ser*, **19**: 765–782 (2003)

[12] Yuan X. Lagrange stability for asymmetric Duffing equations. *Nonlinear Anal*, **43**: 137–151 (2001)

[13] Alonso J M, Ortega R. Roots of unity and unbounded motions of an asymmetric oscillatior. *J Differ Equ*, **143**: 201–220 (1998)

[14] Fabry C, Mawhin J. Oscillations of a forced asymmetric oscillator at resonance. *Nonlinearity*, **13**: 493–505 (2000)

[15] Yang X. Unboundedness of the large solutions of some asymmetric oscillators at resonance. *Math Nachr*, **276**: 89–102 (2004)

[16] Fabry C, Fonda A. Unbounded motions of perturbed isochronous hamiltonian systems at resonance. *Adv Nonlinear Stud*, **5**: 351–373 (2005)

[17] Wang Z. Irrational rotation numbers and unboundedness of Liénard equations with asymmetric nonlinearities. *Proc Amer Math Soc*, **131**: 523–531 (2003)

[18] Ortega R. Boundedness in a piecewise linear oscillator and a variant of the small twist theorem. *Proc London Math Soc*, **79**: 381–413 (1999)

[19] Lloyd N G. Degree Theory. Cambridge: University Press, 1978