Physica D 240 (2011) 1629–1635

Contents lists available at SciVerse ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

The influence of magnetic field on short-wavelength instability of Riemann ellipsoids

K.A. Mizerski [a,*], K. Bajer [b]

[a] Department of Applied Mathematics, University of Leeds, Woodhouse Lane, Leeds LS2 9JT, UK

[b] Institute of Geophysics, University of Warsaw, Pasteura 7, 02-093 Warsaw, Poland

ARTICLE INFO

Article history:
Available online 17 February 2011

Keywords:
Magneto-elliptic instability
Riemann ellipsoids
Tidal synchronization

ABSTRACT

We address the question of stability of the so-called S-type Riemann ellipsoids, i.e. a family of Euler flows in gravitational equilibrium with the vorticity and background rotation aligned along the principal axis perpendicular to the flow. The Riemann ellipsoids are the simplest models of self-gravitating, tidally deformed stars in binary systems, with the ellipticity of the flow modelling the tidal deformation. By the use of the WKB theory we show that mathematically the problem of stability of Riemann ellipsoids with respect to short-wavelength perturbations can be reduced to the problem of magneto-elliptic instability in rotating systems, studied previously by Mizerski and Bajer [K.A. Mizerski, K. Bajer, The magneto-elliptic instability of rotating systems, J. Fluid Mech. 632 (2009) 401–430]. In other words the equations describing the evolution of short-wavelength perturbations of the Riemann ellipsoids considered in Lagrangian variables are the same as those for the evolution of the magneto-elliptic-rotational (*MER*) waves in unbounded domain. This allowed us to use the most unstable *MER* eigenmodes found in Mizerski et al. [K.A. Mizerski, K. Bajer, H.K. Moffatt, The α-effect associated with elliptical instability, J. Fluid Mech., 2010 (in preparation)] to provide an estimate of the characteristic tidal synchronization time in binary star systems. We use the idea of Tassoul [J.-L. Tassoul, On synchronization in early-type binaries, Astrophys. J. 322 (1987) 856–861] and that the interactions between perturbations significantly increase the effective viscosity and hence the energy dissipation in an Ekman-type boundary layer at the surface of the star. The results obtained suggest that if the magnetic field generated by (say) the secondary component of a binary system is strong enough to affect the flow dynamics in the primary, non-magnetized component, the characteristic tidal synchronization time can be significantly reduced.

1. Introduction

The so-called (magneto-)elliptic instability first identified by [1,2], by inclusion of the effect of a uniform magnetic field perpendicular to the plane of the basic flow, was related to the problem of turbulence generation and hence momentum transport in accretion discs by [3]. Earlier [4,5] by the use of the WKB theory for short-wavelength perturbations related the elliptic instability in the presence of background rotation (first analysed by [6,7]) to bounded domains such as the S-type Riemann ellipsoids. The Riemann ellipsoids are usually thought of as the simplest models of stars and hence their results can be interpreted in the context of influence of tidal deformation of stars in binary systems on their stability. Here we apply the theory of magneto-elliptic instability in rotating systems developed by [8] to Riemann ellipsoids and we also make an attempt to estimate the characteristic tidal synchronization time in binary star systems with the inclusion of the effect of the magnetic field.

The results obtained in [8] concern flows in unbounded space. In a finite mass of fluid we would expect the growth and propagation of the MER (*Magneto–Elliptic–Rotational*) waves to be little affected by distant boundaries, especially when the wavelength is short compared with the size of the body of fluid. This intuition was given a firm mathematical basis by [4,5], who analysed the stability of the S-type Riemann ellipsoids [9] with respect to perturbations with very small spatial scale. These are finite fluid masses with ellipsoidal free surface (stationary in a rotating frame of reference) and linear internal flow. Self-gravitation of the fluid balances the centrifugal force which makes a steady-state free surface possible. [4] used the geometrical optics approximation to show that the S-type Riemann ellipsoids are prone to the short-wave elliptical instability. Hence, the latter is not a peculiarity of unbounded flows but an ubiquitous phenomenon affecting elliptical flows of finite extent.

* Corresponding author. Tel.: +44 1133435177.
E-mail addresses: krzysztof.mizerski@gmail.com, kamiz@maths.leeds.ac.uk (K.A. Mizerski).

0167-2789/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.physd.2011.02.006

If the S-type Riemann ellipsoid is made of a perfectly conducting fluid, then adding uniform axial magnetic field gives a steady MHD flow ('magnetic Riemann ellipsoid'). In Section 2 we present the analysis analogous to that of [4] showing that the magnetic Riemann ellipsoids are prone to the short-wave MER instability. More precisely, by the use of the WKB method we show, that the system of equations describing the evolution of short-wavelength perturbations of the magnetic Riemann ellipsoids, in Lagrangian variables associated with the unperturbed orbits reduces to the system of equations describing the evolution of MER waves in unbounded domain given in [8]. The disturbances are localized along the unperturbed streamlines, which is the reason why the influence of the boundaries on the evolution of short-wavelength perturbations is negligible and the results from an unbounded domain can be directly employed in the problem at hand.

Therefore, more generally, we can expect that the unstable MER waves, including the energy-carrying ones (finite group velocity) and the ones generating the α-effect (cf. [10], where the problem of magnetic dynamo action associated with the MER modes was studied), are likely to be excited in elliptical vortices of finite extent, at least those of sufficiently large size in all directions.

In Section 3 we use the theory of [11,12] who provided an estimate for the tidal synchronization time assuming the formation of an Ekman-type boundary layer at the surface of the star with enhanced dissipation via Reynolds stresses created by tidal perturbations. The degree of nonsynchronisation is small, which justifies the linearization in the equations describing the evolution of disturbances. The same assumptions were also exploited by [13, 14]. The models of [15–18] for angular momentum transfer in stars were also based on the development of instabilities and their interactions producing torque via Reynolds stresses.

In the problem analysed here the star is permeated by a uniform and stationary magnetic field given beforehand and hence it is not generated by the analysed star via dynamo action. This corresponds to a situation in which only one, say the secondary component of a binary system is magnetized (i.e. generates magnetic field via dynamo action) and this field influences the dynamics in the interior of the primary component, affecting the structure of tidal modes. The problem involving the magnetic dynamo action in the primary star is far more complicated. However, in our recent study [10] the MER modes were proved capable of generating the α-effect and hence the possibility of incorporation of the mean field theory into the Riemann ellipsoids problem analysed in here is currently being investigated.

2. The magnetic Riemann ellipsoids

We will now formulate short-wavelength stability problem for the Riemann ellipsoids in the presence of a uniform magnetic field perpendicular to the plane of the flow. As mentioned above, we will use the theory developed for the non-magnetic case by [4] and generalize their results to include the effect of external magnetic field. The aim is to investigate the possible influence of the magnetic field in binary systems, generated by, say, the primary component, on the stability of the flow in the interior of the secondary component and via the generated instability on the characteristic tidal synchronization time. It has to be said that the influence of the tidal potential on the gravitational equilibrium of the analysed ellipsoidal figure has not been studied. However the tidal forces lead to departure from axial symmetry of the shape of a star, and the resulting strain in the flow in the surface boundary layer leads to development of instabilities. The ellipticity of the flow is a simplest model of such a tidal deformation leading to instability development and this is why the elliptical instability is often referred to as the tidal instability (cf. [19–21]). In this section we demonstrate how the analytical and numerical results of our previous study [8], where we investigated the problem of magneto-elliptic instability in rotating, unbounded systems, can be directly translated to a problem of stability of stellar interiors. To achieve this, we perform a perturbative analysis of a family of flows called incompressible S-type Riemann ellipsoids, in the limit of short horizontal wavelengths of the perturbations, with the use of the WKB method, also called the geometrical optics approximation [22–24]. The Riemann ellipsoids are models of non-axially symmetric rotating self-gravitating masses. Though as models of astrophysical objects they are idealized, by not taking into account all processes occurring inside these objects, such as convection or thermo-nuclear reactions (the compressibility of the fluid is neglected also), they nevertheless play an important role in the theory of stability and equilibrium of the stars [9,4]. In the magnetic case, they are a family of stationary solutions of the following equations:

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p - 2\boldsymbol{\Omega}\times\mathbf{u} + \nabla\left(\Psi + \frac{1}{2}\left(\boldsymbol{\Omega}\times\mathbf{x}\right)^2\right) + \frac{1}{\mu\rho}\operatorname{curl}\mathbf{B}\times\mathbf{B}, \tag{1}$$

$$\frac{D\mathbf{B}}{Dt} = (\mathbf{B}\cdot\nabla)\,\mathbf{u}, \tag{2}$$

$$\nabla\cdot\mathbf{u} = 0, \qquad \nabla\cdot\mathbf{B} = 0, \tag{3}$$

$$\Psi(\mathbf{x}, t) = G\int_{D_t}\frac{\rho}{|\mathbf{x}-\mathbf{y}|}\mathrm{d}\mathbf{y}, \tag{4}$$

where the first equation is the Navier–Stokes equation with the Lorentz force, the second is the induction equation and in the third the law of mass conservation and the Gauss' law are given. $\Psi(\mathbf{x}, t)$ is the gravitational potential of the rotating mass, and the operator $\frac{D}{Dt}$ is the substantial derivative. In (4) D_t is the volume occupied by the fluid at time t. The Riemann ellipsoids are a special family of solutions of the above system of equations, with uniform magnetic field $\mathbf{B} = \mathbf{B}_0 = \mathbf{Const}$ and velocity field linear in Cartesian coordinates, with $\rho = const$, $\boldsymbol{\Omega} = \Omega\hat{e}_z$ and the vorticity $\boldsymbol{\gamma} = \gamma\hat{e}_z$ parallel to the vector $\boldsymbol{\Omega}$. Additionally we assume that the rest of the space, outside the ellipsoid, is an insulator. These solutions are of the form:

$$\mathbf{u}_0(\mathbf{x}) = \omega\left(-\frac{a_1}{a_2}y, \frac{a_2}{a_1}x, 0\right), \qquad \mathbf{B}_0 = B_0\hat{e}_z, \tag{5}$$

$$p_0(\mathbf{x}) = P_0\left(1 - \frac{x^2}{a_1^2} - \frac{y^2}{a_2^2} - \frac{z^2}{a_3^2}\right), \tag{6}$$

where a_1, a_2 (and a_3) are the principle half-axes of the ellipsoid, $\gamma = \omega\left(\frac{a_1}{a_2} + \frac{a_2}{a_1}\right)$ (In the previous notation of [8] $\frac{a_1}{a_2}$ corresponds to the parameter E), and the modified pressure (gas pressure + magnetic pressure) vanishes at the surface. Substitution of $\mathbf{u}_0(\mathbf{x})$, $p_0(\mathbf{x})$, and $\mathbf{B}_0$ into Eq. (1) allows us to express P_0, ω and Ω in terms of the half-axes a_1, a_2 and a_3. Not every choice of a pair $\left(\frac{a_2}{a_1}, \frac{a_3}{a_1}\right)$ corresponds to a stationary solution of the form described, since the ellipsoid needs to be in a gravitational equilibrium; nevertheless the Rossby number

$$Ro = \frac{\gamma}{\Omega} = \frac{\omega}{\Omega}\left(\frac{a_1}{a_2} + \frac{a_2}{a_1}\right), \tag{7}$$

can take all real values. A detailed description of the characteristics of these solutions is given in [9] as well as in [4,5].

The stability of the solutions $\mathbf{u}_0(\mathbf{x})$, $\mathbf{B}_0(\mathbf{x})$ and $p_0(\mathbf{x})$, $\Psi(\mathbf{x}, t)$ is determined by the linearized equations for the perturbations

$\mathbf{u}'(\mathbf{x},t)$, $\mathbf{B}'(\mathbf{x},t)$, $p'(\mathbf{x},t)$ and $\psi(\mathbf{x},t)$ of all four fields,

$$\frac{D\mathbf{u}'}{Dt} = -\widehat{A}\mathbf{u}' - 2\widehat{O}\mathbf{u}' - \nabla p' + \nabla\psi + \mathrm{curl}\mathbf{B}' \times \mathbf{B}_0, \tag{8}$$

$$\frac{D\mathbf{B}'}{Dt} = \widehat{A}\mathbf{B}' + (\mathbf{B}_0\cdot\nabla)\mathbf{u}', \tag{9}$$

$$\nabla\cdot\mathbf{u}' = 0, \qquad \nabla\cdot\mathbf{B}' = 0, \tag{10}$$

$$\psi(\mathbf{x},t) = \frac{1}{\pi}\int_{\partial D_t}\frac{\xi_n}{|\mathbf{x}-\mathbf{y}|}\mathrm{d}\sigma_y, \tag{11}$$

where

$$\widehat{A} = \omega\begin{bmatrix} 0 & -\frac{a_1}{a_2} & 0 \\ \frac{a_2}{a_1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \widehat{O} = \begin{bmatrix} 0 & -\Omega & 0 \\ \Omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{12}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}_0\cdot\nabla, \tag{13}$$

and as in [4], (where analogous analysis as presented here was performed, but for the non-magnetic case $\mathbf{B} = 0$), the unit of time was chosen as $(\pi G\rho)^{-1/2}$. The magnetic field was scaled with $1/\sqrt{(\mu\rho)}$. In Eq. (11) ξ_n is the normal component at the surface of the domain of the Lagrangian displacement $\boldsymbol{\xi}(\mathbf{x},t)$ in the perturbed flow $\mathbf{u}_0(\mathbf{x}) + \mathbf{u}'(\mathbf{x},t)$ of a fluid element, that would be at $\mathbf{x}$ at time t in the unperturbed flow $\mathbf{u}_0(\mathbf{x})$. This quantity is determined by the equation

$$\frac{D\boldsymbol{\xi}}{Dt} - (\boldsymbol{\xi}\cdot\nabla)\mathbf{u}_0 = \mathbf{u}' \tag{14}$$

(see [25,9]).

To perform the stability analysis of the Riemann ellipsoids in the presence of the magnetic field we will use the WKB method (see [4]). This means that we will seek solutions of the system of Eqs. (8)–(11) with a very small spatial scale of the following form

$$\begin{aligned}\mathbf{u}'(\mathbf{x},t,\varepsilon) &= \mathrm{e}^{\mathrm{i}\Phi(\mathbf{x},t)/\varepsilon}\left[\mathbf{u}^{(0)}(\mathbf{x},t) + \varepsilon\mathbf{u}^{(1)}(\mathbf{x},t)\right] \\ &\quad + \varepsilon\mathbf{u}^{r}(\mathbf{x},t,\varepsilon), \\ \mathbf{B}'(\mathbf{x},t,\varepsilon) &= \mathrm{e}^{\mathrm{i}\Phi(\mathbf{x},t)/\varepsilon}\left[\mathbf{B}^{(0)}(\mathbf{x},t) + \varepsilon\mathbf{B}^{(1)}(\mathbf{x},t)\right] \\ &\quad + \varepsilon\mathbf{B}^{r}(\mathbf{x},t,\varepsilon), \\ p'(\mathbf{x},t,\varepsilon) &= \mathrm{e}^{\mathrm{i}\Phi(\mathbf{x},t)/\varepsilon}\left[p^{(0)}(\mathbf{x},t) + \varepsilon p^{(1)}(\mathbf{x},t)\right] + \varepsilon p^{r}(\mathbf{x},t,\varepsilon), \\ \boldsymbol{\xi}'(\mathbf{x},t,\varepsilon) &= \mathrm{e}^{\mathrm{i}\Phi(\mathbf{x},t)/\varepsilon}\left[\boldsymbol{\xi}^{(0)}(\mathbf{x},t) + \varepsilon\boldsymbol{\xi}^{(1)}(\mathbf{x},t)\right] + \varepsilon\boldsymbol{\xi}^{r}(\mathbf{x},t,\varepsilon),\end{aligned} \tag{15}$$

where $0 < \varepsilon \ll 1$ is a measure of the spatial scale, $\Phi(\mathbf{x},t)$ is a real function, and $\mathbf{u}^{(j)}(\mathbf{x},t)$, $\mathbf{B}^{(j)}(\mathbf{x},t)$, $p^{(j)}(\mathbf{x},t)$ and $\boldsymbol{\xi}^{(j)}(\mathbf{x},t)$, for $j = 1, 2, r$ are complex-valued functions. In general the proof of the correctness of analysis performed within the frame of the WKB approximation is done by finding suitable upper estimates for the remainders ($j = r$), to ensure that in the case of unbounded growth of the leading order terms of the expansion series in ε, the remainders are not capable of cancelling this growth. Such proofs are usually done by constructing energetic estimates. In this paper, however, we do not attempt to perform such a proof and we will simply limit ourselves to the analysis of the first two terms of the WKB expansion series and so the remainders will not play any role in the further analysis.

For reasons that will become clear later we may assume now that on the surface of the domain both terms of the expansion series of the Lagrange displacement, $\boldsymbol{\xi}^{(0)}(\mathbf{x},t)$ and $\boldsymbol{\xi}^{(1)}(\mathbf{x},t)$, vanish. It then follows that the two initial terms of the expansion series of the perturbation of the gravitational potential vanish as well and we may write

$$\psi(\mathbf{x},t,\varepsilon) = \varepsilon\psi^{r}(\mathbf{x},t,\varepsilon) = \frac{1}{\pi}\int_{\partial D_t}\frac{\varepsilon\xi_n^{r}}{|\mathbf{x}-\mathbf{y}|}\mathrm{d}\sigma_y. \tag{16}$$

To relate the results of [8] to Riemann ellipsoids we will assume further that the external magnetic field is small, namely that $\mathbf{B}_0 \equiv \epsilon\mathcal{B}_0 = O(\epsilon)$,[1] which means that we consider a case in which the magnetic energy is much smaller than the kinetic energy in the flow. This assumption is, in general, only justified on grounds of simplicity, however, the magnetic field generated by one component of a binary system is, in fact, expected to only weakly affect the companion star. Substituting now the expressions for the perturbations (15) into Eqs. (8)–(10) and equating the terms of order ϵ^{-1} we obtain

$$\frac{D\Phi}{Dt}\mathbf{u}^{(0)} + \mathbf{k}p^{(0)} = 0, \tag{17}$$

$$\frac{D\Phi}{Dt}\mathbf{B}^{(0)} = 0, \tag{18}$$

$$\mathbf{k}\cdot\mathbf{u}^{(0)} = 0, \qquad \mathbf{k}\cdot\mathbf{B}^{(0)} = 0, \tag{19}$$

where $\mathbf{k} = \nabla\Phi$. Next, equating the terms of order ϵ^0 leads to

$$\begin{aligned}&\frac{D\mathbf{u}^{(0)}}{Dt} + \left(\widehat{A} + 2\widehat{O}\right)\mathbf{u}^{(0)} + \nabla p^{(0)} - k_z\mathcal{B}_0\mathbf{B}^{(0)} \\ &\quad = -\mathrm{i}\mathbf{k}\left(p^{(1)} + \mathcal{B}_0 B_z^{(0)}\right) - \mathrm{i}\frac{D\Phi}{Dt}\mathbf{u}^{(1)},\end{aligned} \tag{20}$$

$$\frac{D\mathbf{B}^{(0)}}{Dt} - \widehat{A}\mathbf{B}^{(0)} - k_z\mathcal{B}_0\mathbf{u}^{(0)} = -\mathrm{i}\frac{D\Phi}{Dt}\mathbf{B}^{(1)}, \tag{21}$$

$$\nabla\cdot\mathbf{u}^{(0)} = -\mathrm{i}\mathbf{k}\cdot\mathbf{u}^{(1)}, \qquad \nabla\cdot\mathbf{B}^{(0)} = -\mathrm{i}\mathbf{k}\cdot\mathbf{B}^{(1)}. \tag{22}$$

By taking the dot product of Eq. (17) with vector $\mathbf{k}$ we get

$$p^{(0)} = 0, \qquad \frac{D\Phi}{Dt} = 0, \tag{23}$$

and similarly, by taking a dot product of Eq. (20) with vector $\mathbf{k}$ we can eliminate the pressure term to get

$$\begin{aligned}&\frac{D\mathbf{u}^{(0)}}{Dt} + \left(\widehat{A} + 2\widehat{O}\right)\mathbf{u}^{(0)} - k_z\mathcal{B}_0\mathbf{B}^{(0)} \\ &\quad - \mathbf{k}\frac{\mathbf{k}}{k^2}\cdot\left\{\frac{D\mathbf{u}^{(0)}}{Dt} + \left(\widehat{A} + 2\widehat{O}\right)\mathbf{u}^{(0)}\right\} = 0.\end{aligned} \tag{24}$$

To obtain an equation for the wave vector $\mathbf{k}$ we calculate its substantial derivative,

$$\frac{D\mathbf{k}}{Dt} = \nabla\left(\frac{D\Phi}{Dt}\right) - \widehat{A}^T\mathbf{k} = -\widehat{A}^T\mathbf{k}. \tag{25}$$

So finally we obtain the following system of equations, governing the evolution of the leading order terms of short-wavelength perturbations of the Riemann ellipsoids penetrated by a weak, external magnetic field

$$\begin{aligned}&\frac{D\mathbf{u}^{(0)}}{Dt} + \left(\widehat{A} + 2\widehat{O}\right)\mathbf{u}^{(0)} - 2\mathbf{k}\frac{\mathbf{k}}{k^2}\cdot\left(\widehat{A} + \widehat{O}\right)\mathbf{u}^{(0)} \\ &\quad - k_z\mathcal{B}_0\mathbf{B}^{(0)} = 0,\end{aligned} \tag{26}$$

$$\frac{D\mathbf{B}^{(0)}}{Dt} - \widehat{A}\mathbf{B}^{(0)} - k_z\mathcal{B}_0\mathbf{u}^{(0)} = 0, \tag{27}$$

$$\frac{D\mathbf{k}}{Dt} = -\widehat{A}^T\mathbf{k}, \tag{28}$$

supplemented by the conditions (19) (though it is a simple task to prove, that it is enough to apply the solenoidal conditions (19) only at initial instant, and then for all times $t > 0$ they

[1] For $\mathbf{B}_0 = O(1)$ there exist Alfvèn-like solutions of the type $\mathbf{u}^{(0)} = \pm\mathbf{B}^{(0)}$, $\frac{D\Phi}{Dt} = \pm k_z B_0$.

must hold in consequence of Eqs. (26)–(28)). In Lagrangian variables $\mathbf{x}_0$, associated with the unperturbed orbits, for which the ordinary equation $\frac{d\mathbf{x}}{dt} = \mathbf{u}_0(\mathbf{x})$ is linear, the above equations form exactly the same system of ordinary, linear, differential equations as obtained for magneto-elliptic-rotational (*MER*) instability in unbounded domain in [8]. As pointed out by several other authors [24,4] the leading order approximations $\left(\mathbf{u}^{(0)}, \mathbf{B}^{(0)}\right)$ can be localized by a proper choice of initial data. If the initial data is non-zero only on a chosen tube of streamlines it remains non-zero only within this tube for $t > 0$.[2] It can be easily shown that the same is true for the next order terms $\left(\mathbf{u}^{(1)}, \mathbf{B}^{(1)}\right)$, which are also localized if $\left(\mathbf{u}^{(0)}, \mathbf{B}^{(0)}\right)$ are. We infer therefore, that if we assume $\mathbf{u}^{(0)} = \mathbf{u}^{(1)} = 0$ and $\mathbf{B}^{(0)} = \mathbf{B}^{(1)} = 0$ on ∂D at $t = 0$, then this boundary condition persists for all $t > 0$. The same is true for $\boldsymbol{\xi}^{(0)}$ and $\boldsymbol{\xi}^{(1)}$ since (23) holds and therefore $\boldsymbol{\xi}^{(j)}$ and $\mathbf{u}^{(j)}$ for $j = 1, 2$ are related by Eq. (14). This explains the previous assumption $\psi(\mathbf{x}, t, \varepsilon) = \varepsilon\psi^r(\mathbf{x}, t, \varepsilon)$.

We have managed, therefore, to prove the applicability of the magneto-elliptic-rotational instability theory, derived in [8], to the problem of magnetic Riemann ellipsoids analysed here. Hence to obtain the unstable solutions in the limit of small ellipticity of the flow, i.e. $\zeta = \frac{1}{2}\left(\frac{a_1}{a_2} - \frac{a_2}{a_1}\right) \ll 1$ (assuming without loss of generality $\frac{a_1}{a_2} \geq 1$) we are allowed to take the unstable MER modes found in [10], localized in the aforementioned way, and in order to get back to the Euler variables use the following transformation[3]

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} \cos\omega t & \dfrac{a_1}{a_2}\sin\omega t & 0 \\ -\dfrac{a_2}{a_1}\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{29}$$

The assumption of small ellipticity of the flow corresponds to small tidal deformation of a star in a binary and is crucial to make analytical progress. This is allowed for the equilibrium solutions in the form of Riemann ellipsoids and simply means that the ratio of the third semiaxis to the first one, $\frac{a_3}{a_1}$, can only take values from about 0.3 to about 1 (cf. [4]). Furthermore, consistently with the theory describing the tidal synchronization in binaries developed by [11,12] we also make the assumption of small degree of nonsynchronism and hence we assume that ω is small when compared to the background rotation Ω, with the latter being interpreted as the angular velocity of the star in its orbiting motion around the centre of mass of the binary. We now proceed to calculation of an estimate of the tidal synchronization time in binaries and determination of the influence which the magnetic field may have on the characteristic timescale of synchronization.

3. The effect of the magnetic field on tidal synchronization time in binaries

The theory of [11], later developed by [12], provides an estimate for the characteristic tidal synchronization time, τ, through an assumption that the dissipation of energy occurs in a thin boundary layer of Ekman type at the surface of the star. The viscosity and thus the dissipative effects are enhanced and dominated by the Reynolds stresses created by the tidal perturbations, which are small in the limit of small degree of nonsynchronism assumed in the theory, what justifies the linearization in the perturbation equations. The molecular viscosity is assumed to be negligible. The synchronization time $\tau \sim \delta_E^{-1}$ is inversely proportional to the thickness of the Ekman layer and hence to the square root of the locally vertical (i.e. radial) turbulent viscosity $(\nu_V)^{1/2}$. The latter is obtained by a standard Reynolds stress model (cf. [26]) i.e. from the relation $\sigma_{yz} = \sigma_{zy} \approx \rho\omega\nu_V$, where σ_{yz} is a component of the Reynolds stress tensor. Additionally, as often done in the literature (see e.g. [27]), the magnetic stress tensor generated by the magnetic field perturbations is also assumed to contribute to the turbulent viscosity, in an analogous way.

Hence, to estimate the tidal synchronization time in the presence of magnetic field it only remains to obtain an estimate for the turbulent viscosity for the three classes of unstable MER modes, calculated in [10]. The unstable MER modes are generated via resonances of three types: (i) the *hydro–hydro* resonance between hydrodynamic modes, (ii) the *magnetic–magnetic* resonance between two magnetic modes (which vanish in the absence of the magnetic field) and (iii) the *mixed hydro–magnetic* resonance between the hydrodynamic and magnetic modes. The latter is the only case leading to propagation of energy (cf. [10]) while the first type modes, i.e. the hydro–hydro type, are the most unstable ones and hence are most likely to be the most energetic ones. We will only consider the form of the unstable MER modes at leading order in the ellipticity ζ, in other words for circular basic flow[4] and for consistency it is required that $1 \gg \zeta \gg \epsilon$. Hence to leading order in ζ the semiaxes $a \doteq a_1 \approx a_2$ are equal. The disturbances are assumed to be localized near the surface in the vicinity of some colatitude θ_0, which does not need to be specified. The form of the time-dependence of the perturbations can be easily obtained in Lagrangian variables by the use of the Floquet theory (cf. [10]). However, in Eulerian variables additional time-dependence is introduced via the transformation (29). Nevertheless, adopting the turbulent diffusion model based on Reynolds stresses given in [26] and an analogous model for the contribution to turbulent viscosity from the magnetic stress tensor, a good estimate for the locally vertical turbulent viscosity coefficient is obtained by considering only the contribution from the 'yz' component of the Reynolds stress tensor at initial instant, i.e.,

$$\begin{aligned} \nu_V &\approx \frac{1}{\omega}\left\langle -\mathfrak{Re}u'_y\mathfrak{Re}u'_z + \mathfrak{Re}B'_y\mathfrak{Re}B'_z\right\rangle\Big|_{t=0} \\ &= \frac{1}{2\omega}\mathfrak{Re}\left\langle -u_y^{(0)}u_z^{(0)*} + B_y^{(0)}B_z^{(0)*}\right\rangle\Big|_{t=0}, \end{aligned} \tag{30}$$

where $\langle\cdot\rangle = \left(1/D^3\right)\int_{x-D/2}^{x+D/2}\int_{y-D/2}^{y+D/2}\int_{z-D/2}^{z+D/2}(\cdot)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$ (and by choice $D = N/k_z$ and $1 \ll N \in \mathbb{N}$) is the spatial average and the spatial scale separation between the averaged quantities and the perturbations together with slow time-dependence of the basic flow frequency ω is assumed. The upper star denotes a complex conjugate. After [12], the characteristic tidal synchronization time is proportional to

$$\tau \sim \Omega^{-1}\left(\frac{\nu_V}{\Omega a^2}\right)^{-1/2}. \tag{31}$$

The first case of resonance between two hydrodynamic modes, i.e. modes that persist in the absence of the magnetic field,

2 In other words, if at initial instant, at point $\mathbf{x}_0$ in D we choose $\mathbf{u}^{(0)} = 0$, $\mathbf{B}^{(0)} = 0$, then they stay zero for all $t > 0$ at this point, and as a consequence of linearity of the equation $\frac{d\mathbf{x}}{dt} = \mathbf{u}_0(\mathbf{x})$, along the whole streamline passing through $\mathbf{x}_0$.

3 We emphasize that the 'z' coordinate is untouched by this transformation. This, together with the fact that the vertical wave number of the perturbations is constant (see [8]), justifies the choice of the spatial average in the next section for the analysis of the influence of tidal forces on changes in the angular momentum (i.e the characteristic tidal synchronization time) of a star in a binary, since the 'z'-average of the perturbations is zero.

4 As shown by [10] in the limit of small ellipticity ζ the first-order asymptotic analysis and calculation of the first-order corrections to the complex growth rates of the MER modes allows one to obtain only the leading order forms of the eigenmodes. In other words the first-order corrections to the propagator of the evolution equation and the growth rate only specify a certain eigenvector, i.e. the unstable mode, from the two-dimensional eigenspace associated with the growth rate in the case of circular vortex $\zeta = 0$.

generates the most unstable MER waves and hence most energetic ones. Therefore, the estimate of the turbulent viscosity and the tidal synchronization time with the use of the hydro–hydro modes is most likely to be meaningful. For the sake of completeness analogous estimates in the remaining two resonant cases are given in the Appendix.

The form of the modes with the largest growth rate at time $t = 0$, found in [10], is

$$\mathbf{u}^{(0)} = \frac{1}{k_0 \sin\vartheta}\begin{bmatrix} \cos\vartheta\,(1+\mathrm{i}) \\ \mathrm{i}+1 \\ -\sin\vartheta\,(1+i) \end{bmatrix}, \tag{32}$$

$$\mathbf{B}^{(0)} = \frac{H\cos\vartheta}{k_0 \sin\vartheta}\begin{bmatrix} \cos\vartheta\,(1-\mathrm{i}) \\ \mathrm{i}-1 \\ -\sin\vartheta\,(1-\mathrm{i}) \end{bmatrix}, \tag{33}$$

where $2\pi k_0^{-1}$ is the wavelength of the perturbations, $H = k_0\mathcal{B}_0/\sqrt{\mu_0\rho}\omega$, $\mathcal{R} = \Omega/\omega \gg 1$ by the assumption of small degree of nonsynchronism, and

$$\cos\vartheta = \frac{1}{(1+\mathcal{R}) + \sqrt{(1+\mathcal{R})^2 + H^2}}, \tag{34}$$

with $k_0 \cos\vartheta$ being the vertical wave number of the perturbations. In general the magnetic field is also assumed strong enough so that the rotation rate ω is small compared to $k_0\mathcal{B}_0/\sqrt{\mu_0\rho}$.[5] This leads to the following estimates for the turbulent viscosity and the characteristic tidal synchronization time.

$$\nu_V \approx \frac{\omega}{k_0^2}\left[1 + \frac{\Gamma^2}{\left(1+\sqrt{1+\Gamma^2}\right)^2}\right], \tag{35}$$

$$\tau \sim \frac{k_0 a}{(\omega\Omega)^{1/2}}\left[1 + \frac{\Gamma^2}{\left(1+\sqrt{1+\Gamma^2}\right)^2}\right]^{-1/2}, \tag{36}$$

and $\Gamma = H/\mathcal{R} = k_0\mathcal{B}_0/\sqrt{\mu_0\rho}\Omega$. The synchronization time τ depends on the parameter Γ which measures the strength of the magnetic field with respect to the background rotation. The function $\Gamma^2/\left(1+\sqrt{1+\Gamma^2}\right)^2$ is monotonically increasing for positive Γ. Hence our results suggest that the presence of magnetic field tends to decrease the characteristic tidal synchronization time by a factor between 1 for $\Gamma = 0$ and about 1/2 for $\Gamma \gg 1$, i.e. for strong magnetic field. It must be noted, however, that in the previous section we have already assumed weak magnetic fields, of order ϵ (where ϵ is the measure of the wavelengths of perturbations). Thus the above statement about the magnetic fields being strong might be interpreted as $\epsilon^{-1} \gg \Gamma \gg 1$. Analogous estimates obtained with the use of the unstable modes generated via the magnetic–magnetic resonance (see the Appendix) led to qualitatively the same conclusions, although, surprisingly, in that case the characteristic tidal synchronization time is most effectively decreased by weak magnetic fields. This observation is somewhat reminiscent of the result of [28] that weak magnetic field necessarily leads to generation of short-wavelength waves, which play an important role in damping of the stellar pulsation. The case of mixed hydro-magnetic resonance modes was also studied in the Appendix and it was shown that at least within the kept order of accuracy they do not contribute to the turbulent viscosity. Thus, although the mixed resonance is the only resonance creating MER waves that propagate energy [10], their influence on the characteristic tidal synchronization time is expected to be negligible.

[5] The analysis stays valid for any ratio of the newly introduced small parameters such as $\mathcal{R}^{-1}$, H^{-1} and the Ekman number to ϵ.

4. Conclusions

We have studied the influence of an external magnetic field on the stability of Riemann ellipsoids with respect to short-wavelength perturbations. Our study was a generalization of a previous work of [4,5] to include the effect of a uniform magnetic field aligned with the axis of rotation of the ellipsoid. By the use of the WKB approximation we have shown that in the weak field case the system of evolution equations for the short-wavelength perturbations, when considered in the Lagrangian variables associated with the unperturbed orbits, reduces the system of equations describing the evolution of inertial waves in the problem of magneto-elliptic-rotational (MER) instability in unbounded domain, as in [8].

The Riemann ellipsoids are very simple models of rotating masses in gravitational equilibrium. We have exploited this idea to model tidally and rotationally deformed stars in binary systems with the S-type Riemann ellipsoids, i.e. with the flow vorticity aligned with the background rotation vector. The ellipticity of the flow is a simplest way to model the tidal deformation of a star and the rotational deformation was obtained by taking the principal semiaxis, aligned with rotation and vorticity, of a different size than the other two. For disturbances localized near the surface, in the limit of small tidal deformations and small degree of nonsynchronism of the axial and orbital motions of a star in a binary system, we have provided an estimate of the characteristic tidal synchronization time when the dissipation of energy is predominantly caused by the presence of MER waves in the system. We have used the theory of [11,12], in which the viscous dissipation, generated predominantly via Reynolds stresses, is assumed to occur only in a thin Ekman boundary layer at the surface of the star, to study the influence of magnetic field on the characteristic time necessary to synchronize the axial motion with the orbital motion of a star in a binary system. We found that the presence of magnetic field increases the energy dissipation via the magnetic stress tensor and hence the characteristic tidal synchronization time is shortened. We studied the effect of all three types of unstable MER modes (cf. [8,10]) and found that for the most unstable MER waves destabilized via the resonance between two hydrodynamic modes, and for those destabilized through a resonance between two magnetic modes (which are present only in the magnetic case) the turbulent viscosity is indeed increased by the presence of magnetic field. Interestingly, the magnetic–magnetic modes tend to dissipate the energy of the axial motion more efficiently for small amplitudes of the magnetic fields (a result analogous to [28] who found that in the presence of weak magnetic field short-wavelength waves provide a source of damping of the stellar pulsation). The mixed resonance between the hydrodynamic and magnetic modes leads to a negligibly small increase of dissipation and does not influence the tidal synchronization time at the leading order of accuracy. We stress here, however, that the results were obtained only for short-wavelength perturbations, since only in this limit can the problem of stability of the magnetic Riemann ellipsoids be reduced to the problem of magneto-elliptic-rotational instability in unbounded domain (thus the exact same form of the MER modes as obtained by [10] could be used here to estimate the tidal synchronization time). This is clearly a limitation of the presented theory, since in principle, perturbations with longer wavelengths could also play an important role in the dynamics.

Furthermore, it should be noted that in the configuration studied in here the background magnetic field should be interpreted as an external field for the analysed star, possibly generated by the secondary component of the binary system. The primary component modelled by the Riemann ellipsoid is assumed to be non-magnetized. As shown by [10] for unbounded domain the mixed type resonance is the only case leading to unstable modes possessing non-zero helicity and hence it is the only case for which the magnetic dynamo can occur via the so-called α-effect. Therefore, if we were to consider the effect of magnetic field generated by the primary star via dynamo action, the MER modes generated via the mixed resonance would certainly play an important role in the dynamics. This topic is a subject of current investigation.

There also exists another class of strongly unstable MER modes, not generated via a resonance mechanism. The so-called *horizontal modes* (see [7,8]), which propagate along the 'z' axis, and were recently proved by [10] to be capable of generating the α-effect, typically have very large growth rates and hence may often dominate the dynamics. However, they exist only for $\mathcal{R} = \Omega/\omega < 0$, i.e. if the orbital and axial motions are opposite. For small tidal (elliptical) deformations, the range of values of the parameter $\mathcal{R}$ in which the horizontal instability occurs is very narrow. Therefore the horizontal modes are unlikely to play a significant role in the dynamics of tidal synchronization and thus were neglected in the presented analysis.

Finally, it must be clearly stated that we did not consider any interactions between the three types of MER waves, which can also contribute to the total, locally vertical, turbulent viscosity and thus influence the tidal synchronization time. However, as already stated above, the most unstable modes should also be the most energetic ones and hence their interactions are expected to contribute most significantly to the analysed phenomenon of tidal synchronization in binaries.

Acknowledgements

This work was supported by STFC. We are grateful to Keith Moffatt for his valuable remarks. KM would like to thank Ellen Zweibel for fruitful discussions. The effort of the anonymous referees in providing constructive comments is gratefully acknowledged.

Appendix. Other resonances

Here we calculate the locally vertical turbulent viscosity and the characteristic tidal synchronization time associated with the other two types of unstable MER modes. First we consider the case of the mixed resonance between hydrodynamic and magnetic modes, which is the only resonance type generating unstable waves that propagate energy (cf. [10]). In this case

$$\cos\vartheta = \frac{1}{\sqrt{(1+\mathcal{R})^2 + H^2}}. \tag{A.1}$$

For mixed resonance modes the most unstable ones, at time $t = 0$, take the form (cf. [10])

$$\mathbf{u}^{(0)} = \lambda \frac{1}{k_0 \sin\vartheta} \begin{bmatrix} \cos\vartheta\,(\varpi_1 + \mathrm{i}\varpi_3) \\ -\varpi_3 + \mathrm{i}\varpi_1 \\ -\sin\vartheta\,(\varpi_1 + \mathrm{i}\varpi_3) \end{bmatrix}, \tag{A.2}$$

$$\mathbf{B}^{(0)} = \lambda \frac{H\cos\vartheta}{k_0 \sin\vartheta} \begin{bmatrix} \cos\vartheta\,(1+\mathrm{i}) \\ -1+\mathrm{i} \\ -\sin\vartheta\,(1+\mathrm{i}) \end{bmatrix}, \tag{A.3}$$

where $\varpi_1 = (1+\mathcal{R})\cos\vartheta + 1$, $\varpi_3 = (1+\mathcal{R})\cos\vartheta - 1$ and $\lambda = \exp[\mathrm{i}2\pi\,(1+\mathcal{R})\cos\vartheta]$. For such fields, the '$yz$' component of the Reynolds stress tensor at $t = 0$ is exactly zero. However, we can consider the 'xz' component instead. Nevertheless, because of our assumption of small degree of nonsynchronism, $\mathcal{R} \gg 1$, using (30) with $u_y^{(0)}$ and $B_y^{(0)}$ replaced by $u_x^{(0)}$ and $B_x^{(0)}$ respectively we conclude that the locally vertical turbulent viscosity in the mixed hydro-magnetic resonance case is negligibly small,

$$\nu_V^{(HM)} \approx 0, \tag{A.4}$$

which would lead to very long tidal synchronization times. For the last case of resonance between two magnetic modes we have

$$\cos\vartheta = \frac{1}{\sqrt{(1+\mathcal{R})^2 + H^2} - (1+\mathcal{R})}, \tag{A.5}$$

and within the family of magnetic–magnetic modes the most unstable ones at time $t = 0$ are

$$\mathbf{u}^{(0)} = \frac{1}{k_0 \sin\vartheta} \begin{bmatrix} \cos\vartheta\,(\mathrm{i}-1) \\ \mathrm{i}-1 \\ -\sin\vartheta\,(\mathrm{i}-1) \end{bmatrix}, \tag{A.6}$$

$$\mathbf{B}^{(0)} = \frac{H\cos\vartheta}{k_0 \sin\vartheta} \begin{bmatrix} -\cos\vartheta\,(\mathrm{i}+1) \\ 1+i \\ \sin\vartheta\,(\mathrm{i}+1) \end{bmatrix}, \tag{A.7}$$

(where, again, the last two equations were taken directly from [10]). Now, with the use of (30) and (31) we find the estimates for the turbulent viscosity and the tidal synchronization time associated with the unstable magnetic–magnetic modes

$$\nu_V^{(MM)} \approx \frac{\omega}{k_0^2}\left[1 + \frac{\Gamma^2}{\left(\sqrt{1+\Gamma^2} - 1\right)^2}\right], \tag{A.8}$$

$$\tau^{(MM)} \sim \frac{k_0 a}{(\omega\Omega)^{1/2}}\left[1 + \frac{\Gamma^2}{\left(\sqrt{1+\Gamma^2} - 1\right)^2}\right]^{-1/2}. \tag{A.9}$$

The last equation also suggests, as in the case of hydro–hydro resonance, that the presence of the magnetic field tends to decrease the characteristic tidal synchronization time. Interestingly, the modes generated via the magnetic–magnetic resonance seem to increase the turbulent viscosity and thus decrease the tidal synchronization time more effectively for small magnetic field, i.e. small Γ (the function $\Gamma^2/\left(\sqrt{1+\Gamma^2} - 1\right)^2$ is monotonically decreasing for positive Γ). In fact in the expression for turbulent viscosity in (A.8) blows up in the limit $\Gamma \longrightarrow 0$, however, as explained in [8] the magnetic–magnetic resonance operates only for $H^2 \geq 3 + 2\mathcal{R}$, which in the case at hand corresponds to $\Gamma^2 \gtrsim 2/\mathcal{R}$. Thus for finite $\mathcal{R}$ as Γ tends to zero the growth rate of the modes excited via the magnetic–magnetic resonance decreases and at $\Gamma \sim 2/\mathcal{R}$ it reaches zero, in other words this resonance type no longer leads to instability and hence it does not contribute to the turbulent viscosity. For $\Gamma \approx 2/\mathcal{R}$ formula (A.9) gives a lower bound for $\tau^{(MM)}$.

References

[1] B.J. Bayly, Three-dimensional instability of elliptical flow, Phys. Rev. Lett. 57 (17) (1986) 2160–2163.

[2] R.T. Pierrehumbert, Universal short-wave instability of two-dimensional eddies in an inviscid fluid, Phys. Rev. Lett. 57 (17) (1986) 2157–2159.

[3] N.R. Lebovitz, E. Zweibel, Magnetoelliptic instabilities, Astrophys. J. 609 (2004) 301–312.

[4] N.R. Lebovitz, A. Lifschitz, Short-wavelength instabilities of Riemann ellipsoids, Phil. Trans. R. Soc. A 354 (1709) (1996) 927–950.

[5] N.R. Lebovitz, A. Lifschitz, New global instabilities of the Riemann ellipsoids, Astrophys. J. 458 (1996) 699–713.

[6] A.D.D. Craik, The stability of unbounded two- and three-dimensional flows subject to body forces: some exact solutions, J. Fluid Mech. 198 (1989) 275–292.

[7] T. Miyazaki, Elliptical instability in a stably stratified rotating fluid, Phys. Fluids A 5 (11) (1993) 2702–2709.

[8] K.A. Mizerski, K. Bajer, The magnetoelliptic instability of rotating systems, J. Fluid Mech. 632 (2009) 401–430.

[9] S. Chandrasekhar, Ellipsoidal Figures of Equilibrium, Yale University Press, New Haven, CT, 1969.

[10] K.A. Mizerski, K. Bajer, H.K. Moffatt, The α-effect associated with elliptical instability, J. Fluid Mech., 2010 (in preparation).

[11] J.-L. Tassoul, On synchronization in early-type binaries, Astrophys. J. 322 (1987) 856–861.

[12] J.-L. Tassoul, M. Tassoul, A time-dependent model for synchronization in close binaries, Astrophys. J. 359 (1990) 155–163.
[13] J.-L. Tassoul, On orbital circularization in detached close binaries, Astrophys. J. 324 (1988) L71–L73.
[14] J.-L. Tassoul, On orbital circularization in massive close binaries, Astrophys. J. 358 (1990) 196–198.
[15] H. Ando, Kelvin–Helmholtz instabilities and their application to B-type variables, Mon. Not. R. Astron. Soc. 197 (1981) 1139–1152.
[16] H. Ando, Active picture of rotation, Astron. Astrophys. 108 (1982) 7–13.
[17] H. Ando, Wave-rotation interaction in stars, Publ. Astron. Soc. Japan 35 (1983) 343–353.
[18] P. Goldreich, P.D. Nicholson, Tidal friction in early-type stars, Astrophys. J. 342 (1989) 1079–1084.
[19] W.V.R. Malkus, An experimental study of global instabilities due to the tidal (elliptical) distortion of a rotating elastic cylinder, Geophys. Astrophys. Fluid Dyn. 48 (1989) 123–134.
[20] L. Lacaze, W. Herreman, M. Le Bars, S. Le Dizès, P. Le Gal, Magnetic field induced by elliptical instability in a rotating spheroid, Geophys. Astrophys. Fluid Dyn. 100 (2006) 299–317.
[21] J. Arkani-Hamed, B. Seyed-Mahmoud, K.D. Aldridge, R.E. Baker, Tidal excitation of elliptical instability in the Martian core: possible mechanism for generating the core dynamo, J. Geophys. Res. 113 (2008) E06003.
[22] V.P. Maslov, The Complex WKB Method for Nonlinear Equations I: Linear Theory, Birkhäuser, 1994.
[23] K.S. Eckhoff, On stability for symmetric hyperbolic systems, I, J. Differential Equations 40 (1) (1981) 94–115.
[24] A. Lifschitz, E. Hameiri, Local stability conditions in fluid dynamics, Phys. Fluids A 3 (11) (1991) 2644–2651.
[25] P.G. Drazin, W.H. Reid, Hydrodynamic Stability, Cambridge University Press, 2004.
[26] J. Pedlosky, Geophysical Fluid Dynamics, second ed., Springer-Verlag, New York, 1987.
[27] L.L. Kitchatinov, V.V. Pipin, G. Ruediger, Turbulent viscosity, magnetic diffusivity, and heat conductivity under the influence of rotation and magnetic field, Astron. Nachr. 315 (2) (1994) 157–170.
[28] P.H. Roberts, A.M. Soward, The effect of a weak magnetic field on stellar oscillations, Mon. Not. R. Astron. Soc. 205 (1983) 1171–1189.