J Geom Anal (2014) 24:2135–2185
DOI 10.1007/s12220-013-9411-z

Asymptotics of ACH-Einstein Metrics

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Received: 3 April 2012 / Published online: 10 September 2013

Abstract We study the boundary asymptotics of ACH metrics which are formally Einstein. In terms of the partially integrable almost CR structure induced on the boundary at infinity, existence and uniqueness of such formal asymptotic expansions are studied. It is shown that there always exist formal solutions to the Einstein equation if we allow logarithmic terms, and that a local CR-invariant tensor arises as the obstruction to the existence of a log-free solution. Some properties of this new CR invariant, the CR obstruction tensor, are discussed.

Keywords ACH metrics · The Einstein equation · Partially integrable almost CR manifolds · The CR obstruction tensor

Mathematics Subject Classification Primary 32V05 · Secondary 53A55

1 Introduction

Asymptotically complex hyperbolic metrics, or ACH metrics for short, that we study here were introduced by Epstein, Melrose, and Mendoza in a study [10] of the resolvent of the Laplacian of complete Kähler metrics of the form $\partial\overline{\partial}\log(1/r)$ on a bounded strictly pseudoconvex domain, where r is a boundary defining function. The purpose of this paper is to discuss the formal asymptotic expansion of ACH-Einstein metrics at the boundary and geometry of the CR structures that they induce. The integrability condition satisfied by those CR structures is weaker than the classical one; Tanno [28] called it the partial integrability condition. This condition is also natural from the viewpoint of the theory of parabolic geometries [7, 8].

Communicated by John M. Lee.

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The relation between the boundary behavior of Cheng–Yau's complete Kähler–Einstein metric [9] on a bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^{n+1}$ and the CR structure of the boundary is a classical object of CR geometry. Following the pioneering work of Fefferman [12] on the zero boundary value problem of the complex Monge–Ampère equation, to which construction of complete Kähler–Einstein metrics reduces, Lee and Melrose [24] proved that its solution admits an asymptotic expansion at $\partial\Omega$ including logarithmic terms. Graham [16] showed that this expansion is determined by the local CR geometry of $\partial\Omega$ up to the ambiguity of one real scalar-valued function on $\partial\Omega$, and identified the coefficient of the first logarithmic term as the only obstruction to the existence of a log-free solution.

As for ACH metrics, a perturbation result on the existence of solutions of the Einstein equation is obtained by Biquard [3], and the existence of asymptotic expansions is studied by Biquard and Herzlich [5]. Furthermore, in the 4-dimensional case, the asymptotics is investigated more closely by the same authors [4] to obtain a Burns–Epstein type formula. Our formal analysis applies to any dimension, and so it may serve as a tool to extend [4] to the higher-dimensional case.

Another interesting way to see the study of ACH metrics is comparing it with that of AH (asymptotically hyperbolic) metrics, which we recall briefly. A Riemannian metric g defined on the interior $\mathring{X}$ of an $(n+1)$-dimensional smooth manifold-with-boundary X is said to be *asymptotically hyperbolic* if $\rho^2 g$ nondegenerately extends up to ∂X, where ρ is a boundary defining function, and moreover if $|d\rho|_{\rho^2 g} = 1$ at ∂X. An AH metric induces a conformal structure $[h]$ on $M = \partial X$, namely, the conformal class of the pullback of $\rho^2 g$. The n-dimensional conformal manifold $(M, [h])$ is called the conformal infinity of the AH metric g, or of the AH manifold (X, g), and its geometry is studied in connection with the boundary asymptotics of AH-Einstein metrics.

The odd- and even-dimensional conformal geometries have substantially different characters, and a similarity to CR geometry occurs for the latter one. Among the central objects studied in recent even-dimensional conformal geometry are the Fefferman–Graham obstruction tensor $\mathcal{O}_{ij}$ [13, 14] (for $n \ge 4$) and Branson's Q-curvature [6], and our work brings their CR counterparts to light. The conformal obstruction tensor $\mathcal{O}_{ij}$ arises as the obstruction to the existence of a log-free formal solution of the Einstein equation for AH metrics. In the same spirit we shall construct a CR version, which will be denoted by $\mathcal{O}_{\alpha\beta}$. On the other hand, our analysis of ACH-Einstein metrics is sufficient for doing the same construction for CR geometries as the scattering-theoretic definition [19] of the Q-curvature; one has only to apply the discussion of Guillarmou and Sà Barreto [20] to our metric. This is already done for integrable CR manifolds by Hislop, Perry, and Tang [21] using the asymptotics of complete Kähler–Einstein metrics obtained in [12], but the ACH approach has an advantage that it can extend the notion of CR Q-curvature naturally to the case of partially integrable almost CR manifolds. On this line, just like in the conformal case [17], one can show that the CR obstruction tensor $\mathcal{O}_{\alpha\beta}$ is equal to the variation of the total CR Q-curvature with respect to modifications of partially integrable almost CR structures preserving the contact distribution. The details about the CR Q-curvature will be discussed elsewhere.

By definition a $(2n+1)$-dimensional almost CR manifold $(M, T^{1,0})$, where $T^{1,0}$ is the CR holomorphic tangent bundle, is *partially integrable* if and only if

$$\big[C^\infty\big(M, T^{1,0}\big), C^\infty\big(M, T^{1,0}\big)\big] \subset C^\infty\big(M, T^{1,0} \oplus \overline{T^{1,0}}\big). \tag{1.1}$$

The sum $T^{1,0} \oplus \overline{T^{1,0}}$ is identical to the complexification $H_{\mathbb{C}}$ of a certain real subbundle H of TM. (We may also describe an almost CR structure on M as a pair (H, J), where $J \in \operatorname{End} H$, $J^2 = -1$, and $T^{1,0}$ is the i-eigenbundle of J.) The nonintegrability of $(M, T^{1,0})$ is measured by the *Nijenhuis tensor* $N \in C^\infty(M, H^* \otimes H^* \otimes H)$ defined by

$$N(X, Y) := \overline{\Pi^{1,0}}\big[\Pi^{1,0}X, \Pi^{1,0}Y\big] + \Pi^{1,0}\big[\overline{\Pi^{1,0}}X, \overline{\Pi^{1,0}}Y\big], \quad X, Y \in C^\infty(M, H),$$

where $\Pi^{1,0}$ and $\overline{\Pi^{1,0}}$ are the projections from $H_{\mathbb{C}}$ onto $T^{1,0}$ and $\overline{T^{1,0}}$, respectively. We extend N complex bilinearly (such extensions will be done in the sequel without notice). Given a local frame $\{Z_\alpha\}$ of $T^{1,0}$, we put $Z_{\overline{\alpha}} = \overline{Z_\alpha}$ and write $N(Z_\alpha, Z_\beta) = N_{\alpha\beta}{}^{\overline{\gamma}} Z_{\overline{\gamma}}$.

A partially integrable almost CR manifold $(M, T^{1,0})$ is said to be *nondegenerate* if H is a contact distribution. In this case the conormal bundle $E \subset T^*M$ of H is orientable, and hence $E^\times := E \setminus$ (zero section) splits into two $\mathbb{R}^+$-bundles. We fix one of them and call its sections *pseudohermitian structures.* A choice of a pseudohermitian structure θ defines the *Levi form* h by

$$h(X, Y) := d\theta(X, JY), \quad X, Y \in C^\infty(M, H). \tag{1.2}$$

Thanks to the nondegeneracy and the partial integrability, h is a nondegenerate hermitian form. For another pseudohermitian structure $\hat{\theta} = e^{2u}\theta$, we have $\hat{h} = e^{2u}h$. In particular, the signature (p, q) of the Levi form, $p + q = n$, is independent of the choice of θ. Once we fix a pseudohermitian structure, $h_{\alpha\overline{\beta}} = h(Z_\alpha, Z_{\overline{\beta}})$ and its dual $h^{\alpha\overline{\beta}}$ allows us to lower and raise indices of various tensors.

According to [10], ACH metrics are a certain class of Θ-metrics. Although we shall detail the basics of ACH metrics in Sect. 1, we include here a brief account of relevant definitions. Let X be a C^∞-smooth manifold-with-boundary, and Θ a smooth section of $T^*X|_{\partial X}$ such that the 1-form $\iota^*\Theta$ on ∂X is nowhere vanishing, where $\iota\colon \partial X \hookrightarrow X$ is the inclusion map. We call a conformal class $[\Theta]$ of such Θ a *Θ-structure*, and a pair $(X, [\Theta])$ a *Θ-manifold*. Associated with it is the *Θ-tangent bundle* ${}^\Theta TX$, which is derived from the usual tangent bundle TX by blowing up the zero section over ∂X in a manner determined by the Θ-structure. By definition *Θ-metrics* are nondegenerate fiber metrics of ${}^\Theta TX$. Let $(X, [\Theta])$ be a $(2n+2)$-dimensional Θ-manifold and suppose that a nondegenerate partially integrable almost CR structure $T^{1,0}$ is given on $M = \partial X$. We assume that they are *compatible* in the sense that $\iota^*[\Theta]$ agrees with the conformal class $[\theta]$ of pseudohermitian structures of $(M, T^{1,0})$. By the fact that $\ker \iota^*[\Theta] = H \subset TM$ is a contact distribution, there is a natural filtration $K_2 \subset K_1 \subset {}^\Theta TX|_{\partial X}$ by subbundles, where K_1 is of rank $2n+1$ and K_2 of rank 1. Any Θ-metric g with some positivity condition induces an orthogonal decomposition ${}^\Theta TX|_{\partial X} = R \oplus K_1$, $K_1 = K_2 \oplus L$. The bundle L is identified with H up to a conformal factor, and thus we have another decomposition $L_{\mathbb{C}} = L^{1,0} \oplus \overline{L^{1,0}}$. The definition of the notion of ACH metrics that induces the CR structure $T^{1,0}$ is given in Sect. 1 in terms of these ingredients. By a *distinguished local frame* $\{W_\infty, W_0, W_\alpha, W_{\overline{\alpha}}\}$ for an ACH metric we mean a local frame of ${}^{\Theta}TX$ near ∂X such that, if restricted to ∂X, W_∞ generates R, W_0 generates K_2, and W_1, $\dots$, W_n span $L^{1,0}$. When we simply say "ACH metrics," they are always smooth up to the boundary.

For example, if $(M, T^{1,0})$ is an arbitrary nondegenerate partially integrable almost CR manifold, $X = M \times [0, \infty)_\rho$ carries a standard Θ-structure which is compatible with $T^{1,0}$. Then

$$g_\theta = \frac{4}{\rho^2} d\rho^2 + \frac{1}{\rho^4}\theta^2 + \frac{1}{\rho^2} h$$

gives a standard model for ACH metrics, where θ is any pseudohermitian structure on $(M, T^{1,0})$. For an arbitrary Θ-manifold $(X, [\Theta])$ with $\partial X = M$, a Θ-metric g on X is an ACH metric inducing $T^{1,0}$ if and only if, for some (and hence for any) choice of θ and for some smooth $[\Theta]$-preserving diffeomorphism Φ between neighborhoods of the boundaries of $M \times [0, \infty)$ and X, $\Phi^* g \sim g_\theta$ holds in the sense that their difference is $O(\rho)$. This is an immediate consequence of Proposition 2.7. If T is a vector field on M which is transverse to H, the set of vector fields $\{\rho\partial_\rho, \rho^2 T, \rho Z_\alpha, \rho Z_{\overline{\alpha}}\}$ on $M \times (0, \infty)$ extends to a frame of the Θ-tangent bundle of $M \times [0, \infty)$.

The first main theorem in this paper is on the existence of an approximate solution of the Einstein equation. For any ACH metric g, its Ricci tensor is naturally defined as a symmetric 2-tensor over ${}^{\Theta}TX$. We define the tensor E by

$$E := \mathrm{Ric} + \frac{1}{2}(n+2)g.$$

Theorem 1.1 *Let $(X, [\Theta])$ be a $(2n+2)$-dimensional Θ-manifold and $T^{1,0}$ a compatible nondegenerate partially integrable almost CR structure on $M = \partial X$. Then there exists an ACH metric g inducing $T^{1,0}$ for which*

$$\begin{aligned} E_{\infty\infty} &= O\bigl(\rho^{2n+4}\bigr), \quad E_{\infty 0} = O\bigl(\rho^{2n+4}\bigr), \quad E_{\infty\alpha} = O\bigl(\rho^{2n+3}\bigr), \\ E_{00} &= O\bigl(\rho^{2n+4}\bigr), \quad E_{0\alpha} = O\bigl(\rho^{2n+3}\bigr), \\ E_{\alpha\overline{\beta}} &= O\bigl(\rho^{2n+3}\bigr), \quad E_{\alpha\beta} = O\bigl(\rho^{2n+2}\bigr) \end{aligned} \tag{1.3}$$

with respect to any distinguished local frame $\{W_\infty, W_0, W_\alpha, W_{\overline{\alpha}}\}$ of ${}^{\Theta}TX$ near the boundary, where ρ is any boundary defining function of X.

Note that the condition (1.3) is independent of the choice of a distinguished local frame and a boundary defining function.

Construction of better approximate solutions is obstructed by a tensor $\mathcal{O}_{\alpha\beta}$ on the boundary, which is called the *CR obstruction tensor* and is defined as follows. Let g be any ACH metric satisfying (1.3) and $\theta \in \iota^*[\Theta]$ a pseudohermitian structure on ∂X. By a $[\Theta]$-preserving diffeomorphism Φ, near the boundary we identify g

with a Θ-metric on $M \times [0, \infty)$ that is equal to g_θ modulo $O(\rho)$. We set

$$\mathcal{O}_{\alpha\beta} := \left(\rho^{-2n-2} E(\rho Z_\alpha, \rho Z_\beta)\right)\big|_{\partial X}$$

in terms of the tensor E. This is well defined, i.e., this does not depend on the choice of g, and is a natural pseudohermitian invariant of $(M, T^{1,0}, \theta)$. We shall prove that for $\hat{\theta} = e^{2u}\theta$ it holds that

$$\hat{\mathcal{O}}_{\alpha\beta} = e^{-2nu} \mathcal{O}_{\alpha\beta}. \tag{1.4}$$

Let ζ be the section of the CR canonical bundle $K = \bigwedge^{n+1} (\overline{T^{1,0}})^\perp$ of M associated with θ in such a way that Farris's volume normalization condition [11]

$$\theta \wedge (d\theta)^n = i^{n^2} n! (-1)^q \theta \wedge (T \lrcorner \zeta) \wedge (T \lrcorner \overline{\zeta}), \tag{1.5}$$

where the signature of the Levi form is (p, q), is satisfied. We define the density-weighted version of the CR obstruction tensor by

$$\boldsymbol{\mathcal{O}}_{\alpha\beta} := \mathcal{O}_{\alpha\beta} \otimes |\zeta|^{2n/(n+2)} \in \mathcal{E}_{(\alpha\beta)}(-n, -n).$$

Similarly we set $\boldsymbol{A}_{\alpha\beta} = A_{\alpha\beta}$, $\boldsymbol{N}_{\alpha\beta}{}^{\overline{\gamma}} = N_{\alpha\beta}{}^{\overline{\gamma}}$, where A is the pseudohermitian torsion tensor associated with a choice of θ, and indices of such density-weighted tensors are lowered and raised using $\boldsymbol{h}_{\alpha\overline{\beta}} = h_{\alpha\overline{\beta}} \otimes |\zeta|^{-2/(n+2)}$ and its dual $\boldsymbol{h}^{\alpha\overline{\beta}}$. Then we have the following results, the first of which is just another expression of (1.4).

Theorem 1.2 (1) *The density-weighted CR obstruction tensor $\boldsymbol{\mathcal{O}}_{\alpha\beta}$ is a CR invariant.*

(2) *For an integrable CR manifold, $\boldsymbol{\mathcal{O}}_{\alpha\beta}$ vanishes.*

(3) *Let $\boldsymbol{D}^{\alpha\beta}$ be a differential operator $\mathcal{E}_{(\alpha\beta)}(-n, -n) \to \mathcal{E}(-n-2, -n-2)$ defined by*

$$\boldsymbol{D}^{\alpha\beta} = \nabla^\alpha \nabla^\beta - i \boldsymbol{A}^{\alpha\beta} - \boldsymbol{N}^{\gamma\alpha\beta} \nabla_\gamma - \boldsymbol{N}^{\gamma\alpha\beta}{}_{,\gamma}.$$

Then this is a CR-invariant operator and we have $\boldsymbol{D}^{\alpha\beta} \boldsymbol{\mathcal{O}}_{\alpha\beta} - \boldsymbol{D}^{\overline{\alpha}\overline{\beta}} \boldsymbol{\mathcal{O}}_{\overline{\alpha}\overline{\beta}} = 0$.

Since the integrability condition for almost CR manifolds is automatically satisfied when $n = 1$, the obstruction tensor does not appear in this dimension. This fact was previously observed by Biquard–Herzlich [4, Corollary 5.4]. On the other hand, for $n \geq 2$, there exist partially integrable almost CR manifolds with $\boldsymbol{\mathcal{O}}_{\alpha\beta} \neq 0$, as we will see in Sect. 6.

We shall also investigate how well the solution is improved if we introduce logarithmic terms to ACH metrics. A function $f \in C^0(X) \cap C^\infty(\mathring{X})$ is said to be an element of $\mathcal{A}(X)$ if it admits an asymptotic expansion of the form

$$f \sim \sum_{q=0}^{\infty} f^{(q)} (\log \rho)^q, \qquad f^{(q)} \in C^\infty(X) \tag{1.6}$$

for any boundary defining function ρ. If $f \in \mathcal{A}(X)$, then the Taylor expansions of $f^{(q)}$ at ∂X are uniquely determined. A *singular ACH metric* is a Θ-metric g with

$g_{IJ} \in \mathcal{A}(X)$ satisfying the same condition for usual ACH metrics. Then the components of its Ricci tensor also belong to $\mathcal{A}(X)$, and hence so do those of the tensor $E = \mathrm{Ric} + \frac{1}{2}(n+2)g$. For any $p \in \partial X$, we say that $f \in \mathcal{A}(X)$ vanishes to the infinite order at p if and only if all the coefficients $f^{(q)}$ have the vanishing Taylor expansions at p. A tensor over ${}^{\Theta}TX$ vanishes to the infinite order at p if and only if all of its components vanish to the infinite order at p.

Theorem 1.3 *Let $(X, [\Theta])$ and $(M, T^{1,0})$ be as in Theorem 1.1 and $p \in \partial X$. Then there exists a singular ACH metric for which E vanishes to the infinite order at p. Furthermore, if $\mathcal{O}_{\alpha\beta}(p) = 0$, then there exists such an ACH metric with no logarithmic terms.*

When M is the boundary of a bounded strictly pseudoconvex domain Ω, the result above has a somewhat curious implication. Recall that Graham showed in [16] that the obstruction to the existence of a smooth solution to the zero boundary value problem for the complex Monge–Ampère equation on Ω is one scalar-valued function on $\partial\Omega$. On the other hand, by Theorem 1.2 (2), the second assertion of Theorem 1.3 applies to this case. Our result says that in the ACH category we do not need logarithmic terms to get a metric that is Einstein, at least at any given point on $\partial\Omega$. The author believes that there is some framework that can capture both the scalar-valued obstruction for integrable CR manifolds and the CR obstruction tensor at the same time. This might be an interesting topic of further study.

The relation of our Theorems 1.1 and 1.2 to Biquard's work [3] is vague but worth mentioning. Consider the case where $X = B^{n+1}$ is the unit ball in $\mathbb{C}^{n+1}$, $M = S^{2n+1} = \partial B^{n+1}$, and let g_0 be the complex hyperbolic metric on B^{n+1}. Biquard constructs an ACH-Einstein metric on B^{n+1} with Hölder $C^{k+2,\alpha}$ regularity, where $k \geq 0$ and $0 < \alpha < 1$ (the reader should be careful to Biquard's definition of ACH metrics), for given partially integrable almost CR structure on S^{2n+1} with standard underlying contact structure and with Levi form sufficiently close to the standard one in the $C^{k+2,\alpha}$ topology. He solves the Einstein equation under the harmonic gauge condition, i.e., his metric g is a solution to

$$\Phi(g) := \mathrm{Ric}(g) + \frac{1}{2}(n+2)g + \mathrm{div}_g^*\left(\mathrm{div}_{g_0} g + \frac{1}{2} d\, \mathrm{tr}_{g_0} g\right) = 0.$$

The key analysis toward Biquard's result is that the linearization of Φ at g_0 is an isomorphism

$$(d\Phi)_{g_0} \colon C_\delta^{k+2,\alpha}\left(\mathrm{Sym}^2 T^* B^{n+1}\right) \cong C_\delta^{k,\alpha}\left(\mathrm{Sym}^2 T^* B^{n+1}\right), \quad k \text{ arbitrary}, \tag{1.7}$$

if $0 < \delta < 2n+2$. Here $C_\delta^{k,\alpha}$ denotes the space of $C^{k,\alpha}$ sections with $O(\rho^\delta)$ decay. Once this isomorphism is established for some $\delta > 0$, then the implicit function theorem completes the construction of an Einstein metric. The isomorphism (1.7) for δ large is used if one wants to know the higher asymptotic behavior of the solution; if we have $g_1 \in C^{k+2,\alpha}$ such that $\Phi(g_1) \in C_\delta^{k,\alpha}$, then (1.7) shows that g_1 is modified to a genuine solution to $\Phi(g) = 0$ by adding a $C_\delta^{k+2,\alpha}$ term. Here the need for approximate solutions emerges. Although we will adapt a different gauge to prove Theorem 1.1 in this article, if harmonic-gauge approximate solutions also exist up to the same order $\delta = 2n+2$, as is certainly the case for AH metrics (compare [18] and [17]), then one can explicitly identify Biquard's solution up to $O(\rho^{2n+2-\epsilon})$ ambiguity for arbitrarily small $\epsilon > 0$. The coincidence of the critical order of the isomorphism (1.7) and that of our approximate solutions is very natural.

Our result contradicts a work of Seshadri [26], which states that there are a "primary" scalar-valued obstruction function and a "secondary" 1-tensor obstruction to the existence of ACH-Einstein metrics without logarithmic terms. Despite the fact that there is a difference in the definition of ACH metrics, the conflict is not because of it. The work [26] contains some crucial calculation errors in Sect. 4, where the computation of the Ricci tensor is carried out. Nevertheless, the influence of Seshadri's paper on our analysis is obvious; if it were not for it, this work should have been much harder to complete.

The paper is organized as follows. We first recall some basic facts about ACH metrics in Sect. 2. In Sect. 3 we quickly develop a theory of pseudohermitian geometry for partially integrable almost CR manifolds. After studying how the Ricci tensor depends on the metric in Sects. 4 and 5, we prove Theorem 1.1 and Theorem 1.2 in Sect. 6. In Sect. 7, we calculate the first variation of $\mathcal{O}_{\alpha\beta}$ with respect to the modification of partially integrable almost CR structure on the Heisenberg group from the flat one and verify that a generic small modification gives rise to nonvanishing CR obstruction tensor. Section 8 is devoted to an investigation of singular ACH metrics and the proof of Theorem 1.3.

In this paper the word "smooth" means infinite differentiability. The Einstein summation convention is used throughout. Parentheses surrounding indices denote the symmetrization. Our convention for the exterior product $\omega \wedge \eta$ of 1-forms is $(\omega \wedge \eta)(X, Y) = \omega(X)\eta(Y) - \omega(Y)\eta(X)$, while for the symmetric product $\omega\eta$ we observe $(\omega\eta)(X, Y) = \frac{1}{2}(\omega(X)\eta(Y) + \omega(Y)\eta(X))$.

I would like to express my gratitude to Kengo Hirachi for guidance to this interesting research area and continuous encouragement. I also wish to thank Takao Akahori, Olivier Biquard, Charles Fefferman, Robin Graham, Colin Guillarmou, Hiraku Nozawa, Raphaël Ponge, and Neil Seshadri for helpful advice and discussions. This work is partially supported by Grant-in-Aid for JSPS Fellows (22-6494).

2 Θ-Structures and ACH Metrics

Let X be a smooth manifold-with-boundary and $\Theta \in C^\infty(\partial X, T^*X|_{\partial X})$ such that $\iota^*\Theta$ is a nowhere vanishing 1-form on ∂X, where $\iota\colon \partial X \hookrightarrow X$ is the inclusion map. Then a Lie subalgebra $\mathcal{V}_\Theta$ of $C^\infty(X, TX)$ is defined as follows: For any boundary defining function ρ, a vector field V belongs to $\mathcal{V}_\Theta$ if and only if

$$V \in \rho C^\infty(X, TX), \qquad \tilde{\Theta}(V) \in \rho^2 C^\infty(X).$$

Here $\tilde{\Theta} \in C^\infty(X, T^*X)$ is any extension of Θ. It is clear that $\mathcal{V}_\Theta$ depends only on the conformal class $[\Theta]$ of Θ, which we call a *Θ-structure* on X. A pair $(X, [\Theta])$ is called a *Θ-manifold*.

Now consider a $(2n+2)$-dimensional Θ-manifold $(X, [\Theta])$. There is a canonical vector bundle ${}^{\Theta}TX$ of rank $2n+2$ over X, whose sections are the elements of $\mathcal{V}_{\Theta}$. Over the interior of X it is identified with the usual tangent bundle TX. To illustrate the structure near $p \in \partial X$, let $\{N, T, Y_j\} = \{N, T, Y_1, \dots, Y_{2n}\}$ be a local frame of TX in a neighborhood of p dual to a certain coframe of the form $\{d\rho, \tilde{\Theta}, \alpha^j\}$, where $\tilde{\Theta}$ is an extension of some $\Theta \in [\Theta]$. Then any $V \in \mathcal{V}_{\Theta}$ is, near p, expressed as

$$V = a\rho N + b\rho^2 T + c^j \rho Y_j, \quad a, b, c^j \in C^{\infty}(X). \tag{2.1}$$

Hence $\{\rho N, \rho^2 T, \rho Y_j\}$ extends to a frame of ${}^{\Theta}TX$ near $p \in \partial X$. The dual frame of the bundle ${}^{\Theta}T^*X := ({}^{\Theta}TX)^*$ is $\{d\rho/\rho, \tilde{\Theta}/\rho^2, \alpha^j/\rho\}$. A fiber metric of ${}^{\Theta}TX$, which is not necessarily positive-definite, is called a *Θ-metric*.

Example 2.1 Let $\Omega \subset \mathbb{C}^{n+1}$ be a bounded strictly pseudoconvex domain. Then the boundary $M = \partial\Omega$ carries a natural strictly pseudoconvex CR structure. If $r \in C^{\infty}(\overline{\Omega})$ is a boundary defining function and $\tilde{\theta} := \frac{i}{2}(\partial r - \overline{\partial} r)$, then $\theta := \iota^*\tilde{\theta}$ is a pseudohermitian structure on M, where $\iota\colon M \hookrightarrow \overline{\Omega}$ is the inclusion map. We consider the following complete Kähler metric G on Ω:

$$G = 4\sum_{j,k} \frac{\partial^2}{\partial z^j \partial \overline{z}^k}\left(\log \frac{1}{r}\right) dz^j d\overline{z}^k.$$

Let ξ be the unique $(1,0)$ vector field satisfying $\xi \lrcorner \partial\overline{\partial} r = 0 \mod \overline{\partial} r$, $\partial r(\xi) = 1$ and $\nu := \operatorname{Re}\xi$, $\tau := 2\operatorname{Im}\xi$. Then $dr(\nu) = 1$, $dr(\tau) = 0$ and $\tilde{\theta}(\nu) = 0$, $\tilde{\theta}(\tau) = 1$. We set $\xi \lrcorner \partial\overline{\partial} r = \kappa \overline{\partial} r$, or $\kappa = \partial\overline{\partial} r(\xi, \overline{\xi})$. Furthermore, we let $\xi_1, \dots, \xi_n$ be $(1,0)$ vector fields spanning $\ker \partial r \subset T^{1,0}\mathbb{C}^{n+1}$ near M and $\xi_{\overline{\alpha}} := \overline{\xi_{\alpha}}$. The coframe $\{dr, \tilde{\theta}, \tilde{\theta}^{\alpha}, \tilde{\theta}^{\overline{\alpha}}\}$ is defined as the dual of $\{\nu, \tau, \xi_{\alpha}, \xi_{\overline{\alpha}}\}$. Then equations (1.3) and (1.4) of [16] read, for some set of functions $\{\tilde{h}_{\alpha\overline{\beta}}\}$,

$$G = 4(1 - r\kappa)\frac{\partial r \overline{\partial} r}{r^2} + 4\tilde{h}_{\alpha\overline{\beta}} \frac{\tilde{\theta}^{\alpha}\tilde{\theta}^{\overline{\beta}}}{r} = (1 - r\kappa)\frac{dr^2 + 4\tilde{\theta}^2}{r^2} + 4\tilde{h}_{\alpha\overline{\beta}}\frac{\tilde{\theta}^{\alpha}\tilde{\theta}^{\overline{\beta}}}{r} \tag{2.2}$$

and

$$\partial\overline{\partial} r = \kappa \partial r \wedge \overline{\partial} r - \tilde{h}_{\alpha\overline{\beta}} \tilde{\theta}^{\alpha} \wedge \tilde{\theta}^{\overline{\beta}} = i\kappa dr \wedge \tilde{\theta} - \tilde{h}_{\alpha\overline{\beta}} \tilde{\theta}^{\alpha} \wedge \tilde{\theta}^{\overline{\beta}}. \tag{2.3}$$

The *square root* $X := \overline{\Omega}_{1/2}$ of $\overline{\Omega}$ in the sense of [10] is defined as in the following way. As a topological manifold, X is identical with $\overline{\Omega}$. The smooth structure on X is given in such a way that the identity maps $\mathring{X} \to \Omega$, $\partial X \to M$ are diffeomorphisms and $\rho := \sqrt{r/2}$ is a boundary defining function for X. Let $i_{1/2}\colon X \to \overline{\Omega}$ be the identity map and $\tilde{\Theta}$, $\tilde{\Theta}^{\alpha}$, $\tilde{\Theta}^{\overline{\alpha}}$ the pullbacks of $\tilde{\theta}$, $\tilde{\theta}^{\alpha}$, $\tilde{\theta}^{\overline{\alpha}}$ by $i_{1/2}$. The conformal class $[\Theta]$ of $\Theta := \tilde{\Theta}|_{\partial X}$ is independent of the choice of r because that of $\tilde{\theta}|_{\partial\Omega}$ is independent. Then G lifts to the following metric on $\mathring{X}$:

$$g := i_{1/2}^* G = 4\big(1 - 2\rho^2\kappa\big)\frac{d\rho^2}{\rho^2} + \big(1 - 2\rho^2\kappa\big)\frac{\tilde{\Theta}^2}{\rho^4} + 2\tilde{h}_{\alpha\overline{\beta}}\frac{\tilde{\Theta}^{\alpha}}{\rho}\frac{\tilde{\Theta}^{\overline{\beta}}}{\rho}. \tag{2.4}$$

Since $\{d\rho/\rho, \tilde{\Theta}/\rho^2, \tilde{\Theta}^\alpha/\rho, \tilde{\Theta}^{\overline{\alpha}}/\rho\}$ is a local frame of ${}^\Theta T^*_{\mathbb{C}} X$, this expression shows that g extends to a positive-definite Θ-metric on $(X, [\Theta])$.

Let F_p, $p \in \partial X$, be the set of vector fields of the form (2.1) with $a(p) = b(p) = c^j(p) = 0$. Then the fiber ${}^\Theta T_p X$ is naturally identified with the quotient vector space $\mathcal{V}_\Theta / F_p$. Since F_p is an ideal, ${}^\Theta T_p X$ is a Lie algebra, which is called the *tangent algebra* at p. In the sequel we always further assume that

$$\ker \iota^*[\Theta] \subset T(\partial X) \text{ is a contact distribution on } \partial X;$$

then the derived series of ${}^\Theta T_p X$ consists of the following subalgebras:

$$K_{1,p} := \langle \rho^2 T, \rho Y_1, \dots, \rho Y_{2n} \rangle / F_p, \qquad K_{2,p} := \langle \rho^2 T \rangle / F_p.$$

Collecting these subspaces we obtain the subbundles K_1 and K_2 of ${}^\Theta TX|_{\partial X}$.

ACH metrics generalize the Θ-metrics constucted from complete Kähler metrics as illustrated in Example 2.1. The characterizing features are completely described in terms of the boundary value of g. Our first two assumptions are that

$$\left| \frac{d\rho}{\rho} \right|_g = \frac{1}{2} \quad \text{at } \partial X \tag{2.5}$$

and

$$g \text{ is positive-definite on } K_2. \tag{2.6}$$

It is clear that (2.5) is independent of the choice of a boundary defining function ρ. The condition (2.6) implies that if we pull $\rho^4 g$, regarded as a section of $\operatorname{Sym}^2 T^*X$, back to ∂X then it is equal to the square of some contact form that belongs to $\iota^*[\Theta]$. If there is a fixed Θ-metric g satisfying these two conditions, then for any $p \in \partial X$ there is a unique orthogonal decomposition

$${}^\Theta T_p X = R_p \oplus K_{1,p}, \quad K_{1,p} = K_{2,p} \oplus L_p. \tag{2.7}$$

The subbundle of ${}^\Theta TX|_{\partial X}$ whose fiber at p is L_p is denoted by L.

Let $H \subset T(\partial X)$ be the kernel of $\iota^*[\Theta]$. Given a boundary defining function ρ, there is a vector-bundle isomorphism

$$\lambda_\rho \colon H \to L, \qquad Y_p \mapsto \pi_p(\rho Y \bmod F_p), \tag{2.8}$$

where $Y \in C^\infty(X, TX)$ is any extension of $Y_p \in H_p$ and $\pi_p \colon K_{1,p} \to L_p$ is the projection with respect to the decomposition (2.7). By a *compatible almost CR structure* for $[\Theta]$ we mean any nondegenerate partially integrable almost CR structure $T^{1,0}$ on ∂X for which the conformal class $[\theta]$ of pseudohermitian structures is equal to $\iota^*[\Theta]$.

Definition 2.2 Let $(X, [\Theta])$ be a Θ-manifold. An *ACH metric* on $(X, [\Theta])$ is a Θ-metric g satisfying (2.5), (2.6), and the following additional conditions:

(i) For any $p \in \partial X$, if $r_p \in R_p$ is the vector such that $(d\rho/\rho)_p(r_p) = 1$, then the map $L_p \to {}^\Theta T_p X$, $Z_p \mapsto [r_p, Z_p]$, is the identity map onto L_p;

(ii) There is a compatible almost CR structure $T^{1,0}$ such that, for some (hence for any) boundary defining function ρ and a pseudohermitian structure $\theta \in \iota^*[\Theta]$ characterized by $\iota^*(\rho^4 g) = \theta^2$, via (2.8) $g|_L$ agrees with the Levi form on H determined by θ.

The condition (i) above is independent of the choice of ρ. On (ii), the assumptions of partial integrability and nondegeneracy for $T^{1,0}$ are not restrictive here, since if $\lambda_\rho^*(g|_L) = (d\theta)|_H(\cdot, J\cdot)$ holds for an almost CR structure (H, J) on $M = \partial X$ satisfying $\ker \iota^*[\Theta] = H$, then $(d\theta)|_H(\cdot, J\cdot)$ is symmetric and hence hermitian, which implies that (H, J) is partially integrable, and its nondegeneracy is nothing but the contact condition for $\iota^*[\Theta]$ that we keep imposing. Furthermore, because of the contact condition, (H, J) is unique. We say that $T^{1,0}$ is the *induced partially integrable almost CR structure* at the boundary at infinity of the *ACH manifold* $(X, [\Theta], g)$.

Remark 2.3 Let g be a Θ-metric on $(X, [\Theta])$ satisfying (2.5) and (2.6). We further assume that we have a local frame $\{N, T, Y_j\}$ of TX around $p \in \partial X$, which is dual to $\{d\rho, \tilde{\Theta}, \alpha^j\}$ for an extension $\tilde{\Theta}$ of some $\Theta \in [\Theta]$, such that $d\tilde{\Theta}(N, Y_j) = -\tilde{\Theta}([N, Y_j]) = O(\rho)$ and $R_p = \langle \rho N\rangle / F_p$. Then, since $r_p = (\rho N)_p$ and $[\rho N, \rho^2 T] = 2\rho^2 T$, $[\rho N, \rho Y_j] = \rho Y_j \mod F_p$, the map $L_p \to {}^\Theta T_p X$, $Z_p \mapsto [r_p, Z_p]$ is the identity if and only if $L_p = \langle \rho Y_1, \dots, \rho Y_{2n}\rangle / F_p$.

Proposition 2.4 *Let $\Omega \subset \mathbb{C}^{n+1}$ be a bounded strictly pseudoconvex domain and $T^{1,0}$ the induced CR structure on $M = \partial\Omega$. Then, for any choice of a boundary defining function $r \in C^\infty(\overline{\Omega})$, the Θ-metric (2.4) on the square root of $\overline{\Omega}$ is an ACH metric inducing $T^{1,0}$.*

Proof We follow the same notation as in Example 2.1 and let $\{N, \tilde{T}, \tilde{Z}_\alpha, \tilde{Z}_{\overline{\alpha}}\}$ be the dual of $\{d\rho, \tilde{\Theta}, \tilde{\Theta}^\alpha, \tilde{\Theta}^{\overline{\alpha}}\}$. It is obvious from (2.4) that (2.5), (2.6), and (ii) of Definition 2.2 are satisfied. Since (2.4) also shows $R_p = \langle \rho N\rangle / F_p$ and $(L_p)_{\mathbb{C}} = \langle \rho\tilde{Z}_1, \dots, \rho\tilde{Z}_n, \rho\tilde{Z}_{\overline{1}}, \dots, \rho\tilde{Z}_{\overline{n}}\rangle / F_p$, by Remark 2.3 we only have to check $d\tilde{\Theta}(N, \tilde{Z}_\alpha) = O(\rho)$ to prove that (i) holds. It follows from (2.3) that $d\tilde{\theta}$ does not contain a $dr \wedge \tilde{\theta}^\alpha$ term, which implies that $d\tilde{\Theta}(N, \tilde{Z}_\alpha) = 0$. □

For any Θ-metric on X satisfying (2.5), (2.6), and a choice of a contact form in $\iota^*[\Theta]$, there is a special boundary defining function, which is called a *model boundary defining function,* as shown below.

Lemma 2.5 *Let $(X, [\Theta])$ be a Θ-manifold and g a Θ-metric satisfying (2.5), (2.6). Then, for any $\theta \in \iota^*[\Theta]$, there exists a boundary defining function ρ such that*

$$\left|\frac{d\rho}{\rho}\right|_g = \frac{1}{2} \quad \text{near } \partial X \tag{2.9}$$

and $\iota^*(\rho^4 g) = \theta^2$. The germ of ρ along ∂X is unique.

Proof This is given in [20], but for the reader's convenience we include a proof. Let ρ' be any boundary defining function and set $\rho = e^{\psi}\rho'$. Then $|d\rho/\rho|_g = 1/2$ is equivalent to

$$\frac{2X_{\rho'}}{\rho'}\psi + \rho\left|\frac{d\psi}{\rho'}\right|_g^2 = \frac{1}{\rho'}\left(\frac{1}{4} - \left|\frac{d\rho'}{\rho'}\right|_g^2\right), \tag{2.10}$$

where $X_{\rho'} = \sharp_g(d\rho'/\rho')$ is the dual of $d\rho'/\rho'$ with respect to g. If we express $X_{\rho'}$ in the form (2.1), then the assumption (2.5) implies that $a = 1/4$ on ∂X. Hence (2.10) is a noncharacteristic first-order PDE. After prescribing the boundary value of ψ so that $\iota^*(\rho^4 g) = \theta^2$ is satisfied, we obtain a unique solution of (2.10) near ∂X. □

Fix any contact form $\theta \in \iota^*[\Theta]$ on $M = \partial X$. Let ρ be a model boundary defining function associated with θ and $X_\rho := \sharp_g(d\rho/\rho)$. We consider the smooth map induced by the flow Fl_t of the vector field $4X_\rho/\rho$, which is transverse to ∂X:

$$\Phi\colon \big(\text{an open neighborhood of } M \times \{0\} \text{ in } M \times [0,\infty)\big) \to X, \qquad (p,t) \mapsto \mathrm{Fl}_t(p).$$

The manifold-with-boundary $M \times [0,\infty)_t$, whose boundary $M \times \{0\}$ is identified with M, carries a standard Θ-structure, which is given by extending $[\theta]$ in such a way that $[\theta](\partial_t) = 0$. Since $\tilde{\Theta}(4X_\rho/\rho) = 4\rho g(d\rho/\rho, \tilde{\Theta}/\rho^2) = O(\rho)$, we conclude that Φ is a *Θ-diffeomorphism*, i.e., a diffeomorphism preserving Θ-structures, onto its image. By the construction $t\partial_t$ is orthogonal to $\ker(dt/t)$ with respect to the induced Θ-metric Φ^*g, and we also see that $t = \Phi^*\rho$, which implies that t is a model boundary defining function for Φ^*g and θ.

Definition 2.6 Let $(M, T^{1,0})$ be a nondegenerate partially integrable almost CR manifold and give $M \times [0,\infty)_\rho$ the standard Θ-structure. Let θ be a pseudohermitian structure on $(M, T^{1,0})$. Then a *normal-form ACH metric g* for $(M, T^{1,0})$ and θ is an ACH metric defined near the boundary of $M \times [0,\infty)_\rho$ satisfying the following conditions:

(i) $\rho\partial_\rho$ is orthogonal to $\ker(d\rho/\rho)$ with respect to g;
(ii) ρ is a model boundary defining function for g and θ;
(iii) the induced partially integrable almost CR structure is $T^{1,0}$.

What we have proved is that, for any choice of $\theta \in \iota^*[\Theta]$, any ACH metric on X is, if it is restricted to some neighborhood of the boundary, identified with a normal-form ACH metric for $(M, T^{1,0})$ and θ via a boundary-fixing Θ-diffeomorphism, where $T^{1,0}$ is the partially integrable almost CR structure on $M = \partial X$ induced by the original ACH metric.

Proposition 2.7 *Let $(M, T^{1,0})$ be a nondegenerate partially integrable almost CR manifold and $X \subset M \times [0,\infty)_\rho$ an open neighborhood of $M = M \times \{0\}$ carrying the standard Θ-structure. Let $\{Z_\alpha\}$ in general denote a local frame of $T^{1,0}$, $\{\theta^\alpha\}$ a family of 1-forms on M satisfying $\theta^\beta(Z_\alpha) = \delta_\alpha{}^\beta$ and $\theta^{\overline{\alpha}} = \overline{\theta^\alpha}$. We fix a pseudohermitian structure θ. The 1-forms θ, θ^α, and $\theta^{\overline{\alpha}}$ are extended in such a way that they annihilate*

∂_ρ and are constant in the ρ-direction. Then a Θ-metric g on X is a normal-form ACH metric for $(M, T^{1,0})$ and θ if and only if it is of the form

$$g = 4\left(\frac{d\rho}{\rho}\right)^2 + g_{00}\left(\frac{\theta}{\rho^2}\right)^2 + 2g_{0A}\frac{\theta}{\rho^2}\frac{\theta^A}{\rho} + g_{AB}\frac{\theta^A}{\rho}\frac{\theta^B}{\rho}, \tag{2.11}$$

where the indices A, B run $\{1, \dots, n, \overline{1}, \dots, \overline{n}\}$, and satisfies

$$g_{00}|_M = 1, \qquad g_{0\alpha}|_M = 0, \qquad g_{\alpha\overline{\beta}}|_M = h_{\alpha\overline{\beta}} \quad \text{and} \quad g_{\alpha\beta}|_M = 0, \tag{2.12}$$

where $h_{\alpha\overline{\beta}}$ is the Levi form associated with θ.

Proof The condition $\rho\partial_\rho \perp_g \ker(d\rho/\rho)$, together with (2.10), implies that g is of the form (2.11). In order for ρ to be a model boundary defining function for θ, g_{00} must be 1 at M. By Remark 2.3, the condition (i) in Definition 2.2 is equivalent to $g_{0\alpha}|_M = 0$ in this situation. The given almost CR structure $T^{1,0}$ is the one in (ii) of Definition 2.2 if and only if $g_{\alpha\overline{\beta}}|_M = h_{\alpha\overline{\beta}}$ and $g_{\alpha\beta}|_M = 0$. □

3 Pseudohermitian Geometry

Let $(M, T^{1,0})$ be a nondegenerate partially integrable almost CR manifold. In the presence of a fixed pseudohermitian structure θ, there is a canonical direct sum decomposition of $T_{\mathbb{C}}M$:

$$T_{\mathbb{C}}M = \mathbb{C}T \oplus T^{1,0} \oplus T^{0,1}.$$

Here T, the *Reeb vector field,* is characterized by

$$\theta(T) = 1, \qquad T \lrcorner d\theta = 0.$$

If $\{Z_\alpha\}$ is a local frame of $T^{1,0}$, the *admissible coframe* $\{\theta^\alpha\}$ is defined in such a way that $\theta^\alpha(Z_\beta) = \delta_\beta{}^\alpha$ and $\theta^\alpha|_{\mathbb{C}T\oplus T^{0,1}} = 0$. This makes $\{\theta, \theta^\alpha, \theta^{\overline{\alpha}}\}$ into the dual coframe of $\{T, Z_\alpha, Z_{\overline{\alpha}}\}$. The index 0 is used for components corresponding with T or θ.

The Tanaka–Webster connection [27, 29] can be extended to the case of partially integrable almost CR structures as stated in the following proposition.

Proposition 3.1 *On a nondegenerate partially integrable almost CR manifold $(M, T^{1,0})$ with a fixed pseudohermitian structure θ, there is a unique connection ∇ on TM satisfying the following conditions*:

(i) *H, T, J, h are all parallel with respect to ∇;*
(ii) *The torsion tensor $\Theta(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$ satisfies*

$$\begin{cases} \Theta(X, Y) + \Theta(JX, JY) = 2\,d\theta(X, Y)T, & X, Y \in \Gamma(H), \\ \Theta(T, JX) = -J\Theta(T, X), & X \in \Gamma(H). \end{cases} \tag{3.1}$$

The components $\Theta_{\alpha\beta}{}^{0}$, $\Theta_{\alpha\beta}{}^{\gamma}$, $\Theta_{\alpha\beta}{}^{\overline{\gamma}}$ of the torsion are not visible in (3.1). Following the argument in the integrable case the first two are shown to be zero. One immediately sees from the definition that the last one is related to the Nijenhuis tensor by

$$\Theta_{\alpha\beta}{}^{\overline{\gamma}} = -N_{\alpha\beta}{}^{\overline{\gamma}} \qquad \bigl(\text{and } \Theta_{\overline{\alpha}\overline{\beta}}{}^{\gamma} = -N_{\overline{\alpha}\overline{\beta}}{}^{\gamma}\bigr).$$

The other nonzero components of the torsion are

$$\Theta_{\alpha\overline{\beta}}{}^{0} = ih_{\alpha\overline{\beta}}, \qquad \Theta_{0\alpha}{}^{\overline{\beta}} = -\Theta_{\alpha 0}{}^{\overline{\beta}} =: A_{\alpha}{}^{\overline{\beta}}$$

and their complex conjugates. We call $A_{\alpha}{}^{\overline{\beta}}$ the *Tanaka–Webster torsion tensor*.

Remark 3.2 There is another generalization of the Tanaka–Webster connection to the partially integrable case given by Tanno [28], which is also used in [1], [2], and [26]. Our generalization is different from it in that ours preserves J, which facilitates the whole argument below, and that $\Theta_{\alpha\beta}{}^{\overline{\gamma}}$ is generally nonzero instead. It seems that our connection is first considered by Mizner [25].

The first structure equation is as follows:

$$d\theta = ih_{\alpha\overline{\beta}}\theta^{\alpha} \wedge \theta^{\overline{\beta}}, \tag{3.2}$$

$$d\theta^{\gamma} = \theta^{\alpha} \wedge \omega_{\alpha}{}^{\gamma} - A_{\overline{\alpha}}{}^{\gamma}\theta^{\overline{\alpha}} \wedge \theta - \frac{1}{2}N_{\overline{\alpha}\overline{\beta}}{}^{\gamma}\theta^{\overline{\alpha}} \wedge \theta^{\overline{\beta}}. \tag{3.3}$$

Let $\{\omega_{\alpha}{}^{\beta}\}$ be the connection forms of the Tanaka–Webster connection. Without any modification, the proof of Lemma 2.1 in [23] applies to the partially integrable case and we obtain the following lemma.

Lemma 3.3 *In a neighborhood of any point $p \in M$, there exists a frame $\{Z_{\alpha}\}$ of $T^{1,0}$ for which $\omega_{\alpha}{}^{\beta}(p) = 0$ holds.*

Such a local frame is used to prove various identities between exterior derivatives and covariant derivatives. For example, the exterior derivative of a $(1,0)$-form $\sigma = \sigma_{\alpha}\theta^{\alpha}$ is given by

$$d\sigma = \sigma_{\alpha,\beta}\theta^{\beta} \wedge \theta^{\alpha} + \sigma_{\alpha,\overline{\beta}}\theta^{\overline{\beta}} \wedge \theta^{\alpha} + \sigma_{\alpha,0}\theta \wedge \theta^{\alpha} - A_{\overline{\beta}}{}^{\alpha}\sigma_{\alpha}\theta^{\overline{\beta}} \wedge \theta - \frac{1}{2}N_{\overline{\beta}\overline{\gamma}}{}^{\alpha}\sigma_{\alpha}\theta^{\overline{\beta}} \wedge \theta^{\overline{\gamma}}.$$

Here covariant derivatives of tensors are denoted by indices after commas. This notation will be used in the sequel. In the case of covariant derivatives of a scalar-valued function we omit the comma; e.g., $\nabla_{\alpha}u = u_{\alpha}$ and $\nabla_{\overline{\beta}}\nabla_{\alpha}u = u_{\alpha\overline{\beta}}$.

Proposition 3.4 *We have*

$$A_{\alpha\beta} = A_{\beta\alpha}, \tag{3.4}$$

$$N_{\alpha\beta\gamma} + N_{\beta\alpha\gamma} = 0, \qquad N_{\alpha\beta\gamma} + N_{\beta\gamma\alpha} + N_{\gamma\alpha\beta} = 0. \tag{3.5}$$

Proof By differentiating (3.2) and considering types we obtain (3.4) and $N_{[\alpha\beta\gamma]} = 0$ (where the square brackets denote skew-symmetrization). The first identity of (3.5) is obvious from the definition of the Nijenhuis tensor, and it thereby proves the second one. □

Lemma 3.5 *The second covariant derivatives of a scalar-valued function u satisfy the following*:

$$u_{\alpha\overline{\beta}} - u_{\overline{\beta}\alpha} = ih_{\alpha\overline{\beta}}u_0, \qquad u_{\alpha\beta} - u_{\beta\alpha} = -N_{\alpha\beta}{}^{\overline{\gamma}}u_{\overline{\gamma}}, \qquad u_{0\alpha} - u_{\alpha 0} = A_\alpha{}^{\overline{\beta}}u_{\overline{\beta}}. \tag{3.6}$$

Proof The same argument as the one in [23] applies to our case. □

Next we shall study the curvature $R^{\mathrm{TW}}(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$. If we set $\Pi_\alpha{}^\beta = d\omega_\alpha{}^\beta - \omega_\alpha{}^\gamma \wedge \omega_\gamma{}^\beta$, it holds that $R^{\mathrm{TW}}(X,Y)Z_\alpha = \Pi_\alpha{}^\beta(X,Y)Z_\beta$. We put

$$\begin{aligned}\Pi_{\alpha\overline{\beta}} &= R_{\alpha\overline{\beta}\sigma\overline{\tau}}\theta^\sigma \wedge \theta^{\overline{\tau}} + W_{\alpha\overline{\beta}\gamma}\theta^\gamma \wedge \theta + W_{\alpha\overline{\beta}\overline{\gamma}}\theta^{\overline{\gamma}} \wedge \theta \\ &\quad + V_{\alpha\overline{\beta}\sigma\tau}\theta^\sigma \wedge \theta^\tau + V_{\alpha\overline{\beta}\overline{\sigma}\overline{\tau}}\theta^{\overline{\sigma}} \wedge \theta^{\overline{\tau}},\end{aligned} \tag{3.7}$$

where $V_\alpha{}^\beta{}_{(\sigma\tau)} = V_\alpha{}^\beta{}_{(\overline{\sigma}\overline{\tau})} = 0$. Since $\nabla h = 0$ we have $\Pi_{\alpha\overline{\beta}} + \Pi_{\overline{\beta}\alpha} = 0$, and hence

$$R_{\alpha\overline{\beta}\sigma\overline{\tau}} = R_{\overline{\beta}\alpha\overline{\tau}\sigma}, \qquad W_{\alpha\overline{\beta}\overline{\gamma}} = -W_{\overline{\beta}\alpha\overline{\gamma}}, \qquad V_{\alpha\overline{\beta}\sigma\tau} = -V_{\overline{\beta}\alpha\sigma\tau}. \tag{3.8}$$

We substitute (3.7) into the exterior derivative of (3.3) and compare the coefficients to obtain

$$R_{\alpha\overline{\beta}\sigma\overline{\tau}} - R_{\sigma\overline{\beta}\alpha\overline{\tau}} = -N_{\alpha\sigma}{}^{\overline{\gamma}}N_{\overline{\tau}\gamma\overline{\beta}}, \tag{3.9a}$$

$$W_{\alpha\overline{\beta}\gamma} = A_{\alpha\gamma,\overline{\beta}} - N_{\gamma\sigma\alpha}A_{\overline{\beta}}{}^{\sigma}, \qquad V_{\alpha\overline{\beta}\sigma\tau} = \frac{i}{2}(h_{\sigma\overline{\beta}}A_{\alpha\tau} - h_{\tau\overline{\beta}}A_{\alpha\sigma}) + \frac{1}{2}N_{\sigma\tau\alpha,\overline{\beta}}. \tag{3.9b}$$

The component $R_{\alpha\overline{\beta}\rho\overline{\sigma}}$ is called the *Tanaka–Webster curvature tensor*. We put $R_{\alpha\overline{\beta}} := R_\gamma{}^\gamma{}_{\alpha\overline{\beta}}$ and $R := R_\alpha{}^\alpha$. It is seen from the first identity of (3.8) that $R_{\alpha\overline{\beta}} = R_{\overline{\beta}\alpha}$, and from (3.9a) we have

$$R_\alpha{}^\gamma{}_{\gamma\overline{\beta}} = R_{\alpha\overline{\beta}} - N_{\alpha\sigma\tau}N_{\overline{\beta}}{}^{\tau\sigma}. \tag{3.10}$$

As we have discussed above, a choice of a pseudohermitian structure θ defines the Tanaka–Webster connection and supplies various pseudohermitian invariants. If a certain pseudohermitian invariant is also conserved by any change of pseudohermitian structure, it is called a *CR invariant*. To investigate such invariants, we need the transformation law of the connection and relevant quantities.

Proposition 3.6 *Let θ and $\hat{\theta} = e^{2u}\theta$, $u \in C^\infty(M)$, be two pseudohermitian structures on a nondegenerate partially integrable almost CR manifold $(M, T^{1,0})$. Then, the Tanaka–Webster connection forms, the torsions, and the Ricci tensors are related as follows*:

$$\hat{\omega}_\alpha{}^\beta = \omega_\alpha{}^\beta + 2\big(u_\alpha\theta^\beta - u^\beta\theta_\alpha\big) + 2\delta_\alpha{}^\beta u_\gamma\theta^\gamma + 2i\big(u^\beta{}_\alpha + 2u_\alpha u^\beta + 2\delta_\alpha{}^\beta u_\gamma u^\gamma\big)\theta, \tag{3.11}$$

$$\hat{A}_{\alpha\beta} = A_{\alpha\beta} + i(u_{\alpha\beta} + u_{\beta\alpha}) - 4iu_\alpha u_\beta + i(N_{\gamma\alpha\beta} + N_{\gamma\beta\alpha})u^\gamma, \tag{3.12}$$

$$\hat{R}_{\alpha\overline{\beta}} = R_{\alpha\overline{\beta}} - (n+2)(u_{\alpha\overline{\beta}} + u_{\overline{\beta}\alpha}) - \big(u_\gamma{}^\gamma + u^\gamma{}_\gamma + 4(n+1)u_\gamma u^\gamma\big)h_{\alpha\overline{\beta}}. \tag{3.13}$$

Proof The new Reeb vector field is $\hat{T} = e^{-2u}(T - 2iu^\alpha Z_\alpha + 2iu^{\overline{\alpha}} Z_{\overline{\alpha}})$ and the admissible coframe dual to $\{Z_\alpha\}$ is $\{\hat{\theta}^\alpha = \theta^\alpha + 2iu^\alpha\theta\}$. To establish (3.11) and (3.12), it is enough to check that

$$d\hat{h}_{\alpha\overline{\beta}} = \hat{h}_{\gamma\overline{\beta}}\hat{\omega}_\alpha{}^\gamma + \hat{h}_{\alpha\overline{\gamma}}\hat{\omega}_{\overline{\beta}}{}^{\overline{\gamma}}$$

and

$$d\hat{\theta}^\gamma = \hat{\theta}^\alpha \wedge \hat{\omega}_\alpha{}^\gamma - \hat{h}^{\gamma\overline{\beta}}\hat{A}_{\overline{\alpha}\overline{\beta}}\hat{\theta}^{\overline{\alpha}} \wedge \hat{\theta} - \frac{1}{2}N_{\overline{\alpha}\overline{\beta}}{}^\gamma\hat{\theta}^{\overline{\alpha}} \wedge \hat{\theta}^{\overline{\beta}}.$$

They are shown straightforwardly using (3.6).

We compute $\hat{\Pi}_\gamma{}^\gamma = d\hat{\omega}_\gamma{}^\gamma$ modulo $\hat{\theta}^\alpha \wedge \hat{\theta}^\beta$, $\hat{\theta}^{\overline{\alpha}} \wedge \hat{\theta}^{\overline{\beta}}$, $\hat{\theta}$, or equivalently, modulo $\theta^\alpha \wedge \theta^\beta$, $\theta^{\overline{\alpha}} \wedge \theta^{\overline{\beta}}$, θ. By the first identity of (3.6) we obtain that, modulo $\hat{\theta}^\alpha \wedge \hat{\theta}^\beta$, $\hat{\theta}^{\overline{\alpha}} \wedge \hat{\theta}^{\overline{\beta}}$, $\hat{\theta}$,

$$\begin{aligned}\hat{\Pi}_\gamma{}^\gamma &\equiv \Pi_\gamma{}^\gamma - \big[(n+2)(u_{\alpha\overline{\beta}} + u_{\overline{\beta}\alpha}) + \big(u_\gamma{}^\gamma + u_{\overline{\gamma}}{}^{\overline{\gamma}} + 4(n+1)u_\gamma u^\gamma\big)h_{\alpha\overline{\beta}}\big]\theta^\alpha \wedge \theta^{\overline{\beta}} \\ &\equiv \big[R_{\alpha\overline{\beta}} - (n+2)(u_{\alpha\overline{\beta}} + u_{\overline{\beta}\alpha}) - \big(u_\gamma{}^\gamma + u_{\overline{\gamma}}{}^{\overline{\gamma}} + 4(n+1)u_\gamma u^\gamma\big)h_{\alpha\overline{\beta}}\big]\hat{\theta}^\alpha \wedge \hat{\theta}^{\overline{\beta}}.\end{aligned}$$

This proves (3.13). □

Finally, we sketch the concept of density bundles following [15]. Let us assume that we have fixed a complex line bundle $E(1,0)$ over M together with a duality between $E(1,0)^{\otimes(n+2)}$ and the CR canonical bundle K. Such a choice may not exist globally, but locally it does; when we use density bundles we restrict our scope to the local theory. Then $E(w,0)$ is the w-th tensor power of $E(1,0)$, and we set

$$E\big(w,w'\big) = E(w,0) \otimes E\big(0,w'\big), \quad w, w' \in \mathbb{Z},$$

where $E(0,w') := \overline{E(w',0)}$. We call $E(w,w')$ the *density bundle of biweight* (w,w'). Since there is a specified isomorphism $E(-n-2,0) \cong K$, we can define a connection ∇ on $E(w,w')$ so that it is compatible with the Tanaka–Webster connection on K. The space of local sections of $E(w,w')$ is denoted by $\mathcal{E}(w,w')$.

Farris [11] observed that, if ζ is a locally defined nonvanishing section of K, there is a unique pseudohermitian structure θ satisfying (1.5). If we replace ζ with $\lambda\zeta$, $\lambda \in C^\infty(M, \mathbb{C}^\times)$, then θ is replaced by $|\lambda|^{2/(n+2)}\theta$. We set

$$|\zeta|^{2/(n+2)} = \zeta^{1/(n+2)} \otimes \overline{\zeta}^{1/(n+2)} \in \mathcal{E}(-1,-1),$$

which is independent of the choice of the $(n+2)$-nd root of ζ and is in one-to-one correspondence with θ, and define $|\zeta|^{-2/(n+2)} \in \mathcal{E}(1,1)$ as its dual. Then we obtain a CR-invariant section $\boldsymbol{\theta} := \theta \otimes |\zeta|^{-2/(n+2)}$ of $T^*M \otimes E(1,1)$.

The Levi form h is a section of the bundle $(T^{1,0})^* \otimes (T^{0,1})^*$, which is also denoted by $E_{\alpha\overline{\beta}}$ using abstract indices α and $\overline{\beta}$. Since $h_{\alpha\overline{\beta}}$ and θ have the same scaling factor, $\boldsymbol{h}_{\alpha\overline{\beta}} := h_{\alpha\overline{\beta}} \otimes |\zeta|^{-2/(n+2)} \in \mathcal{E}_{\alpha\overline{\beta}}(1,1)$ is a CR-invariant section of $E_{\alpha\overline{\beta}}(1,1) := E_{\alpha\overline{\beta}} \otimes E(1,1)$. Its dual is $\boldsymbol{h}^{\alpha\overline{\beta}} \in \mathcal{E}^{\alpha\overline{\beta}}(-1,-1)$. Indices of density-weighted tensors are lowered and raised by $\boldsymbol{h}_{\alpha\overline{\beta}}$ and $\boldsymbol{h}^{\alpha\overline{\beta}}$.

One can show that $\nabla\boldsymbol{\theta}$ and $\nabla\boldsymbol{h}$ are both zero. To see this it is enough to show that $\nabla|\zeta|^2 = 0$, which follows from $\nabla h = 0$. For details see the proof of Proposition 2.1 in [15].

The density-weighted versions of the Nijenhuis tensor, the Tanaka–Webster torsion tensor, and the curvature tensor are defined by

$$\boldsymbol{N}_{\alpha\beta}{}^{\overline{\gamma}} := N_{\alpha\beta}{}^{\overline{\gamma}} \in \mathcal{E}_{\alpha\beta}{}^{\overline{\gamma}}, \qquad \boldsymbol{A}_{\alpha\beta} := A_{\alpha\beta} \in \mathcal{E}_{\alpha\beta},$$
$$\boldsymbol{R}_{\alpha\overline{\beta}\sigma\overline{\tau}} := R_{\alpha\overline{\beta}\sigma\overline{\tau}} \otimes |\zeta|^{-2/(n+2)} \in \mathcal{E}_{\alpha\overline{\beta}\sigma\overline{\tau}}(1,1).$$

When dealing with density-weighted tensors, we let ∇_α, $\nabla_{\overline{\alpha}}$, and $\nabla_{\mathbf{0}}$ denote the components of ∇ relative to θ^α, $\theta^{\overline{\alpha}}$, and $\boldsymbol{\theta}$. Since the transformation law (3.11) of the Tanaka–Webster connection forms does not contain the Nijenhuis tensor, equation (2.7) and Proposition 2.3 in [15] also hold in the partially integrable case. Using them we can derive the transformation law of any covariant derivative of any density-weighted tensor.

4 Some Low-Order Terms of Einstein Metrics

Let $(M, T^{1,0})$ be a nondegenerate partially integrable almost CR manifold with a fixed pseudohermitian structure θ and $X \subset M \times [0,\infty)_\rho$ an open neighborhood of $M = M \times \{0\}$. We take a local frame

$$\left\{\rho\partial_\rho, \rho^2 T, \rho Z_\alpha, \rho Z_{\overline{\alpha}}\right\} \tag{4.1}$$

of ${}^\Theta TX$, where T is the Reeb vector field associated with θ and $\{Z_\alpha\}$ is a local frame of $T^{1,0}$, both extended constantly in the ρ-direction. The corresponding indices are $\infty, 0, 1, \dots, n, \overline{1}, \dots, \overline{n}$. The local frame (4.1) is denoted by $\{\boldsymbol{Z}_I\}$ if needed.

Rule for the index notation *The following rule is observed in the sequel, except in the proof of Proposition 6.5*:

- $\alpha, \beta, \gamma, \sigma, \tau$ *run* $\{1, \dots, n\}$ *and* $\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\sigma}, \overline{\tau}$ *run* $\{\overline{1}, \dots, \overline{n}\}$;
- i, j, k *run* $\{0, 1, \dots, n, \overline{1}, \dots, \overline{n}\}$;
- I, J, K, L *run* $\{\infty, 0, 1, \dots, n, \overline{1}, \dots, \overline{n}\}$.

Lowercase Greek indices and their complex conjugates are raised and lowered by the Levi form unless otherwise stated.

We consider a normal-form ACH metric g on X. By Proposition 2.7, the ACH condition for g is equivalent to

$$\begin{aligned} &g_{\infty\infty} = 4, \qquad g_{\infty 0} = 0, \qquad g_{\infty\alpha} = 0, \\ &g_{00} = 1 + O(\rho), \qquad g_{0\alpha} = O(\rho), \qquad g_{\alpha\overline{\beta}} = h_{\alpha\overline{\beta}} + O(\rho), \qquad g_{\alpha\beta} = O(\rho), \end{aligned} \tag{4.2}$$

where $h_{\alpha\overline{\beta}}$ is the Levi form. We shall compute the Ricci tensor of g and the tensor $E := \mathrm{Ric} + \frac{1}{2}(n+2)g$. Our goal in this section is the following proposition. By abuse of notation, in what follows we use the same symbol for a tensor on M and its constant extension in the ρ-direction.

Proposition 4.1 *The tensor E_{IJ} of a normal-form ACH metric g is $O(\rho^3)$ if and only if*

$$\begin{aligned} &g_{00} = 1 + O(\rho^3), \qquad g_{0\alpha} = O(\rho^3), \\ &g_{\alpha\overline{\beta}} = h_{\alpha\overline{\beta}} + \rho^2 \Phi_{\alpha\overline{\beta}} + O(\rho^3), \qquad g_{\alpha\beta} = \rho^2 \Phi_{\alpha\beta} + O(\rho^3), \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} \Phi_{\alpha\overline{\beta}} &= -\frac{2}{n+2}\left(R_{\alpha\overline{\beta}} - 2N_{\alpha\sigma\tau} N_{\overline{\beta}}{}^{\tau\sigma} - \frac{1}{2(n+1)}\left(R - 2N_{\gamma\sigma\tau} N^{\gamma\tau\sigma}\right) h_{\alpha\overline{\beta}} \right), \\ \Phi_{\alpha\beta} &= -2i A_{\alpha\beta} - \frac{2}{n}\left(N_{\gamma\alpha\beta,}{}^{\gamma} + N_{\gamma\beta\alpha,}{}^{\gamma}\right). \end{aligned} \tag{4.4}$$

The functions φ_{ij} are defined by

$$g_{00} = 1 + \varphi_{00}, \qquad g_{0\alpha} = \varphi_{0\alpha}, \qquad g_{\alpha\overline{\beta}} = h_{\alpha\overline{\beta}} + \varphi_{\alpha\overline{\beta}}, \qquad g_{\alpha\beta} = \varphi_{\alpha\beta}. \tag{4.5}$$

The totality of (φ_{ij}) is seen as a symmetric 2-tensor on M with coefficients in $C^\infty(X)$ using the frame $\{T, Z_\alpha, Z_{\overline{\alpha}}\}$. Hence the action of the Tanaka–Webster connection operator ∇ on (φ_{ij}) is naturally defined.

We define a connection $\overline{\nabla}$ on TX by setting $\overline{\nabla}_Z W = \nabla_Z W$ for vector fields Z, W on M and

$$\overline{\nabla}\partial_\rho = 0, \qquad \overline{\nabla}_{\partial_\rho} T = \overline{\nabla}_{\partial_\rho} Z_\alpha = 0.$$

The connection forms of $\overline{\nabla}$ with respect to the frame $\{\boldsymbol{Z}_I\}$ are given by

$$\overline{\omega}_\infty{}^\infty = \frac{d\rho}{\rho}, \qquad \overline{\omega}_0{}^0 = 2\frac{d\rho}{\rho}, \qquad \overline{\omega}_\alpha{}^\beta = \omega_\alpha{}^\beta + \delta_\alpha{}^\beta \frac{d\rho}{\rho}, \tag{4.6}$$

where $\omega_\alpha{}^\beta$ are the connection forms of ∇ with respect to $\{Z_\alpha\}$. The torsion $\overline{\Theta}$ is

$$\begin{aligned}
&\overline{\Theta}_{IJ}{}^{\infty} = \overline{\Theta}_{\infty\infty}{}^{0} = \overline{\Theta}_{0I}{}^{0} = \overline{\Theta}_{\infty I}{}^{\gamma} = \overline{\Theta}_{00}{}^{\gamma} = \overline{\Theta}_{0\alpha}{}^{\beta} = \overline{\Theta}_{\alpha\overline{\beta}}{}^{\gamma} = \overline{\Theta}_{\alpha\beta}{}^{\gamma} = 0, \\
&\overline{\Theta}_{\alpha\overline{\beta}}{}^{0} = ih_{\alpha\overline{\beta}}, \qquad \overline{\Theta}_{0\alpha}{}^{\overline{\beta}} = \rho^2 A_\alpha{}^{\overline{\beta}}, \qquad \overline{\Theta}_{\alpha\beta}{}^{\overline{\gamma}} = -\rho N_{\alpha\beta}{}^{\overline{\gamma}};
\end{aligned} \tag{4.7}$$

the Ricci tensor of $\overline{\nabla}$, defined by $\overline{R}_{IJ} := \overline{R}_I{}^K{}_{KJ}$, is given by

$$\begin{aligned}
&\overline{R}_{\infty I} = \overline{R}_{I\infty} = \overline{R}_{0I} = \overline{R}_{I0} = 0, \\
&\overline{R}_{\alpha\overline{\beta}} = \rho^2\big(R_{\alpha\overline{\beta}} - N_{\alpha\sigma\tau} N_{\overline{\beta}}{}^{\tau\sigma}\big), \qquad \overline{R}_{\alpha\beta} = \rho^2\big(i(n-1)A_{\alpha\beta} + N_{\gamma\beta\alpha,}{}^{\gamma}\big).
\end{aligned} \tag{4.8}$$

We sometimes reinterpret a tensor on X as a set of tensors on M with coefficient in $C^\infty(X)$. For example, a symmetric 2-tensor S_{IJ} is also regarded as the composed object of a scalar-valued function $S_{\infty\infty}$, a 1-tensor $S_{\infty i}$, and a 2-tensor S_{ij}, with coefficients in $C^\infty(X)$. Thus ∇ can be applied to $S_{IJ} = (S_{\infty\infty}, S_{\infty i}, S_{ij})$. Let $\#(I_1, \ldots, I_N) := N +$ (the number of 0's in $I_1, \ldots, I_N$). Then, from (4.6) we have the following formulae:

$$\overline{\nabla}_\infty S_{IJ} = \big(\rho\partial_\rho - \#(I,J)\big)S_{IJ}, \qquad \overline{\nabla}_0 S_{IJ} = \rho^2\nabla_0 S_{IJ}, \qquad \overline{\nabla}_\alpha S_{IJ} = \rho\nabla_\alpha S_{IJ}. \tag{4.9}$$

We set $\nabla^g_{Z_J} Z_I = \overline{\nabla}_{Z_J} Z_I + D_I{}^K{}_J Z_K$, where ∇^g is the Levi-Civita connection of g. Then the Ricci tensor of g is given by

$$\mathrm{Ric}_{IJ} = \overline{R}_{IJ} + \overline{\nabla}_K D_I{}^K{}_J - \overline{\nabla}_J D_I{}^K{}_K - D_I{}^L{}_K D_J{}^K{}_L + D_I{}^L{}_J D_L{}^K{}_K. \tag{4.10}$$

Thus the calculation of the Ricci tensor essentially reduces to that of $D_I{}^K{}_J$. We can compute $D_{IKJ} := g_{KL} D_I{}^L{}_J$ by the formula

$$D_{IKJ} = \frac{1}{2}(\overline{\nabla}_I g_{JK} + \overline{\nabla}_J g_{IK} - \overline{\nabla}_K g_{IJ} + \overline{\Theta}_{IKJ} + \overline{\Theta}_{JKI} + \overline{\Theta}_{IJK}),$$

where $\overline{\Theta}_{IJK} := g_{KL}\overline{\Theta}_{IJ}{}^L$. The result is given in Table 1.

To prove Proposition 4.1, it is enough to calculate everything modulo $O(\rho^3)$. However, for later use, we shall carry out more precise computation. What we allow ourselves to neglect are

(N1) any term at least quadratic in $\varphi_{ij,k\ldots}$ with $O(1)$ coefficients,
(N2) any term linear in $\varphi_{00,k\ldots}$, $\varphi_{0\alpha,k\ldots}$, $\varphi_{0\overline{\alpha},k\ldots}$, or $\varphi_{\alpha\overline{\beta},k\ldots}$ with $O(\rho)$ coefficients which vanish in the case of the Heisenberg group with standard pseudohermitian structure, and
(N3) any term linear in $\varphi_{\alpha\beta,k\ldots}$ or $\varphi_{\overline{\alpha}\overline{\beta},k\ldots}$ with $O(\rho^2)$ coefficients which vanish in the case of the Heisenberg group with standard pseudohermitian structure.

Modulo terms of type (N1), g^{IJ} is given by

$$
\begin{aligned}
&g^{\infty\infty} \equiv \frac{1}{4}, \qquad g^{\infty 0} \equiv g^{\infty\alpha} \equiv 0,\\
&g^{00} \equiv 1 - \varphi_{00}, \qquad g^{0\alpha} \equiv -\varphi_0{}^{\alpha}, \qquad g^{\alpha\overline{\beta}} \equiv h^{\alpha\overline{\beta}} - \varphi^{\alpha\overline{\beta}}, \qquad g^{\alpha\beta} \equiv -\varphi^{\alpha\beta}.
\end{aligned}
\tag{4.11}
$$

By these formulae and Table 1, we compute $D_I{}^K{}_J$ modulo terms of type (N1)–(N3) using the equality $D_I{}^K{}_J = g^{KL} D_{ILJ}$. Table 2 is the result.

Finally, we can show the following formulae for the tensor E. We define the sub-laplacian by $\Delta_b := -(\nabla^{\alpha}\nabla_{\alpha} + \nabla^{\overline{\alpha}}\nabla_{\overline{\alpha}})$.

Lemma 4.2 *The tensor E of an ACH metric g is, modulo terms of type* (N1)–(N3),

$$
\begin{aligned}
E_{\infty\infty} &\equiv -\frac{1}{2}\rho\partial_\rho(\rho\partial_\rho - 4)\varphi_{00} - \rho\partial_\rho(\rho\partial_\rho - 2)\varphi_\alpha{}^{\alpha},\\
E_{\infty 0} &\equiv \frac{1}{2}\rho(\rho\partial_\rho + 1)\big(\nabla^{\alpha}\varphi_{0\alpha} + \nabla^{\overline{\alpha}}\varphi_{0\overline{\alpha}}\big) - \rho^2(\rho\partial_\rho + 1)\nabla_0\varphi_\alpha{}^{\alpha},\\
E_{\infty\alpha} &\equiv -\frac{1}{2}i(\rho\partial_\rho + 1)\varphi_{0\alpha} - \frac{1}{2}\rho(\rho\partial_\rho - 1)\nabla_\alpha\varphi_{00} - \rho^2\partial_\rho\nabla_\alpha\varphi_\beta{}^{\beta}\\
&\quad + \frac{1}{2}\rho^2\partial_\rho\big(\nabla^{\overline{\beta}}\varphi_{\alpha\overline{\beta}} + \nabla^{\beta}\varphi_{\alpha\beta}\big) + \frac{1}{2}\rho^2\partial_\rho N_\alpha{}^{\overline{\beta}\overline{\gamma}}\varphi_{\overline{\beta}\overline{\gamma}}\\
&\quad + \frac{1}{2}\rho^2(\rho\partial_\rho - 1)\nabla_0\varphi_{0\alpha},\\
E_{00} &\equiv -\frac{1}{8}\big((\rho\partial_\rho)^2 - (2n+4)\rho\partial_\rho - 4n\big)\varphi_{00} + \frac{1}{2}(\rho\partial_\rho - 2)\varphi_\alpha{}^{\alpha}\\
&\quad + i\rho\big(\nabla^{\alpha}\varphi_{0\alpha} - \nabla^{\overline{\alpha}}\varphi_{0\overline{\alpha}}\big) + \frac{1}{2}\rho^2\Delta_b\varphi_{00} + \rho^3\big(\nabla_0\nabla^{\alpha}\varphi_{0\alpha} + \nabla_0\nabla^{\overline{\alpha}}\varphi_{0\overline{\alpha}}\big)\\
&\quad - \rho^4\nabla_0\nabla_0\varphi_\alpha{}^{\alpha},\\
E_{0\alpha} &\equiv \rho^3 A_{\alpha\beta,}{}^{\beta} + \rho^3 N_\alpha{}^{\overline{\beta}\overline{\gamma}} A_{\overline{\beta}\overline{\gamma}} - \frac{1}{8}(\rho\partial_\rho + 1)(\rho\partial_\rho - 2n - 3)\varphi_{0\alpha}\\
&\quad + \frac{3}{4}i\rho\nabla_\alpha\varphi_{00} + \frac{1}{2}i\rho\nabla_\alpha\varphi_\beta{}^{\beta} - i\rho\nabla^{\overline{\beta}}\varphi_{\alpha\overline{\beta}} + \frac{1}{2}\rho^2\Delta_b\varphi_{0\alpha} - \frac{1}{2}i\rho^2\nabla_0\varphi_{0\alpha}\\
&\quad + \frac{1}{2}\rho^2\big(\nabla_\alpha\nabla^{\beta}\varphi_{0\beta} + \nabla_\alpha\nabla^{\overline{\beta}}\varphi_{0\overline{\beta}}\big) - \rho^3\nabla_0\nabla_\alpha\varphi_\beta{}^{\beta}\\
&\quad + \frac{1}{2}\rho^3\big(\nabla_0\nabla^{\overline{\beta}}\varphi_{\alpha\overline{\beta}} + \nabla_0\nabla^{\beta}\varphi_{\alpha\beta}\big),\\
E_{\alpha\overline{\beta}} &\equiv \rho^2 R_{\alpha\overline{\beta}} - 2\rho^2 N_\alpha{}^{\overline{\gamma}}{}_\rho N_{\overline{\beta}}{}^{\rho}{}_{\overline{\gamma}} - \frac{1}{8}\big((\rho\partial_\rho)^2 - (2n+2)\rho\partial_\rho - 8\big)\varphi_{\alpha\overline{\beta}}\\
&\quad + \frac{1}{8}h_{\alpha\overline{\beta}}(\rho\partial_\rho - 4)\varphi_{00} + \frac{1}{4}h_{\alpha\overline{\beta}}\rho\partial_\rho\varphi_\gamma{}^{\gamma} + i\rho(\nabla_\alpha\varphi_{0\overline{\beta}} - \nabla_{\overline{\beta}}\varphi_{0\alpha})
\end{aligned}
$$

Table 1 D_{IKJ} for a normal-form ACH metric g. $D_{0K\infty}$ and $D_{\alpha K\infty}$ are omitted, and can be complemented by the relations $D_{0K\infty} = D_{\infty K0}$ and $D_{\alpha K\infty} = D_{\infty K\alpha}$

Type	Value
$D_{\infty\infty\infty}$	-4
$D_{\infty 0\infty}$	0
$D_{\infty\alpha\infty}$	0
$D_{\infty\infty 0}$	0
$D_{\infty 00}$	$-2 + \frac{1}{2}(\rho\partial_\rho - 4)\varphi_{00}$
$D_{\infty\alpha 0}$	$\frac{1}{2}(\rho\partial_\rho - 3)\varphi_{0\alpha}$
$D_{\infty\infty\alpha}$	0
$D_{\infty 0\alpha}$	$\frac{1}{2}(\rho\partial_\rho - 3)\varphi_{0\alpha}$
$D_{\infty\overline{\beta}\alpha}$	$-h_{\alpha\overline{\beta}} + \frac{1}{2}(\rho\partial_\rho - 2)\varphi_{\alpha\overline{\beta}}$
$D_{\infty\beta\alpha}$	$\frac{1}{2}(\rho\partial_\rho - 2)\varphi_{\alpha\beta}$
$D_{0\infty 0}$	$2 - \frac{1}{2}(\rho\partial_\rho - 4)\varphi_{00}$
D_{000}	$\frac{1}{2}\rho^2\nabla_0\varphi_{00}$
$D_{0\alpha 0}$	$-\frac{1}{2}\rho\nabla_\alpha\varphi_{00} + \rho^2(\nabla_0\varphi_{0\alpha} + A_\alpha{}^{\overline{\beta}}\varphi_{0\overline{\beta}})$
$D_{0\infty\alpha}$	$-\frac{1}{2}(\rho\partial_\rho - 3)\varphi_{0\alpha}$
$D_{00\alpha}$	$\frac{1}{2}\rho\nabla_\alpha\varphi_{00}$
$D_{0\overline{\beta}\alpha}$	$\frac{i}{2}h_{\alpha\overline{\beta}} + \frac{i}{2}h_{\alpha\overline{\beta}}\varphi_{00} + \frac{1}{2}\rho(\nabla_\alpha\varphi_{0\overline{\beta}} - \nabla_{\overline{\beta}}\varphi_{0\alpha}) + \frac{1}{2}\rho^2(\nabla_0\varphi_{\alpha\overline{\beta}} + A_\alpha{}^{\overline{\gamma}}\varphi_{\overline{\beta}\overline{\gamma}} + A_{\overline{\beta}}{}^{\gamma}\varphi_{\alpha\gamma})$
$D_{0\beta\alpha}$	$\rho^2 A_{\alpha\beta} + \frac{1}{2}\rho(\nabla_\alpha\varphi_{0\beta} - \nabla_\beta\varphi_{0\alpha} - N_{\alpha\beta}{}^{\overline{\gamma}}\varphi_{0\overline{\gamma}}) + \frac{1}{2}\rho^2(\nabla_0\varphi_{\alpha\beta} + A_\alpha{}^{\overline{\gamma}}\varphi_{\beta\overline{\gamma}} + A_\beta{}^{\overline{\gamma}}\varphi_{\alpha\overline{\gamma}})$
$D_{\alpha\infty 0}$	$-\frac{1}{2}(\rho\partial_\rho - 3)\varphi_{0\alpha}$
$D_{\alpha 00}$	$\frac{1}{2}\rho\nabla_\alpha\varphi_{00} - \rho^2 A_\alpha{}^{\overline{\beta}}\varphi_{0\overline{\beta}}$
$D_{\alpha\overline{\beta}0}$	$\frac{i}{2}h_{\alpha\overline{\beta}} + \frac{i}{2}h_{\alpha\overline{\beta}}\varphi_{00} + \frac{1}{2}\rho(\nabla_\alpha\varphi_{0\overline{\beta}} - \nabla_{\overline{\beta}}\varphi_{0\alpha}) + \frac{1}{2}\rho^2(\nabla_0\varphi_{\alpha\overline{\beta}} - A_\alpha{}^{\overline{\gamma}}\varphi_{\overline{\beta}\overline{\gamma}} + A_{\overline{\beta}}{}^{\gamma}\varphi_{\alpha\gamma})$
$D_{\alpha\beta 0}$	$\frac{1}{2}\rho(\nabla_\alpha\varphi_{0\beta} - \nabla_\beta\varphi_{0\alpha} - N_{\alpha\beta}{}^{\overline{\gamma}}\varphi_{0\overline{\gamma}}) + \frac{1}{2}\rho^2(\nabla_0\varphi_{\alpha\beta} - A_\alpha{}^{\overline{\gamma}}\varphi_{\beta\overline{\gamma}} + A_\beta{}^{\overline{\gamma}}\varphi_{\alpha\overline{\gamma}})$
$D_{\alpha\infty\overline{\beta}}$	$h_{\alpha\overline{\beta}} - \frac{1}{2}(\rho\partial_\rho - 2)\varphi_{\alpha\overline{\beta}}$
$D_{\alpha 0\overline{\beta}}$	$\frac{i}{2}h_{\alpha\overline{\beta}} + \frac{i}{2}h_{\alpha\overline{\beta}}\varphi_{00} + \frac{1}{2}\rho(\nabla_\alpha\varphi_{0\overline{\beta}} + \nabla_{\overline{\beta}}\varphi_{0\alpha}) - \frac{1}{2}\rho^2(\nabla_0\varphi_{\alpha\overline{\beta}} + A_\alpha{}^{\overline{\gamma}}\varphi_{\overline{\beta}\overline{\gamma}} + A_{\overline{\beta}}{}^{\gamma}\varphi_{\alpha\gamma})$
$D_{\alpha\overline{\gamma}\overline{\beta}}$	$\frac{i}{2}(h_{\alpha\overline{\gamma}}\varphi_{0\overline{\beta}} + h_{\alpha\overline{\beta}}\varphi_{0\overline{\gamma}}) + \frac{1}{2}\rho(\nabla_\alpha\varphi_{\overline{\beta}\overline{\gamma}} + \nabla_{\overline{\beta}}\varphi_{\alpha\overline{\gamma}} - \nabla_{\overline{\gamma}}\varphi_{\alpha\overline{\beta}} - N_{\overline{\beta}\overline{\gamma}}{}^{\sigma}\varphi_{\alpha\sigma})$
$D_{\alpha\gamma\overline{\beta}}$	$-\frac{i}{2}(h_{\gamma\overline{\beta}}\varphi_{0\alpha} - h_{\alpha\overline{\beta}}\varphi_{0\gamma}) + \frac{1}{2}\rho(\nabla_\alpha\varphi_{\gamma\overline{\beta}} + \nabla_{\overline{\beta}}\varphi_{\alpha\gamma} - \nabla_\gamma\varphi_{\alpha\overline{\beta}} - N_{\alpha\gamma}{}^{\overline{\sigma}}\varphi_{\overline{\beta}\overline{\sigma}})$
$D_{\alpha\infty\beta}$	$-\frac{1}{2}(\rho\partial_\rho - 2)\varphi_{\alpha\beta}$
$D_{\alpha 0\beta}$	$-\rho^2 A_{\alpha\beta} + \frac{1}{2}\rho(\nabla_\alpha\varphi_{0\beta} + \nabla_\beta\varphi_{0\alpha} + N_{\alpha\beta}{}^{\overline{\gamma}}\varphi_{0\overline{\gamma}}) - \frac{1}{2}\rho^2(\nabla_0\varphi_{\alpha\beta} + A_\alpha{}^{\overline{\gamma}}\varphi_{\beta\overline{\gamma}} + A_\beta{}^{\overline{\gamma}}\varphi_{\alpha\overline{\gamma}})$
$D_{\alpha\overline{\gamma}\beta}$	$\frac{i}{2}(h_{\alpha\overline{\gamma}}\varphi_{0\beta} + h_{\beta\overline{\gamma}}\varphi_{0\alpha}) + \frac{1}{2}\rho(\nabla_\alpha\varphi_{\beta\overline{\gamma}} + \nabla_\beta\varphi_{\alpha\overline{\gamma}} - \nabla_{\overline{\gamma}}\varphi_{\alpha\beta} - N_{\alpha\beta}{}^{\overline{\sigma}}\varphi_{\overline{\gamma}\overline{\sigma}})$
$D_{\alpha\gamma\beta}$	$-\rho N_{\alpha\gamma\beta} + \frac{1}{2}\rho(\nabla_\alpha\varphi_{\beta\gamma} + \nabla_\beta\varphi_{\alpha\gamma} - \nabla_\gamma\varphi_{\alpha\beta} - N_{\alpha\beta}{}^{\overline{\sigma}}\varphi_{\gamma\overline{\sigma}} - N_{\alpha\gamma}{}^{\overline{\sigma}}\varphi_{\beta\overline{\sigma}} - N_{\beta\gamma}{}^{\overline{\sigma}}\varphi_{\alpha\overline{\sigma}})$

$$
\begin{aligned}
&- \frac{1}{4}i\rho^2 h_{\alpha\overline{\beta}}\nabla_0\varphi_{00} - \frac{1}{2}i\rho^2 h_{\alpha\overline{\beta}}\nabla_0\varphi_\gamma{}^\gamma - \frac{1}{2}\rho^2\nabla_\alpha\nabla_{\overline{\beta}}\varphi_{00} - \rho^2\nabla_\alpha\nabla_{\overline{\beta}}\varphi_\gamma{}^\gamma \\
&+ \frac{1}{2}\rho^2\big(\Delta_b\varphi_{\alpha\overline{\beta}} + \nabla_\alpha\nabla^\gamma\varphi_{\overline{\beta}\gamma} + \nabla_\alpha\nabla^{\overline{\gamma}}\varphi_{\overline{\beta}\overline{\gamma}} + \nabla_{\overline{\beta}}\nabla^{\overline{\gamma}}\varphi_{\alpha\overline{\gamma}} + \nabla_{\overline{\beta}}\nabla^\gamma\varphi_{\alpha\gamma}\big) \\
&+ \frac{1}{2}\rho^3(\nabla_0\nabla_\alpha\varphi_{0\overline{\beta}} + \nabla_0\nabla_{\overline{\beta}}\varphi_{0\alpha}) - \frac{1}{2}\rho^4\nabla_0\nabla_0\varphi_{\alpha\overline{\beta}},
\end{aligned}
$$

Table 2 ${D_I}^K{}_J$ for a normal-form ACH metric g. ${D_0}^K{}_\infty$ and ${D_\alpha}^K{}_\infty$ are omitted, and can be complemented by the relations ${D_0}^K{}_\infty = {D_\infty}^K{}_0$ and ${D_\alpha}^K{}_\infty = {D_\infty}^K{}_\alpha$

Type	Value (modulo terms of type (N1)–(N3))
${D_\infty}^\infty{}_\infty$	-1
${D_\infty}^0{}_\infty$	0
${D_\infty}^\alpha{}_\infty$	0
${D_\infty}^\infty{}_0$	0
${D_\infty}^0{}_0$	$-2 + \frac{1}{2}\rho\partial_\rho\varphi_{00}$
${D_\infty}^\alpha{}_0$	$\frac{1}{2}(\rho\partial_\rho + 1){\varphi_0}^\alpha$
${D_\infty}^\infty{}_\alpha$	0
${D_\infty}^0{}_\alpha$	$\frac{1}{2}(\rho\partial_\rho - 1)\varphi_{0\alpha}$
${D_\infty}^\beta{}_\alpha$	$-{\delta_\alpha}^\beta + \frac{1}{2}\rho\partial_\rho{\varphi_\alpha}^\beta$
${D_\infty}^{\overline{\beta}}{}_\alpha$	$\frac{1}{2}\rho\partial_\rho{\varphi_\alpha}^{\overline{\beta}}$
${D_0}^\infty{}_0$	$\frac{1}{2} - \frac{1}{8}(\rho\partial_\rho - 4)\varphi_{00}$
${D_0}^0{}_0$	$\frac{1}{2}\rho^2\nabla_0\varphi_{00}$
${D_0}^\alpha{}_0$	$-\frac{1}{2}\rho\nabla^\alpha\varphi_{00} + \rho^2\nabla_0{\varphi_0}^\alpha$
${D_0}^\infty{}_\alpha$	$-\frac{1}{8}(\rho\partial_\rho - 3)\varphi_{0\alpha}$
${D_0}^0{}_\alpha$	$-\frac{i}{2}\varphi_{0\alpha} + \frac{1}{2}\rho\nabla_\alpha\varphi_{00}$
${D_0}^\beta{}_\alpha$	$\frac{i}{2}{\delta_\alpha}^\beta + \frac{i}{2}{\delta_\alpha}^\beta\varphi_{00} - \frac{i}{2}{\varphi_\alpha}^\beta + \frac{1}{2}\rho(\nabla_\alpha{\varphi_0}^\beta - \nabla^\beta\varphi_{0\alpha}) + \frac{1}{2}\rho^2\nabla_0{\varphi_\alpha}^\beta$
${D_0}^{\overline{\beta}}{}_\alpha$	$\rho^2{A_\alpha}^{\overline{\beta}} - \frac{i}{2}{\varphi_\alpha}^{\overline{\beta}} + \frac{1}{2}\rho(\nabla_\alpha{\varphi_0}^{\overline{\beta}} - \nabla^{\overline{\beta}}\varphi_{0\alpha}) + \frac{1}{2}\rho^2\nabla_0{\varphi_\alpha}^{\overline{\beta}}$
${D_\alpha}^\infty{}_0$	$-\frac{1}{8}(\rho\partial_\rho - 3)\varphi_{0\alpha}$
${D_\alpha}^0{}_0$	$-\frac{i}{2}\varphi_{0\alpha} + \frac{1}{2}\rho\nabla_\alpha\varphi_{00}$
${D_\alpha}^\beta{}_0$	$\frac{i}{2}{\delta_\alpha}^\beta + \frac{i}{2}{\delta_\alpha}^\beta\varphi_{00} - \frac{i}{2}{\varphi_\alpha}^\beta + \frac{1}{2}\rho(\nabla_\alpha{\varphi_0}^\beta - \nabla^\beta\varphi_{0\alpha}) + \frac{1}{2}\rho^2\nabla_0{\varphi_\alpha}^\beta$
${D_\alpha}^{\overline{\beta}}{}_0$	$-\frac{i}{2}{\varphi_\alpha}^{\overline{\beta}} + \frac{1}{2}\rho(\nabla_\alpha{\varphi_0}^{\overline{\beta}} - \nabla^{\overline{\beta}}\varphi_{0\alpha}) + \frac{1}{2}\rho^2\nabla_0{\varphi_\alpha}^{\overline{\beta}}$
${D_\alpha}^\infty{}_{\overline{\beta}}$	$\frac{1}{4}h_{\alpha\overline{\beta}} - \frac{1}{8}(\rho\partial_\rho - 2)\varphi_{\alpha\overline{\beta}}$
${D_\alpha}^0{}_{\overline{\beta}}$	$\frac{i}{2}h_{\alpha\overline{\beta}} + \frac{1}{2}\rho(\nabla_\alpha\varphi_{0\overline{\beta}} + \nabla_{\overline{\beta}}\varphi_{0\alpha}) - \frac{1}{2}\rho^2\nabla_0\varphi_{\alpha\overline{\beta}}$
${D_\alpha}^\gamma{}_{\overline{\beta}}$	$\frac{i}{2}{\delta_\alpha}^\gamma\varphi_{0\overline{\beta}} + \frac{1}{2}\rho(\nabla_\alpha{\varphi_{\overline{\beta}}}^\gamma + \nabla_{\overline{\beta}}{\varphi_\alpha}^\gamma - \nabla^\gamma\varphi_{\alpha\overline{\beta}}) - \frac{1}{2}\rho{N_{\overline{\beta}}}^{\gamma\sigma}\varphi_{\alpha\sigma}$
${D_\alpha}^{\overline{\gamma}}{}_{\overline{\beta}}$	$-\frac{i}{2}{\delta_{\overline{\beta}}}^{\overline{\gamma}}\varphi_{0\alpha} + \frac{1}{2}\rho(\nabla_\alpha{\varphi_{\overline{\beta}}}^{\overline{\gamma}} + \nabla_{\overline{\beta}}{\varphi_\alpha}^{\overline{\gamma}} - \nabla^{\overline{\gamma}}\varphi_{\alpha\overline{\beta}}) - \frac{1}{2}\rho{N_\alpha}^{\overline{\gamma\sigma}}\varphi_{\overline{\beta}\overline{\sigma}}$
${D_\alpha}^\infty{}_\beta$	$-\frac{1}{8}(\rho\partial_\rho - 2)\varphi_{\alpha\beta}$
${D_\alpha}^0{}_\beta$	$-\rho^2A_{\alpha\beta} + \frac{1}{2}\rho(\nabla_\alpha\varphi_{0\beta} + \nabla_\beta\varphi_{0\alpha}) - \frac{1}{2}\rho{N_{\alpha\beta}}^{\overline{\gamma}}\varphi_{0\overline{\gamma}} - \frac{1}{2}\rho^2\nabla_0\varphi_{\alpha\beta}$
${D_\alpha}^\gamma{}_\beta$	$\frac{i}{2}({\delta_\alpha}^\gamma\varphi_{0\beta} + {\delta_\beta}^\gamma\varphi_{0\alpha}) + \frac{1}{2}\rho(\nabla_\alpha{\varphi_\beta}^\gamma + \nabla_\beta{\varphi_\alpha}^\gamma - \nabla^\gamma\varphi_{\alpha\beta}) - \frac{1}{2}\rho{N_{\alpha\beta}}^{\overline{\sigma}}{\varphi_{\overline{\sigma}}}^\gamma$
${D_\alpha}^{\overline{\gamma}}{}_\beta$	$-\rho{N_\alpha}^{\overline{\gamma}}{}_\beta + \frac{1}{2}\rho(\nabla_\alpha{\varphi_\beta}^{\overline{\gamma}} + \nabla_\beta{\varphi_\alpha}^{\overline{\gamma}} - \nabla^{\overline{\gamma}}\varphi_{\alpha\beta})$

$$
\begin{aligned}
E_{\alpha\beta} \equiv{}& in\rho^2A_{\alpha\beta} + \rho^2\big({N_{\gamma\alpha\beta,}}^\gamma + {N_{\gamma\beta\alpha,}}^\gamma\big) - \rho^4A_{\alpha\beta,0} - \frac{1}{8}\rho\partial_\rho(\rho\partial_\rho - 2n - 2)\varphi_{\alpha\beta} \\
& - \frac{1}{2}\rho^2\nabla_\alpha\nabla_\beta\varphi_{00} - \rho^2\nabla_\alpha\nabla_\beta{\varphi_\gamma}^\gamma + \frac{1}{2}\rho^2\big(\Delta_b\varphi_{\alpha\beta} + \nabla_\alpha\nabla^{\overline{\gamma}}\varphi_{\beta\overline{\gamma}} \\
& + \nabla_\alpha\nabla^\gamma\varphi_{\beta\gamma} + \nabla_\beta\nabla^{\overline{\gamma}}\varphi_{\alpha\overline{\gamma}} + \nabla_\beta\nabla^\gamma\varphi_{\alpha\gamma} + 2i\nabla_0\varphi_{\alpha\beta}\big) \\
& + \frac{1}{2}\rho^3(\nabla_0\nabla_\alpha\varphi_{0\beta} + \nabla_0\nabla_\beta\varphi_{0\alpha}) - \frac{1}{2}\rho^4\nabla_0\nabla_0\varphi_{\alpha\beta}\,.
\end{aligned}
$$

Table 3 $\overline{\nabla}_K D_I{}^K{}_J$ for a normal-form ACH metric g

Type	Value (modulo terms of type (N1)–(N3))
$\overline{\nabla}_K D_\infty{}^K{}_\infty$	1
$\overline{\nabla}_K D_\infty{}^K{}_0$	$\frac{1}{2}\rho^3\partial_\rho\nabla_0\varphi_{00} + \frac{1}{2}\rho(\rho\partial_\rho + 1)(\nabla^\alpha\varphi_{0\alpha} + \nabla^{\overline{\alpha}}\varphi_{0\overline{\alpha}})$
$\overline{\nabla}_K D_\infty{}^K{}_\alpha$	$\frac{1}{2}\rho^2(\rho\partial_\rho - 1)\nabla_0\varphi_{0\alpha} + \frac{1}{2}\rho^2\partial_\rho(\nabla^{\overline{\beta}}\varphi_{\alpha\overline{\beta}} + \nabla^\beta\varphi_{\alpha\beta})$
$\overline{\nabla}_K D_0{}^K{}_0$	$-\frac{3}{2} - \frac{1}{8}(\rho\partial_\rho - 3)(\rho\partial_\rho - 4)\varphi_{00} + \frac{1}{2}\rho^2\Delta_b\varphi_{00} + \rho^3(\nabla_0\nabla^\alpha\varphi_{0\alpha} + \nabla_0\nabla^{\overline{\alpha}}\varphi_{0\overline{\alpha}})$ $+\frac{1}{2}\rho^4\nabla_0\nabla_0\varphi_{00}$
$\overline{\nabla}_K D_0{}^K{}_\alpha$	$-\frac{1}{8}(\rho\partial_\rho - 2)(\rho\partial_\rho - 3)\varphi_{0\alpha} + \frac{i}{2}\rho\nabla_\alpha\varphi_{00} - \frac{i}{2}\rho(\nabla^{\overline{\beta}}\varphi_{\alpha\overline{\beta}} + \nabla^\beta\varphi_{\alpha\beta})$ $+\frac{1}{2}\rho^2\Delta_b\varphi_{0\alpha} + \frac{1}{2}\rho^2(\nabla_\alpha\nabla^\beta\varphi_{0\beta} + \nabla_\alpha\nabla^{\overline{\beta}}\varphi_{0\overline{\beta}})$ $+\rho^3 A_{\alpha\beta,}{}^\beta + \frac{1}{2}\rho^3\nabla_0\nabla_\alpha\varphi_{00} + \frac{1}{2}\rho^3(\nabla_0\nabla^\beta\varphi_{\alpha\beta} + \nabla_0\nabla^{\overline{\beta}}\varphi_{\alpha\overline{\beta}})$
$\overline{\nabla}_K D_\alpha{}^K{}_{\overline{\beta}}$	$-\frac{1}{4}h_{\alpha\overline{\beta}} - \frac{1}{8}(\rho\partial_\rho - 1)(\rho\partial_\rho - 2)\varphi_{\alpha\overline{\beta}} + \frac{i}{2}\rho(\nabla_\alpha\varphi_{0\overline{\beta}} - \nabla_{\overline{\beta}}\varphi_{0\alpha})$ $+\frac{1}{2}\rho^2(\Delta_b\varphi_{\alpha\overline{\beta}} + \nabla_\alpha\nabla^\gamma\varphi_{\overline{\beta}\gamma} + \nabla_\alpha\nabla^{\overline{\gamma}}\varphi_{\overline{\beta}\overline{\gamma}} + \nabla_{\overline{\beta}}\nabla^{\overline{\gamma}}\varphi_{\alpha\overline{\gamma}} + \nabla_{\overline{\beta}}\nabla^\gamma\varphi_{\alpha\gamma})$ $+\frac{1}{2}\rho^3(\nabla_0\nabla_\alpha\varphi_{0\overline{\beta}} + \nabla_0\nabla_{\overline{\beta}}\varphi_{0\alpha}) - \frac{1}{2}\rho^4\nabla_0\nabla_0\varphi_{\alpha\overline{\beta}}$
$\overline{\nabla}_K D_\alpha{}^K{}_\beta$	$\rho^2 N_{\gamma\alpha\beta,}{}^\gamma - \rho^4 A_{\alpha\beta,0} - \frac{1}{8}(\rho\partial_\rho - 1)(\rho\partial_\rho - 2)\varphi_{\alpha\beta} + \frac{i}{2}\rho(\nabla_\alpha\varphi_{0\beta} + \nabla_\beta\varphi_{0\alpha})$ $+\frac{1}{2}\rho^2(\Delta_b\varphi_{\alpha\beta} + \nabla_\alpha\nabla^{\overline{\gamma}}\varphi_{\beta\overline{\gamma}} + \nabla_\alpha\nabla^\gamma\varphi_{\beta\overline{\gamma}} + \nabla_\beta\nabla^{\overline{\gamma}}\varphi_{\alpha\overline{\gamma}} + \nabla_\beta\nabla^\gamma\varphi_{\alpha\gamma}) + i\rho^2\nabla_0\varphi_{\alpha\beta}$ $+\frac{1}{2}\rho^3(\nabla_0\nabla_\alpha\varphi_{0\beta} + \nabla_0\nabla_\beta\varphi_{0\alpha}) - \frac{1}{2}\rho^4\nabla_0\nabla_0\varphi_{\alpha\beta}$

Table 4 $\overline{\nabla}_J D_I{}^K{}_K$ for a normal-form ACH metric g

Type	Value (modulo terms of type (N1)–(N3))
$\overline{\nabla}_\infty D_\infty{}^K{}_K$	$2n + 3 + \frac{1}{2}\rho\partial_\rho(\rho\partial_\rho - 1)\varphi_{00} + \rho\partial_\rho(\rho\partial_\rho - 1)\varphi_\alpha{}^\alpha$
$\overline{\nabla}_0 D_\infty{}^K{}_K$	$\frac{1}{2}\rho^3\partial_\rho\nabla_0\varphi_{00} + \rho^3\partial_\rho\nabla_0\varphi_\alpha{}^\alpha$
$\overline{\nabla}_\alpha D_\infty{}^K{}_K$	$\frac{1}{2}\rho^2\partial_\rho\nabla_\alpha\varphi_{00} + \rho^2\partial_\rho\nabla_\alpha\varphi_\beta{}^\beta$
$\overline{\nabla}_0 D_0{}^K{}_K$	$\frac{1}{2}\rho^4\nabla_0\nabla_0\varphi_{00} + \rho^4\nabla_0\nabla_0\varphi_\alpha{}^\alpha$
$\overline{\nabla}_\alpha D_0{}^K{}_K$	$\frac{1}{2}\rho^3\nabla_0\nabla_\alpha\varphi_{00} + \rho^3\nabla_0\nabla_\alpha\varphi_\beta{}^\beta$
$\overline{\nabla}_{\overline{\beta}} D_\alpha{}^K{}_K$	$\frac{1}{2}\rho^2\nabla_\alpha\nabla_{\overline{\beta}}\varphi_{00} + \rho^2\nabla_\alpha\nabla_{\overline{\beta}}\varphi_\gamma{}^\gamma + \frac{i}{2}\rho^2 h_{\alpha\overline{\beta}}\nabla_0\varphi_{00} + i\rho^2 h_{\alpha\overline{\beta}}\nabla_0\varphi_\gamma{}^\gamma$
$\overline{\nabla}_\beta D_\alpha{}^K{}_K$	$\frac{1}{2}\rho^2\nabla_\alpha\nabla_\beta\varphi_{00} + \rho^2\nabla_\alpha\nabla_\beta\varphi_\gamma{}^\gamma$

Proof Using Table 2 we compute, modulo terms of type (N1)–(N3),

$$\overline{\nabla}_K D_I{}^K{}_J, \qquad \overline{\nabla}_J D_I{}^K{}_K, \qquad D_I{}^L{}_K D_J{}^K{}_L, \quad \text{and} \quad D_I{}^L{}_J D_L{}^K{}_K$$

to obtain Tables 3, 4, 5 and 6. Then, from (4.8) and (4.10), the lemma follows. $\square$

Table 5 $D_I{}^L{}_K D_J{}^K{}_L$ for a normal-form ACH metric g

Type	Value (modulo terms of type (N1)–(N3))
$D_\infty{}^L{}_K D_\infty{}^K{}_L$	$2n+5-2\rho\partial_\rho\varphi_{00}-2\rho\partial_\rho\varphi_\alpha{}^\alpha$
$D_\infty{}^L{}_K D_0{}^K{}_L$	$-\rho^2\nabla_0\varphi_{00}-\rho^2\nabla_0\varphi_\alpha{}^\alpha$
$D_\infty{}^L{}_K D_\alpha{}^K{}_L$	$\frac{i}{2}(\rho\partial_\rho+1)\varphi_{0\beta}-\rho\nabla_\alpha\varphi_{00}-\rho\nabla_\alpha\varphi_\beta{}^\beta-\frac{1}{2}\rho(\rho\partial_\rho-2)N_\alpha{}^{\overline{\beta}\overline{\gamma}}\varphi_{\overline{\beta}\overline{\gamma}}$
$D_0{}^L{}_K D_0{}^K{}_L$	$-\frac{1}{2}(n+4)+(\rho\partial_\rho-n-2)\varphi_{00}+\varphi_\alpha{}^\alpha-i\rho(\nabla^\alpha\varphi_{0\alpha}-\nabla^{\overline{\alpha}}\varphi_{0\overline{\alpha}})$
$D_0{}^L{}_K D_\alpha{}^K{}_L$	$-\rho^3N_\alpha{}^{\overline{\beta}\overline{\gamma}}A_{\overline{\beta}\overline{\gamma}}+\frac{1}{4}(3\rho\partial_\rho-2n-5)\varphi_{0\alpha}+\frac{i}{2}\rho(\nabla^{\overline{\beta}}\varphi_{\alpha\overline{\beta}}-\nabla^\beta\varphi_{\alpha\beta})-\frac{i}{2}\rho N_\alpha{}^{\overline{\beta}\overline{\gamma}}\varphi_{\overline{\beta}\overline{\gamma}}$ $+\frac{i}{2}\rho^2\nabla_0\varphi_{0\alpha}$
$D_\alpha{}^L{}_K D_{\overline{\beta}}{}^K{}_L$	$\rho^2N_\alpha{}^{\overline{\gamma}}{}_\rho N_{\overline{\beta}}{}^\rho{}_{\overline{\gamma}}+\frac{1}{2}(\rho\partial_\rho-2)\varphi_{\alpha\overline{\beta}}+\frac{1}{2}h_{\alpha\overline{\beta}}\varphi_{00}-\frac{i}{2}\rho(\nabla_\alpha\varphi_{0\overline{\beta}}-\nabla_{\overline{\beta}}\varphi_{0\alpha})$
$D_\alpha{}^L{}_K D_\beta{}^K{}_L$	$\frac{1}{2}\rho\partial_\rho\varphi_{\alpha\beta}+i\rho^2A_{\alpha\beta}+\frac{i}{2}\rho(\nabla_\alpha\varphi_{0\beta}+\nabla_\beta\varphi_{0\alpha})$

Table 6 $D_I{}^L{}_J D_L{}^K{}_K$ for a normal-form ACH metric g

Type	Value (modulo terms of type (N1)–(N3))
$D_\infty{}^L{}_\infty D_L{}^K{}_K$	$2n+3-\frac{1}{2}\rho\partial_\rho\varphi_{00}-\rho\partial_\rho\varphi_\alpha{}^\alpha$
$D_\infty{}^L{}_0 D_L{}^K{}_K$	$-\rho^2\nabla_0\varphi_{00}-2\rho^2\nabla_0\varphi_\alpha{}^\alpha$
$D_\infty{}^L{}_\alpha D_L{}^K{}_K$	$-\frac{1}{2}\rho\nabla_\alpha\varphi_{00}-\rho\nabla_\alpha\varphi_\beta{}^\beta+\rho N_\alpha{}^{\overline{\beta}\overline{\gamma}}\varphi_{\overline{\beta}\overline{\gamma}}$
$D_0{}^L{}_0 D_L{}^K{}_K$	$-\frac{1}{2}(2n+3)+\frac{1}{4}\rho\partial_\rho\varphi_{00}+\frac{1}{2}\rho\partial_\rho\varphi_\alpha{}^\alpha+\frac{1}{8}(2n+3)(\rho\partial_\rho-4)\varphi_{00}$
$D_0{}^L{}_\alpha D_L{}^K{}_K$	$\frac{1}{8}(2n+3)(\rho\partial_\rho-3)\varphi_{0\alpha}+\frac{i}{4}\rho\nabla_\alpha\varphi_{00}+\frac{i}{2}\rho\nabla_\alpha\varphi_\beta{}^\beta-\frac{i}{2}\rho N_\alpha{}^{\overline{\beta}\overline{\gamma}}\varphi_{\overline{\beta}\overline{\gamma}}$
$D_\alpha{}^L{}_{\overline{\beta}} D_L{}^K{}_K$	$-\frac{1}{4}(2n+3)h_{\alpha\overline{\beta}}+\frac{1}{8}(2n+3)(\rho\partial_\rho-2)\varphi_{\alpha\overline{\beta}}+\frac{1}{8}h_{\alpha\overline{\beta}}\rho\partial_\rho\varphi_{00}+\frac{1}{4}h_{\alpha\overline{\beta}}\rho\partial_\rho\varphi_\gamma{}^\gamma$ $+\frac{i}{4}\rho^2h_{\alpha\overline{\beta}}\nabla_0\varphi_{00}+\frac{i}{2}\rho^2h_{\alpha\overline{\beta}}\nabla_0\varphi_\gamma{}^\gamma$
$D_\alpha{}^L{}_\beta D_L{}^K{}_K$	$\frac{1}{8}(2n+3)(\rho\partial_\rho-2)\varphi_{\alpha\beta}$

Since by definition φ_{ij} is $O(\rho)$, from Lemma 4.2 we have

$$\begin{aligned}
E_{\infty\infty}&=\frac{3}{2}\varphi_{00}+\varphi_\alpha{}^\alpha+O(\rho^2), \qquad E_{\infty 0}=O(\rho^2),\\
E_{\infty\alpha}&=-i\varphi_{0\alpha}+O(\rho^2),\\
E_{00}&=\frac{3}{8}(2n+1)\varphi_{00}-\frac{1}{2}\varphi_\alpha{}^\alpha+O(\rho^2), \qquad E_{0\alpha}=\frac{1}{2}(n+1)\varphi_{0\alpha}+O(\rho^2),\\
E_{\alpha\overline{\beta}}&=\frac{1}{8}(2n+9)\varphi_{\alpha\overline{\beta}}-\frac{3}{8}h_{\alpha\overline{\beta}}\varphi_{00}+\frac{1}{4}h_{\alpha\overline{\beta}}\varphi_\gamma{}^\gamma+O(\rho^2),\\
E_{\alpha\beta}&=\frac{1}{8}(2n+1)\varphi_{\alpha\beta}+O(\rho^2).
\end{aligned} \tag{4.12}$$

These identities show that all φ_{ij} must be $O(\rho^2)$ in order for E_{IJ} to be $O(\rho^2)$. If $\varphi_{ij}=O(\rho^2)$, by repeating this process we obtain the following, which immediately show Proposition 4.1.

$$
\begin{aligned}
E_{\infty\infty} &= 2\varphi_{00} + O(\rho^3), \qquad E_{\infty 0} = O(\rho^3), \qquad E_{\infty\alpha} = -\frac{3}{2} i\varphi_{0\alpha} + O(\rho^3), \\
E_{00} &= \frac{1}{2}(2n+1)\varphi_{00} + O(\rho^3), \qquad E_{0\alpha} = \frac{3}{8}(2n+1)\varphi_{0\alpha} + O(\rho^3), \\
E_{\alpha\overline{\beta}} &= \rho^2 R_{\alpha\overline{\beta}} - 2\rho^2 N_{\alpha}{}^{\overline{\gamma}}{}_{\rho} N_{\overline{\beta}}{}^{\rho}{}_{\overline{\gamma}} + \frac{1}{2}(n+2)\varphi_{\alpha\overline{\beta}} - \frac{1}{4} h_{\alpha\overline{\beta}}\varphi_{00} \\
&\quad + \frac{1}{2} h_{\alpha\overline{\beta}} \varphi_{\gamma}{}^{\gamma} + O(\rho^3), \\
E_{\alpha\beta} &= in\rho^2 A_{\alpha\beta} + \rho^2 (N_{\gamma\alpha\beta,}{}^{\gamma} + N_{\gamma\beta\alpha,}{}^{\gamma}) + \frac{1}{2} n\varphi_{\alpha\beta} + O(\rho^3).
\end{aligned}
\tag{4.13}
$$

5 Higher-Order Perturbation

Taking over the setting from the last section, we next introduce a perturbation in g and see what happens to the tensor E. Let $m \geq 1$ be a fixed integer and ψ_{ij} a 2-tensor on M with coefficients in $C^\infty(X)$ such that

$$
\begin{aligned}
\psi_{00} &= O(\rho^{m+2}), \qquad \psi_{0\alpha} = O(\rho^{\max\{m+1,3\}}), \\
\psi_{\alpha\overline{\beta}} &= O(\rho^{m+2}), \qquad \psi_{\alpha\beta} = O(\rho^{\max\{m,3\}}).
\end{aligned}
$$

Let g be a normal-form ACH metric satisfying (4.3) and consider another metric g' with the following components with respect to $\{\boldsymbol{Z}_I\} = \{\rho\partial_\rho, \rho^2 T, \rho Z_\alpha, \rho Z_{\overline{\alpha}}\}$:

$$
g'_{ij} = g_{ij} + \psi_{ij}\,. \tag{5.1}
$$

Note that g' also satisfies (4.3). We can read off from Lemma 4.2 the amount to which the tensor E changes, which is denoted by δE_{IJ}. For example, we have

$$
\begin{aligned}
\delta E_{\infty\alpha} &= -\frac{1}{2} i(\rho\partial_\rho + 1)\psi_{0\alpha} + \frac{1}{2}\rho^2 \partial_\rho \nabla^\beta \psi_{\alpha\beta} + \frac{1}{2}\rho N_\alpha{}^{\overline{\beta}\overline{\gamma}} \rho\partial_\rho \psi_{\overline{\beta}\overline{\gamma}} + O(\rho^{m+2}), \\
\delta E_{0\alpha} &= -\frac{1}{8}\big((\rho\partial_\rho)^2 - (2n+2)\rho\partial_\rho - 2n - 3\big)\psi_{0\alpha} + O(\rho^{m+2}), \\
\delta E_{\alpha\beta} &= -\frac{1}{8}\rho\partial_\rho(\rho\partial_\rho - 2n - 2)\psi_{\alpha\beta} + O(\rho^{m+1}).
\end{aligned}
\tag{5.2}
$$

In the same way we can compute $\delta E_{\infty\infty}$, $\delta E_{\infty 0}$, δE_{00}, and $\delta E_{\alpha\overline{\beta}}$ modulo $O(\rho^{m+2})$. But we want them to be given modulo one order higher. In this section we shall prove the following.

Proposition 5.1 *The components $\delta E_{\infty\infty}$, $\delta E_{\infty 0}$, δE_{00}, $\delta E_{\alpha\overline{\beta}}$ are given by, modulo $O(\rho^{m+3})$,*

$$
\delta E_{\infty\infty} \equiv -\frac{1}{2}\rho\partial_\rho(\rho\partial_\rho - 4)\psi_{00} - \rho\partial_\rho(\rho\partial_\rho - 2)\psi_\alpha{}^\alpha
$$

$$+\frac{1}{2}\rho^2(\rho\partial_\rho)^2\big(\Phi^{\alpha\beta}\psi_{\alpha\beta}+\Phi^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}}\big), \tag{5.3a}$$

$$\delta E_{\infty 0}\equiv\frac{1}{2}\rho(\rho\partial_\rho+1)\big(\nabla^\alpha\psi_{0\alpha}+\nabla^{\overline{\alpha}}\psi_{0\overline{\alpha}}\big)-\frac{1}{2}\rho^3\partial_\rho\big(A^{\alpha\beta}\psi_{\alpha\beta}+A^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}}\big), \tag{5.3b}$$

$$\begin{aligned}\delta E_{00}\equiv{}&-\frac{1}{8}\big((\rho\partial_\rho)^2-(2n+4)\rho\partial_\rho-4n\big)\psi_{00}+\frac{1}{2}(\rho\partial_\rho-2)\psi_\alpha{}^\alpha\\&+i\rho\big(\nabla^\alpha\psi_{0\alpha}-\nabla^{\overline{\alpha}}\psi_{0\overline{\alpha}}\big)-\frac{1}{4}\rho^3\partial_\rho\big(\Phi^{\alpha\beta}\psi_{\alpha\beta}+\Phi^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}}\big),\end{aligned} \tag{5.3c}$$

$$\begin{aligned}\delta E_\alpha{}^\alpha\equiv{}&\frac{1}{8}n(\rho\partial_\rho-4)\psi_{00}-\frac{1}{8}\big((\rho\partial_\rho)^2-(4n+2)\rho\partial_\rho-8\big)\psi_\alpha{}^\alpha\\&-i\rho\big(\nabla^\alpha\psi_{0\alpha}-\nabla^{\overline{\alpha}}\psi_{0\overline{\alpha}}\big)\\&-\frac{1}{8}\rho^2\big((n-2)\rho\partial_\rho+(2n+4)\big)\big(\Phi^{\alpha\beta}\psi_{\alpha\beta}+\Phi^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}}\big)\\&+\frac{1}{2}\rho^2\big(\nabla^\alpha\nabla^\beta\psi_{\alpha\beta}+\nabla^{\overline{\alpha}}\nabla^{\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}}\big)+\frac{1}{2}\rho^2\big(N^{\gamma\alpha\beta}{}_{,\gamma}\psi_{\alpha\beta}+N^{\overline{\gamma}\overline{\alpha}\overline{\beta}}{}_{,\overline{\gamma}}\psi_{\overline{\alpha}\overline{\beta}}\big)\\&+\frac{1}{2}\rho^2\big(N^{\gamma\alpha\beta}\nabla_\gamma\psi_{\alpha\beta}+N^{\overline{\gamma}\overline{\alpha}\overline{\beta}}\nabla_{\overline{\gamma}}\psi_{\overline{\alpha}\overline{\beta}}\big),\end{aligned} \tag{5.3d}$$

$$\begin{aligned}\mathrm{tf}(\delta E_{\alpha\overline{\beta}})\equiv{}&-\frac{1}{8}\big((\rho\partial_\rho)^2-(2n+2)\rho\partial_\rho-8\big)\,\mathrm{tf}(\psi_{\alpha\overline{\beta}})\\&+i\rho\,\mathrm{tf}(\nabla_\alpha\psi_{0\overline{\beta}}-\nabla_{\overline{\beta}}\psi_{0\alpha})+\rho^2\,\mathrm{tf}(\Psi_{\alpha\overline{\beta}}),\end{aligned} \tag{5.3e}$$

where $\delta E_\alpha{}^\alpha$ *is the trace of* $\delta E_{\alpha\overline{\beta}}$ *with respect to* $h_{\alpha\overline{\beta}}$, tf *denotes the trace-free part, and*

$$\begin{aligned}\Psi_{\alpha\overline{\beta}}={}&\frac{1}{4}(\rho\partial_\rho-2)\big(\Phi_\alpha{}^{\overline{\gamma}}\psi_{\overline{\beta}\overline{\gamma}}+\Phi_{\overline{\beta}}{}^{\gamma}\psi_{\alpha\gamma}\big)+\frac{1}{2}\big(\nabla^{\overline{\gamma}}\nabla_\alpha\psi_{\overline{\beta}\overline{\gamma}}+\nabla^\gamma\nabla_{\overline{\beta}}\psi_{\alpha\gamma}\big)\\&-N_\alpha{}^{\overline{\gamma}\overline{\sigma}}{}_{,\overline{\gamma}}\psi_{\overline{\beta}\overline{\sigma}}-N_{\overline{\beta}}{}^{\gamma\sigma}{}_{,\gamma}\psi_{\alpha\sigma}+N_\alpha{}^{\overline{\gamma}\overline{\sigma}}(\nabla_{\overline{\beta}}\psi_{\overline{\gamma}\overline{\sigma}}-\nabla_{\overline{\sigma}}\psi_{\overline{\beta}\overline{\gamma}})\\&+N_{\overline{\beta}}{}^{\gamma\sigma}(\nabla_\alpha\psi_{\gamma\sigma}-\nabla_\sigma\psi_{\alpha\gamma}).\end{aligned}$$

Proof First, let

$$\nabla^{g'}_{\boldsymbol{Z}_J}\boldsymbol{Z}_I=\overline{\nabla}_{\boldsymbol{Z}_J}\boldsymbol{Z}_I+D'_I{}^K{}_J\boldsymbol{Z}_K$$

and $D'_{IKJ}:=g'_{KL}D'_I{}^L{}_J$. Then $\delta D_{IKJ}=D'_{IKJ}-D_{IKJ}$ is given in Table 7, which is read off immediately from Table 1.

Next, we compute $\delta D_I{}^K{}_J:=D'_I{}^K{}_J-D_I{}^K{}_J$. To do this we need the knowledge of the following quantities: D_{IKJ} modulo $O(\rho^3)$, g^{IJ} modulo $O(\rho^3)$, and $\delta g^{IJ}:=g'^{IJ}-g^{IJ}$ modulo $O(\rho^{m+3})$. They can be read off from Table 1, (4.3), and (4.11). Namely, D_{IKJ} mod $O(\rho^3)$ are given by

$$D_{\infty\infty\infty}\equiv-4,\qquad D_{\infty 0\infty}\equiv 0,\qquad D_{\infty\alpha\infty}\equiv 0,$$

Table 7 $\delta D_{IKJ} = D'_{IKJ} - D_{IKJ}$ for a perturbation (5.1)

Type	Value (modulo $O(\rho^{m+3})$)
$\delta D_{\infty\infty\infty}$	0
$\delta D_{\infty 0\infty}$	0
$\delta D_{\infty\alpha\infty}$	0
$\delta D_{\infty\infty 0}$	0
$\delta D_{\infty 00}$	$\frac{1}{2}(\rho\partial_\rho - 4)\psi_{00}$
$\delta D_{\infty\alpha 0}$	$\frac{1}{2}(\rho\partial_\rho - 3)\psi_{0\alpha}$
$\delta D_{\infty\infty\alpha}$	0
$\delta D_{\infty 0\alpha}$	$\frac{1}{2}(\rho\partial_\rho - 3)\psi_{0\alpha}$
$\delta D_{\infty\overline{\beta}\alpha}$	$\frac{1}{2}(\rho\partial_\rho - 2)\psi_{\alpha\overline{\beta}}$
$\delta D_{\infty\beta\alpha}$	$\frac{1}{2}(\rho\partial_\rho - 2)\psi_{\alpha\beta}$
$\delta D_{0\infty 0}$	$-\frac{1}{2}(\rho\partial_\rho - 4)\psi_{00}$
δD_{000}	0
$\delta D_{0\alpha 0}$	0
$\delta D_{0\infty\alpha}$	$-\frac{1}{2}(\rho\partial_\rho - 3)\psi_{0\alpha}$
$\delta D_{00\alpha}$	0
$\delta D_{0\overline{\beta}\alpha}$	$\frac{i}{2}h_{\alpha\overline{\beta}}\psi_{00} + \frac{1}{2}\rho(\nabla_\alpha\psi_{0\overline{\beta}} - \nabla_{\overline{\beta}}\psi_{0\alpha}) + \frac{1}{2}\rho^2(A_\alpha{}^{\overline{\gamma}}\psi_{\overline{\beta}\overline{\gamma}} + A_{\overline{\beta}}{}^{\gamma}\psi_{\alpha\gamma})$
$\delta D_{0\beta\alpha}$	$\frac{1}{2}\rho(\nabla_\alpha\psi_{0\beta} - \nabla_\beta\psi_{0\alpha} - N_{\alpha\beta}{}^{\overline{\gamma}}\psi_{0\overline{\gamma}}) + \frac{1}{2}\rho^2\nabla_0\psi_{\alpha\beta}$
$\delta D_{\alpha\infty\overline{\beta}}$	$-\frac{1}{2}(\rho\partial_\rho - 2)\psi_{\alpha\overline{\beta}}$
$\delta D_{\alpha 0\overline{\beta}}$	$\frac{i}{2}h_{\alpha\overline{\beta}}\psi_{00} + \frac{1}{2}\rho(\nabla_\alpha\psi_{0\overline{\beta}} + \nabla_{\overline{\beta}}\psi_{0\alpha}) - \frac{1}{2}\rho^2(A_\alpha{}^{\overline{\gamma}}\psi_{\overline{\beta}\overline{\gamma}} + A_{\overline{\beta}}{}^{\gamma}\psi_{\alpha\gamma})$
$\delta D_{\alpha\overline{\gamma}\overline{\beta}}$	$\frac{i}{2}h_{\alpha\overline{\beta}}\psi_{0\overline{\gamma}} + \frac{i}{2}h_{\alpha\overline{\gamma}}\psi_{0\overline{\beta}} + \frac{1}{2}\rho(\nabla_\alpha\psi_{\overline{\beta}\overline{\gamma}} - N_{\overline{\beta}\overline{\gamma}}{}^{\sigma}\psi_{\alpha\sigma})$
$\delta D_{\alpha\gamma\overline{\beta}}$	$\frac{i}{2}h_{\alpha\overline{\beta}}\psi_{0\gamma} - \frac{i}{2}h_{\gamma\overline{\beta}}\psi_{0\alpha} + \frac{1}{2}\rho(\nabla_{\overline{\beta}}\psi_{\alpha\gamma} - N_{\alpha\gamma}{}^{\overline{\sigma}}\psi_{\overline{\beta}\overline{\sigma}})$
$\delta D_{\alpha\infty\beta}$	$-\frac{1}{2}(\rho\partial_\rho - 2)\psi_{\alpha\beta}$
$\delta D_{\alpha 0\beta}$	$\frac{1}{2}\rho(\nabla_\alpha\psi_{0\beta} + \nabla_\beta\psi_{0\alpha} - N_{\alpha\beta}{}^{\overline{\gamma}}\psi_{0\overline{\gamma}}) - \frac{1}{2}\rho^2\nabla_0\psi_{\alpha\beta}$
$\delta D_{\alpha\overline{\gamma}\beta}$	$\frac{i}{2}h_{\alpha\overline{\gamma}}\psi_{0\beta} + \frac{i}{2}h_{\beta\overline{\gamma}}\psi_{0\alpha} - \frac{1}{2}\rho(\nabla_{\overline{\gamma}}\psi_{\alpha\beta} + N_{\alpha\beta}{}^{\overline{\sigma}}\psi_{\overline{\gamma}\overline{\sigma}})$
$\delta D_{\alpha\gamma\beta}$	$\frac{1}{2}\rho(\nabla_\alpha\psi_{\beta\gamma} + \nabla_\beta\psi_{\alpha\gamma} - \nabla_\gamma\psi_{\alpha\beta})$

$$
\begin{aligned}
&D_{\infty\infty 0} \equiv 0, && D_{\infty 00} \equiv -2, && D_{\infty\alpha 0} \equiv 0, && \\
&D_{\infty\infty\alpha} \equiv 0, && D_{\infty 0\alpha} \equiv 0, && D_{\infty\overline{\beta}\alpha} \equiv -h_{\alpha\overline{\beta}}, && D_{\infty\beta\alpha} \equiv 0, \\
&D_{0\infty 0} \equiv 2, && D_{000} \equiv 0, && D_{0\alpha 0} \equiv 0, && \\
&D_{0\infty\alpha} \equiv 0, && D_{00\alpha} \equiv 0, && D_{0\overline{\beta}\alpha} \equiv \frac{1}{2}ih_{\alpha\overline{\beta}}, && D_{0\beta\alpha} \equiv \rho^2 A_{\alpha\beta}, \\
&D_{\alpha\infty 0} \equiv 0, && D_{\alpha 00} \equiv 0, && D_{\alpha\overline{\beta}0} \equiv \frac{1}{2}ih_{\alpha\overline{\beta}}, && D_{\alpha\beta 0} \equiv 0, \\
&D_{\alpha\infty\overline{\beta}} \equiv h_{\alpha\overline{\beta}}, && D_{\alpha 0\overline{\beta}} \equiv \frac{1}{2}ih_{\alpha\overline{\beta}}, && D_{\alpha\overline{\gamma}\overline{\beta}} \equiv 0, && D_{\alpha\gamma\overline{\beta}} \equiv 0, \\
&D_{\alpha\infty\beta} \equiv 0, && D_{\alpha 0\beta} \equiv -\rho^2 A_{\alpha\beta}, && D_{\alpha\overline{\gamma}\beta} \equiv 0, && D_{\alpha\gamma\beta} \equiv -\rho N_{\alpha\gamma\beta};
\end{aligned}
$$

g^{IJ} mod $O(\rho^3)$ are

$$g^{\infty\infty} \equiv \frac{1}{4}, \qquad g^{\infty 0} \equiv g^{\infty\alpha} \equiv 0, \qquad g^{00} \equiv 1, \qquad g^{0\alpha} \equiv 0,$$
$$g^{\alpha\overline{\beta}} \equiv h^{\alpha\overline{\beta}} - \rho^2 \Phi^{\alpha\overline{\beta}}, \qquad g^{\alpha\beta} \equiv -\rho^2 \Phi^{\alpha\beta};$$

δg^{IJ} mod $O(\rho^{m+3})$ are

$$\begin{aligned} &\delta g^{\infty\infty} \equiv \delta g^{\infty 0} \equiv g^{\infty\alpha} \equiv 0, \qquad \delta g^{00} \equiv -\psi_{00}, \qquad \delta g^{0\alpha} \equiv -\psi_0{}^{\alpha}, \\ &\delta g^{\alpha\overline{\beta}} \equiv -\psi^{\alpha\overline{\beta}} + \rho^2\big(\Phi^{\alpha}{}_{\overline{\gamma}}\psi^{\overline{\beta}\overline{\gamma}} + \Phi^{\overline{\beta}}{}_{\gamma}\psi^{\alpha\gamma}\big), \\ &\delta g^{\alpha\beta} \equiv -\psi^{\alpha\beta} + \rho^2\big(\Phi^{\alpha}{}_{\gamma}\psi^{\beta\gamma} + \Phi^{\beta}{}_{\gamma}\psi^{\alpha\gamma}\big). \end{aligned} \tag{5.4}$$

Since Table 7 and (5.4) shows that δg^{IJ} and δD_{IKJ} are both $O(\rho^{\max\{m,3\}})$, we have $\delta D^{KL} \cdot \delta D_{ILJ} = O(\rho^{m+3})$ and hence

$$\delta D_I{}^K{}_J \equiv g^{KL} \cdot \delta D_{ILJ} + \delta g^{KL} \cdot D_{ILJ} \quad \text{mod } O\big(\rho^{m+3}\big),$$

where $\delta D_I{}^K{}_J := D'_I{}^K{}_J - D_I{}^K{}_J$. Thus we obtain Table 8.

On the other hand, Table 2 shows that, modulo $O(\rho^3)$,

$$\begin{aligned} &D_\infty{}^\infty{}_\infty \equiv -1, \quad D_\infty{}^0{}_\infty \equiv 0, \quad D_\infty{}^\alpha{}_\infty \equiv 0, \\ &D_\infty{}^\infty{}_0 \equiv 0, \quad D_\infty{}^0{}_0 \equiv -2, \quad D_\infty{}^\alpha{}_0 \equiv 0, \\ &D_\infty{}^\infty{}_\alpha \equiv 0, \quad D_\infty{}^0{}_\alpha \equiv 0, \quad D_\infty{}^\beta{}_\alpha \equiv -\delta_\alpha{}^\beta + \rho^2\Phi_\alpha{}^\beta, \quad D_\infty{}^{\overline{\beta}}{}_\alpha \equiv \rho^2\Phi_\alpha{}^{\overline{\beta}}, \\ &D_0{}^\infty{}_0 \equiv \frac{1}{2}, \quad D_0{}^0{}_0 \equiv 0, \quad D_0{}^\alpha{}_0 \equiv 0, \\ &D_0{}^\infty{}_\alpha \equiv 0, \quad D_0{}^0{}_\alpha \equiv 0, \quad D_0{}^\beta{}_\alpha \equiv \frac{1}{2}i\delta_\alpha{}^\beta - \frac{1}{2}i\rho^2\Phi_\alpha{}^\beta, \\ &D_0{}^{\overline{\beta}}{}_\alpha \equiv -\frac{1}{2}i\rho^2\Phi_\alpha{}^{\overline{\beta}} + \rho^2 A_\alpha{}^{\overline{\beta}}, \\ &D_\alpha{}^\infty{}_0 \equiv 0, \quad D_\alpha{}^0{}_0 \equiv 0, \quad D_\alpha{}^\beta{}_0 \equiv \frac{1}{2}i\delta_\alpha{}^\beta - \frac{1}{2}i\rho^2\Phi_\alpha{}^\beta, \quad D_\alpha{}^{\overline{\beta}}{}_0 \equiv -\frac{1}{2}i\rho^2\Phi_\alpha{}^{\overline{\beta}}, \\ &D_\alpha{}^\infty{}_{\overline{\beta}} \equiv \frac{1}{4}h_{\alpha\overline{\beta}}, \quad D_\alpha{}^0{}_{\overline{\beta}} \equiv \frac{1}{2}ih_{\alpha\overline{\beta}}, \quad D_\alpha{}^\gamma{}_{\overline{\beta}} \equiv 0, \quad D_\alpha{}^{\overline{\gamma}}{}_{\overline{\beta}} \equiv 0, \\ &D_\alpha{}^\infty{}_\beta \equiv 0, \quad D_\alpha{}^0{}_\beta \equiv -\rho^2 A_{\alpha\beta}, \quad D_\alpha{}^\gamma{}_\beta \equiv 0, \quad D_\alpha{}^{\overline{\gamma}}{}_\beta \equiv -\rho N_\alpha{}^{\overline{\gamma}}{}_\beta. \end{aligned} \tag{5.5}$$

Using Table 8 and (5.5), we compute

$$\begin{aligned} &\overline{\nabla}_K\big(\delta D_I{}^K{}_J\big), \qquad \overline{\nabla}_J\big(\delta D_I{}^K{}_K\big), \qquad D_I{}^L{}_K \cdot \delta D_J{}^K{}_L, \\ &D_I{}^K{}_L \cdot \delta D_K{}^L{}_L, \quad \text{and} \quad D_K{}^L{}_L \cdot \delta D_I{}^K{}_J, \end{aligned}$$

Table 8 $\delta D_I{}^K{}_J = D'_I{}^K{}_J - D_I{}^K{}_J$ for a perturbation (5.1)

Type	Value (modulo $O(\rho^{m+3})$)
$\delta D_\infty{}^\infty{}_\infty$	0
$\delta D_\infty{}^0{}_\infty$	0
$\delta D_\infty{}^\alpha{}_\infty$	0
$\delta D_\infty{}^\infty{}_0$	0
$\delta D_\infty{}^0{}_0$	$\frac{1}{2}\rho\partial_\rho\psi_{00}$
$\delta D_\infty{}^\alpha{}_0$	$\frac{1}{2}(\rho\partial_\rho+1)\psi_0{}^\alpha$
$\delta D_\infty{}^\infty{}_\alpha$	0
$\delta D_\infty{}^0{}_\alpha$	$\frac{1}{2}(\rho\partial_\rho-1)\psi_{0\alpha}$
$\delta D_\infty{}^\beta{}_\alpha$	$\frac{1}{2}\rho\partial_\rho\psi_\alpha{}^\beta - \frac{1}{2}\rho^3\partial_\rho\Phi^{\beta\gamma}\psi_{\alpha\gamma} - \rho^2\Phi_{\alpha\gamma}\psi^{\beta\gamma}$
$\delta D_\infty{}^{\overline{\beta}}{}_\alpha$	$\frac{1}{2}\rho\partial_\rho\psi_\alpha{}^{\overline{\beta}} - \frac{1}{2}\rho^3\partial_\rho\Phi^{\gamma\overline{\beta}}\psi_{\alpha\gamma} - \rho^2\Phi_{\alpha\overline{\gamma}}\psi^{\overline{\beta}\overline{\gamma}}$
$\delta D_0{}^\infty{}_0$	$-\frac{1}{8}(\rho\partial_\rho-4)\psi_{00}$
$\delta D_0{}^0{}_0$	0
$\delta D_0{}^\alpha{}_0$	0
$\delta D_0{}^\infty{}_\alpha$	$-\frac{1}{8}(\rho\partial_\rho-3)\psi_{0\alpha}$
$\delta D_0{}^0{}_\alpha$	$-\frac{i}{2}\psi_{0\alpha}$
$\delta D_0{}^\beta{}_\alpha$	$\frac{i}{2}\delta_\alpha{}^\beta\psi_{00} - \frac{i}{2}\psi_\alpha{}^\beta + \frac{1}{2}\rho(\nabla_\alpha\psi_0{}^\beta - \nabla^\beta\psi_{0\alpha})$ $-\frac{1}{2}\rho^2(A_{\alpha\gamma}\psi^{\beta\gamma} - A^{\beta\gamma}\psi_{\alpha\gamma}) + \frac{i}{2}\rho^2(\Phi^{\beta\gamma}\psi_{\alpha\gamma} + \Phi_{\alpha\gamma}\psi^{\beta\gamma})$
$\delta D_0{}^{\overline{\beta}}{}_\alpha$	$-\frac{i}{2}\psi_\alpha{}^{\overline{\beta}} + \frac{1}{2}\rho(\nabla_\alpha\psi_0{}^{\overline{\beta}} - \nabla^{\overline{\beta}}\psi_{0\alpha} - N_\alpha{}^{\overline{\beta}\overline{\gamma}}\psi_{0\overline{\gamma}}) + \frac{1}{2}\rho^2\nabla_0\psi_\alpha{}^{\overline{\beta}}$ $+\frac{i}{2}\rho^2(\Phi^{\gamma\overline{\beta}}\psi_{\alpha\gamma} + \Phi_{\alpha\overline{\gamma}}\psi^{\overline{\beta}\overline{\gamma}})$
$\delta D_\alpha{}^\infty{}_{\overline{\beta}}$	$-\frac{1}{8}(\rho\partial_\rho-2)\psi_{\alpha\overline{\beta}}$
$\delta D_\alpha{}^0{}_{\overline{\beta}}$	$\frac{1}{2}\rho(\nabla_\alpha\psi_{0\overline{\beta}} + \nabla_{\overline{\beta}}\psi_{0\alpha}) - \frac{1}{2}\rho^2(A_\alpha{}^{\overline{\gamma}}\psi_{\overline{\beta}\overline{\gamma}} + A_{\overline{\beta}}{}^\gamma\psi_{\alpha\gamma})$
$\delta D_\alpha{}^\gamma{}_{\overline{\beta}}$	$\frac{i}{2}\delta_\alpha{}^\gamma\psi_{0\overline{\beta}} + \frac{1}{2}\rho(\nabla_\alpha\psi_{\overline{\beta}}{}^\gamma - N_{\overline{\beta}}{}^{\gamma\sigma}\psi_{\alpha\sigma})$
$\delta D_\alpha{}^{\overline{\gamma}}{}_{\overline{\beta}}$	$-\frac{i}{2}\delta_{\overline{\beta}}{}^{\overline{\gamma}}\psi_{0\alpha} + \frac{1}{2}\rho(\nabla_{\overline{\beta}}\psi_\alpha{}^{\overline{\gamma}} - N_\alpha{}^{\overline{\gamma}\overline{\sigma}}\psi_{\overline{\beta}\overline{\sigma}})$
$\delta D_\alpha{}^\infty{}_\beta$	$-\frac{1}{8}(\rho\partial_\rho-2)\psi_{\alpha\beta}$
$\delta D_\alpha{}^0{}_\beta$	$\frac{1}{2}\rho(\nabla_\alpha\psi_{0\beta} + \nabla_\beta\psi_{0\alpha}) - \frac{1}{2}\rho(N^{\overline{\gamma}}{}_{\alpha\beta} + N^{\overline{\gamma}}{}_{\beta\alpha})\psi_{0\overline{\gamma}} - \frac{1}{2}\rho^2\nabla_0\psi_{\alpha\beta}$
$\delta D_\alpha{}^\gamma{}_\beta$	$\frac{i}{2}\delta_\alpha{}^\gamma\psi_{0\beta} + \frac{i}{2}\delta_\beta{}^\gamma\psi_{0\alpha} - \frac{1}{2}\rho\nabla^\gamma\psi_{\alpha\beta} - \frac{1}{2}\rho(N^{\overline{\sigma}}{}_{\alpha\beta} + N^{\overline{\sigma}}{}_{\beta\alpha})\psi_{\overline{\sigma}}{}^\gamma$
$\delta D_\alpha{}^{\overline{\gamma}}{}_\beta$	$\frac{1}{2}\rho(\nabla_\alpha\psi_\beta{}^{\overline{\gamma}} + \nabla_\beta\psi_\alpha{}^{\overline{\gamma}} - \nabla^{\overline{\gamma}}\psi_{\alpha\beta})$

all modulo $O(\rho^{m+3})$. The result is Tables 9–13. From these tables and

$$\begin{aligned}\delta E_{IJ} \equiv{}& \frac{1}{2}(n+2)\delta g_{IJ} + \overline{\nabla}_K\big(\delta D_I{}^K{}_J\big) - \overline{\nabla}_J\big(\delta D_I{}^K{}_K\big)\\ &- D_I{}^L{}_K\cdot\delta D_J{}^K{}_L - D_J{}^L{}_K\cdot\delta D_I{}^K{}_L\\ &+ D_I{}^L{}_J\cdot\delta D_L{}^K{}_K + D_L{}^K{}_K\cdot\delta D_I{}^L{}_J \quad \bmod O\big(\rho^{m+3}\big),\end{aligned}$$

we can verify Proposition 5.1. □

Table 9 $\overline{\nabla}_K(\delta D_I{}^K{}_J)$ for a perturbation (5.1)

Type	Value (modulo $O(\rho^{m+3})$)
$\overline{\nabla}_K(\delta D_\infty{}^K{}_\infty)$	0
$\overline{\nabla}_K(\delta D_\infty{}^K{}_0)$	$\frac{1}{2}\rho(\rho\partial_\rho+1)(\nabla^\alpha\psi_{0\alpha}+\nabla^{\overline{\alpha}}\psi_{0\overline{\alpha}})$
$\overline{\nabla}_K(\delta D_0{}^K{}_0)$	$-\frac{1}{8}(\rho\partial_\rho-3)(\rho\partial_\rho-4)\psi_{00}$
$\overline{\nabla}_K(\delta D_\alpha{}^K{}_{\overline{\beta}})$	$-\frac{1}{8}(\rho\partial_\rho-1)(\rho\partial_\rho-2)\psi_{\alpha\overline{\beta}}+\frac{i}{2}\rho(\nabla_\alpha\psi_{0\overline{\beta}}-\nabla_{\overline{\beta}}\psi_{0\alpha})$ $+\frac{1}{2}\rho^2\nabla^{\overline{\gamma}}(\nabla_\alpha\psi_{\overline{\beta}\overline{\gamma}}-N_{\overline{\beta}\overline{\gamma}}{}^\sigma\psi_{\alpha\sigma})+\frac{1}{2}\rho^2\nabla^\gamma(\nabla_{\overline{\beta}}\psi_{\alpha\gamma}-N_{\alpha\gamma}{}^{\overline{\sigma}}\psi_{\overline{\beta}\overline{\sigma}})$

Table 10 $\overline{\nabla}_J(\delta D_I{}^K{}_K)$ for a perturbation (5.1)

Type	Value (modulo $O(\rho^{m+3})$)
$\overline{\nabla}_\infty(\delta D_\infty{}^K{}_K)$	$\frac{1}{2}\rho\partial_\rho(\rho\partial_\rho-1)\psi_{00}+\rho\partial_\rho(\rho\partial_\rho-1)\psi_\alpha{}^\alpha$ $-\frac{1}{2}\rho^2(\rho\partial_\rho+1)(\rho\partial_\rho+2)(\Phi^{\alpha\beta}\psi_{\alpha\beta}+\Phi^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}})$
$\overline{\nabla}_0(\delta D_\infty{}^K{}_K)$	0
$\overline{\nabla}_0(\delta D_0{}^K{}_K)$	0
$\overline{\nabla}_{\overline{\beta}}(\delta D_\alpha{}^K{}_K)$	0

Table 11 $D_I{}^L{}_K\cdot\delta D_J{}^K{}_L$ for a perturbation (5.1)

Type	Value (modulo $O(\rho^{m+3})$)
$D_\infty{}^L{}_K\cdot\delta D_\infty{}^K{}_L$	$-\rho\partial_\rho\psi_{00}-\rho\partial_\rho\psi_\alpha{}^\alpha+\rho^2(\rho\partial_\rho+1)(\Phi^{\alpha\beta}\psi_{\alpha\beta}+\Phi^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}})$
$D_\infty{}^L{}_K\cdot\delta D_0{}^K{}_L$	$-\frac{i}{2}\rho^2(\Phi^{\alpha\beta}\psi_{\alpha\beta}-\Phi^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}})$
$D_0{}^L{}_K\cdot\delta D_\infty{}^K{}_L$	$\frac{i}{2}\rho^2(\Phi^{\alpha\beta}\psi_{\alpha\beta}-\Phi^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}})+\frac{1}{2}\rho^2(A^{\alpha\beta}\rho\partial_\rho\psi_{\alpha\beta}+A^{\overline{\alpha}\overline{\beta}}\rho\partial_\rho\psi_{\overline{\alpha}\overline{\beta}})$
$D_0{}^L{}_K\cdot\delta D_0{}^K{}_L$	$\frac{1}{2}(\rho\partial_\rho-n-2)\psi_{00}+\frac{1}{2}\psi_\alpha{}^\alpha-\frac{i}{2}\rho(\nabla^\alpha\psi_{0\alpha}-\nabla^{\overline{\alpha}}\psi_{0\overline{\alpha}})$ $-\frac{1}{4}\rho^2(\Phi^{\alpha\beta}\psi_{\alpha\beta}+\Phi^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}})$
$D_\alpha{}^L{}_K\cdot\delta D_{\overline{\beta}}{}^K{}_L$	$\frac{1}{4}(\rho\partial_\rho-2)\psi_{\alpha\overline{\beta}}+\frac{1}{4}h_{\alpha\overline{\beta}}\psi_{00}+\frac{i}{2}\rho\nabla_{\overline{\beta}}\psi_{0\alpha}-\frac{1}{4}\rho^2(\rho\partial_\rho-2)\Phi_\alpha{}^{\overline{\gamma}}\psi_{\overline{\beta}\overline{\gamma}}$ $-\frac{i}{2}\rho^2(A_\alpha{}^{\overline{\gamma}}\psi_{\overline{\beta}\overline{\gamma}}+A_{\overline{\beta}}{}^\gamma\psi_{\alpha\gamma})-\frac{1}{2}\rho^2N_\alpha{}^{\overline{\gamma}\overline{\sigma}}(\nabla_{\overline{\beta}}\psi_{\overline{\gamma}\overline{\sigma}}+\nabla_{\overline{\gamma}}\psi_{\overline{\beta}\overline{\sigma}}-\nabla_{\overline{\sigma}}\psi_{\overline{\beta}\overline{\gamma}})$

Proposition 5.2 *Let g be a normal-form ACH metric satisfying* (4.3) *and g' given by* (5.1). *Then*,

$$\delta E_{\infty\infty}=-\frac{1}{2}(m+2)(m-2)\psi_{00}-m(m+2)\psi_\alpha{}^\alpha+\frac{1}{2}m^2\rho^2\big(\Phi^{\alpha\beta}\psi_{\alpha\beta}+\Phi^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}}\big)+O\big(\rho^{m+3}\big), \tag{5.6a}$$

$$\delta E_{\infty 0}=\frac{1}{2}(m+2)\rho\big(\nabla^\alpha\psi_{0\alpha}+\nabla^{\overline{\alpha}}\psi_{0\overline{\alpha}}\big)-\frac{1}{2}m\rho^2\big(A^{\alpha\beta}\psi_{\alpha\beta}+A^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}}\big)+O\big(\rho^{m+3}\big), \tag{5.6b}$$

Table 12 $D_I{}^K{}_L \cdot \delta D_K{}^L{}_L$ for a perturbation (5.1)

Type	Value (modulo $O(\rho^{m+3})$)
$D_\infty{}^K{}_\infty \cdot \delta D_K{}^L{}_L$	$-\frac{1}{2}\rho\partial_\rho\psi_{00} - \rho\partial_\rho\psi_\alpha{}^\alpha + \frac{1}{2}\rho^2(\rho\partial_\rho + 2)(\Phi^{\alpha\beta}\psi_{\alpha\beta} + \Phi^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}})$
$D_\infty{}^K{}_0 \cdot \delta D_K{}^L{}_L$	0
$D_0{}^K{}_0 \cdot \delta D_K{}^L{}_L$	$\frac{1}{4}\rho\partial_\rho\psi_{00} + \frac{1}{2}\rho\partial_\rho\psi_\alpha{}^\alpha - \frac{1}{4}\rho^2(\rho\partial_\rho + 2)(\Phi^{\alpha\beta}\psi_{\alpha\beta} + \Phi^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}})$
$D_\alpha{}^K{}_{\overline{\beta}} \cdot \delta D_K{}^L{}_L$	$\frac{1}{8}h_{\alpha\overline{\beta}}\rho\partial_\rho\psi_{00} + \frac{1}{4}h_{\alpha\overline{\beta}}\rho\partial_\rho\psi_\gamma{}^\gamma - \frac{1}{8}\rho^2 h_{\alpha\overline{\beta}}(\rho\partial_\rho + 2)(\Phi^{\sigma\tau}\psi_{\sigma\tau} + \Phi^{\overline{\sigma}\overline{\tau}}\psi_{\overline{\sigma}\overline{\tau}})$

Table 13 $D_K{}^L{}_L \cdot \delta D_I{}^K{}_J$ for a perturbation (5.1)

Type	Value (modulo $O(\rho^{m+3})$)
$D_K{}^L{}_L \cdot \delta D_\infty{}^K{}_\infty$	0
$D_K{}^L{}_L \cdot \delta D_\infty{}^K{}_0$	0
$D_K{}^L{}_L \cdot \delta D_0{}^K{}_0$	$\frac{1}{8}(2n+3)(\rho\partial_\rho - 4)\psi_{00}$
$D_K{}^L{}_L \cdot \delta D_\alpha{}^K{}_{\overline{\beta}}$	$\frac{1}{8}(2n+3)(\rho\partial_\rho - 2)\psi_{\alpha\overline{\beta}}$

$$\delta E_{\infty\alpha} = -\frac{1}{2}i(m+2)\psi_{0\alpha} + \frac{1}{2}m\rho\nabla^\beta\psi_{\alpha\beta} + \frac{1}{2}m\rho N_\alpha{}^{\overline{\beta}\overline{\gamma}}\psi_{\overline{\beta}\overline{\gamma}} + O(\rho^{m+2}), \tag{5.6c}$$

$$\begin{aligned}\delta E_{00} = &-\frac{1}{8}(m^2 - 2nm - 8n - 4)\psi_{00} + \frac{1}{2}m\psi_\alpha{}^\alpha + i\rho(\nabla^\alpha\psi_{0\alpha} - \nabla^{\overline{\alpha}}\psi_{0\overline{\alpha}}) \\ &-\frac{1}{4}\rho^2 m(\Phi^{\alpha\beta}\psi_{\alpha\beta} + \Phi^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}}) + O(\rho^{m+3}),\end{aligned} \tag{5.6d}$$

$$\delta E_{0\alpha} = -\frac{1}{8}(m+2)(m-2n-2)\psi_{0\alpha} + O(\rho^{m+2}), \tag{5.6e}$$

$$\begin{aligned}\delta E_\alpha{}^\alpha = &\frac{1}{8}n(m-2)\psi_{00} - \frac{1}{8}(m^2 - (4n-2)m - 8n - 8)\psi_\alpha{}^\alpha \\ &+ (O(\rho^{m+2}) \textit{ terms depending on } \psi_{0\alpha} \textit{ and } \psi_{\alpha\beta}) \\ &+ O(\rho^{m+3}),\end{aligned} \tag{5.6f}$$

$$\begin{aligned}\mathrm{tf}(\delta E_{\alpha\overline{\beta}}) = &-\frac{1}{8}(m^2 - 2nm - 2n - 9)\,\mathrm{tf}(\psi_{\alpha\overline{\beta}}) \\ &+ (O(\rho^{m+2}) \textit{ terms depending on } \psi_{0\alpha} \textit{ and } \psi_{\alpha\beta}) + O(\rho^{m+3}),\end{aligned} \tag{5.6g}$$

$$\delta E_{\alpha\beta} = -\frac{1}{8}m(m-2n-2)\psi_{\alpha\beta} + O(\rho^{m+1}). \tag{5.6h}$$

Proof This follows from (5.2), (5.3a)–(5.3e), and the fact that the Euler vector field $\rho\partial_\rho$ acts on an $O(\rho^m)$ function as, modulo $O(\rho^{m+1})$, a scalar multiplication by m. □

6 Approximate Solutions and Obstruction

By using the results in Sects. 4 and 5, in this section we construct a normal-form ACH metric for which $E = \mathrm{Ric} + \frac{1}{2}(n+2)g$ vanishes to as high order as possible. First we observe the contracted Bianchi identity satisfied by the tensor E.

Lemma 6.1 *Let $m \geq 1$ be a positive integer. Suppose that g is a normal-form ACH metric satisfying*

$$\begin{aligned}
E_{\infty\infty} &= O\big(\rho^{m+2}\big), & E_{\infty 0} &= O\big(\rho^{m+2}\big), & E_{\infty\alpha} &= O\big(\rho^{\max\{m+1,3\}}\big),\\
E_{00} &= O\big(\rho^{m+2}\big), & E_{0\alpha} &= O\big(\rho^{\max\{m+1,3\}}\big), & &\\
E_{\alpha\overline{\beta}} &= O\big(\rho^{m+2}\big), & E_{\alpha\beta} &= O\big(\rho^{\max\{m,3\}}\big). & &
\end{aligned}$$

Then we have

$$\begin{aligned}
O\big(\rho^{m+3}\big) = {}& (m-4n-2)E_{\infty\infty} - 4(m-2)E_{00} - 8mE_{\alpha}{}^{\alpha}\\
&+ 8\rho\big(\nabla^{\alpha}E_{\infty\alpha} + \nabla^{\overline{\alpha}}E_{\infty\overline{\alpha}}\big)\\
&+ 4\rho^2(m-2)\big(\Phi^{\alpha\beta}E_{\alpha\beta} + \Phi^{\overline{\alpha}\overline{\beta}}E_{\overline{\alpha}\overline{\beta}}\big),
\end{aligned} \tag{6.1a}$$

$$\begin{aligned}
O\big(\rho^{m+3}\big) = {}& (m-2n-2)E_{\infty 0} + 4\rho\big(\nabla^{\alpha}E_{0\alpha} + \nabla^{\overline{\alpha}}E_{0\overline{\alpha}}\big)\\
&+ 4\rho^2\big(A^{\alpha\beta}E_{\alpha\beta} + A^{\overline{\alpha}\overline{\beta}}E_{\overline{\alpha}\overline{\beta}}\big),
\end{aligned} \tag{6.1b}$$

$$O\big(\rho^{m+2}\big) = 2(m-2n-2)E_{\infty\alpha} + 4\rho\nabla^{\beta}E_{\alpha\beta} - 4iE_{0\alpha} + 4\rho N_{\alpha}{}^{\overline{\beta}\overline{\gamma}}E_{\overline{\beta}\overline{\gamma}}. \tag{6.1c}$$

Proof We have the contracted Bianchi identity $g^{IJ}\nabla^{g}{}_{K}\mathrm{Ric}_{IJ} = 2g^{IJ}\nabla^{g}{}_{I}\mathrm{Ric}_{JK}$, where ∇^{g} is the Levi-Civita connection determined by g. Since ∇^{g} is a metric connection we further have

$$g^{IJ}\nabla^{g}{}_{K}E_{IJ} = 2g^{IJ}\nabla^{g}{}_{I}E_{JK}.$$

In terms of the extended Tanaka–Webster connection $\overline{\nabla}$ and the tensor D, we can rewrite this identity as

$$g^{IJ}\big(\overline{\nabla}_{K}E_{IJ} - 2D_{I}{}^{L}{}_{K}E_{JL}\big) = 2g^{IJ}\big(\overline{\nabla}_{I}E_{JK} - D_{J}{}^{L}{}_{I}E_{LK} - D_{K}{}^{L}{}_{I}E_{JL}\big),$$

or equivalently,

$$0 = g^{IJ}\big(\overline{\nabla}_{K}E_{IJ} - 2\overline{\nabla}_{I}E_{JK} + 2D_{I}{}^{L}{}_{J}E_{KL} - 2\overline{\Theta}_{IK}{}^{L}E_{JL}\big),$$

where $\overline{\Theta}$ is the torsion form of $\overline{\nabla}$. Since $g^{0\alpha} = O(\rho^3)$ and $E_{IJ} = O(\rho^m)$, we obtain

$$\begin{aligned}
O\big(\rho^{m+3}\big) = {}& g^{\infty\infty}\big(\overline{\nabla}_{K}E_{\infty\infty} - 2\overline{\nabla}_{\infty}E_{\infty K} + 2D_{\infty}{}^{L}{}_{\infty}E_{KL} - 2\overline{\Theta}_{\infty K}{}^{L}E_{\infty L}\big)\\
&+ g^{00}\big(\overline{\nabla}_{K}E_{00} - 2\overline{\nabla}_{0}E_{0K} + 2D_{0}{}^{L}{}_{0}E_{KL} - 2\overline{\Theta}_{0K}{}^{L}E_{0L}\big)
\end{aligned}$$

$$
\begin{aligned}
&+2g^{\beta\overline{\gamma}}\big(\overline{\nabla}_K E_{\beta\overline{\gamma}}-\overline{\nabla}_\beta E_{\overline{\gamma}K}-\overline{\nabla}_{\overline{\gamma}}E_{\beta K}+\big(D_\beta{}^L{}_{\overline{\gamma}}+D_{\overline{\gamma}}{}^L{}_\beta\big)E_{KL}\\
&-\overline{\Theta}_{\beta K}{}^L E_{\overline{\gamma}L}-\overline{\Theta}_{\overline{\gamma}K}{}^L E_{\beta L}\big)\\
&+g^{\beta\gamma}\big(\overline{\nabla}_K E_{\beta\gamma}-2\overline{\nabla}_\beta E_{\gamma K}+2D_\beta{}^L{}_\gamma E_{KL}-2\overline{\Theta}_{\beta K}{}^L E_{\gamma L}\big)\\
&+g^{\overline{\beta}\overline{\gamma}}\big(\overline{\nabla}_K E_{\overline{\beta}\overline{\gamma}}-2\overline{\nabla}_{\overline{\beta}}E_{\overline{\gamma}K}+2D_{\overline{\beta}}{}^L{}_{\overline{\gamma}}E_{KL}-2\overline{\Theta}_{\overline{\beta}K}{}^L E_{\overline{\gamma}L}\big).
\end{aligned}
$$

Substituting $K=\infty$, $K=0$, and $K=\alpha$ into this formula, in view of (4.7), (4.9), and (5.5) we find that

$$
\begin{aligned}
O\big(\rho^{m+3}\big)&=(\rho\partial_\rho-4n-4)E_{\infty\infty}-4(\rho\partial_\rho-4)E_{00}-8(\rho\partial_\rho-2)E_\alpha{}^\alpha\\
&\quad+8\rho\big(\nabla^\alpha E_{\infty\alpha}+\nabla^{\overline{\alpha}}E_{\infty\overline{\alpha}}\big)\\
&\quad+4\rho^2(\rho\partial_\rho-2)\big(\Phi^{\alpha\beta}E_{\alpha\beta}+\Phi^{\overline{\alpha}\overline{\beta}}E_{\overline{\alpha}\overline{\beta}}\big),
\end{aligned}
\tag{6.2a}
$$

$$
\begin{aligned}
O\big(\rho^{m+3}\big)&=(\rho\partial_\rho-2n-4)E_{\infty 0}+4\rho\big(\nabla^\alpha E_{0\alpha}+\nabla^{\overline{\alpha}}E_{0\overline{\alpha}}\big)\\
&\quad+4\rho^2\big(A^{\alpha\beta}E_{\alpha\beta}+A^{\overline{\alpha}\overline{\beta}}E_{\overline{\alpha}\overline{\beta}}\big),
\end{aligned}
\tag{6.2b}
$$

$$
O\big(\rho^{m+2}\big)=2(\rho\partial_\rho-2n-3)E_{\infty\alpha}+4\rho\nabla^\beta E_{\alpha\beta}-4iE_{0\alpha}+4\rho N_\alpha{}^{\overline{\beta}\overline{\gamma}}E_{\overline{\beta}\overline{\gamma}},
\tag{6.2c}
$$

which imply (6.2a)–(6.2c). □

Let

$$
a(I,J)=\begin{cases}3, & (I,J)=(\infty,\infty),(\infty,0),(0,0),(\alpha,\overline{\beta}),\\ 2, & (I,J)=(\infty,\alpha),(0,\alpha),\\ 1, & (I,J)=(\alpha,\beta).\end{cases}
\tag{6.3}
$$

The next theorem proves Theorem 1.1.

Theorem 6.2 *Let $(M,T^{1,0})$ be a nondegenerate partially integrable almost CR manifold and θ any pseudohermitian structure. Then there exists a normal-form ACH metric g for $(M,T^{1,0})$ which satisfies*

$$
E_{IJ}=O\big(\rho^{2n+1+a(I,J)}\big)
\tag{6.4}
$$

with respect to the frame (4.1) of ${}^\Theta TX$. For such a metric, each g_{ij} is uniquely determined modulo $O(\rho^{2n+1+a(i,j)})$.

Proof By Proposition 4.1 we already have a normal-form ACH metric $g^{(0)}$ satisfying $E_{IJ}=O(\rho^3)$ for every I, J, with $O(\rho^3)$ ambiguity in each component $g^{(0)}_{ij}$. We shall inductively show that there exists a normal-form ACH metric $g^{(m)}$ satisfying

$$
E_{IJ}=O\big(\rho^{\max\{m+a(I,J),3\}}\big)
\tag{6.5}
$$

for each m, $m = 1, \ldots, 2n+1$, and for such $g^{(m)}$ its components $g^{(m)}_{ij}$ are unique modulo $O(\rho^{\max\{m+a(i,j),3\}})$.

Suppose we have a normal-form ACH metric $g^{(m-1)}$ that satisfies (6.5) for $m-1$ as well as (4.3). Consider a new ACH metric $g^{(m)}$ given by $g^{(m)}_{ij} = g^{(m-1)}_{ij} + \psi_{ij}$, where ψ_{ij} is such that $\psi_{ij} = O(\rho^{\max\{m-1+a(i,j),3\}})$. Then $\delta E = E' - E$ is given in Proposition 5.2. In view of (5.6e) and (5.6h) we can determine $\psi_{0\alpha}$ mod $O(\rho^{m+2})$ and $\psi_{\alpha\beta}$ mod $O(\rho^{\max\{m+1,3\}})$ so that $E^{(m)}_{0\alpha} = O(\rho^{m+2})$ and $E^{(m)}_{\alpha\beta} = O(\rho^{\max\{m+1,3\}})$ hold, because the exponents $-\frac{1}{8}(m+2)(m-2n-2)$ and $-\frac{1}{8}m(m-2n-2)$ are nonzero for $m = 1, \ldots, 2n+1$. After that, by a similar reasoning using (5.6g), we can determine $\operatorname{tf}(\psi_{\alpha\overline{\beta}})$ mod $O(\rho^{m+3})$ so that $\operatorname{tf}(E^{(m)}_{\alpha\overline{\beta}}) = O(\rho^{m+3})$ hold. Next we see (5.6d) and (5.6f) as a system of linear equations for ψ_{00} and $\psi_\alpha{}^\alpha$. The determinant of the coefficients is

$$\begin{aligned}
&\begin{vmatrix} -\frac{1}{8}\left(m^2 - 2nm - 8n - 4\right) & \frac{1}{2}m \\ \frac{1}{8}n(m-2) & -\frac{1}{8}\left(m^2 - (4n-2)m - 8n - 8\right) \end{vmatrix} \\
&\quad = \frac{1}{64}(m+2)(m+4)(m-2n-2)(m-4n-2), \qquad (6.6)
\end{aligned}$$

which shows that this system is nondegenerate for $m = 1, \ldots, 2n+1$. Hence we can determine ψ_{00} and $\psi_\alpha{}^\alpha$, both modulo $O(\rho^{m+3})$, so that $E^{(m)}_{00} = O(\rho^{m+3})$ and $E^{(m)}{}_\alpha{}^\alpha = O(\rho^{m+3})$ hold. Thus we have attained $E_{ij} = O(\rho^{\max\{m+a(i,j),3\}})$, and if $g^{(m-1)}_{ij}$ are unique up to $O(\rho^{\max\{m-1+a(i,j),3\}})$, the desired uniqueness result holds for $g^{(m)}_{ij}$.

Finally, we check that $g^{(m)}$ is determined in such a way that it satisfies (6.5) for $I = \infty$, too. This is done by using Lemma 6.1. In fact, for $g^{(m)}$, $E^{(m)}_{\infty 0} = O(\rho^{m+3})$ and $E^{(m)}_{\infty\alpha} = O(\rho^{m+2})$ should hold, because in (6.1b) and (6.1c) the terms on the right-hand sides are, except the first terms in each identity, already $O(\rho^{m+3})$ and $O(\rho^{m+2})$, respectively, and the coefficients of the first terms are both nonzero. Similarly, (6.1a) shows that $E^{(m)}_{\infty\infty} = O(\rho^{m+3})$. Hence the induction is complete. □

In spite of the success of the inductive determination of g_{ij} up to the stage in the theorem above, the next step cannot be executed, as (5.6e) and (5.6h) indicate; the freedom of the choice of g satisfying (6.4) does not affect the ρ^{2n+2}-term coefficient of $E_{\alpha\beta}$ and the ρ^{2n+3}-term coefficient of $E_{0\alpha}$. So we define

$$\mathcal{O}_{\alpha\beta} := \left(\rho^{-2n-2} E_{\alpha\beta}\right)\big|_{\rho=0} \qquad (6.7)$$

and call it the *obstruction tensor* associated with $(M, T^{1,0}, \theta)$. In fact, the condition $E_{\alpha\overline{\beta}} = O(\rho^{2n+4})$ on the metric from which $\mathcal{O}_{\alpha\beta}$ is computed can be weakened to $E_{\alpha\overline{\beta}} = O(\rho^{2n+3})$, for the $O(\rho^{2n+3})$ ambiguity in $\operatorname{tf}(g_{\alpha\overline{\beta}})$ emerging from that does not have any effect on the ρ^{2n+2}-term coefficient of $E_{\alpha\beta}$ as (5.6h) shows. This fact further implies that we can use any approximately Einstein ACH metric g that Theorem 1.1 claims its existence, because if ρ is a model boundary defining function for g and θ, then there is a boundary-fixing Θ-diffeomorphism Φ such that Φ^*g is a normal-form ACH metric for which the second coordinate function is equal to $\Phi^*\rho$, and its Einstein tensor vanishes to the same order as that of g does.

The ρ^{2n+3}-term coefficient of $E_{0\alpha}$ is not a new obstruction, since by (6.1c) we have

$$\left(\rho^{-2n-3}E_{0\alpha}\right)\big|_M = -i\nabla^\beta \mathcal{O}_{\alpha\beta} - iN_\alpha{}^{\overline{\beta}\overline{\gamma}}\mathcal{O}_{\overline{\beta}\overline{\gamma}}. \tag{6.8}$$

Proposition 6.3 *Let θ and $\hat{\theta} = e^{2u}\theta$, $u \in C^\infty(M)$, be two pseudohermitian structures on $(M, T^{1,0})$. Then*

$$\hat{\mathcal{O}}_{\alpha\beta} = e^{-2nu}\mathcal{O}_{\alpha\beta}, \tag{6.9}$$

where $\mathcal{O}_{\alpha\beta}$ is the obstruction tensor for $(M, T^{1,0}, \theta)$ and $\hat{\mathcal{O}}_{\alpha\beta}$ is that for $(M, T^{1,0}, \hat{\theta})$.

Proof Let $(X, [\Theta])$ be a Θ-manifold such that $\partial X = M$ and $\iota^*[\Theta]$ is the conformal class of the pseudohermitian structures on M, and take any ACH metric g satisfying the condition in Theorem 1.1. If ρ is a model boundary defining function for θ and $\hat{\rho} = e^\psi\rho$, $\psi \in C^\infty(X)$, is one for $\hat{\theta}$, then we have $\psi|_M = u$ by the condition $\iota^*(\hat{\rho}^4 g) = \hat{\theta}^2$. Hence, if $\{\tilde{Z}_\alpha\}$ is any extension of a local frame of $T^{1,0}$, we have $\hat{\mathcal{O}}_{\alpha\beta} = (\hat{\rho}^{-2n-2}E(\hat{\rho}\tilde{Z}_\alpha, \hat{\rho}\tilde{Z}_\beta))|_M = e^{-2nu}(\rho^{-2n-2}E(\rho\tilde{Z}_\alpha, \rho\tilde{Z}_\beta))|_M = e^{-2nu}\mathcal{O}_{\alpha\beta}$. □

The proposition above implies that the density-weighted version of the obstruction tensor

$$\boldsymbol{\mathcal{O}}_{\alpha\beta} := \mathcal{O}_{\alpha\beta} \otimes |\zeta|^{2n/(n+2)} \in \mathcal{E}_{(\alpha\beta)}(-n, -n)$$

is a CR-invariant tensor, where $\mathcal{E}_{(\alpha\beta)}$ denotes the space of local sections of $\mathrm{Sym}^2(T^{1,0})^*$.

Next we recall (6.8). Let us also look at a similar result

$$\left(\rho^{-2n-3}\left(\nabla^\alpha E_{0\alpha} + \nabla^{\overline{\alpha}}E_{0\overline{\alpha}}\right)\right)\big|_M = -A^{\alpha\beta}\mathcal{O}_{\alpha\beta} - A^{\overline{\alpha}\overline{\beta}}\mathcal{O}_{\overline{\alpha}\overline{\beta}},$$

which follows from (6.1b). Combining these identities we obtain

$$D^{\alpha\beta}\mathcal{O}_{\alpha\beta} - D^{\overline{\alpha}\overline{\beta}}\mathcal{O}_{\overline{\alpha}\overline{\beta}} = 0, \tag{6.10}$$

where

$$D^{\alpha\beta} = \nabla^\alpha\nabla^\beta - iA^{\alpha\beta} - N^{\gamma\alpha\beta}\nabla_\gamma - N^{\gamma\alpha\beta}{}_{,\gamma}. \tag{6.11}$$

Replacing N, A with $\boldsymbol{N}$, $\boldsymbol{A}$ and taking contractions with respect not to h but to $\boldsymbol{h}$, we obtain a differential operator $\boldsymbol{D}^{\alpha\beta} : \mathcal{E}_{(\alpha\beta)}(-n, -n) \to \mathcal{E}(-n-2, -n-2)$ between density-weighted bundles. Then we have $\boldsymbol{D}^{\alpha\beta}\boldsymbol{\mathcal{O}}_{\alpha\beta} - \boldsymbol{D}^{\overline{\alpha}\overline{\beta}}\boldsymbol{\mathcal{O}}_{\overline{\alpha}\overline{\beta}} = 0$. Furthermore, $\boldsymbol{D}^{\alpha\beta}$ belongs to a one-parameter family of CR-invariant differential operators, as we shall describe in the following proposition.

Proposition 6.4 *Let* $(M, T^{1,0})$ *be a nondegenerate partially integrable almost CR manifold. Let*

$$\boldsymbol{D}_t^{\alpha\beta} : \mathcal{E}_{(\alpha\beta)}(-n,-n) \to \mathcal{E}(-n-2,-n-2), \quad t \in \mathbb{C}$$

be a one-parameter family of differential operators defined by, in terms of any pseudohermitian structure θ,

$$\boldsymbol{D}_t^{\alpha\beta} = \nabla^\alpha \nabla^\beta - i\boldsymbol{A}^{\alpha\beta} - (1+tn)\boldsymbol{N}^{\gamma\alpha\beta}\nabla_\gamma - \big(1+t(n+1)\big)\boldsymbol{N}^{\gamma\alpha\beta}{}_{,\gamma}. \tag{6.12}$$

Then this is well defined, i.e., the right-hand side of (6.12) *is independent of* θ.

Proof This can be checked by using equation (2.7) and Proposition 2.3 of [15], as we have remarked at the end of Sect. 3. The details are left to the reader. □

The next proposition finishes the proof of Theorem 1.2.

Proposition 6.5 *The obstruction tensor* $\mathcal{O}_{\alpha\beta}$ *for a nondegenerate* (*integrable*) *CR manifold vanishes.*

Proof Since $\mathcal{O}_{\alpha\beta}$ is a certain polynomial of derivatives of pseudohermitian torsion and curvature, using the formal embedding (see, e.g., [22]) we can reduce the problem to the case of a (small piece of) nondegenerate real hypersurface $M \subset \mathbb{C}^{n+1}$. In this proof we use indices j, k for components with respect to the complex coordinates $(z^1, \ldots, z^{n+1})$.

Let r be Fefferman's approximate solution of the complex Monge–Ampère equation [12], i.e., a smooth defining function of M such that $J(r) = 1 + O(r^{n+2})$, where

$$J(r) := (-1)^{n+1} \det \begin{pmatrix} r & \partial r/\partial \overline{z}^k \\ \partial r/\partial z^j & \partial^2 r/\partial z^j \partial \overline{z}^k \end{pmatrix}.$$

We set $\tilde{\theta} := \frac{i}{2}(\partial r - \overline{\partial} r)$ and $\theta := \iota^* \tilde{\theta}$, where $\iota \colon M \hookrightarrow \mathbb{C}^{n+1}$ is the inclusion. On $\Omega = \{r > 0\}$, we consider the Kähler metric G in Example 2.1 given by Fefferman's approximate solution r. It is easily verified that $\det(G_{j\overline{k}}) = r^{-(n+2)} J(r)$, and the usual formula for the Ricci tensor of a Kähler metric shows that

$$\mathrm{Ric}(G)_{j\overline{k}} = -\frac{1}{2}(n+2) G_{j\overline{k}} + \frac{\partial^2}{\partial z^j \partial \overline{z}^k} \log J(r).$$

Observe that, if we set $\log J(r) = r^{n+2} f$,

$$\begin{aligned} \partial\overline{\partial} \log J(r) &= (n+2)(n+1) r^n f \partial r \wedge \overline{\partial} r + (n+2) r^{n+1} \\ &\quad \times (f \partial\overline{\partial} r + \partial f \wedge \overline{\partial} r + \partial r \wedge \overline{\partial} f) + r^{n+2} \partial\overline{\partial} f. \end{aligned} \tag{6.13}$$

We use the same notation as in Example 2.1. Since κ is a real-valued function, $\tau \lrcorner \partial\overline{\partial} r = -i(\xi - \overline{\xi}) \lrcorner \partial\overline{\partial} r = -i(\kappa \overline{\partial} r + \kappa \partial r) = -i\kappa dr$. Therefore $T \lrcorner d\theta = T \lrcorner \iota^*(-i\partial\overline{\partial} r) = \iota^*(-\kappa dr) = 0$, where T is the restriction of τ to M. This shows that T

is the Reeb vector field on M associated with θ. By restricting $\xi_1, \dots, \xi_n$ to M, we obtain a local frame $\{Z_1, \dots, Z_n\}$ of $T^{1,0}M$.

We identify a (one-sided) neighborhood of M in $\overline{\Omega}$ with $M \times [0, \epsilon)$ by

$$M \times [0, \epsilon) \to \overline{\Omega}, \qquad (p, s) \mapsto \mathrm{Fl}_s(p),$$

where Fl_s is the flow generated by ν. In view of the fact that s is equal to the pullback of r, we write r instead of s in the sequel. The constant extensions of T and Z_α in the r-direction are also denoted by T and Z_α. Then obviously $T = \tau + O(r)$, $Z_\alpha = \xi_\alpha + O(r)$. By (6.13), $E(G) = \mathrm{Ric}(G) + \frac{1}{2}(n+2)G$ satisfies

$$\begin{aligned}
&E(G)(\nu, \nu) = E(G)\left(\frac{1}{2}(\xi + \overline{\xi}), \frac{1}{2}(\xi + \overline{\xi})\right) = \frac{1}{2}E(G)(\xi, \overline{\xi}) = O\big(r^n\big),\\
&E(G)(\tau, \tau) = E(G)\big(-i(\xi - \overline{\xi}), -i(\xi - \overline{\xi})\big) = 2E(G)(\xi, \overline{\xi}) = O\big(r^n\big),\\
&E(G)(\nu, \tau) = 0, \qquad E(G)(\nu, \xi_\alpha) = O\big(r^{n+1}\big), \qquad E(G)(\tau, \xi_\alpha) = O\big(r^{n+1}\big),\\
&E(G)(\xi_\alpha, \xi_{\overline{\beta}}) = O\big(r^{n+1}\big), \qquad E(G)(\xi_\alpha, \xi_\beta) = 0.
\end{aligned}$$

Hence, with respect to the local frame $\{\partial_r = \nu, T, Z_\alpha, Z_{\overline{\alpha}}\}$ of $T_{\mathbb{C}}(M \times [0, \epsilon))$, we have

$$\begin{aligned}
&E(G)_{\infty\infty} = O\big(r^n\big), \qquad E(G)_{\infty 0} = O\big(r^{n+1}\big), \qquad E(G)_{\infty\alpha} = O\big(r^{n+1}\big),\\
&E(G)_{00} = O\big(r^n\big), \qquad E(G)_{0\alpha} = O\big(r^{n+1}\big),\\
&E(G)_{\alpha\overline{\beta}} = O\big(r^{n+1}\big), \qquad E(G)_{\alpha\beta} = O\big(r^{n+2}\big).
\end{aligned}$$

Therefore the induced ACH metric g on the square root of $M \times [0, \epsilon)$ in the sense of [10] satisfies, with respect to the frame $\{\rho\partial_\rho, \rho^2 T, \rho Z_\alpha, \rho Z_{\overline{\alpha}}\}$,

$$\begin{aligned}
&E_{\infty\infty} = O\big(\rho^{2n+4}\big), \qquad E_{\infty 0} = O\big(\rho^{2n+6}\big), \qquad E_{\infty\alpha} = O\big(\rho^{2n+5}\big),\\
&E_{00} = O\big(\rho^{2n+4}\big), \qquad E_{0\alpha} = O\big(\rho^{2n+5}\big),\\
&E_{\alpha\overline{\beta}} = O\big(\rho^{2n+4}\big), \qquad E_{\alpha\beta} = O\big(\rho^{2n+6}\big).
\end{aligned}$$

Hence g satisfies (6.4). Moreover, since $E_{\alpha\beta} = O(\rho^{2n+3})$, it follows that $\mathcal{O}_{\alpha\beta} = 0$. □

7 On the First Variation of the CR Obstruction Tensor

In this section, we calculate the first-order term of the obstruction tensor with respect to a variation from the standard CR structure on the Heisenberg group. First, we introduce a tensor that describes a modification of partially integrable almost CR structures.

Proposition 7.1 *Let* $(M, T^{1,0})$ *be a nondegenerate partially integrable almost CR manifold and* $\{Z_\alpha\}$ *a local frame of the bundle* $T^{1,0}$. *Let* $\mu_\alpha{}^{\overline{\beta}} \in \mathcal{E}_\alpha{}^{\overline{\beta}}$ *and set*

$$\hat{Z}_\alpha := Z_\alpha + \mu_\alpha{}^{\overline{\beta}} Z_{\overline{\beta}};$$

$\{\hat{Z}_\alpha\}$ *defines a new almost CR structure on* M *without changing the contact distribution* H. *Then this is partially integrable if and only if*

$$\mu_{\alpha\beta} = \mu_{\beta\alpha},$$

where the upper index is lowered by the Levi form of $(M, T^{1,0})$ *associated with any pseudohermitian structure.*

Proof The new almost CR structure is partially integrable if and only if

$$\theta\big([\hat{Z}_\alpha, \hat{Z}_\beta]\big) = \theta\big(\big[Z_\alpha + \mu_\alpha{}^{\overline{\sigma}} Z_{\overline{\sigma}}, Z_\beta + \mu_\beta{}^{\overline{\tau}} Z_{\overline{\tau}}\big]\big) = 0,$$

where θ is any pseudohermitian structure for $(M, T^{1,0})$. Since $\theta([Z_\alpha, Z_\beta]) = \theta([Z_{\overline{\sigma}}, Z_{\overline{\tau}}]) = 0$, this is equivalent to

$$\theta\big([Z_{\overline{\sigma}}, Z_\beta]\big)\mu_\alpha{}^{\overline{\sigma}} + \theta\big([Z_\alpha, Z_{\overline{\tau}}]\big)\mu_\beta{}^{\overline{\tau}} = 0,$$

or $\mu_{\alpha\beta} - \mu_{\beta\alpha} = 0$. □

Let $M = \mathcal{H}$ be the $(2n+1)$-dimensional Heisenberg group and θ the standard contact form. Then the obstruction tensor $\mathcal{O}_{\alpha\beta}$ with respect to θ is a functional of partially integrable almost CR structures on $\ker\theta$. For the standard CR structure we have $\mathcal{O}_{\alpha\beta} = 0$. We shall compute the derivative of $\mathcal{O}_{\alpha\beta}$ at the standard CR structure in the direction of $\mu_{\alpha\beta}$, where the second index of $\mu_{\alpha\beta}$ is understood to be lowered by the Levi form associated with θ. The differentials of various quantities at the standard CR structure will be indicated by the bullet $\bullet$.

Proposition 7.2 *Consider* $h_{\alpha\overline{\beta}}$, $N_{\alpha\beta\gamma}$, $A_{\alpha\beta}$, *and* $R_{\alpha\overline{\beta}}$ *associated with the standard contact form* θ *on the sphere. Then, their differentials at the standard CR structure are as follows*:

$$h^\bullet_{\alpha\overline{\beta}} = 0, \qquad N^\bullet_{\alpha\beta\gamma} = \nabla_\alpha \mu_{\beta\gamma} - \nabla_\beta \mu_{\alpha\gamma},$$
$$A^\bullet_{\alpha\beta} = -\nabla_0 \mu_{\alpha\beta}, \qquad R^\bullet_{\alpha\overline{\beta}} = -\nabla_\alpha \nabla^{\overline{\sigma}} \mu_{\overline{\beta}\overline{\sigma}} - \nabla_{\overline{\beta}} \nabla^\tau \mu_{\alpha\tau}.$$

Proof Since the both sides of the four equalities are all tensorial, we may take any frame to derive them. Let $\{Z_\alpha\}$ be a local frame of $T^{1,0}$ of the Heisenberg group such that

$$[Z_\alpha, Z_{\overline{\beta}}] = -i h_{\alpha\overline{\beta}} T, \qquad [Z_\alpha, Z_\beta] = [Z_\alpha, T] = 0$$

and

$$h_{\alpha\overline{\beta}} = \begin{cases} 1, & \text{if } \alpha = \beta, \\ 0, & \text{otherwise,} \end{cases}$$

where T is the Reeb vector field associated with θ. Then the differentials of the Lie brackets are given by

$$\begin{aligned} [\hat{Z}_\alpha, \hat{Z}_{\overline{\beta}}]^\bullet &= \big(\nabla_\alpha \mu_{\overline{\beta}}{}^{\sigma}\big) Z_\sigma - \big(\nabla_{\overline{\beta}} \mu_\alpha{}^{\overline{\tau}}\big) Z_{\overline{\tau}}, \\ [\hat{Z}_\alpha, \hat{Z}_\beta]^\bullet &= \big(\nabla_\alpha \mu_\beta{}^{\overline{\gamma}} - \nabla_{\overline{\beta}} \mu_\alpha{}^{\overline{\gamma}}\big) Z_{\overline{\gamma}}, \\ [\hat{Z}_\alpha, T]^\bullet &= -\big(\nabla_0 \mu_\alpha{}^{\overline{\gamma}}\big) Z_{\overline{\gamma}}. \end{aligned}$$

They immediately show that $h^\bullet_{\alpha\overline{\beta}} = 0$ and $N^\bullet{}_{\alpha\beta}{}^{\overline{\gamma}} = \nabla_\alpha \mu_\beta{}^{\overline{\gamma}} - \nabla_\beta \mu_\alpha{}^{\overline{\gamma}}$. The first structure equation (3.3) implies

$$A^\bullet{}_\alpha{}^{\overline{\beta}} = \theta^{\overline{\beta}}\big([\hat{Z}_\alpha, T]^\bullet\big) = -\nabla_0 \mu_\alpha{}^{\overline{\beta}}.$$

Similarly, we have

$$\omega^\bullet{}_\alpha{}^\beta (Z_{\overline{\gamma}}) = -\nabla_\alpha \mu_{\overline{\gamma}}{}^{\beta}, \qquad \omega^\bullet{}_\alpha{}^\beta (T) = 0,$$

and this together with $\omega^\bullet_{\alpha\overline{\beta}} + \omega^\bullet_{\overline{\beta}\alpha} = (dh_{\alpha\overline{\beta}})^\bullet = 0$ implies $\omega^\bullet{}_\alpha{}^\beta (Z_\gamma) = \nabla^\beta \mu_{\alpha\gamma}$. From (3.7) we have

$$R^\bullet_{\alpha\overline{\beta}} = Z_\alpha \omega^\bullet{}_\gamma{}^\gamma (Z_{\overline{\beta}}) - Z_{\overline{\beta}} \omega^\bullet{}_\gamma{}^\gamma (Z_\alpha) - \omega^\bullet{}_\gamma{}^\gamma \big([Z_\alpha, Z_{\overline{\beta}}]\big) = -\nabla_\alpha \nabla^{\overline{\gamma}} \mu_{\overline{\beta}\overline{\gamma}} - \nabla_{\overline{\beta}} \nabla^\gamma \mu_{\alpha\gamma}.$$

This completes the proof. □

Let g be a normal-form ACH metric for θ satisfying the condition in Theorem 6.2. Let

$$g_{00} = 1 + \varphi_{00}, \qquad g_{0\alpha} = \varphi_{0\alpha}, \qquad g_{\alpha\overline{\beta}} = h_{\alpha\overline{\beta}} + \varphi_{\alpha\overline{\beta}}, \qquad g_{\alpha\beta} = \varphi_{\alpha\beta}.$$

Then, as seen in Theorem 6.2,

$$\varphi[m]_{ij} := \frac{1}{m!}\big(\partial_\rho^m \varphi_{ij}\big)\big|_{\rho=0}, \qquad m \le 2n + 1 + a(i, j)$$

are uniquely determined. For the standard CR structure they completely vanish. We shall observe the differentials $\varphi[m]^\bullet_{ij}$ of $\varphi[m]_{ij}$. For notational convenience, we set $\varphi[m]_{ij} := 0$ for $m \le 0$ and

$$\chi_k(m) := \begin{cases} 1, & m = k, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 7.3 *The differentials $\varphi[m]^\bullet_{ij}$ of $\varphi[m]_{ij}$ at the standard CR structure satisfy*

$$
\begin{aligned}
0 &= -\frac{1}{8}\big(m^2-(2n+4)m-4n\big)\varphi[m]^\bullet_{00}+\frac{1}{2}(m-2)\varphi[m]^\bullet{}_\alpha{}^\alpha \\
&\quad + i\big(\nabla^\alpha\varphi[m-1]^\bullet_{0\alpha}-\nabla^{\overline{\alpha}}\varphi[m-1]^\bullet_{0\overline{\alpha}}\big)+\frac{1}{2}\Delta_b\varphi[m-2]^\bullet_{00} \\
&\quad + \big(\nabla_0\nabla^\alpha\varphi[m-3]^\bullet_{0\alpha}+\nabla_0\nabla^{\overline{\alpha}}\varphi[m-3]^\bullet_{0\overline{\alpha}}\big)-\nabla_0\nabla_0\varphi[m-4]^\bullet{}_\alpha{}^\alpha, \\
0 &= -\chi_3(m)\nabla_0\nabla^\beta\mu_{\alpha\beta}-\frac{1}{8}(m+1)(m-2n-3)\varphi[m]^\bullet_{0\alpha}+\frac{3i}{4}\nabla_\alpha\varphi[m-1]^\bullet_{00} \\
&\quad + \frac{i}{2}\nabla_\alpha\varphi[m-1]^\bullet{}_\beta{}^\beta-i\nabla^{\overline{\beta}}\varphi[m-1]^\bullet_{\alpha\overline{\beta}}+\frac{1}{2}\Delta_b\varphi[m-2]^\bullet_{0\alpha} \\
&\quad - \frac{i}{2}\nabla_0\varphi[m-2]^\bullet_{0\alpha}+\frac{1}{2}\big(\nabla_\alpha\nabla^\beta\varphi[m-2]^\bullet_{0\beta}+\nabla_\alpha\nabla^{\overline{\beta}}\varphi[m-2]^\bullet_{0\overline{\beta}}\big) \\
&\quad - \nabla_0\nabla_\alpha\varphi[m-3]^\bullet{}_\beta{}^\beta+\frac{1}{2}\big(\nabla_0\nabla^{\overline{\beta}}\varphi[m-3]^\bullet_{\alpha\overline{\beta}}+\nabla_0\nabla^\beta\varphi[m-3]^\bullet_{\alpha\beta}\big), \\
0 &= -\chi_2(m)\big(\nabla_\alpha\nabla^{\overline{\gamma}}\mu_{\overline{\beta}\overline{\gamma}}+\nabla_{\overline{\beta}}\nabla^\gamma\mu_{\alpha\gamma}\big)-\frac{1}{8}\big(m^2-(2n+2)m-8\big)\varphi[m]^\bullet_{\alpha\overline{\beta}} \\
&\quad + \frac{1}{8}h_{\alpha\overline{\beta}}(m-4)\varphi[m]^\bullet_{00}+\frac{1}{4}h_{\alpha\overline{\beta}}m\varphi[m]^\bullet{}_\gamma{}^\gamma \\
&\quad + i\big(\nabla_\alpha\varphi[m-1]^\bullet_{0\overline{\beta}}-\nabla_{\overline{\beta}}\varphi[m-1]^\bullet_{0\alpha}\big)-\frac{i}{4}h_{\alpha\overline{\beta}}\nabla_0\varphi[m-2]^\bullet_{00} \\
&\quad - \frac{i}{2}h_{\alpha\overline{\beta}}\nabla_0\varphi[m-2]^\bullet{}_\gamma{}^\gamma-\frac{1}{2}\nabla_\alpha\nabla_{\overline{\beta}}\varphi[m-2]^\bullet_{00}-\nabla_\alpha\nabla_{\overline{\beta}}\varphi[m-2]^\bullet{}_\gamma{}^\gamma \\
&\quad + \frac{1}{2}\Delta_b\varphi[m-2]^\bullet_{\alpha\overline{\beta}}+\frac{1}{2}\big(\nabla_\alpha\nabla^\gamma\varphi[m-2]^\bullet_{\overline{\beta}\gamma}+\nabla_\alpha\nabla^{\overline{\gamma}}\varphi[m-2]^\bullet_{\overline{\beta}\overline{\gamma}} \\
&\quad + \nabla_{\overline{\beta}}\nabla^{\overline{\gamma}}\varphi[m-2]^\bullet_{\alpha\overline{\gamma}}+\nabla_{\overline{\beta}}\nabla^\gamma\varphi[m-2]^\bullet_{\alpha\gamma}\big) \\
&\quad + \frac{1}{2}\big(\nabla_0\nabla_\alpha\varphi[m-3]^\bullet_{0\overline{\beta}}+\nabla_0\nabla_{\overline{\beta}}\varphi[m-3]^\bullet_{0\alpha}\big)-\frac{1}{2}\nabla_0\nabla_0\varphi[m-4]^\bullet_{\alpha\overline{\beta}}, \\
0 &= -\chi_2(m)\big(\Delta_b\mu_{\alpha\beta}+\nabla_\alpha\nabla^\gamma\mu_{\beta\gamma}+\nabla_\beta\nabla^\gamma\mu_{\alpha\gamma}+2i\nabla_0\mu_{\alpha\beta}\big)+\chi_4(m)\nabla_0\nabla_0\mu_{\alpha\beta} \\
&\quad - \frac{1}{8}m(m-2n-2)\varphi[m]^\bullet_{\alpha\beta}-\frac{1}{2}\nabla_\alpha\nabla_\beta\varphi[m-2]^\bullet_{00}-\nabla_\alpha\nabla_\beta\varphi[m-2]^\bullet{}_\gamma{}^\gamma \\
&\quad + \frac{1}{2}\Delta_b\varphi[m-2]^\bullet_{\alpha\beta}+\frac{1}{2}\big(\nabla_\alpha\nabla^{\overline{\gamma}}\varphi[m-2]^\bullet_{\beta\overline{\gamma}}+\nabla_\alpha\nabla^\gamma\varphi[m-2]^\bullet_{\beta\gamma} \\
&\quad + \nabla_\beta\nabla^{\overline{\gamma}}\varphi[m-2]^\bullet_{\alpha\overline{\gamma}}+\nabla_\beta\nabla^\gamma\varphi[m-2]^\bullet_{\alpha\gamma}\big) \\
&\quad + i\nabla_0\varphi[m-2]^\bullet_{\alpha\beta}+\frac{1}{2}\big(\nabla_0\nabla_\alpha\varphi[m-3]^\bullet_{0\beta}+\nabla_0\nabla_\beta\varphi[m-3]^\bullet_{0\alpha}\big) \\
&\quad - \frac{1}{2}\nabla_0\nabla_0\varphi[m-4]^\bullet_{\alpha\beta},
\end{aligned}
$$

Table 14 Terms appearing in the differentials $\varphi[m]^\bullet_{ij}$ of the coefficients of the approximate normal-form ACH-Einstein metric

Type	Terms
$\varphi[2l]^\bullet_{00}$	$\Delta_b^k\nabla_0^{l-1-k}\nabla^\alpha\nabla^\beta\mu_{\alpha\beta}$, $\Delta_b^k\nabla_0^{l-1-k}\nabla^{\overline{\alpha}}\nabla^{\overline{\beta}}\mu_{\overline{\alpha\beta}}$
$\varphi[2l+1]^\bullet_{0\alpha}$	$\Delta_b^k\nabla_0^{l-k}\nabla^\beta\mu_{\alpha\beta}$, $\Delta_b^k\nabla_0^{l-1-k}\nabla_\alpha\nabla^\sigma\nabla^\tau\mu_{\sigma\tau}$, $\Delta_b^k\nabla_0^{l-1-k}\nabla_\alpha\nabla^{\overline{\sigma}}\nabla^{\overline{\tau}}\mu_{\overline{\sigma\tau}}$
$\varphi[2l]^\bullet_{\alpha\overline{\beta}}$	$\Delta_b^k\nabla_0^{l-1-k}\nabla_\alpha\nabla^{\overline{\sigma}}\mu_{\overline{\beta\sigma}}$, $\Delta_b^k\nabla_0^{l-1-k}\nabla_{\overline{\beta}}\nabla^\sigma\mu_{\alpha\sigma}$,
	$\Delta_b^k\nabla_0^{l-2-k}\nabla_\alpha\nabla_{\overline{\beta}}\nabla^\sigma\nabla^\tau\mu_{\sigma\tau}$, $\Delta_b^k\nabla_0^{l-2-k}\nabla_{\overline{\beta}}\nabla_\alpha\nabla^{\overline{\sigma}}\nabla^{\overline{\tau}}\mu_{\overline{\sigma\tau}}$,
	$h_{\alpha\overline{\beta}}\Delta_b^k\nabla_0^{l-1-k}\nabla^\sigma\nabla^\tau\mu_{\sigma\tau}$, $h_{\alpha\overline{\beta}}\Delta_b^k\nabla_0^{l-1-k}\nabla^{\overline{\sigma}}\nabla^{\overline{\tau}}\mu_{\overline{\sigma\tau}}$
$\varphi[2l]^\bullet_{\alpha\beta}$	$\Delta_b^k\nabla_0^{l-k}\mu_{\alpha\beta}$, $\Delta_b^k\nabla_0^{l-1-k}\nabla_{(\alpha}\nabla^\sigma\mu_{\beta)\sigma}$,
	$\Delta_b^k\nabla_0^{l-2-k}\nabla_\alpha\nabla_\beta\nabla^\sigma\nabla^\tau\mu_{\sigma\tau}$, $\Delta_b^k\nabla_0^{l-2-k}\nabla_\alpha\nabla_\beta\nabla^{\overline{\sigma}}\nabla^{\overline{\tau}}\mu_{\overline{\sigma\tau}}$

where in each equality m takes any nonnegative integer and ∇ denotes the Tanaka–Webster connection for the standard contact form θ.

Proof This follows from Lemma 4.2, because terms of type (N1)–(N3), which are neglected in the formulae recorded in that lemma, are at least quadratic in $\mu_{\alpha\beta}$. By setting $E_{IJ} = O(\rho^{2n+1+a(I,J)})$, the Taylor expansions of the last four equalities in Lemma 4.2 give the claimed formulae, thanks to Proposition 7.2. □

In principle we can calculate all $\varphi[m]^\bullet_{ij}$ using the recurrence formulae above. It is easy to see that $\varphi[m]^\bullet_{00} = \varphi[m]^\bullet_{\alpha\overline{\beta}} = \varphi[m]^\bullet_{\alpha\beta} = 0$ for m odd and $\varphi[m]^\bullet_{0\alpha} = 0$ for m even, and each nonzero $\varphi[m]^\bullet_{ij}$ is a linear combination over $\mathbb{C}$ of covariant derivatives of $\mu_{\alpha\beta}$ which are given in Table 14. As a result the differential $\mathcal{O}^\bullet_{\alpha\beta}$ of the obstruction tensor is a linear combination of

$$\Delta_b^k\nabla_0^{n+1-k}\mu_{\alpha\beta}, \qquad \Delta_b^k\nabla_0^{n-k}\nabla_{(\alpha}\nabla^\sigma\mu_{\beta)\sigma},$$
$$\Delta_b^k\nabla_0^{n-1-k}\nabla_\alpha\nabla_\beta\nabla^\sigma\nabla^\tau\mu_{\sigma\tau} \quad\text{and}\quad \Delta_b^k\nabla_0^{n-1-k}\nabla_\alpha\nabla_\beta\nabla^{\overline{\sigma}}\nabla^{\overline{\tau}}\mu_{\overline{\sigma\tau}},$$

which are linearly independent if $n \geq 2$.

Proposition 7.4 *Let $n \geq 2$ and*

$$\mathcal{O}^\bullet_{\alpha\beta} = \sum_{k=0}^{n+1} a_k\Delta_b^k\nabla_0^{n+1-k}\mu_{\alpha\beta} + \sum_{k=0}^{n} b_k\Delta_b^k\nabla_0^{n-k}\nabla_{(\alpha}\nabla^\sigma\mu_{\beta)\sigma}$$
$$+ \sum_{k=0}^{n-1} c_k\Delta_b^k\nabla_0^{n-1-k}\nabla_\alpha\nabla_\beta\nabla^\sigma\nabla^\tau\mu_{\sigma\tau} + \sum_{k=0}^{n-1} d_k\Delta_b^k\nabla_0^{n-1-k}\nabla_\alpha\nabla_\beta\nabla^{\overline{\sigma}}\nabla^{\overline{\tau}}\mu_{\overline{\sigma\tau}}.$$

Then $a_{n+1} = (-1)^{n+1}/(n!)^2$.

Proof The last equality in Lemma 7.3 and Table 14 show

$$0 \equiv -\chi_2(2l)\Delta_b\mu_{\alpha\beta} - \frac{1}{2}l(l-n-1)\varphi[2l]^\bullet_{\alpha\beta} + \frac{1}{2}\Delta_b\varphi[2l-2]^\bullet_{\alpha\beta}$$

modulo $\Delta_b^k\nabla_0^{l-k}\mu_{\alpha\beta}$, $k<l$, and

$$\Delta_b^k\nabla_0^{l-1-k}\nabla_{(\alpha}\nabla^\sigma\mu_{\beta)\sigma}, \qquad \Delta_b^k\nabla_0^{l-2-k}\nabla_\alpha\nabla_\beta\nabla^\sigma\nabla^\tau\mu_{\sigma\tau},$$
$$\Delta_b^k\nabla_0^{l-2-k}\nabla_\alpha\nabla_\beta\nabla^{\overline{\sigma}}\nabla^{\overline{\tau}}\mu_{\overline{\sigma\tau}}.$$

Hence we have $\varphi[2]^\bullet_{\alpha\beta} \equiv (2/n)\Delta_b\mu_{\alpha\beta}$ and

$$\varphi[2l]^\bullet_{\alpha\beta} \equiv -\frac{1}{l(n+1-l)}\Delta_b\varphi[2l-2]^\bullet_{\alpha\beta}.$$

This immediately shows that

$$\varphi[2l]^\bullet_{\alpha\beta} \equiv \frac{2}{n}\cdot\frac{-1}{2(n-1)}\cdot\frac{-1}{3(n-2)}\cdot\cdots\cdot\frac{-1}{l(n+1-l)}\Delta_b^l\mu_{\alpha\beta}, \quad l=1,2,\dots,n.$$

Then we use the last equality in Lemma 4.2 to see

$$\begin{aligned}\mathcal{O}^\bullet_{\alpha\beta} &\equiv \frac{1}{2}\Delta_b\varphi[2n]^\bullet_{\alpha\beta} \equiv \frac{1}{2}\cdot\frac{2}{n}\cdot\frac{-1}{2(n-1)}\cdot\frac{-1}{3(n-2)}\cdot\cdots\cdot\frac{-1}{n\cdot 1}\Delta_b^{n+1}\mu_{\alpha\beta}\\ &\equiv \frac{(-1)^{n+1}}{(n!)^2}\Delta_b^{n+1}\mu_{\alpha\beta},\end{aligned}$$

which implies the claim. □

Corollary 7.5 *Let $n \ge 2$. Then there is a partially integrable almost CR structure on the $(2n+1)$-dimensional Heisenberg group, arbitrarily close to the standard one, for which the obstruction tensor does not vanish.*

8 Formal Solutions Involving Logarithmic Singularities

Let X be a C^∞-smooth manifold-with-boundary and $\rho \in C^\infty(X)$ a boundary defining function. We say that a function $f \in C^0(X)\cap C^\infty(\mathring{X})$ belongs to $\mathcal{A}(X)$, or simply $\mathcal{A}$, if it admits an asymptotic expansion of the form (1.6). By this we mean that for any $m \ge 0$,

$$r_N := f - \sum_{q=0}^{N} f^{(q)}(\log\rho)^q \in C^m(X) \quad \text{and} \quad r_N = O\big(\rho^m\big)$$

holds for sufficiently large N. The Taylor expansions of $f^{(q)}$ at ∂X are uniquely determined; we write $f \in \mathcal{A}^m$ if $f^{(q)} = O(\rho^m)$, $q \ge 0$, and $\mathcal{A}^\infty := \bigcap_{m=0}^\infty \mathcal{A}^m$. The usage of the symbol $\mathcal{A}^m$ is similar to that of $O(\rho^m)$; for example, $f = f_0 + \mathcal{A}^m$ means that $f - f_0 \in \mathcal{A}^m$. One can show that $\mathcal{A}$ is closed under multiplication, and that if $f \in \mathcal{A}$ and f is nonzero everywhere then $f^{-1} \in \mathcal{A}$. Furthermore, $\mathcal{A}$ is closed under the actions of totally characteristic linear differential operators, i.e., noncommutative polynomials of smooth vector fields tangent to the boundary.

As in Sects. 4–6, again in this section X is an open neighborhood of M in $M \times [0, \infty)$, where $(M, T^{1,0})$ is a nondegenerate partially integrable almost CR manifold. We fix a pseudohermitian structure θ and consider (nonsmooth) Θ-metrics of the form (2.11) with $g_{ij} \in \mathcal{A}$ satisfying (2.12), which we call *singular normal-form ACH metrics* for $(M, T^{1,0})$ and θ.

All the calculations regarding the Ricci tensor go in the same way as in Sects. 4 and 5 except that, while on the space of smooth $O(\rho^m)$ functions $\rho\partial_\rho$ behaves as a mere "m times" operator modulo $O(\rho^{m+1})$, it is no longer the case when $O(\rho^m)$ and $O(\rho^{m+1})$ are replaced by $\mathcal{A}^m$ and $\mathcal{A}^{m+1}$. Nevertheless, since $\mathcal{A}$ is closed under the actions of totally characteristic operators, the Ricci tensors for singular normal-form ACH metrics have expansions of the form (1.6) with respect to the frame $\{\rho\partial_\rho, \rho^2 T, \rho Z_\alpha, \rho Z_{\overline{\alpha}}\}$.

Proposition 8.1 *There exists a singular normal-form ACH metric g satisfying*

$$E_{IJ} = \mathcal{A}^{2n+1+a(I,J)}, \tag{8.1}$$

where $a(I, J)$ is defined by (6.3). The components g_{ij} are uniquely determined, and do not contain logarithmic terms, modulo $\mathcal{A}^{2n+1+a(i,j)}$.

Proof This can be proved by following the argument in Sects. 4, 5, and the first half of Sect. 6 again. We shall include here a detailed account of the following fact only, which is a version of Proposition 4.1: $E_{IJ} = \mathcal{A}^3$ if and only if

$$\begin{aligned} &g_{00} = 1 + \mathcal{A}^3, \qquad g_{0\alpha} = \mathcal{A}^3, \qquad g_{\alpha\overline{\beta}} = h_{\alpha\overline{\beta}} + \rho^2 \Phi_{\alpha\overline{\beta}} + \mathcal{A}^3, \\ &g_{\alpha\beta} = \rho^2 \Phi_{\alpha\beta} + \mathcal{A}^3, \end{aligned}$$

where $\Phi_{\alpha\overline{\beta}}$ and $\Phi_{\alpha\beta}$ are defined by (4.4). Then the rest of the proof goes similarly.

Let g be given. If we define φ_{ij} by (4.5), then Lemma 4.2 is again valid. Take $N \geq 1$ large enough so that φ_{ij} and E_{IJ} for given g are of the form

$$\varphi_{ij} = \sum_{q=0}^{N} \varphi_{ij}^{(q)} (\log \rho)^q + \mathcal{A}^3, \quad \varphi_{ij}^{(q)} \in C^\infty(X),$$

and

$$E_{IJ} = \sum_{q=0}^{N} E_{IJ}^{(q)} (\log \rho)^q + \mathcal{A}^3, \quad E_{IJ}^{(q)} \in C^\infty(X).$$

Then by Lemma 4.2 we have the same identities as (4.12) between $E_{IJ}^{(N)}$ and $\varphi_{ij}^{(N)}$; namely, the following holds for $q=N$:

$$\begin{aligned}
&E_{\infty\infty}^{(q)}=\frac{3}{2}\varphi_{00}^{(q)}+\varphi^{(q)}{}_{\alpha}{}^{\alpha}+O(\rho^2),\\
&E_{\infty 0}^{(q)}=O(\rho^2),\qquad E_{\infty\alpha}^{(q)}=-i\varphi_{0\alpha}^{(q)}+O(\rho^2),\\
&E_{00}^{(q)}=\frac{3}{8}(2n+1)\varphi_{00}^{(q)}-\frac{1}{2}\varphi^{(q)}{}_{\alpha}{}^{\alpha}+O(\rho^2),\qquad E_{0\alpha}^{(q)}=\frac{1}{2}(n+1)\varphi_{0\alpha}^{(q)}+O(\rho^2),\\
&E_{\alpha\overline{\beta}}^{(q)}=\frac{1}{8}(2n+9)\varphi_{\alpha\overline{\beta}}^{(q)}-\frac{3}{8}h_{\alpha\overline{\beta}}\varphi_{00}^{(q)}+\frac{1}{4}h_{\alpha\overline{\beta}}\varphi^{(q)}{}_{\gamma}{}^{\gamma}+O(\rho^2),\\
&E_{\alpha\beta}^{(q)}=\frac{1}{8}(2n+1)\varphi_{\alpha\beta}^{(q)}+O(\rho^2).
\end{aligned}$$

Hence $\varphi_{ij}^{(N)}$ must be $O(\rho^2)$ so as to make $E_{IJ}^{(N)}=O(\rho^2)$. If $\varphi_{ij}^{(q)}=O(\rho^2)$, $q_0+1\le q\le N$, then the identities above hold for $q=q_0$, which shows that $E_{IJ}^{(q_0)}=O(\rho^2)$ is equivalent to $\varphi_{ij}^{(q_0)}=O(\rho^2)$. Hence we conclude that $E_{IJ}=\mathcal{A}^2$ if and only if $\varphi_{ij}=\mathcal{A}^2$.

Next, again by Lemma 4.2 we see that the following is true for $q=N$:

$$\begin{aligned}
&E_{\infty\infty}^{(q)}=2\varphi_{00}^{(q)}+O(\rho^3),\qquad E_{\infty 0}^{(q)}=O(\rho^3),\qquad E_{\infty\alpha}^{(q)}=-\frac{3}{2}i\varphi_{0\alpha}^{(q)}+O(\rho^3),\\
&E_{00}^{(q)}=\frac{1}{2}(2n+1)\varphi_{00}^{(q)}+O(\rho^3),\qquad E_{0\alpha}^{(q)}=\frac{3}{8}(2n+1)\varphi_{0\alpha}^{(q)}+O(\rho^3),\\
&E_{\alpha\overline{\beta}}^{(q)}=\frac{1}{2}(n+2)\varphi_{\alpha\overline{\beta}}^{(q)}-\frac{1}{4}h_{\alpha\overline{\beta}}\varphi_{00}^{(q)}+\frac{1}{2}h_{\alpha\overline{\beta}}\varphi^{(q)}{}_{\gamma}{}^{\gamma}+O(\rho^3),\\
&E_{\alpha\beta}^{(q)}=\frac{1}{2}n\varphi_{\alpha\beta}^{(q)}+O(\rho^3).
\end{aligned}$$

An inductive argument shows that $E_{IJ}^{(q)}=O(\rho^3)$, $1\le q\le N$, if and only if $\varphi_{ij}^{(q)}=O(\rho^3)$, $1\le q\le N$. Finally, the same identities as (4.13) hold for $E_{IJ}^{(0)}$ and $\varphi_{ij}^{(0)}$, which imply that $\varphi_{ij}^{(0)}$ must satisfy $\varphi_{00}^{(0)}=O(\rho^3)$, $\varphi_{0\alpha}^{(0)}=O(\rho^3)$, $\varphi_{\alpha\overline{\beta}}^{(0)}=\rho^2\Phi_{\alpha\overline{\beta}}+O(\rho^3)$, and $\varphi_{\alpha\beta}^{(0)}=\rho^2\Phi_{\alpha\beta}+O(\rho^3)$ as desired. □

Let $\overline{g}$ be such a normal-form ACH metric, and for specificity, let its components $\overline{g}_{ij}$ be polynomials of degree $2n+a(i,j)$ in ρ, which are uniquely determined. We set

$$\overline{E}_{IJ}=\rho^{2n+1+a(I,J)}F_{IJ}+O(\rho^{2n+2+a(I,J)}),\tag{8.2}$$

where F_{IJ} is constant in the ρ-direction. We already know that $F_{\alpha\beta}=\mathcal{O}_{\alpha\beta}$ and $F_{0\alpha}=-i\nabla^{\beta}\mathcal{O}_{\alpha\beta}-iN_{\alpha}{}^{\overline{\beta}\overline{\gamma}}\mathcal{O}_{\overline{\beta}\overline{\gamma}}$. Set

$$u:=-\frac{1}{n+1}\left(F_{\infty 0}-i\nabla^{\alpha}F_{\infty\alpha}+i\nabla^{\overline{\alpha}}F_{\infty\overline{\alpha}}\right).$$

Theorem 8.2 *Let κ be any smooth function and $\lambda_{\alpha\beta}$ a smooth symmetric* 2*-tensor satisfying*

$$D^{\alpha\beta}\lambda_{\alpha\beta} - D^{\overline{\alpha}\overline{\beta}}\lambda_{\overline{\alpha}\overline{\beta}} = iu. \tag{8.3}$$

Then there is a singular normal-form ACH metric g satisfying $E_{IJ} = \mathcal{A}^\infty$ and

$$\frac{1}{(2n+4)!}\big(\partial_\rho^{2n+4} g_{00}^{(0)}\big)\big|_M = \kappa, \qquad \frac{1}{(2n+2)!}\big(\partial_\rho^{2n+2} g_{\alpha\beta}^{(0)}\big)\big|_M = \lambda_{\alpha\beta}, \tag{8.4}$$

where $g_{ij} \sim \sum_{q=0}^\infty g_{ij}^{(q)}(\log\rho)^q$ is the asymptotic expansion of g_{ij}. The components g_{ij} are uniquely determined modulo $\mathcal{A}^\infty$ by the condition above.

As is clear from the proof below, Theorem 8.2 also holds in the following formal sense. Let $p \in M$, κ a smooth function, and $\lambda_{\alpha\beta}$ a tensor satisfying (8.3) to the infinite order at p. Then there exists a singular normal-form ACH metric g satisfying (8.4) and $E_{IJ} = \mathcal{A}^\infty$ to the infinite order at p, and the Taylor expansions of $g_{ij}^{(q)}$ at p are uniquely determined by those of κ and $\lambda_{\alpha\beta}$. On the other hand, there is a formal power series solution to (8.3) by the Cauchy–Kovalevskaya theorem. Hence, by Borel's lemma, we have $\lambda_{\alpha\beta}$ solving (8.3) to the infinite order at p and prove the first statement of Theorem 1.3. We do not know whether (8.3) is solvable in the category of smooth tensors.

Remark 8.3 The appearance of a formally undetermined term $\lambda_{\alpha\beta}$ at the $(2n+2)$-nd order generalizes a result of Biquard and Herzlich [4, Corollary 5.4] in the case $n = 1$.

The first step to prove Theorem 8.2 is the following.

Lemma 8.4 *There exists a singular normal-form ACH metric g satisfying*

$$\begin{aligned} &E_{\infty\infty} = \mathcal{A}^{2n+4}, \qquad E_{\infty 0} = \mathcal{A}^{2n+4}, \qquad E_{\infty\alpha} = \mathcal{A}^{2n+3}, \\ &E_{00} = \mathcal{A}^{2n+4}, \qquad E_{0\alpha} = \mathcal{A}^{2n+4}, \qquad E_{\alpha\overline{\beta}} = \mathcal{A}^{2n+4}, \qquad E_{\alpha\beta} = \mathcal{A}^{2n+3} \end{aligned}$$

and the followings not containing logarithmic terms:

$$\begin{aligned} &E_{\infty 0} \bmod \mathcal{A}^{2n+5}, \qquad E_{\infty\alpha} \bmod \mathcal{A}^{2n+4}, \\ &E_{00} \bmod \mathcal{A}^{2n+5}, \qquad E_{\alpha\overline{\beta}} \bmod \mathcal{A}^{2n+5}. \end{aligned}$$

Any such metric g is of the form

$$\begin{aligned} g_{00} &= \overline{g}_{00} + \psi_{00}^{(0)} + \psi_{00}^{(1)}\log\rho + \psi_{00}^{(2)}(\log\rho)^2 + \mathcal{A}^{2n+5}, \\ g_{0\alpha} &= \overline{g}_{0\alpha} + \psi_{0\alpha}^{(0)} + \psi_{0\alpha}^{(1)}\log\rho + \mathcal{A}^{2n+4}, \\ g_{\alpha\overline{\beta}} &= \overline{g}_{\alpha\overline{\beta}} + \psi_{\alpha\overline{\beta}}^{(0)} + \psi_{\alpha\overline{\beta}}^{(1)}\log\rho + \psi_{\alpha\overline{\beta}}^{(2)}(\log\rho)^2 + \mathcal{A}^{2n+5}, \\ g_{\alpha\beta} &= \overline{g}_{\alpha\beta} + \psi_{\alpha\beta}^{(0)} + \psi_{\alpha\beta}^{(1)}\log\rho + \mathcal{A}^{2n+3}, \end{aligned}$$

where $\psi_{ij}^{(q)} = O(\rho^{2n+1+a(i,j)})$. *Furthermore, among* $\psi_{ij}^{(q)}$,

$$\psi_{00}^{(2)}, \quad \psi_{\alpha\overline{\beta}}^{(2)}, \quad \psi_{0\alpha}^{(1)}, \quad \mathrm{tf}\big(\psi_{\alpha\overline{\beta}}^{(1)}\big), \quad \psi_{\alpha\beta}^{(1)} \quad \textit{and} \quad \frac{1}{2}n\psi_{00}^{(1)} + (n+1){\psi^{(1)}}_{\alpha}{}^{\alpha}$$

are uniquely determined modulo $O(\rho^{2n+2+a(i,j)})$. *In particular, if* $\mathcal{O}_{\alpha\beta} = 0$ *then they are zero modulo* $O(\rho^{2n+2+a(i,j)})$.

Proof We shall determine when

$$g_{ij} = \overline{g}_{ij} + \sum_{q=0}^{N} \psi_{ij}^{(q)} (\log\rho)^q, \quad \psi_{ij}^{(q)} = O\big(\rho^{2n+1+a(i,j)}\big) \tag{8.5}$$

enjoys the condition imposed. By (5.2) and (5.3a)–(5.3e), which are also valid here if $O(\rho^{m'})$ is replaced by $\mathcal{A}^{m'}$, the difference δE_{IJ} between the tensors E of $\overline{g}$ and g is of the form

$$\delta E_{IJ} = \sum_{q=0}^{N} \delta E_{IJ}^{(q)} (\log\rho)^q + \mathcal{A}^{2n+2+a(I,J)}.$$

We may assume $N \geq 3$. Then, by (5.2) we have $\delta E_{0\alpha}^{(N)} = O(\rho^{2n+4})$ and $\delta E_{\alpha\beta}^{(N)} = O(\rho^{2n+3})$, which imply that $E_{0\alpha}^{(N)} = O(\rho^{2n+4})$, $E_{\alpha\beta}^{(N)} = O(\rho^{2n+3})$ already hold, and

$$\begin{aligned} \delta E_{0\alpha}^{(q-1)} &= -\frac{1}{4}q(n+2)\psi_{0\alpha}^{(q)} + O\big(\rho^{2n+4}\big), \\ \delta E_{\alpha\beta}^{(q-1)} &= -\frac{1}{4}q(n+1)\psi_{\alpha\beta}^{(q)} + O\big(\rho^{2n+3}\big) \end{aligned} \tag{8.6}$$

for $q = N$. This shows that $E_{0\alpha}^{(N-1)} = O(\rho^{2n+4})$, $E_{\alpha\beta}^{(N-1)} = O(\rho^{2n+3})$ if and only if $\psi_{0\alpha}^{(N)} = O(\rho^{2n+4})$, $\psi_{\alpha\beta}^{(N)} = O(\rho^{2n+3})$. Since (8.6) holds for $q = q_0$ if $\psi_{0\alpha}^{(q)} = O(\rho^{2n+4})$ and $\psi_{\alpha\beta}^{(q)} = O(\rho^{2n+3})$ for $q_0 + 1 \leq q \leq N$, inductively we verify that $E_{0\alpha} = \mathcal{A}^{2n+4}$, $E_{\alpha\beta} = \mathcal{A}^{2n+3}$ if and only if $\psi_{0\alpha}^{(q)} = O(\rho^{2n+4})$, $\psi_{\alpha\beta}^{(q)} = O(\rho^{2n+3})$, $2 \leq q \leq N$ and

$$\psi_{0\alpha}^{(1)} = \frac{4}{n+2}\rho^{2n+3}F_{0\alpha} + O\big(\rho^{2n+4}\big), \qquad \psi_{\alpha\beta}^{(1)} = \frac{4}{n+1}\rho^{2n+2}F_{\alpha\beta} + O\big(\rho^{2n+3}\big). \tag{8.7}$$

Next, from (5.3c)–(5.3e) we have $n\delta E_{00}^{(N)} - 2\delta {E^{(N)}}_{\alpha}{}^{\alpha} = O(\rho^{2n+5})$ and

$$\begin{aligned} \delta E_{00}^{(q)} &= \frac{1}{2}n\psi_{00}^{(q)} + (n+1){\psi^{(q)}}_{\alpha}{}^{\alpha} + O\big(\rho^{2n+5}\big), \\ \mathrm{tf}\big(\delta E_{\alpha\overline{\beta}}^{(q)}\big) &= -\frac{1}{2}n\,\mathrm{tf}\big(\psi_{\alpha\overline{\beta}}^{(q)}\big) + O\big(\rho^{2n+5}\big), \\ n\delta E_{00}^{(q-1)} - 2\delta {E^{(q-1)}}_{\alpha}{}^{\alpha} &= -\frac{1}{4}q(n+3)\big(n\psi_{00}^{(q)} - 2{\psi^{(q)}}_{\alpha}{}^{\alpha}\big) + O\big(\rho^{2n+5}\big) \end{aligned}$$

for $q = N$. Hence both $\psi^{(N)}_{00}$ and $\psi^{(N)}_{\alpha\overline{\beta}}$ must be $O(\rho^{2n+5})$. Inductively we show that, in order for us to have $E^{(q)}_{00} = O(\rho^{2n+5})$, $E^{(q)}_{\alpha\overline{\beta}} = O(\rho^{2n+5})$, $2 \le q \le N$, it is necessary and sufficient that $\psi^{(q)}_{00}$, $\psi^{(q)}_{\alpha\overline{\beta}}$, $3 \le q \le N$, and $\frac{1}{2}n\psi^{(2)}_{00} + (n+1){\psi^{(2)}}_{\alpha}{}^{\alpha}$, $\operatorname{tf}(\psi^{(2)}_{\alpha\overline{\beta}})$ are all $O(\rho^{2n+5})$.

Again by (5.3c)–(5.3e), modulo $O(\rho^{2n+4})$ terms which linearly depend on $\psi^{(2)}_{00}$, $\psi^{(1)}_{0\alpha}$ and $\psi^{(1)}_{\alpha\beta}$,

$$\delta E^{(1)}_{00} \equiv \frac{1}{2}n\psi^{(1)}_{00} + (n+1){\psi^{(1)}}_{\alpha}{}^{\alpha} + O\big(\rho^{2n+5}\big),$$

$$\operatorname{tf}\big(\delta E^{(1)}_{\alpha\overline{\beta}}\big) \equiv -\frac{1}{2}n\operatorname{tf}\big(\psi^{(1)}_{\alpha\overline{\beta}}\big) + O\big(\rho^{2n+5}\big),$$

$$n\delta E^{(1)}_{00} - 2\delta {E^{(1)}}_{\alpha}{}^{\alpha} \equiv -\frac{1}{2}(n+3)\big(n\psi^{(2)}_{00} - 2{\psi^{(2)}}_{\alpha}{}^{\alpha}\big) + O\big(\rho^{2n+5}\big).$$

Therefore $\psi^{(2)}_{00}$, ${\psi^{(2)}}_{\alpha}{}^{\alpha}$, $\operatorname{tf}(\psi^{(1)}_{\alpha\overline{\beta}})$, and $\frac{1}{2}n\psi^{(1)}_{00} + (n+1){\psi^{(1)}}_{\alpha}{}^{\alpha}$ are uniquely determined modulo $O(\rho^{2n+5})$ by the requirement $E^{(1)}_{00} = O(\rho^{2n+5})$, $E^{(1)}_{\alpha\overline{\beta}} = O(\rho^{2n+5})$.

For g_{ij} satisfying all the restrictions we have found above, $E_{\infty 0}$ and $E_{\infty\alpha}$ do not contain logarithmic terms modulo $\mathcal{A}^{2n+5}$ and $\mathcal{A}^{2n+4}$, respectively; one can show this fact by (6.2b) and (6.2c), or by (6.10). If $\mathcal{O}_{\alpha\beta} = 0$, (8.7) implies that $\psi^{(1)}_{0\alpha}$ and $\psi^{(1)}_{\alpha\beta}$ are zero, and hence $\psi^{(2)}_{00}$, ${\psi^{(2)}}_{\alpha}{}^{\alpha}$, $\operatorname{tf}(\psi^{(1)}_{\alpha\overline{\beta}})$, and $\frac{1}{2}n\psi^{(1)}_{00} + (n+1){\psi^{(1)}}_{\alpha}{}^{\alpha}$ are also zero. □

The rest of the proof of Theorem 8.2 consists of two parts, in the first of which we finish constructing a singular normal-form ACH metric satisfying $E_{IJ} = \mathcal{A}^{2n+2+a(I,J)}$, and in the second we go through an inductive argument to achieve $E_{IJ} = \mathcal{A}^{\infty}$.

Proof of Theorem 8.2 Let g be a singular normal-form ACH metric we have obtained in Lemma 8.4. By (5.3b), (5.2), and (8.7) we have

$$\begin{aligned}
\delta E^{(0)}_{\infty 0} &= (n+2)\rho\big(\nabla^{\alpha}\psi^{(0)}_{0\alpha} + \nabla^{\overline{\alpha}}\psi^{(0)}_{0\overline{\alpha}}\big) - (n+1)\rho^2\big(A^{\alpha\beta}\psi^{(0)}_{\alpha\beta} + A^{\overline{\alpha}\overline{\beta}}\psi^{(0)}_{\overline{\alpha}\overline{\beta}}\big)\\
&\quad + \rho^{2n+4}\left(\frac{2}{n+2}\big(\nabla^{\alpha}F_{0\alpha} + \nabla^{\overline{\alpha}}F_{0\overline{\alpha}}\big) - \frac{2}{n+1}\big(A^{\alpha\beta}F_{\alpha\beta} + A^{\overline{\alpha}\overline{\beta}}F_{\overline{\alpha}\overline{\beta}}\big)\right)\\
&\quad + O\big(\rho^{2n+5}\big),\\
\delta E^{(0)}_{\infty\alpha} &= -i(n+2)\psi^{(0)}_{0\alpha} + (n+1)\rho\nabla^{\beta}\psi^{(0)}_{\alpha\beta} + (n+1)\rho {N_{\alpha}}^{\overline{\beta}\overline{\gamma}}\psi^{(0)}_{\overline{\beta}\overline{\gamma}}\\
&\quad - \rho^{2n+3}\left(\frac{2}{n+2}iF_{0\alpha} - \frac{2}{n+1}\big(\nabla^{\beta}F_{\alpha\beta} + {N_{\alpha}}^{\overline{\beta}\overline{\gamma}}F_{\overline{\beta}\overline{\gamma}}\big)\right) + O\big(\rho^{2n+4}\big).
\end{aligned}$$

If we set $\psi^{(0)}_{0\alpha} = \rho^{2n+3}\nu_{\alpha} + O(\rho^{2n+4})$ and $\psi^{(0)}_{\alpha\beta} = \rho^{2n+2}\mu_{\alpha\beta} + O(\rho^{2n+3})$, then attaining $E^{(0)}_{\infty 0} = O(\rho^{2n+5})$ and $E^{(0)}_{\infty\alpha} = O(\rho^{2n+4})$ is equivalent to solving the following system of PDEs:

$$\begin{cases} (n+2)(\nabla^{\alpha}\nu_{\alpha}+\nabla^{\overline{\alpha}}\nu_{\overline{\alpha}})-(n+1)(A^{\alpha\beta}\mu_{\alpha\beta}+A^{\overline{\alpha}\overline{\beta}}\mu_{\overline{\alpha}\overline{\beta}}) \\ \quad =-F_{\infty 0}-\frac{2}{n+2}(\nabla^{\alpha}F_{0\alpha}+\nabla^{\overline{\alpha}}F_{0\overline{\alpha}})+\frac{2}{n+1}(A^{\alpha\beta}F_{\alpha\beta}+A^{\overline{\alpha}\overline{\beta}}F_{\overline{\alpha}\overline{\beta}}), \\ -i(n+2)\nu_{\alpha}+(n+1)\nabla^{\beta}\mu_{\alpha\beta}+(n+1)N_{\alpha}{}^{\overline{\beta}\overline{\gamma}}\mu_{\overline{\beta}\overline{\gamma}} \\ \quad =-F_{\infty\alpha}+\frac{2}{n+2}iF_{0\alpha}-\frac{2}{n+1}(\nabla^{\beta}F_{\alpha\beta}+N_{\alpha}{}^{\overline{\beta}\overline{\gamma}}F_{\overline{\beta}\overline{\gamma}}). \end{cases} \tag{8.8}$$

If we substitute the second equation into the first one and use $F_{\alpha\beta}=\mathcal{O}_{\alpha\beta}$ and (6.10), the system is reduced to $D^{\alpha\beta}\mu_{\alpha\beta}-D^{\overline{\alpha}\overline{\beta}}\mu_{\overline{\alpha}\overline{\beta}}=iu$. Hence, by setting $\mu_{\alpha\beta}=\lambda_{\alpha\beta}$ and determining ν_{α} by (8.8) we achieve $E^{(0)}_{\infty 0}=O(\rho^{2n+5})$ and $E^{(0)}_{\infty\alpha}=O(\rho^{2n+4})$.

Having fixed $\psi^{(0)}_{0\alpha}$ and $\psi^{(0)}_{\alpha\beta}$, now we may determine $\psi^{(1)}_{00}$, $\psi^{(1)}{}_{\alpha}{}^{\alpha}$, $\mathrm{tf}(\psi^{(0)}_{\alpha\overline{\beta}})$, and $\frac{1}{2}n\psi^{(0)}_{00}+(n+1)\psi^{(0)}{}_{\alpha}{}^{\alpha}$ modulo $O(\rho^{2n+5})$ so that $E^{(0)}_{00}$, $E^{(0)}_{\alpha\overline{\beta}}$ are $O(\rho^{2n+5})$ by observing (5.3c)–(5.3e). It automatically holds that $E_{\infty\infty}=\mathcal{A}^{2n+5}$ by (6.2a). Although $\frac{1}{2}n\psi^{(0)}_{00}+(n+1)\psi^{(0)}{}_{\alpha}{}^{\alpha}$ is fixed, $\psi^{(0)}_{00}$ (or $\psi^{(0)}{}_{\alpha}{}^{\alpha}$) is remaining to be free, so we prescribe it by $\psi^{(0)}_{00}=\rho^{2n+4}\kappa+O(\rho^{2n+5})$.

We have shown that there is a singular normal-form ACH metric satisfying $E_{IJ}=\mathcal{A}^{2n+2+a(I,J)}$, and that if we impose the condition (8.4) then g_{ij} are unique modulo $\mathcal{A}^{2n+2+a(i,j)}$. Let $m\geq 2n+3$ and suppose that g is a singular normal-form ACH metric satisfying $E_{IJ}=\mathcal{A}^{m-1+a(I,J)}$. We set

$$g'_{ij}=g_{ij}+\sum_{q=0}^{N}\psi^{(q)}_{ij}(\log\rho)^{q},$$

where $\psi^{(q)}_{ij}=O(\rho^{m-1+a(i,j)})$, and shall prove that $\psi^{(q)}_{ij}$ mod $O(\rho^{m+a(i,j)})$ may be uniquely determined so that $E'_{IJ}=\mathcal{A}^{m+a(I,J)}$ holds. Then the induction works and we obtain the theorem.

By replacing N with a larger one if necessary, we express $\delta E=E'-E$ as

$$\delta E_{IJ}=\sum_{q=0}^{N}\delta E^{(q)}_{IJ}(\log\rho)^{q}+\mathcal{A}^{m+a(I,J)}.$$

Then by (5.2) and (5.3a)–(5.3e) we have, modulo terms linearly depending on $\psi^{(q+2)}_{ij}$ or $\psi^{(q+1)}_{ij}$,

$$\begin{aligned} \delta E^{(q)}_{00} \equiv & -\frac{1}{8}\big(m^{2}-2nm-8n-4\big)\psi^{(q)}_{00}+\frac{1}{2}m\psi^{(q)}{}_{\alpha}{}^{\alpha} \\ & +\big(O\big(\rho^{m+2}\big)\text{ terms depending on }\psi^{(q)}_{0\alpha}\text{ and }\psi^{(q)}_{\alpha\beta}\big) \\ & +O\big(\rho^{m+3}\big) \end{aligned} \tag{8.9a}$$

$$\delta E^{(q)}_{0\alpha} \equiv -\frac{1}{8}(m+2)(m-2n-2)\psi^{(q)}_{0\alpha} + O\big(\rho^{m+2}\big), \tag{8.9b}$$

$$\begin{aligned}\delta E^{(q)}{}_{\alpha}{}^{\alpha} \equiv{}& \frac{1}{8}n(m-2)\psi^{(q)}_{00} - \frac{1}{8}\big(m^2-(4n-2)m-8n-8\big)\psi^{(q)}{}_{\alpha}{}^{\alpha} \\ &+ \big(O\big(\rho^{m+2}\big) \text{ terms depending on } \psi^{(q)}_{0\alpha} \text{ and } \psi^{(q)}_{\alpha\beta}\big) \\ &+ O\big(\rho^{m+3}\big),\end{aligned} \tag{8.9c}$$

$$\begin{aligned}\operatorname{tf}\big(\delta E^{(q)}_{\alpha\overline{\beta}}\big) \equiv{}& -\frac{1}{8}\big(m^2-2nm-2n-9\big)\operatorname{tf}\big(\psi^{(q)}_{\alpha\overline{\beta}}\big) \\ &+ \big(O\big(\rho^{m+2}\big) \text{ terms depending on } \psi^{(q)}_{0\alpha} \text{ and } \psi^{(q)}_{\alpha\beta}\big) \\ &+ O\big(\rho^{m+3}\big),\end{aligned} \tag{8.9d}$$

$$\delta E^{(q)}_{\alpha\beta} \equiv -\frac{1}{8}m(m-2n-2)\psi^{(q)}_{\alpha\beta} + O\big(\rho^{m+1}\big). \tag{8.9e}$$

By (6.6), if $m \neq 4n+2$, we may determine $\psi^{(N)}_{ij}$, $\psi^{(N-1)}_{ij}$, …, $\psi^{(0)}_{ij}$ inductively so that $E'_{ij} = \mathcal{A}^{m+a(i,j)}$ hold. Then by (6.2a)–(6.2c) it automatically holds that $E'_{\infty\infty} = \mathcal{A}^{m+3}$, $E'_{\infty 0} = \mathcal{A}^{m+3}$ and $E'_{\infty\alpha} = \mathcal{A}^{m+2}$. If $m = 4n+2$, instead of (8.9c) we use

$$\begin{aligned}\delta E^{(q)}_{\infty\infty} \equiv{}& -8n(n+1)\psi^{(q)}_{00} - 8(n+1)(2n+1)\psi^{(q)}{}_{\alpha}{}^{\alpha} \\ &+ \big(O\big(\rho^{4n+4}\big) \text{ terms depending on } \psi^{(q)}_{\alpha\beta}\big) + O\big(\rho^{4n+5}\big),\end{aligned}$$

which holds modulo $\psi^{(q+2)}_{ij}$ and $\psi^{(q+1)}_{ij}$. We may determine $\psi^{(N)}_{ij}$, $\psi^{(N-1)}_{ij}$, …, $\psi^{(0)}_{ij}$ inductively so that $E'_{\infty\infty} = \mathcal{A}^{4n+5}$, $E'_{00} = \mathcal{A}^{4n+5}$, $E'_{0\alpha} = \mathcal{A}^{4n+4}$, $\operatorname{tf}(E'_{\alpha\overline{\beta}}) = \mathcal{A}^{4n+5}$, and $E'_{\alpha\beta} = \mathcal{A}^{4n+4}$. By (6.2a)–(6.2c), we obtain $E'{}_{\alpha}{}^{\alpha} = \mathcal{A}^{4n+3}$, $E'_{\infty 0} = \mathcal{A}^{4n+5}$, and $E'_{\infty\alpha} = \mathcal{A}^{4n+4}$. □

Finally, we shall discuss constructing a completely log-free solution when $\mathcal{O}_{\alpha\beta} = 0$. We set

$$\begin{aligned}v :={}& -F_{00} + \frac{2}{n}F_{\alpha}{}^{\alpha} - \frac{1}{n}\big(\nabla^{\alpha}F_{\infty\alpha} + \nabla^{\overline{\alpha}}F_{\infty\overline{\alpha}}\big) + \frac{2}{n(n+2)}i\big(\nabla^{\alpha}F_{0\alpha} - \nabla^{\overline{\alpha}}F_{0\overline{\alpha}}\big) \\ &- \frac{2}{n(n+1)}(\nabla^{\alpha}\nabla^{\beta}F_{\alpha\beta} + \nabla^{\overline{\alpha}}\nabla^{\overline{\beta}}F_{\overline{\alpha}\overline{\beta}} + N^{\gamma\alpha\beta}\nabla_{\gamma}F_{\alpha\beta} + N^{\overline{\gamma}\overline{\alpha}\overline{\beta}}\nabla_{\overline{\gamma}}F_{\overline{\alpha}\overline{\beta}} \\ &+ N^{\gamma\alpha\beta}{}_{,\gamma}F_{\alpha\beta} + N^{\overline{\gamma}\overline{\alpha}\overline{\beta}}{}_{,\overline{\gamma}}F_{\overline{\alpha}\overline{\beta}}).\end{aligned}$$

Theorem 8.5 *Suppose that $\mathcal{O}_{\alpha\beta} = 0$. Let κ be a smooth function and $\lambda_{\alpha\beta}$ a smooth symmetric 2-tensor satisfying*

$$\begin{cases} D^{\alpha\beta}\lambda_{\alpha\beta} - D^{\overline{\alpha}\overline{\beta}}\lambda_{\overline{\alpha}\overline{\beta}} = iu, \\ D^{\alpha\beta}_{-2/n}\lambda_{\alpha\beta} + D^{\overline{\alpha}\overline{\beta}}_{-2/n}\lambda_{\overline{\alpha}\overline{\beta}} = v. \end{cases} \tag{8.10}$$

Then there is a normal-form ACH metric g, which is free of logarithmic terms, satisfying $E_{IJ} = \mathcal{A}^\infty$ and

$$\frac{1}{(2n+4)!}\left(\partial_\rho^{2n+4} g_{00}\right)\big|_M = \kappa, \qquad \frac{1}{(2n+2)!}\left(\partial_\rho^{2n+2} g_{\alpha\beta}\right)\big|_M = \lambda_{\alpha\beta}. \tag{8.11}$$

The Taylor expansions of the components g_{ij} at ∂X are unique.

Again this theorem also holds in the formal sense. Since the principal parts of $D^{\alpha\beta}$ and $D^{\alpha\beta}_{-2/n}$ agree, the system (8.10) is formally solvable at any given point; in fact, if one arbitrarily prescribes the components of $\lambda_{\alpha\beta}$ except λ_{11}, for example, and writes $\lambda_{11} = \mu + i\nu$ where μ and ν are real-valued, then (8.10) can be regarded as a system of PDEs for μ and ν and the Cauchy–Kovalevskaya theorem can be applied to this system. Thus we can show the second statement of Theorem 1.3.

Proof If $\mathcal{O}_{\alpha\beta} = 0$, then a (potentially) singular normal-form ACH metric g satisfying the conditions in the statement of Lemma 8.4 is of the form

$$\begin{aligned}
g_{00} &= \overline{g}_{00} + \psi^{(0)}_{00} + \psi^{(1)}_{00} \log\rho + \mathcal{A}^{2n+5},\\
g_{0\alpha} &= \overline{g}_{0\alpha} + \psi^{(0)}_{0\alpha} + \mathcal{A}^{2n+4},\\
g_{\alpha\overline{\beta}} &= \overline{g}_{\alpha\overline{\beta}} + \psi^{(0)}_{\alpha\overline{\beta}} + \frac{1}{n} h_{\alpha\overline{\beta}} {\psi^{(1)}}_\gamma{}^\gamma \log\rho + \mathcal{A}^{2n+5},\\
g_{\alpha\beta} &= \overline{g}_{\alpha\beta} + \psi^{(0)}_{\alpha\beta} + \mathcal{A}^{2n+3}.
\end{aligned}$$

Here $\frac{1}{2} n\psi^{(1)}_{00} + (n+1){\psi^{(1)}}_\alpha{}^\alpha = O(\rho^{2n+5})$ should hold. After prescribing $\psi^{(0)}_{0\alpha}$ and $\psi^{(0)}_{\alpha\beta}$, the potential log-term coefficients $\psi^{(1)}_{00}$ and ${\psi^{(1)}}_\alpha{}^\alpha$ are determined by requiring $nE^{(0)}_{00} - 2{E^{(0)}}_\alpha{}^\alpha = O(\rho^{2n+5})$. So let us look at the dependence of $nE^{(0)}_{00} - 2{E^{(0)}}_\alpha{}^\alpha$ on $\psi^{(0)}_{\alpha\beta}$. Using (5.3c) and (5.3d) again, we obtain

$$\begin{aligned}
&n\delta E^{(0)}_{00} - 2\delta {E^{(0)}}_\alpha{}^\alpha\\
&\quad= -\frac{1}{2}(n+2)\left(n\psi^{(1)}_{00} - 2{\psi^{(1)}}_\alpha{}^\alpha\right)\\
&\qquad + i(n+2)\rho\left(\nabla^\alpha\psi^{(0)}_{0\alpha} - \nabla^{\overline{\alpha}}\psi^{(0)}_{0\overline{\alpha}}\right) - \frac{1}{2}n\rho^2\left(\Phi^{\alpha\beta}\psi^{(0)}_{\alpha\beta} + \Phi^{\overline{\alpha}\overline{\beta}}\psi^{(0)}_{\overline{\alpha}\overline{\beta}}\right)\\
&\qquad - \rho^2\left(\nabla^\alpha\nabla^\beta\psi^{(0)}_{\alpha\beta} + \nabla^{\overline{\alpha}}\nabla^{\overline{\beta}}\psi^{(0)}_{\overline{\alpha}\overline{\beta}} + N^{\gamma\alpha\beta}\nabla_\gamma\psi^{(0)}_{\alpha\beta} + N^{\overline{\gamma}\overline{\alpha}\overline{\beta}}\nabla_{\overline{\gamma}}\psi^{(0)}_{\overline{\alpha}\overline{\beta}}\right.\\
&\qquad \left. + {N^{\gamma\alpha\beta}}_{,\gamma}\psi^{(0)}_{\alpha\beta} + {N^{\overline{\gamma}\overline{\alpha}\overline{\beta}}}_{,\overline{\gamma}}\psi^{(0)}_{\overline{\alpha}\overline{\beta}}\right) + O\left(\rho^{2n+5}\right).
\end{aligned}$$

Hence, if we can set $\psi^{(0)}_{0\alpha}$ and $\psi^{(0)}_{\alpha\beta}$ appropriately, then $\psi^{(1)}_{00} - 2{\psi^{(1)}}_\alpha{}^\alpha$ must be $O(\rho^{2n+5})$, and hence both $\psi^{(1)}_{00}$ and ${\psi^{(1)}}_\alpha{}^\alpha$ must be $O(\rho^{2n+5})$. Let $\psi^{(0)}_{0\alpha} = \rho^{2n+3}\nu_\alpha + O(\rho^{2n+4})$ and $\psi^{(0)}_{\alpha\beta} = \rho^{2n+2}\mu_{\alpha\beta} + O(\rho^{2n+3})$. Combined with (8.8), the equations to be solved are

$$\begin{cases} (n+2)(\nabla^\alpha \nu_\alpha + \nabla^{\overline{\alpha}} \nu_{\overline{\alpha}}) - (n+1)(A^{\alpha\beta}\mu_{\alpha\beta} + A^{\overline{\alpha}\overline{\beta}}\mu_{\overline{\alpha}\overline{\beta}}) \\ \quad = -F_{\infty 0} - \frac{2}{n+2}(\nabla^\alpha F_{0\alpha} + \nabla^{\overline{\alpha}} F_{0\overline{\alpha}}) + \frac{2}{n+1}(A^{\alpha\beta}F_{\alpha\beta} + A^{\overline{\alpha}\overline{\beta}}F_{\overline{\alpha}\overline{\beta}}), \\ \qquad -i(n+2)\nu_\alpha + (n+1)\nabla^\beta \mu_{\alpha\beta} + (n+1)N_\alpha{}^{\overline{\beta}\overline{\gamma}}\mu_{\overline{\beta}\overline{\gamma}} \\ \quad = -F_{\infty\alpha} + \frac{2}{n+2} i F_{0\alpha} - \frac{2}{n+1}(\nabla^\beta F_{\alpha\beta} + N_\alpha{}^{\overline{\beta}\overline{\gamma}} F_{\overline{\beta}\overline{\gamma}}), \\ i(n+2)(\nabla^\alpha \nu_\alpha - \nabla^{\overline{\alpha}} \nu_{\overline{\alpha}}) - \frac{1}{2} n(\Phi^{\alpha\beta}\mu_{\alpha\beta} + \Phi^{\overline{\alpha}\overline{\beta}}\mu_{\overline{\alpha}\overline{\beta}}) - \nabla^\alpha\nabla^\beta \mu_{\alpha\beta} - \nabla^{\overline{\alpha}}\nabla^{\overline{\beta}}\mu_{\overline{\alpha}\overline{\beta}} \\ \qquad - N^{\gamma\alpha\beta}\nabla_\gamma \mu_{\alpha\beta} - N^{\overline{\gamma}\overline{\alpha}\overline{\beta}}\nabla_{\overline{\gamma}}\mu_{\overline{\alpha}\overline{\beta}} - N^{\gamma\alpha\beta}{}_{,\gamma}\mu_{\alpha\beta} - N^{\overline{\gamma}\overline{\alpha}\overline{\beta}}{}_{,\overline{\gamma}}\mu_{\overline{\alpha}\overline{\beta}} \\ \quad = -nF_{00} + 2F_\alpha{}^\alpha. \end{cases} \tag{8.12}$$

By substituting the second equation into the other two and using (4.4), the system is reduced to

$$\begin{cases} D^{\alpha\beta}\mu_{\alpha\beta} - D^{\overline{\alpha}\overline{\beta}}\mu_{\overline{\alpha}\overline{\beta}} = iu, \\ D^{\alpha\beta}_{-2/n}\mu_{\alpha\beta} + D^{\overline{\alpha}\overline{\beta}}_{-2/n}\mu_{\overline{\alpha}\overline{\beta}} = v. \end{cases}$$

So we set $\mu_{\alpha\beta} = \lambda_{\alpha\beta}$ and determine ν_α by the second equation of (8.12). Then $E_{\infty 0} = \mathcal{A}^{2n+5}$, $E_{\infty\alpha} = \mathcal{A}^{2n+4}$, and $nE_{00} - 2E_\alpha{}^\alpha = \mathcal{A}^{2n+5}$ are solved by $\psi^{(1)}_{00} = O(\rho^{2n+5})$, $\psi^{(1)}{}_\alpha{}^\alpha = O(\rho^{2n+5})$. As before, $\mathrm{tf}(\psi^{(0)}_{\alpha\overline{\beta}})$ mod $O(\rho^{2n+5})$ and $\frac{1}{2}n\psi^{(0)}_{00} + (n+1)\psi^{(0)}{}_\alpha{}^\alpha$ mod $O(\rho^{2n+5})$ are uniquely determined so that $E_{00} = \mathcal{A}^{2n+5}$, $E_{\alpha\overline{\beta}} = \mathcal{A}^{2n+5}$. We set $\psi^{(0)}_{00} = \rho^{2n+4}\kappa + O(\rho^{2n+5})$. By (6.2a) we have $E_{\infty\infty} = \mathcal{A}^{2n+5}$.

Now we have constructed a normal-form ACH metric g, which is log-free, satisfying $E_{IJ} = \mathcal{A}^{2n+2+a(I,J)}$ and (8.11) in a unique way. After that we once again follow the latter half of the proof of Theorem 8.2 to determine all the higher-order terms of g_{ij}. No logarithmic terms occur in this process. □

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