DECREASING THE ERRORS OF MEASUREMENT CHANNELS BY AVERAGING THEIR READINGS

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The method of decreasing the variance of measurement channel errors in dynamic measurements by introducing additional channel readings and then averaging them is known as the speed—accuracy tradeoff [1, 2]. The necessary conditions for this method to be applicable are a reserve of speed on the part of the measurement channel and access to computational devices with negligible errors. However, these conditions are not sufficient.

On the one hand, the introduction of additional channel readings, provides new measurement information, which can be used to increase the accuracy of the channel; on the other hand, each additional channel reading comes endowed with its own error, and the averaging algorithm has methodological error. The search for optimal relations between these factors and conditions for efficient utilization of the method continues to frustrate researchers [3]. The successful out come of this quest depends largely on the particular model of summation of the elementary errors from the above-cited factors.

In the present article we propose a solution of this timely problem on the basis of a general approach to the determination of errors in measurement systems, which has demonstrated its advantages in error calculations for analog measurement channels [4], the reconstruction of signals from discrete readings of their values [5], and statistical measurements [6]. In application to our stated problem, this approach essentially calls for the error of the measurement system {measurement channel + computing device} to be treated as an indivisible unit whole, and the methodological averaging error to be treated as a special case of system error, without errors in the measurement channels.

Let the measured signal $x(t)$ be described by a normal stochastic process, which is nonstationary in the expected value $m(t)$ and is stationary in the covariance function $R(t_2 - t_1)$. The process is dicretized uniformly with step T by the measurement channel, whose i-th reading x_i is encumbered only with a random error variance σ_δ^2. The step T is chosen either with allowance for the channel error by the quantum [7] and information [8] criteria or without regard for the error by the frequency [9] and correlation [10] criterion of deviation from the function to be reconstructed [11].

We partition the step T into an even number 2d of intervals, $d = 1, 2, \ldots$, and single out the averaged readings $x_{i(-d)}, \ldots, x_{i(-1)}, x_{i1}, \ldots, x_{id}$ around the channel readings x_i in the interval $t_i - T/2 \le t \le t_i + T/2$. The measurement information contained in the averaged channel readings is utilized to the fullest if averaging is preceded by reconstruction of the signal between readings according to the conditional expected value $m(t|x_{i(-d)}, \ldots, x_{id})$, followed by averaging over the time t [5, 6]. The average reading x_{i0} of the measurement channel is then equal to

$$x_{i0} = \frac{1}{T} \int_{t_i - T/2}^{t_i + T/2} m(t|x_{i(-d)}, \ldots, x_{id})\,dt. \tag{1}$$

If the averaged reading x_{i0} differs from (1), the expected value $m_{\delta 0}$ of the error of the averaged reading is obtained equal to

$$m_{\delta 0} = x_{i0} - \frac{1}{T} \int_{t_i - T/2}^{t_i + T/2} m(t|x_{i(-d)}, \ldots, x_{id})\,dt. \tag{2}$$

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The variance $\sigma^2_{\delta 0}$ of the error of the averaged reading (1) is expressed in terms of the covariance function $R_{\delta x}(t^*, t^{**}|x_{i(-d)},\ldots, x_{id})$ of the error of signal reconstruction from the averaged readings and is equal to

$$\sigma^2_{\delta 0} = \frac{1}{T^2} \int\limits_{t_i-T/2}^{t_i+T/2} \int\limits_{t_i-T/2}^{t_i+T/2} R_{\delta x}(t^*, t^{**}|x_{i(-d)},\ldots, x_{id})dt^* dt^{**}. \tag{3}$$

If the discretization step T is chosen with allowance for the channel error, it needs to be refined after the averaged reading (1) is obtained. If it is chosen with allowance for signal reconstruction from the discrete channel readings, a signal with the same function should be reconstructed between the averaged readings. Ideally, the reconstruction should be based on the conditional expected value.

As an example, we consider a Wiener process with covariance function [4]

$$R(t_2-t_1)=\sigma^2-N|t_2-t_1|/4,$$

where the variance $\sigma^2 = \infty$, and N/2 is the intensity of the energy spectrum of white noise representing the first derivative of the Wiener process.

The signal can be reconstructed on the interval $t_i - T/2 \leq t \leq t_i + T/2$ by interpolation and extrapolation. If the channel errors are uncorrelated and an interpolation algorithm is used to reconstruct the Wiener process, Eqs. (1) and (3) have the form

$$\left.\begin{aligned} x_{i0} &= \frac{1}{4d} \sum_{j=-d}^{d-1} \{[x_{ij}-m(t_{ij})]+[x_{i(j+1)}-m(t_{(j+1)})]\}+ \\ &+ \frac{1}{T} \int\limits_{t_i-T/2}^{t_i+T/2} m(t)dt; \\ \sigma^2_{\delta 0} &= NT/96d^2+\sigma^2_{\delta}/4d. \end{aligned}\right\} \tag{4}$$

The result for extrapolation is analogous:

$$\left.\begin{aligned} x_{i0} &= \frac{1}{2d} \sum_{j=-d}^{d-1} [x_{ij}-m(t_{ij})]+ \\ &+ \frac{1}{T} \int\limits_{t_i-T/2}^{t_i+T/2} m(t)dt; \\ \sigma^2_{\delta 0} &= NT/24d^2+\sigma^2_{\delta}/2d. \end{aligned}\right\} \tag{5}$$

It is evident from Eqs. (4) and (5) that the introduction of the averaged channel readings incurs a methodological error proportional to NT. For large values of NT and small 2d this causes $\sigma^2_{\delta 0}$ to increase relative to the error variance σ^2_{δ} for a single channel reading. As 2d is increased, the variance $\sigma^2_{\delta 0}$ decreases until eventually it becomes smaller than $\sigma^2_{\delta 0}$. Assuming that $\sigma^2_{\delta 0} < \sigma^2_{\delta}$, we find the values of 2d at which it becomes advisable to introduce averaging with interpolation:

$$2d>(1+ \sqrt{1+2NT/3\sigma^2_{\delta}})/4 \tag{6}$$

and with extrapolation:

$$2d>(1+ \sqrt{1+2NT/3\sigma^2_{\delta}})/2. \tag{7}$$

Substituting the minimum value 2d = 2 in Eqs. (6) and (7), we obtain a relation between the intensity of the energy spectrum of the signal, the discretization step, and the variance of the channel; this relation can then be used to diminish the average error of the channel for any increase in the speed 2d for interpolation:

$$NT/2\sigma^2_{\delta} <36 \tag{8}$$

and for extrapolation:

$$NT/2\sigma_\delta^2 < 6. \qquad (9)$$

It is evident from expressions (6)-(9) that the interpolation algorithm used to average channel readings requires having the speed 2nd and having six times the range of ratios $NT/2\sigma_\delta^2$.

Investigations have shown that since the covariant function of the channel errors varies from $-\sigma_\delta^2$ to $+\sigma_\delta^2$, the resulting aperture of the variance $\sigma_{\delta 0}^2$, the resulting aperture of the variance $\sigma_{\delta 0}^2$ of the error of the averaged channel reading is equal to $(4d - 1)\sigma_\delta^2/2d$ for interpolation and $(2d - 1)\sigma_\delta^2/d$ for extrapolation. The difference between these apertures is equal to $\sigma_\delta^2/2d$ and decreases as the speed is increased.

It is somewhat more complicated to obtain similar recommendations for other stochastic processes, because the expressions for $\sigma_{\delta 0}^2$ are transcendental. The above-formulated general approach to their analysis is the only possible alternative in this case.

Similar investigations for a Markov stochastic process with covariance function [4]

$$R(t_2-t_1)=\sigma^2 e^{-\alpha|t_2-t_1|},$$

where α is the dynamic transmissibility, have shown that the transition from extrapolation to interpolation cuts $\sigma_{\delta 0}^2$ in half for $\alpha T/2d = 0$. Extrapolation and interpolation are equivalent for $\alpha T/2d > 10$.

In the measurement of deterministic signals we have $m(t) = x(t)$ and $R(t_2 - t_1) = \sigma^2 = 0$ [4]. Substituting these values of σ_δ^2 in Eqs. (2) and (3), we obtain the characteristics of the errors of the averaged channel reading in the form

$$m_{\delta 0} = x_{i0} - \frac{1}{T}\int\limits_{t_i-T/2}^{t_i+T/2} x(t)dt \;;\; \sigma_{\delta 0}^2 = 0. \qquad (10)$$

The expected value $m_{\delta 0}$ in Eq. (10) depends on the algorithm used to reconstruct the signal between the averaged channel readings. For example, in the case of extrapolation by a zeroth-degree polynomial we have

$$m_{\delta 0} = \frac{1}{2d}\sum_{j=-d}^{d-1} x_{ij} - \frac{1}{T}\int\limits_{t_i-T/2}^{t_i+T/2} x(t)dt. \qquad (11)$$

If the measurement channel is free of systematic error, $x_{ij} = x(t_{ij})$, and Eq. (11) is the expected value $m_{\delta OM}$ of the methodological error of the averaged channel reading. For example, if the measured signal is $x(t) = X_0 + X_m \sin(\omega t + \psi)$, where X_0 is a constant component, X_m is the amplitude, ω is the angular frequency, and ψ is the initial phase, $m_{\delta OM}$ has the following form for

$$m_{\delta OM} = X_m \sin\frac{\omega T}{2}\left(\frac{1}{2d}\operatorname{cosec}\frac{\omega T}{4d} - \frac{2}{\omega T}\right)\sin(\omega t_i + \psi). \qquad (12)$$

A result similar to (12) has been obtained previously [1]. According to Eqs. (11) and (12), averaging the readings of the measurement channel in the measurement of deterministic signals does not guarantee a reduction in its error.

Consequently, the efficacy of increasing the speed of a measurement channel with subsequent averaging depends significantly on the technique used to reconstruct the signal between the averaged readings. Only the transition from extrapolation to interpolation of a Wiener process reduces the required speed of the channel by one half at the same accuracy level and increases sixfold the aperture of the ratios $NT/2\sigma_\delta^2$ for which averaging is recommended.