manuscripta math. 96, 97 – 112 (1998)

Fumiharu Kato

Functors of log Artin rings

Received: 4 August 1997 / Revised version: 13 January 1998

Abstract. This paper gives a generalization of the theory of functors of Artin rings in the framework of log geometry. In the final section we apply it to the log smooth deformation theory.

1. Introduction

This short note presents the generalization of the classical theory of functors of Artin rings [4] in the framework of log geometry. The main theorem is the pro-representability criteria which will be given in Theorem 3.13 below. By this, we have the general principle of handling several kinds of local functors in log geometry. One of such functors is the log smooth deformation functor. The log smooth deformation theory has been originally introduced by Kawamata-Namikawa in [3] and generalized in [1].

These theories were developed in terms of the classical theory of functors of Artin rings; however, it would be more natural to construct it by considering functors on the category of Artin rings with log structures, for it is the theory of liftings to log schemes. In the final section of this note we will revise the log smooth deformation theory by applying our theory of functors of log Artin rings. By this, moreover, we can generalize the theory in [1] (cf. Remark 4.6). The goal of this direction is the moduli theory in log geometry, which can be expected to be more fruitful than that in the classical algebraic geometry, for it is considered to give nice compactifications of classical moduli. In the forthcoming paper the author will develope, using this theory, the moduli theory of log curves. Another important functor might be the "log Picard functor", of which the general definition seems still missing. The author hopes to discuss it in a future paper.

This work has been done when the author visited Institut Henri Poincaré in 1997. The author is grateful to IHP for the nice hospitality. The author thanks

F. Kato: Graduate School of Mathematics, Kyushu University, 6-10-1 Hakozaki Higashi-ku, Fukuoka 812, Japan.
e-mail: fkato@math.kyushu-u.ac.jp

Mathematics Subject Classification (1991): Primary 13D10; Secondary 14D15, 16S80

Dr. T. Kajiwara for valuable discussions. Finally, the author is grateful to the referee for his careful reading and valuable suggestions.

Conventions and notation: Here is the list of general conventions and notation which will be valid throughout this note:

(1) Monoids and rings are always assumed to be commutative and unital; in particular, homomorphisms of them are assumed to carry units to units. Rings are often regarded as monoids with respect to multiplication.
(2) For a monoid M the subgroup of all invertible elements of M is denoted by $M^\times$, and we write $\overline{M} = M/M^\times$ (cf. [1, pp. 319] for quotient of monoids).
(3) For a local ring R the maximal ideal is denoted by $\mathfrak{m}_R$.
(4) For a log scheme X the log structure of X is denoted by $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$. (Sheaves on schemes are considered with respect to étale topology unless otherwise specified.)
(5) The sheaf $\mathcal{M}_X/\mathcal{O}_X^\times$ of monoids, for a log scheme X, is denoted by $\overline{\mathcal{M}}_X$ (by definition of log structures, this notation is consistent with (2)).

2. Category of log Artin rings

First we recall some terminologies on monoids. A monoid M is said to be integral if the natural homomorphism $M \to M^{\mathrm{gp}}$ to the associated group is injective. A monoid M is said to be fine if it is integral and finitely generated.

Let us define the categories which we will work on throughout. We fix the following data:

(D1) A complete Noetherian local ring Λ with the residue field $k := \Lambda/\mathfrak{m}_\Lambda$.
(D2) Homomorphisms $\alpha_\Lambda : M_\Lambda \to \Lambda$ and $\alpha_0 : M_0 \to k$ of monoids such that $\alpha_\Lambda^{-1}(\Lambda^\times) \cong \Lambda^\times$ and $\alpha_0^{-1}(k^\times) \cong k^\times$.
(D3) A homomorphism $\rho_\Lambda : M_\Lambda \to M_0$ of monoids such that $\alpha_0 \circ \rho_\Lambda = \mathrm{res}_\Lambda \circ \alpha_\Lambda$, where $\mathrm{res}_\Lambda : \Lambda \to k$ is the residue map.

The first datum gives the category C_Λ (resp. $\mathsf{C}^\wedge_\Lambda$) of Artin (resp. complete Noetherian) local Λ-algebras having the residue field k (cf. [4]). We require that these data satisfy the following conditions:

(C1) The monoids M_Λ and M_0 have splittings $M_\Lambda \cong \alpha_\Lambda^{-1}(\Lambda^\times) \oplus P$ and $M_0 \cong \alpha_0^{-1}(k^\times) \oplus Q$, where P and Q are toric unit-free monoids.
(C2) There exists an injective homomorphism $\sigma : P \to Q$ such that, through the splittings in (C1), $\rho_\Lambda(\alpha_\Lambda^{-1}(u), p) = (\alpha_0^{-1}(\mathrm{res}_\Lambda(u)), \sigma(p))$ for $u \in \Lambda^\times$ and $p \in P$.
(C3) $Q \setminus \sigma(P)$ is an ideal of Q.
(C4) The induced group homomorphism $\sigma^{\mathrm{gp}} : P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$ of $\sigma : P \to Q$ has torsion-free cokernel.

Here a monoid M is said to be *toric* if it is fine and the associated group Q^{gp} is torsion-free, and a monoid M is said to be *unit-free* if $M^{\times} = 1$. A subset I of a monoid M is called an *ideal* if for any $x \in I$ and any $a \in M$ we have $ax \in I$. Note that in (C3) we allow the particular case $Q \setminus \sigma(P) = \emptyset$. (The meaning of the data (D1)–(D3) and the justification of the conditions (C1)–(C4) will be explained in the context after Definition 2.1.)

Then the categories we are now going to work on are defined as follows:

Definition 2.1. *The category LC_Λ (resp. $\mathsf{LC}^\wedge_\Lambda$) is the category of which an object is a pair (A, M) consisting of a ring A in C_Λ (resp. $\mathsf{C}^\wedge_\Lambda$) and an integral monoid M together with homomorphisms $\alpha\colon M \to A$, $M_\Lambda \to M$, and $\rho\colon M \to M_0$ of monoids such that:*

(1) $\alpha^{-1}(A^{\times}) \cong A^{\times}$.

(2) The diagram

$$\begin{tikzcd} M_\Lambda \arrow[r] \arrow[d, "\alpha_\Lambda"'] & M \arrow[r, "\rho"] \arrow[d, "\alpha"'] & M_0 \arrow[d, "\alpha_0"] \\ \Lambda \arrow[r] & A \arrow[r] & k \end{tikzcd}$$

commutes, and the composite of the two arrows in the first row coincides with the ρ_Λ given in (D3).

(3) The diagram

$$\begin{tikzcd} A^{\times} \arrow[r] \arrow[d, "\alpha^{-1}"'] & k^{\times} \arrow[d, "\alpha_0^{-1}"] \\ M \arrow[r] & M_0 \end{tikzcd}$$

is co-Cartesian (i.e., $M \oplus_{A^{\times}} k^{\times} \cong M_0$).

(4) The homomorphism α maps all elements in $K = M \setminus \mathrm{Im}(M_\Lambda \oplus_{\Lambda^{\times}} A^{\times})$ (which is easily seen to be an ideal of M) to zero.

A morphism in LC_Λ (resp. $\mathsf{LC}^\wedge_\Lambda$) is a commutative diagram

$$\begin{tikzcd} M' \arrow[r] \arrow[d, "\alpha'"'] & M \arrow[d, "\alpha"] \\ A' \arrow[r] & A \end{tikzcd}$$

of homomorphisms of monoids such that $A' \to A$ is a morphism in C_Λ (resp. $\mathsf{C}^\wedge_\Lambda$) and that all the possible induced diagrams are commutative.

The definition of these categories calls for some explanations. First we note that giving those data (D2) and (D3) satisfying (C1) and (C2) is equivalent to giving a morphism $S_0 \to T$ of fine log schemes underlain by $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} \Lambda$ having the global chart modeled on the homomorphism

$\sigma : P \to Q$ (cf. [2, (2.9) (2)]). These assumptions on log structures are practical, and have been imposed in order to avoid unnecessary complication.

Then giving an Artinian ring A in C_Λ together with a homomorphism $\alpha : M \to A$ of monoids having those conditions (1), (2), and (3) in Definition 2.1 is equivalent to giving a fine log schemes S over T together with an exact closed immerstion $S_0 \hookrightarrow S$ over T of which the underlying morphism comes from an object in C_Λ. Moreover, the log structure on S also has a global chart modeled on Q due to the following lemma:

Lemma 2.2. *Let $S \hookrightarrow S'$ be a thickening of finite order (cf. [1, 3.5]) of fine log schemes underlain by a surjection $A' \to A$ in C_Λ. Suppose that a global chart $Q \to \mathcal{M}_S$ of S modeled on a toric unit-free monoid is given. Then this chart lifts to a global chart $Q \to \mathcal{M}_{S'}$ of S'.*

Proof. It is not difficult to see the following facts: (1) The morphism $h : \mathcal{M}_{S'} \to \mathcal{M}_S$ is surjective and exact (i.e., $(h^{\mathrm{gp}})^{-1}(\mathcal{M}_S) = \mathcal{M}_{S'}$). (2) The sequence $1 \to \mathcal{J} \to \mathcal{M}_{S'}^{\mathrm{gp}} \to \mathcal{M}_S^{\mathrm{gp}} \to 1$ is exact, where $\mathcal{J}$ is the kernel of $\mathcal{O}_{S'}^\times \to \mathcal{O}_S^\times$. Set $M = \Gamma(\mathrm{Spec}\, A, \mathcal{M}_S)$ and $M' = \Gamma(\mathrm{Spec}\, A', \mathcal{M}_{S'})$. By Hilbert's Theorem 90 the morphism $M' \to M$ is also exact and surjective having the kernel J, the global section of $\mathcal{J}$. Since Q^{gp} is free, the splitting $Q \to M$ lifts to that of M'. Finally we have to check that the induced morphism $S' \to \mathrm{Spec}\,\mathbf{Z}[Q]$ is strict, where $\mathrm{Spec}\,\mathbf{Z}[Q]$ is equipped with the canonical log structure (cf. [1, 2.5]). This is done by Lemma 2.3 (2) below. □

Lemma 2.3. (1) *Let $f : X \to Y$ be a morphism of log schemes with X integral. Then the morphism f is strict if and only if the induced morphism $f^{-1}\overline{\mathcal{M}}_Y \to \overline{\mathcal{M}}_X$ of sheaves on X is an isomorphism.*

(2) *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of log schemes with Y integral. Suppose that the underlying morphism of f is surjective. Then, if $g \circ f$ and f are strict, so is g.*

Proof. (2) follows from (1). The proof of (1) is straightforward and is left to the reader. □

Secondary, the conditions (C3), (C4), and (4) in Definition 2.1 have been posed in connection with automorphisms of log structures. Usually the automorphisms of log structures are so complicated that it is sometimes difficult to control them; what seems the most problematic is the fact that in general they do not lift with respect to thickenings. For example, consider the standard point $\mathbf{N} \to \mathbf{Z}/p\mathbf{Z} = \mathbf{F}_p$ (see Example 2.11 for the definition of the standard point) and its extension $\mathbf{N} \to \mathbf{Z}/p^2\mathbf{Z}$ sending $1 \mapsto p$. Then it is easily seen that every non-trivial automorphism of the log structure $\mathbf{N} \to \mathbf{Z}/p\mathbf{Z}$ never lift to that of $\mathbf{N} \to \mathbf{Z}/p^2\mathbf{Z}$. The advantage of having those conditions (C3) and (C4) is that we can actually control the automorphisms of log structures because of the following lemma:

Lemma 2.4. *For an object* $(\alpha\colon M \to A)$ *in* $\mathsf{LC}^{\wedge}_{\Lambda}$*, the group of all automorphisms of* M *fixing* M_{Λ}*,* M_0*, and* A *is canonically isomorphic to* $\mathrm{Hom}_{\mathbf{Z}}(\mathrm{Coker}(P^{\mathrm{gp}} \to Q^{\mathrm{gp}}), J_A)$ *where* $J_A = \mathrm{Ker}(A^{\times} \to k^{\times})$.

Proof. Each automorphism as in the lemma defines an automorphism of M^{gp} as an extension of Q^{gp} by $A^{\times}$ which induces trivial automorphisms both as extenstion of $\mathrm{Im}P^{\mathrm{gp}}$ by $A^{\times}$ and as that of Q^{gp} by $k^{\times}$. Hence it associates canonically an element in $\mathrm{Hom}_{\mathbf{Z}}(Q^{\mathrm{gp}}/\mathrm{Im}P^{\mathrm{gp}}, J_A)$. Conversely, starting from a homomorphism $Q^{\mathrm{gp}}/\mathrm{Im}P^{\mathrm{gp}} \to J_A$, we get an automorphism of M^{gp} and of M, since $M \to Q$ is exact. We have to check that this automorphism is compatible with $\alpha\colon M \to A$. But we can actually do it, using the fact that α maps the element outside the image of $M_{\Lambda} \oplus_{\Lambda^{\times}} A^{\times}$ to zero. □

From the injectivity of σ in (C2), combining with (C3) and (C4), we have the following:

Lemma 2.5. *For any object* A *in* $\mathsf{C}^{\wedge}_{\Lambda}$*, an object* $(\alpha\colon M \to A)$ *in* $\mathsf{LC}^{\wedge}_{\Lambda}$ *underlain by* A *exists uniquely up to isomorphisms.*

Proof. The existence of such $(\alpha\colon M \to A)$ is clear since we can always take $\lambda^{\dagger}\colon Q \to A$ which maps elements in $Q \setminus \sigma(P)$ to zero such that the induced diagram

$$\begin{array}{ccccc} P \oplus \Lambda^{\times} & \longrightarrow & Q \oplus A^{\times} & \longrightarrow & Q \oplus k^{\times} \\ \downarrow & & \downarrow & & \downarrow \\ \Lambda & \longrightarrow & A & \longrightarrow & k \end{array}$$

commutes, where the arrows in the first row are induced by $\sigma\colon P \to Q$ and $\mathrm{id}\colon Q \to Q$, respectively. We write the object in $\mathsf{LC}^{\wedge}_{\Lambda}$ obtained in this way by $(\alpha^{\dagger}\colon M^{\dagger} \to A)$. Now take any $(\alpha\colon M \to A)$ as in the lemma. By Lemma 2.2, M has a splitting $M \cong Q \oplus A^{\times}$, and hence we have $\lambda\colon Q \to A$ which lifts given $Q \to k$. Then, taking the difference of the two homomorphisms $\lambda^{\dagger} \circ \sigma$ and $\lambda \circ \sigma$, we get an element in $\mathrm{Hom}(P, A^{\times}) = \mathrm{Hom}(P^{\mathrm{gp}}, A^{\times})$. By (C4), the map $\mathrm{Hom}(Q^{\mathrm{gp}}, A^{\times}) \to \mathrm{Hom}(P^{\mathrm{gp}}, A^{\times})$ is surjective, and hence, we have a map $Q \to A^{\times}$ which induces an isomorphism $M^{\dagger} \xrightarrow{\sim} M$ of monoids which commutes with $M_{\Lambda} \to M^{\dagger}$ and $M_{\Lambda} \to M$. By the same reason as in the proof of Lemma 2.4, this isomorphism of monoids gives an isomorphism of $(\alpha^{\dagger}\colon M^{\dagger} \to A)$ and $(\alpha\colon M \to A)$ in $\mathsf{LC}^{\wedge}_{\Lambda}$. □

Finally, we should remark on what the definition of $\mathsf{LC}^{\wedge}_{\Lambda}$ means: Let $\mathsf{pro\text{-}LC}_{\Lambda}$ be the category of pro-objects of LC_{Λ}.

Lemma 2.6. *The category* $\mathsf{LC}^{\wedge}_{\Lambda}$ *is canonically a full subcategory of* $\mathsf{pro\text{-}LC}_{\Lambda}$ *of which objects are projective systems* $(M_i \to R_i)_{i \in \mathbf{N}}$ *consisting of objects in* LC_{Λ} *indexed by the ordered set of non-negative integers such that:*

(1) *Each* $R_i \to R_{i-1}$ *is surjective and* $M_i \oplus_{R_i^\times} R_{i-1}^\times \cong M_{i-1}$ *for all sufficiently large* i.
(2) *The induced morphism* $\mathfrak{m}_{R_i}/(\mathfrak{m}_\Lambda + \mathfrak{m}_{R_i}^2) \to \mathfrak{m}_{R_{i-1}}/(\mathfrak{m}_\Lambda + \mathfrak{m}_{R_{i-1}}^2)$ *is an isomorphism for all sufficiently large* i.

In particular, any object $(\alpha\colon M \to R)$ *in* $\mathsf{LC}_\Lambda^\wedge$ *has a global chart* $Q \to M$ *which lifts the given chart* $Q \to M_0$ *of* $(\alpha_0\colon M_0 \to k)$.

Proof. Let $M \to R$ be an object in $\mathsf{LC}_\Lambda^\wedge$. Then for each $i \in \mathbf{N}$ the ring $R_i = R/\mathfrak{m}_R^{i+1}$ with the induced monoid $M_i = M \oplus_{R^\times} R_i^\times$ defines an object in LC_Λ, and the system $(M_i \to R_i)$ belongs to $\text{pro-}\mathsf{LC}_\Lambda$ which satisfies those conditions in the lemma. Conversely, given a system $(M_i \to R_i)$ as above, we can construct, due to Lemma 2.2, a compatible system of charts $Q \to M_i$ for sufficiently large i which gives compatible splittings $M_i = Q \oplus R_i^\times$. Then the projective limit M has the induced global chart $Q \to M$. The projective limit R of rings R_i is classically known to belong to $\mathsf{C}_\Lambda^\wedge$. Then we easily see that $M \to R$ and induced homomorphisms define an object in $\mathsf{LC}_\Lambda^\wedge$. □

In the rest of this section, we fix some notation and terminologies on the categories we have just defined, and establish some elementary properties of them. Let us write from now on an object in $\mathsf{LC}_\Lambda^\wedge$ like $\mathcal{R} = (M \to R)$, or even only $\mathcal{R}$ if no confusion may occur.

Proposition 2.7. *Any morphism* $\mathcal{R}' \to \mathcal{R}$ *in* $\mathsf{LC}_\Lambda^\wedge$ *is strict, i.e., the corresponding morphism of log schemes is strict.*

Proof. If $\mathcal{R}$ belongs to LC_Λ, the proposition follows from Lemma 2.3 (2). In general, for an object $\mathcal{R} = (M \to R)$ in $\mathsf{LC}_\Lambda^\wedge$ let us denote by $\mathcal{R}_i$ the induced object $(M_i = M \oplus_{R^\times} R_i^\times \to R_i)$ in LC_Λ, where $R_i = R/\mathfrak{m}_R^{i+1}$ for any $i \in \mathbf{N}$. For a morphism $\mathcal{R}' \to \mathcal{R}$, the induced morphisms $\mathcal{R}'_i \to \mathcal{R}_i$ belongs to LC_Λ and hence is strict. By Lemma 2.6 we have a chart $Q \to M'$ of $\mathcal{R}'$ which lifts that of M_0, where $\mathcal{R}' = (M' \to R')$. Since $\mathcal{R}' \to \mathcal{R}'_i$ and $\mathcal{R}'_i \to \mathcal{R}_i$ are strict, the composite $Q \to M' \to M_i$ defines a chart of $\mathcal{R}_i$, and then defines a compatible system of splittings $M_i \cong R_i^\times \oplus Q$. It follows by taking the limit that the composite $Q \to M' \to M$ gives a chart and hence $\mathcal{R}' \to \mathcal{R}$ is strict as desired. □

Let $\mathsf{v}\colon \mathsf{LC}_\Lambda^\wedge \to \mathsf{C}_\Lambda^\wedge$ be the forgetful functor. The following terminologies will be useful later.

Definition 2.8. *A morphism* $\varphi\colon \mathcal{B} \to \mathcal{A}$ *in* $\mathsf{LC}_\Lambda^\wedge$ *is said to be a surjection (resp. small extension) if the morphism* $\mathsf{v}(\varphi)$ *in* $\mathsf{C}_\Lambda^\wedge$ *is a surjection (resp. small extension).*

By v the category $\mathsf{LC}_\Lambda^\wedge$ is co-fibered over $\mathsf{C}_\Lambda^\wedge$. In particular, for any homomorphism $u\colon R \to R'$ in $\mathsf{C}_\Lambda^\wedge$ and any $\mathcal{R} = (M \to R)$ over R, there exists the *direct image* $u_*\mathcal{R}$ given by R' and the morphism induced by the push-out $M \oplus_{R^\times} R'^\times \to R$. By Proposition 2.7 we get:

Corollary 2.9. *The category* $\mathsf{LC}^{\wedge}_{\Lambda}$ *is a co-fibered groupoid (cf. [5, Exposé VI]) over* $\mathsf{C}^{\wedge}_{\Lambda}$.

Let us denote from now on the fiber of v over A by $\mathsf{LC}^{\wedge}_{\Lambda}(A)$; in case A is in C_{Λ} we can also denote it by $\mathsf{LC}_{\Lambda}(A)$. For two objects $\mathcal{A}'$ and $\mathcal{A}$ in a common fiber $\mathsf{LC}^{\wedge}_{\Lambda}(A)$ the set of all morphisms $\varphi\colon \mathcal{A}' \to \mathcal{A}$ such that $\mathsf{v}(\varphi) = \mathrm{id}_A$ is denoted by $\mathrm{Isom}_A(\mathcal{A}', \mathcal{A})$. By Lemma 2.5, $\mathrm{Isom}_A(\mathcal{A}', \mathcal{A})$ is non-empty. In particular, we write $\mathrm{Aut}_A(\mathcal{A}) := \mathrm{Isom}_A(\mathcal{A}, \mathcal{A})$. By Lemma 2.4 we have the canonical isomorphism

$$\mathrm{Aut}_A(\mathcal{A}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Z}}(\mathrm{Coker}(P^{\mathrm{gp}} \to Q^{\mathrm{gp}}), J_A). \tag{2.1}$$

The fiber $\mathsf{LC}_{\Lambda}(k)$ may be already rich. Indeed, any character $\chi \in \mathrm{Hom}_{\mathbf{Z}}(Q^{\mathrm{gp}}, k^{\times})$ associates the *twisted object* $k^{(\chi)}$ in $\mathsf{LC}_{\Lambda}(k)$ given by the data $\alpha_0\colon M_0 \to k$, $\rho_0^{(\chi)}\colon M_0 \to M_0$, and the composite $\rho_{\Lambda} \circ (\rho_0^{(\chi)})^{-1}\colon M_{\Lambda} \to M_0$, where $\rho_0^{(\chi)}$ maps $(q, u) \in Q \oplus k^{\times} \cong M_0$ to $(q, u \cdot \chi(q))$.

Lemma 2.10. *For any* $*, *' \in \mathsf{LC}_{\Lambda}(k)$*, the set* $\mathrm{Isom}_k(*', *)$ *consists of one element.*

Proof. The lemma follows from (1) $\mathrm{Isom}_k(k^{(1)}, *) \neq \emptyset$ for any $* \in \mathsf{LC}_{\Lambda}(k)$, and (2) $\mathrm{Aut}_k(k^{(1)}) = 1$, both of which are easy to see. □

Example 2.11. Let us consider the standard point over the field k, that is, the field k with the log structure $\mathbf{N} \oplus k^{\times} \to k$ sending (n, u) to $0^n \cdot u$ (i.e., $Q = \mathbf{N}$).

(1) If the log structure on Λ is trivial (i.e., $P = \{1\}$), then the log structure of every object $\alpha\colon M \to A$ in $\mathsf{LC}^{\wedge}_{\Lambda}$ is hollow, i.e., α maps $M \setminus \alpha^{-1}(A^{\times})$ to zero.
(2) If Λ has the log structure induced by $P = \mathbf{N} \to \Lambda$ which maps 1 to an element in $\mathfrak{m}_{\Lambda}$ and $\sigma = \mathrm{id}$, then the morphism $S_0 \to T$ is strict.

3. Pro-representability of functors

We shall consider a covariant set-valued functor $F\colon \mathsf{LC}_{\Lambda} \to \mathsf{Sets}$ on the category of log Artin rings such that $F(k^{(1)})$ consists of one element, or equivalently (due to Lemma 2.10) $F(*)$ consists of one element for any $* \in \mathsf{LC}_{\Lambda}(k)$. Our strategy of investigating the functor F is to estimate it by a functor of the classical category. Given a functor F as above, we define a functor $\mathsf{v}_*F\colon \mathsf{C}_{\Lambda} \to \mathsf{Sets}$ by

$$\mathsf{v}_*F(A) = \{(\mathcal{A}, \xi) \mid \mathcal{A} \in \mathsf{LC}_{\Lambda}(A), \xi \in F(\mathcal{A})\}/\sim$$

for any object A in C_{Λ}, where $(\mathcal{A}, \xi) \sim (\mathcal{A}', \xi')$ if and only if there exists $\varphi \in \mathrm{Isom}_A(\mathcal{A}, \mathcal{A}')$ such that $\xi' = F(\varphi)(\xi)$. By Lemma 2.10 the set $\mathsf{v}_*F(k)$ consists of one element.

Properly speaking, we should formulate the situation more generally in terms of 2-categories; viz., we should consider a co-fibered groupoid F over LC_Λ such that $\mathsf{F}(k^{(1)}) = \mathsf{e}$, where e is a category in which for any two objects there exists one and only one morphism between them (cf. [5, Exposé VI]). (In this note, however, we will not work in such a general situation because it may be incongruous with the aim of this short note.) In this situation, the push-out $\mathsf{v}_*\mathsf{F}$ is obviously a co-fibered groupoid over C_Λ having the same property. If the functor associated to F is F, then the functor associated to $\mathsf{v}_*\mathsf{F}$ is nothing but v_*F, which justifies our notation.

The associated functor v_*F is expected to be a good estimate of F; in fact, we have:

Proposition 3.1. *If the functor F is pro-represented by $\mathcal{R}$, then the functor v_*F is pro-represented by R, where $R = \mathsf{v}(\mathcal{R})$.*

Proof. Denote by $\mathrm{h}_{\mathcal{R}} = \mathrm{Hom}(\mathcal{R}, -)$ the pro-representable functor on LC_Λ. Let $A \in \mathsf{C}_\Lambda$. Then any element $[\mathcal{A}, \varphi] \in \mathsf{v}_*\mathrm{h}_{\mathcal{R}}(A)$ gives the underlying homomorphism $R \to A$ of φ. From this ring homomorphism we can recover the isomorphism class $[\mathcal{A}, \varphi]$ by the log structure on $\mathcal{R}$. □

So we define:

Definition 3.2. *A pair $(\mathcal{R}, \xi)$ of $\mathcal{R} \in \mathsf{LC}^\wedge_\Lambda$ and $\xi \in F(\mathcal{R})$ is called a pseudo-pro-representative of F if $(R, [\mathcal{R}, \xi])$ pro-represents v_*F, where $R = \mathsf{v}(\mathcal{R})$.*

The pseudo-pro-representative is, however, not really a pro-representative in general: For a functor F on LC_Λ we define another functor $\overline{F}$ on LC_Λ by $\overline{F}(\mathcal{A}) := F(\mathcal{A})/\mathrm{Aut}_A(\mathcal{A})$; this is actually a functor since every morphism in $\mathsf{LC}^\wedge_\Lambda$ is strict. Note that there exists a natural bijective map $\overline{F}(\mathcal{A}) \xrightarrow{\sim} \mathsf{v}_*F(A)$ sending $[\xi] \mapsto [\mathcal{A}, \xi]$ (cf. Lemma 2.5). For example, fix $\mathcal{R} \in \mathsf{LC}^\wedge_\Lambda$ and consider the functor $\overline{\mathrm{h}}_{\mathcal{R}}$. Then $\overline{\mathrm{h}}_{\mathcal{R}}$ is not pro-represented unless the associated group homomorphism $\sigma^{\mathrm{gp}}\colon P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$ of $\sigma\colon P \to Q$ is surjective (or equivalently, $\mathrm{Aut}_A(\mathcal{A}) = 1$ for any $\mathcal{A}$) while $\mathsf{v}_*\overline{\mathrm{h}}_{\mathcal{R}}$ is obviously pro-represented by the ring $R = \mathsf{v}(\mathcal{R})$. Obviously the groups $\mathrm{Aut}_A(\mathcal{A})$ cause the difficulties of this kind. So we are lead to consider the following class of functors:

Definition 3.3. *A Functor F on LC_Λ is said to be rigid if for any $\mathcal{A} \in \mathsf{LC}_\Lambda$ and any $\varphi \in \mathrm{Aut}_A(\mathcal{A})$ the bijective map $F(\varphi)\colon F(\mathcal{A}) \to F(\mathcal{A})$ has no fixed element unless $\varphi = \mathrm{id}$.*

In particular, if $\sigma^{\mathrm{gp}}\colon P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$ is surjective, then every functor is rigid.

Proposition 3.4. (1) *Any pro-representable functor on LC_Λ is rigid.*
(2) *Let F be a rigid functor on LC_Λ. Then any pseudo-pro-representative of F pro-represents F.*

Proof. Let $\theta\colon \mathcal{R} \to \mathcal{A}$ be a morphism in $\mathsf{LC}^\wedge_\Lambda$ and $\varphi\colon \mathcal{A} \to \mathcal{A}$ an automorphism over A. Set $\mathcal{R} = (N \to R)$ and $\mathcal{A} = (M \to A)$. Suppose $\varphi \circ \theta = \theta$. Since θ is strict, we have a commutative diagram of monoids

$$\begin{array}{ccc} N \oplus_{R^\times} A^\times & \xrightarrow{\sim} & M \\ \| & & \downarrow \\ N \oplus_{R^\times} A^\times & \xrightarrow{\sim} & M, \end{array}$$

where the horizontal arrows are induced by θ and the right vertical arrow by φ. Hence we get $\varphi = \mathrm{id}$ and then (1) is proved. Let $(\mathcal{R}, \xi)$ be a pseudo-pro-representative of F. To show that the induced morphism $\mathrm{h}_\mathcal{R} \to F$ is an isomorphism, take $\eta \in F(\mathcal{A})$. Then $[\mathcal{A}, \eta] \in \mathsf{v}_*F(A)$ comes from a homomorphism $u\colon R \to A$ in $\mathsf{C}^\wedge_\Lambda$, i.e., $[u_*\mathcal{R}, \eta'] = [\mathcal{A}, \eta]$, where η' is the image of ξ under $F(\mathcal{R}) \to F(u_*\mathcal{R})$. Since F is rigid, there exists a unique isomorphism $u_*\mathcal{R} \to \mathcal{A}$ which sends η' to η. Thus we get a unique morphism $\mathcal{R} \to u_*\mathcal{R} \to \mathcal{A}$ which sends ξ to η. Hence (2) is proved. $\square$

Next, let us discuss hulls. We shall first establish the concept of tangent spaces. Let $\mathcal{A}' \to \mathcal{A} \leftarrow \mathcal{A}''$ be a diagram in LC_Λ with the underlying diagram of rings $A' \to A \leftarrow A''$. Set $\mathcal{A}' = (M' \to A')$, $\mathcal{A} = (M \to A)$, and $\mathcal{A}'' = (M'' \to A'')$. Then it can be checked that the induced homomorphism $M' \times_M M'' \to A' \times_A A''$ defines an object in LC_Λ, denoted by $\mathcal{A}' \times_\mathcal{A} \mathcal{A}''$. It is also easy to see that $\mathcal{A}' \times_\mathcal{A} \mathcal{A}''$ gives the fiber product of $\mathcal{A}' \to \mathcal{A} \leftarrow \mathcal{A}''$ in LC_Λ.

For a finite dimensional k-vector space V, we denote by $k[V]$ the ring $k \oplus V$ with V square zero. The ring $k[V]$ belongs to C_Λ and has the section $k \hookrightarrow k[V]$. By this section we can define the log structure on $k[V]$ by $M_0 \oplus_{k^\times} k[V]^\times \to k[V]$, and then we have an object in LC_Λ denoted by $k[V]_0$. In particular, to the ring of dual numbers $k[\epsilon] := k[E]/(E^2)$ we associate the object $k[\epsilon]_0$.

Suppose F is a functor on LC_Λ satisfying

$$F(k[V]_0 \times_k k[W]_0) \xrightarrow{\sim} F(k[V]_0) \times F(k[W]_0) \tag{3.1}$$

for any vector spaces V and W. Then as usual the morphism $k[V]_0 \times_k k[V]_0 \to k[V]_0$ sending $(v, w) \mapsto v + w$ $(v, w \in V)$ and $k[V]_0 \to k[V]_0$ sending $v \mapsto av$ $(v \in V)$ for $a \in k$ define a structure of k-vector space on $F(k[V]_0)$ (cf. [4, 2.10]). In case $k[V] = k[\epsilon]$ we denote $\mathrm{t}_F := F(k[\epsilon]_0)$ and call it the *tangent space* of F.

Set $k[\epsilon]_0 = (M_{\epsilon,0} \to k[\epsilon])$ and $k[V]_0 = (M_{V,0} \to k[V])$. We fix a chart $Q \to M_0$ of M_0. Then it induces charts $Q \to M_{\epsilon,0}$ and $Q \to M_{V,0}$ by the inclusions $M_0 \hookrightarrow M_{\epsilon,0}$ and $M_0 \hookrightarrow M_{V,0}$, respectively. Then depending on these choices of charts, each morphism $k[\epsilon] \to k[V]$ in C_Λ extends uniquely to a morphism $k[\epsilon]_0 \to k[V]_0$ in LC_Λ by the homomorphism $M_{\epsilon,0} \to M_{V,0}$ of monoids preserving charts. Thus we get a non-canonical map $\mathrm{Hom}_{\mathsf{C}_\Lambda}(k[\epsilon], k[V])\ (\cong V) \to \mathrm{Hom}_{\mathsf{LC}_\Lambda}(k[\epsilon]_0, k[V]_0)$ which induces a map

$$\mathrm{t}_F \otimes_k V \to F(k[V]_0). \tag{3.2}$$

Since $F(k[V])$ is isomorphic to $\dim_k V$ copies of t_F, we easily see that the map (3.2) is an isomorphism.

Now let F be a functor as in the beginning of this section. For a diagram $\mathcal{A}' \to \mathcal{A} \leftarrow \mathcal{A}''$ in LC_Λ, consider the induced map

$$F(\mathcal{A}' \times_{\mathcal{A}} \mathcal{A}'') \longrightarrow F(\mathcal{A}') \times_{F(\mathcal{A})} F(\mathcal{A}''). \tag{3.3}$$

Then we consider the following conditions:

($\mathrm{H}_1^{\log}$) (3.3) is surjective whenever $\mathcal{A}'' \to \mathcal{A}$ is a small extension (cf. Definition 2.8).
($\mathrm{H}_2^{\log}$) (3.3) is a bijection when $\mathcal{A} = k^{(1)}$ and $\mathcal{A}'' = k[\epsilon]_0$.

If F has these two properties, then we can show that F satisfies (3.1) for any vector spaces V and W, and then $F(k[V]_0)$ has a natural structure of k-vector space (by induction with respect to $\dim_k V$). Moreover, by an argument similar to that in [4, (2.15)], we have:

Lemma 3.5. *Suppose F satisfies the conditions ($\mathrm{H}_1^{\log}$) and ($\mathrm{H}_2^{\log}$). Then for any small extension $\theta\colon \mathcal{A}' \to \mathcal{A}$ with $I := \mathrm{Ker}(\mathsf{v}(\theta))$, the k-linear space $F(k[I]_0)$ acts transitively on the set $\{\xi' \in F(\mathcal{A}') \mid F(\theta)(\xi') = \xi\}$ for any fixed $\xi \in F(\mathcal{A})$.*

Definition 3.6. *A morphism $F \to G$ of functors on LC_Λ is said to be formally smooth, if for any small extension $\mathcal{B} \to \mathcal{A}$ the map $F(\mathcal{B}) \longrightarrow F(\mathcal{A}) \times_{G(\mathcal{A})} G(\mathcal{B})$ is surjective.*

Lemma 3.7. *For any functor F, the natural morphism $F \to \overline{F}$ is formally smooth.*

Proof. The lemma follows from the fact that any automorphism lifts with respect to small extension due to (2.1). □

Lemma 3.8. *Consider the following conditions:*

(1) *$\varphi\colon F \to G$ is formally smooth.*
(2) *$\overline{\varphi}\colon \overline{F} \to \overline{G}$ is formally smooth.*
(3) *$\mathsf{v}_*\varphi\colon \mathsf{v}_*F \to \mathsf{v}_*G$ is formally smooth in the usual sense.*

Then we have the implications:

$$(1) \implies (2) \iff (3).$$

If F, G, and $\overline{G}$ satisfy ($\mathrm{H}_1^{\log}$) *and* ($\mathrm{H}_2^{\log}$)*, and*

$$\mathrm{Ker}(\mathrm{t}_F \to \mathrm{t}_{\overline{F}}) \longrightarrow \mathrm{Ker}(\mathrm{t}_G \to \mathrm{t}_{\overline{G}})$$

is surjective, then we have also $(2) \Rightarrow (1)$.

Proof. The equivalence of (2) and (3) follows from the fact that the natural map $\overline{F}(\mathcal{A}) \to \mathsf{v}_*F(A)$ by $[\xi] \mapsto [\mathcal{A}, \xi]$ is a bijection by Lemma 2.5. (1) $\Rightarrow$ (2) is straightforward. Let us prove the other part. Suppose (2) and the conditions as in the lemma. Let $\theta\colon \mathcal{B} \to \mathcal{A}$ be any small extension in LC_Λ. We are going to see that the mapping $F(\mathcal{B}) \to F(\mathcal{A}) \times_{G(\mathcal{A})} G(\mathcal{B})$ by $\zeta \mapsto (F(\theta)(\zeta), \varphi(\mathcal{B})(\zeta))$ is surjective. Suppose an element $(\xi, \eta) \in F(\mathcal{A}) \times_{G(\mathcal{A})} G(\mathcal{B})$ is given. Set $\eta^\dagger = G(\theta)(\eta)$. Let $\overline{\xi} \in \overline{F}(\mathcal{A})$ and $\overline{\eta} \in \overline{G}(\mathcal{B})$ be the image of ξ and η under $F(\mathcal{A}) \to \overline{F}(\mathcal{A})$ and $G(\mathcal{B}) \to \overline{G}(\mathcal{B})$, respectively. Since $\overline{\varphi}\colon \overline{F} \to \overline{G}$ is formally smooth, we can take $\overline{\zeta} \in \overline{F}(\mathcal{B})$ such that $\overline{F}(\theta)(\overline{\zeta}) = \overline{\xi}$ and $\overline{\varphi}(\mathcal{B})(\overline{\zeta}) = \overline{\eta}$. Since $F \to \overline{F}$ is formally smooth by Lemma 3.7, the set

$$F(\mathcal{B}; \overline{\xi}, \overline{\zeta}) := \{\zeta' \in F(\mathcal{B}) \mid F(\theta)(\zeta') = \xi \text{ and } \overline{\zeta}' = \overline{\zeta}\}$$

is non-empty , where $\overline{\zeta}'$ is the image of ζ' under $F(\mathcal{B}) \to \overline{F}(\mathcal{B})$. Set

$$G(\mathcal{B}; \eta^\dagger, \overline{\eta}) := \{\eta' \in G(\mathcal{B}) \mid G(\theta)(\eta') = \eta^\dagger \text{ and } \overline{\eta}' = \overline{\eta}\},$$

where $\overline{\eta}'$ is the image of η' under $G(\mathcal{B}) \to \overline{G}(\mathcal{B})$. Set $I = \mathrm{Ker}(\mathsf{v}(\theta))$. By Lemma 3.5, one sees that the group $\mathrm{Ker}(F(k[I]_0) \to \overline{F}(k[I]_0))$ (resp. $\mathrm{Ker}(G(k[I]_0) \to \overline{G}(k[I]_0))$) acts transitively on $F(\mathcal{B}; \xi, \overline{\zeta})$ (resp. $G(\mathcal{B}; \eta^\dagger, \overline{\eta})$). Moreover, the map $F(\mathcal{B}; \xi, \overline{\zeta}) \to G(\mathcal{B}; \eta^\dagger, \overline{\eta})$ given by $\varphi(\mathcal{B})$ is equivariant with the group homomorphism

$$\mathrm{Ker}(F(k[I]_0) \to \overline{F}(k[I]_0)) \longrightarrow \mathrm{Ker}(G(k[I]_0) \to \overline{G}(k[I]_0)).$$

This last homomorphism is surjective by the assumption and the isomorphy of (3.2). Hence there exists $\zeta \in F(\mathcal{B}; \xi, \overline{\zeta})$ such that $\varphi(\mathcal{B})(\zeta) = \eta$. □

Definition 3.9. *Let F be a functor on LC_Λ. A pair $(\mathcal{R}, \xi)$ consisting of $\mathcal{R} \in \mathsf{LC}_\Lambda^\wedge$ and $\xi \in F(\mathcal{R})$ is called a pseudo-hull of F if $(R, [\mathcal{R}, \xi])$ is a hull of v_*F, where $R = \mathsf{v}(\mathcal{R})$.*

Clearly the pseudo-hull is not really a "hull" of F; the induced morphism $\mathrm{h}_{\mathcal{R}} \to F$ is not formally smooth in general unless F is rigid. Besides, the map $\mathrm{t}_{\mathcal{R}} \to \mathrm{t}_F$ is not bijective: For example, the functor $\overline{\mathrm{h}}_{\mathcal{R}}$ considered in §2 has a pseudo-hull (in fact, a pseudo-pro-representative) $\mathcal{R}$, but $\mathrm{Hom}(\mathcal{R}, k[\epsilon]_0) \to \mathrm{Hom}(\mathcal{R}, k[\epsilon]_0)/\mathrm{Aut}_{k[\epsilon]}(k[\epsilon]_0)$ is not bijective in general. Again, automorphisms are the nuisances!

Definition 3.10. *A functor F on LC_Λ is said to be rigid in the first order if for any $\varphi \in \mathrm{Aut}_{k[\epsilon]}(k[\epsilon]_0)$ the bijective map $F(\varphi)\colon \mathrm{t}_F \to \mathrm{t}_F$ has no fixed point unless $\varphi = \mathrm{id}$.*

Definition 3.11. *Let F be a functor on LC_Λ. A pair $(\mathcal{R}, \xi)$ with $\mathcal{R} \in \mathsf{LC}^\wedge_\Lambda$ and $\xi \in F(\mathcal{R})$ is called a hull of F if the induced morphism $\mathrm{h}_\mathcal{R} \to F$ is formally smooth and the map $\mathrm{t}_\mathcal{R} \to \mathrm{t}_F$ is bijective.*

Then:

Proposition 3.12. (1) *Any functor on LC_Λ having a hull is rigid in the first order.*

(2) *If F is rigid in the first order and satisfies ($\mathrm{H}_1^{\log}$) and ($\mathrm{H}_2^{\log}$), then any pseudo-hull of F is a hull.*

Proof. The proof of the first assertion is quite similar to that of Proposition 3.4. For the second assertion, let $\mathcal{R}$ be a pseudo-hull of F. We first notice that, by the rigidity in the first order, the map $\mathrm{t}_\mathcal{R} \to \mathrm{t}_F$ is bijective. It is easily seen that $\overline{F}$ has the properties ($\mathrm{H}_1^{\log}$) and ($\mathrm{H}_2^{\log}$). Then the formally smoothness of $\mathrm{h}_\mathcal{R} \to F$ follows from Lemma 3.8. □

Now we state the criteria for pro-representability. We shall consider, for a functor F defined on LC_Λ, the following conditions in addition to ($\mathrm{H}_1^{\log}$) and ($\mathrm{H}_2^{\log}$) as above:

($\mathrm{H}_3^{\log}$) $\dim_k \mathrm{t}_F < +\infty$.

($\mathrm{H}_4^{\log}$) For any small extension $\mathcal{A}' \to \mathcal{A}$, the map

$$F(\mathcal{A}' \times_\mathcal{A} \mathcal{A}') \longrightarrow F(\mathcal{A}') \times_{F(\mathcal{A})} F(\mathcal{A}')$$

is bijective.

Theorem 3.13. *Let $F\colon \mathsf{LC}_\Lambda \to \mathsf{Sets}$ be a functor such that $F(k^{(1)})$ consists of one element.*

(1) *F has a pseudo-hull if and only if $\overline{F}$ satisfies ($\mathrm{H}_1^{\log}$), ($\mathrm{H}_2^{\log}$), and ($\mathrm{H}_3^{\log}$). F has a hull if and only if F satisfies ($\mathrm{H}_1^{\log}$), ($\mathrm{H}_2^{\log}$), and ($\mathrm{H}_3^{\log}$), and is rigid in the first order.*

(2) *F is pseudo-pro-representable if and only if $\overline{F}$ satisfies ($\mathrm{H}_1^{\log}$), ($\mathrm{H}_2^{\log}$), ($\mathrm{H}_3^{\log}$) and ($\mathrm{H}_4^{\log}$). F is pro-representable if and only if F satisfies ($\mathrm{H}_1^{\log}$), ($\mathrm{H}_2^{\log}$), ($\mathrm{H}_3^{\log}$) and ($\mathrm{H}_4^{\log}$), and is rigid.*

Proof. We can prove the assertions on pseudo-hulls and pseudo-pro-representatives easily by [4, (2.11)] making use of the natural bijection $\overline{F}(\mathcal{A}) \xrightarrow{\sim} \mathsf{v}_* F(A)$ by Lemma 2.5. For the other parts of the theorem, first we note that if

F satisfies the conditions ($\mathrm{H}_1^{\log}$), ($\mathrm{H}_2^{\log}$), ($\mathrm{H}_3^{\log}$) (resp. and ($\mathrm{H}_4^{\log}$)), then $\overline{F}$ also satisfies these conditions; indeed, ($\mathrm{H}_1^{\log}$), ($\mathrm{H}_2^{\log}$), and ($\mathrm{H}_4^{\log}$) can be verified without so much pain. ($\mathrm{H}_3^{\log}$) is obvious, for there exists a surjective map $\mathrm{t}_F \to \mathrm{t}_{\overline{F}}$. Then the rest assertions follow from Proposition 3.4 (1) (2) and Proposition 3.12 (1) (2). □

4. Example: log smooth deformation

Let us fix the data (D1), (D2), and (D3) as in §2 and denote the log schemes corresponding to $(M_\Lambda \to \Lambda)$ and $(M_0 \to k)$ by T and S_0, respectively. In this section it is more convenient to work in terms of log schemes; as we have seen in §2, objects in $\mathsf{LC}_\Lambda^\wedge$ correspond to certain log schemes over T together with exact closed immersion from S_0 over T. For $\mathcal{A} \in \mathsf{LC}_\Lambda^\wedge$ the associated log scheme is denoted by $\operatorname{Spec} \mathcal{A}$ (e.g., $S_0 = \operatorname{Spec} k^{(1)}$).

Let $f_0\colon X_0 \to S_0$ be a log smooth morphism of fine log schemes.

Definition 4.1. (cf. [1]) *Log smooth lifting.*

(1) *For any $\mathcal{A}$ in LC_Λ set $S = \operatorname{Spec} \mathcal{A}$ and let $\iota\colon S_0 \hookrightarrow S$ be the associated exact closed immersion over T. Then a log smooth lifting of f_0 to S is a pair (f, φ) consisting of a log smooth morphism $f\colon X \to S$ of fine log schemes and a morphism $\varphi\colon X_0 \to X$ which makes the diagram*

$$\begin{array}{ccc} X_0 & \xrightarrow{\varphi} & X \\ {\scriptstyle f_0}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ S_0 & \xrightarrow[\iota]{} & S \end{array}$$

Cartesian in the category of fine log schemes.

(2) *Let $(f\colon X \to S, \varphi)$ and $(f'\colon X' \to S, \varphi')$ be two log smooth liftings of f_0 to a common $S = \operatorname{Spec} \mathcal{A}$. Then these log smooth liftings are said to be isomorphic if there exists a morphism $\Phi\colon X \to X'$ over S such that $\Phi \circ \varphi = \varphi'$.*

The morphism Φ in (2) is actually an isomorphism since it is strict and underlying morphisms of log smooth liftings are flat. We define the functor $D_{X_0/S_0}\colon \mathsf{LC}_\Lambda \to \mathsf{Sets}$ by setting

$$D_{X_0/S_0}(\mathcal{A}) = \text{the set of isomorphism classes of log smooth liftings of } f_0 \text{ to } \operatorname{Spec} \mathcal{A}.$$

Then by [1, 8.6] we know that

$$\mathrm{t}_{D_{X_0/S_0}} \cong \mathrm{H}^1(X_0, \theta_{X_0/S_0}),$$

where θ_{X_0/S_0} is the $\mathcal{O}_{X_0}$-dual of the sheaf of log differentials $\omega^1_{X_0/S_0}$. First we should know when the functor D_{X_0/S_0} is rigid:

Proposition 4.2. *Let $\widetilde{\mathcal{N}}_{X_0}$ be the log structure on the underlying scheme of X_0 obtained by pulling-back the log structure of Λ, and let $\mathcal{N}_{X_0}$ be the image of $\widetilde{\mathcal{N}}_{X_0} \to \mathcal{M}_{X_0}$, which gives a sub-log structure of $\mathcal{M}_{X_0}$. Suppose f_0 is integral and that $f_{0*}\mathcal{O}_{X_0} = \mathcal{O}_{S_0}$. Then:*

(1) *D_{X_0/S_0} is rigid in the first order if and only if*

$$\mathrm{Hom}_{\mathbf{Z}}(\overline{\mathcal{M}}^{\mathrm{gp}}_{X_0}/\overline{\mathcal{N}}^{\mathrm{gp}}_{X_0}, \mathcal{O}_{X_0}) = 0,$$

where we regard $\mathcal{O}_{X_0}$ as a sheaf of additive groups.

(2) *D_{X_0/S_0} is rigid if and only if, for any $\mathcal{A}$ in LC_Λ and any log smooth lifting $f: X \to S$ of f_0 to* $\mathrm{Spec}\,\mathcal{A}$, *we have*

$$\mathrm{Hom}_{\mathbf{Z}}(\overline{\mathcal{M}}^{\mathrm{gp}}_{X_0}/\overline{\mathcal{N}}^{\mathrm{gp}}_{X_0}, \mathcal{J}) = 0,$$

where $\mathcal{J} = \mathrm{Ker}(\mathcal{O}^\times_X \to \mathcal{O}^\times_{X_0})$.

Note that, if $(\Lambda, M_\Lambda) \to k^{(1)}$ is strict, then D_{X_0/S_0} is always rigid. To prove the proposition, it is convenient for us to have the following lemma:

Lemma 4.3. *Let $j: X \to X'$ be a thickening of order* 1. *Suppose there exists a sub-log structures $\mathcal{N}_X \subseteq \mathcal{M}_X$ and $\mathcal{N}_{X'} \subseteq \mathcal{M}_{X'}$ such that $j^*\mathcal{N}_{X'} = \mathcal{N}_X$, $\mathcal{K}_X = \mathcal{M}_X \setminus \mathcal{N}_X$ and $\mathcal{K}_{X'} = \mathcal{M}_{X'} \setminus \mathcal{N}_{X'}$ are ideals, and that α_X and $\alpha_{X'}$ maps $\mathcal{K}_X$ and $\mathcal{K}_{X'}$ to zero, respectively. Then the set of all automorphisms $\sigma: X' \to X'$ which leaves X, the underlying scheme of X', and the sheaf $\mathcal{N}_{X'}$ invariant is canonically isomorphic to* $\mathrm{Hom}_{\mathbf{Z}}(\overline{\mathcal{M}}^{\mathrm{gp}}_X/\overline{\mathcal{N}}^{\mathrm{gp}}_X, \mathcal{J})$, *where* $\mathcal{J} = \mathrm{Ker}(\mathcal{O}^\times_{X'} \to \mathcal{O}^\times_X)$.

Proof. It is easy to see that such an automorphism $\sigma: X' \to X'$ gives rise to an element in $\mathrm{Hom}_{\mathbf{Z}}(\overline{\mathcal{M}}^{\mathrm{gp}}_X/\overline{\mathcal{N}}^{\mathrm{gp}}_X, \mathcal{J})$. Conversely, given an element in $\mathrm{Hom}_{\mathbf{Z}}(\overline{\mathcal{M}}^{\mathrm{gp}}_X/\overline{\mathcal{N}}^{\mathrm{gp}}_X, \mathcal{J})$, we easily get an automorphism $\mathcal{M}_{X'} \to \mathcal{M}_{X'}$ since $\mathcal{M}_{X'} \to \overline{\mathcal{M}}_{X'}$ is exact. We need to check this automorphism is compatible with $\alpha_{X'}$. This can be done by the similar reason as in the proof of Lemma 2.4. □

Proof of Proposition 4.2. We prove (1). Set $S = \mathrm{Spec}\,k[\epsilon]_0$ and let $(f: X \to S, \varphi)$ be a log smooth lifting. We look at the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_{\mathbf{Z}}(\overline{\mathcal{M}}^{\mathrm{gp}}_{X_0}/f_0^{-1}\overline{\mathcal{M}}^{\mathrm{gp}}_{S_0}, \mathcal{O}_{X_0}) &\longrightarrow \mathrm{Hom}_{\mathbf{Z}}(\overline{\mathcal{M}}^{\mathrm{gp}}_{X_0}/\overline{\mathcal{N}}^{\mathrm{gp}}_{X_0}, \mathcal{O}_{X_0}) \\ &\longrightarrow \mathrm{Hom}_{\mathbf{Z}}(f_0^*\overline{\mathcal{M}}^{\mathrm{gp}}_{S_0}/f_0^*\overline{\mathcal{N}}^{\mathrm{gp}}_{S_0}, \mathcal{O}_{X_0}), \end{aligned}$$

where $\mathcal{N}_{S_0}$ is the sub-log structure on S_0 defined similarly to $\mathcal{N}_{X_0}$. Then we have

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Z}}(f_0^*\overline{\mathcal{M}}^{\mathrm{gp}}_{S_0}/f_0^*\overline{\mathcal{N}}^{\mathrm{gp}}_{S_0}, \mathcal{O}_{X_0}) &= \mathrm{Hom}_{\mathbf{Z}}(\overline{\mathcal{M}}^{\mathrm{gp}}_{S_0}/\overline{\mathcal{N}}^{\mathrm{gp}}_{S_0}, f_{0*}\mathcal{O}_{X_0}) \\ &= \mathrm{Hom}_{\mathbf{Z}}(\overline{\mathcal{M}}^{\mathrm{gp}}_{S_0}/\overline{\mathcal{N}}^{\mathrm{gp}}_{S_0}, \mathcal{O}_{S_0}), \end{aligned}$$

where the last set is nothing but the set of all automorphisms of $k[\epsilon]_0$ (cf. Lemma 2.4). The resulting map

$$\gamma\colon \mathrm{Hom}_{\mathbf{Z}}(\overline{\mathcal{M}}^{\mathrm{gp}}_{X_0}/\overline{\mathcal{N}}^{\mathrm{gp}}_{X_0}, \mathcal{O}_{X_0}) \to \mathrm{Hom}_{\mathbf{Z}}(\overline{\mathcal{M}}^{\mathrm{gp}}_{S_0}/\overline{\mathcal{N}}^{\mathrm{gp}}_{S_0}, \mathcal{O}_{S_0})$$

has the following meaning: Let $\Phi\colon X \to X$ be an automorphism of X fixing X_0 and the sheaf $\mathcal{N}_X$, where $\mathcal{N}_X$ is defined similarly to $\mathcal{N}_{X_0}$. This Φ is mapped by γ to an automorphism $\phi\colon S \to S$ fixing S_0 and the sheaf $\mathcal{N}_S$ (defined similarly to $\mathcal{N}_{S_0}$) that makes the diagram

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ S & \xrightarrow[\phi]{} & S \end{array}$$

commute. Moreover, since f is a log smooth lifting, this diagram is Cartesian in the category of fine log schemes. The rigidity in the first order is equivalent to that for any such diagram we have $\phi = \mathrm{id}$, and hence, $\Phi = \mathrm{id}$. This is equivalent to that the domain of the homomorphism γ is zero. Hence we obtain (1). The second part can be proved similarly by the induction with respect to the length of A. Note that $f_{0*}\mathcal{O}_{X_0} = \mathcal{O}_{S_0}$ implies $f_*\mathcal{O}_X = \mathcal{O}_S$ for any log smooth lifting $f\colon X \to S$ since the underlying morphism of f is flat by the integrality of f_0. □

The following theorem can be proved by an argument similar to that in [1, §9] and the results we have so far obtained:

Theorem 4.4. *Suppose that f_0 is integral, the underlying morphism of f_0 is proper, and that $f_{0*}\mathcal{O}_{X_0} = \mathcal{O}_{S_0}$. Then:*

(1) *The functor D_{X_0/S_0} has a pseudo-hull. (Actually, D_{X_0/S_0} itself satisfies ($\mathrm{H}_1^{\log}$), ($\mathrm{H}_2^{\log}$), and ($\mathrm{H}_3^{\log}$).)*
(2) *The functor D_{X_0/S_0} has a hull if and only if*

$$\mathrm{Hom}_{\mathbf{Z}}(\overline{\mathcal{M}}^{\mathrm{gp}}_{X_0}/\overline{\mathcal{N}}^{\mathrm{gp}}_{X_0}, \mathcal{O}_{X_0}) = 0.$$

In particular, if $(\Lambda, M_\Lambda) \to k^{(1)}$ is strict, then D_{X_0/S_0} has a hull.
(3) *Suppose D_{X_0/S_0} is rigid. Then the hull of D_{X_0/S_0} pro-represents D_{X_0/S_0} if and only if, for each small extension $\mathcal{A}' \to \mathcal{A}$, and each log smooth lifting (f', φ') to $\mathcal{A}'$, every automorphism of $(f', \varphi') \otimes_{\mathcal{A}'} \mathcal{A}$ is induced by an automorphism of (f', φ'). In particular, if D_{X_0/S_0} is rigid and*

$$\mathrm{H}^0(X_0, \theta_{X_0/S_0}) = 0,$$

then D_{X_0/S_0} is pro-representable.

Example 4.5. Let X_0 be a d-semistable normal crossing variety over k and put on it the log structure of semistable type (cf. [1, §11]). We consider the log smooth deformation of $f_0\colon X_0 \to S_0$, where S_0 is the standard point (cf. Example 2.11).

(1) If the log structure on Λ is trivial, then every log smooth lifting of f_0 is underlain by a an "equi-singular" flat deformation, i.e., the one étale locally written by $\operatorname{Spec} A[z_0, \ldots, z_n]/(z_0 \cdots z_d)$. (In this case the log smooth deformation functor is not necessarily rigid in the first order.)
(2) If the log structure on Λ is defined as in Example 2.11 (2), then the log smooth deformation functor is rigid and log smooth liftings of f_0 behave similarly to those discussed in [1, §11].

Remark 4.6. The log smooth deformation of normal crossing varieties explained in Example 4.5 (2) has been first introduced by Kawamata-Namikawa [3]. The paper [1] generalized this idea to the general theory of log smooth deformations. But in [1] (and also [3]) the theory of categories of log Artin rings has not been explicitely treated, although the idea of its special case has been certainly used. In our term, the situation in [3] and [1] was set up by putting $\Lambda = R[[P]]$ (the completion of $R[P]$ along the ideal $R[P\setminus\{1\}]+\mathfrak{m}_R$) and $P = Q$. In particular, the deformation functors in those papers are rigid. In this sense, the log smooth deformation theory explained in this section generalize that of [1].

References

[1] Kato, F.: Log smooth deformation theory. Tohoku Math. J. **48**, 317–354 (1996)
[2] Kato, K.: Logarithmic structures of Fontaine–Illusie. In: *Algebraic Analysis, Geometry, and Number Theory*, (J.-I. Igusa, ed.). Johns Hopkins Univ., 1988, pp. 191–224
[3] Kawamata, Y. and Namikawa, Y.: Logarithmic deformations of normal crossing varieties and smoothing of degenerate Calabi-Yau varieties. Invent. Math. **118**, 395–409 (1994)
[4] Schlessinger, M.: Functors of Artin rings. Trans. Am. Math. Soc. **130**, 208–222 (1968)
[5] Grothendieck, A., Deligne, P., and Katz, N.: *Groupes de monodromie en géométrie algébrique*. Lecture Notes in Math. **288**, **340**, Berlin–Heidelberg–New York: Springer-Verlag, 1972–1973