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ON POSITIVE QUATERNIONIC KÄHLER MANIFOLDS WITH CERTAIN SYMMETRY RANK

BY

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ABSTRACT

Let M be a positive quaternionic Kähler manifold of real dimension $4m$. In this paper we show that if the symmetry rank of M is greater than or equal to $[m/2]+3$, then M is isometric to $\mathbf{HP}^m$ or $Gr_2(\mathbf{C}^{m+2})$. This is sharp and optimal, and will complete the classification result of positive quaternionic Kähler manifolds equipped with symmetry. The main idea is to use the connectedness theorem for quaternionic Kähler manifolds with a group action and the induction arguments on the dimension of the manifold.

1. Introduction and main results

The concept of the symmetry rank sym-rank(M) of a Riemannian manifold (M,g) was first introduced by Grove and Searle in [12], in order to measure the amount of symmetry of M. Here the symmetry rank is defined as the rank of the isometry group Isom(M,g). Equivalently, it can be defined as the largest number r such that a r-dimensional torus acts effectively and isometrically on M.

Grove and Searle showed in [12] that the symmetry rank sym-rank(M) of a positively curved manifold is less than or equal to $\left[\frac{\dim M+1}{2}\right]$ and that the maximal rank case holds if and only if the manifold M is diffeomorphic to the

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unit sphere or the complex projective space or a lens space. In fact, this result seemed to be motivated by the homeomorphism classification of 4-dimensional positively curved manifold with an isometric effective S^1-action by Hsiang and Kleiner in [15]. In his paper [21], Rong gave various classification results of positively curved manifolds with almost maximal symmetry rank case. Recently there has been a remarkable progress on the the classification of positively curved manifolds with an isometric torus action. Such a progress is mainly due to the connectedness theorem and its applications by Wilking in [23], Fang, Mendonça and Rong in [6] and Fang and Rong in [7].

A compact quaternionic Kähler manifold M is a Riemannian manifold of real dimension $4m$ whose holonomy group is contained in the Lie group $Sp(m)Sp(1)$ in $SO(4m)$ for $m \geq 2$. Such a manifold is called **positive** if it has the positive scalar curvature. It is known that every quaternionic Kähler manifold is Einstein. So it is customary to define a 4-dimensional quaternionic Kähler manifold to be both Einstein with non-zero scalar curvature and self-dual. While many complete, non-compact, non-symmetric quaternionic Kähler manifolds with negative scalar curvature are known to exist, so far the only known examples of compact positive quaternionic Kähler manifolds are symmetric. Moreover, a theorem of Alekseevsky asserts that there are no other compact homogeneous positive quaternionic Kähler manifolds (see, e.g., [1]).

Hitchin proved in [14] that every positive quaternionic Kähler 4-manifold must be isometric to $\mathbf{CP}^2$ and S^4. In case of dimension 8, Poon and Salamon showed in [20] that every positive quaternionic Kähler manifold should be isometric to $\mathbf{HP}^2$, $Gr_2(\mathbf{C}^4)$ or $G_2/SO(4)$, i.e. Wolf spaces. In their paper [19], Podesta and Verdiani showed that a positive quaternionic Kähler manifold which is acted on isometrically by a compact Lie group with a hypersurface orbit is a symmetric space. See also the paper [5] of Dancer and Swann. Moreover, in [13] Herrera and Herrera gave the classification of positive quaternionic Kähler 12-dimensional manifolds under an isometric S^1-action. In particular, according to a result of Lebrun and Salamon in [17], every positive quaternionic Kähler manifold M is simply connected and the second homotopy group π_2 is a finite group with 2-torsion, trivial or $\mathbf{Z}$. More precisely, M is isometric to $\mathbf{HP}^m$ (resp., $Gr_2(\mathbf{C}^{m+2})$) if $\pi_2(M) = 0$ (resp., $\pi_2(M) = \mathbf{Z}$). (See [8] and the references therein for more results.)

On the other hand, it is an interesting problems to classify positive quaternionic Kähler manifolds in terms of the rank of its isometry group. In particular, in [3] Bielawski classified positive quaternionic Kähler manifolds of dimension $4m$ with isometry rank $m+1$. In [9] Fang gave an interesting classification result of positive quaternionic Kähler manifolds with symmetry using an extension of the connectedness theorem of Wilking, and independently Fang, Mendonça and Rong for positively curved manifolds as follows.

Theorem 1.1: *Let M be a positive quaternionic Kähler manifold of dimension $4m$. Then the isometry group* $\mathrm{Isom}(M)$ *has a symmetry rank* sym-rank (M) *at most $m+1$. Moreover, if the symmetry rank* sym-rank (M) *is greater than or equal to $\frac{m+6}{2}$, then M is isometric to $\mathbf{HP}^m$ or $Gr_2(\mathbf{C}^{m+2})$.*

If m is even, the theorem is sharp, since the symmetry rank of the oriented real Grassmannian $\widetilde{Gr_4}(\mathbf{R}^{m+4})$ is $\frac{m+4}{2}$ and $\widetilde{Gr_4}(\mathbf{R}^{m+4})$ is not one of the cases Theorem 1.1. Note also that the symmetry rank equals $m+1$ if M is $\mathbf{HP}^m$ or $Gr_2(\mathbf{C}^{m+2})$. In the same paper Fang conjectured that Theorem 1.1 would be improved a little bit further, if m is odd.

The aim of the present paper is to confirm his conjecture. The main strategy of this paper is the connectedness theorem in the presence of a Lie group acting isometrically and fixing a quaternionic Kähler submanifold pointwise. Note that in [8] Fang has already established the connectedness theorem for positive quaternionic manifolds without the presence of a Lie group action. In this paper we extend his connectedness theorem to the case that admits a Lie group action, following the line of Wilking in [23].

More precisely, in this paper we prove the following theorem which is Conjecture 1.3 in [9]:

Theorem 1.2: *Let M be a positive quaternionic Kähler manifold of dimension $4m$. If the symmetry rank* sym-rank (M) *is greater than or equal to $\left[\frac{m}{2}\right]+3$, then M is isometric to $\mathbf{HP}^m$ or $Gr_2(\mathbf{C}^{m+2})$.*

If m is odd, this theorem is sharp again, since the symmetry rank of $\widetilde{Gr_4}(\mathbf{R}^{m+4})$ is $\frac{m+3}{2}$ and $\widetilde{Gr_4}(\mathbf{R}^{m+4})$ is not one of the cases Theorem 1.2.

We organize this paper as follows. In Section 2, we prove the connectedness theorem in the presence of a Lie group acting isometrically and fixing a quaternionic Kähler submanifold pointwise. As remarked above, the connectedness theorem for positive quaternionic manifolds without a Lie group action has already been obtained in [8]. For the proof of the case that admits a Lie group action, it turns out that it suffices to closely follow the work of Wilking in [23]. Section 3 deals with positive quaternionic Kähler manifold M with an isometric S^1-action. An important result of this section is Proposition 3.1 which is equivalent to Lemma 4.1 in [9]. For the sake of completeness, we will give a relatively detailed proof of it. In Section 4, we give a proof of Theorem 1.2 using the induction arguments on the dimension of positive quaternionic Kähler manifolds.

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2. The Connectedness Theorem in the presence of symmetry

The goal of this section is to prove Theorem 2.1. It is called the **Connectedness Theorem** which will play a key role in the proof of Theorem 1.2. We use the same notations as in [9].

From now on, a Lie group G will always be assumed to be compact and every action is isometric and effective. Recall also that a map $f : N \to M$ between two manifolds is called **h-connected** if the induced map $f_* : \pi_i(N) \to \pi_i(M)$ is an isomorphism for all $i < h$ and an epimorphism for $i = h$. If f is an imbedding this is equivalent to saying that up to a homotopy, M can be obtained from $f(N)$ by attaching cells of dimension $\geq h + 1$.

Now we are ready to prove the main result of this section

Theorem 2.1: *Let M be a positive quaternionic Kähler manifold of dimension $4m$. If N is a quaternionic Kähler submanifold of dimension $4n$, then the inclusion $N \hookrightarrow M$ is $(2n - m + 1)$-connected. Furthermore, if there is a Lie group G acting isometrically on M and a pointwise fixing N, then the inclusion map is $(2n - m + 1 + \delta(G))$-connected, where $\delta(G)$ is the dimension of the principal orbit of G.*

Proof. This is an adaptation of the proof of Theorem A in [8] and Theorem 1 in [23], and the first part of it is due to Fang. Thus we will highlight only distinct points, compared to the proof in [8] or [23].

Let $\iota : N \to M$ be the inclusion, and let $f : L \to M \times M$ denote the map $f = (\iota, \iota)$, where $L = N \times N$. Let $P(M)$ denote the space of all piecewise smooth paths in M. We then define the space $P(M, f)$ of all piecewise smooth paths (x, γ) in $L \times P(M)$, where $x \in L$ and $\gamma : [0, 1] \to M$ such that γ is piecewise smooth and such that $f(x) = (\gamma(0), \gamma(1))$. The topology of $P(M, f)$ is induced from that of $L \times P(M)$.

As usual we then define the energy function E by

$$E : P(M, f) \to \mathbf{R}, \quad E(x, \gamma) = \int_0^1 \|\dot\gamma(t)\|^2 \, dt.$$

Then the set of paths having energy less than or equal to c may be approximated by a finite dimensional manifold of broken geodesics. Notice that N is naturally imbedded into $P(M, f)$ as the set of constant paths and that $E^{-1}(0) = N$. Then we want to show that the inclusion

$$N \hookrightarrow P(M, f)$$

is $(2n - m + 1 + \delta(G))$-connected.

To do so, notice first that the critical points of the energy function are geodesics (x, γ) such that $(\dot\gamma(0), -\dot\gamma(1))$ is perpendicular to $f_*(T_x L)$ at $f(x) = (\gamma(0), \gamma(1))$. Then we estimate the index of the energy function along such a geodesic (x, γ) from below, as follows. Indeed, let W be a piecewise smooth and normal parallel vector field along γ such that

$$(W(0), W(1)) \in f_*(T_x L) \tag{2.1}$$

at $f(x) = (\gamma(0), \gamma(1))$. Then as in [8] we may assume that IW, JW and KW are all parallel and satisfy the condition (2.1) at $f(x) = (\gamma(0), \gamma(1))$, since f is a quaternionic imbedding.

Suppose that a Lie group G acts isometrically on M fixing N pointwise. Then choose a G-invariant metric $\langle \cdot, \cdot \rangle$ on G. Consider the diagonal free action of G on $M \times G$, and for $\lambda > 0$ let

$$(M, g_\lambda) = ((M, g) \times (G, \lambda \langle \cdot, \cdot \rangle))/G,$$

where g_λ denotes the induced orbit metric of G on M. In order to obtain a better estimate for the connectedness in the presence of the symmetry, we need to calculate the index of E on $\mathcal{W}$ spanned by piecewise smooth and normal parallel vector fields W along γ which are orthogonal to the G-orbit of γ. This is the main reason we use this kind of complementary construction. In other words, we need to use an orbit metric on M, and g_λ on M for a sufficiently small positive λ will play such a role.

Then the following observation of Theorem 3.20 of O'Neill in [4] will be useful in a later discussion. That is, if the sectional curvature of (M, g) is positive, then the sectional curvature of (M, g_λ) is positive as well (see below for more details), and the geodesic $\gamma : [0,1] \to (M, g)$ orthogonal to the G-orbits is also a geodesic of length $\|\gamma'(0)\|_g$ with respect to the metric g_λ. Moreover, for sufficiently small λ the index of E along (x,γ) with respect to the metric g is equal to the index of E along (x,γ) with respect to the metric g_λ.

Now we need to consider two cases, depending on the isotropy group of $\gamma(t)$: trivial or nontrivial. If the isotropy group of $\gamma(t)$ is trivial for some t, then we let $\mathcal{W}$ be the quaternionic vector space spanned by piecewise smooth and normal parallel vector fields W along γ such that

(a) $(W(0), W(1)) \in f_*(T_xL)$ for $f(x) = (\gamma(0), \gamma(1))$,
(b) W, IW, JW and KW are all orthogonal to $G(\gamma(t))$ for all $t \in [0,1]$.

Then clearly the quaternionic dimension of $\mathcal{W}$ satisfies

$$\dim_{\mathbb{H}} \mathcal{W} \geq 2n - m + 1 + \delta(G).$$

Further, as in Section 2 of [23], it does not depend on the choice of λ.

Now it remains to calculate the index of the second variation Q of E along (x,γ). To do so, it suffices to show that the index of E is greater than or equal to $\dim_{\mathbb{H}} \mathcal{W}$. Since the imbedding f is totally geodesic, for a piecewise smooth and normal vector field W along γ with respect to the metric g such that $(W(0), W(1)) \in f_*(T_xL)$ at $f(x) = (\gamma(0), \gamma(1))$ the second variation formula of E along a critical point (x,γ) (see, e.g., p.17 in [8]) yields

(2.2)
$$\begin{aligned}
&\frac{1}{2}(Q(W,W) + Q(IW,IW) + Q(JW,JW) + Q(KW,KW)) \\
&\qquad = -\int_0^1 (\langle R(\dot\gamma, W)W, \dot\gamma\rangle + \langle R(\dot\gamma, IW)IW, \dot\gamma\rangle + \langle R(\dot\gamma, JW)JW, \dot\gamma\rangle \\
&\qquad\qquad + \langle R(\dot\gamma, KW)KW, \dot\gamma\rangle)\, dt \\
&\qquad = -\int_0^1 \frac{\mu}{m+2} \|W\|^2 \cdot \|\dot\gamma\|^2\, dt < 0,
\end{aligned}$$

where μ is the positive Einstein constant of M, and we use the identity in the Proposition 1.2 of [8].

Moreover, the first line of (2.2) for the vector field W in the space $\mathcal{W}$ with respect to the metric g_λ decreases as λ goes to zero. To be precise, by the formula of O'Neill for the submersion (Theorem 3.20 in [4]) we have

$$K_g(\dot\gamma(t), W) = K_{g_\lambda}(\dot\gamma(t)^H, W^H) + \frac{3\lambda^2}{4}\|[\dot\gamma(t)^H, W^H]^V\|^2_{\langle\cdot,\cdot\rangle},$$

where K denotes the sectional curvature, $\dot\gamma(t)^H$ and W^H denote the horizontal lifts of $\dot\gamma(t)$ and W, respectively, and $[\dot\gamma(t)^H, W^H]^V$ denotes the vertical lift of $[\dot\gamma(t)^H, W^H]$ for the submersion

$$(M, g) \times (G, \lambda\langle\cdot,\cdot\rangle) \to ((M \times G)/G, g) = (M, g). \tag{2.3}$$

This implies that the index induced from the second variation of E along (x, γ) for the metric g_λ is still negative definite on $\mathcal{W}$ for sufficiently small λ, since the metric g_λ is the induced orbit metric on M and the horizontal lift of a tangent vector on M in the space $\mathcal{W}$ for the submersion (2.3) can be identified with a tangent vector field which is orthogonal to the G-orbit on M. Thus the index of E restricted to the subspace $\mathcal{W}$ is greater than or equal to $\dim_{\mathbb{H}} \mathcal{W}$. Since the index of E is greater than or equal to the index of E restricted to the subspace $\mathcal{W}$, we can see that the index of E is greater than or equal to $(2n-m+1+\delta(G))$.

Next, if the isotropy group of $\gamma(t)$ is a nontrivial proper subgroup of G for all t, then there exists a subgroup H of G such that γ lies in a component F of the fixed point set of H. Let l be the codimension of F in M. Then the codimension of N in F is given by $m - n - l$. Thus if we use the induction on the dimension m, we may assume that the index of E is greater than or equal to

$$2n - (m - l) + 1 + \delta(Z) = 2n - m + l + \delta(Z) + 1 \geq 2n - m + 1 + \delta(G),$$

where Z denotes the normalizer of H in G and $\delta(Z)$ is the dimension of a principal Z-orbit in F, and we used the inequality $\delta(Z) \geq \delta(G) - l$. Hence the result follows again.

Finally, notice that there exists a Morse Bott function E' on $P(M, f)$ or a finite dimensional approximation of $E^{-1}([0, c])$, C^∞-close to E such that $E = E'$ in a neighborhood of $E^{-1}(0)$ and any critical point in $P(M, f)\backslash N$ of E' is non-degenerate and has index greater than or equal to $2n - m + 1 + \delta(G)$. This implies that up to homotopy, $P(M, f)$ can be obtained from N by attaching cells of dimension greater than or equal to $2n-m+1+\delta(G)$. Hence the inclusion $N \hookrightarrow P(M, f)$ is $(2n - m + \delta(G))$-connected.

On the other hand, the relative homotopy group $\pi_i(M,N)$ satisfies

$$\pi_i(M,N) \cong \pi_{i-1}(P(M,f),N)$$

for all i. Since $\pi_i(P(M,f),N) = 0$ for all $i = 0,1,2,\ldots,2n-m+\delta(G)$, it follows from the homotopy exact sequence for a pair that the inclusion map $N \hookrightarrow M$ is $(2n-m+1+\delta(G))$-connected. ∎

3. Quaternionic Kähler manifolds with an isometric S^1-action

The goal of this section is to prove Proposition 3.1 which deals with positive quaternionic Kähler manifold M with an effective isometric S^1-action. To do so, we need a concept of the moment map on a positive quaternionic Kähler manifold M with an isometric S^1-action. (See [2] for more details.) For the existence of quaternion-Kähler moment map, see [10].

Let X be the Killing vector field generated by the S^1-action, and let $\overline{X}$ be its dual 1-form with respect to the Riemannian metric. Let S^2H be the bundle given by the adjoint representation of $Sp(1)$. Then the bundle S^2H has the Lie aglebra $\mathbf{sp}(1)$ as the fiber and has the local basis $\{I_1.I_2,I_3\}$, corresponding to the three elements $i,j,k \in Sp(1)$, that are three almost complex structures such that $I_1I_2 = -I_2I_1 = I_3$. Let ω_1, ω_2, and ω_3 be the locally defined 2-forms associated to the three almost complex structures I_1, I_2 and I_3, respectively. Let Ω be the closed non-degenerate 4-form defined by

$$\Omega = \sum_{i=1}^{3} \omega_i \wedge \omega_i.$$

Then the moment map μ on M is defined to be a section of S^2H solving the following equation

$$\nabla\mu = \sum_{i=1}^{3} \overline{I_i X} \otimes I_i \quad (\text{or } d\mu = \iota_X\Omega), \tag{3.1}$$

where ∇ denotes the Levi-Civita connection of the Riemannian metric. Since the moment map takes its values in a fiber bundle, $\mu^{-1}(0)$ means the inverse image of the zero section.

The following proposition plays an important role in the proof of Theorem 1.2. We remark that the proof given below is a detailed extension of the proof of Lemma 4.1 in [9]. We give it here for the sake of completeness.

PROPOSITION 3.1: *Let M be a positive quaternionic Kähler manifold of dimension $4m$ with an effective isometric S^1-action ($m \geq 3$). Let μ be the moment map defined by (3.1). Let N be a fixed point component of codimension 4 in M of the S^1-action which is contained in $\mu^{-1}(0)$. Then we have the following two cases:*

(1) *if $b_2(M) = 0$ then M is isometric to $\mathbf{HP}^m$;*
(2) *if $b_2(M) \neq 0$ then M is isometric to $Gr_2(\mathbf{C}^{m+2})$.*

Proof. To prove it, we need a series of lemmas:

LEMMA 3.2: *Assume that the second Betti number $b_2(M)$ is equal to 0. The following identity holds:*

$$P_t(M) - P_t(N) = \sum_F t^{\lambda_F} P_t(F), \tag{3.2}$$

where the sum is taken over all fixed point components F outside $\mu^{-1}(0)$, λ_F denotes the Morse index of F, and P_t denotes the Poincaré polynomial for the cohomology with rational coefficients.

Proof. Note first from Lemma 3.6 of [9] that N coincides with $\mu^{-1}(0)$. Lemma 3.6 of [9] is a consequence of the result of circle actions in hyperKähler case which is analogous to the stratification theorem of Lerman and Sjamaar in symplectic geometry; see [18]. Thus the set $\mu^{-1}(0)$ is in fact a quaternionic Kähler submanifold of M. However, in general, $\mu^{-1}(0)$ is not a smooth submanifold of M unless S^1 acts on it with only finite isotropy groups; see [2]. This together with Proposition 3.6 of [2] implies that the function $f = \|\mu\|^2$ is non-degenerate in the sense of Bott and that all critical submanifolds, except the minimum $\mu^{-1}(0)$, always have even Morse index greater than or equal to $2m$. It is also true from Proposition 4.3 of [2] that the non-degenerate Morse function f is equivariantly perfect over the rationals in the sense of

$$\hat{P}_t(M) = \hat{P}_t(\mu^{-1}(0)) + \sum_F t^{\lambda_F} \hat{P}_t(F), \tag{3.3}$$

where $\hat{P}_t$ denotes the equivariant Poincaré polynomial for the equivariant cohomology with rational coefficients. On the other hand, since by Lemma 2.2 in [2] we have

$$H^*_{S^1}(M; \mathbf{Q}) = H^*(M; \mathbf{Q}) \otimes H^*(BS^1; \mathbf{Q}),$$

we obtain $\hat{P}_t(M) = P_t(M)P_t(BS^1)$. It is also easy to see that $\hat{P}_t(F) = P_t(F)P_t(BS^1)$ and $\hat{P}_t(N) = P_t(N)P_t(BS^1)$. Hence we have the required identity from (3.3). This completes the proof. ∎

Lemma 3.3: *Assume that the second Betti number $b_2(M)$ is equal to* 0.

(1) *If the fixed point component F outside $\mu^{-1}(0)$ has dimension greater than 0, then $\dim_{\mathbf{R}} F + \lambda_F$ is less than or equal to $4m-4$;*
(2) *if the fixed point component F outside $\mu^{-1}(0)$ has dimension equal to 0, then F is an isolated fixed point with the Morse index λ_F equal to $4m$ or less than $4m-2$.*

Proof. Since $b_2(M) = 0$, the left hand side of (3.2) is of the form

$$\begin{aligned} t^{4m} &+ b_2(M)t^{4m-2} + \text{lower order terms of degree} \leq 4m-4 \\ &= t^{4m} + \text{lower order terms of degree} \leq 4m-4. \end{aligned} \tag{3.4}$$

Suppose that $\dim_{\mathbf{R}} F + \lambda_F$ equals $4m-2$. Then the right hand side of (3.2) is of the form $t^{4m-2} + \cdots$, which is a contradiction to (3.4). Next suppose that $\dim_{\mathbf{R}} F + \lambda_F$ is equals to $4m$. Then, since F is a compact Kähler submanifold of M by Proposition 3.4 in [2] and thus has nontrivial Betti number at every even degree, the right hand side of (3.2) is of the form $t^{4m} + ct^{4m-2} + \cdots$ for some positive integer c. This is again a contradiction to (3.4). This completes the proof of (1).

The proof of (2) is similar. ∎

Finally we are ready to give a proof of the proposition. To do so, note that the inclusion $N \to M$ is a $(2m-1)$-equivalence, since the Morse index λ_F is even and greater than or equal to $2m$. Next apply Morse theory together with Lemma 3.3 to deduce that M is homotopic to $N \cup e^{\lambda_i} \cup e^{4m}$, where $2m \leq \lambda_i \leq \dim_{\mathbf{R}} F + \lambda_i \leq 4m-4$ and e^{λ_i} denotes the λ_i-cell. Hence $H^{4m-2}(M, N; \mathbf{Z}) = 0$ and also we have

$$H_2(M; \mathbf{Z}) = H_2(M-N; \mathbf{Z}) = H^{4m-2}(M, N; \mathbf{Z}) = 0,$$

where we use the fact that N is of codimension 4 in the second equality and the Alexander duality in the third one. But this implies that $\pi_2(M) = 0$. Hence M is isometric to $\mathbf{HP}^m$ by a result of Lebrun and Salamon in [17].

On the other hand, if $b_2(M)$ is not zero, then again we can apply the result of Lebrun and Salamon in [17] to deduce that M is isometric to $Gr_2(\mathbf{C}^{m+2})$. This completes the proof. ∎

4. Proof of Main Result

In this section we give a proof of Theorem 1.2 by combining Proposition 3.1 with the induction arguments on the dimension of the manifold. To do so, it is obvious that it suffices to prove the following theorem.

Theorem 4.1: *Let M be a positive quaternionic Kähler manifold of dimension $4m$ with a positive odd integer m. If* sym-rank $(M) \geq (m+5)/2$, *then M is isometric to $\mathbf{HP}^m$ or $Gr_2(\mathbf{C}^{m+2})$.*

Proof. Throughout the proof, we shall assume without loss of generality that $m \geq 3$.

Assume that the rank of M is r. So we may assume that T^r acts isometrically on M. Since the Euler characteristic χ of a positive quaternionic Kähler manifold is always positive [17] and coincides with the Euler characteristic of the fixed-point set [16], there exists a fixed point x in M. Thus we can consider the isotropy representation of T^r at the fixed point x. Let N denote a quaternionic Kähler submanifold of M passing through x fixed by a circle subgroup of T^r obtained from considering the isotropy representation. Then N is totally geodesic and quaternionic Kähler by a theorem of Grove in [11]. If N is of codimension 4, then N should be contained in the set $\mu^{-1}(0)$ by [2], provided $m \geq 3$. So we are done by Proposition 3.1. Otherwise, we need to use the induction on the dimension m of the positive quaternionic Kähler manifold M.

Since m is odd, it follows from Theorem 1.1 that we have

$$2m+2 \leq \dim N \leq 4m-8 \tag{4.1}$$

with $\operatorname{rank}(N) = r-1 \geq (m+3)/2$. Moreover, since N is fixed by a circle subgroup of T^r the inclusion map $N \hookrightarrow M$ is $(\frac{1}{2}\dim N - m + 2)$-connected by Theorem 2.1. Since

$$\frac{1}{2}\dim N - m + 2 \geq m+1-m+2 = 3,$$

the inclusion is at least 3-connected. Hence we can conclude that we have $\pi_2(N) \cong \pi_2(M)$.

To begin the induction on the dimension of M, it suffices to first consider the case M is of dimension 20 with rank ≥ 10. (Note that if $\dim M$ is less than or equal to 12 then there exist no quaternionic submanifolds N of codimension 8 with rank $(N) \geq 3$.) In this case we have $\dim N = 12$ with $\mathrm{rank}(N) = r - 1 \geq 4$ by the equation (4.1). But then N should be isometric to $\mathbf{HP}^3$ or $Gr_2(\mathbf{C}^5)$ (see, e.g., [13]). Hence $\pi_2(M)$ is 0 or $\mathbf{Z}$. This implies that M should also be isometric to $\mathbf{HP}^4$ or $Gr_2(\mathbf{C}^6)$ by a classification of Lebrun and Salamon in [17]. Now assume that the theorem is true for all $l \leq m-1$ with $m \geq 7$. Then it follows from (4.1) that again we have a quaternionic Kähler submanifold N of dimension $4(m-j)$ $(j = 2, \ldots, (m-1)/2)$ with $\mathrm{rank}(N) \geq (m+3)/2$. Since we have

$$\mathrm{rank}\,(N) \geq \frac{1}{2}(m+3) \geq \begin{cases} \frac{1}{2}(m-j+5), & \text{if } j = \text{even} \\ \frac{1}{2}(m-j+6), & \text{if } j = \text{odd} \end{cases}$$

for all $j = 2, \ldots, (m-1)/2$, it follows from the induction hypothesis on the dimension and Theorem 1.1 (in [9]) that N should be isometric to $\mathbf{HP}^{m-j}$ or $Gr_2(\mathbf{C}^{2+m-j})$. Since $\pi_2(N) \cong \pi_2(M)$, this in turn implies that M is isometric to $\mathbf{HP}^m$ or $Gr_2(\mathbf{C}^{2+m})$. This completes the proof. ∎

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