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CONSTRUCTIVE
APPROXIMATION

Frames Adapted to a Phase-Space Cover

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Abstract We construct frames adapted to a given cover of the time–frequency or time-scale plane. The main feature is that we allow for quite general and possibly irregular covers. The frame members are obtained by maximizing their concentration in the respective regions of phase-space. We present applications in time–frequency, wavelet, and Gabor analysis.

Keywords Phase-space · Localization operator · Frame · Short-time Fourier transform · Time–frequency analysis · Time-scale analysis

Mathematics Subject Classification 42C15 · 42C40 · 41A30 · 41A58 · 40H05 · 47L15

1 Introduction

A time–frequency representation of a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ is a function defined on $\mathbb{R}^d \times \mathbb{R}^d$ whose value at $z = (x, \xi)$ represents the influence of the frequency ξ near x. The short-time Fourier transform (STFT) is a standard choice for such a representation, popular in analysis, and signal processing. It is defined, by means of an adequate smooth and fast-decaying window function $\varphi \in \mathcal{S}(\mathbb{R}^d)$, as

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$$V_\varphi f(z) = \int_{\mathbb{R}^d} f(t)\overline{\varphi(t-x)}e^{-2\pi i\xi t}\mathrm{d}t, \quad z=(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d. \tag{1}$$

If the window φ is normalized by $\|\varphi\|_2 = 1$, the distribution f can be resynthesized from its time–frequency content by

$$f(t) = V_\varphi^* V_\varphi f(t) = \int_{\mathbb{R}^d\times\mathbb{R}^d} V_\varphi f(x,\xi)\varphi(t-x)e^{2\pi i\xi t}\mathrm{d}x\mathrm{d}\xi, \qquad t\in\mathbb{R}^d. \tag{2}$$

This representation is extremely redundant. One of the aims of time–frequency analysis is to provide a representation of an arbitrary signal as a linear combination of elementary time–frequency atoms, which form a less redundant dictionary. The standard choice is to let these atoms be time–frequency shifts of a single window function φ, thus providing a uniform partition of the time–frequency plane. The resulting systems of atoms are known as *Gabor frames*. However, in certain applications, atomic decompositions adapted to a less regular pattern may be required (see, for example, [4,15,32,46]).

For example, a time–frequency partition may be derived from perceptual considerations. For audio signals, this means that low frequency bins are given a finer resolution than bins in high regions, where better time-resolution is often desirable, cp. [48,52]. Such a partition is schematically depicted in the left plot of Fig. 1.

More irregular partitions may be desirable whenever the frequency characteristics of an analyzed signal change over time and require adaptation in both time *and* frequency. For example, adaptive partitions obtained from information theoretic criteria were suggested in [39,41]. In such a situation, partitions as irregular as shown in the right plot of Fig. 1 can be appropriate.

In this article, we consider the following problem. Given a—possibly irregular—cover of the time–frequency plane $\mathbb{R}^{2d}$, we wish to construct a frame for $L^2(\mathbb{R}^d)$ with atoms whose time–frequency concentration follows the shape of the cover members. This allows us to vary the trade-off between time and frequency resolution along the time–frequency plane. The adapted frames are constructed by selecting, for each member of a given cover, a family of functions maximizing their concentration in the corresponding region of the time–frequency domain, or phase-space. These functions can be obtained as eigenfunctions of time–frequency localization operators, as we now describe.

Given a bounded measurable set $\Omega\subseteq\mathbb{R}^{2d}$ in the time–frequency plane, the *time–frequency localization operator* H_Ω is defined by masking the coefficients in (2), cf. [16,17],

$$H_\Omega f(t) = V_\varphi^*(1_\Omega V_\varphi f)(t) = \int_\Omega V_\varphi f(x,\xi)\varphi(t-x)e^{2\pi i\xi t}\mathrm{d}x\mathrm{d}\xi. \tag{3}$$

H_Ω is self-adjoint and trace-class, so we can consider its spectral decomposition

$$H_\Omega f = \sum_{k=1}^{\infty}\lambda_k^\Omega\langle f,\phi_k^\Omega\rangle\phi_k^\Omega, \tag{4}$$

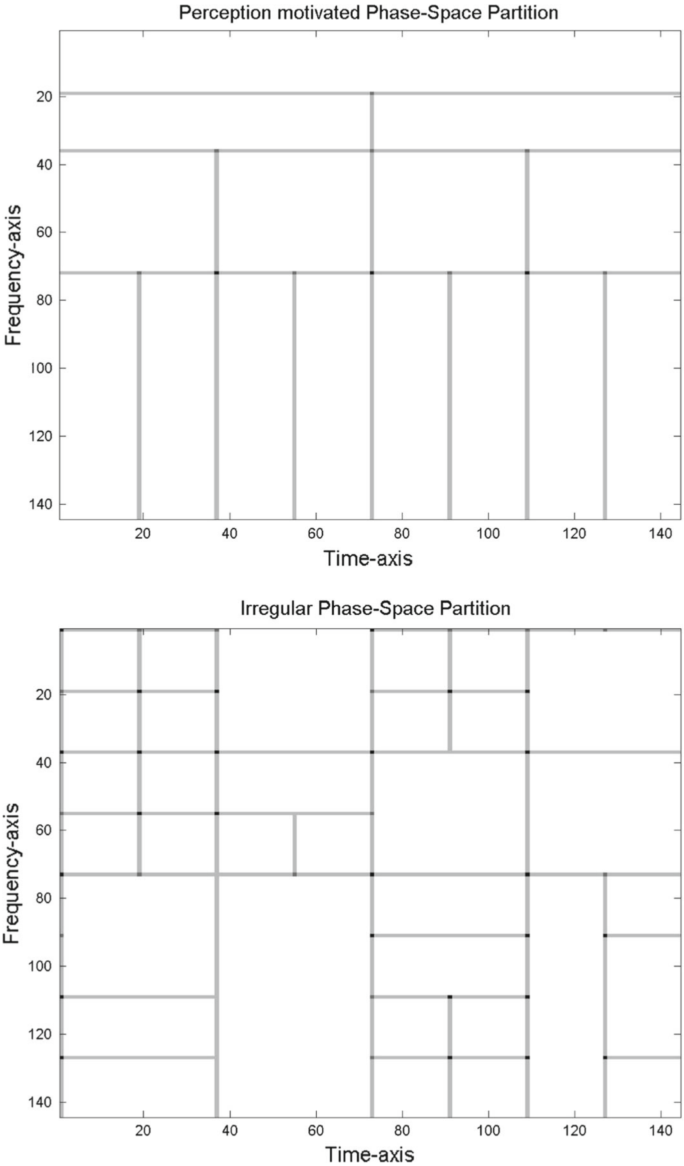

Fig. 1 Partitions in time–frequency

where the eigenvalues are indexed in descending order. Since $\langle H_\Omega f, f\rangle = \left\langle 1_\Omega V_\varphi f, V_\varphi f\right\rangle = \int_\Omega \left|V_\varphi f\right|^2$, the first eigenfunction of H_Ω is optimally concentrated inside Ω in the following sense:

$$\int_\Omega \left|V_\varphi \phi_1^\Omega(z)\right|^2 \, \mathrm{d}z = \max_{\|f\|_2=1} \int_\Omega \left|V_\varphi f(z)\right|^2 \, \mathrm{d}z.$$

More generally, it follows from the Courant minimax principle, see, e.g., [45, Sect. 95], that the first N eigenfunctions of H_Ω form an orthonormal set in $L^2(\mathbb{R}^d)$ that maximizes the quantity $\sum_{j=1}^N \int_\Omega |V_\varphi \phi_j^\Omega(z)|^2\, dz$ among all orthonormal sets of N functions in $L^2(\mathbb{R}^d)$. In this sense, their time–frequency profile is optimally adapted to Ω. Figure 2 illustrates this principle by showing some time–frequency boxes Ω along with the STFT and real part of the corresponding localization operator's first eigenfunctions.

Based on these observations, we propose the following construction of frames. Let $\{\Omega_\gamma : \gamma \in \Gamma\}$ be a cover of $\mathbb{R}^{2d}$ consisting of bounded measurable sets. In order to construct a frame adapted to the cover, we select, for each region Ω_γ, the first N_γ eigenfunctions $\phi^1_{\Omega_\gamma}, \dots, \phi^{N_\gamma}_{\Omega_\gamma}$ of the operator H_{Ω_γ}. We will prove that there is a number $\alpha > 0$ such that if $N_\gamma \geq \alpha\,|\Omega_\gamma|$, then the collection of all the chosen eigenfunctions spans $L^2(\mathbb{R}^d)$ in a stable fashion. (Here $|\Omega_\gamma|$ is the Lebesgue measure of Ω_γ.) Note that the condition $N_\gamma \geq \alpha\,|\Omega_\gamma|$ is in accordance with the uncertainty principle, which roughly says that for each time–frequency region Ω_γ there are only approximately $|\Omega_\gamma|$ degrees of freedom.

We allow for covers that are arbitrary in shape as long as they satisfy the following mild admissibility condition. An indexed set $\{\Omega_\gamma : \gamma \in \Gamma\}$, is said to be an *admissible cover* of $\mathbb{R}^{2d}$ if the following conditions hold:

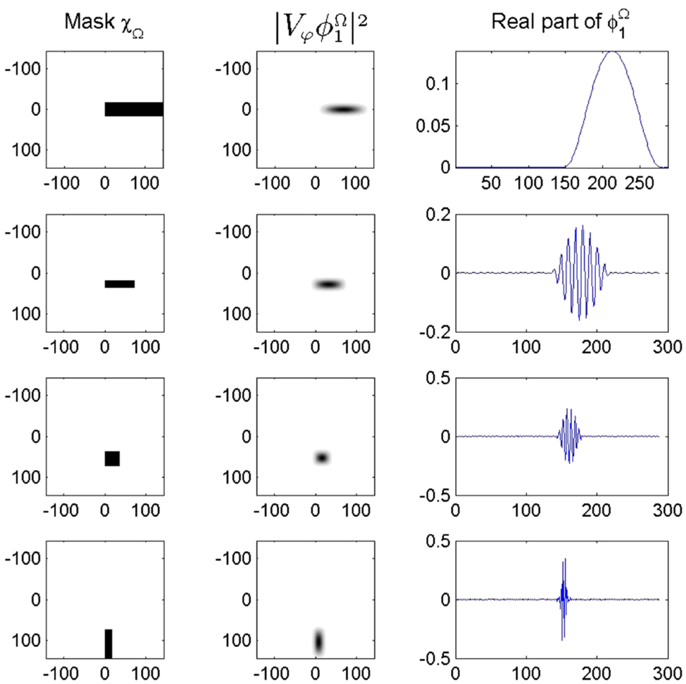

Fig. 2 Four different *rectangular masks* in the time–frequency domain and the first eigenfunctions of the corresponding localization operators. *Middle plots* show the absolute value squared of the STFT and *right plots* show the real part

- $\Gamma \subseteq \mathbb{R}^{2d}$ and $\sup_{z\in\mathbb{R}^{2d}} \#(\Gamma \cap B_1(z)) < +\infty$.
- For each $\gamma \in \Gamma$, Ω_γ is a measurable subset of $\mathbb{R}^{2d}$.
- $\bigcup_{\gamma\in\Gamma} \Omega_\gamma = \mathbb{R}^{2d}$.
- There exists $R > 0$ such that

$$\Omega_\gamma \subseteq B_R(\gamma), \quad \gamma \in \Gamma. \tag{5}$$

Under this condition, we prove the following:

Theorem 1.1 *Let $\left\{\Omega_\gamma : \gamma \in \Gamma\right\}$ be an admissible cover of $\mathbb{R}^{2d}$. Then there exists a constant $\alpha > 0$ such that for every choice of numbers $\{N_\gamma : \gamma \in \Gamma\} \subseteq \mathbb{N}$ satisfying*

$$\alpha \left|\Omega_\gamma\right| \leq N_\gamma \text{ and } \sup_{\gamma\in\Gamma} N_\gamma < +\infty,$$

the family of functions $\left\{\lambda_k^{\Omega_\gamma} \phi_k^{\Omega_\gamma} : \gamma \in \Gamma, 1 \leq k \leq N_\gamma\right\}$, obtained from the eigenfunctions and eigenvalues of the operators H_{Ω_γ}—cf. (3) and (4)—is a frame of $L^2(\mathbb{R}^d)$. That is, for some constants $0 < A \leq B < +\infty$, the following frame inequality holds

$$A\|f\|_2^2 \leq \sum_{\gamma\in\Gamma} \sum_{k=1}^{N_\gamma} \left|\left\langle f, \lambda_k^{\Omega_\gamma} \phi_k^{\Omega_\gamma} \right\rangle\right|^2 \leq B\|f\|_2^2, \quad f \in L^2(\mathbb{R}^d).$$

Theorem 1.1 is proved at the end of Sect. 5.2. We also show that if an admissible cover $\left\{\Omega_\gamma : \gamma \in \Gamma\right\}$ satisfies the additional *inner regularity condition*:

- There exist $r > 0$ such that

$$B_r(\gamma) \subseteq \Omega_\gamma, \quad \gamma \in \Gamma,$$

then the statement of Theorem 1.1 remains valid if the functions $\lambda_k^{\Omega_\gamma} \phi_k^{\Omega_\gamma}$ are replaced by their unweighted versions $\phi_k^{\Omega_\gamma}$ (see Theorem 5.10). In this case, $\inf_\gamma \left|\Omega_\gamma\right| \geq \left|B_\gamma(0)\right|$, so the lower bound on N_γ in the hypothesis of Theorem 1.1 can be expressed as $N_\gamma \geq \tilde{\alpha}$, for some constant $\tilde{\alpha} > 0$.

While Theorem 1.1 was our main motivation, it is just a sample of our results. The introduction of an abstract model for phase-space provides sufficient flexibility to obtain variants of Theorem 1.1 in the context of time-scale analysis (Theorem 6.4) and of discrete time–frequency representations (Theorem 6.5).

1.1 Technical Overview

The proofs of the main results in this paper are based on two major observations. First, the norm-equivalence[1]

$$\|f\|_2^2 \approx \sum_{\gamma\in\Gamma} \|H_{\eta_\gamma} f\|_2^2, \quad f \in L^2(\mathbb{R}^d), \tag{6}$$

[1] For two nonnegative functions $f, g : X \to (0, +\infty)$, the statement $f \approx g$ means that there exist constants $c, C \in (0, +\infty)$ such that $cf(x) \leq g(x) \leq Cf(x)$ for all $x \in X$.

holds for a family of time–frequency localization operators

$$H_{\eta_\gamma} f(t) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \eta_\gamma(x,\xi) V_\varphi f(x,\xi) \varphi(t-x) e^{2\pi i \xi t} \mathrm{d}x \mathrm{d}\xi$$

provided that the symbols $\eta_\gamma : \mathbb{R}^{2d} \to [0, +\infty)$ satisfy

$$\sum_{\gamma \in \Gamma} \eta_\gamma(z) \approx 1 \tag{7}$$

and the enveloping condition

$$\eta_\gamma(z) \leq g(z-\gamma), \text{ for some } g \in L^1(\mathbb{R}^{2d}) \text{ and } \gamma \in \Gamma, \text{ with } \Gamma \subseteq \mathbb{R}^{2d} \text{ a lattice.} \tag{8}$$

The inequalities (6) were first proved in [19] for symbols of the form $\eta_\gamma(z) = g(z-\gamma)$ and $\Gamma = \mathbb{Z}^{2d}$, then for a general lattice in [20], and finally for fully irregular symbols satisfying (8) in [47]. It is interesting to note that the proofs in [19,20] are *based* on the observation that under condition (7), the norm-equivalence (6) is equivalent to the fact that finitely many eigenfunctions of the operator H_g generate a multi-window Gabor frame over the lattice Γ. The proof of the general case in [47] does not explicitly involve the eigenfunctions of the operators H_{η_γ}, nor does it rely on tools specific to Gabor frames on lattices. Thus, the question arises whether it is also possible in the irregular case to construct a frame consisting of finite sets of eigenfunctions of the operators H_{η_γ}. Here, this question is given a positive answer.

Second, the observation that (6) remains valid when the operators H_{η_γ} are replaced by finite rank approximations $H^\varepsilon_{\eta_\gamma}$ obtained by thresholding their eigenvalues, cf. Theorem 5.3, is the core of the proof of our main results. This finite rank approximation is in turn achieved by proving that the operators H_{η_γ} behave "globally" like projectors. More precisely, in Proposition 5.1, we obtain the following extension of (6):

$$\|f\|_2^2 \approx \sum_{\gamma \in \Gamma} \|H_{\eta_\gamma} f\|_2^2 \approx \sum_{\gamma \in \Gamma} \|(H_{\eta_\gamma})^2 f\|_2^2. \tag{9}$$

This will allow us to "localize in phase-space" the L^2-norm estimates relating $H^\varepsilon_{\eta_\gamma}$ and H_{η_γ}.

Note that, in general, the operators H_{η_γ} have infinite rank even if η_γ is the characteristic function of a compact set (see Lemma 5.8). Consequently, $\|H_{\eta_\gamma} f\|_2 \not\approx \|(H_{\eta_\gamma})^2 f\|_2$ and therefore the global properties of the family $\left\{(H_{\eta_\gamma})^2 : \gamma \in \Gamma\right\}$ are crucial to prove (9). While the squared operators $(H_{\eta_\gamma})^2$ are not time–frequency localization operators, their time–frequency localizing behavior is preserved under conditions as given by (8), and they are the prototypical example of a family of operators that is *well spread in the time–frequency plane*. The latter notion is defined in Sect. 3, and we exploit the fact that the tools from [47] are valid for these operator families.

For clarity, we choose to accentuate the case of time–frequency analysis, but all the proofs are carried out in an abstract setting that yields, for example, analogous consequences in time-scale analysis.

1.2 Organization

The article is organized as follows. Section 2 motivates the abstract model to study phase-space covers and gives the main examples to keep in mind. Section 2.3 formally introduces the abstract model for phase-space, and Sect. 3 presents certain key technical notions, in particular the properties required for a family of localization operators to exhibit an almost-orthogonality property. In Sect. 4, we first prove our results in the context of time–frequency analysis, where some technical problems of the abstract setting do not arise. In addition, in the context of time–frequency analysis, we are able to extend the result on phase-space adapted frames from $L^2(\mathbb{R}^d)$ to an entire class of Banach spaces, the modulation spaces, by exploiting spectral invariance results for pseudodifferential operators. Theorem 5.5 in Sect. 5 is the general version of Theorem 1.1 stated above. This latter result is proved after Theorem 5.5 as an application. Section 6 develops the results in the abstract setting. These are then applied to time-scale analysis in Sect. 6.1. Finally, Sect. 6.2 contains an additional application of the abstract results to time–frequency analysis, this time using Gabor multipliers, which are time–frequency masking operators related to a discrete time–frequency representation (Gabor frame). The atoms thus obtained maximize their time–frequency concentration with respect to a weight on a discrete time–frequency grid, and the resulting frames are relevant in numerical applications.

For clarity, the presentation of the results highlights the case of time–frequency analysis, which was our main motivation. Most of the technicalities in Sect. 2.3 are irrelevant to that setting (although they are relevant for time-scale analysis). The reader interested mainly in time–frequency analysis is encouraged to jump directly to Sect. 4 and then go back to Sects. 2.3 and 3 having a clear example in mind.

The article also contains two appendices providing auxiliary results related to the almost-orthogonality tools from [47] and spectral invariance of pseudodifferential operators.

2 Phase-Space

In this section, we introduce an abstract model for phase-space. We first provide some motivation and the main examples to keep in mind.

2.1 The Time–Frequency Plane as an Example of Phase-Space

In time–frequency analysis, a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ is studied by means of a time–frequency representation, e.g., the STFT $V_\varphi f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, cf. (1). In signal processing, where f is called a signal, the domain $\mathbb{R}^d$ is referred to as *signal space*, while $\mathbb{R}^d \times \mathbb{R}^d$ is referred to as *phase-space*. This terminology is inspired by mechanics, where the position of a freely moving particle is described by a point x in the configuration space $\mathbb{R}^d$, while (x, ξ) describes a pair of position–momentum variables belonging to the phase-space $\mathbb{R}^d \times \mathbb{R}^d$.

Since f can be re-synthesized from $V_\varphi f$—cf. (2)—all its properties can in principle be reformulated as properties of $V_\varphi f$. However, not every function on $\mathbb{R}^d \times \mathbb{R}^d$ is the STFT of a distribution on $\mathbb{R}^d$. Indeed, if we let $S := V_\varphi L^2(\mathbb{R}^d)$ be the image of $L^2(\mathbb{R}^d)$ under V_φ, it turns out that S is a *reproducing kernel space* (see [33, Chaps. 3, 11]). This means that S is a closed subspace of $L^2(\mathbb{R}^{2d})$ consisting of continuous functions and that for all $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$ the evaluation functional $S \ni F \mapsto F(x, \xi) \in \mathbb{C}$ is continuous. In particular, $S \subsetneq L^2(\mathbb{R}^d)$.

The fact that $S := V_\varphi L^2(\mathbb{R}^d)$ is a "small" subspace of $L^2(\mathbb{R}^d)$ is important to understanding the problem studied in this article. In designing a frame for $L^2(\mathbb{R}^d)$ with a prescribed phase-space profile, the challenge lies in the fact that the shapes we design in $\mathbb{R}^{2d}$ must correspond to functions in the small subspace S. The role of time–frequency localization operators is crucial and will be detailed in the rest of this section.

For $m \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, the *time–frequency localization operator* with symbol m is

$$H_m f(t) = \int_{\mathbb{R}^d \times \mathbb{R}^d} V_\varphi f(x, \xi) m(x, \xi) \varphi(t - x) e^{2\pi i \xi t} \mathrm{d}x \mathrm{d}\xi, \quad t \in \mathbb{R}^d. \tag{10}$$

If $m = 1_\Omega$ is the characteristic function of a set Ω, we write H_Ω instead of H_{1_Ω}, cf. (3).

The STFT (with respect to a fixed normalized window $\varphi \in L^2(\mathbb{R}^d)$) defines a map $V_\varphi : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$. Its adjoint $V_\varphi^* : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^d)$ is given by

$$V_\varphi^* F(t) = \int_{\mathbb{R}^d \times \mathbb{R}^d} F(x, \xi) \varphi(t - x) e^{2\pi i \xi t} \mathrm{d}x \mathrm{d}\xi, \quad t \in \mathbb{R}^d.$$

The inversion formula in (2) says that $V_\varphi^* V_\varphi = I_{L^2(\mathbb{R}^d)}$. Hence V_φ is an isometry on $L^2(\mathbb{R}^d)$. Consider a symbol $m \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and the time–frequency localization operator from (10). Using V_φ and V_φ^*, the definition reads

$$H_m f = V_\varphi^*(m V_\varphi f), \quad f \in L^2(\mathbb{R}^d). \tag{11}$$

The fact that V_φ is an isometry with range S implies that $V_\varphi V_\varphi^* = P_S$ is the orthogonal projection $L^2(\mathbb{R}^{2d}) \to S$. Explicitly, for $F \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$,

$$P_S F(x', \xi') = \int_{\mathbb{R}^d \times \mathbb{R}^d} F(x, \xi) \left\langle \varphi(\cdot - x) e^{2\pi i \xi \cdot}, \varphi(\cdot - x') e^{2\pi i \xi' \cdot} \right\rangle_{L^2(\mathbb{R}^d)} \mathrm{d}x \mathrm{d}\xi. \tag{12}$$

It follows from (11) that

$$(V_\varphi H_m V_\varphi^*) V_\varphi f = P_S(m V_\varphi f), \quad f \in L^2(\mathbb{R}^d).$$

Since $F = V_\varphi f$ is the generic form of a function in S, we obtain that

$$(V_\varphi H_m V_\varphi^*)F = P_S(mF), \quad F \in S.$$

This means that the time–frequency localization operator H_Ω is unitarily equivalent to the operator $M_m : S \to S$ given by

$$M_m F := P_S(mF), \qquad F \in S. \tag{13}$$

The operator M_m consists of multiplication by m followed by projection onto S. We will call these operators *phase-space* multipliers: they apply a mask m to a function $F = V_\varphi f \in S$, typically yielding a function $m \cdot F \notin S$, and then provide the best L^2 approximation of $m \cdot F$ within S. (These operators are sometimes also called Toeplitz operators for the STFT.)

2.2 Other Transforms

The interpretation of time–frequency localization operators as phase-space multipliers (multiplication followed by projection) is central to this article. We will consider a general setting where the role of the STFT can be replaced by other transforms. An important example is the wavelet transform of a function $L^2(\mathbb{R}^d)$ with respect to an adequate window $\psi \in \mathcal{S}(\mathbb{R}^d)$,

$$W_\psi f(x,s) = s^{-d/2} \int_{\mathbb{R}^d} f(t)\overline{\psi\left(\frac{t-x}{s}\right)}\mathrm{d}t, \quad x \in \mathbb{R}^d, s > 0. \tag{14}$$

(see Sect. 6.1 for details.) Analogously to time–frequency analysis, we still call $\mathbb{R}^d \times \mathbb{R}_+$ the phase-space associated with the wavelet transform. This terminology is justified by the fact that there is a formal analogy between the two contexts. The range of the wavelet transform $W_\psi L^2(\mathbb{R}^d)$ is, under suitable assumptions on ψ, a reproducing kernel subspace of $L^2(\mathbb{R}^d \times \mathbb{R}_+, s^{-(d+1)}\mathrm{d}x\mathrm{d}s)$, and time-scale localization operators are defined analogously to the time–frequency localization operators (see Sect. 6.1 for explicit formulas).

The similarity between the time–frequency and time-scale contexts stems from the fact that both are associated with representations of a locally compact group. In the former case, the Heisenberg group acts on $L^2(\mathbb{R}^d)$ by translations and modulations, while in the latter, the affine group acts by translations and dilations. The theory of *coorbit spaces* [25,26] treats this situation in general, studying the transform associated with the representation coefficients of a group action and associating a range of function spaces to it. The model for abstract phase-space to be introduced in Sect. 2.3 is largely inspired by [25]. It allows for the simultaneous treatment of various settings, since it makes no explicit reference to an integral transform. The main ingredients are a "big" space E called the environment and a "small" subspace called the atomic space. In the time–frequency example, $E = L^2(\mathbb{R}^{2d})$ and $S = V_\varphi L^2(\mathbb{R}^d)$. Similarly, the case of time-scale analysis (wavelets) uses $E = L^2(\mathbb{R}^d \times \mathbb{R}_+, s^{-(d+1)}\mathrm{d}x\mathrm{d}s)$ and $S = W_\psi L^2(\mathbb{R}^d)$. Building on the heuristics provided by time–frequency and time-scale analysis, we think of S as a collection of phase-space representations for functions in $L^2(\mathbb{R}^d)$, while the environment E is big enough for certain operations, such as pointwise multiplication by arbitrary bounded measurable functions, to be well defined. One central assumption of the model in Sect. 2.3 is then the existence of a projector $P : E \to S$, so that one can consider phase-space multipliers like in (13).

The price to pay for this unified approach is a certain level of technicality. The Euclidean space is not suitable any more as a model for the domain of the functions in S: we need to consider a general locally compact group. To provide an easily accessible example, the main concepts will be spelled out in the concrete case of time–frequency analysis.

2.3 Abstract Phase-Space

2.3.1 Locally Compact Groups and Function Spaces

Throughout this article, $\mathcal{G}$ will be a locally compact, σ-compact, topological group with modular function Δ. The left Haar measure of a set $X \subseteq \mathcal{G}$ will be denoted by $|X|$. Integration will always be considered with respect to the left Haar measure. For $x \in \mathcal{G}$, we denote by L_x and R_x the operators of left and right translation, defined by $L_x f(y) = f(x^{-1}y)$ and $R_x f(y) = f(yx)$. We also consider the involution $f^\vee(x) = f(x^{-1})$.

Given two nonnegative functions f, g, we write $f \lesssim g$ if there exists a constant $C \geq 0$ such that $f \leq Cg$. We say that $f \approx g$ if both $f \lesssim g$ and $g \lesssim f$. The characteristic function of a set A will be denoted by 1_A.

A set $\Gamma \subseteq \mathcal{G}$ is called *relatively separated* if for some (or any) $V \subseteq \mathcal{G}$ relatively compact neighborhood of the identity, the quantity—called the *spreadness of* Γ

$$\rho(\Gamma) = \rho_V(\Gamma) := \sup_{x \in \mathcal{G}} \#(\Gamma \cap xV) \tag{15}$$

is finite, i.e., if the amount of elements of Γ that lie in any left translate of V is uniformly bounded.

The following definition introduces a class of function spaces on $\mathcal{G}$. The Lebesgue spaces $L^p(\mathcal{G})$ are natural examples.

Definition 2.1 (*Banach function spaces*) A Banach space E is called a solid, translation invariant BF space if it satisfies the following:

(i) E is continuously embedded into $L^1_{\mathrm{loc}}(\mathcal{G})$, the space of complex-valued locally integrable functions on $\mathcal{G}$.

(ii) Whenever $f \in E$ and $g : \mathcal{G} \to \mathbb{C}$ is a measurable function such that $|g(x)| \leq |f(x)|$ a.e., it is true that $g \in E$ and $\|g\|_E \leq \|f\|_E$.

(iii) E is closed under left and right translations (i.e., $L_x E \subseteq E$ and $R_x E \subseteq E$, for all $x \in \mathcal{G}$) and the following relations hold with the corresponding norm estimates

$$L^1_u(\mathcal{G}) * E \subseteq E \text{ and } E * L^1_v(\mathcal{G}) \subseteq E,$$

where $u(x) := \|L_x\|_{E\to E}$, $v(x) := \Delta(x^{-1})\|R_{x^{-1}}\|_{E\to E}$.

Definition 2.2 (*Admissible weights for a BF space*) Given a solid, translation invariant BF space E, a continuous function $w : \mathcal{G} \to (0, +\infty)$ satisfying

$$\begin{aligned} w(x) &= \Delta(x^{-1})w(x^{-1}), \\ w(xy) &\leq w(x)w(y) \text{ (submultiplicativity)}, \\ w(x) &\geq C_{E,w} \max\left\{u(x),\ u(x^{-1}),\ v(x),\ \Delta(x^{-1})v(x^{-1})\right\}, \\ &\qquad \text{where } u(x) := \|L_x\|_{E\to E},\ v(x) := \Delta(x^{-1})\|R_{x^{-1}}\|_{E\to E}, \end{aligned} \tag{16}$$

for some constant $C_{E,w} > 0$ is called an *admissible weight* for E.

If w is admissible for E, it follows that $w(x) \gtrsim 1$, $L^1_w * E \subseteq E$, and $E * L^1_w \subseteq E$, and the constants in the corresponding norm estimates depend only on $C_{E,w}$, cf. [25].

For a solid translation invariant BF space E and $\Gamma \subseteq \mathcal{G}$ a relatively separated set, we construct discrete versions E_{d} as follows. Fix V, a symmetric relatively compact neighborhood of the identity, and let

$$E_{\mathrm{d}} = E_{\mathrm{d}}(\Gamma) := \left\{ c \in \mathbb{C}^\Gamma \,\middle|\, \sum_{\gamma\in\Gamma} |c_\gamma|\, 1_{\gamma V} \in E \right\}, \text{ with norm}$$

$$\left\| \left(c_\gamma\right)_{\gamma\in\Gamma} \right\|_{E_{\mathrm{d}}} := \left\| \sum_{\gamma\in\Gamma} |c_\gamma|\, 1_{\gamma V} \right\|_E .$$

The definition depends on V, but a different choice of V yields the same space with equivalent norm (this is a consequence of the right invariance of E, see, for example, [44, Lemma 2.2]). For $E = L^p_w$, the corresponding discrete space $E_{\mathrm{d}}(\Gamma)$ is just $\ell^p_w(\Gamma)$, where the (admissible) weight w is restricted to the set Γ.

We next define the left *Wiener amalgam space* with respect to a solid, translation invariant BF space E. Let V again be a symmetric, relatively compact neighborhood of the identity. For a locally bounded function $f : \mathcal{G} \to \mathbb{C}$, consider the left *local maximum function* defined by

$$f^\#(x) := \operatorname*{esssup}_{y\in V} |f(xy)| = \|f \cdot (L_x 1_V)\|_\infty, \quad x \in \mathcal{G},$$

and similarly the right local maximum function is given by $f_\#(x) \coloneqq \operatorname{esssup}_{y\in V} |f(yx)| = \|f \cdot (R_x 1_V)\|_\infty$.

Definition 2.3 (*Wiener amalgam spaces*)

$$W(L^\infty, E)(\mathcal{G}) := \left\{ f \in L^\infty_{\mathrm{loc}}(\mathcal{G}) \,\middle|\, f^\# \in E \right\},$$

with norm $\|f\|_{W(L^\infty,E)} := \|f^\#\|_E$. The right Wiener amalgam space $W_R(L^\infty, E)$ is defined similarly, this time using the norm $\|f\|_{W_R(L^\infty,E)} := \|f_\#\|_E$.

A different choice of V yields the same spaces with equivalent norms (see, for example, [22, Theorem 1] or [23]). When E is a weighted L^p space on $\mathbb{R}^d$, the corresponding amalgam space coincides with the classical $L^\infty - \ell^p$ amalgam space [30,38]. In the present article, we will be mainly interested in the spaces $W(L^\infty, L^1_w)$ and $W_R(L^\infty, L^1_w)$, for which we need the following facts.

Proposition 2.4 *The spaces $W(L^\infty, L^1_w)$ and $W_R(L^\infty, L^1_w)$ are convolution algebras. That is, the relations $W(L^\infty, L^1_w) * W(L^\infty, L^1_w) \hookrightarrow W(L^\infty, L^1_w)$ and $W_R(L^\infty, L^1_w) * W_R(L^\infty, L^1_w) \hookrightarrow W_R(L^\infty, L^1_w)$ hold together with the corresponding norm estimates.*

Proof The left amalgam space satisfies the (translation invariance) relation $L^1_w * W(L^\infty, L^1_w) \hookrightarrow W(L^\infty, L^1_w)$. This is a particular case of [26, Theorem 7.1] and can also be readily deduced from the definitions. Since $W(L^\infty, L^1_w) \hookrightarrow L^1_w$, the statement about $W(L^\infty, L^1_w)$ follows. The involution $f^\vee(x) = f(x^{-1})$ maps $W_R(L^\infty, L^1_w)$ isometrically onto $W(L^\infty, L^1_w)$ (because $V = V^{-1}$) and satisfies $(f * g)^\vee = g^\vee * f^\vee$. Hence the statement about the right amalgam space follows from the one about the left one. $\square$

2.3.2 The Model for Abstract Phase-Space

In the abstract model for phase-space, we consider a solid BF space E (called the environment) and a certain distinguished subspace S_E, which is the range of an idempotent operator P. The precise form of the model is taken from [47] and is designed to fit the theory in [25] (see also [14,29,42]). We list a number of ingredients in the form of two assumptions: (A1) and (A2).

(A1)
- E is a solid, translation invariant BF space, called *the environment*.
- w is an admissible weight for E.
- S_E is a closed complemented subspace of E, called *the atomic subspace*.
- Each function in S_E is continuous.

(A2) P is an operator and K is a nonnegative function satisfying the following:
- $P : W(L^1, L^\infty_{1/w}) \to L^\infty_{1/w}$ is a (bounded) linear operator,
- $P(E) = S_E$ and $P(f) = f$, for all $f \in S_E$,
- $K \in W(L^\infty, L^1_w) \cap W_R(L^\infty, L^1_w)$,
- For $f \in W(L^1, L^\infty_{1/w})$,

$$|P(f)(x)| \le \int_{\mathcal{G}} |f(y)|\, K(y^{-1}x)\mathrm{d}y, \quad x \in \mathcal{G}. \tag{17}$$

When $E = L^2(\mathcal{G})$, we will additionally assume the following:

(A3) $P : L^2(\mathcal{G}) \to S_{L^2}$ is the orthogonal projection.

Note that Assumption (A2) means that the retraction $E \to S_E$ is given by an operator dominated by right convolution with a kernel in $W(L^\infty, L^1_w) \cap W_R(L^\infty, L^1_w)$.

Example 2.5 The discussion in Sect. 2.1 provides the main example of the abstract model. If $\varphi \in \mathcal{S}(\mathbb{R}^d)$, we can let $\mathcal{G} = \mathbb{R}^d \times \mathbb{R}^d$, $E = L^2(\mathbb{R}^{2d})$, and $S = S_E = V_\varphi L^2(\mathbb{R}^d)$. For this choice, $w \equiv 1$ is an admissible weight. We also let $P = P_S : L^2(\mathbb{R}^{2d}) \to S_E$ be the orthogonal projection. To see that P satisfies (A2), note that from (12),

$$\begin{aligned} \left|P_S F(x', \xi')\right| &\le \int\limits_{\mathbb{R}^d\times\mathbb{R}^d} |F(x,\xi)| \left|\left\langle \varphi(\cdot - x)e^{2\pi i \xi\cdot}, \varphi(\cdot - x')e^{2\pi i \xi'\cdot}\right\rangle\right| \mathrm{d}x\mathrm{d}\xi \\ &= \int\limits_{\mathbb{R}^d\times\mathbb{R}^d} |F(x,\xi)| \left|V_\varphi\varphi(x'-x, \xi'-\xi)\right| \mathrm{d}x\mathrm{d}\xi. \end{aligned}$$

It is easy to see that the fact that $\varphi \in \mathcal{S}(\mathbb{R}^d)$ implies that $K := \left|V_\varphi\varphi\right| \in W(L^\infty, L^1)(\mathbb{R}^{2d})$.

For the remainder of Sect. 2.3, we assume (A1) and (A2). Under these conditions, the following holds:

Proposition 2.6 *[47, Proposition 3]*

(a) P *boundedly maps* E *into* $W(L^\infty, E)$.
(b) $S_E \hookrightarrow W(L^\infty, E)$.
(c) If $f \in W(L^1, L^\infty_{1/w})$, *then* $\|P(f)\|_{L^\infty_{1/w}} \lesssim \|f\|_{W(L^1,L^\infty_{1/w})}\|K\|_{W_R(L^\infty,L^1_w)}$.
(d) If $f \in W(L^1, L^\infty)$, *then* $\|P(f)\|_{L^\infty} \lesssim \|f\|_{W(L^1,L^\infty)}\|K\|_{W_R(L^\infty,L^1_w)}$.

Remark 2.7 Since $w \gtrsim 1$, $L^\infty \hookrightarrow L^\infty_{1/w}$.

Remark 2.8 The estimates in Proposition 2.6 hold uniformly for all the spaces E with the same weight w and the same constant $C_{E,w}$ (cf. (16)). This is relevant to the applications, where the same projection P is used with different spaces E and corresponding subspaces $S_E = P(E)$.

2.3.3 Phase-Space Multipliers

Recall the projector $P : E \to S_E$ from (A2). For $m \in L^\infty(\mathcal{G})$, the *phase-space multiplier* with *symbol m* is the operator $M_m : S_E \to S_E$ defined by

$$M_m(f) := P(mf), \quad f \in S_E.$$

It is bounded by Proposition 2.6 and the solidity of E:

$$\|M_m(f)\|_E \lesssim \|m\|_\infty \|f\|_E, \quad f \in S_E. \tag{18}$$

Example 2.9 As discussed in Sect. 2.1, time–frequency localization operators (10) are unitarily equivalent via the STFT to phase-space multipliers, with E, S_E, and P as in Example 2.5.

In the context of time-scale analysis, one can define operators analogously to time–frequency localization operators using the wavelet transform in (14) instead of the STFT. These are called time-scale localization operators or wavelet multipliers [18, 40,53]. With an adequate choice of E, S_E, and P, time-scale localization operators are unitarily equivalent to phase-space multipliers. This is developed in Sect. 6.1.

For future reference, we note some Hilbert-space properties of phase-space multipliers (when $E = L^2(\mathcal{G})$) which are well known for time–frequency localization operators, cf. [7,8,27,54].

Proposition 2.10 *Let $E = L^2(\mathcal{G})$, and assume (A1), (A2), and (A3). Then the following hold:*

(a) Let $m_1, m_2 \in L^\infty(\mathcal{G})$ be real-valued. If $m_1 \le m_2$ a.e., then $M_{m_1} \le M_{m_2}$ as operators. In particular, if m is nonnegative and bounded, then M_m is a positive operator.
(b) Let $m \in L^1(\mathcal{G}) \cap L^\infty(\mathcal{G})$ be nonnegative. Then $M_m : S_{L^2} \to S_{L^2}$ is trace-class and $\operatorname{trace}(M_m) \lesssim \|m\|_1$.

Proof To prove (a), let $f \in S_{L^2}$ and note that by (A3),

$$\left\langle M_{m_1} f, f\right\rangle = \langle P(m_1 f), f\rangle = \langle m_1 f, f\rangle = \int_{\mathcal{G}} m_1(x)\, |f(x)|^2 \,\mathrm{d}x$$
$$\le \int_{\mathcal{G}} m_2(x)\, |f(x)|^2 \,\mathrm{d}x = \left\langle M_{m_2} f, f\right\rangle.$$

Let us now prove (b). For $f \in S_{L^2}$ and $x \in \mathcal{G}$, by Assumption (A2) (cf. 17),

$$|f(x)| \le \int_{\mathcal{G}} |f(y)|\, K(y^{-1}x)\mathrm{d}y = \int_{\mathcal{G}} |f(y)|\, K^\vee(x^{-1}y)\mathrm{d}y \le \|f\|_2 \|K^\vee\|_2,$$

which is finite because $K^\vee \in W_R(L^\infty, L^1_w) \subseteq L^2$. Hence $f(x) = \langle f, E_x\rangle$ for some function $E_x \in S_{L^2}$ with $\|E_x\|_2 \le \|K^\vee\|_2$.

Let $\{e_k\}_k$ be an orthonormal basis of S_{L^2}. Since by (a), M_m is a positive operator, it suffices to check that $\sum_k \langle M_m e_k, e_k\rangle \lesssim \|m\|_1$ (see for example [49, Theorem 2.14]). To this end, note that for $x \in \mathcal{G}$,

$$\sum_k |e_k(x)|^2 = \sum_k |\langle e_k, E_x\rangle|^2 = \|E_x\|^2 \le \|K^\vee\|_2^2.$$

Hence,

$$\sum_k \langle M_m e_k, e_k\rangle = \sum_k \int_{\mathcal{G}} m(x)\,|e_k(x)|^2 \leq \|K^\vee\|_2^2 \|m\|_1.$$

This completes the proof. □

3 Well-Spread Families of Operators

Throughout this section, we assume (A1) and (A2) from Sect. 2.3.2. Recall the notion of phase-space multiplier $M_m : S_E \to S_E$, $M_m(f) = P(mf)$, from Sect. 2.3.3. If $P : E \to S_E$ is the projector from (A2) and $m \in L^\infty(\mathcal{G})$ is a symbol, then (A1) and (A2) intuitively imply that a phase-space multiplier spreads the mass of f in a controlled way. Indeed, using the bound in (A2), we obtain

$$|M_m f(x)| = |P(mf)(x)| \leq \int_{\mathcal{G}} |m(y)|\,|f(y)|\,K(y^{-1}x)dy, \quad x \in \mathcal{G}. \tag{19}$$

Hence, if m is known to be concentrated in a certain region of $\mathcal{G}$, $M_m(f)$ will be also.

One of the main technical insights of this article is the fact that some important tools used in the investigation of families of phase-space multipliers only depend on estimates such as (19). To formalize this observation, we now introduce the key concept of families of operators that are *well-spread* in phase-space, implying that the operators are dominated by product–convolution operators centered at suitably distributed nodes γ.

Definition 3.1 (*Well-spread family of operators*) Let $\Gamma \subseteq \mathcal{G}$ be a relatively separated set, $\Theta \in W(L^\infty, L^1_w) \cap W_R(L^\infty, L^1_w)$ and $g \in W_R(L^\infty, L^1_w)$ be nonnegative functions. A family of operators $\{T_\gamma : S_E \to S_E : \gamma \in \Gamma\}$ is called *well-spread with envelope* (Γ, Θ, g) if the following pointwise estimate holds

$$\left|T_\gamma f(x)\right| \leq \int_{\mathcal{G}} g(\gamma^{-1}y)\,|f(y)|\,\Theta(y^{-1}x)\mathrm{d}y, \quad \gamma \in \Gamma,\ x \in \mathcal{G}. \tag{20}$$

If we do not want to emphasize the role of the envelope, we say that $\{T_\gamma : \gamma \in \Gamma\}$ is a well-spread family of operators, assuming the existence of an adequate envelope.

The advantage of working with well-spread families instead of just families of phase-space multipliers is that well-spreadness is stable under various operations, e.g., finite composition. This will be essential to the proofs of our main results.

The canonical example of a well-spread family of operators is a family of phase-space multipliers $\left\{ M_{\eta_\gamma} \,\middle|\, \gamma \in \Gamma \right\}$ associated with an adequate family of symbols.

Definition 3.2 (*Well-spread family of symbols*) A family of (measurable) symbols $\left\{ \eta_\gamma : \mathcal{G} \to \mathbb{C} \,\middle|\, \gamma \in \Gamma \right\}$ is called *well-spread* if

- $\Gamma \subseteq \mathcal{G}$ is a relatively separated set, and
- there is a function $g \in W_R(L^\infty, L^1_w)$ such that $\left|\eta_\gamma(x)\right| \le g(\gamma^{-1}x)$, $x \in \mathcal{G}$, $\gamma \in \Gamma$.

The pair (Γ, g) is called an envelope for $\left\{\eta_\gamma : \gamma \in \Gamma\right\}$.

Together with the properties of the convolution kernel K dominating the projection onto S_E—cf. (A1) and (A2)—we immediately obtain a family of well-spread operators.

Proposition 3.3 *Let $\left\{\eta_\gamma : \gamma \in \Gamma\right\}$ be a well-spread family of symbols. Then the corresponding family of phase-space multipliers $\left\{M_{\eta_\gamma} : \gamma \in \Gamma\right\}$ is well-spread.*

Proof It follows readily from the definitions and Assumptions (A1) and (A2) that if (Γ, g) is an envelope for $\left\{\eta_\gamma : \gamma \in \Gamma\right\}$, then (Γ, K, g) is an envelope for $\left\{M_{\eta_\gamma} : \gamma \in \Gamma\right\}$. □

The reason we introduce the concept of well-spread families of operators is that composition of phase-space multipliers usually fails to yield a phase-space multiplier. However, the estimate in (20) is stable under various operations. In this article, we will be mainly interested in finite composition, and we have the following:

Proposition 3.4 *Let $\left\{\eta_\gamma : \gamma \in \Gamma\right\}$ be a well-spread family of symbols. Then the family of operators $\left\{(M_{\eta_\gamma})^2 : \gamma \in \Gamma\right\}$ is well-spread.*

Proof If (Γ, g) is an envelope for $\left\{\eta_\gamma : \gamma \in \Gamma\right\}$, then

$$\begin{aligned}
\left|(M_{\eta_\gamma})^2 f(x)\right| &\le \int_{\mathcal{G}} g(\gamma^{-1}y) \left|M_{\eta_\gamma} f(y)\right| K(y^{-1}x)\mathrm{d}y \\
&\le \int_{\mathcal{G}} \int_{\mathcal{G}} g(\gamma^{-1}y)g(\gamma^{-1}z) \left|f(z)\right| K(z^{-1}y)K(y^{-1}x)\mathrm{d}y\mathrm{d}z \\
&\le \|g\|_\infty \int_{\mathcal{G}} g(\gamma^{-1}z) \left|f(z)\right| (K * K)(z^{-1}x)\mathrm{d}z.
\end{aligned}$$

Hence, if we set $\Theta := K * K$, it follows that $(\Gamma, \Theta, \|g\|_\infty g)$ is an envelope for $\left\{(M_{\eta_\gamma})^2 : \gamma \in \Gamma\right\}$. Since K belongs to $W(L^\infty, L^1_w) \cap W_R(L^\infty, L^1_w)$, by Proposition 2.4, so does Θ. □

3.1 Almost-Orthogonality Estimates

We now introduce one of the key estimates used in this article, which can be seen as a generalization of (6).

Theorem 3.5 *Let $\left\{T_\gamma : \gamma \in \Gamma\right\}$ be a well-spread family of operators. Suppose that the operator $\sum_\gamma T_\gamma : S_E \to S_E$ is invertible. Then the following norm-equivalence holds:*

$$\|f\|_E \approx \|(\|T_\gamma f\|_{L^2(\mathcal{G})})_{\gamma \in \Gamma}\|_{E_{\mathrm{d}}}, \quad f \in S_E. \tag{21}$$

Remark 3.6 The implicit constants depend only on $\|K\|_{W(L^\infty,L^1_w)}$, $\|K\|_{W_R(L^\infty,L^1_w)}$, $\|\Theta\|_{W(L^\infty,L^1_w)}$, $\|\Theta\|_{W_R(L^\infty,L^1_w)}$, $\|g\|_{W_R(L^\infty,L^1_w)}$, $\rho(\Gamma)$ (cf. 15), and $C_{E,w}$ (cf. 16) and the norms of $\sum_\gamma T_\gamma$ and $(\sum_\gamma T_\gamma)^{-1}$.

Theorem 3.5, in its version for families of phase-space multipliers associated with well-spread families of symbols, is proved in [47]. However, the arguments involved work unaltered for the case of general well-spread families of operators. Indeed, Definition 3.1 is tailored to the requirements of the proof in [47]. For completeness, we sketch a proof of Theorem 3.5 in Appendix 1.

Theorem 3.5 is an almost-orthogonality principle: because of the phase-space localization of the family $\{T_\gamma : \gamma \in \Gamma\}$, the effects of each individual operator within the sum $\sum_\gamma T_\gamma$ decouple.

We also point out that the L^2-norm in (21) can be replaced by any solid translation invariant norm (see [47]). However, the by far most important case is the one of L^2. Indeed, in this article we exploit (21) to extrapolate certain thresholding estimates from L^2 to other Banach spaces, cf. Theorem 4.5.

As a consequence of Theorem 3.5, we get the following fact:

Corollary 3.7 *Let $\{T_\gamma : \gamma \in \Gamma\}$ be a well-spread family of operators. Suppose that $E = L^2(\mathcal{G})$ and that the operator $\sum_\gamma T_\gamma : S_{L^2} \to S_{L^2}$ is invertible. Then so is the operator $\sum_\gamma T_\gamma^* T_\gamma : S_{L^2} \to S_{L^2}$.*

Proof By Theorem 3.5, the invertibility of $\sum_\gamma T_\gamma$ implies that for $f \in S_{L^2}$,

$$\|f\|_2^2 \approx \sum_{\gamma\in\Gamma} \|T_\gamma f\|_2^2 = \left\langle \sum_{\gamma\in\Gamma} T_\gamma^* T_\gamma f, f \right\rangle.$$

Hence $\sum_\gamma T_\gamma^* T_\gamma$ is positive definite and therefore invertible on S_{L^2}. □

Remark 3.8 Consider a well-spread family of self-adjoint operators $\{T_\gamma : \gamma \in \Gamma\}$. Since $(\sum_\gamma T_\gamma)^2 = \sum_\gamma (T_\gamma)^2 + \sum_{\gamma,\gamma'/\gamma\neq\gamma'} T_\gamma T_{\gamma'}$ and $(\sum_\gamma T_\gamma)^2$ is invertible if and only if $\sum_\gamma T_\gamma$ is, Corollary 3.7 says that the invertibility of $(\sum_\gamma T_\gamma)^2$ implies that of its "diagonal part" $\sum_\gamma (T_\gamma)^2$. This gives further support to the interpretation of Theorem 3.5 as an almost-orthogonality principle.

4 The Case of Time–Frequency Analysis

In this section, we show how time–frequency analysis fits the abstract model from Sect. 2.3 and also illustrate the notions from Sect. 3 in this particular case. We first introduce the relevant class of function spaces.

4.1 Modulation Spaces

Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ be a nonzero function normalized by $\|\varphi\|_2 = 1$, and recall that the STFT of a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ is the function $V_\varphi f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ given by

$$V_\varphi f(z) = \int_{\mathbb{R}^d} f(t)\overline{\varphi(t-x)} e^{-2\pi i \xi t} \mathrm{d}t, \quad z = (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d.$$

Modulation spaces are defined by imposing weighted L^p-norms on the STFT, as we now describe. As weight, we consider continuous functions $v : \mathbb{R}^{2d} \to (0, +\infty)$ such that $v(z+z') \le C v(z)(1+|z'|)^N$, for some constants $N, C > 0$ and all $z, z' \in \mathbb{R}^{2d}$. Such functions are called *polynomially moderated weights*. For a polynomially moderated weight v and $p \in [1, +\infty]$, the *modulation space* $M^p_v(\mathbb{R}^d)$ is

$$M^p_v(\mathbb{R}^d) := \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \,\middle|\, V_\varphi f \in L^p_v(\mathbb{R}^{2d}) \right\}.$$

The space $M^p_v(\mathbb{R}^d)$ is given the norm

$$\|f\|_{M^p_v(\mathbb{R}^d)} := \|V_\varphi f\|_{L^p_v(\mathbb{R}^{2d})} = \left(\int_{\mathbb{R}^{2d}} |f(z)|^p \, v(z)^p \mathrm{d}z \right)^{1/p},$$

with the usual modification for $p = +\infty$. A different choice of φ yields the same space with an equivalent norm, $M^p_v(\mathbb{R}^d)$ is a Banach space, and $V_\varphi : M^p_v(\mathbb{R}^d) \to V_\varphi(M^p_v(\mathbb{R}^d))$ defines an isometric isomorphism, where $V_\varphi(M^p_v(\mathbb{R}^d))$ is considered as a subspace of $L^p_v(\mathbb{R}^{2d})$ (see [25, Proposition 4.3] or [33, Chap. 11]). The unweighted modulation space $M^2(\mathbb{R}^d)$ coincides with $L^2(\mathbb{R}^d)$.

The assumption that φ is a Schwartz function can be relaxed by considering the following classes of weights.

Definition 4.1 (*Admissible TF weights*) A continuous even function $w : \mathbb{R}^{2d} \to (0, +\infty)$ is called an *admissible TF weight* if it satisfies the following:

(i) w is submultiplicative, i.e., $w(z+z') \le w(z)w(z')$, for all $z, z' \in \mathbb{R}^{2d}$;
(ii) $w(z) \le C(1+|z|)^N$, for some constants $C, N > 0$ and all $z \in \mathbb{R}^{2d}$;
(iii) w satisfies the following condition—known as the GRS-condition after Gelfand, Raikov and Shilov [31]

$$\lim_{n\to\infty} w(nz)^{1/n} = 1, \quad z \in \mathbb{R}^{2d}.$$

The polynomial weights $w(z) = (1+|z|)^s$ with $s \ge 0$ are examples of admissible TF weights. A second weight $v : \mathbb{R}^{2d} \to (0, +\infty)$ is said to be w-moderated if

$$v(z+z') \le C_v w(z) v(z'), \quad z, z' \in \mathbb{R}^{2d}, \tag{22}$$

for some constant C_v. Note that in this case, v is polynomially moderated because w is dominated by a polynomial. (Do not confuse the notion of admissible TF weight with the one of being a weight *admissible for a certain BF space*—cf. Definition 2.2.)

Let w be an admissible TF weight, and let $\varphi \in M^1_w(\mathbb{R}^d)$ be nonzero. For every w-moderated weight v and $p \in [1, +\infty]$, the modulation space $M^p_v(\mathbb{R}^d)$ can be described as

$$M^p_v(\mathbb{R}^d) = \left\{ f \in M^\infty_{1/w}(\mathbb{R}^d) \,\middle|\, V_\varphi f \in L^p_v(\mathbb{R}^{2d}) \right\},$$

and $\|V_\varphi f\|_{L^p_v(\mathbb{R}^{2d})}$ is an equivalent norm on it (see [33, Chap. 11]). Hence, the assumption $\varphi \in M^1_w(\mathbb{R}^d)$ is enough to treat the class of all modulation spaces M^p_v, with v a w-moderated weight and $p \in [1, +\infty]$. (The GRS-condition above does not play a role in what was presented so far; it is required for the technical results used in Sects. 4.3, 4.4.)

4.2 Phase-Space and Modulation Spaces

Extending Example 2.5, in this section we explain how modulation spaces fit the model of Sect. 2.3.

Let w be an admissible TF weight, $\varphi \in M^1_w(\mathbb{R}^d)$ normalized by $\|\varphi\|_2 = 1$, and v a w-moderated weight. To apply the model, we let $\mathcal{G} = \mathbb{R}^{2d}$ with the usual group structure and $E := L^p_v(\mathbb{R}^{2d})$. Since v is w-moderated, we deduce readily that w is admissible for E.

We also let $S_E := V_\varphi M^p_v(\mathbb{R}^d)$. The fact that S_E is closed within $L^p_v(\mathbb{R}^{2d})$ follows from the fact that $V_\varphi : M^p_v(\mathbb{R}^d) \to S_E$ is an isometric isomorphism and $M^p_v(\mathbb{R}^d)$ is complete. As projection $P : L^p_w(\mathbb{R}^{2d}) \to V_\varphi M^p_v(\mathbb{R}^d)$, we consider the operator composition of V_φ with its formal adjoint: $P = V_\varphi V^*_\varphi$. As discussed in Sect. 2.1 and Example 2.5, this operator is well defined and satisfies

$$\left|PF(x', \xi')\right| \leq \int\limits_{\mathbb{R}^d \times \mathbb{R}^d} |F(x, \xi)| \left|V_\varphi \varphi(x' - x, \xi' - \xi)\right| \mathrm{d}x\mathrm{d}\xi, \quad F \in L^p_v(\mathbb{R}^{2d}).$$

The fact that $\varphi \in M^1_w(\mathbb{R}^d)$ in principle means that $V_\varphi\varphi \in L^1_w(\mathbb{R}^{2d})$, but it also follows that $V_\varphi\varphi \in W(L^\infty, L^1_w)(\mathbb{R}^{2d})$—see for example [33, Proposition 12.1.11].

Hence, (A2)–(A3) are satisfied with $K := \left|V_\varphi\varphi\right|$. (Note that since $\mathcal{G} = \mathbb{R}^{2d}$ is abelian, the left and right amalgam spaces coincide.)

Recall the time–frequency localization operator with symbol $m \in L^\infty(\mathbb{R}^{2d})$ formally given by $H_m f = V^*_\varphi(m V_g f)$, cf. (10). In Sect. 2.1, we discussed the operator $H_m : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$. Analogous considerations apply to $H_m : M^p_v(\mathbb{R}^d) \to M^p_v(\mathbb{R}^d)$ and show that $V_\varphi H_m V^*_\varphi F = P(mF)$ for all $F \in V_\varphi(M^p_v(\mathbb{R}^d))$. This means that under the isometric isomorphism $V_\varphi : M^p_v(\mathbb{R}^d) \to V_\varphi(M^p_v(\mathbb{R}^d))$, the time–frequency localization operator H_m corresponds to phase-space multiplier M_m from Sect. 2.3.3.

Remark 4.2 In this article, we will be concerned with time–frequency localization operators with bounded symbols m. However, the condition that m be bounded is by no means necessary for $H_m : M_v^p(\mathbb{R}^d) \to M_v^p(\mathbb{R}^d)$ to be bounded. See, for example, [6, 12, 13, 27] for sharper boundedness results for time–frequency localization operators.

4.3 Spectral Invariance

We present a technical result that says that for a certain class of operators, the property of being invertible on $L^2(\mathbb{R}^d)$ automatically implies invertibility on a range of modulation spaces. The following proposition is obtained by combining known spectral invariance results for pseudodifferential operators [34, 35, 50, 51]. For convenience of the reader, in Appendix 2 we provide the necessary background results and give a proof based on those results.

Proposition 4.3 *Let $w : \mathbb{R}^{2d} \to (0, +\infty)$ be an admissible TF weight—cf. Definition 4.1—and let $\varphi \in M_w^1(\mathbb{R}^d)$ be nonzero. Let v be a w-moderated weight, and let $p \in [1, +\infty]$. Let $T : M_v^p(\mathbb{R}^d) \to M_v^p(\mathbb{R}^d)$ be an operator satisfying the enveloping condition*

$$\left|V_\varphi T(f)(z)\right| \leq \int_{\mathbb{R}^{2d}} \left|V_\varphi f(z')\right| H(z - z')\mathrm{d}z', \quad z \in \mathbb{R}^{2d}, \tag{23}$$

for some function $H \in L_w^1(\mathbb{R}^{2d})$. Assume that $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is invertible. Then $T : M_v^p(\mathbb{R}^d) \to M_v^p(\mathbb{R}^d)$ is invertible.

4.4 Families of Operators Well-Spread in the Time–Frequency Plane

We now illustrate the concepts and results from Sect. 3 in the case of time–frequency analysis.

Let $w : \mathbb{R}^{2d} \to (0, +\infty)$ be an admissible TF weight and $\varphi \in M_w^1(\mathbb{R}^d)$ with $\|\varphi\|_2 = 1$. Setting $\mathcal{G} = \mathbb{R}^{2d}$ in (15), we call a set $\Gamma \subseteq \mathbb{R}^{2d}$ relatively separated if $\sup_{z \in \mathbb{R}^{2d}} \#(\Gamma \cap B_1(z)) < +\infty$. Analogously to Definition 3.2, a family of symbols $\left\{\eta_\gamma : \mathbb{R}^{2d} \to \mathbb{C} \,\middle|\, \gamma \in \Gamma\right\}$ is *well-spread* (relative to w) if Γ is relatively separated and there exists $g \in W(L^\infty, L_w^1)(\mathbb{R}^{2d})$ such that

$$\left|\eta_\gamma(z)\right| \leq g(z - \gamma), \quad z \in \mathbb{R}^{2d}, \gamma \in \Gamma.$$

Similarly, analogously to Definition 3.1, a family of operators $\left\{H_\gamma : M_{1/w}^\infty(\mathbb{R}^d) \to M_{1/w}^\infty(\mathbb{R}^d) \gamma \in \Gamma\right\}$ is said to be *well-spread in the time–frequency plane* (relative to (φ, w)) if there exists an envelope (Γ, Θ, g), with $\Gamma \subseteq \mathbb{R}^{2d}$ a relatively separated set and $\Theta, g \in W(L^\infty, L_w^1)(\mathbb{R}^{2d})$ such that

$$\left|V_\varphi H_\gamma f(z)\right| \le \int_{\mathbb{R}^{2d}} g(z'-\gamma)\left|V_\varphi f(z')\right| \Theta(z-z')\mathrm{d}z', \quad z \in \mathbb{R}^{2d}, \gamma \in \Gamma. \tag{24}$$

Note that $\{H_\gamma : \gamma \in \Gamma\}$ is well spread in the TF plane if and only if the family of operators $\left\{ V_\varphi H_\gamma V_\varphi^* : V_\varphi(M^\infty_{1/w}(\mathbb{R}^d)) \to V_\varphi(M^\infty_{1/w}(\mathbb{R}^d)) \,\middle|\, \gamma \in \Gamma \right\}$ is well spread in the sense of Sect. 3.

Note also that if $\{H_\gamma : \gamma \in \Gamma\}$ is well spread in the time–frequency plane, then, because of (24), for all w-moderated weights v and $p \in [1, +\infty]$, each operator H_γ maps $M^p_v(\mathbb{R}^d)$ into $M^p_v(\mathbb{R}^d)$ with a norm bound independent of γ,

$$\|H_\gamma f\|_{M^p_v} \le C_v \|g\|_\infty \|\Theta\|_{L^1_w} \|f\|_{M^p_v}.$$

Remark 4.4 In parallel to Proposition 3.3, if $\{\eta_\gamma : \gamma \in \Gamma\}$ is a well-spread family of symbols, then the corresponding family of time–frequency localization operators $\{H_{\eta_\gamma} : \gamma \in \Gamma\}$ is well-spread in the time–frequency plane.

In the case of time–frequency analysis, we can strengthen Theorem 3.5 by means of the spectral invariance result in Proposition 4.3. Due to this result, the invertibility assumption in Theorem 3.5 can be replaced by assuming invertibility on $L^2(\mathbb{R}^d)$.

Theorem 4.5 *Let $w : \mathbb{R}^{2d} \to (0, +\infty)$ be an admissible TF weight, and let $\varphi \in M^1_w(\mathbb{R}^d)$. Let $\{H_\gamma : \gamma \in \Gamma\}$ be well spread in the TF plane (relative to (φ, w)).*

Suppose that the operator $\sum_\gamma H_\gamma : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is invertible. Then, for all w-moderated weights v and for all $1 \le p \le +\infty$, $\sum_\gamma H_\gamma : M^p_v \to M^p_v$ is invertible and the following norm-equivalence holds:

$$\|f\|_{M^p_v} \approx \left(\sum_{\gamma \in \Gamma} \|H_\gamma f\|^p_{L^2(\mathbb{R}^d)} v(\gamma)^p \right)^{1/p}, \quad f \in M^p_v(\mathbb{R}^d),$$

with the usual modification for $p = \infty$.

Remark 4.6 The estimates hold uniformly for $1 \le p \le +\infty$ and any family of weights having a uniform constant C_v (cf. 22).

Proof of Theorem 4.5 Let v be a w-moderated weight and $1 \le p \le +\infty$. By hypothesis $\sum_\gamma H_\gamma : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is invertible. In order to apply Proposition 4.3 to this operator, we let $f \in M^p_v(\mathbb{R}^d)$ and use (24) to estimate

$$\begin{aligned} \left| V_\varphi \Big(\sum_{\gamma \in \Gamma} H_\gamma f \Big)(z) \right| &\le \int_{\mathbb{R}^{2d}} \left|V_\varphi f(z')\right| \Theta(z-z') \sum_{\gamma \in \Gamma} g(z'-\gamma)\mathrm{d}z' \\ &\lesssim \|g\|_{W(L^\infty, L^1_w)} \int_{\mathbb{R}^{2d}} \left|V_\varphi f(z')\right| \Theta(z-z')\mathrm{d}z'. \end{aligned}$$

Since $\Theta \in W(L^\infty, L^1_w)(\mathbb{R}^{2d})$, it follows from Proposition 4.3 that $\sum_\gamma H_\gamma : M^p_v(\mathbb{R}^d) \to M^p_v(\mathbb{R}^d)$ is invertible.

We use the notation $S^p_v := V_\varphi(M^p_v(\mathbb{R}^d))$. Let $T_\gamma := V_\varphi H_\gamma V^*_\varphi : S^p_v \to S^p_v$. The family $\{T_\gamma : \gamma \in \Gamma\}$ is well spread (in the sense of Sect. 3). Since $\sum_\gamma H_\gamma : M^p_v(\mathbb{R}^d) \to M^p_v(\mathbb{R}^d)$ is invertible, so is $\sum_\gamma T_\gamma : S^p_v \to S^p_v$. Therefore, we can apply Theorem 3.5, and the conclusion follows. □

5 Frames of Eigenfunctions of Time–Frequency Localization Operators

In this section, we prove the main results on constructing frames with a prescribed time–frequency profile. Throughout this section, let $w : \mathbb{R}^{2d} \to (0, +\infty)$ be an admissible TF weight and $\varphi \in M^1_w(\mathbb{R}^d)$ be normalized by $\|\varphi\|_2 = 1$.

5.1 Thresholding Eigenvalues

Let $\{\eta_\gamma : \gamma \in \Gamma\}$ be a family of nonnegative functions on $\mathbb{R}^{2d}$ that is well spread (relative to w), and consider the corresponding family of time–frequency localization operators $H_{\eta_\gamma} f = V^*_\varphi(\eta_\gamma V_\varphi f)$. Since each η_γ is nonnegative and belongs to $L^1(\mathbb{R}^{2d})$, H_{η_γ} is positive and trace-class, and $\mathrm{trace}(H_{\eta_\gamma}) = \|\eta_\gamma\|_1$, see [7,8,27,54] and Proposition 2.10. Hence $H_{\eta_\gamma} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ can be diagonalized as

$$H_{\eta_\gamma} f = \sum_{k \geq 1} \lambda^\gamma_k \left\langle f, \phi^\gamma_k \right\rangle \phi^\gamma_k, \quad f \in L^2(\mathbb{R}^d),$$

where $\left\{ \phi^\gamma_k \,\middle|\, k \in \mathbb{N} \right\}$ is an orthonormal subset of $L^2(\mathbb{R}^d)$—possibly incomplete if $\ker(H_{\eta_\gamma}) \neq \{0\}$—and $(\lambda^\gamma_k)_k$ is a nonincreasing sequence of nonnegative real numbers.

The time–frequency profile of the functions $\left\{ \phi^\gamma_k \,\middle|\, k \in \mathbb{N} \right\}$ is optimally adapted to the mask η_γ in the following sense. For each $N \in \mathbb{N}$, the set $\{\phi^\gamma_1, \dots, \phi^\gamma_N\}$ is an orthonormal set maximizing the quantity

$$\sum_{k=1}^{N} \int_{\mathbb{R}^{2d}} \eta_\gamma(z) \left|V_\varphi f_k(z)\right|^2 \, \mathrm{d}z \tag{25}$$

among all orthonormal sets $\{f_1, \dots, f_N\} \subseteq L^2(\mathbb{R}^d)$.

Moreover, since $\eta_\gamma \in L^1_w$ and $\varphi \in M^1_w$, $\phi^\gamma_k \in M^1_w$ provided that $\lambda^\gamma_k \neq 0$ (see, for example, [20, Lemma 5]).

For every $\varepsilon > 0$, we define the operator $H^\varepsilon_{\eta_\gamma}$ by applying a threshold to the eigenvalues of H_{η_γ},

$$H^\varepsilon_{\eta_\gamma} f := \sum_{k:\lambda^\gamma_k > \varepsilon} \lambda^\gamma_k \left\langle f, \phi^\gamma_k \right\rangle \phi^\gamma_k, \quad f \in L^2(\mathbb{R}^d). \tag{26}$$

Hence,

$$\|H^{\varepsilon}_{\eta_\gamma} f\|_2 \le \|H_{\eta_\gamma} f\|_2 \le \|H^{\varepsilon}_{\eta_\gamma} f\|_2 + \varepsilon \|f\|_2, \quad f \in L^2(\mathbb{R}^d). \tag{27}$$

As a first step to analyze the effect of the thresholding operation $H_{\eta_\gamma} \mapsto H^{\varepsilon}_{\eta_\gamma}$, we show that the operators $\{H_{\eta_\gamma} : \gamma \in \Gamma\}$ behave "globally" like projectors. Since these operators may have infinite rank, in general $\|(H_{\eta_\gamma})^2 f\|_2 \not\approx \|H_{\eta_\gamma} f\|_2$. However, we prove the following:

Proposition 5.1 *Let $w : \mathbb{R}^{2d} \to (0, +\infty)$ be an admissible TF weight, and let $\varphi \in M^1_w(\mathbb{R}^d)$. Let $\{\eta_\gamma : \gamma \in \Gamma\}$ be a well-spread family of nonnegative symbols on $\mathbb{R}^{2d}$ (relative to w) with $\sum_\gamma \eta_\gamma \approx 1$. Then for all w-moderated weights v and all $p \in [1, +\infty]$,*

$$\|f\|_{M^p_v} \approx \left(\sum_{\gamma\in\Gamma} \|H_{\eta_\gamma} f\|^p_{L^2(\mathbb{R}^d)} v(\gamma)^p \right)^{1/p} \approx \left(\sum_{\gamma\in\Gamma} \|(H_{\eta_\gamma})^2 f\|^p_{L^2(\mathbb{R}^d)} v(\gamma)^p \right)^{1/p}, \quad f \in M^p_v(\mathbb{R}^d).$$

Remark 5.2 The estimate $\|f\|_{M^p_v} \approx \|(\|H_{\eta_\gamma} f\|_2)_\gamma\|_{\ell^p_v}$ is contained in [19,20] for the case of families of symbols consisting of lattice translates of a single function, and in [47] in the present generality. The norm-equivalence involving $(H_{\eta_\gamma})^2 f$ is new.

Proof of Proposition 5.1 Since the symbols η_γ satisfy $m := \sum_\gamma \eta_\gamma \ge A$ for some constant $A > 0$, it follows that

$$\left\langle \sum_{\gamma\in\Gamma} H_{\eta_\gamma} f, f \right\rangle = \langle H_m f, f\rangle = \left\langle V^*_\varphi (m V_\varphi f), f \right\rangle$$
$$= \left\langle m V_\varphi f, V_\varphi f \right\rangle = \int_{\mathbb{R}^{2d}} m(z) \left|V_\varphi f(z)\right|^2 dz \ge A \|V_\varphi f\|_2^2 = A\|f\|_2^2.$$

Hence, $\sum_\gamma H_{\eta_\gamma} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is invertible. Now Proposition 3.3 and Theorem 4.5 yield the estimate involving $\|f\|_{M^p_v}$ and $\|(\|H_{\eta_\gamma} f\|_2)_\gamma\|_{\ell^p_v}$.

For the estimate involving the squared operators, note that Corollary 3.7 implies that $\sum_\gamma (H_{\eta_\gamma})^2 : S_{L^2} \to S_{L^2}$ is also invertible. According to Proposition 3.4, the family $\{(H_{\eta_\gamma})^2 : \gamma \in \Gamma\}$ is also well spread. Hence, a second application of Theorem 4.5 concludes the proof. □

We now state and prove that a sufficiently fine thresholding of the eigenvalues of the operators $\{H_{\eta_\gamma} : \gamma \in \Gamma\}$ still decomposes the family of modulation spaces.

Theorem 5.3 *Let $w : \mathbb{R}^{2d} \to (0, +\infty)$ be an admissible TF weight, and let $\varphi \in M^1_w(\mathbb{R}^d)$. Let $\{\eta_\gamma : \gamma \in \Gamma\}$ be a family of nonnegative symbols on $\mathbb{R}^{2d}$ that is well spread (relative to w) and such that $\sum_\gamma \eta_\gamma \approx 1$. Let v be a w-moderated weight. Then there exist constants $0 < c \le C < +\infty$ such that for all sufficiently small $\varepsilon > 0$ and all $p \in [1, +\infty]$,*

$$c\|f\|_{M^p_v} \le \left(\sum_{\gamma\in\Gamma} \|H^{\varepsilon}_{\eta_\gamma} f\|^p_{L^2(\mathbb{R}^d)} v(\gamma)^p \right)^{1/p} \le C\|f\|_{M^p_v}, \quad f \in M^p_v(\mathbb{R}^d),$$

with the usual modification for $p = \infty$.

The choice of ε *and the estimates are uniform for* $1 \le p \le +\infty$ *and any family of weights having a uniform constant* C_v *(cf. 22).*

Proof Given $\varepsilon > 0$ and $f \in M_v^p(\mathbb{R}^d)$, we apply (27) to $H_{\eta_\gamma}(f)$ to obtain

$$\|(H_{\eta_\gamma})^2 f\|_2 \le \|H_{\eta_\gamma}^\varepsilon H_{\eta_\gamma} f\|_2 + \varepsilon \|H_{\eta_\gamma} f\|_2.$$

Since $H_{\eta_\gamma}^\varepsilon$ and H_{η_γ} commute, $\|H_{\eta_\gamma}^\varepsilon H_{\eta_\gamma} f\|_2 = \|H_{\eta_\gamma} H_{\eta_\gamma}^\varepsilon f\|_2 \lesssim \|H_{\eta_\gamma}^\varepsilon f\|_2$. Hence,

$$\|(H_{\eta_\gamma})^2 f\|_2 \lesssim \|H_{\eta_\gamma}^\varepsilon f\|_2 + \varepsilon \|H_{\eta_\gamma} f\|_2.$$

Taking ℓ_v^p norm on γ yields

$$\|(\|(H_{\eta_\gamma})^2 f\|_2)_\gamma\|_{\ell_v^p} \lesssim \|(\|H_{\eta_\gamma}^\varepsilon f\|_2)_\gamma)\|_{\ell_v^p} + \varepsilon \|(\|H_{\eta_\gamma} f\|_2)_\gamma\|_{\ell_v^p}.$$

Using the estimates in Proposition 5.1, we get

$$\|f\|_{M_v^p} \le C\|(\|H_{\eta_\gamma}^\varepsilon f\|_2)_\gamma\|_{\ell_v^p} + c\varepsilon \|f\|_{M_v^p}$$

for some constants c, C. Hence,

$$(1 - c\varepsilon)\|f\|_{M_v^p} \le C\|(\|H_{\eta_\gamma}^\varepsilon f\|_2)_\gamma\|_{\ell_v^p}.$$

This gives the desired lower bound (for all $0 < \varepsilon < 1/c$). The upper bound follows from the first inequality in (27) and Proposition 5.1. □

Remark 5.4 Note that the proof of Theorem 5.3 only uses the estimate in (27) and the fact that H_{η_γ} and $H_{\eta_\gamma}^\varepsilon$ commute. Hence, more general thresholding rules besides the one in (26) can be used.

5.2 Frames of Eigenfunctions

Finally, we obtain the desired result on frames of eigenfunctions.

Theorem 5.5 *Let* $w : \mathbb{R}^{2d} \to (0, +\infty)$ *be an admissible TF weight, and let* $\varphi \in M_w^1(\mathbb{R}^d)$. *Let* $\{\eta_\gamma : \gamma \in \Gamma\}$ *be a family of nonnegative symbols on* $\mathbb{R}^{2d}$ *that is well spread (relative to* w*) and such that* $\sum_\gamma \eta_\gamma \approx 1$. *Let* v *be a* w*-moderated weight. Then there exists a constant* $\alpha > 0$ *such that, for every choice of finite subsets of eigenfunctions of* H_{η_γ} $\left\{\phi_k^\gamma \,\middle|\, \gamma \in \Gamma, 1 \le k \le N_\gamma\right\}$ *with*

$$\alpha \|\eta_\gamma\|_1 \le N_\gamma \text{ and } \sup_{\gamma \in \Gamma} N_\gamma < +\infty,$$

the following frame estimates hold simultaneously for all $1 \leq p \leq +\infty$*, with the usual modification for* $p = \infty$*:*

$$\|f\|_{M_v^p} \approx \left(\sum_{\gamma \in \Gamma} \sum_{k=1}^{N_\gamma} \left|\langle f, \lambda_k^\gamma \phi_k^\gamma \rangle\right|^p v(\gamma)^p \right)^{1/p}, \qquad f \in M^p(\mathbb{R}^d).$$

Moreover, α *can be chosen uniformly for any class of weights* v *having a uniform constant* C_v *(cf. 22).*

Before proving Theorem 5.5, we make some remarks.

Remark 5.6 As opposed to other constructions that partition either the time or frequency domain (see, e.g., [4,9,10,24]), the symbols η_γ partition the time–frequency plane simultaneously.

Remark 5.7 Note that if (Γ, Θ, g) is an envelope for $\{\eta_\gamma : \gamma \in \Gamma\}$, since $\|\eta_\gamma\|_1 \leq \|g\|_1$, in Theorem 5.5 it is always possible to make a uniform choice $N_\gamma = N_0$.

Proof of Theorem 5.5 For every $\gamma \in \Gamma$ and $\varepsilon > 0$, let $I_\gamma^\varepsilon := \{k \in \mathbb{N}/\lambda_k^\gamma > \varepsilon\}$, which is a finite set. Using Theorem 5.3, Proposition 5.1, and the orthonormality of the eigenfunctions, we can find a value of $\varepsilon > 0$ such that

$$\begin{aligned}\|f\|_{M_v^p} &\approx \left(\sum_{\gamma \in \Gamma} \left(\sum_{k \in \mathbb{N}} \left|\langle f, \lambda_k^\gamma \phi_k^\gamma \rangle\right|^2 \right)^{p/2} v(\gamma)^p \right)^{1/p} \\ &\approx \left(\sum_{\gamma \in \Gamma} \left(\sum_{k \in I_\gamma^\varepsilon} \left|\langle f, \lambda_k^\gamma \phi_k^\gamma \rangle\right|^2 \right)^{p/2} v(\gamma)^p \right)^{1/p},\end{aligned}$$

with the usual modification for $p = +\infty$. This implies that for any choice of subsets of indices $J_\gamma^\varepsilon \supseteq I_\gamma^\varepsilon$, we also have

$$\|f\|_{M_v^p} \approx \left(\sum_{\gamma \in \Gamma} \left(\sum_{k \in J_\gamma^\varepsilon} \left|\langle f, \lambda_k^\gamma \phi_k^\gamma \rangle\right|^2 \right)^{p/2} v(\gamma)^p \right)^{1/p}. \tag{28}$$

Furthermore, since $\sum_k \lambda_k^\gamma = \operatorname{trace}(H_{\eta_\gamma}) = \|\eta_\gamma\|_1$, we have $\#I_\gamma^\varepsilon \leq \varepsilon^{-1} \sum_k \lambda_k^\gamma = \varepsilon^{-1}\|\eta_\gamma\|_1$. Hence, setting $\alpha := \varepsilon^{-1}$, we ensure that for $N_\gamma \geq \alpha\|\eta_\gamma\|_1$, the set $J_\gamma^\varepsilon := \left\{ k \in \mathbb{N} \,\middle|\, 1 \leq k \leq N_\gamma \right\}$ contains I_γ^ε and therefore satisfies (28). Let $N := \sup_\gamma N_\gamma$. By hypothesis, $N < \infty$. Since $\#J_\gamma^\varepsilon = N_\gamma \leq N$, it follows that

$$\left(\sum_{k \in J_\gamma^\varepsilon} \left|\langle f, \lambda_k^\gamma \phi_k^\gamma \rangle\right|^2 \right)^{1/2} \approx \left(\sum_{k \in J_\gamma^\varepsilon} \left|\langle f, \lambda_k^\gamma \phi_k^\gamma \rangle\right|^p \right)^{1/p},$$

with a constant that depends on N (and the usual modification for $p = +\infty$). Combining the last estimate with (28), we obtain the desired conclusion. □

As an application of Theorem 5.5, we can now prove Theorem 1.1 (stated in the Introduction).

Proof of Theorem 1.1 Let $\{\Omega_\gamma : \gamma \in \Gamma\}$ be an admissible cover of $\mathbb{R}^{2d}$, as defined in the Introduction—cf. (5). Let $\eta_\gamma := 1_{\Omega_\gamma}$ be the characteristic function of Ω_γ. The fact that $\{\Omega_\gamma : \gamma \in \Gamma\}$ covers $\mathbb{R}^{2d}$ implies that $\sum_\gamma \eta_\gamma = \sum_\gamma 1_{\Omega_\gamma} \geq 1$. In addition, if we let $g := 1_{B_R(0)}$, it follows from (5) that $\{\eta_\gamma : \gamma \in \Gamma\}$ is well spread with envelope (Γ, g) and constant weight $w \equiv 1$. We note that $\left|\Omega_\gamma\right| = \|\eta_\gamma\|_1$ and apply Theorem 5.5 with $p = 2$ and $v \equiv 1$. (Recall that $M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$.) □

5.3 Inner Regularity

Finally, we derive a variant of Theorem 5.5, where, under an inner regularity assumption on the family of symbols, we renormalize the frame of eigenfunctions so that each frame element has norm 1. To this end, we first prove the following lemma, which may be of independent interest:

Lemma 5.8 *Let $\varphi \in L^2(\mathbb{R}^d) \setminus \{0\}$, and let $\Omega \subseteq \mathbb{R}^d$ be a measurable set with nonempty interior. Then the time–frequency localization operator $H_\Omega : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$,*

$$H_\Omega f(t) = \int_\Omega V_\varphi f(x, \xi)\varphi(t - x)e^{2\pi i \xi t} \mathrm{d}x\mathrm{d}\xi, \quad t \in \mathbb{R}^d,$$

has infinite rank.

Proof The proof is based on the fact that the STFT of Hermite functions are weighted polyanalytic functions (cf. [1–3]) and therefore cannot vanish on a ball.

Suppose, for the sake of contradiction, that H_Ω has rank $n - 1 < +\infty$. Let $h_1, \ldots, h_n \in L^2(\mathbb{R}^d)$ be multidimensional Hermite functions of order $\leq n$. For example, if $g_1, \ldots, g_n \subseteq L^2(\mathbb{R})$ are the first one-dimensional Hermite functions, we can let $h_k \in L^2(\mathbb{R}^d)$ be the tensor product $h_k(x_1, \ldots, x_d) := g_k(x_1)g_1(x_2)\ldots g_1(x_d)$. Let V_n be the subspace of $L^2(\mathbb{R}^d)$ spanned by $h_1, \ldots, h_n$.

Since V_n has dimension n, it follows that there exists some nonzero $f = \sum_{k=1}^n c_k h_k \in V_n$ such that $H_\Omega f = 0$. Consequently,

$$0 = \langle H_\Omega f, f\rangle = \int_\Omega \left|V_\varphi f(z)\right|^2 \mathrm{d}z,$$

and therefore $V_\varphi f \equiv 0$ on Ω. With the notation $(x, \xi) =: z \in \mathbb{C}^n$ and $m(z) = e^{-x\cdot\xi i + \pi|z|^2}$, let $F(z) := m(z)V_f\varphi(\bar{z})$. Since $V_\varphi f(z) = \overline{V_f\varphi(-z)}$, it follows that F vanishes on $\Omega' := -\overline{\Omega}$. We will show that $F \equiv 0$. Since m never vanishes, this will imply that $f \equiv 0$, thus yielding a contradiction.

The function $F(z) = \sum_k c_k m(z) V_{h_k}\varphi(-z)$ is a polyanalytic function of order (at most) n, i.e., $(\partial/\partial_{\overline{z}})^\beta F \equiv 0$ for every multi-index $|\beta| \le n$, see [1–3]. A polyanalytic function that vanishes on a set of nonempty interior must vanish identically. For $d = 1$, this can be proved directly by induction on n or deduced from much sharper uniqueness results (see [5]). The case of general dimension d reduces to $d = 1$ by fixing $d - 1$ variables of F and applying the one-dimensional result. □

We also quote the following intertwining property:

Lemma 5.9 *For $z = (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$, consider the time–frequency shift*

$$\pi(z) f(t) := e^{2\pi i \xi t} f(t - x), \quad f \in L^2(\mathbb{R}^d).$$

For $m \in L^\infty(\mathbb{R}^{2d})$, let $H_m : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be the time–frequency localization operator with symbol m, i.e., $H_m f := V_\varphi^*(m V_\varphi f)$. Then

$$\pi(z) H_m \pi(z)^* = H_{m(\cdot - z)}, \quad z = (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d, m \in L^\infty(\mathbb{R}^{2d}).$$

The proof of Lemma 5.9 is a straightforward calculation, see [19, Lemma 2.6]. Using Lemma 5.8, we obtain a variant of Theorem 5.5.

Theorem 5.10 *Let $w : \mathbb{R}^{2d} \to (0, +\infty)$ be an admissible TF weight, and let $\varphi \in M^1_w(\mathbb{R}^d)$. Let $\{\eta_\gamma : \gamma \in \Gamma\}$ be a family of nonnegative symbols on $\mathbb{R}^{2d}$ that is well spread (relative to w) and such that $\sum_\gamma \eta_\gamma \approx 1$. Let v be a w-moderated weight. Assume in addition that there exists a ball B_r and a constant $c > 0$ such that*

$$\eta_\gamma(z) \ge c 1_{B_r}(z - \gamma), \quad z \in \mathbb{R}^{2d}, \ \gamma \in \Gamma. \tag{29}$$

Then there exists a constant $\alpha > 0$ such that, for every choice of finite subsets of eigenfunctions $\left\{ \phi^\gamma_k \,\middle|\, \gamma \in \Gamma, 1 \le k \le N_\gamma \right\}$ with $\inf_\gamma N_\gamma \ge \alpha$ and $\sup_\gamma N_\gamma < +\infty$, the following frame estimates hold simultaneously for all $1 \le p \le +\infty$, with the usual modification for $p = \infty$:

$$\|f\|_{M^p_v} \approx \left(\sum_{\gamma \in \Gamma} \sum_{k=1}^{N_\gamma} \left| \langle f, \phi^\gamma_k \rangle \right|^p v(\gamma)^p \right)^{1/p}, \qquad f \in M^p_v(\mathbb{R}^d).$$

Moreover, α can be chosen uniformly for any class of weights v having a uniform constant C_v (cf. 22).

Before proving Theorem 5.10, we make some remarks.

Remark 5.11 When η_γ is the characteristic function of a set Ω_γ, the condition in (29) holds whenever the sets satisfy: $B_r(\gamma) \subseteq \Omega_\gamma \subseteq B_R(\gamma)$, with $R > r > 0$ and Γ a relatively separated set.

Remark 5.12 The frame in Theorem 5.10 comprises the first N_γ elements of each of the orthonormal sets $\{\phi_k^\gamma : k \geq 1\}$. These first N_γ functions are the ones that are best concentrated, according to the weight η_γ. This resembles the problem studied in [46]. However, the results there require very precise information on the frames being pieced together and hence do not apply here.

Remark 5.13 In the language of [11,43], Theorem 5.10 shows that the subspaces spanned by the finite families of eigenfunctions form a *stable splitting* or *fusion frame*. From an application point of view, it is useful to have orthogonal projections onto subspaces with time–frequency concentration in a prescribed area of the time–frequency plane.

Remark 5.14 When η_γ is the characteristic function of a set $\Omega + \gamma$ and Γ is a lattice, Theorem 5.10 reduces to the main technical result in [20]. The proof there does not adapt to the irregular context, since it relies on the use of rotation algebras (noncommutative tori). The proof we give here resorts instead to the almost-orthogonality techniques from [47] (cf. Theorem 3.5) together with spectral invariance results for pseudodifferential operators with symbols in the Sjöstrand class [34,35,50,51].

Proof of Theorem 5.10 First note that (29), together with the well-spreadness condition, implies that $\|\eta_\gamma\|_1 \approx 1$ (the constants, of course, depend on r, R, and c). Hence, the condition $\alpha\|\eta_\gamma\|_1 \leq N_\gamma$ required by Theorem 5.5 can be granted by simply requiring $\tilde{\alpha} \leq N_\gamma$ to hold with a different constant $\tilde{\alpha}$.

By Theorem 5.5, we have that $\|f\|_{M_v^p} \approx (\sum_{\gamma\in\Gamma} \sum_{k=1}^{N_\gamma} |\langle f, \lambda_k^\gamma \phi_k^\gamma\rangle|^p\, v(\gamma)^p)^{1/p}$. Hence, it suffices to show that $\lambda_k^\gamma \approx 1$ for $1 \leq k \leq N_\gamma$.

The upper bound follows from the well-spreadness condition, because if (Γ, Θ, g) is an envelope for $\{H_{\eta_\gamma} : \gamma \in \Gamma\}$, then all the singular values of H_{η_γ} are bounded by $\|H_{\eta_\gamma}\|_{2\to 2} \leq \|g\|_\infty$, cf. (18).

Let $N := \sup_\gamma N_\gamma$. By Lemma 5.8, the localization operator H_{B_r} has infinite rank. Hence, the nonzero eigenvalues of H_{B_r} form an infinite nonincreasing sequence $\lambda_k^r > 0$, $k \geq 1$. From (29), it follows that $H_{\eta_\gamma} \geq cH_{B_r+\gamma}$ (cf. Proposition 2.10). By Lemma 5.9, the sequence of eigenvalues of $H_{B_r+\gamma}$ coincides with the one of H_{B_r}. Consequently, for $1 \leq k \leq N_\gamma$,

$$\lambda_k^\gamma \geq c\lambda_k^r \geq c\lambda_{N_\gamma}^r \geq c\lambda_N^r > 0,$$

as desired. □

Remark 5.15 For the results in this section, the abstract setting of Sect. 2.3 allows for the replacement of ℓ_v^p by more general normed spaces. Indeed, the results derived in the abstract setting cover modulation spaces defined with respect to general translation-invariant solid spaces, cf. [25].

6 Frames of Eigenfunctions: General L^2 Estimates

In this section, we prove results similar to the ones in Sect. 5, but this time in the abstract setting of Sect. 3. We work only with the space L^2 instead of treating a class of Banach spaces. The reason for this restriction is that in Sect. 5 we used tools from the theory of pseudodifferential operators to extend certain results from $L^2(\mathbb{R}^d)$ to a range of modulation spaces, and those tools are not available in the abstract setting. The proofs in this section are, mutatis mutandis, the same as in Sect. 4 and will just be sketched.

Let $\mathcal{G}$ be a locally compact, σ-compact group. Let $E = L^2(\mathcal{G})$, and let us assume that (A1), (A2), and (A3) from Sect. 2.3.2 hold. Let a well-spread family $\{\eta_\gamma : \gamma \in \Gamma\}$ of nonnegative functions on $\mathcal{G}$ be given. We consider the corresponding family of phase-space multipliers,

$$M_{\eta_\gamma} f = P(\eta_\gamma f), \quad f \in S_{L^2}.$$

Since each η_γ is nonnegative and belongs to $L^1(\mathcal{G})$, according to Proposition 2.10, the corresponding operator $M_{\eta_\gamma} : S_{L^2} \to S_{L^2}$ is positive and trace-class, and $\operatorname{trace}(M_{\eta_\gamma}) \lesssim \|\eta_\gamma\|_1$. Let $M_{\eta_\gamma} : S_{L^2} \to S_{L^2}$ be diagonalized as

$$M_{\eta_\gamma} f = \sum_{k \geq 1} \lambda_k^\gamma \left\langle f, \phi_k^\gamma \right\rangle \phi_k^\gamma, \qquad f \in S_{L^2},$$

where $\left\{ \phi_k^\gamma \;\middle|\; k \geq 1 \right\}$ is an orthonormal subset of S_{L^2} and $\left\{ \lambda_k^\gamma \;\middle|\; k \geq 1 \right\}$ is decreasing. Let us define

$$M_{\eta_\gamma}^\varepsilon f = \sum_{k : \lambda_k^\gamma > \varepsilon} \lambda_k^\gamma \left\langle f, \phi_k^\gamma \right\rangle \phi_k^\gamma, \quad f \in S_{L^2}.$$

If M_{η_γ} has finite rank, then $\lambda_k^\gamma = 0$ for $k \gg 1$, and the choice of the corresponding eigenfunctions is arbitrary.

We now derive results similar to Theorems 5.3 and 5.5, but this time in the current abstract setting.

Theorem 6.1 *Under Assumptions (A1), (A2), and (A3), let* $\{\eta_\gamma : \gamma \in \Gamma\}$ *be a well-spread family of nonnegative symbols such that* $\sum_\gamma \eta_\gamma \approx 1$. *Then there exist constants* $0 < c \leq C < +\infty$ *such that for all sufficiently small* $\varepsilon > 0$,

$$c\|f\|_2^2 \leq \sum_{\gamma \in \Gamma} \|M_{\eta_\gamma}^\varepsilon f\|_2^2 \leq C\|f\|_2^2, \quad f \in S_{L^2}.$$

Furthermore, there exists a constant $\alpha > 0$ *such that, for every choice of numbers* $\{N_\gamma : \gamma \in \Gamma\} \subseteq \mathbb{N}$ *satisfying* $\alpha\|\eta_\gamma\|_1 \leq N_\gamma$ *and* $\sup_\gamma N_\gamma < +\infty$, *the family*

$$\left\{ \lambda_k^\gamma \phi_k^\gamma \;\middle|\; \gamma \in \Gamma, 1 \leq k \leq N_\gamma \right\}, \tag{30}$$

formed from eigenfunctions and eigenvalues of the operator M_{η_γ}, *is a frame of* S_{L^2}.

Before proving Theorem 6.1, we make some remarks.

Remark 6.2 Note again that when η_γ is the characteristic function of a set Ω_γ, we are picking $\approx \left|\Omega_\gamma\right|$ eigenfunctions from each phase-space multiplier. Here, $\left|\Omega_\gamma\right|$ is the Haar measure of Ω_γ.

Remark 6.3 The operator M_{η_γ} may have finite rank (for example if $\mathcal{G}$ is a discrete group and η_γ is the characteristic function of a finite set). In this case, the choice of the eigenfunctions associated to the singular value zero is irrelevant, since in (30) these are multiplied by zero.

Proof of Theorem 6.1 We parallel the proofs in Sect. 5. Since $\sum_\gamma M_{\eta_\gamma} = M_{\sum_\gamma \eta_\gamma}$ and $\sum_\gamma \eta_\gamma \geq A > 0$, it follows from Proposition 2.10 that $\sum_\gamma M_{\eta_\gamma}$ is positive definite and therefore invertible. Theorem 3.5 consequently yields

$$\sum_{\gamma\in\Gamma} \|M_{\eta_\gamma} f\|^2_{L^2(\mathcal{G})} \approx \|f\|_2^2, \quad f \in S_{L^2}.$$

In addition, Proposition 3.4, Corollary 3.7, and a second application of Theorem 3.5 yield

$$\sum_{\gamma\in\Gamma} \|M^2_{\eta_\gamma} f\|^2_{L^2(\mathcal{G})} \approx \|f\|_2^2, \quad f \in S_{L^2}.$$

The thresholded operators $M^\varepsilon_{\eta_\gamma}$ satisfy

$$\|M^\varepsilon_{\eta_\gamma} f\|_2 \leq \|M_{\eta_\gamma} f\|_2 \leq \|M^\varepsilon_{\eta_\gamma} f\|_2 + \varepsilon\|f\|_2, \qquad f \in S_{L^2}.$$

Applying this to $M_{\eta_\gamma} f$ and noting that M_{η_γ} and $M^\varepsilon_{\eta_\gamma}$ commute gives

$$\begin{aligned}
\|M^2_{\eta_\gamma} f\|_2 &\leq \|M^\varepsilon_{\eta_\gamma} M_{\eta_\gamma} f\|_2 + \varepsilon\|M_{\eta_\gamma} f\|_2,\\
&= \|M_{\eta_\gamma} M^\varepsilon_{\eta_\gamma} f\|_2 + \varepsilon\|M_{\eta_\gamma} f\|_2,\\
&\lesssim \|M^\varepsilon_{\eta_\gamma} f\|_2 + \varepsilon\|M_{\eta_\gamma} f\|_2, \quad f \in S_{L^2}.
\end{aligned}$$

Putting all these inequalities together gives

$$\left(\sum_{\gamma\in\Gamma} \|M^\varepsilon_{\eta_\gamma} f\|_2^2\right)^{1/2} \lesssim \|f\|_2 \lesssim \left(\sum_{\gamma\in\Gamma} \|M^\varepsilon_{\eta_\gamma} f\|_2^2\right)^{1/2} + \varepsilon\|f\|_2, \quad f \in S_{L^2}.$$

This implies that, for $0 < \varepsilon \ll 1$, $(\sum_{\gamma\in\Gamma} \|M^\varepsilon_{\eta_\gamma} f\|_2^2)^{1/2} \approx \|f\|_2$, as claimed. The fact that the system in (30) is a frame of S_{L^2} now follows like in Theorem 5.5, this time using Proposition 2.10 to estimate: $\#\left\{\lambda_k^\gamma : \lambda_k^\gamma > \varepsilon\right\} \leq \varepsilon^{-1}\operatorname{trace}(M_{\eta_\gamma}) \lesssim \varepsilon^{-1}\|\eta_\gamma\|_1$. □

6.1 Application to Time-Scale Analysis

We now show how to apply Theorem 6.1 to time-scale analysis. Let $\psi : \mathbb{R}^d \to \mathbb{C}$ be a Schwartz-class radial function with several vanishing moments. The *wavelet transform* of a function $f \in L^2(\mathbb{R}^d)$ with respect to ψ is defined by

$$W_\psi f(x,s) = s^{-d/2} \int_{\mathbb{R}^d} f(t) \overline{\psi\left(\frac{t-x}{s}\right)} \mathrm{d}t, \quad x \in \mathbb{R}^d, s > 0.$$

If ψ is properly normalized (and we assume so), W_ψ maps $L^2(\mathbb{R}^d)$ isometrically into $L^2(\mathbb{R}^d \times \mathbb{R}_+, s^{-(d+1)}\mathrm{d}x\mathrm{d}s)$. For a measurable bounded symbol $m : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{C}$, the *wavelet multiplier* $\mathrm{WM}_m : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is defined as

$$\mathrm{WM}_m f(t) = \int_0^{+\infty} \int_{\mathbb{R}^d} m(x,s) W_\psi f(x,s) \pi(x,s)\psi(t) \mathrm{d}x \frac{\mathrm{d}s}{s^{d+1}}, \quad t \in \mathbb{R}^d, \tag{31}$$

where $\pi(x,s)\psi(t) := s^{-d/2}\psi\left(\frac{t-x}{s}\right)$. (Here, the integral converges in the weak sense.) Note that $\mathrm{WM}_m = W_\psi^*(mW_\psi F)$. The operator WM_m is also known as a *wavelet localization operator* [16–18].

In order to apply the model from Sect. 2.3.2, we consider the affine group $\mathcal{G} = \mathbb{R}^d \times \mathbb{R}_+$, where multiplication is given by $(x,s)\cdot(x',s') = (x+sx', ss')$. The Haar measure in $\mathcal{G}$ is given by $|X| = \int_X s^{-(d+1)}\mathrm{d}x\mathrm{d}s$, and the modular function is given by $\Delta(x,s) = s^{-d}$.

We let $E := L^2(\mathcal{G})$ and $S_E := W_\psi L^2(\mathbb{R}^d)$. In complete analogy to the time–frequency analysis case, we let $P := W_\psi W_\psi^* : L^2(\mathcal{G}) \to W_\psi L^2(\mathbb{R}^d)$ be the orthogonal projection and $K := \left|W_\psi \psi\right|$. We further let $w(x,s) := \max\left\{1, s^d\right\}$. The kernel K belongs to $W(L^\infty, L^1_w) \cap W_R(L^\infty, L^1_w)$ if ψ has sufficiently many vanishing moments (see [37, Sect. 4.2]).

As an example of a well-spread family of symbols, we consider the characteristic functions of a cover of $\mathcal{G}$ by irregular boxes. Let us take as centers the points

$$\Gamma := \left\{ \gamma_{j,l} := (l2^j, 2^j) \,\middle|\, j \in \mathbb{Z}, l \in \mathbb{Z}^d \right\},$$

and consider a family of boxes around $(0,1) \in \mathbb{R}^d \times \mathbb{R}_+$,

$$V_{j,l} := [-a^1_{j,l}/2, a^1_{j,l}/2] \times \cdots \times [-a^d_{j,l}/2, a^d_{j,l}/2] \times [(b_{j,l})^{-1}, b_{j,l}], \tag{32}$$

where $0 \le a^i_{j,l} \le a < +\infty$, $i = 1, \ldots, d$ and $0 < b^{-1} \le b_{j,l} \le b < +\infty$. Let us set

$$U_{j,l} := \gamma_{j,l} V_{j,l}, \quad j \in \mathbb{Z}, l \in \mathbb{Z}^d. \tag{33}$$

The family of characteristic functions $\{1_{U_{j,l}} : j \in \mathbb{Z}, l \in \mathbb{Z}^d\}$ is well spread, with envelope (Γ, g), where $\Gamma := \left\{ \gamma_{j,l} := (l2^j, 2^j) \,\middle|\, j \in \mathbb{Z}, l \in \mathbb{Z}^d \right\}$ and $g := 1_{[-a/2,a/2]^d \times [b^{-1},b]}$.

Note that $\|1_{U_{j,l}}\|_{L^1(\mathcal{G})} = |U_{j,l}| = |V_{j,l}| = \frac{1}{d} \prod_{i=1}^d a_{j,l}^i \cdot [(b_{j,l})^d - (b_{j,l})^{-d}]$. Theorem 6.1 yields the following.

Theorem 6.4 *Suppose that the sets $\left\{ U_{j,l} \,\middle|\, j \in \mathbb{Z}, l \in \mathbb{Z}^d \right\}$ from (32) and (33) cover $\mathbb{R}^d \times \mathbb{R}_+$.*

For each $j \in \mathbb{Z}, l \in \mathbb{Z}^d$, let the wavelet multiplier $\mathrm{WM}_{1_{U_{j,l}}}$—cf. (31)—have eigenfunctions $\{\phi_{j,l}^k : k \geq 1\}$ with corresponding eigenvalues $\{\lambda_{j,l}^k : k \geq 1\}$ ordered decreasingly.

Then there exists a constant $\alpha > 0$ such that for every choice of numbers $\{N_{j,l} : j \in \mathbb{Z}, l \in \mathbb{Z}^d\} \subseteq \mathbb{N}$ satisfying

$$\alpha \left|U_{j,l}\right| \leq N_{j,l} \text{ and } \sup_{j,l} N_{j,l} < +\infty,$$

the family $\left\{ \lambda_{j,l}^k \phi_{j,l}^k \,\middle|\, j \in \mathbb{Z}, l \in \mathbb{Z}^d, 1 \leq k \leq N_{j,l} \right\}$ is a frame of $L^2(\mathbb{R}^d)$.

6.2 Application to Gabor Analysis

Let us consider a window $\varphi \in M^1(\mathbb{R}^d)$ with $\|\varphi\|_2 = 1$ and a (full rank) lattice $\Lambda \subseteq \mathbb{R}^{2d}$, i.e., $\Lambda = P\mathbb{Z}^{2d}$, where $P \in \mathbb{R}^{2d \times 2d}$ is an invertible matrix. The *Gabor system* associated with φ and Λ is the collection of functions

$$\mathcal{G}_{\varphi,\Lambda} := \left\{ \varphi_\lambda(t) := e^{2\pi i \xi t} \varphi(t - x) \,\middle|\, \lambda = (x, \xi) \in \Lambda \right\}.$$

We assume that this collection of functions is a *tight-frame*. This means that for some constant $A > 0$, every function $f \in L^2(\mathbb{R}^d)$ admits the expansion

$$f = A \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\varphi \rangle \, \pi(\lambda)\varphi. \tag{34}$$

In this section, we show how to apply the abstract results from Sect. 6 to obtain frames consisting of functions f whose coefficients $\langle f, \pi(\lambda)\varphi \rangle$ have a prescribed profile.

For a bounded sequence $m : \Lambda \to \mathbb{C}$, the *Gabor multiplier* $\mathrm{GM}_m : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is defined by applying the mask m to the frame expansion in (34),

$$\mathrm{GM}_m f = A \sum_{\lambda \in \Lambda} m(\lambda) \, \langle f, \pi(\lambda)\varphi \rangle \, \pi(\lambda)\varphi. \tag{35}$$

(see [27] for a survey on Gabor multipliers; see also [21,36].) If $m \geq 0$, the first N eigenfunctions of GM_m form an orthonormal set in $L^2(\mathbb{R}^d)$ that maximizes the quantity

$$\sum_{j=1}^{N}\sum_{\lambda\in\Lambda} m(\lambda)\left|\langle f_j, \pi(\lambda)\varphi\rangle\right|^2$$

among all orthonormal sets of N functions $\{f_1, \ldots, f_N\} \subseteq L^2(\mathbb{R}^d)$.

Let us show how the abstract setting of Sect. 2.3.2 can be applied. The discussion is analogous to Example 2.5. We let $\mathcal{G} = \Lambda$, considered as a group, and $E := \ell^2(\Lambda)$. Consider the *analysis operator* $C_{\Lambda,\varphi} : L^2(\mathbb{R}^d) \to \ell^2(\Lambda)$ given by $C_{\Lambda,\varphi} f(\lambda) = \sqrt{A}\langle f, \pi(\lambda)\varphi\rangle$. Let $S_{\ell^2} := C_{\Lambda,\varphi}(L^2(\mathbb{R}^d))$. Since we assume that $\mathcal{G}_{\varphi,\Lambda}$ is a tight frame, the operator $C_{\Lambda,\varphi} : L^2(\mathbb{R}^d) \to \ell^2(\Lambda)$ is an isometry—cf. (34). The orthogonal projection $P : \ell^2(\Lambda) \to S_{\ell^2}$ is then $P = C_{\Lambda,\varphi}C^*_{\Lambda,\varphi}$ and is therefore represented by the matrix $\kappa(\mu, \lambda) = A\langle\varphi_\lambda, \varphi_\mu\rangle$. Consequently,

$$|\kappa(\mu, \lambda)| = A|\langle\varphi_\lambda, \varphi_\mu\rangle| = A|V_\varphi\varphi(\mu - \lambda)|, \quad \mu, \lambda \in \Lambda.$$

Since $\varphi \in M^1(\mathbb{R}^d)$, $C_{\Lambda,\varphi}$ maps $M^1(\mathbb{R}^d)$ into $\ell^1(\Lambda)$ (see for example [28]) and we conclude that $K := A\left|V_\varphi\varphi_{|\Lambda}\right| = \sqrt{A}\left|C_{\Lambda,\varphi}\varphi\right| \in \ell^1(\Lambda) = W(\ell^\infty, \ell^1)(\Lambda)$. Hence, (A1), (A2), and (A3) from Sect. 2.3 are satisfied with $\mathcal{G} = \Lambda$, $E = \ell^2(\Lambda)$, and $w \equiv 1$.

In addition, note that the Gabor multiplier in (35) satisfies $\mathrm{GM}_m f = C^*_{\Lambda,\varphi}(mC_{\Lambda,\varphi} f)$. Hence $C_{\Lambda,\varphi}\mathrm{GM}_m C^*_{\Lambda,\varphi} : S_{\ell^2} \to S_{\ell^2}$ is a phase-space multiplier with symbol m—cf. Sect. 2.3.3.

As an example of a well-spread family of symbols on Λ, we may now consider a well-spread family $\left\{\eta_\gamma : \gamma \in \Gamma\right\}$ of nonnegative symbols defined on $\mathbb{R}^{2d}$, where $\Gamma \subseteq \Lambda$ is a relatively separated subset of $\mathbb{R}^d$ and $\sum_\gamma \eta_\gamma \approx 1$, and restrict each η_γ to Λ. As an application of 6.1, we obtain the following result:

Theorem 6.5 *Let $\varphi \in M^1(\mathbb{R}^d)$ with $\|\varphi\|_2 = 1$ and $\Lambda \subseteq \mathbb{R}^{2d}$ be a lattice. Let $\left\{\eta_\gamma : \gamma \in \Gamma\right\}$ be a well-spread family of nonnegative symbols defined on $\mathbb{R}^{2d}$ such that $\sum_\gamma \eta_\gamma \approx 1$. Let us restrict each η_γ to Λ and consider the corresponding Gabor multiplier GM_{η_γ}—cf. (35)—having eigenfunctions $\left\{\phi^\gamma_k : k \geq 1\right\}$ ordered decreasingly with respect to the corresponding eigenvalues λ^γ_k.*

Then there exists a constant $\alpha > 0$ such that for every choice of numbers $\{N_\gamma : \gamma \in \Gamma\} \subseteq \mathbb{N}$ satisfying

$$\alpha\|\eta_\gamma\|_{\ell^1(\Lambda)} \leq N_\gamma \text{ and } \sup_\gamma N_\gamma < +\infty,$$

the family $\left\{\lambda^\gamma_k\phi^\gamma_k \,\middle|\, \gamma \in \Gamma, 1 \leq k \leq N_\gamma\right\}$ is a frame of $L^2(\mathbb{R}^d)$.

Remark 6.6 While Theorem 5.5 provides frames for $L^2(\mathbb{R}^d)$ consisting of functions having a spectrogram that is optimally adapted to a given weight on $\mathbb{R}^{2d}$—cf. (25), Theorem 6.5 provides frame elements where the profile of the coefficients associated with the discrete expansion in (34) is optimized with respect to a weight on Λ.

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Appendix 1: Proof of Theorem 3.5

In this appendix, we prove Theorem 3.5. The proof is essentially contained in [47], but is not explicitly stated in the required generality. We therefore show how to derive Theorem 3.5 from some technical lemmas in [47].

Remark 6.7 We quote simplified versions of some statements in [47]. The article [47] considers a technical variant of the amalgam space $W_R(L^\infty, L^1_w)(\mathcal{G})$, called the weak amalgam space $W_R^{\text{weak}}(L^\infty, L^1_w)$ (see [47, Sect. 2.4]), which we do not wish to introduce here. By [47, Proposition 1], $L^1(\mathcal{G}) \hookrightarrow W_R^{\text{weak}}(L^\infty, L^1_w)(\mathcal{G}) \hookrightarrow W_R(L^\infty, L^1_w)(\mathcal{G})$. Some results from [47] that we quote assume that a certain function g belongs to $W_R(L^\infty, L^1_w)(\mathcal{G})$ and are proved in [47] under the weaker assumption: $g \in W_R^{\text{weak}}(L^\infty, L^1_w)$.

We quote the following estimate:

Lemma 6.8 *[25, Lemma 3.8], [47, Lemma 2] Let E be a solid, translation invariant BF space, let w be an admissible weight for it, and let $\Gamma \subseteq \mathcal{G}$ be a relatively separated set. Then for every $f \in E$ and $g \in W_R(C_0, L^1_w)$, the sequence $(\langle f, L_\lambda g\rangle)_{\lambda\in\Lambda}$ belongs to $E_{\mathrm{d}}(\Lambda)$ and*

$$\|(\langle f, L_\lambda g\rangle)_\lambda\|_{E_{\mathrm{d}}} \lesssim \|f\|_E \|g\|_{W_R(L^\infty, L^1_w)}.$$

The implicit constants depend on the spreadness $\rho(\Gamma)$—cf. (15).

Suppose that Assumptions (A1) and (A2) from Sect. 2.3.2 hold.

For a solid, translation invariant BF space E, we consider an L^2-valued version of $E_{\mathrm{d}}(\Gamma)$,

$$E_{\mathrm{d},L^2} = E_{\mathrm{d},L^2}(\Gamma) := \left\{ (f_\gamma)_{\gamma\in\Gamma} \in (L^2(\mathcal{G}))^\Gamma \;\middle|\; (\|f_\gamma\|_{L^2})_{\gamma\in\Gamma} \in E_{\mathrm{d}}(\Gamma) \right\},$$

and endow it with the norm $\|(f_\gamma)_{\gamma\in\Gamma}\|_{E_{\mathrm{d},L^2}} := \|(\|f_\gamma\|_{L^2})_{\gamma\in\Gamma}\|_{E_{\mathrm{d}}}$.

Let $\left\{ T_\gamma \;\middle|\; \gamma \in \Gamma \right\}$ be a well-spread family of operators—cf. Sect. 3. Let $U \subseteq \mathcal{G}$ be a relatively compact neighborhood of the identity. Consider the operators C_T and S_U formally defined by

$$C_T(f) := (T_\gamma(f))_{\gamma\in\Gamma}, \quad f \in S_E, \tag{36}$$

$$S_U((f_\gamma)_{\gamma\in\Gamma}) := \sum_{\gamma\in\Gamma} P(f_\gamma) 1_{\gamma U}, \quad f_\gamma \in L^2(\mathcal{G}), \tag{37}$$

where $1_{\gamma U}$ denotes the characteristic function of the set γU. These operators satisfy the following mapping properties.

Proposition 6.9 *Assume (A1) and (A2), and let $\left\{ T_\gamma \;\middle|\; \gamma \in \Gamma \right\}$ be a well-spread family of operators. Then the operators C_T and S_U in (36) and (37) satisfy the following:*

(a) The analysis operator C_T maps S_E boundedly into $E_{\mathrm{d},L^2}(\Gamma)$.
(b) For every relatively compact neighborhood of the identity U, and every sequence $F \equiv (f_\gamma)_\gamma \in E_{\mathrm{d},L^2}$, the series defining $S_U(F)$ converge absolutely in $L^2(\mathcal{G})$ at every point. Moreover, the operator S_U maps $E_{\mathrm{d},L^2}(\Gamma)$ boundedly into E (with a bound that depends on U).

Proof Part (b) is proved in [47, Proposition 4 (b)] under a weaker hypothesis. Part (a) is a slight variant of [47, Proposition 4 (a)]; for completeness we give a full argument. Let (Γ, Θ, g) be an envelope for $\left\{ T_\gamma \,\middle|\, \gamma \in \Gamma \right\}$.

Let $f \in S_E$. Since η_γ is bounded, $f\eta_\gamma \in E$. By the definition of well-spread family (cf. 20),

$$\left|T_\gamma f(x)\right| \le \int_{\mathcal{G}} |f(y)|\, g(\gamma^{-1}y)\Theta(y^{-1}x)\mathrm{d}y = \big(|f|\, L_\gamma g\big) * \Theta(x).$$

By Young's inequality $L^1 * L^2 \hookrightarrow L^2$, we have

$$\|T_\gamma f\|_2 \le \|\Theta\|_2 \int_{\mathcal{G}} |f(y)|\, g(\gamma^{-1}y)\mathrm{d}y \lesssim \|\Theta\|_{W(L^\infty, L^1_w)} \int_{\mathcal{G}} |f(y)|\, g(\gamma^{-1}y)\mathrm{d}y.$$

Now Lemma 6.8 yields $\|C_T(f)\|_{E_{\mathrm{d},L^2}} \lesssim \|f\|_E \|g\|_{W_R(L^\infty, L^1_w)}$, as desired. □

Remark 6.10 Note that in the last proof, the use of the L^2 norm is somewhat arbitrary; a number of other function norms could have been used instead (cf. [47, Proposition 4]).

Now we prove the key approximation result (cf. [47, Theorem 1]).

Theorem 6.11 *Assume (A1) and (A2), and let $\left\{ T_\gamma \,\middle|\, \gamma \in \Gamma \right\}$ be a well-spread family of operators. Given $\varepsilon > 0$, there exists U_0, a relatively compact neighborhood of e such that for all $U \supseteq U_0$,*

$$\left\| \sum_{\gamma \in \Gamma} T_\gamma f - S_U C_T f \right\|_E \le \varepsilon \|f\|_E, \quad f \in S_E. \tag{38}$$

Remark 6.12 The neighborhood U_0 can be chosen uniformly for any class of spaces E having the same weight w and the same constant $C_{E,w}$ (cf. (16)).

Concerning the parameters in Assumptions (A1) and (A2) and (20), the choice of U_0 only depends on $\|K\|_{W(L^\infty, L^1_w)}$, $\|K\|_{W_R(L^\infty, L^1_w)}$, $\|\Theta\|_{W(L^\infty, L^1_w)}$, $\|\Theta\|_{W_R(L^\infty, L^1_w)}$, $\|g\|_{W_R(L^\infty, L^1_w)}$, and $\rho(\Gamma)$ (cf. (15)).

Proof of Theorem 6.11 Let $f \in S_E$, and let U be a relatively compact neighborhood of e. Because of the inclusion $S_E \hookrightarrow W(L^\infty, E)$ in Proposition 2.6, it suffices to dominate the left-hand side of (38) by $\varepsilon \|f\|_{W(L^\infty, E)}$.

Note that since $T_\gamma f \in S_E$, $S_U C_T f(x) = \sum_{\gamma\in\Gamma} T_\gamma f(x) 1_{\gamma U}(x)$. Hence, using (20), let us estimate

$$\left|\sum_{\gamma\in\Gamma} T_\gamma f(x) - S_U C_T f(x)\right| = \left|\sum_{\gamma\in\Gamma} 1_{\gamma(\mathcal{G}\setminus U)}(x) T_\gamma(f)(x)\right| \\ \leq \sum_{\gamma\in\Gamma} \int_{\mathcal{G}} |f(y)|\, g(\gamma^{-1}y)\Theta(y^{-1}x) 1_{\gamma(\mathcal{G}\setminus U)}(x)\mathrm{d}y.$$

The rest of the proof is carried out exactly as in [47, Theorem 1]. Indeed, the proof there only depends on the estimate just derived.[2] (The definition of well-spread family of operators was tailored so that the proof in [47, Theorem 1] would still work.) □

Finally, we can prove Theorem 3.5.

Proof of Theorem 3.5 Let $\{T_\gamma : \gamma \in \Gamma\}$ be a well-spread family of operators, and suppose that the operator $\sum_\gamma T_\gamma : S_E \to S_E$ is invertible. We have to show that for $f \in S_E$, $\|f\|_E \approx \|C_T(f)\|_{E_{\mathrm{d},L^2}(\Gamma)}$. The estimate $\|C_T(f)\|_{E_{\mathrm{d},L^2}(\Gamma)} \lesssim \|f\|_E$ is proved in Proposition 6.9 (a). To establish the second inequality, consider the operator $P S_U C_T : S_E \to S_E$. Then for $f \in S_E$,

$$\left\|\sum_{\gamma\in\Gamma} T_\gamma f - P S_U C_T f\right\|_E = \left\|P \sum_{\gamma\in\Gamma} T_\gamma f - P S_U C_T f\right\|_E \lesssim \left\|\sum_{\gamma\in\Gamma} T_\gamma f - S_U C_T f\right\|_E .$$

This estimate, together with Theorem 6.11, implies that $\|\sum_\gamma T_\gamma - P S_U C_T\|_{S_E\to S_E} \to 0$ as U grows to $\mathcal{G}$. Hence, there exists U such that $P S_U C_T$ is invertible on S_E. Consequently, for $f \in S_E$, $\|f\|_E \approx \|P S_U C_T f\|_E \lesssim \|C_T(f)\|_{E_{\mathrm{d},L^2}(\Gamma)}$. Here we have used the boundedness of S_U—contained in Proposition 6.9 (b)—and the boundedness of $P : E \to W(L^\infty, E) \hookrightarrow E$—contained in Proposition 2.6. □

Appendix 2: Pseudodifferential Operators and Proof of Proposition 4.3

The Weyl transform of a distribution $\sigma \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ is an operator σ^w that is formally defined on functions $f : \mathbb{R}^d \to \mathbb{C}$ as

$$\sigma^w(f)(x) := \int_{\mathbb{R}^d\times\mathbb{R}^d} \sigma\left(\frac{x+y}{2}, \xi\right) e^{2\pi i(x-y)\xi} f(y)\mathrm{d}y\mathrm{d}\xi, \quad x \in \mathbb{R}^d.$$

The fundamental results in the theory of pseudodifferential operators provide conditions on σ for the operator σ^w to be well defined and bounded on various function spaces. We now quote some results about pseudodifferential operators acting on modulation spaces—cf. Sect. 4.1.

[2] The function Θ is called H in the proof [47, Theorem 1].

In [34,35], it was shown that modulation spaces on $\mathbb{R}^{2d}$ serve as symbol classes to study pseudodifferential operators acting on modulation spaces on $\mathbb{R}^d$, recovering and extending classical results from Sjöstrand [50,51]. We quote the following simplified version of [35, Theorems 4.1, 4.6, and Corollaries 3.3, 4.7]. (The GRS-condition for admissible weights in Sect. 4.1 is important here.)

Theorem 6.13 *Let w be an admissible TF weight—cf. Definition 4.1—and let $\varphi \in M^1_w(\mathbb{R}^d)$ be nonzero. Let us denote $\widetilde{w}(z_1, z_2) = w(-z_2, z_1)$. Then the following statements hold true:*

(i) *If $\sigma \in M^{\infty,1}_{\widetilde{w}}(\mathbb{R}^{2d})$, then σ^w is bounded on $M^p_v(\mathbb{R}^d)$ for all w-moderated weights v and all $p \in [1, +\infty]$.*
(ii) *If $\sigma \in M^{\infty,1}_{\widetilde{w}}(\mathbb{R}^{2d})$ and σ^w is invertible as an operator on $L^2(\mathbb{R}^d)$, then σ^w is invertible as an operators on $M^p_v(\mathbb{R}^d)$ for all w-moderated weights v and all $p \in [1, +\infty]$.*
(iii) *Let $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ be a linear and continuous operator. For $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$ let us denote $\varphi_{(x,\xi)}(t) := e^{2\pi i \xi t}\varphi(t - x)$. If there exists a function $H \in L^1_w(\mathbb{R}^{2d})$ such that*

$$\left|\left\langle T(\varphi_{(x,\xi)}), \varphi_{(x',\xi')}\right\rangle\right| \le H(x' - x, \xi' - \xi), \quad (x, \xi), (x', \xi') \in \mathbb{R}^d \times \mathbb{R}^d,$$

then there exists $\sigma \in M^{\infty,1}_{\widetilde{w}}(\mathbb{R}^{2d})$ such that $T = \sigma^w$ on $\mathcal{S}(\mathbb{R}^d)$.

As an application of Theorem 6.13, we now prove Proposition 4.3.

Proof of Proposition 4.3 With the notation of Theorem 6.13, we use (23) to estimate

$$\begin{aligned}
\left|\left\langle T(\varphi_{(x,\xi)}), \varphi_{(x',\xi')}\right\rangle\right| &= \left|V_\varphi T(\varphi_{(x,\xi)})(x', \xi')\right| \\
&\le \int_{\mathbb{R}^{2d}} \left|V_\varphi \varphi_{(x,\xi)}(z'')\right| H((x', \xi') - z'')\mathrm{d}z'' \\
&= \int_{\mathbb{R}^{2d}} \left|V_\varphi \varphi(z'' - (x, \xi))\right| H((x', \xi') - z'')\mathrm{d}z'' \\
&= (H * \left|V_\varphi \varphi\right|)((x', \xi') - (x, \xi)).
\end{aligned}$$

Since $\varphi \in M^1_w(\mathbb{R}^d)$ and $H \in L^1_w(\mathbb{R}^{2d})$, we deduce that $H * \left|V_\varphi \varphi\right| \in L^1_w(\mathbb{R}^{2d})$.

Hence, Theorem 6.13 implies that there exists $\sigma \in M^{\infty,1}_{\widetilde{w}}(\mathbb{R}^{2d})$ such that $T \equiv \sigma^w$ on $\mathcal{S}(\mathbb{R}^d)$. Since both operators are bounded on $L^2(\mathbb{R}^d)$, it follows that $T \equiv \sigma^w$ on $L^2(\mathbb{R}^d)$. By hypothesis, $T = \sigma^w : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is invertible. A new application of Theorem 6.13 implies that $\sigma^w : M^p_v(\mathbb{R}^d) \to M^p_v(\mathbb{R}^d)$ is invertible. It is tempting to conclude that then $T : M^p_v(\mathbb{R}^d) \to M^p_v(\mathbb{R}^d)$ is invertible because it "is" σ^w. If $p < +\infty$, that conclusion is indeed correct because both operators coincide on the dense space $\mathcal{S}$. The case $p = +\infty$ requires some care. We now discuss this in detail.

We note that $M^p_v(\mathbb{R}^d) \hookrightarrow M^\infty_{1/w}(\mathbb{R}^d)$ and use the facts that $M^\infty_{1/w}(\mathbb{R}^d)$ can be identified with the dual-space of the (separable) Banach space $M^1_w(\mathbb{R}^d)$ and that $\mathcal{S}$ is dense in $M^{\infty}_{1/w}(\mathbb{R}^d)$ with respect to the weak* topology (see [33, Chap. 11]). To conclude that $T = \sigma^w$ on $M^p_v(\mathbb{R}^d)$, we show that both operators are continuous with respect to the weak* topology of $M^{\infty}_{1/w}$.

Let $f \in M^p_v(\mathbb{R}^d)$, and let us show that $T(f) = \sigma^w(f)$. Let $\{f_k : k \in \mathbb{N}\} \subseteq \mathcal{S}(\mathbb{R}^d)$ be a sequence such that $f_k \longrightarrow f$ in the weak* topology of $M^{\infty}_{1/w}(\mathbb{R}^d)$. The operator $\sigma^w : M^{\infty}_{1/w}(\mathbb{R}^d) \to M^{\infty}_{1/w}(\mathbb{R}^d)$ is weak* continuous because it is the adjoint of the operator $\overline{\sigma}^w : M^1_w(\mathbb{R}^d) \to M^1_w(\mathbb{R}^d)$. Hence $T(f_k) = \sigma^w(f_k) \longrightarrow \sigma^w(f)$ in the weak* topology of $M^{\infty}_{1/w}(\mathbb{R}^d)$. Let us note that this implies that

$$V_\varphi T(f_k)(z) \longrightarrow V_\varphi \sigma^w(f)(z), \text{ as } k \longrightarrow +\infty, \text{ for all } \quad z \in \mathbb{R}^{2d}. \tag{39}$$

Indeed, if $z = (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$, the function $\varphi_{(x,\xi)} := e^{2\pi i \xi \cdot}\varphi(\cdot - x)$ belongs to $M^1_w(\mathbb{R}^d)$ and consequently $V_\varphi T(f_k)(z) = \langle T(f_k), \varphi_{(x,\xi)}\rangle \longrightarrow \langle \sigma^w(f), \varphi_{(x,\xi)}\rangle = V_\varphi \sigma^w(f)(z)$. Similarly, since $f_k \longrightarrow f$ in the weak* topology of $M^{\infty}_{1/w}$, we know that $V_\varphi f_k(z) \longrightarrow V_\varphi f(z)$ for all $z \in \mathbb{R}^{2d}$.

Using the enveloping condition in (23), we estimate for $z \in \mathbb{R}^{2d}$:

$$\left|V_\varphi T(f)(z) - V_\varphi T(f_k)(z)\right| \leq \int_{\mathbb{R}^{2d}} \left|V_\varphi(f - f_k)(z')\right| H(z - z')\mathrm{d}z'.$$

The integrand in the last expression tends to 0 pointwise as $k \longrightarrow +\infty$. In order to apply Lebesgue's dominated convergence theorem, we show that the integrand is dominated by an integrable function. Since $H \in L^1_w(\mathbb{R}^{2d})$, it suffices to show that $\sup_k \|V_\varphi(f - f_k)\|_{L^{\infty}_{1/w}} < +\infty$. This is true because $\|V_\varphi(f - f_k)\|_{L^{\infty}_{1/w}} = \|f - f_k\|_{M^{\infty}_{1/w}}$ and weak*-convergent sequences are bounded. Hence, Lebesgue's dominated convergence theorem can be applied, and we conclude that $V_\varphi T(f_k)(z) \longrightarrow V_\varphi T(f)(z)$ for all $z \in \mathbb{R}^{2d}$. Combining this with (39), we conclude that $V_\varphi T(f) \equiv V_\varphi \sigma^w(f)$. Hence $T(f) = \sigma^w(f)$, as desired. □

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