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VARIETIES WITH A FINITE NUMBER OF SUBQUASIVARIETIES

M. V. Sapir

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INTRODUCTION

Following [1], the lattice of subquasivarieties of a quasivariety $\mathfrak{M}$ of universal algebras will be called its Q-lattice. The main purpose of the present paper is to give a description of the locally finite varieties of semigroups with a finite Q-lattice. We will also give the corresponding descriptions for groups and rings (they can easily be obtained from certain assertions in [2-5]). Various facts concerning quasivarieties with finite Q-lattices can be found in [1, 6].

We will use the standard notation of [7-9]. In particular, Z(m) denotes the cyclic group of order m. The variety of all Abelian groups of exponent m is denoted by $\mathfrak{A}_m$. The wreath product of groups A and B is denoted by A wr B. The semigroup S with an adjoined zero is denoted by S^0. The quasivariety (variety) generated by the semigroup S is denoted by q(S) (v(S)).

We will also use the following notation of [10]. Let **L**, **R**, **N**, and **I** denote, respectively, the following two-element semigroups: a left zero semigroup, a right zero semigroup, a zero semigroup, and a commutative band; let **P** denote the semigroup defined on the set $\{\varepsilon, \omega, \pi\}$ with the following multiplication: $\varepsilon^2 = \varepsilon$, $\varepsilon \cdot \omega = \omega$, and all other products are equal to π; let **Q** denote the semigroup dual to **P**.

Let us recall some other definitions. A group is called an "A-group" if all its Sylow subgroups are Abelian. A ring is called a "ring of index 2" if all its nilpotent subrings have zero multiplication. A semigroup is called a "Clifford semigroup" if it is a union of groups. A universal algebra is called "Q-critical" if it does not belong to the quasivariety generated by its proper subalgebras. Suppose S is a semigroup and $\{X_a | a \in S\}$ a family of pairwise disjoint sets such that $X_a \cap S = \{a\}$ for each $a \in S$. We define an operation on the set $\bigcup_{a \in S} X_a$ by putting

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$xy = ab$ for any $x \in X_a$, $y \in X_b$; the resulting groupoid T is a semigroup, which is called an "inflation" of S; if each X_a except for some X_b is a singleton and $X_b = \{b, b'\}$, where $b' \notin S$, then we call T an "inflation of S by means of the pair (b, b')."

Our main results are the following two theorems.

THEOREM 1. A locally finite variety of semigroups has a finite Q-lattice if and only if it is generated by a finite semigroup of one of the following 20 types:

G, $L \times G$, $R \times G$, $L \times R \times G$, where G is an A-group;

$I \times H$, $L \times I \times H$, $R \times I \times H$, $L \times R \times I \times H$, where $H \in \mathfrak{A}_m \mathfrak{A}_n$, $(m, n) = 1$, and $p^2 \nmid m$, $p^6 \nmid n$ for any prime p;

$N \times C$, $L \times N \times C$, $R \times N \times C$, $L \times R \times N \times C$, where $C \simeq Z(k)$, $k \geqslant 1$;

$N \times I \times D$, $L \times N \times I \times D$, $R \times N \times I \times D$, $L \times R \times N \times I \times D$, where $D \simeq Z(m)$ and $p^6 \nmid m$ for any prime p;

$P \times E$, $R \times P \times E$, $Q \times E$, $L \times Q \times E$, where $E \simeq Z(l)$ and $p^3 \nmid l$ for any prime p.

THEOREM 2. A locally finite variety of semigroups has a finite Q-lattice if and only if it consists of finitely approximable semigroups and contains no semigroups of the following six forms: $I \times Z(p^6)$, $P \times Z(p^3)$, $Q \times Z(p^3)$, $N \times (Z(p) \operatorname{wr} Z(q))$, $I \times (Z(p) \operatorname{wr} Z(q)) \times (Z(r) \operatorname{wr} Z(p))$, $I \times (Z(p) \operatorname{wr} Z(q)) \times Z(p^2)$, where p, q, and r are primes.

A complete description of the varieties of finitely approximable (f.a.) semigroups can be found in [10] (see Lemma 3.1 below).

We also have

THEOREM 3. For a locally finite variety $\mathfrak{M}$ of groups (associative rings) the following conditions are equivalent:

1) $\mathfrak{M}$ has a finite Q-lattice;

2) $\mathfrak{M}$ consists of f.a. groups (rings);

3) $\mathfrak{M}$ consists only a finite number of Q-critical groups (rings);

4) $\mathfrak{M}$ is generated by a finite A-group (a finite ring of index 2).

We will use the following assertion, which is a simple special case of Theorem 8 of [6].

LEMMA 0.1. If a locally finite variety of universal algebras has a finite Q-lattice, then it consists of f.a. algebras.

Theorem 3 shows that in the case of groups and (associative) rings the converse is true. However, as can be seen from Theorem 2, in the case of semigroups it is already false. Furthermore, in Part 1 of Sec. 1 we will show that there exists a variety of f.a. semigroups with a continual Q-lattice.

Let us prove Theorem 3. The implication 1) → 2) follows from Lemma 0.1. The implication 2) → 4) in the case of groups was proved in [2]; in the case of rings it was announced in the report of I. V. L'vov to the Algebra and Logic Seminar of the Institute of Mathematics, Siberian Branch, Academy of Sciences of the USSR (see [4]). The implication 3) → 1) follows from the obvious fact that any locally finite quasivariety is generated by its Q-critical algebras. Let us prove 4) → 3). Suppose $\mathfrak{M}$ is a variety of groups that is generated by a finite A-group. It was proved in [3] (Lemma 8) that there exists a number n, depending only on $\mathfrak{M}$, such that any group in $\mathfrak{M}$ can be approximated by subgroups of orders at most n. Therefore, each Q-critical group in $\mathfrak{M}$ has order at most n. This proves 4) → 3) in the case of groups. The proof in the case of rings is analogous, but instead of [3] we refer to [5].

The rest of the paper is devoted to the proof of Theorem 2. We obtain Theorem 1 as a consequence of Theorem 2 and the main results of [10]. The proof of Theorem 2 is divided into three sections: in the first we prove necessity, in the second some results on finite Abelian groups with distinguished subgroups, and in the third sufficiency.

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SECTION 1

In view of Lemma 0.1, to prove necessity in Theorem 2 we need only show that the varieties generated by the semigroups of the forms listed in Theorem 2 have infinite Q-lattices. Suppose $\mathfrak{M}$ is such a variety.

There are six possible cases, so our exposition is naturally divided into six parts.

Part 1. $\mathfrak{M} = v(I \times \tilde{Z}(p^6))$, where p is a prime.

We will prove that in this case $\mathfrak{M}$ contains a continuum of subquasivarieties. For each natural number n consider the following Abelian p-group A_n and subgroup B_n (the operation is written additively):

$$A_n = \langle a_i, b_i, c_i \,|\, |a_i| = p^6, |b_i| = p^4, |c_i| = p^2, i = 1, 2, \ldots, n \rangle;$$
$$B_n = \langle p^2a_i + pb_i + c_i, p^2b_{j+1} + pc_j, p^2b_1 + pc_n \,|\, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n-1 \rangle.$$

Suppose $x \in X_n = \{a_i, b_i, c_i \,|\, 1 \leqslant i \leqslant n\}$, $d \in A_n$. The coefficient of x in the decomposition of d with respect to the basis X_n will be called the projection of d on x.

LEMMA 1.1. Suppose (m, n) = 1 and φ is a homomorphism of A_m into A_n such that $\varphi(B_m) \subseteq B_n$. Then $\varphi(p^5(a_1 - a_2)) = 0$.

Proof. For any natural number k we denote by [k] the smallest natural number congruent to k modulo n. Suppose $\varphi(a_1) = \sum_1^n \alpha_i a_i + d_1$, where the α_i are integers, $d_1 \in A_n$, and the projections of d_1 on all the a_i, $1 \le i \le n$, are multiples of p. We will prove by induction on t from 1 to m that

$$\begin{aligned} \varphi(a_t) &= \sum \alpha_i a_{[t+i-1]} + d_t; \\ \varphi(b_t) &= \sum \alpha_i b_{[t+i-1]} + e_t; \\ \varphi(c_t) &= \sum \alpha_i c_{[t+i-1]} + f_t, \end{aligned} \tag{1}$$

where d_t, e_t, f_t are elements of A_n whose projections on a_i, b_i, c_i, respectively, are multiples of p for $1 \le i \le n$.

Basis of the induction: t = 1. The element $\varphi(p^2a_1 + pb_1 + c_1)$ belongs to B_n by hypothesis. Therefore,

$$\varphi(p^2a_1 + pb_1 + c_1) = \sum p^2\alpha_i a_i + p^2d_1 + p\varphi(b_1) + \varphi(c_1) =$$
$$= \sum \varepsilon_i(p^2a_i + pb_i + c_i) + \sum \delta_j(p^2b_{j+1} + pc_j) + \delta_n(p^2b_1 + pc_n)$$

for certain integers ε_i and δ_j. Comparing the projections of the left- and right-hand sides of the last equality on a_i, $1 \le i \le n$, we obtain $\varepsilon_i \equiv \alpha_i \pmod p$. Thus,

$$p\varphi(b_1) + \varphi(c_1) = \sum_1^n \alpha_i(pb_i + c_i) + u, \tag{2}$$

where $u \in A_n$ and the projections of u on each b_i and c_i are multiples of p. Multiplying both sides of (2) by p^2, we obtain $p^3\varphi(b_1) = \sum \alpha_i p^3 b_i$, hence $\varphi(b_1) = \sum \alpha_i b_i + e_1$, where e_1 possesses the desired property. Therefore, in view of (2), $\varphi(c_1) = \sum \alpha_i c_i + f_1$, where f_1 is as desired. The basis of the induction is proved.

Suppose (1) is true for some t < m. We will prove that (1) is then true for t + 1. By hypothesis, $\varphi(p^2b_{t+1} + pc_t) \in B_n$. Consequently,

$$\varphi(p^2b_{t+1} + pc_t) = p^2\varphi(b_{t+1}) + \sum p\alpha_i c_{[t+i-1]} + pf_t = \sum \varepsilon_i(p^2a_i + pb_i + c_i) + \sum \delta_j(p^2b_{j+1} + pc_j) + \delta_n(p^2b_1 + pc_n)$$

for certain ε_i and δ_j. Comparing the orders of the elements on the left and right, we see that $p^2 \mid \varepsilon_i$ for all i from 1 to n. By the induction assumption, the projection of f_t on c_i is a multiple of p for each i. Consequently, $\alpha_i \equiv \delta_{[t+i-1]} \pmod p$. It follows that $\varphi(b_{t+1}) = \sum \alpha_i b_{[t+i]} + e_{t+1}$, where $e_{t+1} \in A_n$ and the projections of e_{t+1} on all b_i are multiples of p. If we now consider $\varphi(p^2a_{t+1} + pb_{t+1} + c_{t+1})$, we obtain exactly as in the proof of the basis of induction that

$$\varphi(a_{t+1}) = \sum \alpha_i a_{[t+i]} + d_{t+1}, \quad \varphi(c_{t+1}) = \sum \alpha_i c_{[t+i]} + f_{t+1},$$

where d_{t+1} and f_{t+1} possess the desired properties. Thus, we have completed the induction step and have proved (1).

By hypothesis, the element $\varphi(p^2b_1 + pc_m)$ belongs to B_n. Therefore, using (1), we obtain that

$$\sum p^2\alpha_i b_i + \sum p\alpha_i c_{[m+i-1]} = \sum \varepsilon_i(p^2a_i + pb_i + c_i) + \sum \delta_j(p^2b_{j+1} + pc_j) + \delta_n(p^2b_1 + pc_n) - p^2e_1 - pf_m$$

for certain ε_i and δ_j. As above, we see that $p^2 | \varepsilon_i$ for each i from 1 to n. Therefore,

$$\sum p^2\alpha_i b_i + \sum p\alpha_i c_{[m+i-1]} = \sum_1^n \delta_i (p^2 b_{[i+1]} + pc_i) + u,$$

where $u \in A_n$, the projections of u on the b_i are multiples of p^3, and the projections of u on the c_i are multiples of p^2 for all i. Take any natural number t from 1 to n. Comparing the projections of the left- and right-hand sides of the last equality on $c_{[m+t-1]}$ and then on $b_{[m+t]}$, we obtain $\alpha_t \equiv \delta_{[m+t-1]} \equiv \alpha_{[m+t]} \pmod p$. If we now replace t successively by m + t, 2m + t, . . . , km + t, we obtain $\alpha_t \equiv \alpha_{[km+t]} \pmod p$ for any natural number k. By hypothesis, m and n are relatively prime. Therefore, [km + t] ranges over the whole set $\{1, 2, \ldots, n\}$ as k ranges over the natural sequence. Consequently, $\alpha_i \equiv \alpha_j \pmod p$ for all i and j. It follows that $\varphi(p^5(a_1 - a_2)) = p^5(d_1 - d_2) = 0$, as required. Lemma 1.1 is proved.

We will now show that the variety $\mathfrak{M}$ has a continual Q-lattice. Suppose $n \geqslant 1$, $a \in A_n$. Let $\bar{a}$ denote the image of a under the natural homomorphism of A_n onto A_n / B_n. We define a semigroup S_n on the set $A_n \cup A_n/B_n$ by preserving the operations within A_n and A_n / B_n and putting $a + \bar{b} = \bar{b} + a = \overline{a + b}$ for any $a \in A_n$, $\bar{b} \in A_n/B_n$. Clearly, S_n is a commutative band of the groups A_n and A_n / B_n, and A_n / B_n is an ideal of S_n. Therefore, S_n is embedded in the product $A_n^0 \times (A_n / B_n)$. Since A_n and A_n / B_n are finite Abelian groups of exponent p^6, they can be approximated by the group $Z(p^6)$. Consequently, S_n can be approximated by the semigroup $Z^0(p^6)$. The latter is the Rees quotient semigroup of $\mathbf{I} \times Z(p^6)$ relative to the ideal $\{0\} \times Z(p^6)$ and, therefore, belongs to $\mathfrak{M}$. Consequently, for any n. Denote by x_n the element $p^5(a_1 - a_2)$ of A_n, and by e_n the zero of A_n. Suppose φ is an arbitrary homomorphism of A_n into A_m, where (n, m) = 1. Since a group cannot have a homomorphism onto $\mathbf{I}$, there are two possibilities: 1) $\varphi(A_n) \subseteq A_m/B_m$; 2) $\varphi(A_n) \subseteq A_m$. Assume that 1) holds. Then $\varphi(x_n) = \varphi(x_n) + \bar{e}_m = \varphi(x_n + \bar{e}_n) = \varphi(\bar{x}_n + \bar{e}_n) = \varphi(\bar{e}_n) = \bar{e}_m$. Therefore, $\varphi(x_n) = \varphi(e_n)$. Now suppose 2) holds and $b \in B_n$. Then $\overline{\varphi(b)} = \varphi(b) + \bar{e}_m = \varphi(b + \bar{e}_n) = \varphi(\bar{e}_n) = \bar{e}_m$, hence $\varphi(b) \in B_m$. Thus, $\varphi(B_n) \subseteq B_m$ and, in view of Lemma 1.1, $\varphi(x_n) = e_m = \varphi(e_n)$. Consequently, x_n and e_n are fused by any homomorphism of A_n into A_m. Now consider the set K of all prime numbers. Since K consists of pairwise relatively prime numbers, it follows from what was proved above that for any n and m in K, n ≠ m, the elements x_n and e_n of S_n are fused by any homomorphism of S_n into S_m. Therefore, S_n cannot be approximated by semigroups in $\{S_i | i \in K \backslash n\}$ for any n. Therefore, quasivarieties generated by distinct subsets of $\{S_i | i \in K\}$ are distinct. They are all contained in $\mathfrak{M}$. Consequently, the Q-lattice of the variety $\mathfrak{M}$ is continual.

Part 2. $\mathfrak{M} = v(\mathbf{P} \times Z(p^3))$, where p is a prime.

For each n ≥ 1 we define an Abelian p-group A_n and subgroups B_n, C_n, and D_n as follows:

$$A_n = \langle a_i, b_i, c_i \,||a_i| = |b_i| = p^3, |c_i| = p, i = 1, 2, \ldots, n\rangle;$$
$$B_n = \langle pa_i, b_i, c_i | 1 \leqslant i \leqslant n\rangle;$$
$$C_n = \langle pa_i, pb_i + c_i | 1 \leqslant i \leqslant n\rangle;$$
$$D_n = \langle p^2a_i + p^2b_i, \; pa_j + pb_{j+1} + c_{j+1}, \; pa_n + pb_1 + c_1 | 1 \leqslant i \leqslant n, \; 1 \leqslant j \leqslant n-1\rangle.$$

Clearly, $D_n < C_n < B_n$ for each n ≥ 1. By analogy with Lemma 1.1 we can prove

LEMMA 1.2. Suppose (m, n) = 1 and φ is a homomorphism of A_m into A_n such that $\varphi(B_m) \subseteq B_n$, $\varphi(C_m) \subseteq C_n$, $\varphi(D_m) \subseteq D_n$. Then $\varphi(p^2(a_1 - a_2)) = 0$.

For each n ≥ 1 we define a binary relation σ_n on the semigroup $A_n \times \mathbf{P}^0$ by putting $(x, a)\sigma_n(y, b)$ if and only if either $a = b = \varepsilon$, x = y, or $a = b = \omega$, $x - y \in C_n$, or $a = b = \pi$, $x - y \in B_n$, or $a = b = 0$, $x - y \in D_n$. It can be verified directly that σ_n is a congruence. We denote the quotient semigroup $(A_n \times \mathbf{P}^0)/\sigma_n$ by S_n. Clearly, $S_n \in \mathfrak{M}$. As in Part 1, it can be shown that distinct subsets of $\{S_i | i \in K\}$ generate distinct quasivarieties. Thus, in this case also the Q-lattice of the variety $\mathfrak{M}$ is continual.

Part 3. $\mathfrak{M} = v(\mathbf{Q} \times Z(p^3))$, where p is a prime.

This case is dual to the preceding one.

Part 4. $\mathfrak{M} = v(\mathbf{N} \times (Z(p) \operatorname{wr} Z(q)))$, where p and q are primes.

If p = q, then Z(p) wr Z(q) is a nilpotent non-Abelian group. Therefore, in view of Theorem 3, the Q-lattice of the variety $\mathfrak{M}$ is infinite. We may therefore assume that p ≠ q. Suppose G is a minimal non-Abelian group in $\mathfrak{M}$. Then $G = A \leftthreetimes B$, where A is an elementary Abelian p-group, $B = \langle b \rangle$ is a cyclic group of order q, and $G' = A$ is the monolith of G. We fix a nonidentity element $a \in A$ and a natural number t greater than 1. Let $\{f_1, \ldots, f_m\}$ be the set of all nonidentity t-dimensional vectors over B whose first nonidentity component is equal to b. Then obviously $m = m(t) = (q^t - 1)/(q - 1)$. Let M_t be the t × m matrix whose i-th column is f_i,

$1 \le i \le m$. Let $e_1, e_2, \ldots, e_t$ be the row vectors of this matrix, enumerated from top to bottom. Let F_t denote the direct product of m copies of G, i.e., $F_t = \prod_1^m G_i$, where $G \simeq G_i = A_i \leftthreetimes B_i, A_i \simeq A$, $B \simeq B_i = \langle b_i \rangle$. We will assume that each G_i is naturally embedded in F_t. We denote $\prod_1^m A_i$ by A_t. We denote by E_t the subgroup of $\prod_1^m B_i$ generated by the vectors $e_1, e_2, \ldots, e_t$, and we denote by H_t the subgroup A_tE_t of F_t. Let a_i be the representative of the fixed element a in A_i, $1 \le i \le m$. Put $x_t = (a_1, \ldots, a_m)$. Finally, let π_i $(1 \le i \le m)$ denote the projection of F_t onto G_i.

LEMMA 1.3. Suppose γ is an endomorphism of the group H_t such that $\gamma(x_t) = x_t$. Then $|\gamma(H_t)| \ge t$.

Proof. Suppose, to the contrary, that γ is an endomorphism of H_t such that $\gamma(x_t) = x_t$ and $|\gamma(H_t)| < t$. Then any power γ^k of γ has the same properties. Since some power of any endomorphism of a finite group is an idempotent endomorphism, we may assume without loss of generality that $\gamma^2 = \gamma$. Then obviously $\gamma(H_t) \cap \operatorname{Ker}(\gamma) = 1$. Consequently, $H_t = \operatorname{Ker}(\gamma) \leftthreetimes \gamma(H_t)$. Moreover, $x_t \in \gamma(H_t)$. We denote $\operatorname{Ker}(\gamma)$ by U and $\gamma(H_t)$ by V. Since the orders of A_t and E_t are relatively prime, it follows from the Schur–Zassenhaus theorem that all complements of A_t in H_t are conjugate. Then it is easy to see that $U = (U \cap A_t) \leftthreetimes (U \cap E_t)$. The subgroup $U \cap A_t$ is normal in H_t, being the intersection of two normal subgroups. We will prove that

$$U \cap A_t = \prod A_i \quad (i \in J), \tag{4}$$

where $J = \{i \mid \pi_i(U \cap A_t) \ne 1\}$. Suppose $1 \le i \le m$ and $\pi_i(U \cap A_t) \ne 1$. Suppose also that $u \in U \cap A_t$, $\pi_i(u) \ne 1$, and the number of nonidentity components of u is minimal among the elements of $U \cap A_t$ with this property. Assume there exists $j \ne i$ such that $\pi_j(u) \ne 1$. Let $f_i = (b^{s_1}, b^{s_2}, \ldots, b^{s_t})$, $f_j = (b^{k_1}, b^{k_2}, \ldots, b^{k_t})$. Since $f_i \ne f_j$ and the first nonidentity components of these vectors are equal to b, the vectors $(s_1, s_2, \ldots, s_t)$ and $(k_1, k_2, \ldots, k_t)$ are not proportional as elements of the space of t-dimensional vectors over Z/qZ. Consequently, there exist natural numbers $\alpha_1, \alpha_2, \ldots, \alpha_t$ such that

$$\alpha_1 s_1 + \ldots + \alpha_t s_t \not\equiv 0 \pmod q,\ \alpha_1 k_1 + \ldots + \alpha_t k_t \equiv 0 \pmod q. \tag{5}$$

Consider $v = e_1^{\alpha_1} \ldots e_t^{\alpha_t}$ in E_t and $u_1 = [u, v]$. The latter lies in $U \cap A_t$, since $U \cap A_t \lhd H_t$. It follows from (5) that $\pi_i(u_1) = [\pi_i(u)]^{-1} b^{-\alpha_1 s_1 - \ldots - \alpha_t s_t} \pi_i(u) b^{\alpha_1 s_1 + \ldots + \alpha_t s_t} \ne 1$, otherwise $\pi_i(u_1)$ would lie in the center of G_i, but G_i has no center because it is monolithic; $\pi_j(u_1) = [\pi_j(u)]^{-1} b^0 \pi_j(u) b^0 = 1$. Moreover, if $1 \le r \le m$ and $\pi_r(u) = 1$, then obviously $\pi_r(u_1) = 1$. Thus, the number of nonidentity components of the vector u_1 is at least one less than the corresponding number for u, which contradicts the choice of u. Therefore, u has only one nonidentity component, i.e., $1 \ne u \in A_i \cap U$. But $U \cap A_i \lhd G_i$. Therefore, the monolith of G_i, the subgroup A_i, is contained in $U \cap A_t$. Thus, we have shown that if $1 \le i \le m$ and $\pi_i(U \cap A_t) \ne 1$, then $A_i \subseteq U$. This implies (4).

Now consider V. Since $V \cap A_t$ is a normal Sylow p-subgroup of V, it follows that $V = (V \cap A_t)V_q$, where V_q is a Sylow q-subgroup of V. We may assume without loss of generality that $V_q \subseteq E_t$. Note that $H_t = UV$ implies

$$E_t = (U \cap E_t)V_q. \tag{6}$$

The subgroup $V \cap A_t$ can be viewed as a module over $(Z/pZ)V_q$. By Maschke's theorem, this module is completely reducible. Therefore, it is a direct sum of minimal submodules. It follows that $V \cap A_t$ is a direct product of minimal normal subgroups $T_1, T_2, \ldots, T_k$ of V. For each $j = 1, \ldots, k$ we denote by I_j the set $\{i \mid \pi_i(T_j) \ne 1\}$. Since $x_t = (a_1, \ldots, a_m) \in V$, it follows that $\bigcup_j I_j = \{1, \ldots, m\}$, and since $|V| < t$, we have $k < t$.

Suppose $1 \le j \le k$ and $i, h \in I_j$. We will prove that if $\pi_i(v) = 1$, then $\pi_h(v) = 1$ for any $v \in V_q$. Indeed, suppose $\pi_i(v) = 1$, $\pi_h(v) \ne 1$ for some $v \in V_q$. Then obviously $1 \ne [T_j, v] \subseteq T_j \cap \prod_{r \ne i} A_r$. This contradicts the minimality of the normal subgroup T_j of V.

We will now prove there exists $\nu \in \{1, \ldots, m\}$ such that $1 = \pi_\nu(V_q)$. We may assume without loss of generality that $f_i = (1, \ldots, 1, b, 1, \ldots, 1)$ for $i = 1, \ldots, t$, where the b occurs in the i-th place. Since $k < t$ and $\bigcup_1^k I_j = \{1, \ldots, m\}$, there exist j, $1 \le j \le k$, and distinct i and r, $1 \le i$, $r \le t$, such that $i, r \in I_j$. We may assume, again without loss of generality, that $i = j = 1$, $r = 2$. If $\pi_2(V_q) = 1$, then as the desired ν we can take 2. Suppose $\pi_2(V_q) \ne 1$ and v is an element of V_q such that $\pi_2(v) \ne 1$. Suppose $\pi_1(v) = s_1$, $\pi_2(v) = s_2$, and r is such that $rs_2 \equiv 1 \pmod q$. In view of the choice of the set $\{f_i \mid 1 \le i \le m\}$, it contains the vector $(b, b^{-rs_1}, 1, \ldots, 1)$. Let ν be the index of this vector. We will prove that ν is the desired number. Take $u = e_1^{k_1} e_2^{k_2} \ldots e_t^{k_t}$. Assume that $\pi_\nu(u) \ne 1$. Then

$$\pi_1\left(u^{rs_2}v^{-rk_2}\right)=b^{(k_1s_2-k_2s_1)r}=b^{k_1-k_2s_1r}=\pi_v(u)\neq 1.$$

But it is easy to see that $\pi_2(u^{rs_2}v^{-rk_2})=1$, which is impossible by what was proved in the previous paragraph, inasmuch as 1 and 2 belong to the same I_l. Since $\bigcup_1^k I_j=\{1, 2, \ldots, m\}$, it follows that $v\in I_j$ for some j, $1 \le j \le k$. Then, as was shown above, $\pi_i(V_q) = 1$ for any $i\in I_j$. Since $U\cap V=1$, we also have $T_j\cap U=1$. Therefore, $J\not\supseteq I_j$. Take $i\in I_j-J$. Since $\pi_i(V_q) = 1$, it follows that V_q centralizes A_i. But $U\cap E_t$ also centralizes A_i, inasmuch as A_i and U are two normal subgroups of H_t whose intersection is the identity subgroup. Therefore, in view of (6), all of E_t centralizes A_i. This, in turn, means that the i-th column of the matrix M_t, i.e., f_i, consists of ones, which contradicts the choice of the set $\{f_i \mid 1 \le i \le m\}$. Lemma 1.3 is proved.

Let S_t denote a semigroup that is an inflation of H_t by means of the pair $(x_t, x_t^!)$, where $x_t^!\notin H_t$. We will prove that S_t cannot be approximated by subsemigroups of orders less than t. Indeed, suppose γ is an endomorphism of S_t such that $\gamma(x_t^!) \neq \gamma(x_t)$ and $|\gamma(S_t)| < t$. Assume that $\gamma(x_t')\in H_t$. Then $\gamma(x_t) = \gamma(x_t)^{p+1} = \gamma(x_t^!)^{p+1}$. It follows that $\gamma(x_t^!) = \gamma(x_t)$, a contradiction. Therefore, $\gamma(x_t^!) = x_t^!$. Consequently, $\gamma(x_t) = \gamma[(x_t^!)^{p+1}] = (x_t^!)^{p+1} = x_t$. Thus, $\gamma(x_t) = x_t$ and $|\gamma(H_t)| < t$, which is impossible by Lemma 1.3.

Finally, we will prove that the Q-lattice of the variety $\mathfrak{M}$ is infinite. Since S_t is an inflation of H_t, it follows that S_t is a subdirect product of H_t and some zero semigroup [11]. But the group H_t is, by construction, a subdirect product of m copies of the group G in $\mathfrak{M}$. Therefore, S_t belongs to $\mathfrak{M}$ for any $t \ge 1$. Assume there are only a finite number of distinct quasivarieties $q(S_t)$, $t \ge 1$. Then there exist natural numbers t and r such that $r > |S_t|$ and $q(S_r) = q(S_t)$. This equality means that S_r is approximated by S_t and, conversely, S_t is approximated by S_r. Therefore, S_r can be approximated by subsemigroups of orders less than $|S_t| < r$, which, as we have just shown, is impossible.

Part 5. $\mathfrak{M}=v(\mathrm{I}\times(Z(p)\operatorname{wr}Z(q))\times(Z(r)\operatorname{wr}Z(p)))$, where p, q, r are primes.

As in Sec. 4, we may assume that $p \neq q$ and $p \neq r$. Let C be a minimal non-Abelian $\{r, p\}$-group in $\mathfrak{M}$. Then C is a monolithic group and the quotient group of C by its monolith is cyclic of order p. We denote the monolith of C by M and take an arbitrary element $c\in C\backslash M$.

Fix $t \ge 2$. Let H_t and x_t be the same as in Part 4. We define a homomorphism φ of C into H_t by putting $\varphi(c^kM) = \{x_t^k\}$ for any k from 1 to p. Clearly, $\operatorname{Ker}(\varphi) = M$. We define a semigroup R_t on the set $C\cup H_t$ by preserving the operations within C and H_t and putting $xy = \varphi(x)y$ and $yx = y\varphi(x)$ for any $x\in C$ and $y\in H_t$. Clearly, $R_t\in\mathfrak{M}$. We will prove that R_t cannot be approximated by subsemigroups of orders less than t. It will then follow, as in Part 4, that the Q-lattice of $\mathfrak{M}$ is infinite. Assume there exists an endomorphism γ of the semigroup R_t such that $|\gamma(R_t)| < t$ and $\gamma(b) \neq \gamma(1)$, where b is an arbitrary nonidentity element of M. If $\gamma(C)\subseteq H_t$, then obviously $\gamma(M) = \gamma(\operatorname{Ker}(\varphi)) = \gamma(1)$ and so $\gamma(b) = \gamma(1)$, which is impossible. Thus, $\gamma(C)\subseteq C$ and $\gamma(H_t)\subseteq H_t$. Let γ_1 denote the restriction of γ to C. If $\operatorname{Ker}(\gamma_1)\neq 1$, then $M\subseteq\operatorname{Ker}(\gamma_1)$, since M is the monolith of C, and therefore $\gamma(b) = \gamma(1) = 1$. Thus, γ_1 is an automorphism. It follows that $\gamma^n(b) \neq \gamma^n(1)$ and $|\gamma^n(R_t)| < t$ for any $n \ge 1$. We may therefore assume without loss of generality that γ acts identically on C. Consequently, $\gamma(x_t) = \gamma(\varphi(c)) = \gamma(c\cdot\varphi(1)) = c\cdot\gamma(\varphi(1)) = \varphi(c) = x_t$. By Lemma 1.3, $|\gamma(H_t)| > t$, which contradicts the choice of γ.

Part 6. $\mathfrak{M}=v(\mathrm{I}\times(Z(p)\operatorname{wr}Z(q))\times Z(p^2))$, where p and q are primes.

As in Part 5, $\mathfrak{M}$ contains a monolithic group whose quotient group by the monolith is cyclic of order p. Such a group is $Z(p^2)$. Therefore, the proof of the fact that the Q-lattice of $\mathfrak{M}$ is infinite can be carried out as in Part 5.

SECTION 2

This section contains some facts about finite Abelian groups that will be needed later. We will employ the following notation. Throughout this section the operation will be written additively. Suppose A is a finite Abelian p-group with basis X. Let $X^{(i)}$ denote the set of all elements of X of order p^i. Suppose $x\in X$. To indicate that x belongs to the set $X^{(i)}$ it will be convenient to write $x^{(i)}$ instead of x. We have $A=\langle x\rangle+\langle X\backslash x\rangle$. Let π_X^x denote the projection of A on $\langle x\rangle$ corresponding to this decomposition.

From here up to Proposition 2.1, A is a finite Abelian p-group, $p^5A = 0$, and N is a subgroup of A.

LEMMA 2.1. Assume that A has no direct summand of exponent at most p^2. Then there exist bases X of A and $Y = \{y_1, \ldots, y_k\}$ of N such that for any i from 1 to k one of the following conditions holds: a) $y_i = p^{m_i}\times a_i^{(5)}$; b) $y_i = p^{m_i}b_i^{(4)}$; c) $y_i = p^{m_i}c_i^{(3)}$; d) $y = p^{m_i}a_i^{(5)} + p^{m_i-1}c_i^{(3)}$; e) $y_i = p^{m_i}a_i^{(5)} + p^{m_i-1}c_i^{(3)}$, $y_{i+1} = p^{m_i+n_i}c$; f) $i-1$ satisfies e), where a_i, b_i, c_i are distinct elements of X for distinct i.

Proof. Suppose A is a minimal counterexample and N is a subgroup of A of smallest rank for which there are no bases of the desired kind. Then it is easy to see that A contains no nonzero subgroups U and V such that

$$A = U \dotplus V,\ N = (N \cap U) + (N \cap V). \tag{7}$$

Let $Y = \{y_1, \ldots, y_k\}$ be a basis of N, where $k > 1$. We may assume without loss of generality that y_k has smallest order among the elements of Y. Let $|y_k| = p^r$. Let N_1 denote the subgroup $\langle Y \backslash y_k \rangle$ of N. The rank of N_1 is $k-1$. We may therefore assume that there has been chosen in A a basis $X = \{a_i^{(5)}, b_j^{(4)}, c_h^{(3)} \mid 1 \le i \le s, 1 \le j \le t, 1 \le h \le u\}$ such that the bases X in A and $Y_1 = Y \backslash y_k$ in N_1 satisfy the conclusion of Lemma 2.1. Let $y_k = \sum_1^s \alpha_i a_i + \sum_1^t \beta_i b_i + \sum_1^u \gamma_i c_i$ be the decomposition of y in the basis X. We consider only the case where

$$p^r > \left|\sum_1^t \beta_i b_i\right|,\quad p^r > \left|\sum_1^u \gamma_i c_i\right|; \tag{8}$$

the other cases are handled analogously. It follows from (8) that all α_i and β_i are divisible by p^{5-r}. Suppose $\alpha_i = p^{5-r}\alpha_i'(1 \le i \le s)$, $\beta_j = p^{5-r}\beta_j'(1 \le j \le t)$. Take $a = a^{(5)} \in X$. Assume that $\pi_X^a(N_1) \ne 0$, i.e., $\pi_X^a(Y_1) \ne 0$. It is easy to verify that $p^4 a \in N_1$. Since $p^{r-1}y_k = p^{r-1}\sum_1^s \alpha_i a_i \notin N_1$, there exists an integer j between 1 and s such that $|\alpha_j a_j| = p^r$ and $p^{r-1}\alpha_j a_j \notin N_1$, hence $p^4 a_j \notin N_1$. Therefore, $\pi_X^{a_j}(N_1) = 0$. We now replace in X the element a_j by $a = \sum_1^s \alpha_i' a_i + \sum_1^t \beta_i' b_i$. We obviously obtain a new basis of A. Let us denote it by X_1. Clearly, X_1 and Y_1 satisfy the conclusion of Lemma 2.1. We have $y_k = p^{5-r}a + \sum_1^u \gamma_i c_i$. We may assume without loss of generality that no γ_i, $1 \le i \le u$, is a multiple of p^{5-r}. Therefore, $|\gamma_i c_i| > p^{r-2}$ for $1 \le i \le u$. On the other hand, in view of (8), $p^r > |\gamma_i c_i|$. Consequently, $|\gamma_i c_i| = p^{r-1}$ $(1 \le i \le u)$.

If some j between 1 and $k-1$ satisfies condition a) or b) in the conclusion of Lemma 2.1, then condition (7) holds for the subgroups $\langle a_j \rangle$ and $\langle X_1 \backslash a_j \rangle$ (respectively, $\langle b_j \rangle$ and $\langle X \backslash b_j \rangle$), which is impossible. Consequently, no j from 1 to $k-1$ satisfies condition a) or b).

Assume there exists j, $1 \le j \le k-1$, satisfying condition c). Since the order $|y_k|$ is minimal among the orders of elements of Y, we have $p^r \le p^{3-m_j}$, hence $\left|\sum_1^u \gamma_i c_i\right| < p^{3-m_j}$. Consequently, for some γ we have $\pi_{X_1}^{c_j}(y_k - \gamma c_j) = 0$. Then the subgroups $\langle c_j \rangle$ and $\langle X_1 \backslash c_j \rangle$ satisfy (7), which is impossible. Thus, no j from 1 to $k-1$ satisfies c). The cases where some j, $1 \le j \le k-1$, satisfies d), e), or f) are handled analogously. It follows that $k = 1$, i.e., $N = \langle y_k \rangle$. Since $|\gamma_i c_i| = p^{r-1}(1 \le i \le u)$, each γ_i is divisible by p^{4-r}. Suppose $\gamma_i = p^{4-r}\gamma_i'$ for some γ_i'. Replace in X_1 the element c_i (i is an arbitrary integer between 1 and u) by $\sum_1^u \gamma_j' c_j = c$. We obtain a new basis in A. Let us denote it by X_2. Since $y_k = p^{5-r}a + p^{4-r}c$, the bases Y in N and X_2 in A satisfy the conclusion of the lemma, which is contrary to the choice of N and A. Lemma 2.1 is proved.

Analogously we can prove

LEMMA 2.2. Suppose $p^3A = 0$. Then there exist bases X in A and $Y = \{y_1, \ldots, y_k\}$ in N such that for any i, $1 \le i \le k$, one of the following conditions holds: a) $y_i = p^{m_i}a_i^{(3)}$; b) $y_i = p^{m_i}b_i^{(2)}$; c) $y_i = c_i^{(1)}$; d) $y_i = pa_i^{(3)} + c_i^{(1)}$, where a_i, b_i, c_i are distinct elements of X for distinct i.

LEMMA 2.3. There exist in A subgroups U and V such that $A = U \dotplus V, N = (U \cap N) + (V \cap N)$, and $|V| < p^{15}$.

Proof. Suppose A is a minimal counterexample and N is a subgroup of A for which there are no such U and V. We will assume that A contains no complemented subgroups of exponent p^4 or p^2. The proof of the lemma in the general case differs from that given here only in the amount of calculation.

Note that A obviously contains no nonzero subgroups U and V such that (7) holds, and $|A| \le p^{15}$.

It follows from Lemma 2.1 that there exist bases X in A and $Y = \{y_1, \ldots, y_n\}$ in N such that for any i from 1 to n one of the following conditions holds: 1) $y_i = a_i^{(s)} + d_i$; 2) $y_i = p^{s-1}a_i^{(s)} + d_i$; 3) $y_i = p^m a_i^{(s)}$; 4) $y_i = p^k a_i^{(5)} + p^{k-1}b_i^{(3)}$; 5) $y_i = p^3 a_i^{(5)} + d_i$; 6) $y_i = p^3 a_i^{(5)} + p^2 b_i^{(5)} + d_i$; 7) $y_i = pa_i^{(5)} + b_i^{(3)}$ and $y_{i+1} = pb_i^{(3)} + d_i$; 8) condition

7) holds for $i-1$; 9) $y_i = p^2 a_i^{(5)} + d_i$; 10) $y_i = pb_i^{(3)} + d_i$; 11) $y_i = p^2 a_i^{(5)} + pb_i^{(3)} + d_i$; 12) $y_i = p^2 a_i^{(5)} + pb_i^{(3)} + d_i$ and $y_{i+1} = p^2 b_i$; 13) condition 12) holds for $i-1$; 14) $y_i = pa_i^{(5)} + d_i$. Here $0 \neq d_i \in \langle X^1 \rangle$ and a_i, b_i are distinct elements of X for distinct i, and $1 \le s \le 5$, $0 \le m \le s-1$, $1 \le k \le 3$. We will call such bases compatible. We will show that none of these conditions can be satisfied and will thereby arrive at a contradiction.

If N were to contain a subgroup $U \neq 0$ complemented in A, then U and its complement in A would satisfy (7), which is impossible. Therefore, conditions 1) and 2) cannot hold for any i.

If 3) holds for an i, $1 \le i \le n$, we put $U = \langle a_i \rangle$, $V = \langle X \backslash a_i \rangle$. Obviously, (7) holds for U and V. Moreover, $U \neq 0 \neq V$, since $p^s = |U| \le p^5 < p^{15} \le |A|$. Therefore, condition 3) also cannot hold.

Condition 4) can be eliminated analogously.

Suppose i, $1 \le i \le n$, satisfies condition 5). We may, of course, assume that $d_i \in X$. Since $|d|_i = p$, for any j from 1 to n there exists a number δ_j such that $\pi_X^{d_i}(y_j - \delta_j y_i) = 0$. Replace in Y the element y_j by $y_j - \delta_j y_i$, $1 \le j \le n$. Also, replace certain elements of X in accordance with the following rule. Suppose $j (1 \le j \le n)$ satisfies condition m). If $m \in \{5, 6\}$, then replace a_j by $a_j - \delta_j b_i$; if $m \in \{9, 11, 12\}$, then replace a_j by $a_j - p\delta_j a_i$; if $m = 14$, replace a_j by $a_j - p^2 \delta_j a_i$; if $m \in \{8, 10\}$, replace b_j by $b_j - \delta_j p^2 a_i$. We thus obtain new bases $\tilde{X}$ in A and $\tilde{Y}$ in N. It is easy to see that $\tilde{X}$ and $\tilde{Y}$ are compatible and $d_i \in \tilde{X}$, $\pi_{\tilde{X}}^{d_i}(v) = 0$ for any v in $\tilde{Y} \backslash y_i$. It follows easily that the subgroups $U = \langle a_i, d_i \rangle$ and $V = \langle \tilde{X} \backslash a_i \backslash d_i \rangle$ satisfy (7). Thus, condition 5) cannot hold for any i. Conditions 6)–8) can be eliminated analogously.

Suppose 9) holds for some i, $1 \le i \le n$. We may again assume that $d_i \in X$. We also assume without loss of generality that the numbers $1, 2, \ldots, k < n$ and only these satisfy 10). It is possible that this set of numbers is empty. Then we will assume that $k = 0$.

Suppose that for any j from 1 to k we have

$$\pi_X^{d_i}(y_j) = 0. \tag{9}$$

Then, as is easily seen, there exist compatible bases X_1 in A and Y_1 in N such that $y_i \in Y_1$, a_i, $d_i \in X_1$, and (9) holds for all j, $1 \le j \le n$, except $j = i$. It follows that $\langle a_i, d_i \rangle$ and $\langle X_1 \backslash a_i \backslash d_i \rangle$ satisfy (7), which is impossible.

Thus, (9) is false for some j between 1 and k. We assume without loss of generality that j is 1. We may also assume that (9) is true for all $2 \le j \le k$. Consider the group $A_1 = \langle b_2, \ldots, b_k, X^{(1)} \rangle$ and its subgroup $N_1 = \langle y_2, \ldots, y_k \rangle$. Since $p^3 A_1 = 0$, we can apply Lemma 2.2 to A_1 and N_1. Then there exist bases $P = \{t_2^{(3)}, \ldots, t_k^{(3)}, r_1^{(1)}, \ldots, r_s^{(1)}\}$ in A_1 and $Q = \{q_2, \ldots, q_k\}$ in N_1 such that for each j, $2 \le j \le k$, one of the following three conditions holds: α) $q_j = p^{m_j} t_j$; β) $q_j = r_j$; γ) $q_j = pt_j + r_j$, where t_j, r_j are distinct for distinct j. We replace in X the subset $\{b_2, \ldots, b_k\} \cup X^{(1)}$ by P, and in Y the subset $\{y_2, \ldots, y_k\}$ by Q. We obviously obtain new compatible bases X_1 in A and Y_1 in N. If some j, $2 \le j \le k$, satisfies condition α) or β), then it satisfies condition 3), which is impossible by what was proved. Therefore, each j from 2 to k satisfies γ), i.e.,

$$\begin{gathered} q_2 = pt_2 + r_2, \\ \cdots\cdots \\ q_k = pt_k + r_k, \end{gathered}$$

where all of the t_j and r_j are distinct elements of X_1 for distinct j. Suppose

$$y_1 = pb_1 + \delta d_i + \sum_2^k \gamma_j r_j + d,$$

where δ and all of the γ_j are integers and $d \in \langle Y_1 \backslash \{t_j | 2 \le j \le k\} \rangle$. We replace in X_1 the element b_1 by $t_1 = b_1 - \sum_2^k \gamma_j t_j$, and in Y_1 the element y_1 by $y_1 - \sum_2^k \gamma_j q_j$. Denote the new bases of A and N by X_2 and Y_2, respectively. They are obviously compatible. If $d = 0$, then $U = \langle a_i, t_1, d_i \rangle$ and $V = \langle X_2 \backslash a_i \backslash t_1 \backslash d_i \rangle$ satisfy (7), and $U \neq \{0\} \neq V$, inasmuch as $|U| = p^9 < p^{15} \le |A|$. Therefore, $d \neq 0$. We may assume that $d \in X_2^{(1)}$. We replace d in X_2 by $d + \delta d_i$ and denote $(X_2 \backslash d) \cup \{d + \delta d_i\}$ by X_3. Clearly, the bases X_3 and Y_2 are compatible and

$$\pi_{X_3}^{d_i}(y) = 0$$

for all $y \in Y_2$ satisfying condition 10). But we have already examined this case. Thus, no i, $1 \le i \le n$, satisfies condition 9).

The remaining cases 10)-14) can now be eliminated one by one as in the case of condition 5). Lemma 2.3 is proved.

LEMMA 2.4. Suppose $p^2A = 0$ and N_1, N_2, N_3 are subgroups of A such that $N_1 \subseteq N_2 \subseteq N_3$. Then there exist in A subgroups U and V such that $A = U + V$, $N_i = (N_i \cap U) + (N_i \cap V)$, $i = 1, 2, 3$, and V is cyclic, hence $|V| \leq p^2$.

Proof. Assume the assertion of Lemma 2.4 does not hold for the group A and its subgroups N_1, N_2, N_3 $(N_1 \subseteq N_2 \subseteq N_3)$. Suppose $\{x_1, \ldots, x_m\}$ is a basis of N_1. Denote $\langle x_1, \ldots, x_{m-1}\rangle$ by N. We may assume without loss of generality that A contains no nonzero subgroups U and V such that $A = U + V$ and $N_i = (U \cap N_i) + (V \cap N_i)$, $1 \leq i \leq 3$, but there exists a family of cyclic subgroups $\{C_i = \langle c_i \rangle | i = 1, 2, \ldots, k\}$ such that $A = \sum_1^k{}^{\cdot} C_i$, $N_2 = \sum_1^k{}^{\cdot} (C_i \cap N_2)$, $N_3 = \sum_1^k{}^{\cdot} (C_i \cap N_3)$, $N = \sum_1^k{}^{\cdot} (C_i \cap N)$. Then N_1 contains no subgroup complemented in A, hence $x_m = \alpha_1 pc_1 + \ldots + \alpha_k pc_k$ for certain integers α_i. If $N \neq 0$, then $N \cap C_i = \langle pc_i \rangle$ for some i, $1 \leq i \leq k$. It follows easily that $N_1 = (N_1 \cap C_i) + (N_1 \cap \sum_{i \neq j} C_j)$. The corresponding equalities for A, N_2, and N_3 are true in view of the choice of the family $\{C_i | 1 \leqslant i \leqslant k\}$. The resulting contradiction shows that $N = 0$. Clearly, $\alpha_i pc_i \neq 0$ for all i, $1 \leq i \leq k$. Put $N_4 = A$. To each i from 1 to k we assign a number $\varphi(i)$, $1 \leq \varphi(i) \leq 4$, such that $C_i \subseteq N_{\varphi(i)}$, $C_i \not\subseteq N_{\varphi(i)-1}$. Assume for definiteness that $\varphi(1) \leq \varphi(2)$. Then, as is easily seen, in the role of the "forbidden" U and V we can take $\langle \alpha_1 c_1 + \alpha_2 c_2 \rangle$ and $\sum_{j \neq 1} C_j$, respectively. Lemma 2.4 is proved.

Proposition 2.1. Suppose A is a finite Abelian group of exponent m, and suppose N_1, N_2, N_3, F are subgroups, $N_1 \leq N_2 \leq N_3$, $|A : F| = f$, and a is an element of A. Then:

A. There exist in A subgroups U and V such that $A = U + V$, $a \in V$, and the order $|V|$ is bounded in terms of m.*

B. If $p^s \nmid m$ for any prime p, then there exist in A subgroups U and V such that $A = U + V$, $N_1 = (U \cap N_1) + (V \cap N_1)$, $a \in V$, and $U \leq F$, where the order $|V|$ is bounded in terms of m and f.

C. If $p^3 \nmid m$ for any prime p, then there exist in A subgroups U and V such that $A = U + V$, $N_i = (U \cap N_i) + (V \cap N_i)$, $1 \leqslant i \leqslant 3$, $U \leqslant F$, and the order $|V|$ is bounded in terms of m and f.

Proof. A. By Lemma 4 of [3], $\langle a \rangle$ is contained in a subgroup that is complemented in A and whose order is bounded in terms of m. We can take this subgroup as V; in the role of U we can take any complement of V in A.

B. It follows easily from Lemma 2.3 that A contains subgroups $C_1, \ldots, C_k$ such that $A = \sum{}^{\cdot} C_i$, $N_1 = \sum{}^{\cdot} (C_i \cap N_i)$, $|C_i| \leqslant m^3$ $(1 \leqslant i \leqslant k)$. Let π_i denote the projection of A on $C_i (1 \leq i \leq k)$. Suppose $R \leq A$, $|R| = r$. We partition the set $\{1, 2, \ldots, k\}$ into classes D_α as follows. We put two integers i and j, $i < j$, into the same class D_α if there exists an isomorphism φ_{ij} of C_i onto C_j such that $\varphi_{ij}\pi_i(R) = \pi_j(R)$, $\varphi_{ij}(N_1 \cap C_i) = N_1 \cap C_j$. In addition, the isomorphisms φ_{ij} can, of course, be chosen so that $\varphi_{ij}\varphi_{jn} = \varphi_{in}$. Clearly, the number of classes D_α is bounded in terms of r and m. For each D_α consider the sum $\sum_{i \in D_\alpha} C_i$. We denote the diagonal of this sum by S_α, and the subgroup $\sum_\alpha S_\alpha$ by S. We now discard exactly one number from each D_α and denote the sum of the remaining C_i by T. It is easy to verify that $R \leqslant S$, $A = T + S$, $N_1 = (T \cap N_1) + (S \cap N_1)$. It is also clear that the order $|S|$ is bounded in terms of r and m.

We will construct such T and S for $R = \langle a \rangle$. Let $F \cap T = E$ and $N_1 \cap T = N$.

Assume first that $E \cap N = 0$. Since $|T/E + N| < f$, it follows from Lemma 5 of [3] that $(E + N)/N$ contains subgroup X/N complemented in T/N whose index is bounded in terms of m and f. Let Y/N be a complement of X/N in T/N. Then $T = (X \cap E) + Y$ and $N \subseteq Y$. Moreover, the order $|Y|$ is bounded in terms of m and f.

Now suppose $E \cap N \neq 0$. By what was proved in the previous paragraph, T contains $X \subseteq E$ and Y such that $T = X \dotplus Y$, $X \cap Y = E \cap N$, $N < Y$, and $|Y/N \cap E|$ is bounded in terms of m and f. Again applying Lemma 5 of [3], we see that $E \cap N$ contains a subgroup K complemented in Y whose index in Y is bounded in terms of m and f. Let Y_1 be a complement of K in Y. Then $T = X + Y_1$, $X \cap Y_1 = E \cap N \cap Y_1$, and $N = (N \cap X) + (N \cap Y)$. As shown in

*I.e., does not exceed a number depending only on m.

the first paragraph of the proof of assertion B, there exist in X subgroups X_1 and X_2 such that $X = X_1 + X_2$, $N \cap X = (N \cap X_1) + (N \cap X_2)$, $X \cap Y_1 \subseteq X_2$, and $|X_2|$ is bounded in terms of m and $|X \cap Y_1|$ and, therefore, in terms of m and f. Then as the desired U and V we can take X_1 and $X_2 + Y_1 + S$, respectively.

The proof of C is similar to that of B.

Proposition 2.1 is proved.

SECTION 3

In this section we will prove Theorem 1 and sufficiency in Theorem 2. We will need the description of the varieties of f.a. semigroups in [10]. The description is given in the next lemma. Recall that a semigroup is called orthodox if the set of its idempotents is a subsemigroup; a semigroup is called normal (right normal, left normal) if it satisfies the identity xyzt = xzyt (xyz = yxz, xyz = xzy).

LEMMA 3.1. The following conditions for a variety of semigroups $\mathfrak{M}$ are equivalent: a) $\mathfrak{M}$ consists of f.a. semigroups; b) $\mathfrak{M}$ is generated by a finite semigroup of one of the following 20 forms: $\mathbf{G}$, $\mathbf{L}\times\mathbf{G}$, $\mathbf{R}\times\mathbf{G}$, $\mathbf{L}\times\mathbf{R}\times\mathbf{G}$, $\mathbf{N}\times\mathbf{G}$, $\mathbf{L}\times\mathbf{N}\times\mathbf{G}$, $\mathbf{R}\times\mathbf{N}\times\mathbf{G}$, $\mathbf{L}\times\mathbf{R}\times\mathbf{N}\times\mathbf{G}$, $\mathbf{I}\times\mathbf{G}$, $\mathbf{L}\times\mathbf{I}\times\mathbf{G}$, $\mathbf{R}\times\mathbf{I}\times\mathbf{G}$, $\mathbf{L}\times\mathbf{R}\times\mathbf{I}\times\mathbf{G}$, $\mathbf{N}\times\mathbf{I}\times\mathbf{G}$, $\mathbf{L}\times\mathbf{N}\times\mathbf{I}\times\mathbf{G}$, $\mathbf{R}\times\mathbf{N}\times\mathbf{I}\times\mathbf{G}$, $\mathbf{L}\times\mathbf{R}\times\mathbf{N}\times\mathbf{I}\times\mathbf{G}$, $\mathbf{P}\times\mathbf{C}$, $\mathbf{Q}\times\mathbf{C}$, $\mathbf{L}\times\mathbf{Q}\times\mathbf{C}$, $\mathbf{R}\times\mathbf{P}\times\mathbf{C}$, where $\mathbf{G}$ is an A-group and $\mathbf{C}$ is a cyclic group; c) $\mathfrak{M}$ consists either of orthodox semigroups that are inflations of normal bands of f.a. groups, or of right normal (left normal) semigroups that are bands of inflations of Abelian groups of finite exponent.

Now suppose $\mathfrak{M}$ is a variety of f.a. semigroups that does not contain semigroups of the forms mentioned in Theorem 2. We will prove that the Q-lattice of $\mathfrak{M}$ is finite. The proof is in three parts.

Part 1. $\mathfrak{M}$ consists of inflations of orthodox normal bands of groups.

Here we have the following lemma.

LEMMA 3.2. Suppose S is a Q-critical semigroup in $\mathfrak{M}$. Then either S is isomorphic to one of the semigroups $\mathbf{L}$, $\mathbf{R}$, $\mathbf{L}^0$, $\mathbf{R}^0$, $\mathbf{I}$, or S is an inflation of an A-group G and $|S\backslash G| \leqslant 1$, or S is an inflation of a commutative band of two A-groups G_1, G_2 and $|S\backslash(G_1 \cup G_2)| \leqslant 1$.

Proof. Note first that S is finite, inasmuch as $\mathfrak{M}$ is a locally finite variety. Therefore, S cannot be approximated by proper subsemigroups. Let a, b be distinct elements of S that cannot be separated by a homomorphism onto a proper subsemigroup of S. In view of the choice of $\mathfrak{M}$, it follows that S is an inflation of an orthodox normal band J of groups G_i, $i \in J$. Therefore, S is the band of inflations T_i of the groups G_i $(i \in J)$.

Assume that $a \in T_i$, $b \in T_j$, $i \neq j$. Then a and b are separated by the natural homomorphism of S onto J. However, as is easily seen, J is isomorphic to the semigroup E of idempotents of S. Therefore, a and b are separated by a homomorphism of S onto E. Consequently, S = E. Then S is a commutative band I of rectangular components S_α, $\alpha \in I$.

If $a \in S_\alpha$, $b \in S_\beta$, and $\alpha \neq \beta$, then a and b are separated by a homomorphism onto the semigroup $\mathbf{I}$, which is obviously embedded in S. Consequently, S $\simeq$ $\mathbf{I}$.

Suppose a, $b \in S_\alpha$ for some $\alpha \in I$. Let $z \in S$. It follows easily from the normality of S that the left translation l_z and right translation r_z of S that correspond to z are endomorphisms of S.

Suppose $z \in S_\alpha$. Since S_α is a rectangular band, there occurs at least one of the following two cases: $za \neq zb$ or $az \neq bz$. Suppose the first case occurs (the second is handled analogously). Then a and b are separated by the endomorphism l_z. Therefore, l_z is an automorphism of S. Since $l_z(x) = zx = z \cdot zx = l_z(zx)$ $(x \in S)$, it follows that $x = zx$ for any $x \in S$. Consequently, S_α is a right zero semigroup, each element of S_α is a left unity of S, and α is the unity in I.

If $S \in S_\alpha$, then S is approximated by the semigroup $\mathbf{R}$, hence S $\simeq$ $\mathbf{R}$. We may, therefore, assume that $S \neq S_\alpha$.

Take $\beta \in I$, $\beta \neq \alpha$, and $x \in S_\beta$. If $xa \neq xb$, then l_x is an automorphism, hence $xa = a$, which is impossible inasmuch as $xa \in S_\beta \neq S_\alpha$. Therefore, $xa = xb$ for all $x \in S\backslash S_\alpha$. By Theorem IV.4.3 of [11], there exists in S_β an element x such that $xa = ax = x$. Then $xb = bx = x$. Consequently, the set $\{a, b, x\}$ is a subsemigroup of S isomorphic to $\mathbf{R}^0$. By Theorem IV.4.6 of [11], S is approximated by the semigroups $\mathbf{L}^0$ and $\mathbf{R}^0$. From this and the fact that a and b are right zeros for one another it follows that these elements are separated by a homomorphism onto $\mathbf{R}^0$. Consequently, S $\simeq$ $\mathbf{R}^0$.

It remains to consider the case where a, $b \in T_i$ for some $i \in J$. Suppose e is the unity of the group G_i. We define a mapping φ_e of S into itself by putting

$$\varphi_e(z) = \begin{cases} z, & \text{if} \quad z \in T_i, \\ eze, & \text{if} \quad z \notin T_i. \end{cases}$$

We will prove that φ_e is an endomorphism. Since S is finite and is an inflation of a band of groups, S satisfies the identity $x^2 = x^{2+d}$ for some $d \geq 1$ and the identities $x^{d+1}y = xy$ and $xy^{d+1} = xy$. Since the set of idempotents of S is a normal subsemigroup, S satisfies the identity $x^d y^d z^d t^d = x^d z^d y^d t^d$. Consequently, for any x and t in S we have $exeete = ex^{d+1}ete = exx^d ete = (ex)^{d+1} x^d e^d te = exex^d te = exexx^{d-1} te = ex(ex)^{d+1} x^{d-1} te = ex(ex)^d ex^d te = exe(ex)^d x^d te = (ex)^{d+1} x^d te = exte$. Therefore, $\varphi_e(x)\varphi_e(t) = \varphi_e(xt)$ for any $x,\ t \in S \setminus T_i$. Suppose $x \in T_i$. The semigroup S is an inflation of the band of groups $G = \bigcup_{j \in J} G_j$ by means of the family of sets $\{X_c | c \in G\}$. Suppose $x \in X_c$. Then $c \in G_i$, $ece = c$. Consequently, for any $t \in S$ we have $xete = cete = ecete = ecte = exte = xte$ and, analogously, $etex = etxe = etx$. Therefore, $\varphi_e(t)\varphi_e(x) = \varphi_e(tx)$ for any $t \in S \setminus T_i$. It is also clear that $\varphi_e(tx) = tx = \varphi_e(t)\varphi_e(x)$ for any $t,\ x \in T_i$. Consequently, φ_e is indeed an endomorphism. Obviously, $\varphi_e(a) = a \neq b = \varphi_e(b)$. Then φ_e is an automorphism. Since for any z in S we have $\varphi_e(eze) = \varphi_e(z)$, it follows that $eze = z$. Consequently, for any two idempotents f and g of S we have $fg = efege = efge = egfe = gf$. Therefore, the idempotents of S form a commutative band. It follows that J is a commutative band and i is its unity.

Let j denote the zero of the semigroup J and f the unity of the group G_j (possibly $j = i$, if $S = T_i$). Since S is an inflation of the commutative band of groups G, it follows that f lies in the center of G. Therefore, the mapping ψ of S into itself defined by

$$\Psi(z) = \begin{cases} z, & \text{if} \quad z \in T_i, \\ fz, & \text{if} \quad z \notin T_i \end{cases}$$

is an endomorphism separating a and b. Consequently, ψ is an automorphism. Therefore, J contains at most two elements (i and j).

Assume that $|S \setminus G| \geqslant 2$ and x, y are distinct elements of $S \setminus G$. Consider the following two mappings of S into itself:

$$\gamma(z) = \begin{cases} z, & \text{if} \quad z \neq x, \\ x^{d+1}, & \text{if} \quad z = x; \end{cases} \qquad \eta(z) = \begin{cases} z, & \text{if} \quad z \neq y, \\ y^{d+1}, & \text{if} \quad z = y. \end{cases}$$

Since S is an inflation of G, both γ and η are endomorphisms. It is easy to see that $\gamma(S)$ and $\eta(S)$ are proper subsemigroups of S and that a and b are separated by either γ or η, which contradicts the fact that S is Q-critical. Therefore, $|S \setminus G| \leqslant 1$.

Thus, G is either a group or a commutative band of two groups and $|S \setminus G| \leqslant 1$. It remains to observe that, by Theorem 3, all subgroups of S are A-groups. Lemma 3.2 is proved.

We will also need the following five lemmas.

LEMMA 3.3. Suppose T is a Clifford semigroup, $t \in T$, $t' \notin T$, and S is an inflation of T by means of the pair (t, t'). Suppose also there exists an endomorphism ε of T such that $\varepsilon(t) = t$ and $|\varepsilon(T)| < n - 1$, where n is a fixed natural number greater than 1. Then S can be approximated by subsemigroups of orders less than n if and only if T has this same property.

Proof. Denote by φ an endomorphism of S onto T that fixes T and sends t' into t.

Assume that S can be approximated by subsemigroups of orders less than n and x, y are distinct elements of T. Then there exists an endomorphism γ of S such that $\gamma(x) \neq \gamma(y)$ and $|\gamma(S)| < n$. Since t' is not a group element of S, it follows that $\gamma(T) \subseteq T$. Consequently, x and y are separated by an endomorphism of T onto a subsemigroup of order less than n.

Conversely, suppose T can be approximated by subsemigroups of orders less than n and x, y ($x \neq y$) are elements of S. Assume that $\{x, y\} \neq \{t, t'\}$. Then $\varphi(x) \neq \varphi(y)$. Since $\varphi(x),\ \varphi(y) \in T$, there exists an endomorphism δ of T onto a subsemigroup of order less than n that separates these elements. Then the endomorphism $\delta\varphi$ separates x and y, and $|\delta\varphi(S)| < n$. Now assume that $\{x, y\} = \{t, t'\}$, i.e., $x = t$ and $y = t'$, for example. Consider the mapping $\gamma: S \to S$ acting on T like ε and sending t' into t'. This mapping is an endomorphism, inasmuch as $\varepsilon(t) = t$. Moreover, $\gamma(x) \neq \gamma(y)$ and $|\gamma(S)| = |\varepsilon(T)| + 1 < n$. Lemma 3.3 is proved.

Suppose S is a commutative band of the groups G_1 and G_2, and G_2 is an ideal of S. The mapping φ_S of G_1 into G_2 sending x into e_2x, where e_2 is the unity of G_2, is easily seen to be a homomorphism. Following [11], we will call φ_S a structural homomorphism of S.

LEMMA 3.4. Suppose S is a commutative band of the groups G_1 and G_2, where G_2 is an ideal of S, and T is a commutative band of the groups H_1 and H_2, where H_2 is an ideal of T. Suppose $\psi_1 : G_1 \to H_1$ and $\psi_2 : G_2 \to H_2$ are homomorphisms such that $\varphi_T\psi_1 = \psi_2\varphi_S$, i.e., the diagram

$$\begin{array}{ccc} G_1 & \xrightarrow{\psi_1} & H_1 \\ \downarrow{\scriptstyle\varphi_S} & & \downarrow{\scriptstyle\varphi_T} \\ G_2 & \xrightarrow{\psi_2} & H_2 \end{array} \tag{10}$$

is commutative. Then the mapping $\gamma : S \to T$ that acts on G_1 like ψ_1 and on G_2 like ψ_2 is a homomorphism. Conversely, any homomorphism $\gamma : S \to T$ such that $\gamma(G_1) \subseteq H_1$ can be constructed in this way. If $\gamma(G_1) \subseteq H_2$, then $\gamma(\mathrm{Ker}\,\varphi_S) = f_2$, where f_2 is the unity of H_2.

The proof uses a standard argument.

LEMMA 3.5. Suppose $\mathfrak{N}$ is a locally finite variety of A-groups and $\mathfrak{N}$ contains no groups of the following forms: $Z(p^6)$, $(Z(p) \mathrm{wr} Z(q)) \times Z(p^2)$, $(Z(p) \mathrm{wr} Z(q)) \times (Z(r) \mathrm{wr} Z(p))$ for primes p, q, r. Then $\mathfrak{N} \subseteq \mathfrak{A}_m\mathfrak{A}_n$, where $(m, n) = 1$, $p^2 \nmid m$, and $p^6 \nmid n$ for all primes p.

Proof. By Theorem 54.42 of [9], the varieties $\mathfrak{A}_m\mathfrak{A}_n$ are hereditarily finitely based. We may therefore assume that all proper subvarieties of $\mathfrak{N}$ lie in $\mathfrak{A}_m\mathfrak{A}_n$ for certain relatively prime m and n, where $p^2 \nmid m$ and $p^6 \nmid n$ for any prime p. Suppose G is a monolithic group in $\mathfrak{N}\backslash\mathfrak{A}_m\mathfrak{A}_n$ with monolith M. Then $\mathfrak{N} = v(G)$. One of the following two cases is possible: M is an elementary Abelian p-group or M is a direct power of a non-Abelian simple group H. Let us consider each of these cases.

1. M is an elementary Abelian p-group.

By a theorem of Taunt [12], $[C_G(M)]' \cap M = 1$. Since M is the monolith of G, it follows that $[C_G(M)]' = 1$. Consequently, $C_G(M) = S_p$, where S_p is a Sylow p-subgroup of G. Therefore, $S_p \lhd G$, hence $G = S_p \leftthreetimes H$ for some $H < G$. Since $p \nmid |H|$, it follows that v(H) is a proper subvariety of $\mathfrak{N}$. Therefore, $H \in \mathfrak{A}_m\mathfrak{A}_n$, in view of the choice of $\mathfrak{N}$. Thus, $H = A \leftthreetimes B$, where A and B are Abelian groups of exponents $m_1 | m$ and $n_1 | n$, respectively. Suppose $q \,|\, |A|$ and A_q is the Sylow q-subgroup of A, $x \in A_q$, $|x| = q$. Since $C_G(M) = S_p$, it follows that $M \cdot \langle x \rangle$ is a non-Abelian subgroup of G. It belongs to and generates $\mathfrak{A}_p\mathfrak{A}_q$, in view of Theorem 54.41 of [9]. Consequently, $\mathfrak{N}$ contains the group $Z(p) \mathrm{wr} Z(q)$ for any prime $q \,|\, |A|$. If H were non-Abelian, then by an analogous argument $\mathfrak{N}$ would contain the group $Z(q) \mathrm{wr} Z(r)$ for certain $q \,|\, |A|$ and $r \,|\, |B|$; then $\mathfrak{N}$ would contain the group $(Z(q) \mathrm{wr} Z(r)) \times (Z(p) \mathrm{wr} Z(q))$, which is impossible. Consequently, H is an Abelian group of exponent m_1n_1. Thus, $G \in \mathfrak{A}_{p^k}\mathfrak{A}_{m_1n_1}$, where p^k is the exponent of S_p. This contradicts the choice of G.

2. M is a direct power of a simple non-Abelian group H.

Suppose $p \,|\, |H|$ and S_p is a Sylow p-subgroup of H. If $C_H(S_p) = N_H(S_p)$, then, by a theorem of Newman [13], S_p would have a normal complement in H, which would contradict the simplicity of H. Therefore, $C_H(S_p) \neq N_H(S_p)$. Take $x \in N_H(S_p)\backslash C_H(S_p)$, $|x| = q^k$, where q is a prime. Then v(H) contains the variety $\mathfrak{A}_p\mathfrak{A}_q$, hence $\mathfrak{N} \ni Z(p) \mathrm{wr} Z(q)$. Thus, for any $p \,|\, |H|$ there exists a prime q such that $Z(p) \mathrm{wr} Z(q) \in \mathfrak{N}$. This easily implies the existence of primes p, q, r such that $(Z(p) \mathrm{wr} Z(q)) \times (Z(r) \mathrm{wr} Z(p))$ belongs to $\mathfrak{N}$. Lemma 3.5 is proved.

LEMMA 3.6. Suppose A and B are Abelian groups of exponent m and $p^6 \nmid m$ for any prime p. Suppose φ is a homomorphism of A into B, and let $x \in A$. Then there exist endomorphisms ψ of A and η of B such that $\varphi\psi = \eta\varphi$, $\psi(x) = x$, and the orders $|\psi(A)|$ and $|\eta(B)|$ are bounded in terms of m.

Proof. We will denote Ker (φ) by N. Assume first that the mapping φ is surjective. By Proposition 2.1, there exist in A subgroups U and V such that $A = U \times V$, $N = (N \cap U) \times (N \cap V)$, $x \in V$, $|V|$ is bounded in terms of m. Then $B = \varphi(U) \times \varphi(V)$. Let ψ denote the projection of A onto V, and η the projection of B onto $\varphi(V)$. It is easy to see that these mappings ψ and η satisfy the conclusion of the lemma.

Now assume that φ is not surjective. We define a mapping φ_1 of $A \times B$ into B by putting $\varphi_1(a, b) = \varphi(a)b$ for any $a \in A$ and $b \in B$. It is easy to see that φ_1 is a surjective homomorphism, i.e., we are in the situation covered by the previous case. Consequently, there exist endomorphisms ψ_1 of $A \times B$ and η_1 of B such that

$$\varphi_1\psi_1 = \eta_1\varphi_1, \tag{11}$$

$\psi_1(x, 1) = (x, 1)$, and the orders $|\psi_1(A \times B)|$ and $|\eta_1(B)|$ are bounded in terms of m. Let π denote the projection of $A \times B$ onto A, and γ the natural embedding of A into $A \times B$. It is easy to see that the diagrams

$$\begin{array}{ccc} A \xrightarrow{\gamma} A\times B & & A\times B \xrightarrow{\pi} A \\ \varphi\downarrow \quad \downarrow\varphi_1 & \text{and} & \varphi_1\downarrow \quad \downarrow\varphi \\ B \xrightarrow{\mathrm{id}_B} B & & B \xrightarrow{\mathrm{id}_B} B \end{array}$$

are commutative, where id_B is the identity automorphism of B. It follows from this and (11) that $\varphi(\pi\psi_1\gamma) = \eta_1\varphi$. Moreover, $\pi\psi_1\gamma(x) = x$ and the orders $|\pi\psi_1\gamma(A)|$ and $|\eta_1(B)|$ are bounded in terms of m. Thus, as ψ and η we can take $\pi\psi_1\gamma$ and η_1. Lemma 3.6 is proved.

LEMMA 3.7. Suppose G and H are groups in $\mathfrak{A}_m\mathfrak{A}_n$, where $(m, n) = 1$, $p^2 \nmid m$, and $p^6 \nmid n$ for any prime p. Suppose φ is a homomorphism of G into H, and let $x \in \mathrm{Ker}(\varphi) = N$. Then there exist endomorphisms ψ of G and η of H such that

$$\varphi\psi = \eta\varphi, \tag{12}$$

$\psi(x) \neq 1$, and the orders $|\psi(G)|$ and $|\eta(H)|$ are bounded in terms of m and n.

Proof. Assume there exist no endomorphisms ψ of G and η of H such that (12) holds, $\psi(x) \neq 1$, and either $\psi(G) \neq G$ or $\eta(H) \neq H$. We will show that the orders $|G|$ and $|H|$ are bounded in terms of m and n. From this will obviously follow the assertion of Lemma 3.7.

Suppose $G = A \leftthreetimes B$, $H = C \leftthreetimes D$, where $A, C \in \mathfrak{A}_m$, $B, D \in \mathfrak{A}_n$. According to the Schur–Zassenhaus theorem, all complements of A in G are conjugate, hence $N = (N \cap A) \leftthreetimes (N \cap B)$. Suppose $x = ab$, where $a \in A$, $b \in B$. The subgroup A can be viewed as a module over $(\mathbb{Z}/m\mathbb{Z})B$. Since $p^2 \nmid m$ for any prime p and since $(m, n) = 1$, this module is completely reducible. Assume that $A \neq 1$ and A is a nonminimal normal subgroup of G. Then it is easy to see that $A = A_1 \times A_2$, $A_1 \neq 1$, A_1 and A_2 are normal in G, $a \in A_1$, $A_2 \leqslant N$ for certain A_1 and A_2. Then $G = A_1 \leftthreetimes A_2B$. Let ψ_1 denote the projection of G onto A_2B, and η the projection of H onto D. It is easy to see that $\varphi\eta = \psi_1\varphi$ and $\psi_1(x) \neq 1$, but $\psi_1(G) \neq G$, which is impossible by assumption. Consequently, either $A = 1$ or A is a minimal normal subgroup of G. We consider only the second case; the first is handled analogously.

Suppose $N \cap A = 1$. Let ψ denote the projection of G onto B. It is easy to verify that (12) holds for ψ and the η defined in the previous paragraph. Moreover, $\psi(x) \neq 1$, since $x \notin A$. Therefore, it follows from our assumption that $\psi(G) = G$ and $\eta(H) = H$, i.e., G and H are Abelian. By Lemma 3.6, the orders $|G|$ and $|H|$ are bounded in terms of m and n. Thus, we may assume that $A \subseteq N$. Then (12) holds for id_G and η. Therefore, η is an automorphism, hence $H = D$, i.e., H is an Abelian group. Then, as in the proof of Lemma 3.6, everything can be reduced to the case where φ is surjective.

The variety $\mathfrak{A}_m\mathfrak{A}_n$ is a Cross variety [9]. By Lemma 52.21 of [9], the orders of the chief factors of groups in $\mathfrak{A}_m\mathfrak{A}_n$ are bounded in terms of m and n. Consequently, $|A|$ is bounded in terms of m and n. Denote $C_B(A)$ by F, and the index $|G:F|$ by f. By Proposition 2.1, there exist in B two subgroups U and V such that $B = U \times V$, $N \cap B = (N \cap U) \times (N \cap V)$, $b \in V$, $U \subseteq F$, and the order $|V|$ is bounded in terms of n and f, hence in terms of m and n. Since $U < F = C_B(A)$, it follows that $U \lhd G$. Therefore, $G = U \leftthreetimes AV$, $N = (N \cap U) \leftthreetimes (N \cap AV)$. Since φ is surjective, $H = \varphi(U) \times \varphi(AV)$. Let π denote the projection of G onto AV, and τ the projection of H onto $\varphi(AV)$. It is easy to see that $\varphi\pi = \tau\varphi$. Moreover, $\pi(x) = x \neq 1$ and the orders $|AV|$ and $|\varphi(AV)|$ are bounded in terms of m and n. It follows from our assumption that π and τ are automorphisms, hence the orders of G and H themselves are bounded in terms of m and n. Lemma 3.7 is proved.

Let us now turn to our variety $\mathfrak{M}$ and prove that $\mathfrak{M}$ contains only a finite number of Q-critical semigroups. This will imply that the Q-lattice of $\mathfrak{M}$ is finite.

Suppose S is a Q-critical semigroup in $\mathfrak{M}$ not isomorphic to $\mathbf{L}$, $\mathbf{L}^0$, $\mathbf{R}$, $\mathbf{R}^0$, $\mathbf{I}$. By Lemma 3.2, S is a semigroup of one of the following four types: 1) a finite A-group; 2) an inflation of a finite A-group G, where $|S\backslash G| = \{x\}$; 3) a finite commutative band of two groups G_1 and G_2; 4) an inflation of a semigroup G that is a finite commutative band of two groups G_1 and G_2, where $|S\backslash G| = \{x\}$.

The number of Q-critical semigroups of type 1) in $\mathfrak{M}$ is finite by Theorem 3.

Suppose S is a Q-critical semigroup of type 2). Then G is Abelian, otherwise v(G) contains $Z(p)\,\mathrm{wr}\,Z(q)$ for certain primes p and q, hence $\mathfrak{M} \ni N \times (Z(p)\,\mathrm{wr}\,Z(q))$, which is impossible. By Proposition 2.1, there exist in G subgroups U and V such that $G = U \times V$, $x^{d+1} \in V$, where d is the exponent of G and $|V|$ is bounded in terms of d. Let φ denote the projection of G onto V. It is easy to see that φ satisfies the conditions of Lemma 3.3 with n depending only on d. Moreover, the finite Abelian group G can be approximated by cyclic subgroups of orders at most d. Therefore, by Lemma 3.3, S can be approximated by subsemigroups of orders less than n + d. Therefore, $|S|$ is bounded in terms of d.

Suppose S is a Q-critical semigroup in $\mathfrak{M}$ of type 3). Since $\mathbf{I} < S$, it follows that $\mathfrak{M}$ cannot contain groups of the forms listed in the statement of Lemma 3.5. Therefore, G_1 and G_2 belong to $\mathfrak{A}_m\mathfrak{A}_n$, where $(m, n) = 1$, $p^2 \nmid m$, $p^4 \nmid n$ for any prime p, and $mn < d$. Let ω be the mapping of S into G_2 defined by

$$\omega(z)=\begin{cases}\varphi_{sz}, & \text{if} \quad z\in G_1,\\ z, & \text{if} \quad z\in G_2.\end{cases}$$

It is easy to verify that ω is an endomorphism of S. Since, by what has been proved, $\mathfrak{M}$ contains only a finite number of Q-critical groups, G_2 can be approximated by subgroups whose orders are bounded in terms of $\mathfrak{M}$. Consequently, any two elements a, b of S such that $\omega(a) \neq \omega(b)$ can be separated by a homomorphism of S onto a subgroup of G_2 whose order is bounded in terms of $\mathfrak{M}$. Suppose a and b are distinct elements of S and $\omega(a) = \omega(b)$. If one of them lies in G_1 and the other in G_2, then a and b can be separated by a homomorphism into the semigroup $\mathbf{I}$ embedded in S. Suppose $a, b \in G_1$ (the case $a, b \in G_2$ is impossible, since ω acts on G_2 identically). Then $x = ab^{-1} \in \operatorname{Ker}(\varphi_s)$. By Lemma 3.7, there exist endomorphisms ψ of G_1 and η of G_2 such that $\varphi_s\eta = \psi\varphi_s$, $\psi(x) \neq 1$, i.e., $\psi(a) \neq \psi(b)$, and the orders $|\psi(G_1)|$ and $|\eta(G_2)|$ are bounded in terms of m and n. By Lemma 3.4, there then exists an endomorphism γ of S whose restriction to G_1 agrees with ψ and whose restriction to G_2 agrees with η. Therefore, $\gamma(a) \neq \gamma(b)$ and $|\gamma(S)|$ is bounded in terms of m and n. Since S is Q-critical and the elements a, b were chosen arbitrarily, the order of S itself is bounded in terms of m and n, hence in terms of d. Thus, $\mathfrak{M}$ contains only a finite number of Q-critical semigroups of type 3).

Suppose S is a Q-critical semigroup in $\mathfrak{M}$ of type 4). Then $\mathfrak{M}$ contains $\mathbf{N}$ and $\mathbf{I}$. Therefore, the groups in $\mathfrak{M}$ are Abelian and d is not divisible by the sixth power of a prime. By Lemma 3.6, there exist endomorphisms ψ of G_1 and η of G_2 such that $\varphi_G\psi = \eta\varphi_G$, $\psi(x^{d+1}) = x^{d+1}$, and the orders $|\psi(G_1)|$ and $|\eta(G_2)|$ are bounded in terms of d, where d is the exponent of the group part of $\mathfrak{M}$. By Lemma 3.4, the mapping $\gamma: G \to G$ that agrees with ψ on G_1 and with η on G_2 is an endomorphism. Suppose a and b are distinct elements of G. If one of them lies in G_1 and the other in G_2, then a and b can be separated by a homomorphism onto $\mathbf{I}$. If a and b belong to G_2, they can be separated by a homomorphism of G onto a cyclic group. Suppose a and b lie in G_1. It follows from Lemmas 3.6 and 3.4 that there exists an endomorphism δ of G such that $\delta(ab^{-1}) = ab^{-1} \neq 1$, i.e., $\delta(a) \neq \delta(b)$, and $|\delta(G)|$ is bounded in terms of d. Thus, G can be approximated by subsemigroups whose orders are bounded in terms of d. Therefore, all conditions of Lemma 3.3 hold for S with some n depending only on d. Consequently, S can be approximated by subsemigroups of orders less than n. Thus, $\mathfrak{M}$ contains only a finite number of Q-critical semigroups of type 4).

Part 2. $\mathfrak{M}$ consists of right normal semigroups that are bands of inflations of Abelian groups of finite exponent.

Here we have the following

LEMMA 3.8. Suppose S is a Q-critical semigroup in $\mathfrak{M}$. Then S is isomorphic to one of the semigroups $\mathbf{R}$, $\mathbf{R}^0$, $\mathbf{I}$, $\mathbf{P}$; or S is a cyclic group; or S is an inflation of a finite Abelian group G and $|S\backslash G| = 1$; or S is a commutative band of two finite Abelian groups G_1 and G_2; or S is an inflation of a commutative band of two finite Abelian groups; or S is a commutative band of semigroups G and F, where G is a finite Abelian group, F is an inflation of a finite Abelian group H, and F is an ideal of S; or S is a chain of three finite semigroups A, H, and F, where A and F are Abelian groups, H is an inflation of an Abelian group G, F is the zero of the band, and A its unity element.

The proof is analogous to that of Lemma 3.2.

We will prove that the number of Q-critical semigroups in $\mathfrak{M}$ is finite. It is easy to see that it suffices to show that $\mathfrak{M}$ has only a finite number of Q-critical semigroups of the last type listed in Lemma 3.8. Suppose S is a Q-critical semigroup in $\mathfrak{M}$ of such a type and a, b are elements of S that cannot be separated by a homomorphism of S onto a proper subsemigroup of S. Let e, f, i denote the unity elements of the groups A, F, G, respectively, and d the exponent of the group part of $\mathfrak{M}$. Then it is easy to verify that raising to the $(d + 1)$-th power is an endomorphism of S onto $C = A \cup G \cup F$. Therefore, $a^{d+1} = b^{d+1}$ and C is a Clifford subsemigroup of S. It follows that either $a \notin C$ or $b \notin C$. Suppose $a \notin C$. Then a and b lie in H and $a \in H\backslash C$. If a has no left divisors in S, then the mapping

$$\delta(x)=\begin{cases}ix, & \text{if} \quad x\in AS,\\ x, & \text{if} \quad x\notin AS\end{cases}$$

is a homomorphism of S onto $H \cup F$ separating a and b, which is impossible. Therefore, a has left divisors in S. They, as is easily seen, lie in A. Consequently, $ea = a$. Analogously, $eb = b$. If $ga \in C$ for some $g \in A$, then

$g^{-1}ga \in C$. But then $a = ea = g^{-1}ga \in C$, which is impossible. Therefore, $Aa \subseteq H\backslash G$. Thus, A acts on the set Aa by left translations. Let B be the kernel of the corresponding representation. Since A is Abelian, B coincies with the stabilizer of a. If $Aa \neq H\backslash G$, then the mapping $\sigma : S \to S$ defined by

$$\sigma(x) = \begin{cases} x, & \text{if} \quad x \in A \cup Aa \cup G \cup F, \\ ix, & \text{if} \quad x \in H \setminus G \setminus Aa \end{cases}$$

is a homomorphism of S onto a proper subsemigroup of S and separates a and b. Therefore, $Aa = H\backslash G$. Let φ_1 and φ_2 be the structural homomorphisms of the commutative bands $A \cup G$ and $A \cup F$, respectively. As in the proof of Lemma 3.6, we can reduce everything to the case where φ_1 and φ_2 are subjective. Let C = Ker (φ_1), D = Ker (φ_2). It is easy to see that B ≤ C ≤ D. Since $\mathbf{P} \in \mathfrak{M}$, it follows that d is not divisible by the third power of a prime. If $b \in Aa$, choose an element g of A such that ga = b; if $b \notin Aa$, let g be an arbitrary nonunity element of A. By assertion C of Proposition 2.1, there exist in A subgroups U and V such that $A = U \times V$, $B = (B \cap U) \times (B \cap V)$, $C = (C \cap U) \times (C \cap V)$, $D = (D \cap U) \times (D \cap U)$, $g \in V$, and $|V|$ is bounded in terms of d. Since φ_1 and φ_2 are subjective, it follows that $A/B = UB/B \times VB/B$, $G = \varphi_1(U) \times \varphi_1(V)$, $F = \varphi_2(U) \times \varphi_2(V)$. Let π_1, π_2, π_3, π_4 denote, respectively, the projections of A onto V, A/B onto VB/B, G onto φ_1(V), and F onto φ_2(V). We define a mapping τ of S into S by putting

$$\tau(x) = \begin{cases} \pi_1(x), & \text{if} \quad x \in A, \\ \pi_2(x)\,a, & \text{if} \quad x \in Aa, \\ \pi_3(x), & \text{if} \quad x \in G, \\ \pi_4(x), & \text{if} \quad x \in F. \end{cases}$$

It is easy to verify that τ is an endomorphism of S, $\tau(a) \neq \tau$(b), and $|\tau(S)|$ is bounded in terms of d. In view of the choice of a and b, it follows that τ is an automorphism. Therefore, |S| is bounded in terms of d.

Part 3. $\mathfrak{M}$ consists of left normal semigroups that are bands of inflations of Abelian groups of finite exponent.

This case is dual to the preceding one.

Theorem 2 is completely proved.

Let us prove Theorem 1. Necessity. Suppose $\mathfrak{M}$ is a locally finite variety of semigroups with a finite Q-lattice. If $\mathfrak{M}$ contains **I**, then, by Theorem 2 and Lemma 3.5, all groups in $\mathfrak{M}$ are contained in the variety $\mathfrak{A}_m\mathfrak{A}_n$, where (m, n) = 1, $p^2 \nmid m$, and $p^6 \nmid n$ for any prime p. If $\mathfrak{M}$ contains N, then, by Theorem 2, $\mathfrak{M}$ cannot contain the variety $\mathfrak{A}_p\mathfrak{A}_q$ for any p and q, hence all groups in $\mathfrak{M}$ are Abelian. If $\mathfrak{M}$ contains **P** or **Q**, then, by Theorem 2, the exponent of the group part of $\mathfrak{M}$ is not divisible by the third power of a prime. From these assertions and part b) of Lemma 3.1 it follows at once that $\mathfrak{M}$ is generated by one of the semigroups listed in the statement of Theorem 1.

Sufficiency. It is easy to verify that the variety generated by one of the semigroups listed in the statement of Theorem 1 cannot contain the semigroups listed in the statement of Theorem 2.

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A CLASS OF SHARP INEQUALITIES FOR POLYNOMIALS, MOMENTS, AND ANALYTIC FUNCTIONS

S. Ya. Khavinson

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INTRODUCTION

It was shown in [1-3] that, in many cases, the polynomial yielding the usual best approximation in the uniform metric is extremal, and, in some specific way, it is the solution to a complicated "perturbed" extremal problem if it is assumed that the perturbations are sufficiently small. In this complicated extremal problem, not only the magnitude of the deviation of the polynomial from the function being approximated, but also the magnitudes of the coefficients of the approximating polynomial are investigated. Some further results in this direction may be found in [4]. However, the results obtained in [1-4] were established in a form which is too restricted for many concrete best-approximation problems. It is a fact that these results are used essentially as so-called "clearing" theorems (see, for example, [5]) in the theory of best approximation and are formulated in conformity with the cleared Chebyshev set. Meanwhile, in manyproblems on the approximation of complex-valued functions (for example, on the circle), the set of points at which the absolute value of the difference between the approximated function and the polynomial of best approximation attains its maximum is expressed in "uncleared" form – e.g., it consists of the entire circle.

Insofar as the possibility of invariance of the extremal property for the polynomial of best approximation is concerned, in the complex approximation problem the large size of the aforementioned set permits yet a greater chance – a fact completely omitted from consideration in the works cited above. This fact meanwhile allows us to obtain some interesting sharp inequalities, not only for polynomials, but also for analytic functions. (We recall, e.g., that the extremal fractions in the well-known Caratheodory–Fejer problem (see, e.g., [6, pp. 305-310]) have constant absolute value over all unit circles.) In particular, we obtain a sharpening of an interpolation inequality due to Dzhrbashyan [7] for analytic functions of the class H_1 in the unit disk, and we establish the fact that, under definite conditions, our inequality is sharp.

The contents of this article are divided into four sections: In Sec. 1, we briefly present some information we need from [1-3]. In Sec. 2, we establish a basic theorem to the effect that the polynomial of best approximation (in the uniform metric) also remains extremal in some complex extremal problems. As already mentioned above, this theorem is presented in such a form as to make it applicable to a series of situations which fail to satisfy the requirements of the corresponding theorems of [1-3]. (Making use of a happenstance, we remark that in [3] the proofs of Theorems 18-22 are incorrect, since Lemmas 3 and 4, formulated earlier in the article, are not true; this defect, however, is easily remedied.)

The fact of "overstability" – the invariance of an extremal element – in the problems under consideration is essentially connected with the properties of the uniform metric and is not observed in the case of the L_p metric.

In Sec. 3, on the structure provided by the basic results of Sec. 2, we strengthen the extremal property of the polynomial of best approximation in a series of known problems, and we give a dual interpretation of these facts for moments and integrals of Cauchy–Stieltjes type. In Sec. 4, we present some sharp inequalities for analytic functions of the class H_1, satisfying additional conditions.

1. PRELIMINARY CONSIDERATIONS

Let E be a locally convex real or complex linear topological space, E* the conjugate space of E, and p(x) a continuous prenorm on E. Let E_n be the n-dimensional linear topological space consisting of the points $(\lambda) = (\lambda_1, \ldots, \lambda_n)$, and $p_1(\lambda) = p_1(\lambda_1, \ldots, \lambda_n)$ is a continuous prenorm on E_n. We consider the space E_n to be

V. V. Kuibyshev Civil Engineering Institute, Moscow. Translated from Sibirskii Matematicheskii Zhurnal, Vol. 22, No. 6, pp. 188-207, November-December, 1981. Original article submitted February 25, 1980.