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DECAY RATES IN A PIEZOELECTRIC STRIP

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Abstract—Saint-Venant end effects can be characterized mathematically by the so-called decay analysis. With this decay analysis, Saint-Venant's principle can be extended to modern materials and structures beyond the traditional homogeneous and isotropic elastic solids. In the present paper, the decay rate in a piezoelectric strip is considered. In order to overcome the difficulties raised by general anisotropy, the Stroh formalism is applied here for the two-dimensional decay analysis in piezoelectricity.

1. INTRODUCTION

The decay analysis is a mathematical technique for studying the Saint-Venant's principle, which is considered as one of the cornerstones of linear elasticity. Since Saint-Venant's principle was proposed over a century ago, his statement of the principle is apparently too ambiguous to apply to some modern materials. Understanding the principle from rigorous mathematical fundamentals began in the 1960s; since then the establishment of Saint-Venant's principle based upon a rigorous mathematical formulation has been an active research topic. Thorough reviews about research progress in establishing Saint-Venant's principle were given by Horgan and Knowles [1] and a follow-up paper by Horgan [2].

The decay rate technique can turn the qualitative Saint-Venant's statement into a quantitative description via the so-called eigen-expansion. In the present paper, let us consider a non-dimensionalized strip bounded in $x_1 \in [0, \infty)$ and $x_2 \in [-1, 1]$. For this configuration, a generic field solution can be expanded as:

$$F(x_1, x_2) = F_0(x_1, x_2) + \sum_{k=1}^{\infty} C_k e^{-\lambda_k x_1} F_k(x_2) \tag{1.1}$$

where $F_0(x_1, x_2)$ is the so-called Saint-Venant solution, which is non-decaying with respect to the coordinate x_1, and all the terms in the summation represent exponential decay. In other words, all the $\mathrm{Re}(\lambda) < 0$ are dropped for a finite solution. In all these decay terms, the first term with smallest value of $\mathrm{Re}(\lambda)$ is the slowest decay term. People refer to this λ as the decay rate [2] which will decide the decay distance of end effects.

The present paper introduces the Saint-Venant's principle into piezoelectric materials by finding the decay rate in a piezoelectric strip. Since plate-shaped piezoelectric devices have been widely used in resonators (quartz) and sensor, the analysis for a strip configuration is of great theoretical as well as practical importance. In the present paper, we focus on a linear piezoelectric material whose constitutive equation is given by:

$$\sigma = C\gamma - eE, \qquad D = e\gamma + \varepsilon E \tag{1.2}$$

where C is fourth rank elastic stiffness tensor, ε the second rank permittivity tensor and e the third rank piezoelectricity tensor. When the piezoelectricity tensor vanishes, any electro-mechanical problem is decoupled into an anisotropic elastic one and a dielectric one. It is apparent from equation (1.2) that the piezoelectric material must be anisotropic because the piezoelectric tensor, being of rank three, would vanish for isotropic materials.

Due to the anisotropy of the piezoelectric material, analysis of piezoelectric solids is, in general, difficult. Some results for piezoelectric plate vibrations [3] illustrate the great complexity introduced by the anisotropy. Recent development of two-dimensional anisotropic elasticity via Stroh's formalism [4, 5] makes the two-dimensional analysis for piezoelectric solids possible. In the following sections, the Stroh's formalism for the anisotropic elasticity will be extended to the piezoelectric strip. Although there are other approaches in the literature, the Stroh formulation has been preferred in modern research in anisotropic elasticity. The topics range from dislocations [4, 5], and surface waves [6] and to interfacial cracks [7, 8]. A detailed review of this approach and its applications is beyond the scope of the present paper, but references on these topics with Stroh's notation can be found in above mentioned works. A modified Stroh's formulation for piezoelectricity was applied by Barnett and Lothe [9] and Suo *et al.* [10] in their dislocation and crack studies, and by Lothe and Barnett [11, 12] to surface waves in piezoelectric half-spaces. In the following sections, Stroh's formulation is applied to the decay analysis.

2. STROH'S FORMULATION FOR PIEZOELECTRIC MATERIALS

2.1 Basic equations for piezoelectricity

In a rectangular coordinate system the linear piezoelectric solid is described by:
Constitutive laws:

$$\sigma_{ij} = C_{ijkl}\gamma_{kl} - e_{kij}E_k \qquad (2.1)$$

$$D_i = e_{ikl}\gamma_{kl} + \varepsilon_{ik}E_k$$

where σ_{ij}, γ_{ij}, D_i and E_k are stress, strain, electric displacement induction and electric field, respectively.
Deformation relations:

$$\gamma_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k}), \qquad (2.2)$$

$$E_k = -\varphi_{,k}$$

where u_k and φ are mechanical displacement and electric potential.
Equilibrium equations:

$$\sigma_{ij,i} = 0,$$

$$D_{i,i} = 0 \qquad (2.3)$$

where no body force and electric charge are assumed. Substituting equations (2.1) and (2.2) into equation (2.3) yields:

$$(C_{ijkl}u_k + e_{lji}\varphi)_{,li} = 0$$

$$(e_{ikl}u_k - \varepsilon_{il}\varphi)_{,li} = 0. \qquad (2.4)$$

If all the field variables are independent of the third coordinate, say x_3, solution can be assumed as:

$$\mathbf{U} = \{u_k, \varphi\}^T = \mathbf{a}f(\zeta_1 x_1 + \zeta_2 x_2), \qquad (2.5)$$

where, without loss generality,

$$\zeta_1 = 1, \qquad \zeta_2 = p. \qquad (2.6)$$

A direct substitution of equation (2.5) into (2.4) gives

$$(C_{\alpha ik\beta}a_k + e_{\alpha i\beta}a_4)\zeta_\alpha\zeta_\beta = 0$$

$$(e_{\alpha k\beta}a_k - \varepsilon_{\alpha\beta}a_4)\zeta_\alpha\zeta_\beta = 0. \qquad (2.7)$$

For non-zero **a**'s, we must have:

$$\det\begin{bmatrix} C_{\alpha jk\beta}\zeta_\alpha\zeta_\beta & e_{\alpha j\beta}\zeta_\alpha\zeta_\beta \\ e_{\alpha k\beta}\zeta_\alpha\zeta_\beta & -\varepsilon_{\alpha\beta}\zeta_\alpha\zeta_\beta \end{bmatrix}=0. \tag{2.8}$$

This is an eigenvalue problem for p. It can be proved that the eigenvalue p cannot be purely real due to the positive definiteness of tensors C_{ijkl} and ε_{ij} [10]. Four pairs of p can be arranged as:

$$p_{I+4}=\bar{p}_I, \qquad (I=1,2,3 \text{ and } 4)$$

$$p_I=\alpha_I+i\beta_I, \qquad \beta_I>0. \tag{2.9}$$

Corresponding to eigenvalues $p_I=\alpha_I+i\beta_I$, there are four independent eigenvectors which form a 4×4 matrix, namely:

$$\mathbf{A}=\{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4\}. \tag{2.10}$$

The complex conjugates

$$\bar{\mathbf{A}}=\{\bar{\mathbf{a}}_1,\bar{\mathbf{a}}_2,\bar{\mathbf{a}}_3,\bar{\mathbf{a}}_4\} \tag{2.11}$$

are the eigenvectors corresponding to $p_{I+4}=\bar{p}_I$. It is noted that matrix **A** is non-singular for distinct eigenvalues. However, **A** may be singular in those cases in which nonsemisimple degeneracy of the eigenvalues occurs; such cases can be treated by constructing generalized eigenvectors. We will take **A** as non-singular in the present work. The degenerate cases will be discussed elsewhere.

2.2 Eigenexpansion for the piezoelectric strip

With the concept expressed in equation (1.1), the decay terms in the displacement and electric potential can be expressed as:

$$\mathbf{U}=\sum_{k=1}^{\infty} C_k e^{-\lambda_k x}\mathbf{U}^{(k)}(y). \tag{2.12}$$

This is called the eigen-expansion for the decay analysis in which λ_k and $\mathbf{U}^{(k)}$ are eigenvalues and eigenfunctions. From the previous subsection, we know that the solution can be written in terms of complex variables $x+p_I y$ and their complex conjugates. A general solution is the superposition of the eight independent solutions corresponding to these eight eigenvalues, p_I and $\bar{p}_I$, i.e.

$$\mathbf{U}^{(k)}(y)=(\mathbf{A}\langle e^{-\lambda_k p_I y}\rangle\mathbf{q}_k+\bar{\mathbf{A}}\langle e^{-\lambda_k \bar{p}_I y}\rangle\mathbf{h}_k) \tag{2.13}$$

where

$$\langle e^{-\lambda p_I y}\rangle=\mathrm{diag}\{e^{-\lambda p_1 y}, e^{-\lambda p_2 y}, e^{-\lambda p_3 y}, e^{-\lambda p_4 y}\}. \tag{2.14}$$

The eigenvalue λ, $\mathbf{q}_k$ and $\mathbf{h}_k$ will be determined from the homogeneous boundary conditions along the upper and lower surfaces ($y=\pm 1$) of the strip.

Upon the substituting the displacement and electric potential into the constitutive equations, the stress and induction components can be expressed as:

$$\mathbf{t}^{(k)}(y)=\{\sigma_{2i},D_2\}^T=\lambda_k(\mathbf{B}\langle e^{-\lambda_k p_I y}\rangle\mathbf{q}_k+\bar{\mathbf{B}}\langle e^{-\lambda_k \bar{p}_I y}\rangle\mathbf{h}_k) \tag{2.15}$$

$$\{\sigma_{1i},D_1\}^T=\lambda_k(\mathbf{BP}\langle e^{-\lambda_k p_I y}\rangle\mathbf{q}_k+\overline{\mathbf{BP}}\langle e^{-\lambda_k \bar{p}_I y}\rangle\mathbf{h}_k) \tag{2.16}$$

where

$$\mathbf{B}=\{\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3,\mathbf{b}_4\}, \qquad \mathbf{P}=\mathrm{diag}\{p_1,p_2,p_3,p_4\}. \tag{2.17}$$

With the equations in Section 2.1, one has

$$\mathbf{b}_j=(C_{2jk\beta}a_k+e_{\beta j2}a_4)\zeta_\beta, \qquad \mathbf{b}_4=(-\varepsilon_{1\beta}a_4+e_{2k\beta}a_k)\zeta_\beta. \tag{2.18}$$

This section can be rewritten by using an "eight-dimensional formulation" as Barnett and Lothe [9], which is given in the Appendix.

It is worthwhile to emphasize that there are two eigenvalue problems in the method presented here. The first one equation (2.8) is due to the anisotropy of the piezoelectric material. The eigenvalues are determined solely by material constants, and the eigen-space is spanned by eight eigenvectors. On the hand, the second eigenvalue problem is introduced by the decay analysis [equation (2.12)]. There are an infinite number of eigenvalues and eigenfunctions for this eigenvalue problem. Determination of the eigenvalue and eigenfunction for the second problem will depend on the boundary conditions at $y = \pm 1$.

In the eigenexpansion equation (2.12), the eigenvalue λ_k with the smallest real part is of interest for studies of Saint-Venant's principle. We will focus on this term and drop the superscripts and subscript on the eigenvalue and eigenfunction to avoid unnecessary notation.

3. BOUNDARY CONDITIONS AND DETERMINATION OF THE EIGENVALUES λ_k

3.1 Boundary conditions along the upper and lower surfaces of the strip

For a traditional Saint-Venant problem, the boundary conditions along the upper and lower surface of the strip are that these surfaces be traction-free. However, there are other alternatives for homogeneous boundary conditions along the surfaces. Wang *et al.* [13] posed a total of *eight* possible boundary conditions for pure elastic problems with mixed displacement and traction components. For the present piezoelectric strip, the boundary conditions along the upper and lower surfaces can be one of the following *sixteen* alternatives. The boundary conditions are formed by selecting only one variable from the following groups:

$$(\sigma_{21}, u_1), \qquad (\sigma_{22}, u_2), \qquad (\sigma_{23}, u_3), \qquad (D_2, \varphi). \tag{3.1}$$

For instance, one of the choices is:

$$\sigma_{21} = 0, \qquad \sigma_{22} = 0, \qquad \sigma_{23} = 0, \qquad \varphi = 0 \quad \text{at} \quad y = 1/-1. \tag{3.2}$$

A generalized matrix representation of the homogeneous boundary condition is

$$\mathbf{I}_u \mathbf{U}'_x + \mathbf{I}_t \mathbf{t} = \mathbf{0} \tag{3.3}$$

where $\mathbf{I}_u$ and $\mathbf{I}_t$ are 4×4 diagonal matrices whose diagonal elements are either one or zero. As an example, the matrices for equation (3.2) are:

$$\mathbf{I}_u = \text{diag}\{0, 0, 0, 1\}$$
$$\mathbf{I}_t = \text{diag}\{1, 1, 1, 0\}. \tag{3.4}$$

3.2 Determination of the eigenvalue λ

Substituting equations (2.13) and (2.15) into equation (3.3) for a given boundary condition, one obtains:

$$\mathbf{K}_+ \langle e^{-\lambda p} \rangle \mathbf{q} + \bar{\mathbf{K}}_+ \langle e^{-\lambda \bar{p}} \rangle \mathbf{h} = 0 \qquad y = 1 \tag{3.5}$$

$$\mathbf{K}_- \langle e^{\lambda p} \rangle \mathbf{q} + \bar{\mathbf{K}}_- \langle e^{\lambda \bar{p}} \rangle \mathbf{h} = 0 \qquad y = -1 \tag{3.6}$$

where

$$\mathbf{K} = \mathbf{I}_u \mathbf{A} + \mathbf{I}_t \mathbf{B}. \tag{3.7}$$

It is noted that the lower and upper surface may be posted different boundary conditions. Thus, $\mathbf{K}_+$ and $\mathbf{K}_-$ are not necessarily the same.

It can be proved that matrix $\mathbf{K}$ is non-singular. The proof is similar to the lower dimensional anisotropic elasticity case which was provided by Wang *et al.* [13]. Bearing this in mind, we find the eigenvalue equation of the eigenvalue λ permitting non-trivial $\mathbf{q}$ and $\mathbf{h}$ to be

$$\det(\mathbf{K}_+\langle e^{-2\lambda p}\rangle\mathbf{K}_-^{-1} - \bar{\mathbf{K}}_+\langle e^{-2\lambda\bar{p}}\rangle\bar{\mathbf{K}}_-^{-1}) = 0. \tag{3.8}$$

In general, the solution of equation (3.8) involves numerical procedures. Some simple examples are discussed in the next section.

3.3 Boundary conditions at the end ($x = 0$)

End conditions are needed to determined the participation factors C_k in the eigen-expansion equation (2.12) with aid of orthogonality of the eigenfunctions. The end condition for a traditional Saint-Venant's principle is one of "self balance". For instance,

$$\int_{-1}^{+1} \sigma_{1i}(0, x_2)\,dx_2 = 0, \qquad \int_{-1}^{+1} x_2\sigma_{11}(0, x_2)\,dx_2 = 0 \tag{3.9}$$

is the self-balanced end loading in the elasticity. However, this condition is not necessarily valid for cases in which the boundary conditions along the upper and lower surface are displacement boundary conditions. Since our primary interest of the present paper is find the decay rate, such boundary condition will not be studied in the present work. The orthogonality of the eigenfunction is discussed in the Appendix where the eight-dimensional matrix scheme is introduced. A detailed discussion of the end condition will be presented elsewhere.

4. EXAMPLES OF DECAY RATE IN PIEZOELECTRIC STRIP

4.1 Decay rate in a dielectric strip

When the piezoelectric tensor vanishes, the problem decouples into pure anisotropic elastic and dielectric ones. The former, the elastic strip, has been considered by Wang *et al.* [13] for a general anisotropic two-dimensional elastic strip via Stroh's formulation. The problem of a dielectric strip, which has not been reported in the literature, will be presented here.

For case of $\mathbf{e} = \mathbf{0}$, the eigenvalue problem equation (2.7) is simplified as

$$C_{\alpha ik\beta}\zeta_\alpha\zeta_\beta a_k = 0,$$

and

$$\varepsilon_{\alpha\beta}\zeta_\alpha\zeta_\beta a_4 = 0. \tag{4.1}$$

The corresponding eigenvectors are in the form of:

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_e & 0 \\ 0 & a_4 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} \mathbf{B}_e & 0 \\ 0 & b_4 \end{pmatrix} \tag{4.2}$$

where $\mathbf{A}_e$ and $\mathbf{B}_e$ are 3×3 matrices for anisotropic elasticity. The dielectric problem is formed by equation (4.1b) and scalars in equation (4.2). The eigenvalue corresponding to the anisotropic dielectricity is obtained from equation (4.1b) as

$$\varepsilon_{11} + 2\varepsilon_{12}p + \varepsilon_{22}p^2 = 0, \tag{4.3}$$

whose roots are:

$$p_4 = -\frac{\varepsilon_{12}}{\varepsilon_{22}} + i\sqrt{\frac{\varepsilon_{11}}{\varepsilon_{22}} - \left(\frac{\varepsilon_{12}}{\varepsilon_{22}}\right)^2}, \qquad p_8 = \bar{p}_4. \tag{4.4}$$

The imaginary part of p_4 is positive because the permitivity tensor is positive definite.

For the electric potential version decay analysis, the boundary conditions along the upper and lower surfaces of the strip are:

$$\varphi = 0, \qquad \text{at} \qquad y = \pm 1. \tag{4.5}$$

After some straightforward calculations, the equation for the eigenvalue equation (3.8) corresponding to dielectricity part is obtained as:

$$e^{-2\lambda p_4} - e^{-2\lambda \bar{p}_4} = 0. \tag{4.6}$$

The solution is:

$$2\lambda \operatorname{Im}(p_4) = n\pi. \tag{4.7}$$

The smallest root, which is normally called the decay rate, is

$$\lambda = \frac{\pi}{2 \operatorname{Im}(p_4)}. \tag{4.8}$$

It is noted that degree of the electromechanical coupling in a piezoelectric material can be described by a dimensionless parameter formed by the three types of moduli, which is generally in the range:

$$\frac{e}{\sqrt{\varepsilon C}} = 0.1 \sim 1. \tag{4.9}$$

Weakly coupled material, such as quartz which is widely used as frequency filter and resonator has moduli on the order of (see Salt [14]):

$$C \sim 10^{11}\ \mathrm{N/m^2}, \qquad \varepsilon \sim 10^{-11}\ \mathrm{F/m^2}, \qquad e \sim 10^{-1}\ \mathrm{C/m^2}. \tag{4.10}$$

It is seen that:

$$\frac{e}{\sqrt{\varepsilon C}} \sim 0.1.$$

The decay rate for this kind of weakly coupled piezoelectric material can be approximated by the decoupled elasticity and dielectricity.

On the other hand, the strongly electromechanical coupled material, such as lead–zirconate–titanate (say PZT-5H), has moduli on the order of:

$$C \sim 10^{11}\ \mathrm{N/m^2}, \qquad \varepsilon \sim 10^{-8}\ \mathrm{F/m^2}, \qquad e \sim 10\ \mathrm{C/m^2}. \tag{4.11}$$

The appropriate dimensionless parameter is in the order of:

$$\frac{e}{\sqrt{\varepsilon C}} \sim 0.3.$$

For this kind of material, the decay rate has to be calculated based upon the fully coupled formulation.

4.2 Decay rate in a piezoelectric material with transverse symmetry around the poling axis

The above mentioned PZT-5H belongs to this material category. Assuming the x–y plane is the isotropic plane, the material constants of this material are:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (c_{11}-c_{12})/2 \end{bmatrix} \begin{bmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{33} \\ 2\gamma_{32} \\ 2\gamma_{13} \\ 2\gamma_{12} \end{bmatrix} - \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

and

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{33} \\ 2\gamma_{32} \\ 2\gamma_{13} \\ 2\gamma_{12} \end{bmatrix} + \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}.$$

Let us consider a two-dimensional problem in the (x, y)-plane, the in-plane deformation (u_x, u_y) is decoupled from the anti-plane field (u_z, φ). The former is identical to an elastic problem. We focus on the latter. We find that:

$$p_3 = p_4 = i, \qquad \bar{p}_7 = \bar{p}_8 = -i \tag{4.12}$$

and

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{2e} & 0 \\ 0 & \mathbf{A}_p \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \mathbf{B}_{2e} & 0 \\ 0 & \mathbf{B}_p \end{bmatrix} \tag{4.13}$$

where the right upper 2×2 matrices correspond to in-plane deformation, while the left lower corner matrices correspond to the coupled anti-plane deformation and electric field. It is straightforward to deduce that:

$$\mathbf{A}_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{B}_p = i \begin{bmatrix} c_{44} & e_{15} \\ e_{15} & -\varepsilon_{11} \end{bmatrix}. \tag{4.14}$$

The boundary conditions along the upper and lower surfaces are taken as:

$$\varphi = 0, \quad \text{and} \quad \sigma_{23} = 0. \tag{4.15}$$

Thus,

$$\mathbf{K}_p = \mathbf{I}_u \mathbf{A}_p + \mathbf{I}_t \mathbf{B}_p = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + i \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_{44} & e_{15} \\ e_{15} & -\varepsilon_{11} \end{bmatrix} = \begin{bmatrix} ic_{44} & ie_{15} \\ 0 & 1 \end{bmatrix}. \tag{4.16}$$

The equation for the eigenvalues, equation (3.8), is reduced to:

$$e^{2\lambda i} - e^{-2\lambda i} = 0. \tag{4.17}$$

The solution is:

$$2\lambda = n\pi. \tag{4.18}$$

With a decay rate

$$\lambda_{\text{smallest}} = \pi/2. \tag{4.19}$$

5. SUMMARY AND REMARK

Saint-Venant's end effect in a piezoelectric strip is studied via the eigen-expansion equation (1.1). The Stroh formalism is applied here to overcome the complexity raised by the anisotropy associated with the piezoelectric effect. In a general piezoelectric material, numerical procedures will be involved in solving equations (2.8) and (3.8) since there is no closed form solution available for general anisotropy.

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REFERENCES

[1] C. O. HORGAN and J. KNOWLES, *Adv. Appl. Mech.* **23**, 180 (1983).
[2] C. O. HORGAN, *Appl. Mech. Rev.* **42**, 295 (1989).
[3] H. F. TIERSTEN, *Linear Piezoelectric Plate Vibrations*. Plenum Press, New York (1969).
[4] A. N. STROH, *Phil. Mag.* **3**, 625 (1958).
[5] A. N. STROH, *J. Math. Phys.* **41**, 77 (1962).
[6] D. M. BARNETT and J. LOTHE, *Proc. R. Soc. Lond.* **A402**, 135 (1985).
[7] T. C. T. TING, *Int. J. Solids Structures* **22**, 965 (1986).
[8] Z. SUO, *Proc. R. Soc. Lond.* **A427**, 331 (1990).
[9] D. M. BARNETT and J. LOTHE, *Physica Statist. Solidi* (*b*) **67**, 105 (1975).
[10] Z. SUO, C. M. KUO, D. M. BARNETT and J. R. WILLIS, *J. Phys. Mech. Solids* **40**, 739 (1992).
[11] J. LOTHE and D. M. BARNETT, *Physica Norwegica* **8**, 239 (1976).
[12] J. LOTHE and D. M. BARNETT, *J. Appl. Phys.* **47**, 1799 (1976).
[13] M. Z. WANG, T. C. T. TING and G. YAN, *Q. Appl. Math.* **L1**, 283 (1993).
[14] D. SALT, *Hy-Q: Handbook of Quartz Crystal Devices*. Van Nostrand-Reinhold (1987).

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APPENDIX

Eight-Dimensional Representation of Piezoelectricity

A more rigorous and computational useful formulation can be developed out via an eight-dimensional scheme which was introduced by Barnett and Lothe [9]. In the following eight-dimensional formulation, the lowercase subscripts take on the range 1, 2 and 3, while the uppercase subscripts take on the range 1, 2, 3 and 4. We introduce some auxiliary quantities in order to do the matrix calculation, namely

$$Z_{Mn} = \begin{cases} \gamma_{mn} & M = 1, 2, 3 \\ -E_n & M = 4 \end{cases} \tag{A1}$$

$$\Sigma_{Mn} = \begin{cases} \sigma_{mn} & M = 1, 2, 3 \\ D_n & M = 4 \end{cases} \tag{A2}$$

$$U_M = \begin{cases} u_m & M = 1, 2, 3 \\ \phi & M = 4 \end{cases} \tag{A3}$$

$$E_{iJMn} = \begin{cases} C_{ijmn} & J, M = 1, 2, 3 \\ e_{nij} & J = 1, 2, 3; M = 4 \\ e_{imn} & J = 4; M = 1, 2, 3 \\ -\varepsilon_{in} & J, M = 4 \end{cases} \tag{A4}$$

It should be pointed out that these quantities are not tensors, and one has to be careful when transforming coordinates.

With these new matrices, the constitutive equation [equations (2.1) and (2.2)] is written as

$$\Sigma_{iJ} = E_{iJMn} Z_{Mn} = E_{iJMn} U_{M,n}. \tag{A5}$$

The equilibrium equation [equation (2.3)] is written as:

$$\Sigma_{iJ,i} = 0. \tag{A6}$$

To satisfy equation (A6), a stress function $\Phi = \{\Phi_1, \Phi_2, \Phi_3, \Phi_4\}^T$ is introduced:

$$\Sigma_{1J} = -\Phi_{J,2} \qquad \Sigma_{2J} = \Phi_{J,1} \tag{A7}$$

Substituting equation (A7) into equation (A6), one obtains:

$$\begin{aligned} \mathbf{QU}_{,1} + \mathbf{RU}_{,2} &= -\Phi_{,2} \\ \mathbf{R}^T\mathbf{U}_{,1} + \mathbf{TU}_{,2} &= \Phi_{,1} \end{aligned} \tag{A8}$$

where

$$Q_{JM} = E_{1JM1}, \qquad R_{JM} = E_{1JM2}, \qquad T_{JM} = E_{2JM2}. \tag{A9}$$

Equation (A8) can be rewritten in an eight-dimensional form as

$$\frac{\partial \mathbf{w}}{\partial y} = \mathbf{N} \frac{\partial \mathbf{w}}{\partial x} \tag{A10}$$

where

$$\mathbf{w} = \begin{pmatrix} \mathbf{U} \\ \Phi \end{pmatrix}, \qquad \mathbf{N} = \begin{pmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{pmatrix} \tag{A11}$$

$$\begin{aligned} \mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^T, &\qquad \mathbf{N}_2 = \mathbf{T}^{-1} = \mathbf{N}_2^T \\ \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T - \mathbf{Q} &= \mathbf{N}_3^T. \end{aligned} \tag{A12}$$

The inverse of matrix $\mathbf{T}$ is obtained based on the following argument. From equation (A4) and equation (A9), we have

$$\mathbf{T} = \begin{pmatrix} \mathbf{T}_e & \mathbf{e} \\ \mathbf{e}^T & -\varepsilon_{22} \end{pmatrix} \tag{A13}$$

where the upper left corner of the matrix, $\mathbf{T}_e$, is made using part of the elastic tensor which is 3×3 positive definite [7]; the lower right corner stems from the dielectric tensor and is seem to be a negative scalar; and $\mathbf{e}$ is formed by components of piezoelectric tensor. It is found that

$$\mathbf{T}^{-1} = \begin{pmatrix} \mathbf{T}_e^{-1}(\mathbf{I} + q\mathbf{e}\mathbf{e}^T\mathbf{T}_e^{-1}) & -q\mathbf{T}_e^{-1}\mathbf{e} \\ -q\mathbf{e}^T\mathbf{T}_e^{-1} & q \end{pmatrix} \tag{A14}$$

where, using the positive definiteness of $\mathbf{T}_e$,

$$q = \frac{1}{-\varepsilon_{22} - \mathbf{e}^T\mathbf{T}_e^{-1}\mathbf{e}} < 0. \tag{A15}$$

In order to diagonalize equation (A10), the following eigenvalue problem is considered:

$$\mathbf{N}\xi = p\xi \tag{A16}$$

The eigenvalue is obtained by solving:

$$\det(\mathbf{N} - p\mathbf{I}) = 0. \tag{A17}$$

The explicit form of (A17) is the exactly same as equation (2.8). When we consider a decay field, the field is expanded as:

$$\mathbf{w}(x_1, x_2) = \sum_{k=1}^{\infty} C_k \mathbf{w}^{(k)}(x_2)\mathrm{e}^{-\lambda_k x_1}. \tag{A18}$$

Substituting (A18) into (A10), we have

$$\frac{\mathrm{d}\mathbf{w}^{(k)}}{\mathrm{d}x_2} = -\lambda_k \mathbf{N}\mathbf{w}^{(k)}. \tag{A19}$$

Solution of equation (A19) is of the form

$$\mathbf{w}^{(k)}(x_2) = \mathrm{e}^{-\lambda_k p x_2}\xi \tag{A20}$$

where ξ is the eigenvector in equation (A16).

The orthogonality of the eigenfunctions in equation (A18) is shown by considering:

$$\frac{\mathrm{d}}{\mathrm{d}x_2}((\mathbf{w}^{(-m)})^T\mathbf{J}\mathbf{w}^{(k)}) = (\lambda_m - \lambda_k)(\mathbf{w}^{(-m)})^T\mathbf{J}\mathbf{N}\mathbf{w}^{(k)} \tag{A21}$$

where

$$\mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}, \qquad \mathbf{N}^T\mathbf{J} = \mathbf{J}\mathbf{N} \tag{A22}$$

have been used. Integrating both sides of equation (A21) with respect to x_2 from $(-1, 1)$, we find that

$$\{\mathbf{U}^{(-m)T}\Phi^{(k)} + \Phi^{(-m)T}\mathbf{U}^{(k)}\}_{-1}^{1} = (\lambda_m - \lambda_k)\int_{-1}^{1} (\mathbf{w}^{(-m)})^T\mathbf{J}\mathbf{N}\mathbf{w}^{(k)}\,\mathrm{d}x_2 \tag{A23}$$

The left-hand side vanishes because of the side boundary conditions listed in equation (3.1). Thus, the orthogonality of the eigen-functions is obtained. This derivation is similar to that of Wang *et al.* [13] for the pure anisotropic elasticity problem. With this orthogonality condition, the participation factors C_k in equation (A18) can be determined. However, when we are only concerned with the decay rate in the strip, only the first eigenvalue is needed, and the participation factors are unimportant.