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Smooth Irrotational Flows in the Large to the Euler–Poisson System in $\mathbf{R}^{3+1}$

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Abstract: A simple two-fluid model to describe the dynamics of a plasma is the Euler–Poisson system, where the compressible electron fluid interacts with its own electric field against a constant charged ion background. The plasma frequency produced by the electric field plays the role of "mass" term to the linearized system. Based on this "Klein–Gordon" effect, we construct global smooth irrotational flows with small velocity for the electron fluid.

1. Introduction

A plasma is a collection of moving electrons and ions. At high frequencies, a simple-fluid model for a plasma breaks down. The electrons and ions tend to move independently, and charge separations occur. The greater inertia of the ions implies that they will be unable to follow the rapid fluctuation of the fluid, only electrons partake in the motion. The ions merely provide a uniform background of positive charge. One of the simplest two-fluid model for a plasma is the Euler–Poisson system

$$\partial_t n + \nabla \cdot (nu) = 0, \tag{1}$$
$$\partial_t u + u \cdot \nabla u + \frac{1}{m_e n}\nabla p(n) = \frac{e}{m_e}\nabla\phi$$

with the electric field $\nabla\phi$ which satisfies the Poisson system

$$\Delta\phi = 4\pi e(n - n_0), \text{ with } |\phi| \to 0, \text{ as } |x| \to \infty. \tag{2}$$

Here, the electrons of charge e and mass m_e are described by a density $n(t,x)$ and an average velocity $u(t,x)$. The constant equilibrium-charged density of ions and electrons is $\pm e n_0$. We assume the pressure is

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$$p(n) = An^{\gamma}$$

with $\gamma > 1$, and A is a constant (The case $\gamma = 1$ can be treated too). Throughout this paper, we consider an irrotational flow, that is

$$\nabla \times u \equiv 0 \tag{3}$$

which is invariant for all time. Therefore, we deduce, from (2), (1) and (3), that the dynamic equation of $\nabla\phi$ is

$$\partial_t \nabla\phi = -4\pi e \nabla\Delta^{-1}\nabla \cdot [nu] = -4\pi e\{n_0 u + \nabla\Delta^{-1}\nabla \cdot [(n - n_0)u]\}. \tag{4}$$

We study the dynamic problem of (1) and (4), together with the constraint (2) at time $t = 0$. It follows that the Poisson equation (2) holds for all time. There is a quiet fluid equilibrium for (1) and (2) of electrons: $n \equiv n_0$, $u \equiv 0$ ($E \equiv 0$).

In the absence of the electric field $\nabla\phi$, the Euler–Poisson system reduces to the well-known Euler equations for compressible fluids. Despite many important progresses over the years (especially in 1-D), the existence and uniqueness of global solutions in the 3+1 dimension remains an outstanding open problem.

Consider smooth, irrotational initial data which are small perturbations of a quiet fluid $n \equiv n_0$, $u \equiv 0$. In general, it is well-known that singularity (shock waves) develops in finite time for the pure Euler equations [Si1]. On the other hand, although it seems more complicated, in contrast, we demonstrate that these initial data lead to smooth, irrotational solutions to the Euler–Poisson system for all time.

Theorem 1. *Let $\rho(x) \in C_c^{\infty}(\mathbf{R}^3)$ and vector value function $\upsilon(x) \in C_c^{\infty}(\mathbf{R}^3)$ with*

$$\nabla \times \upsilon = \text{(irrotationality)}, \qquad \int_{\mathbf{R}^3} \rho \mathrm{dx} = \text{(neutrality)}.$$

Then there exist $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, there exist unique smooth solutions $(n^{\epsilon}(t,x), u^{\epsilon}(t,x))$ to the Euler–Poisson system (1), (4) and (2) for $0 \le t < \infty$ with initial data $(\epsilon\rho + n_0, \epsilon\upsilon)$. Moreover, $n^{\epsilon}(t,x) - n_0$ and $u^{\epsilon}(t,x)$ decays, uniformly in $t \ge 0$, $x \in \mathbf{R}^3$, as $(1+t)^{-p}$ for any $1 < p < \frac{3}{2}$.

In particular, (3) holds for radially symmetric data $\rho = \overline{\rho}(|x|)$, $\upsilon(x) = \overline{\upsilon}(|x|)\frac{x}{|x|}$, with scalar functions $\overline{\rho} \in C_c^{\infty}(\mathbf{R}^3)$ and $\overline{\upsilon} \in C_c^{\infty}(\mathbf{R}^3 \backslash \{0\})$. The neutral condition is sufficient for some non-local requirements for the data. More general statements can be found in Theorem 9.

For the pure Euler equations, the life span of classical irrotational flow with initial data $(\epsilon\rho + n_0, \epsilon\upsilon)$ is of the order $O(\exp(\frac{1}{\epsilon}))$ [Si2], since irrotational solutions of the linearized Euler equations decay like that of $(1+t)^{-1}$, which is not integrable. On the other hand, due to the interaction with its own electric field, the linearized Euler–Poisson system for irrotational flows take the form

$$\partial_{tt} n - \frac{1}{m_e} p'(n_0)\Delta n + \frac{4\pi e^2 n_0}{m_e} n = 0,$$
$$\partial_{tt} u - \frac{1}{m_e} p'(n_0)\Delta u + \frac{4\pi e^2 n_0}{m_e} u = 0.$$

The new "mass" term $\frac{4\pi e^2 n_0}{m_e}$, which is absent in the pure Euler system, comes from (3) and (4). Notice that $\frac{4\pi e^2 n_0}{m_e} = \omega_p^2$, where ω_p, the plasma frequency for plasma oscillations, characterizes one of the fundamental features of a plasma. It is well-known [MSW, Str] that the linear Klein–Gordon equation has a better decay rate of $(1+t)^{-3/2}$. In 1985, global smooth solutions with small amplitude to general scalar, quasi-linear Klein–Gordon equations were constructed independently by Klainerman [K] and Shatah [Sh] via two different methods. Due to the non-local complications (2), it is difficult to directly employ the vector field method of Klainerman. Instead, we modify Shatah's method of a normal form in a L^p ($p>1$) setting, and use a $L^p - L^\infty$ estimate of [N] to construct global solutions.

For pure Euler equations for compressible fluids in $\mathbf{R}^{3+1}$, global weak solutions with radial symmetry and large amplitude have been constructed [CG, MMU], outside the ball $|x| \geq 1$. There are some works for the Euler–Poisson system (1), (4) and (2). Cordier et al [CDMS] constructed interesting steady states solutions in one space dimension. On the other hand, many mathematicians have made contributions to the related Euler–Poisson model in semiconductor physics with a momentum relaxation. See [CW, DM, G, Pe, PRV, WC and Z] for more references on that subject.

2. Reformulation of the Problem

For notational simplicity, we set all physical constants $e, m_e, 4\pi$ and A to be one. It is convenient to introduce new variables to simplify the forth-coming energy estimates. We consider smooth irrotational flows near the equilibrium $n \equiv n_0, u \equiv 0$ and $\nabla\phi \equiv 0$. As in [Si2], if (n,u) is a smooth solution of the Euler–Poisson system, we define

$$m(t,x) = \frac{2}{\gamma-1}[(\frac{n(\frac{t}{c_0}, x)}{n_0})^{(\gamma-1)/2} - 1], v(t,x) = \frac{1}{c_0}u(\frac{t}{c_0}, x),\ \psi(t,x) = \phi(\frac{t}{c_0}, x)$$

with the sound speed $c_0 = \sqrt{\gamma} n_0^{(\gamma-1)/2}$, and $n = n_0(\frac{\gamma-1}{2}m+1)^{2/(\gamma-1)}$. Notice that $\nabla \times v \equiv 0$ from (3). In terms of new variables, the Euler–Poisson system (1), (4) takes the form

$$\begin{aligned} &\partial_t m + \nabla\cdot v + v\cdot\nabla m + \frac{\gamma-1}{2}m\nabla\cdot v = 0,\\ &\partial_t v + \nabla m + v\nabla v + \frac{\gamma-1}{2}m\nabla m = c_0^{-2}\nabla\psi,\\ &\partial_t\nabla\psi = -n_0 v - n_0\nabla\Delta^{-1}\nabla\cdot\{[(\frac{\gamma-1}{2}m+1)^{2/(\gamma-1)} - 1]v\}, \end{aligned} \tag{5}$$

with the constraint

$$\Delta\psi = n_0[(\frac{\gamma-1}{2}m+1)^{2/(\gamma-1)} - 1] \equiv n_0[m - h(m)],$$

where h, as defined, is a smooth function satisfying $h(0)=0$ and $h'(0)=0$. Since $\nabla\times v \equiv 0$, by using the Poisson equation and taking one more derivative of (5), we obtain

$$\begin{aligned} (\partial_{tt} - \Delta + m_0)m &= \nabla\cdot[v\nabla v + \frac{\gamma-1}{2}m\nabla m]\\ &\quad -\partial_t[v\cdot\nabla m + \frac{\gamma-1}{2}m\nabla\cdot v] + m_0 h(m),\\ (\partial_{tt} - \Delta + m_0)v &= \nabla[v\cdot\nabla m + \frac{\gamma-1}{2}m\nabla\cdot v] - \partial_t[v\nabla v + \frac{\gamma-1}{2}m\nabla m]\\ &\quad -m_0\nabla\Delta^{-1}\nabla\cdot\{[m-h(m)]v\}, \end{aligned} \tag{6}$$

where $m_0 = c_0^{-2} n_0$. Notice that the right handside in (6) is formally second order in m and v. In order to further simplify the notation, we define

$$w = (w^0, w^1, w^2, w^3)^T = (m, v)^T,$$

where T is the transpose. Let $\partial_0 = \partial_t = \partial_{x_0}$ and $\partial_j = \partial_{x_j}$ for $1 \le j \le 3$. And we use the standard convention that the Latin letter i, j, k runs from 1 to 3, while the Greek letters μ, ν runs from 0 to 3. In term of w, (5) takes the form

$$\partial_0 w + A_j(w)\partial_j w = (0, c_0^{-2}\nabla\psi)^T, \tag{7}$$

where A_j are symmetric matrices of

$$\begin{pmatrix} w^j & (\frac{\gamma-1}{2}w^0 + 1)\mathbf{e_j^T} \\ (\frac{\gamma-1}{2}w^0 + 1)\mathbf{e_j} & w^j\mathbf{I} \end{pmatrix},$$

see [Si1]. We now rewrite (6) in term of w, and separate the non-local term $m_0\nabla\Delta^{-1}\nabla\cdot\{[m - h(m)]v\}$ from the other. We obtain

$$(\partial_{tt} - \Delta + m_0)w = f(w, \partial w, \partial^2 w) \equiv s(w, \partial w, \partial^2 w) + g(w, \partial w, \partial^2 w). \tag{8}$$

Here the singular, non-local function $s = (s^\mu)$, with

$$s^0 = 0, \quad \text{and } s^l = -m_0\Delta^{-1}\partial_{lk}\{[w^0 - h(w^0)]w^k\},$$

while $g = (g^\mu)$ is the rest of the nonlinear terms. Notice that g is a smooth function of $w, \partial w, \partial^2 w$, with $g(0,0,0) = 0$ and *no* dependence on $\partial_{00}w$. We define $|w(t)|_{k,p}$ as the standard *spatial* Sobolev norm of order k. We use, for simplicity, ∂^α and ∂^i to denote multi-space-time and space derivatives, with lengths α and i respectively. We also use the Einstein summation convention from time to time. We also define

$$\| w(t) \|_{k,p} = |w(t)|_{k,p} + |\partial_0 w(t)|_{k-1,p} \tag{9}$$

for $1 \le p \le \infty$. We first prove

Lemma 2. *Let $k \ge 3$ and $w(t)$ be a solution to (8) with $\| w(t) \|_{k,\infty} \le 1$. Then*

$$\| f(w(t)) \|_{k,p} \le C \| w \|_{[k/2]+2,\infty} (\| w \|_{k+2,p} + \| w \|^2_{k+2,2p}), \tag{10}$$

where $1 < p < \infty$, and $[k/2]$ is the largest integer that does not exceed $k/2$.

Proof. Recall the non-local term s is a sum of products of Riesz transforms of $m_0[w^0 - h(w^0)]w$. By the L^p boundedness of the Riesz transformation [Ste], for $1 < p < \infty$, we have

$$|f(w)|_{k,p} \le C|[w^0 - h(w^0)]w|_{k,p} + |g(w, \partial w, \partial^2 w)|_{k,p}.$$

Notice that g and $[w^0 - h(w^0)]w$ are smooth and second order. By the product rule, the above is majorized by

$$C(|w|_{[k/2],\infty}|w|_{k,p} + \| w \|_{[k/2]+2,\infty} \| w \|_{k+2,p}) \le C \| w \|_{[k/2]+2,\infty} \| w \|_{k+2,p}, \tag{11}$$

since g does not depend on $\partial_{00}w$, and $\| w(t) \|_{k,\infty} \le 1$, for $k \ge 3$. Moreover,

$$\begin{aligned} |\partial_0 f(w(t))|_{k-1,p} &\le C|\partial_0\{[w^0 - h(w^0)]w\}|_{k-1,p} + |\partial_0 g|_{k-1,p} \\ &\le C \| w \|_{[k/2]+1,\infty} \| w \|_{k,p} + |g_\nu \partial_0 \partial^\nu w|_{k-1,p}. \end{aligned}$$

Here ν is a multi-index with $|\nu| \leq 2$, and g_ν is the partial derivative of $g(w, \partial w, \partial^2 w)$ with respect to $\partial^\nu w$. We estimate the last term. If ν does not contain any t (or x_0) derivative, we apply (11) with g_ν

$$|g_\nu \partial_0 \partial^\nu w|_{k-1,p} = |\partial^{k-1-j}(g_\nu)\partial^j \partial_0 \partial^\nu w|_p \leq C \parallel w \parallel_{[k/2]+2,\infty} \parallel w \parallel_{k+2,p},$$

by separating two cases $0 \leq j \leq [\frac{k}{2}] - 1$ and $j \geq [\frac{k}{2}]$ (i.e. $k-1-j \leq [\frac{k}{2}]$). On the other hand, if ν contains one x_0 derivative, then $\partial^\nu = \partial_0 \partial^i$, with $0 \leq i \leq 1$. Substituting (8), we get

$$|g_\nu \partial_0 \partial^\nu w|_{k-1,p} = |g_\nu \partial^i \partial_{00} w|_{k-1,p} \leq |g_\nu \partial^i \{(\Delta - m_0) w\}|_{k-1,p} + |g_\nu \partial^i f(w)|_{k-1,p}.$$

Similarly, separating two cases $0 \leq j \leq [\frac{k}{2}] - 1$ and $k-1-j \leq [\frac{k}{2}]$ yields

$$\begin{aligned} |g_\nu \partial^i \{(\Delta - m_0) w\}|_{k-1,p} &\leq C|\partial^{k-1-j}(g_\nu)\partial^{j+i}\{(\Delta - m_0)w\}| \\ &\leq C \parallel w \parallel_{[k/2]+2,\infty} \parallel w \parallel_{k+2,p} . \end{aligned}$$

We use Holder's inequality and (11) to estimate the second term by

$$\begin{aligned} |g_\nu \partial^i f(w)|_{k-1,p} &\leq C|g_\nu|_{k-1,2p}|f(w)|_{k,2p} \leq C \parallel w \parallel_{k+1,2p} \parallel w \parallel_{[k/2]+2,\infty} \parallel w \parallel_{k+2,2p} \\ &\leq C \parallel w \parallel_{[k/2]+2,\infty} \parallel w \parallel^2_{k+2,2p} . \end{aligned}$$

The lemma thus follows from (11) and the above two estimates. □

3. The Normal Forms

Now we define a normal form transformation for (8). The goal is to construct a new variable $\omega = (\omega^\mu)$, such that $(\partial_{tt} - \Delta + m_0)\omega$ is *cubic* in w. Thus we can apply the linear L^∞ decay estimate for ω. We follow the construction of Shatah [Sh] to define

$$\omega^\mu = w^\mu + [(w^\alpha, \partial_0 w^\alpha), B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T], \tag{12}$$

with summation over $0 \leq \alpha, \beta \leq 3$. Here $B^\mu_{\alpha\beta}$ is a 2×2 matrix to be determined, and

$$\begin{aligned} [V_1, B^\mu_{\alpha\beta}, V_2^T](x) &\equiv \int_{\mathbf{R}^3 \times \mathbf{R}^3} V_1(z) B^\mu_{\alpha\beta}(x-y, x-z) V_2^T(y) dy dz \\ &\equiv \frac{1}{(2\pi)^6} \int_{\mathbf{R}^3 \times \mathbf{R}^3} e^{ix\cdot(\xi+\eta)} \mathcal{F}(V_1)(\eta) \mathcal{F}(B^\mu_{\alpha\beta})(\xi, \eta) \mathcal{F}(V_2^T)(\xi) d\xi d\eta \end{aligned} \tag{13}$$

for any two 1×2 functions V_1 and V_2. The second equation in (13) follows directly from a Fourier transform $\mathcal{F}$ with respect to z, y as well as *both* $x-y$ and $x-z$. Here the Fourier transform $\mathcal{F}$ is defined as

$$\mathcal{F}(V)(\sigma) = \int_{\mathbf{R}^l} e^{-ix\cdot\sigma} V(x) dx, \ \mathcal{F}^{-1}(V)(x) = \frac{1}{(2\pi)^l} \int_{\mathbf{R}^l} e^{ix\cdot\sigma} V(\sigma) d\sigma$$

for any integer $l > 0$ and $V \in \mathcal{S}(\mathbf{R}^l)$.

We compute $(\partial_{tt} - \Delta + m_0)\omega^\mu$ for a smooth solution w of (8). We first compute the most complicated term $\partial_{00}\omega^\mu$. By (13), (8) and (12), we have

$$\begin{aligned}
&\partial_0[(w^\alpha, \partial_0 w^\alpha), B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T] \\
&= [(\partial_0 w^\alpha, \partial_{00} w^\alpha), B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T] + [(w^\alpha, \partial_0 w^\alpha), B^\mu_{\alpha\beta}, (\partial_0 w^\beta, \partial_{00} w^\beta)^T] \\
&= [(\partial_0 w^\alpha, (\Delta - m_0) w^\alpha + f^\alpha), B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T] \\
&\quad + [(w^\alpha, \partial_0 w^\alpha), B^\mu_{\alpha\beta}, (\partial_0 w^\beta, (\Delta - m_0) w^\beta + f^\beta)^T].
\end{aligned}$$

By taking one more t derivative, we obtain

$$\begin{aligned}
&\partial_{00}[(w^\alpha, \partial_0 w^\alpha), B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T] \\
&= [((\Delta - m_0) w^\alpha + f^\alpha, (\Delta - m_0)\partial_0 w^\alpha + \partial_0 f^\alpha), B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T] \\
&\quad + 2[(\partial_0 w^\alpha, (\Delta - m_0) w^\alpha + f^\alpha), B^\mu_{\alpha\beta}, (\partial_0 w^\beta, (\Delta - m_0) w^\beta + f^\beta)^T] \\
&\quad + [(w^\alpha, \partial_0 w^\alpha)^T, B^\mu_{\alpha\beta}, ((\Delta - m_0) w^\beta + f^\beta, (\Delta - m_0)\partial_0 w^\beta + \partial_0 f^\beta)^T].
\end{aligned}$$

We now separate second order terms from higher order terms. Notice that

$$\nabla_{y,z} B^\mu_{\alpha\beta}(x-y, x-z) = -\nabla_{1,2} B^\mu_{\alpha\beta}(x-y, x-z), \tag{14}$$

Integrating by part over the y, z variables, we simplify the above from (13) ($\Delta - m_0$ is self-adjoint)

$$\begin{aligned}
&\partial_{00}[(w^\alpha, \partial_0 w^\alpha), B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T] \\
&= [(w^\alpha, \partial_0 w^\alpha), (\Delta_2 - m_0) B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T] \\
&\quad + 2[(w^\alpha, \partial_0 w^\alpha), \mathcal{C} B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T] \\
&\quad + [(w^\alpha, \partial_0 w^\alpha), (\Delta_1 - m_0) B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T] + R^\mu_1.
\end{aligned}$$

Here $\mathcal{C} B^\mu_{\alpha\beta} \equiv \begin{pmatrix} (\Delta_1 - m_0)(\Delta_2 - m_0) B^\mu_{\alpha\beta 22} & (\Delta_2 - m_0) B^\mu_{\alpha\beta 21} \\ (\Delta_1 - m_0) B^\mu_{\alpha\beta 12} & B^\mu_{\alpha\beta 11} \end{pmatrix}$ with entries $B^\mu_{\alpha\beta ij}$ and

$$\begin{aligned}
R^\mu_1 &= [(f^\alpha, \partial_0 f^\alpha), B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T] \\
&\quad + 2[(0, f^\alpha), B^\mu_{\alpha\beta}, (\partial_0 w^\beta, (\Delta - m_0) w^\beta + f^\beta)^T] \\
&\quad + 2[(\partial_0 w^\alpha, (\Delta - m_0) w^\alpha + f^\alpha), B^\mu_{\alpha\beta}, (0, f^\beta)^T] \\
&\quad + [(w^\alpha, \partial_0 w^\alpha)^T, B^\mu_{\alpha\beta}, (f^\beta, \partial_0 f^\beta)^T]
\end{aligned}$$

is the third order remainder. From the definition of $[\cdot, \cdot]$ in (13),

$$\Delta_x[(w^\alpha, \partial_0 w^\alpha), B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T] = [(w^\alpha, \partial_0 w^\alpha), (\nabla_1 + \nabla_2)^2 B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T],$$

where ∇_i denotes the gradient operator with respect to i^{th} argument of $B^\mu_{\alpha\beta}$, $i = 1, 2$. From (7) and (12), ω satisfies:

$$\begin{aligned}
(\partial_{tt} - \Delta + m_0)\omega^\mu &= (\partial_{tt} - \Delta + m_0) w^\mu + [(w^\alpha, \partial_0 w^\alpha), \mathcal{L} B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T] + R^\mu_1 \\
&= f^\mu + [(w^\alpha, \partial_0 w^\alpha), \mathcal{L} B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T] + R^\mu_1. \qquad (15)
\end{aligned}$$

Here $\mathcal{L} B^\mu_{\alpha\beta}$ is defined as

$$\begin{aligned}\mathcal{L}B^{\mu}_{\alpha\beta} &\equiv \{(\Delta_2 - m_0) + (\Delta_1 - m_0) - (\nabla_1 + \nabla_2)^2 + m_0\}B^{\mu}_{\alpha\beta} + 2CB^{\mu}_{\alpha\beta}\\ &= \{\Delta_2 + \Delta_1 - m_0 - (\nabla_1 + \nabla_2)^2\}B^{\mu}_{\alpha\beta} + 2\mathcal{C}\mathcal{B}^{\mu}_{\alpha\beta}.\end{aligned}$$

We now expand f^μ as a sum of a quadratic and a higher order part

$$f^\mu = [(w^\alpha, \partial_0 w^\alpha), f^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T] + R^\mu_2. \tag{16}$$

Here $f^\mu_{\alpha\beta} = s^\mu_{\alpha\beta} + g^\mu_{\alpha\beta}$ is the kernel of the quadratic part. Moreover, $g^\mu_{\alpha\beta}(y,z) = g^\mu_{\alpha\beta kl}\partial^l_y\delta(y)\partial^k_z\delta(z)$, $g^\mu_{\alpha\beta kl}$ are constant 2×2 matrices, $k + l \le 2$ and δ is the Dirac mass; from (8), the non-local part $s^\mu_{\alpha\beta}$ is

$$[V_1, s^l_{0k}, V_2^T](x) = -m_0\partial_{lk}\Delta^{-1}\{V_1^0 V_2^k\} = m_0\partial_{lk}(-\Delta)^{-1}[V_1, \mathbf{e}_{0k}\delta(y)\delta(z), V_2^T]$$

for $1 \le k, l \le 3$, and $s^\mu_{\alpha\beta} \equiv 0$ otherwise. From (6), the third order term R^μ_2 in (16) is

$$R^0_2 = m_0\{h(w^0) - \frac{1}{2}h''(0)(w^0)^2\} \equiv h_1(w^0),\ R^l_2 = m_0\Delta^{-1}\partial_{lk}[h(w^0)w^k].$$

Let $\mathcal{I}_a = (-\Delta)^{-a/2}$ be the Riesz potential of order $-\infty < a < 3$ [Ste], which is defined in terms of the Fourier transform

$$\mathcal{F}\,(\mathcal{I}_a(\cdot))(\sigma) = |\sigma|^{-a}\mathcal{F}\,(\cdot).$$

As a function of x, it follows from the Fourier transform that for $a < 3$,

$$\mathcal{I}_a(e^{ix\cdot(\xi+\eta)}) = |\xi + \eta|^{-a}e^{ix\cdot(\xi+\eta)}. \tag{17}$$

From the second equation in (13), and (17) with $a = 2$, $[V_1, s^l_{0k}, V_2^T](x)$ equals

$$-\frac{m_0}{(2\pi)^6}\int_{\mathbf{R}^3\times\mathbf{R}^3} e^{ix\cdot(\xi+\eta)}\mathcal{F}(V_1)(\eta)\frac{(\xi_k + \eta_k)(\xi_l + \eta_l)}{|\xi + \eta|^2}\mathbf{e}_{0k}\mathcal{F}(V_2^T)(\xi)d\xi d\eta. \tag{18}$$

In order to eliminate all second order terms of w in (15), we let

$$\mathcal{L}B^\mu_{\alpha\beta} \equiv -f^\mu_{\alpha\beta}. \tag{19}$$

Therefore (15) becomes

$$(\partial_{tt} - \Delta + m_0)\omega^\mu = R^\mu_1 + R^\mu_2, \tag{20}$$

with only third order terms of w left. In order to achieve this goal, we need to solve (19) for $B^\mu_{\alpha\beta}$. Recall the fundamental theorem due to Shatah [Sh]:

Theorem 3. *(a). Let $D(y,z)$, $\mathcal{F}\,(D)(\xi,\eta)$ be 2×2 matrices of distributions, then there exists distributions $B(y,z)$, such that $\mathcal{L}B \equiv D$ in the sense of distributions. Moreover, the Fourier transform of B satisfies*

$$\mathcal{F}(B)(\xi,\eta) = Q(\xi,\eta)\mathcal{F}(D)(\xi,\eta),$$

where $Q(\xi,\eta)$ is C^∞ and $|Q(\xi,\eta)| \le C(1 + |\xi| + |\eta|)^6$.
(b). Furthermore, assume $D = D_{kl}\partial^l_y\delta(y)\partial^k_z\delta(z)$, with multi-index k, l, and constant matrices $D_{kl.}$ Then there exists integer $N > 0$, such that

$$|[\partial^i V_1, B, \partial^j V_2^T]|_p \le C|\partial^i V_1|_{4N,p_1}|\partial^j V_2|_{4N,p_2}$$

for any V_1, V_2, $1 \le p \le \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and any two multi-index i, j .

We can not directly apply part (b) of Shatah's Theorem to solve (19), since the non-local term $\mathcal{F}(s^{\mu}_{\alpha\beta})$ in (18) has singularity at $\xi + \eta = 0$. We use the Riesz potential $\mathcal{I}_a$ to smooth out near $\xi + \eta = 0$. We first observe

Lemma 4. *Let* $\langle\xi\rangle = \sqrt{1 + |\xi|^2}$, $0 < a \le \frac{3}{2}$, *and* $3 + a < N$. *Then*

$$\int_{\mathbf{R}^3} \frac{\xi_k\xi_l}{|\xi|^{2-a}}\langle\xi\rangle^{-N} e^{iy\cdot\xi} d\xi \in L^1(\mathbf{R}^3_y).$$

Proof. For $N > 3 + a$, since $\frac{\xi_k\xi_l}{|\xi|^{2-a}}\langle\xi\rangle^{-N} \in L^1(\mathrm{R}^3_\xi)$, $\int \frac{\xi_k\xi_l}{|\xi|^{2-a}}\langle\xi\rangle^{-N} e^{iy\cdot\xi} d\xi \in L^\infty(\mathrm{R}^3_y)$. Hence it suffices to show $\int \frac{\xi_k\xi_l}{|\xi|^{2-a}}\langle\xi\rangle^{-N} e^{iy\cdot\xi} d\xi \in L^1\{|y| \ge 1\}$. Without loss of generality, we may assume $|y_1| = \max_{1\le i\le 3}\{|y_i|\}$. Integrating by part over the ξ variable, we have

$$\begin{aligned}\left| y_1^3 \int_{\mathbf{R}^3} \frac{\xi_k\xi_l}{|\xi|^{2-a}}\langle\xi\rangle^{-N} e^{iy\cdot\xi} d\xi \right| &= c\left| \int_{\mathbf{R}^3} \partial^3_{\xi_1}\{\frac{\xi_k\xi_l}{|\xi|^{2-a}}\langle\xi\rangle^{-N}\} e^{iy\cdot\xi} d\xi \right| \\ &= c|\int_{\mathbf{R}^3} \langle\xi\rangle^{-N}|\xi|^{a-3}\theta(\xi) e^{iy\cdot\xi} d\xi|,\end{aligned}$$

where $\theta(\xi)$ is a bounded, smooth function, and c some numerical constant. Since $a > 0$, $\langle\xi\rangle^{-N}|\xi|^{a-3}\theta(\xi) \in L^p(\mathrm{R}^3_\xi)$ for $1 \le p < \frac{3}{3-a} \le 2$. Hence from the Hausdroff–Young inequality,

$$|\int \langle\xi\rangle^{-N}|\xi|^{a-3}\theta(\xi) e^{iy\cdot\xi} d\xi| \in L^{p'}(\mathbf{R}^3_y)$$

for $\frac{1}{p} + \frac{1}{p'} = 1$. Finally, since $|y_1|^{-3} \le 3|y|^{-3} \in L^p\{|y| \ge 1\}$ for any $p > 1$,

$$\begin{aligned}&\left| \int \frac{\xi_k\xi_l}{|\xi|^{2-a}}\langle\xi\rangle^{-N} e^{iy\cdot\xi} d\xi \right|_{1\{|y|\ge 1\}} \\ &= |y_1|^{-3} \int \langle\xi\rangle^{-N}|\xi|^{a-3}\theta(\xi) e^{iy\cdot\xi} d\xi|_{1,\{|y|\ge 1\}} \\ &\le C\{\int_{|y|\ge 1} |y|^{-3p}\}^{1/p}|\int \langle\xi\rangle^{-N}|\xi|^{a-3}\theta(\xi) e^{iy\cdot\xi} d\xi|_{p'} < \infty\end{aligned}$$

from Holder's inequality. □

Now we generalize Theorem 3 to solve (19) for $B^{\mu}_{\alpha\beta}$. Notice now $p \ne 1, \infty$.

Theorem 5. *There exists distributions* $B^{\mu}_{\alpha\beta}$ *such that* $\mathcal{L}B^{\mu}_{\alpha\beta} \equiv -f^{\mu}_{\alpha\beta}$ *for* $0 \le \mu, \alpha, \beta \le 3$, *with*

$$\mathcal{F}(B^{\mu}_{\alpha\beta})(\xi, \eta) = Q(\xi, \eta)\mathcal{F}(f^{\mu}_{\alpha\beta}),$$

$|Q(\xi, \eta)| \le C(1 + |\xi| + |\eta|)^6$ *as in Theorem 3. Moreover, for any* $1 < p < \infty$, *there exists* $N > 0$ *such that*

$$|[\partial^i V_1, B^{\mu}_{\alpha\beta}, \partial^j V_2^T]|_p \le C\{|\partial^i V_1|_{4N,p_1}|\partial^j V_2|_{4N,p_2} + |\partial^i V_1|_{4N,r_1}|\partial^j V_2|_{4N,r_2}\}$$

with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, $\frac{1}{r} = \frac{1}{p} + \frac{a}{n}$, *for any* $0 < a \le \frac{3}{2}$, *and multi-indices* i, j.

Proof. From part (a) of Theorem 3, there is $B^\mu_{\alpha\beta}$ such that $\mathcal{L}B^\mu_{\alpha\beta} \equiv -f^\mu_{\alpha\beta}$. Taking the Fourier transform, by (18) we get

$$\mathcal{F}(B^\mu_{\alpha\beta})(\xi,\eta) = Q(\xi,\eta)\{m_0 \frac{(\xi_k+\eta_k)(\xi_l+\eta_l)}{|\xi+\eta|^2}\mathbf{e}_{0k} - \mathcal{F}(g^\mu_{\alpha\beta})\}.$$

By Theorem 3, for $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, the $g-$part is easily estimated as

$$|[\partial^i V_1, \mathcal{F}^{-1}\{Q\mathcal{F}(g^\mu_{\alpha\beta})\}, \partial^j V_2^T]|_p \le C|\partial^i V_1|_{4N,p_1}|\partial^j V_2|_{4N,p_2}.$$

It suffices to estimate the first singular term. By the Hardy-Littlewood-Sobolev inequality, for $\frac{1}{r} = \frac{1}{p} + \frac{a}{n}$, $1 < p < \infty, 0 < a \le \frac{3}{2}$,

$$\begin{aligned}&\left|[\partial^i V_1, \mathcal{F}^{-1}\{Q\frac{(\xi_k+\eta_k)(\xi_l+\eta_l)}{|\xi+\eta|^2}\}\mathbf{e}_{0k}, \partial^j V_2^T]\right|_p\\ \le &\left|\mathcal{I}_a[\partial^i V_1, \mathcal{F}^{-1}\{Q\frac{(\xi_k+\eta_k)(\xi_l+\eta_l)}{|\xi+\eta|^2}\}\mathbf{e}_{0k}, \partial^j V_2^T]\right|_r.\end{aligned}$$

From (17), (18) and the second equation of (13), the above is equivalent to

$$\begin{aligned}&\left|\frac{1}{(2\pi)^6}\int_{R^3\times R^3} e^{ix\cdot(\xi+\eta)}\mathcal{F}(\partial^i V_1)(\eta)\frac{(\xi_k+\eta_k)(\xi_l+\eta_l)}{|\xi+\eta|^{2-a}}\mathbf{e}_{0k}\mathcal{F}(\partial^j V_2^T)(\xi)d\xi d\eta\right|_r\\ =\ &|[\partial^i V_1, \mathcal{F}^{-1}\{|\xi+\eta|^{a-2}(\xi_k+\eta_k)(\xi_l+\eta_l)Q\}\mathbf{e}_{0k}, \partial^j V_2^T]|_r. \qquad (21)\end{aligned}$$

It thus suffices to bound (21) by $C|\partial^i V_1|_{4N,r_1}|\partial^j V_2|_{4N,r_2}$. Let $\langle\sigma\rangle = \sqrt{1+\sigma^2}$. From (14) and (13), we rewrite the right-hand side of (21) as

$$\begin{aligned}&\Big|[\partial^i V_1, \{\langle\partial_1+\partial_2\rangle^{2N}\langle\partial_1-\partial_2\rangle^{2N}\}\\ &\quad\mathcal{F}^{-1}\{\frac{(\xi_k+\eta_k)(\xi_l+\eta_l)Q}{\langle\xi+\eta\rangle^{2N}\langle\xi-\eta\rangle^{2N}|\xi+\eta|^{2-a}}\}[\acute{}\mathbf{e}_{0k}, \partial^j V_2^T]\Big|_r\\ =&\Big|[\partial^i V_1, \{\langle\partial_y+\partial_z\rangle^{2N}\langle\partial_y-\partial_z\rangle^{2N}\}\\ &\quad\mathcal{F}^{-1}\{\frac{(\xi_k+\eta_k)(\xi_l+\eta_l)Q}{\langle\xi+\eta\rangle^{2N}\langle\xi-\eta\rangle^{2N}|\xi+\eta|^{2-a}}\}\mathbf{e}_{0k}, \partial^j V_2^T]\Big|_r\\ \le &\, C_{n_1,n_2}\left|[\partial^{i+n_1}V_1, \mathcal{F}^{-1}\{\frac{(\xi_k+\eta_k)(\xi_l+\eta_l)Q}{\langle\xi+\eta\rangle^{2N}\langle\xi-\eta\rangle^{2N}|\xi+\eta|^{2-a}}\}e_{0k}, \partial^{j+n_2}V_2^T]\right|_r,\\ &\quad\text{(integration by part)},\end{aligned}$$

where summations over $0 \le n_1, n_2 \le 4N$. By a change of variable $x-y = y', x-z = z'$ in (13), the above is majorized by ($\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$),

$$C_{n_1,n_2}|\partial^{i+n_1}V_1|_{r_1}|\mathcal{F}^{-1}\{\frac{(\xi_k+\eta_k)(\xi_l+\eta_l)Q}{\langle\xi+\eta\rangle^{2N}\langle\xi-\eta\rangle^{2N}|\xi+\eta|^{2-a}}\}\mathbf{e}_{0k}|_1|\partial^{j+n_2}V_2^T|_{r_2}.\ \text{(Holder)}$$

We now only need to show that for some integer $N > 0$, $\mathcal{F}^{-1}\{\frac{(\xi_k+\eta_k)(\xi_l+\eta_l)Q}{\langle\xi+\eta\rangle^{2N}\langle\xi-\eta\rangle^{2N}|\xi+\eta|^{2-a}}\} \in L^1(\mathrm{R}^3_y\times\mathrm{R}^3_z)$. Notice that $\mathcal{F}^{-1}\{Q\langle\xi+\eta\rangle^{-N}\langle\xi-\eta\rangle^{-N}\} \in L^1(\mathrm{R}^3_y\times\mathrm{R}^3_z)$, for N large. By Young's inequality, it thus suffices to show

$$\left|\mathcal{F}^{-1}\{\frac{(\xi_k+\eta_k)(\xi_l+\eta_l)}{|\xi+\eta|^{2-a}\langle\xi+\eta\rangle^N\langle\xi-\eta\rangle^N}\}\right|_{1(\mathbf{R}^3_y\times\mathbf{R}^3_z)}<\infty.$$

But from the definition of $\mathcal{F}^{-1}$, this is equivalent to

$$\left|\int_{\mathbf{R}}^3\times\mathbf{R}^3\frac{(\xi_k+\eta_k)(\xi_l+\eta_l)}{|\xi+\eta|^{2-a}\langle\xi+\eta\rangle^N\langle\xi-\eta\rangle^N}e^{iy\cdot\xi+iz\cdot\eta}d\xi d\eta\right|_{1(\mathbf{R}^3_y\times\mathbf{R}^3_z)}$$
$$=c\left|\int_{\mathbf{R}}^3\times\mathbf{R}^3\frac{\xi'_k\xi'_l}{|\xi'|^{2-a}\langle\xi'\rangle^N\langle\eta'\rangle^N}e^{i(y+z)\cdot\xi'+i(y-z)\cdot\eta'}d\xi'd\eta'\right|_{1(\mathbf{R}^3_y\times\mathbf{R}^3_z)}.$$
$$(\xi+\eta=2\xi',\xi-\eta=2\eta').$$

By further changing variables $y+z=y'$, and $y-z=z'$, we estimate above by

$$C\int_{\mathbf{R}^3\times\mathbf{R}^3}\left|\int_{\mathbf{R}^3\times\mathbf{R}^3}\frac{\xi'_k\xi'_l}{|\xi'|^{2-a}\langle\xi'\rangle^N\langle\eta'\rangle^N}e^{iy'\cdot\xi'+iz'\cdot\eta'}d\xi'd\eta'\right|dy'dz'$$
$$=C\int_{\mathbf{R}^3}\left|\int_{\mathbf{R}^3}\frac{\xi'_k\xi'_l}{|\xi'|^{2-a}\langle\xi'\rangle^N}e^{iy'\cdot\xi'}d\xi'\right|dy'\times\int_{\mathbf{R}^3}|\int_{\mathbf{R}^3}\langle\eta'\rangle^{-N}e^{iz'\cdot\eta'}d\eta'|dz'\text{ (Fubini)}$$
$$\leq C\int_{\mathbf{R}^3}|\int_{\mathbf{R}^3}\langle\eta'\rangle^{-N}e^{iz'\cdot\eta'}d\eta'|dz'<\infty,$$

where we have used Lemma 4. We thus deduce our theorem. □

4. The L^∞ Decay Estimate

In this section, we derive the L^∞ decay estimate for the solution w of (8). We first state the $L^\infty-L^p$ estimates for the linear Klein–Gordon equation.

Lemma 6. *Let* $(\partial_{tt}-\Delta+1)\omega=0$. *Then for* $1\leq p\leq 2$, $l\geq 1$,

$$\|\omega(t)\|_{l,\infty}\leq C(1+t)^{-3/2+3/p'}\|\omega(0)\|_{4+l,p}.$$

Proof. Recall the space time norm in (9). The $L^\infty-L^1$ estimate is standard, see [MSW]. Let $\langle\xi\rangle=\sqrt{1+\xi^2}$, we have

$$\mathcal{F}(\omega(t))(\xi)=\mathcal{F}(\omega(0))\cos\langle\xi\rangle t+\mathcal{F}(\partial_0\omega(0))\frac{\sin\langle\xi\rangle t}{\langle\xi\rangle},$$
$$\mathcal{F}(\partial_0\omega(t))(\xi)=\mathcal{F}(\partial_0\omega(0))\cos\langle\xi\rangle t-\mathcal{F}(\omega(0))\langle\xi\rangle\sin\langle\xi\rangle t.$$

For $2\leq p'<\infty$, from Corollary 5.1, 5.2 of [N],

$$\left|\mathcal{F}^{-1}(\frac{\exp\{-i\langle\xi\rangle t\}}{\langle\xi\rangle^3})\right|_{p'}\leq Ct^{-3/2+3/p'}.$$

Hence by the Young's inequality for convolutions ($\frac{1}{p'}+\frac{1}{p}=1$), for $t\geq 1$,

$$\begin{aligned}\|\omega(t)\|_{1,\infty} &= |\omega(t)|_{1,\infty} + |\partial_0\omega(t)|_\infty \\ &\le C\left|\mathcal{F}^{-1}\left(\frac{\exp\{-i\langle\xi\rangle t\}}{\langle\xi\rangle^3}\right)\right|_{p'} \{|\langle\xi\rangle^4\mathcal{F}(\omega(0))|_p + |\langle\xi\rangle^3\mathcal{F}\,(\partial_0\omega(0))|_p\} \\ &\le Ct^{-3/2+3/p'} \parallel \omega(0) \parallel_{4,p}\,.\end{aligned}$$

On the other hand, for $t \le 1$, since $\mathcal{F}^{-1}(\frac{\exp\{-i\langle\xi\rangle t\}}{\langle\xi\rangle^4}) \in L^1 \cap L^\infty$, it follows that $\|\omega(t)\|_{1,\infty} \le C \parallel \omega(0) \parallel_{5,p}$. Hence the lemma is valid when $l = 1$. We deduce the lemma by taking $l-1$ more spatial derivatives. This lemma can also be proven by an interpolation between the $L^\infty - L^1$ estimate and the energy estimate for high derivatives. □

We now define, for positive integer $l > 0$,

$$|w|_X \equiv \parallel w \parallel_{2l,2} \quad |w|_Y \equiv \parallel w \parallel_{l,\infty} \quad |w|_Z \equiv \parallel w \parallel_{l+4,p.},$$
$$|||w||| \equiv \sup_{t>0}[|w|_X + |\nabla\psi|_X + (1+t)^{3/2-3/p'}|w|_Y],$$
$$|||w|||_{T_*} \equiv \sup_{0<t<T_*}[|w|_X + |\nabla\psi|_X + (1+t)^{3/2-3/p'}|w|_Y],$$

with $1 < p < \frac{6}{5}$, or $6 < p' < \infty$. Notice that $|w|_Y \le C|w|_X$ for l large, and $|w|_Y \le C|w|_Z$, both from the Sobolev Imbedding Theorem. The solutions w which we construct satisfy $|||w||| < \infty$. We first derive an a priori bound for $|w|_Y$.

Theorem 7 (L^∞ decay estimate). *Let* $|w(0)|_Y + |w(0)|_Z = \epsilon_0$, *and* $|||w|||_{T_*} \le 1$. *There is* $l_0 > 0$, *such that if* $l \ge l_0$, $0 \le t \le T_*$.

$$(1+t)^{3/2-3/p'}|w(t)|_Y \le C(\epsilon_0 + |||w|||_{T_*}^2).$$

Proof. By the standard Duhamel principle, we apply Lemma 6 (with 1 replaced by m_0) to (20) to get

$$|\omega(t)|_Y \le C(1+t)^{-3/2+3/p'}|\omega(0)|_Z + \int_0^t (1+t-\tau)^{-3/2+3/p'}|R_1 + R_2|_{l+3,p}(\tau)d\tau.$$

Step 1. Estimate $|w(t)|_Y$. Notice that from the normal form transformation (12),

$$|w^\mu(t)|_Y \le |\omega^\mu(t)|_Y + |[(w^\alpha, \partial_0 w^\alpha), B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T]|_Y.$$

We estimate the quadratic term above. By the Sobolev Imbedding Theorem,

$$|[(w^\alpha, \partial_0 w^\alpha), B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T]|_{l,\infty} \le C|[(w^\alpha, \partial_0 w^\alpha), B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T]|_{l+1,q}$$

for some $q > n > 2$. Choose $\frac{1}{r} + \frac{a}{n} = \frac{1}{q}$, a small, and $r > 2$. Repeatedly using (14) as well as integrating by parts over y and z, from Theorem 5, we obtain

$$\begin{aligned}&|[(w^\alpha, \partial_0 w^\alpha), B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T]|_{l+1,q} = C_{ij}|[\partial^i(w^\alpha, \partial_0 w^\alpha), B^\mu_{\alpha\beta}, \partial^j(w^\beta, \partial_0 w^\beta)^T]|_q \\ &\le C_{ij}\{\parallel \partial^i w^\alpha \parallel_{4N+1,q_1(i)}\parallel \partial^j w^\beta \parallel_{4N+1,q_2(j)} + \parallel \partial^i w^\alpha \parallel_{4N+1,r_1(i)}\parallel \partial^j w^\beta \parallel_{4N+1,r_2(j)}\},\end{aligned}$$

summation over multi-index i and j with $0 \le i+j \le l+1$, C_{ij} constants. We choose

$$\frac{1}{q_1(i)} = \frac{i}{(i+j)q}, \frac{1}{q_2(j)} = \frac{j}{(i+j)q}; \ \frac{1}{r_1(i)} = \frac{i}{(i+j)r}, \frac{1}{r_2(j)} = \frac{j}{(i+j)r}. \tag{22}$$

Apply Nirenberg–Gagliardo inequality (interpolation between q, ∞ and r, ∞) to each i, j to get

$$\begin{aligned} &|[(w^\alpha, \partial_0 w^\alpha), B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T]|_{l+1,q} \\ &\leq C_{ij} \parallel w \parallel_{4N+1,\infty} [\parallel w \parallel_{4N+l+2,q} + \parallel w \parallel_{4N+l+2,r}] \\ &\leq C|w|_Y |w|_X, \end{aligned} \tag{23}$$

where $\parallel w \parallel_{4N+l+2,q} + \parallel w \parallel_{4N+l+2,r} \leq C|w|_X$, for $q, r > 2$, $l \geq l_0$ from Sobolev Theorem. Now we estimate for $q > n$,

$$\begin{aligned} &|\partial_0[(w^\alpha, \partial_0 w^\alpha), B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T]|_{l,\infty} \\ &\leq C|\partial_0[(w^\alpha, \partial_0 w^\alpha), B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T]|_{l+1,q} \\ &\leq C|[(\partial_0 w^\alpha, (\Delta - m_0)w^\alpha + f^\alpha), B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T]|_{l+1,q} \\ &+C|[(w^\alpha, \partial_0 w^\alpha), B^\mu_{\alpha\beta}, (\partial_0 w^\beta, (\Delta - m_0)w^\beta + f^\beta)^T]|_{l+1,q}. \end{aligned}$$

It suffices to estimate the first term only, which is bounded by

$$\begin{aligned} &C\{|[(\partial_0 w^\alpha, (\Delta - m_0)w^\alpha), B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T]|_{l+1,q} \\ &\quad +|[(0, f^\alpha), B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T]|_{l+1,q}\} \equiv I_1 + I_2. \end{aligned}$$

Repeating the same proof of (23) with the same choice of (22), we obtain

$$\begin{aligned} I_1 &\leq C_{ij}\{\parallel \partial^i w^\alpha \parallel_{4N+2,q_1(i)} \parallel \partial^j w^\beta \parallel_{4N+2,q_2(j)} \\ &\quad + \parallel \partial^i w^\alpha \parallel_{4N+2,r_1(i)} \parallel \partial^j w^\beta \parallel_{4N+2,r_2(j)}\} \\ &\leq C \parallel w \parallel_{4N+2,\infty} [\parallel w \parallel_{4N+l+3,q} + \parallel w \parallel_{4N+l+3,r}] \leq C|w|_Y |w|_X, \end{aligned}$$

since $q, r > 2$. Applying Theorem 5 with $p_1 = p_2 = 2q$ and $r_1 = r_2 = 2r$, we estimate I_2 as

$$\begin{aligned} I_2 \leq\ & C_{ij}\{|\partial^i f^\alpha|_{4N,2q} \parallel \partial^j w^\beta \parallel_{4N+1,2q} + |\partial^i w^\alpha|_{4N,2r} \parallel \partial^j w^\beta \parallel_{4N+1,2r}\} \\ \leq\ & C\{|f|_{4N+l+1,2q} \parallel w \parallel_{4N+l+1,2q} + |f|_{4N+l+1,2r} \parallel w \parallel_{4N+l+1,2q}\} \leq C|w|_Y |w|_X, \end{aligned}$$

where $|f|_{4N+l+1,2q} + |f|_{4N+l+1,2r} \leq C|w|_Y$ from Lemma 2. In summary, we combine I_1 and I_2 to get

$$|w^\mu(t)|_Y \leq |\omega^\mu(t)|_Y + C|w(t)|_Y |w(t)|_X. \tag{24}$$

Step 2. The estimate of $|\omega(0)|_Z$. Notice that from (12),

$$|\omega^\mu(0)|_Z \leq |w^\mu(0)|_Z + |[(w^\alpha, \partial_0 w^\alpha), B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T]|_Z(0).$$

Applying Theorem 5 with $p_1 = p_2 = 2p$ and $r_1 = r_2 = 2r$, we get

$$\begin{aligned} &|[(w^\alpha, \partial_0 w^\alpha), B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T]|_{l+4,p}(0) \\ \leq\ & C_{ij} \parallel \partial^i w^\alpha(0) \parallel_{4N+1,2p} \parallel \partial^j w^\beta(0) \parallel_{4N+1,2p} \\ & +C_{ij} \parallel \partial^i w^\alpha(0) \parallel_{4N+1,2r} \parallel \partial^j w^\beta(0) \parallel_{4N+1,2r} \\ \leq\ & C|w(0)|^2_X, \end{aligned} \tag{25}$$

since $2p, 2r > 2$, for $l \geq l_0$ and $0 \leq i + j \leq l + 1$. Here $\frac{1}{r} + \frac{a}{n} = \frac{1}{q}$, $r > 2$, and a small. Similarly, with the same choice of p_1 and r_1, we estimate

$$\begin{aligned}
& |\partial_0[(w^\alpha, \partial_0 w^\alpha), B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T]|_{l+3,p}(0) \\
\leq\ & C|[(\partial_0 w^\alpha, (\Delta - m_0)w^\alpha + f^\alpha), B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T]|_{l+3,p}(0) \\
+\ & C|[(w^\alpha, \partial_0 w^\alpha), B^\mu_{\alpha\beta}, (\partial_0 w^\beta, (\Delta - m_0)w^\beta + f^\beta)^T]|_{l+3,p}(0) \qquad (26) \\
\leq\ & C\{\| w(0) \|_{4N+l+5,2p} + |f(0)|_{4N+l+3,2p} + \| w(0) \|_{4N+l+5,2r} + |f(0)|_{4N+l+3,2r}\}^2 \\
\leq\ & C|w(0)|^2_X
\end{aligned}$$

for l large, where we have used the estimate for f in Lemma 2. In summary, from (25) and (26)

$$|\omega^\mu(0)|_Z \leq |w^\mu(0)|_Z + C|w(0)|^2_X. \qquad (27)$$

Step 3. The estimate of $|R_1 + R_2|_{l+3,p}$.

We first estimate R_2. Recall R_2 consists of terms of $\partial_{ij}\Delta^{-1}[h(w^0)w^j]$ and $h_1(w^0)$. By the L^p boundedness of Riesz transform [Ste], $(1 < p < \infty)$

$$|\partial_{ij}\Delta^{-1}[h(w^0)w^j]|_{l+3,p} \leq C|h(w^0)w^j|_{l+3,p} \leq C|w|_Y|w|^2_X$$

for l large. Similarly, $|h_1(w^0)|_{l+3,p} \leq C|w|_Y|w|^2_X$. Hence $|R_2|_{l+3,p} \leq C|w|_Y|w|^2_X$.

Now we estimate R_1. From Theorem 5 with $p_i = 2p, r_i = 2r$ for $i = 1, 2$,

$$\begin{aligned}
|R^\mu_1|_{l+3,p} \leq\ & |[(f^\alpha, \partial_0 f^\alpha), B^\mu_{\alpha\beta}, (w^\beta, \partial_0 w^\beta)^T]|_{l+3,p} \\
& +2|[(0, f^\alpha), B^\mu_{\alpha\beta}, (\partial_0 w^\beta, (\Delta - m_0)w^\beta + f^\beta)^T]|_{l+3,p} \\
& +2|[(\partial_0 w^\alpha, (\Delta - m_0)w^\alpha + f^\alpha), B^\mu_{\alpha\beta}, (0, f^\beta)^T]|_{l+3,p} \\
& +|[(w^\alpha, \partial_0 w^\alpha), B^\mu_{\alpha\beta}, (\partial_0 f^\beta, f^\beta)^T]|_{l+3,p} \\
\leq\ & C\{\| f \|_{4N+l+4,2s} \| w \|_{4N+l+4,2s} \\
& +|f|_{4N+l+3,2s}[\| w \|_{4N+l+5,2s} + |f|_{4N+l+3,2s}]\},
\end{aligned}$$

where $s = 2p$ or $s = 2r > 2$. Now from Lemma 2, $\| f \|_{4N+l+4,2s} \leq C|w|_X|w|_Y$. Applying Sobolev Imbedding Theorem ($2s > 2$) to terms with w yields($l \geq l_0$),

$$|R^\mu_1|_{l+3,p} \leq C|w|_Y|w|^2_X. \qquad (28)$$

Combining (27) and (28), we now have

$$\begin{aligned}
|\omega(t)|_Y \leq\ & C(1+t)^{-3/2+3/p'}|\omega(0)|_Z + \int_0^t (1+t-\tau)^{-3/2+3/p'}|R_1 + R_2|_{l+4,p}(\tau)d\tau \\
\leq\ & C\{(1+t)^{-3/2+3/p'}\epsilon_0 + \int_0^t (1+t-\tau)^{-3/2+3/p'}|w(\tau)|_Y|w(\tau)|^2_X d\tau\} \\
\leq\ & C\{(1+t)^{-3/2+3/p'}\epsilon_0 + |||w|||^3_{T_*}\int_0^t (1+t-\tau)^{-3/2+3/p'}(1+\tau)^{-3/2+3/p'}d\tau\} \\
\leq\ & C(1+t)^{-3/2+3/p'}[\epsilon_0 + |||w|||^3_{T_*}],
\end{aligned}$$

since $p' > 6$. Notice that $(1+t)^{3/2-3/p'}|w(t)|_X|w(t)|_Y \leq C|||w|||^2_{T_*}$, from (24), we conclude, for $0 \leq t \leq T_*$,

$$\begin{aligned}
(1+t)^{3/2-3/p'}|w(t)|_Y \leq\ & (1+t)^{3/2-3/p'}[|\omega(t)|_Y + C|w(t)|_X|w(t)|_Y] \qquad (29) \\
\leq\ & C[\epsilon_0 + |||w|||^2_{T_*}].
\end{aligned}$$

5. The Energy Estimate and Global Existence

We now derive the high-order energy estimate.

Lemma 8 (Energy Estimate). *Let* $|w(0)|_X \leq \epsilon_0$ *and* $|||w|||_{T_*} < \infty$ *for* $0 \leq t \leq T_*$, *then*

$$|w(t)|_X^2 + |\nabla\psi(t)|_X^2 \leq \epsilon_0^2 + C|||w|||_{T_*}^3. \tag{30}$$

Proof. We take the derivative in (7) with $\partial^\alpha = \partial_0\partial^{l-1}$ or $\partial^\alpha = \partial^l$. Taking the vector inner product with $\partial^\alpha w$, we have

$$\begin{aligned}&\langle \partial_0\partial^\alpha w, \partial^\alpha w\rangle + \langle A_j(w)\partial_j\partial^\alpha w, \partial^\alpha w\rangle \\ &= c_0^{-2}\langle \partial^\alpha\nabla\psi, \partial^\alpha v\rangle + c_{\alpha\beta}\langle \partial^\beta A_j(w)\partial^{\alpha-\beta}\partial_j w, \partial^\alpha w\rangle,\end{aligned}$$

where summation over $|\beta| < |\alpha|$, and $\langle w_1, w_2\rangle = w_1^T w_2$, for some constants $c_{\alpha\beta}$. Since A_j is symmetric, we rewrite the above as

$$\begin{aligned}&\partial_0\langle \partial^\alpha w, \partial^\alpha w\rangle + \partial_j\langle A_j(w)\partial^\alpha w, \partial^\alpha w\rangle \\ &= \frac{2}{c_0^2}\langle \partial^\alpha\nabla\psi, \partial^\alpha v\rangle + 2c_{\alpha\beta}\langle \partial^\beta A_j(w)\partial^{\alpha-\beta}\partial_j w, \partial^\alpha w\rangle \\ &\quad + \langle \partial_j A_j(w)\partial^\alpha w, \partial^\alpha w\rangle.\end{aligned} \tag{31}$$

We now take ∂^α in the last equation in (5),

$$\partial_t\partial^\alpha\nabla\psi = -n_0\partial^\alpha v - n_0\nabla\Delta^{-1}\nabla\cdot\partial^\alpha\{[(\frac{\gamma-1}{2}m+1)^{2/(\gamma-1)} - 1]v\}.$$

Plugging $\partial^\alpha v$ back into (31) and integrating over $[0,t]\times\mathrm{R}^3$, we obtain

$$\begin{aligned}&|\partial^\alpha w(t)|_2^2 + \frac{1}{c_0^2 n_0}|\partial^\alpha\nabla\psi(t)|_2^2 \\ \leq\ & \epsilon_0^2 + C\int_0^t |\partial^\alpha\nabla\psi|_2|\nabla\Delta^{-1}\nabla\cdot\partial^\alpha\{[(\frac{\gamma-1}{2}m+1)^{2/(\gamma-1)} - 1]v\}|_2 d\tau \\ &+ C\int_0^t |\partial^\beta A_j(w)\partial^{\alpha-\beta}\partial_j w|_2|\partial^\alpha w|_2 d\tau \\ \leq\ & \epsilon_0^2 + C\int_0^t |w(\tau)|_Y|w(\tau)|_X^2 d\tau,\end{aligned} \tag{32}$$

by the L^2 boundedness of the Riesz transform and $|\partial^\beta A_j(w)\partial^{\alpha-\beta}\partial_j w|_2 \leq C|w|_Y|w|_X$ for $|\alpha| > |\beta|$. We thus further estimate the above by

$$\epsilon_0^2 + C|||w|||_{T_*}^3\int_0^t (1+t)^{-3/2+3/p'}d\tau \leq \epsilon_0^2 + C|||w|||_{T_*}^3. \qquad \square$$

Now are ready for the global existence theorem.

Theorem 9 (Global Existence). *Let $l \geq l_0$ and l_0 be large enough. Let $|w(0)|_Z + |w(0)|_X = \epsilon_0$, and*

$$\partial_j w^i(0) = \partial_i w^j(0) \text{ (irrotationality).}$$

There exists a unique global solution $w(t)$ of the Euler–Poisson system (5) with

$$|||w(t)||| = \sup_{t>0}[|w|_X + |\nabla\psi|_X + (1+t)^{3/2-3/p'}|w|_Y] < \infty,$$

provided ϵ_0 small enough.

Proof. For $l \geq 3/2+1$, based on the energy estimate (32) and standard approximations, the existence of a local regular solution $w(t) \in C([0,T_*),X)$ of (6) follows the standard method of [Ka]. We then combine (29) and (30) to obtain

$$|||w(t)|||_{T_*} \leq C\epsilon_0 + C(|||w(t)|||_{T_*}^2 + |||w(t)|||_{T_*}^{3/2}),$$

provided $|||w|||_{T_*} \leq 1$. It follows from the bootstrap argument that $|||w(t)|||_{T_*} \leq C\epsilon_0$, if ϵ_0 is small and $T_* = \infty$. □

We remark that the condition $|w(0)|_Z + |w(0)|_X = \epsilon_0$ is implicit for the first order Euler–Poisson system. Not explicitly given as an initial condition, $\partial_0 w(0)$ is determined by (5). This implies that the non-local term $\nabla\psi$ is not only in L^2, but also has to be in L^p for p near 1. Equivalently,

$$\nabla\Delta^{-1}(n(0,x) - n_0) \in L^p.$$

This, however, can be achieved by a natural "neutral condition" for $n - n_0$ as follows.

Proof of Theorem 1. Let $\rho = n(0,x) - n_0 \in C_c^\infty$ and $\int \rho = 0$, we show $\nabla\Delta^{-1}\rho \in L^p$, for any $1 < p < 3/2$. Then $|w(0)|_Z < \infty$ follows from (5). Equivalently, we show that $\mathcal{I}_1(\rho) \in L^p$. Here $\mathcal{I}_1$ is the Riesz potential of order one. Since [Ste]

$$\begin{aligned}\mathcal{I}_1(\rho)(x) &= c\int_{\mathbf{R}^3} \rho(x-y)|y|^{-2}dy \\ &= c\int_0^\infty \{r^2 \int_{|\xi|=1} \rho(x - r\xi)d\xi\} r^{-2}dr \text{ (spherical coordinates)} \\ &= 2c\int_0^\infty \{\int_0^r \tau^2 \int_{|\xi|=1} \rho(x-\tau\xi)d\xi d\tau\} r^{-3}dr\end{aligned}$$

via an integration by part over the radial variable r. Here c is some numerical constant. Let supp $\rho(z) \subseteq \{|z| \leq d\}$. We claim that for $|x| > d$,

$$\mathcal{I}_1(\rho)(x) = 2c\int_{|x|-d}^{|x|+d} \{\int_0^r \tau^2 \int_{|\xi|=1} \rho(x-\tau\xi)d\xi d\tau\} r^{-3}dr. \tag{33}$$

To prove (33), we show that the support of $\int_0^r \tau^2 \int_{|\xi|=1} \rho(x-\tau\xi)d\xi d\tau$ is in $[|x|-d, |x|+d]$. In fact, if $r \leq |x|-d$, then $|x-\tau\xi| \geq |x|-\tau \geq |x|-(|x|-d) = d$, hence $\rho(x-\tau\xi) = 0$. It follows

$$\int_0^r \tau^2 \int_{|\xi|=1} \rho(x-\tau\xi)d\xi d\tau = 0.$$

On the other hand, if $r \geq |x| + d$, then for $\tau \geq r$, $|x - \tau\xi| \geq \tau - |x| \geq d$. Hence $\rho(x - \tau\xi) = 0$. Therefore

$$\begin{aligned}
&\int_0^r \tau^2 \int_{|\xi|=1} \rho(x - \tau\xi) d\xi d\tau \\
&= \int_0^r \tau^2 \int_{|\xi|=1} \rho(x - \tau\xi) d\xi d\tau + \int_r^\infty \tau^2 \int_{|\xi|=1} \rho(x - \tau\xi) d\xi d\tau \\
&= \int_0^\infty \tau^2 \int_{|\xi|=1} \rho(x - \tau\xi) d\xi d\tau \\
&= \int_{\mathbf{R}^3} \rho(x - y) dy = 0,
\end{aligned}$$

for $r \geq |x| + d$, from the neutral condition $\int \rho = 0$. We therefore proved (33).

Now from (33) and

$$\left| \int_0^r \tau^2 \int_{|\xi|=1} \rho(x - \tau\xi) d\xi d\tau \right| = \left| \int_{|y| \leq r} \rho(x - y) dy \right| \leq C d^3 |\rho|_\infty,$$

we deduce that $|\mathcal{I}_1(\rho)|(x) \leq C|x|^{-3}$ for $|x|$ large, thus $\mathcal{I}_1(\rho) \in L^p(\{|x| \geq 1\})$ for $1 < p < 3/2$. By the Hardy-Littlewood-Sobolev's inequality, $\mathcal{I}_1(\rho) \in L^2_{loc}(\mathbf{R}^3)$.

Hence $\mathcal{I}_1(\rho) \in L^p(\mathrm{R}^3)$ for any $1 < p < 3/2$. $\square$

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