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Asymptotic Expansions in the Central Limit Theorem for a Special Class of m-Dependent Random Fields II — Lattice Case

By LOTHAR HEINRICH of Freiberg

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1. Introduction

This paper continues the investigation of EDGEWORTH expansions for m-dependent random fields (RF's) which has been started in [6]. We essentially adhere to the notation introduced in [6], however, for better readability we will recall (or redefine) those parts of the notation used here. As in [6] a family of random variables (RV's) X_z, $z \in Z^d = \{0, \pm 1, \pm 2, \ldots\}^d$, defined on $(\Omega, \mathfrak{F}, P)$ is said to be an $(m_1, \ldots, m_d)$-*dependent RF* if for any finite $U, V \subset Z^d$ the random vectors $(X_u)_{u \in U}$ and $(X_v)_{v \in V}$ are independent whenever $\min_{u \in U,\, v \in V} |u_j - v_j| > m_j$ for at least one $j \in \{1, \ldots, d\}$. Here and throughout, if no confusion is possible, z_j denotes the j-th component of $z \in Z^d$. In case $m_1 = \cdots = m_d = m$ $(\geqq 0)$ the RF is usually called *m-dependent*. In the following we list a number of notations and abbreviations (always assuming that the quantities in question exist):

$$|V| = \text{card } V, \quad M_1 = \prod_{j=1}^{d} (m_j + 1), \quad X_z^{(a,b)} = X_z 1_{\{a \leqq |X_z| < b\}},$$
$$(0 \leqq a < b \leqq \infty),$$

where $1_A(\cdot)$ denotes the indicator function of the event $A \in \mathfrak{F}$,

$$S_V^{(a,b)} = \sum_{z \in V} X_z^{(a,b)}, \quad S_V = S_V^{(0,\infty)}, \quad B_V^2 = D^2 S_V, \quad L_{kV}^{(a,b)} = B_V^{-k} \sum_{z \in V} E\, |X_z^{(a,b)}|^k$$

$$L_{kV} = L_{kV}^{(0,\infty)} \quad (k \geqq 1), \qquad \Gamma_k(S_V) = \frac{1}{i^k} \frac{d^k}{dt^k} \ln E e^{itS_V}|_{t=0},$$

$$V_z(p, q) = \{y \in Z^d : z_j + q m_j \leqq y_j$$
$$\leqq z_j + (p - q)(m_j + 1) - 1, j = 1, \ldots, d\}, \qquad p, q \in Z^1,$$

$$\mathfrak{Z}(p) = \{z \in Z^d : z_j = p(m_j + 1) k_j + 1, k_j \in Z^1, j = 1, \ldots, d\},$$

$$p = 1, 2, \ldots,$$

$$Y_z^{(p)} = \sum_{y \in V \cap V_z(p,0)} X_y\,^{1)}, \qquad \varphi(x) = (2\pi)^{-1/2} e^{-x^2/2},$$

1) Note that in [6] $Y_z^{(p)}$ is an abbreviation for another sum.

$$q_{kV}(x) = \sum_{l=1}^{k} \frac{(-1)^{k+2l}}{l!\,B_V^{k+2l}} \sum_{\substack{k_1+\cdots+k_l=k\\ k_i\ge 1}} \prod_{i=1}^{l} \frac{\Gamma_{k_i+2}(S_V)}{(k_i+2)!} \frac{d^{k+2l}}{dx^{k+2l}} \varphi(x), \qquad k \geqq 1,$$

$$P_{kV}(t) = \sum_{l=1}^{k} \frac{(-1)^{k+2l}}{l!\,B_V^{k+2l}} \sum_{\substack{k_1+\cdots+k_l=k\\ k_i\ge 1}} \prod_{i=1}^{l} \frac{\Gamma_{k_i+2}(S_V)}{(k_i+2)!}, \qquad k \geqq 1,$$

where the latter two terms are connected by the identity

$$\int_{-\infty}^{\infty} e^{itx} q_{kV}(x)\,dx = e^{-t^2/2} P_{kV}(t), \quad \text{see e.g. [11]}. \tag{1.1}$$

Throughout $C(.), C_0(.), C_1(.), \ldots$ denote positive constants (only depending on the quantities indicated in the parenthesis) which may be different from one expression to another. For a detailed review of recent developments of the central limit theory for weakly dependent RF's the reader is referred to [6]. In addition to the references therein we mention three further papers – [2], [7], [13] – dealing with rates of convergence in the central limit theorem (CLT) for m-dependent RF's. An interesting result obtained in [13] by using STEIN's technique is the following global estimate:

$$\int_{-\infty}^{\infty} \left| P(S_V - ES_V < xB_V) - \int_{-\infty}^{x} \varphi(y)\,dy \right| dx \leqq C_0(d, \delta)\,(m+1)^{d(1+\delta)}\,L_{2+\delta,V} \tag{1.2}$$

for $0 < \delta \leqq 1$.

However, to the author's knowledge, the corresponding (uniform) bound of the integrand in (1.2) is still unproved for $d \geqq 2$. This problem could be solved up to now only for a special case of $(m_1, \ldots, m_d)$-dependent RF's, namely, if the X_z's can be defined as functions of independent RV's, more precisely, if

$$X_z = f_z(\xi_y : y \in V_z(1, 0)), \qquad z \in V_n = \mathop{\times}_{i=1}^{d} \{1, 2, \ldots, n_i\}, \tag{1.3}$$

where $f_z \mid K^{M_1} \to R^1$, $z \in V_n$, is a family of BOREL measurable mappings (called "*window functions*" in [2]) and ξ_z, $z \in Z^d$, is a field of independent RV's on $(\Omega, \mathfrak{F}, P)$ taking values in the measurable space $[K, \mathfrak{K}]$. Further, assume that $n_j = n_j(n)$, $m_j = m_j(n)$, $j = 1, \ldots, d$, are non-decreasing sequences satisfying $1 \leqq \min\limits_{1\le j\le d} n_j/(m_j+1) \to \infty$ as $n \to \infty$. For notational convenience we write S_n, B_n, L_{kn} instead of S_{V_n}, B_{V_n}, L_{kV_n}. To express the magnitude of the error in the normal approximation of S_n in terms of $|V_n|$ we further suppose

$$B_n^2 \geqq C_1 |V_n|, \qquad \max_{z\in V_n} D^2 Y_z^{(1)} \leqq C_2 M_1 \text{ [2]} \tag{1.4}$$

and, for some integer $s \geqq 3$,

$$\sum_{z\in V_n} E\,|X_z|^s \leqq C_3\,|V_n|, \tag{1.5}$$

$$\frac{1}{|V_n|} \sum_{z\in V_n} E\,|X_z - EX_z|^s\, 1_{\{|X_z - EX_z| \ge |V_n|^{1/2(s-1)}\}} \xrightarrow[n\to\infty]{} 0. \tag{1.6}$$

[2]) In the definition of $Y_z^{(p)}$ the set V must be replaced by V_n if the underlying RF is formed by the RV's (1.3)

Under (1.4) and (1.5) (for $s = 3$) uniform and non-uniform BERRY-ESSEEN bounds in the CLT for S_n were obtained in [5] (Theorem 6) (see also Corollary 2 in [2]) and [6] (Theorem 5). The main results in [6] concern EDGEWORTH expansions of the distribution function of S_n and its derivatives under certain conditional CRAMÉR-PETROV type conditions. First versions of such conditions were introduced into summation theory of weakly dependent RV's by V. A. STATULEVIČIUS (see [15], [16]) in the case of MARKOV dependent RV's and in [1], [4] for RV's admitting the representation (1.3) for $d = 1$. For strictly stationary sequences and fields of RV's of the form (1.3) (with $f_z = f$, $n_j = n$, $m_j = m$ fixed, $(\xi_z)_{z\in Z^d}$ are i.i.d.) under some natural conditions F. GÖTZE and C. HIPP (see [7] and [2]) have shown the condition $\lim\limits_{n\to\infty} Ee^{itS_n} = 0$ for every $t \in R^1 \setminus \{0\}$ be necessary and sufficient for the validity of the EDGEWORTH expansion of length two. It was further shown there that the latter condition is satisfied if and only if either

(i) $$v(t) := E\left|E\left(\exp\left(it \sum_{z\in\{1,\dots,m+1\}^d} X_z\right)\middle\|\,\xi_y; y \in \{1,\dots,m,m+2,\dots,2m+1\}^d\right)\right| < 1$$

for all $t \in R^1 \setminus \{0\}$

or

(ii) $$E\exp\left(it_0k \sum_{z\in\{1,\dots,m\}^d} X_z\right) = 0$$

for $k \in Z^1 \setminus \{0\}$, where $t_0 = \inf\{t > 0: v(t) = 1\}$. [3])

The main goal of the present paper is to establish higher-order normal approximations for the probabilities $P(S_n = N)$, $N \in Z^1$, provided that the RV's (1.3) are integer-valued (see Theorem 1 and 3 in Section 4). For this purpose an additional condition must be imposed on the RF $\{X_z, z \in V_n\}$ to ensure a sufficiently rapid decrease (as $n \to \infty$) of the term $\int\limits_{\varepsilon<|t|\le\pi} \left|\frac{d^s}{dt^s} Ee^{itS_n}\right| dt$, $\varepsilon > 0$, $s \ge 3$ (see Lemma 2). A detailed discussion of this condition is contained in Section 3. The formulation and the proofs of the main theorems are given in Section 4. Finally, in Section 5, some examples will be considered which fit into our general "window model". But first of all we prove a general inequality which has been formulated in [6] (as Lemma 7) and used there to show a non-uniform BERRY-ESSEEN bound.

2. A general truncation inequality for $(m_1, \dots, m_d)$-dependent RF's

Define, for $0 \le a \le b \le \infty$ and $x \in R^1$,

$$\Delta_V^{(a,b)}(x) = |P(S_V^{(0,a)} < xB_V) - P(S_V^{(0,b)} < xB_V)|$$

and

$$\alpha = \Big(\sum_{z\in V} EX_z^2\Big)^{1/2}, \quad \beta = (1 + |x|)\, B_V,$$

$$\gamma = \Big(\sum_{z\in V} E\,|X_z^{(\alpha,\beta)}|^p\Big)^{1/p}, \quad \delta = |x|\, B_V/8 \cdot 3^d M_1$$

[3]) In [17] conditions for the same type of RF's are given to hold the local CLT for lattice distributions.

Lemma 1. *Let $\{X_z, z \in V \subset Z^d\}$ be an $(m_1, \ldots, m_d)$-dependent RF satisfying $EX_z^2 < \infty$ and $EX_z = 0$, $z \in V$. Then, for every integer $p \geqq 2$ and $x \in R^1$,*

$$(2.1) \qquad \Delta_V^{(\alpha,\infty)}(x) \leqq \sum_{z\in V} P(|X_z| \geqq \beta) + C(p, d) \left(\frac{M_1}{1 + |x|}\right)^p L_{pV}^{(\alpha,\beta)},$$

where the second summand on the right-hand side vanishes if $\alpha > \beta$.

For $d = 1$ and $m_1 = 0$ the inequality (2.1) is a little stronger than the assertion of Lemma 6 of Chapter VI in [11] (p. 148). Since the proving idea of Lemma 1 is quite different from the convolution technique employed to show the corresponding result for independent RV's in [11] the proof of (2.1) might be of own interest. Before a detailed proof of (2.1) will be given we first state two consequences of Lemma 1. Together with

$$\sum_{z\in V} P(|X_z| \geqq \beta) \leqq (1 + |x|)^{-p} L_{pV}^{(\beta,\infty)}$$

Lemma 1 immediately leads to

Corollary 1. *Let the conditions of Lemma 1 and $E|X_z|^p < \infty$, $z \in V$, for some integer $p \geqq 2$ be satisfied. Then, for $x \in R^1$,*

$$(2.2) \qquad \Delta_V^{(\alpha,\infty)}(x) \leqq \frac{1 + C(p, d)}{(1 + |x|)^p} M_1^p L_{pV}^{(\alpha,\beta)}.$$

Corollary 2. *Let the conditions of Lemma 1 and $E|X_z|^p < \infty$, $z \in V$, for some integer $p \geqq 1$ be satisfied. Then*

$$(2.3) \qquad \int_{-\infty}^{\infty} (1 + |x|)^{p-1} \Delta_V^{(\alpha,\infty)}(x)\, dx \leqq 2pL_{pV}^{(B_V,\infty)} + 2C(p + 1, d)\, M_1^{p+1} L_{pV}^{(\alpha,\infty)}.$$

Proof of Corollary 2. In (2.1) we replace p by $p + 1$. Substituting $1 + x$ by x and applying FUBINI's theorem we find that

$$\int_0^\infty (1 + x)^{p-1} P\big(|X_z| \geqq (1 + x)\, B_V\big)\, dx \leqq \frac{p}{B_V} \int_{|y|\geqq B_V} |y|^p\, dP(X_z < y)$$

and

$$\int_0^\infty \frac{1}{(1 + x)^2} \int_{\alpha\leqq|y|\leqq(1+x)B_V} |y|^{p+1}\, dP(X_z < y)\, dx \leqq \int_{|y|\geqq\alpha} |y|^p\, dP(X_z < y).$$

Since the right-hand side of (2.1) is an even function in x the estimate (2.3) is completely proved.

Proof of Lemma 1. The starting point is the following identity:

$$(2.4) \qquad \Delta_V^{(a,b)}(x) = \Bigg|P\left(S_V^{(0,a)} \underset{(\leqq)}{\geqq} xB_V, \bigcup_{z\in V} \{a \leqq |X_z| < b\}\right) - P\left(S_V^{(0,b)} \underset{(\leqq)}{\geqq} xB_V, \bigcup_{z\in V} \{a \leqq |X_z| < b\}\right)\Bigg| \quad \text{for } x \underset{(<)}{\geqq} 0.$$

This implies

$$\Delta_V^{(a,b)}(x) \leqq \sum_{z \in V} P(a \leqq |X_z| < b)$$

and so, letting $a = \beta$ and $b = \infty$, (2.1) is trivially shown when $\alpha > \beta$. From now on we suppose $\alpha \leqq \beta$ and for reasons of symmetry we only verify Lemma 1 for $x \geqq 0$. For doing this we distinguish five cases.

Case (i): $0 \leqq x \leqq 1$

For all $x \in R^1$ we have

$$\Delta_V^{(\alpha,\infty)}(x) \leqq \sum_{z \in V} P(|X_z| \geqq \beta) + \left(\frac{B_V}{\alpha}\right)^p L_{pV}^{(\alpha,\beta)}, \tag{2.5}$$

and, in view of $B_V^2 \leqq M_1\alpha^2$, it follows

$$\Delta_V^{(\alpha,\infty)}(x) \leqq \sum_{z \in V} P(|X_z| \geqq \beta) + 2^p \left(\frac{M_1}{1+x}\right)^p L_{pV}^{(\alpha,\beta)}$$

for $0 \leqq x \leqq 1$.

Case (ii): $\delta \leqq \gamma$, $x \geqq 1$

By definition of δ and γ,

$$1 \leqq \left(\frac{\gamma}{\delta}\right)^p \leqq \left(\frac{16 \cdot 3^d \cdot M_1}{1+x}\right)^p L_{pV}^{(\alpha,\beta)}.$$

Hence, (2.1) is evidently valid with $C(p, d) = (16 \cdot 3^d)^p$.

Case (iii): $\gamma < \delta \leqq \alpha$, $x \geqq 1$

Clearly,

$$\frac{B_V}{\alpha} \leqq \frac{B_V}{\delta} = \frac{8 \cdot 3^d \cdot M_1}{x} \leqq 16 \cdot 3^d \frac{M_1}{1+x}.$$

Together with (2.5) we get (2.1) with $C(p, d) = (16 \cdot 3^d)^p$.

Case (iv): $\gamma \leqq \alpha < \delta$, $x \geqq 1$

From (2.4) it is easily seen that

$$\Delta_V^{(\alpha,\infty)}(x) \leqq \Delta_V^{(\alpha,\delta)}(x) + \sum_{z \in V} P(|X_z| \geqq \delta)$$

and

$$\Delta_V^{(\alpha,\delta)}(x) \leqq \sum_{z \in \mathfrak{Z}(1)} P\left(S_V^{(0,\alpha)} \geqq \frac{1}{2} x B_V, \bigcup_{y \in V \cap V_z(1,0)} \{\alpha \leqq |X_y| < \delta\}\right)$$

$$+ \sum_{z \in \mathfrak{Z}(1)} P\left(S_V^{(\alpha,\delta)} \geqq \frac{1}{2} x B_V, \bigcup_{y \in V \cap V_z(1,0)} \{\alpha \leqq |X_y| < \delta\}\right).$$

The definition of δ entails

$$\left|\sum_{y \in V \cap V_z(1,-1)} X_y^{(0,\alpha)}\right| \leqq \frac{1}{8} x B_V \quad \text{and} \quad \left|\sum_{y \in V \cap V_z(1,-1)} X_y^{(\alpha,\delta)}\right| \leqq \frac{1}{8} x B_V$$

with probability 1 for $z \in \mathfrak{Z}(1)$.

By virtue of the $(m_1, \ldots, m_d)$-dependence we find that

$$P\left(S_V^{(\alpha,\delta)} \geqq \frac{1}{2} xB_V, \bigcup_{y\in V\cap V_z(1,0)} \{\alpha \leqq |X_y| < \delta\}\right)$$

$$\leqq P\left(S_V^{(\alpha,\delta)} - \sum_{y\in V\cap V_z(1,0)} X_y^{(\alpha,\delta)} \geqq \frac{3}{8} xB_V, \bigcup_{y\in V\cap V_z(1,0)} \{\alpha \leqq |X_y| < \delta\}\right)$$

$$\leqq P\left(S_V^{(\alpha,\delta)} \geqq \frac{1}{4} xB_V\right) \sum_{y\in V\cap V_z(1,0)} P(\alpha \leqq |X_y| < \delta).$$

The same inequality holds when $S_V^{(\alpha,\delta)}$ is replaced by $S_V^{(0,\alpha)}$. This gives

$$\Delta_V^{(\alpha,\delta)}(x) \leqq \left(P\left(S_V^{(0,\alpha)} \geqq \frac{1}{4} xB_V\right) + P\left(S_V^{(\alpha,\delta)} \geqq \frac{1}{4} xB_V\right)\right) \sum_{z\in V} P(\alpha \leqq |X_z| < \delta). \tag{2.6}$$

In the next step we have to estimate $E(S_V^{(0,\alpha)})^{2p}$ and $E(S_V^{(\alpha,\delta)})^{2p}$. Therefore we rewrite $S_V^{(a,b)}$ as follows:

$$S_V^{(a,b)} = \sum_{j=0}^{d} \sum_{\substack{j_1+\cdots+j_d=j\\ j_i\in\{0,1\}}} \sum_{z\in\mathfrak{Z}(j_1,\ldots,j_d)} Y_z^{(a,b)},$$

where

$$\mathfrak{Z}(j_1, \ldots, j_d) = \mathop{\times}_{i=1}^{d} \{(2k_i + j_i)(m_i + 1), k_i \in Z^1\}$$

and

$$Y_z^{(a,b)} = \sum_{y\in V\cap V_z(1,0)} X_y^{(a,b)}.$$

Using the elementary inequality $|x_1 + \cdots + x_n|^r \leqq n^{r-1}(|x_1|^r + \cdots + |x_n|^r)$ we get

$$E(S_V^{(0,\alpha)})^{2p} \leqq 2^{d(2p-1)} \sum_{j=0}^{d} \sum_{\substack{j_1+\cdots+j_d=j\\ j_i\in\{0,1\}}} E\left(\sum_{z\in\mathfrak{Z}(j_1,\ldots,j_d)} Y_z^{(0,\alpha)}\right)^{2p}$$

and as a consequence of the mutual independence of the RV's $Y_z^{(0,\alpha)}$, $z \in \mathfrak{Z}(j_1, \ldots, j_d)$, the $2p$-th moment of their sum is equal to

$$\sum_{l=1}^{2p} \sum_{\substack{p_1+\cdots+p_l=2p\\ p_i\geqq 1}} \binom{2p}{p_1, \ldots, p_l} \frac{1}{l!} \sum_{\substack{z_1,\ldots,z_l\in\mathfrak{Z}(j_1,\ldots,j_d)\\ z_i\neq z_j}} \prod_{i=1}^{l} E(Y_{z_i}^{(0,\alpha)})^{p_i}.$$

Since, for $q = 1, 2, \ldots$ (using $EX_y = 0$ for $q = 1$),

$$|E(Y_z^{(0,\alpha)})^q| \leqq M_1^{q-1}\alpha^{q-2} \sum_{z\in V\cap V_z(1,0)} EX_y^2$$

the combination of the above estimates yields

$$E(S_V^{(0,\alpha)})^{2p} \leqq \alpha^{2p} M_1^{2p-1} 4^{dp} C(p), \tag{2.7}$$

where

$$C(p) = \sum_{l=1}^{2p} \frac{1}{l!} \sum_{\substack{p_1+\cdots+p_l=2p\\ p_i\geqq 1}} \binom{2p}{p_1, \ldots, p_l}.$$

In like manner one can show that

$$|E(Y_z^{(\alpha,\delta)})^q| \leqq M_1^{q-1} \sum_{y \in V \cap V_z(1,0)} E\,|X_y^{(\alpha,\delta)}|^p \begin{cases} \delta^{q-p} & \text{for } p < q \leqq 2p \\ \alpha^{q-p} & \text{for } 1 \leqq q \leqq p \end{cases}.$$

Since at most one $p_i > p$ and $\gamma \leqq \alpha < \delta$ we obtain

$$\sum_{\substack{z_1,\ldots,z_l \in \mathfrak{Z}(j_1,\ldots,j_d) \\ z_i \neq z_j}} \prod_{i=1}^{l} E(Y_{z_i}^{(\alpha,\delta)})^{p_i} \leqq M_1^{2p-l} \alpha^{2p-p_i-p(l-1)} \delta^{p_i-p} \gamma^{lp} \leqq M_1^{2p-l} \alpha^p \delta^p$$

showing that

$$E(S_V^{(\alpha,\delta)})^{2p} \leqq (\alpha\delta)^p\, M_1^{2p-1} 4^{pd} C(p)\,. \tag{2.8}$$

From the estimates (2.6)–(2.8) we conclude that

$$\begin{aligned} \Delta_V^{(\alpha,\delta)}(x) &\leqq 2 \cdot 4^{dp} C(p)\, M_1^{2p} \frac{(\alpha\delta)^p\, 4^{2p}}{(xB_V)^{2p}\, \alpha^p} \sum_{z \in V} E\,|X_z^{(\alpha,\delta)}|^p \\ &\leqq 2^{2p+1} \left(\frac{4}{3}\right)^{dp} C(p)\, M_1^p (1+x)^{-p} \sum_{z \in V} E\,|X_z^{(\alpha,\beta)}|^p. \end{aligned}$$

Finally, together with the estimate

$$\sum_{z \in V} P(|X_z| \geqq \delta) \leqq \sum_{z \cdots} P(|X_z| \geqq \beta) + \left(\frac{16 \cdot 3^d M_1}{1+x}\right)^p L_{pV}^{(\alpha,\beta)} \tag{2.9}$$

(2.1) is proved for $\gamma \leqq \alpha < \delta$.

Case (v): $\alpha < \gamma < \delta$, $x \geqq 1$

Obviously, we have

$$\Delta_V^{(\alpha,\infty)}(x) \leqq \Delta_V^{(\alpha,\gamma)}(x) + \Delta_V^{(\gamma,\delta)}(x) + \Delta_V^{(\delta,\infty)}(x) \tag{2.10}$$

and the relation (2.4) yields

$$\Delta_V^{(\alpha,\gamma)}(x) \leqq (xB_V)^{-2p} \max\{E(S_V^{(0,\alpha)})^{2p}, E(S_V^{(0,\gamma)})^{2p}\}$$

and

$$\begin{aligned} \Delta_V^{(\gamma,\delta)}(x) &\leqq P\left(S_V^{(0,\gamma)} \geqq \frac{1}{2}\, xB_V\right) + P\left(S_V^{(\gamma,\delta)} \geqq \frac{1}{2}\, xB_V\right) \\ &\leqq 4^p (xB_V)^{-2p} \left(E(S_V^{(0,\gamma)})^{2p} + E(S_V^{(\gamma,\delta)})^{2p}\right). \end{aligned}$$

In the same way as before one can verify the following two estimates:

$$E(S_V^{(0,\gamma)})^{2p} \leqq 4^{dp} C(p)\, M_1^{2p-1} \gamma^{2p}$$

and

$$E(S_V^{(\gamma,\delta)})^{2p} \leqq (\gamma\delta)^p\, 4^{dp} C(p)\, M_1^{2p-1}.$$

Hence, taking into account the assumption $\alpha < \gamma < \delta$, we find that

$$\Delta_V^{(\alpha,\gamma)}(x) \leqq \left(\frac{4}{3}\right)^{dp} C(p) \left(\frac{\gamma}{1+x}\right)^p M_1^{p-1}$$

and

$$\Delta_V^{(\gamma,\delta)}(x) \leqq 2\left(\frac{4}{3}\right)^{dp} C(p)\left(\frac{\gamma}{1+x}\right)^p M_1^{p-1}.$$

These two estimates, (2.9), and (2.10) immediately imply (2.1). Thus, the proof of Lemma 1 is completed.

3. A sufficient condition for the validity of local limit theorems for lattice random variables

We confine ourselves to the consideration of RF's formed by integer-valued RV's of type (1.3). We note the obvious fact that the apparently more general class of RV's $f_z(\xi_y: y \in V_z(1,0))$ which take values on the lattice $\{a_z + kh, k \in Z^1\}$ (for some real numbers a_z and $h > 0$) can be transformed into a family of integer-valued RV's. Even in the simplest case of i.i.d. RV's it is well-known that, in order to obtain asymptotic expansions for the probabilities $P(S_n = N)$, $N \in Z^1$, the maximal span of the lattice on which X_z is concentrated must be equal to 1, i.e. $\min\limits_{0\leq r\leq q-1} P(X_z \not\equiv r (\text{mod } q)) > 0$ for $q = 2, 3, \ldots$ (see [11], Chapter 7).

Later in this section we shall see that the latter condition alone does not always imply the local CLT

$$B_nP(S_n = N) - \varphi((N - ES_n)/B_n) \xrightarrow[n\to\infty]{} 0 \tag{3.1}$$

for (stationary) m-dependent sequences and fields.

Next we introduce some further notations and give a slightly modified version of Lemma 3 from [6]. For $p = 1, 2, \ldots$ we put

$$U_z(p) = \{y \in Z^d: z_j - p(m_j + 1) - m_j + 1 \leqq y_j \leqq z_j + m_j, j = 1, \ldots, d\},$$

$$g_z^{(p)}(t) = E\,|E(e^{itY_z^{(p)}} \,\|\, \mathfrak{F}_z^{(p)})|^{2^d}\,{}^{4)},$$

$$(B_n^{(p)})^2 = \sum_{z\in\mathfrak{Z}(p)} E(Y_z^{(p)} - E(Y_z^{(p)} \,\|\, \mathfrak{F}_z^{(p)}))^2,$$

where $\mathfrak{F}_z^{(p)}$ denotes the σ-algebra generated by the family of independent RV's

$$\left\{\xi_y, y \in \left(\bigcup_{u\in V_n\cap V_z(p,0)} V_u(1,0)\right) \setminus V_z(p+1,1)\right\}.$$

Lemma 2 (see [6]). *Let the RV's* (1.3) *fulfil the condition* $E\,|X_z|^r < \infty$ *for some integer* $r \geqq 0$. *Then, for all* $t \in R^1$ *and* $p \geqq 1$,

$$\left|\frac{d^r}{dt^r} Ee^{itS_n}\right| \leqq |V_n|^{r-1} \sum_{z\in V_n} E\,|X_z|^r \max_{z_1,\ldots,z_r\in\mathfrak{Z}(p)} \prod_{z\in\mathfrak{Z}(p)\setminus\bigcup_{i=1}^r U_{z_i}(p)} (g_z^{(p)}(t))^{2^{-d}}. \tag{3.2}$$

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[4]) $E(X \,\|\, \mathfrak{F})$ denotes the conditional expectation of the RV X with respect to the σ-algebra $\mathfrak{F}$

The inequalities $|\mathfrak{Z}(p) \cap U_z(p)| \leqq \left[\dfrac{p+2}{p}\right]^d$, $z \in V_n$, $p = 1, 2, \ldots$, and $x \leqq e^{x-1}$, $x \in R^1$, lead to a somewhat cruder version of (3.2):

$$\left|\frac{d^r}{dt^r} Ee^{itS_n}\right| \leqq e^{r\left[\frac{p+2}{p}\right]^d} |V_n|^{r-1} \sum_{z\in V_n} E|X_z|^r \exp\left\{-\frac{1}{2^d} \sum_{z\in\mathfrak{Z}(p)} \left(1 - g_z^{(p)}(t)\right)\right\}. \tag{3.3}$$

We now formulate the main result of this section.

Lemma 3. *Let the RV's* (1.3) *be concentrated on* Z^1 *and satisfy* $E|X_z|^s < \infty$, $z \in V_n$, *for some integer* $s \geqq 3$. *Further assume that, for some integer* $p \geqq 1$ *and some* $\varepsilon \in (0, \pi M_1^{(s-1)/(s-2)})$,

$$\frac{1}{N_n^2 \ln\left(|V_n|^{s-1/2} M_1^{1-s}\right)} \sum_{z\in\mathfrak{Z}(p)} E \min_{0\leqq r\leqq q-1} P\left(Y_z^{(p)} \not\equiv r (\mathrm{mod}\ q) \parallel \mathfrak{F}_z^{(p)}\right) \xrightarrow[n\to\infty]{} \infty \tag{3.4}$$

for $q = 2, 3, \ldots, N_n$, *where* $N_n = N_n(s, p, \varepsilon)$ *is the smallest integer equal to or greater than*

$$M_1^{(s-1)/(s-2)} \max\left\{\frac{1}{2\pi\varepsilon}, \left(\frac{(4p^d)^{s-1} \sum_{z\in V_n} E|X_z|^s}{(B_n^{(p)})^2}\right)^{1/(s-2)}\right\}.$$

Then, as $n \to \infty$,

$$\int_{\varepsilon M_1^{-(s-1)/(s-2)} \leqq |t| \leqq \pi} \exp\left\{-\frac{1}{2^d} \sum_{z\in\mathfrak{Z}(p)} \left(1 - g_z^{(p)}(t)\right)\right\} dt = o\left(M_1^{s-1} |V_n|^{-s+1/2}\right). \tag{3.5}$$

Proof of Lemma 3. For brevity let $I_n = I_n(s, p, \varepsilon)$ denote the integral on the left-hand side of (3.5). Then, after some standard manipulations, we obtain

$$I_n \leqq 2\pi \int_{N_n^{-1} \leqq |t| \leqq 1/2} \exp\left\{-\frac{1}{2^{d-1}} \sum_{z\in\mathfrak{Z}(p)} h_z^{(p)}(t)\right\} dt,$$

where

$$\begin{aligned} h_z^{(p)}(t) &= \frac{1}{2}\left(1 - E\left|E\left(e^{i2\pi t Y_z^{(p)}} \parallel \mathfrak{F}_z^{(p)}\right|^2\right)\right. \\ &= E \sum_{k_1,k_2\in Z^1} \sin^2\left(\pi(k_1 - k_2)\, t\right) P\left(Y_z^{(p)} = k_1 \parallel \mathfrak{F}_z^{(p)}\right) P\left(Y_z^{(p)} = k_2 \parallel \mathfrak{F}_z^{(p)}\right). \end{aligned}$$

The basic idea of the proof consists in a partition of the interval $\left[N_n^{-1}, \dfrac{1}{2}\right]$ in sufficiently small non-overlapping subintervals on which a positive lower bound of $h_z^{(p)}(t)$ is obtainable. (In the same way one can find corresponding bounds of $h_z^{(p)}(t)$ on $\left[-\dfrac{1}{2}, N_n^{-1}\right]$.) It should be noted that similar splitting techniques have been used to prove local CLT's for independent and MARKOV-dependent lattice RV's (see [11], [15]). For this end we denote by $N_n^{-1} = t_0 < t_1 < \cdots < t_{r-1} < t_r = \dfrac{1}{2}$ the ordered set of rational numbers having the form $\frac{a}{q}$, where a and q are mutually prime, $2 \leqq q \leqq N_n$, and $1 \leqq a \leqq [N_n/2]$, and we set

$$J_0 = \left[t_0, \frac{1}{2}(t_0 + t_1)\right], \quad J_r = \left[\frac{1}{2}(t_{r-1} + t_r), t_r\right], \quad \text{and}$$

$$J_l = \left[\frac{1}{2}(t_{j-1} + t_j), \frac{1}{2}(t_j + t_{j+1})\right]$$

for $l = 1, 2, \ldots, r - 1$.

Now let us consider an arbitrary fixed interval J_l containing the point $t_l = \frac{a}{q}$. It is a well-known fact from number theory (see e.g. [10]) that each $t \in J_l$ permits the estimate $\left|t - \frac{a}{q}\right| \leqq \frac{1}{aN_n}$. JORDAN's inequality, that is $|\sin \pi x| \geqq 2|x|$ for $|x| \leqq 1/2$, allows the following estimate:

$$\begin{aligned} h_z^{(p)}(t) &\geqq E \sum_{\substack{k_1,k_2 \in Z^1 \\ |k_1-k_2| \leqq N_n/2}} \sin^2 \pi \left(\frac{a}{q}(k_1 - k_2) + \left(t - \frac{a}{q}\right)(k_1 - k_2)\right) \\ &\quad \times P(Y_z^{(p)} = k_1 \parallel \mathfrak{F}_z^{(p)})\, P(Y_z^{(p)} = k_2 \parallel \mathfrak{F}_z^{(p)}) \\ &\geqq q^{-2} A_z^{(p)} + 4\left(t - \frac{a}{q}\right)^2 B_j^{(p)}, \end{aligned}$$

where

$$A_z^{(p)} = E \sum_{\substack{k_1 k_2 \in Z^1 \\ |k_1-k_2| \leqq N_n/2,\, k_1 \not\equiv k_2 (\mathrm{mod}\, q)}} P(Y_z^{(p)} = k_1 \parallel \mathfrak{F}_z^{(p)})\, P(Y_z^{(p)} = k_2 \parallel \mathfrak{F}_z^{(p)})$$

and

$$B_z^{(p)} = E \sum_{\substack{k_1,k_2 \in Z^1 \\ |k_1-k_2| \leqq N_n/2,\, k_1 \equiv k_2 (\mathrm{mod}\, q)}} (k_1 - k_2)^2\, P(Y_z^{(p)} = k_1 \parallel \mathfrak{F}_z^{(p)})\, P(Y_z^{(p)} = k_2 \parallel \mathfrak{F}_z^{(p)}).$$

Elementary calculations yield

$$\begin{aligned} &\frac{1}{4} N_n^2 A_z^{(p)} + B_z^{(p)} \\ &\geqq 2E\big(Y_z^{(p)} - E(Y_z^{(p)} \parallel \mathfrak{F}_z^{(p)})\big)^2 - E \sum_{\substack{k_1,k_2 \in Z^1 \\ |k_1-k_2| > N_n/2}} (k_1 - k_2)^2 \\ &\quad \times P(Y_z^{(p)} = k_1 \parallel \mathfrak{F}_z^{(p)})\, P(Y_z^{(p)} = k_2 \parallel \mathfrak{F}_z^{(p)}) \\ &\geqq 2E\big(Y_z^{(p)} - E(Y_z^{(p)} \parallel \mathfrak{F}_z^{(p)})\big)^2 - \frac{(4p^d M_1)^{s-1}}{N_n^{s-2}} \sum_{z \in V_n \cap V_z(p,0)} E|X_z|^s. \end{aligned}$$

and so, by definition of N_n,

$$\sum_{z \in \mathfrak{Z}(p)} \left(\frac{N_n^2}{4} A_z^{(p)} + B_z^{(p)}\right) \geqq (B_n^{(p)})^2.$$

Now we distinguish two cases. First suppose that

$$\sum_{z\in\mathfrak{Z}(p)} A_z^{(p)} \geqq 2N_n^2(B_n^{(p)})^2.$$

As an immediate consequence we get

$$\int_{J_l} \exp\left\{-\frac{1}{2^{d-1}} \sum_{z\in\mathfrak{Z}(p)} h_z^{(p)}(t)\right\} dt \leqq \lambda(J_l) \exp\left\{-\frac{1}{2^{d-2}} (B_n^{(p)})^2\right\},$$

where $\lambda(.)$ designates the LEBESGUE measure on R^1.

Since

$$\sum_{z\in\mathfrak{Z}(p)} E \min_{0\leqq r\leqq q-1} P\big(Y_z^{(p)} \not\equiv r (\mathrm{mod}\ q) \parallel \mathfrak{F}_z^{(p)}\big) \leqq 2(B_n^{(p)})^2 \tag{3.6}$$

it is easy to see from (3.4) that there exists a null sequence $\varepsilon_n(p)$ such that

$$\int_{J_l} \exp\left\{-\frac{1}{2^{d-1}} \sum_{z\in\mathfrak{Z}(p)} h_z^{(p)}(t)\right\} dt \leqq \lambda(J_l)\, \varepsilon_n(p)\, M_1^{s-1}\, |V_n|^{-s+1/2}. \tag{3.7}$$

To treat the second case, when $\sum_{z\in\mathfrak{Z}(p)} B_z^{(p)} \geqq (B_n^{(p)})^2/2$, we first consider those t-values lying in the interval $J_l' = \left\{t\in J_l : \left|t-\frac{a}{q}\right| \leqq \frac{\delta_n}{q}\right\}$, where $\delta_n = \frac{q}{B_n^{(p)}} \big(\ln \varrho_n(q) \times \ln (|V_n|^{s-1/2} M_1^{1-s})\big)^{1/2}$ and $\varrho_n(q)$ stands for the whole expression in (3.4) growing to infinity.

Again applying JORDAN's inequality we find that

$$h_z^{(p)}(t) \geqq q^{-2} E \sum_{\substack{k_1,k_2\in Z^1\\ |k_1-k_2|\leqq(2\delta_n)^{-1}\\ k_1\not\equiv k_2(\mathrm{mod}\,q)}} P(Y_z^{(p)} = k_1 \parallel \mathfrak{F}_z^{(p)})\, P(Y_z^{(p)} = k_2 \parallel \mathfrak{F}_z^{(p)}).$$

Therefore, by definitions of $B_n^{(p)}$, δ_n, and $\varrho_n(q)$, we have

$$\begin{aligned}
\sum_{z\in\mathfrak{Z}(p)} h_z^{(p)}(t) &\geqq q^{-2} \sum_{z\in\mathfrak{Z}(p)} E \sum_{\substack{k_1,k_2\in Z^1\\ k_1\not\equiv k_2(\mathrm{mod}\,q)}} P(Y_z^{(p)} = k_1 \parallel \mathfrak{F}_z^{(p)})\, P(Y_z^{(p)} = k_2 \parallel \mathfrak{F}_z^{(p)}) \\
&\quad - 8q^{-2}\delta_n^2(B_n^{(p)})^2 \geqq N_n^{-2} \sum_{z\in\mathfrak{Z}(p)} E \sum_{r=0}^{q-1} \sum_{k\in Z^1} \\
&\quad \times P(Y_z^{(p)} = kq + r \parallel \mathfrak{F}_z^{(p)}) \min_{0\leqq r\leqq q-1} \sum_{\substack{k_2\in Z^1\\ k_2\not\equiv r(\mathrm{mod}\,q)}} P(Y_z^{(p)} = k_2 \parallel \mathfrak{F}_z^{(p)}) \\
&\quad - 8 \ln \varrho_n(q) \ln (|V_n|^{s-1/2} M_1^{1-s}) \\
&= \big(\varrho_n(q) - \ln \varrho_n(q)\big) \ln (|V_n|^{s-1/2} M_1^{1-s}).
\end{aligned}$$

By virtue of condition (3.4) it then follows the existence of a null sequence $\varepsilon_n'(q)$, as $n\to\infty$, such that

$$\int_{J_l'} \exp\left\{-\frac{1}{2^{d-1}} \sum_{z\in\mathfrak{Z}(p)} h_z^{(p)}(t)\right\} dt \leqq \lambda(J_l')\, \varepsilon_n'(q)\, M_1^{s-1}\, |V_n|^{-s+1/2} \tag{3.8}$$

for $q = 2, 3, \ldots, N_n$.

Finally let us consider the case when $t \in J_l \setminus J_l'$ $(\neq \emptyset)$. Obviously, we have

$$\sum_{z\in\mathfrak{Z}(p)} h_z^{(p)}(t) \geqq 4\left(t-\frac{a}{q}\right)^2 \sum_{z\in\mathfrak{Z}(p)} B_z^{(p)}$$
$$\geqq 2q^{-2}\delta_n^2(B_n^{(p)})^2 = 2\ln \varrho_n(q)\ln(|V_n|^{s-1/2} M_1^{1-s})$$

proving that, for $q = 2, 3, \ldots, N_n$,

$$\text{(3.9)} \qquad \int\limits_{J_l\setminus J_l'} \exp\left\{-\frac{1}{2^{d-1}} \sum_{z\in\mathfrak{Z}(p)} h_z^{(p)}(t)\right\} dt \leqq \lambda(J_l\setminus J_l')\, \varepsilon_n''(q)\, M_1^{s-1}\, |V_n|^{-s+1/2},$$

where $\varepsilon_n''(q)$ tends to zero as $n \to \infty$.

It remains to summarize the estimates (3.7)–(3.9) which give

$$M_1^{1-s}\, |V_n|^{s-1/2}\, I_n \xrightarrow[n\to\infty]{} 0$$

completing the proof of Lemma 3.

We proceed by considering some special cases of Lemma 3.

Remark 1. Suppose that the RV's (1.3) are integer-valued and uniformly bounded, i.e. $P(|X_z| \leqq C < \infty) = 1$, $z \in V_n$. Then condition (3.4) of Lemma 3 can be replaced by

$$\text{(3.10)} \qquad \frac{\sum\limits_{z\in\mathfrak{Z}(p)} E \min\limits_{0\leqq r\leqq q-1} P(Y_z^{(p)} \not\equiv r(\mathrm{mod}\ q) \,\|\, \mathfrak{F}_z^{(p)})}{\ln(|V_n|^{s-1/2} M_1^{1-s})\, M_1^{2(s-1)/(s-2)}} \xrightarrow[n\to\infty]{} \infty$$

for $q = 2, 3, \ldots, 1 + [\max\{4Cp^d, (2\pi\varepsilon)^{-1} M_1^{(s-1)/(s-2)}\}]$.

In fact, since $|X_z| \leqq C$ we get $|Y_z^{(p)}| \leqq Cp^d M_1$ and, therefore, all steps of the foregoing proof remain unchanged if N_n is chosen to be the smallest integer satisfying

$$N_n \geqq \max\{4Cp^d M_1, M_1^{(s-1)/(s-2)}/2\pi\varepsilon\}.$$

Remark 2. Under the additional assumptions (1.5) and

$$\text{(3.11)} \qquad (B_n^{(p)})^2 \geqq C_0(d, p)\, |V_n| \quad \text{for some } p \geqq 1$$

the condition (3.4) is fulfilled if (3.10) holds for $q = 2, 3, \ldots, 1 + [M_1^{(s-1)/(s-2)} \max (2\pi\varepsilon)^{-1}, (C_3(4p^d)^{s-1}/C_0(d, p))^{1/(s-2)}]$, where the constant C_3 stems from (1.5).

Remark 3. If in addition to (1.5) and (3.11) the number M_1 does not depend on n then condition (3.4) takes on the form

$$\text{(3.12)} \qquad \frac{1}{\ln|V_n|} \sum_{z\in\mathfrak{Z}(p)} E \min_{0\leqq r\leqq q-1} P(Y_z^{(p)} \not\equiv r(\mathrm{mod}\ q) \,\|\, \mathfrak{F}_z^{(p)}) \xrightarrow[n\to\infty]{} \infty$$

for $q = 2, 3, \ldots$

The relation (3.11) seems to be fairly natural as the following result reveals:

Lemma 4. *If the RV's (1.3) fulfil the condition (1.4) (with constants C_1 and C_2) then (3.11) is valid for every integer $p \geqq 4 \cdot 12^d C_2/C_1$, where $C_0(d, p) = C_1/2$.*

Proof of Lemma 4. Arguing in the same way as in the proof of Lemma 4 in [6] we obtain

$$\sum_{z\in\mathfrak{Z}(p)} E\big(E(Y_z^{(p)} \,\|\, \mathfrak{F}_z^{(p)}) - EY_z^{(p)}\big)^2 \leqq 2^d\big(p^d - (p-2)^d\big)\, |V_n \cap \mathfrak{Z}(p)| \max_{z\in\mathfrak{Z}(1)} D^2 Y_z^{(1)}$$

and

$$\Big|B_n^2 - \sum_{z\in\mathfrak{Z}(p)} D^2 Y_z^{(p)}\Big| \leqq 2^{d-1}\big((p+2)^d - (p-2)^d\big)\, |V_n \cap \mathfrak{Z}(p)| \max_{z\in\mathfrak{Z}(1)} D^2 Y_z^{(1)}.$$

Since $E\big(X - E(X \,\|\, \mathfrak{F})\big)^2 = D^2X - E\big(E(X \,\|\, \mathfrak{F}) - EX\big)^2$, we arrive at

$$|B_n^2 - (B_n^{(p)})^2| \leqq 2^{d-1}\big((p+2)^d + 2p^d - 3(p-2)^d\big)\, |V_n \cap \mathfrak{Z}(p)| \max_{z\in\mathfrak{Z}(1)} D^2 Y_z^{(1)}.$$

By virtue of the estimates $(p+2)^d - (p-2)^d \leqq 2 \cdot 3^d p^{d-1}$, $p^d - (p-2)^d \leqq 3^d p^{d-1}$ and $|V_n \cap \mathfrak{Z}(p)| \leqq \prod_{i=1}^{d} \big(1 + [(n_i - 1)/p(m_i + 1)]\big) \leqq 2^d\, |V_n|/p^d M_1$ we have

$$|B_n^2 - (B_n^{(p)})^2| \leqq 2 \cdot 12^d\, |V_n| \max_{z\in\mathfrak{Z}(1)} D^2 Y_z^{(1)}/pM_1,$$

whence, by (1.4), the assertion of Lemma 4 follows.

In view of the later applications the following special case of Lemma 3 is of particular importance. For brevity put $z_0 = (1, \ldots, 1)$,

$$W(p) = \bigcup_{y\in V_{z_0}(p,0)} V_y(1, 0) \setminus V_{z_0}(p + 1, 1) \quad \text{with}$$

$$|W(p)| = \prod_{i=1}^{d} \big((p+1)(m_i+1) - 1\big) - \prod_{i=1}^{d} \big((p-1)(m_i+1) + 1\big).$$

Lemma 5. *Let $\{\xi_z, z \in Z^d\}$ be a family of i.i.d. RV's taking on values in a measurable space $[K, \mathfrak{K}]$ and let $f \mid K^{M_1} \to Z^1$ be a* Borel *measurable function (with respect to $\mathfrak{K}^{M_1}$), where $m_1, \ldots, m_d$ are fixed non-negative integers. Put $X_z = f\big(\xi_y;\, y \in V_z(1, 0)\big)$, $z \in V_n$, and assume that $E\, |X_{z_0}|^s < \infty$ for some integer $s \geqq 3$. Further, suppose that for some $p \geqq 1$ there exists a set $B \in \mathfrak{K}^{|W(p)|}$ such that $P\big((\xi_z)_{z\in W(p)} \in B\big) > 0$ and the RV*

$$\sum_{z\in V_{z_0}(p,0)} f\big(x_y;\, y \in V_z(1, 0)\big)\big|_{x_y = \xi_y;\, y\in V_{z_0}(p+1,1)}$$

is concentrated on an integer lattice with maximal span 1 for all $|W(p)|$-tuples $(x_y)_{y\in W(p)} \in B$.

Then there exists a constant $\alpha > 0$ depending on d, s, p, M_1, ε etc. such that

$$\int_{\varepsilon M_1^{-(s-1)/(s-2)} \leqq |t| \leqq \pi} \exp\Big\{-2^{-d} \sum_{z\in\mathfrak{Z}(p)} \big(1 - g_z^{(p)}(t)\big)\Big\}\, dt \leqq e^{-\alpha|V_n|} \tag{3.13}$$

for every $\varepsilon > 0$ and sufficiently large n.

Proof of Lemma 5. Without any difficulties it is seen that the condition imposed on the behaviour of the lattice RV $Y_{z_0}^{(p)}$ implies that

$$E \min_{0\leqq r\leqq q-1} P(Y_{z_0}^{(p)} \not\equiv r \ (\mathrm{mod}\ q) \,\|\, \mathfrak{F}_{z_0}^{(p)}) > 0 \quad \text{for} \quad q = 2, 3, \ldots, \tag{3.14}$$

where (3.14) is only needed for q-values less than or equal to the bounded integer N_n defined in Lemma 3. In analogy to the proof of Lemma 3 we find that $1 - g_{z_0}^{(p)}(t) > 0$ for all $t \in [\varepsilon M_1^{-(s-1)/(s-2)}, \pi]$.

The assertion of Lemma 5 then follows by noting that

$$\sum_{z\in S(p)} \left(1 - g_z^{(p)}(t)\right) \geqq \left(1 - g_{z_0}^{(p)}(t)\right) \prod_{i=1}^{d} \left[\frac{n_i}{p(m_i + 1)}\right].$$

At the end of this section we discuss two examples to get a feeling for the conditions (3.4) and (3.14), respectively.

Example 1. Put $d = 1$, $m = 1$, and $f(x, y) = (x + 1)\,y$, $x, y \in R^1$, and let the distribution of the i.i.d. RV's $\xi_1, \xi_2, \ldots$ by given by $P(\xi_1 = 1) = P(\xi_2 = 2) = 1/2$.

Obviously $X_1 = f(\xi_1, \xi_2)$ is concentrated on the lattice $\{2, 3, 4, 6\}$ having maximal span 1. On the other hand the maximal span of each of the lattices generated by the RV's

$$f(1, \xi_2) + f(\xi_2, 1) = 3\xi_2 + 1\,, \qquad f(2, \xi_2) + f(\xi_2, 2) = 5\xi_2 + 2\,,$$

$$f(1, \xi_2) + f(\xi_2, 2) = 4\xi_2 + 2\,, \qquad f(2, \xi_2) + f(\xi_2, 1) = 4\xi_2 + 1$$

is greater than 1. Hence (3.14) is not satisfied for $p = 1$. However,

$$P\big(f(1, \xi_2) + f(\xi_2, \xi_3) + f(\xi_3, \xi_4) + f(\xi_4, 1) \in \{8, 11, 14, 15, 19\}\big) = 1\,,$$

i.e. (3.14) as well as the corresponding condition of Lemma 5 (with $B = \{(1, 1)\}$) holds for $p = 2$.

Example 2. Put $d = 1$, $m = 1$, and $f(x, y) = x + 3y$, $x, y \in R^1$, and let the common distribution of the i.i.d. RV's $\xi_1, \xi_2, \ldots$ be defined by $P(\xi_1 = 0) = P(\xi_2 = 1) = 1/2$.

Then, for each $p \geqq 1$, the possible values of

$$Y_1^{(p)} = \sum_{i=1}^{2p} f(\xi_i, \xi_{i+1}) = \xi_1 + 4(\xi_2 + \cdots + \xi_{2p}) + 3\xi_{2p+1}$$

form a lattice with maximal span 1, whereas each of the RV's

$$Y_1^{(p)}|_{\xi_1=\xi_{2p+1}=0}\,, \quad Y_1^{(p)}|_{\xi_1=\xi_{2p+1}=1}\,, \quad Y_1^{(p)}|_{\xi_1=0,\xi_{2p+1}=1}\,, \quad Y_1^{(p)}|_{\xi_1=1,\xi_{2p+1}=0}\,,$$

generates a lattice with maximal span 4. Therefore (3.14) is not fulfilled. We now show that indeed the local CLT (3.1) fails. Since $g(t) = Ee^{it\xi_1} = \frac{1}{2}(1 + e^{it})$ we get $Ee^{itS_n} = g(t)\,\big(g(4t)\big)^{n-1}\,g(3t)$ implying that

$$Ee^{i\pi S_n/2} = \frac{1}{2} \quad \text{and} \quad \varliminf_{n\to\infty} \int\limits_{|t-(\pi/2)|\leqq\varepsilon} |Ee^{itS_n}|\,dt \geqq C(\varepsilon) > 0$$

for all $\varepsilon > 0$.

Moreover, since $P\big(S_n \equiv 2(\mathrm{mod}\ 4)\big) = 0$ and $ES_n = 2n$, $B_n^2 = 4n - \frac{3}{2}$ for $n \geqq 1$ one can easily check that the relation (3.1) is impossible for $N = 4\left(\left[\frac{n}{2}\right] + \left[\sqrt{n}\right]\right) + 2$.

4. Asymptotic expansions in the local CLT for $(m_1, \ldots, m_d)$-dependent lattice random variables

The results of the preceding section together with Lemma 2 in [6] enable us to derive the desired asymptotic (EDGEWORTH) expansions of the difference (3.1) in a relatively simple way. For this end put

$$x_n(N) = (N - ES_n)/B_n, \quad N \in Z^1, \qquad f_n(t) = Ee^{it(S_n - ES_n)/B_n},$$

$$p_{sn}(t) = e^{-t^2/2}\left(1 + \sum_{k=1}^{s-2} P_{kV_n}(t)\right), \qquad \varphi_{sn}(x) = \varphi(x) + \sum_{k=1}^{s-2} q_{kV_n}(x).$$

By standard manipulations we obtain

$$2\pi B_n\big(ix_n(N)\big)^r P(S_n = N) = \int\limits_{|t| \le \pi B_n} e^{-itx_n(N)} \frac{d^r}{dt^r} f_n(t)\, dt$$

provided that $E\,|X_z|^r < \infty$ for $r = 0, 1, 2, \ldots$

Analogously, by integrating the left-hand side of (1.1) r times by parts we arrive at the equality

$$\varphi_{sn}(x) = \frac{1}{2\pi(ix)^r} \int\limits_{-\infty}^{\infty} e^{-itx} \frac{d^r}{dt^r} p_{sn}(x)\, dt.$$

Splitting the domain of integration we are lead to the estimate

$$2\pi\, |x_n(N)|^r \left|P(S_n = N) - \varphi_{sn}\big(x_n(N)\big)\right| \le I_1^{(r)} + I_2^{(r)} + I_3^{(r)} + I_4^{(r)},$$

where

$$I_1^{(r)} = \int\limits_{|t| \le T_n^{(1)}} \left|\frac{d^r}{dt^r}\big(f_n(t) - p_{sn}(t)\big)\right| dt, \qquad I_2^{(r)} = \int\limits_{|t| > T_n^{(1)}} \left|\frac{d^r}{dt^r} p_{sn}(t)\right| dt,$$

$$I_3^{(r)} = \int\limits_{T_n^{(1)} \le |t| \le T_n^{(2)}} \left|\frac{d^r}{dt^r} f_n(t)\right| dt, \qquad I_4^{(r)} = \int\limits_{T_n^{(2)} \le |t| \le \pi B_n} \left|\frac{d^r}{dt^r} f_n(t)\right| dt.$$

Here, $T_n^{(1)} = \delta(n)\, |V_n|^{(s-2)/2(s-1)}/M_1$ and $T_n^{(2)} = \varepsilon B_n/M_1^{(s-1)/(s-2)}$, where $\varepsilon > 0$ is chosen sufficiently small and $\delta(n)$ tends to 0 in such a way that $\delta(n) \ge (M_1/|V_n|^{(s-2)/2(s-1)})^{1/2}$.

Now assume that the conditions (1.4)–(1.6) are satisfied. Then, as it was shown in the proof of Theorem 1 in [6], the following estimates hold for $r = 0, 1, \ldots, s$:

$$|V_n|^{(s-2)/2} M_1^{1-s} \int\limits_{|t| \le T_n^{(1)}} \left|\frac{d^r}{dt^r}\big(f_n(t) - p_{sn}(t)\big)\right| dt \xrightarrow[n \to \infty]{} 0 \tag{4.1}$$

$$\left|\frac{d^r}{dt^r} f_n(t)\right| \le C_1\, |V_n|^{r/2}\, e^{-t^2/6 \cdot 2^d} \quad \text{for } |t| \le T_n^{(2)}$$

and

$$\left|\frac{d^r}{dt^r} p_{sn}(t)\right| \le C_2(1 + |t|^{3(s-2)+r})\, e^{-t^2/2} \quad \text{for all } t \in R^1.$$

Using the additional assumption

$$(4.2)\qquad M_1^{s-1+\delta}/|V_n|^{(s-2)/2} \xrightarrow[n\to\infty]{} 0 \quad \text{for some } \delta > 0$$

we can deduce that, as $n \to \infty$,

$$(4.3)\qquad I_j^{(r)} = o(M_1^{s-1}/|V_n|^{(s-2)/2}) \quad \text{for } j = 1, 2, 3.$$

It remains to treat the term

$$I_4^{(r)} = B_n^{1-r} \int\limits_{sM_1^{-(s-1)/(s-2)} \leqq |t| \leqq \pi} \left| \frac{d^r}{dt^r} E e^{it(S_n - ES_n)} \right| dt.$$

From Lemma 2, (1.4), and (1.5) we derive the following estimate:

$$I_4^{(r)} \leqq C_3 \, |V_n|^{(r+1)/2} \int\limits_{sM_1^{-(s-1)/(s-2)} \leqq |t| \leqq \pi} \exp \left\{ - \frac{1}{2^d} \sum_{z \in \mathfrak{Z}(p)} \left(1 - g_z^{(p)}(t)\right) \right\} dt$$

for $r = 0, 1, \ldots, s$.

This combined with the statement of Lemma 3 and (4.3) (for $r = 0$ and $r = s$) proves the following

Theorem 1. *Let the family of integer-valued RV's* (1.3) *satisfy the conditions* (1.4)–(1.6) *and* (4.2) *for some integer* $s \geqq 3$. *Further, assume that condition* (3.4) *of Lemma* 3 *is fulfilled.*

Then, as $n \to \infty$,

$$\sup_{N \in Z^1} \left(1 + |x_n(N)|\right)^s \left|B_n P(S_n = N) - \varphi_{sn}(x_n(N))\right| = o(M_1^{s-1} \, |V_n|^{-(s-2)/2}).$$

The statement of Theorem 1 contains a number of special cases which are of interest in themselves. Two of them we shall formulate separately.

Theorem 2. *Let the assumptions of Theorem* 1 *with exception of* (1.6) (*which can be dropped*) *be satisfied for* $s = 3$.

Then

$$\left|B_n P(S_n = N) - \varphi(x_n(N))\right| \leqq \frac{C(p, d) \, M_1^2}{\left(1 + |x_n(N)|\right)^3 \, |V_n|^{1/2}}$$

for every $N \in Z^1$ *and* $n = 1, 2, \ldots$

The non-uniform Berry-Esseen inequality stated in Theorem 2 is an immediate consequence of Theorem 1 (for $s = 3$) and the fact that

$$q_{1V_n}(x) = \frac{\Gamma_3(S_n)}{6\sqrt{2\pi}\, B_n^3} (x^3 - 3x) \, e^{-x^2/2}$$

is bounded by $C_1(1 + |x|)^{-3} M_1^2 L_{3n} \leqq C_2(1 + |x|)^{-3} M_1^2 \, |V_n|^{-1/2}$ (see e.g. (3.7) in [6]). Finally, the proof of Theorem 2 is completed with the remark that the left-hand term in (4.1) remains at least bounded (implying large 0 instead of small o in Theorem 1) if only the validity of (1.4) and (1.5) is required.

Finally, we turn to the special case of stationary $(m_1, \ldots, m_d)$-dependent RF's which already appeared in Lemma 4. Under the assumptions of Lemma 4 the conditions (1.5) and (1.6) are trivially satisfied and (1.4) can be equivalently expressed by

$$\lim_{n\to\infty} \frac{B_n^2}{|V_n|} = \sum_{z\in V_{z_1}(0,-1)} E(X_{z_1} - EX_{z_1})(X_z - EX_z) > 0 \tag{4.4}$$

for $z_1 = (m_1 + 1, \ldots, m_d + 1)$. Note that the limit always exists. Thus, we arrive at the following

Theorem 3. *Suppose that the assumptions of Lemma* 5 (*entailing the strict stationarity of the RF* (1.3)) *and* (4.4) *are satisfied. Then the assertion of Theorem* 1 *holds.*

5. Some applications

This concluding section deals with the application of the above results to some concrete models for which certain characteristics of interest can be expressed by sums of RV's having the form (1.3) (see [8], [9], [12], [13]). We confine ourselves to a brief description of the models and the verification of (3.4) or (3.14), respectively.

Model 1. Let $\xi_1, \xi_2, \ldots$ be i.i.d. non-negative RV's with common distribution function $F(x) = P(\xi_1 < x)$ defining an ordinary renewal point process $\{\xi_1 + \xi_2 + \cdots + \xi_k,\ k \geqq 1\}$. Define $X_k = 1_{\{\xi_k + \cdots + \xi_{k+m} < L\}}$ $k \geqq 1$, where L is a fixed positive number.

Thus, $X_k = 1$ as soon as $m + 2$ points are clustered within an interval of length L and $S_n = X_1 + \cdots + X_n$ counts the number of such clusters among the first $n + m$ points. For simplicity take $m = 1$ and let there exist a sufficiently small $\varepsilon \in \left(0, \frac{L}{2}\right)$ such that $0 < F(\varepsilon) < F(L - \varepsilon) < F(L) < 1$. Then, for $(x, y) \in B := [0, \varepsilon) \times [L - \varepsilon\ L)$,

$$P(1_{\{x+\xi_2<L\}} + 1_{\{\xi_2+y<L\}} = 0) = 1 - F(L - x) > 0$$

and

$$P(1_{\{x+\xi_2<L\}} + 1_{\{\xi_2+y<L\}} = 1) = F(L - x) - F(L - y) > 0.$$

Thus, the 1-dependent RV's $X_1, \ldots, X_n$ satisfy condition (3.14) which is equivalent to

$$\int_0^L \int_x^L \min\big(1 - F(L - x) + F(L - y),$$
$$F(L - x) - F(L - y)\big)\, dF(y)\, dF(x) > 0.$$

Model 2. Let $\xi_1, \xi_2, \ldots$ be independent RV's, uniformly distributed on (0, 1), and set $X_k = 1_{\{\xi_k < \xi_{k+1} < \cdots < \xi_{k+m}\}}$. Then $S_n = X_1 + \cdots + X_n$ is equal to the number of (possibly overlapping) strongly increasing sequence segments of length $m + 1$. Condition (3.14) can be easily verified.

Model 3. (see [8]) Let $\xi_1, \xi_2, \ldots$ be independent, uniformly distributed letters from an alphabet with $N \geqq 2$ letters. Further, let $\alpha_1, \ldots, \alpha_{m+1}$ be a fixed sequence of $m + 1$ letters and define

$$X_k = 1_{\{\xi_k = \alpha_1\}} \cdot \cdots \cdot 1_{\{\xi_{k+m} = \alpha_{m+1}\}}, \qquad k \geqq 1,$$

so that $S_n = X_1 + \cdots + X_n$ counts the number of "words" $(\alpha_1, \ldots, \alpha_{m+1})$ formed by $m + 1$ consecutive elements of the sequence $\xi_1, \ldots, \xi_{n+m}$. Since, for $\xi_1, \ldots, \xi_m \in \{1, \ldots, N\} \setminus \{\alpha_1\}$ and $\xi_{m+k} = \alpha_k$, $k = 2, \ldots, m + 1$,

$$\sum_{k=1}^{m+1} 1_{\{\xi_k=\alpha_1\}} \cdot 1_{\{\xi_{k+1}=\alpha_2\}} \cdot \cdots \cdot 1_{\{\xi_{k+m}=\alpha_{m+1}\}} = 1_{\{\xi_{m+1}=\alpha_1\}}$$

the validity of (3.14) follows.

Model 4. Let $\xi_1, \xi_2, \ldots$ denote a sequence of discrete i.i.d. RV's. For every pair i, j $(i \neq j)$ and a fixed positive integer m, let

$$d_{ij}(m) = 1_{\{\xi_i=\xi_j\}} \cdot 1_{\{\xi_{i+1}=\xi_{j+1}\}} \cdot \cdots \cdot 1_{\{\xi_{i+m-1}=\xi_{j+m-1}\}}.$$

In other words $d_{i,i+k} = 1$ whenever any arbitrary sequence segment $(\xi_i, \ldots, \xi_{i+m-1})$ is repeated at $(\xi_{i+k}, \ldots, \xi_{i+k+m})$, that is e.g.

$$d_{i,i+1}(m) = 1 \quad \text{iff} \quad \xi_i = \xi_{i+1} = \cdots = \xi_{i+m}.$$

Thus, $S_n = \sum_{i=1}^{n} d_{i,i+1}(m)$ gives the total number of sequence segments of just one value repeated at least $m + 1$ times. It is easily checked that (3.14) is fulfilled if ξ_1 takes on at least three distinct values. In [13] the RV's $\xi_1, \xi_2, \ldots$ were identified with an arbitrary DNA or RNA base sequence in order to study frequencies of repeats of base sequence segments of various length.

Model 5. Let $\left\{\xi_z, z \in V_n = \{1, 2, \ldots, n\}^d\right\}$ be a field of binary i.i.d. RV's with $P(\xi_z = 1) = 1 - P(\xi_z = 0) = p > 0$, which can be interpreted as particles randomly distributed on Z^d. For some background to this model the reader is referred to [12] and references therein. In spatial statistics one is interested in the (asymptotic) distribution of the number of pairs (of particles) (ξ_y, ξ_z) located on V_n with a fixed Euclidean distance ϱ, say $\varrho \in \left\{1, \sqrt{2}, \ldots, \sqrt{d}, \ldots\right\}$. In other words one has to determine the asymptotic behaviour of the probabilities

$$p_n(\varrho, N) = P\left(S_n = \frac{1}{2} \sum_{\substack{y,z \in V_n \\ (y_1-z_1)^2+\cdots+(y_d-z_d)^2=\varrho^2}} \xi_y \cdot \xi_z = N\right)$$

for $N = 0, 1, \ldots$ To avoid number theoretical complications we consider merely the simplest case $\varrho = 1$. In this case we can rewrite S_n as follows:

$$S_n = \sum_{z \in V_n} f_z(\xi_z, \xi_{z(1)}, \ldots, \xi_{z(d)}),$$

where $z = (z_1, \ldots, z_d)$, $z(j) = (z_1, \ldots, z_j + 1, \ldots, z_d)$ for $j = 1, \ldots, d$, and

$$f_z(x_0, x_1, \ldots, x_d) = x_0 \sum_{\substack{j=1 \\ z_j \neq n}}^{d} x_j.$$

Note the obvious fact that, for $z \in \{1, 2, \ldots, n - 1\}^d$, $f_z(x_0, x_1, \ldots, x_d)$ does not depend on z, that is, it is equal to $x_0(x_1 + \cdots + x_d)$. After some simple calculations we get

$$ES_n = p^2 \left(1 + d \sum_{k=0}^{d-1} \binom{d-1}{k} (n - 1)^{d-k}\right)$$

and, for given $\xi_{(2,1,\ldots,1)} = 1$ and $\xi_y = 0$, $y \in \{1, 2, 3\}^d \setminus \{(2, 2, \ldots, 2), (2, 1, \ldots, 1)\}$,

$$\sum_{z \in \{1,2\}^d} \xi_z(\xi_{z(1)} + \cdots + \xi_{z(d)}) = \xi_{(2,2,\ldots,2)},$$

so that the condition (3.12) (and hence (3.14)) holds.

To conclude with we apply Theorem 1 to the above sum S_n when $\varrho = 1$ and $d = 2$.

Theorem 4. *Let the assumptions of Model 5 be satisfied. Then, for $d = 2$ and some integer $s \geqq 3$, we have*

$$(1 + |x_n(N)|)^s n^{s-2} \left| p_n(1, N) - \varphi(x_n(N)) - \sum_{k=1}^{s-2} q_{kV_n}(x_n(N)) \right| \xrightarrow[n\to\infty]{} \to 0$$

uniformly for all $N = 0, 1, 2, \ldots$, where $x_n(N) = N - p^2(n^2 - (n-1)^2)/B_n$ and $\lim_{n\to\infty} B_n^2/2p^2(1-p)(1+7p)\,n^2 = 1$.

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Sektion Mathematik
Bergakademie Freiberg
Bernhard-von-Cotta-Straße 2
DDR-Freiberg
9200