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FIELD-THEORY MODELS THAT ADMIT ALTERNATIVE LAGRANGIAN FORMULATIONS. I

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Some theorems are proved concerning the structure of Lagrangian systems of differential equations for which the Lagrangian is not uniquely determined. In particular, a complete description is obtained for bi-Lagrangian systems of second order with hyperregular Lagrangians quadratic in the derivatives.

1. Introduction

In this paper, we consider the question of the uniqueness of the determination of the Lagrangian from field equations. This problem is part of the inverse problem of the calculus of variation, which was posed already at the end of the last century in studies of Helmholtz. However, despite this very respectable age, the problem is still far from complete solution, and the obtained results have a very special nature. A review of results on the inverse problem of the calculus of variation can be found, for example, in [1].

We introduce the following notation. Let

$$\frac{\delta L}{\delta u^a} = G_a[u] = G_a(\underset{k}{u}, \underset{k-1}{u}, \ldots, \underset{1}{u}, u, x) = 0 \tag{1}$$

be a Lagrangian system of differential equations of order k. Here, $u = (u^a) = (u^a(x)) = (u^a(x^1, \ldots, x^d))$ are unknown functions, $a = 1, \ldots, n$,

$$\underset{1}{u} = (u_\alpha{}^a) = \left(\frac{\partial u^a}{\partial x^\alpha}\right), \quad \underset{2}{u} = (u^a_{\alpha\beta}) = \left(\frac{\partial^2 u^a}{\partial x^\alpha \partial x^\beta}\right),$$

etc., L is the Lagrangian, and

$$\frac{\delta}{\delta u^a} = \frac{\partial}{\partial u^a} + \sum_{l \geqslant 1} (-)^l \partial_{\alpha_1} \ldots \partial_{\alpha_l} \frac{\partial}{\partial u^a_{\alpha_1 \ldots \alpha_l}}$$

is the Euler operator. Below, we shall also use the upper case Greek letters Γ and Λ to denote multiple indices formed from indices denoted by Greek letters. Summation over repeated multiple indices will be understood. For example, the last formula can be expressed in the form

$$\frac{\delta}{\delta u^a} = (-\partial)_\Gamma \frac{\partial}{\partial u_\Gamma{}^a}.$$

If some function F vanishes on all solutions of Eqs. (1), we shall write $F \overset{0}{=} 0$.

The following question arises naturally: How uniquely is the function L determined from Eqs. (1)? Of course, there is always the trivial nonuniqueness associated with the possibility of going over to the equivalent Lagrangian

$$L \to \tilde{L} = cL + \partial_\alpha J^\alpha, \quad c = \text{const}, \tag{2}$$

without any change of Eqs. (1). However, for some systems of differential equations the arbitrariness in the choice of the Lagrangian is much greater.

We shall say that the system of equations (1) is bi-Lagrangian if there exist a function $\tilde{L} = \tilde{L}[u]$ and matrix $\lambda = (\lambda_a^b[u])$ such that

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$$\tilde{G}_a \equiv \frac{\delta \tilde{L}}{\delta u^a} = \lambda_a{}^b \frac{\delta L}{\delta u^b}, \tag{3}$$

where $\lambda_a^b \neq c\delta_a^b$, c = const. Note that if $\tilde{L}$ and λ satisfy (3), then $\tilde{L} + cL$ and $\lambda_a^b + c\delta_a^b$ also satisfy (3). Therefore, without loss of generality we can assume that det $\lambda \neq 0$ and, therefore, the equations $\tilde{G} = 0$ and Eqs. (1) are equivalent.

Bi-Lagrangian systems are interesting for many reasons. In connection with the study of finite-dimensional completely integrable systems, they were considered, for example, in [2], and in connection with nonuniqueness in the quantization procedure in [3–5]. It was recently established in [6] that in bi-Lagrangian systems there always exists a symmetric recursive operator, this making it possible to obtain hidden symmetries and hidden conservation laws for the considered systems of differential equations.

The problem of describing all bi-Lagrangian systems of given order is evidently very complicated. So far as I know, important results have hitherto been obtained only for the cases k = 2, d = 1 and k = 2, n = 1. In the case k = 2, d = 1 we may mention [7,8], in which the inverse problem of the calculus of variation was investigated and an effective algorithm permitting the construction of all alternative Lagrangians for each given equation was constructed. In the case k = 2, n = 1 it was shown in [9] that the Lagrangian L in (1) is unique if the subsidiary condition of regularity with repect to $\underset{1}{u}$ at the origin is imposed. We note that this last condition is not given in the formulation of the corresponding theorem in [9] but is used essentially in the proof (cf. the Remark after Theorem 2 below).

In this paper, we give a complete description of hyperregular bi-Lagrangian systems of second order with Lagrangians quadratic in the derivatives for d > 1. (The definition of the concept of hyperrgularity will be given below.) In addition, we prove in the process a theorem establishing the constancy of the quantities Sp λ^m for d > 1, thereby extending to the multidimensional case the results of [10] for d = 1. From this it is also easy to deduce uniqueness of the Lagrangian for scalar hyperregular second-order equations. Finally, we give a complete description of the bi-Lagrangian systems (3) of arbitrary order in the case λ_a^b = const.

2. Hyperregular Bi-Lagrangian Systems of Second Order

In this section $L = L(\underset{1}{u}, u, x)$, $\tilde{L} = \tilde{L}(\underset{1}{u}, u, x)$. Therefore, Eqs. (1) and (3) are linear in $\underset{2}{u}$, and the matrix λ in (3) depends only on $\underset{1}{u}$, u, and x. We write Eqs. (1) in the form

$$G_a = g_{ab}^{\alpha\beta} u_{\alpha\beta}^b + F_a(\underset{1}{u}, u, x) = 0, \quad g_{ab}^{\alpha\beta} = \frac{1}{2}\left(\frac{\partial^2 L}{\partial u_\alpha^a \partial u_\beta^b} + \frac{\partial^2 L}{\partial u_\beta^a \partial u_\alpha^b}\right). \tag{4}$$

We shall say that the Lagrangian L is hyperregular if there exists a matrix $g_{\alpha\beta}^{ab}$ such that

$$g_{ac}^{\alpha\gamma} g_{\gamma\beta}^{cb} = \delta_\beta^\alpha \delta_a^b. \tag{5}$$

This definition differs slightly from the standard definition of hyperregularity (see, for example [11]), but it agrees with it for the majority of interesting examples.

THEOREM 1: 1) For hyperregular Lagrangians of first order, the matrix λ in Eqs. (3) satisfies equations of the Lax type

$$\partial_\alpha \lambda - [\Gamma_\alpha, \lambda] \overset{0}{=} 0, \tag{6}$$

where $\Gamma_\alpha = \Gamma_\alpha(\underset{2}{u}, \underset{1}{u}, u, x)$ are certain matrices that are determined below;

2) if d > 1, then Sp λ^m = const, m = 1, 2, ...

Remark. If d = 1, then it follows from (6) that Sp λ^m is an integral of the motion for all m. This result is already known (see [10]).

Proof. We define the differential operator G$^\cdot$ by the formula

$$G^\cdot \cdot f = \frac{d}{d\varepsilon} G[u + \varepsilon f]|_{\varepsilon=0}, \qquad f = (f^1(x), \ldots, f^n(x)).$$

It is readily seen that

$$G^{`}=\left(\frac{\partial G_a}{\partial u_\Gamma{}^b}\partial_\Gamma\right).$$

Let $(G^{`})^*$ be the formally conjugate operator

$$(G^{`})^*=(-\partial)_\Gamma\frac{\partial G_a}{\partial u_\Gamma{}^b}.$$

It is well known that if the system of equations (1) is Lagrangian then

$$G^{`}=(G^{`})^*. \tag{7}$$

Equations (7) are usually called the Helmholtz conditions (see [1]).

We define the matrices

$$g^{\alpha\beta}=(g^{\alpha\beta}_{ab}),\quad g_{\alpha\beta}=(g^{ab}_{\alpha\beta}),\quad A^\alpha=(A^\alpha_{ab})=\left(\frac{\partial G_a}{\partial u^b_\alpha}\right),\quad B=(B_{ab})=\left(\frac{\partial G_a}{\partial u^b}\right).$$

It follows from Eqs. (7) that

$$g^{\alpha\beta}=\overset{T}{g}{}^{\alpha\beta}, \tag{8}$$

$$2\partial_\beta g^{\alpha\beta}=\overset{T}{A}{}^\alpha+A^\alpha, \tag{9}$$

$$\partial_\alpha\partial_\beta g^{\alpha\beta}=\partial_\alpha\overset{T}{A}{}^\alpha+B-\overset{T}{B}. \tag{10}$$

We now write down the Helmholtz conditions for Eqs. (3). Let

$$C^\alpha=(C^\alpha_{ab})=\left(\frac{\partial\lambda_a{}^c}{\partial u^b_\alpha}G_c\right),\quad D=(D_{ab})=\left(\frac{\partial\lambda_a{}^c}{\partial u^b}G_c\right).$$

Then, replacing G by λG in Eqs. (7), we obtain

$$g^{\alpha\beta}\overset{T}{\lambda}=\lambda g^{\alpha\beta}, \tag{11}$$

$$2\partial_\beta(\lambda g^{\alpha\beta})=\overset{T}{A}{}^\alpha\overset{T}{\lambda}+\lambda A^\alpha+C^\alpha+\overset{T}{C}{}^\alpha, \tag{12}$$

$$\partial_\alpha\partial_\beta(\lambda g^{\alpha\beta})-\partial_\alpha(\overset{T}{A}{}^\alpha\lambda)=\lambda B-\overset{T}{B}\overset{T}{\lambda}+\partial_\alpha C^\alpha+D-\overset{T}{D}. \tag{13}$$

From (8), (9), and (12), we obtain

$$2\partial_\beta(\lambda)g^{\alpha\beta}=\overset{T}{A}{}^\alpha\overset{T}{\lambda}-\lambda A^\alpha+C^\alpha+\overset{T}{C}{}^\alpha. \tag{14}$$

Note that $C^\alpha\overset{0}{=}0$. Therefore, multiplying (14) from the right by $g_{\beta\gamma}$ and using (11), we obtain

$$\partial_\alpha\lambda-[{}^1/{}_2\overset{T}{A}{}^\beta g_{\beta\alpha},\lambda]\overset{0}{=}0, \tag{15}$$

and this proves the first assertion of the theorem. The second is a consequence of (15) and the following lemma.

<u>LEMMA 1.</u> If $d>1$, $f=f(\underset{1}{u},u,x)$ and

$$\partial_\alpha f\overset{0}{=}0, \tag{16}$$

then f = const.

Indeed, Eqs. (16) can be rewritten in the form

$$\partial_\alpha f=c_\alpha{}^a G_a, \tag{17}$$

where $c^a_\alpha=c^a_\alpha(\underset{1}{u},u,x)$ are certain functions. Comparing in (17) the terms proportional to $\underset{2}{u}$, we obtain

$$\delta^\beta_\alpha\frac{\partial f}{\partial u^a_\gamma}+\delta^\gamma_\alpha\frac{\partial f}{\partial u^a_\beta}=c^b_\alpha g^{\beta\gamma}_{ba}. \tag{18}$$

Using the hyperregularity of the considered equations, we eliminate c^b_α from (18). We obtain

$$(d^2 + d - 2)\, g^{ab}_{\alpha\beta} \frac{\partial f}{\partial u^b_\beta} = 0,$$

whence for d > 1

$$\frac{\partial f}{\partial u^a_\alpha} = 0. \tag{19}$$

Comparing further (17)-(19), we obtain the proposition of the lemma. The proof of Theorem 1 is complete.

THEOREM 2. For scalar hyperregular second-order equations in the case d > 1 the Lagrangian is uniquely determined (up to equivalence).

Proof. Theorem 2 is a direct corollary of the previous theorem. For in the case n = 1, the matrix λ reduces to a single function, which is constant in accordance with the second assertion of Theorem 1, and this completes the proof.

Remark. The condition of hyperregularity is essential. For example, the equation

$$\partial_\alpha \left(\frac{g^{\alpha\beta}(x)\, u_{\alpha\beta}}{\sqrt{{}^1/_2 g^{\mu\nu}(x)\, u_\mu u_\nu}} \right) = 0$$

has the infinite set of inequivalent Lagrangians

$$L_f = f(u) \sqrt{{}^1/_2 g^{\alpha\beta}(x)\, u_\alpha u_\beta}$$

with arbitrary function f = f(u).

3. Bi-Lagrangian Theories with Constant Matrix λ

In this section, we describe all bi-Lagrangian theories for which in Eqs. (3) the matrix λ_a^b = const. In the following section, we shall show that the problem of describing all hyperregular bi-Lagrangian equations of second order with Lagrangians quadratic in the derivatives reduces to this case.

In Eqs. (3) we make the point change of variables

$$u \to \bar{u} = \bar{u}(u, x), \quad x \to x. \tag{20}$$

Then the matrix λ transforms as a tensor of rank (1, 1):

$$\lambda_a{}^b \to \bar{\lambda}_a{}^b = \frac{\partial u^c}{\delta \bar{u}^a} \lambda_c{}^d \frac{\partial \bar{u}^b}{\partial u^d}. \tag{21}$$

Indeed, if $\bar{L}[\bar{u}] = L[u]$, then

$$\frac{\delta \bar{L}}{\delta \bar{u}^a} = \frac{\partial u^b}{\partial \bar{u}^a} \frac{\delta L}{\delta u^b} \tag{22}$$

(see [1]). Substituting (22) and the analogous formula for $\tilde{L}$ in (3), we obtain (21).

We shall say that the system of equations (1) is reducible if there exists a point transformation of the variables (19) such that in the new variables the original system (1) decomposes into two uncoupled systems of differential equations. In other words, the system (1) is reducible if $\bar{L}[\bar{u}] = \bar{L}_1[\bar{u}^1, \ldots, \bar{u}^s] + \bar{L}_2[\bar{u}^{s+1}, \ldots, \bar{u}^n]$ for some $s \geqq 1$.

LEMMA 2. If the matrix λ in (1) is constant and has two different eigenvalues μ_1 and μ_2, and

$$\mu_1 \neq \mu_2^*, \tag{23}$$

then the system (1) is reducible.

Proof. Suppose the matrix λ has at least two eigenvalues μ_1, μ_2, and $\mu_1 \neq \mu_2^*$. Making the point change of variables

$$u^a \to \bar{u}^a = T_b u^b, \quad T_b = \text{const}, \tag{24}$$

we can reduce the matrix λ in accordance with (21) to the block-diagonal form

$$\lambda = \begin{pmatrix} \lambda^{(1)} & 0 \\ 0 & \lambda^{(2)} \end{pmatrix}, \tag{25}$$

where $\lambda^{(i)}$ are $n_i \times n_i$ matrices that do not have common eigenvalues, i = 1, 2. Indeed, by the transformation $\lambda \to T\lambda T^{-1}$ the real matrix λ can be reduced to the real canonical Jordan form

$$\lambda=\begin{pmatrix}\lambda_{(1)} & & 0\\ & \lambda_{(2)} & \\ & & \ddots \\ 0 & & \lambda_{(l)}\end{pmatrix},$$

where each block matrix $\lambda_{(j)}$ has either only one real eigenvalue or only one pair of complex-conjugate eigenvalues (see [12]). Combining all the blocks containing the matrices with eigenvalues μ_1, u_1^* in one block $\lambda^{(1)}$, and the remaining blocks in the block $\lambda^{(2)}$, we obtain (25).

Suppose the matrix λ has already been reduced to the form (25). In the Helmholtz condition (7) we replace G by λG. Since by hypothesis the matrix λ is constant,

$$\lambda G^{`}=G^{`}\overset{T}{\lambda}, \tag{26}$$

whence

$$\lambda_a{}^c\frac{\partial G_c}{\partial u_\Gamma^b}=\frac{\partial G_a}{\partial u_\Gamma^c}\lambda_b{}^c. \tag{27}$$

Suppose $i_1, j_1 = 1, \ldots, n_1$, $i_2, j_2 = 1, \ldots, n_2$. In (27), we set $a = i_1$, $b = i_2$. Then by virtue of (24),

$$\lambda_{i_1}^{(1)\,j_1}\frac{\partial G_{j_1}}{\partial u_\Gamma^{i_2}}=\frac{\partial G_{i_1}}{\partial u_\Gamma^{j_2}}\lambda_{i_2}^{(2)\,j_2}. \tag{28}$$

Formula (28) can be regarded as the matrix equation

$$\lambda^{(1)}T^\Gamma=T^\Gamma(\lambda^{(2)})^T, \tag{29}$$

where T^Γ is the $n_1 \times n_2$ matrix

$$T^\Gamma=\left(\frac{\partial G_{i_1}}{\partial u_\Gamma^{i_2}}\right).$$

However, it is well known that if in (28) the matrices $\lambda^{(1)}$ and $\lambda^{(2)}$ do not have common eigenvalues, the $T^\Gamma = 0$ (see [13]). It follows directly from this that the system (1) is reducible. The lemma is proved.

Thus, if the system (1) is reducible, then the matrix λ is either one real eigenvalue μ or a pair of complex-conjugate eigenvalues μ and μ^*. As we have already noted, the matrix λ is determined up to the transformations

$$\lambda_a{}^b\to c_1\lambda_a{}^b+c_2\delta_a{}^b,\quad c_1, c_2=\text{const}. \tag{30}$$

Therefore, without loss of reality we can assume that in the first case $\mu = 0$, and in the second $\mu = i$.

If $\mu = 0$, then there exists an integer q_1 such that $\lambda^{q_1} = 0$ (see [13]). We set $q_2=\left[\frac{q_1}{2}\right]+1$, $\lambda_1=\lambda^{q_1}$. Then $\lambda_1^2 = 0$.

We now note that if the matrix λ satisfies (26), then the matrix λ^q for any integer q also satisfies (26). But Eqs. (26) are the necessary and sufficient condition for the original equations (1) to be bi-Lagrangian. Therefore, if $(L, \tilde{L}, \lambda)$ satisfy (3), λ = const, and the system (1) is irreducible, then there exist a Lagrangian $\tilde{L}_1$ and matrix λ_1 such that $(L, \tilde{L}_1, \lambda_1)$ also satisfy (3) and $\lambda_1^2 = 0$.

We consider the case $\mu = i$. In this case, the matrix λ^2 has the unique eigenvalue -1 and either the previous arguments apply or $\lambda^2 = -E$ (where E is the unit matrix).

We summarize the obtained results:

LEMMA 3. Let $(L, \tilde{L}, \lambda)$ satisfy (3), the matrix λ be constant, and the system (1) irreducible. Then there exist a Lagrangian $\tilde{L}_1$ and a constant matrix λ_1 such that $(L, \tilde{L}_1, \lambda_1)$ also satisfy (3), and either $\lambda_1^2 = 0$ or $\lambda_1^2 = -E$.

Let m be an integer satisfying $2m \leq n$. Let the indices i and j take the values

1, ..., m, and the index p the values 1, ..., n − 2m (for n > 2m). We set $v^i = u^i$, $w^i = u^{n-m+i}$, $t^p = u^{m+p}$ (for n > 2m), $z^j = w^j + iv^j$ ($i^2 = -1$).

The following theorem solves the problem of describing bi-Lagrangian systems with constant matrix λ in Eqs. (3).

THEOREM 3. Let (1) be a bi-Lagrangian system and the matrix λ in Eqs. (3) be constant. Then there exists a change of variables of the form (24) such that the Lagrangian L reduces either to the form

$$L = L_0[v, t] + w^i \frac{\delta L_1}{\delta v^i}[v], \tag{31}$$

where $L_0[v, t]$ and $L_1[v]$ are certain functions of x, u, $\underset{1}{u}$, ..., or to the form

$$L = \operatorname{Re} \hat{L}[z], \tag{32}$$

where $\hat{L}$ is an analytic function of z, $\underset{1}{z}$, ...

Proof. Obviously, it is sufficient to prove the theorem for irreducible systems. In accordance with Lemmas 2 and 3, we can restrict the treatment to two cases:

1. The matrix λ has the unique eigenvalue 0 and $\lambda^2 = 0$.
2. The matrix λ has the eigenvalues ±i and $\lambda^2 = -E$.

Making a linear change of variables of the type (24), the matrix λ is reduced in the first case to the form

$$\lambda = \begin{pmatrix} 0 & \overbrace{E}^{m} \\ 0 & 0 \end{pmatrix} \}m. \tag{33}$$

We now consider the Helmholtz conditions (7) for Eqs. (3) with matrix (33). We can readily see that Eqs. (7) for the system $\tilde{G} = 0$ are equivalent to the equations

$$(-\partial)_\Gamma \left(\eta^b \frac{\partial \tilde{G}_b}{\partial u^a_\Gamma}\right) = \partial_\Gamma (\eta^b) \frac{\partial \tilde{G}_a}{\partial u^b_\Gamma}, \tag{34}$$

which are satisfied for any set of functions $\eta^a = \eta^a[u]$. We set $\varphi^i = \eta^i$, $\mu^p = \eta^{m+p}$, $\xi^i = \eta^{n-m+i}$. Since by definition $\tilde{G}_a \equiv 0$ for a > m, it follows from (34) that for a = m + p and a = n − m + i we obtain

$$(-\partial)_\Gamma \left(\varphi^j \frac{\partial \tilde{G}_j}{\partial t^p_\Gamma}\right) = 0, \quad (-\partial)_\Gamma \left(\varphi^j \frac{\partial \tilde{G}_j}{\partial w^i_\Gamma}\right) = 0.$$

Because φ^i are arbitrary, it follows from this that $\tilde{G}$ does not depend on t^p_Γ, w^i_Γ. Taking into account this fact, we obtain from (33) for a = i

$$(-\partial)_\Gamma \left(\varphi^j \frac{\partial \tilde{G}_j}{\partial v^i_\Gamma}\right) = \partial_\Gamma (\varphi^j) \frac{\partial \tilde{G}_i}{\partial v^j_\Gamma}. \tag{35}$$

By virtue of the Helmholtz conditions, it follows from this that there exists a function $L_1 = L_1[v]$ such that

$$\tilde{G}_i = \frac{\delta L_1}{\delta v^i}[v]. \tag{36}$$

On the other hand, by definition

$$\tilde{G}_i = \frac{\delta L}{\delta w^i}[v, t, w]. \tag{37}$$

Comparing (36) and (37), we obtain (31).

We now consider the case $\lambda^2 = -E$. By a linear change of the variables (24), the matrix λ can be reduced to the form

$$\lambda = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \begin{matrix} \}m \\ \}m \end{matrix}. \tag{38}$$

Let $G_j^{(1)} = G_j$, $G_j^{(2)} = G_{m+j}$. Using (26) and (38), we obtain

$$\frac{\partial G_i^{(2)}}{\partial v_\Gamma^j}=\frac{\partial G_i^{(1)}}{\partial w_\Gamma^i},\quad \frac{\partial G_i^{(2)}}{\partial w_\Gamma^j}=-\frac{\partial G_i^{(1)}}{\partial v_\Gamma^i}. \tag{39}$$

In Eqs. (39) we readily recognize the Cauchy–Riemann equations for the functions $\tilde{G}_j=G_j^{(1)}+iG_j^{(2)}$. Therefore $\tilde{G}_j=\tilde{G}_j[z]$. From this it is easy to show the validity of (32). The theorem is proved.

Remark. As Lagrangian $\tilde{L}$ in the case (31) we can take the function

$$\tilde{L}=cL+L_1,\quad c=\text{const},$$

which corresponds to the matrix $\lambda+cE$, where λ is determined by (33). In the case (32), $\tilde{L}=\operatorname{Im}\hat{L}$. However, there may also exist other alternative Lagrangians. A complete description of all pairs $(L,\tilde{L})$ that satisfy (3) with constant matrix λ will be given in the following paper.

4. Bi-Lagrangian Systems with Hyperregular First-Order Lagrangians Quadratic in the Derivatives

We consider bi-Lagrangian systems with Lagrangians of the form

$$L={}^1/_2(g_{ab}^{\alpha\beta}(u,x)+h_{ab}^{\alpha\beta}(u,x))u_\alpha^a u_\beta^b+f_a^\alpha(u,x)u_\alpha^a+V(u,x), \tag{40}$$

where

$$g_{ab}^{\alpha\beta}=g_{ab}^{\beta\alpha}=g_{ba}^{\alpha\beta},\quad h_{ab}^{\alpha\beta}=-h_{ab}^{\beta\alpha}=h_{ba}^{\beta\alpha},$$

the matrix $g_{ab}^{\alpha\beta}$ satisfying (5).

THEOREM 4. Let (1) be a bi-Lagrangian system with Lagrangian (40) and $d>1$. Then there exists a point transformation of the variables of the form (20) such that the matrix $\lambda\to\bar{\lambda}=\text{const}$ (see (21)). In the variables in which $\lambda=\text{const}$, the Lagrangian (40) is equivalent to either the Lagrangian (31) or the Lagrangian (32) with functions L_0, L_1, $\hat{L}$ that depend on the derivatives of not higher than first order and are quadratic in $\underset{1}{v}$, $\underset{1}{t}$, $\underset{1}{z}$.

Proof. We again consider Eqs. (15). By Theorem 1, the matrix λ satisfies Eqs. (6). In the considered case, the quantities A^α do not depend on $\underset{2}{u}$. Therefore, it follows from (15) that

$$u_{\alpha\beta}^c\frac{\partial\lambda_a{}^b}{\partial u_\beta^c}=0.$$

Repeating the arguments in the proof of Lemma 1, we obtain

$$\frac{\partial\lambda_a{}^b}{\partial u_\alpha^c}=0. \tag{41}$$

Further, the matrices A^α can be expressed in the form

$$A_{ab}^\alpha=\left(\frac{\partial\eta_{ac}^{\alpha\beta}}{\partial u^b}+\frac{\partial\eta_{ib}^{\alpha\beta}}{\partial u^c}-\frac{\partial\eta_{bc}^{\alpha\beta}}{\partial u^a}\right)u_\beta^c+\frac{\partial\eta_{ac}^{\beta\alpha}}{\partial x^\beta}+\left(\frac{\partial f_a{}^\alpha}{\partial u^b}-\frac{\partial f_b{}^\alpha}{\partial u^a}\right), \tag{42}$$

where

$$\eta_{ab}^{\alpha\beta}=g_{ab}^{\alpha\beta}+h_{ab}^{\alpha\beta}.$$

We substitute (42) in (15) and, taking into account (41), compare the terms linear in $\underset{1}{u}$. We obtain

$$\frac{\partial\lambda_a{}^b}{\partial u^c}\delta_\alpha^\beta-(\Gamma_\alpha^\beta)_{ca}^d\lambda_d{}^b+(\Gamma_\alpha^\beta)_{cd}^b\lambda_a{}^d=0, \tag{43}$$

where

$$(\Gamma_\alpha^\beta)_{bc}^a=\frac{1}{2}g_{\gamma\alpha}^{da}\left(\frac{\partial\eta_{db}^{\gamma\beta}}{\partial u^c}+\frac{\partial\eta_{dc}^{\beta\gamma}}{\partial u^b}-\frac{\partial\eta_{ca}^{\gamma\beta}}{\partial u^d}\right).$$

We define the quantities

$$\Gamma_{bc}^a=\frac{1}{d}(\Gamma_\alpha^\alpha)_{bc}^a.$$

It is readily seen that for each fixed x the quantities Γ_{bc}^a transform under the change of variables (20) as the coefficients of some affine connection ∇ without torsion. Bearing in mind that λ_a^b transforms as a tensor of rank (1, 1) under the same transformations (see (21)), we obtain from (43)

$$\nabla\lambda=0. \tag{44}$$

But it follows from (44) by Shirokov's theorem [14] that there exists a change of variables of the type (20) such that λ_a^b in the new variables does not depend on u. In other words, in the new variables λ_a^b depends only on x. Further, making the linear transformation (24) with matrix T that depends on x, we can for each x reduce λ to the canonical Jordan form. In this form, each element of the matrix λ is either equal to zero, or unity, or to the real or the imaginary part of some eigenvalue of the matrix λ (see [12]). But in accordance with Theorem 1 the eigenvalues of the matrix λ do not depend on x, and this proves the first part of the theorem. The second part is a direct corollary of the first and Theorem 3. Theorem 4 is proved.

Example. We describe all irreducible hyperregular bi-Lagrangian systems with the Lagrangian (40) for n = 2, d > 1. By Theorem 4, the Lagrangian of every such system belongs, up to equivalence and point transformations of the variables, to one of the two following types:

1) $$L=\mathrm{Re}\,\hat{L}[z],\quad z=u^2+iu^1,\quad \hat{L}[z]={}^1/_2 g^{\alpha\beta}(z,x)z_\alpha z_\beta+V(z,x);$$

2) $$L=\frac{1}{2}\left(g_1^{\alpha\beta}(u^1,x)+u^2\frac{\partial g_2^{\alpha\beta}}{\partial u^1}(u^1,x)\right)u_\alpha^1 u_\beta^1+g_2^{\alpha\beta}(u^1,x)u_\alpha^1 u_\beta^2+V_1(u^1,x)u^2+V_0(u^1,x).$$

In the first case, the alternative Lagrangian can be taken to be Im $\hat{L}[z]$, in the second,

$$c_1L+c_2\left({}^1/_2 g_2^{\alpha\beta}(u^1,x)u_\alpha^1 u_\beta^1+\int du^1V_1(u^1,x)\right),\ c_1,c_2=\mathrm{const}.$$

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