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ON THE TRACES OF SOBOLEV FUNCTIONS ON THE BOUNDARY OF A CUSP WITH A HÖLDER SINGULARITY

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Abstract: For the Sobolev classes W_p^1 on a "zero" cusp with a Hölder singularity at the vertex, we consider the question of compactness of the embedding of the traces of Sobolev functions into the Lebesgue classes on the boundary of the cusp.

Keywords: Sobolev space, embedding theorem, trace

Since recently, rather popular is the studying of some generalized function classes of the Sobolev type on metric spaces. One of such generalizations is the function classes introduced by Hajłasz [1]; henceforth we call them the *Sobolev–Hajłasz spaces* and denote by $HW_p^1(X,d,\mu)$, where X is a metric space, d is the metric, and μ is a Borel measure.

In this article we consider a model example of application of the results for the Sobolev–Hajłasz spaces to studying some properties of the classical Sobolev spaces on the Euclidean domains whose boundary has isolated Hölder singularities.

We are interested in the compactness of embedding of the traces of Sobolev functions in the Lebesgue classes on the boundary of the "zero" cusp. There is still no complete description of the space of traces in the framework of the Sobolev–Hajłasz spaces; however, we can expect that the Sobolev–Hajłasz space on the boundary will be close to the space of traces in the sense that it is compactly embedded into the same Lebesgue classes as the space of traces.

The following two arguments suggest that this closure of the corresponding spaces is possible:

1. For domains $G \subset \mathbb{R}^n$ with smooth boundary, the space of the traces of functions of the class $W_p^1(G)$ coincides with the Besov space $B_p^{1-1/p}(\partial G)$ (see [2]). In this event

$$B_p^{1-1/p}(\partial G) \subset HW_p^1(\partial G, |*|^{1-1/p}, H^{n-1}) \subset B_{p-\varepsilon}^{1-1/(p-\varepsilon)}(\partial G)$$

for every $\varepsilon > 0$, where $|*|$ is the Euclidean metric and H^{n-1} is the Hausdorff measure on ∂G.

2. Under some constraints on the Hölder exponent α, the classical Sobolev space $W_p^1(G_\alpha)$ and the Sobolev–Hajłasz space $HW_p^1(G_\alpha, |*|, dx)$ on the "zero" cusps G_α with Hölder singularities at the vertex of the cusp coincide [3]. Moreover, in the scale of Sobolev–Hajłasz spaces, there is a rather exact embedding theorem for the restrictions of functions to sets of less "dimension" [4].

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1. The Sobolev–Hajłasz Spaces

We state some results that are needed in the sequel.

Suppose that (X,d) is a metric space with finite diameter and μ is a finite regular Borel measure on X. We denote by $B(a,r)$ the open ball of radius r and center $a \in X$.

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A function $g : X \to [0,\infty)$ is *admissible* for a μ-measurable function $u : X \to \overline{R}$ if there is a set $E \subset X$ such that $\mu(E) = 0$ and the inequality

$$|u(x) - u(y)| \le d(x,y)(g(x) + g(y)) \tag{1}$$

holds for all $x, y \in X \setminus E$.

Denote the set of all admissible functions for u by $D(u)$ and put $D_p(u) = D(u) \cap L_p(\mu)$ for $p \ge 1$.

Define the function spaces $HL^1_p(X,d,\mu)$ and $HW^1_p(X,d,\mu)$ as follows:

$$HL^1_p(X,d,\mu) = \{u : X \to \overline{R} \mid D_p(u) \ne \varnothing\},$$
$$HW^1_p(X,d,\mu) = \bigl\{u \in L_p(\mu) \mid u \in HL^1_p(X,d,\mu)\bigr\}.$$

The spaces $HL^1_p(X,d,\mu)$ and $HW^1_p(X,d,\mu)$ coincide as sets of functions [1].

Introduce the seminorm in $HL^1_p(X,d,\mu)$ and the norm in $HW^1_p(X,d,\mu)$ by the equalities

$$\bigl\|u \mid HL^1_p\bigr\| = \inf_{g \in D_p(u)} \|g \mid L_p\|, \quad \bigl\|u \mid HW^1_p\bigr\| = \|u \mid L_p\| + \bigl\|u \mid HL^1_p\bigr\|.$$

In the Euclidean domains $G \subset \mathbb{R}^n$ admitting a bounded extension operator $\mathrm{Ext} : W^1_p(G) \to W^1_p(\mathbb{R}^n)$, the Sobolev space $W^1_p(G)$ and Sobolev–Hajłasz space $HW^1_p(G, |*|, dx)$ coincide as sets of functions and their norms are equivalent [1]. Moreover, for a function u in $HW^1_p(G, |*|, dx)$ an admissible function g can be found as the maximal function of the modulus of the gradient of u; i.e., $g(x) = CM(|\nabla u|)(x)$ and

$$M(h)(x) = \sup_{r < \mathrm{diam}(G)} \frac{1}{|Q(x,r)|} \int\limits_{Q(x,r)} h(y)\, dy,$$

where $Q(x,r)$ is the cube with side $2r$ centered at x and $|Q(x,r)|$ is the n-dimensional Lebesgue measure of the cube.

The measure μ is *s-regular* if the estimate $\mu(B(x,r)) \ge br^s$ holds for an arbitrary ball whose radius does not exceed the diameter of X. Moreover, the exponent s plays the role of "dimension" of the metric space (X,d) in analogs of the classical Sobolev embedding theorems. The following assertion is proven in [1]:

Proposition 1. *Suppose that $1 < p < \infty$ and the measure μ is s-regular. Then the Sobolev–Hajłasz space $HW^1_p(X,d,\mu)$ is continuously embedded into $u \in L_q(\mu)$, where*

(1) $1 \le q \le \frac{ps}{s-p}$ for $p < s$;

(2) $1 \le q < \infty$ for $p = s$;

(3) $1 \le q \le \infty$ for $p > s$.

Henceforth we suppose that the measure μ satisfies the "doubling condition"

$$\mu(B(x,2r)) \le C_d \mu(B(x,r))$$

for all $x \in X$ and $r > 0$.

Every measure satisfying the doubling condition is s-regular with exponent $s = \log_2 C_d$.

If $1 < p \le \infty$ then a function u belongs to the Sobolev–Hajłasz space if and only if Poincaré's inequality

$$\int\limits_{B(x,r)} |u - u_{B(x,r)}| d\mu \le r \int\limits_{B(x,r)} g\, d\mu$$

holds for some function $g \in L_p(\mu)$ and all $x \in X$ and $r > 0$ [5].

Validity of Poincaré's inequality for the functions of $HW^1_p(X,d,\mu)$ and the result of Proposition 1 enable us to restate Theorems 8.2 and 8.3 of [6] in convenient form:

Proposition 2. *For $1 < p < s$ an arbitrary norm-bounded sequence $\{u_i\}$ in $HW_p^1(X,d,\mu)$ contains a subsequence that converges in $L_q(\mu)$ to some function $u \in L_q(\mu)$ for all $1 \le q < \frac{ps}{s-p}$.*

Suppose that a subset $E \subset X$ and a measure ν satisfying the doubling condition are such that the estimate $\nu(B(x,r)) \le Cr^{-\alpha}\mu(B(x,r))$ is valid for an arbitrary ball $B(x,r)$ centered at $x \in E$, where $0 < \alpha < s$. Then the Lebesgue points of $f \in HL_p^1(X,d,\mu)$ with $p > \alpha$ are ν-almost all points of E; moreover, at these points we can define f by the equality

$$f(x) = \lim_{r\to 0} \frac{1}{\mu(B(x,r))} \int\limits_{B(x,r)} f\, d\mu.$$

Moreover, the traces of functions in the Sobolev–Hajłasz spaces $HL_p^1(X,d,\mu)$ on E belong to the corresponding Hölder classes which can be considered as the Sobolev–Hajłasz spaces with respect to the Hölder metric. The following result was obtained in [4]:

Proposition 3. *Let $1 < p < s$ and $0 < \alpha < \min(s,p)$. Then $L_p^1(X,d,\mu)$ is continuously embedded into $L_q^1(E, d^{1-\gamma}, \nu)$, where $\frac{\alpha}{p} < \gamma < \frac{s}{p}$ and $q \le \frac{p(s-\alpha)}{s-\gamma p}$.*

2. Connection Between the Sobolev Spaces and Sobolev–Hajłasz Spaces on the "Zero" Cusps

Denote the points of $\mathbb{R}^n$ by (x,y), where $x \in R$ and $y \in \mathbb{R}^{n-1}$. Given $1 \le \alpha < \infty$, define the cusp $G_\alpha \subset \mathbb{R}^n$ as

$$G_\alpha = \{(x,y) \in \mathbb{R}^n \mid 0 < x < 1,\ 0 < y_k < x^\alpha,\ k = 1,\dots,n-1\}.$$

Put $\Lambda = 1 + (n-1)\alpha$. Since the estimate $|B(0,r) \cap G_\alpha| \sim Cr^\Lambda$ holds for the balls centered at the vertex of the cusp, the exponent Λ often plays the role of the "dimension" of G_α in various estimates.

For $\alpha = 1$ we obtain a domain with Lipschitz boundary; the Sobolev space $W_p^1(G_1)$ and Sobolev–Hajłasz space $HW_p^1(G_1, |*|, dx)$ coincide; their norms are equivalent and an admissible function g for a function $u \in HW_p^1(G_1, |*|, dx)$ can be found from the maximal function of the modulus of the gradient of u; i.e., $g(x) = CM(|\nabla u|)(x)$.

For $\alpha > 1$ the boundary of the cusp G_α has a Hölder singularity at the vertex. It is well known that for G_α no extension is possible of the functions $u \in W_p^1(G_\alpha)$ to the whole $\mathbb{R}^n$ with preservation of the class. However, we can show that, for sufficiently large values of the summability exponent p, the spaces $W_p^1(G_\alpha)$ and $HW_p^1(G_\alpha, |*|, dx)$ coincide.

The embedding $HW_p^1(G_\alpha, |*|, dx) \subset W_p^1(G_\alpha)$ holds always [1] and, to prove the reverse embedding, we use the fact that, although there is no bounded extension operator for functions of $W_p^1(G_\alpha)$ with preservation of the class, for some $1 < q < p$ there is a bounded extension operator $\mathrm{Ext} : W_p^1(G_\alpha) \to W_q^1(G_1)$ [7].

Unfortunately, the extension operator in [7] is not quite appropriate for obtaining the necessary estimates. Therefore, using the scheme of [3] and simple geometric arguments, we construct another extension of a function from G_α to G_1 which fulfils our requirements.

For $1 \le k \le n-1$ consider the system of nested cusps:

$$P_k = \{(x,y) \in \mathbb{R}^n \mid 0 < x < 1,\ 0 < y_i < x,\ 1 \le i \le k,\ 0 < y_j < x^\alpha,\ k+1 \le j \le n-1\}.$$

Note that $G_\alpha \subset P_1$ and $P_{n-1} = G_1$.

First, extend the function $u \in W_p^1(G_\alpha)$ to P_1. Let $(x, y_1, y_2, \dots, y_{n-1}) \in P_1$ and $y_1 = m_1 x^\alpha + t_1$, where $m_1 = \left[\frac{y_1}{x^\alpha}\right]$ is the integer part. Put

$$v_1(x, y_1, y_2, \dots, y_{n-1}) = u(x, y_1^*, y_2, \dots, y_{n-1}),$$

where $y_1^* = t_1$ for even values of m_1 and $y_1^* = x^\alpha - t_1$ for odd values of m_1.

On each segment parallel to the axis OY_1 the values of v_1 are defined by doubling the values of u by symmetry with respect to the points at which the value y_1 is a multiple of x^α. Thus, the extension is carried out successively by symmetry to the layers in which $lx^\alpha < y_1 < (l+1)x^\alpha$. The number of the resulting layers depends on the proximity of x to the origin and is equivalent to $x^{1-\alpha}$. Each straight line parallel to the coordinates axes and intersecting the cusp P_1 contains only finitely many "gluing" points of v_1 which represents the extension of u to P_1; therefore, v_1 belongs to $ACL(P_1)$.

Repeating the above procedure, we construct the extension from P_k to P_{k+1} and eventually the function v defined in the Lipschitz cusp G_1 by the condition $v(x,y) = u(x,y^*)$, where $y_k = m_k x^\alpha + t_k$, $y_k^* = t_k$ for the even values of m_k and $y_k^* = x^\alpha - t_k$ for the odd values of m_k.

The function v belongs to $ACL(G_1)$; moreover, we can easily find its derivatives

$$\left|\frac{\partial v}{\partial y_k}(x,y)\right| = \left|\frac{\partial u}{\partial y_k}(x,y^*)\right|; \quad \frac{\partial v}{\partial x}(x,y) = \frac{\partial u}{\partial x}(x,y^*) + \sum_{k=1}^{n-1} \frac{\partial u}{\partial y_k}(x,y^*)\frac{\partial y_k^*}{\partial x}(x,y).$$

Using the fact that we always have $y_k < x$ for $(x,y) \in G_1$, we obtain

$$\left|\frac{\partial y_k^*}{\partial x}(x,y)\right| \leq 2\alpha m_k x^{\alpha-1} = 2\alpha \left[\frac{y_k}{x^\alpha}\right] \frac{x^\alpha}{y_k}\frac{y_k}{x} \leq 2\alpha.$$

Consequently,

$$|\nabla v(x,y)| \leq C_0 |\nabla u(x,y^*)|.$$

Denote by D_x the section of G_1 by the hyperplane orthogonal to OX and passing through $(x,0)$; and by E_x, the section of G_α by the same hyperplane. It follows from the construction of the extension that

$$\int_{D_x} |\nabla v(x,y)|^q dy \leq C x^{(1-\alpha)(n-1)} \int_{E_x} |\nabla u(x,y)|^q \, dy.$$

Using Hölder's inequality for $q < p$, we easily obtain the following estimate for the integral of the modulus of the gradient of v over G_1:

$$\begin{aligned}\int_{G_1} |\nabla v(x,y)|^q \, dxdy &= \int_0^1 \Big(\int_{D_x} |\nabla v(x,y)|^q \, dy \Big) \, dx \\ &\leq C \int_0^1 x^{(1-\alpha)(n-1)} \Big(\int_{E_x} |\nabla u(x,y)|^q \, dy \Big) \, dx \\ &= C \int_{G_\alpha} x^{(1-\alpha)(n-1)} |\nabla u(x,y)|^q \, dxdy \leq C \|u \mid L_p^1(G_\alpha)\|^q \Big(\int_0^1 x^{-s} \, dx \Big)^{\frac{p-q}{p}},\end{aligned}$$

where $s = \frac{(n-1)(\alpha q - p)}{p-q}$. The last integral converges for $q < \frac{np}{1+(n-1)\alpha}$. Consequently, for $p > \frac{1+(n-1)\alpha}{n} = \frac{\Lambda}{n}$ the function v belongs to $W_q^1(G_1)$ for some $q > 1$. Observe that the resulting constraints on the summability exponents agree with the results of [7].

The spaces $W_q^1(G_1)$ and $HW_q^1(G_1, |*|, dx)$ coincide, and an admissible function g for a function v can be found from the maximal function of the modulus of the gradient of v; i.e., $g = CM(|\nabla v|) \in L_q(G_1)$. To prove that u belongs to $HW_p^1(G_\alpha, |*|, dx)$, it suffices to show that $M(|\nabla v|) \in L_p(G_\alpha)$. Of course, the fact that a function belongs to $L_q(G_1)$ does not imply in general that it belongs to $L_p(G_\alpha)$; however, in the given case the function v is not arbitrary and its values are obtained by doubling the values of u over the corresponding layers; this is what enables us to obtain the necessary estimates.

With each point $z = (x, y) \in G_\alpha$ we associate the number $r(z)$ such that

$$M(|\nabla v|)(z) \le \frac{2}{|Q(z, r(z))|} \int\limits_{Q(z,r(z))} |\nabla v| \, dt d\tau,$$

where $t \in R,\ \tau \in \mathbb{R}^{n-1}$.

Split G_α into disjoint subsets by putting

$$A = \{(x, y) \in G_\alpha \mid r(x, y) > 1/4\}, \quad B = \{(x, y) \in G_\alpha \setminus A \mid r(x, y) < x/2\},$$

$$D_k = \{(x, y) \in G_\alpha \setminus A \mid 2^{k-1}x \le r(x, y) < 2^k x, \quad k = 0, 1, \dots\}.$$

The function $M(|\nabla v|)$ is bounded on A, since

$$M(|\nabla v|)(z) \le \frac{2}{|Q(z, r(z))|} \int\limits_{Q(z,r(z))} |\nabla v| \, dt d\tau \le C_1 \int\limits_{G_1} |\nabla v| \, dt d\tau$$
$$\le C_2 \|\nabla v \mid L_q(G_1)\| \le C_3 \|\nabla u \mid L_p(G_\alpha)\|$$

for $z \in A$. Consequently,

$$\int\limits_A (M(|\nabla v|))^p \, dx dy \le C \|\nabla u \mid L_p(G_\alpha)\|^p.$$

Denote by μ the restriction of the n-dimensional Lebesgue measure to G_α; i.e., $\mu(E) = |E \cap G_\alpha|$. Let $z = (x, y) \in B$ and $r(z) < x^\alpha$. Then

$$\int\limits_{Q(z,r(z))} |\nabla v| \, dt d\tau \le C_0 \int\limits_{Q(z,r(z))} |\nabla u| \, d\mu.$$

Hence,

$$M(|\nabla v|)(z) \le \frac{2}{|Q(z, r(z))|} \int\limits_{Q(z,r(z))} |\nabla v| \, dt d\tau \le \frac{2C_0}{\mu(Q(z, r(z))} \int\limits_{Q(z,r(z))} |\nabla u| \, d\mu.$$

If $x^\alpha \le z(z) < \frac{x}{2}$ then

$$\int\limits_{Q(z,r(z))} |\nabla v| \, dt d\tau \le C_1 \left(\frac{r + x^\alpha}{(x - r)^\alpha} \right)^{n-1} \int\limits_{Q(z,r(z))} |\nabla u| \, d\mu$$

and

$$\frac{\mu(Q(z, r(z))}{|Q(z, r(z))|} \le C_2 \frac{x^{\alpha(n-1)}}{r^{n-1}}.$$

Thus,

$$M(|\nabla v|)(z) \le \frac{2}{|Q(z, r(z))|} \int\limits_{Q(z,r(z))} |\nabla v| \, dt d\tau \le \frac{C_3}{\mu(Q(z, r(z))} \int\limits_{Q(z,r(z))} |\nabla u| \, d\mu.$$

Consequently, the estimate

$$M(|\nabla v|)(z) \le \widetilde{C} \mathscr{M}(|\nabla u|)(z)$$

holds for every point $z = (x, y) \in B$, where

$$\mathscr{M}(|\nabla u|)(z) = \sup_{r>0} \frac{1}{\mu(Q(z, r))} \int\limits_{Q(z,r)} |\nabla u| d\mu. \qquad (2)$$

Since the measure μ satisfies the doubling condition in G_α, the maximal operator (2) is bounded in $L_p(G_\alpha)$ for $p > 1$ [8] and thereby $M(|\nabla v|) \in L_p(B)$.

Suppose that a point $z = (x, y)$ lies in D_k. Then $x + r(z) < 2^{k+1}x$ and

$$M(|\nabla v|)(z) \leq \frac{2}{|Q(z, r(z))|} \int\limits_{Q(z,r(z))} |\nabla v| \, dt d\tau \leq \frac{C}{(2^k x)^n} \int\limits_0^{2^{k+1}x} t^{(n-\Lambda)} \, dt \int\limits_{E_t} |\nabla u| \, d\tau;$$

moreover, the estimate is independent of y.

Since $r(z) \leq \frac{1}{4}$, we find that $x \leq 2^{-(k+1)}$ for every point $z = (x, y) \in D_k$. Executing the change of variable $w = 2^{k+1}x$ in the integral, we obtain

$$\int\limits_{D_k} (M(|\nabla v|))^p \, dx dy \leq \int\limits_0^{2^{-(k+1)}} x^{\Lambda-1} \bigg(\frac{C}{(2^k x)^n} \int\limits_0^{2^{k+1}x} t^{(n-\Lambda)} \, dt \int\limits_{E_t} |\nabla u| \, d\tau \bigg)^p \, dx$$

$$\leq \frac{C_1}{2^{k\Lambda}} \int\limits_0^1 \bigg(\int\limits_0^w h(t) \, dt \bigg)^p w^{\Lambda - np - 1} dw,$$

where $h(t) = t^{(n-\Lambda)} \int_{E_t} |\nabla u| \, d\tau$.

Since $\Lambda - np < 0$, using Hardy's inequality and Hölder's inequality successively, we arrive at the estimate

$$\int\limits_{D_k} (M(|\nabla v|))^p \, dx dy \leq \frac{C_2}{2^{k\Lambda}} \int\limits_0^1 (th(t))^p t^{\Lambda - np - 1} \, dt$$

$$= \frac{C_2}{2^{k\Lambda}} \int\limits_0^1 t^{(\Lambda-1)(1-p)} \bigg(\int\limits_{E_t} |\nabla u| \, d\tau \bigg)^p \, dt \leq \frac{C_2}{2^{k\Lambda}} \int\limits_{G_\alpha} |\nabla u|^p \, d\tau dt.$$

Summing the above estimates, we find that

$$\int\limits_{G_\alpha} (M(|\nabla v|))^p \, dx dy = \int\limits_A (M(|\nabla v|))^p \, dx dy + \int\limits_B (M(|\nabla v|))^p \, dx dy$$

$$+ \sum_k \int\limits_{D_k} (M(|\nabla v|))^p \, dx dy \leq \widetilde{C} \int\limits_{G_\alpha} |\nabla u|^p \, dx dy.$$

Thus, for $p > \Lambda/n$ the function $M(|\nabla v|)$ is admissible for the function u in G_α and belongs to $L_p(G_\alpha)$. Hence, the Sobolev space $W_p^1(G_\alpha)$ and Sobolev–Hajłasz space $HW_p^1(G_\alpha, |*|, dx)$ coincide, and their norms are equivalent.

3. Compactness of the Embedding of Traces on the Boundary of a Cusp

Let $\alpha \geq 1$ and $\alpha < p < \Lambda$. Since $\alpha \geq \Lambda/n$, the Sobolev space $W_p^1(G_\alpha)$ and Sobolev–Hajłasz space $HW_p^1(G_\alpha, |*|, dx)$ coincide, and we can use the assertions of Propositions 1–3.

Denote by μ the restriction of the n-dimensional Lebesgue measure to $\overline{G}_\alpha$ and by ν the restriction of the $(n-1)$-dimensional Hausdorff measure to the boundary of G_α. Inequality (1) in the definition of the Sobolev–Hajłasz spaces must hold only almost everywhere with respect to the measure μ; therefore, we can assume that, initially, the underlying metric space of HW_p^1 is $\overline{G}_\alpha$, i.e., the closure of the cusp.

For an arbitrary point $x \in \partial G_\alpha$ and an arbitrary ball $B(x, r)$, $r < \operatorname{diam} G_\alpha$ we have the estimate $\nu(B(x, r)) \leq C r^{-\alpha} \mu(B(x, r))$. By [4], for $u \in HW_p^1(\overline{G}_\alpha, |*|, \mu)$, the Lebesgue points of u with respect to the measure μ are ν-almost all points $x \in \partial G_\alpha$. Therefore, the values of the trace v of u on ∂G_α can be determined as the limit of the mean values of u with respect to μ; i.e.,

$$v(x) = \lim_{r\to 0} \frac{1}{\mu(B(x,r))} \int\limits_{B(x,r)} u\, d\mu.$$

The measures μ and ν satisfy the doubling condition and the "dimension" of $\overline{G}_\alpha$ defined with respect to μ is equal to Λ. Introduce the new metric on $\overline{G}_\alpha$ by the equality $d(x,y) = |x-y|^{1-\gamma}$, where $\frac{\alpha}{p} < \gamma < 1$.

By Proposition 3, $HL^1_p(\overline{G}_\alpha, |*|, \mu)$ is continuously embedded in $HL^1_r(\partial G_\alpha, d, \nu)$, where $r \le \frac{p(\Lambda-\alpha)}{\Lambda-\gamma p}$.

The proofs of Theorems 1 and 3 of [3] imply the existence of a function g_γ such that the following inequality holds for ν-almost all points $x \in \partial G_\alpha$ and μ-almost all points $y \in G_\alpha$:

$$|v(x) - u(y)| \le (d(x,y))^{1-\gamma}(g_\gamma(x) + g_\gamma(y)); \tag{3}$$

moreover,

$$\|g_\gamma \mid L_r(\partial G_\alpha, \nu)\| \le C_1 \big\|u \mid HL^1_p(G_\alpha, |*|, \mu)\big\|,$$

$$\|g_\gamma \mid L_p(G_\alpha, \mu)\| \le C_2 \big\|u \mid HL^1_p(G_\alpha, |*|, \mu)\big\|.$$

Integrating (3) with respect to the variable y over the whole cusp G_α with respect to the measure μ and dividing the result by $\mu(G_\alpha)$, we obtain

$$|v(x)| \le C_3 g_\gamma(x) + C_4 \big\|u \mid HW^1_p(G_\alpha, |*|, \mu)\big\|;$$

consequently,

$$\bigg(\int\limits_{\partial G_\alpha} |v|^r\, d\nu\bigg)^{1/r} \le C\big\|u \mid HW^1_p(G_\alpha, |*|, \mu)\big\|.$$

Thus, the trace operator

$$\mathrm{Tr} : HW^1_p(G_\alpha, |*|, \mu) \to HW^1_r(\partial G_\alpha, d, \nu)$$

is bounded.

The "dimension" of ∂G_α defined by the measure ν and the metric d is equal to $s_1 = \frac{\Lambda-\alpha}{1-\gamma}$.

By Proposition 1, $HW^1_r(\partial G_\alpha, d, \nu)$ is continuously embedded into $L_{q_0}(\partial G_\alpha, d, \nu)$, where $q_0 = \frac{rs_1}{s_1-r}$, and, by Proposition 2, is compactly embedded into $L_q(\partial G_\alpha, d, \nu)$ for $1 \le q < q_0$.

Recalculating the summability exponents, we obtain the following assertion for the classical Sobolev spaces:

Theorem. *Let $\alpha \ge 1$ and $p > \alpha$. The trace operator $\mathrm{Tr} : W^1_p(G_\alpha) \to L_q(\partial G_\alpha, \nu)$ is compact for*
(1) $1 \le q < p\frac{\Lambda-\alpha}{\Lambda-p}$, when $p < \Lambda$;
(2) $1 \le q < \infty$, when $p \ge \Lambda$.

The second assertion of the theorem is a consequence of the first, since the domain G_α is bounded and $W^1_p(G_\alpha)$ is continuously embedded into $W^1_{p^*}(G_\alpha)$ for all $p^* < p$.

Exactness of the estimate for the exponent q in the assertion (1) of the theorem can be verified by a simple example.

Example. Consider the sequence of Lipschitz functions $\{u_k\}$ defined at $(x,y) \in G_\alpha$ by the condition

$$u_k(x,y) = k^{-1+\Lambda/p}\begin{cases} 1, & 0 < x \le 1/2k, \\ 2(1-kx), & 1/2k \le x \le 1/k, \\ 0, & x \ge 1/k. \end{cases}$$

Since $|u_k| \le k^{-1+\Lambda/p}$, while $|\nabla u_k| = 2k^{\Lambda/p}$ for $1/2k \le x \le 1/k$ and $|\nabla u_k| = 0$ otherwise, we have $\|\nabla u_k \mid L_p(G_\alpha)\| \le C_0$ and $\|u_k \mid L_p(G_\alpha)\| \le C_1 k^{-1}$. Hence, the sequence $\{u_k\}$ is norm-bounded in $W_p^1(G_\alpha)$.

Denote by v_k the trace of u_k on the boundary of G_α. Then $v_k(x,y) = k^{-1+\Lambda/p}$ for $0 < x \le 1/2k$ and $v_k(x,y) = 0$ for $x \ge 1/k$.

Let $E_k = \partial G_\alpha \cap B(0, 1/2k)$. If $q_0 = p\frac{\Lambda-\alpha}{\Lambda-p}$ then

$$\|v_k \mid L_{q_0}(\partial G_\alpha, \nu)\|^{q_0} \ge k^{q_0(-1+\Lambda/p)} \int\limits_{E_k} d\nu \ge C > 0.$$

Since the sequence $\{v_k\}$ converges to zero almost everywhere on ∂G_α and $\|v_k \mid L_{q_0}(\partial G_\alpha, \nu)\| \ge C_0 > 0$, the given sequence has no subsequence convergent in $L_{q_0}(\partial G_\alpha, \nu)$.

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