DIFFUSION OF ELECTRONS IN COHERENT LANGMUIR WAVE PACKETS

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A new diffusion coefficient for electrons repeatedly interacting with coherent Langmuir wave packets is proposed. The model accounts for regular islands embedded in a region of connected stochasticity, since these islands have a profound effect on the diffusion process. The diffusion coefficient has a resonant form which has been explicitly calculated incorporating the finite decay times of the correlation functions. Theoretical predictions compare well with numerical results derived from solutions of the equation of motion.

In this Letter we report on numerical and analytical studies of the interaction between electrons and localized, coherent Langmuir wave structures. The existence of such nonlinear, soliton-like objects has been well established in theory and experiments (e.g. review [1]). In particular our work has been prompted by recent theoretical developments in the strong Langmuir turbulence in which Doolen et al. [2] proposed a mechanism of localization and cavitation formation on preexisting ion density modulations. A coherent version of this mechanism has found application in the 1-D models describing laser–plasma interaction physics [3,4]. In this theory Langmuir waves produced by stimulated Raman scattering (SRS) can nonresonantly interact with ion waves produced by simultaneous stimulated Brillouin scattering (SBS) and develop localized, periodic structures in the fields of the electron plasma waves.

Each structure in this periodic sequence of localized fields can be approximated by the following analytical expression

$$E(x,t)=E_0\,\mathrm{sech}\left(E_0 x\frac{1}{\lambda_D}\frac{1}{(8\pi n k_B T)^{1/2}}\beta\right)\times\cos(\omega_p t-k_L x)\,, \tag{1}$$

where $\beta=1/2\sqrt{3}$, k_L is the electron plasma wave wave-number, $\omega_p=(4\pi e^2 n_e/m_e)^{1/2}$, and $\lambda_D=(k_B T_e/4\pi e^2 n_e)^{1/2}$. For $k_L=0$ expression (1) is a stationary solution to the Zakharov equations [1]. The periodicity can be introduced into eq. (1) by representing the field in terms of wave packets of the form

$$E(x,t)=\sum_n E_n\cos(\omega_p t-k_n x)\,, \tag{2}$$

where $k_n=k_L+nk_A$, $n=0,\pm1,\pm2,\ldots$ and $k_A=2\pi/l$ defines periodicity length l. Eq. (2) has the form of a Bloch function. Fig. 1 shows a particular choice of the wave packet for $E_0^2/8\pi n_e k_B T_e=0.53$, $k_A=2k_L=0.09k_D$. This choice could be relevant for models of SRS and SBS at $n_{cr}/4$ (n_{cr} is the critical density at which the laser frequency $\omega_0=\omega_p$). In order to adequately represent the localized fields we have used 12 modes with amplitudes E_n. The insert in fig. 1

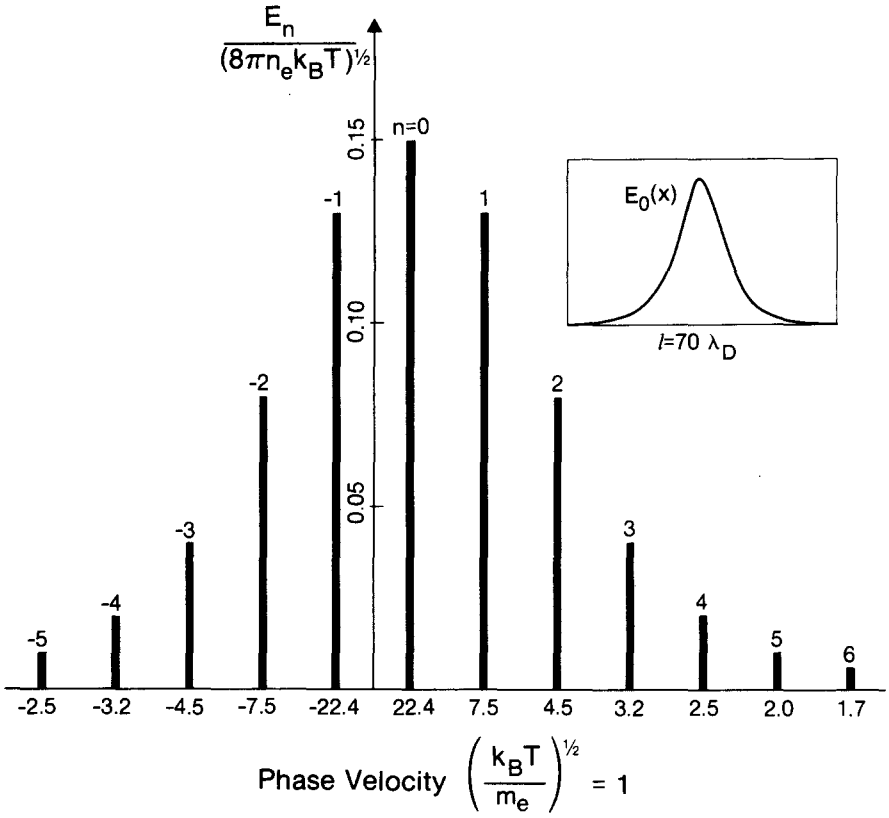

Fig. 1. The spectrum of the Langmuir wave packet (2). The insert shows the field envelope of a single element, over the interval l, in the periodic sequence.

shows that the electric field amplitude is well localized within the periodicity length.

The main purpose of this study is to investigate a statistical description of electrons interacting with the localized fields (2). We describe a diffusion model which allows for the existence of regular islands within the region of stochasticity in phase space. This model uses, as one of its parameters, the finite decay times of the correlation functions.

The problem of electrons interacting with sharply localized Langmuir fields was analyzed in ref. [5]. It was found that if the localized fields had random phases the quasi-linear diffusion gave a good description of the evolution of the particle distribution function. A similar conclusion was reached by Fuchs et al. [6] for coherent wave packets with a large nunber of overlapping modes. Our results show that for realistic wave packets of Langmuir waves (2) which have a small number of modes, quasi-linear theory provides only a crude approximation for the diffusion coefficient.

We begin our analysis with numerical solutions of the equation of motion

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} x(t) = -\frac{e}{m_e} \sum_n E_n \cos(\omega_p t - k_n x) \,. \qquad (3)$$

The results are presented in fig. 2 which shows the phase plane constructed by mapping numerical solutions every $T_p = 2\pi/\omega_p$. The initial velocities spanned the region between the phase velocities of the faster modes in the wave packet (2). Our choice of plasma parameters illustrates quite well all possible kinds of particle trajectories including regions of connected stochasticity, and trapped and untrapped orbits.

Except for the modes $n=0$, -1, all the modes in

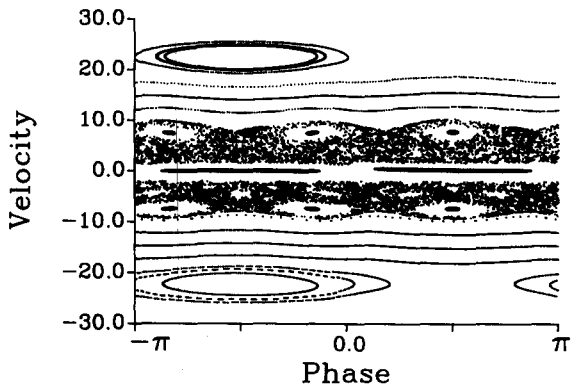

Fig. 2. The Poincaré surface of section plot, based on numerical solutions to the equations of motion (3). Points are mapped every $T_p = 2\pi/\omega_p$. Velocity is in units of the thermal velocity and phase is given in units of $2\pi/k_L$. The length of the cell is $2l$ (note that $2k_L = k_A$).

the wave packet (3) satisfy the overlapping criterion. If particles are given initial velocities within the interval defined by the overlapping resonances, a large region of connected stochasticity is evident. The low velocity bounding the region of stochasticity is determined by the regular trajectories due to the ponderomotive potential. In fig. 2 the two orbits seen in the vicinity of $v=0$ are due to particles which are reflected by the ponderomotive potential of the two localized structures, one in the center of the box and the other on the edges. The high velocity boundary is determined by the regular trajectories of untrapped particles. The blank area around zero velocity $|v| < 1.5v_{th}$ is bounded by regular orbits and is therefore inaccessible to particles from the stochastic region. The same is true for the regular islands embedded in the region of mixing. These islands are particularly large for modes $n=1, -2$ ($v_{ph} = \pm 7.5v_{th}$). We have introduced particles in these islands in order to show the trapped orbits. Finally, fig. 2 shows two large resonances ($n=0, -1$) at velocities $v = \pm 22.4v_{th}$ separated by regular orbits from the stochastic region.

Based on the map in fig. 2 one can make two important predictions for the evolution of an initial maxwellian ($v_{th}=1$) distribution function. First, the bulk of the particles move in the ponderomotive potential well. They are trapped between localized fields and undergo total reflection. In effect the distribution function, averaged over phases, does not change in the velocity range $|v| \lesssim 1.5v_{th}$. Secondly, those electrons which start at low velocities in the stochastic region and diffuse to higher velocities will not get trapped in the potential wells formed by modes $n=1, \pm 2, -3$. However, the existence of these islands still influences the nonadiabatic trajectories, particularly in the vicinity of the velocity bounds of separatrices when particles are rapidly accelerated and have high diffusion rates. This feature will be clarified in the calculations of the diffusion coefficient.

In order to introduce a statistical description into the dynamical system described above we will concentrate on the region of mixing as shown in fig. 2 and try to construct a diffusion coefficient. Diffusion of electrons by coherent wave packets has been discussed recently in great detail by Fuchs et al. [6]. In the weak-field regime, i.e. when the trapping time for the waves is greater than autocorrelation time, and when the overlapping criterion is satisfied, the quasi-linear theory gives an adequate description of the diffusion process. There are certain similarities between our problem and the model analyzed in ref. [6]. Unfortunately, only the slowest modes in the wave packet (2) ($|n| \geq 4$) satisfy the conditions required for the validity of the quasi-linear theory, as discussed in ref. [6]. The remainder of the spectrum (cf. fig. 1) corresponds to large amplitude modes, for which the trapping time becomes less than the time of one pass of a resonant electron through the localized field. The modes still overlap (except $n=0, -1$) but give fairly large adiabatic islands in the phase space (cf. fig. 2). We stress again that our particular choice of conditions for this example illustrates most of the features for the particle orbits. A more localized soliton will give larger mixing region (and vice versa), but will still have the same set of orbits which are represented in fig. 2.

In the derivation of the diffusion coefficient [7,8] one arrives at the formula

$$D(v) = \int_0^\infty \mathrm{d}t \, \tfrac{1}{2} \sum_n \left(\frac{e}{m_e}\right)^2 |E_n|^2 \times \langle \exp\{\mathrm{i}[\varphi_{k_n}(t) - \varphi_{k_n}(0)]\} \rangle , \quad (4)$$

where the averages $\langle\ \rangle$ are taken with respect to initial phases and $\varphi_{k_n}(t) = k_n x(t) - \omega_p t$ is evaluated along a particle trajectory. In quasi-linear theory [7],

$x(t)$ is approximated by a free-streaming trajectory leading to the well known singular result for the diffusion coefficient

$$D_{QL}=\tfrac{1}{2}\pi\left(\frac{e}{m_e}\right)^2\sum_n |E_{k_n}|^2\delta(\omega_p-k_n v)\,.$$

One might argue that in the presence of mixing and in the case of very short autocorrelation times for particles within the localized fields, the quasi-linear description is a valid model. The presence of stochastic instability in the particle motion and the existence of a dense population of overlapping modes, makes it possible to use a transition to a continuous representation $\sum_n \to (1/\Delta k)\int \mathrm{d}k$, for which

$$D_{QL}=\left(\frac{e}{m_e}\right)^2 \tfrac{1}{4}l|E|^2_{k=\omega/v}\frac{1}{|v|}\,. \tag{5}$$

The above arguments proved to work well for the cases discussed in ref. [6]. In our model, which represents typical Langmuir wave packets, the transit time of electrons is comparable to the trapping time. Overlapping modes leave large regular islands and consequently we need to examine eq. (4) more carefully.

We start by abandoning the assumption of free trajectories and introduce a discrete map as an approximation for the motion of particles

$$v^{n+1}=v^n-\frac{e}{m_e}E_k\cos\varphi_k^n\,\Delta t^n\,, \tag{6}$$

$$\varphi_k^{n+1}=\varphi_k^n+kl-\omega_p\Delta t^n\,, \tag{7}$$

where $\Delta t^n=\frac{1}{2}l(1/v^n+1/v^{n+1})$ is the approximate time of one pass through the localized field during the nth collision. In writing eq. (6) we have assumed (cf. ref. [6]) that in the zero order approximation electrons interact with one mode for which $v^n\approx\omega_p/k$ and that during the acceleration the phase in eq. (6) changes along the free-streaming orbit. With eqs. (6) and (7) describing motion of electrons, one can write for the correlation function

$$C=\langle\exp\{\mathrm{i}[\varphi_{k_n}(t)-\varphi_{k_n}(0)]\}\rangle$$

$$\approx\left\langle\exp\left(\mathrm{i}\sum_{r=1}^{m}(\varphi^r_{k_n}-\varphi^{r-1}_{k_n})\right)\right\rangle\,. \tag{8}$$

Simplifying even further the time evolution of $\varphi^r_{k_n}$ by linearizing the map with respect to the strength of the field, we can rewrite eq. (8) as

$$C_m\approx\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\varphi^0_{k_n}\exp\Big(-m\Delta t^0\omega_p+mk_nl$$

$$-s_{k_n}\sum_{r=1}^{m}[1+2(m-r)]\cos\varphi^{r-1}\Big)\,, \tag{9}$$

where $s_{k_n}=\frac{1}{2}(e/m_e)E_{k_n}(\Delta t^0)^2$ and $\Delta t^0=l/v^0$ is the time of one pass, based on the initial velocity.

Fig. 3 presents comparison between results of the approximate map (9) and the numerical results for eq. (8) based on solutions to the equation of motion. In the numerical studies the initial conditions put particles on stochastic orbits and averages were taken with respect to time, rather than phases. The cor-

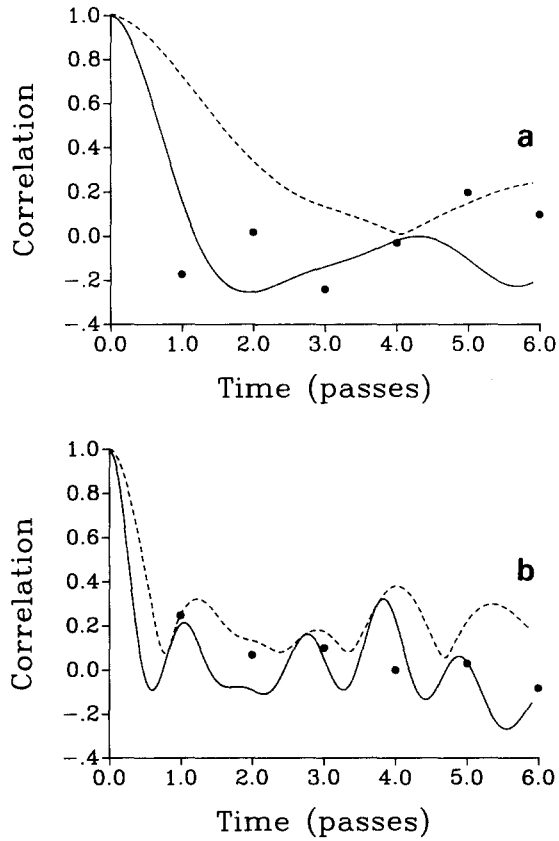

Fig. 3. Correlation functions C (eq. (8)) calculated from the numerical solutions of the equations of motion (3) (solid curve). The circles are obtained from the approximate map (9). The dashed line shows an estimate of the envelope of the correlation function. (a) corresponds to $v=7.18$, (b) $v=5.81$.

relation function for $v_0=7.16$ in fig. 3 corresponds to rather long decay times in terms of number of passes. The initial velocity of this trajectory is closer to the phase velocity of the wave with a regular island. This particle experiences smaller transfer of momentum and longer decay time for the correlation functions as compared to, for example, a particle with velocity $v_0=5.81$ (fig. 3b), which is mixing on much faster time scale.

The decay times for the calculation of the correlation function can be approximated by $\tau_c=\Delta t^0/\alpha$, where $\Delta t^0=l/v$ and the coefficient α varies slightly depending on the mode. Due to the exponential decay of a correlation function in time one can calculate the integral in eq. (4) and obtain a time independent diffusion coefficient. As one might expect the modification of the trajectories will also change the characteristic frequency of $C(t)$,

$$C(t, v) \approx e^{i\Omega t - t/\tau_c}, \tag{10}$$

where for the free orbit approximation one has $\Omega=kvt-\omega_p$. We propose that the nth mode in the wave packet (2), which gives a regular island in the region of mixing contributes two characteristic frequencies to the correlation function. These frequencies are given by $\Omega_n=k_nvt-\omega_p\pm\delta\omega_n$ where $\delta\omega_n$ is of the order of the half width of the undisturbed resonance i.e. $\delta\omega_n\approx 2\sqrt{k_n(e/m_e)E_n}$. This approximation leads to a diffusion coefficient of the following form

$$D(v)=\tfrac{1}{4}\sum_n\left(\frac{e}{m_e}\right)^2|E_{k_n}|^2 \times\left(\frac{v\alpha_n/l}{(v\alpha_n/l)^2+(\omega_p+\delta\omega_n-k_nv)^2} + \frac{v\alpha_n/l}{(v\alpha_n/l)^2+(\omega_p-\delta\omega_n-k_nv)^2}\right), \tag{11}$$

where we have used the explicit form of the correlation decay time, $\tau_c^n=l/v\alpha_n$. The resonant structure in eq. (11) indicates an enhancement of the diffusion coefficient for particles in the regions where the separatrices of adjacent modes are closest.

The numerical results for the diffusion coefficient calculated for one pass of electrons through the localized field were obtained from

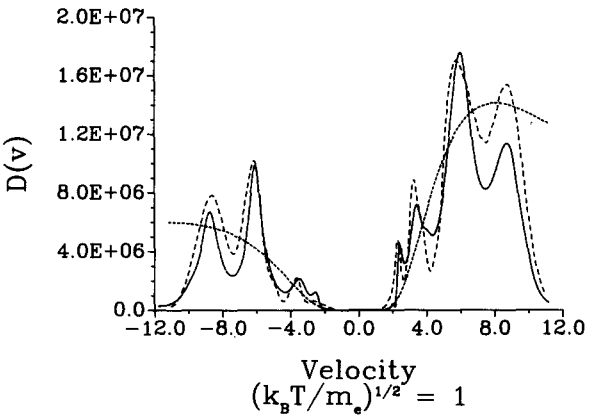

Fig. 4. Diffusion coefficients. Solid curve – theoretical predictions based on eq. (11), long dash – numerical results (12), short dash – quasi-linear theory (5).

$$D(v^0)=\frac{1}{2\Delta t^0}\langle[v(\Delta t^0)-v^0]^2\rangle, \tag{12}$$

where $\Delta t^0=l/v^0$ is the time of one pass. The ensemble of particles for a given velocity v^0 is constructed by introducing particles at different times into the field. By this procedure and by appropriate choices of the initial phases we ensure that only stochastic trajectories contribute to the values of $D(v^0)$. The results are shown in fig. 4 together with quasi-linear results (5) and values given by our model (11). The resonant structure is clearly visible with minima at the phase velocities of the modes. The theoretical result (11) reproduces the results from the numerical solution of eq. (3) reasonably well with the coefficient $\alpha_n\approx 2$ estimated from numerical and analytical studies of the correlation function (8) for most of the modes except $n=1, -2$. The latter components of the spectrum correspond to the largest islands and have $\alpha_n=1.5$.

In fact one can use the numerical results of $D(v)$ (11) to estimate proper decay times τ_c as well as the frequency shifts $\delta\omega_n$, for which $\delta\omega_n=2\sqrt{k_n(e/m_e)E_{k_n}}$ is only a first order approximation. We know that the true extent of the islands is different due to nonlinear coupling between resonances. Our model of the diffusion coefficient gives a good approximation to the numerical results except perhaps near the regular orbits which bound the region of connected stochasticity. In these regions the

model (11) gives results which are too high. We have found that in order to use the diffusion model to accurately predict the temporal evolution of the distribution function, the diffusion coefficient must be truncated at these regular orbits.

The quasi-linear theory gives the right order of magnitude for the diffusion coefficient, but does not reproduce the resonant structures and also extends beyond the stochastic region. A modified quasi-linear theory with a cutoff of the diffusion coefficient or velocities corresponding to the first invariant tori seen in fig. 2, gives good agreement for the heating rates and an evolution of the distribution function. We will report on calculations of the time evolution of the distribution in a later paper. Finally we also observe (cf. ref. [6]) that when the particles fill the accessible velocity interval defined by the regular orbits, the diffusion coefficient should change and consequently a second diffusion coefficient will be required to adequately describe the long time evolution of the distribution function.

In conclusion we have found a general form of the diffusion coefficient for electrons repeatedly interacting with coherent localized Langmuir waves. This model gives satisfactory results even when the wave packet has only a few modes and the region of connected stochasticity, defined by the wave overlap, has large size regular islands which are inaccessible to the stochastic trajectories. The presence of these islands leads to an enhancement of the diffusion coefficient at velocities $(\omega_p \pm \delta\omega_n)/k_n$, where $\delta\omega_n$ is defined by the width of a resonance. In order to construct an accurate model the decay times of the correlation functions were estimated using approximate mapping and numerical solutions to the equations of motion.

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