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PRIMER VECTOR THEORY APPLIED TO THE LINEAR RELATIVE-MOTION EQUATIONS

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SUMMARY

Primer vector theory is used in analysing a set of linear relative-motion equations—the Clohessy–Wiltshire (C/W) equations—to determine the criteria and necessary conditions for an optimal N-impulse trajectory. Since the state vector for these equations is defined in terms of a linear system of ordinary differential equations, all fundamental relations defining the solution of the state and costate equations and the necessary conditions for optimality can be expressed in terms of elementary functions. The analysis develops the analytical criteria for improving a solution by (1) moving any dependent or independent variable in the initial and/or final orbit, and (2) adding intermediate impulses. If these criteria are violated, the theory establishes a sufficient number of analytical equations. The subsequent satisfaction of these equations will result in the optimal position vectors and times of an N-impulse trajectory.

The solution is examined for the specific boundary conditions of (1) fixed-end conditions, two-impulse, and time-open transfer; (2) an orbit-to-orbit transfer; and (3) a generalized rendezvous problem. A sequence of rendezvous problems is solved to illustrate the analysis and the computational procedure.

KEY WORDS Optimal N-impulse space trajectories Primer vector theory Clohessy–Wiltshire equations Optimal rendezvous trajectories

1. INTRODUCTION

In the early 1960s, the solution of an optimal trajectory, based on the assumption that the thrusting acceleration is replaced by an impulse, received considerable attention as a method of performing mission analysis studies. Lawden[1] developed the necessary conditions for an optimal impulsive trajectory. He examined the limiting conditions on an optimal finite-thrust solution wherein the thrust magnitude was unconstrained but bounded between maximum and minimum values. These results are known as Lawden's necessary conditions for an optimal impulsive trajectory. They specify the conditions that must be satisfied by the primer vector and its derivative on a candidate impulsive trajectory.

Lion and Handelsman[2] later established the criteria and necessary optimal conditions for a fixed-time impulsive trajectory whereby a reference trajectory could be improved (i.e. the cost function or the sum of the magnitudes of the applied impulses could be decreased). Jezewski and Rozendaal[3] combined the Lion and Handelsman results with a conjugated gradient iterator[4] to produce an algorithm that generates optimal impulsive trajectories in an efficient and rapid manner.

In the middle 1970s, Jezewski[5] extended this analysis to generate a general differential cost function for the two-body problem. This function defines the gradient structure and cost function for (1) any set of boundary conditions when the applicable constraints are specified, and (2) equality and inequality constraints on both the state and control variables. By this means, completely general, two-body, N-impulse, optimal trajectories can be generated for any set of constraints that can be expressed mathematically.

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This study applies primer vector theory to a set of relative-motion equations—the C/W equations—to determine the necessary conditions for an optimal N-impulse trajectory. In previous studies, Prussing[6] applied primer vector theory to the problem of obtaining multiple-impulse, fixed-time, rendezvous solutions between coplanar circular orbits. Edelbaum[7] and Jones[8] used element formulations to determine a general optimal set of impulses for every solution as long as the linear assumptions were not violated. These and other similar studies construct primer vector rendezvous solutions that satisfy the necessary conditions by choosing and iterating directly on the constants of the solution.

In this study, completely general, optimal, N-impulse transfers between given boundary conditions are obtained for the linear system of equations. The general rendezvous problem is just one subset of these solutions. There are two general restrictions on this solution: (1) the reference solution and all intermediate trajectories must be within the linear neighbourhood of the reference circular orbit, and (2) the maximum number of impulses cannot exceed the number of state variables in the solution (see Neustadt[9]). The technique for achieving a solution differs from those previously mentioned and is similar to the method outlined in Reference 5. A general differential cost function is developed in terms of a set of all possible independent parameters. The specific problem to be solved is selected from this set; the coefficients of the specific set of parameters are required to be zero to achieve an optimal solution. This general method has proved to be highly successful in the non-linear problem.

In the primer vector theory, the solution of the state and costate equations is used along with a scalar function known as the adjoint equation to develop the criteria and necessary conditions for a candidate trajectory to be optimal. Since the C/W differential equations are a linear system of ordinary differential equations, they can be analytically integrated in terms of elementary functions. Also, the costate differential equations (the costate vector is canonically conjugate to the state vector) are also linear and can be integrated in terms of elementary functions. Hence, all fundamental relations defining the state, costate, and necessary conditions for optimality are known analytically in terms of elementary functions.

Section 2 will proceed to define the differential equations for the state and costate vectors in a convenient form and to integrate these equations in terms of elementary functions. A scalar function known as the adjoint equation will be developed from these differential equations and will be used in the subsequent development of the necessary conditions for optimality.

In Section 3, a cost function is defined and, after obtaining its derivative and using the adjoint equation, a general differential cost function is developed for the C/W equations. This function defines the necessary conditions to be satisfied by the C/W equations for an N-impulse trajectory for any changes in (1) the initial and final orbits, (2) the initial and final times, and (3) the times and position vectors of any intermediate impulses. The criterion for adding intermediate impulses is developed in Section 3.1, and subsequent analysis is performed to determine the position vectors for these impulses.

Finally, the general differential cost function is examined for the specific boundary conditions of (1) fixed-end conditions, two-impulse, time-open transfer; (2) orbit-to-orbit transfer; and (3) a generalized rendezvous problem. A number of example rendezvous problems are solved to illustrate the analysis.

2. APPLICABLE EQUATIONS

2.1. *State vector equations*

The development of the linear relative-motion differential equations can be obtained from Clohessy and Wiltshire.[10] A particular form of these quations, known as Hill's equations[11] or the

C/W equations, are

$$
\begin{aligned}
&\ddot{x} - 2\omega\dot{y} = 0 \\
&\ddot{y} + 2\omega\dot{x} - 3\omega^2 y = 0 \\
&\ddot{z} + \omega^2 z = 0
\end{aligned}
\tag{1}
$$

where ω is the constant angular velocity of a coordinate system which is assumed to be located in a circular orbit of a given radius magnitude. The coordinates y and z are along the radius vector and the normal to the orbital plane, respectively, and the coordinate x completes the triad of the right-handed system. These equations, which describe the motion of a particle with respect to this coordinate system, can be analytically integrated such that the solution is described in terms of the boundary conditions on the trajectory and the solution time.

Now defining the state vector

$$\mathbf{S}^{\mathrm{T}} = (\mathbf{R}^{\mathrm{T}}, \mathbf{V}^{\mathrm{T}}) \tag{2}$$

where

$$\mathbf{R}^{\mathrm{T}} = (x, y, z), \quad \mathbf{V} = \dot{\mathbf{R}}$$

and the superscript T refers to the transpose. The solution to equation (1) can be expressed as

$$\mathbf{S}(\tau) = \mathbf{A}(\tau, 0)\,\mathbf{S}(0) \tag{3}$$

where the relative time τ is the difference in the absolute times t (for example, $\tau_1 = t_2 - t_1$), and the matrix $\mathbf{A}$ (a function only of ω and τ) maps the state at one time into another time. If a_{ij} $(i, j = 1, 2, \ldots, 6)$ are the elements of this matrix, then the only non-zero terms are

$$
\begin{array}{lll}
a_{11} = 1, & a_{22} = 4 - 3c, & a_{33} = c \\
a_{12} = 6(\omega\tau - s), & a_{24} = 2(c-1)/\omega, & a_{36} = s/\omega \\
a_{14} = (4s - 3\omega\tau)/\omega, & a_{25} = s/\omega, & \\
a_{15} = 2(1-c)/\omega, & & a_{63} = -\omega s \\
 & a_{52} = 3\omega s, & a_{66} = c \\
a_{42} = 6\omega(1-c), & a_{54} = -2s, & \\
a_{44} = 4c - 3, & a_{55} = c, & \\
a_{45} = 2s, & &
\end{array}
$$

where s and c represent $\sin(\omega\tau)$ and $\cos(\omega\tau)$, respectively. If we write equation (1) in first-order form using equation (2), the C/W differential equations can be expressed as

$$\dot{\mathbf{S}} = \mathbf{F}(\mathbf{R}, \mathbf{V}) \tag{4}$$

where the vector $\mathbf{F}$ is defined as

$$\mathbf{F}^{\mathrm{T}} = (v_x, v_y, v_z, 2\omega v_y, 3\omega^2 y - 2\omega v_x, -\omega^2 z) \tag{5}$$

Equation (4) will be used in Sections 2.2 and 2.3 to develop the costate equations and the adjoint equations, all of which will be used to subsequently develop the necessary conditions for an optimal N-impulse C/W trajectory.

2.2. *Costate vector equations*

The costate equations, or the Lagrange multiplier equations, are a system of equations adjoint to the state equations. In differential form, they are defined as

$$\dot{\boldsymbol{\lambda}} = -\left(\frac{\partial \mathbf{F}}{\partial \mathbf{S}}\right)^{\mathrm{T}} \boldsymbol{\lambda} \tag{6}$$

where the vector $\mathbf{F}$ is given in equation (5).

Developing these differential equations, we obtain

$$\begin{aligned} \dot{\lambda}_1 &= 0, & \dot{\lambda}_4 &= -\lambda_1 + 2\omega\lambda_5 \\ \dot{\lambda} &= -3\omega^2 \lambda_5, & \dot{\lambda}_5 &= -\lambda_2 - 2\omega\lambda_4 \\ \dot{\lambda}_3 &= \omega^2 \lambda_6, & \dot{\lambda}_6 &= -\lambda_3 \end{aligned}$$

Since these equations are linear, they can be immediately integrated to obtain

$$\mathbf{Q} = \begin{bmatrix} -\alpha_1/3\omega \\ \alpha_2 c + \alpha_3 s + 2(\alpha_4 + \alpha_1 \tau) \\ \omega(\alpha_5 s - \alpha_6 c) \end{bmatrix} \tag{7a}$$

$$\mathbf{P} = \begin{bmatrix} \frac{-2}{3\omega}(\alpha_2 c + \alpha_3 s) - (\alpha_4 + \alpha_1 \tau)/\omega \\ (\alpha_2 s - \alpha_3 c - 2\alpha_1/\omega)/3\omega \\ \alpha_5 c + \alpha_6 s \end{bmatrix} \tag{7b}$$

where the vectors $\mathbf{Q}$ and $\mathbf{P}$ (primer vector) are defined by the relation

$$\boldsymbol{\lambda}^{\mathrm{T}} = (\mathbf{Q}^{\mathrm{T}}, \mathbf{P}^{\mathrm{T}}) \tag{8}$$

and the constant coefficients $\alpha_i (i = 1, 2, ..., 6)$ are boundary conditions computed as follows. From Reference 5, the primer vector at the time of an impulse is defined as a unit vector in the direction of the impulse, or

$$\mathbf{P} = \frac{\Delta \mathbf{V}}{|\Delta \mathbf{V}|} \tag{9}$$

If we designate $\mathbf{P}_1^{\mathrm{T}} = \mathbf{P}^{\mathrm{T}}(t_1) = (l_1, l_2, l_3)$ and similarly, $\mathbf{P}_2^{\mathrm{T}} = \mathbf{P}^{\mathrm{T}}(t_2) = (m_1, m_2, m_3)$, then the in-plane elements of the coefficient vector $\boldsymbol{\alpha}$ in equation (7) are determined as

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \mathbf{B}^{-1} \begin{bmatrix} l_1 \\ l_2 \\ m_1 \\ m_2 \end{bmatrix} \tag{10a}$$

where the only non-zero elements b_{ij} $(i,j = 1, 2, ..., 4)$ of the matrix $\mathbf{B}$ [determined from equation

(7b)] are

$$
\begin{aligned}
b_{12} &= -2/3\omega, & b_{21} &= b_{12}/\omega \\
b_{14} &= -1/\omega, & b_{23} &= -1/3\omega \\
b_{31} &= -\tau/\omega, & b_{41} &= b_{21} \\
b_{32} &= -2c/3\omega, & b_{42} &= -b_{23}\, s \\
b_{33} &= -2s/3\omega, & b_{43} &= b_{23}\, c \\
b_{34} &= b_{14}
\end{aligned}
$$

Since the differential equations for the state vector are uncoupled in the z-direction, the out-of-plane elements of the coefficient vector α are determined as

$$
\begin{aligned}
\alpha_5 &= l_3 \\
\alpha_6 &= (m_3 - l_3\, c)/s
\end{aligned}
\tag{10b}
$$

Note that the out-of-plane solution is not valid for $\omega\tau = n\pi$, $n = 0, 1, \ldots$. Knowing the value of the vector $\boldsymbol{\alpha}$, one can determine the primer vector $\mathbf{P}$ and its complementary vector $\mathbf{Q}$ analytically from equation (7) for any time t.

The costate differential equations could also have been obtained from a Hamiltonian approach. If we define a Hamiltonian H as

$$
H = \mathbf{P}^{\mathrm{T}} \dot{\mathbf{V}} + \mathbf{Q}^{\mathrm{T}} \dot{\mathbf{R}} \tag{11}
$$

then the state and costate C/W differential equations can be obtained from the canonical form

$$
\dot{\mathbf{R}}^{\mathrm{T}} = \frac{\partial H}{\partial \mathbf{Q}}, \quad \dot{\mathbf{V}}^{\mathrm{T}} = \frac{\partial H}{\partial \mathbf{P}}
$$

and

$$
\dot{\mathbf{Q}}^{\mathrm{T}} = -\frac{\partial H}{\partial \mathbf{R}}, \quad \dot{\mathbf{P}}^{\mathrm{T}} = -\frac{\partial H}{\partial \mathbf{V}}
$$

In later sections, we will have occasions to use not only the concept of a Hamiltonian, but also its mathematical definition [equation (11)].

2.3. *Adjoint equation*

We need one further mathematical relationship before we can proceed with the development of the criteria for improving a C/W trajectory and the necessary conditions for an optimal N-impulse solution. A functional relationship is required between the costate vector and the perturbations in the state vector.

From equation (4), the differential equation for the variation in the state vector can be expressed as

$$
\delta \dot{\mathbf{S}} = \left(\frac{\partial \mathbf{F}}{\partial \mathbf{S}} \right) \delta \mathbf{S} \tag{12}
$$

where δ represents the contemporaneous variational operator. Note that because the vector $\mathbf{F}$ is linear in the state vector, equation (12) represents the exact variational differential equation and not a truncated expansion. Premultiplying equation (12) by $\boldsymbol{\lambda}^{\mathrm{T}}$ and equation (6) by $\delta \mathbf{S}^{\mathrm{T}}$ and adding, we obtain

$$
\boldsymbol{\lambda}^{\mathrm{T}} \delta \dot{\mathbf{S}} + \delta \mathbf{S}^{\mathrm{T}} \dot{\boldsymbol{\lambda}} = \boldsymbol{\lambda}^{\mathrm{T}} \left(\frac{\partial \mathbf{F}}{\partial \mathbf{S}} \right) \delta \mathbf{S} - \delta \mathbf{S}^{\mathrm{T}} \left(\frac{\partial \mathbf{F}}{\partial \mathbf{S}} \right)^{\mathrm{T}} \boldsymbol{\lambda}
$$

The right-hand side of this equation can be easily verified to be zero. The left-hand side of this equation can be expressed as the exact differential

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\lambda}^{\mathrm{T}}\,\delta\mathbf{S}) = 0$$

which implies that on any C/W arc the scalar product of the costate vector and the variational state vector is a constant, or

$$\boldsymbol{\lambda}^{\mathrm{T}}\,\delta\mathbf{S} = \text{constant} \tag{13}$$

Equation (13) will be referred to as the adjoint equation for the C/W equations.

3. COST FUNCTION AND THE NECESSARY CONDITIONS FOR OPTIMALITY

The cost function, J, is defined as the sum of the magnitudes of the applied impulse vectors. Let us assume that a reference solution consists of three impulses with the following notation: at the time t_1, the impulse vector is $\Delta\mathbf{V}_1$; at the time t_2, the impulse vector is $\Delta\mathbf{V}_2$; and at an intermediate time t_m, $(t_1 < t_m < t_2)$, the impulse vector is $\Delta\mathbf{V}_m$. A three-impulse solution has been assumed for the analysis because the results can be (1) easily reduced to a two-impulse solution by the removal of the intermediate impulse, and (2) readily extended to more than three impulses by additional intermediate impulses. This reference three-impulse cost function is

$$J = |\Delta\mathbf{V}_1| + |\Delta\mathbf{V}_m| + |\Delta\mathbf{V}_2| \tag{14}$$

where $\Delta\mathbf{V} = \mathbf{V}^+ - \mathbf{V}^-$ and the superscripts minus and plus refer to the evaluation immediately before and after an event, respectively. Consider a perturbed three-impulse trajectory. The cost function can be defined as

$$J' = |\Delta\mathbf{V}_1 + (\delta\mathbf{V}_1^+ - \delta\mathbf{V}_1^-)| + |\Delta\mathbf{V}_m + (\delta\mathbf{V}_m^+ - \delta\mathbf{V}_m^-)| + |\Delta\mathbf{V}_2 + (\delta\mathbf{V}_2^+ - \delta\mathbf{V}_2^-)|$$

where the velocities on the perturbed solution differ from those on the reference solution by the following amounts.

$$\begin{array}{ll} \delta\mathbf{V}_1^- \text{ at the time } t_1^-, & \delta\mathbf{V}_m^+ \text{ at the time } t_m^+ \\ \delta\mathbf{V}_1^+ \text{ at the time } t_1^+, & \delta\mathbf{V}_2^- \text{ at the time } t_2^- \\ \delta\mathbf{V}_m^- \text{ at the time } t_m^-, & \delta\mathbf{V}_2^+ \text{ at the time } t_2^+ \end{array}$$

Defining a differential cost function as

$$\mathrm{d}J = J' - J$$

we obtain to first order

$$\mathrm{d}J = \frac{\Delta\mathbf{V}_1^{\mathrm{T}}}{|\Delta\mathbf{V}_1|}(\delta\mathbf{V}_1^+ - \delta\mathbf{V}_1^-) + \frac{\Delta\mathbf{V}_m^{\mathrm{T}}}{|\Delta\mathbf{V}_m|}(\delta\mathbf{V}_m^+ - \delta\mathbf{V}_m^-) + \frac{\Delta\mathbf{V}_2^{\mathrm{T}}}{|\Delta\mathbf{V}_2|}(\delta\mathbf{V}_2^+ - \delta\mathbf{V}_2^-)$$

Using equation (9), evaluated at the times t_1, t_m, and t_2 in this equation, we have

$$\mathrm{d}J = \mathbf{P}_1^{\mathrm{T}}(\delta\mathbf{V}_1^+ - \delta\mathbf{V}_1^-) + \mathbf{P}_m^{\mathrm{T}}(\delta\mathbf{V}_m^+ - \delta\mathbf{V}_m^-) + \mathbf{P}_2^{\mathrm{T}}(\delta\mathbf{V}_2^+ - \delta\mathbf{V}_2^-) \tag{15}$$

The terms $\mathbf{P}_1^{\mathrm{T}}\,\delta\mathbf{V}_1^+$ and $\mathbf{P}_2^{\mathrm{T}}\,\delta\mathbf{V}_2^-$ can be eliminated from this equation by evaluating the adjoint equation [equation (13)] on the two arcs separating the impulse

$$(\mathbf{P}_1, \delta\mathbf{V}_1^+) = (\mathbf{Q}_m^-, \delta\mathbf{R}_m^-) + (\mathbf{P}_m^-, \delta\mathbf{V}_m^-) - (\mathbf{Q}_1, \delta\mathbf{R}_1^+)$$

$$(\mathbf{P}_2, \delta\mathbf{V}_2^-) = (\mathbf{Q}_m^+, \delta\mathbf{R}_m^+) + (\mathbf{P}_m^+, \delta\mathbf{V}_m^+) - (\mathbf{Q}_2, \delta\mathbf{R}_2^-)$$

For the convenience of notation, we shall use interchangeably the notation

$$(\mathbf{X}, \mathbf{Y}) \equiv \mathbf{X}^{\mathrm{T}} \mathbf{Y}$$

Using these relations in equation (15), we obtain

$$\mathrm{d}J = -(\mathbf{Q}_1, \delta\mathbf{R}_1^+) - (\mathbf{P}_1, \delta\mathbf{V}_1^-) + (\mathbf{Q}_2, \delta\mathbf{R}_2^-) + (\mathbf{P}_2, \delta\mathbf{V}_2^+) + (\mathbf{Q}_m^-, \delta\mathbf{R}_m^-) - (\mathbf{Q}_m^+, \delta\mathbf{R}_m^+) \tag{16}$$

The terms involving the vector $\mathbf{P}_m$ in equation (15) have been cancelled since the primer vector is continuous at the intermediate impulse, i.e. $P_m^- = P_m^+ = P_m$. To the first order, the variations in the position and velocity vectors are

$$\delta\mathbf{R} = \mathrm{d}\mathbf{R} - \mathbf{V}\,\mathrm{d}t$$

$$\delta\mathbf{V} = \mathrm{d}\mathbf{V} - \dot{\mathbf{V}}\,\mathrm{d}t$$

Evaluating these equations at the times t_1^+, t_m^-, t_m^+ and t_2^- and using the results in equation (16), we obtain

$$\begin{aligned} \mathrm{d}J = & -(\mathbf{Q}_1^{\mathrm{T}} \mathrm{d}\mathbf{R}_1 + \mathbf{P}_1^{\mathrm{T}} \mathrm{d}\mathbf{V}_1^-) + (\mathbf{Q}_2^{\mathrm{T}} \mathrm{d}\mathbf{R}_2 + \mathbf{P}_2^{\mathrm{T}} \mathrm{d}\mathbf{V}_2^+) \\ & + H_1^+\, dt_1 - H_2^-\, \mathrm{d}t_2 + (H_m^+ - H_m^-)\,\mathrm{d}t_m - (\mathbf{Q}_m^+ - \mathbf{Q}_m^-)^{\mathrm{T}} \mathrm{d}\mathbf{R}_m \end{aligned} \tag{17}$$

where H is the Hamiltonian function defined by equation (11). In the derivation of this equation, we have used the identity

$$\mathbf{P}^{\mathrm{T}}(\dot{\mathbf{V}}^+ - \dot{\mathbf{V}}^-) \equiv 0$$

which may be established by using the definitions of the vectors $\dot{\mathbf{V}}$ and $\mathbf{P}$ from equations (1) and (9), respectively. Equation (17) will be known as the general differential cost function for the C/W equations. The first two terms on the right of this equation represent the variations in the cost function due to changes in the initial and final state, respectively. We shall have more to say about these two terms when we deal with specific boundary conditions. The third and fourth terms in equation (17) are the variations in the cost function because of changes in the initial and final times, respectively. If we are departing the initial orbit at the optimal time and arriving on the final orbit at the optimal time, and there are no other constraints, the coefficients of $\mathrm{d}t_1$ and $\mathrm{d}t_2$, respectively, will be zero. The last two terms in equation (17) are the variations in the cost function because of changes in the intermediate impulse times and position vectors, respectively. For this linear problem, the total number of impulses cannot exceed the number of state variables in the solution (see Neustadt[9]), and hence, there can be no more than four intermediate impulses. Note that on an optimal trajectory, the Hamiltonian and the vector $\mathbf{Q}$ must be continuous across the intermediate impulses. The vector $\mathbf{P}$ has already been required to be continuous by virtue of its definition [equation (9)].

As previously stated, equation (17) defines the general differential cost function for the C/W equations. From this expression, it can be determined how any two-impulse trajectory or any two-impulse segment of an N-impulse trajectory ($N \leqslant 6$) can be improved: by changes in (1) the initial state vector or time, (2) the final state vector or time, or (3) the intermediate impulse times and position vectors.

3.1. *Criterion for an intermediate impulse*

Consider a reference trajectory J, consisting of two impulses between fixed-boundary conditions. Consider a perturbed trajectory J' between the same boundary conditions but consisting of three impulses. The intermediate impulse occurs at a time t_m and at a position vector $\mathbf{R}_m + \delta\mathbf{R}_m$. The vector $\delta\mathbf{R}_m$ is the perturbation from the reference trajectory at the time t_m. We shall attempt to use a comparison between these two trajectories to determine under what conditions the perturbed trajectory (the one with three impulses) will have a lower cost than the reference trajectory. From equation (14), the difference in cost between the two solutions can be expressed as

$$\mathrm{d}J = J' - J = \mathbf{P}_1^{\mathrm{T}}\,\delta\mathbf{V}_1^{+} + |\,\delta\mathbf{V}_m^{+} - \delta\mathbf{V}_m^{-}\,| - \mathbf{P}_2^{\mathrm{T}}\,\delta\mathbf{V}_2^{-} \tag{18}$$

since $\delta\mathbf{V}_1^{-}$ and $\delta\mathbf{V}_2^{+}$ are zero by definition of the boundary conditions. Evaluating the adjoint equation [equation (13)] on the two segments of the perturbed trajectory, we have

$$(\mathbf{P}_1, \delta\mathbf{V}_1^{+}) = (\mathbf{Q}_m^{-}, \delta\mathbf{R}_m^{-}) + (\mathbf{P}_m^{-} \cdot \delta\mathbf{V}_m^{-})$$

$$(\mathbf{P}_2, \delta\mathbf{V}_2^{-}) = (\mathbf{Q}_m^{+}, \delta\mathbf{R}_m^{+}) + (\mathbf{P}_m^{+} \cdot \delta\mathbf{V}_m^{+})$$

since $\delta\mathbf{R}_1 = \delta\mathbf{R}_2 = \mathbf{0}$ by virtue of the boundary conditions.

Using these relationships in equation (18), we have

$$\mathrm{d}J = (\mathbf{Q}_m^{-}, \delta\mathbf{R}_m^{-}) - (\mathbf{Q}_m^{+}, \delta\mathbf{R}_m^{+}) + |\,\delta\mathbf{V}_m^{+} - \delta\mathbf{V}_m^{-}\,| + (\mathbf{P}_m^{-}, \delta\mathbf{V}_m^{-}) - (\mathbf{P}_m^{+}, \delta\mathbf{V}_m^{+}) \tag{19}$$

But the costate vectors **P** and **Q** are evaluated on the reference trajectory and hence are continuous, or

$$\mathbf{P}_m^{-} = \mathbf{P}_m^{+} = \mathbf{P}_m$$

$$\mathbf{Q}_m^{-} = \mathbf{Q}_m^{+} = \mathbf{Q}_m$$

Also, the variations in the position vector at the intermediate times are

$$\delta\mathbf{R}_m^{-} = \mathrm{d}\mathbf{R}_m - \mathbf{V}_m^{-}\,\mathrm{d}t_m$$

$$\delta\mathbf{R}_m^{+} = \mathrm{d}\mathbf{R}_m - \mathbf{V}_m^{+}\,\mathrm{d}t_m$$

Using these results in equation (19), we obtain

$$\mathrm{d}J = |\,\delta\mathbf{V}_m^{+} - \delta\mathbf{V}_m^{-}\,| - \mathbf{P}_m^{\mathrm{T}}(\delta\mathbf{V}_m^{+} - \delta\mathbf{V}_m^{-}) \tag{20}$$

since $\mathrm{d}t_m$ is zero by definition of the perturbed trajectory.

Equation (20) is homogeneous in the intermediate impulse vector $\delta\mathbf{V}_m^{+} - \delta\mathbf{V}_m^{-}$ for if we define a scalar v and a unit vector **L** as

$$v = |\,\delta\mathbf{V}_m^{+} - \delta\mathbf{V}_m^{-}\,|$$

$$\mathbf{L} = \frac{\delta\mathbf{V}_m^{+} - \delta\mathbf{V}_m^{-}}{v}$$

then equation (20) may be expressed as

$$\mathrm{d}J = v(1 - \mathbf{P}_m^{\mathrm{T}}\,\mathbf{L}) \tag{21}$$

The criterion for an intermediate impulse can now be established: if $|\mathbf{P}_m| > 1$, then $\mathrm{d}J < 0$, and $J > J'$; or, the reference trajectory cost is greater than the perturbed trajectory cost. The reference trajectory cost can be improved by applying an intermediate impulse at the time t_m in the direction of $\mathbf{P}_m$. The greatest decrease in cost will occur if the intermediate impulse is applied at the time when $|\mathbf{P}_m|$ is a maximum.

Conversely, since the primer vector is defined as a unit vector in the direction of an impulse, its direction will become poorly defined whenever the magnitude of the impulse tends to zero. Therefore, an initial, intermediate, or final impulse should be removed from the solution whenever its magnitude is less than a specified tolerance.

Finally, if a reference trajectory exists for which $|\mathbf{P}_m| > 1$ at some time t_m, how much should this trajectory be perturbed, and in which direction, such that the perturbed trajectory has a lower cost?

From equation (3), evaluated on the two segments of the perturbed trajectory, we have

$$\mathbf{S}^+(t_1) = \mathbf{A}(t_1, t_m)\mathbf{S}^-(t_m)$$
$$\mathbf{S}^-(t_2) = \mathbf{A}(t_2, t_m)\mathbf{S}^+(t_m)$$

Partitioning the matrix $\mathbf{A}$ as

$$\mathbf{A} = \begin{bmatrix} \boldsymbol{\phi}_{11} & \boldsymbol{\phi}_{12} \\ \boldsymbol{\phi}_{21} & \boldsymbol{\phi}_{22} \end{bmatrix}$$

The position vectors on the two segments can be expressed as

$$\mathbf{R}_1 = \boldsymbol{\phi}_{11}(t_1, t_m)\mathbf{R}_m + \boldsymbol{\phi}_{12}(t_1, t_m)\mathbf{V}_m^-$$
$$\mathbf{R}_2 = \boldsymbol{\phi}_{11}(t_2, t_m)\mathbf{R}_m + \boldsymbol{\phi}_{12}(t_2, t_m)\mathbf{V}_m^+$$

Solving for the vector $\mathbf{V}_m^+$ and $\mathbf{V}_m^-$, we have

$$\mathbf{V}_m^+ = \boldsymbol{\phi}_{12}^{-1}(t_2, t_m)(\mathbf{R}_2 - \boldsymbol{\phi}_{11}(t_2, t_m)\mathbf{R}_m)$$
$$\mathbf{V}_m^- = \boldsymbol{\phi}_{12}^{-1}(t_1, t_m)(\mathbf{R}_1 - \boldsymbol{\phi}_{11}(t_1, t_m)\mathbf{R}_m)$$

where it can be ascertained that $\boldsymbol{\phi}_{12}^{-1}$ exists if $\omega\tau \neq 2n\pi$, $n = 0, 1, \ldots$.

Subtracting the two velocity vectors at the time t_m, we obtain

$$\mathbf{V}_m^+ - \mathbf{V}_m^- = \Delta\mathbf{V}_m = \boldsymbol{\rho}_2 - \boldsymbol{\rho}_1 + \mathbf{M}\mathbf{R}_m = v\frac{\mathbf{P}_m}{|\mathbf{P}_m|} \tag{22}$$

where

$$\boldsymbol{\rho}_1 = \boldsymbol{\phi}_{12}^{-1}(t_1, t_m)\mathbf{R}_1$$
$$\boldsymbol{\rho}_2 = \boldsymbol{\phi}_{12}^{-1}(t_2, t_m)\mathbf{R}_2$$

and the matrix $\mathbf{M}$ is

$$\mathbf{M} = \boldsymbol{\phi}_{12}^{-1}(t_1, t_m)\boldsymbol{\phi}_{11}(t_1, t_m) - \boldsymbol{\phi}_{12}^{-1}(t_2, t_m)\boldsymbol{\phi}_{11}(t_2, t_m)$$

Solving for the intermediate position vector $\mathbf{R}_m$ from equation (22), we have

$$\mathbf{R}_m = \mathbf{M}^{-1}\left[v\frac{\mathbf{P}_m}{|\mathbf{P}_m|} + \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2\right] \tag{23}$$

The only unknown in this equation is the scalar v, the magnitude of the initial intermediate impulse. Since the solution will be determined by a numerical approach, the actual value is not significant to the final solution, as long as it remains small relative to the initial cost J.

3.2. *Boundary conditions*

3.2.1. *Fixed-end conditions, two-impulse transfer.* Consider the problem of finding the time-optimal, two-impulse, transfer trajectory between fixed initial and final state vectors. Because intermediate impulses are excluded, t_m and $\mathbf{R}_m$ are fixed and therefore,

$$\mathrm{d}t_m = 0 \quad \text{and} \quad \mathrm{d}\mathbf{R}_m = \mathbf{0}$$

Also, because the initial state vector $(\mathbf{R}_1, \mathbf{V}_1^-)$ and the final state vector $(\mathbf{R}_2, \mathbf{V}_2^+)$ are fixed, we have

$$\mathrm{d}\mathbf{R}_1 = \mathrm{d}\mathbf{V}_1^- = \mathrm{d}\mathbf{R}_2 = \mathrm{d}\mathbf{V}_2^+ = \mathbf{0}$$

When one uses these results in equation (17), the differential cost function reduces to

$$\mathrm{d}J = H_1^+\,\mathrm{d}t_1 - H_2^-\,\mathrm{d}t_2$$

But because we are dealing with a two-impulse trajectory, the Hamiltonian is continuous (see Lawden[1]). Thus,

$$H_1^+ = H_2^- = H$$

Also, if we define the transfer time as

$$\tau = t_2 - t_1$$

the differential cost function for a time-open two-impulse transfer between fixed initial and final states is

$$\mathrm{d}J = -H\,\mathrm{d}\tau \tag{24}$$

Equation (24) indicates that the cost function J will be an extremal when $H = 0$. Since H is defined in terms of elementary functions [equation (11)], the optimal time to transfer can be readily computed. The results of this solution will agree with those obtained in Reference 12.

3.2.2. *Time-open, orbit-to-orbit transfer.* Consider the problem of transferring between two orbits, both fixed in shape and orientation, in which the transfer time and the departure and arrival times from the initial to the final orbits (respectively) are free. What are the necessary conditions for optimality to be satisfied for this solution?

The differentials of the initial and final state vectors can be expressed as

$$\mathrm{d}\mathbf{R}_1 = \mathbf{V}_1^-\,\mathrm{d}\tau_0 \qquad \mathrm{d}\mathbf{R}_2 = \mathbf{V}_2^+\,\mathrm{d}\tau_F$$
$$\mathrm{d}\mathbf{V}_1^- = \dot{\mathbf{V}}_1^-\,\mathrm{d}\tau_0 \qquad \mathrm{d}\mathbf{V}_2^+ = \dot{\mathbf{V}}_2^+\,\mathrm{d}\tau_F$$

where τ_0 and τ_F are time measurements in the initial and final orbits, respectively. When one uses these results in equation (17), the differential cost function can be expressed as

$$\mathrm{d}J = -H_1^-\,\mathrm{d}\tau_0 + H_2^+\,\mathrm{d}\tau_F + H_1^+\,\mathrm{d}t_1 - H_2^-\,\mathrm{d}t_2 + (H_m^+ - H_m^-)\,\mathrm{d}t_m - (\mathbf{Q}_m^+ - \mathbf{Q}_m^-)^{\mathrm{T}}\,\mathrm{d}\mathbf{R}_m \tag{25}$$

Since all the differentials in this equation are independent and non-zero, the cost function J will be an extremal when all the coefficients of the differentials are zero. For an N-impulse problem, there are $4(N-1)$ equations in terms of $4(N-1)$ unknowns, τ_0, τ_F, t_1, t_2 and the $(N-2)$ intermediate times and position-vectors. The optimal solution can be readily computed since all the coefficients are expressible in terms of elementary functions.

3.2.3. *Rendezvous.* As a final example, let us determine the necessary conditions for an optimal, N-impulse, C/W rendezvous trajectory. This problem can be characterized as a time-open transfer. However, the problem is functionally dependent of the motion in the initial and final orbits. That is, the coasting times in the initial and final orbits, τ_0 and τ_F (respectively) are not independent of the departure and arrival times, t_1 and t_2 (respectively), but are constrained by the relationships

$$\mathrm{d}\tau_0 = \mathrm{d}t_1$$
$$\mathrm{d}\tau_F = \mathrm{d}t_2$$

Using these relationships in equation (25) and the definition of the Hamiltonian from equation (11), we obtain

$$\mathrm{d}J = \mathbf{Q}_1^{\mathrm{T}}(\mathbf{V}_1^+ - \mathbf{V}_1^-)\,\mathrm{d}t_1 + \mathbf{Q}_2^{\mathrm{T}}(\mathbf{V}_2^+ - \mathbf{V}_2^-)\,\mathrm{d}t_2 + (H_m^+ - H_m^-)\,\mathrm{d}t_m - (\mathbf{Q}_m^+ - \mathbf{Q}_m^-)^{\mathrm{T}}\,\mathrm{d}\mathbf{R}_m$$

But $\mathbf{V}_1^+ - \mathbf{V}_1^- = |\Delta\mathbf{V}_1|\mathbf{P}_1$, and $\mathbf{V}_2^+ - \mathbf{V}_2^- = |\Delta\mathbf{V}_2|\mathbf{P}_2$. Therefore, the generalized differential cost function for a C/W rendezvous is

$$dJ = |\Delta\mathbf{V}_1|(\mathbf{P}_1^T\mathbf{Q}_1)dt_1 + |\Delta\mathbf{V}_2|(\mathbf{P}_2^T\mathbf{Q}_2)dt_2 + (H_m^+ - H_m^-)dt_m - (\mathbf{Q}_m^+ - \mathbf{Q}_m^-)^T d\mathbf{R}_m \quad (26)$$

Note that for an optimal departure and arrival time from the initial and final orbit (respectively), the primer vector **P** and its complementary vector **Q** must be orthogonal. The vector **Q**, however, is not the negative derivative of the vector **P**, as can be verified in equation (7).

4. EXAMPLE PROBLEM

As an example to illustrate the analysis and computational procedure, a sequence of rendezvous trajectories will be computed. The targer T is assumed to be at rest and at the origin (see Figure 1) of

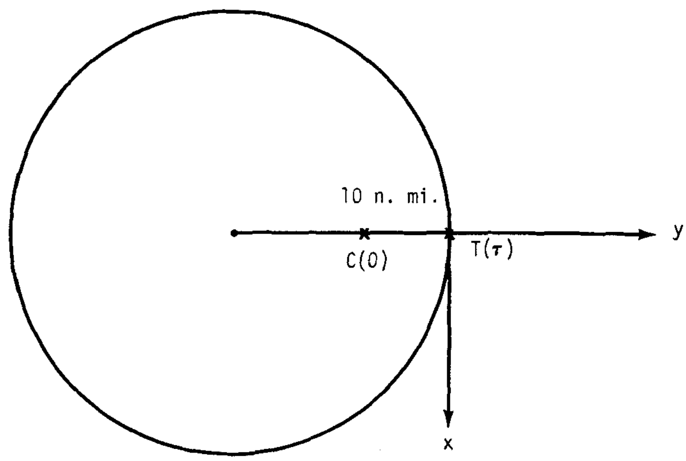

Figure 1. Positions of chase and target vehicles

a coordinate system located in a circular orbit with an altitude of 267 n. mi. at time τ. The chase vehicle C is at rest (with respect to the target) and is displaced 10 n. mi. along the negative y-axis at time 0. Thus, the boundary conditions for the problem are

$$\mathbf{R}^T(0) = (0, -10 \text{ n. mi.}, 0), \quad \mathbf{V}^T(0) = (0, 0, 0)$$

$$\mathbf{R}^T(\tau) = (0, 0, 0), \quad \mathbf{V}^T(\tau) = (0, 0, 0)$$

Let us now examine a sequence of rendezvous trajectories between these boundary conditions for various rendezvous times τ. In Figure 2, the cost of these solutions is indicated by the curve labelled J_2(NC) (the sum of the magnitudes of the two impulses with no initial cost). The costate variables **P** and **Q**, and specifically the primer magnitude $|\mathbf{P}|$, for any one of these solutions can be computed from equation (7) and may appear as illustrated in Figure 3. From equation (9), the magnitude is unity at each of the impulse times and in general will be less than unity on the interval $(0, \tau)$. However, if the coefficent of dt_1 [equation (26)] is computed, it will be determined to be non-zero, and for this problem it will also be positive. This implies that the cost of the solution can be reduced by a negative coast of duration τ_0 computed from the zero of a function f given as

$$f(\tau_0) = |\Delta\mathbf{V}_1|(\mathbf{P}_1^T\mathbf{Q}_1) = 0$$

In Figure 2, the two-impulse cost function with an optimal initial cost J_2(WC) is also plotted. Note

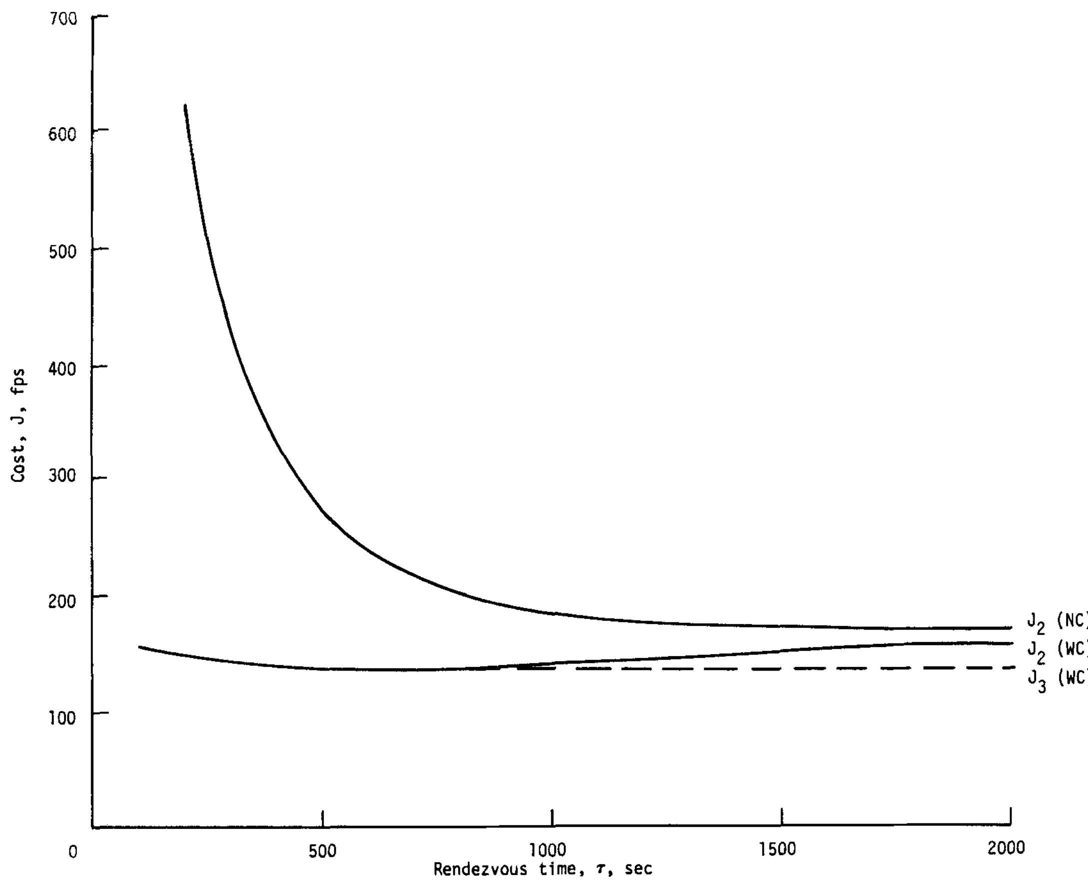

Figure 2. Clohessy–Wiltshire (C/W) rendezvous cost

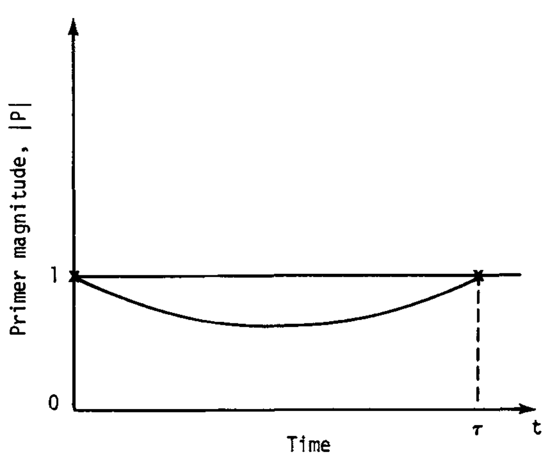

Figure 3. Two-impulse primer magnitude time history

that the cost has been significantly reduced by the introduction of an optimal initial coast. In Figure 4, the optimal initial coast time τ_0 is plotted against the rendezvous time τ. Note that the rendezvous time has been maintained, i.e. $\tau_0 + \tau_1 = \tau$ where τ_1 is the transfer time between the two impulses.

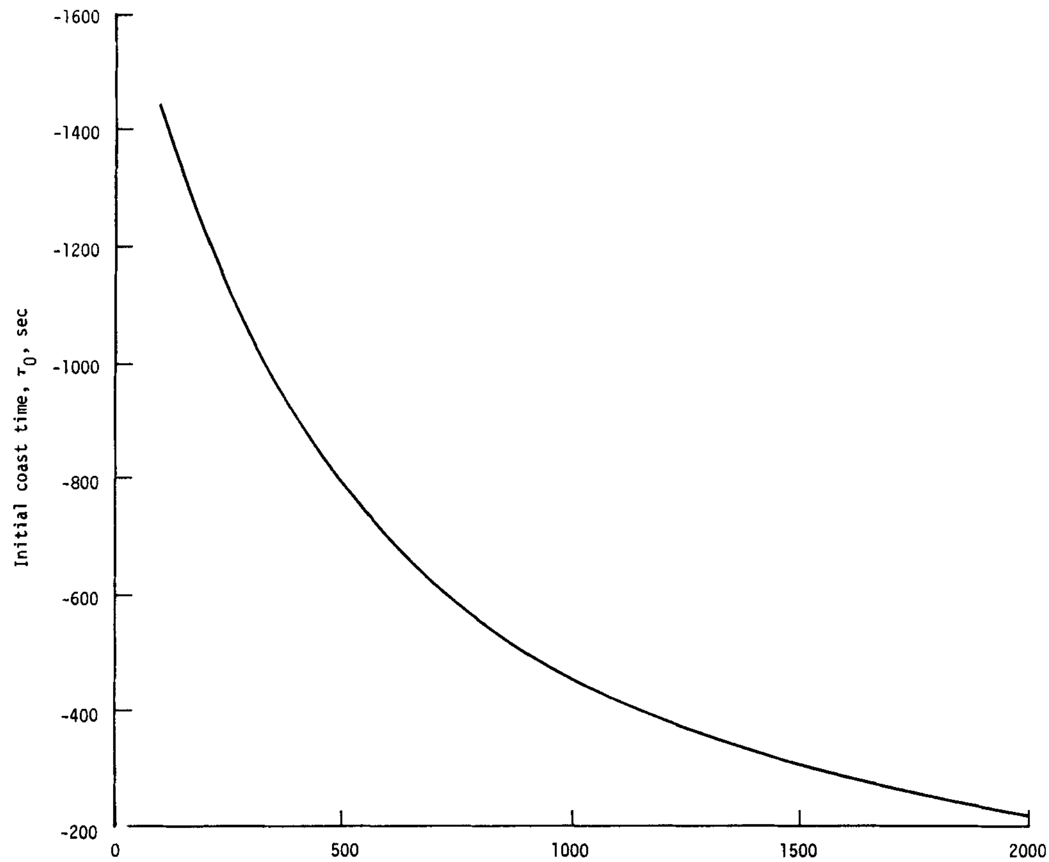

Figure 4. Optimal initial coast for a two-impulse C/W rendezvous

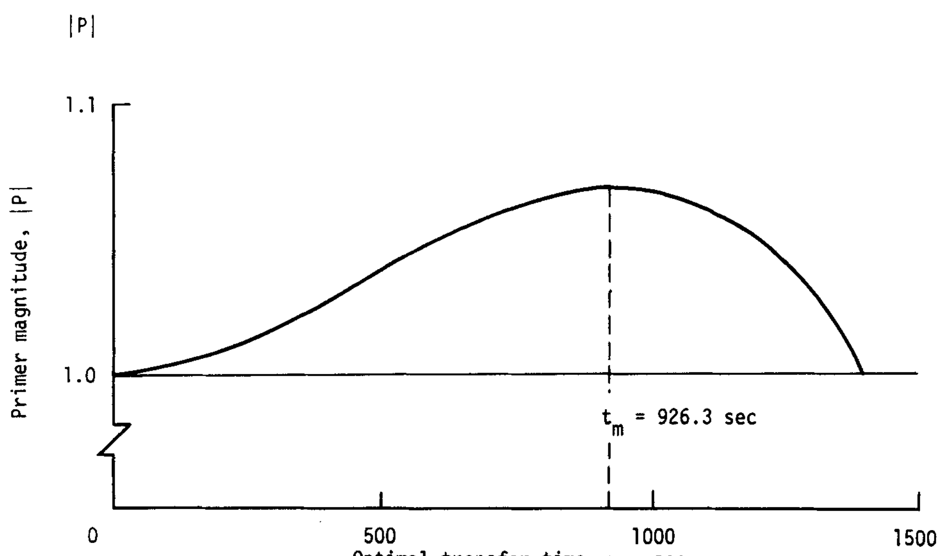

Figure 5. Primer magnitude time history for a C/W, optimal, two-impulse rendezvous with $\tau = 1{,}000$ s

The question now arises, is it possible to further improve the optimal two-impulse trajectory? If we examine the primer magnitude time history for one of these optimal two-impulse solutions, i.e. $\tau = 1{,}000$ s, the primer magnitude will appear as illustrated in Figure 5. The primer magnitude is unity at the two-impulse times $t_1 = 0$, $t_2 = \tau_1 = 1{,}450{\cdot}3$ s and is noted to be greater than unity on the interval. This indicates [see equation (21)] that this solution can be improved by adding an impulse at the time where the primer magnitude is a maximum ($t_m = 926{\cdot}3$ s). The starting position vector for this impulse is given in equation (23). Hence, a three-impulse solution can be computed, and the optimal values for the coast time, arc times, and intermediate position vector can be computed from the zeros of the coefficients of the respective variables in equation (26).

The optimal three-impulse trajectory cost with an optimal initial coast J_3(WC) is also plotted (dashed curve) in Figure 2. Note that the optimal three-impulse cost is always less than or equal to the optimal two-impulse cost, but that these solutions may not always exist. In Figure 2, note that for rendezvous times prior to 655 s, three-impulse solutions do not exist. For this problem, when three-impulse solutions did exist, the cost was a constant value of 134·7 fps. The arc times between the three impulses are illustrated in Figure 6. Note once again that the rendezvous time is

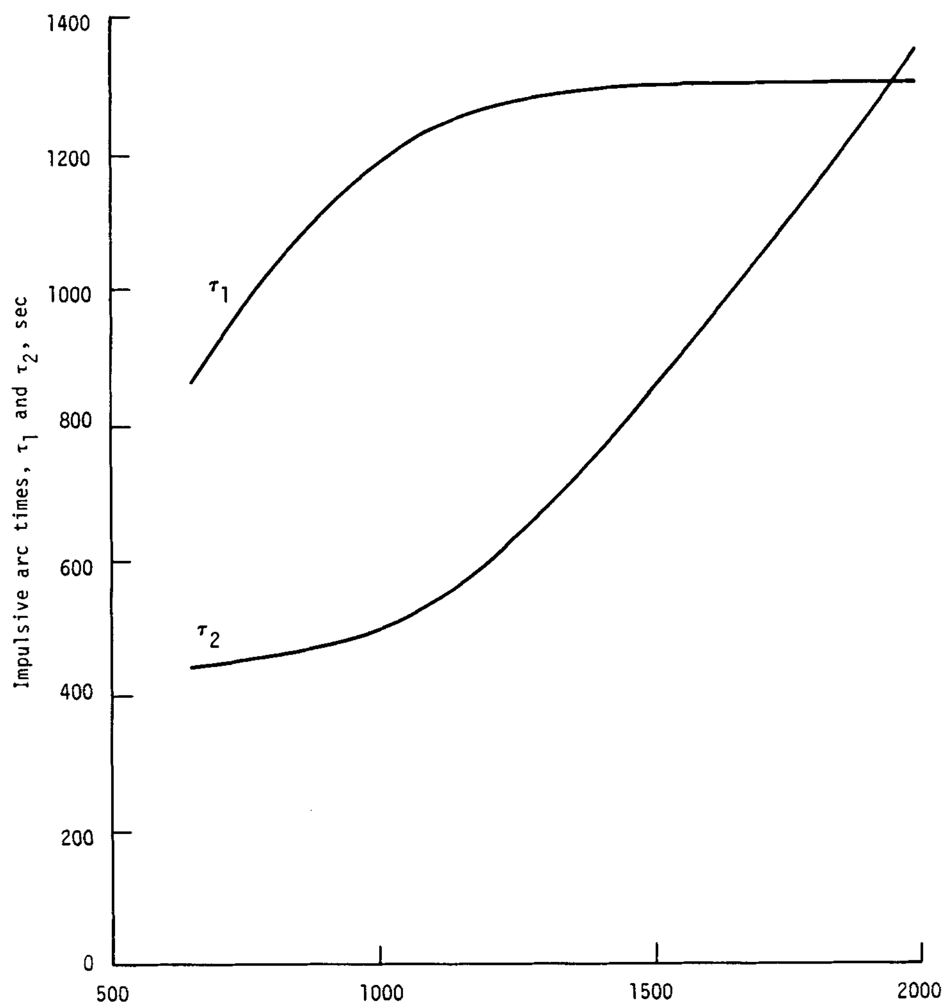

Figure 6. Optimal three-impulse arc time

maintained, i.e. $\tau_0 + \tau_1 + \tau_2 = \tau$. An examination of the primer magnitude time history for each of the two-impulse segments of the three-impulse solution indicates that the cost function cannot be further improved by additional impulses.

Finally, a significant observation about rendezvous trajectories is that the large variations in cost with rendezvous time can be essentially eliminated by the introduction of optimal coast and the optimal number of impulses.

5. CONCLUDING REMARKS

Primer vector theory has been used to develop an efficient method for analysing and computing optimal N-impulse trajectories that are described by a set of linear relative-motion equations. All fundamental relations (state and costate vectors and the necessary conditions for optimality) are defined in terms of elementary functions. The criterion for adding intermediate impulses or removing any impulse is established by a simple examination of the primer vector. A differential cost function, which was developed in terms of a set of all possible independent parameters, was examined for a number of specific problems. For the generalized rendezvous problem, it was determined that for an optimal time of departure from the initial orbit and arrival on the final orbit, the primer vector and its complementary vector must be orthogonal.

REFERENCES

1. Lawden, D. F., *Optimal Trajectories for Space Navigation*, Butterworths, London, 1963.
2. Lion, P. M. and H. L. Handelsman, 'Primer vector on fixed-time impulsive trajectories', *AIAA J.*, **6**, 127–132 (1968).
3. Jezewski, D. J. and H. L. Rozendaal, 'An efficient method for calculating optimal free-space N-impulse trajectories', *AIAA J.*, **6**, 2160–2165 (1968).
4. Johnson, I. L., 'The Davidon–Fletcher–Powell penalty method: a generalized iterative technique for solving parameter optimization problems', *NASA TN D-8251* (1976).
5. Jezewski, D. J., 'Primer vector theory and application', *NASA TR R-454* (1975).
6. Prussing, J. E., 'Optimal multiple-impulse orbital rendezvous', *Report TE-20*, MIT Experimental Astronomy Laboratory (1967).
7. Edelbaum, T. N., 'Minimum impulse transfers in the near vicinity of a circular orbit', *J. Astronautical Sci.*, **14**, 67–73 (1967).
8. Jones, J. B., 'Optimal rendezvous in the neighbourhood of a circular orbit', *J. Astronautical Sci.*, **24**, 55–90 (1976).
9. Neustadt, L. W., 'Optimization, a moment problem, and nonlinear programming', *J. SIAM Control*, ser. A, **2**, 33–52 (1964).
10. Clohessy, W. H. and R. S. Wiltshire, 'Terminal guidance system for satellite rendezvous', *J. Aerospace Sci.*, **27**, 653–658 (1960).
11. Szebehely, V., *Theory of Orbits*, Academic Press, New York, 1967.
12. Jezewski, D. J. and J. D. Donaldson, 'An analytic approach to optimal rendezvous using Clohessy–Wiltshire equations', *J. Astronautical Sci.*, **XXVII**, 293–310 (1979).