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# Stability criterion for a class of nonlinear fractional differential systems☆

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ABSTRACT

Stability analysis of nonlinear fractional differential systems has been an open problem since the 1990s of the last century. Apparently, Lyapunov's second method seems to be invalid for nonlinear fractional differential systems (equations). In this paper, we are concerned with this open problem and have solved it partly. Based on Lyapunov's second method, a novel stability criterion for a class of nonlinear fractional differential system is derived. Our result is simple, global and theoretically rigorous. The conditions to guarantee the stability of the nonlinear fractional differential system are convenient for testing. Compared with the stability criteria in the literature, our criterion is straightforward and suitable for application. Several examples are provided to illustrate the applications of our result.



## 1. Introduction

In this paper, we consider the nonlinear fractional differential system

$$D_{0,t}^{\alpha}x(t) = f(x), \quad t \ge 0 \tag{1.1}$$

$$x(0) = x_0 \tag{1.2}$$

where $0 < \alpha < 1$, $D_{0,t}^{\alpha}x(t)$ denotes Caputo's fractional derivative with the lower limit 0 for the function $x(t)$, $f \in C^1(\Omega)$ with $f(0) = 0$, $\Omega \subset \mathbb{R}$ is a domain that contains the origin $x = 0$, $x_0 \in \Omega$.

The fractional calculus is a generalization of the traditional integer-order calculus. In the past few decades, numerous papers and monographs have been devoted to fractional differential equations (systems) because many physical phenomena such as memory and hereditary, etc., can be properly described with the help of fractional derivative. For more details on the basic theory of fractional calculus, one can see the monographs [1–5], the papers [6–10] and the references therein.

Stability analysis of a system is of importance in control theory. At present, most of the known results on stability analysis of fractional differential systems concentrate on the stability of linear fractional differential systems. In [11], Matignon has given a well-known stability criterion for a linear fractional autonomous differential system with constant coefficient matrix $A$. The criterion is that the stability is guaranteed iff the roots of the eigenfunction of the system lie outside the closed

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angular sector $|\arg(\lambda(A))| \leq \frac{\pi}{2}\alpha$, which generalized the result for the integer case $\alpha = 1$. Later, Matignon's stability criterion was developed by several scholars. Deng et al. [12] generalized the system to a linear fractional differential system with multi-orders and multiple delays, in which the characteristic polynomial is introduced by the Laplace transform method. In [13], LMI (linear matrix inequality) was used in the stability analysis of the linear fractional differential system. In 2010, Qian [14] et al. investigated the stability of a linear system with Riemann–Liouville's derivative. The robust stability for fractional order linear time-invariant interval systems with uncertain parameters FO-LTI was considered in [15]. Based on linear matrix inequalities methods, a criterion for checking asymptotical stability of this class of systems was derived.

Lyapunov's second method is an effective tool to analyze the stability of nonlinear integer-order differential systems without solving state equations. Recently, the nonlinear fractional differential system (1.1) has been discussed in several Refs. [16–20] and some results have been derived by using Lyapunov's method. However, compared with the stability criteria for nonlinear integer-order differential systems, the developments of nonlinear fractional differential systems are unsatisfactory. Now we introduce some developments of the stability of nonlinear fractional differential systems in detail.

In [18], the authors studied the nonlinear fractional differential system

$$ {}^{c}D^{\alpha}_{0,t}x(t) = f(t, x) \tag{1.3}$$

with initial condition $x(0) = x_0 \in \mathbb{R}$, where $0 < \alpha < 1$ and $f(t, 0) = 0$. One of their criteria is as follows.

**Theorem 1.1** (*[18]*)**.** *Let $x = 0$ be an equilibrium point for the system (1.3) and $\mathbb{D} \subset \mathbb{R}^n$ be a domain containing the origin. Let $V(t, x(t)) : [0, +\infty) \times \mathbb{D} \to \mathbb{R}$ be a continuously differentiable function and locally Lipschitz with respect to x such that*

$$\alpha_1\|x\|^a \leq V(t, x(t)) \leq \alpha_2\|x\|^{ab}, \tag{1.4}$$

$$D^{\alpha}_{0,t}V(t, x(t)) \leq -\alpha_3\|x\|^{ab}, \tag{1.5}$$

*where $t \geq 0$, $x \in \mathbb{D}$, $\alpha \in (0, 1)$, $\alpha_1, \alpha_2, \alpha_3$, a and b are arbitrary positive constants. Then $x = 0$ is Mittag-Leffler stable. If the assumptions hold globally on $\mathbb{R}^n$, then $x = 0$ is globally Mittag-Leffler stable.*

Theorem 1.1 shows if the state $x(t)$ of the system (1.3) satisfies conditions (1.4)–(1.5), then $x = 0$ is Mittag-Leffler stable. However, calculating $D^{\alpha}_{0,t}V(t, x(t))$ in (1.5) will involve $\dot{x}(t)$ of the system (1.3), which is not easy. Furthermore, the conditions (1.4)–(1.5) are not straightforward.

Very recently, the authors in [17] have applied the fractional comparison principle to discussing the Mittag-Leffler stability of the nonlinear system (1.3) with the Riemann–Liouville fractional derivative. One of the main results is as follows.

**Theorem 1.2** (*[17]*)**.** *Suppose that $x = 0$ is an equilibrium point of (1.3) and $f(t, x)$ in (1.3) satisfies*

$$|f(t, x)| \leq g(t, |x|), \tag{1.6}$$

*where $g \in C(J, \mathbb{R}^+)$ is monotone increasing in u for each t with $g(t, 0) = 0$. Consider the fractional-order differential equation*

$$D^{\alpha}_{t_0}u(t) = g(t, u), \qquad u(t_0) = u_0. \tag{1.7}$$

*If the zero solution $u = 0$ of (1.7) is a Mittag-Leffler system, then the zero solution $x = 0$ of (1.3) is also a Mittag-Leffler system whenever $u_0 > |x_0|$.*

By Theorem 1.2, in order to judge the stability of the zero solution $x = 0$ of Eq. (1.3), it is necessary to judge the stability of the zero solution $u = 0$ of Eq. (1.7) first. However, there had not been effective methods to determine if Eq. (1.7) was stable.

In 2011, based on the frequency distributed fractional integrator model, the authors in [20] considered the nonlinear fractional differential system

$$D^{\alpha}_{0,t}x = ax^3 + bx \tag{1.8}$$

with $a > 0$, $b < 0$. The results are as follows: the considered system (1.8) is stable if $ax^2 + b < 0$. However, when the right side of (1.8) becomes complex, using the frequency distributed method in paper [20] to analyze the stability of the system (1.8) may become difficult.

Apparently, Lyapunov's second method seems to be invalid for nonlinear fractional differential systems. Stability analysis of nonlinear fractional differential systems has been an open problem since the 1990s of the last century. This is our motivation to study this subject. In this paper, we will solve this open problem partly.

The rest of this paper is organized as follows. In Section 2, some preliminaries are provided. In Section 3, we present our main result of this paper. In Section 4, several illustrative examples are provided to show the application of our results.

## 2. Preliminaries

In this section, we will recall some definitions and lemmas which will be used later.

**Definition 2.1** ([1]). Given an interval $[a, b]$ of $\mathbb{R}$, the fractional order integral of a function $f \in L^1([a, b], \mathbb{R})$ of order $\alpha \in \mathbb{R}^+$ is defined by

$$I_{a,t}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad t \in [a, b],\ \alpha > 0, \tag{2.1}$$

where $\Gamma$ is the Gamma function.

**Definition 2.2** ([1]). Suppose that a function $f$ is defined on the interval $[a, b]$. Caputo's fractional derivative of order $\alpha$ with lower limit $a$ for $f$ is defined as

$$\begin{aligned} D_{a,t}^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds \\ &= I_{a,t}^{n-\alpha} f^{(n)}(t), \quad t \in [a, b], \end{aligned} \tag{2.2}$$

where $0 < n - 1 < \alpha \le n$.

Particularly, when $0 < \alpha \le 1$, it holds

$$D_{a,t}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} \dot{f}(s) ds = I_{a,t}^{1-\alpha} \dot{f}(t), \quad t \in [a, b]. \tag{2.3}$$

**Definition 2.3.** A continuous function $u(t) : [0, +\infty) \to [0, +\infty)$ is said to be a class-$K$ function if $u(t)$ is strictly increasing and $u(0) = 0$.

**Lemma 2.1** ([1]). *Let $0 < \alpha < 1$, $p \ge \alpha$ and $f(t)$ be continuous on $[a, +\infty)$. Then it holds*

$$D_{a,t}^{p}(I_{a,t}^{\alpha} f(t)) = D_{a,t}^{p-\alpha} f(t). \tag{2.4}$$

**Property 2.1.** *Let $0 < \alpha < 1$ and $f(t) \ge 0$ on $[a, b]$. Then it holds*

$$I_{a,t}^{\alpha} f(t) \ge 0, \quad t \in [a, b]. \tag{2.5}$$

We quote a result of existence and uniqueness of the global solution for the nonlinear fractional dynamical system

$$D_{0,t}^{\alpha} x(t) = f(t, x(t)), \tag{2.6}$$

$$x(0) = x_0, \tag{2.7}$$

which is a precondition in the development of this paper.

**Lemma 2.2** (*Existence and Uniqueness [5]*). *Consider the system (2.6)–(2.7). Suppose $0 < \alpha < 1$, $\Omega \subset \mathbb{R}$ is a domain that contains the origin $x = 0$, $x_0 \in \Omega$. Suppose further $f(t, x) : [0, +\infty) \times \Omega \to \mathbb{R}$ is continuous and satisfies a Lipschitz condition in $x$ with a Lipschitz constant $L > 0$. Then there exists a unique function $x(t) \in C[0, +\infty)$ satisfying the system (2.6)–(2.7).*

## 3. Main results

In this section, we will give the stability criterion for the equilibrium point $x = 0$ of the system (1.1). Let us begin with a lemma which is significant in this paper.

**Lemma 3.1.** *Suppose that $0 < \alpha < 1$, $h(t) \in C[0, +\infty)$, $g(t)$ is defined on $[0, +\infty)$ and $h(t) \cdot g(t) \le 0$, $t \in [0, +\infty)$. Then $g(t) \cdot I_{0,t}^{1-\alpha} h(t) \le 0$, $t \in [0, +\infty)$.*

**Proof.** Let $h(t)$ be positive and negative in turns on the intervals $[0, t_1], [t_1, t_2], \ldots, [t_i, t_{i+1}], \ldots$. Without the loss of generality, we assume that $h(t) \ge 0$ on $[0, t_1]$. By the assumption that $h(t) \cdot g(t) \le 0$, it follows $g(t) \le 0$, $t \in [0, t_1]$. By Property 2.1, $h(t) \ge 0$ implies $I_{0,t}^{1-\alpha} h(t) \ge 0$ for $t \in [0, t_1]$. Consequently, it holds

$$g(t) \cdot I_{0,t}^{1-\alpha} h(t) \le 0, \quad t \in [0, t_1]. \tag{3.1}$$

Now we assume $h(t) \le 0, t \in [t_1, t_2]$. Then, $g(t) \ge 0$ and $I_{t_1,t}^{1-\alpha} h(t) \le 0, t \in [t_1, t_2]$. It follows

$$g(t) I_{t_1,t}^{1-\alpha} h(t) \le 0, \quad t \in [t_1, t_2]. \tag{3.2}$$

Combining (3.1) with (3.2) yields

$$\begin{aligned} g(t) I_{0,t}^{1-\alpha} h(t) &= g(t) \left[ I_{0,t_1}^{1-\alpha} h(t) + I_{t_1,t}^{1-\alpha} h(t) \right] \\ &= g(t) I_{0,t_1}^{1-\alpha} h(t) + g(t) I_{t_1,t} h(t) \\ &\le 0, \quad t \in [0, t_2]. \end{aligned} \tag{3.3}$$

Now suppose that

$$g(t) \cdot I_{0,t}^{1-\alpha} h(t) \le 0, \quad t \in [0, t_n]. \tag{3.4}$$

We will prove

$$g(t) I_{0,t}^{1-\alpha} h(t) \le 0, \quad t \in [0, t_{n+1}]. \tag{3.5}$$

Assume that $h(t) \le 0$ for $t \in [t_n, t_{n+1}]$. Then $g(t) \ge 0, t \in [t_n, t_{n+1}]$. As a result, it follows

$$g(t) I_{t_n,t}^{1-\alpha} h(t) \le 0, \quad t \in [t_n, t_{n+1}]. \tag{3.6}$$

Considering (3.4) and (3.6), we get

$$g(t) I_{0,t}^{1-\alpha} h(t) = g(t) I_{0,t_n}^{1-\alpha} h(t) + g(t) I_{t_n,t}^{1-\alpha} h(t) \le 0, \quad t \in [0, t_{n+1}]. \tag{3.7}$$

By the method of mathematical induction, it holds that

$$g(t) I_{0,t}^{1-\alpha} h(t) \le 0, \quad t \in [0, +\infty). \tag{3.8}$$

This completes the proof. □

Now we present the stability criterion for the system (1.1).

**Theorem 3.1.** *Consider the system (1.1). $\Omega \subset \mathbb{R}$ is a domain that contains the origin $x = 0$. Suppose further that $f(x) \in C^1(\Omega)$ with $f(0) = 0$. If $x \cdot f(x) \le 0$, then the equilibrium point $x = 0$ is stable. Further, if $x \ne 0$ implies $x \cdot f(x) < 0$, then the equilibrium point $x = 0$ is asymptotically stable.*

**Proof.** Choose a Lyapunov function $V(x) = \int_0^x f(s)ds$. Obviously, $V(0) = 0$ and $V(x) \le 0$. So $V(x)$ is negative definite. It follows

$$\dot{V}_t(x)|_{(1.1)} = f(x) \cdot \dot{x}(t), \tag{3.9}$$

where $x(t)$ satisfies the system (1.1). By Definition 2.2 or the equality (2.3), the system (1.1) is equivalent to

$$I_{0,t}^{1-\alpha} \dot{x}(t) = f(x). \tag{3.10}$$

Inserting (3.10) into (3.9) yields

$$\dot{V}_t(x)|_{(1.1)} = \dot{x}(t) I_{0,t}^{1-\alpha} \dot{x}(t). \tag{3.11}$$

By Lemma 3.1, it follows that $\dot{V}_t(x)|_{(1.1)} \ge 0$. So $\dot{V}_t(x)$ is positive definite. Thus, the equilibrium point $x = 0$ of the system (1.1) is stable.

Obviously, if $x \ne 0$ implies $x \cdot f(x) < 0$, then $f(x) \not\equiv 0$. It follows that the state $x(t)$ of the system (1.1) satisfies $x(t) \not\equiv constant$. Therefore, $\dot{x}(t) \not\equiv 0$. Thus, $\dot{x}(t) I_{0,t}^{1-\alpha} \dot{x}(t) > 0$. So the system (1.1) is asymptotically stable. This completes the proof. □

**Remark 3.1.** (i) Compared with Theorems 1.1 and 1.2, Theorem 3.1 is simple, global and theoretically rigorous, which is convenient for application.

(ii) Consider the nonlinear integer-order differential system

$$\dot{x} = f(x) \tag{3.12}$$

with $f(0) = 0$. The following result is well known: if $x \cdot f(x) \le 0$, then the equilibrium point $x = 0$ of the system (3.12) is stable. Compared with Theorem 3.1, we extend the stability criterion from the nonlinear integer-order differential system (3.12) to the nonlinear fractional differential system (1.1).

## 4. Illustrative examples

**Example 4.1.** Consider the system

$$D_{0,t}^{\alpha}x(t) = -x^{2n+1}(t), \tag{4.1}$$

where $n$ is a positive integer and $0 < \alpha < 1$. Denote $f(x) = -x^{2n+1}(t)$. It holds $xf(x) = -x^{2n+2}(t) \leq 0$. Obviously, $x \neq 0$ implies $xf(x) < 0$. By Theorem 3.1, the equilibrium point $x = 0$ of the system (4.1) is asymptotically stable.

**Example 4.2.** Consider the system

$$D_{0,t}^{\alpha}x = -2x - x\sin x, \tag{4.2}$$

where $0 < \alpha < 1$. Denote $f(x) = -2x - x\sin x$. Obviously, $xf(x) = -x^2(2 + \sin x) \leq 0$, and $x \neq 0$ implies $xf(x) < 0$. By Theorem 3.1, it follows that the equilibrium point $x = 0$ of the system (4.2) is asymptotically stable.

**Example 4.3** (*Example 7.4 in [18]*)**.** Consider the system (1.1)–(1.2), where $x = 0$ is the equilibrium point of the system (1.1). By Theorem 3.1, if $xf(x) \leq 0$, then the equilibrium point $x = 0$ is stable. If $x \neq 0$ implies $x \cdot f(x) < 0$, then the equilibrium point $x = 0$ is asymptotically stable.

**Remark 4.1.** The answer for this problem in [18] is as follows: if $\|x\|_2 \leq \tilde{l}\|f(x)\|_2$ ($\tilde{l} > 0$ and $\|\cdot\|$ denotes the 2-norm), and $f(x)\frac{df(x)}{dx}\dot{x} \leq 0$, then the equilibrium point $x = 0$ is stable. Compared with the conditions in [18] to guarantee the stability for the system (1.1), our conditions are more simple, straightforward and convenient for testing.

**Example 4.4.** Consider the system (1.8). Denote $f(x) = ax^3 + bx$. Then, $xf(x) = ax^4 + bx^2 = x^2(ax^2 + b)$. By Theorem 3.1, if $ax^2 + b \leq 0$, then the equilibrium point $x = 0$ is stable.

**Remark 4.2.** Compared with the frequency distributed method in [20], our method is simple.

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## References

[1] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1993.
[2] A.A. Kilbas, Hari M. Srivastava, Juan J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier Science B. V, Amsterdam, 2006.
[3] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Differential Equations, John Wiley, New York, 1993.
[4] V. Lakshmikantham, S. Leela, J. Vasundhara, Devi. Theory of Fractional Dynamic Systems, Cambridge Scientific Pub., Cambridge, UK, 2009.
[5] K. Diethelm, The Analysis of Fractional Differential Equations, Springer, Heidelberg, Dordrecht, London, New York, 2010.
[6] V. Lakshmikantham, A.S. Vatsala, Basic theory of fractional differential equations, Nonlinear Anal. 69 (2008) 2677–2682.
[7] V. Lakshmikantham, Theorem of fractional functional differential equations, Nonlinear Anal. 69 (2008) 3337–3343.
[8] Y. Zhou, F. Jiao, Existence of mild solutions for fractional neutral evolution equations, Comput. Math. Appl. 59 (2010) 1063–1077.
[9] Y. Zhou, F. Jiao, Nonlocal Cauchy problem for fractional evolution equations, Nonlinear Anal. RWA 11 (2010) 4465–4475.
[10] Zaid M. Odibat, Analytic study on linear systems of fractional differential equations, Comput. Math. Appl. 59 (2010) 1171–1183.
[11] D. Matignon, Stability results for fractional differential equations with applications to control processing, Comput. Eng. Syst. Appl. 2 (1996) 963–968. Multiconference, IMACS, IEEE-SMC, Lille, France.
[12] W. Deng, C. Li, J. Lu, Stability analysis of linear fractional differential system with multiple time delays, Nonlinear Dynam. 48 (2007) 409–416.
[13] J. Sabatier, M. Moze, C. Farges, LMI stability conditions for fractional systems, Comput. Math. Appl. 59 (2010) 1594–1609.
[14] D. Qian, C. Li, R.P. Agarwal, Patricia J.Y. Wong, Stability analysis of fractional differential systems with Riemann–Liouville derivative, Math. Comput. Modelling 52 (2010) 862–872.
[15] Z. Liao, C. Peng, W. Li, Y. Wang, Robust stability analysis for a class of fractional order systems with uncertain parameters, J. Franklin Inst. 348 (2011) 1101–1113.
[16] J. Wang, L. Lv, Y. Zhou, New concepts and results in stability of fractional differential equations, Commun. Nonlinear Sci. Numer. Simul. 17 (2012) 2530–2538.
[17] S.K. Choi, N. Koo, The monotonic preperty and stability of solution of fractional differential equations, Nonlinear Anal. 74 (2011) 6530–6536.
[18] Y. Li, Y. Chen, I. Pudlubny, Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability, Comput. Math. Appl. 59 (2010) 1810–1821.
[19] F. Chen, J.J. Nieto, Y. Zhou, Global attractivity for nonlinear fractional differential equations, Nonlinear Anal. RWA 13 (2012) 287–298.
[20] J.C. Trigeassou, N. Maamri, J. Sabatier, A. Oustaloup, A Lyapunov approach to the stability of fractional differential equations, Signal Process. 91 (2011) 437–445.