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Nyman type theorem in convolution Sobolev algebras

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Abstract We characterize the dense ideals of certain convolution Sobolev algebras, on the positive half-line, as those ideals I which satisfy the Nyman conditions $Z(I) = \emptyset$ and $\gamma(I) = 0$. Here $Z(I)$ is the hull of I and $\gamma(I) := \inf\{\inf \operatorname{supp}(f) : f \in I\}$.

Keywords Convolution · Sobolev algebra · Dense ideal · Nyman theorem · Fractional derivation

1 Introduction

A non-zero ideal of the convolution Banach algebra $L^1(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$ if and only if the zero set of the ideal is empty. This is a form of the well known Wiener Tauberian theorem in $L^1(\mathbb{R}^n)$; see [15, 16]. The $L^1(\mathbb{R}^+)$ analog of the Wiener theorem was obtained by B. Nyman [14]. The result reads as follows.

Theorem (Nyman) *Given a closed ideal I of $L^1(\mathbb{R}^+)$, then $I = L^1(\mathbb{R}^+)$ if and only if $Z(I) = \emptyset$ and $\gamma(I) = 0$.*

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In the statement, $\gamma(I) := \inf\{\gamma(f) : f \in I\}$ with $\gamma(f) := \inf \operatorname{supp}(f)$ if $f \neq 0$, $\gamma(0) := \infty$; and $Z(I)$ is the hull, or zero-set, of I defined by $Z(I) := \{z \in \overline{\mathbb{C}^+} : \mathcal{L}f(z) = 0 \ (f \in I)\}$ where $\mathbb{C}^+ = \{z \in \mathbb{C} : \Re z > 0\}$ and $\mathcal{L}$ is the Laplace transform.

Since the closed ideals of $L^1(\mathbb{R}^n)$ and $L^1(\mathbb{R}^+)$ are exactly the closed subspaces of these algebras which remain invariant under translations, it is clear that both the Wiener's and Nyman's theorems can be regarded as results about completeness of translates of functions on $\mathbb{R}^n$ and $\mathbb{R}^+$ respectively . In this setting, the extension of the Nyman theorem to weighted convolution algebras, and even in the L^p case, for a fairly general class of weights, has been a long-standing problem only solved quite recently; see [3].

In the present paper we deal with the problem of density of ideals in a class of convolution Banach algebras (subalgebras of $L^1(\mathbb{R}^+)$) $\mathcal{T}_+^{(\alpha)}(t^\alpha)$, $\alpha > 0$, of Sobolev type. These algebras are defined using Weyl fractional derivation, and can be seen as consistent generalizations of $L^1(\mathbb{R}^+)$. They arise in connection with the abstract Cauchy problem in the following way.

Let X be a Banach space and let $\mathcal{B}(X)$ be the Banach algebra of bounded linear operators on X. Let A be a closed linear operator on X with domain $D(A) \subseteq X$. The (formal) solution $u : [0, \infty) \longrightarrow D(A)$ of the Cauchy equation

$$\begin{cases} u'(t) = Au(t), & t \geq 0, \\ u(0) = x, & x \in D(A) \end{cases}$$

is given by $u(t)x = T_0(t)x$ $(t \geq 0)$, where $T_0(t) := e^{tA}$, $t > 0$. When the Cauchy problem is *well-posed*, $T_0(t)$ is a C_0-semigroup of bounded operators on X. On the other hand, there is a bijection between bounded C_0-semigroups on X and bounded homomorphisms $\Theta_0 : L^1(\mathbb{R}^+) \to \mathcal{B}(X)$, implemented by the formula

$$\Theta_0(f) = \int_0^\infty f(t)\, T_0(t)\, dt \quad (f \in L^1(\mathbb{R}^+)).$$

It may well happen, when the Cauchy problem is *ill-posed*, that the operators e^{tA} are unbounded. Then, in many relevant cases the formula

$$T_\alpha(t)x := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, e^{sA} x\, ds \quad (x \in D(A),\, t > 0)$$

defines, for some $\alpha > 0$, a strongly continuous family of bounded operators $(T_\alpha(t))_{t\geq 0} \subseteq \mathcal{B}(X)$ which is called α-times integrated semigroup generated by A (see [2, 10] and references therein). Similarly as above, there exists a correspondence between α-times integrated semigroups $(T_\alpha(t))_{t\geq 0}$ satisfying $\sup_{t\geq 0} t^{-\alpha}\|T_\alpha(t)\| < \infty$ and bounded homomorphisms $\Theta_\alpha : \mathcal{T}_+^{(\alpha)}(t^\alpha) \to \mathcal{B}(X)$. The link between $T_\alpha(t)$ and Θ_α is implemented by the integral formula

$$\Theta_\alpha(f)x = \int_0^\infty W_+^\alpha f(t)\; T_\alpha(t)x\, dt \quad (x \in X;\ f \in \mathcal{T}_+^{(\alpha)}(t^\alpha)).$$

Moreover, for $\alpha = 0$ we have that $\mathcal{T}_+^{(0)}(t^0) = L^1(\mathbb{R}^+)$; see [13].

Thus the convolution algebra $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ plays for α-times integrated semigroups (of polynomial growth t^α) a similar role as $L^1(\mathbb{R}^+)$ plays in the setting of uniformly bounded operator semigroups. It seems appropriate to investigate the properties of the Banach algebra $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ along the same lines as it has been done for the algebra $L^1(\mathbb{R}^+)$ in many respects.

In this paper we are concerned with closed ideals of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$. As a matter of fact, the space of regular maximal ideals of this algebra is isomorphic to the set $\overline{\mathbb{C}^+}$, and the Gelfand transform is the Laplace transform. We characterize here the primary closed ideals at infinity of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ in terms of a Nyman-type theorem (Theorem 3.1 below):

Theorem *Let $\alpha > 0$. Given a closed ideal I of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$, then $I = \mathcal{T}_+^{(\alpha)}(t^\alpha)$ if and only if $Z(I) = \emptyset$ and $\gamma(I) = 0$.*

Besides the original argument (for $\alpha = 0$) in [14], there are other proofs of the Nyman's theorem to be found in papers like [4, 6, 9]. To prove our theorem for $\alpha > 0$ we follow the general scheme considered in [4], which must be substantially and largely adapted to our framework at certain places. Apart from the inherent difficulty to handle (fractional) derivatives, the main conceptual obstacle to approach the density problem in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$, $\alpha > 0$, is probably the fact that these Banach algebras *are not invariant* under translations.

We need to deal with convolution, not only on $\mathbb{R}^+$, but on the whole real line $\mathbb{R}$ indeed. For this, we consider a fractional (regular) Banach algebra $\mathcal{T}^{(\alpha)}(|t|^\alpha)$ which was introduced in [7] to study integrated groups. Also, the extension to the case $\alpha > 0$ of the functional and complex methods of [4] is based upon non trivial results on fractional derivation. Section 2 below contains the fractional calculus and basic properties of the Sobolev algebras which are required in the paper. Section 3 is devoted to establish the Nyman-type result (Theorem 3.1). The proof is rather long and then this section is divided into several subsections, or stages, for the sake of clarification. Finally, in Sect. 4 we point out some final remarks to the theorem concerning the (non-) invariance of ideals.

Throughout the paper we are using the constant convention, that is, we employ quite often the same letter to refer to a constant which may well be different from a line to another in the same argument.

2 Sobolev algebras defined by fractional derivation

The Banach algebra $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ introduced in [7] is defined later on. We choose a presentation of this algebra which is slightly different, though equivalent, from that one of [7] because it is more suitable for our purposes here.

Let $\alpha > 0$. It is readily seen that the integral (fractional) operator $W_+^{-\alpha}: C_c^\infty[0,\infty) \to C_c^\infty[0,\infty)$ defined by

$$W_+^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^\infty (s-t)^{\alpha-1} f(s)\,ds \quad (t \geq 0,\ f \in C_c^\infty[0,\infty))$$

is one-to-one and surjective (and continuous for the usual topology in $C_c^\infty[0,\infty)$). Thus the *fractional derivative* operator $W_+^\alpha\colon C_c^\infty[0,\infty)\to C_c^\infty[0,\infty)$ is defined as the inverse operator of $W_+^{-\alpha}$. As a matter of fact, we have that

$$W_+^\alpha f(t)=(-1)^n\frac{d^n}{dt^n}W_+^{-(n-\alpha)}f(t),\quad t>0,\ f\in C_c^\infty[0,\infty)$$

for $n:=[\alpha]+1$, where $[\alpha]$ is the integer part of α. Also, $W_+^m f=(-1)^m\, d^m f/dt^m$ if m is a non-negative integer.

We are going to extend the operator W_+^α. Let denote $L^1(t^\alpha)$ the Banach space of measurable functions over $\mathbb{R}^+ := (0,\infty)$, which are integrable with respect to the weight t^α, provided with the usual norm. For every $F\in L^1(t^\alpha)$, by Fubini's theorem we have

$$\int_0^\infty\int_x^\infty (y-x)^{\alpha-1}|F(y)|\,dy\,dx=\frac{1}{\alpha}\int_0^\infty y^\alpha|F(y)|\,dy<\infty,$$

from which it follows that the function

$$W_+^{-\alpha}F(x):=\frac{1}{\Gamma(\alpha)}\int_x^\infty (y-x)^{\alpha-1}F(y)\,dy$$

is defined a.e. $x\in\mathbb{R}^+$. Moreover, $W_+^{-\alpha}F$ is an element of $L^1(\mathbb{R}^+)$ with norm $\|W_+^{-\alpha}F\|_1\le\Gamma(\alpha+1)^{-1}\|F\|_{L^1(t^\alpha)}$. In other words, the operator $W_+^{-\alpha}: C_c^\infty[0,\infty) \longrightarrow C_c^\infty[0,\infty)$ extends to a bounded operator (denoted in the same manner) $W_+^{-\alpha}$: $L^1(t^\alpha)\to L^1(\mathbb{R}^+)$.

Definition 2.1 For any $\alpha>0$, let $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ denote the Banach space $W_+^{-\alpha}(L^1(t^\alpha))$ endowed with the norm ν_α obtained as the image of the norm of $L^1(t^\alpha)$ through the operator $W_+^{-\alpha}\colon L^1(t^\alpha)\to L^1(\mathbb{R}^+)$.

Thus we have that the surjective operator $W_+^{-\alpha}\colon L^1(t^\alpha)\to\mathcal{T}_+^{(\alpha)}(t^\alpha)$ is an isometry. Let $W_+^\alpha\colon\mathcal{T}_+^{(\alpha)}(t^\alpha)\to L^1(t^\alpha)$ be the inverse mapping (isometry) of $W^{-\alpha}$. Clearly, $W_+^\alpha\colon\mathcal{T}_+^{(\alpha)}(t^\alpha)\to L^1(t^\alpha)$ extends the fractional derivation operator defined formerly, and every $f\in\mathcal{T}_+^{(\alpha)}(t^\alpha)$ is characterized as an element of $L^1(\mathbb{R}^+)$ for which there exists a unique element in $L^1(t^\alpha)$, which we denote by $W_+^\alpha f$, such that

$$f(x)=\frac{1}{\Gamma(\alpha)}\int_x^\infty (y-x)^{-\alpha-1}W_+^\alpha f(y)\,dy\quad \text{a.e. } x>0.$$

In particular, $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ is a Banach space with respect to the norm

$$\nu_\alpha(f)=\int_0^\infty |W_+^\alpha f(t)|\,t^\alpha\,dt,$$

which contains $C_c^\infty[0,\infty)$ and $\mathcal{S}[0,\infty):=\mathcal{S}(\mathbb{R})|_{[0,\infty)}$ as dense subspaces. Here $\mathcal{S}(\mathbb{R})$ is the Schwarz class on $\mathbb{R}$.

It is readily seen that the continuous inclusions

$$\mathcal{T}_+^{(\beta)}(t^\beta) \hookrightarrow \mathcal{T}_+^{(\alpha)}(t^\alpha) \hookrightarrow \mathcal{T}_+^{(0)}(t^0) = L^1(\mathbb{R}^+), \tag{2.1}$$

hold whenever $\beta > \alpha > 0$. Finally, the Banach space $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ is moreover a Banach algebra with respect to the convolution on $\mathbb{R}^+$

$$f * g(x) := \int_0^x f(x-y)g(y)\,dy, \quad \text{a.e. } x > 0;\ f, g \in \mathcal{T}_+^{(\alpha)}(t^\alpha).$$

For this we mean that the convolution is jointly continuous with respect to the norm ν_α; see [7, p. 16]. The Banach algebras $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ are defined for fractional α in [7] and [13]. For integer α, see [1].

Example 2.2 (i) For $w \in \mathbb{C}$, put $e_w(t) = e^{-wt}$, $(t \in \mathbb{R})$. The restriction of e_w to $[0, \infty)$ belongs to $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ for all $\alpha > 0$, whenever $\Re w > 0$, with

$$W_+^\alpha(e_w)(t) = w^\alpha e^{-wt} \quad (t \geq 0): \tag{2.2}$$

$$W_+^\alpha(e_w)(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n}\left(e^{-wt}\int_0^\infty x^{n-\alpha-1}e^{-wx}\,dx\right) = w^\alpha e^{-wt}.$$

(ii) Set $f(x) := x^{-1/2}e^{-x}$ for $x > 0$. Then $f \in \mathcal{T}_+^{(1)}(t)$ with $W_+^1 f(x) = ((x/2) + 1)e^{-x}$, $x > 0$. Let $r > 0$. It is readily seen that if the translated function $\delta_r * f \equiv f(\,\cdot\, - r)\chi_{(r,\infty)}$ were in $\mathcal{T}_+^{(1)}(t)$ then there should be $W_+^1(\delta_r * f)(x) = (\frac{1}{2}(x - r)^{-3/2} + (x-r)^{-1/2})e^{r-x}\chi_{(r,\infty)}(x)$, for $x > 0$. Hence $\delta_r * f$ does not belong to $\mathcal{T}_+^{(1)}(t)$.

The function in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ of Example 2.2(i) will be important in Lemma 3.6 below. Functions like that one of Example 2.2(ii) tell us that, unlike $L^1(\mathbb{R}^+)$, the algebras $\mathcal{T}_+^{(\alpha)}(t^\alpha)$, so their ideals, are not invariant under translations, for $\alpha > 0$.

We shall need to deal with Banach algebras similar to $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ but living on the whole real line. The definition is as follows. Firstly, let $\mathcal{T}_-^{(\alpha)}((-t)^\alpha)$ be a copy of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$, defined this time on $(-\infty, 0)$. So $\mathcal{T}_-^{(\alpha)}((-t)^\alpha)$ is in particular the completion of $C_c^\infty(-\infty, 0]$ in the norm

$$\nu_\alpha(f) := \int_{-\infty}^0 |W_-^\alpha f(s)|\,(-s)^\alpha\,ds, \quad f \in C_c^\infty(-\infty, 0],$$

where

$$W_-^\alpha f(t) := \frac{d^n}{dt^n} W_-^{-(n-\alpha)} f(t), \quad t < 0,\ n = [\alpha] + 1,$$

and $W_-^{-(n-\alpha)} f$ is given by

$$W_-^{-\beta} f(t) := \frac{1}{\Gamma(\beta)} \int_{-\infty}^t (t-s)^{\beta-1} f(s)\,ds, \quad t \leq 0,\ \text{if } \beta > 0.$$

In this case $W_-^m f = d^m f/dt^m$ if m is a non-negative integer. As regards the analog to Example 2.2(i), note that for $\Re w < 0$ the function $e_w(t) = e^{-wt}$, $(t \le 0)$ (which we continue denoting by e_w) belongs to $\mathcal{T}_-^{(\alpha)}((-t)^\alpha)$ for all $\alpha > 0$. It satisfies

$$W_-^\alpha(e_w)(t) = (-w)^\alpha e^{-wt} \quad (t < 0). \tag{2.3}$$

Let now $\mathcal{T}^{(\alpha)}(|t|^\alpha)$ denote the direct sum $\mathcal{T}_-^{(\alpha)}((-t)^\alpha) \oplus \mathcal{T}_+^{(\alpha)}(t^\alpha)$ endowed with the norm

$$\nu_\alpha(f) := \nu_\alpha(f_-) + \nu_\alpha(f_+), \quad f \in \mathcal{T}^{(\alpha)}(|t|^\alpha),$$

where $f_- := f\chi_{(-\infty,0)}$, $f_+ := f\chi_{(0,\infty)}$. Then the Banach space $\mathcal{T}^{(\alpha)}(|t|^\alpha)$ is a convolution Banach algebra on $\mathbb{R}$ with respect to the norm ν_α, such that

$$\mathcal{T}^{(\beta)}(|t|^\beta) \hookrightarrow \mathcal{T}^{(\alpha)}(|t|^\alpha) \hookrightarrow L^1(\mathbb{R}); \tag{2.4}$$

see [7, pp. 19, 20]. We shall write $W^\alpha f := W_-^\alpha f_- + W_+^\alpha f_+$, so that

$$\nu_\alpha(f) = \int_{-\infty}^\infty |W^\alpha f(t)||t|^\alpha dt, \quad \forall f \in \mathcal{T}^{(\alpha)}(|t|^\alpha).$$

Next we show that the (convolution) algebra of smooth test functions $C_c^\infty(\mathbb{R})$ is dense in $\mathcal{T}^{(\alpha)}(|t|^\alpha)$.

Proposition 2.3 *Let $\alpha > 0$. Then the space $C_c^\infty(\mathbb{R})$ is dense in $\mathcal{T}^{(\alpha)}(|t|^\alpha)$. Consequently, the Schwarz class $\mathcal{S}(\mathbb{R})$ is also densely contained in $\mathcal{T}^{(\alpha)}(|t|^\alpha)$.*

Proof Take $n = [\alpha] + 1$. Note that the norm of $\mathcal{T}^{(n)}(|t|^n)$ is given by $\int_{-\infty}^\infty |t|^n |f^{(n)}(t)|\,dt$ where $f^{(n)}$ is the usual derivative of $f \in C_c^\infty(\mathbb{R})$.

Let $\varphi \in C_c^\infty(\mathbb{R})$, $\varphi \ge 0$, $\int \varphi = 1$. A standard method of approximation tells us that $\lim_{\varepsilon\to 0} h * \varphi_\varepsilon = h$ for all $h \in L^1(\mathbb{R}; 1 + |t|^n)$ in its usual norm, where $\varphi_\varepsilon(x) := \varepsilon^{-1}\varphi(x/\varepsilon)$; $x \in \mathbb{R}$, $\varepsilon > 0$ (that is, $(\varphi_\varepsilon)_{0<\varepsilon<1}$ is a bounded approximate identity, or summability kernel, for $L^1(\mathbb{R}; 1 + |t|^n)$). Take $f \in C_c^\infty[0,\infty)$. Then, for $x \in \mathbb{R}$,

$$\begin{aligned}(f * \varphi_\varepsilon)^{(n)}(x) &= (f * \varphi_\varepsilon^{(n)})(x) \\ &= \int_0^\infty f(y)\varphi_\varepsilon^{(n)}(x-y)\,dy = f(0^+)\varphi_\varepsilon^{(n-1)}(x) + (f' * \varphi_\varepsilon^{(n-1)})(x),\end{aligned}$$

whence

$$(f * \varphi_\varepsilon)^{(n)}(x) = \sum_{k=0}^{n-1} f^{(k)}(0^+)(\varphi_\varepsilon)^{(n-k-1)}(x) + (f^{(n)} * \varphi_\varepsilon)(x).$$

Hence,

$$\begin{aligned}\nu_n(f * \varphi_\varepsilon - f) &= \|(f * \varphi_\varepsilon - f)^{(n)}\|_{L^1(\mathbb{R};|t|^n)} \\ &\le \|f^{(n)} * \varphi_\varepsilon - f^{(n)}\|_{L^1(\mathbb{R};1+|t|^n)}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=0}^{n-1}|f^{(k)}(0^+)|\int_{-\infty}^{\infty}|(\varphi_\varepsilon)^{(n-k-1)}(x)|\,|x|^n\,dx\\
&=\|f^{(n)}*\varphi_\varepsilon-f^{(n)}\|_{L^1(\mathbb{R};1+|t|^n)}\\
&+\sum_{k=0}^{n-1}|f^{(k)}(0^+)|\left(\int_{-\infty}^{\infty}|(\varphi)^{(n-k-1)}(y)|\,|y|^n\,dy\right)\varepsilon^{k+1}.
\end{aligned}
$$

The above estimate tells us that $\lim_{\varepsilon\to 0} f*\varphi_\varepsilon=f$ in $\mathcal{T}_+^{(n)}(t^n)$ and therefore by (2.1) we obtain that $\lim_{\varepsilon\to 0} f*\varphi_\varepsilon=f$ in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$, for every $f\in C_c^\infty[0,\infty)$. Since $C_c^\infty[0,\infty)$ is dense in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ we derive that any function in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$, so in particular every function of $\mathcal{S}[0,\infty)$, can be approximated by functions of $C_c^\infty(\mathbb{R})$ in the norm ν_α. Analogously, the same fact holds for the subalgebras $\mathcal{T}_-^{(\alpha)}((-t)^\alpha)$ and $\mathcal{S}(-\infty,0]$. Thus we have that $C_c^\infty(\mathbb{R})$, so $\mathcal{S}(\mathbb{R})$, is dense in $\mathcal{T}^{(\alpha)}(|t|)^\alpha)$, as we wanted to show. □

The fact that $\mathcal{S}(\mathbb{R})$ is densely contained in $\mathcal{T}^{(\alpha)}(|t|^\alpha)$ is important in order to consider in the sequel the elements of the dual space of $\mathcal{T}^{(\alpha)}(|t|^\alpha)$ as tempered distributions on the real line. More precisely, via the isometry $W_+^{-\alpha}: L^1(t^\alpha)\to\mathcal{T}_+^{(\alpha)}(t^\alpha)$, the functionals L of the dual space $\mathcal{T}_+^{(\alpha)}(t^\alpha)'$ can be identified with the measurable functions $\phi:\mathbb{R}^+\to\mathbb{C}$ defined a.e. on $\mathbb{R}^+$ and such that $t^{-\alpha}\phi(t)\in L^\infty(\mathbb{R}^+)$. In this way the duality between $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ and $\mathcal{T}_+^{(\alpha)}(t^\alpha)'$ is implemented by the formula

$$
L_\phi(f)=\int_0^\infty \phi(t)W_+^\alpha f(t)\,dt\quad (f\in\mathcal{T}_+^{(\alpha)}(t^\alpha);\,\phi\equiv L_\phi\in\mathcal{T}_+^{(\alpha)}(t^\alpha)').
$$

By restriction, each $L\equiv L_\phi$ of $\mathcal{T}_+^{(\alpha)}(t^\alpha)'$ gives rise to an element of $\mathcal{T}^{(\alpha)}(|t|^\alpha)'$: If $L_\phi(f)=0$ for all $f\in C_c^\infty(\mathbb{R})$ then $L_\phi(g)=0$ for every $g\in C_c^\infty[0,\infty)$, since every g in $C_c^\infty[0,\infty)$ extends to a function in $C_c^\infty(\mathbb{R})$. On the other hand, since $\mathcal{S}(\mathbb{R})$ is dense in $\mathcal{T}^{(\alpha)}(|t|^\alpha)$, every element T in $\mathcal{T}^{(\alpha)}(|t|^\alpha)'$, so $L\in\mathcal{T}_+^{(\alpha)}(t^\alpha)'$ in particular, can be interpreted as a *tempered distribution*

$$
\mathcal{S}(\mathbb{R})\overset{i}{\hookrightarrow}\mathcal{T}^{(\alpha)}(|t|^\alpha)\xrightarrow{T}\mathbb{C},
$$

which we continue denoting by T.

The above discussion is also valid for the algebra $\mathcal{T}_-^{(\alpha)}((-t)^\alpha)$.

Some relationships between duality and convolution are now in order. Put $\widetilde{h}(x):=h(-x)$ for any function $h:\mathbb{R}\to\mathbb{C}$ and $x\in\mathbb{R}$. For $\varphi\in\mathcal{T}^{(\alpha)}(|t|^\alpha)'$ and $f\in\mathcal{T}^{(\alpha)}(|t|^\alpha)$, we define the functional $L_\varphi * f$ by

$$
(L_\varphi * f)(g):=L_\varphi(\widetilde{f}*g),\qquad g\in\mathcal{T}^{(\alpha)}(|t|^\alpha).
$$

Clearly, $L_\varphi * f$ is in $\mathcal{T}^{(\alpha)}(|t|^\alpha)'$ and therefore it can be interpreted like a tempered distribution. Also,

$$
L_\varphi * f=(L_{\widetilde{\varphi}}*\widetilde{f})^{\sim}: \tag{2.5}
$$

$$
\begin{aligned}
(L_\varphi * f)(g) = L_\varphi(\tilde{f} * g) &= \int_{-\infty}^{\infty} W^\alpha(\tilde{f} * g)(t)\,\varphi(t)\,dt \\
&= \int_{-\infty}^{\infty} W^\alpha(\widetilde{f} * g)(-t)\,\varphi(-t)\,dt = \int_{-\infty}^{\infty} W^\alpha((f * \widetilde{g})^{\sim})(-t)\,\varphi(-t)\,dt \\
&= \int_{-\infty}^{\infty} W^\alpha(f * \widetilde{g})(t)\,\varphi(-t)\,dt = L_{\widetilde{\varphi}}(f * \widetilde{g}) = (L_{\widetilde{\varphi}} * \widetilde{f})(\widetilde{g}),
\end{aligned}
$$

since $(f * g)^{\sim} = \widetilde{f} * \widetilde{g}$ and $W^\alpha \widetilde{h}(-t) = W^\alpha h(t)$.

3 Nyman type theorem for the algebra $\mathcal{T}_+^{(\alpha)}(t^\alpha)$

For a locally integrable function f on $\mathbb{R}^+$, $f \in L^1_{loc}(\mathbb{R}^+)$, and a subset $S \subseteq L^1_{loc}(\mathbb{R}^+)$ put $\gamma(f) := \inf(\operatorname{supp} f)$, $\gamma(0) = \infty$, and $\gamma(S) := \inf\{\gamma(f) : f \in S\}$. If $f \in L^1(\mathbb{R}^+)$ let $Z(f)$ denote the zero set of f, that is, $Z(f) := \{z \in \overline{\mathbb{C}^+}, \mathcal{L}f(z) = 0\}$. Let $\alpha \geq 0$. For a closed ideal I of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ the zero set of I is defined by

$$
Z(I) := \bigcap_{f \in I} Z(f) \ = \{z \in \overline{\mathbb{C}^+} : \mathcal{L}f(z) = 0,\ \forall f \in I\}.
$$

We prove in this section the main result of the paper.

Theorem 3.1 *Let I be a closed ideal of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$. Then $I = \mathcal{T}_+^{(\alpha)}(t^\alpha)$ if and only if $Z(I) = \emptyset$ and $\gamma(I) = 0$.*

The theorem was established by Nyman in 1951, for $L^1(\mathbb{R}^+)$ (i.e., for $\alpha = 0$). There are several proofs of the result in this case. Here we shall follow the argument done in [4, pp. 196–201], which suits well to our setting under appropriate modifications.

It is clear that $Z(\mathcal{T}_+^{(\alpha)}(t^\alpha)) = \emptyset$ and $\gamma(\mathcal{T}_+^{(\alpha)}(t^\alpha)) = 0$, therefore we really must show only the converse implication. Let I be a closed ideal in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ such that $Z(I) = \emptyset$ and $\gamma(I) = 0$. We shall use duality and the Hahn-Banach theorem to prove that $I = \mathcal{T}_+^{(\alpha)}(t^\alpha)$. So take φ supported on $(-\infty, 0]$ such that $\widetilde{\varphi} \in \mathcal{T}_+^{(\alpha)}(t^\alpha)'$ and assume that

$$
0 = L_{\widetilde{\varphi}}(f) := \int_0^\infty W_+^\alpha f(t)\,\varphi(-t)\,dt \quad (f \in I).
$$

We want to get $L_{\widetilde{\varphi}} \equiv 0$. For this, our first goal is to show that the tempered distribution $L_\varphi * f$ defined prior to (2.5) is supported on $[0, \infty)$ for every $f \in I$. At this point it would be useful to apply invariance of ideals under translations (see [4, p. 198]) but, as it has been observed in Example 2.2(ii), such an invariance does not hold in the algebra $\mathcal{T}_+^{(\alpha)}(t^\alpha)$. Instead, we proceed as follows.

3.1 Invariance of ideals in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$

Lemma 3.2 *Let $k \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ and let $\widetilde{\phi} \in \mathcal{T}_+^{(\alpha)}(t^\alpha)'$. Then there exists $W_+^\alpha k * \phi$, it is continuous and bounded over $[0, \infty)$ and satisfies*

$$\sup_{t \geq 0} |(W_+^\alpha k * \phi)(t)| \leq \sup_{x \leq 0} \left(|x|^{-\alpha} |\phi(x)|\right) \nu_\alpha(k).$$

Proof First take $k \in C_c^\infty[0, \infty)$. Clearly, there exists $W_+^\alpha k * \phi$ and it is continuous and bounded over $[0, \infty)$. In fact,

$$\begin{aligned}|(W_+^\alpha k * \phi)(t)| &\leq \int_{-\infty}^{\infty} |W_+^\alpha k(s)|\, |\phi(t-s)|\, ds \\ &\leq C \int_t^\infty |W_+^\alpha k(s)|\, s^\alpha \frac{(s-t)^\alpha}{s^\alpha}\, ds \leq C\, \nu_\alpha(k)\end{aligned}$$

where $C = \sup_{x \leq 0}(|x|^{-\alpha}|\phi(x)|)$.

For arbitrary $k \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ the result follows by density. □

Now we get the wished-for result on $L_\varphi * f$.

Proposition 3.3 *For every $f \in I$ and $g \in \mathcal{T}^{(\alpha)}(|t|^\alpha)$,*

$$\int_0^\infty W_+^\alpha (f * g)(t)\, \varphi(-t)\, dt = \int_0^\infty g(-t) (W_+^\alpha f * \varphi)(t)\, dt \tag{3.1}$$

and therefore

$$(L_\varphi * f)(g) = \int_0^\infty g(t)\, (W_+^\alpha f * \varphi)(t)\, dt. \tag{3.2}$$

Proof By Lemma 3.2 the second integral in (3.1) exists. Also, if $g \in \mathcal{T}^{(\alpha)}(|t|^\alpha)$ then $g_+ \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ by definition. Hence $f * g_+ \in I$ so we have $L_{\widetilde{\varphi}}(f * g_+) = 0$ by hypothesis. Furthermore

$$W_+^\alpha (f * g_-)(t) = (W_+^\alpha f * g_-)(t) \quad \forall t > 0$$

by [7, Lemma 1.6]. Thus we have

$$\int_0^\infty W_+^\alpha (f * g)(t)\, \varphi(-t)\, dt = \int_0^\infty (W_+^\alpha f * g_-)(t)\, \varphi(-t)\, dt.$$

Now, notice that

$$\begin{aligned}&\int_0^\infty \int_t^\infty |W_+^\alpha f(y)|\, |g_-(t-y)|\, |\varphi(-t)|\, dy\, dt \\ &\quad \leq C \int_0^\infty \int_t^\infty t^\alpha\, |W_+^\alpha f(y)|\, |g_-(t-y)|\, dy\, dt\end{aligned}$$

$$\leq C \int_0^\infty \left((\,\cdot\,)^\alpha \, |W_+^\alpha f| * |g_-| \right)(t)\, dt$$
$$\leq C \|(\,\cdot\,)^\alpha W_+^\alpha f\|_1 \, \|g_-\|_1 \leq C \nu_\alpha(f)\, \nu_\alpha(g) < \infty.$$

Thus we can apply Fubini's theorem and obtain

$$\int_0^\infty (W_+^\alpha f * g_-)(t)\, \varphi(-t)\, dt = \int_0^\infty g(-t)(W_+^\alpha f * \varphi)(t)\, dt.$$

Finally, to show (3.2) it suffices to observe that $(L_\varphi * f)(g) = L_{\tilde{\varphi}}(f * \widetilde{g})$ by (2.5), and then apply (3.1) with $\widetilde{g}$ instead of g. □

Remark 3.4 From (3.2) we have that the functional $(L_\varphi * f)$ is the (temperated) distribution associated with the bounded and continuous function $h := (W_+^\alpha f * \varphi)|_{[0,\infty)}$. So the support $\operatorname{supp}(L_\varphi * f)$ lies in $[0, \infty)$. This property will play a key role in the sequel.

3.2 Juxtaposition and analytical extension

Take $f \in I$. Put $h := (W_+^\alpha f * \varphi)|_{[0,\infty)}$ as before. Let consider the Laplace transforms $H = \mathcal{L}(h)$, $F = \mathcal{L}(f)$, $\Phi = \mathcal{L}(L_\varphi)$. Note that for $z \in \mathbb{C}^- := \{w \in \mathbb{C} : \Re w < 0\}$, by (2.3),

$$\mathcal{L}(L_\varphi)(z) := L_\varphi(e_z) = \int_{-\infty}^0 W_-^\alpha(e_z)(t)\, \varphi(t)\, dt$$
$$= \int_0^\infty W_+^\alpha(e_{-z})(s)\, \varphi(-s)\, ds = (-z)^\alpha \mathcal{L}(\varphi)(z). \tag{3.3}$$

Define

$$\Psi(z) = \begin{cases} \Phi(z) & \text{if } z \in \mathbb{C}^-, \\ \frac{H(z)}{F(z)} & \text{if } z \in \mathbb{C}^+ \setminus Z(f). \end{cases}$$

The idea is to have "$H = \Phi F$", that is, "$h = \varphi * f$" formally. Then we must prove that Ψ admits an entire extension to $\mathbb{C}$. For this we use the localization method carried on in [4, pp. 198–200] for $\alpha = 0$. The case $\alpha > 0$ requires several technical devices in addition.

Suppose for a moment that $F(0) \neq 0$ and choose $\delta > 0$ such that $F(z) \neq 0$ for all $z \in D(0; 4\delta) \cap \overline{\mathbb{C}^+}$. Here $D(0; 4\delta)$ is the open disc centered at 0 with radius 4δ. Take $\Lambda \in C_c^\infty(\mathbb{R})$ such that $\operatorname{supp} \Lambda \subset (-\delta, \delta)$, and let $\lambda \in \mathcal{S}(\mathbb{R})$ such that $\widehat{\lambda} = \Lambda$ where $\widehat{\lambda}$ denotes the Fourier transform of λ. Put $V := [-i\delta, i\delta]$ and define

$$(\Psi \cdot \Lambda)(z) := \int \Psi(z + iy)\, \Lambda(y)\, dy, \quad z \in D(0; \delta) \setminus V.$$

Then $\Psi \cdot \Lambda$ can be extended continuously to V.

Proposition 3.5 *For* Ψ, Λ *and* λ *as above and* $s \in [-\delta, \delta]$, *we have*

$$\lim_{\Re z<0,\, z\to is} (\Psi \cdot \Lambda)(z) = \lim_{\Re z>0,\, z\to is} (\Psi \cdot \Lambda)(z) = 2\pi\, L_{\varphi}(e_{is}\widetilde{\lambda}).$$

The proof of the proposition relies on several lemmata.

Lemma 3.6 *Let* $\alpha > 0$. *Then, for every* $g \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ *and* $k \in \mathbb{R}^+$,

(i) $e_k g \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ *and* $\nu_\alpha(e_k g) \le C_\alpha \nu_\alpha(g)$.
(ii) $\lim_{k\to 0^+}(e_k g) = g$ *in* $\mathcal{T}_+^{(\alpha)}(t^\alpha)$.

Proof First take $g \in C_c^\infty[0,\infty)$. We apply the following formula of fractional derivation of the pointwise product with the exponential function, which is given in [8, p. 338]:

$$W_+^\alpha(e^{-kx}g(x)) = e^{-kx}W_+^\alpha g(x) + ke^{-kx}\int_0^\infty \zeta_\alpha(ky)W_+^\alpha g(y+x)\,dy$$

for $x, k > 0$. Here ζ_α is the function

$$\zeta_\alpha(y) = \sum_{n=0}^\infty \binom{n+\alpha}{n+1}\frac{y^n}{n!}e^{-y}, \qquad y \ge 0.$$

This function is bounded for $0 < \alpha \le 1$ and is $\zeta_\alpha(y) = O(y^{\alpha-1})$ as $y \to \infty$, for $\alpha > 1$ [8, p. 338]. Therefore there exists $C_\alpha > 0$ such that $|\zeta_\alpha(y)| \le C_\alpha(1+y^\alpha)$ for all $y > 0$ and $\alpha > 0$. Thus we have

$$\begin{aligned}
&\int_0^\infty kx^\alpha e^{-kx}\int_0^\infty |\zeta_\alpha(ky)|\,|W_+^\alpha g(y+x)|\,dy\,dx\\
&\quad\le C_\alpha\left[\left(\int_0^\infty ke^{-kx}dx\right) + \left(\int_0^\infty k^{\alpha+1}x^\alpha e^{-kx}dx\right)\right]\\
&\quad\quad\times\int_0^\infty (x+y)^\alpha\left|W_+^\alpha g(y+x)\right|\,dy
\end{aligned}$$

whence we obtain

$$\nu_\alpha(e_k g) \le \nu_\alpha(g) + C_\alpha[1+\Gamma(\alpha+1)]\nu_\alpha(g) = C'_\alpha\nu_\alpha(g).$$

Let now $g \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ arbitrary and let $(g_j)_j \subseteq C_c^\infty[0,\infty)$ a sequence such that $\lim_j g_j = g$ in ν_α, which implies that $\lim_j g_j = g$ in $L^1(\mathbb{R}^+)$. Then $\lim_j(e_k g_j) = e_k g$ in $L^1(\mathbb{R}^+)$ for every $k > 0$. On the other hand, $\nu_\alpha(e_k(g_j - g_l)) \le C\,\nu_\alpha(g_j - g_l)$ for all j, l as shown above. Hence $(e_k g_j)_j$ is a Cauchy sequence in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$. By uniqueness of the limit we obtain that $\lim_j(e_k\,g_j) = e_k g$ in the norm ν_α, whence

$$\nu_\alpha(e_k g) = \lim_j \nu_\alpha(e_k\,g_j) \le C\lim_j \nu_\alpha(g_j) = C\nu_\alpha(g).$$

Finally, it is readily seen that for any $g \in C_c^\infty[0,\infty)$, $\lim_{k\to 0^+} e_k g = g$ in $\mathcal{S}([0,\infty))$ and therefore in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$. Then using density (and the triangle inequality),

$$\lim_{k\to 0^+} e_k g = g \;\text{ in }\; \mathcal{T}_+^{(\alpha)}(t^\alpha), \qquad \text{for all } g \in \mathcal{T}_+^{(\alpha)}(t^\alpha). \qquad \square$$

The proof of the following lemma is standard.

Lemma 3.7 *Let $(\eta_j)_{j\ge 0} \subseteq \mathcal{T}^{(\alpha)}(|t|^\alpha)$ such that $\lim_j \eta_j = \eta_0$ in $\mathcal{T}^{(\alpha)}(|t|^\alpha)$ and suppose that $\widehat{\eta_j}(y) \neq 0$ $(j \ge 0)$, $\forall y \in E$, where $E \subseteq \mathbb{R}$ is compact. Then there exist $(g_j)_{j\ge 0} \subseteq \mathcal{T}^{(\alpha)}(|t|^\alpha)$ such that $\widehat{g_j}(y) = [\widehat{\eta_j}(y)]^{-1}$ $(j \ge 0)$ for all $y \in E$, and $\lim_j g_j = g_0$ in $\mathcal{T}^{(\alpha)}(|t|^\alpha)$.*

Proof The Banach algebra $\mathcal{T}^{(\alpha)}(|t|^\alpha)$ is regular on its spectrum $\mathbb{R}$, see [7]. So the result follows by continuity of the inversion mapping in the quotient Banach algebra $\mathcal{T}^{(\alpha)}(|t|^\alpha)/I(E)$ where $I(E) := \{g \in \mathcal{T}^{(\alpha)}(|t|^\alpha) : \widehat{g}(y) = 0\ (y \in E)\}$. $\square$

The next lemma is [4, Lemma 6.3].

Lemma 3.8 *Let $\eta_j, \eta \in L^1(\mathbb{R})$ with $\lim_j \eta_j = \eta$ in $L^1(\mathbb{R})$, and let $\mu_j, \mu \in L^\infty(\mathbb{R})$ with $\|\mu_j\|_\infty \le \|\mu\|_\infty$ $(\forall j)$ and $\lim_j \mu_j = \mu$ pointwise. Then*

$$\lim_j (\eta_j * \mu_j)(0) = (\eta * \mu)(0).$$

Proof of Proposition 3.5 Let first consider $\Re z < 0$. Then

$$\begin{aligned}(\Psi\cdot\Lambda)(z+is) &= \int_{-\infty}^{\infty} \Psi(z+ir)\,\Lambda_s(r)\,dr \\ &= \int_{-\infty}^{\infty} L_\varphi(e_{z+ir})\,\Lambda_s(r)\,dr = L_\varphi\left(\int_{-\infty}^{\infty} \Lambda_s(r)\,e_{z+ir}\,dr\right)\end{aligned}$$

where $\Lambda_s(r) = \Lambda(r-s)$. By inversion of the Fourier transform theorem,

$$\int_{-\infty}^{\infty} e^{-zt} e^{-irt} \Lambda_s(r)\,dr = 2\pi\, e^{-zt} e^{-ist} \lambda(-t), \qquad t \in \mathbb{R}.$$

So we get

$$\lim_{z\to 0} (\Psi\cdot\Lambda)(z+is) = \lim_{z\to 0} L_\varphi(2\pi\, e_{z+is}\,\widetilde{\lambda}) = 2\pi\, L_\varphi(e_{is}\,\widetilde{\lambda}),$$

since $\lim_{z\to 0} e_{z+is}\,\widetilde{\lambda} = e_{is}\,\widetilde{\lambda}$ in $\mathcal{S}([0,\infty))$.

Now, let $z_n = a_n + ib_n \in \mathbb{C}^+$ with $|z_n| \le \delta$. Recall that we have defined formerly $h := (W_+^\alpha f * \varphi)|_{[0,\infty)}$. Put $h_n := e_{a_n} h$, $f_n := e_{a_n} f$. Note that $h_n, f_n \in L^1(\mathbb{R})$.

Then

$$(\Psi\cdot\Lambda)(z_n+is) = \int_{-\infty}^{\infty} \Psi(a_n+iy)\,\Lambda(y-b_n-s)\,dy$$

$$= \int_{-\infty}^{\infty} \frac{(\mathcal{L}h)(a_n+iy)}{(\mathcal{L}f)(a_n+iy)}\,(\delta_{b_n+s}*\Lambda)(y)\,dy$$

$$= \int_{-\infty}^{\infty} \frac{\widehat{h}_n(y)}{\widehat{f}_n(y)}\,(\delta_{b_n+s}*\Lambda)(y)\,dy.$$

Note that $\mathrm{supp}\,(\delta_{b_n+s}*\Lambda) \subseteq [-(b_n+s+\delta), b_n+s+\delta] \subseteq [-3\delta, 3\delta]$.

By Lemma 3.6, $(f_n)_{n\ge 1} \subseteq \mathcal{T}_+^{(\alpha)}(t^\alpha)$ and $\lim_n f_n = f$ in the norm ν_α. By Lemma 3.7 there exists a sequence $(g_n)_{n\ge 0} \subseteq \mathcal{T}^{(\alpha)}(|t|^\alpha)$ such that $\widehat{f_n}(y)\,\widehat{g_n}(y) = 1$ for all $y \in E := [-3\delta, 3\delta]$ and $\lim_n g_n = g_0$ in ν_α. Taking Fourier transforms it is readily seen that $\widetilde{f} * \widetilde{g}_0 * (e_{is}\widetilde{\lambda}) = e_{is}\widetilde{\lambda}$.

Thus

$$\begin{aligned}(\Psi\cdot\Lambda)(z_n+is) &= \int_{-3\delta}^{3\delta} \widehat{h}_n(y)\,\widehat{g}_n(y)\,[e_{-i(b_n+s)}\lambda]\widehat{\ }(y)\,dy\\ &= \int_{-\infty}^{\infty} \big(h_n * g_n * [e_{-i(b_n+s)}\lambda]\big)\widehat{\ }(y)\,dy\\ &= 2\pi(h_n * g_n * [e_{-i(b_n+s)}\lambda])(0)\end{aligned}$$

by the inversion formula for the Fourier transform.

Moreover, $h_n \in L^\infty(\mathbb{R})$, $\|h_n\|_\infty \le \|h\|_\infty$ ($n \in \mathbb{N}$), and $\lim_n h_n = h$ pointwise. By Lemma 3.8,

$$\lim_n 2\pi(h_n * g_n * [e_{-i(b_n+s)}\lambda])(0) = 2\pi(h * g_0 * [e_{-is}\lambda])(0).$$

Finally,

$$\begin{aligned}&2\pi(h * g_0 * [e_{-is}\lambda])(0)\\ &\quad= 2\pi\int_{-\infty}^{\infty} h(t)\,(g_0 * (e_{-is}\lambda))(-t)\,dt = 2\pi\int_0^{\infty} h(t)\,(\widetilde{g}_0 * (e_{is}\widetilde{\lambda}))(t)\,dt\end{aligned}$$

and the last integral is equal, by (3.2) in Proposition 3.3, to

$$2\pi(L_\varphi * f)(\widetilde{g}_0 * (e_{is}\widetilde{\lambda})) = 2\pi L_\varphi(\widetilde{f} * \widetilde{g}_0 * (e_{is}\widetilde{\lambda})) = 2\pi L_\varphi(e_{is}\widetilde{\lambda})$$

as we wanted to show. $\square$

We have proved that there exists a continuous extension of $\Psi\cdot\Lambda$ to $i\mathbb{R}\cap D(0;\delta)$, and consequently $\Psi\cdot\Lambda$ is analytic in $D(0;\delta)$. Next we show that Ψ is holomorphic in $D(0;\delta)$. Choose a sequence $\Lambda_n \in C_c^\infty(\mathbb{R})$ such that $\Lambda_n \ge 0$, $\mathrm{supp}\,\Lambda_n \subset (-\frac{1}{n}, \frac{1}{n})$ and $\int \Lambda_n = 1$ for $n > 1/\delta$. Let

$$\Psi_n(z) := (\Psi\cdot\Lambda_n)(z), \quad z \in D(0;\delta)\setminus V.$$

Lemma 3.9 *For Ψ and Ψ_n as above,*

(i) *$|x|^{\alpha+1}\Psi_n(x+iy)$ is uniformly bounded in the disc $D(0,\delta)$.*

(ii) $\lim_n \Psi_n(z) = \Psi(z)$ *for all* $z \in D(0,\delta) \setminus V$.

Proof (i) If $z = x + iy$, $x < 0$,

$$|\Psi_n(x+iy)| \le \left(\sup_{|s|<\delta+\frac{1}{n}} |\mathcal{L}(L_\varphi)(x+is)| \right) \left| \int_{-\frac{1}{n}}^{\frac{1}{n}} \Lambda_n(t)\,dt \right|$$
$$= \sup_{|s|<\delta+\frac{1}{n}} |(-(x+is))^\alpha \mathcal{L}\varphi(x+is)|,$$

by (3.3). Also,

$$|\mathcal{L}\varphi(z)| \le \|s^{-\alpha}\varphi\|_\infty \int_0^\infty t^\alpha e^{-|\Re z|t}\,dt = \|s^{-\alpha}\varphi\|_\infty \frac{\Gamma(\alpha+1)}{|\Re z|^{\alpha+1}},$$

and therefore $|\Psi_n(x+iy)| \le C(3\delta)^\alpha |x|^{-(\alpha+1)}$.

If $z = x + iy$, $x > 0$,

$$|\Psi_n(x+iy)| \le \sup_{|s|\le\delta+\frac{1}{n}} \left| \frac{\mathcal{L}(h)(x+is)}{(\mathcal{L}f)(x+is)} \right|$$

where $\mathcal{L}f$ is bounded below in $\overline{D}(0;2\delta)$ by a constant $C_\delta > 0$ and $|\mathcal{L}h(x+iy)| \le Cx^{-1}$ since h is bounded.

Putting both cases above together we obtain that $|x|^{\alpha+1}\Psi_n(x+iy)$ is uniformly bounded in $D(0;\delta)$.

(ii) This is straightforward. □

In the conditions given in the previous lemma the function Ψ turns out to be analytic in $D(0,\delta)$. The proof of this fact can be done similarly to [4, Lemma 6.2]. The argument consists of applying the maximum modulus theorem and the Montel theorem on normal families to the sequence $(z-z_1)^m(z-z_2)^m\Psi_n(z)$. Here z_1, z_2 , in $D(0,\delta) \cap i\mathbb{R}$, are antipodes with respect to the origin, and m is a fixed integer greater than $\alpha + 1$.

Finally, if τ is any point in $\mathbb{R}$ and D is a disc centered at $i\tau$, we can make a translation to $\tau = 0$ taking an even function in $C_c^\infty(\mathbb{R})$ of the type of those Λ formerly used. In this way we conclude that Ψ is holomorphic in $\mathbb{C}$ with the possible exception of $Z(f)$, $f \in I$. Further, Ψ is independent of $f \in I$ on $\mathbb{C}^-$, so on $\mathbb{C}$, and $Z(I) = \emptyset$ by hypothesis. Hence, Ψ is entire.

3.3 Compact support of the functional L_φ

The aim of this subsection is to show that L_φ has compact support. First we shall see that Ψ is of exponential type by applying the following Krein's theorem, whose statement is taken from [4, Theorem 7.4] by convenience.

Lemma 3.10 (Krein theorem) *Let G be an entire function such that*:

(i) *there exists* $\sigma \geq 0$ *such that* G *is a quotient of bounded analytic functions separately on each of the half-planes*

$$\{z \in \mathbb{C},\, \Re z < -\sigma\} \quad \textit{and} \quad \{z \in \mathbb{C},\, \Re z > \sigma\},$$

(ii) *there exists* $m \geq 0$ *such that*

$$\int\int_{\mathbb{C}^- \cup \mathbb{C}^+} \frac{\log^+ |G(z)|}{1 + |z|^m}\, dx\, dy < \infty, \qquad z = x + iy.$$

Then G *is of exponential type.*

Proposition 3.11 Ψ *is an entire function of exponential type*; *in consequence*, L_φ *is a distribution of compact support*.

Proof It is clear that the function Ψ satisfies condition (i) of Lemma 3.10. As regards (ii), note that if $\Re z < 0$ then $|\Psi(z)| \leq C|z|^\alpha |\Re z|^{-(\alpha+1)}$ whence $\log^+ |\Psi(z)| \leq \log^+(C|z|^\alpha) + \log^+(|\Re z|^{-(\alpha+1)})$. So the boundedness of the integral on $\mathbb{C}^-$ in Lemma 3.10, for $G = \Psi$ and $m > 2$, follows readily. For $\Re z > 0$ we have $\log^+ |\Psi(z)| = \log^+(|H(z)||F(z)|^{-1}) \leq \log^+ |H(z)| + \log^+(|F(z)|^{-1})$. Then the finiteness of the integral of Lemma 3.10(ii) on $\mathbb{C}^+$, for the term $\log^+ |H(z)|$, follows as above whereas for the term $\log^+(|F(z)|^{-1})$ it follows by [4, Proposition 7.5] with $m = 4$. Hence Φ is of exponential type.

To finish the proof, we apply the Paley-Wiener theorem for Laplace transforms of distributions, such as it is to be found in [12, Theorem II], to the exponential function $\Psi = \mathcal{L}(L_\varphi)$. Then we conclude that L_φ is a distribution with compact support. $\square$

3.4 Titchmarsh-Lions theorem for distributions

Let $\gamma(L_\varphi) := \inf(\operatorname{supp} L_\varphi)$, and put $-a = \gamma(L_\varphi)$, so that $a \geq 0$ and $\operatorname{supp} L_\varphi \subseteq [-a, 0]$. We want to prove that $a = 0$.

Take $\rho \in C_c^\infty(\mathbb{R})$ and $r := \gamma(\rho)$. Let $f \in I$. By duality on $\mathcal{S}(\mathbb{R})$, for example, it is easy to show that

$$(L_\varphi * f) * \rho = (L_\varphi * \rho) * f.$$

Since $L_\varphi * f \equiv h$ has support in $[0, \infty)$ we have $\gamma((L_\varphi * f) * \rho) \geq r$. Also, L_φ being a compactly supported distribution and ρ a test function, the distribution $L_\varphi * \rho$ belongs to $C_c^\infty(\mathbb{R})$ [15, p. 156]. Therefore, by the well known Titchmarsh theorem (see [4, Theorem 3.10], for instance)

$$r \leq \gamma((L_\varphi * \rho) * f) = \gamma(L_\varphi * \rho) + \gamma(f).$$

Since $\gamma(I) = 0$ we obtain $\gamma(L_\varphi * \rho) \geq r$.

Moreover, there exists an extension of the Titchmarsh theorem to distributions of compact support, even in higher dimensions and given in terms of convex hulls, due to J.L. Lions [11, Theorem 1]. Applying this theorem in our one dimensional case to the distributions L_φ and ρ, it yields

$$r \leq \gamma(L_\varphi * \rho) = \gamma(L_\varphi) + \gamma(\rho) = -a + r.$$

Thus $0 \le -a$ and therefore $a = 0$ as required.

Final stage of the proof of Theorem 3.1 We have seen that L_φ is a distribution concentrated at $x = 0$. Hence there exists $m \in \mathbb{N} \cup \{0\}$ such that

$$L_\varphi(g) = \sum_{k=0}^{m} c_k\, g^{(k)}(0), \quad \forall g \in \mathcal{S}(\mathbb{R}).$$

In particular by choosing an extension g in $\mathcal{S}(\mathbb{R})$ of $e_z \chi_{[-1,1)}$ we have that $\mathcal{L}(L_\varphi)(z)$ is a polynomial in $z \in \mathbb{C}$. Since

$$|\mathcal{L}(L_\varphi)(z)| \le C\, \frac{|z|^\alpha}{|\Re z|^{\alpha+1}}, \quad z \in \mathbb{C}^-,$$

then

$$|\mathcal{L}(L_\varphi)(x)| \le C|x|^{-1} \to 0, \quad \text{as } x \to -\infty.$$

Therefore $(-z)^\alpha \mathcal{L}(\varphi)(z) = \mathcal{L}(L_\varphi)(z) = 0$, and then $\varphi = 0$ almost everywhere by injectivity of the Laplace transform. We have finished the proof of Theorem 3.1. □

4 Final remarks

For $x, \alpha \ge 0$, put $\mathcal{I}_x^\alpha := \{f \in \mathcal{T}_+^{(\alpha)}(t^\alpha) : \gamma(f) \ge x\}$. It is straightforward to show that the set $\mathcal{I}_x^\alpha$ is a (proper, when $x > 0$) closed ideal of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ and that $Z(\mathcal{I}_x) = \emptyset$. We call $\mathcal{I}_x^\alpha$ *standard* ideal.

Theorem 3.1 implies immediately the following corollary.

Corollary 4.1 *Let I be an ideal of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$. Then I is dense in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ if and only if $Z(I) = \emptyset$ and $\gamma(I) = 0$.*

Proof If $\xi \in Z(I)$ then by continuity $\xi \in Z(\overline{I})$ and so $\overline{I} \ne \mathcal{T}_+^{(\alpha)}(t^\alpha)$. If $\gamma(I) = x > 0$ then $\overline{I} \subseteq \mathcal{I}_x^\alpha$ because $\lim f_n = f$ in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ entails $\lim f_n = f$ in $L^1(\mathbb{R}^+)$, which in turn implies $\lim f_{n_k} = f$ a.e. for some subsequence (n_k). Conversely, $Z(I) = \emptyset$ and $\gamma(I) = 0$ imply respectively that $Z(\overline{I}) = \emptyset$ and $\gamma(\overline{I}) = 0$ since $I \subseteq \overline{I}$. By Theorem 3.1, $\overline{I} = \mathcal{T}_+^{(\alpha)}(t^\alpha)$. □

Let us now raise the following question.

Question Let I be a closed ideal of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ such that $Z(I) = \emptyset$. Does it imply that I is standard?

It is well known that the answer is positive for $\alpha = 0$, that is, in $L^1(\mathbb{R}^+)$. This can be seen by noticing that, if $x = \gamma(J)$, then $J_x := \delta_{-x} * J$ is a closed ideal of $L^1(\mathbb{R}^+)$ for which $\gamma(J_x) = 0$ and $Z(J_x) = \emptyset$. Hence, by the Nyman's theorem $J_x = L^1(\mathbb{R}^+)$. It follows that $J = \delta_x * J_x = \delta_x * L^1(\mathbb{R}^+) = \mathcal{I}_x^\alpha$.

The previous argument does not work for $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ with $\alpha > 0$. In fact, δ_x is not continuous on $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ for $x > 0$, and so $\delta_{-x} * I$ is not closed. Alternatively, what is underlying here is the fact that the argument works for $\alpha = 0$ because in this case $\mathcal{I}_x^0 = \delta_x * L^1(\mathbb{R}^+)$, whereas $\mathcal{I}_x^\alpha = [\delta_x * \mathcal{T}_+^{(\alpha)}(t^\alpha)] \cap \mathcal{T}_+^{(\alpha)}(t^\alpha)$ when $\alpha > 0$.

Thus we have that the non-stability of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ under the action of the Dirac masses δ_x implies troubles in such a simple question as the previous one. In other words, the non-invariance of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ under translations is a delicate matter which seems to deserve a closer look.

To conclude the paper, we point out that the following family $(R_t^{\alpha-1})_{t>0}$ of Riesz kernels is a suitable substitute of $(\delta_x)_{x>0}$ for the algebra $\mathcal{T}_+^{(\alpha)}(t^\alpha)$.

For $\beta > 0$, put

$$R_t^{\beta-1}(s) := \begin{cases} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} & \text{if } 0 \le s < t, \\ 0 & \text{if } s \ge t. \end{cases}$$

Then $R_t^{\beta-1}$ is in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$, with $W_+^\alpha R_t^{\beta-1} = R_t^{\beta-1-\alpha}$, if $\beta > \alpha > 0$. Moreover, $R_t^{\alpha-1} * f \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ and there exists $C_\alpha > 0$ such that $\nu_\alpha(R_t^{\alpha-1} * f) \le C_\alpha\, t^\alpha\, \nu_\alpha(f)$ for every $f \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ and $t > 0$. This fact means that $R_t^{\alpha-1}$ is a multiplier of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ for every $t > 0$. Also, the mapping $t \mapsto R_t^{\alpha-1} * f$, $[0, \infty) \to \mathcal{T}_+^{(\alpha)}(t^\alpha)$ is norm-continuous. For these properties, see [7, p. 17].

Then the convolution product in the algebra $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ can be expressed in terms of the Riesz kernels.

Lemma 4.2 *Let $\alpha > 0$. For all $f, g \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ we have*

$$f * g = \int_0^\infty W_+^\alpha f(t)\, R_t^{\alpha-1} * g\, dt,$$

in the norm of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$.

Proof Let $f \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$. For $s > 0$,

$$\begin{aligned} f(s) &= W_+^{-\alpha}(W_+^\alpha f)(s) \\ &= \frac{1}{\Gamma(\alpha)} \int_s^\infty (t-s)^{\alpha-1} W_+^\alpha f(t)\, dt = \int_0^\infty R_t^{\alpha-1}(s)\, W_+^\alpha f(t)\, dt. \end{aligned}$$

Hence, if $g \in C_c^\infty[0, \infty)$ and $t > 0$, we have

$$\begin{aligned} f * g(t) &= \int_0^\infty \left(\int_0^\infty R_u^{\alpha-1}(s)\, W_+^\alpha f(u)\, du \right) g(t-s)\, ds \\ &= \int_0^\infty W_+^\alpha f(u) \left(\int_0^\infty R_u^{\alpha-1}(s)\, g(t-s)\, ds \right) du \\ &= \int_0^\infty W_+^\alpha f(u)\, (R_u^{\alpha-1} * g)(t)\, du. \end{aligned}$$

Moreover,

$$\nu_\alpha(f * g) \le \int_0^\infty |W_+^\alpha f(t)|\, \nu_\alpha(R_t^{\alpha-1} * g)\, dt \le C_\alpha t^\alpha\, \nu_\alpha(g)\nu_\alpha(f)$$

by properties prior to the lemma. Finally we apply density. □

Now, the closed ideals of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ can be characterized as the closed subspaces of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ which are invariant under the action of the Riesz kernels $R_t^{\alpha-1}$, $t > 0$.

Proposition 4.3 *Let I be a closed subspace of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$. Then I is an ideal of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ if and only if*

$$R_t^{\alpha-1} * f \in I \quad \textit{for all } f \in I,\ t > 0.$$

Proof Suppose that I is a closed ideal of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$. Then I is a Banach module over $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ with respect to the convolution in $\mathbb{R}^+$. Further, $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ has a bounded approximate identity ([13, p. 36]) and therefore $\mathcal{T}_+^{(\alpha)}(t^\alpha) * I$ is dense in I. By Cohen's theorem ([5, p. 96]), we have $\mathcal{T}_+^{(\alpha)}(t^\alpha) * I = I$. Then the (Banach) module action of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ on I can be extended to the action of the Banach algebra $\mathrm{Mul}(\mathcal{T}_+^{(\alpha)}(t^\alpha))$ of multipliers of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ by defining $T(f) = T(g) * h$, for every $f \in I, T \in \mathrm{Mul}(\mathcal{T}_+^{(\alpha)}(t^\alpha))$, where $f = g * h$, for some $g \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ and $h \in I$ (see [5, Proposition 5.2]). Since the kernel $R_t^{\alpha-1}$ is a multiplier of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ we obtain that $R_t^{\alpha-1} * f \in I$ for any $f \in I$ and $t > 0$.

Conversely, if $R_t^{\alpha-1} * f \in I$ for every $f \in I$ and $t > 0$ then it is enough to apply Lemma 4.2 to derive that I is an ideal. □

In accordance with the above proposition, Theorem 3.1 can be expressed in terms of completeness of sets of functions in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$.

Corollary 4.4 *Let $\mathfrak{S}$ be a collection of functions in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$. Then the right α-times integrated translates $R_t^{\alpha-1} * f$, with $t \ge 0$ and $f \in \mathfrak{S}$, span a dense subspace of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ if and only if*

(i) *for each $z \in \mathbb{C}^+$ there exists an $f \in \mathfrak{S}$ with $\mathcal{L}f(z) \ne 0$, and*
(ii) *there is no interval $(0, \varepsilon)$, $\varepsilon > 0$, such that all functions in $\mathfrak{S}$ vanish a.e. on it.*

Proof It is enough to apply Theorem 3.1 and Proposition 4.3. □

Remark 4.5 There exist weighted convolution Banach algebras which are natural generalizations of the algebras $\mathcal{T}_+^{(\alpha)}(t^\alpha)$, $\alpha > 0$ (and of course *are not invariant under translations* either): Let ω be a positive, increasing, continuous submultiplicative weight on $[0, \infty)$. Put $\omega_\alpha(t) := t^\alpha \omega(t)$ for $t, \alpha > 0$. Let define the Banach space

$\mathcal{T}_+^{(\alpha)}(\omega_\alpha)$ as the completion of $C_c^\infty[0,\infty)$ in the norm

$$\nu_{\alpha,\omega}(f) := \int_0^\infty |W_+^\alpha f(t)|\,\omega_\alpha(t)\,dt.$$

This space is a Banach algebra for the convolution, and is continuously contained in $L^1(\mathbb{R}^+;\omega)$; see [7].

For a large class of weights, it is to be expected that the approach carried on in this paper, in combination with methods of the theory of weighted Banach algebras, can be helpful in order to look for Nyman type theorems and/or to study the primary ideals of the algebras $\mathcal{T}_+^{(\alpha)}(\omega_\alpha)$. See [3, 9] and references therein. The present paper must be understood as a first step in such a line of research.

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