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On some notions of convergence for n-tuples of operators

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The aim of this paper is to show that we can extend the notion of *convergence in the norm-resolvent sense* to the case of several unbounded noncommuting operators (and to quaternionic operators as a particular case) using the notion of *S*-resolvent operator. With this notion, we can define bounded functions of unbounded operators using the *S*-functional calculus for *n*-tuples of noncommuting operators. The same notion can be extended to the case of the *F*-resolvent operator, which is the basis of the *F*-functional calculus, a monogenic functional calculus for *n*-tuples of commuting operators. We also prove some properties of the *F*-functional calculus, which are of independent interest. Copyright © 2013 John Wiley & Sons, Ltd.

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1. Introduction

To introduce the problem we face in this paper, we recall the following classical facts that hold for bounded operators on a complex Banach space. Denote by $\{P_n\}_{n\in\mathbb{N}}$ a sequence of densely defined unbounded linear operators. The notion of convergence in the norm-resolvent sense (see, for example, [1, p. 284]) is the right tool to study the convergence of suitable functions of the sequence of operators $\{P_n\}_{n\in\mathbb{N}}$. In fact, let X be a real or complex Banach space and denote by $B(X)$ the space of all bounded linear operators endowed with its natural norm. We recall that if $P, P_n \in B(X)$, for all $n \in \mathbb{N}$, then under suitable conditions we have that $P_n \to P$ in the norm if and only if $R(\lambda, P_n) \to R(\lambda, P)$ strongly, where $R(\lambda, P)$ denotes the resolvent operator of P. This means that, in the case of unbounded operators, we can use the convergence in the norm resolvent sense to study the convergence of suitable functions f of operators $f(P_n) \to f(P)$. The functions $f(P)$ and $f(P_n)$ can be defined by the Riesz–Dunford functional calculus [2].

The aim of this paper is to show that we can extend the notion of *convergence in the norm-resolvent sense* to the case of several unbounded noncommuting operators using the S-resolvent operator $S^{-1}(s, T)$, which is the resolvent operator used to define the S-functional calculus, see the book [3] and the references therein. This new functional calculus works for unbounded noncommuting operators and for the set of slice monogenic functions [3], in particular of radially holomorphic functions [4]. It also works, with suitable modifications, for quaternionic operators thus providing a tool to form functions of quaternionic operators, as needed in quaternionic quantum mechanics, see [5]. Moreover, the same notion of convergence in the norm-resolvent sense can be given for the F-resolvent operator, which is the resolvent operator associated with the so-called F-functional calculus. This calculus is based on the Fueter mapping theorem in integral form and is a monogenic functional calculus [6]. It has been introduced in [7] and works for n-tuple of commuting operators. We recall that by the Fueter mapping theorem, a slice monogenic function f is transformed into a monogenic function $\breve{f}$ [8] by the formula $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$ where Δ is the Laplace operator in dimension $n+1$. The integral version of the Fueter mapping theorem is obtained by the Cauchy formula for slice monogenic functions, applying the operator $\Delta^{\frac{n-1}{2}}$ to the Cauchy kernel (for the Fueter mapping theorem, see [9–11]).

It is also worth to mention that the S-resolvent operator plays an important role in the definition of the quaternionic version of the counterpart of the operator $(I-zA)^{-1}$ and in the realization $s(z) = D + zC(I-zA)^{-1}B$ for Schur multipliers. In fact, when A is a quaternionic matrix and p is a quaternion, then $(I-pA)^{-1}$ has to be replaced by $p^{-1}S^{-1}(p^{-1}, A)$, where $S^{-1}(p^{-1}, A)$ is the right S-resolvent operator associated to A. Note that in this note, we consider the left S-resolvent operator, but the left and right resolvents share similar

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properties [3]. For Schur analysis in the slice hyperholomorphic setting, see the papers [12–14], and for an overview of Schur analysis, see [15, 16].

In order to provide some basic facts on the F-functional calculus and on the S-functional calculus, we need the following definitions.

Let $\mathbb{R}_n$ be the real Clifford algebra over n imaginary units $e_1, \dots, e_n$ satisfying the relations $e_ie_j + e_je_i = 0$, $i \neq j$, $e_i^2 = -1$. An element in the Clifford algebra will be denoted by $\sum_A e_A x_A$, where $A = \{i_1 \dots i_r\} \in \mathcal{P}\{1, 2, \dots, n\}$, $i_1 < \dots < i_r$ is a multi-index and $e_A = e_{i_1} e_{i_2} \dots e_{i_r}$, $e_\emptyset = 1$. An element $(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ will be identified with the element $x = x_0 + \underline{x} = x_0 + \sum_{j=1}^n x_j e_j \in \mathbb{R}_n$ called, in short, paravector, and the real part x_0 of x will also be denoted by $Re(x)$. The norm of $x \in \mathbb{R}^{n+1}$ is defined as $|x|^2 = x_0^2 + x_1^2 + \dots + x_n^2$. The conjugate of x is defined by $\bar{x} = x_0 - \underline{x} = x_0 - \sum_{j=1}^n x_j e_j$. Let $\mathbb{S} = \{\underline{x} = e_1 x_1 + \dots + e_n x_n \mid x_1^2 + \dots + x_n^2 = 1\}$; for $I \in \mathbb{S}$, we obviously have $I^2 = -1$. Given an element $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$, let us set $I_x = \underline{x}/|\underline{x}|$ if $\underline{x} \neq 0$, and given an element $x \in \mathbb{R}^{n+1}$, the set

$$[x] := \left\{ y \in \mathbb{R}^{n+1} \;:\; y = x_0 + I|\underline{x}|,\ I \in \mathbb{S} \right\}$$

is an $(n-1)$-dimensional sphere in $\mathbb{R}^{n+1}$. The vector space $\mathbb{R} + I\mathbb{R}$ passing through 1 and $I \in \mathbb{S}$ will be denoted by $\mathbb{C}_I$, and an element belonging to $\mathbb{C}_I$ will be indicated by $u + Iv$, for $u, v \in \mathbb{R}$. We now recall the definition of the class of functions for which the F-functional and the S-functional calculi apply.

Definition 1.1 (Slice monogenic functions)
Let $U \subseteq \mathbb{R}^{n+1}$ be an open set and let $f : U \to \mathbb{R}_n$ be a real differentiable function. Let $I \in \mathbb{S}$ and let f_I be the restriction of f to the complex plane $\mathbb{C}_I$. We say that f is a (left) slice monogenic function, or s-monogenic function, if for every $I \in \mathbb{S}$, we have

$$\frac{1}{2}\left(\frac{\partial}{\partial u} + I\frac{\partial}{\partial v}\right) f_I(u + Iv) = 0.$$

We will denote the space of slice monogenic function on an open set U as $\mathcal{SM}(U)$.

The domains on which slice monogenic functions have a Cauchy formula are defined in the succeeding text.

Definition 1.2
Let $U \subseteq \mathbb{R}^{n+1}$ be a domain, we say that

(i) U is an s-domain if $U \cap \mathbb{R}$ is nonempty and if $\mathbb{C}_I \cap U$ is a domain in $\mathbb{C}_I$ for all $I \in \mathbb{S}$.
(ii) U is axially symmetric if, for all $u + Iv \in U$, the whole $(n-1)$-sphere $[u + Iv]$ is contained in U.

The Cauchy formula for slice monogenic functions and the integral version of the Fueter mapping theorem are the main tools to define the S-functional calculus and the F-functional calculus for n-tuples of operators, respectively.
We now describe the functional setting in which we will work.

Let V be a real Banach space and denote by V_n the two-sided Banach $\mathbb{R}_n$–module $V \otimes \mathbb{R}_n$. An element v in V_n is of the type $\sum_A v_A \otimes e_A$ where A is a multi-index. We denote by $\mathcal{B}(V)$ the space of bounded $\mathbb{R}$-homomorphisms of the real Banach space V to itself endowed with the natural norm denoted by $\|\cdot\|_{\mathcal{B}(V)}$. We define $\|v\|_{V_n} = \sum_A \|v_A\|_V$, and we denote by $\mathcal{B}(V_n)$ the space of all bounded operators of the form $T = \sum_A T_{(A)} e_A$; the subset $\mathcal{B}^{0,1}(V_n)$ denotes the space of all bounded operators of the form $T = T_{(0)} + \sum_{j=1}^n T_{(j)} e_j$ where $T_{(j)} \in \mathcal{B}(V)$ for $j = 0, 1, \dots, n$. We define $\|T\|_{\mathcal{B}(V_n)} = \sum_A \|T_{(A)}\|_{\mathcal{B}(V)}$, and in particular, $\|T\|_{\mathcal{B}^{0,1}(V_n)} = \sum_{j=0}^n \|T_{(j)}\|_{\mathcal{B}(V)}$. When no confusion arises, we will write $\|T\|$ instead of $\|T\|_{\mathcal{B}(V_n)}$ or $\|T\|_{\mathcal{B}^{0,1}(V_n)}$.

The outline of the paper is the following.

In Section 2, we recall the F-functional calculus, and then we state and prove the main results on the converge in the F-resolvent sense. In a subsection, we also prove some properties of the F-functional calculus for bounded operators, and we introduce the F-resolvent equation. Finally, in Section 3 , we show the analogous result on the convergence in the S-resolvent sense for n-tuples on non-necessarily commuting operators.

2. The convergence in the F-resolvent sense

In this section, we will consider n-tuples of commuting operators, in paravector form, that is operators $T = T_{(0)} + \sum_{j=1}^n T_{(j)} e_j$ where $T_{(j)} \in \mathcal{B}(V)$ for $j = 0, 1, \dots, n$, and we will denote the set of these operators as $\mathcal{BC}^{0,1}(V_n)$. We define $\overline{T} = T_{(0)} - \sum_{j=1}^n T_{(j)} e_j$, and so $T + \overline{T} = 2T_0$ and $T\overline{T} = \overline{T}T = T_{(0)}^2 + \sum_{j=1}^n T_{(j)}^2$. The notion of *convergence in the F-resolvent sense* has been inspired by the F-functional calculus for n-tuples of commuting operators introduced in [7]. So in order to prove our main results, we need to recall some preliminaries on the F-functional calculus [7].

Definition 2.1 (The F-spectrum and the F-resolvent sets)
Let $T \in \mathcal{BC}^{0,1}(V_n)$. We define the F-spectrum $\sigma_F(T)$ of T as

$$\sigma_F(T) = \left\{ s \in \mathbb{R}^{n+1} \;:\; s^2\mathcal{I} - s\left(T + \overline{T}\right) + T\overline{T} \;\; \textit{is not invertible in} \;\; \mathcal{BC}(V_n) \right\}.$$

The F-resolvent set $\rho_F(T)$ is defined by

$$\rho_F(T) = \mathbb{R}^{n+1} \setminus \sigma_F(T).$$

We now recall two important properties if the F-spectrum for bounded n-tuples of commuting operators.

Theorem 2.2 (Compactness of the F-spectrum)
Let $T \in \mathcal{BC}^{0,1}(V_n)$. Then the F-spectrum $\sigma_F(T)$ is a compact nonempty set. Moreover, $\sigma_F(T)$ is contained in $\{s \in \mathbb{R}^{n+1} : |s| \leq \|T\| \}$.

Theorem 2.3 (Structure of the F-spectrum)
Let $T \in \mathcal{BC}^{0,1}(V_n)$ and let $p = p_0 + p_1 I \in [p_0 + p_1 I] \subset \mathbb{R}^{n+1} \setminus \mathbb{R}$, such that $p \in \sigma_F(T)$. Then all the elements of the $(n-1)$-sphere $[p_0 + p_1 I]$ belong to $\sigma_F(T)$. Thus, the F-spectrum consists of real points and/or $(n-1)$-spheres.

The notion of F-resolvent operator is inspired by the Fueter mapping theorem in integral form.

Definition 2.4 (F-resolvent operator)
Let n be an odd number, $s \in \mathbb{R}^{n+1}$ and let $T \in \mathcal{BC}^{0,1}(V_n)$. For $s \in \rho_F(T)$, we define the F-resolvent operator by

$$F_n(s,T) := \gamma_n \left(s\mathcal{I} - \overline{T}\right) \left(s^2\mathcal{I} - s\left(T + \overline{T}\right) + T\overline{T}\right)^{-\frac{n+1}{2}},$$

where the constants γ_n are given by

$$\gamma_n := (-1)^{(n-1)/2} 2^{(n-1)/2} (n-1)! \left(\frac{n-1}{2}\right)!.$$

Definition 2.5

- Let $T \in \mathcal{BC}^{0,1}(V_n)$ and let $U \subset \mathbb{R}^{n+1}$ be an axially symmetric s-domain that contains the F-spectrum $\sigma_F(T)$ and such that $\partial(U \cap \mathbb{C}_I)$ is union of a finite number of rectifiable Jordan curves for every $I \in \mathbb{S}$.
- Let W be an open set in $\mathbb{R}^{n+1}$. A function $f \in \mathcal{SM}(W)$ is said to be locally s-monogenic on $\sigma_F(T)$ if there exists a domain $U \subset \mathbb{R}^{n+1}$, as discussed earlier and such that $\overline{U} \subset W$, on which f is s-monogenic.
- We will denote by $\mathcal{SM}_{\sigma_F(T)}$ the set of locally s-monogenic functions on $\sigma_F(T)$.

The following result, together with the compactness of the spectrum (for bounded operators) and the axial symmetry of the F-spectrum, is crucial in order to define the F-functional calculus.

Theorem 2.6
Let n be an odd number, $T \in \mathcal{BC}^{0,1}(V_n)$, let $f \in \mathcal{SM}_{\sigma_F(T)}$ and $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$. Let U be an open set, containing $\sigma_F(T)$, as in Definition 2.5. Then, the integral

$$\frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} F_n(s,T)\, ds_I\, f(s), \quad ds_I = ds/I \tag{1}$$

is independent of $I \in \mathbb{S}$ and of the open set U.

Definition 2.7 (The F-functional calculus)
Let n be an odd number, $T \in \mathcal{BC}^{0,1}(V_n)$. Let U be an open set, containing $\sigma_F(T)$, as in Definition 2.5. Suppose that $f \in \mathcal{SM}_{\sigma_F(T)}$ and let $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$. We define the F-functional calculus as

$$\breve{f}(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} F_n(s,T)\, ds_I\, f(s).$$

Now, we can give the definition of convergence in the F-resolvent set.

Definition 2.8
Let n be an odd number. Let $\{T_m\}_{m\in\mathbb{N}}$ and T belong to $\mathcal{BC}^{0,1}(V_n)$, suppose that $\rho_F(T) = \rho_F(T_m)$ for all $m \in \mathbb{N}$. We say that T_m converges to T in the norm F-resolvent sense if $F_n(s,T_m) \to F_n(s,T)$ as $m \to \infty$ for all $s \in \rho_F(T)$.

Our main result is the following.

Theorem 2.9
Let n be an odd number. Let $\{T_m\}_{m\in\mathbb{N}}$ and T be elements in $\mathcal{BC}^{0,1}(V_n)$, suppose that $\rho_F(T) = \rho_F(T_m)$ for all $m \in \mathbb{N}$. Then $T_m \to T$ in the norm if and only if $T_m \to T$ in the norm F-resolvent sense.

Proof
Set the position

$$\mathcal{Q}_s(T) := \left(s^2\mathcal{I} - s\left(T + \overline{T}\right) + T\overline{T}\right)^{-1}, \quad s \in \rho_F(T),$$

with this notation, the F-resolvent operator can be written as

$$F_n(s,T) := \gamma_n \left(s\mathcal{I} - \overline{T}\right) \mathcal{Q}_s(T)^{\frac{n+1}{2}}.$$

First, we prove that if $T_m \to T$ in the norm then $T_m \to T$ in the norm F-resolvent sense. For all $s \in \rho_F(T)$, we consider the difference

$$\begin{aligned} F_n(s,T) - F_n(s,T_m) &= \gamma_n \left(s\mathcal{I} - \overline{T}\right) \mathcal{Q}_s(T)^{\frac{n+1}{2}} - \gamma_n \left(s\mathcal{I} - \overline{T_m}\right) \mathcal{Q}_s(T_m)^{\frac{n+1}{2}} \\ &= \gamma_n \left(s\mathcal{I} - \overline{T}\right) \mathcal{Q}_s(T)^{\frac{n+1}{2}} - \gamma_n \left(s\mathcal{I} - \overline{T_m}\right) \mathcal{Q}_s(T)^{\frac{n+1}{2}} \\ &\quad + \gamma_n \left(s\mathcal{I} - \overline{T_m}\right) \mathcal{Q}_s(T)^{\frac{n+1}{2}} - \gamma_n \left(s\mathcal{I} - \overline{T_m}\right) \mathcal{Q}_s(T_m)^{\frac{n+1}{2}} \\ &= \gamma_n \left(\overline{T_m} - \overline{T}\right) \mathcal{Q}_s(T)^{\frac{n+1}{2}} \\ &\quad - \gamma_n \left(s\mathcal{I} - \overline{T_m}\right) \left[\mathcal{Q}_s(T)^{\frac{n+1}{2}} - \mathcal{Q}_s(T_m)^{\frac{n+1}{2}}\right]. \end{aligned}$$

Take the norm

$$\begin{aligned} \|F_n(s,T) - F_n(s,T_m)\| \le |\gamma_n| \Big[&\|\mathcal{Q}_s(T)^{\frac{n+1}{2}}\| \, \|T - T_m\| \\ &+ \|\mathcal{Q}_s(T)^{\frac{n+1}{2}} - \mathcal{Q}_s(T_m)^{\frac{n+1}{2}}\| \, \|T_m - \bar{s}\mathcal{I}\| \Big] \end{aligned}$$

and observe that

$$\begin{aligned} \mathcal{Q}_s(T) - \mathcal{Q}_s(T_m) &= \mathcal{Q}_s(T_m) \left(s\left((\overline{T} - \overline{T}_m) + (T - T_m)\right) + T\overline{T} - T_m\overline{T}_m\right) \mathcal{Q}_s(T) \\ &= \mathcal{Q}_s(T_m) \left(s\left((\overline{T} - \overline{T}_m) + (T - T_m)\right) + T\left(\overline{T} - \overline{T}\right)\right. \\ &\quad \left. + (T - T_m)\overline{T}_m\right) \mathcal{Q}_s(T). \end{aligned}$$

To show that, for m large, there exists a positive constant C_s such that $\|Q_s(T_m)\| \le C_s$, we consider the aforementioned relation written as

$$\mathcal{Q}_s(T) = \mathcal{Q}_s(T_m) + \mathcal{Q}_s(T_m) \left(s\left((\overline{T} - \overline{T}_m) + (T - T_m)\right) + T\overline{T} - T_m\overline{T}_m\right) \mathcal{Q}_s(T),$$

and also

$$\mathcal{Q}_s(T) = \mathcal{Q}_s(T_m) \left[\mathcal{I} - \left(s\left((\overline{T}_m - \overline{T}) + (T_m - T)\right) + T_m\overline{T}_m - T\overline{T}\right) \mathcal{Q}_s(T)\right].$$

Now, we set

$$G_m := \left(s\left((\overline{T}_m - \overline{T}) + (T_m - T)\right) + T_m\overline{T}_m - T\overline{T}\right) \mathcal{Q}_s(T)$$

and we recall that if m is sufficiently large, then $\|G_m\| < 1$, and the operator $(\mathcal{I} - G_m)^{-1}$ exists and is continuous. From

$$\mathcal{Q}_s(T_m) = \mathcal{Q}_s(T)(\mathcal{I} - G_m)^{-1} \tag{2}$$

we deduce that $\mathcal{Q}_s(T_m)$ is a bounded operator for m large. This means that

$$\begin{aligned} \|\mathcal{Q}_s(T) - \mathcal{Q}_s(T_m)\| \le \|\mathcal{Q}_s(T)(\mathcal{I} - G_m)^{-1}\| \Big(&|s| \left(\|\overline{T} - \overline{T}_m\| + \|T - T_m\|\right) \\ &+ \|T\|(\|\overline{T} - \overline{T}\| + \|T - T_m\|\|\overline{T}_m\|\Big) \|\mathcal{Q}_s(T)\| \end{aligned}$$

so, there exists a positive constant $C(s)$ that does not depend on m, such that

$$\|\mathcal{Q}_s(T) - \mathcal{Q}_s(T_m)\| \le C(s)\|T - T_m\|. \tag{3}$$

Observe now that $h(n) := \frac{n+1}{2}$ is a natural number because n is odd. So we write

$$\begin{aligned} \mathcal{Q}_s(T)^h - \mathcal{Q}_s(T_m)^h &= \mathcal{Q}_s(T)^h - \mathcal{Q}_s(T)^{h-1}\mathcal{Q}_s(T_m) + \mathcal{Q}_s(T)^{h-1}\mathcal{Q}_s(T_m) - \mathcal{Q}_s(T_m)^h \\ &= \mathcal{Q}_s(T)^{h-1}(\mathcal{Q}_s(T) - \mathcal{Q}_s(T_m)) + \mathcal{Q}_s(T)^{h-1}(\mathcal{Q}_s(T_m) - \mathcal{Q}_s(T_m)). \end{aligned}$$

By recurrence, there exists a positive constant $K_n(s)$ that depends on s and n such that

$$\|\mathcal{Q}_s(T)^{h(n)} - \mathcal{Q}_s(T_m)^{h(n)}\| \le K_n(s)\|T - T_m\|$$

and so, for $\|T - T_m\| \to 0$, we have $\|F_n(s,T) - F_n(s,T_m)\| \to 0$, and the first part of the statement follows.
Conversely, suppose that $\|F_n(s,T) - F_n(s,T_m)\| \to 0$ for all $s \in \rho_F(T)$; we have to show that $\|T - T_m\| \to 0$.

For $\ell \in \mathbb{N}$, we set

$$\mathcal{P}_{\ell,n}(x) := \Delta^{\frac{n-1}{2}} x^\ell. \tag{4}$$

Thanks to Theorem 4.19 in [7] if n is an odd number and $T \in \mathcal{BC}^{0,1}(V_n)$, then

$$\mathcal{P}_{\ell,n}(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} F_n(s,T)\, ds_I\, s^{\ell}$$

and the integral does not depend on the open set U nor on $I \in \mathbb{S}$. Because the monogenic polynomials (4) are homogeneous of degree n, we can choose $\tilde{\ell} = \tilde{\ell}(n)$ that depends on n, such that there exists a positive constant $C_{\tilde{\ell}(n)}$ so that

$$\|T - T_m\| \le C_{\tilde{\ell}(n)} \|\mathcal{P}_{\tilde{\ell},n}(T) - \mathcal{P}_{\tilde{\ell},n}(T_m)\|.$$

We observe that

$$\mathcal{P}_{\tilde{\ell},n}(T) - \mathcal{P}_{\tilde{\ell},n}(T_m) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} [F_n(s,T) - F_n(s,T_m)]\, ds_I\, s^{\tilde{\ell}}.$$

We know that the F-spectrum is a compact and nonvoid set for bounded operators, and the function $f(s) = s^{\tilde{\ell}}$ is continuous on every closed Jordan curve $\partial(U \cap \mathbb{C}_I)$ for $I \in \mathbb{S}$, so there exists a positive constant K_n that does not depend on m, such that

$$\begin{aligned}\|T - T_m\| &\le C_{\tilde{\ell}(n)} \|\mathcal{P}_{\tilde{\ell},n}(T) - \mathcal{P}_{\tilde{\ell},n}(T_m)\| \\ &\le C_{\tilde{\ell}(n)} K_n \max_{I\in\mathbb{S}} \max_{s\in\partial(U\cap\mathbb{C}_I)} \|F_n(s,T) - F_n(s,T_m)\|,\end{aligned}$$

and this concludes the proof. □

Theorem 2.10
Let n be an odd number. Let $\{T_m\}_{m\in\mathbb{N}}$ and T be elements in $\mathcal{BC}^{0,1}(V_n)$; suppose that $\rho_F(T) = \rho_F(T_m)$ for all $m \in \mathbb{N}$. Suppose that

$$\|F_n(s,T) - F_n(s,T_m)\| \to 0 \quad \text{for} \quad m \to \infty,$$

for all $s \in \rho_F(T)$. If $f \in \mathcal{SM}_{\sigma_F(T)}$, then

$$\|\breve{f}(T) - \breve{f}(T_m)\| \to 0 \ \text{ for } \ m \to \infty.$$

Proof
Because

$$\breve{f}(T) - \breve{f}(T_m) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} [F_n(s,T) - F_n(s,T_m)]\, ds_I\, f(s)$$

the argument used in the last part of the proof of the previous theorem gives the statement. □

We conclude this section with some properties of the F-functional calculus for bounded operators.

2.1. *Some properties of the F-functional for bounded operators*

Here we prove some properties of the F-functional calculus. The following results are easy to verify

Proposition 2.11
Let n be an odd number. Let f and $g \in \mathcal{SM}_{\sigma_F(T)}$ and let $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$ and $\breve{g}(x) = \Delta^{\frac{n-1}{2}} g(x)$. Then, we have

$$\left(\breve{f} + \breve{g}\right)(T) = \breve{f}(T) + \breve{g}(T), \qquad \left(\breve{f}\lambda\right)(T) = \breve{f}(T)\lambda, \qquad \textit{for all } \ \lambda \in \mathbb{R}_n.$$

Proof
The equalities follow directly from the definition. □

Proposition 2.12
Let n be an odd number and let $T \in \mathcal{BC}(V_n)$. Let $f(s) = \sum_{\ell\ge 0} s^{\ell} a_{\ell}$ where $a_{\ell} \in \mathbb{R}_n$ be such that $f \in \mathcal{SM}_{\sigma_F(T)}$ and let $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$. Then, we have

$$\breve{f}(T) = \sum_{\ell\ge 0} \mathcal{P}_{\ell,n}(T) a_{\ell}.$$

where $\mathcal{P}_{\ell,n}(T)$ has been obtained by replacing x by T in the polynomials$\mathcal{P}_{\ell,n}(x) := \Delta^{\frac{n-1}{2}} x^{\ell}$.

Proof
The series $\sum_{\ell\geq 0} s^\ell a_\ell$ converges in a ball $B(0,R)$, for suitable $R>0$, that contains $\sigma_F(T)$. So we can choose a ball $B_\varepsilon := \{\, s\,:\, |s| \leq \|T\| + \varepsilon\,\}$ for a sufficiently small $\varepsilon > 0$ such that $B_\varepsilon \subset B(0,R)$. Because the series converges uniformly on ∂B_ε, we have

$$\begin{aligned}\breve{f}(T) &= \frac{1}{2\pi}\int_{\partial(B_\varepsilon\cap\mathcal{C}_I)} F_n(s,T)\, ds_I \sum_{\ell\geq 0} s^\ell a_\ell\\ &= \frac{1}{2\pi}\sum_{\ell\geq 0}\int_{\partial(B_\varepsilon\cap\mathbb{C}_I)} F_n(s,T)\, ds_I\, s^\ell a_\ell\\ &= \frac{1}{2\pi}\sum_{\ell\geq 0}\int_{\partial(B_\varepsilon\cap\mathbb{C}_I)} \sum_{k\geq n-1}\mathcal{P}_{k,n}(T)s^{-1-k}\, ds_I\, s^\ell a_\ell\\ &= \sum_{\ell\geq 0}\mathcal{P}_{\ell,n}(T)a_\ell,\end{aligned}$$

here, we have used the fact that the F-resolvent operator $F_n(s,T)$ admits the power series expansion $\sum_{\ell\geq n-1}\mathcal{P}_{k,n}(T)s^{-1-k}$, and it converges on $\partial(B_\varepsilon)$. □

Theorem 2.13 (Continuity)
Let n be an odd number and let $T\in\mathcal{BC}(V_n)$. Let $f_m\in\mathcal{SM}_{\sigma_F(T)}$, $m\in\mathbb{N}$ and let $W\supset\sigma_F(T)$ be a domain as in Definition 2.5. Suppose that f_m converges uniformly to f on $W\cap\mathbb{C}_I$, for some $I\in\mathbb{S}$, then $f_m(T)$ converges to $f(T)$ in $\mathcal{BC}(V_n)$.

Proof
Let U be a axially symmetric s-domain such that $\overline{U}\subset W$ and assume that $\partial(U\cap\mathbb{C}_I)$ consists of a finite number of rectifiable Jordan arcs. Then $f_m\to f$ converges uniformly on $\partial(U\cap\mathbb{C}_I)$, and consequently, the sequence whose elements are

$$\breve{f}_m(T) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} F_n(s,T)\, ds_I\, f_m(s)$$

converges, in the uniform topology of operators, to the operator

$$\breve{f}(T) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} F_n(s,T)\, ds_I\, f(s).$$

□

We conclude this section with the equation for the F-resolvent operator.

Theorem 2.14 (The F-resolvent equation)
Let n be an odd number and let $T\in\mathcal{BC}(V_n)$. Let $s\in\rho_F(T)$ then $F_n(s,T)$ satisfies the equation

$$F_n(s,T)s - TF_n(s,T) = \gamma_n\mathcal{Q}_s(T)^{\frac{n-1}{2}}. \tag{5}$$

Proof
By definition,

$$F_n(s,T) := \gamma_n\left(s\mathcal{I}-\overline{T}\right)\mathcal{Q}_s(T)^{\frac{n+1}{2}},$$

where

$$\mathcal{Q}_s(T) := \left(s^2\mathcal{I} - s\left(T+\overline{T}\right) + T\overline{T}\right)^{-1},\quad s\in\rho_F(T)$$

so

$$F_n(s,T)s := \gamma_n\left(s\mathcal{I}-\overline{T}\right)s\mathcal{Q}_s(T)^{\frac{n+1}{2}}$$

and

$$TF_n(s,T) := \gamma_n\left(Ts-T\overline{T}\right)\mathcal{Q}_s(T)^{\frac{n+1}{2}},$$

taking the difference, we obtain

$$F_n(s,T)s - TF_n(s,T) = \gamma_n\left(s^2\mathcal{I} - s\left(T+\overline{T}\right) + T\overline{T}\right)\mathcal{Q}_s(T)^{\frac{n+1}{2}}$$

so that we get the statement. □

Remark 2.15
In the case $n = 1$, we have to interpret $F_n(s, T)$ as

$$S_C^{-1}(s,T) = \left(s\mathcal{I} - \overline{T}\right) \mathcal{Q}_s(T)$$

but the $S_C^{-1}(s,T)$ resolvent operator (for more details, see [17]) is associated with an n-tuple of commuting operators, and we obtain the S_C- resolvent equation

$$S_C^{-1}(s,T)s - TS_C^{-1}(s,T) = \mathcal{I},$$

where

$$S_C^{-1}(s,T) := \left(s\mathcal{I} - \overline{T}\right)\left(s^2\mathcal{I} - s\left(T + \overline{T}\right) + T\overline{T}\right)^{-1}$$

because $\gamma_1 \mathcal{Q}_s(T)^0 = \mathcal{I}$.

3. The convergence in the *S*-resolvent sense

The *S*-functional calculus is based on the notions of *S*-spectrum and of *S*-resolvent operator. We recall them for the sake of completeness.

Definition 3.1 (The *S*-spectrum and the *S*-resolvent set)
Let $T \in \mathcal{B}^{0,1}(V_n)$ and $s \in \mathbb{R}^{n+1}$. We define the *S*-spectrum $\sigma_S(T)$ of T as

$$\sigma_S(T) = \{s \in \mathbb{R}^{n+1} \ : \ T^2 - 2\, s_0 T + |s|^2\mathcal{I} \ \textit{ is not invertible in } \ \mathcal{B}(V_n)\}.$$

The *S*-resolvent set $\rho_S(T)$ is defined by

$$\rho_S(T) = \mathbb{R}^{n+1} \setminus \sigma_S(T).$$

Definition 3.2 (The *S*-resolvent operator)
Let $T \in \mathcal{B}^{0,1}(V_n)$ and $s \in \rho_S(T)$. We define the *S*-resolvent operator as

$$S^{-1}(s,T) := -\left(T^2 - 2s_0T + |s|^2\mathcal{I}\right)^{-1}(T - \bar{s}\mathcal{I}).$$

Definition 3.3
We say that $U \subset \mathbb{R}^{n+1}$ is an admissible set if

(i) it is an axially symmetric s-domain that contains the *S*-spectrum $\sigma_S(T)$ of $T \in \mathcal{B}_n^{0,1}(V_n)$,
(ii) $\partial(U \cap \mathbb{C}_I)$ is a union of a finite number of rectifiable Jordan curves for every $I \in \mathbb{S}$.

Definition 3.4 (Locally s-monogenic on $\sigma_S(T)$)
Suppose that U is admissible and $\overline{U}$ is contained in a domain of s-monogenicity of a function f.

(a) Then such a function f is said to be locally s-monogenic on $\sigma_S(T)$.
(b) We will denote by $\mathcal{M}_{\sigma_S(T)}$ the set of locally s-monogenic functions on $\sigma_S(T)$.

Definition 3.5 (The *S*-functional calculus)
Let $T \in \mathcal{B}^{0,1}(V_n)$ and $f \in \mathcal{M}_{\sigma_S(T)}$. Let $U \subset \mathbb{R}^{n+1}$ be an admissible set and let $ds_I = ds/I$ for $I \in \mathbb{S}$. We define

$$f(T) := \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S^{-1}(s,T)\, ds_I\, f(s).$$

The operator $f(T)$ is well defined because the integral does not depend on U nor on $I \in \mathbb{S}$. In particular for $h \in \mathbb{N}$, we have

$$T^h = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S^{-1}(s,T)\, ds_I\, s^h.$$

Definition 3.6
Let $\{T_m\}_{m\in\mathbb{N}}$ and T belong to $\mathcal{B}^{0,1}(V_n)$, suppose that $\rho_S(T) = \rho_S(T_m)$ for all $m \in \mathbb{N}$. We say that T_m converges to T in the norm S-resolvent sense if $S^{-1}(s,T_m) \to S^{-1}(s,T)$ as $m \to \infty$, for all $s \in \rho_S(T)$.

Another main result is the following:

Theorem 3.7
Let $T_m, m \in \mathbb{N}$ and T be elements in $\mathcal{B}^{0,1}(V_n)$; suppose that $\rho_S(T) = \rho_S(T_m)$ for all $m \in \mathbb{N}$. Then $T_m \to T$ in the norm if and only if $T_m \to T$ in the norm S-resolvent sense.

Proof
Because the proof is similar to the case of the F-resolvent, we simply show the main points. First, we prove that if $T_m \to T$ in the norm, then $T_m \to T$ in the norm S-resolvent sense. For $s \in \rho_S(T)$, we will use, for the sake of simplicity, the operator defined by

$$Q_s(T) := \left(T^2 - 2s_0 T + |s|^2 \mathcal{I}\right)^{-1}.$$

Using this notation, we get

$$\begin{aligned} S^{-1}(s,T) - S^{-1}(s,T_m) = & - Q_s(T)\,((T - \bar{s}\mathcal{I}) \\ & - (T_m - \bar{s}\mathcal{I})) - (Q_s(T) - Q_s(T_m))\,(T_m - \bar{s}\mathcal{I}). \end{aligned}$$

Take the norm

$$\|S^{-1}(s,T) - S^{-1}(s,T_m)\| \leq \|Q_s(T)\|\,\|T - T_m\| + \|Q_s(T) - Q_s(T_m)\|\,\|T_m - \bar{s}\mathcal{I}\| \tag{6}$$

and observe that

$$\begin{aligned} Q_s(T) - Q_s(T_m) &= Q_s(T_m)\left(T_m^2 - 2s_0 T_m - T^2 + 2s_0 T\right) Q_s(T) \\ &= Q_s(T_m)\,(T_m(T_m - T) + (T_m - T)T + 2s_0(T - T_m))\,Q_s(T). \end{aligned} \tag{7}$$

Now, we have to show that, for m large, there exists a positive constant C_s such that $\|Q_s(T_m)\| \leq C_s$. We recall that if A is a bounded operator and if $\|A\| < 1$, then the operator $(\mathcal{I} - A)^{-1}$ exists and is continuous. If we set

$$A := \left(T^2 - 2s_0 T - T_m^2 + 2s_0 T_m\right) Q_s(T),$$

then we have that the following identity

$$Q_s(T_m) = Q_s(T)\left[\mathcal{I} - \left(T^2 - 2s_0 T - T_m^2 + 2s_0 T_m\right) Q_s(T)\right]^{-1} \tag{8}$$

holds, for m large, because $\|\left(T^2 - 2s_0 T - T_m^2 + 2s_0 T_m\right) Q_s(T)\| \to 0$ as $m \to \infty$. From (6), taking into account (7) and (8), we get the first part of the statement.

Conversely, suppose that $\|S^{-1}(s,T) - S^{-1}(s,T_m)\| \to 0$ for all $s \in \rho_S(T)$; we have to show that $\|T - T_m\| \to 0$. We observe that

$$T - T_m = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} [S^{-1}(s,T) - S^{-1}(s,T_m)]\, ds_I\, s.$$

We know that the S-spectrum is a compact and nonvoid set for bounded operators [3], the function $f(s) = s$ is continuous on every closed Jordan curve $\partial(U \cap \mathbb{C}_I)$ for $I \in \mathbb{S}$, so there exists a positive constant K such that

$$\|T - T_m\| \leq K \max_{I \in \mathbb{S}} \max_{s \in \partial(U \cap \mathbb{C}_I)} \|S^{-1}(s,T) - S^{-1}(s,T_m)\|,$$

and this concludes the proof. □

Theorem 3.8
Let $\{T_m\}_{m \in \mathbb{N}}$ and T be elements in $\mathcal{B}^{0,1}(V_n)$; suppose that $\rho_S(T) = \rho_S(T_m)$ for all $m \in \mathbb{N}$. Suppose that

$$\|S^{-1}(s,T) - S^{-1}(s,T_m)\| \to 0 \quad \text{for} \quad m \to \infty,$$

for all $s \in \rho_S(T)$. If $f \in \mathcal{SM}_{\sigma_S(T)}$, then

$$\|f(T) - f(T_m)\| \to 0 \ \text{ for } \ m \to \infty.$$

Proof
The statement can be proved as Theorem 2.10. □

In the quaternionic case, Theorem 3.7 turns out to be as follows:

Theorem 3.9
Let $\{H_m\}_{m \in \mathbb{N}}$ and H be bounded quaternionic operators on a quaternionic two-sided Banach space and suppose that $\rho_S(H) = \rho_S(H_m)$, for all $m \in \mathbb{N}$. Then $H_m \to H$ in the norm if and only if $H_m \to H$ in the norm S-resolvent sense.

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