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About some numerical approaches for mixed integral equations

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ABSTRACT

In this paper we tackle on mixed Volterra–Fredholm integral equations, as in linear as in non linear cases. To the aim to obtain numerical solutions of these models, the authors propose in the linear case the direct collocation method using a p-order quasi interpolating spline class and in the nonlinear case the fixed point method based on polynomial approximation built by Schauder tensor bases. The advantages of both methods are outlined and their convergence is studied. Numerical results confirm the theoretical statements.

1. Introduction

Integral Volterra–Fredholm equations of first and second kind are models for many real problems in different areas of science as mathematical physics, biology, economy and mechanics. For example, particular models describe the development of an epidemic or contact problems in mechanics of continuous.

In particular mixed Volterra–Fredholm models are related to spatio/temporal situations, where the Fredholm component is considered in the space, while the integral Volterra component is considered in the time.

Many references can be found in [1].

In this work we consider the following mixed Volterra–Fredholm integral models:

$$\lambda u(x) = f(x) + \int_a^x k_1(x,y,u(y))dy + \int_a^b k_2(x,y,u(y))dy, \quad x \in [a,b], \tag{1}$$

where $\lambda \in \mathbb{R}\setminus\{0\}, f : [a,b] \to \mathbb{R}, k_1, k_2 : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ and $u : [a,b] \to \mathbb{R}$, with f and k_i $(i = 1,2)$ known functions and u an unknown function satisfying (1).

To be more precise, let $f : [a,b] \to \mathbb{R}$ be a continuous function and let $k_1,\ k_2 : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ be continuous functions satisfying a global Lipschitz condition at their third variables, i.e., there exist $M_1,\ M_2 > 0$ such that

$$\left.\begin{array}{l} x,y \in [a,b] \\ v,w \in \mathbb{R} \\ j = 1,2 \end{array}\right\} \Rightarrow |k_j(x,y,v) - k_j(x,y,w)| \leqslant M_j|v-w| \tag{2}$$

and $|\lambda| > (M_1 + M_2)(b-a)$.

Our aim is to provide numerical methods which approximate a continuous function $u : [a,b] \to \mathbb{R}$ solution of (1).

The linear case of (1),

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$$\lambda u(x) = f(x) + \int_a^x h_1(x,y)u(y)dy + \int_a^b h_2(x,y)u(y)dy, \quad x \in [a,b], \tag{3}$$

has been numerically dealt in a recent work [2]. Specifically in [2], taking into account the peculiarity of the mixed integral equation, we tried to afford the solution of (3) through a direct as well as an iterative numerical method, respectively: collocation and fixed point method. Both kinds of methods were proposed for a particular class of approximating functions. Namely, the first is a collocation method based on a linear spline class approximation; the second one is a fixed point method built on Schauder piecewise linear bases.

In [2] we have also outlined pro and con of both methods and some open problems. One of them experimentally arises: by increasing in both methods respectively the evaluation and iteration numbers, the result's precision does not improve. Surely predictable conditioning problems of the collocation method and algorithmic instability related to fixed point method appear. Furthermore, in [2] we underlined that the linearity of the model is a restriction to our proposals.

In this work we tackle on above mentioned problems using suitable modification and generalization of two methods in [2]:

(i) To overcome the first problem we propose for the model (3) a particular collocation method based on p-order ($p > 2$) quasi interpolating (q.i.) spline approximation. The method allows to improve the numerical solution precision while maintaining a low dimension of collocation system.
(ii) To effectively face the nonlinear problem we propose an iterative method of fixed point based on tensor Schauder bases.

The paper is organized as it follows: the first Subsection of Section 2 summarizes the collocation method described in [2] and analyses the related conditioning number, the second Subsection of Section 2 describes and analyses the collocation method based on the q.i. spline of order $p \geqq q2$; Section 3 states the fixed point method related to Schauder bases for the nonlinear case; finally, Section 4 presents some numerical results for both methods to confirm the theoretical statements.

2. P-order spline direct method

Let us consider the linear model (3).

2.1. Conditioning number related to spline linear collocation method

In this section we summarize the genesis and peculiarity of the direct method in [2] built on linear B-spline approximation, the so called Variation-Diminishing Schoenberg (VDS) splines and we analyse the related conditioning problem.

Firstly, let us recall some background on linear VDS splines (see for example [2]).

Let $n > 1$ and $\Pi_n := \{a = t_1 = t_2 < t_3 < \ldots < t_n < t_{n+1} = t_{n+2} = b\}$ be a partition of the interval $[a,b]$ with $H_n := \max_{2 \leqslant j \leqslant n}(t_{j+1} - t_j), H_n \to 0$ as $n \to \infty$.

Π_n is assumed as mesh of the set of normalized B-splines $B_{i,p}$ $(i = 1,2,\ldots,n)$ of order $p = 2$ defined by the following relation:

$$B_{i,2}(x) = \frac{x - t_i}{t_{i+1} - t_i} B_{i,1}(x) + \frac{t_{i+2} - x}{t_{i+2} - t_{i+1}} B_{i+1,1}(x), \tag{4}$$

$$B_{i,1}(x) = \begin{cases} 1, & t_i \leqslant x < t_{i+1}, \\ 0, & \text{otherwise} \end{cases}$$

according that, for $i = 1$ and $i = n$, the second and the first term in (4) hold respectively.

The set of $B_{i,2}(x)$ $(i = 1,2,\ldots,n)$ can be considered as basis of S_{2,Π_n} (order 2 spline space, associated to Π_n). In particular, the VDS splines $\mathcal{S}_n g \in S_{2,\Pi_n}$, with $g \in C[a,b]$, are defined as:

$$\mathcal{S}_n g := \sum_{i=1}^{n} g(t_{i+1}) B_{i,2}(x). \tag{5}$$

$\mathcal{S}_n$ is a projector operator that reproduces exactly a polynomial of first degree ($\mathcal{S}_n P = P,\ P \in \mathcal{P}_2$) and also all linear spline functions ($\mathcal{S}_n S = S,\ S \in S_{2,\Pi_n}$) [3].

Rewriting (3) as

$$[\lambda I + \widetilde{\mathcal{K}}]u = f, \tag{6}$$

where:

-I is the identity operator in $C[a,b]$

-$\widetilde{\mathcal{K}}$ is the following bounded and compact operator:

$$\widetilde{\mathcal{K}} g = \widetilde{\mathcal{K}}_1 g + \mathcal{K}_2 g.$$

-$\widetilde{\mathcal{K}}_1$ and $\mathcal{K}_2$ are defined as follows:

$$\widetilde{\mathcal{K}}_1 g(x) := -\int_a^b \widetilde{h}_1(x,y)g(y)dy, \quad x \in [a,b],$$
$$\mathcal{K}_2 g(x) := -\int_a^b h_2(x,y)g(y)dy, \quad x \in [a,b]$$

and

$$\widetilde{h}_1(x,s) := \begin{cases} h_1(x,s), & \text{if} \quad a \leqslant s \leqslant x \\ 0, & \text{if} \quad s > x \end{cases},$$

then the collocation system on a set of distinct collocation points ξ_k $(k = 1,2,\ldots,n)$, chosen in (a,b), is the following one

$$[\lambda I + \widetilde{\mathcal{K}}]w_n(\xi_k) = f(\xi_k), \quad k = 1,2,\ldots,n \tag{7}$$

being

$$w_n(x) = \sum_{i=1}^{n} \bar{u}_i B_{i,2}(x)$$

and the $\bar{u}_i$ are the approximated values of function u in t_{i+1} $(i = 1,2,\ldots,n)$.

Using the norm of operator as in [5] it follows:

Theorem 1. *Denoting by A the coefficients matrix in system (7) it follows that*

$$cond(A) \leqslant \left(1 + 2\left[\frac{n+1}{2}\right]\right)||(\lambda I + \widetilde{\mathcal{K}})^{-1}||(\lambda + ||\widetilde{\mathcal{K}}||). \tag{8}$$

Proof. System (7) can be rewritten as

$$\mathcal{S}_n(\lambda I + \widetilde{\mathcal{K}})w_n = \mathcal{S}_n f.$$

Taking into account the $\mathcal{S}_n$ operator peculiarities and easily adapting some results in [5], the following relation holds:

$$||A^{-1}|| \leqslant ||\mathcal{S}_n||\left|\left|(\lambda I + \mathcal{S}_n\widetilde{\mathcal{K}})^{-1}\right|\right||\Gamma_n^{-1}||,$$

where Γ_n is the following matrix:

$$\Gamma_n = \left[B_{i,2}(\xi_k)\right]_{i,k=1}^{n}$$

with ξ_k belonging to the interval (a,b).

Since for n sufficiently large $(\lambda I + \mathcal{S}_n\widetilde{\mathcal{K}}) \to (\lambda I + \widetilde{\mathcal{K}})$, the original problem and its approximating equation are approximately the same in their conditioning. So it is significant to analyse the behavior of $||\mathcal{S}_n||$ and $||\Gamma_n^{-1}||$.

Precisely it follows that $||\mathcal{S}_n|| = 1$ and assuming for a given matrix $M \in \mathbb{R}^{n\times n}$ the row norm so defined

$$||M||t := \max_{1\leqslant i\leqslant n}\sum_{j=1}^{n}|m_{i,j}|,$$

it can be easily shown that

$$||\Gamma_n|| = 1, \quad ||\Gamma_n^{-1}|| = 1 + 2\left[\frac{n+1}{2}\right].$$

Moreover it follows that

$$||A|| \leqslant (\lambda + ||\widetilde{\mathcal{K}}||),$$

where

$$||\widetilde{\mathcal{K}}|| := \max_{a\leqslant t\leqslant b}\int_a^b \left|(\widetilde{h}_1(t,s) + h_2(t,s))\right|ds.$$

So it follows that

$$cond(A) \leqslant ||A^{-1}||||A|| \leqslant \left(1 + 2\left[\frac{n+1}{2}\right]\right)||(\lambda I + \widetilde{\mathcal{K}})^{-1}||(\lambda + ||\widetilde{\mathcal{K}}||). \quad \blacksquare$$

Consequently, an obvious dependence of the conditioning number on system dimension arises.

2.2. Direct model spline cubic approximation

Taking into account (8) and experimental results in [2], we intend to improve the method's precision with lower dimension of collocation system. Hereafter we present the numerical model suitable for (3) based on the collocation method using approximating splines, in particular the so called modified q.i. splines of order $p > 2$ (see for example [8,10]). In the following subsection we recall the definitions and peculiarities of p-order *q.i.* spline bases.

2.2.1. About p-order q.i. B-spline

Let $X_m := \{x_0 = a < x_1 < \ldots < x_m < x_{m+1} = b\}$ be a partition of the interval $[a,b]$ with $H_m := \max_{0\leqslant j\leqslant m}(x_{j+1} - x_j)$, $H_m \to 0$ as $m \to \infty$ and let $\{d_j : j = 0,1,\ldots,m+1\}$ be a vector of positive integers where $d_0 = d_{m+1} = p$ and $d_j \leqslant p-1$, $j = 1,2,\ldots,m$.

We set $n + p := \sum_{j=0}^{m+1} d_j$ and define $\Pi_n = \{t_i : i = 1,2,\ldots,n+p\}$ as the nondecreasing sequence obtained from X_m by repeating x_j exactly d_j times, $j = 0,1,\ldots,m+1$.

The t_i are assumed as nodes of the locally uniform spline space [3]:

$S_{p,\Pi_n} := \{g : g|(x_j, x_{j+1}), \in \mathcal{P}_{p-1}, j = 0,\ldots,m,$ (where $\mathcal{P}_{p-1}$ is the set of polynomials of degree less than p) and $g^{(i)}(x_j^+) = g^{(i)}(x_j^-)$, $i = 0,1,\ldots,p-d_j-1; j = 1,2,\ldots,m\}$ and the end points are $p-$fold nodes.

Thus S_{p,Π_n} is the set of polynomials splines of order p with nodes at x_j $(j = 0,1,\ldots,m+1)$ of multiplicity $d_j \leqslant p-1$ $(j = 1,2,\ldots,m)$, consequently every spline in S_{p,Π_n} is in $C[a,b]$.

The set of the normalized B-splines $B_{i,p}$ $(i = 1,2,\ldots,n)$ of order p defined by the following recurrence relation:

$$B_{i,p}(x) = \frac{x - t_i}{t_{i+p-1} - t_i} B_{i,p-1}(x) + \frac{t_{i+p} - x}{t_{i+p} - t_{i+1}} B_{i+1,p-1}(x), \tag{9}$$

$$B_{i,1}(x) = \begin{cases} 1, & t_i \leqslant x < t_{i+1}, \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

is considered as a basis for the spline space S_{p,Π_n}.

Let $d_j = 1$ and T be a set of τ_{ij} $(i = 1,2,\ldots,n;\ j = 1,2,\ldots,p)$ q.i. points belonging for each $i = 1,2,\ldots,n$ to a subset of $[t_i, t_{i+p}]$ and such that $\tau_{ij} \neq \tau_{ih}$ for $j \neq h$. The q.i. spline operator $\mathcal{S}_n g \in S_{p,\Pi_n}$, with $g \in C[a,b]$, is defined as

$$\mathcal{S}_n g(x) := \sum_{i=1}^{n} B_{i,p}(x) \sum_{j=1}^{p} v_{ij} g(\tau_{ij}), \tag{11}$$

where

$$v_{ij} := \sum_{\mu=j}^{p} \frac{\alpha_{i\mu}}{\prod_{\substack{s=1 \\ s\neq j}}^{\mu} (\tau_{ij} - \tau_{is})}, \quad \alpha_{ij} := \sum_{k=1}^{j} (-1)^{j-k} \frac{(k-1)!(p-k)!}{(p-1)!} c_{i,k-1} d_{i,j-k}, \tag{12}$$

with $c_{i,k-1} = symm_{k-1}(t_{i+1},\ldots,t_{i+p-1})$, $d_{i,j-k} = symm_{j-k}(\tau_{i1},\ldots,\tau_{i,j-1})$ (see [3]).

$\mathcal{S}_n$ is a projector operator [6] that reproduces exactly a polynomial of $p-1$ degree and also all p order spline functions, that is,

$$\mathcal{S}_n P = P, \quad P \in \mathcal{P}_p, \quad \mathcal{S}_n S = S, \quad S \in S_{p,\Pi_n}. \tag{13}$$

Moreover (see for example [7]), as we assume $\mathcal{S}_n \widetilde{\mathcal{K}} g = \widetilde{\mathcal{K}} \mathcal{S}_n g$ with $g \in C[a,b]$, it follows that

$$||\widetilde{\mathcal{K}} - \mathcal{S}_n \widetilde{\mathcal{K}}|| \to 0 \quad \text{as } n \to \infty. \tag{14}$$

2.2.2. Collocation method

Applying to (6) the collocation method on a set of distinct collocation points ξ_k $(k = 1,2,\ldots,N)$ in (a,b), where N is the number of distinct elements of the set T, and using q.i. spline approximation, we obtain (7), where in this case

$$w_n(x) = \sum_{i=1}^{n} B_{i,p}(x) \sum_{j=1}^{p} v_{ij} \bar{u}_i \tag{15}$$

and the $\bar{u}_i$ are the approximated values of function u in $\tau_{ij} \in T$ $(i = 1,2,\ldots,n;\ j = 1,2,\ldots,p)$.

A choice of τ_{ij}, so that (13) and (14) are satisfied, is proposed in [7].

The optimal algorithmically choice (see [9]) of q.i. points set:

$$T := \left\{ \begin{array}{c} \tau_{i1} := \eta_1, \quad i = 1,\ldots,n \\ \tau_{i2} := \eta_{i-1}, \tau_{i3} := \eta_{i+1}, \ldots, \tau_{ip} := \eta_{i-(-1)^{l\left[\frac{p}{2}\right]}}, \quad i = \left[\frac{p}{2}\right] + 1, \ldots, n - \left[\frac{p}{2}\right] \end{array} \right\}, \tag{16}$$

where $\eta_i := \frac{t_{i+1}+\ldots+t_{i+p-1}}{p-1}$, $i = 1,2,\ldots,n$, with a suitable choice of the remaining nodes of T, also satisfies (13) and (14) (see [6]).

2.2.3. Remarks about the convergence

It is possible to generalize to the case $p > 2$ the remarks in [2] about the convergence.

To be more precise, applying the bounded projector operator 11 to 6 and to (7) with w_n as in (15) we obtain

$$\mathcal{S}_n(\lambda I + \widetilde{\mathcal{K}})u = \mathcal{S}_n f,$$
$$\mathcal{S}_n(\lambda I + \widetilde{\mathcal{K}})w_n = \mathcal{S}_n f$$

and consequently

$$(\lambda I + \mathcal{S}_n\widetilde{\mathcal{K}})(u - w_n) = \lambda(I - \mathcal{S}_n)u. \tag{17}$$

As $||\widetilde{\mathcal{K}} - \mathcal{S}_n\widetilde{\mathcal{K}}|| \to 0$ as $n \to \infty$, and following the proof of Theorem 1 in [4], we can state from (17) that

$$\sup_{n\geqslant N}||(\lambda I + \mathcal{S}_n\widetilde{K})^{-1}|| \leqslant M < \infty,$$

for n sufficiently large.

It follows that $||u - w_n|| \to 0$, as $n \to \infty$, exactly with the same rate of convergence as $||u - \mathcal{S}_n u||$ does.

3. Schauder bases iterative method

In this section, we consider the nonlinear integral equation (1). As the notations in the previous sections to this more general nonlinear framework, we write $\mathcal{K}_1$ and $\mathcal{K}_2$ for the operators on $C[a,b]$

$$\mathcal{K}_1 v(x) := -\int_a^x k_1(x,y,v(y))dy, \quad v \in C[a,b], \quad x \in [a,b]$$

and

$$\mathcal{K}_2 v(x) := -\int_a^b k_2(x,y,v(y))dy, \quad v \in C[a,b], \quad x \in [a,b],$$

respectively, and $\mathcal{K} := \mathcal{K}_1 + \mathcal{K}_2$. For a compact and Hausdorff topological space Ω we denote by $C(\Omega)$ the Banach space of those real valued functions defined on Ω and endowed with its maximum norm. Then, it is clear that $u \in C[a,b]$ is a solution of the nonlinear integral Eq. (1) if, and only if, it is a fixed point of the operator $\mathcal{N} : C[a,b] \to C[a,b]$ given, for each $v \in C[a,b]$, by

$$\mathcal{N}v := \frac{1}{\lambda}(f - \mathcal{K}v).$$

As a straightforward consequence of the Banach fixed point theorem, if

$$\rho := \frac{(M_1 + M_2)(b-a)}{|\lambda|} < 1,$$

where M_1, M_2 are the Lipschitz constants in (2), then $\mathcal{N}$ has one and only one fixed point u, and for each $v \in C[a,b]$ and $m \geqslant 1$ we have that

$$||\mathcal{N}^m v - u|| \leqslant \frac{\rho^m}{1-\rho}||\mathcal{N}v - v|| \tag{18}$$

and thus,

$$\lim_n ||\mathcal{N}^n v - u|| = 0.$$

Easy examples show that it is not always possible to calculate the sequence of iterations $\{\mathcal{N}^n v\}_{n\geqslant 1}$, and for this very reason, a numerical method is needed in order to approximate the fixed point of $\mathcal{N}$. We develop such an iterative method by means of the use of Schauder bases in an adequate Banach space associated in a natural way with the nonlinear mixed integral Eq. (1).

3.1. Elementary facts on Schauder bases

The proposed method is based on the use of Schauder bases and extends that developed in [2] for the linear mixed Volterra–Fredholm integral equation: on the one hand, the linear case satisfies clearly the Lispschitz conditions (2); on the other hand, in [2] a concrete Schauder basis is considered, unlike our analysis, which works for any Schauder basis. In fact, the basis in [2] is univariate, what restricts its application to a specific kind of equation. Let us also point out that Schauder bases have been successfully used in the numerical treatment of integral, integro–differential or differential equations (see [11–14]).

Let us recall that a Schauder basis in a Banach space E is a sequence $\{\chi_n\}_{n\geqslant 1}$ in E such that any $x \in E$ has a unique representation of the form

$$\lim_n \left\| x - \sum_{k=1}^{n} t_k \chi_k \right\| = 0,$$

where $\{t_n\}_{n\geqslant 1}$ is a sequence of real numbers. Given $m \geqslant 1$, the m–th biorthogonal functional $\chi_m^* : E \to \mathbb{R}$ is defined as

$$\chi_m^*(x) := t_m,$$

where $x \in E$ and $\lim_n \|x - \sum_{k=1}^{n} t_k \chi_k\| = 0$, and the mth projection is the finite–rank linear operator on E given by

$$Q_m(x) := \sum_{k=1}^{m} \chi_k^*(x)\chi_k, \quad x \in E.$$

It follows from the Baire category theorem [15, Theorem 2.1] that for all $m \geqslant 1$ the biorthogonal functional χ_m^* and the projection Q_m are continuous, and from the definition of these functionals and projections, that for all $x \in E$

$$\lim_n \|Q_n(x) - x\| = 0. \tag{19}$$

3.2. The iterative numerical method

Let us now introduce our numerical method for approximating the unique solution of the mixed nonlinear integral Eq. (1). Our approach combines an iterative scheme with some facts related to an arbitrary Schauder basis in the Banach space $C[a,b]^2$.

We describe in a concise way the proposed numerical method. We start from three data: a Schauder basis $\{\chi_n\}_{n\geqslant 1}$ in $C[a,b]^2$, with sequence of associated projections $\{Q_n\}_{n\geqslant 1}$, a sequence of positive integers $\{j_n\}_{n\geqslant 1}$ and a function $v \in C[a,b]$. We define recursively the sequence of approximate functions:

$$\mathtt{AF}_0(x) := v(x), \quad x \in [a,b] \tag{20}$$

and if $m \geqslant 1$

$$\mathtt{AF}_m(x) := \frac{1}{\lambda} f(x),$$

$$-\frac{1}{\lambda}\left(\int_a^x Q_{j_m}(\Upsilon_{m-1}^{(1)}(x,y))dy + \int_a^b Q_{j_m}(\Upsilon_{m-1}^{(2)}(x,y))dy \right), \quad x \in [a,b], \tag{21}$$

where for $j = 1, 2$,

$$\Upsilon_{m-1}^{(j)}(x,y) := k_j(x,y,\mathtt{AF}_{m-1}(y)), \quad x,y \in [a,b]. \tag{22}$$

By the triangle inequality we have that

$$\|\mathtt{AF}_m - u\| \leqslant \|\mathtt{AF}_m - \mathcal{N}^m v\| + \|\mathcal{N}^m v - u\|$$

and since (18) gives an upper bound for the second summand in the right–hand side, it remains to bound the first one in order to control the error $\|\mathtt{AF}_m - u\|$. With this in mind, we deal with estimating the norm $\|\mathtt{AF}_m - \mathcal{N}^m v\|$.

Lemma 2. *Let $\lambda \neq 0, f \in C[a,b]$ and let $k_1,\ k_2 \in C[a,b]^2$ satisfying the Lipschitz conditions (2), with Lipschitz constants M_1 and M_2, respectively, and let*

$$\rho := (M_1 + M_2)(b-a)/|\lambda|.$$

Assume in addition that $v \in C[a,b]$, $\{j_n\}_{n\geqslant 1}$ is a sequence of positive integers and that $\{\chi_n\}_{n\geqslant 1}$ is a Schauder basis in $C[a,b]^2$ whose sequence of associated projections is $\{Q_n\}_{n\geqslant 1}$. If for each $m \geqslant 1$, $\mathtt{AF}_m$ is the approximate function defined by (20)–(22), then

$$\|\mathtt{AF}_m - \mathcal{N}^m v\| \leqslant \frac{(b-a)}{|\lambda|} \sum_{k=1}^{m} \left(\|\Upsilon_{k-1}^{(1)} - Q_{j_k}(\Upsilon_{k-1}^{(1)})\| + \|\Upsilon_{k-1}^{(2)} - Q_{j_k}(\Upsilon_{k-1}^{(2)})\| \right) \rho^{m-k}.$$

Proof. We proceed inductively on m. For $m = 1$, given $x \in [a,b]$ it is clear that

$$\begin{aligned} |\mathtt{AF}_1(x) - \mathcal{N}v(x)| &\leqslant \frac{1}{|\lambda|}\int_a^x |\Upsilon_0^{(1)}(x,y) - Q_{j_1}(\Upsilon_0^{(1)}(x,y))|dy + \frac{1}{|\lambda|}\int_a^b |\Upsilon_0^{(2)}(x,y) - Q_{j_1}(\Upsilon_0^{(2)}(x,y))|dy \\ &\leqslant \frac{b-a}{|\lambda|}\left(\|\Upsilon_0^{(1)} - Q_{j_1}(\Upsilon_0^{(1)})\| + \|\Upsilon_0^{(2)} - Q_{j_1}(\Upsilon_0^{(2)})\| \right) \end{aligned}$$

so,

$$||\mathrm{AF}_1 - \mathcal{N}v|| \leqslant \frac{b-a}{|\lambda|}\left(||\Upsilon_0^{(1)} - Q_{j_1}(\Upsilon_0^{(1)})|| + ||\Upsilon_0^{(2)} - Q_{j_1}(\Upsilon_0^{(2)})||\right).$$

Now, let the announced inequality holds for $m-1$. Then it also is valid for m, because taking into account the induction hypothesis as well as the fact that for all $g, h \in C[a,b], ||\mathcal{N}g - \mathcal{N}h|| \leqslant \rho||g-h||$, we arrive, for all $x \in [a,b]$, at

$$\begin{aligned}
|\mathrm{AF}_m(x) - \mathcal{N}^m v(x)| &\leqslant |\mathrm{AF}_m(x) - \mathcal{N}\mathrm{AF}_{m-1}(x)| + ||\mathcal{N}\mathrm{AF}_{m-1} - \mathcal{N}^m v|| \\
&\leqslant \frac{1}{|\lambda|}\int_a^x |\Upsilon_{m-1}^{(1)}(x,y) - Q_{j_m}(\Upsilon_{m-1}^{(1)}(x,y))|dy + \frac{1}{|\lambda|}\int_a^b |\Upsilon_{m-1}^{(2)}(x,y) - Q_{j_m}(\Upsilon_{m-1}^{(2)}(x,y))|dy + \rho||\mathrm{AF}_{m-1} - \mathcal{N}^{m-1}v|| \\
&\leqslant \frac{(b-a)}{|\lambda|}\left(||\Upsilon_{k-1}^{(1)} - Q_{j_m}(\Upsilon_{m-1}^{(1)})|| + ||\Upsilon_{m-1}^{(2)} - Q_{j_m}(\Upsilon_{m-1}^{(2)})||\right) \\
&\quad + \frac{(b-a)}{|\lambda|}\rho\sum_{k=1}^{m-1}\left(||\Upsilon_{k-1}^{(1)} - Q_{j_k}(\Upsilon_{k-1}^{(1)})|| + ||\Upsilon_{k-1}^{(2)} - Q_{j_k}(\Upsilon_{k-1}^{(2)})||\right)\rho^{m-1-k} \\
&= \frac{(b-a)}{|\lambda|}\sum_{k=1}^{m}\left(||\Upsilon_{k-1}^{(1)} - Q_{j_k}(\Upsilon_{k-1}^{(1)})|| + ||\Upsilon_{k-1}^{(2)} - Q_{j_k}(\Upsilon_{k-1}^{(2)})||\right)\rho^{m-k}
\end{aligned}$$

and thus the arbitrariness of $x \in [a,b]$ implies the announced inequality. □

Let us observe that in the preceding lemma, the condition $\rho < 1$ is not required.

Now we are in a position to guarantee that for a suitable choice of $m \geqslant 1$ and $j_1, \ldots, j_m \geqslant 1$, AF_m is as closed as desired to the fixed point of the operator $\mathcal{N}$:

Theorem 3. *Suppose that $\lambda \neq 0, f \in C[a,b]$ and that $k_1,\ k_2 \in C[a,b]^2$ satisfy the Lipschitz conditions (2), with respective Lipschitz constants M_1 and M_2, and in such a way that*

$$(M_1 + M_2)(b-a)/|\lambda| < 1.$$

Let u be the unique solution of Eq. (1) and let $\{\chi_n\}_{n\geqslant 1}$ be a Schauder basis in $C[a,b]^2$ with sequence of associated projections $\{Q_n\}_{n\geqslant 1}$. Then for each $\varepsilon > 0$ and each $v \in C[a,b]$, there exist $m \geqslant 1$ and $j_1, \ldots, j_m \geqslant 1$ such that

$$||\mathrm{AF}_m - u|| < \varepsilon,$$

where AF_m is the approximate function defined by (20)–(22).

Proof. Since

$$\rho := (M_1 + M_2)(b-a)/|\lambda| < 1,$$

let $m \geqslant 1$ such that

$$\frac{\rho^m}{1-\rho}||\mathcal{N}v - v|| < \frac{\varepsilon}{2}. \tag{23}$$

On the other hand, in view of the convergence property (19), we can find $j_1, \ldots, j_m \geqslant 1$ with

$$\frac{(b-a)}{|\lambda|}\sum_{k=1}^{m}\left(||\Upsilon_{k-1}^{(1)} - Q_{j_k}(\Upsilon_{k-1}^{(1)})|| + ||\Upsilon_{k-1}^{(2)} - Q_{j_k}(\Upsilon_{k-1}^{(2)})||\right)\rho^{m-k} < \frac{\varepsilon}{2}. \tag{24}$$

Finally, (23), (24), Lemma 2 and (18) imply

$$||\mathrm{AF}_m - u|| \leqslant ||\mathrm{AF}_m - \mathcal{N}^m v|| + ||\mathcal{N}^m v - u|| < \varepsilon$$

and the proof is complete. □

4. Numerical results

In what follows we present numerical results for some Volterra–Fredholm integral equations, by using the numerical methods presented above.

4.1. Numerical examples for p-order spline direct method

Let us consider

$$\lambda u(x) = f(x) + \int_0^x h_1(x,s)u(s)\mathrm{d}s + \int_0^1 h_2(x,s)u(s)\mathrm{d}s, \quad x \in [0,1],$$

to the aim to verify the meaningfulness of the choice of an approximating spline class of order $p > 2$, we give some results related to the same test functions as in [2], comparing them in terms of precision of results, using the same number of elements of T.

Precisely we propose two examples:

(1) $h_1(x,s) = h_2(x,s) = e^{(x+s)}$ for Example 4.

(2) $h_1(x,s) = h_2(x,s) = \sin(x)\cos(s)$ for Example 5.

For the collocation method we consider:

- the mesh points as equispaced points in $[0,1]$ with nodes of p-multiplicity in 0 and 1,
- the mesh points all simple in $(0,1)$,
- the q.i. points as (16),
- the number of the collocation points equal to that of the distinct q.i. points.

The algorithm is implemented in MATLAB 7.3®.

In Tables 1–4 we compare for $p = 2$ and $p = 4$ the maximun absolute value of the errors related to the approximation of unknown function in a suitable subset of n points in $(0,1)$.

Example 1. *For* $\lambda = 15, u(x) = x^3$ and

$$f(x) = 15x^3 - e^x(-2e + 12 + e^x(x^3 - 3x^2 + 6x - 6)), \qquad x \in [0,1],$$

the numerical results appears in Table 1. Analogously, Table 2 includes the numerical experiments when $\lambda = 15, u(x) = e^x$ and

$$f(x) = \frac{1}{2}e^x(-e^2 - e^{2x} + 32), \quad x \in [0,1].$$

If we would obtain 10^{-10} precision order with $p = 2$, in the case of Table 2, the rounding error does not make reliable the solution of the collocation system, while, in the case $p = 4$, we achieve the goal by $n = 101$.

Table 1
Example 1 – $u(x) = x^3$.

n	$p = 2$	$p = 4$
11	7.5 E−3	2.5 E−15

Table 2
Example 1 – $u(x) = e^x$.

n	$p = 2$	$p = 4$
11	3.3 E−3	6.6 E−5
101	3.2 E−5	8.0 E−10

Table 3
Example 2 – $u(x) = x^2$.

n	$p = 2$	$p = 4$
11	2.7 E−3	5.5 E−16

Table 4
Example 2 – $u(x) = e^x$.

n	$p = 2$	$p = 4$
11	3.1 E−3	6.6 E−6
101	3.1 E−5	7.9 E−10
201	7.6 E−6	5.0 E−11

Example 2. Now $\lambda = 1$, $u(x) = x^2$ and

$$f(x) = x^2 \sin x(x^2 \sin x + 2x\cos x - 2\sin x - \sin 1 + 2\cos 1), \quad x \in [0,1],$$

with the corresponding results in Table 3, and $\lambda = 1$, $u(x) = e^x$ and

$$f(x) = e^x - \sin x(e^x(\sin x + \cos x) + e(\sin 1 + \cos 1) - 2)/2, \quad x \in [0,1],$$

4.2. Numerical testing for the Schauder bases iterative method

Let us consider the non linear case in (1). We include an example to ensure the efficiency and simplicity of the proposed method.

We emphasize that the approximation results for the fixed point method in Section 3.2 are stated for an arbitrary Schauder basis in $C[a,b]^2$. In order to obtain concrete numerical approximations of the solution of a nonlinear mixed Volterra–Fredholm integral equation, we shall fix a concrete bivariate Schauder basis.

The Schauder basis considered here is that obtained as the bivariate basis derived from the usual Schauder basis in $C[a,b]$, also known as the Faber-Schauder system. To be more precise, let us recall that given a sequence $\{t_n\}_{n\geqslant 1}$ of distinct points in $[a,b]$ such that $t_1 = a$ and $t_2 = b$, the usual Schauder basis $\{\zeta_n\}_{n\geqslant 1}$ in $C[a,b]$ is defined by

$$\zeta_1(t) := 1, \quad t \in [a,b],$$

while for $n \geqslant 1, \zeta_n$ is the piecewise linear continuous function on $[a,b]$ with nodes at $\{t_k : 1 \leqslant k \leqslant n\}$, given by

$$\zeta_n(t_n) = 1$$

and for $k < n$

$$\zeta_n(t_k) = 0.$$

We denote by $\{\zeta_n^*\}_{n\geqslant 1}$ and $\{P_n\}_{n\geqslant 1}$ its associated sequences of biorthogonal and projections, respectively.

The usual Schauder basis $\{\chi_n\}_{n\geqslant 1}$ in $C[a,b]^2$, which is what we set in our numerical experiments, is the corresponding bivariate tensor basis of $\{\zeta_n\}_{n\geqslant 1}$ ([16] and [17]): if [] denotes "integer part" and $\sigma : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ is the bijective mapping given by

$$\sigma(n) := \begin{cases} (\sqrt{n}, \sqrt{n}), & \text{if } [\sqrt{n}] = \sqrt{n} \\ (n - [\sqrt{n}]^2, [\sqrt{n}] + 1), & \text{if } 0 < n - [\sqrt{n}]^2 \leqslant [\sqrt{n}], \\ ([\sqrt{n}] + 1, n - [\sqrt{n}]^2 - [\sqrt{n}]), & \text{if } [\sqrt{n}] < n - [\sqrt{n}]^2 \end{cases}$$

then

$$\chi_n(t,s) := \zeta_p(t)\zeta_q(s), \quad t,s \in [a,b],$$

whenever $\sigma(n) = (p,q)$. $\{\chi_n^*\}_{n\geqslant 1}$ and $\{Q_n\}_{n\geqslant 1}$ stand for the respective sequences of biorthogonal functionals and projections. It is a well-known fact that the sequence $\{\chi_n^*\}_{n\geqslant 1}$ is easily determined by finite linear combinations of evaluations at adequate known functions, and therefore so does $\{Q_n\}_{n\geqslant 1}$. More specifically, for each $g \in C[a,b]^2$,

$$\chi_1^*(g) = g(t_1, t_1)$$

and for $n \geqslant 2$ and $\sigma(n) = (p,q)$,

$$\chi_n^*(g) = g(t_p, t_q) - \sum_{k=1}^{n-1} \chi_k^*(g)\chi_k(t_p, t_q). \tag{25}$$

Now we describe the parameters of the numerical method. In what follows we assume that $[a,b] = [0,1]$, because such is the case in the concrete nonlinear example below. The subset $\{t_n\}_{n\geqslant 1}$ chosen for constructing the usual Schauder basis in $C[0,1]^2$ is defined by

$$t_0 = 0, \quad t_1 = 1$$

and for $n \geqslant 1$,

$$t_{n+1} = \frac{2k+1}{2^{l+1}}$$

provided that $n = 2^l + k + 1$, where $0 \leqslant k < 2^l$ are positive integers. Once we have one of the data defining the approximating sequence $\{\mathtt{AF}_m\}_{m\geqslant 1}$ given by (20)–(22), we choose the other two: the initial function is

$$\mathtt{AF}_0 = \frac{f}{|\lambda|}$$

and for all $m \geqslant 1$, we take $j_m = j^2$ for simplicity in the implementation, where j is a given integer.

Table 5
Example 3 – $u(x) = 1 - x$.

x	$j = 9$	$j = 17$
	$\lvert \mathrm{AF}_3(x) - u(x) \rvert$	$\lvert \mathrm{AF}_3(x) - u(x) \rvert$
0	2.9 E−4	7.4 E−5
0.125	3.1 E−4	7.7 E−5
0.25	3.4 E−4	8.1 E−5
0.375	3.6 E−4	8.6 E−5
0.5	3.9 E−4	9.1 E−5
0.625	4.2 E−4	9.7 E−5
0.75	4.5 E−4	1.0 E−4
0.875	4.9 E−4	1.1 E−4
1	5.3 E−4	1.2 E−4

We include in the following numerical Example 6, for $j = 9$ and $j = 17$, the absolute errors committed in certain representative points $x \in [0, 1]$, when we approximate the exact solution u by the approximate function AF_h where $h = 3$.

Example 3. Let us consider the nonlinear mixed integral equation, similar to that in [18, Example 2],

$$4\,u(x) = f(x) + \int_0^x \sin(x-y)\cos u(y)dy + \int_0^1 (x-y)(1+u(y)^2)dy, \qquad x \in [0,1],$$

where for $x \in [0, 1]$,

$$f(x) = \frac{1}{12}(55 - 64x + 6(\sin 1)x\cos x - 6(\cos 1)x\sin x - 6(\sin 1)\sin x).$$

Its exact solution is $u(x) = 1 - x$ and the numerical results obtained from the iterative method are taken in Table 5 for the parameters previously commented. The algorithms associated with the numerical method have been performed using Mathematica 7.

Let us emphasize that the choice for AF_0 seems to be irrelevant, since after some iterations its influence in the approximations disappears. Let us also point out that this iterative numerical method has the advantage of being very easy to implement. The approximating functions are the sum of a known function and integrals of piecewise bivariate polynomials of degree 2. As immediately follows from (25), the calculation of the coefficients of such polynomials just requires linear combinations of several evaluations of the basic functions at adequate points.

5. Final remarks

In this work we presented two numerical methods for solving mixed Volterra–Fredholm integral problems. In particular we proposed, to tackle on the linear cases, a direct collocation method based on q.i. spline of order of precision $p \geqslant q2$. In this way a high degree of accuracy is easily ensured using collocation systems of minimum order, overcoming consequently rounding errors problems.

In the nonlinear cases we suggested a fixed point method derived from suitable properties of a Schauder basis in the Banach space of continuous functions $C[a, b]^2$, well suited to define and apply tensorial schemes, that in addition is very easily implemented.

The tables of the numerical results confirm the theoretical statements.

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