Irreducible Realization in Canonical Forms

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ABSTRACT: *This paper describes a computational method for the realization of a finite dimensional, time invariant dynamical system whose description is available in terms of a transfer matrix or input–output data. The realization is irreducible and relies for its computation on the properties of a Hankel matrix, and obtains a state variable description of the system in either controllable or observable canonical forms.*

I. Introduction

The problem of realizing a state variable model of minimal dimension for a time invariant, finite dimensional dynamical system when its description is available in terms of a transfer matrix, input–output data or impulse–response matrix, are well investigated in the literature (**1–13**). Basically, there are two approaches to the problem: one which uses the properties of polynomial matrices, and the other preferred by researchers which formulates and solves the problem via the Hankel matrix and its properties. Few workers (**13**) have devoted themselves to obtaining the irreducible realization directly in state variable canonical forms with which it is easy to study the performance characteristics of the system because these forms generally have a minimum number of parameters (**14**). The realization in (**13**) is not in controllable or observable canonical forms [see (**12, 15, 16**)] which are well adapted to the study of linear multivariable systems.

The aim of the present paper is to obtain an irreducible realization of a time invariant, finite dimensional dynamical system whose description is given in terms of a transfer matrix in either controllable or observable canonical forms. The method is based for its formulation and solution upon the Hankel matrix associated with the transfer function. The set of independent vectors of this matrix is employed to define another set of basis vectors which are used to describe a computational algorithm for its realization.

II. System Description and Preliminaries

Let a linear finite dimensional, time invariant, completely controllable and completely observable, dynamical system be represented by the state variable description:

$$\begin{aligned} \dot{x}(t) &= Ax(t)+Bu(t) \\ y(t) &= Cx(t)+Eu(t), \end{aligned} \tag{2.1}$$

where $x(t)$ is an n-state vector, $y(t)$ is a p-output vector, $u(t)$ an m-input vector, A, B, C and E are matrices of compatible dimensions. It is easy to see from (2.1) that the transfer function description of the same system is:

$$T(s) = C(sI-A)^{-1}B+E, \tag{2.2}$$

where $T(s)$ is a matrix of order $p \times m$ and is a rational function of s. $T(s)$ is called a *proper transfer* matrix if E is not a null matrix otherwise it is called *strictly proper.* If $T(s)$ is given as a proper transfer matrix whose elements are rational function of s, the problem of realization is to find A, B, C and E.

Definition 1. The realization $(\tilde{A}, \tilde{B}, \tilde{C}, E)$ of a rational transfer matrix $T(s)$ is said to be in controllable canonical forms if

$$\tilde{A} = [A_{ij}], \quad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ \cdot \\ \cdot \\ \cdot \\ \tilde{B}_m \end{bmatrix}, \quad \tilde{C} = [\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_m], \tag{2.3}$$

where $i, j = 1, 2, \ldots, m$; every element A_{ij} of $\tilde{A}$ is a matrix of order $d_i \times d_j$ having the form:

$$A_{ii} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_{ii}(1) & a_{ii}(2) & a_{ii}(3) & \cdots & a_{ii}(d_i) \end{bmatrix}, \quad A_{ij} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & & \cdot \\ a_{ij}(1) & a_{ij}(2) & \cdots & a_{ij}(d_j) \end{bmatrix};$$

$$i \neq j; \quad \text{(2.3a,b)}$$

every element $\tilde{B}_i$ of $\tilde{B}$ is a matrix of order $d_i \times m$ having non-zero elements only in its last row where they begin from the i-th column:

$$[0 \ldots 0, 1, b_{ii+1}, \ldots, b_{im}], \quad i = 1, 2, \ldots, m;$$

the forms of $\tilde{C}$ and E are non-specific but $\tilde{C}_i$ in $\tilde{C}$ is a matrix of order $p \times d_i$ having the elements:

$$\tilde{C}_i = \begin{bmatrix} c_{11}(i) & c_{12}(i) & \cdots & c_{id_i}(i) \\ c_{21}(i) & c_{22}(i) & \cdots & c_2 d_i(i) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & & \cdot \\ c_{p1}(i) & c_{p2}(i) & \cdots & c_{pd_i}(i) \end{bmatrix}, \quad i = 1, 2, \ldots, m. \tag{2.3c}$$

Here d_i's are the controllability indices.

We shall be concerned here only with systems where in B the columns b_i, $i=1,2,\dots,m$ and in C the rows c_i, $i=1,2,\dots,p$ are linearly independent.

Definition 2. The realization $(\tilde{A}, \tilde{B}, \tilde{C}, E)$ of a rational transfer matrix $T(s)$ is said to be in observable canonical forms if

$$\tilde{A}=[A_{ij}],\quad \tilde{B}=\begin{bmatrix}\tilde{B}_1\\ \tilde{B}_2\\ \cdot\\ \cdot\\ \cdot\\ \tilde{B}_p\end{bmatrix}\quad \text{and}\quad \tilde{C}=[\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_p], \tag{2.4}$$

where $i, j=1,2,\dots,p$; every element A_{ij} of $\tilde{A}$ is a matrix of order $\tilde{d}_i\times\tilde{d}_j$ having the form:

$$A_{ii}=\begin{bmatrix}0 & 0 & \cdots & 0 & a_{ii}(1)\\ 1 & 0 & \cdots & 0 & a_{ii}(2)\\ 0 & 1 & \cdots & 0 & a_{ii}(3)\\ \cdot & \cdot & & \cdot & \cdot\\ \cdot & \cdot & \cdots & \cdot & \cdot\\ \cdot & \cdot & & \cdot & \cdot\\ 0 & 0 & \cdots & 1 & a_{ii}(\tilde{d}_i)\end{bmatrix},\quad A_{ij}=\begin{bmatrix}0 & 0 & \cdots & a_{ij}(1)\\ 0 & 0 & \cdots & a_{ij}(2)\\ \cdot & \cdot & & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & & \cdot\\ 0 & 0 & \cdots & a_{ij}(\tilde{d}_j)\end{bmatrix};\quad i\neq j; \tag{2.4a,b}$$

the form of $\tilde{B}$ is non-specific, where every element $\tilde{B}_i$ is a matrix of order $\tilde{d}_i\times m$ having the elements

$$\tilde{B}_i=\begin{bmatrix}b_{11}(i) & b_{12}(i) & \cdots & b_{1m}(i)\\ b_{21}(i) & b_{22}(i) & \cdots & b_{2m}(i)\\ \cdot & \cdot & & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & & \cdot\\ b_{\tilde{d}_i1}(i) & b_{\tilde{d}_i2}(i) & \cdots & b_{\tilde{d}_im}(i)\end{bmatrix},\quad i=1,2,\dots,p; \tag{2.4c}$$

the element $\tilde{C}_i$ of $\tilde{C}$ is a matrix of order $p\times\tilde{d}_i$ having non-zero elements only in its last column where they begin from the i-th row:

$$[0\dots0, 1, c_{i+1,i}\dots c_{pi}]^T.$$

The form of E is non-specific. $\tilde{d}_i$'s are here observability indices.

To determine E given $T(s)$ as a proper transfer matrix one has to take the limit of $T(s)$ as $s\to\infty$ giving

$$E=\lim_{s\to\infty} T(s). \tag{2.5}$$

Definition 3. The part left after subtracting E as obtained in (2.5) from the given $T(s)$ is called *reduced transfer* matrix and is designated by $G(s)$. Thus

$$G(s)=T(s)-\lim_{s\to\infty} T(s)=C(sI-A)^{-1}B, \tag{2.6}$$

which is a strictly proper rational matrix as is easy to visualize.

The problem of realization thus becomes to determine A, B, C from the reduced transfer matrix $G(s)$. Expanding the right hand side of (2.6) about $s=0$, we get

$$G(s)=\sum_{i=0}^{\infty} H_i s^{-(i+1)}, \quad \text{where} \quad H_i=CA^iB. \tag{2.7}$$

If $T(s)$ is given H_i's are determined by means of (2.6) and (2.7).

Definition 4. The triple (A, B, C) is a realization of the reduced transfer matrix $G(s)$ if and only if $CA^iB=H_i$, $i=0, 1, \ldots,$ and is irreducible if and only if A, B is completely controllable and A, C is completely observable.

By virtue of (2.7) a sequence of Hankel matrices $\mathcal{H}_{ba}$ for all $a, b=0, 1, \ldots$ is defined by

$$\mathcal{H}_{ba}=\begin{bmatrix} H_0 & H_1 & \cdots & H_{a-1} \\ H_1 & H_2 & \cdots & H_a \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & & \cdot \\ H_{b-1} & H_b & \cdots & H_{a+b-2} \end{bmatrix} \tag{2.8}$$

The rank of $\mathcal{H}_{ba}$ determines the order or degree of realization i.e. order of A in view of the following theorem proved in (7).

Theorem I

Let $G(s)$ be a strictly proper rational matrix and let r be the degree of least common multiple of the denominators (*lcd*) of the elements in $G(s)$, and let b and a be the first integers such that rank $\mathcal{H}_{ab}=\text{rank}\ \mathcal{H}_{rr}=n$. Then n is the degree of minimum realization of $G(s)$, a is the observability index (max $\tilde{d}_i$) and b is controllability index (max d_i) of the minimal realization (A, B, C).

Let us denote by $h_i(j)$ for $i=0, 1, \ldots, r-1$ and $j=1, 2, \ldots, m$ the j-th column in the i-th column blocks headed by H_i in (2.8) when $a=b=r$. Then considering the matrix

$$\mathcal{H}_{rr}=[h_0(1), h_0(2), \ldots, h_0(m); h_1(1), h_1(2), \ldots, h_1(m); h_2(1), \ldots, h_{r-1}(m)] \tag{2.9}$$

we can pick $h_i(j)$ if it is linearly independent of the vectors preceding it in (2.9) and group the vectors thus selected having the same value for j giving

$$[h_0(1), h_1(1), \ldots, h_{\delta_1-1}(1); h_0(2), \ldots, h_{\delta_2-1}(2); h_0(3), \ldots, h_{\delta_m-1}(m)]. \tag{2.10}$$

The δ_i's $i=1, \ldots, m$ are called *column indices* of $\mathcal{H}_{rr}$. Similarly considering rows we can determine row indices $\tilde{\delta}_i$'s $i = 1, \ldots, p$ of $\mathcal{H}_{rr}$. With these ideas we can present our next theorem proved in (**17**).

Theorem II

Let (A, B, C) be a realization of the reduced transfer matrix $G(s)$ and $\mathcal{H}_{rr}$ be the Hankel matrix associated with it where r is a degree of its *lcd*. Then column (row) indices of $\mathcal{H}_{rr}$ are the same as the invariant controllability (observability) indices of the pair $[A, B]$ ($[A, C]$) and *vice versa*.

As a consequence of the above two theorems we get:

Theorem III

If (A, B, C) is a realization of $G(s)$ as given in (2.6) then rank of the matrix $\mathscr{C}_r$ or $\mathcal{O}_r$ defined by

$$\mathscr{C}_r = [B, AB, \ldots, A^{r-1}B], \quad \mathcal{O}_r = [C', A'C', \ldots, A'^{r-1}C']' \tag{2.11}$$

where a prime denotes the transpose of a matrix and r is a degree of *lcd* of $G(s)$ is n.

III. Realization in Controllable Canonical Forms

Let us consider the realization in controllable canonical forms. By the fact that (2.1) is completely controllable it is possible to construct a non-singular matrix Q (**15**) such that

$$AQ = Q\tilde{A}, \quad B = Q_m\tilde{B}_m \tag{3.1a,b}$$

where $\tilde{B}_m$ is a matrix of order $m \times n$ having i-th row equal to the $k_i (= \sum_{j=1}^{i} d_j)$-th row of $\tilde{B}$ in (2.3) $i = 1, 2, \ldots, m$ and Q_m is a matrix of order $n \times m$ whose i-th column is the same as the k_i-th column in Q, $i = 1, 2, \ldots, m$. Since $\tilde{B}_m^{-1}$ is an upper triangular matrix having unit elements on the main diagonal, we have from (3.1b) denoting by β_{ki} the remaining elements of $\tilde{B}_m^{-1}$ and by q_i the i-th column of Q:

$$q_{k_i} = b_i + \sum_{k=1}^{i-1} \beta_{ki} b_k. \tag{3.2}$$

Let $p_0(i) = \mathcal{O}_r q_{k_i}$, then premultiplying both sides of (3.2) by $\mathcal{O}_r$ and observing from (2.8) and (2.9) that $h_0(i) = \mathcal{O}_r b_i$ for all $i = 1, 2, \ldots, m$ we get

$$p_0(i) = h_0(i) + \sum_{k=1}^{i-1} \beta_{ki} h_0(k). \tag{3.3}$$

By definition of column indices we can write

$$h_{d_i}(i) = \sum_{j=0}^{d_i-1} \sum_{k=1}^{m} \alpha_{jk} h_j(k) + \sum_{k=1}^{i-1} \alpha_{d_ik} h_{d_i}(k). \tag{3.4}$$

Premultiplying both sides of (3.2) by $\mathcal{O}_r A^{d_i}$ and comparing the result with (3.4) it is easily concluded that $\beta_{ki} = -\alpha_{d_ik}$.

Definition 5. Let g_j's represent the columns of a matrix G and let $g_1(i)$ be the first non-zero element from the top in the i-th row of its first column. Adding suitable multiples of g_1 to the remaining columns the elements in their i-th rows can be reduced to zero. If $g_2(k)$ is the first non-zero element from the top in the k-th row of second column of G so reduced, then this column can be similarly employed to reduce to zero k-th row elements of all columns but for the first and the second. If any column has zero elements in course of computation one has to pass on to the next column. This process is continued till all columns are exhausted. These operations are called elementary column transformations. Elementary row transformations are defined similarly.

Let $H(=\mathcal{H}_{d+1,d+1},\ d=\max d_i)$ and a unit matrix of order $(d+1)m$ be subjected to elementary column transformations of H whereby they are reduced to $\tilde{H}$ and T. Then it is a simple result of linear algebra that $HT=\tilde{H}$. Here T is an upper triangular matrix having unit diagonal elements and is referred to a transforming matrix of H. Let t_{ij} be any element of T. Then taking into account (2.9) and (3.4) we can write

$$\beta_{ki}=t_{md_i+k,md_i+i};\ k=1,2,\ldots,i-1;\ i=1,2,\ldots,m. \tag{3.5}$$

If we define $\mathcal{C}_r A^j q_{k_i}=p_j(i)$, then it is evident from (3.2), (3.3) and Theorem III that $p_j(i)$, $i=1,2,\ldots,m$; $j=0,1,\ldots,d_i-1$ constitute a set of n linearly independent vectors.

We now write (3.1a) in terms of its columns for all $i=1,2,\ldots,m$ giving us:

$$\begin{aligned} Aq_{k_{i-1}+1}&=\sum_{j=1}^{m}a_{ji}(1)q_{k_j},\\ Aq_{k_{i-1}+2}&=q_{k_{i-1}+1}+\sum_{j=1}^{m}a_{ji}(2)q_{k_j},\\ &\ \ \vdots\\ Aq_{k_i-1}&=q_{k_i-2}+\sum_{j=1}^{m}a_{ji}(d_i-1)q_{k_j},\\ Aq_{k_i}&=q_{k_i-1}+\sum_{j=1}^{m}a_{ji}(d_i)q_{k_j}. \end{aligned} \tag{3.6}$$

Substituting q_{k_i-1} from the last equation into one preceding it and continuing this for q_{k_i-2}, q_{k_i-3} in this way we get:

$$q_{k_i-s}=A^s q_{k_i}-\sum_{r=0}^{s-1}A^{s-1-r}\sum_{j=1}^{m}a_{ji}(d_i-r)q_{k_j},\ s=1,\ldots,d_i-1 \tag{3.7a}$$

and

$$A^{d_i}q_{k_i}=\sum_{r=1}^{d_i}A^{d_i-r}\sum_{j=1}^{m}a_{ji}(d_i-r+1)q_{k_j}. \tag{3.7b}$$

Premultiplying both sides of (3.7b) by $\mathcal{O}_r$ there results

$$p_{d_i}(i)=\sum_{r=1}^{d_i}\sum_{j=1}^{m} a_{ji}(d_i-r+1)p_{d_i-r}(j), \tag{3.8}$$

where in view of (3.2)

$$p_j(i)=h_j(i)+\sum_{k=1}^{i-1}\beta_{ki}h_j(k),\ i=1,2,\ldots,m;\ j=0,1,\ldots,d_i. \tag{3.9}$$

Knowing H_j's $j=0,1,\ldots,$ one determines β_{ki}'s with the aid of (3.5) and $p_j(i)$'s by (3.9) and constructs the following matrix:

$$\mathcal{P}_d=\mathcal{P}_{d+1,d+1}=[p_0(1),p_0(2),\ldots,p_0(m);p_1(1),p_1(2),\ldots,p_1(m);\\ p_2(1),\ldots,p_d(m)] \tag{3.10}$$

and subjects it to elementary column transformations. Let T_d be a transforming matrix of order $(d+1)\ m$. Then in view of (3.8) (md_i+i)-th column of T_d determines $a_{ji}(d_i-r+1)$ $j=1,\ldots,m$; $r=1,\ldots,d_i$. Again premultiplying both sides of (3.7b) by $\mathcal{O}_rA^s$ it follows that for $s=0,1,\ldots,d-d_i$;

$$p_{d_i+s}(i)=\sum_{r=1}^{d_i}\sum_{j=1}^{m} a_{ji}(d_i-r+1)p_{d_i-r+s}(j). \tag{3.11}$$

The above relation shows that for all $s=0,1,\ldots,d-d_i$; $i=1,\ldots,m$, the $\{m(d_i+s)+i\}$-th columns in T_d can be arranged to have the same elements in the rows $ms+p$, $p=1,2,\ldots,md_i+i$.

Let us now define:

$P_j=CA^j[q_{k_1},\ldots,q_{k_m}]$, a matrix of order $p\times m$ i.e.

$$\tilde{C}\tilde{A}^j\tilde{B}=P_j\tilde{B}_m, \tag{3.12}$$

$$C_j=\begin{bmatrix} c_{1d_1-d+j}(1) & c_{1d_2-d+j}(2) & \cdots & c_{1d_m-d+j}(m)\\ c_{2d_1-d+j}(1) & c_{2d_2-d+j}(2) & \cdots & c_{2d_m-d+j}(m)\\ \vdots & \vdots & \cdots & \vdots\\ c_{pd_1-d+j}(1) & c_{pd_2-d+j}(2) & \cdots & c_{pd_m-d+j}(m)\end{bmatrix} \tag{3.13}$$

$$A_j=\begin{bmatrix} a_{11}(d_1-d+j) & a_{12}(d_2-d+j) & \cdots & a_{1m}(d_m-d+j)\\ a_{21}(d_1-d+j) & a_{22}(d_2-d+j) & \cdots & a_{2m}(d_m-d+j)\\ \vdots & \vdots & \cdots & \vdots\\ a_{m1}(d_1-d+j) & a_{m2}(d_2-d+j) & \cdots & a_{mm}(d_m-d+j)\end{bmatrix} \tag{3.14}$$

for $j=1,2,\ldots,d$. Introducing (3.12) into (3.10) we can write $\mathcal{P}_dT_d=\tilde{\mathcal{P}}_d$ in the form:

$$\begin{bmatrix} P_0 & P_1 & \cdots & P_d \\ P_1 & P_2 & \cdots & P_{d+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & & \cdot \\ P_d & P_{d+1} & \cdots & P_{2d} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1d} & -A_1 \\ & T_{22} & \cdots & T_{2d} & -A_2 \\ & & \ddots & \cdot & \cdot \\ & 0 & & \cdot & \cdot \\ & & & \cdot & \cdot \\ & & & T_{dd} & -A_d \\ & & & & I \end{bmatrix} = \tilde{\mathscr{P}}_d \tag{3.15}$$

which is evident from (3.8) and (3.11) where in $\tilde{\mathscr{P}}_d$ the submatrix constituted by the last m columns is a null matrix, T_{ij}'s are submatrices of order $m \times m$, I is a unit matrix of order m and A_j's are defined in (3.14).

Equating the $(k_i - s)$-th columns on both sides of $CQ = \tilde{C}$ for all $i = 1, 2, \ldots, m$ we get

$$C[q_{k_1-s}, q_{k_2-s}, \ldots, q_{k_m-s}] = C_{d-s}, \tag{3.16}$$

where $s = 0, 1, \ldots, d-1$, q_i's having non-positive subscripts are considered zero vectors. Substituting from (3.7a) and taking into account of (3.12) and (3.14) we get

$$P_{d-j} - \sum_{i=0}^{d-j-1} P_j A_{i+j+1} = C_j, \quad j = 1, \ldots, d. \tag{3.17}$$

Referring to (3.15) the above Eq. (3.17) conforms to the following representation by block triangular matrices:

$$\begin{bmatrix} & & & P_0 \\ & & P_0 & P_1 \\ 0 & & \cdot & \cdot \\ & & \cdot & \cdot \\ & \ddots & \cdot & \cdot \\ 0 \;\; P_0 & \cdots & P_{d-2} & P_{d-1} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1d} & -A_1 \\ & T_{22} & \cdots & T_{2d} & -A_2 \\ & & \ddots & \cdot & \cdot \\ & & & \cdot & \cdot \\ & 0 & & \cdot & \cdot \\ & & & T_{dd} & -A_d \\ & & & & I \end{bmatrix} = \begin{bmatrix} & & & C_d \\ & & X & C_{d-1} \\ 0 & & \cdot & \cdot \\ & & \cdot & \cdot \\ & \ddots & \cdot & \cdot \\ & & X & C_2 \\ 0 \;\; X & \cdots & X & C_1 \end{bmatrix} \tag{3.18}$$

Observe that in view of (3.2) $\tilde{B}_m^{-1}$ is an upper triangular matrix $[\beta_{ki}]$ defined by (3.5) having unit diagonal elements.

Theorem IV

The triple $(\tilde{A}, \tilde{B}, \tilde{C})$ is a realization of the reduced transfer matrix $G(s)$ in controllable canonical form if and only if there exist matrices A_j's $(j = 1, \ldots, d)$ of order $m \times m$, C_j's $(j = 1, \ldots, d)$ and P_j's $(j = 0, 1, 2, \ldots)$ both of order $p \times m$ satisfying

$$P_{d-j} - \sum_{i=0}^{d-j-1} P_i A_{i+j+1} = C_j, \quad j = 1, 2, \ldots, d;$$

and

$$P_{d+j} - \sum_{i=0}^{d-1} P_{i+j} A_{i+1} = 0 \quad \text{for all} \quad j \geq 0. \tag{3.19}$$

Proof: Given $G(s)$ and the realization $(\tilde{A}, \tilde{B}, \tilde{C})$, the non-singular upper triangular matrix $\tilde{B}_m$ is defined with the non-zero rows of $\tilde{B}$ as described in (2.3) and P_j's are determined as in (3.12). Defining A_j's and C_j's as in (3.13) and (3.14) the necessity of the theorem follows as a result of (3.15) and (3.18) and the discussion associated with them.

The proof of sufficiency is similar to Theorem 4 in (**17**), only P_i's in the present case must replace H_i's in that theorem and hence omitted.

Theorem V. The realization $(\tilde{A}, \tilde{B}, \tilde{C})$ as defined by the controllable canonical form in Theorem IV is irreducible.

Referring to Theorem I, it immediately follows from Theorem 6 in (**7**) that $(\tilde{A}, \tilde{B}, \tilde{C})$ is an irreducible realization with 'a' as observability index and 'b' as controllability index.

Theorem VI

In the irreducible realization $(\tilde{A}, \tilde{B}, \tilde{C})$ in controllable canonical forms, as defined in Theorem IV, the non-fixed parameters describing $\tilde{A}$ are minimum in number.

Proof: To prove the Theorem, we observe as stated in (**14**) that the element A_{ji} in the canonical form (2.3) is in terms of minimum number of parameters if its non-fixed parameters are $a_{ji}(k)$, $k = 1, 2, \ldots, \min(d_i, d_j)$. Assuming $d_i > d_j$, we note from (3.9) that $p_0(j), \ldots, p_{d_j-1}(j)$ are linearly independent and hence their coefficients $a_{ji}(1), \ldots, a_{ji}(d_j)$ in (3.8) are in general non-zero but those $a_{ji}(d_j+1), \ldots, a_{ji}(d_i)$ associated with the dependent vectors $p_{d_j}(j), \ldots, p_{d_i-1}(j)$ must be zero for all $i, j = 1, 2, \ldots, m$. This completes the proof.

Example 1. Let

$$T(s) = \begin{bmatrix} \dfrac{1}{(s+1)^2} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{1}{(s+1)(s+2)} & \dfrac{1}{(s+2)^2} \end{bmatrix}.$$

We note that here $E = 0$, $T(s) = G(s)$ and $r = 4$. Constructing $\mathscr{H}_{44}$ and its transforming matrix T, we get

$$\underset{\mathcal{H}_{44}}{\begin{bmatrix} 0 & 0 & 1 & 1 & -2 & -3 & 3 & 7 \\ 0 & 1 & 1 & -1 & -3 & 0 & 7 & 4 \\ 1 & 1 & -2 & -3 & 3 & 7 & -4 & -15 \\ 1 & -1 & -3 & 0 & 7 & 4 & -15 & -16 \\ -2 & -3 & 3 & 7 & -4 & -15 & 5 & 31 \\ -3 & 0 & 7 & 4 & -15 & -16 & 31 & 48 \\ 3 & 7 & -4 & -15 & 5 & 31 & -6 & -63 \\ 7 & 4 & -15 & -16 & 31 & 48 & -63 & -128 \end{bmatrix}} \underset{T}{\begin{bmatrix} 1 & -1 & 3 & -1 & 0 & 0 & 2 & 0 \\ & 1 & -1 & 2 & 1 & 0 & 0 & 0 \\ & & 1 & -1 & 2 & -1 & 5 & 0 \\ & & & 1 & 0 & 2 & 0 & 0 \\ & \mathbf{0} & & & 1 & -1 & 4 & -1 \\ & & & & & 1 & 0 & 2 \\ & & & & & & 1 & -1 \\ & & & & & & & 1 \end{bmatrix}} \underset{\tilde{\mathcal{H}}_{44}}{\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -3 & 3 & -2 & 0 & -1 & 0 & 0 & 0 \\ 3 & 4 & -2 & 0 & 4 & 0 & 0 & 0 \\ 7 & -3 & 2 & 0 & 5 & 0 & 0 & 0 \end{bmatrix}}$$

Hence $d_1 = 3$, $d_2 = 1$. Because $m = 2$, in view of (3.5), $\beta_{12} = t_{34} = -1$. By virtue of (3.9) we compute $\mathscr{P}_d$ in (3.10) and construct the first matrix in the left side of (3.15) and (3.18) which after elementary column transformation give us:

$$\begin{bmatrix} & \mathbf{0} & & & & & 0 & 0 \\ & & & & & & 0 & 1 \\ & & & & 0 & 0 & 1 & 0 \\ & & & & 0 & 1 & 1 & -2 \\ \mathbf{0} & & 0 & 0 & 1 & 0 & -2 & -1 \\ & & 0 & 1 & 1 & -2 & -3 & 3 \end{bmatrix} \qquad \begin{array}{c} \begin{bmatrix} & \mathbf{0} & & & 0 & 0 \\ & & & & 0 & 1 \\ \hline & & 0 & 0 & 1 & 0 \\ & & 0 & 1 & 1 & 0 \\ \hline \mathbf{0} & 0 & 0 & 1 & 0 & 2 & 0 \\ & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ C_3 \\ \\ C_2 \\ \\ C_1 \end{array}$$

$$\begin{array}{cccccccc} p_0(1) & p_0(2) & p_1(1) & p_1(2) & p_2(1) & p_2(2) & p_3(1) & p_3(2) \end{array}$$
$$\begin{bmatrix} 0 & 0 & 1 & 0 & -2 & -1 & 3 & 4 \\ 0 & 1 & 1 & -2 & -3 & 3 & 7 & -3 \\ 1 & 0 & -2 & -1 & 3 & 4 & -4 & -11 \\ 1 & -2 & -3 & 3 & 7 & -3 & -15 & -1 \\ -2 & -1 & 3 & 4 & -4 & -11 & 5 & 26 \\ -3 & 3 & 7 & -3 & -15 & -1 & 31 & 17 \\ 3 & 4 & -4 & -11 & 5 & 26 & -6 & -57 \\ 7 & -3 & -15 & -1 & 31 & 17 & -63 & -65 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 1 & 1 & 0 & 2 & 0 \\ & 1 & -1 & 2 & 1 & 0 & 0 & 0 \\ & & 1 & 0 & 2 & 1 & 5 & 0 \\ & & & 1 & 0 & 2 & 0 & 0 \\ & & & & 1 & 0 & 4 & 1 \\ & \mathbf{0} & & & & 1 & 0 & 2 \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{bmatrix} \begin{array}{l} \\ -A_1 \\ \\ -A_2 \\ \\ -A_3 \\ \\ \\ \end{array} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -3 & 3 & -2 & 0 & -1 & 0 & 0 & 0 \\ 3 & 4 & -2 & 0 & 4 & 0 & 0 & 0 \\ 7 & -3 & 2 & 0 & 5 & 0 & 0 & 0 \end{bmatrix}$$

and where

$$K=\begin{array}{c c} \begin{array}{cc} \overset{n_1}{\longleftrightarrow} & \overset{n_2}{\longleftrightarrow} \end{array} & \\ \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} & \begin{array}{c} \updownarrow 1 \\ \updownarrow 1 \end{array} \end{array}, \quad \tilde{K}=\begin{array}{c c} \begin{array}{cccc} \overset{n_1}{\longleftrightarrow} & \overset{n_2}{\longleftrightarrow} & \overset{n_1}{\longleftrightarrow} & \overset{n_2}{\longleftrightarrow} \end{array} & \\ \begin{bmatrix} K_{11} & K_{12} & 0 & 0 \\ 0 & 0 & K_{21} & K_{22} \end{bmatrix} & \begin{array}{c} \updownarrow 1 \\ \updownarrow 1 \end{array} \end{array} \tag{19}$$

$$\tilde{G}=\begin{array}{c c} \begin{array}{cc} \overset{m_1}{\longleftrightarrow} & \overset{m_2}{\longleftrightarrow} \end{array} & \\ \begin{bmatrix} G_{11} & 0 \\ G_{21} & 0 \\ 0 & G_{12} \\ 0 & G_{22} \end{bmatrix} & \begin{array}{c} \updownarrow n_1 \\ \updownarrow n_2 \\ \updownarrow n_1 \\ \updownarrow n_2 \end{array} \end{array} = \begin{array}{c c} \begin{array}{cc} \overset{m_1}{\longleftrightarrow} & \overset{m_2}{\longleftrightarrow} \end{array} & \\ \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} & \begin{array}{c} \updownarrow n \\ \updownarrow n \end{array} \end{array}.$$

The remaining sub-matrices in Eq. (18) do not depend on K_{ij} or on G_{ij} and are given in the Appendix.

It will now be demonstrated that the local controller gains K^0 and the local observer gains G_i^0 that minimize (13) are given by

$$K^0(t)=-R^{-1}B'M_x^0(t) \tag{20}$$

$$G_i^0=P_i^0\bar{C}_i'R_{Ni}^{-1}, \quad i=1,2 \tag{21}$$

where

$$-\dot{M}_x^0=M_x^0A+A'M_x^0-M_x^0BR^{-1}B'M_x^0+Q$$
$$M_x^0(T)=0$$

$$\begin{aligned} \dot{P}_i^0&=AP_i^0+P_i^0A'-P_i^0\bar{C}_iR_{Ni}^{-1}\bar{C}_iP_i^0+(\bar{\bar{B}}_N\bar{\bar{B}}_N')_i \\ P_i(0)&=E\{x(0)x'(0)\}. \end{aligned} \tag{23}$$

To establish this result note that the evaluation of Eq. (13) for the closed-loop system (17), yields

$$\bar{V}=\tfrac{1}{2}E\left\{[x,e]'\begin{bmatrix} M_x & M_{xe} \\ M_{xe}' & M_e \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix} + \int_t^T tr\left\{\begin{bmatrix} \bar{\bar{B}}_N' & \bar{\bar{B}}_N' \\ 0 & -\tilde{C}_N'\tilde{G}' \end{bmatrix}\cdot\begin{bmatrix} M_x & M_{xe} \\ M_{xe}' & M_e \end{bmatrix}\begin{bmatrix} \bar{\bar{B}}_N & 0 \\ \bar{\bar{B}}_N & -\tilde{G}\tilde{C}_N \end{bmatrix}\right\}d\tau\right. \tag{24}$$

where

$$-\dot{M}_x=M_x(A+BK)+(A+BK)'M_x+Q+K'RK \tag{25}$$

$$-\dot{M}_{xe}=M_x(-B\tilde{K})+M_{xe}(\tilde{A}-\tilde{G}\tilde{C})+(A+BK)'M_{xe}-K'R\tilde{K} \tag{26}$$

$$-\dot{M}_e=M_{xe}'(-B\tilde{K})+M_e(\tilde{A}-\tilde{G}\tilde{C})-\tilde{K}'B'M_{xe}+(\tilde{A}-\tilde{G}\tilde{C})'M_e+\tilde{K}'R\tilde{K} \tag{27}$$

with boundary conditions,

$$M_x(T)=0, \quad M_{xe}(T)=0, \quad M_e(T)=0. \tag{28}$$

Since the positive semi-definite matrix Q appears only in (25), the optimum gain K is given by the same value as in the deterministic case, (20), with M_x^0 satisfying Eq. (22).

of the denominators of its elements and form $\mathcal{H}_{rr}$ as defined in (2.8) by expanding it according to (2.7).

(3) Subject $\mathcal{H}_{rr}$ to elementary row (or column) operations as described in Definition 5 and find the observability (or controllability) indices $\tilde{d}_i$'s $i=1,2,\ldots,p$ (or d_i's, $i=1,2,\ldots,m$) following Theorem II. Let T be the transforming matrix such that $TH=\tilde{H}$ (or $HT=\tilde{H}$) where $H=\mathcal{H}_{\tilde{d}+1,\tilde{d}+1}$, $\tilde{d}=\max \tilde{d}_i$ (or $H=\mathcal{H}_{d+1,d+1}$, $d=\max d_i$). Note that T and H are in general different in the process of obtaining observable and controllable forms, but for notational convenience they are shown to be the same. Further assume

$$\tilde{\beta}_{ik}=t_{p\tilde{d}_i+i,p\tilde{d}_i+k};k=1,2,\ldots,i-1;\; i=1,2,\ldots,p, \tag{4.2}$$

(or Eq. (3.5)) and define linearly independent n row (or column) vectors

$$\tilde{p}_j(i)=\tilde{h}_j(i)+\sum_{k=1}^{i-1}\tilde{\beta}_{ik}\tilde{h}_j(k),\, i=1,2,\ldots,p;\, j=0,1,\ldots,\tilde{d}_i, \tag{4.3}$$

where $\tilde{h}_j(i)$ denotes the i-th row of j-th row-block of H: see 2.8 (or define as (3.9)).

(4) With the aid of vectors as defined in (4.3) for j now varying up to $\tilde{d}$ (or in (3.9) for j varying up to d) a matrix $\mathcal{P}_{\tilde{d}}$ (or $\mathcal{P}_d$) is constructed and row (or column) transformed to $\tilde{\mathcal{P}}_{\tilde{d}}$ (or $\tilde{\mathcal{P}}_d$) by the transforming matrix $T_{\tilde{d}}$ (or T_d) such that $T_{\tilde{d}}\,\mathcal{P}_{\tilde{d}}=\tilde{\mathcal{P}}_{\tilde{d}}$ (or $\mathcal{P}_dT_d=\tilde{\mathcal{P}}_d$) is in the form:

$$\begin{bmatrix} T_{11} & & & & \\ T_{21} & T_{22} & & 0 & \\ \cdot & \cdot & & & \\ \cdot & \cdot & \ddots & & \\ \cdot & \cdot & & & \\ T_{\tilde{d}1} & T_{\tilde{d}2} & \cdots & T_{\tilde{d}\tilde{d}} & \\ -A_1 & -A_2 & \cdots & -A_{\tilde{d}} & I \end{bmatrix}\begin{bmatrix} P_0 & P_1 & \cdots & P_{\tilde{d}} \\ P_1 & P_2 & \cdots & P_{\tilde{d}+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & & \cdot \\ P_{\tilde{d}} & P_{\tilde{d}+1} & \cdots & P_{2\tilde{d}} \end{bmatrix}=\tilde{\mathcal{P}}_{\tilde{d}} \tag{4.4}$$

where in $\tilde{\mathcal{P}}_{\tilde{d}}$ the submatrix constituted by the last p rows is a null matrix and A_i's are defined by

$$A_i=\begin{bmatrix} a_{11}(\tilde{d}_1-\tilde{d}+i) & a_{12}(\tilde{d}_1-\tilde{d}+i) & \ldots & a_{1p}(\tilde{d}_1-\tilde{d}+i) \\ a_{21}(\tilde{d}_2-\tilde{d}+i) & a_{22}(\tilde{d}_2-\tilde{d}+i) & \ldots & a_{2p}(\tilde{d}_2-\tilde{d}+i) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & & \cdot \\ a_{p1}(\tilde{d}_p-\tilde{d}+i) & a_{p2}(\tilde{d}_p-\tilde{d}+i) & \ldots & a_{pp}(\tilde{d}_p-\tilde{d}+i) \end{bmatrix},$$

$$i=1,2,\ldots,\tilde{d}, \tag{4.5}$$

in which the elements $a_{ij}(k)$'s are identified with those given in (2.4a,b) (or (3.15) and (3.14) the elements $a_{ij}(k)$'s are identified with those given in (2.3a,b)).

(5) Having obtained the transforming matrix $T_{\tilde{d}}$ (or T_d) and knowing P_i's $i=0, 1, \ldots, \tilde{d}-1$ (or P_i's $i=0, 1, \ldots, d-1$) as in (4.3) (or in (3.15)) we construct

$$\begin{bmatrix} T_{11} & & & & \\ T_{21} & T_{22} & & 0 & \\ \vdots & \vdots & \ddots & & \\ T_{\tilde{d}1} & T_{\tilde{d}2} & \cdots & T_{\tilde{d}\tilde{d}} & \\ -A_1 & -A_2 & \cdots & -A_{\tilde{d}} & I \end{bmatrix} \begin{bmatrix} & & & & 0 \\ & 0 & & 0 & P_0 \\ & & & P_0 & P_1 \\ & & \ddots & \vdots & \vdots \\ P_0 & P_1 & \cdots & P_{\tilde{d}-2} & P_{\tilde{d}-1} \end{bmatrix} = \begin{bmatrix} & & & & 0 \\ & 0 & & 0 & X \\ & & & X & X \\ & & \ddots & \vdots & \vdots \\ B_{\tilde{d}} & B_{\tilde{d}-1} & \cdots & B_2 & B_1 \end{bmatrix} \tag{4.6}$$

(or as given in (3.18)) where B_i's (or C_j's) are defined

$$B_i = \begin{bmatrix} b_{\tilde{d}_1-\tilde{d}+i,1}(1) & b_{\tilde{d}_1-\tilde{d}+i,2}(1) & \cdots & b_{\tilde{d}_1-\tilde{d}+i,m}(1) \\ b_{\tilde{d}_2-\tilde{d}+i,1}(2) & b_{\tilde{d}_2-\tilde{d}+i,2}(2) & \cdots & b_{\tilde{d}_2-\tilde{d}+i,m}(2) \\ \vdots & \vdots & & \vdots \\ b_{\tilde{d}_p-\tilde{d}+i,1}(p) & b_{\tilde{d}_p-\tilde{d}+i,2}(p) & \cdots & b_{\tilde{d}_p-\tilde{d}+i,m}(p) \end{bmatrix}, \quad i=1,2,\ldots,\tilde{d} \tag{4.7}$$

(or as in (3.13)) in which $b_{ij}(k)$'s (or $c_{ij}(k)$'s) are identified with those given in (2.4c) (or (2.3c)).

(6) Define $\tilde{\beta}=[\tilde{\beta}_{ik}]$, a lower triangular matrix (or $\beta=[\beta_{ki}]$, an upper triangular matrix) by the elements as defined in (4.2) (or (3.5)) with unit diagonal elements. The i-th column (or row) of $\tilde{\beta}^{-1}$ (or β^{-1}) determines $\tilde{k}_i=\tilde{d}_1+\ldots+\tilde{d}_i$-th column (or $k_i=d_1+\ldots+d_i$-th row) of $\tilde{B}$ (or $\tilde{C}$).

Proof: The algorithm is a direct consequence of Theorem VII (or Theorems IV, V, and VI.

Example 2. Obtain an observable canonical form of the transfer matrix:

$$T(s)=\frac{1}{(s+1)^3}\begin{bmatrix} 4s^2+8s+11 & 5s^2+10s+7 \\ 7s^2+14s+28 & 5s^2+10s+11 \end{bmatrix}.$$

It is easy to see that here $E=0$, $T(s)=G(s)$ and $r=3$. The matrix $\mathscr{H}_{33}$ is constructed according to step 2 of the Algorithm and is subjected to elementary row transformations as follows:

$$
\begin{bmatrix}
1 & & & & & \\
-\frac{7}{4} & 1 & & & \mathbf{0} & \\
1 & 0 & 1 & & & \\
-3 & 1 & -3 & 1 & & \\
\frac{12}{5} & -\frac{9}{5} & 2 & 0 & 1 & \\
0 & 0 & -3 & 1 & -3 & 1
\end{bmatrix}
\begin{bmatrix}
4 & 5 & -4 & -5 & 11 & 7 \\
7 & 5 & -7 & -5 & 28 & 11 \\
-4 & -5 & 11 & 7 & -25 & -11 \\
-7 & -5 & 28 & 11 & -70 & -23 \\
11 & 7 & -25 & -11 & 46 & 17 \\
28 & 11 & -70 & -23 & 133 & 41
\end{bmatrix}
=
\begin{bmatrix}
4 & 5 & -4 & -5 & 11 & 7 \\
(0 & -3 & 0 & 3 & 7 & -1)\frac{5}{4} \\
0 & 0 & 7 & 2 & -14 & -4 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -28 & -8 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

whence observability indices are $\tilde{d}_1 = 3$, $\tilde{d}_2 = 1$. Because $p = 2$, in view of (4.2) $\tilde{\beta}_{21} = t_{43} = -3$. Then according to steps 4 and 5 of the Algorithm combining (4.4) and (4.6), we obtain

$$
\underbrace{\begin{bmatrix}
1 & & & & & & & \\
\frac{5}{4} & 1 & & & & \mathbf{0} & & \\
1 & 0 & 1 & & \cdot^{\cdot^{\cdot}} & & & \\
0 & 1 & 0 & 1 & & & & \\
-3 & -1.8 & 2 & 0 & 1 & & & \\
0 & 0 & 0 & 1 & 0 & 1 & & \\
\hline
1 & 0 & 3 & 0 & 3 & 0 & 1 & \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1
\end{bmatrix}}_{A_1 \;\; A_2 \;\; A_3}^{T_{\tilde{d}}}
\overbrace{\left[\begin{array}{cccccc|cccccccc}
 & & & & & \mathbf{0} & 4 & 5 & -4 & -5 & 11 & 7 & -25 & -11 \\
 & \mathbf{0} & & & & & -5 & -10 & 5 & 10 & -5 & -10 & 5 & 10 \\
 & & & \ddots & 4 & 5 & -4 & -5 & 11 & 7 & -25 & -11 & 46 & 17 \\
 & & & & -5 & -10 & 5 & 10 & -5 & -10 & 5 & 10 & -5 & -10 \\
 & & 4 & 5 & -4 & -5 & 11 & 7 & -25 & -11 & 46 & 17 & -74 & -25 \\
 & & -5 & -10 & 5 & 10 & -5 & -10 & 5 & 10 & -5 & -10 & 5 & 10 \\
4 & 5 & -4 & -5 & 11 & 7 & -25 & -11 & 46 & 17 & -74 & -25 & 109 & 35 \\
-5 & -10 & 5 & 10 & -5 & -10 & 5 & 10 & -5 & -10 & 5 & 10 & -5 & -10
\end{array}\right]}^{\mathscr{P}_{\tilde{d}}}
$$

$$
= \underbrace{\left[\begin{array}{cc|cc|cc|cccccccc}
 & & & & & \mathbf{0} & 4 & 5 & -4 & -5 & 11 & 7 & -25 & -11 \\
 & \mathbf{0} & & & & & (0 & -3 & 0 & 3 & 7 & -1 & -21 & -3)\frac{5}{4} \\
 & & & \ddots & 4 & 5 & 0 & 0 & 7 & 2 & -14 & -4 & 21 & 6 \\
 & & & & -5 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 4 & 5 & 4 & 5 & 0 & 0 & 0 & 0 & -28 & -8 & 84 & 24 \\
 & & -5 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
4 & 5 & 8 & 10 & 11 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-5 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]}_{B_3 \;\; B_2 \;\; B_1}^{\tilde{\mathscr{P}}_{\tilde{d}}},
$$

We also observe that

$$\tilde{\beta}=\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}.$$

Therefore, irreducible realization in observable canonical form is:

$$\tilde{A}=\begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & -3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \tilde{B}=\begin{bmatrix} 11 & 7 \\ 8 & 10 \\ 4 & 5 \\ -5 & -10 \end{bmatrix} \quad \text{and} \quad \tilde{C}=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}.$$

It is interesting to observe that there are other realizations in observable canonical forms of the transfer matrix in Example 2. viz.,

$$\tilde{A}=\begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & -3 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \tilde{B}=\begin{bmatrix} 16 & 17 \\ 3 & 0 \\ 4 & 5 \\ -5 & -10 \end{bmatrix} \quad \text{and} \quad \tilde{C}=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix},$$

although not with minimum number of non-fixed parameters.

V. Conclusion

A computational algorithm is developed in this paper to obtain state variable description of a time invariant finite dimensional, completely controllable and completely observable canonical forms. Furthermore, the obtained canonical form of A will contain a minimum number of non-fixed parameters. The major disadvantage of the proposed method seems to be the evaluation of an inverse of an upper triangular matrix of order m or p which is not difficult in view of a known recursive algorithm (**18**). Whether the properties of polynomial matrices can be applied to obtain the canonical forms will be an interesting topic of future research.

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