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Cominimal Projections in l_∞^n*

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Let $Y \subset l_\infty^n$ be a subspace of codimension two and let $\mathscr{P}(l_\infty^n, Y)$ denote the set of all linear projections from l_∞^n onto Y. A complete characterization of Y for which there exists $P_o \in \mathscr{P}(l_\infty^n, Y)$ such that $\|Id - P_o\| = 1$ will be given. Also an estimate from below of the constant

$$\lambda_I(Y, l_\infty^n) = \inf\{\|Id - P\|: P \in \mathscr{P}(l_\infty^n, Y)\}$$

as well as the formulas for cominimal projections in some particular cases will be presented. © 1999 Academic Press

1. INTRODUCTION

Let X be a normed space and let $Y \subset X$ be a linear subspace of X. A bounded linear operator $P: X \to Y$ is called a *projection* if $Py = y$ for any $y \in Y$. Denote by $\mathscr{P}(X, Y)$ the set of all projections from X onto Y. A projection P_0 is called *cominimal* iff

$$\|Id - P_0\| = \lambda_I(Y, X) = \inf\{\|Id - P\|: P \in \mathscr{P}(X, Y)\}. \tag{1.1}$$

The significance of this notion can be illustrated by the following well known inequality:

$$\begin{aligned}(1 + \|P\|)\operatorname{dist}(x, Y) &\geqslant \|Id - P\| \operatorname{dist}(x, Y)\\ &\geqslant \|(Id - P)(x)\| \geqslant \operatorname{dist}(x, Y)\end{aligned}$$

for every $x \in X \setminus Y$ and $P \in \mathscr{P}(X, Y)$.

This means that if $\|P\|$ or $\|Id - P\|$ is small then Px is a "good" linear replacement of any $x \in X$ in Y. It is easily seen that

$$\|Id - P\| \geqslant 1 \qquad \text{for every} \quad P \in \mathscr{P}(X, Y).$$

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It is also clear that if P_0 is a cominimal projection then

$$\|Id - P_0\| = \operatorname{dist}(Id, \mathscr{P}(X, Y)).$$

For more information concerning minimal and cominimal projections the reader is referred to [CL], [CM1], [CM2], [CP], [FMW], [Fr], [KT]. Also a more complete list of references can be found in [LO]. It is easy to prove that if Y is a hyperplane in X then $\lambda_I(Y, X) = 1$. If codim $Y > 1$ the formulas for cominimal projections as well as the value of the constant $\lambda_I(Y, X)$ are not known apart from some trivial cases.

The aim of this paper is to investigate the constant $\lambda_I(Y, l^n_\infty)$ where l^n_∞ denotes the space $\mathfrak{R}^n$ with the maximum norm and Y is a subspace of l^n_∞ of codimension two. We present a complete characterization of subspaces Y for which $\lambda_I(Y, X) = 1$ (Theorem 3.1). If $\lambda_I(Y, X) > 1$, an estimate from below of this constant will be shown (Theorem 3.5). Also a formula for cominimal projections as well as the exact value of $\lambda_I(Y, X)$ will be determined in some particular cases (Theorem 3.2, Example 3.3 and Theorem 3.9).

Now let us introduce some notions and results which will be of use later. By $S(X)$ we denote the unit sphere in a normed space X and by $ext(X)$ the set of its extreme points. The symbol $\mathscr{L}(X, Y)$ means the space of all linear, continuous mappings from X to Y. If Y is a linear subspace of X we write

$$\mathscr{L}_Y = \{L \in \mathscr{L}(X, Y) \colon L|_Y = 0\}.$$

It is obvious that

$$\lambda_I(Y, X) = \operatorname{dist}(Id - P, \mathscr{L}_Y)$$

for every $P \in \mathscr{P}(X, Y)$.

If $X = l^n_\infty$ the symbol T_{ij}, $i, j \in \{1, 2, ..., n\}$ stands for a transposition

$$T_{ij}(x_1, ..., x_i, ..., x_j, ..., x_n) = (x_1, ..., x_j, ..., x_i, ..., x_n), \tag{1.2}$$

where $x = (x_1, ..., x_n) \in \mathfrak{R}^n$. Now let X be a normed space. For any $x \in X$ set

$$E(x) = \{f \in ext(X^*) \colon f(x) = \|x\|\}.$$

Definition 1.1 [SW, Def. 5.1]. Let X be a real normed space, $x \in X \backslash \{0\}$ and let $Y \subset X$ be an n-dimensional linear subspace. A set $I = \{g^1, ..., g^k\} \subset ext(X^*)$ is called an *I-set* iff there exist positive numbers $\lambda^1, ..., \lambda^k$ such that

$$\sum_{i=1}^{k} \lambda^i g^i|_Y = 0.$$

If moreover $I \subset E(x)$ the I is called an *I-set with respect to x*. An I-set I is said to be *minimal*, if there is no proper subset of I which forms I-set. A minimal I-set I is called *regular* iff $k = n+1$ (by the Caratheodory theorem $n+1$ is the largest possible number (see [Ch])).

The importance of regular I-set is illustrated by

THEOREM 1.2 [SW, Th. 5.8]. *Let X be a real normed space. Let $x \in X \setminus Y$, $y \in Y$. If there exists a regular I-set for $x - y$ then y is a strongly unique best approximation to x in Y.*

From [RS] it immediately follows

THEOREM 1.3 [RS]. *Let X be a finite dimensional normed space. Then*

$$ext(\mathscr{L}^*(X)) = ext(X^*) \otimes ext(X),$$

where $(x^* \otimes x)(L) = x^*(Lx)$ for $x \in X$, $x^* \in X^*$ and $L \in \mathscr{L}(X, X)$.

LEMMA 1.4 (see, e.g., [BC]). *Assume X is a normed space and let $Y \subset X$ be a subspace of codimension k, $Y = \bigcap_{i=1}^{k} \ker g^i$ where $g^i \in X^*$ are linearly independent. Let $P \in \mathscr{P}(X, Y)$. Then there exist $y^1, ..., y^k \in X$ satisfying*

$$g^i(y^j) = \delta_{i,j}, \qquad i, j = 1, ..., k \tag{1.3}$$

such that

$$x - Px = \sum_{i=1}^{k} g^i(x)\, y^i \qquad \text{for} \quad x \in X. \tag{1.4}$$

On the other hand, if $y^1, ..., y^k \in Y$ satisfy (1.3) then the operator $P = Id - \sum_{i=1}^{k} g^i(\cdot)\, y^i$ belongs to $\mathscr{P}(X, Y)$.

LEMMA 1.5. *Let $X = l_\infty^n$ and let $Y = \bigcap_{i=1}^{k} \ker g^i$, $k \leqslant n$, where $g^i \in S(X^*)$ are linearly independent. Let $P \in \mathscr{P}(X, Y)$, $P = Id - \sum_{i=1}^{k} g^i(\cdot)\, y^i$ where $y^i \in \mathfrak{R}^n$, $i \in \{1, ..., k\}$. Then*

$$\|Id - P\| = \max_{i \in \{1, ..., n\}} \left(\sum_{s=1}^{n} \left| \sum_{j=1}^{k} g_s^j y_i^j \right| \right) \tag{1.5}$$

Proof. Let $x \in S(X)$. Then

$$\|(Id - P)(x)\| = \max_{i \in \{1, ..., n\}} \left| \sum_{j=1}^{k} g^j(x)\, y_i^j \right| \leqslant \max_{i \in \{1, ..., n\}} \left(\sum_{s=1}^{n} \left| \sum_{j=1}^{k} g_s^j y_i^j \right| \right).$$

Setting $x=(x_1, x_2, ..., x_n)$ such that

$$x_s = \begin{cases} sgn \sum_{j=1}^{k} g_s^j y_i^j & \text{if } \sum_{j=1}^{k} g_s^j y_i^j \neq 0 \\ 0 & \text{if } \sum_{j=1}^{k} g_s^j y_i^j = 0 \end{cases}$$

for $s=\{1, 2, ..., n\}$, we get the result. ∎

LEMMA 1.6 (see, e.g., [LO, Prop. II.7.1, p. 82]). *Let Y_1, Y_2 be two linear subspaces of a normed space X. Suppose that there is a linear isometry T of X into itself such that $T(Y_1)=Y_2$. Then $\lambda_I(Y_1, X)=\lambda_I(Y_2, X)$.*

Proof. Let us define a mapping Φ from $\mathscr{P}(X, Y_1)$ onto $\mathscr{P}(X, Y_2)$ by

$$\Phi(P) = T \circ P \circ T^{-1}.$$

Since Id commutes with T and T^{-1} it is easy to see that $\|Id - \Phi(P)\| = \|Id - P\|$ for any $P \in \mathscr{P}(X, Y_1)$ which completes the proof.

DEFINITION 1.7. Let X be a normed space and Y_1, Y_2 be two linear subspaces of X. It is said that Y_1 is equivalent up to isometry to Y_2 iff there is a linear isometry T of X into itself such that $T(Y_1)=Y_2$.

2. TECHNICAL LEMMAS

In this section, unless otherwise stated, we assume that $n \in \mathbb{N}$, $n \geqslant 3$.

LEMMA 2.1 (see [Le, Lemma 2.1]). *Let $Y \subset l_\infty^n$ be a subspace of codimension two, $Y = \ker g^1 \cap \ker g^2$ where $g^1, g^2 \in S(l_1^n)$ are linearly independent functionals. Then there is a linear subspace $\tilde{Y} \subset l_\infty^n$ equivalent up to isometry to Y such that $\tilde{Y} = \ker \tilde{g}^1 \cap \ker \tilde{g}^2$, where $\tilde{g}^1, \tilde{g}^2 \in S(l_1^n)$ are of the form $\tilde{g}^1 = (\tilde{g}_1^1, 0, \tilde{g}_3^1, ..., \tilde{g}_n^1)$, $\tilde{g}^2 = (0, \tilde{g}_2^2, \tilde{g}_3^2, ..., \tilde{g}_n^2)$, $\tilde{g}_1^1, \tilde{g}_2^2 > 0$, $\tilde{g}_j^1, \tilde{g}_j^2 \geqslant 0$ for $j \in \{3, ..., n\}$.*

Let $g^1, g^2 \in S(l_1^n)$ be linearly independent functionals such that

$$g^1 = (g_1^1, 0, g_3^1, ..., g_n^1) \tag{2.1}$$

$$g^2 = (0, g_2^2, g_3^2, ..., g_n^2), \tag{2.2}$$

$$g_1^1, g_2^2 > 0, \quad g_j^1, g_j^2 \geqslant 0 \quad \text{and} \quad g_j^1 + g_j^2 > 0 \qquad \text{for} \quad j \in \{1, ..., n\}. \tag{2.3}$$

Hence $Y = \ker g^1 \cap \ker g^2$ is a subspace of codimension two in $\mathfrak{R}^n$. Let $y^1, y^2 \in \mathfrak{R}^n$, satisfy (1.3) and let $P_0 \in \mathscr{P}(l^n_\infty, Y)$ be the projection determined by y^1, y^2 (see Lemma 1.4), which means

$$(Id - P_0)(x) = g^1(x)\, y^1 + g^2(x)\, y^2.$$

For $s \in \{2, ..., n\}$ we put

$$u_k = g^1_1 \sum_{j=k+1}^{n} g^2_j$$

$$v_k = g^2_2 \sum_{i=3}^{k} g^1_i,$$

where by definition $v_2 = u_n = 0$.

Now some useful properties of the functionals g^1 and g^2 will be shown.

Lemma 2.2. *There is only one number* $s \in \{3, ..., n\}$ *satisfying two inequalities*:

$$u_x \leqslant v_s \tag{2.4}$$

$$v_{s-1} < u_{s-1}. \tag{2.5}$$

Proof. The sequence $(u_k)_{k \in \{2, ..., n\}}$ is decreasing, while the sequence $(v_k)_{k \in \{2, ..., n\}}$ is increasing. If neither $u_3 \leqslant v_3$ nor $v_{n-1} < u_{n-1}$, the number

$$s = \min\{k \colon u_k \leqslant v_k\} \in \{4, ..., n-1\}$$

clearly satisfies the lemma. It follows easily that if $u_3 \leqslant v_3$ then $u_{n-1} < v_{n-1}$ and we get $s = 3$. Analogously if $v_{n-1} < u_{n-1}$ then $v_3 < u_3$ and $s = n$. ∎

Lemma 2.3. *There is only one number* $s \in \{3, ..., n\}$ *satisfying two inequalities*:

$$u_s < v_s \tag{2.6}$$

$$v_{s-1} \leqslant u_{s-1}. \tag{2.7}$$

Proof. If neither $u_3 < v_3$ nor $v_{n-1} \leqslant u_{n-1}$, the number

$$s = \min\{k \colon u_k < v_k\} \in \{4, ..., n-1\}$$

satisfies the lemma. If $u_3 < v_3$ then $u_{n-1} < v_{n-1}$ and $s = 3$, if $v_{n-1} \leqslant u_{n-1}$ then $v_3 < u_3$ and $s = n$. ∎

The only s constructed in Lemma 2.2 will be denoted by s_a and the only s constructed in Lemma 2.3 by s_b.

LEMMA 2.4. *There are two possibilities*: $(s_a = s_b)$ *or* $(s_a = s_b - 1)$.

Proof. By definition s_a and s_b we ge that $s_a \leqslant s_b$. If in (2.4) we have $u_{s_a} < v_{s_a}$ then s_a satisfies Lemma 2.3 and we have $s_a = s_b$. If in (2.4) we have $u_{s_a} = v_{s_a}$ then it is easy to check that $u_{s_a+1} < v_{s_a+1}$ and $s_b = s_a + 1$ satisfies Lemma 2.3. ∎

Let $s \in \{3, ..., n\}$ and $g^1, g^2 \in S(l_1^n)$ be linearly independent functionals satisfying (2.1)–(2.3). Suppose

$$\det \begin{bmatrix} g_i^1 & g_j^1 \\ g_i^2 & g_j^2 \end{bmatrix} \neq 0 \tag{2.8}$$

for every $i, j \in \{1, 2, ..., n\}$, $i \neq j$, then we set

$$I = \left\{ i \in \{3, ..., n\} : \frac{g_s^1}{g_s^2} > \frac{g_i^1}{g_i^2} \right\},$$

$$J = \left\{ j \in \{3, ..., n\} : \frac{g_s^1}{g_s^2} < \frac{g_j^1}{g_j^2} \right\}.$$

THEOREM 2.5. *Let*

$$\phi^1 = e_1 \otimes (1, -1, 1, ..., 1)$$
$$\phi^2 = e_2 \otimes (-1, 1, 1, ..., 1)$$
$$\phi^s = e_s \otimes (1, 1, ..., 1)$$
$$\phi_1^i = e_i \otimes (1, 1, ..., 1)$$
$$\phi_2^i = e_i \otimes (-1, 1, 1, ..., 1)$$
$$\phi_1^j = e_j \otimes (1, 1, ..., 1)$$
$$\phi_2^j = e_j \otimes (1, -1, 1, ..., 1)$$

for $i \in I$, $j \in J$, *where* $e_k(x) = x_k$ *for* $x \in \mathfrak{R}^n$ *and* $k \in \{1, ..., n\}$.
Then $\{\phi^1, \phi^2 \phi^s, \phi_1^i, \phi_2^i \phi_1^j, \phi_2^j\}$, $(i \in I, j \in J)$ *is a minimal, regular I-set.*

Proof. Consider the following equation:

$$\lambda^1 \phi^1|_{\mathscr{L}_Y} + \lambda^2 \phi^2|_{\mathscr{L}_Y} + \lambda^s \phi^s|_{\mathscr{L}_Y} + \sum_{i \in I} (\lambda_1^i \phi_1^i|_{\mathscr{L}_Y} + \lambda_2^i \phi_2^i|_{\mathscr{L}_Y})$$
$$+ \sum_{j \in J} (\lambda_1^j \phi_1^j|_{\mathscr{L}_Y} + \lambda_2^j \phi_2^j|_{\mathscr{L}_Y}) = 0 \tag{2.9}$$

with unknown variables λ^1, λ^2, λ^s, λ_1^i, λ_2^i, λ_1^j, λ_2^j, $(i \in I, j \in J)$. Note that $\dim \mathscr{L}_Y = 2(n-2)$ and the mappings $\{g^1(\cdot)\, w^k, g^2(\cdot)\, w^k, \}$, $k \in \{3, ..., n\}$

form basis of $\mathscr{L}_Y$. (Here $w^k = (-g^1_k/g^1_1, -g^2_k/g^2_2, 0, ..., 0, 1, 0, ..., 0) \in \mathfrak{R}^n$ where 1 is equal to the k-th coordinate.)

Fix $\lambda^1 = 1$. Taking the value of the both sides of (2.9) on the elements $\{g^1(\cdot)\, w^s, g^2(\cdot)\, w^s\}$, we get

$$\begin{cases} \lambda^s = \dfrac{g^1_s}{g^1_1} + \lambda^2 \dfrac{g^2_s}{g^2_2}(1 - 2g^1_1) \\ \lambda^s = \dfrac{g^1_s}{g^1_1}(1 - 2g^2_2) + \lambda^2 \dfrac{g^2_s}{g^2_2}. \end{cases} \tag{2.10}$$

Hence

$$\lambda^2 = \frac{g^1_s (g^2_2)^2}{g^2_s (g^1_1)^2} > 0. \tag{2.11}$$

By (2.10) and (2.11)

$$\lambda^s = \frac{g^1_s\, g^2_2}{(g^1_1)^2}\left[g^2_2(1 - g^1_1) + g^1_1(1 - g^2_2)\right] > 0. \tag{2.12}$$

Now let $k \in I \cup J$. Put

$$a^k_1 = \frac{g^1_k}{g^1_1} + \frac{g^2_k\, g^1_s\, g^2_2}{g^2_s (g^1_1)^2}(1 - 2g^1_1)$$

$$a^k_2 = \frac{g^1_k}{g^1_1}(1 - 2g^2_2) + \frac{g^2_k\, g^1_s\, g^2_2}{g^2_s (g^1_1)^2}.$$

Taking the value of the both sides of (2.9) on the elements $\{g^1(\cdot)\, w^i, g^2(\cdot)\, w^i\}$, for $i \in I$ we get

$$\begin{cases} \lambda^i_1 + \lambda^i_2 (1 - 2g^1_1) = a^i_1 \\ \qquad\qquad \lambda^i_1 + \lambda^i_2 = a^i_2. \end{cases}$$

Applying the Cramer rule we get $\lambda^i_1 = W^i_1/W^i$, $\lambda^i_2 = W^i_2/W^i$, where

$$W^i = \det\begin{bmatrix} 1 & 1 - 2g^1_1 \\ 1 & 1 \end{bmatrix} = 2g^1_1 > 0$$

$$W^i_1 = \det\begin{bmatrix} a^i_1 & 1 - 2g^1_1 \\ a^i_2 & 1 \end{bmatrix} = 2\frac{g^1_i}{g^1_1}\left[g^1_1(1 - g^2_2) + g^2_2(1 - g^1_1)\right]$$

$$W^i_2 = \det\begin{bmatrix} 1 & a^i_1 \\ 1 & a^i_2 \end{bmatrix} = 2\frac{g^2_i\, g^1_s\, g^2_2}{g^2_s (g^1_1)^2}\left[g^2_i - \frac{g^1_i\, g^2_s}{g^1_s}\right].$$

It is obvious that $W_1^i > 0$ and $\lambda_1^i > 0$. If $i \in I$ then $W_2^i > 0$ and consequently, $\lambda_2^i > 0$. Taking the value of the both sides of (2.9) on the elements $\{g^1(\cdot)\, w^j, g^2(\cdot)\, w^j\}$, for $j \in J$ we get

$$\begin{cases} \lambda_1^j + \lambda_2^j = a_1^j \\ \lambda_1^j + \lambda_2^j(1 - 2g_2^2) = a_2^j. \end{cases}$$

In the same way we obtain $\lambda_1^j > 0$, $\lambda_2^j > 0$.

It is easy to check that the above constructed I-set is regular and minimal, which gives the result. ∎

3. THE MAIN RESULTS

THEOREM 3.1. *Let $g^1, g^2, ..., g^k \in S(l_1^n)$, $k \leqslant n$, be linearly independent functionals such that $g_j^i \geqslant 0$ for every $i \in \{1, 2, ..., k\}$, $j \in \{1, 2, ..., n\}$, $g_i^i > 0$, $g_j^i = 0$ for every $i, j \in \{1, 2, ..., k\}$, $i \neq j$. Put $Y = \bigcap_{i=1}^k \ker g^i$. Let $y^i \in l_\infty^n$, and $P_0 \in \mathscr{P}(l_\infty^n)$ satisfy* (1.3) *and* (1.4).

Then $\|Id - P_0\| = 1$ if and only if for every $i \neq j$ $supp(g^i) \cap supp(g^j) = \varnothing$, where

$$supp(g^i) = \{k\colon g_k^i \neq 0\}.$$

Moreover if $g_j^i \neq 0$ then for every $t \in \{1, ..., k\}$,

$$y_j^t = \begin{cases} 0 & \text{if} \quad i \neq t \\ 1 & \text{if} \quad i = t. \end{cases} \tag{3.1}$$

Proof. Suppose that $\|Id - P_o\| = 1$. Then by (1.5)

$$1 = \|Id - P_o\| \geqslant |y_j^1 + y_j^2 + \cdots + y_j^k|$$

for every $j \in \{1, 2, ..., n\}$. Since $g^i \in S(l_1^n)$, by (1.3),

$$y_j^1 + y_j^2 + \cdots + y_j^k = 1. \tag{3.2}$$

Note that by Lemma 1.5, for every $i \in \{1, ..., k\}$ and $j \in \{1, ..., n\}$

$$1 = \|Id - P_o\| \geqslant g_1^1 y_j^1 + \cdots + g_{i-1}^{i-1} y_j^{i-1} - g_i^i y_j^i + g_{i+1}^{i+1} y_j^{i+1} + \cdots$$
$$+ g_k^k y_j^k + \sum_{t=k+1}^{n} \sum_{i=1}^{k} g_t^i y_j^i.$$

By (3.2) and the above inequality, $0 \geqslant -2g_i^i y_j^i$ and consequently

$$y_j^i \geqslant 0 \tag{3.3}$$

for $i \in \{1, ..., k\}$ and $j \in \{1, ..., n\}$. Now let $g_j^i > 0$ for some $j \neq i$. Then for $t \in \{1, ..., k\}$, $t \neq i$

$$y_j^t = \left(- \sum_{k:\, g_k^i \neq 0} g_k^i y_k^t \right) \Big/ g_j^i. \tag{3.4}$$

Consequently, by (3.3), for any $t \neq i$ $y_j^t = 0$. In view of (3.2) $y_j^i = 1$, which proves (3.1). Hence for every i such that $g_j^i > 0$, $y_j^i = 1$. By (3.1), there is at most one $i \in \{1, ..., k\}$ with $g_j^i > 0$ which proves that $supp(g^i) \cap supp(g^l) = \emptyset$ if $i \neq l$. Conversely, suppose that $supp(g^i) \cap supp(g^l) = \emptyset$ for $i \neq l$. For any $i \in \{1, ..., k\}$, $j \in \{1, ..., n\}$ define $y_j^i = 1$ if $g_j^i > 0$ and $y_j^i = 0$ in the opposite case. Put $y^i = (y_l^i, ..., y_n^i)$. Since $supp(g^i) \cap supp(g^l) = \emptyset$ for $l \neq i$, $g^i(y^l) = \delta_{i,l}$. Let $P_o \in \mathscr{P}(l_\infty^n, Y)$ be the projection determined by $y^1, ..., y^k$ (see Lemma 1.4). But Lemma 1.5, $\|Id - P_o\| = 1$, which completes the proof. ∎

Now let $n \geqslant 3$, $s = s_a$ (see Lemma 2.4) and $g^1, g^2 \in S(l_1^n)$ be linearly independent functionals satisfying (2.1)–(2.3), and (2.8). Suppose additionally that

$$\frac{g_3^1}{g_3^2} < \frac{g_4^1}{g_4^2} < \cdots < \frac{g_n^1}{g_n^2}. \tag{3.5}$$

Note that if in (2.8) we set $i = 1$ then we have $g_j^2 \neq 0$ for $j \in \{3, ..., n\}$, on the other hand if we set $j = 2$ then $g_i^1 \neq 0$ for $i \in \{3, ..., n\}$.

Put $Y = \ker g^1 \cap \ker g^2$ and $x_s = (g_s^2 g_1^1 + g_s^1 g_2^2)/(g_s^1 \sum_{j=s}^n g_j^2 + g_s^2 \sum_{i=3}^{s-1} g_i^1)$.

THEOREM 3.2. *If*

$$1 + x_s \geqslant \max \left\{ \frac{g_3^2}{g_3^1} + 2g_2^2; \frac{g_n^1}{g_n^2} + 2g_1^1 \right\} \tag{3.6}$$

then

$$d_s = \frac{g_s^2 g_1^1 + g_s^1 g_2^2}{g_s^2 g_1^1 + g_s^1 g_2^2 - 2g_1^1 g_2^2 (g_s^1 \sum_{j=s}^n g_j^2 + g_s^2 \sum_{i=3}^{s-1} g_i^1)} = \lambda_I(Y, l_\infty^n). \tag{3.7}$$

Moreover there is a strongly unique (in particular unique) minimal projection. This projection is determined by the vectors $y^1, y^2 \in \mathfrak{R}^n$ satisfy (1.3) such that

$$y_k^1 = \begin{cases} 0 & \textit{if} \quad k \in \{3, ..., s-1\} \\ \dfrac{g_2^2 \sum_{i=3}^s g_i^1 - g_1^1 \sum_{j=s+1}^n g_j^2}{g_s^2 g_1^1 + g_s^1 g_2^2 - 2 g_1^1 g_2^2 (g_s^1 \sum_{j=s}^n g_j^2 + g_s^2 \sum_{i=3}^{s-1} g_i^1)} & \textit{if} \quad k = s \\ d_s & \textit{if} \quad k \in \{s+1, ..., n\} \end{cases} \tag{3.8}$$

$$y_k^2 = \begin{cases} d_s & \textit{if} \quad k \in \{3, ..., s-1\} \\ \dfrac{g_1^1 \sum_{j=s}^n g_j^2 - g_2^2 \sum_{i=3}^{s-1} g_i^1}{g_s^2 g_1^1 + g_s^1 g_2^2 - 2 g_1^1 g_2^2 (g_s^1 \sum_{j=s}^n g_j^2 + g_s^2 \sum_{i=3}^{s-1} g_i^1)} & \textit{if} \quad k = s \\ 0 & \textit{if} \quad k \in \{s+1, ..., n\}. \end{cases} \tag{3.9}$$

Proof. Consider a system of equations

$$\begin{aligned} &\phi^k(g^1(\cdot)\, y^1 + g^2(\cdot)\, y^2) = d_s \qquad \text{for} \quad k \in \{1, 2, s\} \\ &\phi_l^k(g^1(\cdot)\, y^1 + g^2(\cdot)\, y^2) = d_s \qquad \text{for} \quad k \in \{3, ..., n\} \setminus \{s\}, \qquad l \in \{1, 2\} \end{aligned} \tag{3.10}$$

$$\begin{aligned} g^1(y^1) = g^2(y^2) = 1 \\ g^1(y^2) = g^2(y^1) = 0. \end{aligned} \tag{3.11}$$

By the definition of ϕ^k, ϕ_l^k (see Theorem 2.5), (3.10), (3.11) can be rewritten in the form

$$y_1^1 + (1 - 2g_2^2)\, y_1^2 = d_s \tag{3.12}$$

$$(1 - 2g_1^1)\, y_2^1 + y_2^2 = d_s \tag{3.13}$$

$$y_s^1 + y_s^2 = d_s \tag{3.14}$$

$$\begin{aligned} y_i^1 = 0, \qquad y_i^2 = d_s \qquad \text{for} \quad i \in I \\ y_j^1 = d_s, \qquad y_j^2 = 0 \qquad \text{for} \quad j \in J. \end{aligned} \tag{3.15}$$

From this we get:

$$\begin{aligned} d_s &= \frac{g_s^2 g_1^1 + g_s^1 g_2^2}{g_s^2 g_1^1 + g_s^1 g_2^2 - 2 g_1^1 g_2^2 (g_s^1 \sum_{j=s}^n g_j^2 + g_s^2 \sum_{i=3}^{s-1} g_i^1)} \\ y_s^1 &= \frac{g_2^2 \sum_{i=3}^s g_i^1 - g_1^1 \sum_{j=s+1}^n g_j^2}{g_s^2 g_1^1 + g_s^1 g_2^2 - 2 g_2^2 (g_s^1 \sum_{j=s}^n g_j^2 + g_s^2 \sum_{i=3}^{s-1} g_i^1)} \\ y_s^2 &= \frac{g_1^1 \sum_{j=s}^n g_j^2 - g_2^2 \sum_{i=3}^{s-1} g_i^1}{g_s^2 g_1^1 + g_s^1 g_2^2 - 2 g_1^1 g_2^2 (g_s^1 \sum_{j=s}^n g_j^2 + g_s^2 \sum_{i=3}^{s-1} g_i^1)}. \end{aligned} \tag{3.16}$$

Note that if $s_a = s_b$ then $y^1_s > 0$ and $y^2_s > 0$. If $s_b = s_a + 1$ then $y^1_s = 0$ and $y^2_s > 0$. Let $P_0 \in \mathcal{P}(l^n_\infty, Y)$ be the projection determined by y^1 and y^2. By Lemmas 2.2–2.4 and (3.14) we have

$$\phi^s(Id - P_0) = \|Id - P_0\| = d_s. \tag{3.17}$$

By (3.15) it is easy to see that

$$\phi^k_l(Id - P_0) = \|Id - P_0\| = d_s \qquad \text{for} \quad k \in \{3, ..., n\} \setminus \{s\}, \quad l \in \{1, 2\}. \tag{3.18}$$

Note that

$$y^2_1 = y^1_2 = -\frac{g^1_s \sum_{j=s}^n g^2_j + g^2_s \sum_{i=3}^{s-1} g^1_i}{g^2_s g^1_1 + g^1_s g^2_2 - 2 g^1_1 g^2_2 (g^1_s \sum_{j=s}^n g^2_j + g^2_s \sum_{i=3}^{s-1} g^1_i)}.$$

Hence by (3.12), (3.13) we get

$$\frac{y^1_1}{-y^2_1} = \frac{d_s}{-y^2_1} + 1 - 2g^2_2$$
$$\frac{y^2_2}{-y^1_2} = \frac{d_s}{-y^1_2} + 1 - 2g^1_1.$$

In view of (3.6) and (3.16)

$$\frac{y^1_1}{-y^2_1} \geqslant \frac{g^2_3}{g^1_3}$$
$$\frac{y^2_2}{-y^1_2} \geqslant \frac{g^1_n}{g^2_n},$$

which by (3.5) gives

$$\phi^1(Id - P_0) = \|Id - P_0\| = d_s, \tag{3.19}$$
$$\phi^2(Id - P_0) = \|Id - P_0\| = d_s. \tag{3.20}$$

Consequently, the functionals ϕ^1, ϕ^2, ϕ^s, ϕ^i_1, ϕ^i_2, ϕ^j_1, ϕ^j_2 form a regular *I*-set, with respect to $Id - P_0$.

By Theorem 1.3 ϕ^1, ϕ^2, ϕ^s, ϕ^i_1, ϕ^i_2, ϕ^j_1, $\phi^j_2 \in ext(\mathcal{L}^*(l^n_\infty))$. From (3.17)–(3.20) it follows that this *I*-set is contained in $E(Id - P_0)$.

By Theorem 1.2, 0 is the unique best approximation for $Id - P_0$ in $\mathcal{L}_Y$, which means that $Id - P_0$ is the unique minimal projection and we get (3.7)–(3.9). ∎

EXAMPLE 3.3. 1. Let $n = 3$, $g^1 = (1/3, 0, 2/3)$, $g^2 = (0, 3/4, 1/4)$ satisfy (2.1)–(2.3), (2.8), (3.5), $Y = \ker g^1 \cap \ker g^2$. Then $s = 3$ and $y^1_1 / -y^2_1 = 3 \geqslant g^2_3 / g^1_3$ and $y^2_2 / -y^1_2 = 23/6 \geqslant g^1_3 / g^2_3$ which give (3.6).

By Theorem 3.2 we get $d_3=7/6$ satisfies (3.7) and projection P_0 is cominimal.

2. Put $n=4$, $g^1=(1/3, 0, 1/3, 1/3)$, $g^2=(0, 1/2, 1/3, 1/6)$ satisfy (2.1)–(2.3), (2.8), (3.5), $Y=\ker g^1 \cap \ker g^2$. Note that $s=3$ and $y_1^1/-y_1^2=5/3 \geqslant g_3^2/g_3^1$, $y_2^2/-y_2^1=2 \geqslant g_4^1/g_4^2$ so we have (3.6), $d_3=5/4$ satisfies (3.7) and projection P_0 is cominimal.

3. Put $n=5$, $g^1=(2/3, 0, 1/27, 1/9, 5/27)$, $g^2=(0, 3/4, 1/12, 1/9, 1/18)$ satisfy (2.1)–(2.3), (2.8), (3.5), $Y=\ker g^1 \cap \ker g^2$. Now $s=4$, $y_1^1/-y_1^2=71/11 \geqslant g_3^2/g_3^1$, $y_2^2/y_2^1=437/66 \geqslant g_5^1/g_5^2$ and $d_4=153/131$ satisfies (3.7), and projection P_0 is cominimal.

4. Let $n=7$, $g^1=(95/298, 0, 27/298, 43/298, 27/298, 81/298, 25/298)$, $g^2=(0, 94/200, 21/200, 28/200, 14/200, 34/200, 9/200)$ satisfy (2.1)–(2.3), (2.8), (3.5), $Y=\ker g^1 \cap \ker g^2$. We have $s=4$, $y_1^1/-y_1^2 \approx 1.57938 \geqslant g_3^2/g_3^1$, $y_2^2/-y_2^1 \approx 1.888180 \geqslant g_7^1/g_7^2$, $d_4 \approx 1.24568$ satisfies (3.7), and projection P_0 is cominimal.

Remark 3.4. Note that if $s_b=s_a+1$ then $x_{s_b}=x_{s_a}$. If we assume $s=s_b$ in Theorem 3.2 then $d_{s_a}=d_{s_b}=y_{s_b}^1=y_{s_a}^2$ and $y_{s_a}^1=y_{s_b}^2=0$.

If (3.6) is valid and $s=s_b$ then by Theorem 3.2 $d_{s_b}=\lambda_I(Y, l_\infty^n)$ and we get the cominimal projection from Theorem 3.2 for $s=s_a$.

THEOREM 3.5. *Suppose that* (3.6) *does not hold. Then*

$$1<d_s<\lambda_I(Y, l_\infty^n).$$

Proof. Firstly we show that $1<d_s$. We need only to prove that:

$$g_s^2 g_1^1+g_s^1 g_2^2-2g_1^1 g_2^2\left(g_s^1 \sum_{j=s}^{n} g_j^2+g_s^2 \sum_{i=3}^{s-1} g_i^1\right)>0.$$

Note that

$$\begin{aligned}
&g_s^2 g_1^1+g_s^1 g_2^2-2g_1^1 g_2^2\left(g_s^1 \sum_{j=s}^{n} g_j^2+g_s^2 \sum_{i=3}^{s-1} g_i^1\right)\\
&\quad=g_s^2 g_1^1\left(1-2g_2^2 \sum_{i=3}^{s} g_i^1\right)+g_s^1 g_2^2\left(1-2g_1^1 \sum_{j=s+1}^{n} g_j^2\right)\\
&\quad=g_s^2 g_1^1\left[g_2^2\left(1-\sum_{i=3}^{s} g_i^1\right)+\left(\sum_{j=s}^{n} g_j^2-g_2^2 \sum_{i=3}^{s-1} g_i^1\right)+\sum_{i=3}^{s-1} g_i^2\right]-g_1^1 g_2^2 g_s^1 g_s^2\\
&\qquad+g_s^1 g_2^2\left[g_1^1\left(1-\sum_{j=s+1}^{n} g_j^2\right)+\left(\sum_{i=3}^{s} g_i^1-g_1^1 \sum_{j=s+1}^{n} g_j^2\right)+\sum_{j=s+1}^{n} g_j^1\right].
\end{aligned}$$

Note that

$$-g_1^1 g_2^2 g_s^1 g_s^2 + g_s^1 g_2^2 \left[g_1^1 \left(1 - \sum_{j=s+1}^{n} g_j^2\right) + \left(\sum_{i=3}^{s} g_i^1 - g_1^1 \sum_{j=s+1}^{n} g_j^2\right) + \sum_{j=s+1}^{n} g_j^1 \right]$$
$$= g_s^1 g_2^2 \left[g_1^1 \left(1 - \sum_{j=s}^{n} g_j^2\right) + \left(\sum_{i=3}^{s} g_i^1 - g_1^1 \sum_{j=s+1}^{n} g_j^2\right) + \sum_{j=s+1}^{n} g_j^1 \right].$$

By Lemma 2.2 we get the result.

The inequality $d_s < \lambda_I(Y, l_\infty^n)$ follows from Theorem 2.5, (3.17)–(3.20) and from the fact that if we have a functional F of norm 1 vanishing on L_Y then

$$\lambda_I(Y, l_\infty^n) \geqslant F(Id - P)$$

for any $P \in \mathcal{P}(l_\infty^n, Y)$. ∎

EXAMPLE 3.6. 1. Let $n = 4$, $g^1 = (1/3, 0, 1/3, 1/3)$, $g^2 = (0, 1/2, 4/10, 1/10)$ satisfy (2.1)–(2.3), (2.8), (3.5), $Y = \ker g^1 \cap \ker g^2$. Now $s = 3$, but $y_2^2/-y_2^1 = 32/15 < g_4^1/g_4^2$ so (3.6) does not hold. We get $d_3 = 27/11 < \lambda_I(Y, l_\infty^4) \leqslant 1.39092$.

2. Let $n = 5$, $g^1 = (6/21, 0, 6/21, 5/21, 4/21)$, $g^2 = (0, 3/17, 7/17, 4/17, 3/17)$ satisfy (2.1)–(2.3), (2.8), (3.5), $Y = \ker g^1 \cap \ker g^2$. Now we have $s = 4$ and $y_1^1/-y_1^2 = 1312/1003 < g_3^2/g_3^1$ so (3.6) does not hold.

We get $d_4 = 1547/1311 < \lambda_I(Y, l_\infty^5) \leqslant 1.31580$.

Remark 3.7. If $g^1, g^2 \in S(l_1^n)$ have negative coordinates then by Lemma 2.1 there exist functionals $\tilde{g}^1, \tilde{g}^2 \in S(l_1^n)$ such that (2.1)–(2.3) are satisfied and $\tilde{Y} = \ker \tilde{g}^1 \cap \ker \tilde{g}^2$ is equivalent up to isometry (see Def. 1.7) to $Y = \ker g^1 \cap \ker g^2$.

By Lemma 1.6, $\lambda_I(Y, l_\infty^n) = \lambda_I(\tilde{Y}, l_\infty^n)$. Moreover, if $\tilde{P}_0 \in \mathcal{P}(l_\infty^n, \tilde{Y}\}$ is a cominimal projection then $P_0 = A^{-1} \circ \tilde{P}_0 \circ A \in \mathcal{P}(l_\infty^n, Y)$ is a cominimal projection onto Y. Here A is a linear isometry from l_∞^n onto itself such that $A(Y) = \tilde{Y}$. Also the estimate from below presented in Theorem 3.5 is invariant under linear isometries. Hence by Lemma 2.1, Theorem 3.5 works for any Y with $\lambda_I(Y, l_\infty^n) < 1$. Note that Theorem 3.1 gives a complete characterization of this case.

Remark 3.8. The formula from Theorem 3.2 and the estimate from Theorem 3.5 remain true if (2.8) is not satisfied. It follows easily from the fact that a function

$$(f, g) \to \lambda_I(\ker(f) \cup \ker(g), l_\infty^n)$$

is continuous (where $f, g \in S(l_1^n)$).

Theorem 3.9. If $n=3$ then projection P_0 given by (3.8), (3.9) is cominimal.

Proof. If $n=3$ then

$$\frac{y_1^1}{-y_1^2}-\frac{g_3^2}{g_3^1}=1-2g_2^2+\frac{g_3^2 g_1^1+g_3^1 g_2^2}{g_3^1(1-g_2^2)}=g_2^2\left(\frac{1}{g_3^1}+\frac{1}{g_3^2}-2\right)>0,$$
$$\frac{y_2^2}{-y_2^1}-\frac{g_3^1}{g_3^2}=1-2g_1^1+\frac{g_3^1 g_2^2+g_3^2 g_1^1}{g_3^2(1-g_1^1)}=g_1^1\left(\frac{1}{g_3^1}+\frac{1}{g_3^2}-2\right)>0,$$

which gives the result. ∎

Remark 3.10. If $n=3$ then

$$d_3=\frac{g_2^2 g_3^1+g_1^1 g_3^2}{g_2^2 g_3^1+g_1^1 g_3^2-2g_1^1 g_2^2 g_3^1 g_3^2}.$$

The cominimal projection is determined by the vectors $y^1, y^2 \in l_\infty^3$, satisfying (1.3) such that

$$y_3^1=\frac{g_2^2 g_3^1}{g_2^2 g_3^1+g_1^1 g_3^2-2g_1^1 g_2^2 g_3^1 g_3^2}$$
$$y_3^2=\frac{g_1^1 g_3^2}{g_2^2 g_3^1+g_1^1 g_3^2-2g_1^1 g_2^2 g_3^1 g_3^2}.$$

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