INTEGRABLE HAMILTONIAN SYSTEMS CONNECTED WITH GRADED LIE ALGEBRAS

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In this paper there is given a geometric scheme for constructing integrable Hamiltonian systems based on Lie groups, generalizing the construction of M. Adler. The operation of this scheme is considered for parabolic decompositions of semisimple Lie groups. Fundamental examples of integrable systems are connected with graded Lie algebras. Among them are the generalized periodic chains of Toda, multidimensional tops, and the motion of a point on various homogeneous spaces in a quadratic potential.

Introduction

The goal of the present paper is the construction of finite-dimensional Hamiltonian systems, having a series of characteristic properties, which also explain the term "integrable":

1. The systems have many integrals of motion in involution; in a series of cases complete integrability is proved.

2. The construction of their trajectories reduces to the solution of the factorization problem in an appropriate Lie group.

3. The systems have Lax form, i.e., there exists a commutator representation of the equations of motion.

4. The systems have transparent mechanical meaning: their phase space is a cotangent bundle and the Hamiltonian splits into the sum of the kinetic and potential energies.

The ideas on which the construction given here is based appeared and was developed intensively in the course of the last 15 years. The main ones were introduced by Kirillov [1] and are the Hamiltonian structure in the space dual to a Lie algebra; the scheme of reduction of Hamiltonian systems with a symmetry [2]; the Lax representation [3] and its connection with complete integrability [4]; finally, the remarkable connection established by Adler [5] of the Lax representation with mechanics on orbits. Essentially, our construction is a geometric interpretation of Adler's scheme. At the time it arose there had already been accumulated a lot of material on Lax representations. Lax pairs were found for Toda chains [6-10], the multidimensional top [11], systems of one-dimensional particles, generalizing systems of Calogero and Sutherland [8, 12]. In Novikov [13] there was introduced for the first time an additional spectral parameter, which allowed one to linearize the Lax equations on the Jacobian of the corresponding spectral curve [14]. Certain problems (Calogero—Sutherland systems [15] generalized Toda chains [16]) were considered from the point of view of Hamiltonian reduction by the symmetry group. In our construction the Lax representation, Hamiltonian reduction, and the spectral parameter are united in a general group-theoretic formulation.

The present paper is a detailed and improved account of that part of [17] of the author and M. A. Semenov-Tyan-Shanskii, which concerns classical (i.e., not quantum) systems. Its plan is the following. In Sec. 1 we give the basic geometric construction (Theorem 1). As an example, we consider a decomposition of the algebra of pseudodifferential symbols different from that used by Adler. In Sec. 2 we give an interpretation of Theorem 1 in the language of Hamiltonian reduction and we give the information about orbits of semidirect products needed later. In Sec. 3, Theorem 1 is applied to the parabolic decomposition of semisimple Lie algebras and new integrable systems, generalizing Toda chains, are constructed. The completion of flows of such systems is considered based on Hamiltonian reduction.

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The following sections are devoted to graded Lie algebras and the equations with spectral parameter connected with them. In Sec. 4 the algebraic definitions needed are given, the decompositions to which Theorem 1 is applied are constructed, and the corresponding Banach Lie groups — flow groups — are defined. In Sec. 5, isolated facts concerning Hamiltonian mechanics and the space dual to a graded Lie algebra are collected together: the dimension of an orbit in general position is calculated, various Poisson brackets generating one and the same collection of Lax equations are considered. A simple proof is given of important special cases of Theorem 1 in the graded situation. Section 6 is devoted to concrete examples. Here there are defined generalized periodic Toda chains and systems with compact configuration space of the type of multidimensional tops. Among the new integrable systems we note the motion of a point on various homogeneous spaces with a linear and a quadratic potential, the system of two mutually bilinearly acting tops, the rotation of a top in a linear and a quadratic field. Finally, in Sec. 7, with the help of algebrogeometric methods we prove the completeness of the integrals of motion in the model problem on orbits of general type.

It is necessary to note that the group-theoretic construction of integrable systems undertaken here is closely connected with the algebrogeometric approach of [13, 14, 18, 19]: it is proved by the author and M. A. Semenov-Tyan-Shanskii that the problem of factorization to which the solution of the equations of motion is reduced, in its own right can be solved in terms of the Baker—Akhiezer function connected with the spectral curve. In this way our approach can be considered as a group-theoretic interpretation of the methods of algebraic geometry.

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1. Geometric Construction of Lax Systems

Here we expound a scheme for constructing algebras of functions on the space dual to some Lie algebra, which are in involution with respect to the Berezin—Kirillov—Kostant brackets. This scheme is a geometric generalization of the algebraic construction proposed by Adler [5] and Kostant [20]. Here and throughout the text, Lie groups are denoted by upper case Latin letters, their Lie algebras by the corresponding Gothic ones.

1.1. Kirillov Bracket. We recall the basic facts concerning the Kirillov bracket [2, 21]. Let G be a Lie group, $\mathfrak{g}$ be its Lie algebra, $\mathfrak{g}^*$ be the dual space. With the help of left translations we identify the tangent bundle TG with $G\times\mathfrak{g}$, and the cotangent bundle T^*G with $G\times\mathfrak{g}^*$. On T^*G there is a canonical Poisson bracket. We consider its restriction to the space of left-invariant functions; this space can be identified naturally with the space $\mathcal{F}(\mathfrak{g}^*)$ of functions on $\mathfrak{g}^*$. Here linear functions on $\mathfrak{g}^*$, that is, elements of $\mathfrak{g}$, are Hamiltonians of right translations in T^*G. Their Poisson bracket coincides with the Lie bracket in $\mathfrak{g}$. Thus, the Poisson bracket in the space $\mathcal{F}(\mathfrak{g}^*)$, which we shall call, in what follows, the Kirillov bracket, for linear functions $x, y\in\mathfrak{g}$ is equal to

$$\{x,y\}=[x,y],$$

and by the same token, for arbitrary functions $\varphi,\psi\in\mathcal{F}(\mathfrak{g}^*)$,

$$\{\varphi,\psi\}(\xi)=\xi([d\varphi(\xi),d\psi(\xi)]),\quad \xi\in\mathfrak{g}^*.$$

Let $I(\mathfrak{g}^*)$ be the algebra of invariant functions on $\mathfrak{g}^*$. Obviously functions from $I(\mathfrak{g}^*)$ can be extended to bilaterally invariant functions on T^*G.

LEMMA 1.1.1. The center of the Lie algebra $\mathcal{F}(\mathfrak{g}^*)$ coincides with $I(\mathfrak{g}^*)$.

Whence it is evident that the Kirillov bracket is usually degenerate.

LEMMA 1.1.2. Suppose given on the manifold M a (degenerate) Poisson bracket. We denote by $\Lambda: T^*M\to TM$ the corresponding Hamiltonian operator: $\langle\Lambda_m d\varphi, d\psi\rangle=\{\varphi,\psi\}(m)$, $m\in M$.

Then

a) the distribution $m \mapsto \Lambda_m(T^*_m)$ in M is integrable;

b) on the integral manifolds of this distribution the operator Λ induces a symplectic structure.

Thus, the degenerate Poisson bracket is connected with a fibered symplectic structure. For the Kirillov bracket $\Lambda_\xi(x) = ad^*x \cdot \xi$, hence the integral manifolds coincide with the orbits of the coadjoint action. In other words, upon restriction to orbits the Kirillov bracket becomes nondegenerate.

1.2. Basic Construction. Let us assume that the algebra $\mathfrak{g}$ as a vector space is represented in the form of a linear sum of two of its subalgebras: $\mathfrak{g} = \mathfrak{a} \dotplus \mathfrak{b}$. In this case for the dual space one has the decomposition $\mathfrak{g}^* = \mathfrak{a}^* \dotplus \mathfrak{b}^*$, where $\mathfrak{a}^* = \mathfrak{b}^\perp$, $\mathfrak{b}^* = \mathfrak{a}^\perp$. Let A and B be connected subgroups corresponding to the subalgebras $\mathfrak{a}$ and $\mathfrak{b}$.

We construct the algebra $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{b}$ which is the direct sum of the algebras $\mathfrak{a}$ and $\mathfrak{b}$, and we consider the linear map $\sigma_0 : \mathfrak{g}_0 \to \mathfrak{g}$, $\sigma_0(\alpha \oplus \beta) = \alpha - \beta$. One can assume that the map σ_0 defines on the subspace $\mathfrak{g}$ a new Lie algebra structure; by the same token on $\mathfrak{g}^*$ there are two Kirillov brackets.

THEOREM 1. (i) Functions from $I(\mathfrak{g}^*)$ are in involution with respect to both brackets on $\mathfrak{g}^*$.

(ii) Let $\varphi \in I(\mathfrak{g}^*)$. The equations of motion on $\mathfrak{g}^*$, defined by the Hamiltonian φ with respect to the second Kirillov bracket, have the form

$$\dot{\xi} = ad^* d\varphi(\xi)_- \cdot \xi ,$$

where $d\varphi(\xi)$ is the projection of $d\varphi(\xi)$ onto $\mathfrak{b}$ parallel to $\mathfrak{a}$.

(iii) Let $exp\, t d\varphi(\xi) = a(t)\, b(t)$, where $a(\cdot)$ and $b(\cdot)$ are smooth curves in the groups A and B. The solution of the given equations of motion, starting at the point ξ, has the form

$$\xi(t) = Ad^* b(t) \cdot \xi .$$

Proof. We set $G_0 = A \times B$ and we define a map $\sigma : G_0 \to G$ in the following way: $\sigma(a,b) = ab^{-1}$. Obviously the differential of the map σ is A-invariant and B-equivariant, that is, $d\sigma(a,b) = Ad b \circ d\sigma(e)$. Since $d\sigma(e) = \sigma_0$, we get $d\sigma(a,b) = Ad b \circ \sigma_0$. Whence it follows that σ is an immersion. Hence one can define the map $\sigma^* : T^*G_0 \to T^*G$, by setting on the fiber $T^*_{a,b}$, $\sigma^* = d\sigma(a,b)^{*-1}$. If φ is a function on T^*G, then we denote by φ^σ the function on T^*G_0: $\varphi^\sigma(m) = \varphi(\sigma^* m)$. The map σ^* is a symplectic immersion and $\{\varphi^\sigma, \psi^\sigma\} = (\{\varphi,\psi\})^\sigma$. Now to prove (i) it remains to note that for $\varphi \in I(\mathfrak{g}^*)$ the function φ^σ is left G_0-invariant.

LEMMA 1.2. Trajectories in T^*G, defined by the Hamiltonian $\varphi \in I(\mathfrak{g}^*)$, have the form

$$(g(t), \xi(t)) = (g(0)\, exp\, t d\varphi(\xi), \xi(0)) .$$

Proof. The equations of motion defined by the invariant Hamiltonian φ, have the form $\dot{\xi} = 0$, $\dot{g} = d\varphi(\xi)$, whence also follows the formula for the trajectories.

Taking the preimage of such a trajectory (with $g(0) = e$) in T^*G_0 and calculating its components in the decomposition $T^*G_0 = G_0 \times \mathfrak{g}^*_0$, we get the trajectory of the Hamiltonian

φ^{σ} in $\mathfrak{g}_0^*$. If $\exp t\, d\varphi(\xi) = a(t)\, b(t)$, then $(d\sigma)^*(\exp t\, d\varphi(\xi)) = \sigma_0^* \cdot Ad^* b(t)$. Whence follows (iii). Differentiating the trajectory with respect to t we get (ii). The proof of the theorem is completed.

As is evident from the proof, the map $\varphi \mapsto \varphi^{\sigma}$ is a homomorphism of the algebra of left -invariant and right B-invariant functions on T^*G with Poisson bracket into the algebra $\mathcal{F}(\mathfrak{g}^*)$ with Kirillov bracket.

Remark. If in $\mathfrak{g}$ there is an invariant nondegenerate bilinear form, then one can identify $\mathfrak{g}^*$ with $\mathfrak{g}$, ad^* with the commutator, and the equations of motion acquire Laxian form.

As was remarked in 1.1, Hamiltonian mechanics in $\mathfrak{g}_0^*$ develops on orbits. The orbits in $\mathfrak{g}_0^*$ are products of orbits in $\mathfrak{a}^*$ and in $\mathfrak{b}^*$. In particular, if $f \in \mathfrak{a}^*$ is a character of the algebra $\mathfrak{a}$, that is, a single point orbit, then restricting Theorem 1 to the subspace $f + \mathfrak{b}^*$, we get as a consequence of Theorem 10 and Proposition 12 of [17]:

COROLLARY 1.2. Let f be a character of the algebra $\mathfrak{a}$.

(i) Functions on $\mathfrak{b}^*$ of the form $\varphi_f(\xi) = \varphi(\xi + f)$, $\varphi \in I(\mathfrak{g}^*)$, are in involution with respect to the Kirillov bracket on $\mathfrak{b}^*$.

(ii) The equations of motion on $\mathfrak{b}^*$, given by the Hamiltonian φ_f, have the form

$$\dot{\xi} = ad^*\, d\varphi(\xi + f)_- \cdot (\xi + f).$$

(iii) Let $\exp t\, d\varphi(\xi + f) = a(t) b(t)$, where $a(\cdot)$ and $b(\cdot)$ are smooth curves in the groups A and B. The solution of the given equations of motion, starting at the point ξ, has the form

$$\xi(t) = Ad^* b(t) \cdot (\xi + f) - f$$

or, equivalently,

$$\xi(t) = Ad_B^*\, b(t) \cdot \xi .$$

1.3. Algebraic Proof. We also give a purely algebraic proof of the first two points of Theorem 1. We introduce the following notation: for $x \in \mathfrak{g}$ ($\xi \in \mathfrak{g}^*$) we set $x = x_+ + x_-$, $x_+ \in \mathfrak{a}$, $x_- \in \mathfrak{b}$ ($\xi = \xi_+ + \xi_-$, $\xi_+ \in \mathfrak{a}^*$, $\xi_- \in \mathfrak{b}^*$). In addition, we write $ad^* x \cdot \xi = [x, \xi]^*$.

Proof (ii). The equations of motion on $\mathfrak{g}_0^*$ have the form

$$\dot{\xi} = -ad^*_{\mathfrak{g}_0}\, d\varphi^{\sigma}(\xi) \cdot \xi .$$

If $\mathfrak{g}_0^*$ is identified with $\mathfrak{g}^*$ with the help of σ_0^*, then, as one sees easily, the operation $ad^*_{\mathfrak{g}_0}$ turns into

$$ad^*_{\mathfrak{g}_0} x \cdot \xi = [x_+, \xi_+]^*_{\xi} - [x_-, \xi_-]^*_- ,$$

and the equations of motion on $\mathfrak{g}^*$ with respect to the $\mathfrak{g}_0$-bracket have the form

$$\dot{\xi} = [M_-, \xi_-]^*_- - [M_+, \xi_+]^*_+ , \quad M = d\varphi(\xi) . \tag{1}$$

Now we use the fact that φ is an invariant function, i.e., $[M, \xi]^* = 0$ or

$$[M_+, \xi_+]^* + [M_+, \xi_-]^* + [M_-, \xi_+]^* + [M_-, \xi_-]^* = 0 . \tag{2}$$

But $[M_+, \xi_-]^*_+ = 0$, since $[\mathfrak{a}, \mathfrak{a}^{\perp}]^* \subset \mathfrak{a}^{\perp}$. For the analogous reason $[M_-, \xi_+]^*_+ = [M_-, \xi_+]^*$. Expressing $[M_+, \xi_+]^*_+$ from (2) and substituting into (1), we get

$$\dot{\xi} = [M_-,\xi_-]^*_- + [M_-,\xi_+]^* + [M_-,\xi_-]^*_+ = [M_-,\xi]^* .$$

Whence follows easily the involutivity of the functions $\varphi,\psi \in I(\mathfrak{g}^*)$. In fact,

$$\{\varphi,\psi\}_0(\xi) = <\dot{\xi}, d\psi(\xi)> = <[M_-,\xi]^*, d\psi(\xi)> = -<M_-, [d\psi(\xi),\xi]^*> = 0$$

since $[d\psi(\xi),\xi]^* = 0$.

1.4. Example: a nonstandard decomposition of the algebra of symbols and contractions of Benni's equation.

An algebraic scheme using the decomposition of the Lie algebra and traces of invariant functionals on one of the subalgebras was first proposed by Adler [5] and consists of the first two points of Corollary 1.2, under the additional assumption $f=0$. The basic goal of his paper consisted of the orbital interpretation of the new Hamiltonian mechanics in the space of differential operators, constructed by Gel'fand and Dikii [23], so a principal role in [5] was played by the algebra $\mathfrak{g}$, consisting of formal pseudodifferential symbols and the subalgebras of it consisting of symbols of differentials $(\mathfrak{a})$ and of Volterra $(\mathfrak{b})$ operators. Such a decomposition does not fall into the domain of action of Theorem 1. In fact, the algebra of differential symbols not only does not have nontrivial characters, but even orbits of finite functional dimension. On the other hand, all invariant polynomials on $\mathfrak{g}$ of the form $\operatorname{tr}\varphi(L)$ vanish on $\mathfrak{b}^* \simeq \mathfrak{a}$. By the same token, the scheme of 1.2 becomes empty, and Adler was constrained, due to [23], to turn to a substitute for invariant functionals (to traces of fractional powers), which are defined only in the subspace of symbols of fixed order.

Here we want to note that the possibility of the algebra of symbols is not exhausted by Adler's decomposition, and to illustrate Theorem 1 with the help of another decomposition of the algebra $\mathfrak{g}$.

We recall that the algebra of symbols $\mathfrak{g}$ consists of formal Laurent series $L = \sum_{n \le N(L)} u_n \xi^n$ with functional coefficients and the multiplication

$$L_1 \circ L_2 = \sum_{k=0}^{\infty} \frac{1}{k!} \partial_\xi^k L_1 \partial_x^k L_2 .$$

In $\mathfrak{g}$ there is an invariant trace $\operatorname{tr} L = \int u_{-1}(x)dx$. Invariance means that $\operatorname{tr}[L_1,L_2] = 0$. With the help of the trace one constructs the invariant nondegenerate product

$$(L_1, L_2) = \operatorname{tr}(L_1 \circ L_2).$$

We consider the decomposition $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$, where

$$\mathfrak{a} = \{\sum_{n\ge 1} u_n \xi^n\}, \qquad \mathfrak{b} = \{\sum_{n\le 0} u_n \xi^n\}.$$

In this decomposition the multiplication operators are adjoint to Volterra ones. We have

$$\mathfrak{b}^* \simeq \{\sum_{n \ge -1} u_n \xi^n\}.$$

The algebra $\mathfrak{a}$ as before has no characters, but the restrictions of polynomials $\operatorname{tr}\varphi(L)$ to $\mathfrak{b}^*$ are now nontrivial. The coadjoint action of $\mathfrak{b}$ in $\mathfrak{b}^*$ preserves filtration, and the subspace of symbols with fixed highest term splits into a finite-parameter family of orbits.

We consider an example. Symbols of the form $L=\xi+u+v\xi^{-1}$ fill out an invariant (affine) subspace of $\mathfrak{b}^*$. The Poisson bracket of the variables u and v has the form

$$\{u(x),v(y)\}=\delta'(x-y),\quad \{u(x),u(y)\}=0=\{v(x),v(y)\}.$$

The orbits are given by the conditions $\int u=const$, $\int v=const$. Polynomials of the form $H_n(u,v)=\mathrm{tr}\,L^n$ are in involution. The equations of motion have the form

$$\dot u=\partial_x\frac{\delta H}{\delta v},\qquad \dot v=\partial_x\frac{\delta H}{\delta u}.$$

For the Hamiltonian $H=\frac{1}{3}\mathrm{tr}\,L^3=\int(u^2v+v^2+uv')$ we get the system

$$\begin{cases}\dot u=\partial_x(u^2+2v-u')\\ \dot v=\partial_x(2uv+v').\end{cases}$$

For this system there is a Laxian pair:

$$\dot L=[L,M],\quad\text{where}\quad M=(L^2)_+=\xi^2+2u\xi.$$

We give another analogous example, in which as $\mathfrak{g}$ one takes the "quasiclassical limit" of the algebra of symbols, studied by Lebedev and Manin [22] in connection with Benni's equations. The algebra $\mathfrak{g}$ is the algebra of Laurent series $L=\Sigma u_n\xi^n$ with the ordinary multiplication, and the Lie bracket is defined by the Poisson bracket with respect to the variables x,ξ:

$$[L_1,L_2]=\partial_x L_1\,\partial_\xi L_2-\partial_\xi L_1\,\partial_x L_2.$$

Everything said above carries over to this case with the one correction that for $L=\xi+u+v\xi^{-1}$ we get $\frac{1}{3}\mathrm{tr}\,L^3=\int(u^2v+v^2)$ and the previous system simplifies ($H=\frac{1}{6}\mathrm{tr}\,L^3$):

$$\begin{cases}\dot u=u_xu+v_x\\ \dot v=(uv)_x.\end{cases}$$

In this system we know the contracted system of Benni's equations. $H_n=\mathrm{tr}\,L^n$ is its integral of motion. D. R. Lebedev and Yu. I. Manin were concerned with a more meaningful formulation of Benni's system at times A_n, $n=0,1,\dots$. The conservation laws for it are $\tilde H_n=\mathrm{tr}\,(\xi+A(\xi))^n$, where $A(\xi)=\sum_{n\geq 0}A_n\xi^{n-1}$. The moments of the contracted system $A_n=vu^n$ so that $\tilde H_n=\mathrm{tr}(\xi+\frac{v}{\xi-u})^n$. It turns out that H_n coincides with $\tilde H_n$. In fact,

$$\mathrm{tr}(\xi+v(\xi-u)^{-1})^n=\int dx\int dz\left(z+\frac{v}{z-u}\right)^n=\int dx\int dy\,(y+u+vy^{-1})^n=\mathrm{tr}\,(\xi+u+v\xi^{-1})^n.$$

We note further that the orbits of the algebra $\mathfrak{a}$ in $\mathfrak{a}^*\simeq\{\sum_{n\leq -2}u_n\xi^n\}$ have finite functional dimension. For example, $\mathcal{O}_1=\{w\xi^{-2}\}$ is an orbit. Applying Theorem 1 to the orbits $\mathcal{O}_1\times\{u,v\}$, we get functionals in involution of the form $H_n=\mathrm{tr}\,L^n$, $L=\xi+u+v\xi^{-1}+w\xi^{-2}$ for the Poisson bracket $\{w(x),w(y)\}=\delta'(x-y)$. For the Hamiltonian $H=\frac{1}{6}\mathrm{tr}\,L^3$ we get the system

$$\begin{cases}\dot u=u_xu+v_x\\ \dot v=(uv)_x+\frac{1}{3}w\\ \dot w=\frac{1}{3}u_x.\end{cases}$$

2. Hamiltonian Reduction and Orbits

Here we expose the scheme of Hamiltonian reduction [2], which we use in Sec. 3 for completing flows of generalized Toda chains. In terms of reduction we explain Theorem 1. Then we consider orbits of the semidirect product of a group K and a vector space V. Those of them which have the structure of the cotangent bundle to a K-orbit in V^*, are especially interesting for mechanics.

2.1. Hamiltonian Reduction. Let the connected Lie group G act in a Hamiltonian way on the symplectic manifold (M,ω). This means that

a) for each $x \in \mathfrak{g}$ the corresponding vector field on M is Hamiltonian with Hamiltonian H_x;

b) the map $x \mapsto H_x$ is linear, and

c) $H_{[x,y]} = \{H_x, H_y\}$.

The map of moments $\Phi: M \to \mathfrak{g}^*$ is adjoint to the map $H: \mathfrak{g} \to \mathcal{F}(M)$:

$$< \Phi m, x > = H_x(m).$$

The map of moments G is equivariant.

Let $f \in \mathfrak{g}^*$ be a regular value of the map Φ, so that $M_f = \Phi^{-1}(f)$ is a smooth submanifold in M. We denote by ω_f the restriction of the form ω to M_f, by G_f the stationary subgroup of the element f. The manifold M_f is invariant with respect to the action of G_f, and the orbits of this action are cut out on M_f by the orbits of the group G in M.

LEMMA 2.1.1. The kernel of the form ω_f coincides with the tangent foliation to the G_f-orbits in M_f.

Whence it follows that all G_f-orbits in (a connected component of) the manifold M_f have identical dimension and form a foliation. Let us assume that this foliation is a smooth bundle with base $\overline{M}_f$, which will be called the reduced space.

LEMMA 2.1.2. The 2-form ω_f projects into a closed nondegenerate 2-form $\overline{\omega}_f$ on $\overline{M}_f$.

Hamiltonian mechanics on $\overline{M}_f$ is connected with mechanics on M in the following way. Let φ be a G-invariant function on M. The corresponding Hamiltonian field X_φ is also G-invariant and tangent to the submanifold M_f. Hence there are defined natural projections of the function φ and field X_φ onto $\overline{M}_f$: we denote them by $\overline{\varphi}$ and $\overline{X}_\varphi$. Let $X_{\overline{\varphi}}$ be a Hamiltonian field on $\overline{M}_f$, generated by the Hamiltonian $\overline{\varphi}$.

LEMMA 2.1.3. The flow of the reduced Hamiltonian is the reduction of the original flow: $X_{\overline{\varphi}} = \overline{X}_\varphi$.

COROLLARY 2.1. If φ_1 and φ_2 are G-invariant functions on M, then $\{\overline{\varphi}_1, \overline{\varphi}_2\} = \overline{\{\varphi_1, \varphi_2\}}$.

Example: the action of a Lie group on its cotangent bundle. The action of a group G on a manifold N canonically subordinate to a Hamiltonian action of G on $M = T^*N$: if X is a vector field on N, corresponding to a generator $x \in \mathfrak{g}$, then the Hamiltonian H_x on M is equal to $H_x(m) = < m, X >$. We consider, in particular, two actions of the group G on itself — by right and left translations. We identify the space T^*G with the product $G \times \mathfrak{g}^*$ with the help of left translations. Then the left action of the group G can be written in the form: $h(g,\xi) = (hg, \xi)$, the right: $h \circ (g,\xi) = (gh^{-1}, Ad^* h \cdot \xi)$. The left and right maps of moments have the form

$$\Phi_l(g,\xi) = Ad^* g \cdot \xi, \qquad \Phi_r(g,\xi) = -\xi.$$

As a result of reduction over the point ξ in both cases one gets the reduced manifold isomorphic with the orbit $\mathcal{O}_\xi$.

2.2. Reduction and Adler's Scheme. The basic theorem of 1.2 can be obtained starting from reduction of the space T^*G by the left action of the subgroup A and the right action of the subgroup B. Here the reduced manifold contains the orbit of the algebra $\mathfrak{g}_0$ as an open subset. The flows of biinvariant Hamiltonians on T^*G are complete (Lemma 1.2), so the reduced flows on the reduced manifold are also complete. Thus reduction gives a canonical completion of flows of Laxian systems on orbits in $\mathfrak{g}_0^*$. It is only necessary to verify that the reduced space is actually a smooth manifold, which will be done in the following point for certain generalized Toda chains.

We proceed to precise formulations. We define the action of the group $G_0 = A \times B$ on G in the following way: $(a,b)\cdot g = agb^{-1}$. Then the map $\sigma: G_0 \to G$, $\sigma(a,b) = ab^{-1}$, as well as the induced map $\sigma^*: T^*G_0 \to T^*G$, commute with the action of G_0. Let $\xi \in \mathfrak{g}_0^*$, $\mathcal{O}_\xi$ be an orbit in $\mathfrak{g}_0^*$ passing through the point ξ. We denote by $\overline{M}_\xi$ the space obtained by reducing T^*G by the indicated action of the group G_0 over the point ξ.

Let us suppose that the decomposition $\mathfrak{g} = \mathfrak{a} \dotplus \mathfrak{b}$ has the following property: the map $\sigma: G_0 \to G$ is one-to-one. In particular, the parabolic decompositions, which are discussed in Sec. 3, are like this.

LEMMA 2.2. There is a natural imbedding of the orbit $\mathcal{O}_\xi$ in the space $\overline{M}_\xi$ as an open domain. This imbedding carries the flow of the Hamiltonian φ^σ on $\mathcal{O}_\xi$, $\varphi \in I(\mathfrak{g}^*)$, into the reduced flow of the Hamiltonian φ.

The proof follows from the description of the orbits of the group G_0 in terms of reduction (see end of 2.1).

In combination with the properties of reduction, Lemma 2.2 swallows up Theorem 1, giving, in particular, a simple proof of the involutivity of the reduced functions $\overline{\varphi}$, $\varphi \in I(\mathfrak{g}^*)$, In connection with this, we note that in the first papers the involutivity of the integrals of motion of Laxian systems was proved by calculations. One calculates the symplectic form in given scatterings [4], one directly calculates the Poisson bracket of two integrals [6], or one uses "asymptotic" considerations [8]. Then simpler methods appeared. Kazhdan, Kostant, and Sternberg [15] interpreted the results of Moser [8] and Ol'shanetskii and Perelomov [12] from the point of view of Hamiltonian reduction of systems with a symmetry, using it in the direction opposite to the traditional one: not for simplifying complicated systems, but for complicating simple ones. Here the involutivity of the conservation laws became obvious and a natural explanation was obtained for the "explicit" formulas for the trajectories as a result of the factorization of explicitly given matrices. However, the connection between the general scheme of reduction and the Lax representation, which was observed in isolated examples, appeared accidental.

2.3. Orbits of Semidirect Products. Suppose given a linear representation of the Lie group K in the space V. We denote by $G = K * V$ the corresponding affine group which is the semidirect product of K and V. The Lie algebra $\mathfrak{g}$ decomposes into the semidirect sum $\mathfrak{g} = \mathfrak{k} + V$ and $\mathfrak{g}^* = \mathfrak{k}^* + V^*$.

Let $\xi \in \mathfrak{g}^*$, $\xi = \pi + a$, where $\pi \in \mathfrak{k}^*$, $a \in V^*$.

We introduce the following notation:

K_a is the stationary subgroup of the point a under the Ad^*-action of K in V^*

$P: \mathfrak{k}^* \longrightarrow \mathfrak{k}_a^*$ is the natural projection;

$\Phi_a : T^*K \longrightarrow \mathfrak{k}_a^*$ is the right mapping of moments,

$$\Phi_a(\kappa, \rho) = -P\rho, \quad \kappa \in K, \quad \rho \in \mathfrak{k}^*.$$

We write $M_\pi = \Phi_a^{-1}(-P_\pi)$ and let $\overline{M}_\pi$ be the result of reducing the space T^*K by the right action of the subgroup K_a over the point $P\pi$.

The following proposition generalizes the lemma on p. 97 of [17].

Proposition 2.3. The orbit $\mathcal{O}_\xi$ of the group G in the space $\mathfrak{g}^*$ is isomorphic as a symplectic K-space with the manifold $\overline{M}_\pi$.

Proof. We associate with the vector $u \in V$ the function u_a on the group K defined by the formula $u_a(\kappa) = a(\kappa^{-1}u)$. Obviously, the map $u \longmapsto u_a$ is K-equivariant. We define the Hamiltonian action of the group G on the space T^*K in the following way: the group K acts by left translations, and Hamiltonian actions of the subgroup V are the functions u_a. The corresponding mapping of moments $\Phi : T^*K \longrightarrow \mathfrak{g}^*$ has the form

$$\Phi(\kappa, \rho) = Ad^*\kappa(\rho + a).$$

Obviously, the action of the subgroup K_a by right translations commutes with the action constructed of the group G, so the manifold M_π is G-invariant. Since $M_\pi = K \times (\pi + \mathfrak{k}_a^\perp)$, one has

$$\Phi(M_\pi) = Ad^* K(\pi + a + \mathfrak{k}_a^\perp).$$

It is easy to see that $Ad^*V(a) = \mathfrak{k}_a^\perp$. Whence it follows that $\Phi(M_\pi) = \mathcal{O}_\xi$. To complete the proof it remains to note that the fibers of the map Φ, restricted to M_π, coincide with the orbits of the group $(K_a)_{P\pi}$, which is the stationary subgroup of the point $P\pi$.

COROLLARY 2.3. Let us assume that $\pi = 0$, i.e., $\xi = a$. We denote by $K \cdot a$ the orbit of the point a under the action of the group K. Then the orbit $\mathcal{O}_\xi$ is isomorphic as symplectic K-space with the cotangent bundle $T^*(K \cdot a)$. The isomorphism $T^*(K \cdot a) \longrightarrow \mathcal{O}_\xi$ is given by the map of moments

$$(\kappa \cdot a, \rho) \longmapsto Ad^*\kappa(\rho + a), \quad \rho \in \mathfrak{k}_a^\perp.$$

In this formula the orbit $K \cdot a$ is considered as the homogeneous space K/K_a. If one considers $K \cdot a$ as submanifold of V^*, then the cotangent space $T^*_{\kappa \cdot a}$ becomes a quotient-space of the space V. Let $v \in V$ be a representative of an element from $T^*_{\kappa \cdot a}$. Then to it corresponds the impulse $\rho = -\kappa^{-1} ad^* v(\kappa \cdot a)$ and the point $\eta = -ad^* v(\kappa \cdot a) + \kappa \cdot a$ on the orbit $\mathcal{O}_\xi$.

Suppose given on the group is a biinvariant (pseudo-) Riemannian metric. Then on the space T^*K there is defined a biinvariant quadratic function dual to the metric — the kinetic energy E. If N is a homogeneous K-space, then E projects into a K-invariant "kinetic energy" E_N on T^*N: $E_N = E \circ \Phi$, where $\Phi: T^*N \to \mathfrak{k}^*$ is the map of moments.

In the general case the energy E_N can be degenerate (as a quadratic form). A test for the nondegeneracy of E_N is the reductivity of the stationary subgroup K_n of some (hence any) point $n \in N$. The latter means that the restriction of the (pseudo-) Riemannian metric from K to K_n is nondegenerate. In this case on the homogeneous space N there is defined a K-invariant "orthogonal projection" of the metric from K. It is easy to see that the kinetic energy on T^*N, corresponding to this metric, coincides with E_N.

3. Generalized Toda Chains

A nonclosed Toda chain was, apparently, the first finite-dimensional system studied with the help of the Lax representation [6-8], and the first system for which all previously known results were completely reproduced by Adler's scheme [2, 32]. Here we consider various systems connected with the parabolic decomposition of a Lie algebra, in this sense generalizing classical Toda chains. As a rule these systems are incomplete: in them there are trajectories, going to infinity in a finite time. A canonical method is given for completing such systems, based on Hamiltonian reduction and giving simultaneous completion of the flows of all integral motions.

3.1. Definitions. Let $\mathfrak{g}$ be a semisimple real Lie algebra, $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}_+$ be its Iwasawa decomposition, Δ be a system of roots of the pair $(\mathfrak{g}, \mathfrak{a})$, $\Delta = \Delta_+ \cup \Delta_-$ be a partition into positive and negative roots, compatible with the Iwasawa decomposition: if $\mathfrak{g}_\alpha$ is the eigensubspace corresponding to the root α, then $\mathfrak{n}_+ = \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$. We denote by P the system of simple roots in Δ_+.

Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Then $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}_+$ is a minimal parabolic subalgebra. Other parabolic subalgebras are subalgebras containing $\mathfrak{p}$. For them one has the Langlands decomposition

$$\mathfrak{g} = \mathfrak{l} + \bar{\mathfrak{a}} + \bar{\mathfrak{n}}_+ ,$$

where $\bar{\mathfrak{a}} \subset \mathfrak{a}$, $\bar{\mathfrak{n}}_+ \subset \mathfrak{n}_+$, $\mathfrak{l}$ is a reductive subalgebra, centralizing $\bar{\mathfrak{a}}$ and normalizing $\bar{\mathfrak{n}}_+$.

We define $\mathfrak{g}$-graduation of the algebra $\mathfrak{g}$ as follows. Let the element $x_0 \in \bar{\mathfrak{a}}$ be defined by the conditions $\alpha(x_0) = 1$ for those roots $\alpha \in P$, which are not orthogonal to $\bar{\mathfrak{a}}$. Then the eigenvalues of the operator $ad\, x_0$ are integers. We denote by d_i the eigensubspace corresponding to the eigenvalue i. We have

$$\mathfrak{l} + \bar{a} = d_0, \quad \bar{\mathfrak{n}}_+ = \bigoplus_{i>0} d_i \qquad \bar{\mathfrak{n}}_- = \bigoplus_{i<0} d_i$$
$$\mathfrak{g} = \mathfrak{g}_+ = \bigoplus_{i \geqslant 0} d_i \qquad \mathfrak{g}_- = \bigoplus_{i \leqslant 0} d_i .$$

Now we are ready for the construction of the systems cited in the heading. With the help of the Killing form B we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ and ad^* with ad. Let $\mathfrak{g} = \mathfrak{n}_+ \dotplus \mathfrak{g}_-$ be the parabolic decomposition of the algebra $\mathfrak{g}$, $\mathfrak{g}^* = \mathfrak{n}_- \dotplus \mathfrak{g}_+$ be the dual decomposition, $\mathfrak{n}_+^* \simeq \mathfrak{n}_-$, $\mathfrak{g}_-^* \simeq \mathfrak{g}_+$. It is easy to see that the ad^*-action of the algebra $\mathfrak{g}_-$ in $\mathfrak{g}_+$ does not increase the $\mathfrak{g}$-graduation:

$$ad^* \mathfrak{g}_- \cdot d_j \subset \bigoplus_{i=0}^{j} d_{i_j}.$$

Thus, the subspaces $\mathcal{D}_j = \bigoplus_{i=0} d_i$ are invariant and decompose into G_--orbits.

By what was proved in 2.3, the orbit $\mathcal{O}_e$, passing through the element $e \in d_1$, is isomorphic with the manifold $T^*(L\bar{A} \cdot e)$ as symplectic $L\bar{A}$-space. In addition, restricting the Killing form of the algebra $\mathfrak{g}$ to $\mathcal{O}_e$, we get the $L\bar{A}$-invariant kinetic energy E:

$$E(\xi) = \frac{1}{2} B(\xi, \xi), \qquad \xi \in \mathcal{O}_e \, .$$

We apply Corollary 1.2 to the orbit $f + \mathcal{O}_e$, considering that the characters of the algebra $\mathfrak{n}_+$ are defined by elements $f \in d_{-1}$. Calculating the Killing form on $f + \mathcal{O}_e$, we get a Hamiltonian of the form

$$H = E + V_f \, ,$$

where E is the invariant kinetic energy on $\mathcal{O}_e$, V_f is a potential function on $\mathcal{O}_e$, $V_f(\xi) = B(\xi, f)$.

We formulate Corollary 1.2 in this concrete situation:

Proposition 3.1. (i) The Hamiltonian system on $\mathcal{O}_e$ with Hamiltonian $H = \frac{1}{2} B|_{f+\mathcal{O}_e}$ has integrals of motion in involution of the form $\varphi|_{f+\mathcal{O}_e}$, where $\varphi \in I(\mathfrak{g})$.

(ii) The equations of motion have Laxian form

$$\dot{\xi} = [\xi_-, \xi],$$

where $\xi \in f + \mathcal{O}_e$, ξ_- is the $\mathfrak{g}_-$-component of ξ.

(iii) Trajectories of the system can be obtained by expanding $\exp t\xi$ as a product $\exp t\xi = a(t)\, b(t)$, where $a(t) \in N_+$, $b(t) \in G_-$; the trajectory starting at the point ξ has the form

$$\xi(t) = \mathrm{Ad}\, b(t) \cdot \xi \, .$$

We shall call these systems generalized Toda chains or G-chains.

3.2. Principal Nilpotent Elements. Proposition 3.2. Let the element $f \in d_{-1}$ be such that $\mathfrak{g}_f \cap \mathfrak{n}_+ = \{0\}$, $\mathfrak{g}_f$ is the centralizer of f in $\mathfrak{g}$. Then the adjoint action of the subgroup N_+ on the affine subspace $f + \mathfrak{g}_+$ is free and proper.

Proof. We shall show that the action is free. From the condition imposed on f it follows that the infinitesimal action of the algebra $\mathfrak{n}_+$ is free, i.e., $[n, \xi] = 0$, $n \in \mathfrak{n}_+$, $\xi \in f + \mathfrak{g}_+$ implies $n = 0$. In fact, let $n = \sum_{i>0} n_i$, $\xi = f + \sum_{i \geq 0} \xi_i$ be the graded decomposition. Then $[n, \xi] = 0$ implies

$$[n_1, f] = 0$$
$$\cdots\cdots\cdots$$
$$[n_j, f] + \sum_{i=1}^{j-1} [n_i, \xi_{j-i-1}] = 0 \, .$$

Whence, considering that $[n, f] = 0$ implies $n = 0$, we get inductively that $n_i = 0$. Since N_+ is a nilpotent subgroup, it follows from $\mathrm{Ad}\, g \cdot \xi = \xi$, $g \in N_+$ that $g = \exp n$ and $[n, \xi] = 0$, i.e., $n = 0$ and $g = e$.

Now we shall show that the action of N_+ is proper. This means that if $\xi'_k = \mathrm{Ad}\, g_k \cdot \xi_k$ and here $\xi_k \to \xi$, $\xi'_k \to \xi'$, then the sequence g_k is compact. For the proof we consider the procedure for reconstruction of the element g from the equation $\xi' = Ad\, g \cdot \xi$. Let $g = \exp n$, so $Ad\, g = \exp ad\, n = \sum_{k \geq 0} \frac{1}{k!}(ad\, n)^k$. We have the equation

$$\sum \frac{1}{k!}\,(ad\, n)^k \cdot \xi = \xi' .$$

Decomposing this equation into graded components, $n = \sum n_i$, $\xi = f + \sum \xi_i$, $\xi' = f + \sum \xi_i'$, we get

$$\begin{aligned} &\xi_0 + ad\, n_1 \cdot f = \xi_0' \\ &\xi_1 + (ad\, n_1)^2 \cdot f + ad\, n_1 \cdot \xi_0 + ad\, n_2 \cdot f = \xi_1' \\ &\dots \quad \dots \quad \dots \quad \dots \\ &P_i(n_1, \dots, n_i;\ \xi_0, \dots, \xi_i) + ad\, n_{i+1} \cdot f = \xi_i' . \end{aligned}$$

It follows from the properties of the element f that these relations are uniquely solvable relative to n:

$$n_{i+1} = (ad\, f|_{d_{i+1}})^{-1}\,(P_i(n_1, \dots, n_i; \xi_0, \dots, \xi_i) - \xi_i') .$$

Thus, n depends polynomially on ξ, ξ'. Whence follows the propriety of the action.

For the Borel subalgebra $\mathfrak{q}_+$ in the split case B. Kostant constructed a linear section to the action of the group N_+. Proposition 3.2 is a weakened but more universal version of Theorem 1.2 of [24].

Elements $f \in d_{-1}$ such that $\mathfrak{g}_f \cap \mathfrak{n}_+ = \{0\}$, will, by analogy with [25], be called principal.

We recall that the element $x_0 \in \bar{\mathfrak{a}}$ defines the $\mathfrak{q}$-grading in $\mathfrak{g}$.

LEMMA 3.2. Let $f \in d_{-1}$. If the subspace $ad\, f \cdot d_1 \subset d_0$ contains the element x_0, then f is a principal nilpotent element.

Proof. If $[f, e] = x_0$, $e \in d_1$, then the triple (e, x_0, f) forms a standard basis of the algebra $s\ell(2)$. From the theory of representations of this algebra applied to the ad-representation of it in $\mathfrak{g}$, it follows that if $[f, x] = 0$, then x lies in the sum of the eigenspaces of the operator $ad\, x_0$, corresponding to nonpositive eigenvalues, that is, $x \in \mathfrak{q}_-$.

COROLLARY 3.2 [25]. Let $\mathfrak{g}$ be a splittable semisimple Lie algebra, $\mathfrak{q}$ be its Borel subalgebra. The element $f = \sum_{\alpha \in -P} f_\alpha$, $f_\alpha \in \mathfrak{g}_\alpha$, is principal if and only if all f_α are different from zero.

3.3. Completion of Flows of Generalized Toda Chains. Let $\mathfrak{g} = \mathfrak{n}_+ + \mathfrak{q}_-$ be the parabolic decomposition of the algebra $\mathfrak{g}$, $\mathfrak{g}^* = \mathfrak{n}_- + \mathfrak{q}_+$ be the dual decomposition. We identify the space T^*G with the product $G \times \mathfrak{g}^*$ with the help of right translations. Then the left action of the group G can be written in the form: $h(g, \xi) = (hg, Ad\, h \cdot \xi)$, the right: $h_0(g, \xi) = (gh^{-1}, \xi)$. The left and right mappings of moments have the form

$$\Phi_\ell(g, \xi) = \xi, \quad \Phi_r(g, \xi) = -Ad^* g^{-1} \cdot \xi .$$

Let $\Phi: T^*G \to \mathfrak{n}_- \oplus \mathfrak{q}_+$ be the mapping of moments corresponding to the left action of the subgroup N_+ and the right action of the subgroup Q_-:

$$\Phi(g, \xi) = Pr_{\mathfrak{n}_-}(\xi) \oplus - Pr_{\mathfrak{q}_+}(Ad\, g^{-1} \cdot \xi) .$$

For $f \in \mathfrak{n}_-$, $c \in \mathfrak{q}_+$ we write $M_{f,c} = \Phi^{-1}(f \oplus c)$. Let Q_c be the stationary subgroup of the element c.

Proposition 3.3.1. Let f be a principal nilpotent element. Then

(i) $M_{f,c}$ is a smooth submanifold in T^*G.

(ii) The quotient-space $\overline{M}_{f,c} = N_+ \setminus M_{f,c}/Q_c$ is a smooth symplectic manifold.

Proof. (i) It suffices to prove that $f \oplus c$ is a regular value of the map Φ, i.e., that the differential $d\Phi_{g,\xi}: \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{n}_- \oplus \mathfrak{q}_+$ of the map Φ at the point $(g,\xi) \in M_{f,c}$ is surjective. It is easy to see that

$$d\Phi_{g,\xi}(x,y) = Pr_{\mathfrak{n}_-} y \oplus Pr_{\mathfrak{q}_+} Ad\, g^{-1}([x,\xi]-y).$$

To prove the surjectivity of $d\Phi_{g,\xi}$ it suffices to verify that the map $(x,y) \to [x,\xi]-y$, mapping $\mathfrak{g} \oplus \mathfrak{q}_+$ into $\mathfrak{g}$, is surjective. But if the element $z \in \mathfrak{g}$ is such that $B([x,\xi]-y, z) = 0$ for all $x \in \mathfrak{g}$, $y \in \mathfrak{q}_+$, then $z \in \mathfrak{n}_+$ and $[z,\xi] = 0$. Since $\xi \in f + \mathfrak{q}_+$, and the element f is principal, it follows from this that $z = 0$. Thus (i) is proved.

(ii) It suffices to prove that the group $G_0 = N_+ \times Q_-$ acts on the manifold $G \times (f + \mathfrak{q}_+) \supset M_{f,c}$ freely and properly. This fact follows directly from Proposition 3.2.

It was noted in 2.2 that the orbit $\mathcal{O}_c$ is naturally imbedded in the manifold $\overline{M}_{f,c}$. Let $\mathfrak{q}$ be a minimal parabolic subalgebra in $\mathfrak{g}$, $f = \sum_{\alpha \in -P} f_\alpha$, and all f_α be different from zero. Under these conditions we shall prove the following proposition.

Proposition 3.3.2. The completion of $\overline{M}_{f,c} \setminus \mathcal{O}_c$ is the union of submanifolds of lower dimension and the flow Q is a chain transversal to these submanifolds.

Proof. We use the Bruhat decomposition

$$G = \bigcup_{w \in W} N_+ w Q_- .$$

The orbit $\mathcal{O}_c$ lies over the highest cell $N_+ Q_-$; we shall show that subsets of the reduced manifold lying over the other cells have lower dimension. The manifold $\overline{M}_{f,c}$ is obtained as a result of the factorization of the manifold

$$M_{f,c} = \{(g,\xi) : Pr_{\mathfrak{n}_-} \xi = f,\ Pr_{\mathfrak{q}_+} Ad\, g^{-1} \cdot \xi = -c\}.$$

by the left action of the group N_+ and the right action of the group Q_-. Let $g = nwq$, $\xi = f + y$; $n \in N_+$, $q \in Q_-$, $y \in \mathfrak{q}_+$. Then

$$Ad\, g^{-1} \cdot \xi = -c + x, \quad x \in \mathfrak{n}_-,$$

i.e.,

$$Ad\, w^{-1} \cdot Ad\, n^{-1} \xi = Ad\, q(-c + x)$$

or

$$Ad\, w^{-1}(f + y') = c' + x'.$$

Finally, we get the equation

$$Ad\, w^{-1} \cdot y' - x' = c' - Ad\, w^{-1} f. \tag{*}$$

If the space of solutions of this equation is nonempty, then its dimension is equal to $d(w) = \dim(Ad\, w^{-1} \mathfrak{q}_+ \cap \mathfrak{n}_-)$. On the other hand, the codimension of the cell $N_+ w Q_-$ of G is equal to

$$\dim G - \dim N_+ w Q_- = \dim(w^{-1} N_+ w \cap Q_-) = d(w).$$

Since the group $N_+ \times Q_c$ acts in $M_{f,c}$ freely, it suffices to prove that those points nwq, for which (*) is solvable, i.e., for which $(Ad\, q \cdot c + Ad\, w^{-1} f) \in Ad\, w^{-1} \mathfrak{q}_+ + \mathfrak{n}_-$, form in the cell

$N_+ w Q_-$ a subset of lower dimension. Let $\alpha \in -P$ be such that $w^{-1}\alpha \in \Delta_+$. Then $\mathrm{Ad}\, w^{-1} f_\alpha \notin \mathrm{Ad}\, w^{-1} \mathfrak{g}_+$. If $\beta = w^{-1}\alpha \notin P$, then there are no solutions of (*) at all. Now if $\beta \in P$, then there arises the following condition on q. Let $q = ma\bar{n}$ be the Langlands decomposition, c_β be the component of the element c in the space $\mathfrak{g}_\beta$. Then for the solvability of (*) it is necessary that $\mathrm{Ad}(ma)\cdot c_\beta = \mathrm{Ad}\, w^{-1}\cdot f_\alpha$. This condition singles out in MA a submanifold of codimension $\geqslant 1$, to which corresponds a submanifold of the same codimension in the cell $N_+ w Q_-$. The latter follows from the existence of the unique decomposition $N_+ w Q_- = N_+ w MAN_-^{(w)}$, $N_-^{(w)} \subset N_-$.

Now we shall show that the flow Q is a chain transversal to the "lowest cells." It suffices to prove the transversality of the flow in $M_{f,c}$ (before reduction) whose trajectories have the form $(g(t), \xi(t)) = (g\, \exp t\xi, \xi)$, $(g,\xi) \in M_{f,c}$ according to Lemma 1.2. Let $g \in N_+ w Q_-$, $g = n w q$. The tangent space to the cell $N_+ w Q_-$ (in a right-invariant basis) at the point g is $Tg = n_+ + \mathrm{Ad}(nw)\cdot \mathfrak{q}$. The tangent vector to the trajectory at the point (g,ξ) has the form $(\xi, 0)$, $\xi \in f + \mathfrak{g}_+$. Since for $w \neq 1$ the element f does not lie in the subalgebra $\mathrm{Ad}\, w \mathfrak{g}_-$, the vector ξ does not belong to the space Tg, and the trajectories are not tangent to lowest cells.

<u>3.4. Symmetric Realization of Toda Chains.</u> We consider now "small" Toda chains, connected with the subalgebra $\mathfrak{b} = \mathfrak{a} + \mathfrak{n}_-$; in the split case $\mathfrak{b}$ is a Borel subalgebra. The subalgebra $\mathfrak{b}$ participates in two decompositions: the triangular $\mathfrak{g} = (\mathfrak{n}_+ + \mathfrak{m}) \dotplus \mathfrak{b}$ and the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \dotplus \mathfrak{b}$. It turns out that the Toda chains constructed from these two decompositions are equivalent.

The Toda chain corresponding to the decomposition $\mathfrak{g} = (\mathfrak{n}_+ + \mathfrak{m}) \dotplus \mathfrak{b}$, is defined on the orbit $\mathcal{O}_e$, $e = \sum_{\alpha \in P} e_\alpha$, whose general point has the form $\xi = p + \sum c_\alpha e_\alpha$, $p \in \mathfrak{a}$.

To the decomposition $\mathfrak{g} = \mathfrak{k} \dotplus \mathfrak{b}$ corresponds the symmetric realization $\mathfrak{b}^* \simeq \mathfrak{p}$, where $\mathfrak{g}^* = \mathfrak{g} = \mathfrak{k} \dotplus \mathfrak{p}$ is a Cartan decomposition. The orbit of the chain $\mathcal{O}'_e$ passes through the point $e - \theta e$ (θ is the Cartan automorphism) and its general point has the form

$$\xi' = p' + \sum c'_\alpha (e_\alpha - \theta e_\alpha), \quad p' \in \mathfrak{a}.$$

<u>Proposition 3.4.1.</u> (i) The map $\mathcal{O}'_e \to \mathcal{O}_e$: $p = \frac{1}{2} p'$, $c_\alpha = \frac{1}{4} c'^2_\alpha$ is a symplectic diffeomorphism.

(ii) If $f = -\theta e$, φ is an invariant polynomial on $\mathfrak{g}$ of degree d, then

$$\varphi(\xi') = 2^d \varphi(\xi + f).$$

(iii) Hamiltonian flows which are defined by Hamiltonians $\varphi \in I(\mathfrak{g})$ on the orbit $\mathcal{O}_e$, are complete.

<u>Remark.</u> For an arbitrary translation vector f, (ii) and (iii) are false.

<u>Proof.</u> (i) It follows directly from the explicit form of the Poisson bracket: $\{p, c_\alpha\} = \alpha = \{p', c'_\alpha\}$.

(ii) Let $u = v \cdot \prod_{\alpha \in \Delta} e_\alpha^{k_\alpha}$ be a monomial of degree d, appearing in φ (here v is a monomial in $\mathfrak{a}$). In view of the A-invariance of the polynomial φ, $\sum k_\alpha \cdot \alpha = 0$. If for some $\alpha \in \Delta \setminus (P \cup -P)$ the coefficient $k_\alpha > 0$, then $u(\xi') = u(\xi + f) = 0$. Now if $k_\alpha = 0$ for all $\alpha \in \Delta \setminus (P \cup -P)$, then $u = v \prod_{\alpha \in P}$

$(e_\alpha e_{-\alpha})^{k_\alpha}$, whence $u(\xi') = 2^d u(\xi+f)$.

(iii) This follows from the Iwasawa decomposition.

Remark. Functions $\varphi(\xi)$, $\varphi \in I(\mathfrak{g})$, $\xi \in \mathfrak{p}$, defined on $\mathfrak{b}^* \simeq \mathfrak{p}$, are obviously M-invariant. We turn to the minimal parabolic subalgebra $\mathfrak{q} = \mathfrak{m} + \mathfrak{b}$. Let $\eta \in \mathfrak{q}^*$, $\eta = \pi + \xi$, $\pi \in \mathfrak{m}^*$, $\xi \in \mathfrak{b}^*$. We get that functions on $\mathfrak{q}^*$ of the form $\tilde{\varphi}(\eta) = \varphi(\xi)$ commute with one another and with functions of the form $\tilde{\psi}(\eta) = \psi(\pi)$. Choosing a subalgebra of functions on $\mathfrak{m}^*$, which is commutative with respect to the Poisson bracket, we get an involutive system of functions $\{\tilde{\varphi}, \tilde{\psi}\}$ on $\mathfrak{q}^*$. By somewhat modifying the proof of Theorem 1, one can prove the following proposition.

Proposition 3.4.2. Flows of Hamiltonians $H = \tilde{\varphi} + \tilde{\psi}$, $\varphi \in I(\mathfrak{g})$, $\psi \in \mathcal{F}(\mathfrak{m}^*)$, in the space $\mathfrak{q}^*$ are complete.

3.5. Examples. a) Toda chains connected with the Iwasawa decomposition of split semi-simple Lie algebras are listed in [10]. Recently, they have been studied in detail in [16, 20]. Here they are considered only for illustrating the preceding results. We shall use the Gauss decomposition $\mathfrak{g} = \mathfrak{n}_+ + \mathfrak{b}_-$, $\mathfrak{b}_- = \mathfrak{a} + \mathfrak{n}_-$. Let $B = AN_-$ be a connected component of the Borel subgroup.

The orbit $\mathcal{O}_e \subset \mathfrak{b}_+$, passing through the element $e \in \mathfrak{d}_1$, is isomorphic with $T^*(A \cdot e)$ (Corollary 2.3). We choose nonzero elements $e_\alpha \in \mathfrak{g}_\alpha$ and we set $e = \sum_{\alpha \in P} e_\alpha$. Then the map $A \to A \cdot e$ is one-to-one and $\mathcal{O}_e \simeq T^*A$. The exponential map identifies $\mathfrak{a}$ with A and thus, $T^*\mathfrak{a}$ with T^*A. Corollary 2.3 associates the point $(p,q) \in T^*\mathfrak{a}$ with the point $\xi = p + \sum_{\alpha \in P} e^{\langle q,\alpha \rangle} e_\alpha \in \mathcal{O}_e$. Choosing the translation vector $f \in \mathfrak{d}_{-1}$, we get the Hamiltonian

$$H = \frac{1}{2} B(p,p) + \sum_{\alpha \in P} f_\alpha e^{\langle q,\alpha \rangle}.$$

The equations of motion have the form

$$\dot{\xi} = [\, \xi + f,\ p + f \,].$$

If all $f_\alpha \geq 0$, then the system is complete (Sec. 3.4). Otherwise Proposition 3.3.1 gives a completion of the Toda chain with "improper" signs in the potential (under the condition $f_\alpha \neq 0\ \ \forall \alpha \in P$).

b) By a non-Abelian Toda chain is meant a Hamiltonian system on the phase space $T^*(G^n)$, $G = GL(m, \mathbb{R})$, with Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{n} \operatorname{tr} p_i^2 + \sum_{i=1}^{n-1} \operatorname{tr} g_{i+1} g_i^{-1}, \quad p_i = \dot{g}_i g_i^{-1}.$$

The system has group of symmetries $G \times G$, acting by translations on the configuration space G^n:

$$(g,h) \cdot (g_1, \ldots, g_n) = (g g_1 h^{-1}, \ldots, g g_n h^{-1}).$$

We represent the matrix algebra $\mathfrak{gl}(m,n)$ as the tensor product $\mathfrak{gl}(m,n) = \mathfrak{gl}(m) \otimes \mathfrak{gl}(n)$ and we consider a block parabolic subalgebra $\mathfrak{q} = \mathfrak{gl}(m) \otimes \mathfrak{b}$, where $\mathfrak{b}$ is a Borel subalgebra in $\mathfrak{gl}(n)$. Let Q be the corresponding parabolic subgroup, $Q = LAN$ be the Langlands decomposition. Then $LA = G^n$. It follows from Proposition 2.3 that the orbit $\mathcal{O}_e$ of the Q-chain is the reduction of the space T^*G^n. Let $e = 1_m \otimes e_n$, $f = 1_m \otimes f_n$, where e_n and f_n are "initially given" classical Toda chains in $\mathfrak{gl}(n)$.

Proposition 3.5. (i) The Toda Q-chain on the orbit $\mathcal{O}_e$ (Sec. 3.1) coincides with the reduction of the non-Abelian Toda chain by the right action of the group of symmetries G over the point $0 \in \mathfrak{g}^*$.

(ii) f is a $\mathfrak{g}$-principal element.

We note further that the left action of the group of symmetries G in the initial Toda chain turns in the Q-chain (i.e., after reduction) into the $\mathcal{A}d$-action of the block-diagonal subgroup $G = \mathrm{diag}\, GL(m) \subset GL(mn)$. The non-Abelian Toda chain is incomplete, its kinetic energy is indefinite. The number of independent integrals of motion given in Proposition 3.1 is equal to $mn-1$ and is less than the dimension of the configuration space, which is equal to $m^2(n-1)$. The introduction of a spectral parameter adds new integrals of motion (Sec. 7).

c) We list chains connected with minimal parabolic subalgebras. Complex semisimple Lie algebras, considered as real, reduce to the obvious complexifications of splittable chains.

1) $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{H})$: quaternion Toda chain, analogous to the non-Abelian one. The configuration space is $\mathbb{H}^{*n}/\mathbb{H}^*$, $\mathbb{H}^*$ is the multiplicative group of the quaternions. The Hamiltonian $H = \mathrm{Re}\left[\frac{1}{2}\sum_1^n (\dot{q}_i q_i^{-1})^2 + \sum_1^{n-1} c_i q_{i+1} q_i^{-1}\right]$, c_i are fixed quaternions. The analog of Proposition 3.5 is valid. If $c_i \neq 0$, then by a change of variables one can make $c_i = 1$.

2) $\mathfrak{g} = \mathfrak{so}(m,n)$, $m > n$: the standard Toda chain of length n, interacting with a point on the sphere S^{m-n-1}. The Hamiltonian $H = \frac{1}{2}\sum_1^n p_i^2 - \frac{1}{2}\pi^2 + \sum_1^{n-1} e^{q_{i+1}-q_i} + e^{q_1}(v, f)$. Here $v \in S^{m-n-1}$, (v, f) is the scalar product with the vector $f \in \mathbb{R}^{m-n}$, π is the momentum of a point on the sphere.

3) $\mathfrak{g} = \mathfrak{su}(m,n)$, $m > n$: complex Toda chain, interacting with a point on the sphere $S = S^{2(m-1)-1} \subset \mathbb{C}^{m-n}$. The configuration space is the result of taking the quotient of the product $\mathbb{C}^{*n} \times S$ by the diagonal action of the group $u(1)$. The Hamiltonian (before reduction)

$$H = \mathrm{Re}\left[\frac{1}{2}\sum_1^n (\dot{z}_i z_i^{-1})^2 - \frac{1}{2}\pi^2 + \sum_1^{n-1} z_{i+1} z_i^{-1} + z_1 (f, v)\right],$$

$v \in S$, π is the momentum of a point on the sphere $f \in \mathbb{C}^{m-n}$.

If $m = n$, then the configuration space is $\mathbb{C}^{*n}/u(1)$, the potential $v = \mathrm{Re}\, \Sigma\, z_{i+1} z_i^{-1} + |z_1|^2$.

4) $\mathfrak{g} = \mathfrak{so}^*(2n)$: quaternion-orthogonal chain. The configuration space is $\mathbb{H}^{*\ell}$ for $n = 2\ell+1$ and $\mathbb{H}^{*\ell}/Sp(1)$ for $n = 2\ell$. The Hamiltonian $H = \mathrm{Re}\left[\sum_1^\ell (\dot{q}_i q_i^{-1})^2 + \sum_1^{\ell-1} c_i q_{i+1} q_i^{-1} + V\right]$, where $V = c q_1$ for $n = 2\ell+1$ and $V = c|q_1|^2$ for $n = 2\ell$.

5) $\mathfrak{g} = \mathfrak{sp}(m,n)$, $m > n$: quaternionic chain, interacting with a point on the sphere $S = S^{4(m-n)-1} \subset \mathbb{H}^{m-n}$. The configuration space is the result of taking the quotient of $\mathbb{H}^{*n} \times S$ by the diagonal action of $Sp(1)$. The potential $V = \mathrm{Re}\left[\sum_1^{n-1} c_i q_{i+1} q_i^{-1} + q_1 (f, v)\right]$, $v \in S$, $f \in \mathbb{H}^{m-n}$. If $m = n$, then the configuration space is $\mathbb{H}^{*n}/U(1)$ and $V = \mathrm{Re}\left[\sum_1^{n-1} c_i q_{i+1} q_i^{-1}\right] + c(q_1)$, where $c(q_1)$ is the matricial element of the quaternion q_1 in the orthogonal representation in $\mathbb{R}^3$.

4. Graded Lie Algebras and Groups of Flows

In this section we introduce infinite-dimensional Lie algebras and their decompositions, which play a basic role in what follows. The description of the orbits and Hamiltonian mechanics on them which we need does not require the use of the corresponding infinite-dimensional Lie groups — groups of flows. However, the geometric construction of Sec. 1 allows one to express the solution of the equations of motion in terms of the solution of the problem of factorization in the group; for the sake of this we discuss briefly the necessary properties of the group of flows.

4.1. Affine Lie Algebras.

Let $\mathfrak{g}$ be a (finite-dimensional) Lie algebra. By the affine Lie algebra $\tilde{\mathfrak{g}}$ we shall mean the Lie algebra of Laurent series with coefficients in $\mathfrak{g}$ and pointwise commutator:

$$\tilde{\mathfrak{g}} = \{ x = \sum_{n \leqslant N(x)} x_n z^n,\ x_n \in \mathfrak{g} \},\ [\, x z^m,\ y z^n \,] = [x, y]\, z^{m+n} .$$

In the algebra $\tilde{\mathfrak{g}}$ there are subalgebras $\mathfrak{g}_\pm$, defined in the following way:

$$\mathfrak{g}_+ = \{ \sum_{n \geqslant 0} x_n z^n \}, \qquad \mathfrak{g}_- = \{ \sum_{n \leqslant 0} x_n z^n \} .$$

These subalgebras contain decreasing sequences of ideals of the form $\mathfrak{g}_\pm^k = \mathfrak{g}_\pm \cdot z^{\pm k}$, $k > 0$.

If there is given a subalgebra $\mathfrak{q} \subset \mathfrak{g}$, then we define subalgebras $\mathfrak{q}_\pm \subset \mathfrak{g}_\pm$, by setting $\mathfrak{q}_\pm = \mathfrak{q} + \mathfrak{g}_\pm^1$. Then the decomposition of the algebra $\mathfrak{g}$ into a linear sum of two subalgebras $\mathfrak{g} = \mathfrak{a} \dot{+} \mathfrak{b}$ generates an analogous decomposition for $\tilde{\mathfrak{g}}$: $\tilde{\mathfrak{g}} = \mathfrak{a}_+ \dot{+} \mathfrak{b}_-$.

The space $\tilde{\mathfrak{g}}^*$, dual to the algebra $\tilde{\mathfrak{g}}$, is the space of Laurent series with coefficients in $\mathfrak{g}^*$: the value of the functional $\xi = \Sigma\, \xi_n z^n$ on the element $x = \Sigma\, x_n z^n$ is equal to $\xi(x) = \Sigma\, \xi_n(x_{-n})$. We introduce the notation $< S >_n$ for the n-th coefficient of the series S. Then one can write

$$\xi(x) = < \xi(z)(x(z)) >_0 .$$

The decomposition $\tilde{\mathfrak{g}} = \mathfrak{a}_+ \dot{+} \mathfrak{b}_-$ implies $\tilde{\mathfrak{g}}^* = \mathfrak{a}_+^* \dot{+} \mathfrak{b}_-^*$, where $\mathfrak{a}_+^* = \mathfrak{b}_-^\perp$, $\mathfrak{b}_-^* = \mathfrak{a}_+^\perp$, that is,

$$\mathfrak{a}_+^* = \{ \sum_{n \leqslant 0} \xi_n z^n,\ \xi_0 \in \mathfrak{a}^* = \mathfrak{b}^\perp \},$$

$$\mathfrak{b}_-^* = \{ \sum_{n \geqslant 0} \xi_n z^n,\ \xi_0 \in \mathfrak{b}^* = \mathfrak{a}^\perp \}.$$

The coadjoint action of the algebra $\tilde{\mathfrak{g}}$ in $\tilde{\mathfrak{g}}^*$ is functional in z: $ad^* x \cdot \xi(z) = ad^* x(z) \cdot \xi(z)$. The coadjoint action of the algebra $\mathfrak{b}_-$ in $\mathfrak{b}_-^*$ assumes the form

$$ad^*_{\mathfrak{b}_-} x \cdot \xi = (ad^* x \cdot \xi)_+ ,$$

where $(\cdot)_+$ denotes the projection in $\tilde{\mathfrak{g}}^*$ onto the subspace $\mathfrak{b}_-^*$ parallel to $\mathfrak{a}_+^*$. Whence it is evident that the finite-dimensional subspace

$$\mathfrak{b}_n^* = \{ \sum_0^n \xi_k z^k,\quad \xi_0 \in \mathfrak{b}^* \}$$

is $ad^* \mathfrak{b}_-$-invariant. This is also clear from the fact that the subspace $\mathfrak{b}_n^*$ is naturally dual to the quotient-algebra $\mathfrak{b}_n = \mathfrak{b}_- / \mathfrak{g}_-^{n+1}$ and the ad^*-action of $\mathfrak{b}_-$ in $\mathfrak{b}_n^*$ reduces to the action of the algebra $\mathfrak{b}_n^*$. Thus, the $\mathfrak{b}_-$-orbits in the space $\mathfrak{b}_n^*$, i.e., the integral manifolds of the distribution $\xi \mapsto ad^*_{\mathfrak{b}_-} \mathfrak{b}_- \cdot \xi$, are orbits of a finite-dimensional Lie group with Lie algebra $\mathfrak{b}_n$.

4.2. Symmetrically Graded Lie Algebras. Let θ be an involutive automorphism of the algebra $\mathfrak{g}$, $\mathfrak{k}(\mathfrak{p})$ be its eigensubspace corresponding to the eigenvalue $1(-1)$. We extend θ to an involution $\tilde{\theta}$ of the algebra $\tilde{\mathfrak{g}}$:

$$\tilde{\theta}(xz^n) = (-1)^n \theta(x) z^n .$$

The subalgebra $\tilde{\mathfrak{g}}^\theta \subset \tilde{\mathfrak{g}}$, consisting of $\tilde{\theta}$-invariant elements, will mean the θ-graded Lie algebra:

$$\tilde{\mathfrak{g}}^\theta = \{ x = \Sigma x_n z^n : x_{2k} \in \mathfrak{k},\ x_{2k+1} \in \mathfrak{p} \} .$$

If θ is the Cartan involution in the semisimple algebra $\mathfrak{g}$, then we will also call $\tilde{\mathfrak{g}}^\theta$ symmetrically graded.

The dual space $\tilde{\mathfrak{g}}^{\theta *}$ is naturally imbedded in $\tilde{\mathfrak{g}}^*$ as the subspace of $\tilde{\theta}^*$-invariant elements:

$$\tilde{\mathfrak{g}}^{\theta *} = \{ \xi = \Sigma \xi_n z^n : \xi_{2k} \in \mathfrak{k}^*,\ \xi_{2k+1} \in \mathfrak{p}^* \} .$$

All the constructions of 4.1 carry over to θ-graded algebras.

4.3. Basic Theorem. The algebraic proof of 1.3 carries over word for word to the algebras $\tilde{\mathfrak{g}}$, $\tilde{\mathfrak{g}}^\theta$. In order to formulate an analog of Theorem 1, it remains to describe the algebra of invariant polynomials on $\tilde{\mathfrak{g}}^*$.

Let $\tilde{\varphi}$ be a polynomial on $\tilde{\mathfrak{g}}^*$. By definition, $\tilde{\varphi} \in I(\tilde{\mathfrak{g}}^*)$, if $ad^* d\tilde{\varphi}(\xi) \cdot \xi = 0$ for all $\xi \in \tilde{\mathfrak{g}}^\times$. Let $\varphi \in I(\mathfrak{g}^*)$; we construct polynomials φ_n on $\tilde{\mathfrak{g}}^*$, $n \in \mathbb{Z}$, taking the n-th Fourier coefficient of the series $\varphi(\xi(z))$:

$$\varphi_n(\xi) = \langle \varphi(\xi(z)) \rangle_n .$$

Obviously, $\varphi_n \in I(\tilde{\mathfrak{g}}^*)$ and the polynomials of this form generate the algebra $I(\tilde{\mathfrak{g}}^*)$. The restriction of a polynomial from $I(\tilde{\mathfrak{g}}^*)$ to $\tilde{\mathfrak{g}}^{\theta *} \subset \tilde{\mathfrak{g}}^*$ gives an invariant polynomial on $\tilde{\mathfrak{g}}^{\theta *}$.

We denote by $\tilde{\mathfrak{g}}_0 = \mathfrak{a}_+ \oplus \mathfrak{b}_-$ the direct sum of the subalgebras $\mathfrak{a}_+$ and $\mathfrak{b}_-$ and we identify the space $\tilde{\mathfrak{g}}_0^* = \mathfrak{a}_+^* \oplus \mathfrak{b}_-^*$ with $\tilde{\mathfrak{g}}^*$ with the help of the map $\sigma: \alpha \oplus \beta \longmapsto \alpha - \beta$, thus defining on $\tilde{\mathfrak{g}}^*$ a second Kirillov bracket.

THEOREM 2. (i) Polynomials of the form φ_n, $\varphi \in I(\mathfrak{g}^*)$, are in involution with respect to both brackets on $\tilde{\mathfrak{g}}^*$. (ii) The equations of motion in $\tilde{\mathfrak{g}}^*$, defined by the Hamiltonian φ_n with respect to the second Kirillov bracket, have the form

$$\dot{\xi} = ad^* M_- \cdot \xi ,$$

where M_- is the projection of $M = z^{-n} d\varphi(\xi(z))$ onto $\mathfrak{b}_-$ parallel to $\mathfrak{a}_+$. (iii) If in $\mathfrak{g}$ there is a nondegenerate invariant bilinear form, then the equations of motion assume the commutative Laxian form

$$\dot{\xi} = [M_-, \xi] .$$

In the formulation of Theorem 2 one can replace the algebra $\tilde{\mathfrak{g}}$ by $\tilde{\mathfrak{g}}^\theta$.

We comment on point (iii). If B is an invariant bilinear form in $\mathfrak{g}$, then the formula

$$\tilde{B}_n(x, y) = \langle B(x(z), y(z)) \rangle_n$$

gives an invariant form in $\tilde{\mathfrak{g}}$; if B is nondegenerate, then all $\tilde{B}_n$ are also nondegenerate. In what follows we shall use the form $\tilde{B} = \tilde{B}_0$. With its help one makes the following identifications:

$$\tilde{\mathfrak{g}}^* \simeq \tilde{\mathfrak{g}}, \quad \tilde{\mathfrak{g}}^{\theta *} \simeq \tilde{\mathfrak{g}}^{\theta}, \quad ad^* \simeq ad,$$

$$\mathfrak{a}_+^* \simeq \mathfrak{b}^{\perp} + \mathfrak{g}_-^1, \quad \mathfrak{b}_-^* \simeq \mathfrak{a}^{\perp} + \mathfrak{g}_+^1,$$

$ad^*_{\mathfrak{b}_-} x \cdot \xi = [x, \xi]_+$, + denotes the projection onto $\mathfrak{b}_-^*$ parallel to $\mathfrak{a}_+^*$.

In constructing examples we shall basically use Corollary 1.2 for the semisimple algebra $\mathfrak{g}$. Obviously a character f of the algebra $\mathfrak{a}_+$ has the form $f = f_0 + f_1 z^{-1}$, where f_0 is a character of $\mathfrak{a}$, $f_1 \in [\mathfrak{a}, \mathfrak{g}]^{\perp}$. In particular, if $\mathfrak{a} = \{0\}$, $\mathfrak{a}_+ = \mathfrak{g}_+^1$, then $f = f_1 z^{-1}$, $f_1 \in \mathfrak{g}^*$. We formulate Corollary 1.2 separately:

COROLLARY 4.3. Let f be a character of the algebra $\mathfrak{a}_+$. (i) Polynomials on $\mathfrak{b}_-^*$ of the form $\varphi_n(\xi) = \langle \varphi(\xi(z) + f) \rangle_n$, $\varphi \in I(\mathfrak{g}^*)$, are in involution with respect to the Kirillov bracket on $\mathfrak{b}_-^*$. (ii) Let $\mathfrak{g}$ be reductive. The equations of motion on $\mathfrak{b}_-^*$, defined by the Hamiltonian φ_n, have Laxian form

$$\dot{\xi} = [M_-, \xi + f],$$

where M_- is the projection of $M = z^{-n} d\varphi(\xi(z) + f)$ onto $\mathfrak{b}_-$ parallel to $\mathfrak{a}_+$.

4.4. Groups of Flows. In order to connect the preceding with the theory of analytic Lie groups, we consider a modification of the algebra $\tilde{\mathfrak{g}}$, the Banach Lie algebra $\mathfrak{g}_W$, consisting of absolutely convergent Fourier series with coefficients in $\mathfrak{g}$. Everything said above in relation to the algebra $\tilde{\mathfrak{g}}$ is applicable to $\mathfrak{g}_W$ too. For simplicity we restrict ourselves to the case of matrix groups. We note that in [17] there is an error at this point: Lemma 19 is false and the set ${}^{\circ}G_W$ defined there is not a group.

Let $G \subset GL(n, \mathbb{R})$ be a connected matrix group, $G^{\mathbb{C}} \subset GL(n, \mathbb{C})$ be its complexification, $g \mapsto \bar{g}$ be the complex conjugation in $G^{\mathbb{C}}$. The Lie algebra $\mathfrak{g}$ lies in $\mathrm{Mat}(n, \mathbb{R})$. We denote by W the (real) algebra of absolutely convergent Fourier series with coefficients from $\mathrm{Mat}(n, \mathbb{R})$. We set

$$\mathfrak{g}_W = \{u \in W : u = \Sigma u_n z^n, u_n \in \mathfrak{g}\}.$$

In other words, $\mathfrak{g}_W = \{u \in W : u(z) \in \mathfrak{g}^{\mathbb{C}}, u(\bar{z}) = \overline{u(z)}$ for $|z| = 1\}$. The corresponding Banach Lie groups are the flow groups G_W:

$$G_W = \{g \in W : g(z) \in G^{\mathbb{C}}, g(\bar{z}) = \overline{g(z)} \quad \text{for} \quad |z| = 1\}.$$

We denote by W_+ (W_-) the subalgebra of W, consisting of functions, analytic (antianalytic) inside the unit disk. We set

$$G_{\pm} = \{g \in G_W \cap W_{\pm}, \; g(z) \in G^{\mathbb{C}} \quad \text{for} \quad |z| \leq 1\}$$
$$N_{\pm} = \{g \in G_{\pm} : g(0) = \mathbb{1}\}.$$

It follows from Wiener's theorem that G_W, $G_{\pm}$, $N_{\pm}$ are groups.

LEMMA 4.4. Let $g \in G_W$, $|1 - g|_W < \frac{1}{2}$. Then g decomposes into a product

$$g = n g_-,$$

where $n \in N_+$, $g_- \in G_-$.

Proof. We consider the following curve in the group G_W, joining $\mathbb{1}$ with g:

$$g_t(z) = \exp(t \log g(z)).$$

From $|1-g|_w < \frac{1}{2}$ the estimate $|1-g_t|_w < 1$ follows easily. By Lemma 5.1 from [26], for such functions g_t there exists a decomposition in $GL(n,C)_w$:

$$g_t = n(t)\, g_-(t),$$

where $n(\cdot)$ and $g_-(\cdot)$ are smooth curves in the groups $GL(n)_+, GL(n)_-$, respectively, where $n(t, 0) = \mathbb{1}$. It remains to prove that $n(t) \in N_+$, $g_-(t) \in G_-$.

Differentiating the equation $g_t = n(t)\, g_-(t)$ with respect to t, we get

$$n^{-1}\dot n + \dot g_- g_-^{-1} = n^{-1}\dot g g_-^{-1}.$$

Let P be the natural projection of $\mathcal{W}$ onto $\mathcal{W}_-$. The trajectories $n(t)$, $g(t)$ satisfy the differential equations on the groups $GL(n,\mathbb{C})_\pm$:

$$\dot g_- g_-^{-1} = P(g_- L g_-^{-1}),$$
$$n^{-1}\dot n = (I-P)(n^{-1} R n),$$

where $L(t) = g_t^{-1}\dot g_t$, $R(t) = \dot g_t g_t^{-1}$. Since the corresponding vector fields for all t are tangent to the subgroup $G_\pm$, the trajectories $n(\cdot)$, $g(\cdot)$, starting at $\mathbb{1} \in G_\pm$, lie entirely in $G_\pm$.

COROLLARY 4.4. The map $N_+ \times G_- \to G_w : (n, g_-) \mapsto n g_-$ is one-to-one and covers a neighborhood of the identity in G_w.

Now let A be a subgroup in G. We set $A_\pm = AN_\pm$. For subgroups $A, B \subset G$, satisfying the conditions of 1.2, we construct the subgroups $A_+, B_- \subset G_w$; they satisfy the same conditions. We get the following supplement to Theorem 2:

THEOREM 2. (iv) Let $\exp tM = a(t)\, b(t)$, where $a(\cdot)$, $b(\cdot)$ are smooth curves in the groups A_+, B_-, and $a(0) = \mathbb{1} = b(0)$. Then a solution of the equations of motion $\dot\xi = [M_-, \xi]$ is given by the formula

$$\xi(t) = \mathrm{Ad}\, b(t) \cdot \xi.$$

A finite-dimensional Lax representation with supplementary spectral parameter first appeared in Novikov [13] in describing stationary solutions of the higher equations $K_q\Phi$. However, the Hamiltonian structure of Novikov's equations turned out to be rather complicated and its connection with the structures considered in this and the following sections is unclear. The fact that an involutive system of integrals of motion in both cases is generated by invariants of the Lax matrix points to the existence of such a connection.

It was established by the author and M. A. Semenov-Tyan-Shanskii that the factorization of the matrix-valued function $\exp tM$ can be obtained with the help of the Baker–Akhiezer function connected with the spectral curve of the matrix ξ. Thus, the solution of the equations of motion in principle can be expressed in terms of multidimensional θ-functions. In this way Theorem 2 gives a group-theoretic interpretation of the algebrogeometric methods of [13, 14, 18, 19].

5. Hamiltonian Mechanics in the Space $\tilde{\mathfrak{g}}^*$

In this section there are collected various results on orbits and equations of motion in the spaces $\mathfrak{g}_n^*$ and $\mathfrak{g}_n^{\theta *}$: we calculate the dimension of orbits in general position, study the connection of various Hamiltonian structures, give an elementary proof of special cases of Theorem 2, generalizing the methods of Mishchenko and Fomenko [27, 28].

We recall that $\mathfrak{g}^*_n = \{ \sum_0^n \xi_k z^k, \xi_k \in \mathfrak{g}^* \}$, $\mathfrak{g}^{\theta*}_n = \mathfrak{g}^*_n \cap \mathfrak{g}^{\theta*}$.

5.1. Dimension of Orbits in General Position. Here it is assumed that the the algebra $\mathfrak{g}$ is such that the functional dimension of the algebra $I(\mathfrak{g}^*)$ is equal to the codimension of an orbit in general position in $\mathfrak{g}^*$.

Proposition 5.1.1. The maximal dimension of an orbit in $\mathfrak{g}^*_n$ is equal to $(n+1)d$, where d is the maximal dimension of an orbit in $\mathfrak{g}^*$.

Proof. Let $\dim \mathcal{O}_\alpha = d$, $\alpha \in \mathfrak{g}^*$. Then the orbit in $\mathfrak{g}^*_n$, passing through $\xi = \alpha \cdot z^n$, has dimension $(n+1)d$. In fact, its tangent space $T_\xi = (ad^* \mathfrak{g}_- \cdot \xi)_+ = \sum_{i=0}^n ad^* \mathfrak{g} \cdot \alpha z^i$ has dimension $(n+1)d$, since $\dim(ad^* \mathfrak{g} \cdot \alpha) = d$.

We shall prove the opposite inequality $\dim T_\xi \leqslant (n+1)d$ for $\xi \in \mathfrak{g}^*_n$. The proof goes by induction on n. We consider the space

$$T^1_\xi = (ad^* \mathfrak{g}^1_- \cdot \xi)_+ .$$

Since $T^1_\xi = (ad^* \mathfrak{g}_- \cdot \xi z^{-1})_+$, by the inductive hypothesis $\dim T^1_\xi \leqslant nd$. Since $T_\xi = T^1_\xi + ad^* \mathfrak{g} \cdot \xi$, to prove the estimate $\dim T_\xi \leqslant (n+1)d$ it suffices to prove that

$$\dim (ad^* \mathfrak{g} \cdot \xi / ad^* \mathfrak{g} \cdot \xi \cap T^1_\xi) \leqslant d . \tag{*}$$

Let $\varphi \in I(\mathfrak{g}^*)$; we consider the polynomial φ^n on $\tilde{\mathfrak{g}}^*$ of the form

$$\varphi^n(\xi) = \langle \varphi(\xi \cdot z^{-n}) \rangle_{-n} .$$

Then $\varphi^n \in I(\tilde{\mathfrak{g}}^*)$, i.e., $ad^* d\varphi(\xi) \cdot \xi = 0$. But for $\xi = \sum_0^n \xi_i z^i$ we have $d\varphi^n(\xi) = d\varphi(\xi_n) - x$, $x \in \mathfrak{g}^1_-$. Whence

$$ad^* d\varphi^n(\xi_n) \cdot \xi = (ad^* x \cdot \xi)_+ \in T^1_\xi .$$

If ξ_n is a regular element, i.e., $\dim \mathcal{O}_{\xi_n} = d$, then $\operatorname{codim}_{\mathfrak{g}} \{ d\varphi(\xi_n), \varphi \in I(\mathfrak{g}^*) \} = d$. Whence follows the estimate (*) and the inequality $\dim T_\xi \leqslant (n+1)d$ for those ξ, at which the highest coefficient ξ_n is regular. Since such ξ form an open dense set in $\mathfrak{g}^*_n$, the inductive passage is completed.

Now we calculate the maximal dimension of an orbit of the symmetrically graded algebra $\mathfrak{g}^\theta_-$ in $\mathfrak{g}^{\theta*}_n$ for split semisimple $\mathfrak{g}$. In this case the space $\mathfrak{p}$ contains a $\mathfrak{g}$-regular element.

Proposition 5.1.2. The maximal dimension of an orbit in $\mathfrak{g}^{\theta*}_n$ is equal to

(i) $(s+1)d$, if $n = 2s+1$,

(ii) $s \cdot d + d(\mathfrak{k})$, if $n = 2s$; here $d(\mathfrak{k})$ is the maximal dimension of an orbit in a compact subalgebra $\mathfrak{k}$.

Proof. (i). Let $\xi = \sum_0^n \xi_i z^i \in \mathfrak{g}^{\theta*}_n$, $\alpha = \xi_n$ be a regular element, $\mathfrak{a} = \mathrm{Cent}(\alpha)$ be the Cartan subalgebra in $\mathfrak{g}$, spanned by α, $\mathfrak{m} = \mathfrak{a}^\perp$. We set $\mathfrak{m}_{-n} = \{ \sum_{-n}^0 x_i z^i, x_i \in \mathfrak{m} \}$, $\mathfrak{a}_n = \{ \sum_0^n x_i z^i, x_i \in \mathfrak{a} \}$. It is easy to verify that the map $ad_+ \xi : \mathfrak{m}_{-n} \to T_\xi$, $ad_+ \xi \cdot x = [\xi, x]_+$, has no kernel. Whence, comparing dimensions, we see that $ad_+ \xi$ is an isomorphism. But $[\xi, \mathfrak{m}_{-n}]_+ \cap \mathfrak{a}_n = \{0\}$, that is, T_ξ projects in a one-to-one fashion onto $\mathfrak{m}_n = \{ \sum_0^n x_i z^i, x_i \in \mathfrak{m} \}$. Whence follows easily the inequality $\dim (T_\xi \cap \tilde{\mathfrak{g}}^\theta) \leqslant (s+1)d$, i.e., $\dim \mathcal{O}^\theta_\xi \leqslant (s+1)d$. The opposite inequality is obvious.

The proof of (ii) is analogous.

5.2. Compatible Poisson Brackets on $\mathfrak{g}_n^*$. We consider the decomposition $\tilde{\mathfrak{g}} = \mathfrak{g}_+^1 \dotplus \mathfrak{g}_-$; let $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_+^1 \oplus \mathfrak{g}_-$. We fix a positive $n \in \mathbb{Z}$ and we consider the $ad^* \tilde{\mathfrak{g}}_0$-invariant subspaces in $\tilde{\mathfrak{g}}^*$:

$$\mathcal{E}_k = \mathfrak{g}_n^* \cdot z^{-k}, \quad k = 0, 1, \ldots, n+1.$$

On each of them there is (Theorem 2) a collection of Hamiltonians in involution and Laxian equations with respect to their (degenerate) Poisson brackets. Obviously, multiplication by $z^{k-\ell}$, identifying $\mathcal{E}_k$ with $\mathcal{E}_\ell$, carries one collection of Hamiltonians into the other. It turns out that the equations of motion too are carried by this identification into one another, although the Poisson brackets are completely different.

LEMMA 5.2. Let $\varphi \in I(\mathfrak{g}^*)$. The equation of motion on $\mathcal{E}_0$ defined by the Hamiltonian φ_m, is carried under multiplication by z^{-k} into the equations of motion on $\mathcal{E}_k$, defined by the Hamiltonian $\varphi_m^k(\xi) = \varphi_{k+m}(\xi \cdot z^k)$.

Proof. The equations of motion for the Hamiltonian φ_m have the form

$$\dot{\xi} = ad^* d\varphi_m(\xi)_- \cdot \xi,$$

and for the Hamiltonian φ_m^k

$$\dot{\xi} = ad^* d\varphi_m^k(\xi)_- \cdot \xi.$$

But $d\varphi_m^k(\xi) = z^{-k} d\varphi_m(\xi \cdot z^k)$, whence the lemma also follows.

Thus, the Laxian equations in $\mathfrak{g}_n^*$ are Hamiltonian and in involution with respect to $(n+2)$ different Poisson brackets. This recalls the situation considered by Magri [29], Kulish and Reiman [30], Gel'fand and Dorfman [31], especially in the case $n=1$: here all three brackets are included in a linear family. More precisely, let $\{\cdot,\cdot\}_k$ be the Kirillov bracket on $\mathcal{E}_k$, $k = 0, 1, 2$, and all $\mathcal{E}_k$ be identified with $\mathfrak{g}_1^*$. Then the bracket $\Sigma c_k \{\cdot,\cdot\}_k$ satisfies the Jacobi identity.

Everything said above is also valid for θ-graded algebras under the condition that one considers $\mathcal{E}_k$ with even k.

It is curious that the comparison of brackets in the invariant subspaces $\mathcal{E}_f = f z^{-1} + \mathfrak{g}_0^*$ and $\mathcal{E}_f z^{-1}$ leads to the Mishchenko—Fomenko construction (see 5.3).

5.3. Family of Hamiltonian Structures in $\mathfrak{g}^*$ and Proof of Theorem 2 in Special Cases. In what follows, $\mathfrak{k}$ and $\mathfrak{p}$ denote the eigensubspaces of the involution θ in the algebra $\mathfrak{g}$, corresponding to the eigenvalues 1 and -1, respectively. We denote by $\mathfrak{g}_\theta = \mathfrak{k} \dotplus \mathfrak{p}$ the semidirect sum of the subalgebra $\mathfrak{k}$ and the vector space $\mathfrak{p}$; $\mathfrak{g}_\theta$ is isomorphic with the algebra $\mathfrak{g}_1^\theta = \mathfrak{g}_-^\theta / \mathfrak{g}_-^{\theta,2}$ (Sec. 4). The dual space $\mathfrak{g}_\theta^*$ is identified with $\mathfrak{g}^*$, and also with the invariant subspace $\mathfrak{g}_1^{\theta *}$ in $\mathfrak{g}_-^{\theta *}$. One can assume that on $\mathfrak{g}^*$ there are given two Kirillov brackets, $\{,\}$ and $\{,\}_\theta$, connected with the Lie brackets in $\mathfrak{g}$ and $\mathfrak{g}_\theta$.

We fix an element $f \in \mathfrak{p}^*$. We denote by $\{,\}_f$ a Poisson bracket on $\mathfrak{g}^*$ of Heisenberg type: for linear functions $x, y \in \mathfrak{g}$,

$$\{x, y\}_f = f([x, y]).$$

Proposition 5.3. (i) The bracket $\{,\}_{\alpha,\beta} = \alpha\{,\} + (1-\alpha)\{,\}_\theta + \beta\{,\}_f$ is a Poisson bracket on $\mathfrak{g}^*$ for any $\alpha, \beta \in \mathbb{R}$. (ii) Let $T_{\lambda,\mu}$ be the linear transformation in $\mathfrak{g}^*$,

$$T_{\lambda,\mu}(\pi+s) = \pi+\lambda s+\mu f, \qquad \pi\in k^*, \quad s\in p^*.$$

For a function φ on g^* we set

$$\varphi_{\lambda,\mu}(\xi) = \varphi(T_{\lambda,\mu}(\xi)).$$

If the point $(\alpha,\beta)\in\mathbb{R}^2$ lies on the line passing through the points (α_i,β_i), then for $\lambda_i^{-2}=\alpha_i$, $\lambda_i^{-1}\mu_i=\beta_i$

$$\{\varphi^1_{\lambda_1,\mu_1}, \varphi^2_{\lambda_2,\mu_2}\}_{\alpha,\beta} = 0$$

for any $\varphi^1, \varphi^2\in I(g^*)$.

Proof. (i) It is easy to verify that $T_{\lambda,\mu}$ transforms the bracket $\{,\}$ into the bracket $\{,\}_{\alpha,\beta}$ with indices $\alpha=\lambda^{-2}$, $\beta=\lambda^{-1}\mu$, whence also follows (i). (ii) It is evident from the proof of (i) that the functions $\varphi_{\lambda,\mu}$ lie in the kernel of the bracket $\{,\}_{\alpha,\beta}$ with $\alpha=\lambda^{-2}$, $\beta=\lambda^{-1}\mu$. Hence if $(\alpha,\beta)=a(\alpha_1,\beta_1)+b(\alpha_2,\beta_2)$, $a+b=1$, then

$$\{\varphi^1_{\lambda_1,\mu_1},\varphi^2_{\lambda_2,\mu_2}\}_{\alpha,\beta} = a\{\varphi^1_{\lambda_1,\mu_1},\varphi^2_{\lambda_2,\mu_2}\}_{\alpha_1,\beta_1} + b\{\varphi^1_{\lambda_1,\mu_1},\varphi^2_{\lambda_2,\mu_2}\}_{\alpha_2,\beta_2} = 0.$$

We note that the proof of (ii) is an interpretation of the method of Mishchenko and Fomenko.

Since $\{,\}_{0,0}=\{,\}_\theta$, taking the line $\alpha=\beta$, i.e., $\mu=\lambda^{-1}$, we get an important special case of Corollary 4.3, relating to the subspace $g_1^{\theta *}$.

COROLLARY 5.3.1. Functions of the form

$$\varphi_\lambda(\pi+s) = \varphi(\pi+\lambda s+\lambda^{-1}f), \qquad \varphi\in I(g^*),$$

are in involution with respect to the bracket $\{,\}_\theta$ on g^* (more generally, with respect to brackets from the family

$$\alpha\{,\}+(1-\alpha)\{,\}_\theta+\alpha\{,\}_f).$$

Now if one takes the line $\alpha=\beta+1$, then $\mu=\lambda^{-1}-\lambda$ and we get

COROLLARY 5.3.2 [32]. Functions of the form

$$\varphi_\lambda(\pi+s)=\varphi(\pi+\lambda s+(\lambda^{-1}-\lambda)f), \qquad \varphi\in I(g^*),$$

are in involution with respect to the bracket $\{,\}$ on g^* (more generally, with respect to the family of brackets

$$\alpha\{,\}+(1-\alpha)\{,\}_\theta+(\alpha-1)\{,\}_f).$$

In a slightly different form, Corollary 5.3.2 is contained in Adler and Moerbeke [32], where they use for the Hamiltonian interpretation the Lax representation found by Moser [33] of the equations of motion of a point on an ellipsoid in a central force field.

We note that Corollaries 5.3.1 and 5.3.2 are connected with one another by the translation $\beta\to\beta+1$.

Applying Proposition 5.3 to the line $\alpha=1$, we get a corollary, generalizing Theorem 1.6 of [28]:

COROLLARY 5.3.3. Functions on g^* of the form

$$\varphi_\lambda(\xi) = \varphi(\xi+\lambda f), \quad \varphi\in I(g^*),$$

are in involution with respect to the family of brackets $\{,\}+\beta\{,\}_f$.

Suppose there is a linear family of Poisson brackets

$$\{,\}_\alpha = \{,\}_0 - \alpha\{,\}_1$$

(e.g., the restriction of the family $\{,\}_{\alpha,\beta}$ to a line in the $\{\alpha,\beta\}$ plane). Let the functions φ_α lie in the kernel of the bracket $\{,\}_\alpha$. Setting $\varphi_\alpha = \Sigma\varphi_k\alpha^k$, we get the hierarchy of equations of motion (see [29, 30]):

$$\{\varphi_{k+1},\cdot\}_0 = \{\varphi_k,\cdot\}_1 .$$

In particular, we consider $\{,\}_\alpha = \{,\}_{\alpha,\alpha}$. Then $\{,\}_0 = \{,\}_\theta$, $\{,\}_0 - \{,\}_1 = \{,\} - \{,\}_f$. We note that the last bracket is obtained from $\{,\}$ by translation of the argument by f. The polynomials $\varphi_\alpha = \varphi_{\alpha^{-1/2},\alpha^{1/2}}$, where φ is a polynomial from $I(\mathfrak{g}^*)$, lie in the kernel of the bracket $\{,\}_\alpha$ and expand in series in half-integral powers of α.

LEMMA 5.3. The equations of motion defined by the Hamiltonian φ_k (respectively, φ_α) with respect to the bracket $\{,\}_\theta$, are carried under the substitution $\xi \mapsto \xi - f$ into the equations of motion defined with respect to the bracket $\{,\}$ by the Hamiltonian $H_k = \sum_{i=0}^{\infty}\varphi_{i+k}$ (respectively, $H = (1-\alpha)^{-1}\varphi_\alpha$).

Proof. In order that from the Hamiltonian φ_k one finds the Hamiltonian H_k in the new bracket, it is necessary to solve the equation

$$\{\varphi_k,\cdot\}_0 = \{H_k,\cdot\}_0 - \{H_k,\cdot\}_1 .$$

Obviously $H_k = \sum_{i=0}^{\infty}\varphi_{i+k}$ is a solution. Since $\varphi_\alpha = \Sigma\varphi_k\alpha^k$, to the Hamiltonian φ_α corresponds

$$H_\alpha = \sum_k \alpha^k \sum_{i=0}^{\infty}\varphi_{i+k} = \sum_k \varphi_k \sum_{i=0}^{\infty}\alpha^{k-i} = (1-\alpha)^{-1}\varphi_\alpha .$$

Up to translation of the variable and time dilation, the Hamiltonian φ_α gives one equation with respect to the two brackets $\{,\}$ and $\{,\}_\theta$.

As an example of Corollary 5.3.1, we consider the Euler equations of rotation of an n-dimensional top. The Lax representation with spectral parameter for the Euler equations was found by Manakov [11] and systematically studied by Mishchenko and Fomenko [27, 28]. Let $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{R})$, θ be the Cartan involution, so that $k = so(n)$. Let $I \in p$ be the inertia tensor of the top, $\Omega = ad\,I\cdot(ad\,I^2)^{-1}$ be a linear transformation in k.

The bracket $\{,\}_\theta$ can be restricted to the $ad^*_{\mathfrak{g}_\theta}$-invariant subspace $k^* \simeq k$. In this subspace the equation of the top is generated by the Hamiltonian

$$H(\pi) = -\frac{1}{2}\,\mathrm{tr}\,\pi\cdot\Omega(\pi) ,$$

and the Manakov representation [14] for it has the form

$$\dot{\pi} = [\,\pi + \lambda I^2,\ \Omega(\pi) + \lambda I\,] .$$

We set $f = I^2$ and we consider a polynomial on k of the form $\varphi_2(\pi) = \langle\varphi(I^2+\pi z)\rangle_2$, $\varphi\in I(\mathfrak{g})$. It is easy to verify (see 6.2) that the corresponding Lax equation has the form

$$\dot{\pi} = [\,\pi + I^2 z^{-1},\ ad\,M_0\circ(ad\,I^2)^{-1}\pi + Mz^{-1}\,] ,$$

where $M = d\varphi(I^2)$. Here φ_2 is quadratic in π, $d\varphi_2 = ad\,M(ad\,I^2)^{-1}$ and $\varphi_2(\pi) = -\frac{1}{2}\operatorname{tr}(\pi\cdot d\varphi_2(\pi))$. In particular, if $M = I$, i.e., $d\varphi(I^2) = I$, then $\varphi_2 = H$ and we get the Manakov representation with the substitution $\lambda \mapsto z^{-1}$. The symbolically necessary invariant function can be written as $\varphi(x) = \frac{2}{3}\operatorname{tr} x^{3/2}$. It is also not hard to construct the needed Hamiltonian as a linear combination of polynomials $H = \Sigma c_k H_k$

$$H_k = -\langle \frac{1}{k+1}\operatorname{tr}(I^2+\pi z)^{k+1}\rangle_2 = \sum_{i<j} \frac{a_i^{2k} - a_j^{2k}}{a_i^2 - a_j^2}\pi_{ij}^2, \quad I = \operatorname{diag}(a_1, \ldots, a_n).$$

For this it suffices to solve the linear system of van der Monde type $\sum_1^n c_k I^{2k} = I$.

6. Dynamical Systems in Subspaces of Small Graduation

Here we consider periodic Toda chains and systems $\mathfrak{g}_1^{\theta *}$, describing multidimensional tops in a potential field. The algebra $\mathfrak{g}$ is the real form of a simple Lie algebra.

6.1. Periodic Toda Chains. The construction of chains with respect to a root decomposition was given in 3.1. To apply this procedure to the affine Lie algebra, we write its root system.

We define a differentiation $\mathfrak{D}: \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ by the formula $\mathfrak{D}(xz^n) = n x z^n$ and we form the semidirect sum $\hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \dotplus \mathbb{R}\mathfrak{D}$. Let $\mathfrak{a}$ be a Cartan subalgebra in $\mathfrak{g}$, $\hat{\mathfrak{a}} = \mathfrak{a} \dotplus \mathbb{R}\mathfrak{D}$. Roots of the pair $(\hat{\mathfrak{g}}, \hat{\mathfrak{a}})$ are the nonzero elements ω from $\hat{\mathfrak{a}}^*$ such that the root subspace $\hat{\mathfrak{g}}_\omega = \{x \in \hat{\mathfrak{g}}: ad\,a\cdot x = \omega(a)x\}$ is different from $\{0\}$. The roots have the form $\omega = \alpha + n\gamma$, $n \in \mathbb{Z}$, where α is a root of $(\mathfrak{g}, \alpha)$, the element γ is defined by the equation $\gamma(\mathfrak{D}) = 1$, $\gamma|_{\mathfrak{a}} = 0$. The root ω is positive if $n > 0$ or $n = 0$, $\alpha \in \Delta_+$. The system of simple roots $\hat{P}$ is formed by the roots $\{\alpha, \gamma - \alpha^*\}$, where $\alpha \in P$, α^* is the unique root of maximal height.

We define an element $x_0 \in \hat{\mathfrak{a}}$ by the equation $\omega(x_0) = 1$, $\omega \in \hat{P}$. Let $\mathfrak{D}_j \subset \hat{\mathfrak{g}}$ be the eigensubspace of the operator $ad\,x_0$, corresponding to the eigenvalue $j \in \mathbb{Z}$. The decomposition $\tilde{\mathfrak{g}} = \oplus \mathfrak{D}_j$ defines in $\hat{\mathfrak{g}}$ a graduation according to the height of the root, compatible with the graduation in $\mathfrak{g}$ (3.1). Starting from this we can repeat word for word the arguments of 3.1. We proceed directly to examples (see 3.5).

a) Chains, connected with the Iwasawa decomposition $\tilde{\mathfrak{g}} = \mathfrak{k} + \mathfrak{b}_-$, $\mathfrak{b}_- = \oplus_{j \le 0} \mathfrak{D}_j$, in the split case were listed by Bogoyavlensky [10], starting from other considerations. A more natural construction of such chains on the basis of graded Lie algebras was suggested by the author, Semenov-Tyan-Shanskii, and Frenkel [34]. The subalgebra $\tilde{\mathfrak{k}} = \{\Sigma x_n z^n: \theta(x_n) = x_{-n}\}$ leads to the symmetric realization of the space $\mathfrak{b}_-^*$. In contrast with the nonperiodic case, a system of roots of type BC_n gives a new chain (this was noted by M. A. Ol'shanetskii and A. M. Perelomov) with potential

$$V = \sum_1^{n-1} e^{q_{i+1} - q_i} + e^{q_1} + c\cdot e^{-2q_n}.$$

Using the Gauss decomposition $\tilde{\mathfrak{g}} = n_+ + \mathfrak{b}_-$ and repeating the argument of 3.5a), we get

Proposition 6.1 [34]. The orbit of a generalized periodic Toda chain consists of the the points

$$\xi = p + \sum_{\omega\in\hat{P}} e^{<q,\omega>} e_\omega\,, \quad p,q \in \mathfrak{a}\,,$$

the variables p,q are canonical. Taking a translation vector $f \in \mathcal{D}_1$, we get Hamiltonian $H = \frac{1}{2}B(p,p) + \sum_{\omega\in\hat{P}} f\omega e^{<q,\omega>}$. The equations of motion have the form

$$\dot{\xi} = [\xi + f, \; p + f]\,.$$

If all $f_\omega > 0$, then the system is complete.

The completeness follows from the fact that the surfaces of constant energy are compact: the condition $\Sigma f_\omega e^{<q,\omega>} < c$ implies $|q| < c_1$.

b) A periodic non-Abelian Toda chain is given on the phase space $T^*(G^n)$, $G = GL(m,\mathbb{R})$, by the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n} tr(\dot{g}_i g_i^{-1})^2 + tr\left(\sum_{i=1}^{n-1} g_{i+1} g_i^{-1} + g_1 g_n^{-1}\right)\,.$$

After reduction by the right diagonal action of the group G one gets a system on the orbit of a block parabolic subalgebra in $\mathfrak{gl}(mn,\mathbb{R})$ — see, 3.5b). The Lax matrix $L = \xi + f$,

$$\xi = \sum_1^{n-1} g_{i+1} g_i^{-1} \otimes e_i + g_1 g_n^{-1} \otimes e_{-\alpha^*} z + diag(\dot{g}_i g_i^{-1}),\; f = \Sigma \mathbb{1}_m \otimes e_i + \mathbb{1}_m \otimes e_{\alpha^*} z^{-1}.$$

c) Periodic chains connected with minimal parabolic subalgebras are enumerated on the basis of 3.5c). We give the component V_p of the potentials described there. The complex algebras lead to the obvious complexifications of the split periodic chains.

1) $\mathfrak{g} = sl(n,\mathbb{H})$: periodic quaternionic Toda chain,

$$V_p = \mathrm{Re}\, q_1 q_n^{-1}\,.$$

2) $\mathfrak{g} = so(m,n)$: $V_p = c \cdot e^{-q_{n-1} - q_n}$

3) $\mathfrak{g} = su(m,n)$: $V_p = c \cdot |z_n|^{-2}$

4) $\mathfrak{g} = so^*(2n)$: $V_p = c \cdot |q_n|^{-2}$

5) $\mathfrak{g} = sp(m,n)$: $V_p = c(q_n^{-1})$,

where $c(\cdot)$ is the matrix element orthogonal to the representation $\mathbb{H}^*$ in $\mathbb{R}^3$. Here the configuration space is somewhat changed: for $m > n$ this is $\mathbb{H}^* \times S/\mathcal{U}(1)$, for $m = n$†

6.2. Manakov Representation. In the general Lax formula for the representation $\dot{L} = \pm[L, M_\pm]$, where + (-) denotes projection onto $\mathfrak{g}_+^1$ ($\mathfrak{g}_-$), we single out a case when $M_\pm$ has specially simple structure. The title of this paragraph is explained by the fact that the representation given by S. V. Manakov of the Euler equations of a top has exactly this form.

We consider the decomposition $\tilde{\mathfrak{g}} = \mathfrak{g}_+^1 \dotplus \mathfrak{g}_-$. Let $\xi \in f z^{-1} + \mathfrak{g}_n^*$, $\xi = f z^{-1} + \sum_0^n \xi_i z^i$; $\varphi \in I(\mathfrak{g}^*)$. We consider polynomials on $\tilde{\mathfrak{g}}^*$ of the form $\varphi_k^m(\xi) = <\varphi(\xi \cdot z^m)>_k$. Then $d\varphi_k^m(\xi) = z^{m-k} d\varphi(\xi \cdot z^m)$.

LEMMA 6.2. Let $\varphi \in I(\mathfrak{g}^*)$ be a homogeneous polynomial of degree d, $M_\pm = [d\varphi_k^m(\xi)]_\pm$. Then

(i) If $k - m + 1 - (d-1)(m+n) = 0$, then $M_+ = d\varphi(\xi_n) \cdot z$.

(ii) If $k - m - 1 - (d-1)(m-1) = 0$, then $M_- = d\varphi(f) \cdot z^{-1} + M_0$.

†Some material omitted from Russian original—Publisher.

For a semisimple regular f the element M_0 can be calculated in the following way: let $\mathfrak{a}$ be the Cartan subalgebra f, $\varphi''(f)$ spanned by $\varphi|\mathfrak{a}$, $\xi_0=\alpha+\beta$, where $\alpha\in\mathfrak{a}$, $\beta\in\mathfrak{a}^{\perp}$. Then

$$M_0=\varphi''(f)\cdot\alpha+\operatorname{ad} d\varphi(f)\circ(\operatorname{ad} f)^{-1}\cdot\beta.$$

Analogous formulas are true also for the algebra $\tilde{\mathfrak{g}}^{\theta}$, taking into account the fact that the gradients must be projected onto $\tilde{\mathfrak{g}}^{\theta}$. In particular, in (ii) we get

$$M_0=\operatorname{ad} d\varphi(f)\circ(\operatorname{ad} f)^{-1}\cdot\xi_0.$$

The lemma is proved by simple calculation, which we shall omit.

6.3. Systems, Connected with Symmetrically Graded Algebras. Let θ be a Cartan automorphism in $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{p}$ be the eigenspaces of θ, corresponding to the eigenvalues 1 and -1 respectively, $\mathfrak{g}^{\theta}$ be the symmetrically graded algebra. Here we consider systems in $\mathfrak{g}_-^{\theta *}\simeq\mathfrak{g}_+^{\theta}$ on orbits lying in the subspace $\mathfrak{k}+\mathfrak{p}z$. The algebra $\mathfrak{g}_{\theta}=\mathfrak{g}_-^{\theta}/\mathfrak{g}_-^{\theta,2}$, isomorphic to the semidirect sum $\mathfrak{k}\dotplus\mathfrak{p}$, acts effectively in this subspace. Hence Corollary 3.2 is applicable. The orbit $\mathcal{O}_a$, passing through the point $az\in\mathfrak{p}z$, is isomorphic with the cotangent bundle $T^*(K\cdot a)$ as symplectic K-space, where the isomorphism $T^*(K\cdot a)\to\mathcal{O}_a$ is given by the map of moments $(k\cdot a,\rho)\to k\cdot(\rho+az),\rho\in\mathfrak{k}_a^{\perp}$ or $(s,p)\mapsto[s,p]+sz$, $s=k\cdot a\in K\cdot a$, $p\in\mathfrak{p}$.

We shall be interested in Hamiltonians, quadratic with respect to the momentum $\pi=\operatorname{Ad}k\cdot\rho=[s,p]$. We single out three cases. By $\xi=\pi+sz$ we denote a point on the orbit $\mathcal{O}_a$, $L=\xi+fz^{-1}$, $f\in\mathfrak{p}$.

A) $f=0$, the Hamiltonian can be chosen in the form

$$H(\xi)=\langle\varphi(\xi z^{-1})\rangle_{-2},\quad \varphi\in I(\mathfrak{g}).$$

Obviously, this Hamiltonian is K-invariant. One easily calculates that

$$H(\pi+sz)=\frac{1}{2}B(\pi,\operatorname{ad} d\varphi(s)\circ(\operatorname{ad} s)^{-1}\pi).$$

The Manakov representation has the form $\dot{L}=[L,d\varphi(s)z]$. The Hamiltonian system describes the free motion of a point on the manifold $K\cdot a$ in some K-invariant Riemannian metric.

B) $f\neq 0$, the Hamiltonian is generated by the Killing form

$$H(\pi+sz)=\frac{1}{2}B(\pi,\pi)+B(f,s).$$

The Manakov representation has two forms

$$\dot{L}=-[L,\pi+fz^{-1}],\quad \dot{L}=[L,sz].$$

The system describes the motion of a point on $K\cdot a$ in the standard metric (see end of 2.3) and potential $V(s)=B(f,s)$.

C) General case, $f\neq 0$. c1) $H(\xi)=\langle\varphi(L\cdot z)\rangle_2$, $\varphi\in I(\mathfrak{g})$. Calculation gives

$$H(\xi)=\frac{1}{2}B(\pi,\operatorname{ad} d\varphi(f)\circ(\operatorname{ad} f)^{-1}\pi)+B(d\varphi(f),s).$$

In the representation $\dot{L}=-[L,M_-]$ we have

$$M_-=d\varphi(f)\cdot z^{-1}+\operatorname{ad} d\varphi(f)\cdot(\operatorname{ad} f)^{-1}\pi$$

c2) $H=\langle\varphi(Lz^{-1})\rangle_{-2}$,

$$H(\xi)=\frac{1}{2}B(\pi,\operatorname{ad} d\varphi(s)\circ(\operatorname{ad} s)^{-1}\pi)+B(f,d\varphi(s)).$$

The Manakov representation has the form

$$\dot{L}=[L,d\varphi(s)z].$$

If G is a matrix group, then the elements of the matrix $\kappa \cdot a = \mathfrak{b}$ depend quadratically on the matricial elements of κ (sometimes this dependence reduces to a linear one). Thus, the potential parts of the Hamiltonians being lifted from the orbit $K \cdot a$ to the group K, turn out to be quadratic or linear functions of the matricial elements.

We enumerate the types of orbits of a compact subgroup K in the space $\mathfrak{p}$. We note that if $\mathfrak{g}$ is a complex simple Lie algebra, then the action of K in $\mathfrak{p}$ is equivalent to the adjoint action of K in $\mathfrak{k}$.

<u>Explanation of the Table.</u> Throughout we have in mind the block-diagonal imbedding of $K(n_1)\times\ldots\times K(n_k)$ in $K(n)$, $n=\Sigma n_i$ in combination with the natural imbeddings $\mathcal{U}(n)\subset \mathcal{S}O(2n)$, $\mathcal{U}(n)\subset \mathcal{S}p(n)$, $\mathcal{S}p(n)\subset \mathcal{U}(2n)$. In points 8, 9, 10 the subgroup $\operatorname{diag} K(n_1)\times\ldots\times K(n_k)$, $\Sigma n_i = l$, is situated in $K(m)\times K(n)$ in the following way. In $K(p)$, $p=m,n$, there is singled out the subgroup $K(p-l)\times K(l)$, and the subgroup $K(n_1)\times\ldots\times K(n_k)\subset K(l)$ is located on the diagonal of the product $K(l)\times K(l)\subset K(m)\times K(n)$.

In Table 1 there occur manifolds which are well known in geometry. These are, e.g., the manifold of flags (1, 4, 5), the Lagrangian Grassman manifolds $\mathcal{U}(n)/O(n)$ and $\mathcal{S}p(n)/\mathcal{U}(n)$ (7 and 3), the manifolds of orthogonal complex structures $\mathcal{S}O(2n)/\mathcal{U}(n)$ (2).

We consider separate examples in more detail.

1. It is convenient to consider instead of the algebra $sl(n,\mathbb{R})$ the algebra $gl(n,\mathbb{R})$.

Let $a\in\mathfrak{p}$, $a=\operatorname{diag}(a_1,\ldots,a_n)$ and the diagonal elements be disposed in decreasing order

$$a_1=\ldots=a_{n_1}>a_{n_1+1}=\ldots>a_{n_1+\ldots+n_{k-1}+1}=\ldots=a_n.$$

Then $\mathcal{S}O(n)\cdot a=\mathcal{S}O(n)/\mathcal{S}(O(n_1)\times\ldots\times O(n_k))$ is the manifold of flags in $\mathbb{R}^n$.

TABLE 1

Lie algebra	Type of orbit in $\mathfrak{p}$
1. $sl(n,\mathbb{C})$	$\mathcal{U}(n)/\mathcal{U}(n_1)\times\ldots\times\mathcal{U}(n_k)$, $\Sigma n_i = n$
2. $so(n,\mathbb{C})$	$\mathcal{S}O(n)/\mathcal{U}(n_1)\times\ldots\times\mathcal{U}(n_k)\times\mathcal{S}O(m)$, $m+2\Sigma n_i = n$
3. $sp(n,\mathbb{C})$	$\mathcal{S}p(n)/\mathcal{U}(n_1)\times\ldots\times\mathcal{U}(n_k)\times\mathcal{S}p(m)$, $m+\Sigma n_i = n$
4. $sl(n,\mathbb{R})$	$\mathcal{S}O(n)/\mathcal{S}(O(n_1)\times\ldots\times O(n_k))$, $\Sigma n_i = n$
5. $sl(n,\mathbb{H})$	$\mathcal{S}p(n)/\mathcal{S}p(n_1)\times\ldots\times\mathcal{S}p(n_k)$, $\Sigma n_i = n$
6. $so^*(2n)$	$\mathcal{U}(n)/\mathcal{S}p(n_1)\times\ldots\times\mathcal{S}p(n_k)\times\mathcal{U}(m)$, $m+2\Sigma n_i = n$
7. $sp(n,\mathbb{R})$	$\mathcal{U}(n)/O(n_1)\times\ldots\times O(n_k)\times\mathcal{U}(m)$, $m+\Sigma n_i = n$
8. $su(m,n)$	$\mathcal{U}(m)\times\mathcal{U}(n)/\mathcal{U}(m-l)\times\mathcal{U}(n-l)\times\operatorname{diag}\mathcal{U}(n_1)\times\ldots\times\mathcal{U}(n_k)$ $\Sigma n_i = l$
9. $so(m,n)$	$\mathcal{S}O(m)\times\mathcal{S}O(n)/\mathcal{S}(O(m-l)\times O(n-l)\times\operatorname{diag}O(n_1)\times\ldots\times O(n_k))$ $\Sigma n_i = l$
10. $sp(m,n)$	$\mathcal{S}p(m)\times\mathcal{S}p(n)/\mathcal{S}p(m-l)\times\mathcal{S}p(n-l)\times\operatorname{diag}\mathcal{S}p(n_1)\times\ldots\times\mathcal{S}p(n_k)$ $\Sigma n_i = l$

(A) Motion of an n-dimensional top according to Lagrange. Let $\varphi \in I(\mathfrak{g})$, $d\varphi(a) = b = \mathrm{diag}\,(b_1, \dots, b_n)$. Then

$$H(\pi + az) = \langle \varphi(\pi z^{-1} + a) \rangle_2 = \sum \frac{b_i - b_j}{a_i - a_j}\, \pi_{ij}^2\,, \quad \pi \in so(n).$$

If the polynomial φ is such that $b = a^{1/2}$ $(a \geq 0)$, then we get the Hamiltonian of the top with inertia tensor b. The phase space here is already reduced by the symmetry group of the inertia tensor. The Manakov representation has the form

$$\frac{d}{dt}(\pi + sz) = [\pi + sz\,,\, s^{1/2} z]\,.$$

All Laxian integrals of motion are $SO(n)$-invariant, so to construct a complete system of integrals it is necessary in addition to take a maximal commutative system of functions on $so(n)$.

(B) If $s = \mathrm{Ad}\,u \cdot a = uau^t$, $u \in SO(n)$, then for the potential V in the variables u_{ij} we get

$$V(u) = \mathrm{tr}\, f \cdot uau^t = \sum_{ijk} a_k f_{ij}\, u_{ik} u_{jk}\,.$$

The system describes the motion of a point on the flag manifold (or a symmetric top) in a quadratic potential. The Manakov representation has the form

$$\dot{L} = [L, M]\,, \quad L = \pi + sz + fz^{-1}, \quad M = sz\,.$$

In particular, for $a = \mathrm{diag}\,(1, 0, \dots, 0)$ we get $\mathcal{O}_a \simeq \mathbb{R}P^{n-1}$. In canonical variables x, p on $\mathbb{R}P^{n-1}$ (or on the sphere S^{n-1}), normalized by the condition $|x| = 1$, $(x, p) = 0$, we get $s = x \otimes x$, $\pi = x \wedge p$

$$L = x \wedge p + x \otimes x \cdot z + f z^{-1}$$
$$H = \frac{1}{2} p^2 + \sum f_{ij}\, x_i x_j$$
$$\dot{L} = [L, x \otimes x \cdot z]\,.$$

The motion of a point on the sphere in a quadratic potential was integrated by K. Neiman in 1858 by the Hamilton–Jacobi method. The integrals of motion

$$F_i = x_i^2 + \sum_{j \neq i} \frac{1}{f_i - f_j}\, \pi_{ij}^2$$

(here $f = \mathrm{diag}(f_1, \dots, f_n)$) were found by K. Uhlenbeck and later used widely by Moser [43], who also constructed an L–M pair. The integrals F_i are contained among the invariants of the matrix:

$$F_i = \langle \varphi(Lz) \rangle_2\,,$$

if $d\varphi(a)_{km} = \delta_{ik}\delta_{im}$.

(C) Rotation of a top in a quadratic potential. Choosing the polynomial φ as in case (A), we get for $H(\xi) = \langle \varphi(Lz^{-1}) \rangle_{-2}$

$$H(\pi + sz) = \sum \frac{\pi_{ij}^2}{a_i + a_j} + \mathrm{tr}(f \cdot d\varphi(s))\,.$$

Substituting $s = uau^t$, we get the potential

$$V(u) = \mathrm{tr}(f u\, d\varphi(a)\, u^t) = \sum_{ijk} b_k f_{ij} u_{ik} u_{jk} .$$

The Manakov representation: $\dot{L} = [L, s^{1/2} z]$.

2. $\mathfrak{g} = so(m,n)$, $m \geq n$; $\mathfrak{k} = so(m) + so(n)$. The Cartan subalgebra in $\mathfrak{p}$ consists of matrices $a = E(a_1,\dots,a_n) = \sum a_i (e_{i,m+i} + e_{m+i,i})$, where $(e_{ij})_{km} = \delta_{ik}\delta_{jm}$. Suppose among the numbers a_i there are $n-\ell$ zeros, and the remaining ones are split into k groups of equal numbers with n_i numbers in the i-th group. Then

$$SO(m)\times SO(n)\cdot a \simeq SO(m)\times SO(n)/S(O(m-\ell)\times O(n-\ell)\times \mathrm{diag}\, O(n_1)\times\dots\times O(n_k)).$$

Here we consider only Hamiltonians of type (B).

Let $a_1 = \dots = a_n = 1$. Then $SO(m)\times SO(n)\cdot a = St(m,n)$ is a Stiefel manifold. The potential, expressed in coordinates u_{ij} of an n-frame in $\mathbb{R}^m$, corresponding to a point of $St(m,n)$ has the form

$$V(u) = \sum f_{ij} u_{ij} .$$

The system describes the motion of a point on $St(m,n)$ (or of a reduced symmetric top) in a linear potential. In particular, for $n=1$, $St(m,1) = S^{m-1}$. In canonical variables $\{x, p\}$, $|x| = 1$, $(x,p) = 0$, we get $H = \frac{1}{2}p^2 + (f,x)$. The Lax matrix for the motion of a point on a sphere in a linear potential has the form

$$L = x\wedge p + (xz + fz^{-1}) \otimes e_{m+1} + e_{m+1} \otimes (xz + fz^{-1}),$$

where e_{m+1} is the $(m+1)$-th basis vector.

For $m = n$, $SO(m)\times SO(n)a \simeq SO(n)$. In variables $\{u, \pi\}$ on $T^*SO(n)$ the Lax matrix for the rotation of a symmetric top in a linear potential $V(u) = \mathrm{tr}\, fu$ has the form

$$L = \begin{pmatrix} u\pi u^t & uz + fz^{-1} \\ u^t z + f^t z^{-1} & -\pi \end{pmatrix}.$$

3. $\mathfrak{g} = so(n,n)$, $a = E(a_1,\dots,a_k,0,\dots,0)$, a_i are pairwise different and $a_i \neq 0$. In this case $SO(n)\times SO(n)\cdot a \simeq St(n,k)\times St(n,k)$ and the Hamiltonian of type (B) gives a system of two points on $St(n,k)$ (or of two reduced symmetric tops), interacting by means of the bilinear potential $V(u,v) = \sum a_k f_{ij} u_{ik} u_{jk}$. In particular, for $k=1$ we get a system of two bilinearly interacting points on the $(n-1)$-dimensional sphere. The Lax matrix in canonical variables $(x, p;\ y, q)$ has the form

$$L = \begin{pmatrix} x\wedge p & x\otimes y\cdot z + fz^{-1} \\ y\otimes x\cdot z + f^t z^{-1} & -y\wedge q \end{pmatrix}.$$

Integrals of motion of type c1 for $f = \mathrm{diag}(f_1,\dots,f_n)$ have the form

$$H = \sum_{i,j} (f_i - f_j)^{-2}\left[b_i f_i (\pi_{ij}^2 + \rho_{ij}^2) + 2 b_i f_j \pi_{ij}\rho_{ij}\right] + 2\sum_i b_i x_i y_i ,$$

where $\pi = x\wedge p$, $\rho = y\wedge q$, $E(b_1,\dots,b_n) = d\varphi(E(f_1,\dots,f_n))$.

Hamiltonians of type (C) describe the interaction of nonsymmetric tops. From the examples given the geometric content of Laxian systems in $\mathfrak{g}_1^{\theta *}$ should be clear.

7. Completeness of Systems of Integrals of Motion

The verification of the completeness of a system of integrals of motion constructed from an invariant graded algebra is of considerable difficulty. For systems with number of degrees of freedom equal to the rank of the group (Toda chains, motion in a quadratic potential on a sphere), the completeness is established easily. For orbits in general position in the subspace of zero graduation (e.g., for the Euler equations of a top), completeness was proved in [28]. Complete integrability of a non-Abelian Toda chain was recently proved by Krichever [35].

The most natural path to a proof of complete integrability uses methods of algebraic geometry, which lead to the linearization of the Laxian equations on the Jacobian of the spectral curve of the Lax matrix. In the survey of Dubrovian, Matveev, and Novikov [14] there was proved a theorem which is basic for us on the correspondence between matricial polynomials with fixed spectrum and the Jacobian of the spectral curve. In connection with finite-difference equations this method was developed in [18, 19]. In particular, in [19] was proved the equivalence of a linear motion on the Jacobian and equations of the form $\dot{L}=[L,M_+]$. In [32] graded algebras are introduced and a series of examples are worked out.

We recall the algebrogeometric scheme of solution of Laxian equations, following [19] basically, and then we show how the formula for the trajectories (Theorem 2, (iv)) leads easily to the linearization of the equations. In what follows $\mathfrak{g}=\mathfrak{gl}(m,\mathbb{C})$, $\mathfrak{g}_{n_+,n_-}=\{\sum_{i=-n_-}^{n_+} x_i z^i\}$, $n_\pm \geq 0$. We shall use the notation L instead of ξ, as more customary in this situation. For simplicity we shall assume $L=\sum_{-n_-}^{n_+} l_i z^i$ is an element in general position in $\mathfrak{g}_{n_+,n_-}$. This means that l_{n_+} and l_{n_-} are matrices with simple spectra.

7.1. Spectral Problem. Let $L\in\mathfrak{g}_{n_+,n_-}$, X_L be an affine curve, defined by the equation $\det(L(z)-\lambda)=0$. It will be assumed that the spectrum at a general point is simple and the curve X_L is nonsingular and irreducible. Let $\overline{X}_L$ be a smooth compactification of the curve X_L; the coordinates z,λ be meromorphic functions on $\overline{X}_L$. We denote by $(z)=P^+-P^-$ the divisor of the function z, $P^\pm$ be the effective divisors of degree m. We set $\mathcal{U}_\pm=X\setminus P^\pm$, $X_0=\mathcal{U}_+\cap\mathcal{U}_-$. We shall write X instead of $\overline{X}_L$.

We consider the linear bundle of eigensubspaces of the matrix $L(\cdot)$, defined over X outside ramification points of the function $\lambda(z)$, i.e., where the spectrum of L is simple. It defines a meromorphic map of the curve X into $\mathbb{C}P^{m-1}$, the space of lines in $\mathbb{C}^m$. Since each such map is holomorphic, our bundle extends canonically to a holomorphic bundle E_L over X.

We denote by $\mathcal{L}(\mathcal{U},E)$ the space of regular sections of the bundle E over the domain $\mathcal{U}$; if $\mathcal{U}=X$, then we shall write simply $\mathcal{L}(E)$.

Proposition 7.1.1. Let L be an element in general position in $\mathfrak{g}_{n_+,n_-}$. Then

(i) The genus $g=g(X)$ of the curve X is equal to $\frac{1}{2}m(m-1)(n_++n_-)-m+1$.

(ii) The degree of the bundle E_L^* is equal to $g+m-1$ and $\mathcal{L}(E_L^*(-p^+))=0$.

Proof. (i) The closure of the curve X in the completion $\mathbb{C}P^1\times\mathbb{C}P^1$ of the plane $\mathbb{C}^2=\{z,\lambda\}$ has singularities only at the points $P_+=(0,\infty)$ and $P_-=(\infty,\infty)$. The genus of the nonsingular curve X can be calculated in terms of the bidegree (d_1,d_2) of its equation and the indices $v(P_\pm)$ of the singular points according to the formula

$$g=d_1d_2-(d_1+d_2)+1-v(P_+)-v(P_-).$$

In our case $d_1=m(n_++n_-)$, $d_2=m$, and the indices are easily calculated from the principal part of the equation at the points $P_\pm: v(P_\pm)=\frac{1}{2}m(m-1)n_\pm$. As a result, we get

$$g=\frac{1}{2}m(m-1)(n_++n_-)-m+1.$$

(ii) Since the bundle E_L is immersed in $\mathbb{C}^m$, the linear coordinates in $\mathbb{C}^m$ give an m-dimensional space of sections of the dual bundle E_L^* and we denote this space by V. We also denote by $R=\mathbb{C}[z,z^{-1}]$ the ring of Laurent polynomials.

LEMMA 7.1 [19]. The natural map $r:V\otimes R\to\mathcal{L}(X_0,E_L^*)$ is an isomorphism of R-modules.

Proof of the Lemma. The ring R_0 of functions, regular in X_0, is generated by the functions z,z^{-1}, and λ. The spaces $V\otimes R$ and $\mathcal{L}(X_0,E_L^*)$ are R_0-modules: for $V\otimes R$ this follows from the fact that if $\Psi=(\Psi_1,\ldots,\Psi_m)^t$ is a basis in V, then $\lambda\Psi=L\Psi$. If $r(V\otimes R)$ is a proper submodule in $\mathcal{L}(X_0,E_L^*)$, then, as commutative algebra asserts, one can find a point $x\in X_0$, at which all elements $r(V\otimes R)$ vanish. Since the coordinates Ψ_i do not vanish simultaneously, $r(V\otimes R)=\mathcal{L}(X_0,E_L^*)$. Now we shall show that r has no kernel. This is deduced easily from the following assertion:

$$\circledast \text{ If } \Psi\in V \text{ and } \Psi\cdot z^{-1} \text{ lies in } \mathcal{L}(E_L^*), \text{ then } \Psi=0.$$

We shall prove it. Let the meromorphic section Ψz^{-1} be regular. Then Ψ vanishes at points P^+ and thus all eigenvectors of the matrix ℓ_{n_+} have zero Ψ-coordinate. Since we assume that ℓ_{n_+} has simple spectrum, it follows from this that $\Psi=0$.

Now let us assume that $\sum_{i=n_1}^{n_2}\varphi_i z^i=0$, $\varphi_i\in V$ and among φ_i there is a nonzero section. One can assume that $n_1=-1$ and $\varphi_{-1}\neq0$. Then $\varphi_{-1}z^{-1}=-\sum_0^{n_2}\varphi_i z^i$. The sum on the right side has no poles in P^+, so $\varphi_{-1}z^{-1}\in\mathcal{L}(E_L^*)$, which leads to a contradiction.

It follows from Lemma 7.1 that $V=\mathcal{L}(E_L^*)$. Using $\circledast$, we get $\mathcal{L}(E_L^*(-P^+))=0$. Whence it follows that $\deg E_L^*\leq g+m-1$. To prove the opposite inequality, we consider the twisted bundle $E_K=E_L^*(KP^-)$. For sufficiently large K,

$$\dim\mathcal{L}(E_K)=\deg E_L^*-g+1+Km.$$

Since the space $V\otimes\{\sum_0^K c_i z^i\}$ is contained in $\mathcal{L}(E_K)$, $\dim\mathcal{L}(E_K)\geq(K+1)m$ or $\deg E_L^*\geq g+m-1$.

Now we consider the inverse problem — the reconstruction of the matrix L from the algebraic data: a nonsingular complete curve X of genus g, a meromorphic function z of degree m, $(z)=P^+-P^-$, a meromorphic function λ, regular outside $P^+\cup P^-$, and a linear bundle E over X.

Definition. A linear bundle E of degree $g+n-1$ is called z-regular if $\mathcal{L}(E(-P^+))=0$.

Proposition 7.1.2. If E is regular, then there exist matrices $L\in\widetilde{gl(m,\mathbb{C})}$ for which $\overline{X}_L\simeq X$ and $E_L^*\simeq E$, and all such matrices are $GL(m,\mathbb{C})$-conjugate.

Proof. We write $V=\mathcal{X}(E)$, $X_0=X\setminus(P^+\cup P^-)$, $R=\mathbb{C}[z,z^{-1}]$, $E_k=E(kP^-)$. We shall show that the natural map $\tau: V\otimes R\to\mathcal{X}(X_0,E)$ is an isomorphism of R-modules. In fact, the proof of Lemma 7.1 shows that τ is an imbedding. To prove surjectivity it suffices to verify that $\mathcal{X}(E_k)\subset\tau(V\otimes R)$ for all $k\geqslant 0$. If $\varphi\in\mathcal{X}(E_k)$, then from the regularity of E it follows that one can find $\psi\in V$ such that $\varphi-\psi z^k\in\mathcal{X}(E_{k-1})$. Induction on k completes the argument.

Further, multiplication by the function λ, which is regular in X_0, gives an R-linear operator in $\mathcal{X}(X_0,E)$. Applying τ^{-1}, we get an R-linear operator in $V\otimes R$, that is a Laurent polynomial $L(z)$ with coefficients in $\mathcal{E}nd(V)$. Finally, from the regularity of E it follows that $\dim V=m$. The proof is concluded.

7.2. Dynamics in Algebraic Data. We denote by T_L the set of matrices from $\mathfrak{g}_{n_+,n_-}$, isospectral with L; in other words, T_L is a level set of the algebra of invariant polynomials $I(\tilde{\mathfrak{g}})$. We denote by $\mathcal{J}_k$ the manifold of classes of isomorphic linear bundles over X of degree k, $\mathcal{J}(L)=\mathcal{J}_{-(g+m-1)}$. We have constructed maps $\mathcal{J}: T_L\to\mathcal{J}(L)$, $L'\to E_{L'}$, whose fibers are $GL(m)$-orbits.

The set T_L is invariant with respect to the Lax equations. In their own right these equations are $GL(m)$-invariant, and hence are projected by the map $\mathcal{J}$ onto $\mathcal{J}(L)$. We shall show that Theorem 2 instantly leads to the result of the "direct spectral problem": the Lax equations of motion project into linear equations on $\mathcal{J}(L)$.

We fix a Hamiltonian $\varphi\in I(\tilde{\mathfrak{g}})$. Let $M=d\varphi(L)$ and L_t be a solution of the Hamiltonian Laxian equations $\dot{L}=[L,M_+]$, $L_0=L$. The spectral curve X_{L_t} in the process of evolution is unchanged. We write the change with time of the corresponding bundle E_{L_t}. The algebra R_0 of functions, regular in X_0, coincides with the algebra of polynomials in z,z^{-1}, and λ. Let $\psi(x)\in E_L(x)$ be an eigenvector of the matrix $L(z(x))$, corresponding to a point of the spectrum $x\in X$. Since $[L,M]=0$, we have $M(z(x))\psi(x)=\mu(x)\psi(x)$, where $\mu\in R_0$. We denote by F_t the linear bundle over X, given by the transition function $\exp t\mu$ for the covering $X=\mathcal{U}_+\cup\mathcal{U}_-$.

Proposition 7.2.1. $E_{L_t}=E_L\otimes F_t$.

Proof. Let $\exp tM=g_-^{-1}(t)g_+(t)$ be a solution of the factorization problem, defined a **priori**, for sufficiently small t. The evolution of the matrix L is given by the formula

$$L_t=g_-(t)Lg_-(t)^{-1}=g_+(t)Lg_+(t)^{-1}.$$

Hence the bundle E_{L_t}, as a subbundle in $X\times\mathbb{C}^m$, over the domains $\mathcal{U}_\pm$ can be represented in the form $E_{L_t}=g_\pm(t)E_L$: the functions $g_\pm(t)$ give isomorphisms of E_L and E_{L_t} over $\mathcal{U}_\pm$. Here the transition function in $\mathcal{U}_+\cap\mathcal{U}_-$, distinguishing these isomorphisms, is $g_-(t)^{-1}g_+(t)|_{E_L}=\exp tM|_{E_L}=\exp t\mu$, which also proves the proposition.

It remains to note that F_t is a one-parameter group of linear bundles of degree zero and $E_L\otimes F_t$ is a linear trajectory on $\mathcal{J}(L)$.

It is also true that to each linear equation on $\mathcal{J}(L)$ corresponds a Laxian equation in T_L. This follows from Proposition 7.2.1 and the fact that the regular bundles form a Zariski open set in $\mathcal{J}_{g+m-1}$. We also give an alternative proof of this fact.

Proposition 7.2.2. Any linear equation on $\mathcal{J}(L)$ lifts to a Laxian equation in T_L.

Proof. It is clear from Proposition 7.2.1 that it suffices to prove the following: Any bundle F of degree 0 over X can be given by transition function $\exp \mu$, $\mu \in R_0$, for the covering $\{\mathcal{U}_+, \mathcal{U}_-\}$. Since the domains $\mathcal{U}_\pm$ are affine curves, over each of them the bundle F is trivial, and hence it is given by a transition function φ. Since $\deg F = 0$, φ can be chosen homotopic to a constant (as a map of X_0 into $\mathbb{C}^*$). In this case there exists a single-valued function $\mu = \log \varphi$.

7.3. Complete Integrability. We proceed to the question of the completeness of the system of integrals of motion of Laxian equations on $\tilde{\mathfrak{g}}_0$ — the orbit $\mathcal{O}_L$ of the matrix L. Let for $n_- > 0$, H_L denote the centralizer of the element ℓ_{-n_-} in $GL(m)$ (we note that in this case $\ell'_{-n_-} = \ell_{-n_-}$ for all $L' \in \mathcal{O}_L$), and $H_L = GL(m)$ for $n_- = 0$. We write $T_L^\circ = T_L \cap \mathcal{O}_L$.

Proposition 7.3.1. (i) The Hamiltonian group H_L acts on $\mathcal{O}_L$ and preserves the Laxian equations of motion.

(ii) There is a natural isomorphism of the "reduced torus" T_L°/H onto a Zariski open subset of the Jacobian $\mathcal{J}(L)$.

(iii) If $n_- > 0$, then the algebra $I(\tilde{\mathfrak{g}})$ gives a complete system of integrals of motion and T_L° is a "maximal torus" in $\mathcal{O}_L$. If $n_- = 0$, then a complete system of integrals of motion is obtained by adding to $I(\tilde{\mathfrak{g}})$ a maximal Kirillov-bracket-in-$\mathfrak{gl}(m)$-commutative algebra of functions of $\mathfrak{g}$.

Proof. (i) As noted in [32], for $n_- > 0$ the Hamiltonians of the action of H_L on $\mathcal{O}_L$ are contained in the algebra $I(\tilde{\mathfrak{g}})$. In fact, we consider the polynomials $\varphi_n(L') = \langle \varphi(L' z^{n_-}) \rangle_{n_-}$, $\varphi \in I(\mathfrak{g})$. If $M = d\varphi_n(L')$, $L' \in \mathcal{O}_L$, then $M_- = d\varphi(\ell_{-n_-})$ and the Laxian equation $\dot{L}' = [M_-, L']$ gives an Ad-action of the subgroup $\exp tM_- \subset H_L$. Since ℓ_{-n_-} is a matrix with distinct eigenvalues, these subgroups fill the entire group H_L. In the case $n_- = 0$ the assertion is obvious.

(ii) follows from Propositions 7.1.1 and 7.1.2.

(iii) From the proof of (i) and Proposition 7.2.2 it follows that for $n_- > 0$ the trajectories of Hamiltonians from $I(\mathfrak{g})$, starting at the point L, fill an open part of the level set T_L°. This is equivalent with complete integrability. For $n_- = 0$, the submanifold, filled by trajectories starting at L, is transversal to the $GL(m)$-orbit of the point L. Whence (iii) follows easily.

We note that $\dim \mathcal{O}_L = m(m-1)(n_+ + n_-)$ for $n_- > 0$ and $\dim \mathcal{O}_L = m(m+1)(n_+ + 1)$ for $n_- = 0$ (Proposition 5.1.1). This is consistent with the fact that $\dim \mathcal{J}(L) = \frac{1}{2} m(m+1)(n_+ - n_-) + m + 1$ (Proposition 7.1.1).

Now we consider the same question for the symmetrically graded complex algebra $\tilde{\mathfrak{g}}^\theta = \{L \in \tilde{\mathfrak{g}} : L(-z) = -L(z)^t\}$. If $L \in \tilde{\mathfrak{g}}^\theta$, then on the curve X_L there is an involution $\tau(z, \lambda) = (-z, -\lambda)$. It is proved in [32] that the bundles corresponding to points of the level set $T_L \cap \mathfrak{g}^\theta$, fill an open part of a subtorus in $\mathcal{J}(L)$. Under a suitable isomorphism of $\mathcal{J}(L)$ with $\mathcal{J}_0$ this subtorus goes into the Prym manifold induced by the involution τ_* in $\mathcal{J}_0$,

$$\mathrm{Prym}(\tau) = \{E \in \mathcal{J}_0 : \tau_*(E) = E^*\}.$$

Let $\mathcal{O}_L^\theta$ be the orbit of the point L with respect to the algebra $\tilde{\mathfrak{g}}_0^\theta$, $T_L^\theta = T_L \cap \mathcal{O}_L^\theta$, H_L^θ be the component of the identity of the intersection $H_L \cap O(n, \mathbb{C})$.

Proposition 7.3.2. (i) The Hamiltonian group H_L^θ acts on $\mathcal{O}_L^\theta$ and preserves the Laxian equations of motion.

(ii) There is a natural isomorphism of the "reduced torus" T_L^θ/H_L^θ onto a Zariski open subset in $\mathrm{Prym}(\tau)$.

(iii) If $n_- > 0$, then the algebra $I(\tilde{\mathfrak{g}})$ gives a complete system of integrals of motion and T_L^θ is a "maximal torus" in $\mathcal{O}_L^\theta$. If $n_- = 0$, then a complete system of integrals of motion is obtained by adding to $I(\tilde{\mathfrak{g}})$ a maximal Kirillov-bracket-in-$so(n)$-commutative algebra of functions of ℓ_0.

Proof. The map $\mathcal{J}^\theta : T_L \cap \mathfrak{g}^\theta \to \mathrm{Prym}(\tau)$ described above has the same properties as the map $\mathcal{J} : T_L \to \mathcal{J}(L)$: its fibers are orbits of the group $O(n, \mathbb{C})$, Laxian flows in $T_L \cap \mathfrak{g}^\theta$ go into linear flows on $\mathrm{Prym}(\tau)$ and conversely. Hence the proof of Proposition 7.3.1 carries over to this case. We also give an independent calculation of the dimension of the Prym manifold.

Let $\mathcal{J}^\tau$ be the set of fixed points of the involution τ_*. Obviously, $\dim \mathrm{Prym}(\tau) = g - \dim \mathcal{J}^\tau$. Since $\mathcal{J}^\tau$ is the Jacobian of the quotient-curve $X^\tau = X/\tau$, one has $\dim \mathcal{J}^\tau = g(X^\tau)$. To calculate $g(X^\tau)$ it suffices to know the ramification index of the covering $X \to X^\tau$. Ramification can occur only at points with coordinates $z = 0, \infty$, $\lambda = 0, \infty$. They are exhausted by those $2m$ points which arise upon resolution of singularities in $P_+ = (0, \infty)$, $P_- = (\infty, \infty)$. Thus, the ramification index is equal to $v = v_+ + v_-$. From the Hurwitz formula $2g(X^\tau) = g + 1 - v$ we get $\dim \mathrm{Prym}(\tau) = \frac{1}{2}(g-1) + \frac{1}{4}v$. Here $v_\pm = m$, if $n_\pm$ is odd and $v_\pm = \frac{1}{2}(1 - (-1)^m)$, if $n_\pm$ is even.

All the preceding arguments can be carried out for complex algebras and groups. In the case of a real algebra $\mathfrak{g} = \mathfrak{gl}(m, \mathbb{R})$, to isospectral matrices corresponds the "real part" of the Jacobian $\mathcal{J}(L)$, and the assertions about complete integrability continue to hold.

For Lie algebras different from $\mathfrak{gl}(m)$, one should consider a faithful matrix representation. Laxian motion will occur along a subtorus, which is defined by the symmetry properties of the representation.

7.4. Additional Integrals for a Non-Abelian Toda Chain. As remarked in 3.5, an ordinary $L-M$ pair for a nonclosed non-Abelian Toda chain gives $(mn-1)$ independent integrals of motion, less than the number of degrees of freedom $(n-1)m^2$. The use of the graded algebra allows one to give the same chain with the help of the matrix $L_c = L + C \otimes e_{\alpha*} z^{-1}$, $c \in \mathfrak{gl}(m)$

$$L_c = \begin{pmatrix} p_1 & a_1 & \cdots & cz^{-1} \\ 1 & p_2 & \ddots & \vdots \\ \vdots & & p_{n-1} & a_{n-1} \\ 0 & \cdots & 1 & p_n \end{pmatrix},$$

where $p_i = \dot{g}_i g_i^{-1}$, $a_i = g_{i+1} g_i^{-1}$, $g_i \in GL(m)$. The difference from a periodic chain is that here the orbit lies in the subspace of zero z-graduation.

By analogy with [35] one can assume that the number of independent invariants of the matrix L_c is not less than the genus of the curve X_{L_c}, defined by the equation $\det(L_c(z)-\lambda)=0$. We calculate it. The bidegree of the curve is equal to $(d_1,d_2)=(m,mn)$. Let $c=\mathrm{diag}(c_1,\ldots,c_m)$, c_i be pairwise distinct. The closure of the curve X_L in $\mathbb{C}P^1\times\mathbb{C}P^1$ has a singularity at the point $z=0$, $\lambda=\infty$, where the principal part of the equation is $\prod_{i=1}^{m}(\lambda^n-c_i z^{-1})$. The index of this singular point is equal to $v=\frac{1}{2}m(m-1)n$, whence

$$g=d_1d_2-(d_1+d_2)+1-v=\frac{1}{2}m(m-1)n-m+1\ .$$

Moreover, in a non-Abelian Toda chain there is the symmetry group $\mathrm{diag}\,GL(m)\subset GL(mn)$. Transformations from $GL(m)$ commuting with the matrix C add to the integrals $\langle\varphi(L_c)\rangle_k$, $\varphi\in I(\mathfrak{gl}(mn))$ another $m-1$ integrals in involution.

Concluding Remarks

Here we consider the quantum version of the scheme of 1.2 and we formulate some unsolved problems.

1. Quantization

By quantization of an integrable dynamical system in $\mathfrak{g}^*$ we understand the following. Let $\mathcal{U}_1\subset\mathcal{U}_2\subset\ldots\subset\mathcal{U}(\mathfrak{g})$ be the standard filtration in the enveloping algebra $\mathcal{U}(\mathfrak{g})$, $\beta: S(\mathfrak{g})\to\mathcal{U}(\mathfrak{g})\to\oplus\,\mathcal{U}_i/\mathcal{U}_{i-1}$ be the symmetrization isomorphism. Suppose given a subalgebra $A\subset S(\mathfrak{g})$, commutative with respect to the Kirillov bracket. We introduce on A the filtration by polynomial degree.

Definition. By quantization of the algebra A is meant a homomorphism $q: A\to\mathcal{U}(\mathfrak{g})$ of algebras with filtration such that the corresponding homomorphism of graded algebras coincides with the restriction of β to $\mathrm{grad}(A)$.

In other words, quantization is the realization of Hamiltonians by commuting operators, preserving the highest symbol.

Example. A quantization of the algebra of invariants $A=I(\mathfrak{g}^*)$ is the isomorphism $Q: A\to Z(\mathfrak{g})$, constructed by Harish-Chandra in the semisimple case and by Duflo [36] in the general case.

We keep the notation of 1.2: $\mathfrak{g}=\mathfrak{a}\dot{+}\mathfrak{b}$, $\mathfrak{g}_0=\mathfrak{a}\oplus\mathfrak{b}$, $\sigma_0:\mathfrak{g}_0\to\mathfrak{g}$ is defined by the equation $\sigma_0(\alpha\oplus\beta)=\alpha-\beta$. Let $u\mapsto u^*$ be the anti-involution in $\mathcal{U}(\mathfrak{g})$, generated by multiplication by -1 in $\mathfrak{g}$. We define the map $\tilde{\sigma}:\mathcal{U}(\mathfrak{g}_0)\to\mathcal{U}(\mathfrak{g})$ "induced" by the map σ_0: $\mathcal{U}(\mathfrak{g}_0)=\mathcal{U}(\mathfrak{a})\otimes\mathcal{U}(\mathfrak{b})$ and for $u\in\mathcal{U}(\mathfrak{a})$, $v\in\mathcal{U}(\mathfrak{b})$ we set $\tilde{\sigma}(u\otimes v)=uv^*$,

$$\rho=\tilde{\sigma}^{-1}\ .$$

Let $\sigma_*: S(\mathfrak{g}_0)\to S(\mathfrak{g})$ be an extension of σ_0 to the symmetric algebra. We recall that in Theorem 1 we considered the algebra of Hamiltonians on $\mathfrak{g}_0^*$ of the form $A^\sigma=\sigma_*^{-1}I(\mathfrak{g}^*)$.

Proposition 1. Let $Q: I(\mathfrak{g}^*)\to Z(\mathfrak{g})$ be the Duflo quantization in $\mathfrak{g}^*$. The map

$$q=\rho\circ Q\circ\sigma_*$$

gives a quantization of the algebra A^σ.

Proof. Obviously it suffices to prove that on the center $\mathfrak{Z}(\mathfrak{g})$ the map ρ is a homomorphism. For this we note that it follows from the definition of ρ that for $u\in\mathfrak{U}(\mathfrak{a}), v\in\mathfrak{U}(\mathfrak{b})$, $z\in\mathfrak{U}(\mathfrak{g})$

$$\rho(uzv) = \rho(u)\,\rho(v)\,\rho(z).$$

Hence for $z\in\mathfrak{Z}(\mathfrak{g})$, $x\in\mathfrak{U}(\mathfrak{g})$ we have $\rho(xz)=\rho(x)\,\rho(z)$.

The geometric interpretation of the map $\rho|\mathfrak{z}(\mathfrak{g})$ is that under the change of variables $\sigma: G_0 \to G$ the biinvariant differential operator $z\in\mathfrak{Z}(\mathfrak{g})$ goes into the left-invariant operator $\rho(z)\in\mathfrak{U}(\mathfrak{g}_0)$.

We consider the question of symmetry of the operator $\rho(z), z\in\mathfrak{Z}(\mathfrak{g})$ under a unitary representation of the group G_0, in particular, under quantization on orbits. To this end we introduce the notation $*$ for the antiautomorphism of the enveloping algebra $\mathfrak{U}(\mathfrak{g})(\mathfrak{U}(\mathfrak{g}_0))$, equal to $-id$ on $\mathfrak{g}$ (on $\mathfrak{g}_0$); in the unitary representation, to it corresponds conjugation. Obviously, $\mathfrak{Z}(\mathfrak{g})$ is $*$-invariant. We denote by dg_r, dg_ℓ the right and left Haar measures on G_0, $dg_\ell = \Delta(g)\,dg_r$. We set $2\delta(x)=\frac{d}{dt}\Delta(\exp tx)\big|_{t=0}$, $x\in\mathfrak{g}_0$. Let $\tilde{\delta}$ be the automorphism of the algebra $\mathfrak{U}(\mathfrak{g}_0)$, defined on generators $x\in\mathfrak{g}_0$ by the equation $\tilde{\delta}(x) = x-\delta(x)$. Finally, we set $\gamma = \tilde{\delta}\circ\rho$.

Proposition 2. Let the group G be unimodular. Then the homomorphism $\gamma:\mathfrak{Z}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g}_0)$ is symmetric, i.e., $\gamma(z^*) = \gamma(z)^*$.

Proof. The map $\sigma: G_0 \to G$ carries the Haar measure on G into the left-invariant measure G_0, and thus, the automorphism $*$ in $\mathfrak{Z}(\mathfrak{g})$ into conjugation by the measure dg_ℓ in the regular representation of the algebra $\mathfrak{U}(\mathfrak{g}_0)$. Since $\mathfrak{g}_0$ is the algebra of left-invariant fields, its regular representation is unitary with respect to the right measure dg_r, i.e., to $*$ in $\mathfrak{U}(\mathfrak{g}_0)$ corresponds conjugation by the measure dg_r. But multiplication by $\Delta^{1/2}$ carries one conjugation into the other and the operator $u\in\mathfrak{U}(\mathfrak{g}_0)$ into $\tilde{\delta}(u)$.

For split Toda chains there is more detailed information [17]. The problem of quantization can be solved completely, by constructing the spectral decomposition of the algebra of quantized integrals of motion. It turns out to be unitarily equivalent to the algebra of operators of multiplication by polynomials, invariant with respect to the Weyl group, in the space of square-summable functions on the Cartan subalgebra.

2. Some Unsolved Problems

1. There are two completely integrable systems, quite similar to the systems of Sec. 6, but not included in our scheme. The first is the anharmonic oscillator, defined by the Hamiltonian

$$H(x,p) = \sum(p_i^2 + a_i x_i^2) + \tfrac{1}{2}\left(\sum x_i^2\right)^2.$$

The integrals of motion in involution

$$H_i(x,p) = p_i^2 + a_i x_i^2 + \tfrac{1}{2}x_i^2\sum x_j^2 + \tfrac{1}{2}\sum_{j\neq i}\frac{(x_i p_j - x_j p_i)^2}{a_i - a_j}$$

recall the integrals of motion for the oscillator on the sphere, but the L–M pair has not been found.

The second system is the motion of a particle on the sphere $|x|=1$ in $\mathbb{R}^n$ in the potential $V(x)=\frac{1}{2}(\sum a_i x_i^2+\sum c_i^2 x_i^{-2})$. Moser [33] found an pair in which

$$L=x\wedge p+x\otimes x+x\otimes\frac{c}{x}+\frac{c}{x}\otimes x+A,$$

where $A=diag(a_1,\ldots,a_n)$, $\frac{c}{x}=\left(\frac{c_1}{x_1},\ldots,\frac{c_n}{x_n}\right)$. The orbital interpretation of this pair is missing.

2. For split Toda chains, Kostant [20] defined variables of action-angle type, in which all the Laxian equations are linearized. What does the symplectic form look like in these variables? In particular, will they be real action-angle variables?

3. Generalize the method of completion of flows on the basis of Hamiltonian reduction to the graded case.

4. Is there a connection between Hamiltonian structures of stationary equations [14] and the structures of 5.2 in $\mathfrak{g}_n^*$?

5. Adler [5] proposed a second Hamiltonian structure for a classical Toda chain, with respect to which the previous integrals of motion commute and give the same collection of equations. The problem is the group interpretation of this structure.

7. Quantized Hamiltonian systems of Sec. 6 are easily constructed and have discrete spectra. In [17] there are proposed quantized integrals of motion, but commutativity is not proved, in particular, it is unknown whether they give quantized integrals. It would be interesting, by analogy with the finite-dimensional case, to learn to use representations of affine algebras for finding spectra and eigenfunctions of the quantized system.

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