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The Role of Mathematics in Gravitational Physics
From the Sublime to the Subliminal?[1]

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Abstract

Certain gravitational systems are described for which particular equations of state are singled out by means of purely mathematical considerations. The fact that these equations of state are realistic, and indeed the most relevant to the physical systems under study, suggests that locked into Einstein's field equations lies some information about local physical conditions, and that in order to describe gravitating systems, it is not always necessary to draw on extraneous branches of physics.

§(1): *Prologue*

The classical role of mathematics in physics has been to act as a language and to provide a logical framework for discussion. At a practical level, this framework can be cumbersome, but frequently it is attractive and elegant, and it has justifiably come to be regarded by applied mathematicians as a subject in its own right. This summarizes the "*sublime*" role of mathematics in physics.

However, certain recent results in general relativity indicate that mathematics may be capable of playing a new role. In contrast to the incontestable logic of sublime mathematics, which in principle leads relentlessly to a definite answer to a well-posed physical problem, mathematics can also act by subtle suggestion, and at times this may even occur without our being aware of it. This is the "*subliminal*" role of mathematics in physics. I shall be discussing in this

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essay some examples of ways in which this subliminal role can manifest itself in general relativity.

§(2): *Introduction*

In the study of general relativistic systems, it is customary to suppose that one is perfectly free to specify the equation of state in a rather arbitrary fashion. Knowledge of the equation of state is usually necessary in order to render Einstein's field equations completely deterministic. The usual procedure is to impose an equation of state obtained by means of purely *physical* considerations that are somewhat extraneous to the gravitational aspect of the problem. The aim of the present essay is to show that, at least for certain systems, particular equations of state are singled out by purely *mathematical* considerations alone. The fact that these equations of state are realistic, and moreover that they are the most relevant to the systems under study, suggests that, locked into Einstein's field equations, lies some hidden information about local physical conditions. This further suggests that one's attitude should be to try, by purely mathematical considerations, to unify, rather than divorce, the roles of gravitational and nongravitational physics in determining the properties of a self-gravitating system.

By way of illustration, I shall discuss two quite unrelated systems. The first is a class of anisotropic cosmological models, and the second is a relativistic spherically symmetric star. The ways in which the mathematics acts to single out the equations of state are not the same. In the cosmological case, we consider the effect of "small" changes in the equation of state on the general qualitative behavior of the models. A startling result emerges from purely mathematical arguments: The equations of state[2] $p = 0$, $p = \frac{1}{3}\rho$, and $p = \rho$ are very special, in that the asymptotic behavior of the models (to the past and to the future) can vary dramatically when the equation of state is perturbed; these equations of state are precisely the ones that are usually arrived at by purely physical arguments as being the most relevant for cosmological problems. In the stellar structure case, we find that the requirement that there be a family of stars all similar to each other (in a well-defined mathematical sense) necessitates an equation of state $p = (\gamma - 1)\rho$, where γ is constant. A similar philosophy, adapted to Newtonian stars, will necessitate the equations of state $p = (\gamma - 1)\rho$ or $p = \alpha\rho^{\gamma}$, where α and γ are constants. It is curious that in each case the mathematics singles out those particular equations of state that are the most relevant to the problem.

§(3): *Subliminal Cosmology*

Hawking and Penrose [1] have established some outstanding theorems on the *existence* of singularities in general relativity. In these theorems, a singularity

[2] Here and throughout p denotes the isotropic pressure and ρ the energy density.

is defined abstractly in terms of the incompleteness of causal curves, and so the energy density does not necessarily become infinite at such a singularity. The question of the *nature* of the singularity arising in general relativity is a notoriously difficult one, and little is known. Nevertheless, Ellis and King [2] have obtained some far-reaching results pertaining to the kinds of singularity that are possible in perfect fluid spatially homogeneous anisotropic cosmological models. In the case where the matter flows orthogonally to the hypersurfaces of homogeneity, the situation is fairly straightforward, and a singularity usually results, at which the energy density, ρ, becomes infinite. However, in the case of the "tilted" models, where the matter flow congruence is not aligned with the congruence of geodesics normal to the spatially homogeneous hypersurfaces, it remains a very difficult task to determine with any certainty exactly what sort of singularity will arise. The simplest tilted model is the one that is best understood. This is an anisotropic generalization of the $k = -1$ (open) Robertson–Walker universe, and it is of type V in the Bianchi classification scheme; for the special case where the matter content is dust ($p = 0$), it is known as the Farnsworth [3] solution. I have shown [4] (cf. [5]) that in the case of dust, this model has a strong type of singularity ($\rho \to \infty$), but that for *any* equation of state $p = (\gamma - 1)\rho$, where $1 < \gamma < 2$, the singularity is of a much weaker type. This peculiar property can be understood to some extent from the standpoint of the structural stability of the field equations, i.e., as a purely mathematical artifact induced by the nonlinearity of the differential equations. However, from the physical point of view, a rather curious feature of it is that it occurs at an equation of state that is physically relevant. In a similar way, the asymptotic behavior to the future (i.e., as the universe expands) varies as a function of the equation of state of the matter content, the chief transition now occurring at the physically relevant equation of state $p = \frac{1}{3}\rho$. It is at present unknown whether or not this property is a feature peculiar to the type-V model, but it seems reasonable to speculate that other more general tilted models will exhibit a similar behavior. This may well be the case, since indications are that the higher the symmetry of the model, the less sensitive the qualitative behavior of the solution is to changes in the equation of state (for instance, in the spherically symmetric Robertson-Walker models, the general qualitative behavior is completely independent of the matter content, provided that it is sufficiently realistic).

There is a similar sort of instability present in vacuum models, where the introduction of even a minute amount of matter can drastically alter the general behavior. However, the vacuum solutions are rather exceptional, and their instability comes as no surprise.

§(4): *Subliminal Astrophysics*

We can discuss the structure of relativistic stars in a similar vein, by assuming that, given the equation of state, an entire family of solutions can be generated

[6]. The way in which this similarity is defined is best seen by first considering Newtonian stars. In this case, the structure equations are

$$\frac{dp}{dr} = \frac{-GM\rho}{r^2}$$

and

$$\frac{dM}{dr} = 4\pi r^2 \rho \cdot$$

where M is the total mass enclosed within a radius r (see, e.g., [7]). These equations are supplemented by the conditions $p = 0$ when $\rho = 0$, and $M = 0$ when $r = 0$. It seems physically plausible to demand that a simple family of solutions should exist, the members being related to each other by transformations of the form $r \to \tilde{r} = ar$, $\rho \to \tilde{\rho} = b\rho$; $M \to \tilde{M} = cM$. In this case, one obtains a *polytropic* equation of state: i.e., precisely that which is usually postulated by appeal to extraneous physics.

Because the r coordinate is less well defined in general relativity, one cannot justify such a scaling procedure in the corresponding relativistic problem. However, one can perhaps reasonably require that a family of solutions should exist whose members are related by transformations of the form $r \to \tilde{r}(r)$, $\rho \to \tilde{\rho}(\rho)$, $M \to \tilde{M}(M)$. In this case, one obtains, *of necessity*, precisely that equation of state that is relevant to relativistic stellar structure, namely, $p \propto \rho$. Once again, we see that there seems to be no a priori reason why the mathematics "knows" in advance the physics of the situation.

§(5): *Conclusion*

To summarize, we have seen that the subliminal role is a new and exciting way in which mathematics can act in a physical theory. This role may be tied in with the question of the structural stability of the equations of the theory. Leading on from the examples discussed above, it is tempting to speculate that such considerations may play a decisive part in the examination of a novel concept: the stability of a physical system *under perturbations of the underlying theory*. Such a stability could be considered in a very transparent form by using the Post-parametrized Newtonian formalism (see [8] for an introductory review), or an extension of it. However, since this formalism at its present stage does not encompass all conceivable theories, it would only be capable of exhibiting instabilities, and not stabilities. On the other hand, it may be possible to regard the existence of a physical system as evidence for its stability under general perturbations of the *theory*, this existence acting as a constraint on the set of all physical theories. Such a state of affairs might then prompt one to ask: Is metatheory a better theory?

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