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Maximal and Minimal Vertex-Critical Graphs of Diameter Two

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A graph is vertex-critical if deleting any vertex increases its diameter. We construct, for each $v \geqslant 5$ except $v=6$, a vertex-critical graph of diameter two on v vertices with at least $\frac{1}{2}v^2 - \sqrt{2}\, v^{3/2} + c_1 v$ edges, where c_1 is some constant. We show that such a graph contains at most $\frac{1}{2}v^2 - (\sqrt{2}/2)\, v^{3/2} + c_2 v$ edges, where c_2 is some constant. We also construct, for each $v \geqslant 5$ except $v=6$, a vertex-critical graph of diameter two on v vertices with at most $\frac{1}{2}(5v-12)$ edges. We show that such a graph must contain at least $\frac{1}{2}(5v-29)$ edges. © 1998 Academic Press

1. INTRODUCTION

For standard notation and terminology, we follow those of Bondy and Murty [3]. A graph G has vertex set $V(G)$, edge set $E(G)$, $v(G)$ vertices, and $\varepsilon(G)$ edges. The distance $d(x, y)$ between two vertices x and y of G is defined as the length of a shortest (x, y)-path in G; if there is no path connecting x and y we define $d(x, y)$ to be infinite. The diameter of a graph G, denoted by $D(G)$, is defined to be the maximum distance in G, that is,

$$D(G) = \max_{x,\, y \in V(G)} d(x, y).$$

A graph G is *k-vertex-critical* if

$$D(G-v) > D(G) = k$$

for every $v \in V(G)$.

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Vertex-critical graphs have been studied in [2, 4–8]. Here we only consider 2-vertex-critical graphs. Figure 1 shows some examples of such graphs. Plesník [8] asked whether every 2-vertex-critical graph G satisfies $\varepsilon(G) \leqslant \frac{1}{4}v^2$. Later, Erdös and Howorka [6] constructed a family of graphs, which show that a 2-vertex-critical graph may contain as many as $\frac{1}{2}v^2 - \sqrt{2}\, v^{3/2}$ edges.

Erdös and Howorka [6] considered the graphs G which satisfy the following property ($P(r)$): for each vertex u, there is a set $M(u)$ of r vertices such that u is the only vertex adjacent to all vertices of $M(u)$. Clearly, every 2-vertex-critical graph G satisfies $P(2)$. They proved that a graph with the property $P(r)$ satisfies $\varepsilon(G) \leqslant 1/2v(v-1) - [c_r + o(1)]\, v^{1+1/r}$. They also claimed that $c_2 = \sqrt{2}/2$. However, this claim does not seem to follow from their calculation in [6].

In this paper, we construct, for each $v \geqslant 5$ except $v = 6$, a 2-vertex-critical graph G which satisfies $\varepsilon(G) \geqslant \frac{1}{2}v^2 - \sqrt{2}\, v^{3/2} + c_1 v$ where c_1 is some constant. When $v = \frac{1}{2}k(k-3)$ for some $k \geqslant 5$, our graphs coincide with those found by Erdös and Howorka [6]. More precisely, if F_v denotes the maximum number of edges possible in a 2-vertex-critical graph with v vertices, then we show that, for each $v \geqslant 5$ except $v = 6$,

$$\frac{1}{2}\, v^2 - \sqrt{2}\, v^{3/2} + c_1 v \leqslant F_v \leqslant \frac{1}{2}\, v^2 - \frac{\sqrt{2}}{2}\, v^{3/2} + c_2 v,$$

where c_1 and c_2 are constants.

We also analyze the minimum number of edges possible in a 2-vertex-critical graph with v vertices. Denote this number by f_v. We show that, for each $v \geqslant 5$ except $v = 6$,

$$\tfrac{1}{2}(5v - 29) \leqslant f_v \leqslant \tfrac{1}{2}(5v - 12).$$

Let G be a graph and let x be a vertex of G. We shall use $N_G(x)$ (or simply $N(x)$) to denote the set of vertices which are adjacent to x. We shall write $d_G(x) = |N_G(x)|$, the *degree* of x, and $\delta(G) = \min\{d_G(x) : x \in V(G)\}$, the *minimum degree* of G. If x is not adjacent to some vertex y of G, then

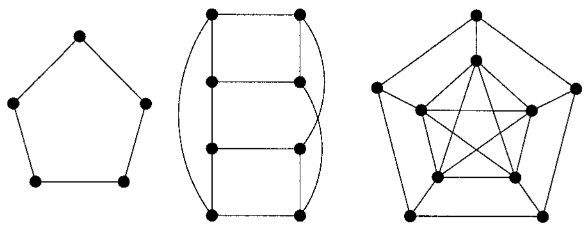

FIG. 1. Some 2-vertex-critical graphs.

we shall call xy a *non-edge* of G. For $S \subseteq V(G)$, we shall use $G\langle S \rangle$ to denote the subgraph of G induced by S.

A *matching* $\mathscr{M}$ of G is a set of edges of G, no two of which share an endvertex. An *augmenting path* (with respect to a matching $\mathscr{M}$) is a path $x_1 x_2 \cdots x_{2k}$ such that $x_i x_{i+1} \in \mathscr{M}$ for each even i. Let X, $Y \subseteq V(G)$. We define $d(X) = \sum_{x \in X} d(x)$, $e(X, Y) = |\{xy \in E(G) : x \in X \text{ and } y \in Y\}|$, and $d_{\max}(X, Y) = \max\{d(x, y) : x \in X \text{ and } y \in Y\}$.

2. THE VALUE F_v

In [6], the following graphs G have been constructed: Let $H = u_1 u_2 \cdots u_k u_1$ be the cycle of length k ($k \geqslant 5$) and let T be the complete graph on $\frac{1}{2}k(k-5)$ vertices with

$$V(T) = \{v_{i,j} : 1 \leqslant i \leqslant k-3,\ j \leqslant k,\ 3 \leqslant j-i \leqslant k-3\}.$$

Construct G from H and T by adding two edges from $v_{i,j}$ to u_i and u_j, respectively, for each $v_{i,j} \in V(T)$. It is easy to see that these are 2-vertex-critical and contain at least $\frac{1}{2}v^2 - \sqrt{2}\, v^{3/2} + c_1 v$ where c_1 is some constant. Hence the following lemma is proved.

LEMMA 2.1. *For each $k \geqslant 5$, there is a 2-vertex-critical graph G with $v = \frac{1}{2}k(k-3)$ and $\varepsilon \geqslant \frac{1}{2}v^2 - \sqrt{2}\, v^{3/2} + c_1 v$, where c_l is some constant.*

LEMMA 2.2. *For each $k \geqslant 6$ and $\frac{1}{2}(k-1)(k-4) < v < \frac{1}{2}k(k-3)$ except $v = 6$, there is a 2-vertex-critical graph G with v vertices and $\varepsilon \geqslant \frac{1}{2}v^2 - \sqrt{2}\, v^{3/2} + c_1 v$, where c_1 is some constant.*

Proof. When $k = 6$ and $\frac{1}{2}(k-1)(k-4) < v < \frac{1}{2}k(k-3)$ except $v = 6$, Fig. 2 shows the constructions.

Let $k \geqslant 7$ and let G' be the 2-vertex-critical graph on $\frac{1}{2}k(k-3)$ vertices constructed above. We obtain a graph G on v vertices as follows: When

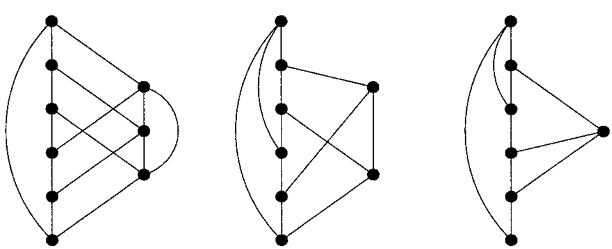

FIG. 2. 2-vertex-critical graphs on 9, 8, 7 vertices.

$\frac{1}{2}(k-1)(k-4)+1<v<\frac{1}{2}k(k-3)$, we identify in G' the vertex $v_{i,i+3}$ with the vertex $v_{i,i+4}$ for each $i=1, 2, ..., \frac{1}{2}k(k-3)-v$. When $k\geqslant 8$ and $v=\frac{1}{2}(k-1)(k-4)+1$, we identify (in G') $v_{i,i+3}$ with $v_{i,i+4}$ for each $i=1, 2, ..., k-4$ and identify $v_{1,k-3}$ with $v_{k-3,k}$. When $k=7$ and $v=\frac{1}{2}(k-1)(k-4)+1=10$, a graph is shown in Fig. 1.

It can be checked that the graphs constructed above are 2-vertex-critical and contain at least $\frac{1}{2}v^2-\sqrt{2}\,v^{3/2}+c_1 v$ edges, where c_1 is some constant. ∎

Combining Lemmas 2.1 and 2.2, we have the following:

THEOREM 2.3. *For each $v\geqslant 5$ except $v=6$, there is a 2-vertex-critical graph on v vertices with $\varepsilon\geqslant\frac{1}{2}v^2-\sqrt{2}\,v^{3/2}+c_1 v$, where c_1 is some constant.*

COROLLARY 2.4. *For each $v\geqslant 5$ except $v=6$, we have*

$$F_v\geqslant\frac{1}{2}v^2-\sqrt{2}\,v^{3/2}+c_1 v,$$

where c_1 is some constant.

Let G be a 2-vertex-critical graph with v vertices. For each vertex $u\in V(G)$, there exists a pair of vertices $u', u''\in V(G-u)$ such that $d_{G-u}(u', u'')\geqslant 3$. Associate each vertex $u\in V(G)$ with such a pair of vertices u', u'' and denote it by $u=c(u'u'')$. Construct an *associated graph* G^* of G as follows: the vertex set of G^* is $V(G)$ and the edge set of G^* is $\{u'u'': u\in V(G)$ and $u=c(u'u'')\}$. The associated graph G^* has v vertices and v edges. In the remainder of this section, we shall always use $\mathscr{M}$ to denote a maximum matching in G^* and write $s=|\mathscr{M}|$.

LEMMA 2.5. *Let G be a 2-vertex-critical graph and let G^*, $\mathscr{M}$ and s be defined as above. The following statements hold.*

(i) *If $xy\in E(G^*)$, then $d_G(x)+d_G(y)\leqslant v-1$.*

(ii) *If $x\in V(G^*)$ and $N_{G^*}(x)=\{y_1, y_2, ..., y_k\}$, then $y_i c(xy_j)$ is a non-edge in G, for all $1\leqslant i\neq j<k$.*

(iii) *G contains at least $\frac{1}{2}s(v-1)$ non-edges.*

(iv) *G contains at least $s(v-2s)$ non-edges.*

Proof. Note that $N_G(x)\cap N_G(y)=\{c(xy)\}$ and thus (i) follows. For (ii), suppose that $y_i c(xy_j)$ is an edge of G. Then $xc(xy_j)\,y_i$ is a path of length 2 in $G-c(xy_i)$ joining x and y_i, *a* contradiction.

Let S denote the set of $2s$ vertices incident with edges in $\mathscr{M}$. To prove (iii) and (iv), we count the number of non-edges of G incident with the vertices in S in two ways. By (i), for each $xy\in\mathscr{M}$, $d_G(x)+d_G(y)\leqslant v-1$. Thus $\sum_{x\in S} d_G(x)\leqslant s(v-1)$. Hence the number of non-edges of G incident with the vertices in S is at least $\frac{1}{2}\sum_{x\in S}(v-1-d_G(x))\geqslant\frac{1}{2}s(v-1)$ and we have (iii). On the other hand, since there are at most $\binom{2s}{2}=s(2s-1)$ non-edges with both endvertices in S, G contains at least $\sum_{x\in S}(v-1-d_G(x))-s(2s-1)\geqslant s(v-2s)$ non-edges incident with the vertices in S and we have (iv). ∎

THEOREM 2.6. *For each $v\geqslant 5$ except $v=6$,*

$$F_v\leqslant\frac{1}{2}v^2-\frac{\sqrt{2}}{2}v^{3/2}+c_2 v,$$

where C_2 is some constant.

Proof. If $s>\sqrt{2v}$, then, by Lemma 2.5 (iii), G contains at least $\sqrt{2v}(v-1)/2$ non-edges and hence the theorem follows. If $(\sqrt{2v}-1)/2\leqslant s\leqslant\sqrt{2v}$, then, by Lemma 2.5 (iv), G contains at least $(\sqrt{2v}-1)/2(v-2\sqrt{2v})$ non-edges. However, this implies that

$$\begin{aligned}\varepsilon&\leqslant\frac{1}{2}v(v-1)-\frac{\sqrt{2v}-1}{2}(v-2\sqrt{2v})\\&\leqslant\frac{1}{2}v^2-\frac{\sqrt{2}}{2}v^{3/2}+2v.\end{aligned}$$

So from now on we assume that $s<(\sqrt{2v}-1)/2$.

Let $S\subseteq V(G^*)$ denote the set of endvertices of edges contained in $\mathscr{M}$ and let $T=V(G^*)-S$. Let $Y=\{y_1, y_2, ..., y_l\}$ consist of vertices of S which are adjacent (in G^*) to at least two vertices in T. Then $\mathscr{M}$ contains no edge with both endvertices in Y, as otherwise G^* would contain a matching of size greater than $\mathscr{M}$, contradicting the maximality of $\mathscr{M}$. Let $Z=\{z_1, z_2, ..., z_l\}$ be the set of vertices such that $y_iz_i\in\mathscr{M}$ for each $i=1, 2, ..., l$. Write $\mathscr{M}'=\{y_iz_i\colon i=1, 2, ..., l\}$.

Let $\mathscr{M}''=\{u_jv_j\colon j=1, 2, ..., p\}$ be a maximal set of edges in $\mathscr{M}-\mathscr{M}'$ which satisfies the following property: v_1 is adjacent (in G^*) to at least two vertices in Z, and for each $j=2, ..., p$, v_j is adjacent (in G^*) to at least two vertices in $Z\cup\{u_1, ..., u_{j-1}\}$. Denote $Z^+=Z\cup\{u_j\colon j=1, ..., p\}$, $Y^+=Y\cup\{v_j\colon j=1, ..., p\}$, and $X=S-(Y^+\cup Z^+)$.

Claim 1. Each vertex $x\in X$ is adjacent (in G^*) to at most one vertex in Z^+ and at most one vertex in T.

It follows immediately from the definition of Y and the the maximality of $\mathscr{M}''$.

Claim 2. In G^*, each vertex of Z^+ can only be adjacent to some vertices in $X \cup Y^+$, i.e., for each $z \in Z^+$, $N_{G^*}(z) \subseteq X \cup Y^+$.

Suppose that $G^*\langle Z\rangle$ contains an edge, say $z'z''$. By the definition of Z^+, we can find two disjoint paths P_1 and P_2, such that $P_1 = u_{j1} v_{j1} u_{j2} v_{j2} u_{j3} \cdots v_{j_a} z_{k_1}$, where $u_{j_1} = z'$, $j_1 > j_2 > \cdots > j_a$ and $Z_{k_1} \in Z$, and $P_2 = u_{i_1} v_{i_1} u_{i_2} v_{i_2} u_{i_3} \cdots v_{i_b} z_{k_2}$, where $u_{i_1} = z''$, $i_1 > i_2 > \cdots > i_a$ and $z_{k_2} \in Z$. Let $t_l, t_2 \in T$ be two distinct vertices adjacent to y_{k_1}, y_{k_2}, respectively. Then we obtained an augmenting path

$$t_1 y_{k_1} z_{k_1} v_{j_a} \cdots v_{j_2} u_{j_2} v_{j_1} u_{j_1} u_{i_1} v_{i_1} u_{i_2} v_{i_2} \cdots v_{i_b} z_{k_2} y_{k_2} t_2,$$

a contradiction to the maximality of $\mathscr{M}$. A similar proof shows that there is no edge from Z^+ to T in G^*. Therefore Claim 2 is proved.

Let G' be the graph obtained from G^* by deleting all edges from the set

$$\{e \colon c(e) \in X\} \cup \{xy \colon x \in X,\ y \in T \cup Z^+ \cup Y^+\} \cup \{x_1 x_2 \colon x_1, x_2 \in X\}.$$

We note that $|E(G')| \geqslant v - 3\,|X| - \frac{1}{2}\,|X|\,(2s - |X|) - \binom{|X|}{2}$, by claim 1.

We now consider the case when $|X| < 2s$ (the case when $|X| = 2s$ will be considered at the end of the proof). Define the values p and q such that $|X| = p\sqrt{v}$ and $2s = q\sqrt{v}$. Then $|X| < 2s < \sqrt{2v-1}$ implies that $0 \leqslant p < q < \sqrt{2}$. Note that by Claim 2, each edge of G' is incident with at least one vertex in Y^+. Orient G' in such a way that all edges between Y^+ and $V(G') - Y^+$ are oriented towards $V(G') - Y^+$. Let D' be the resulting oriented graph.

For each x, let d_x^+ denote the outdegree of x in D'. Let y_i, $i = 1, 2, ..., d_x^+$, be the vertices dominated by x in D'. We obtain $d_x^+(d_x^+ - 1)$ non-edges in G, each of which has one endvertex in $\{y_1, y_2, ..., y_{d_x^+}\}$ and the other endvertex in $\{c(xy_1), c(xy_2), ..., c(xy_{d_x^+})\}$. Adding up all these non-edges over all $x \in Y^+$ we obtain at least $\frac{1}{2} \sum_{x \in Y^+} d_x^+(d_x^+ - 1)$ non-edges in $G - X$ because each non-edge, say wq, can be counted at most twice (once when considering the edge, $e \in E(G^*)$, with $c(e) = w$, and once when considering the edge, $f \in E(G^*)$, with $c(f) = q$). On the other hand, by Lemma 2.5 (i), G contains at least $\frac{1}{2}\,|X|\,(v-1) - \binom{|X|}{2}$ non-edges, each of which has precisely one endvertex in X. Hence, by Cauchy–Schwarz inequality, we obtain the following lower bound on the number of non-edges contained in G:

$$\frac{1}{2} \sum_{x \in Y^+} d_x^+(d_x^+ - 1) + \frac{1}{2}\,|X|\,(v-1) - \binom{|X|}{2}$$

$$\geqslant \frac{\left(v - 3\,|X| - |X|\left(s - \frac{1}{2}\,|X|\right) - \binom{|X|}{2}\right)^2}{2s - |X|} - \frac{1}{2} \sum_{x \in Y^+} d_x^+ + \frac{|X|\,v - |X|^2}{2}$$

$$
\begin{aligned}
&\geqslant \frac{\left(v - \frac{5}{2}\,|X| - |X|\,s\right)^2}{2s - |X|} - \frac{1}{2}\,v + \frac{|X|\,v - |X|^2}{2} \\
&= \frac{\left(v - \frac{5}{2}\,pv^{1/2} - \frac{1}{2}\,pqv\right)^2}{v^{1/2}(q-p)} - \frac{1}{2}\,v + \frac{pv^{3/2} - p^2v}{2} \\
&= \frac{(v(2-pq) - 5pv^{1/2})^2}{4v^{1/2}(q-p)} + \frac{1}{2}\,pv^{3/2} - \frac{1}{2}\,v(p^2+1) \\
&\geqslant \frac{2-pq}{q-p} \times \frac{v^2(2-pq) - 10v^{3/2}p + 25p^2v/(2-pq)}{4v^{1/2}} + \frac{1}{2}\,pv^{3/2} - \frac{3}{2}\,v \\
&\geqslant \frac{2-\sqrt{2}\,p}{\sqrt{2}-p} \times \frac{v^2(2-pq) - 10v^{3/2}p}{4v^{1/2}} + \frac{1}{2}\,pv^{3/2} - \frac{3}{2}\,v \\
&\geqslant \sqrt{2} \times \frac{v^2(2-pq) - 10v^{3/2}p}{4v^{1/2}} + \frac{1}{2}\,pv^{3/2} - \frac{3}{2}\,v \\
&= \sqrt{2}\,v^{3/2}\,\frac{(2-pq)+\sqrt{2}\,p}{4} - v\left(\frac{3}{2} + \frac{5\sqrt{2}\,p}{2}\right) \\
&\geqslant \frac{\sqrt{2}}{2}\,v^{3/2} - \frac{13}{2}\,v.
\end{aligned}
$$

This implies that

$$\varepsilon \leqslant \frac{v(v-1)}{2} - \frac{\sqrt{2}}{2}\,v^{3/2} + \frac{13}{2}\,v = \frac{1}{2}\,v^2 - \frac{\sqrt{2}}{2}\,v^{3/2} + 6v.$$

Finally, we consider the case when $|X| = 2s$ (note $2s < \sqrt{2v} - 1$). Claim 1 implies that G^* contains at most $\binom{|X|}{2} + |X|$ edges. However,

$$\binom{|X|}{2} + |X| < \frac{(\sqrt{2v}-1)^2 + \sqrt{2v} - 1}{2} = \frac{2v - \sqrt{2v}}{2} < v,$$

which is a contradiction since we know that $|E(G^*)| = v$. ∎

Combining Corollary 2.4 and Theorem 2.6, we have the following:

Corollary 2.7. *For each $v \geqslant 5$ except $v = 6$,*

$$\frac{1}{2}\,v^2 - \sqrt{2}\,v^{3/2} + c_1 v \leqslant F_v \leqslant \frac{1}{2}\,v^2 - \frac{\sqrt{2}}{2}\,v^{3/2} + c_2 v,$$

where c_1 and c_2 are some constants.

3. THE VALUE f_v

In this section, we estimate the value of f_v, the minimum number of edges possible in a 2-vertex-critical graphs with v vertices.

LEMMA 3.1. *For each odd $v \geqslant 7$, there is a 2-vertex-critical graph with v vertices and $\frac{1}{2}(5v-17)$ edges.*

Proof. A 2-vertex-critical graph G on 7 vertices consists of

$$V(G) = \{x_i \colon i = 1, 2, ..., 7\}$$

and

$$E(G) = \{x_1x_2, x_1x_3, x_1x_4, x_2x_4, x_2x_5, x_3x_6, x_4x_7, x_5x_6, x_6x_7\}.$$

To construct a 2-vertex-critical graph on $v \geqslant 9$ vertices for each odd v, we add vertices y_j, z_j, $j = 1, 2, ..., \frac{1}{2}(v-7)$ and edges $x_1 y_j, x_1 z_j, y_j z_j, y_j x_5, z_j x_7$, $j = 1, 2, ..., \frac{1}{2}(v-7)$ (see Fig. 3). ∎

Note that the cycle of length 5 is the only 2-vertex-critical graph on 5 vertices.

Figure 1 shows a 2-vertex-critical graph with 8 vertices and 12 edges. The Petersen graph is a 2-vertex-critical graph with 10 vertices and 15 edges. Fig. 4a shows a 2-vertex-critical graph with 12 vertices and 21 edges and Fig. 4b shows how to construct a 2-vertex-critical graph on (even) $v \geqslant 14$ vertices which contains $\frac{1}{2}(5v-12)$ edges. Hence we obtain the following:

LEMMA 3.2. *For each even $v \geqslant 8$, there is a 2-vertex-critical graph on v vertices with $\varepsilon \leqslant \frac{1}{2}(5v-12)$.*

Summarizing all above, we have the following:

THEOREM 3.3. *For each $v \geqslant 5$ except $v = 6$, we have*

$$f_v \leqslant \tfrac{1}{2}(5v-12).$$

To establish a lower bound for f_v, we first prove the following lemma which applies for an arbitrary graph G.

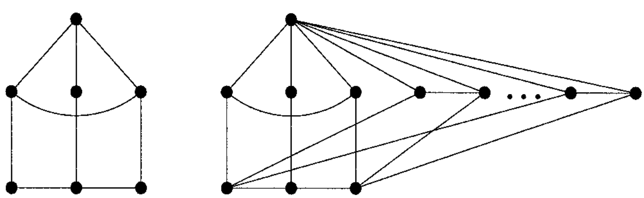

FIG. 3. 2-vertex-critical graphs on (odd) $v \geqslant 7$ vertices.

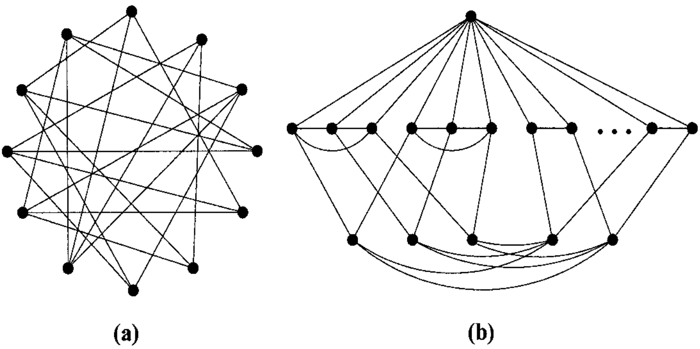

FIG. 4. 2-vertex-critical graphs on (even) $v \geqslant 12$ vertices.

LEMMA 3.4. *Let G be any graph with v vertices and ε edges. Suppose that (X, Y, W) is a partition of $V(G)$ so that the following holds*:

(i) $d(v) \geqslant 2$ *for all* $v \in X \cup Y$.

(ii) $d(v) \geqslant 1$ *for all* $v \in W$.

(iii) $e(x, Y \cup W) \geqslant 1$ *for all* $x \in X$ *and* $e(y, X \cup W) \geqslant 1$ *for all* $y \in Y$.

(iv) *If* $W \neq \varnothing$ *then there exists* $b \in X \cup Y$ *such that* $d_{\max}(b, W) \leqslant 2$.

(v) $d_{\max}(X, Y) \leqslant 2$.

Then $\varepsilon \geqslant \frac{1}{2}(3v - 2\,|W| - 15)$. *Furthermore if* $X = \varnothing$ *or* $Y = \varnothing$ *then* $\varepsilon \geqslant \frac{1}{2}(3v - 2\,|W| - 2)$.

Proof. Assume, without loss of generality, that $|X| \leqslant |Y|$. For every vertex $y \in Y$ we can find a vertex $f(y) \in X \cup W$ such that $yf(y) \in E(G)$, by (iii). Furthermore we may assume that $f(y) \in X$ if $e(y, X) \geqslant 1$. Now denote the set of the $|Y|$ edges of type $yf(y)$ by $\mathscr{M}$. Furthermore define the following:

$$Z_x = \{z \colon e(z, X) \geqslant \tfrac{1}{2}\,|X| - 1 \text{ and } e(z, Y) \geqslant 1\}$$

$$Z = \{z \colon d(z) \geqslant \tfrac{1}{2}\,|X|\}$$

$$R = \{r \colon d(r) \leqslant 2 \text{ and } r \in W\}$$

$$Q = \{q \colon d(q) = 2\} - W$$

$$G^* = G - \mathscr{M}$$

$$U = \{u \colon d_{G^*}(u) = 0 \text{ and } u \in W\}.$$

Claim 1. $N(q) \cap Z \neq \varnothing$ for all $q \in Q$.

Let $q \in Q$ be arbitrary and let $N(q) = \{r_1, r_2\}$. Assume, without loss of generality, that $q \in Y$ (the case when $q \in X$ can be handled analogously, since $|Y| \geqslant |X|$). Now (v) ensures that $X - \{r_1, r_2\} \subseteq N(r_1) \cup N(r_2)$. This implies $d(r_1) + d(r_2) \geqslant |X|$, as $q \in Y$ and qr_1 and qr_2 are edges in G. Therefore at least one of r_1, r_2 is in Z.

An analogous proof applies for Claims 2 and 3 below, and is therefore omitted.

Claim 2. $N(y) \cap Z_x \neq \varnothing$ for all $y \in Y$ with $d(y) = 2$.

Claim 3. $d(N(x)) \geqslant |Y|$ for all $x \in X$.

Claim 4. If b is the vertex defined in (iv), then $e_{G^*}(b, Y) \geqslant |U| - 1$ and $d_{G^*}(Z_x \cup b) \geqslant |U| + \frac{1}{2} |Z_x| \, |X| - |Z_x| - 2$. Furthermore if $U \neq \varnothing$ then $b \in Y$.

Clearly, if $U = \varnothing$, then $e_{G^*}(b, Y) \geqslant |U| - 1 = 1$. So assume that $U \neq \varnothing$ and $U = \{u_1, u_2, ..., u_{|U|}\}$. Let P_{u_i}, $i = 1, 2, ..., |U|$, be a path of length at most two from b to u_i in G. By the definition of U, each P_{u_i} contains a vertex $c_i \in Y$ such that $f(c_i) = u_i$. This implies that $b \in Y$. Now the vertices c_i must be distinct and in $N_{G^*}(b) \cup \{b\}$. Therefore $e_{G^*}(b, Y) \geqslant |U| - 1$.

We now show that $d_{G^*}(Z_x \cup b) \geqslant |U| + \frac{1}{2} |Z_x| \, |X| - |Z_x| - 2$. Note that $e_{G^*}(z, X) \geqslant \frac{1}{2} |X| - 2$ and $d_{G^*}(z) \geqslant \frac{1}{2} |X| - 1$ for all $z \in Y \cap Z_x$, by the definition of $\mathscr{M}$ and Z_x. Furthermore $e_{G^*}(z, X) \geqslant \frac{1}{2} |X| - 1$ for all $z \in Z_x \cap (X \cup W)$. Therefore if $b \in Z_x$ then $d_{G^*}(b \cup Z_x) = e_{G^*}(b, X \cup Y) + d_{G^*}(Z_x - b) \geqslant |U| - 1 + \frac{1}{2} |X| - 2 + (|Z_x| - 1)(\frac{1}{2} |X| - 1) = \frac{1}{2} |Z_x| \, |X| - |Z_x| + |U| - 2$. If $b \notin Z_x$ then $d_{G^*}(b \cup Z_x) \geqslant |U| - 1 + |Z_x| \, (\frac{1}{2} |X| - 1) = \frac{1}{2} |Z_x| \, |X| - |Z_x| + |U| - 1$. This completes the proof of Claim 4.

Below we shall be looking at the graph G, unless otherwise specified. We prove the lemma by considering the following five cases.

Case 1: $|X| \leqslant 3$. By Claim 4 above and the fact that $|Y| = v - |W| - |X|$, we obtain

$$\begin{aligned} 2\varepsilon &\geqslant 2 \, |\mathscr{M}| + d_{G^*}(b) + d_{G^*}(Y - b) + d_{G^*}(W) \\ &\geqslant 2 \, |Y| + (|U| - 1) + (|Y| - 1) + (|W| - |U|) \\ &\geqslant 3(v - |W| - |X|) + |W| - 2 \\ &= 3v - 2 \, |W| - 2 - 3 \, |X|. \end{aligned}$$

Hence $\varepsilon \geqslant \frac{1}{2}(3v - 2 \, |W| - 11)$. Further, if $X = \varnothing$, then $\varepsilon \geqslant \frac{1}{2}(3v - 2 \, |W| - 2)$.

Case 2. $|X|\geqslant 4$, $|Z|\geqslant 4$ *and there is a vertex* $x\in X$ *with* $d(x)=2$. We easily obtain the inequality

$$\begin{aligned}2\varepsilon &\geqslant d(Z)+d(V(G)-Z-(R-Z))+d(R-Z)\\ &\geqslant \tfrac{1}{2}\,|Z|\,|X|+2(v-|Z|-|R-Z|)+|R-Z|\\ &\geqslant 2v-|R-Z|+\tfrac{1}{2}\,|Z|\,|X|-2\,|Z|.\end{aligned}$$

However, in the above bound $d(N(x))$ has been given a value of at most $|X|$ (as $\frac{1}{2}\,|X|\geqslant 2$), whereas we know, by Claim 3, that $d(N(x))\geqslant |Y|$. This implies

$$\begin{aligned}2\varepsilon &\geqslant 2v-|R-Z|+\tfrac{1}{2}\,|Z|\,|X|-2\,|Z|+(|Y|-|X|)\\ &\geqslant 2v-|W|+\tfrac{1}{2}\,|Z|\,|X|-2\,|Z|+(v-|W|-|X|-|X|)\\ &=3v-2\,|W|-8+\tfrac{1}{2}(|X|-4)(|Z|-4|)\\ &\geqslant 3v-2\,|W|-8.\end{aligned}$$

Case 3: $|X|\geqslant 4$, $|Z|\leqslant 3$ *and there is a vertex* $x\in X$ *with* $d(x)=2$. Note that $d(Z)\geqslant |Q|$ by Claim 1:

$$\begin{aligned}2\varepsilon &\geqslant d(V(G)-Z-(Q-Z)-(R-Z))+d(Z)+d(Q-Z)+d(R-Z)\\ &\geqslant 3(v-|Z|-|Q-Z|-|R-Z|)+|Q|+2\,|Q-Z|+|R-Z|\\ &\geqslant 3v-3\,|Z|-2\,|R-Z|\\ &\geqslant 3v-2\,|W|-9.\end{aligned}$$

Case 4: $|X|\geqslant 4$, $|Z_x|\geqslant 6$ *and* $d(x)\geqslant 3$ *for all* $x\in X$. By Claim 4, we obtain

$$\begin{aligned}2\varepsilon &\geqslant 2\,|\mathscr{M}|+d_{G^*}(b\cup Z_x)+d_{G^*}(Y-(b\cup Z_x))+d_{G^*}(W-Z_x)\\ &\geqslant 2\,|Y|+(|U|+\tfrac{1}{2}\,|Z_x|\,|X|-|Z_x|-2)+(|Y|-1-|Z_x\cap Y|)\\ &\quad+(|W|-|U|-|Z_x\cap W|)\\ &\geqslant 3(v-|X|-|W|)+\tfrac{1}{2}\,|Z_x|\,|X|-2\,|Z_x|-3+|W|\\ &\geqslant 3v-2\,|W|+\tfrac{1}{2}\,|Z_x|\,|X|-3\,|X|-2\,|Z_x|-3\\ &=3v-2\,|W|+\tfrac{1}{2}(|X|-4)(|Z_x|-6)-15\\ &\geqslant 3v-2\,|W|-15.\end{aligned}$$

Case 5: $|X| \geqslant 4$, $|Z_x| \leqslant 5$ *and* $d(x) \geqslant 3$ *for all* $x \in X$. By Claim 2, $e(Z_x, Q) \geqslant |Q|$ and by the definition of Z_x we obtain that $e(Z_x, X) \geqslant \frac{1}{2}|Z_x|\,|X| - |Z_x|$. This implies that $d(Z_x \geqslant |Q| + \frac{1}{2}|Z_x|\,|X| - |Z_x|$, as $Q \subseteq Y$. Furthermore as $|X| \geqslant 4$ we get

$$\begin{aligned}
2\varepsilon &\geqslant d(V(G) - Z_x - (Q - Z_x) - (R - Z_x)) + d(Z_x) + d(Q - Z_x) + d(R - Z_x)\\
&\geqslant 3(v - |Q - Z_x| - |Z_x| - |R - Z_x|) + (|Q| + \tfrac{1}{2}|Z_x|\,|X| - |Z_x|)\\
&\quad + 2\,|Q - Z_x| + |R - Z_x|\\
&\geqslant 3v - 2\,|R - Z_x| - 4\,|Z_x| + \tfrac{1}{2}|Z_x|\,|X|\\
&\geqslant 3v - 2\,|W| - 2\,|Z_x|\\
&\geqslant 3v - 2\,|W| - 10. \quad \blacksquare
\end{aligned}$$

THEOREM 3.5 *Let G be a 2-vertex-critical graph with v vertices and ε edges. Then*

$$\varepsilon \geqslant \tfrac{1}{2}(5v - 29).$$

Proof. Note that $\delta(G) \geqslant 2$. We shall consider separately the cases when $\delta(G) = 2$, $\delta(G) = 3$, $\delta(G) = 4$, and $\delta(G) \geqslant 5$.

Assume that $\delta(G) \geqslant 5$. Then $2\varepsilon \geqslant \sum_{x \in V(G)} d(x) \geqslant 5v$ and hence $\varepsilon \geqslant \frac{5}{2}v$.

Assume that $\delta(G) = 4$. Let v be a vertex of degree 4 and let $\{x_1, x_2, x_3, x_4\}$ be the neighbours of v. As the diameter of G is two, every vertex in $V(G) - \{v, x_1, x_2, x_3, x_4\}$ has an edge into $\{x_1, x_2, x_3, x_4\}$. After deleting these $v - 5$ edges (one for each $y \in V(G) - \{v, x_1, x_2, x_3, x_4\}$), the minimum degree of vertices in $V(G) - \{v, x_1, x_2, x_3, x_4\}$ is at least three. Therefore $\varepsilon \geqslant (v - 5) + \frac{3}{2}(v - 5) + |\{vx_1, vx_2, vx_3, vx_4\}| \geqslant \frac{1}{2}(5v - 17)$.

Assume that $\delta(G) = 3$. Let v be any vertex of degree 3 and let $\{x_1, x_2, x_3\}$ be the neighbours of v. Without loss of generality assume that $d_{G-v}(x_2, x_3) \geqslant 3$. Furthermore, assume, without loss of generality, that $x_1x_3 \notin E(G)$ (otherwise $x_1x_2 \notin E(G)$, as $d_{G-v}(x_2x_3 \geqslant 3)$. Since $d_{G-v}(x_2x_3) \geqslant 3$, we can partition $V(G) - \{v, x_1x_2x_3\}$ into the following five sets:

$$\begin{aligned}
X &= N(x_3) - N(x_1)\\
Y_1 &= N(x_2) - N(x_1)\\
W &= N(x_1) \cap N(x_3) - v\\
Y_2 &= N(x_1) \cap N(x_2) - v\\
Y_3 &= N(x_1) - N(x_2) - N(x_3).
\end{aligned}$$

We note that $Y_1 \neq \emptyset$ and $X \neq \emptyset$, by the criticality of x_2 and x_3, respectively. Let x^* be any vertex in X and let G' be the graph obtained from $G - \{v, x_1, x_2, x_3\}$ by adding edges $\{x^*y\colon y \in Y_2, x^*y \notin E(G)\}$. Denote $Y = Y_1 \cup Y_2 \cup Y_3$. Now it can be checked that the graph G' together with its vertex set partitioned into the sets X, Y, W satisfies the conditions of Lemma 3.4. Hence $e(G') \geqslant \frac{1}{2}(3v(G') - 2\,|W| - 15)$. Therefore

$$
\begin{aligned}
\varepsilon &\geqslant \varepsilon(G') - |Y_2| + |X| + |Y_1| + 2\,|W| + 2\,|Y_2| + |Y_3| + |\{vx_1, vx_2, vx_3\}| \\
&\geqslant \tfrac{1}{2}(3(v-4) - 15) + |X| + |Y| + |W| + 3 \\
&\geqslant \tfrac{1}{2}(5(v-4) - 15 + 6) \\
&= \tfrac{1}{2}(5v - 29).
\end{aligned}
$$

Assume that $\delta(G) = 2$. Let v be any vertex of degree 2 and let $\{x_1, x_2\}$ be the neighbours of v. Clearly, we must have $d_{G-v}(x_1, x_2) \geqslant 3$. Thus $V(G) - \{v, x_1, x_2\}$ is partitioned into two sets $X = N(x_2) - v$ and $Y = N(x_1) - v$. Let S consists of all vertices in $V(G) - \{v, x_1, x_2\}$ of degree two in G. We claim that either $S \subseteq X$ or $S \subseteq Y$ or G is a 5-cycle. Indeed, if this is not true, then there exist three vertices s_x, s_y, z with $s_x \in S \cap X$, $s_y \in S \cap Y$ and $z \in X \cup Y - S$. Without loss of generality assume that $z \in X$. If $s_x s_y \in E(G)$, then there is no path of length at most two connecting s_y and z in G, a contradiction. So assume that $s_x s_y \notin E(G)$. Since s_x (resp. s_y) is adjacent to at least one vertex in Y (resp. X), there is no path of length at most two connecting s_x and s_y in G. As our theorem is true when G is a 5-cycle, we assume, without loss of generality, that $S \subseteq Y$.

Let $S' = N(S) - \{x_1\}$. The criticality of vertices of S ensures that $S' \subseteq X$, $|S| = |S'|$ and $e(Y - S, S') = 0$. Since $d(s) = 2$ for each $s \in S$, we obtain that $e(S, Y - S) = 0$ and $e(S, X - S') = 0$. Since every vertex in S has to have a path of length at most two to every vertex in X, we see that every vertex $s \in S'$ is adjacent to all vertices in $X - s$. We now consider separately the two cases: $X - S' = \emptyset$ and $X - S' \neq \emptyset$. Write $s = |S| = |S'|$.

Assume first that $X - S' \neq \emptyset$. We note that $Y - S \neq \emptyset$ as every vertex in $x \in X - S'$ must have an edge to $Y - S$ (because G is critical with respect to x and $X - S'$ has no edges to S). Let $G' = G - S - S' - \{v, x_1, x_2\}$ and let W be all the vertices in G' with degree one in G' (note that $W \subseteq X - S'$). It is easy to check that G' with the partition $V(G') = (X - S' - W, Y - S, W)$ satisfies statements (i)–(v) in Lemma 3.4. Hence $\varepsilon(G') \geqslant (3\varepsilon(G') - 2\,|W| - 15)$. As there are $s + \frac{1}{2}s(s-1)$ vertices in $G\langle S \cup S' \rangle$ and $s\,|X - S'|$ edges between S' and $(X - S')$, we obtain

$$\begin{aligned}
\varepsilon &\geqslant \tfrac{1}{2}(3(v-3-2s)-2\,|W|-15)+s+\tfrac{1}{2}s(s-1)\\
&\quad +s\,|X-S'|+|X|+|Y|+|\{vx_1, vx_2\}|\\
&=\tfrac{1}{2}(3v-9-15)+\tfrac{1}{2}(-6s+2s+s(s-1)+2(s-1)\,|X-S'|)\\
&\quad +|X-S'|-|W|+(v-3)+2\\
&=\tfrac{1}{2}(5v-26)+\tfrac{1}{2}((s-1)\,(s+2\,|X-S'|-4)-4)+|X-S'|-|W|\\
&\geqslant \tfrac{1}{2}(5v-30)+\tfrac{1}{2}(s-1)(s-2)+|X-S'|-|W|.
\end{aligned}$$

If $W \neq X-S'$ then $W \subset X-S'$ and $\varepsilon \geqslant \frac{1}{2}(5v-28)$. If $W = X-S'$ then $\varepsilon(G') \geqslant \frac{1}{2}(3v(G')-2\,|W|-2)$, by Lemma 3.4. Using the same calculations as above, we have $\varepsilon \geqslant \frac{1}{2}(5v-17)$.

Assume now that $X-S' = \varnothing$. Then $Y-S=\varnothing$ as any vertex in $Y-S$ should have had an edge into X (and there are no edges between $Y-S$ and S'). However this implies that $v=2s+3$ and $\varepsilon = 2+s+s+s+|E(G\langle S'\rangle)| = 3s+2+\frac{1}{2}s(s-1)$, as $G\langle S'\rangle$ is a complete graph. This implies that $\varepsilon = \frac{1}{2}(5v-10s-15+6s+4+s(s-1)) = \frac{1}{2}(5v+(s-2)(s-3)-17) \geqslant \frac{1}{2}(5v-17)$. ∎

Corollary 3.6. *For each $v \geqslant 5$ except $v=6$,*

$$f_v \geqslant \tfrac{1}{2}(5v-29).$$

Combining Theorem 3.3 and Corollary 3.6, we have the following:

Corollary 3.7. *For each $v \geqslant 5$ except $v=6$,*

$$\tfrac{1}{2}(5v-29) \leqslant f_v \leqslant \tfrac{1}{2}(5v-12).$$

Note (Added January 1998). After submission, it was brought to our attention that Ando and Egawa [1] proved, independently, that $f_v \geqslant (5n-17)/2$.

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