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Integrability Conditions, Wave Functions, and Conservation Laws for the Relativistic Schrödinger Equations.

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Summary. — The theory of the relativistic Schrödinger equations is further elaborated: the integrability conditions for the existence of a wave function $\psi(x)$ directly lead to the general situation where a wave function («pure state») is not available to describe the physical system.

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1. – Introduction.

According to the modern viewpoint in elementary-particle physics, the success of the gauge theories [1] is due to the fact that they describe the micro-world in a dualistic way: on the one hand, there are particles (to be described by matter fields ψ) and on the other hand there are interactions of the particles (to be described by gauge fields $\mathcal{A}$). For both fields $\{\psi, \mathcal{A}\}$ one postulates a coupled system of equations, obeying some symmetry principles, and if this is done adequately one observes that one can successfully describe a large set of microphysical phenomena, or even all possible phenomena («theories of everything» [2,3]). Clearly, there is no general consense about these far-reaching theories, but the common conviction is that all the different kinds of fundamental interactions should be unified into one single force («Grand Unified Theories» [4]). But when one looks at these unification attempts, one readily recognizes that they mainly refer to only one side of the dualistic pair $\{\psi, \mathcal{A}\}$, namely to the interaction part $\mathcal{A}$. Here, all kinds of interactions have been found to obey one single type of equation of motion, namely the Maxwell equations and its non-Abelian generalizations. As a consequence, the interaction field $\mathcal{A}$ has acquired a unified mathematical meaning for all kinds of interactions ($\rightsquigarrow$ «fibre bundle connection» [5]) such that the different interactions are operating merely in the corresponding subbundles.

However, the situation is not so satisfactory when one looks at the different matter fields ψ. Here, it is a striking fact that there is no generally valid counterpart

of the Maxwell type of equations. Amazingly enough the gauge principle works as well with a Dirac field (obeying a *first-order* equation of motion) as with a Klein-Gordon field (obeying a *second-order* equation of motion). Therefore, the question arises whether one can find a single but generally valid equation of motion for all the different matter fields. This equation would then be formally the same for bosons and fermions, and it would govern the motion of all the different kinds of material particles in a similar way as the (generalized) Maxwell equations govern the dynamics of all the interaction particles (photons, gluons, etc.).

Such a unified equation of motion for matter has recently been proposed [6] and looks as follows:

$$i\hbar c \mathscr{D}_\mu \psi = \mathscr{H}_\mu \psi \,. \tag{1.1}$$

Here, the «wave function» $\psi(x)$ is a section of an N-dimensional complex vector bundle $\mathscr{V}_N$ over pseudo-Riemannian space-time (for the sake of generality), the «Hamiltonian» $\mathscr{H}_\mu$ is a $\mathscr{G}L(N, \boldsymbol{C})$-valued 1-form, and $\mathscr{D}_\mu$ is the gauge-covariant derivative

$$\mathscr{D}_\mu \psi := \partial_\mu \psi + \mathscr{A}_\mu \psi \,. \tag{1.2}$$

By imposing certain constraints upon the Hamiltonian $\mathscr{H}_\mu$, the solutions $\psi(x)$ of the «relativistic Schrödinger equation» (1.1) can be made to obey Dirac's equation

$$i\hbar c \gamma^\mu \mathscr{D}_\mu \psi = Mc^2 \psi \tag{1.3}$$

or the Klein-Gordon equation

$$\mathscr{D}^\mu \mathscr{D}_\mu \psi + \left(\frac{Mc}{\hbar} \right)^2 \psi = 0 \,. \tag{1.4}$$

But before writing down these constraints, let us first mention the integrability condition ensuring the existence of solutions $\psi(x)$ to the basic eq. (1.1); this is given by

$$\mathscr{D}_\mu \mathscr{H}_\nu - \mathscr{D}_\nu \mathscr{H}_\mu + \frac{i}{\hbar c} [\mathscr{H}_\mu, \mathscr{H}_\nu] = i\hbar c \, \mathscr{F}_{\mu\nu} \,, \tag{1.5}$$

where $\mathscr{F}_{\mu\nu}$ is the curvature of the connection $\mathscr{A}_\mu$,

$$\mathscr{F}_{\mu\nu} = \partial_\mu \mathscr{A}_\nu - \partial_\nu \mathscr{A}_\mu + [\mathscr{A}_\mu, \mathscr{A}_\nu] . \tag{1.6}$$

Clearly, the integrability condition (1.5) is the same for all types of matter fields ψ (bosons and fermions), but the other constraints upon the Hamiltonian $\mathscr{H}_\mu$ will be different for different kinds of matter. For instance, if we want to describe Dirac particles we obviously have to impose upon $\mathscr{H}_\mu$ the «Dirac condition»

$$\gamma^\mu \cdot \mathscr{H}_\mu = Mc^2 \cdot \mathbf{1} \tag{1.7}$$

in order that any solution $\psi(x)$ of the relativistic Schrödinger equation (1.1) also fits into the Dirac equation (1.3). Similarly, in order to obey the Klein-Gordon equation

(1.4) one has to require

$$\mathscr{D}^{\mu}\mathscr{H}_{\mu} - \frac{i}{\hbar c}\mathscr{H}^{\mu}\cdot\mathscr{H}_{\mu} = -i\hbar c\left(\frac{Mc}{\hbar}\right)^2\cdot\mathbf{1}, \tag{1.8}$$

which then also guarantees charge conservation

$$\nabla^{\mu}j_{\mu} = 0 \tag{1.9}$$

and therefore is called the «conservation equation». For the Dirac case, the corresponding conservation equation is deduced from the integrability condition (1.5) together with the Dirac condition (1.7) and is found to contain an additional polarization term (Σ) with respect to the Klein-Gordon case, namely

$$\mathscr{D}^{\mu}\mathscr{H}_{\mu} - \frac{i}{\hbar c}\mathscr{H}^{\mu}\cdot\mathscr{H}_{\mu} = -i\hbar c\left(\frac{Mc}{\hbar}\right)^2\cdot\mathbf{1} - i\hbar c\Sigma^{\mu\nu}\cdot\mathscr{F}_{\mu\nu}\,. \tag{1.10}$$

Now, it will appear as a matter of fact that a lot of questions arise with the emergence of a new wave equation, such as the relativistic Schrödinger equation (1.1). Consequently, the concern of the present paper necessarily aims at a further investigation of the *mathematical* properties and implications of the new wave equation (the more *physical* implications have been mentioned elsewhere [7]). Especially the following questions are treated in some detail:

i) Is there any mathematical framework for describing *many-particle* situations?

ii) Is the integrability condition (1.5) *necessary and sufficient* for the general situation?

iii) If the existence of solutions can be guaranteed, what is their *general shape*?

iv) What kind of *conservation laws* are admitted? (Charge, energy-momentum, etc.)

Subsequently, these questions shall be answered in great detail and in short the results are the following:

1'1. *Many particles.* – The very nature of the relativistic Schrödinger equations suggests that the many-particle systems are to be described by a fibre bundle $\mathscr{Y}_N$ which is the Whitney *sum* of the individual particle bundles (not the tensor *product* of standard quantum theory!). Thus, N scalar bosons are described by an N-component wave function $\psi(x)$, or N Dirac particles by a $(4N)$-component wave function ψ, etc.

1'2. *Integrability conditions.* – It turns out that the integrability condition (1.5) is necessary but not sufficient for the existence of such an N-component wave function $\psi(x)$ as solution to the basic equation (1.1). However, the supplementary condition, ensuring also the sufficiency, is readily found (by means of bundle trivialization) and consists in the requirement that the anti-Hermitian part of the Hamiltonian $\mathscr{H}_{\mu}$ commutes with itself (see eq. (2.8) below).

1·3. *Wave functions.* – The general form of the N-component wave function ψ is the usual one of ordinary quantum mechanics, namely there is the «modulus» ($\mathscr{L}$) and the «phase» ($\mathscr{Z}$) of ψ (see eq. (3.14) below). However, both the modulus $\mathscr{L}$ and the phase $\mathscr{Z}$ are space-time–dependent ($N \times N$)-matrices with their product acting on a *constant* reference function ϕ'. Moreover, the «phase factor» $\mathscr{Z}$ is an element of the unitary group $U(N)$ in N dimensions. As in standard quantum mechanics, the wave function $\psi(x)$ is used to obtain the local density $\Delta(x) = \bar{\psi}(x) \cdot \delta \cdot \psi(x)$ of some «observable» δ attributed to the N-particle system. Apart from the above «bracket definition» of the density $\Delta(x)$, one can find this density also through an equivalent «trace definition» $\Delta(x) = \mathrm{tr}(\mathscr{L}\bar{\mathscr{L}} \cdot \delta)$, where $\bar{\mathscr{L}}$ is the Hermitian conjugate of the «modulus» $\mathscr{L}$. In the latter form it becomes obvious that the phase factor $\mathscr{Z}$ does not enter the densities.

1·4. *Conservation laws.* – Concerning the conservation laws, the relativistic Schrödinger equations (1.1) admit two principally distinct kinds, which divide all the material particles into two categories: *Klein-Gordon particles* and *Dirac particles.* These two categories roughly coincide with the standard categories of bosons and fermions, but the distinguishing feature refers here to the specific way in which the conservation of charge is attained. Concretely, in order to formulate some conservation law within the present framework, one needs an operator δ («observable»), which produces the associated density Δ along the procedure mentioned above, and then the conservation law is mathematically expressed in form of a vanishing source for the corresponding density. For example, for charge conservation one needs a velocity operator v_μ, then the current density is defined as $j_\mu = \bar{\psi} \cdot v_\mu \cdot \psi \equiv \mathrm{tr}(\mathscr{L}\bar{\mathscr{L}} \cdot v_\mu)$, and finally this density is required to have vanishing source: $\nabla^\mu j_\mu = 0$! Now it is just the velocity operator v_μ which establishes the two different kinds of matter (Dirac *vs.* Klein-Gordon). Whereas for the Klein-Gordon category the velocity operator is a true dynamical object depending upon the Hamiltonian $\mathscr{H}_\mu$, the Dirac category is based upon a non-dynamical velocity operator ($v_\mu \to \gamma_\mu$) which constitutes an *absolute object* of the theory.

2. – Change of the connection.

The point of departure to obtain the complete integrability conditions for the relativistic Schrödinger equations is a change of the connection $\mathscr{A}_\mu$ in the vector bundle $\mathscr{V}_N$. To this end, we split up the Hamiltonian $\mathscr{H}_\mu$ into its anti-Hermitian parts $\mathscr{K}_\mu (= \overline{\mathscr{K}_\mu})$ and $\mathscr{L}_\mu (= \overline{\mathscr{L}_\mu})$ in the following way:

$$\mathscr{H}_\mu = \hbar c(\mathscr{K}_\mu + i\mathscr{L}_\mu), \tag{2.1}$$

with the Hermitian part $\mathscr{K}_\mu$ being given by

$$\mathscr{K}_\mu = \frac{1}{2\hbar c}(\mathscr{H}_\mu + \overline{\mathscr{H}_\mu}) \tag{2.2}$$

and, analogously, the anti-Hermitian part is built up by the (Hermitian!) object $\mathscr{L}_\mu$,

$$\mathscr{L}_\mu = \frac{1}{2i\hbar c}(\mathscr{H}_\mu - \overline{\mathscr{H}_\mu}). \tag{2.3}$$

In the 1-dimensional case $N = 1$ (single scalar particle), the Hermitian part $\mathscr{K}_\mu$ reduces to the ordinary 1-form k_μ which has been called the «kinetic field» [6] and is responsible for the non-local wave properties of the particle [7]. Similarly, the $\mathscr{G}L(N, \boldsymbol{C})$-valued 1-form $\mathscr{L}_\mu$ is the N-particle generalization of the «localization field» l_μ, which is also an ordinary 1-form for a single scalar particle and describes its localization properties. Thus, the pair $\{\mathscr{K}_\mu, \mathscr{L}_\mu\}$ is a pair of «complementary» variables for the system; but in our present approach to quantum mechanics we do not postulate some commutation relation for these complementary variables, rather we try to deduce a system of coupled differential equations for them (see below).

In the next step, we use the kinetic field $\mathscr{K}_\mu$ in order to introduce a new connection $\overset{\circ}{\mathscr{A}}_\mu$ in the complex vector bundle $\mathscr{V}_N$ through

$$\overset{\circ}{\mathscr{A}}_\mu = \mathscr{A}_\mu + i\mathscr{K}_\mu \equiv \mathscr{A}_\mu + \frac{i}{\hbar}\mathscr{P}_\mu . \tag{2.4}$$

Here we have used a generalized de Broglie relationship between the «wave vector» $\mathscr{K}_\mu$ and the momentum 1-form $\mathscr{P}_\mu$

$$\mathscr{P}_\mu := \hbar\mathscr{K}_\mu . \tag{2.5}$$

The meaning of such a change of connection becomes obvious, when we compute now the curvature $\overset{\circ}{\mathscr{F}}_{\mu\nu}$ of the new connection:

$$\overset{\circ}{\mathscr{F}}_{\mu\nu} := \partial_\mu \overset{\circ}{\mathscr{A}}_\nu - \partial_\nu \overset{\circ}{\mathscr{A}}_\mu + [\overset{\circ}{\mathscr{A}}_\mu, \overset{\circ}{\mathscr{A}}_\nu] = \mathscr{F}_{\mu\nu} + i[\mathscr{D}_\mu \mathscr{K}_\nu - \mathscr{D}_\nu \mathscr{K}_\mu] + [\mathscr{K}_\mu, \mathscr{K}_\nu] . \tag{2.6}$$

Indeed, by exploiting the Hermitian part of the integrability condition (1.5) we readily find here

$$\overset{\circ}{\mathscr{F}}_{\mu\nu} = -[\mathscr{L}_\mu, \mathscr{L}_\nu] \tag{2.7}$$

which properly is a $\mathscr{G}\mathscr{L}^-(N, \boldsymbol{C})$-valued 2-form. Thus, the new curvature $\overset{\circ}{\mathscr{F}}_{\mu\nu}$ exceeds now the original gauge algebra $\mathscr{G}$, which is a subalgebra of $\mathscr{G}\mathscr{L}^-(N, \boldsymbol{C})$, and the question is: what has been gained through that extension of the connection 1-form? (Observe that the algebra $\mathscr{G}\mathscr{L}(N, \boldsymbol{C})$ is the direct (real) sum of its Hermitian subspace $\mathscr{G}\mathscr{L}^+(N, \boldsymbol{C}) = \overline{\mathscr{G}\mathscr{L}^+(N, \boldsymbol{C})}$ and of its anti-Hermitian subalgebra $\mathscr{G}\mathscr{L}^-(N, \boldsymbol{C}) = -\overline{\mathscr{G}\mathscr{L}^-(N, \boldsymbol{C})}$; the latter is isomorphic to the Lie algebra $\mathscr{U}(N)$ of the unitary group $U(N)$ in N dimensions). Now, the answer is that we *require* the new connection $\overset{\circ}{\mathscr{A}}_\mu$ to be *trivial, i.e.* its curvature $\overset{\circ}{\mathscr{F}}_{\mu\nu}$ (2.7) *must* vanish,

$$[\mathscr{L}_\mu, \mathscr{L}_\nu] \equiv 0 . \tag{2.8}$$

Thus, by resorting to the new connection we did not really enlarge the holonomy group but, on the contrary, we contracted it to unity; and we shall readily show that the postulate (2.8), together with the eq. (1.5), is the necessary and sufficient integrability condition for the relativistic Schrödinger equation (1.1).

However, before proceeding to the proof of the completeness of this integrability condition, we want to make a brief side-step to the meaning of the connection change (2.4). As it stands, a connection is a geometric object in some fibre bundle, but it has

also a direct physical meaning because it is closely related to the physical notion of «vector potential» $\boldsymbol{A}_\mu$, being considered as the carrier of the interaction under consideration. The relationship between the geometric and the physical object is

$$\mathscr{A}_\mu = \frac{i}{\hbar c}\,\mathscr{Q}\cdot\boldsymbol{A}_\mu\,, \tag{2.9}$$

where $\mathscr{Q}$ is the «charge operator» ($\rightsquigarrow$ matrix of coupling constants), which is a covariantly constant object

$$\mathscr{D}_\mu\mathscr{Q}\equiv 0 \tag{2.10}$$

commuting with the gauge algebra $\mathscr{G}$, *i.e.*

$$[\mathscr{Q},\mathscr{A}_\mu]=0\,. \tag{2.11}$$

Thus, the change of connection (2.4) $\mathscr{A}_\mu\to\mathring{\mathscr{A}}_\mu$ may also be considered as the well-known procedure of «minimal coupling»

$$\mathscr{P}_\mu\to\mathscr{P}_\mu+\frac{\mathscr{Q}}{c}\cdot\boldsymbol{A}_\mu \tag{2.12}$$

which is used when one wants to couple a particle (either classical or quantum-mechanical) to an external force field. Consequently, the deeper meaning of the «minimal coupling» is nothing else than to generate a *trivial* vector potential!

3. – Construction of the wave function.

With the help of the new connection $\mathring{\mathscr{A}}_\mu$ it is now easy to construct the wave function $\psi(x)$ *explicitly*, and the feasibility of such a construction is equivalent to the completeness of the integrability conditions (1.5) and (2.8).

For a demonstration of this assertion let us first re-write the relativistic Schrödinger equation (1.1) by use of the new connection as

$$\mathring{\mathscr{D}}_\mu\psi=\mathscr{L}_\mu\cdot\psi\qquad(\mathring{\mathscr{D}}_\mu:=\partial_\mu+\mathring{\mathscr{A}}_\mu) \tag{3.1}$$

and from here one readily deduces the following expression for the new curvature $\mathring{\mathscr{F}}_{\mu\nu}$ in terms of the localization field $\mathscr{L}_\mu$:

$$\mathring{\mathscr{F}}_{\mu\nu}=\mathring{\mathscr{D}}_\mu\mathscr{L}_\nu-\mathring{\mathscr{D}}_\nu\mathscr{L}_\mu-[\mathscr{L}_\mu,\mathscr{L}_\nu]\qquad(\mathring{\mathscr{D}}_\mu\mathscr{L}_\nu:=\partial_\mu\mathscr{L}_\nu+[\mathring{\mathscr{A}}_\mu,\mathscr{L}_\nu]-\Gamma^\lambda_{\nu\mu}\mathscr{L}_\lambda)\,. \tag{3.2}$$

However, our result (2.7) immediately implies here

$$\mathring{\mathscr{D}}_\mu\mathscr{L}_\nu-\mathring{\mathscr{D}}_\mu\mathscr{L}_\nu=0\,, \tag{3.3}$$

which is the higher-dimensional ($N>1$) generalization of the gradient condition for a single particle ($N=1$: $\mathscr{L}_\mu\to l_\mu$) in flat space-time [6]:

$$\partial_\mu l_\nu-\partial_\nu l_\mu=0\rightsquigarrow l_\mu=\frac{\partial_\mu l}{l}\,. \tag{3.4}$$

Observe also that the generalized gradient condition (3.3) automatically ensures the validity of the Bianchi identity

$$\overset{\circ}{\mathscr{D}}_\lambda \overset{\circ}{\mathscr{F}}_{\mu\nu} + \overset{\circ}{\mathscr{D}}_\mu \overset{\circ}{\mathscr{F}}_{\nu\lambda} + \overset{\circ}{\mathscr{D}}_\nu \overset{\circ}{\mathscr{F}}_{\lambda\mu} \equiv 0 \tag{3.5}$$

even if the integrability constraint (2.8) is not applied!

However, we want to insist on that constraint and try to introduce now the scalar localization field $\mathscr{L}(x)$ quite analogously to the 1-dimensional case ($N = 1$) of eq. (3.4), *i.e.* we put

$$\mathscr{L}_\mu = (\overset{\circ}{\mathscr{D}}_\mu \mathscr{L}) \cdot \mathscr{L}^{-1} \qquad (\overset{\circ}{\mathscr{D}}_\mu \mathscr{L} := \partial_\mu \mathscr{L} + [\overset{\circ}{\mathscr{A}}_\mu, \mathscr{L}]) . \tag{3.6}$$

Clearly, we must assume here that the matrix $\mathscr{L}$ has an inverse element, but the situation is quite similar to the 1-dimensional case (3.4), *i.e.* we can admit certain submanifolds of space-time where $\mathscr{L}$ becomes singular (corresponding to the zeros of $l(x)$ for $N = 1$, cf. also the remarks below (3.14)) Now we have to check whether the gradient ansatz (3.6) is consistent with the gradient condition (3.3) which implies

$$\overset{\circ}{\mathscr{D}}_\mu \mathscr{L}_\nu - \overset{\circ}{\mathscr{D}}_\nu \mathscr{L}_\mu = - \mathscr{L} \cdot \overset{\circ}{\mathscr{F}}_{\mu\nu} \cdot \mathscr{L}^{-1} . \tag{3.7}$$

Thus, the conclusion is that the localization vector $\mathscr{L}_\mu$ indeed can be deduced from the localization scalar $\mathscr{L}(x)$ whenever the curvature $\overset{\circ}{\mathscr{F}}_{\mu\nu}$ vanishes, *i.e.* when the integrability condition (2.8) holds! In this way, we recognize clearly that our demand for the existence of the localization scalar $\mathscr{L}(x)$ is the true motivation for contracting the holonomy group of the new connection $\overset{\circ}{\mathscr{A}}_\mu$ into unity!

Once the existence of the localization field $\mathscr{L}(x)$ has been guaranteed, one can readily make further progress towards the construction of the wave function $\psi(x)$. Indeed, inserting the gradient ansatz (3.6) into eq. (3.1) yields a simpler equation for ψ, namely

$$\overset{\circ}{\mathscr{D}}_\mu (\mathscr{L}^{-1} \cdot \psi) = 0 . \tag{3.8}$$

This is a very nice result, because it says that in the *trivial* vector bundle $\overset{\circ}{\mathscr{V}}_N$ (being equipped with the *trivial* connection $\overset{\circ}{\mathscr{A}}_\mu$) there exists a covariantly constant section $\phi(x)$

$$\phi = \mathscr{L}^{-1} \cdot \psi . \tag{3.9}$$

(The existence of a constant section of trivial vector bundles is assured by quite general theorems [8].) Furthermore, it is well known that a trivial connection $\overset{\circ}{\mathscr{A}}_\mu$ is always a pure gauge:

$$\overset{\circ}{\mathscr{A}}_\mu = \mathscr{Z}^{-1} \cdot \partial_\mu \mathscr{Z} , \qquad \mathscr{Z} \in U(N) \tag{3.10}$$

and therefore can be gauged off:

$$\overset{\circ}{\mathscr{A}}'_\mu = \mathscr{Z} \cdot \overset{\circ}{\mathscr{A}}_\mu \cdot \mathscr{Z}^{-1} + \mathscr{Z} \cdot \partial_\mu \mathscr{Z}^{-1} \equiv 0 . \tag{3.11}$$

Thus in the new gauge, the wave function reads

$$\phi' = \mathcal{Z} \cdot \phi \tag{3.12}$$

and is constant in the ordinary sense

$$\partial_\nu \phi' \equiv 0 \rightsquigarrow \phi' = \text{const}. \tag{3.13}$$

Collecting the results now, we see that the explicit construction of the wave function ψ is always possible, provided both integrability conditions (1.5) and (2.8) are satisfied; moreover, the general form of the wave function was found as

$$\psi(x) = \mathcal{L}(x) \cdot \mathcal{Z}^{-1}(x) \cdot \phi' \,. \tag{3.14}$$

(This result also holds if $\mathcal{L}$ is a singular matrix). Here ϕ' is a constant element of the typical fibre $\boldsymbol{C}^N$, $\mathcal{Z}(x)$ is an element of the (pseudo-) unitary group $U(N)$ in N dimensions ($\overline{\mathcal{Z}} = \mathcal{Z}^{-1}$) and $\mathcal{L}(x)$ is an $(N \times N)$-matrix, or more precisely: these objects are sections of the appropriate fibre bundles. Thus, if the integrability conditions are respected, the relativistic Schrödinger equation (1.1) admits solutions $\psi(x)$ of the form (3.14) which we take now as a solid foundation for the description of quantum phenomena.

Observe also that the present arguments, leading to the final form of the wave function $\psi(x)$ (3.14), remain valid if $\mathcal{L}(x)$ should turn out as a singular matrix ($\rightsquigarrow \det \mathcal{L} = 0$). The reason is that one can still use the preceding equations containing the inverse matrix $\mathcal{L}^{-1}$ (such as, *e.g.*, (3.6)), however in a form multiplied through by $\mathcal{L}$. For instance, eq. (3.6) would then read

$$\overset{\circ}{\mathcal{D}}_\mu \mathcal{L} = \mathcal{L}_\mu \cdot \mathcal{L} \tag{3.15}$$

which may be considered as the differential equation for the determination of the localization scalar $\mathcal{L}(x)$, when the Hamiltonian $\mathcal{H}_\mu$ (2.1) is already known.

But now the question arises, whether it is possible to base quantum mechanics upon the Hamiltonian $\mathcal{H}_\mu$ alone, whose equations of motion are just the integrability conditions (1.5), (2.8) and the conservation equation (1.8), respectively (1.10). Why do we need a wave function at all in quantum mechanics, when most of the information inherent in that wave function $\psi(x)$ is already present in the Hamiltonian $\mathcal{H}_\mu$ (respectively in the kinetic field $\mathcal{H}_\mu(x)$ and the localization field $\mathcal{L}_\mu(x)$) to be considered as the proper dynamical variable?

4. – Particle-wave duality.

Before we study the question of the necessity of a wave function, we want first to elaborate somewhat more the importance of the Hamiltonian $\mathcal{H}_\mu$ as a dynamical object competing in its relevance with the wave function ψ. In this way, we can develop a better understanding for the interrelationship of both objects ψ and $\mathcal{H}_\mu$ and for their interplay during the production of the typical quantum effects.

According to Bohr [9], the strangeness of the quantum world is due to the fact that two classically irreconciliable features are simultaneously embraced by any quantum object, namely the properties of being both a localized particle and an extended wave train. In non-relativistic quantum mechanics, one tries to account for this fact by

building up the dynamics of the system upon Heisenberg's commutation relation for canonically conjugate variables, such as the position and momentum operators, for any particle. However, the preliminary character of this procedure is best demonstrated by the fact that it is not clear at all how to generalize this dualistic view to *composite* objects in the *relativistic* domain!

In our present approach, there is no difficulty in this respect. In a very natural way, there arise two complementary variables, namely the localization field $\mathscr{L}_\mu$ (2.3) and the kinetic field $\mathscr{K}_\mu$ (2.2) which are the (anti)-Hermitian parts of the Hamiltonian $\mathscr{H}_\mu$ (2.1) and they together build up the wave function ψ (3.14) such that the localization field $\mathscr{L}$ is the «modulus» of ψ and the kinetic field $\mathscr{K}_\mu$ enters the «phase factor» $\mathscr{Z}$ (see the 1-dimensional case $N = 1$ in ref. [6]). Clearly, the entanglement of both complementary aspects into the single object ψ must lead to certain interpretative difficulties with some quantum effects, which, however, could possibly be avoided by keeping apart the complementary subsystems from one another during the computations. This means that we should rather look for the dynamical equations for $\mathscr{K}_\mu$ and $\mathscr{L}$ in place of that for the wave function ψ in order to better understand the localization problems in quantum theory (as a simple demonstration hereof, see the treatment of the Bohm-Aharonov effect [7]).

In this sense, we deduce now from the Hamiltonian dynamics (1.5) and (1.10) together with the supplementary condition (2.8) the field equations for our complementary variables and find for the localization field $\mathscr{L}_\mu$,

$$(4.1)\quad \overset{\circ}{\mathscr{D}}{}^\mu \mathscr{L}_\mu + \mathscr{L}^\mu \mathscr{L}_\mu + \left(\frac{\mathscr{M} c^2}{\hbar c}\right)^2 - \mathscr{K}^\mu \cdot \mathscr{K}_\mu = i[\mathscr{K}^\mu, \mathscr{L}_\mu] - \frac{1}{2}\{\Sigma^{\mu\nu}, \mathscr{F}_{\mu\nu}\},$$

where $\{,\}$ is the anti-commutator as usual

$$(4.2)\qquad \{\Sigma^{\mu\nu}, \mathscr{F}_{\mu\nu}\} \equiv \Sigma^{\mu\nu} \cdot \mathscr{F}_{\mu\nu} + \mathscr{F}_{\mu\nu} \cdot \Sigma^{\mu\nu}.$$

Or, if we prefer to work with the localization scalar $\mathscr{L}$ (3.6) in favour of the 1-form $\mathscr{L}_\mu$, we get

$$(4.3)\quad \overset{\circ}{\mathscr{D}}{}^\mu \overset{\circ}{\mathscr{D}}_\mu \mathscr{L} + \left(\left[\frac{\mathscr{M} c^2}{\hbar c}\right]^2 - \mathscr{K}^\mu \cdot \mathscr{K}_\mu\right) \cdot \mathscr{L} = \left(i[\mathscr{K}^\mu, \mathscr{L}_\mu] - \frac{1}{2}\{\Sigma^{\mu\nu}, \mathscr{F}_{\mu\nu}\}\right) \cdot \mathscr{L}.$$

Quite similarly, one may exploit the Hermitian part of the Hamitonian dynamics for establishing the field equations for the kinetic field $\mathscr{K}_\mu(x)$. First, the conservation equation (1.10) yields here

$$(4.4)\qquad \overset{\circ}{\mathscr{D}}{}^\mu \mathscr{K}_\mu = -\{\mathscr{L}^\mu, \mathscr{K}_\mu\} - \frac{1}{2} i[\Sigma^{\mu\nu}, \mathscr{F}_{\mu\nu}]$$

and then the integrability condition (1.5) readily leads to

$$(4.5)\qquad \overset{\circ}{\mathscr{D}}_\mu \mathscr{K}_\nu - \overset{\circ}{\mathscr{D}}_\nu \mathscr{K}_\mu - i[\mathscr{K}_\mu, \mathscr{K}_\nu] = i\mathscr{F}_{\mu\nu}.$$

In this way we got a set of disentangled but coupled field equations (4.1)-(4.5) for the complementary variables $\mathscr{K}_\mu$ and $\mathscr{L}$, which hold over any (pseudo-)Riemannian space-time and thus there is no longer any difficulty with the notion of complementarity of composite objects in the relativistic domain! Quite similarly to

the case of a single scalar boson [6], we expect also for a system of many particles that the kinetic field $\mathscr{K}_\mu$ is non-zero *far away* from the support of the localization field $\mathscr{L}$. Thus, the typical non-local phenomena of quantum mechanics must also be expected for multi-particle systems (*e.g.*, Bohm-Aharonov effect for a ionized heavy atom). Surely, the most interesting question is here the transition into classical physics for a great number of particles ($N \gg 1$) which may cluster together and form a «dust grain» being held together by its self-interaction forces $\mathscr{F}_{\mu\nu}$ and being then subject as a whole to the classical laws of motion. Though the current state of the art does not admit the rigorous treatment of such difficult problems, it is nevertheless satisfying to observe that the present approach on principle is capable of describing such clustering effects. Here, the clusters may even decouple from one another, but the particles within a cluster may remain coherent: it is expected that for such a situation the Hamiltonian $\mathscr{H}_\mu$ (or equally well $\mathscr{K}_\mu$, $\mathscr{L}_\mu$) become *block diagonalized* so that any cluster is described by the corresponding sub-block Hamiltonian ($\rightsquigarrow$ the Withney sum bundle of the individual *particle* bundles is reduced to the Whitney sum of the individual *cluster* bundles). For bound clusters the dualistic wave-particle properties of any such cluster will then be governed adequately by the corresponding block objects $\mathscr{K}_\mu$, $\mathscr{L}_\mu$.

5. – Densities and conservation laws.

On returning to the question of the necessity of a wave function ψ, raised at the end of sect. 3, it seems immediately obvious that the existence of such an object ψ is indispensable for defining the conservation laws. Remember that the «strongly conserved» quantities (*i.e.* charge, mass, spin etc.) never change and define the individuality of a particle; on the other hand, the «weakly conserved» quantities (such as energy-momentum etc.) are kept constant exclusively in the absence of an external force. In any case, the conservation laws stipulate what is allowed and what is forbidden.

A conservation law is usually expressed mathematically in form of a continuity equation. The most famous example is charge conservation,

$$\nabla^\mu j_\mu \equiv 0\,, \tag{5.1}$$

or, similarly, the energy-momentum conservation,

$$\nabla^\mu T_{\mu\nu} = 0\,. \tag{5.2}$$

The latter reads more generally

$$\nabla^\mu T_{\mu\nu} = \boldsymbol{f}_\nu\,, \tag{5.3}$$

where the emergence of the force density $\boldsymbol{f}_\nu$ signals the «weakness» of energy-momentum conservation. Obviously, the conservation laws are based upon certain densities, such as current density $j_\mu(x)$ or energy-momentum density $T_{\mu\nu}(x)$; and the definition of such a density just requires the use of the wave function ψ together with some operator, understood here as a $\mathscr{G}\mathscr{L}(N, \boldsymbol{C})$-valued form. For instance, the current density j_μ for a general N-particle system is constructed by use of the velocity

operator v_μ as

$$j_\mu = \bar{\psi} \cdot v_\mu \cdot \psi \,, \tag{5.4}$$

or, analogously, the energy-momentum density is built upon the energy-momentum operator $\mathscr{T}_{\mu\nu}$ via

$$T_{\mu\nu} = \bar{\psi} \cdot \mathscr{T}_{\mu\nu} \cdot \psi \,. \tag{5.5}$$

Thus, we arrive at the preliminary result that *the existence of a wave function ψ is required by the conservation laws* and therefore we have to insist upon the validity of those integrability conditions mentioned above (sect. **1, 2**) (see below for the case of doing without any wave function).

Consequently, after having appropriately defined the physical densities carried by the wave field $\psi(x)$, one has to ensure the validity of the desired conservation laws and this requirement will yield some condition upon the corresponding operators. For example, the charge conservation (5.1) imposes the following requirement upon the velocity operator v_μ [6]:

$$\mathscr{D}_\mu v^\mu + \frac{i}{\hbar c} [\overline{\mathscr{H}_\mu} \cdot v^\mu - v^\mu \cdot \mathscr{H}_\mu] = 0 \,, \tag{5.6}$$

which may be recast also into the following alternative form by use of the trivial connection $\overset{\circ}{\mathscr{A}}_\mu$:

$$\overset{\circ}{\mathscr{D}}_\mu v^\mu + \{\mathscr{L}^\mu, v_\mu\} = 0 \,. \tag{5.7}$$

In a similar way, the weak conservation law (5.3) yields for the energy-momentum operator $\mathscr{T}_{\mu\nu}$

$$\mathscr{D}^\mu \mathscr{T}_{\mu\nu} + \frac{i}{\hbar c} [\overline{\mathscr{H}^\mu} \cdot \mathscr{T}_{\mu\nu} - \mathscr{T}_{\mu\nu} \cdot \mathscr{H}^\mu] = f_\nu \,, \tag{5.8}$$

where the force operator f_ν is a $\mathscr{GL}(N, \boldsymbol{C})$-valued 1-form and is related to the force density $\boldsymbol{f}_\nu$ of eq. (5.3) as usual:

$$\boldsymbol{f}_\nu = \bar{\psi} \cdot f_\nu \cdot \psi \,. \tag{5.9}$$

Summarizing results, we see that we can deduce conservation laws for the relativistic Schrödinger equations (1.1) if we already know the Hamiltonian $\mathscr{H}_\mu$, so that we can find the solutions for the corresponding operators, such as, for instance, v_μ from eq. (5.6). For instance, the velocity operators v_μ for a Dirac particle have been found to agree with the Dirac matrices γ_μ which are covariantly constant ($\mathscr{D}_\mu \gamma_\nu \equiv 0$) so that the operator equation (5.6) is solved by the Dirac condition (1.7). Now we want to express the densities also in terms of the matrices $\mathscr{L}$ and $\mathscr{Z}$ through which the general wave function $\psi(x)$ (3.14) had been parametrized.

For that purpose, observe first that the field equation for an arbitrary (real) density $\Delta_\bullet(x)$ due to the corresponding Hermitian operator $\delta_\bullet(x)$ (the dot denotes the

totality of indices) reads

$$\nabla_\mu \Delta_\bullet = \bar{\psi}\left(\mathscr{D}_\mu \delta_\bullet + \frac{i}{\hbar c} [\overline{\mathscr{K}}_\mu \cdot \delta_\bullet - \delta_\bullet \cdot \mathscr{K}_\mu] \right) \psi . \tag{5.10}$$

A special case hereof are the above-mentioned continuity equations, *e.g.*, $\delta_\bullet \to v_\mu$, $\Delta_\bullet \to j_\mu$. This field equation is understood in the sense that the operator derivative $\mathscr{D}_\mu \delta_\bullet$ and the commutator $[\overline{\mathscr{K}}_\mu \cdot \delta_\bullet - \delta_\bullet \cdot \mathscr{K}_\mu]$ are written as a sum of the other operators ($= \mathscr{G}\mathscr{L}(N, \boldsymbol{C})$-valued forms), yielding other densities for which then the analogous derivative may be written down and one ends up with a coupled first-order system for such a set of densities. This system will have a unique solution after one has prescribed the «initial values» of those densities at some reference point x_{in} of space-time. We assume here without loss of generality that this set of $N \times N = N^2$ «initial densities» $\Delta_{\bullet \text{in}} = \Delta_\bullet(x_{\text{in}})$ is «*complete*» in the sense that any possible initial value ψ_{in} at the chosen reference point x_{in} leads to a unique set of initial densities

$$\mathbf{br}\colon \psi_{\text{in}} \overset{(\delta_{\bullet \text{in}})}{\to} \Delta_{\bullet \text{in}} . \tag{5.11}$$

The invertible map **br**, introduced herewith up to an irrelevant $\mathscr{U}(1)$ phase factor, signals that the way from the wave function to the densities is performed by «bracketing» the corresponding operator $\delta_\bullet$ by ψ, such as demonstrated in (5.4), (5.5), or (5.9), respectively.

However, when the wave function is given by the matrices $\mathscr{L}$, $\mathscr{Z}$ (3.14) rather than in the form of a complex vector $\psi(x)$, it is suggestive to use the trace map **tr**,

$$\mathbf{tr}\colon \{\mathscr{Z}, \mathscr{L}\} \overset{(\delta_\bullet)}{\to} \Delta_\bullet \tag{5.12}$$

rather than the bracket map **br** (5.11), *i.e.* we put

$$\Delta_\bullet(x) = \operatorname{tr}(\mathscr{I} \cdot \delta_\bullet(x)) . \tag{5.13}$$

Here, the *intensity matrix* $\mathscr{I}$ has to be determined in terms of the matrices $\mathscr{L}$, $\mathscr{Z}$ (3.14) in such a way that both sets of densities coincide exactly

$$\Delta_\bullet(x) = \bar{\psi}(x) \cdot \delta_\bullet(x) \cdot \psi(x) \equiv \operatorname{tr}(\mathscr{I}(x) \cdot \delta_\bullet(x)) . \tag{5.14}$$

Our guess for the (Hermitian!) intensity matrix is

$$\mathscr{I} = \mathscr{L} \cdot \bar{\mathscr{L}} . \tag{5.15}$$

Observe that the replacement $\mathscr{L} \to \mathscr{L} \cdot \mathscr{Z}^{-1}$ leaves the intensity matrix $\mathscr{I}$ (5.15) invariant and therefore the unitary factor $\mathscr{Z}^{-1}$ in the wave function (3.14) must bear redundant physical significance; a similar argument applies also for a gauge transformation

$$\mathscr{L} \to \mathscr{S}^{-1} \cdot \mathscr{L} \cdot \mathscr{S} , \tag{5.16a}$$

$$\mathscr{Z} \to \mathscr{Z} \cdot \mathscr{S} , \tag{5.16b}$$

$$\delta_\bullet \to \mathscr{S}^{-1} \cdot \delta_\bullet \cdot \mathscr{S} . \tag{5.16c}$$

But now we have to reassure that our guess (5.15) was really consistent. This is readily recognized in two steps: first, the field equation for $\mathscr{L}$ (3.15) is used to obtain

the field equation for the intensity matrix $\mathscr{I}$,

$$\mathscr{D}_\mu \mathscr{I} = -\frac{i}{\hbar c}[\mathscr{H}_\mu \cdot \mathscr{I} - \mathscr{I} \cdot \overline{\mathscr{H}_\mu}], \tag{5.17}$$

which may be considered as the relativistic analogue of the non-relativistic density matrix equation in quantum statistics [10]. If this result is used for the derivative of the density $\Delta_\bullet$ (5.13) we get

$$\nabla_\mu \Delta_\bullet(x) = \mathrm{tr}\left\{\mathscr{I} \cdot \left(\mathscr{D}_\mu \delta_\bullet + \frac{i}{\hbar c}[\overline{\mathscr{H}_\mu} \cdot \delta_\bullet - \delta_\bullet \cdot \mathscr{H}_\mu]\right)\right\} \tag{5.18}$$

in isomorphic agreement with the bracket definition of the density derivatives (5.10). Thus, the conclusion is that the trace densities (5.13) obey the same first-order field equations as do the bracket densities and, consequently, both densities are identical over the whole space-time if they coincide in some single reference point x_{in}. However, in the second step, this coincidence of both densities in a single point can always be achieved by exploiting the arbitrariness of the constant reference state ϕ' of eq. (3.13). This completes the proof of equivalence of the bracket and trace definitions (5.11) and (5.12) for the physical densities!

It may seem now that the physical content of both the bracket and trace approaches is exactly the same. However, one is readily convinced that *the trace approach is the far more general one*; the reason is that it provides us with the possibility of dispensing with the existence of a wave function ψ and thus it enables us to describe quantum systems which do not possess any wave function at all!

For the description of such a more general quantum system, being subject to the (external or internal) forces $\mathscr{F}_{\mu\nu}$, we merely need the Hamiltonian $\mathscr{H}_\mu$ as a solution of the integrability condition (1.5) and the conservation equation (1.8) (or (1.10), respectively) but the supplementary condition (2.8) must not necessarily be satisfied. Consequently, a wave function ψ must not necessarily exist. Nevertheless, the physical densities $\Delta_\bullet(x)$ (as the carrier of all the physical information about the quantum system) may well exist if we first solve the equation of motion (5.17) for the intensity matrix $\mathscr{I}$ and then compute the desired densities $\Delta_\bullet(x)$ by means of the trace prescription (5.13). Clearly, these densities will then obey the equations of motion (5.18) without any reference to the notion of a wave function!

However, in the special case where a wave function ψ exists, there must be some constraint for the densities $\Delta_\bullet(x)$ as a consequence of the additional condition (2.8). What kind of constraint is this? Let us demonstrate the answer for the special situation of Dirac's spinor field where these constraints in question are the well-known «*Fierz identities*»: it has been observed at various occasions [11-13] that the following identities hold:

$$j^\mu j_\mu = -\tilde{j}^\mu \tilde{j}_\mu = \rho^2 + \tilde{\rho}^2, \tag{5.19a}$$

$$j^\mu \tilde{j}_\mu = 0, \tag{5.19b}$$

$$S_{\mu\nu} = \frac{1}{4}\frac{\tilde{\rho}}{\rho^2 + \tilde{\rho}^2}(j_\mu \tilde{j}_\nu - j_\nu \tilde{j}_\mu) - \frac{1}{4}\frac{\rho}{\rho^2 + \tilde{\rho}^2}\varepsilon_{\mu\nu\lambda\sigma} j^\lambda \tilde{j}^\sigma, \tag{5.19c}$$

if these densities of Dirac's spinor field are defined in the usual way:

$$
\left\{\begin{array}{lll}
a) & \text{scalar:} & \rho = \bar{\psi}\cdot\mathbf{1}\cdot\psi \equiv \mathrm{tr}(\mathscr{T}\cdot\mathbf{1}), \\
b) & \text{pseudo-scalar:} & \tilde{\rho} = \bar{\psi}\cdot\varepsilon\cdot\psi \equiv \mathrm{tr}(\mathscr{T}\cdot\varepsilon) \\
 & & \left(\varepsilon := \frac{1}{4!}\,\varepsilon_{\mu\nu\lambda\sigma}\gamma^{\mu}\gamma^{\nu}\gamma^{\lambda}\gamma^{\sigma}\right); \\
c) & \left\{\begin{array}{l}\text{vector:} \\ \text{axial vector:}\end{array}\right. & \begin{array}{l} j_{\mu} = \bar{\psi}\cdot\gamma_{\mu}\cdot\psi \equiv \mathrm{tr}(\mathscr{T}\cdot\gamma_{\mu}), \\ \tilde{j}_{\mu} = i\bar{\psi}\cdot\tilde{\gamma}_{\mu}\cdot\psi \equiv \mathrm{tr}(\mathscr{T}\cdot i\tilde{\gamma}_{\mu}) \\ (\tilde{\gamma}_{\mu} = \varepsilon\cdot\gamma_{\mu}); \end{array} \\
d) & \text{polarization:} & S_{\mu\nu} = \frac{i}{2}\,\bar{\psi}\cdot\Sigma_{\mu\nu}\cdot\psi \equiv \mathrm{tr}\left(\mathscr{T}\cdot\frac{i}{2}\,\Sigma_{\mu\nu}\right) \\
 & & \left(\Sigma_{\mu\nu} := \frac{1}{4}\,[\gamma_{\mu},\,\gamma_{\nu}]\right).
\end{array}\right. \tag{5.20}
$$

In fact, one is easily convinced that the $4\times 4 = 16$ operators $\{\mathbf{1}, \varepsilon, \gamma_{\mu}, i\tilde{\gamma}_{\mu}, (i/2)\Sigma_{\mu\nu}\}$ form a complete set in the sense of the bracket definition (5.11). Now the original question was: What property of the intensity matrix $\mathscr{T}$ ensures the Fierz identities (5.19*a*)-(5.19*c*)? But the answer is now trivial: since a wave function ψ *exists*, the equivalence of the bracket and trace definitions (5.20) demand that the intensity matrix $\mathscr{T}$ is (proportional to) a projector:

$$\mathscr{T} = \psi\otimes\bar{\psi} \tag{5.21}$$

or, putting without loss of generality ($\rho > 0$),

$$\psi = \sqrt{\rho}\,\hat{\psi} \qquad (\bar{\hat{\psi}}\cdot\hat{\psi} = 1), \tag{5.22}$$

one finds

$$\mathscr{T} = \rho\hat{\mathscr{T}}, \qquad \hat{\mathscr{T}}^2 = \hat{\mathscr{T}} \qquad (\hat{\mathscr{T}} := \hat{\psi}\otimes\bar{\hat{\psi}}). \tag{5.23}$$

On the other hand, the trace definitions (5.20) specify the intensity matrix $\mathscr{T}$ as

$$\mathscr{T} = \frac{1}{4}\,(\rho\cdot\mathbf{1} - \tilde{\rho}\cdot\varepsilon + j^{\mu}\gamma_{\mu} - i\tilde{j}^{\mu}\tilde{\gamma}_{\mu} + 4iS^{\mu\nu}\Sigma_{\mu\nu}). \tag{5.24}$$

Thus, introducing this form into the projector property $\hat{\mathscr{T}}^2 = \hat{\mathscr{T}}$ (5.23) and comparing the left- and right-hand sides with respect to the linearly independent Clifford-algebra elements, one just arrives at the Fierz identities (5.19*a*)-(5.19*c*).

Summarizing results, we see that whenever the quantum system is in a «pure state», then the densities have to obey the «Fierz identities» which originally were discovered for Dirac's spinor field ($N = 4$) but evidently must exist for any fibre dimension N.

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