

# Nonlinear Feedback Control of Parabolic Partial Differential Equation Systems with Time-dependent Spatial Domains

Antonios Armaou and Panagiotis D. Christofides

*Department of Chemical Engineering, University of California, Los Angeles,  
California 90095-1592*

E-mail: [pd@seas.ucla.edu](mailto:pd@seas.ucla.edu)

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This paper proposes a methodology for the synthesis of nonlinear finite-dimensional time-varying output feedback controllers for systems of quasi-linear parabolic partial differential equations (PDEs) with time-dependent spatial domains, whose dynamics can be partitioned into slow and fast ones. Initially, a nonlinear model reduction scheme, similar to the one introduced in Christofides and Daoutidis, *J. Math. Anal. Appl.* **216** (1997), 398–420, which is based on combinations of Galerkin's method with the concept of approximate inertial manifold is employed for the derivation of low-order ordinary differential equation (ODE) systems that yield solutions which are close, up to a desired accuracy, to the ones of the PDE system, for almost all times. Then, these ODE systems are used as the basis for the explicit construction of nonlinear time-varying output feedback controllers via geometric control methods. The controllers guarantee stability and enforce the output of the closed-loop parabolic PDE system to follow, up to a desired accuracy, a prespecified response for almost all times, provided that the separation of the slow and fast dynamics is sufficiently large. Differences in the nature of the model reduction and control problems between parabolic PDE systems with fixed and moving spatial domains are identified and discussed. The proposed control method is used to stabilize an unstable steady state of a diffusion-reaction process whose spatial domain changes with time. It is shown to lead to a significant reduction on the order of the stabilizing nonlinear output feedback controller and outperform a nonlinear controller synthesis method that does not account for the variation of the spatial domain. © 1999 Academic Press

**Key Words:** nonlinear model reduction; nonlinear control; diffusion-reaction processes with moving boundary.

## 1. INTRODUCTION

There is a large number of industrial control problems which involve highly nonlinear transport-reaction processes with moving boundaries such as, crystal growth, metal casting, gas-solid reaction systems, and coatings. In these processes, nonlinear behavior typically arises from complex reaction mechanisms and their Arrhenius dependence on temperature, while motion of boundaries is usually a result of phase change, such as chemical reaction, mass and heat transfer, and melting or solidification. The mathematical models of transport-reaction processes with moving boundaries are usually obtained from the dynamic conservation equations and consist of nonlinear parabolic partial differential equations (PDEs) with time-dependent spatial domains.

Research on control of linear-quasi-linear parabolic PDEs has been extensive in the past and has mainly focused on systems with fixed spatial domains (see, for example, the review papers [3, 11, 21] and the references therein). The main feature of parabolic PDE systems is that the eigenspectrum of the spatial differential operator can be partitioned into a finite-dimensional slow one and an infinite-dimensional stable fast complement [15]. This implies that the dominant dynamic behavior of such systems can be approximately described by finite-dimensional systems. Therefore, the standard approach to the control of linear-quasi-linear parabolic PDE systems (e.g., [2, 8]) involves the application of the standard Galerkin's method to the parabolic PDE system to derive ODE systems that accurately describe the dominant dynamics of the PDE system, which are subsequently used as the basis for controller synthesis. The main disadvantage of this approach is that the number of modes that should be retained to derive an ODE system which yields the desired degree of approximation, may be very large (e.g., [1, 5]), thereby leading to complex controller synthesis and high dimensionality of the resulting controllers.

These controller synthesis and implementation problems, together with the need to develop computationally efficient numerical solution algorithms for nonlinear parabolic PDEs, have motivated extensive research efforts on the problem of deriving low-order ODE systems that accurately reproduce the dynamics and solutions of nonlinear parabolic PDEs. The concept of inertial manifold (IM) has provided a natural framework for addressing this problem [27]. If it exists, an IM is a positively invariant, exponentially attracting, finite-dimensional Lipschitz manifold. The IM is an appropriate tool for model reduction because if the trajectories of the PDE system are on the IM, then this system is exactly described by a low-order ODE system (called inertial form). Unfortunately, even for PDE systems for which an IM is known to exist, the computation of the closed-form expression of the IM (and therefore the derivation of the

corresponding inertial form) is a formidable task. Motivated by this, various approaches have been proposed in the literature for the construction of approximations of the inertial manifold (called approximate inertial manifolds (AIMs)) (see, for example, [12–14]). The AIMs are subsequently used for the derivation of approximations of the inertial form that accurately reproduce the solutions and dynamics of the parabolic PDE system.

In the context of control of parabolic PDE systems with fixed spatial domain, the concept of inertial manifold has been used: (a) in [6, 7], to determine the extent to which linear boundary proportional control influences the dynamic and steady-state closed-loop response of a nonlinear parabolic PDE system, and (b) in [20, 24, 25], to address the problem of stabilization of a parabolic PDE with boundary low-order linear output feedback control; while the concept of approximate inertial manifold has been used in [12], to synthesize nonlinear low-order output feedback controllers that enforce closed-loop stability and output tracking in quasi-linear parabolic PDE systems with distributed control action.

Despite the progress on nonlinear control of parabolic PDE systems with fixed spatial domains, few results are available on control and estimation of parabolic PDE systems with time-dependent spatial domains. In this area, important contributions include Wang's work on the synthesis of linear optimal controllers (e.g., [28, 29]) and their application to temperature and thermal gradient regulation in crystal growth processes [30], as well as the synthesis of nonlinear distributed state estimators using stochastic methods in [23].

This paper focuses on a broad class of quasi-linear parabolic PDE systems with time-dependent spatial domains whose dynamics can be partitioned into slow and fast ones. Such systems arise naturally in the modeling of diffusion-reaction processes with moving boundaries. The objective is to develop a general method for the synthesis of low-order (and therefore, practically implementable) nonlinear time-varying output feedback controllers that enforce stability and output tracking in the closed-loop system.

The paper is structured as follows: Initially, the class of parabolic PDE systems considered in the article is given and is formulated as an evolution equation in an appropriate Hilbert space. Then, a nonlinear model reduction scheme, similar to the one introduced in [12], which is based on combination of Galerkin's method with the concept of approximate inertial manifold is employed for the derivation of ODE systems that yield solutions which are close, up to a desired accuracy, to the ones of the PDE system, for almost all times. Then, these ODE systems are used as the basis for the explicit construction of nonlinear time-varying output feedback controllers via geometric control methods. The controllers guarantee stability and enforce the output of the closed-loop parabolic PDE system

to follow, up to a desired accuracy, a prespecified response for almost all times, provided that the separation of the slow and fast dynamics is sufficiently large. Differences in the nature of the model reduction and control problems between parabolic PDE systems with fixed and moving spatial domains are identified and discussed. Finally, the proposed control method is applied to a typical diffusion-reaction process whose spatial domain changes with time, and is shown to lead to a significant reduction on the order of the stabilizing controller, as well as to outperform a nonlinear controller synthesis method which does not account for the variation of the spatial domain.

## 2. PRELIMINARIES

### 2.1. Parabolic Partial Differential Equation Systems with Time-dependent Spatial Domain

We consider quasi-linear parabolic PDE systems with time-dependent spatial domains with the following state-space description,

$$\begin{aligned}\frac{\partial \bar{x}}{\partial t} &= A \frac{\partial \bar{x}}{\partial z} + B \frac{\partial^2 \bar{x}}{\partial z^2} + wb(z, t)u + f(t, \bar{x}), \\ y_{c_i} &= \int_0^{l(t)} c_i(z, t) k \bar{x} dz, \quad i = 1, \dots, l, \\ y_{m_\kappa} &= \int_0^{l(t)} s_\kappa(z, t) \omega \bar{x} dz, \quad \kappa = 1, \dots, q\end{aligned}\tag{1}$$

subject to the boundary conditions,

$$\begin{aligned}C_1 \bar{x}(0, t) + D_1 \frac{\partial \bar{x}}{\partial z}(0, t) &= R_1, \\ C_2 \bar{x}(l(t), t) + D_2 \frac{\partial \bar{x}}{\partial z}(l(t), t) &= R_2,\end{aligned}\tag{2}$$

and the initial condition,

$$\bar{x}(z, 0) = \bar{x}_0(z),\tag{3}$$

where the rate of change of the length of the domain,  $l(t)$ , is governed by the following ordinary differential equation,

$$\frac{dl}{dt} = \mathcal{G} \left( t, l, \int_0^{l(t)} \bar{a} \left( z, t, l, \bar{x}, \frac{\partial \bar{x}}{\partial z} \right) dz \right), \quad (4)$$

where  $\bar{x}(z, t) = [\bar{x}_1(z, t) \cdots \bar{x}_n(z, t)]^T$  denotes the vector of state variables,  $[0, l(t)] \subset \mathbb{R}$  is the domain of definition of the process,  $z \in [0, l(t)]$  is the spatial coordinate,  $t \in [0, \infty)$  is the time,  $u = [u_1 \ u_2 \ \cdots \ u_l]^T \in \mathbb{R}^l$  denotes the vector of manipulated inputs,  $y_{c_i} \in \mathbb{R}$  denotes the  $i$ th controlled output, and  $y_{m_\kappa} \in \mathbb{R}$  denotes the  $\kappa$ th measured output.  $\partial \bar{x} / \partial z$ ,  $\partial^2 \bar{x} / \partial z^2$  denote the first- and second-order spatial derivatives of  $\bar{x}$ ,  $f(t, \bar{x})$ ,  $\mathcal{G}(t, l, \int_0^{l(t)} \bar{a}(z, t, l, \bar{x}, \frac{\partial \bar{x}}{\partial z}) dz)$  are nonlinear vector functions,  $\bar{a}(z, t, l, \bar{x}, \frac{\partial \bar{x}}{\partial z})$  is a nonlinear scalar function,  $k, w, \omega$  are constant vectors,  $A, B, C_1, D_1, C_2, D_2$  are constant matrices,  $R_1, R_2$  are column vectors, and  $\bar{x}_0(z)$  is the initial condition.

$b(z, t)$  is a known smooth vector function of  $(z, t)$  of the form  $b(z, t) = [b_1(z, t) \ b_2(z, t) \ \cdots \ b_l(z, t)]$ , where  $b_i(z, t)$  describes how the control action  $u_i(t)$  is distributed in the interval  $[0, l(t)]$  (e.g., point-distributed actuation),  $c_i(z, t)$  is a known smooth function of  $(z, t)$  which is determined by the desired performance specifications in the interval  $[0, l(t)]$  (e.g. regulation of the entire temperature profile of a crystal or regulation of the temperature at a specific point), and  $s_\kappa(z, t)$  is a known smooth function of  $(z, t)$  which is determined by the location and type of the  $\kappa$ th measurement sensor (e.g., point-distributed sensing). In the case of point actuation (i.e., the control action enters the system at a single point  $z_0$ , with  $z_0 \in [0, l(t)]$ ), the function  $b_i(z, t)$  is taken to be nonzero in a finite spatial interval of the form  $[z_0 - \epsilon, z_0 + \epsilon]$ , where  $\epsilon$  is a small positive real number, and zero elsewhere in  $[0, l(t)]$ . We note that in contrast to the case of parabolic PDE systems defined on a fixed spatial domain [12], we allow the actuator, performance specification, and measurement sensor functions to depend explicitly on time (i.e., moving control actuators and objectives, and measurement sensors). The value of using moving control actuators and sensors in certain applications is illustrated in the example of Section 5. Throughout the paper, we use the order of magnitude notation  $O(\epsilon)$  (i.e.,  $\delta(\epsilon) = O(\epsilon)$  if there exist positive real numbers  $k_1$  and  $k_2$  such that  $|\delta(\epsilon)| \leq k_1|\epsilon|$ ,  $\forall |\epsilon| < k_2$ ).

To simplify the notation of this article, we assume that  $l(t)$  is a known and smooth function of time and develop the proposed nonlinear control method on the basis of the following parabolic PDE system with time-dependent spatial domain (the derivation of the results for systems of the

form of Eqs. (2)–(4) is conceptually similar),

$$\begin{aligned}\frac{\partial \bar{x}}{\partial t} &= A \frac{\partial \bar{x}}{\partial z} + B \frac{\partial^2 \bar{x}}{\partial z^2} + wb(z, t)u + f(t, \bar{x}), \\ y_{c_i} &= \int_0^{l(t)} c_i(z, t) k \bar{x} dz, \quad i = 1, \dots, l, \\ y_{m_\kappa} &= \int_0^{l(t)} s_\kappa(z, t) \omega \bar{x} dz, \quad \kappa = 1, \dots, q\end{aligned}\tag{5}$$

subject to the boundary conditions,

$$\begin{aligned}C_1 \bar{x}(0, t) + D_1 \frac{\partial \bar{x}}{\partial z}(0, t) &= R_1, \\ C_2 \bar{x}(l(t), t) + D_2 \frac{\partial \bar{x}}{\partial z}(l(t), t) &= R_2,\end{aligned}\tag{6}$$

and the initial condition:

$$\bar{x}(z, 0) = \bar{x}_0(z).\tag{7}$$

Our assumptions on the properties of  $l(t)$  are precisely stated below:

*Assumption 1.*  $l(t)$  is a known smooth (i.e.,  $\dot{l}$  exists and is bounded,  $\forall t \in [0, \infty)$ ) function of time which satisfies  $l(t) \in (0, l_{\max}]$ ,  $\forall t \in [0, \infty)$ , where  $l_{\max}$  denotes the maximum length of the spatial domain.

## 2.2. Formulation of the Parabolic Partial Differential Equation System in Hilbert Space

We formulate the system of Eqs. (5)–(7) in a Hilbert space  $\mathcal{H}(t)$  consisting of  $n$ -dimensional vector functions defined on  $[0, l(t)]$  that satisfy the boundary conditions of Eq. (6), with inner product and norm,

$$\begin{aligned}(\omega_1, \omega_2) &= \int_0^{l(t)} (\omega_1(z), \omega_2(z))_{\mathbb{R}^n} dz, \\ \|\omega_1\|_2 &= (\omega_1, \omega_1)^{1/2},\end{aligned}\tag{8}$$

where  $\omega_1, \omega_2$  are two elements of  $\mathcal{H}(t)$  and the notation  $(\cdot, \cdot)_{\mathbb{R}^n}$  denotes the standard inner product in  $\mathbb{R}^n$ . To this end, we define the state function  $x$  on  $\mathcal{H}(t)$  as

$$x(t) = \bar{x}(z, t), \quad t > 0, z \in [0, l(t)],\tag{9}$$

the time-varying operator,

$$\mathcal{A}(t)x = A \frac{\partial \bar{x}}{\partial z} + B \frac{\partial^2 \bar{x}}{\partial z^2} - \dot{l}(t) \frac{\partial \bar{x}}{\partial z},$$

$$x \in D(\mathcal{A}) = \left\{ x \in \mathcal{H}(t) : C_1 \bar{x}(0, t) + D_1 \frac{\partial \bar{x}}{\partial z}(0, t) = R_1, \right. \quad (10)$$

$$\left. C_2 \bar{x}(l(t), t) + D_2 \frac{\partial \bar{x}}{\partial z}(l(t), t) = R_2 \right\},$$

and the input, controlled output, and measurement operators as

$$\mathcal{B}(t)u = wb(t)u, \quad \mathcal{C}(t)x = (c(t), kx), \quad \mathcal{S}(t)x = (s(t), \omega x), \quad (11)$$

where  $c(t) = [c_1(t) \ c_2(t) \ \cdots \ c_l(t)]^T$  and  $s(t) = [s_1(t) \ s_2(t) \ \cdots \ s_q(t)]^T$ , and  $c_i(t) \in \mathcal{H}(t)$ ,  $s_k(t) \in \mathcal{H}(t)$ . The system of Eqs. (5)–(7) can then be written as

$$\begin{aligned} \dot{x} &= \mathcal{A}(t)x + \mathcal{B}(t)u + f(t, x), & x(0) &= x_0, \\ y_c &= \mathcal{C}(t)x, & y_m &= \mathcal{S}(t)x, \end{aligned} \quad (12)$$

where  $f(t, x(t)) = f(t, \bar{x}(z, t))$  and  $x_0 = \bar{x}_0(z)$ . We assume that the nonlinear term  $f(t, x)$  satisfies  $f(t, 0) = 0$  and is also locally Lipschitz continuous uniformly in  $t$ ; i.e., there exist positive real numbers  $a_0, K_0$  such that for any  $x_1, x_2 \in \mathcal{H}(t)$  that satisfy  $\max\{\|x_1\|_2, \|x_2\|_2\} \leq a_0$ , we have that:

$$\|f(t, x_1) - f(t, x_2)\|_2 \leq K_0 \|x_1 - x_2\|_2, \quad \forall t \in [0, \infty). \quad (13)$$

*Remark 1.* In the formulation of the PDE system of Eqs. (5)–(7) in  $\mathcal{H}(t)$ , the time-varying term  $-\dot{l}(t)(\partial \bar{x}/\partial z)$  in the expression of  $\mathcal{A}(t)$  (Eq. (10)) accounts for convective transport owing to the motion of the domain. This term was not present in the expression of the differential operator in the case of parabolic PDE systems with fixed spatial domains (where  $\dot{l}(t) \equiv 0$ ), and makes  $\mathcal{A}(t)$  an explicit function of time.

### 2.3. Singular Perturbation Formulation

In this subsection, we precisely state our assumption that the dynamics of the infinite-dimensional system of Eq. (12) can be partitioned into slow (which are finite dimensional) and fast (which are infinite dimensional) ones. We note that this assumption is usually satisfied by most diffusion-reaction processes (see, for example, [8, 10] and the application of Section 5). Assumption 2 that follows states our requirement.

*Assumption 2.* Let  $\{\phi_j(t)\}$ ,  $j = 1, \dots, \infty$ , be an orthogonal and countable,  $\forall t \in [0, \infty)$ , basis of  $\mathcal{H}(t)$ . Let also  $\mathcal{H}_s(t) = \text{span}\{\phi_1(t), \phi_2(t), \dots, \phi_m(t)\}$  and  $\mathcal{H}_f(t) = \text{span}\{\phi_{m+1}(t), \phi_{m+2}(t), \dots\}$  be two modal subspaces of  $\mathcal{H}(t)$  such that  $\mathcal{H}_s(t) \oplus \mathcal{H}_f(t) = \mathcal{H}(t)$ , and define the orthogonal (pointwise in time) projection operators  $P_s: \mathcal{H}(t) \rightarrow \mathcal{H}_s(t)$  and  $P_f: \mathcal{H}(t) \rightarrow \mathcal{H}_f(t)$  so that the state  $x$  of the system of Eq. (12) can be written as  $x = x_s + x_f = P_s x + P_f x$ . Using this decomposition for  $x$ , we assume that the system of Eq. (12) can be written in the following singularly perturbed form,

$$\begin{aligned} \frac{dx_s}{dt} &= \mathcal{A}_s(t)x_s + \mathcal{B}_s(t)u + f_s(t, x_s, x_f), & x_s(0) &= P_s x_0, \\ \epsilon \frac{\partial x_f}{\partial t} &= \mathcal{A}_{f\epsilon}(t)x_f + \epsilon \mathcal{B}_f(t)u + \epsilon f_f(t, x_s, x_f), & x_f(0) &= P_f x_0, \\ y_c &= \mathcal{C}(t)(x_s + x_f), & y_m &= \mathcal{D}(t)(x_s + x_f), \end{aligned} \quad (14)$$

where  $\mathcal{A}_s(t) = P_s \mathcal{A}(t)$ ,  $\mathcal{B}_s(t) = P_s \mathcal{B}(t)$ ,  $f_s(t, x_s, x_f) = P_s f(t, x) + P_s \mathcal{A}(t)x_f$ ,  $\mathcal{A}_{f\epsilon}(t) = \epsilon P_f \mathcal{A}(t)$  is an unbounded differential operator,  $\mathcal{B}_f(t) = P_f \mathcal{B}(t)$ ,  $f_f(t, x_s, x_f) = P_f f(t, x) + P_f \mathcal{A}(t)x_s$ ,  $\epsilon \ll 1$  and the operators  $\mathcal{A}_s(t)$ ,  $\mathcal{A}_{f\epsilon}(t)$  generate semigroups with growth rates which are of the same order of magnitude.

The notation  $\partial x_f / \partial t$  is used to denote that the state  $x_f$  belongs in an infinite-dimensional space.

*Remark 2.* The statement “the operators  $\mathcal{A}_s(t)$ ,  $\mathcal{A}_{f\epsilon}(t)$  generate semigroups with growth rates which are of the same order of magnitude” in Assumption 2 means that the solutions  $x_s$  and  $x_f$  of the systems  $\dot{x}_s = \mathcal{A}_s(t)x_s$  and  $\dot{x}_f = \mathcal{A}_{f\epsilon}(t)x_f$ , respectively, satisfy:  $|x_s(t)| \leq K|x_s(0)|e^{k_1 t}$ ,  $\|x_f(t)\|_2 \leq K\|x_f(0)\|_2 e^{k_2 t}$ , where  $K, k_1, k_2$  are real numbers, and  $O(k_1) = O(k_2)$ . The norm notation  $|x_s|$  denotes that  $x_s$  belongs in a finite-dimensional Hilbert space. This assumption ensures that the system of Eq. (14) is in the standard singularly perturbed form (the reader may refer to the classic book [19] for results on singular perturbations).

*Remark 3.* In Assumption 2, the basis,  $\{\phi_j(t)\}$ ,  $j = 1, \dots, \infty$ , of  $\mathcal{H}(t)$  can be chosen from standard basis functions sets, or it can be computed by solving an eigenvalue problem of the form  $\mathcal{A}(t)\phi_j(t) = \lambda_j(t)\phi_j(t)$  or by applying Karhunen–Loève expansion on an appropriately chosen ensemble of solutions of the system of Eq. (12) (see [5, 8] for details on Karhunen–Loève expansion). The terms  $P_s \mathcal{A}(t)x_f$  in  $f_s(t, x_s, x_f)$  and  $P_f \mathcal{A}(t)x_s$  in  $f_f(t, x_s, x_f)$  account for the use of, possibly arbitrary, basis function sets.



*Remark 4.* Assumption 2 can be thought of as a natural generalization of the following one (which was introduced in [12] for parabolic PDE systems with fixed spatial domains): “the eigenspectrum of the spatial differential operator,  $\mathcal{A}$  (i.e.,  $\mathcal{A}(t)$  with  $\dot{l} \equiv 0$ ), can be partitioned into a finite set of eigenvalues which are close to the imaginary axis, and an infinite set of eigenvalues which are far in the left-half plane,” to parabolic PDE systems with time-dependent spatial domains. Furthermore, for parabolic PDE systems with fixed spatial domains,  $\epsilon$  was defined as  $\epsilon = |\operatorname{Re} \lambda_1|/|\operatorname{Re} \lambda_{m+1}|$ , where  $\lambda_1, \lambda_{m+1}$  are the eigenvalues of the operator  $\mathcal{A}$ , which are obtained by solving the eigenvalue problem  $\mathcal{A}\phi_j = \lambda_j\phi_j$ , where  $\phi_j$  is an eigenfunction. Generalizing this definition to systems with time-dependent spatial domains, one can define  $\epsilon = \sup_{t \in [0, \infty)} |\operatorname{Re} \lambda_1(t)|/|\operatorname{Re} \lambda_{m+1}(t)|$ , where  $\lambda_1(t)$  is the largest eigenvalue of the matrix  $\mathcal{A}_s(t)$  and  $\lambda_{m+1}(t)$  is the largest eigenvalue of the operator (infinite range matrix)  $P_f\mathcal{A}(t)$ . Moreover, the presence of  $\epsilon$  in the right hand side of the  $x_f$ -subsystem in Eq. (14) is consistent with the development in the case of parabolic PDE systems with fixed spatial domains [12], where a structurally similar system to the one of Eq. (14) was derived by using  $\epsilon = |\operatorname{Re} \lambda_1|/|\operatorname{Re} \lambda_{m+1}|$ .

### 3. NONLINEAR MODEL REDUCTION

In this section, we construct nonlinear low-dimensional ODE systems that accurately reproduce the dynamics and solutions of the infinite-dimensional system of Eq. (14). The construction of the ODE systems is achieved by generalizing a nonlinear model reduction procedure introduced in [12] for systems with fixed spatial domains, to the class of systems of Eq. (14). The nonlinear model reduction procedure is based on a combination of the standard Galerkin’s method with the concept of approximate inertial manifold and provides a characterization of the accuracy of the ODE systems in terms of  $\epsilon$ .

We begin with the introduction of the concepts of inertial manifold and approximate inertial manifold. Our definition of the concept of inertial manifold is a direct generalization of the one used in [12] (see also [27]) for systems with time-invariant differential operators, to systems with time-varying operators (the reader may also refer to [17, 26] for concepts of IMs for infinite-dimensional systems with time-varying forcing inputs). An inertial manifold  $\mathcal{M}(t)$  for the system of Eq. (14) is a subset of  $\mathcal{H}(t)$ , which satisfies the following properties:

- (i)  $\mathcal{M}(t)$  is a finite-dimensional Lipschitz manifold,
- (ii)  $\mathcal{M}(t)$  is a graph of a Lipschitz function  $\Sigma(t, x_s, u, \epsilon)$  mapping  $[0, \infty) \times \mathcal{H}_s(t) \times \mathbb{R}^l \times (0, \epsilon^*]$  into  $\mathcal{H}_f(t)$  and for every solution  $x_s(t), x_f(t)$

of Eq. (14) with  $x_f(0) = \Sigma(0, x_s(0), u, \epsilon)$ , then

$$x_f(t) = \Sigma(t, x_s(t), u, \epsilon), \quad \forall t \geq 0. \quad (15)$$

(iii)  $\mathcal{M}(t)$  attracts every trajectory exponentially.

The evolution of the state  $x_f$  on  $\mathcal{M}(t)$  is given by Eq. (15), while the evolution of the state  $x_s$  is governed by the following finite-dimensional system (called inertial form),

$$\begin{aligned} \frac{dx_s}{dt} &= \mathcal{A}_s(t)x_s + \mathcal{B}_s(t)u + f_s(t, x_s, \Sigma(t, x_s, u, \epsilon)), \\ y_c &= \mathcal{C}(t)(x_s + \Sigma(t, x_s, u, \epsilon)), \quad y_m = \mathcal{S}(t)(x_s + \Sigma(t, x_s, u, \epsilon)). \end{aligned} \quad (16)$$

Assuming that  $u(t)$  is smooth, differentiating Eq. (15) and utilizing Eq. (14),  $\Sigma(t, x_s, u, \epsilon)$  can be computed as the solution of the following partial differential equation,

$$\begin{aligned} \epsilon \frac{\partial \Sigma}{\partial t} + \epsilon \frac{\partial \Sigma}{\partial x_s} [\mathcal{A}_s(t)x_s + \mathcal{B}_s(t)u + f_s(t, x_s, \Sigma)] + \epsilon \frac{\partial \Sigma}{\partial u} \dot{u} \\ = \mathcal{A}_f(t)\Sigma + \epsilon \mathcal{B}_f(t)u + \epsilon f_f(t, x_s, \Sigma), \end{aligned} \quad (17)$$

which  $\Sigma(t, x_s, u, \epsilon)$  has to satisfy for all  $t \in [0, \infty)$ ,  $x_s \in \mathcal{X}_s(t)$ ,  $u \in \mathbb{R}^l$ ,  $\epsilon \in (0, \epsilon^*]$ .

From the complex structure of Eq. (17), it is obvious that the computation of the explicit form of  $\Sigma(t, x_s, u, \epsilon)$  is impossible in most practical applications. To circumvent this problem, a procedure based on singular perturbations (introduced in [12]) is used to compute approximations of  $\Sigma(t, x_s, u, \epsilon)$  (approximate inertial manifolds) and approximations of the inertial form, of desired accuracy. More specifically, the vectors  $\Sigma(t, x_s, u, \epsilon)$  and  $u$  are expanded in a power series in  $\epsilon$ ,

$$\begin{aligned} u &= \bar{u}_0 + \epsilon \bar{u}_1 + \epsilon^2 \bar{u}_2 + \cdots + \epsilon^k \bar{u}_k + O(\epsilon^{k+1}), \\ \Sigma(t, x_s, u, \epsilon) &= \Sigma_0(t, x_s, u) + \epsilon \Sigma_1(t, x_s, u) + \epsilon^2 \Sigma_2(t, x_s, u) + \cdots \\ &\quad + \epsilon^k \Sigma_k(t, x_s, u) + O(\epsilon^{k+1}), \end{aligned} \quad (18)$$

where  $\bar{u}_k, \Sigma_k$  are smooth vector functions. Substituting the expressions of Eq. (18) into Eq. (17), and equating terms of the same power in  $\epsilon$ , one can obtain approximations of  $\Sigma(t, x_s, u, \epsilon)$  up to a desired order. Substituting the  $O(\epsilon^{k+1})$  approximation of  $\Sigma(t, x_s, u, \epsilon)$  and  $u$  into Eq. (16), the

following approximation of the inertial form is obtained,

$$\begin{aligned}
\frac{dx_s}{dt} &= \mathcal{A}_s(t)x_s + \mathcal{B}_s(t)(\bar{u}_0 + \epsilon\bar{u}_1 + \cdots + \epsilon^k\bar{u}_k) \\
&\quad + f_s(t, x_s, \Sigma_0(t, x_s, u) + \epsilon\Sigma_1(t, x_s, u) + \cdots + \epsilon^k\Sigma_k(t, x_s, u)), \\
y_c &= \mathcal{C}(t)(x_s + \Sigma_0(t, x_s, u) + \epsilon\Sigma_1(t, x_s, u) + \cdots \\
&\quad + \epsilon^k\Sigma_k(t, x_s, u)), \\
y_m &= \mathcal{D}(t)(x_s + \Sigma_0(t, x_s, u) + \epsilon\Sigma_1(t, x_s, u) + \cdots \\
&\quad + \epsilon^k\Sigma_k(t, x_s, u)).
\end{aligned} \tag{19}$$

To characterize the discrepancy between the solution of the open-loop finite-dimensional system of Eq. (19) and the solution of the  $x_s$ -subsystem of the open-loop infinite-dimensional system of Eq. (14), we need to impose the following stability requirements on the slow and fast dynamics of the system of Eq. (14).

Assumption 3 states that the system of Eq. (19) with  $u(t) \equiv 0$  and  $\epsilon = 0$  is exponentially stable.

*Assumption 3.* The finite-dimensional system of Eq. (19) with  $u(t) \equiv 0$  and  $\epsilon = 0$  is exponentially stable, in the sense that there exists a smooth Lyapunov function  $V: \mathcal{H}_s(t) \rightarrow \mathbb{R}_{\geq 0}$  and a set of positive real numbers  $(a_1, a_2, a_3, a_4, a_5)$ , such that for all  $x_s \in \mathcal{H}_s(t)$  that satisfy  $|x_s| \leq a_5$ , the following conditions hold

$$\begin{aligned}
a_1|x_s|^2 &\leq V(t, x_s) \leq a_2|x_s|^2, \\
\dot{V}(t, x_s) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_s}[\mathcal{A}_s(t)x_s + f_s(t, x_s, 0)] \leq -a_3|x_s|^2, \\
\left| \frac{\partial V}{\partial x_s} \right| &\leq a_4|x_s|.
\end{aligned} \tag{20}$$

To state our stability requirements of Eq. (14), we write the system of Eq. (14) in the fast time-scale  $\tau = \frac{t}{\epsilon}$  and set  $\epsilon = 0$ , to derive the following infinite-dimensional fast subsystem:

$$\frac{\partial x_f}{\partial \tau} = \mathcal{A}_{f\epsilon}(t)x_f. \tag{21}$$

Assumption 4 that follows states that the above system is exponentially stable, uniformly in  $t \in [0, \infty)$ .

*Assumption 4.* The fast subsystem of Eq. (21) is exponentially stable, uniformly in  $t \in [0, \infty)$ , in the sense that there exists a Lyapunov functional  $W: \mathcal{H}_f(t) \rightarrow \mathbb{R}_{\geq 0}$  and a set of positive real numbers  $(b_1, b_2, b_3, b_4, b_5, b_6)$ , such that for all  $x_f \in \mathcal{H}_f(t)$  that satisfy  $\|x_f\|_2 \leq b_6$ , the following conditions hold

$$\begin{aligned} b_1 \|x_f\|_2^2 &\leq W(t, x_f) \leq b_2 \|x_f\|_2^2, \\ \frac{\partial W}{\partial x_f} \mathcal{A}_{f\epsilon}(t) x_f &\leq -b_3 \|x_f\|_2^2, \\ \left\| \frac{\partial W}{\partial x_f} \right\|_2 &\leq b_4 \|x_f\|_2, \\ \left\| \frac{\partial W}{\partial t} \right\|_2 &\leq b_5 \|x_f\|_2^2. \end{aligned} \tag{22}$$

We are now in a position to state the main result of this section, which characterizes the discrepancy between the solution obtained from the open-loop system of Eq. (19) and the expansion for  $\Sigma(t, x_s, u, \epsilon)$  of Eq. (18) and the solution of the infinite-dimensional open-loop system of Eq. (14), in terms of  $\epsilon$ . The proof is given in the Appendix.

**PROPOSITION 1.** *Consider the system of Eq. (12) with  $u(t) \equiv 0$  and suppose that Assumptions 1–4 hold. Then, there exist positive real numbers  $\mu_1, \mu_2, \epsilon^*$  such that if  $|x_s(0)| \leq \mu_1$ ,  $\|x_f(0)\|_2 \leq \mu_2$ , and  $\epsilon \in (0, \epsilon^*]$ , then the solutions  $x_s(t)$ ,  $x_f(t)$  of the system of Eq. (14) satisfy  $\forall t \in [t_b, \infty)$ ,*

$$\begin{aligned} x_s(t) &= \tilde{x}_s(t) + O(\epsilon^{k+1}), \\ x_f(t) &= \tilde{x}_f(t) + O(\epsilon^{k+1}), \end{aligned} \tag{23}$$

where  $t_b$  is the time required for  $x_f(t)$  to approach  $\tilde{x}_f(t)$ ,  $\tilde{x}_s(t)$  is the solution of Eq. (19) with  $u(t) \equiv 0$ , and  $\tilde{x}_f(t) = \Sigma_0(t, \tilde{x}_s, 0) + \epsilon \Sigma_1(t, \tilde{x}_s, 0) + \epsilon^2 \Sigma_2(t, \tilde{x}_s, 0) + \dots + \epsilon^k \Sigma_k(t, \tilde{x}_s, 0)$ .

*Remark 5.* An estimate of  $\epsilon^*$  can be obtained, in principle, from the proof of the theorem. However, such an estimate is typically conservative, and thus, it is useful to check its appropriateness through computer simulations.

*Remark 6.* Utilizing the result of Proposition 1, one can show that  $x(t) = \tilde{x}(t) + O(\epsilon^{k+1})$ ,  $\forall t \geq t_b$ , where  $x(t)$  is the solution of the open-loop infinite-dimensional system of Eq. (12) and  $\tilde{x}(t) = \tilde{x}_s(t) + \tilde{x}_f(t)$  is the solution obtained from the  $O(\epsilon^{k+1})$  approximation of the open-loop inertial form (Eq. (19)) and the inertial manifold (Eq. (18)).

*Remark 7.* For  $k = 0$ , the expansion of Eq. (18) yields  $\Sigma(t, x_s, u, \epsilon) = \Sigma_0(t, x_s, u) = 0$ , and the corresponding approximate inertial form is

$$\begin{aligned} \frac{dx_s}{dt} &= \mathcal{A}_s(t)x_s + \mathcal{B}_s(t)\bar{u}_0 + f_s(t, x_s, 0), \\ y_c &= \mathcal{C}(t)x_s, \quad y_m = \mathcal{S}(t)x_s. \end{aligned} \quad (24)$$

The above system is identical to the one obtained from a direct application of Galerkin's method to the system of Eq. (14) and does not utilize any information about the structure of the fast subsystem, thus yielding solutions which are only  $O(\epsilon)$  close to the solutions of the open-loop system of Eq. (12) (result of Proposition 1 with  $k = 0$ ). On the other hand, for  $k = 1$ , the expansion of Eq. (18) yields  $\Sigma(t, x_s, u, \epsilon) = \epsilon \Sigma_1(t, x_s, \bar{u}_0) = -\epsilon(\mathcal{A}_{f\epsilon})^{-1}(t)[\mathcal{B}_f(t)\bar{u}_0 + f_f(t, x_s, 0)]$  (note that from Assumption 4 we have that  $(\mathcal{A}_{f\epsilon})^{-1}(t)$  exists and is bounded,  $\forall t \geq 0$ ), and the corresponding approximate inertial form is

$$\begin{aligned} \frac{dx_s}{dt} &= \mathcal{A}_s(t)x_s + \mathcal{B}_s(t)(\bar{u}_0 + \epsilon\bar{u}_1) + f_s(t, x_s, \epsilon\Sigma_1(t, x_s, \bar{u}_0)), \\ y_c &= \mathcal{C}(t)(x_s + \epsilon\Sigma_1(t, x_s, \bar{u}_0)), \\ y_m &= \mathcal{S}(t)(x_s + \epsilon\Sigma_1(t, x_s, \bar{u}_0)). \end{aligned} \quad (25)$$

The above system does utilize information about the structure of the fast subsystem, thereby yielding solutions which are  $O(\epsilon^2)$  close to the solutions of the open-loop system of Eq. (12).

*Remark 8.* The expansion of  $\Sigma(t, x_s, u, \epsilon)$  in a power series in  $\epsilon$  (Eq. (18)) is motivated and validated from the fact that as  $\epsilon \rightarrow 0$  the inertial form of Eq. (19) reduces to the system of Eq. (24) obtained from the standard Galerkin's method, which ensures that the inertial form is well posed with respect to  $\epsilon$ . The expansion of  $u$  in a power series in  $\epsilon$  in Eq. (18) is motivated by our intention to appropriately modify the synthesis of the controller such that the outputs of the  $O(\epsilon^{k+1})$  approximation of the closed-loop inertial form,  $y_{cs_i}$ ,  $i = 1, \dots, l$ , satisfy  $\lim_{t \rightarrow \infty} |y_{cs_i} - v_i| = 0$ , where  $v_i$  is the reference input.

#### 4. NONLINEAR OUTPUT FEEDBACK CONTROL

In this section, we synthesize nonlinear finite-dimensional output feedback controllers that guarantee local exponential stability and force the controlled output of the closed-loop PDE system to follow, up to a desired

accuracy, a prespecified response, provided that  $\epsilon$  is sufficiently small. The output feedback controllers are constructed through combination of state feedback controllers with state observers.

More specifically, we use the system of Eq. (19) to synthesize nonlinear state feedback controllers of the following general form,

$$\begin{aligned} u &= \bar{u}_0 + \epsilon \bar{u}_1 + \epsilon^2 \bar{u}_2 + \cdots + \epsilon^k \bar{u}_k \\ &= p_0(t, x_s) + Q_0(t, x_s)v + \epsilon [p_1(t, x_s) + Q_1(t, x_s)v] + \cdots \\ &\quad + \epsilon^k [p_k(t, x_s) + Q_k(t, x_s)v], \end{aligned} \quad (26)$$

where  $p_0(t, x_s), \dots, p_k(t, x_s)$  are smooth vector functions,  $Q_0(t, x_s), \dots, Q_k(t, x_s)$  are smooth matrices, and  $v \in \mathbb{R}^l$  is the constant reference input vector. The nonlinear controllers are constructed by following a sequential procedure. Specifically, the component  $\bar{u}_0 = p_0(t, x_s) + Q_0(t, x_s)v$  is initially synthesized on the basis of the  $O(\epsilon)$  approximation of the inertial form; then the component  $\bar{u}_1 = p_1(t, x_s) + Q_1(t, x_s)v$  is synthesized on the basis of the  $O(\epsilon^2)$  approximation of the inertial form. In general, at the  $k$ th step, the component  $\bar{u}_k = p_k(t, x_s) + Q_k(t, x_s)v$  is synthesized on the basis of the  $O(\epsilon^{k+1})$  approximation of the inertial form (Eq. (19)). The synthesis of  $[p_\nu(t, x_s), Q_\nu(t, x_s)]$ ,  $\nu = 0, \dots, k$ , so that a nonlinear controller of the form of Eq. (26) guarantees local exponential stability and forces the output of the system of Eq. (19) to follow a desired linear response is performed by utilizing geometric control methods for nonlinear ODEs (the details of the controller synthesis can be found in [16, 22], and are omitted for brevity).

Since measurements of  $\bar{x}(z, t)$  (and thus,  $x_s(t)$ ) are usually not available in practice, we assume that there exists an  $L$  so that the nonlinear dynamical system,

$$\begin{aligned} \frac{d\eta}{dt} &= \mathcal{A}_s(t)\eta + \mathcal{B}_s(t) [\bar{u}_0 + \epsilon \bar{u}_1 + \epsilon^2 \bar{u}_2 + \cdots + \epsilon^k \bar{u}_k] \\ &\quad + f_s(t, \eta, \epsilon \Sigma_1(t, \eta, u) + \epsilon^2 \Sigma_2(t, \eta, u) + \cdots + \epsilon^k \Sigma_k(t, \eta, u)) \\ &\quad + L [y_m - \mathcal{S}(t)(\eta + \epsilon \Sigma_1(t, \eta, u) + \epsilon^2 \Sigma_2(t, \eta, u) + \cdots \\ &\quad + \epsilon^k \Sigma_k(t, \eta, u))] \end{aligned} \quad (27)$$

where  $\eta$  denotes an  $m$ -dimensional state vector, is a local exponential observer for the system of Eq. (19) (i.e., the discrepancy  $|\eta(t) - x_s(t)|$  tends exponentially to zero).

Theorem 1 that follows provides the synthesis formula of the output feedback controller and conditions that guarantee closed-loop stability in

the case of considering an  $O(\epsilon^2)$  approximation of the exact slow system for the synthesis of the controller. The derivation of synthesis formulas for higher order approximations of the output feedback controller is notationally complicated, although conceptually straightforward, and thus, they are omitted for reasons of brevity. To state our result, we need to use the Lie derivative notation and the concepts of relative order and characteristic matrix (which are defined in the Appendix), for the system of Eq. (25). The proof of the theorem is given in the Appendix.

**THEOREM 1.** *Consider the parabolic PDE system of Eq. (12), for which Assumptions 1, 2, and 4 hold. Consider also the  $O(\epsilon^2)$  approximation of the inertial form and assume that its characteristic matrix  $C_1(t, x_s, \epsilon)$  is invertible  $\forall t \in [0, \infty)$ ,  $\forall x_s \in \mathcal{X}_s(t)$ ,  $\forall \epsilon \in (0, \epsilon^*]$ . Suppose also that the following conditions hold:*

1. *The roots of the equation,*

$$\det(B(s)) = 0, \quad (28)$$

*where  $B(s)$  is an  $l \times l$  matrix whose  $(i, j)$ th element is of the form  $\sum_{k=0}^{\tau_i} \beta_{jk}^i s^k$ , lie in the open left-half of the complex plane, where  $\beta_{jk}^i$  are adjustable controller parameters.*

2. *The zero dynamics of the  $O(\epsilon^2)$  approximation of the inertial form are locally exponentially stable.*

*Then, there exist positive real numbers  $\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\epsilon}^*$  such that if  $|x_s(0)| \leq \tilde{\mu}_1$ ,  $\|x_f(0)\|_2 \leq \tilde{\mu}_2$ , and  $\epsilon \in (0, \tilde{\epsilon}^*]$ , and  $\eta(0) = x_s(0)$ , the dynamic output feedback controller,*

$$\begin{aligned} \frac{d\eta}{dt} = & \mathcal{A}_s(t)\eta + \mathcal{B}_s(t)[\bar{u}_0(t, \eta) + \epsilon\bar{u}_1(t, \eta)] \\ & + f_s(t, \eta, \epsilon(\mathcal{A}_{f\epsilon})^{-1}(t)[- \mathcal{B}_f(t)\bar{u}_0(t, \eta) - f_f(t, \eta, 0)]) \\ & + L(y - (\mathcal{S}(t)\eta + \mathcal{S}(t)\epsilon(\mathcal{A}_{f\epsilon})^{-1}(t) \\ & \quad \times [- \mathcal{B}_f(t)\bar{u}_0(t, \eta) - f_f(t, \eta, 0)])) \end{aligned} \quad (29)$$

$$u = \bar{u}_0(t, \eta) + \epsilon\bar{u}_1(t, \eta)$$

$$\begin{aligned} = & \{[\beta_{1r_1} \cdots \beta_{lr_l}]C_0(t, \eta)\}^{-1} \left\{ v - \sum_{i=1}^l \sum_{k=0}^{r_i} \beta_{ik} L_{f_0}^k h_{0i}(t, \eta) \right\} \\ & + \epsilon \{[\beta_{1r_1} \cdots \beta_{lr_l}]C_1(t, \eta, \epsilon)\}^{-1} \left\{ v - \sum_{i=1}^l \sum_{k=0}^{r_i} \beta_{ik} L_{f_1}^k h_{1i}(t, \eta, \epsilon) \right\}. \end{aligned}$$

- (a) *guarantees local exponential stability of the closed-loop system, and*  
 (b) *ensures that the outputs of the closed-loop system satisfy for all*  
 $t \in [t_b, \infty)$ ,

$$y_{c_i}(t) = y_{cs_i}(t) + O(\epsilon^2), \quad i = 1, \dots, l, \quad (30)$$

where  $t_b$  is the time required for the off-manifold fast transients to decay to zero exponentially, and  $y_{cs_i}(t)$  is the solution of:

$$\sum_{i=1}^l \sum_{k=0}^{r_i} \beta_{ik} \frac{d^k y_{cs_i}}{dt^k} = v. \quad (31)$$

*Remark 9.* The implementation of the controller of Eq. (29) requires to explicitly compute the vector function  $\Sigma_1(t, \eta, \bar{u}_0)$ . However,  $\Sigma_1(t, \eta, \bar{u}_0)$  has an infinite-dimensional range and therefore cannot be implemented in practice. Instead a finite-dimensional approximation of  $\Sigma_1(t, \eta, \bar{u}_0)$ , say  $\Sigma_{1_t}(t, \eta, \bar{u}_0)$ , can be derived by keeping the first  $\tilde{m}$  elements of  $\Sigma_1(t, \eta, \bar{u}_0)$  and neglecting the remaining infinite ones. Clearly, as  $\tilde{m} \rightarrow \infty$ ,  $\Sigma_{1_t}(t, \eta, \bar{u}_0)$  approaches  $\Sigma_1(t, \eta, \bar{u}_0)$ . This implies that by picking  $\tilde{m}$  to be sufficiently large, the controller of Eq. (29) with  $\Sigma_{1_t}(t, \eta, \bar{u}_0)$  instead of  $\Sigma_1(t, \eta, \bar{u}_0)$  guarantees stability and enforces the requirement of Eq. (31) in the closed-loop infinite-dimensional system.

*Remark 10.* Note that in the presence of small initialization errors of the observer states (i.e.,  $\eta(0) \neq x_s(0)$ ), uncertainty in the model parameters and external disturbances, although a slight deterioration of the performance may occur, (i.e., the requirement of Eq. (30) will not be exactly imposed in the closed-loop system), the output feedback controller of Theorem 1 will continue to enforce exponential stability and asymptotic output tracking in the closed-loop system. Furthermore, the assumption that the characteristic matrix  $C_1(t, x_s, \epsilon)$  is invertible  $\forall t \in [0, \infty)$ ,  $\forall x_s \in \mathcal{X}_s(t)$ ,  $\forall \epsilon \in (0, \epsilon^*]$  is made to simplify the development and can be relaxed by using dynamic state feedback instead of static state feedback (see [16] for details).

*Remark 11.* The exponential stability of the closed-loop system guarantees that in the presence of small errors spaces of the closed-loop system are finite dimensional, and the controller of Eq. (29) enforces an approximately linear input-output response between  $y_{c_i}$  and  $v_i$ , it is possible to implement a linear error feedback controller around the  $(y_{c_i} - v_i)$  loop to ensure asymptotic offsetless output tracking in the closed-loop system, in the presence of constant unknown process parameters and unmeasured disturbance inputs. (The reader may refer to [9, 31] for results on control of quasi-linear parabolic PDE systems with uncertainty.)



*Remark 12.* The nonlinear controller of Eq. (29) possesses a robustness property with respect to fast and asymptotically stable unmodeled dynamics (i.e., the controller enforces exponential stability and output tracking in the closed-loop system, despite the presence of additional dynamics in the process model, as long as they are stable and sufficiently fast). This property of the controller can be rigorously established by analyzing the closed-loop system with the unmodeled dynamics using singular perturbations, and is of particular importance for many practical applications where unmodeled dynamics often occur due to actuator and sensor dynamics, fast process dynamics, etc.

*Remark 13.* Finally, we note that the validity of the approach that we followed here to synthesize the nonlinear controller of Eq. (29) relies on the large separation (expressed in terms of  $\epsilon$ ) of the slow and fast dynamics of the parabolic PDE system of Eq. (5). This approach is not directly applicable to hyperbolic PDE systems where the eigenmodes cluster along vertical or nearly vertical asymptotes in the complex plane, and thus, the controller has to be modified to compensate for the destabilizing effect of the residual modes (see [4] for details).

## 5. APPLICATION TO A DIFFUSION-REACTION PROCESS WITH MOVING BOUNDARY

We consider a diffusion-reaction process with moving boundary which is described by the following parabolic PDE,

$$\frac{\partial \bar{x}}{\partial t} = \frac{\partial^2 \bar{x}}{\partial z^2} + \beta_T e^{-\gamma/(1+\bar{x})} - \beta_T e^{-\gamma} + \beta_u (b(z, t)u(t) - \bar{x}) \quad (32)$$

subject to the Dirichlet boundary conditions,

$$\bar{x}(0, t) = 0, \quad \bar{x}(l(t), t) = 0 \quad (33)$$

and the initial condition,

$$\bar{x}(z, 0) = \bar{x}_0(z), \quad (34)$$

where  $\bar{x}$  is the state,  $\beta_T$ ,  $\gamma$ ,  $\beta_u$  are dimensionless process parameters,  $b(z, t) = [b_1(z, t) \ b_2(z, t)]$  are the actuator distribution functions, and  $u[u_1 \ u_2]^T$  is the manipulated input vector. The spatial domain is assumed to change according to the relation,

$$l(t) = \pi(1.4 - 0.4e^{(-0.02t^{2.7})}) \quad (35)$$

(it can be easily seen that the above function satisfies the requirements of Assumption 1) and the following typical values were given to the process parameters:

$$\beta_T = 85.0, \quad \beta_U = 2, \quad \gamma = 4.0. \quad (36)$$

For the system of Eq. (32), the differential operator is of the form,

$$\mathcal{A}(t)x = \frac{\partial^2 \bar{x}}{\partial z^2} - l \frac{\partial \bar{x}}{\partial z}, \quad (37)$$

$$x \in D(\mathcal{A}) = \{x \in \mathcal{H}([0, l(t)]; \mathbb{R}); \bar{x}(0, t) = 0; \bar{x}(l(t), t) = 0\},$$

and a countable,  $\forall t \in [0, \infty)$ , orthogonal basis of  $\mathcal{H}(t)$  is

$$\phi_j(z, t) = \sqrt{\frac{2}{l(t)}} \sin\left(j \frac{\pi}{l(t)} z\right), \quad j = 1, \dots, \infty. \quad (38)$$

Note that the above set of basis functions satisfies the conditions of Assumption 2.

A 20th order Galerkin truncation of the system of Eq. (32) was used in our simulations (it was verified that further increase of the order of the Galerkin model provided no substantial improvement on the accuracy of the simulation results). It was found that the operating steady state  $\bar{x}(z, t) = 0$  is an unstable one (Fig. 1 shows the evolution of the open-loop state starting from initial conditions close to the steady state  $\bar{x}(z, t) = 0$ ;

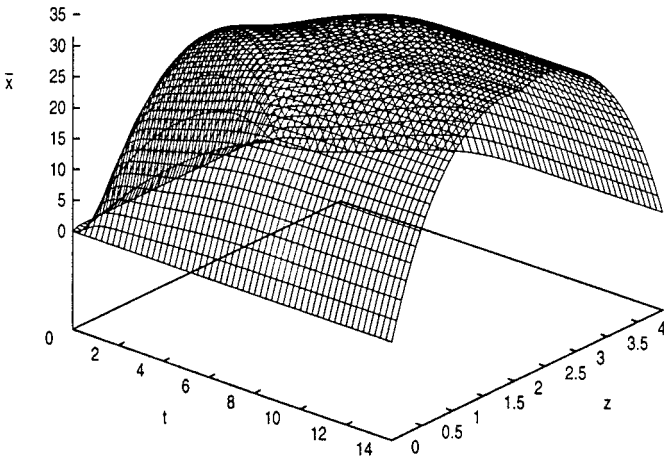


FIG. 1. Evolution of the state of an open-loop system.

the system moves to another stable steady state characterized by a maximum at  $z = 0.5 l(t)$ ). Moreover, the linearization of the system of Eq. (32) around the steady state  $\bar{x}(z, t) = 0$  possesses two positive eigenvalues  $\forall l(t) \in [\pi, 1.4\pi]$ .

The control objective is to stabilize the system at the unstable steady state  $\bar{x}(z, t) = 0$  by employing a nonlinear output feedback controller which uses a point measurement of the state at  $z = 0.7l(t)$  (i.e., moving sensor with  $s(z, t) = \delta(z - 0.7l(t))$ , where  $\delta(\cdot)$  is the Dirac function). Since the maximum open-loop value of  $\bar{x}(z, t)$  occurs for  $z = 0.5 l(t)$  and the first two modes of the process are unstable  $\forall l(t) \in [\pi, 1.4\pi]$ , the controlled outputs were defined as

$$\begin{aligned} y_{c1}(t) &= \int_0^{l(t)} \sqrt{\frac{2}{l(t)}} \sin\left(\frac{\pi}{l(t)} z\right) \bar{x}(z, t) dz, \\ y_{c2}(t) &= \int_0^{l(t)} \sqrt{\frac{2}{l(t)}} \sin\left(2\frac{\pi}{l(t)} z\right) \bar{x}(z, t) dz. \end{aligned} \quad (39)$$

The actuator distribution functions were taken to be  $b_1(z, t) = 1$  (uniform in space, distributed control action) and  $b_2(z, t) = \delta(z - \frac{2}{3}l(t))$  (moving point control actuation).

Several simulation runs were performed to evaluate: (a) the reduction on the order of the controller achieved when the controller is synthesized on the basis of ODE models derived from combination of Galerkin's method with approximate inertial manifolds, and (b) the choice of using moving control actuators and measurement sensors. In all the simulation runs, the process was assumed to be at a nonzero initial condition.

We initially employed the standard Galerkin's method to derive an approximate ODE system that was used for the synthesis of a nonlinear output feedback controller. It was found that the *lowest order* model obtained from the standard Galerkin's method which leads to the synthesis of a controller that stabilizes the open-loop system at  $\bar{x}(z, t) = 0$  is 8 (i.e.,  $m = 8$ ,  $\tilde{m} = 0$ ). Figures 2 and 3 show the evolution of the state of the closed-loop system and the profile of the manipulated input under an 8th order nonlinear output feedback controller of the form of Eq. (29), respectively (the controller parameters are  $\epsilon = 0$ ,  $\beta_{10} = 1.0$ ,  $\beta_{11} = 4.0$ ,  $\beta_{20} = 1.0$ ,  $\beta_{21} = 4.0$ , and  $L = [0.7 \ -25.0 \ -9.6 \ 0.0 \ \cdots \ 0.0]^T$ ). It is clear that this controller stabilizes the state of the system at  $\bar{x}(z, t) = 0$ .

We subsequently used the proposed combination of Galerkin's method with approximate inertial manifolds to derive an ODE system which was used for the synthesis of a nonlinear output feedback controller. For the case of using an  $O(\epsilon^2)$  approximation of the AIM, it was found that the

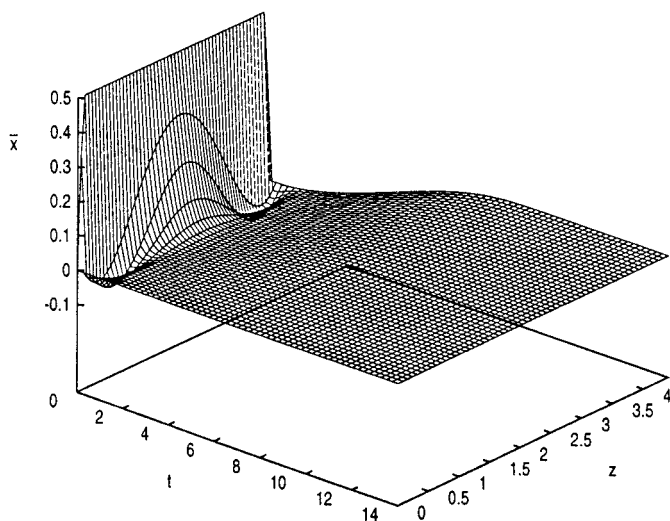


FIG. 2. Evolution of the state of a closed-loop system under an 8th order nonlinear time-varying output feedback controller synthesized on the basis of an ODE system obtained by using standard Galerkin's method.

*lowest order* ODE model, which leads to the synthesis of a controller that stabilizes the open-loop system at  $\bar{x}(z, t) = 0$ , is one of order 5 which uses a 6th order approximation for  $x_f$  (i.e.,  $m = 5$  and  $\tilde{m} = 6$ ). Figures 4 and 5 show the evolution of the state of the closed-loop system and the profile of the manipulated input under a 5th order controller of the form of Eq. (29), respectively (the controller parameters are  $\epsilon = 0.0278$ ,  $\beta_{10} = 1.0$ ,  $\beta_{11} = 4.0$ ,  $\beta_{20} = 1.0$ ,  $\beta_{21} = 4.0$ , and  $L = [0.7 \ -25.0 \ -9.6 \ 0.0 \ \cdots \ 0.0]^T$ ). The controller clearly regulates the system at  $\bar{x}(z, t) = 0$ .

We also implemented on the process a nonlinear output feedback controller which was synthesized on the basis of an 8th order Galerkin truncation of the system of Eq. (32) with  $l(t) = \pi$  (i.e., the domain is assumed to be fixed in the design of the controller). Figure 6 shows the evolution of the state of the closed-loop system and Fig. 7 shows the corresponding manipulated input profiles. Clearly, this controller leads to closed-loop instability because the PDE system becomes "uncontrollable" (i.e. the position of the point control actuator approaches the location of the zero of the second eigenfunction at  $z = 0.5/(5.29)$  which makes the stabilization of the second unstable mode impossible, thereby leading to closed-loop instability). We finally note that similar closed-loop instabilities were observed when: (a) the nonlinear output feedback controller was synthesized on the basis of an 8th order Galerkin truncation of the system

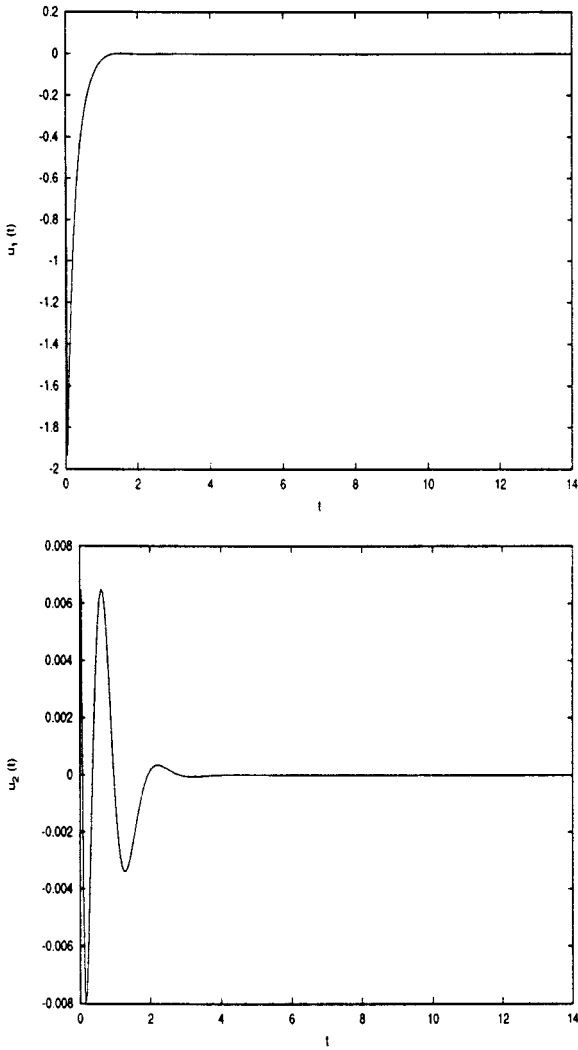


FIG. 3. Manipulated input profiles of 8th order nonlinear time-varying output feedback controller.

of Eq. (32) with  $l(t)$  equal to  $1.2\pi$  and  $1.4\pi$ , and (b) the nonlinear output feedback controller was synthesized on the basis of a 5th order model obtained from the combination of Galerkin's method with AIMS ( $m = 5$ ,  $\tilde{m} = 6$ ), for  $l(t) = \pi, 1.2\pi, 1.4\pi$ .

From the results of the simulation study, it is evident that the combination of Galerkin's method and approximate inertial manifolds to derive

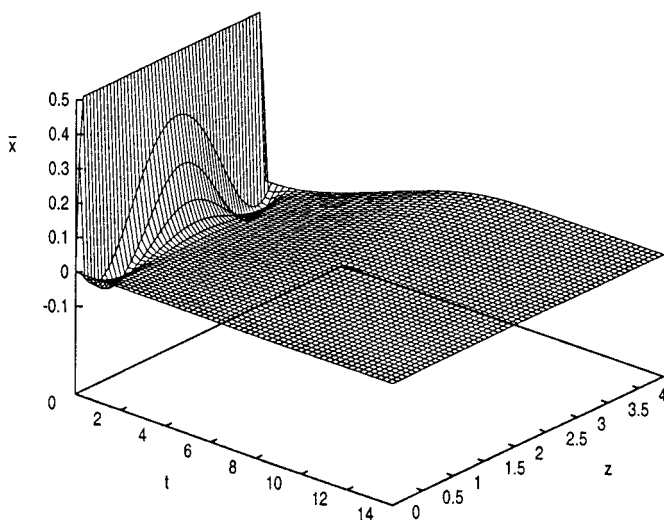


FIG. 4. Evolution of the state of a closed-loop system under a 5th order nonlinear time-varying output feedback controller synthesized on the basis of an ODE system obtained from the combination of Galerkin's method with approximate inertial manifolds.

ODE models used for controller synthesis, leads to a significant reduction on the order of the stabilizing controller, while the use of moving control actuators and sensors allows controlling processes with moving boundaries, which are difficult to control with actuators and sensors placed at fixed locations.

## 6. CONCLUSIONS

In this work, a methodology was developed for the synthesis of nonlinear finite-dimensional time-varying output feedback controllers for systems of quasi-linear parabolic partial differential equations (PDEs) with time-dependent spatial domains, whose dynamics can be separated into slow and fast ones. Initially, a nonlinear model reduction procedure, based on a combination of Galerkin's method with the concept of approximate inertial manifold, was employed for the derivation of ODE systems that yield solutions which are close, up to a desired accuracy, to the ones of the PDE system, for almost all times. Then, these ODE systems were used as the basis for the explicit construction of nonlinear time-varying output feedback controllers via geometric control methods. The controllers guaranteed stability and enforced the output of the closed-loop parabolic PDE

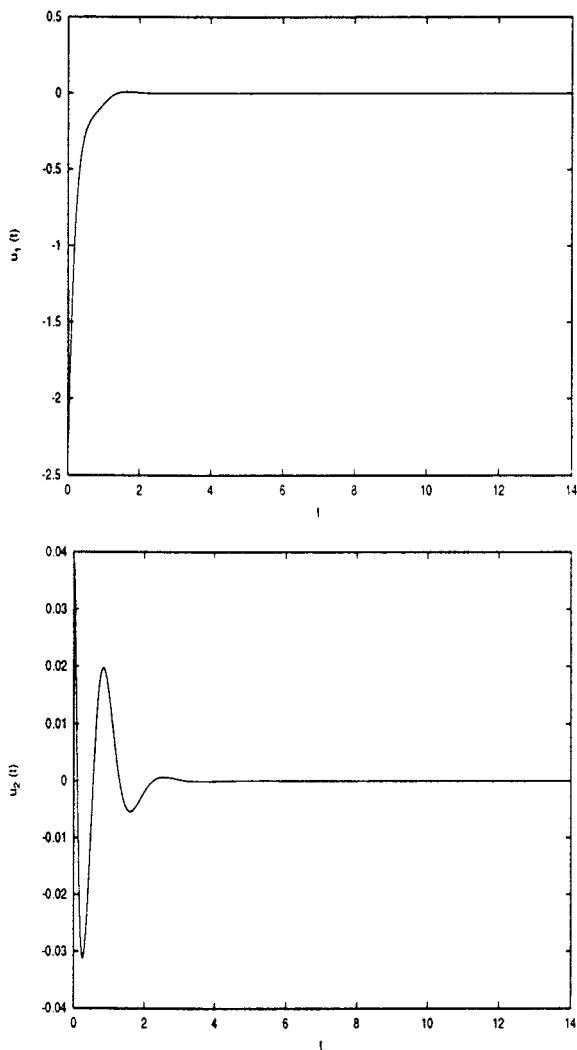


FIG. 5. Manipulated input profiles of 5th order nonlinear time-varying output feedback controller.

system to follow, up to a desired accuracy, a prespecified response for almost all times, provided that the separation of the slow and fast dynamics was sufficiently large. Differences in the nature of the model reduction and control problems between parabolic PDE systems with fixed and moving spatial domains were identified and discussed. The proposed control method was used to stabilize an unstable steady state of a diffu-

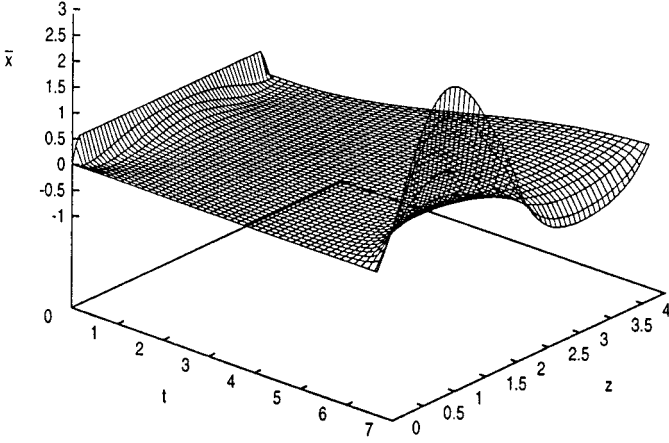


FIG. 6. Evolution of the state of a closed-loop system under an 8th order nonlinear output feedback controller, synthesized under the assumption of fixed spatial domain (i.e.,  $l(t) = \pi, \forall t \geq 0$ ).

sion-reaction process with time-dependent spatial domain, and was shown to lead to a significant reduction on the order of the stabilizing controller, as well as to outperform a controller synthesis method which does not account for the variation of the spatial domain.

## APPENDIX: CONCEPTS OF RELATIVE ORDER AND CHARACTERISTIC MATRIX

First, the Lie derivative of the scalar function  $h_{0_i}(t, x_s)$  with respect to the vector function  $f_0(t, x_s)$  is defined as  $L_{f_0} h_{0_i}(t, x_s) = \partial h_{0_i} / \partial x_s f_0(t, x_s) + \partial h_{0_i} / \partial t$  (this definition of Lie derivative was introduced in [22] and is different than the standard one used in [16] for the case of time-invariant  $h_{0_i}, f_0$ ),  $L_{f_0}^k h_{0_i}(t, x_s)$  denotes the  $k$ th order Lie derivative and  $L_{g_{0_i}} L_{f_0}^k h_{0_i}(t, x_s)$  denotes the mixed Lie derivative. Now, referring to the system of Eq. (24), we set  $\mathcal{A}_s(t)x_s + f_s(t, x_s, 0) = f_0(t, x_s)$ ,  $\mathcal{B}_s(t) = g_0(t, x_s)$ ,  $\mathcal{C}_i(t)x_s = h_{0_i}(t, x_s)$  to obtain

$$\begin{aligned} \frac{dx_s}{dt} &= f_0(t, x_s) + g_0(t, x_s)u, \\ y_{cs_i} &= h_{0_i}(t, x_s). \end{aligned} \tag{40}$$



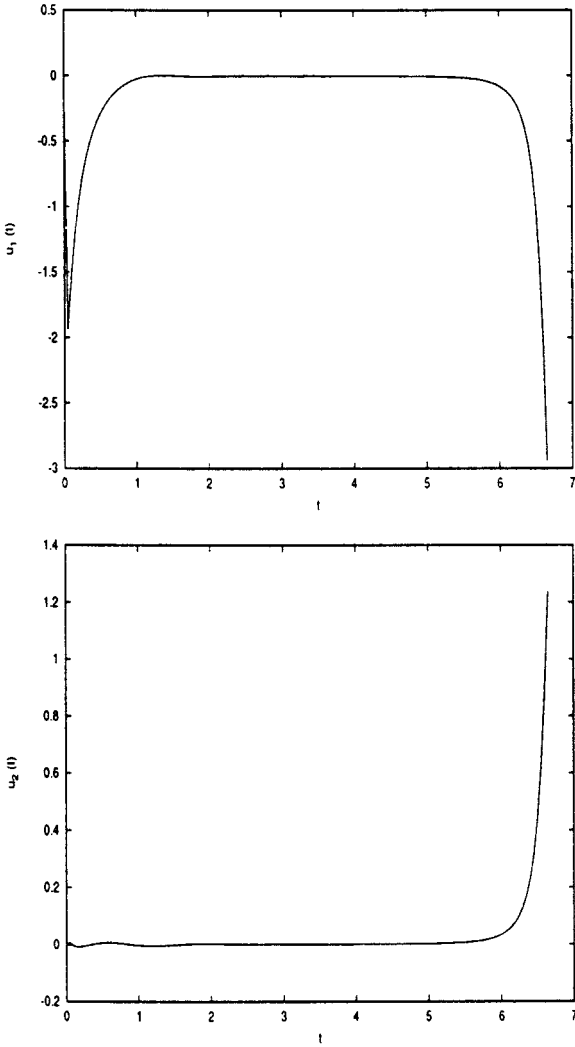


FIG. 7. Manipulated input profiles of an 8th order nonlinear output feedback controller synthesized under the assumption of fixed spatial domain (i.e.,  $l(t) = \pi, \forall t \geq 0$ ).

For the above system, the relative order of the output  $y_{cs_i}$  with respect to the vector of manipulated inputs  $u$  is defined as the smallest integer  $r_i$  for which,

$$\left[ L_{g_{0_1}} L_{f_0}^{r_i-1} h_{0_i}(t, x_s) \cdots L_{g_{0_i}} L_{f_0}^{r_i-1} h_{0_i}(t, x_s) \right] \neq [0 \cdots 0], \quad (41)$$

or  $r_i = \infty$  if such an integer does not exist. Furthermore, the matrix,

$$C_0(t, x_s) = \begin{bmatrix} L_{g_{01}} L_{f_0}^{r_1-1} h_{01}(t, x_s) & \cdots & L_{g_{0l}} L_{f_0}^{r_l-1} h_{0l}(t, x_s) \\ L_{g_{01}} L_{f_0}^{r_2-1} h_{02}(t, x_s) & \cdots & L_{g_{0l}} L_{f_0}^{r_l-1} h_{0l}(t, x_s) \\ \vdots & \cdots & \vdots \\ L_{g_{01}} L_{f_0}^{r_i-1} h_{0i}(t, x_s) & \cdots & L_{g_{0l}} L_{f_0}^{r_l-1} h_{0l}(t, x_s) \end{bmatrix} \quad (42)$$

is the characteristic matrix of the system of Eq. (40).

*Proof of Proposition 1.* The proof of the proposition is obtained by following a two-step approach. In the first step, we show that the system of Eq. (14) is exponentially stable, provided that the initial conditions and  $\epsilon$  are sufficiently small. The exponential stability property is used in the second step to prove closeness of solutions as given in Eq. (23).

*Exponential stability:* The system of Eq. (14) can be equivalently written as

$$\begin{aligned} \frac{dx_s}{dt} &= \mathcal{A}_s(t) x_s + f_s(t, x_s, 0) + [f_s(t, x_s, x_f) - f_s(t, x_s, 0)], \\ \epsilon \frac{\partial x_f}{\partial t} &= \mathcal{A}_{f\epsilon}(t) x_f + \epsilon f_f(t, x_s, x_f). \end{aligned} \quad (43)$$

Let  $\mu_1^*, \mu_2^*$  with  $\mu_1^* \geq a_5$  and  $\mu_2^* \geq b_5$  be two positive real numbers such that if  $|x_s| \leq \mu_1^*$  and  $\|x_f\|_2 \leq \mu_2^*$ , then there exist positive real numbers  $(k_1, k_2, k_3)$  such that]

$$\begin{aligned} |f_s(t, x_s, x_f) - f_s(t, x_s, 0)| &\leq k_1 \|x_f\|_2, \\ \|f_f(t, x_s, x_f)\|_2 &\leq k_2 |x_s| + k_3 \|x_f\|_2. \end{aligned} \quad (44)$$

Pick  $\mu_1 < a_s < \mu_1^*$  and  $\mu_2 < b_6 < \mu_2^*$ .

Consider the smooth time-varying function  $L: \mathcal{H}_s(t) \times \mathcal{H}_f(t) \rightarrow \mathbb{R}_{\geq 0}$ ,

$$L(t, x_s, x_f) = V(t, x_s) + W(t, x_f), \quad (45)$$

where  $V(t, x_s)$  and  $W(t, x_f)$  were defined in Assumptions 3 and 4, respectively, as a Lyapunov function candidate for the system of Eq. (43). From Eqs. (20) and (22), we have that  $L(t, x_s, x_f)$  is positive definite, proper (tends to  $+\infty$  as  $|x_s| \rightarrow \infty$ , or  $\|x_f\|_2 \rightarrow \infty$ ) and decrescent, with respect to

its arguments. Computing the time-derivative of  $L$  along the trajectories of the system of Eq. (43), and using the bounds of Eqs. (20) and (22) and the estimates of Eq. (44), the following expressions can be obtained,

$$\begin{aligned}
\dot{L}(t, x_s, x_f) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_s} \dot{x}_s + \frac{\partial W}{\partial t} + \frac{\partial W}{\partial x_f} \dot{x}_f \\
&= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_s} [\mathcal{A}_s(t) x_s + f_s(t, x_s, 0)] \\
&\quad + \frac{\partial V}{\partial x_s} [f_s(t, x_s, x_f) - f_s(t, x_s, 0)] \\
&\quad + \frac{\partial W}{\partial t} + \frac{1}{\epsilon} \frac{\partial W}{\partial x_f} \mathcal{A}_{f\epsilon}(t) x_f + \frac{\partial W}{\partial x_f} f_f(t, x_s, x_f) \\
&\leq -a_3 |x_s|^2 + a_4 k_1 |x_s| \|x_f\|_2 + b_5 \|x_f\|_2^2 \\
&\quad - \frac{b_3}{\epsilon} \|x_f\|_2^2 + b_4 \|x_f\|_2 (k_2 |x_s| + k_3 \|x_f\|_2) \\
&\leq -a_3 |x_s|^2 + (a_4 k_1 + b_4 k_2) |x_s| \|x_f\|_2 \\
&\quad - \left( \frac{b_3}{\epsilon} - b_4 k_3 - b_5 \right) \|x_f\|_2^2 \\
&\leq - \begin{bmatrix} |x_s| & \|x_f\|_2 \end{bmatrix} \begin{bmatrix} a_3 & -\frac{a_4 k_1 + b_4 k_2}{2} \\ -\frac{a_4 k_1 + b_4 k_2}{2} & \frac{b_3}{\epsilon} - b_4 k_3 - b_5 \end{bmatrix} \\
&\quad \times \begin{bmatrix} |x_s| \\ \|x_f\|_2 \end{bmatrix}. \tag{46}
\end{aligned}$$

Defining

$$\epsilon_1 = \frac{(a_3 - \sqrt{\delta}) b_3}{(a_3 - \sqrt{\delta})(b_5 + b_4 k_3 + \sqrt{\delta}) + ((a_4 k_1 + b_4 k_2)/2)^2},$$

where  $\delta \in \mathbb{R}_{>0}$ , we have that if  $\epsilon \in (0, \epsilon_1]$  then  $\dot{L}(x_s, x_f) \leq -\delta(|x_s|^2 + \|x_f\|_2^2)$ , which from the properties of  $L$  directly implies that the state of the system of Eq. (43) is exponentially stable; i.e., there exists a positive real

number  $\sigma$  such that:

$$\begin{bmatrix} |x_s| \\ \|x_f\|_2 \end{bmatrix} \leq e^{-\sigma t} \begin{bmatrix} |x_s(0)| \\ \|x_f(0)\|_2 \end{bmatrix}. \quad (47)$$

*Closeness of solutions:* We initially show the closeness of solution result for the  $x_f$ -states, and then for the  $x_s$ -states (note that from the first part of the proof  $\epsilon \in (0, \epsilon_1]$ ). To establish the estimate for the  $x_f$ -states, we initially prove that the off manifold transients decay quickly (the decay rate is precisely given below) to the manifold. To this end, we define the error vector  $e_f(t) = x_f(t) - \Sigma(t, x_s, 0, \epsilon)$ , differentiate  $e_f(t)$  with respect to time, and multiply the resulting system with  $\epsilon$ , to obtain the following dynamical system:

$$\frac{\partial e_f}{\partial \tau} = \mathcal{A}_{f\epsilon}(t)e_f + \epsilon f_t(t, x_s, e_f + \Sigma(t, x_s, 0, \epsilon)) - \epsilon f_f(t, x_s, \Sigma(t, x_s, 0, \epsilon)). \quad (48)$$

Referring to the above system with  $\epsilon = 0$ , we have from Assumption 4 that there exists a Lyapunov functional  $\bar{W}: \mathcal{H}_f(t) \rightarrow \mathbb{R}_{\geq 0}$  and a set of positive real numbers  $(\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4, \bar{b}_5, \bar{b}_6)$ , such that for all  $e_f \in \mathcal{H}_f(t)$  that satisfy  $\|e_f\| \leq \bar{b}_6$ , the following conditions hold,

$$\begin{aligned} \bar{b}_1 \|e_f(t(\tau))\|_2^2 &\leq \bar{W}(t, e_f(\tau)) \leq \bar{b}_2 \|e_f(\tau)\|_2^2, \\ \frac{\partial \bar{W}}{\partial e_f} \mathcal{A}_{f\epsilon}(t)e_f &\leq -\bar{b}_3 \|e_f(\tau)\|_2^2, \\ \left\| \frac{\partial \bar{W}}{\partial e_f} \right\|_2 &\leq \bar{b}_4 \|e_f(\tau)\|_2, \\ \left\| \frac{\partial \bar{W}}{\partial t} \right\|_2 &\leq \bar{b}_5 \|e_f(\tau)\|_2^2. \end{aligned} \quad (49)$$

Computing the time-derivative of  $\bar{W}(t, e_f)$  along the trajectories of the system of Eq. (48) and using that  $\|f_f(t, x_s, e_f + \Sigma(t, x_s, 0, \epsilon)) - f_f(t, x_s, \Sigma(t, x_s, 0, \epsilon))\|_2 \leq k_5 \|e_f\|_2$ , where  $k_5$  is a positive real number (which follows from the fact that the states  $(x_s, e_f)$  are bounded), we have

$$\begin{aligned} \frac{\partial \bar{W}}{\partial \tau} &\leq -\bar{b}_3 \|e_f\|_2^2 + \epsilon \bar{b}_5 \|e_f\|_2^2 + \epsilon \bar{b}_4 \|e_f\|_2 k_5 \|e_f\|_2 \\ &\leq -(\bar{b}_3 - \epsilon(\bar{b}_4 k_5 + \bar{b}_5)) \|e_f\|_2^2. \end{aligned} \quad (50)$$

Set  $\epsilon_2 = (\bar{b}_3 - \bar{\delta})/(\bar{b}_4 k_5 + \bar{b}_5)$  and  $\epsilon^* = \min\{\epsilon_1, \epsilon_2\}$ , where  $\bar{\delta} \in \mathbb{R}_{>0}$ . From the above inequality we have that if  $\epsilon \in (0, \epsilon^*]$ , the following bound holds for  $\|e_f(\tau)\|_2$  for all  $t \in [0, \infty)$ ,

$$\|e_f(\tau)\|_2 \leq K_3 \|e_f(0)\|_2 e^{-\gamma'/\epsilon}, \quad (51)$$

where  $K_3, \gamma$  are positive real numbers. From the series expansion of  $\Sigma(t, x_s, 0, \epsilon)$  and the definition of the  $O(\epsilon^{k+1})$ , we also have that,

$$\|\Sigma(t, x_s, 0, \epsilon) - \tilde{x}_f\|_2 \leq \tilde{k}_1 \epsilon^{k+1}, \quad (52)$$

where  $\tilde{x}_f = \Sigma_0(t, \tilde{x}_s, 0) + \epsilon \Sigma_1(t, \tilde{x}_s, 0) + \epsilon^2 \Sigma_2(t, \tilde{x}_s, 0) + \dots + \epsilon^k \Sigma_k(t, \tilde{x}_s, 0)$  and  $\tilde{k}_1$  is a positive number. Combining the bounds of Eqs. (51)–(52), the following bound can be written

$$\|x_f - \tilde{x}_f\|_2 \leq \tilde{k}_1 \epsilon^{k+1} + \tilde{k}_2 e^{-\gamma(t/\epsilon)}, \quad \forall t \in [0, \infty), \quad (53)$$

where  $\tilde{k}_2$  is a positive number. It follows directly that  $x_f = \tilde{x}_f + O(\epsilon^{k+1})$ ,  $\forall t \in [t_b, \infty)$ , where  $t_b$  is the time required for  $x_f$  to approach the inertial manifold (i.e.,  $\tilde{k}_2 e^{-\gamma(t/\epsilon)} = 0$  for  $t \geq t_b$ ).

Defining the error coordinate  $e_s(t) = x_s(t) - \tilde{x}_s(t)$  and differentiating  $e_s(t)$  with respect to time, the following system can be obtained:

$$\frac{de_s}{dt} = \mathcal{A}_s(t) e_s + f_s(t, \tilde{x}_s + e_s, x_f) - f_s(t, \tilde{x}_s, \tilde{x}_f). \quad (54)$$

The representation of the system of Eq. (54) in the fast time-scale  $\tau$  takes the form,

$$\frac{de_s}{d\tau} = \epsilon [\mathcal{A}_s(t) e_s + f_s(t, \tilde{x}_s + e_s, x_f) - f_s(t, \tilde{x}_s, \tilde{x}_f)], \quad (55)$$

where  $(e_s, \tilde{x}_s, t)$  can be considered approximately constant, and thus using that  $|e_s(0)| = 0$  and the continuity of solutions for  $e_s(\tau)$ , the following bound holds,

$$|e_s(\tau)| \leq \epsilon k_7 \tau_b, \quad \forall \tau \in [0, \tau_b), \quad (56)$$

where  $k_7$  is a positive real number and  $\tau_b = t_b/\epsilon = O(1)$ . Based on the fact that  $\tilde{x}_s(t), \tilde{x}_f(t)$  decay exponentially to zero, and from Assumption 3, we have that the system,

$$\frac{de_s}{d\tau} = \mathcal{A}_s(t) e_s + f_s(t, \tilde{x}_s + e_s, \tilde{x}_f) - f_s(t, \tilde{x}_s, \tilde{x}_f) \quad (57)$$

is exponentially stable, which implies that there exists a smooth Lyapunov function  $\tilde{V}: \mathcal{H}_s \rightarrow \mathbb{R}_{\geq 0}$  and a set of positive real numbers  $(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4, \tilde{a}_5)$ , such that for all  $e_s \in \mathcal{H}_s$  that satisfy  $|e_s| \leq \tilde{a}_5$ , the following conditions hold,

$$\begin{aligned} \tilde{a}_1 |e_s|^2 &\leq \tilde{V}(t, e_s) \leq \tilde{a}_2 |e_s|^2, \\ \dot{\tilde{V}}(e_s) &= \frac{\partial \tilde{V}}{\partial t} + \frac{\partial \tilde{V}}{\partial e_s} \left[ \mathcal{A}_s(t) e_s + f_s(t, \tilde{x}_s + e_s, \tilde{x}_f) - f_s(t, \tilde{x}_s, \tilde{x}_f) \right] \\ &\leq -\tilde{a}_3 |e_s|^2, \\ \left| \frac{\partial \tilde{V}}{\partial e_s} \right| &\leq \tilde{a}_4 |e_s|. \end{aligned} \quad (58)$$

From the assumption that  $f_s(t, x_s, x_f)$  is Lipschitz continuous with respect to  $x_s$  and  $x_f$ ,

$$\left| f_s(t, \tilde{x}_s + e_s, x_f) - f_s(t, \tilde{x}_s + e_s, \tilde{x}_f) \right| \leq \tilde{k} \|x_f - \tilde{x}_f\|_2, \quad (59)$$

where  $\tilde{k}$  is a positive number. Substituting the inequality of Eq. (53) to Eq. (59) the following bound is obtained:

$$\left| f_s(t, \tilde{x}_s + e_s, x_f) - f_s(t, \tilde{x}_s + e_s, \tilde{x}_f) \right| \leq \tilde{k} \tilde{k}_1 \epsilon^{k+1} + \tilde{k} \tilde{k}_2 e^{-\gamma(t/\epsilon)}. \quad (60)$$

Computing the time-derivative of  $\tilde{V}(t, e_s)$  along the trajectories of the system of Eq. (54) and using that for  $t \in [0, \infty)$  the bound described in Eq. (60) holds, we have for all  $t \in [0, \infty)$ :

$$\dot{\tilde{V}}(t, e_s) \leq -\tilde{a}_3 |e_s|^2 + \left( \tilde{k}_1 \epsilon^{k+1} + \tilde{k}_2 e^{-\gamma(t/\epsilon)} \right) \tilde{a}_4 |e_s|. \quad (61)$$

From the above inequality, using Corollary 5.3 in [18] and the fact that  $|e_s(0)| = 0$ , we have that the following bound hold for  $|e_s(t)|$  for all  $t \in [0, \infty)$ ,

$$|e_s(t)| \leq \bar{K} e^{-\gamma(t/\epsilon)} + \tilde{K} \epsilon^{k+1}, \quad (62)$$

where  $\bar{K}, \tilde{K}$  are positive real numbers. Since the term  $\bar{K} e^{-\gamma(t/\epsilon)}$  vanishes outside the interval  $[0, t_b)$ , it follows from Eq. (62) that for all  $t \in [t_b, \infty)$ :

$$|e_s(t)| \leq \tilde{K} \epsilon^{k+1}. \quad (63)$$

From the above inequality, the estimate  $x_s(t) = \tilde{x}_s(t) + O(\epsilon^{k+1})$ , for  $t \geq t_b$ , follows directly. ■

*Proof of Theorem 1.* Substituting the output feedback controller of Eq. (29) into the system of Eq. (14), we get

$$\begin{aligned}
\frac{d\eta}{dt} &= \mathcal{A}_s(t)\eta + \mathcal{B}_s(t)(p_0(t, \eta) + Q_0(t, \eta)v \\
&\quad + \epsilon[p_1(t, \eta) + Q_1(t, \eta)v]) \\
&\quad + f_s(t, \eta, \epsilon\Sigma_1(t, \eta, \bar{u}_0)) \\
&\quad + L(\mathcal{S}(t)x_s + \mathcal{S}(t)x_f - \mathcal{S}(t)\eta - \epsilon\mathcal{S}(t)\Sigma_1(t, \eta, \bar{u}_0)), \\
\frac{dx_s}{dt} &= \mathcal{A}_s(t)x_s + \mathcal{B}_s(t)(p_0(t, \eta) + Q_0(t, \eta)v \\
&\quad + \epsilon[p_1(t, \eta) + Q_1(t, \eta)v]) + f_s(t, x_s, x_f), \\
\epsilon\frac{\partial x_f}{\partial t} &= \mathcal{A}_{f\epsilon}(t)x_f + \epsilon\mathcal{B}_f(t)(p_0(t, \eta) + Q_0(t, \eta)v \\
&\quad + \epsilon[p_1(t, \eta) + Q_1(t, \eta)v]) + \epsilon f_f(t, x_s, x_f), \\
y_{c_i} &= \mathcal{C}_i(t)x_s + \mathcal{C}_i(t)x_f, \quad i = 1, \dots, l.
\end{aligned} \tag{64}$$

Performing a two-time-scale decomposition in the above system, the fast subsystem takes the form,

$$\frac{\partial x_f}{\partial \tau} = \mathcal{A}_{f\epsilon}(t)x_f, \tag{65}$$

which is assumed to be exponentially stable (Assumption 4). The  $O(\epsilon^2)$  approximation of the closed-loop inertial form is

$$\begin{aligned}
\frac{d\eta}{dt} &= \mathcal{A}_s(t)\eta + \mathcal{B}_s(t)(p_0(t, \eta) + Q_0(t, \eta)v + \epsilon[p_1(t, \eta) + Q_1(t, \eta)v]) \\
&\quad + f_s(t, \eta, \epsilon\Sigma_1(t, \eta, \bar{u}_0)) + L(\mathcal{S}(t)x_s + \epsilon\mathcal{S}(t)\Sigma_1(t, x_s, \bar{u}_0) \\
&\quad - \mathcal{S}(t)\eta - \epsilon\mathcal{S}(t)\Sigma_1(t, \eta, \bar{u}_0)), \\
\frac{dx_s}{dt} &= \mathcal{A}_s(t)x_s + \mathcal{B}_s(t)(p_0(t, \eta) + Q_0(t, \eta)v \\
&\quad + \epsilon[p_1(t, \eta) + Q_1(t, \eta)v]) + f_s(t, x_s, \epsilon\Sigma_1(t, x_s, \bar{u}_0)), \\
y_{cs_i} &= \mathcal{C}_i(t)x_s + \epsilon\mathcal{C}_i(t)\Sigma_1(t, x_s, \bar{u}_0), \quad i = 1, \dots, l.
\end{aligned} \tag{66}$$

Using the hypothesis  $\eta(0) = x_s(0)$ , the above system can be written as

$$\begin{aligned} \frac{dx_s}{dt} = & \mathcal{A}_s(t)x_s + \mathcal{B}_s(t)(p_0(t, x_s) + Q_0(t, x_s)v \\ & + \epsilon[p_1(t, x_s) + Q_1(t, x_s)v]) \\ & + f_s(t, x_s, \epsilon \Sigma_1(t, x_s, \bar{u}_0)), \\ y_{cs_i} = & \mathcal{C}_i(t)x_s + \epsilon \mathcal{C}_i(t)\Sigma_1(t, x_s, \bar{u}_0), \quad i = 1, \dots, l. \end{aligned} \quad (67)$$

Computing the time-derivatives of the controlled output  $y_{cs_i}$  up to order  $r_i$  and substituting into Eq. (31), one can show that the input-output response of Eq. (31) is enforced in the above closed-loop system. Furthermore, an approach, similar to the one in [22], can be followed to establish that Assumptions 1 and 2 of the theorem guarantee that the system of Eq. (67) is locally exponentially stable. Therefore, we have that the system for Eq. (66) is locally exponentially stable and its outputs  $y_{cs_i}$ ,  $i = 1, \dots, l$ , change according to Eq. (31). A direct application of the result of Proposition 1 then yields that there exist positive real numbers  $\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\epsilon}^*$  such that if  $|x_s(0)| \leq \tilde{\mu}_1$ ,  $\|x_f(0)\|_2 \leq \tilde{\mu}_2$ , and  $\epsilon \in (0, \tilde{\epsilon}^*]$ , the closed-loop infinite-dimensional system is exponentially stable and the relation of Eq. (30) holds. ■

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