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Nonlinear output regulation for invertible nonlinear MIMO systems

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SUMMARY

In this paper, we address the problem of output regulation for a broad class of multi-input multi-output (MIMO) nonlinear systems. Specifically, we consider input–affine systems, which are invertible and input–output linearizable. This class includes, as a trivial special case, the class of MIMO systems which possess a well-defined vector relative degree. It is shown that if a system in this class is strongly minimum phase, in a sense specified in the paper, the problem of output regulation can be solved via partial-state feedback or via (dynamic) output feedback. The result substantially broadens the class of nonlinear MIMO systems for which the problem in question is known to be possible. Copyright © 2015 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The study of output regulation for nonlinear systems was initiated with the pioneering contributions of [1], and later was developed by several authors (see, for instance, [2–4] and references therein). Instrumental, in the analysis of this design problem, is the characterization of the so-called 'zero-error steady-state (controlled invariant) manifold', a manifold, which is rendered invariant by feedback and on which the regulated output vanishes. The characterization of such manifold was initially provided in [1], where its existence was expressed in terms of a pair of equations, that subsequently became known as *regulator equations* and then refined in various forms (see, for instance, [2] and [5]). Basically, the problem of output regulation can be solved by means of a controller that includes an 'internal model' of the exogenous inputs (e.g., [2, 4, 6–8]), whose purpose is to provide a 'feedforward' control that renders the manifold in question invariant and a 'stabilizer', whose purpose is to render the manifold in question asymptotically stable. To the best of the authors' knowledge, the output regulation problem for single-input single-output (SISO) nonlinear systems is pretty well understood and solved under reasonable hypotheses, while this problem for multi-input multi-output (MIMO) nonlinear systems, despite a number of significant encouraging contributions, is still at a rather preliminary stage of development.

There is a special class of MIMO systems for which the extension of the existing theory of regulator design for SISO theory is relatively easy, the class of systems which have the same number of inputs and outputs, possess a well-defined *vector relative degree* and a globally defined *normal form*. Such systems are studied, for instance, in the recent paper [9], where the design of a regulator

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is developed, under the assumption that the zero dynamics are input-to-state stable (ISS) and that the so-called high-frequency gain matrix satisfies a suitable 'positivity condition'. It must be observed, though, that this class of systems is a very special class of MIMO nonlinear systems. In fact, a MIMO nonlinear system may not possess a vector relative degree, the existence of globally defined normal form is a non-generic property because it requires certain vector fields to commute, and the crucial property that the zero dynamics are ISS can only be checked once such normal form is obtained.

In the recent paper [10], we have initiated a systematic study of a much broader class of MIMO nonlinear systems. These are the systems which are invertible (in the sense of [11, 12], see also [13]) and input–output linearizable (i.e., the input–output behavior can be rendered linear by means of static state-feedback law). This is indeed a class of systems that substantially broadens the class of MIMO systems possessing a vector relative degree (which are trivially invertible and input–output linearizable). We stress that, while this class of systems is characterized by the existence of an 'input–output linearizing' feedback law, no explicit use of such law is needed: it is only assumed that such feedback law exists (in fact, explicit use of such law would turn out in a poorly robust design, as such law entails exact cancelation of nonlinear terms). Easily checkable conditions for the existence of such law are known (see [14]) and can be found e.g, in [15]. If a system belongs to this class, the nonlinear inversion algorithm of Hirschorn [11], an enhanced version of the so-called structure algorithm of Silverman [16], can be implemented in a simplified manner and this, in turn, makes it possible to develop a 'partial' normal form that provides the ground for the development – under appropriate 'minimum-phase' assumptions, equivalent to the assumption that the zero-dynamics are ISS – of a method for global/semi-global stabilization via partial-state feedback. In [10], such minimum-phase hypothesis is characterized – similarly to what is carried out in [17, 18] – in coordinate free terms (thus not appealing to the existence of a globally defined normal form) and this extends to this class of MIMO systems some earlier results proved in [19] for SISO systems. If the partial states are not directly available for measurements, a dynamical system driven only by the actual outputs can be designed that, extending in a nontrivial way the results of [20], provides appropriate asymptotic estimates of such partial state (see [21]).

The method proposed in [10] is based on the use of (possibly nonlinear) high-gain feedback from a set of 'redesigned outputs'. As it is known, the use of high-gain output feedback on a MIMO nonlinear system requires appropriate assumptions on the so-called high-frequency gain matrix, the simpler version of which is that this matrix is (or can be rendered, via state-independent changes of coordinates) positive definite. In the paper [10], a rather milder assumption is proposed in this respect, namely that all leading principal minors of such matrix are bounded away from zero, and that the matrix is independent of a limited set of selected state variables. The latter condition is automatically satisfied if such matrix only depends on a vector exogenous variables, and this makes such assumption appealing in the context of the problem output regulation, where such exogenous variables may represent disturbance inputs and/or uncertain parameters.

The purpose of the present paper is to develop a theory of output regulation for this class of MIMO systems. This is achieved by combining the procedure for the design of internal model suggested in [22], and extended in [9] to the case of MIMO possessing a vector relative degree, with the method developed in [10] for the broader class of MIMO systems described in the previous paragraphs. The main reason why such development is not straightforward resides in the fact that, to the end of taking advantage of the weakened assumption on the high-frequency gain matrix introduced in [10], the design of an internal model proposed in [9] must be appropriately enhanced, as shown later in the paper.

As a result, a theory of output regulation is derived for a substantially broader class of MIMO nonlinear systems, those which are invertible, are input–output linearizable (and do not necessarily possess a globally defined normal form), are 'minimum-phase', and have a 'high-frequency gain matrix' that satisfies the assumptions similar to those used in [10]. The paper is organized as follows. Section 2 presents the problem formulation, introduces the basic assumptions and summarizes relevant results from [10]. The main results are given in Section 3, where the internal model is designed (inspired by the result of [3]) and the global stabilization of the zero-error steady-state manifold is discussed. In Section 4, a simple example is presented, and finally, in Section 5, a brief conclusion is drawn.

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2. SETUP AND ASSUMPTIONS

2.1. Problem formulation

This paper deals with the problem of output regulation for a rather general class of nonlinear multivariable nonlinear systems defined by equations of the form

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{x} &= f(w,x) + g(w,x)u \\ y &= h(w,x) \\ y_{\mathrm{m}} &= h_{\mathrm{m}}(w,x) \end{aligned} \tag{1}$$

with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$, regulated output $y \in \mathbb{R}^m$, and measured output $y_{\mathrm{m}} \in \mathbb{R}^p$, in which the variable w models exogenous inputs that might represent reference/commands (to be tracked/rejected) or uncertain parameters. It is assumed that $w \in \mathcal{W} \subset \mathbb{R}^{n_w}$, where $\mathcal{W}$ is a compact set, invariant for the dynamics of $\dot{w} = s(w)$. The mappings $s(w)$, $f(w,x)$, $g(w,x)$, $h(w,x)$, and $h_{\mathrm{m}}(w,x)$ are smooth mappings.

The objective of the paper is to solve a problem of output regulation, that is, to design an appropriate output feedback controller of the form

$$\begin{aligned} \dot{x}_{\mathrm{c}} &= f_{\mathrm{c}}(x_{\mathrm{c}}, y_{\mathrm{m}}) \\ u &= h_{\mathrm{c}}(x_{\mathrm{c}}, y_{\mathrm{m}})\,, \end{aligned} \tag{2}$$

with state $x_{\mathrm{c}} \in \mathbb{R}^{n_{\mathrm{c}}}$, yielding a closed-loop system in which, for any initial condition $(w(0), x(0), x_{\mathrm{c}}(0)) \in \mathcal{W} \times \mathbb{R}^n \times \mathbb{R}^{n_{\mathrm{c}}}$

- the resulting trajectory of (1) and (2) is bounded, and
- $\lim_{t\to\infty} y(t) = 0$.

Remark
The problem statement given earlier seeks boundedness of trajectories and convergence to zero of the regulated output *for all* initial conditions. If this is the case, it is said that the controller (2) solves a problem of *global* output regulation. A weaker version of the problem is the one in which a compact set $X \times X_{\mathrm{c}}$ is given and a regulator of the form (2) is sought such that boundedness of trajectories and convergence to zero of the regulated output hold for all initial conditions $(w(0), x(0), x_{\mathrm{c}}(0)) \in \mathcal{W} \times X \times X_{\mathrm{c}}$. Note that for a problem cast in these terms, a controller (2) solving the problem *depends* on the choice of the compact set $X \times X_{\mathrm{c}}$. This (weaker) version of the problem is usually referred to as *semi-global* output regulation. In what follows, we provide assumptions under which a solution of the problem of global output regulation can be found.[‡] These assumptions – as it will be seen later – entail strong growth condition for certain functions. Such growth conditions are not needed if only semi-global regulation is sought. However, taking these strong assumptions (and, as a byproduct, obtaining global regulation) helps to streamline the exposition. In some remarks here and there along the paper, we will point out how such assumptions could be weakened if just semi-global regulation is sought.

2.2. Invertible and input–output linearizable MIMO nonlinear systems

The class of systems considered in this paper is characterized by the two following assumptions.

Assumption 1
There exists a state feedback law $u = \alpha(w,x) + \beta(w,x)v$, with invertible $\beta(w,x)$, such that the resulting system

[‡] More precisely, we show how global output regulation can be achieved when the control law (2) is driven by a measured output defined as in (4).

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$$
\begin{aligned}
\dot{w} &= s(w) \\
\dot{x} &= f(w,x) + g(w,x)\alpha(w,x) + g(w,x)\beta(w,x)v, \\
y &= h(w,x)
\end{aligned}
$$

has a linear input–output behavior between input v and output y.

Assumption 2
The system is strongly invertible, in the sense of [11, 12].

Systems satisfying Assumption 1 have been thoroughly studied in the literature (see, e.g., Chapter 5 of [15], where necessary and sufficient conditions for the existence of such $\alpha(w,x)$ and $\beta(w,x)$ are determined). We stress that in the present paper, we *do not make use* of the feedback law $u = \alpha(w,x) + \beta(w,x)v$ indicated in Assumption 1, for the simple fact that this law requires access to the *full* state x, and this is not the case in the present setting, where partial state or simply output feedback is used. We only use the Assumption to identify suitable structural properties of such systems. We also stress that the class of systems characterized by these two assumptions contains as a *very special case* the class of multivariable systems that have a well-defined vector relative degree.

2.3. *Output redesign*

A system characterized by Assumptions 1 and 2 does not necessarily have a vector relative degree. However, it is possible to 'redesign' its outputs in such a way as to obtain a system having vector relative $\{1, 1, \dots, 1\}$. [§] As shown in [10], if Assumptions 1 and 2 hold, the implementation of the so-called structure algorithm leads to the definition of a set of integers $m_1, m_2, \dots, m_{k^*}$ satisfying $\sum_{i=1}^{k^*} = m$, a set of integers $N_1 < N_2 < \dots < N_{k^*}$ satisfying $\sum_{i=1}^{k^*} m_i N_i := N \leqslant n$, a partition of y into k^* sub-vectors $y_i \in \mathbb{R}^{m_i}$ and, for each $i = 1, 2, \dots, k^*$, of a sets of smooth $\mathbb{R}^{m_i}$-valued functions $\bar{\xi}^i_1(w,x), \bar{\xi}^i_2(w,x), \dots, \bar{\xi}^i_{N_i}(w,x)$, which can be chosen as a *partial set of state variables*. The $\bar{\xi}^i_j(w,x)$'s obey equations of the form

$$
\begin{aligned}
\dot{\bar{\xi}}^1 &= A^1\bar{\xi}^1 + G_1\bar{\xi}^1_{N_1} \\
\dot{\bar{\xi}}^1_{N_1} &= a^1(w,x) + b^1(w,x)u \\
\dot{\bar{\xi}}^2 &= A^2\bar{\xi}^2 + G_2\bar{\xi}^2_{N_2} + B_2 Q_1(w,x,u) \\
\dot{\bar{\xi}}^2_{N_2} &= a^2(w,x) + b^2(w,x)u \\
&\dots \\
\dot{\bar{\xi}}^k &= A^k\bar{\xi}^k + G_k\bar{\xi}^k_{N_k} + B_k Q_{k-1}(w,x,u) \\
\dot{\bar{\xi}}^k_{N_k} &= a^k(w,x) + b^k(w,x)u \\
&\dots
\end{aligned}
\tag{3}
$$

in which

$$
\begin{aligned}
y_k &= \bar{\xi}^k_1 \\
\bar{\xi}^k &= \mathrm{col}\left(\bar{\xi}^k_1, \bar{\xi}^k_2, \dots, \bar{\xi}^k_{N_k-1}\right) \\
Q_k(w,x,u) &= \begin{pmatrix} a^1(w,x) + b^1(w,x)u \\ \dots \\ a^k(w,x) + b^k(w,x)u \end{pmatrix}
\end{aligned}
$$

[§]This procedure corresponds – in the present general setting – to the procedure by means of which, in a single-input single-output system, the relative degree is lowered to 1 by 'addition of (stable) zeros'.

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for $k = 1, \dots, k^*$. In particular, the $m \times m$ matrix

$$\frac{\partial Q_{k^*}(w,x,u)}{\partial u} = \begin{pmatrix} b^1(w,x) \\ \dots \\ b^{k^*}(w,x) \end{pmatrix},$$

that – in the present (more general) context – plays a role corresponding to that of the so-called high-frequency matrix of a MIMO system having a vector relative degree, *is non-singular* for all (w,x). We stress that (3) is not a full model of the system if N is strictly less than n. For such reason, this is referred to as to a *partial* normal form.

The 'redesigned' output map is defined as $\tilde{h}(w,x) = \mathrm{col}(\tilde{h}_1(w,x), \dots, \tilde{h}_{k^*}(w,x))$ in which

$$\tilde{h}_k(w,x) = c_1^k \bar{\xi}_1^k(w,x) + c_2^k \bar{\xi}_2^k(x) + \ldots + c_{N_k-1}^k \bar{\xi}_{N_k-1}^k(w,x) + \bar{\xi}_{N_k}^k(w,x) \tag{4}$$

and the c_j^k's, for $j = 1, \dots, N_k - 1$, are such that all roots of the polynomial

$$\lambda^{N_k-1} + c_{N_k-1}^k \lambda^{N_k-2} + \ldots + c_2^k \lambda + c_1^k = 0$$

have a negative real part.

In [10], it has been shown that a change of variables of the form

$$\begin{aligned} \xi^1 &= \bar{\xi}^1 \\ \xi^k &= \bar{\xi}^k + M_{k-1} \begin{pmatrix} \tilde{y}_1 \\ \cdots \\ \tilde{y}_{k-1} \end{pmatrix} \qquad \text{for } k = 2, \dots, k^*, \end{aligned} \tag{5}$$

in which M_{k-1} is a suitably defined matrix, transforms the full set of equations (3) into a set of equations that can be split in two subsets: a subset of equations of the form

$$\dot{\xi} = F\xi + G\tilde{y} \tag{6}$$

in which $\xi = \mathrm{col}(\xi^1, \dots, \xi^{k^*})$ and a subset of equations of the form

$$\dot{\tilde{y}} = R(w,x) + B(w,x)u\,, \tag{7}$$

in which

$$B(w,x) = M \frac{\partial Q_{k^*}(w,x,u)}{\partial u}$$

where M is a non-singular constant matrix. Moreover, because of the special choice of the c_j^k's in (4), it turns out that the matrix F in (6) is a Hurwitz matrix. For the actual expressions of F and M, the reader is referred to [10]. The matrix $B(w,x)$ is non-singular for all (w,x), and therefore, (7) shows that the system, with redesigned output $\tilde{y} = \tilde{h}(w,x)$,

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{x} &= f(w,x) + g(w,x)u \\ \tilde{y} &= \tilde{h}(w,x) \end{aligned}$$

has vector relative degree $\{1, 1, \dots, 1\}$. Moreover, because (6) is a stable linear system driven by $\tilde{y}$, it is seen that if $\tilde{y}(t)$ converges to zero, then also all the $\xi^i(t)$'s converge to zero, and consequently, as seen from (5), also all the $\bar{\xi}^i(t)$'s converge to zero. If this is the case, then also the regulated error $y(t)$ converges to zero.

In view of this, it is readily concluded that if a controller of the form

$$\begin{aligned} \dot{x}_c &= f_c(x_c, \tilde{y}) \qquad x_c \in \mathbb{R}^{n_c} \\ u &= h_c(x_c, \tilde{y})\,, \end{aligned} \tag{8}$$

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solves a problem of output regulation for the system (with measured output $\tilde{y}$)

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{x} &= f(w,x) + g(w,x)u \\ y &= h(w,x) \\ y_{\mathrm{m}} &= \tilde{h}(w,x) \quad = \quad \tilde{y}\,, \end{aligned} \tag{9}$$

then the very same controller solves the problem of output regulation for the original system (1).

Controller (8) is a controller of the form (2), driven by the redesigned output $\tilde{y}$. Of course, it would be nice to be able to find a controller driven only by the regulated output y rather than by the redesigned output $\tilde{y}$. Now, $\tilde{y}$ is a linear combination of the various components $\bar{\xi}_j^k$ of the partial set of new coordinates used to define the partial normal form (3). If the latter is not directly accessible for feedback, they can anyway be asymptotically estimated by means of a generalized 'high-gain observer', driven by the actual regulated output y, as described in detail in [10, 21]. The resulting control scheme would be able, though, to yield only semi-global output regulation. To avoid repetitions, the design of such 'high-gain observer' will not be discussed in the present paper, and the reader is referred to the cited literature. Thus, from now on, we concentrate on the problem of solving a problem of output regulation for (9), which is a system having vector relative degree $\{1, 1, \ldots, 1\}$, by means of a controller defined as in (8). Further remarks about the design of a controller driven by the actual regulated output y will be provided at the end of the paper.

2.4. *Regulator equations and minimum-phase assumption*

In view of the previous discussion, we concentrate, from now on, on the solution of a problem of output regulation for system (9), in which we recall y denotes the regulated output, while $\tilde{y}$ denotes the measured output. To this end, we need first to formulate appropriate assumptions. As customary in a problem of output regulation, such assumptions deal with (i) the possibility of creating, in the controlled system, an invariant manifold on which the regulated variable vanishes (the so-called zero-error manifold) and (ii) conditions that make the design of a stabilizer (a controller able to render such manifold attractive) possible.

The assumption that makes the fulfillment of the condition (i) possible is the existence, for the (original) controlled system (1), of a solution of the so-called regulator equations of [1]. To this end, we assume the existence of a smooth map $\pi : \mathcal{W} \to \mathbb{R}^n$ and of a smooth map $\Psi : \mathcal{W} \to \mathbb{R}^m$ such that

$$\begin{aligned} \frac{\partial \pi}{\partial w} s(w) &= f(w,\pi(w)) - g(w,\pi(w))\Psi(w) \\ 0 &= h(w,\pi(w))\,. \end{aligned}$$

Looking at the structure of the Equation (3), it is readily seen that $\bar{\xi}_j^k(w,\pi(w)) = 0$ for all k and j, and that $\Psi(w)$ is the unique solution of

$$Q_{k^*}(w,\pi(w),-\Psi(w)) = \begin{pmatrix} a^1(w,\pi(w)) \\ \cdots \\ a^{k^*}(w,\pi(w)) \end{pmatrix} - \begin{pmatrix} b^1(w,\pi(w)) \\ \cdots \\ b^{k^*}(w,\pi(w)) \end{pmatrix} \Psi(w) = 0\,.$$

Because the redesigned output $\tilde{y}$ satisfies an equation of the form (7), the map $\Psi(w)$ can also be seen as the unique solution of

$$R(w,\pi(w)) - B(w,\pi(w))\Psi(w) = 0\,. \tag{10}$$

It is known from the literature that if such map $\pi(w)$ exists, the resulting map $\Psi(w)$ can be taken as a point of departure for the design of a control structure having the property indicated in (i). Several alternative methods are possible, among which the most general one seems to be the method proposed in [3]. To be able to fulfill also the property indicated in (ii), we assume that the system under consideration is *strongly minimum-phase*, in the sense defined in [10] (see also [17]). In particular, we assume the following:

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Assumption 3
System (9) is uniformly detectable relative to the compact set

$$\mathcal{A} = \{(w, x) \in \mathcal{W} \times \mathbb{R}^n : x = \pi(w)\},$$

with a linear gain function, that is, there exists a class $\mathcal{KL}$ function β_1 and a number $\ell_1 > 0$ such that for every initial conditions ,$(w(0), x(0)) \in \mathcal{W} \times \mathbb{R}^n$ and every input u, the inequality ¶

$$|(w(t), x(t))|_{\mathcal{A}} \leqslant \beta_1(|(w(0), x(0))|_{\mathcal{A}}, t) + \ell_1 \|\tilde{y}\|_{[0,t)} \tag{11}$$

holds along the corresponding solutions $(w(t), x(t))$, as long as they exist.

Remark
The assumption in question is given as a property of the system (9), whose output is the redesigned output $\tilde{y}$. As such, it may seem that the fulfillment of the property in question might be influenced by the process followed to derive, from the given output y, the redesigned output $\tilde{y}$. Actually, this is not the case, and it can be claimed that the property expressed by Assumption 3 is a property of the original system (1). This issue has been thoroughly analyzed in [10] and is not repeated here for obvious reasons.

Remark
If only semi-global output regulation is sought, this assumption could be weakened, replacing (11) by an inequality of the form

$$|(w(t), x(t))|_{\mathcal{A}} \leqslant \beta_1(|(w(0), x(0))|_{\mathcal{A}}, t) + \ell_1(\|\tilde{y}\|_{[0,t)})$$

in which $\ell_1(\cdot)$ is any class $\mathcal{K}$ function, which is linearly bounded at the origin.

We conclude the subsection with an additional assumption that, in a sense, matches the requirement that the gain function in (11) is a linear function and is helpful in obtaining a global convergence result. To this end, consider the function

$$\delta_0(w, x) := R(w, x) - B(w, x)\Psi(w),$$

and bearing in mind (10) observe that

$$\delta_0(w, x) = [R(w, x) - R(w, \pi(w))] - [B(w, x) - B(w, \pi(w))]\Psi(w).$$

It is seen from this that the function $\delta_0(w, x)$ vanishes on the set $\mathcal{A}$, and because all functions involved are smooth functions, it follows that $|\delta_0(w, x)| \leqslant \alpha(|(w, x)|_{\mathcal{A}})$ for some class $\mathcal{K}$ function $\alpha(\cdot)$, which is linearly bounded in a neighborhood of the origin. In what follows, we assume that the whole function $\alpha(\cdot)$ is bounded by a linear function.

Assumption 4
There exists a number $c_0 > 0$ such that

$$|R(w, x) - B(w, x)\Psi(w)| \leqslant c_0 |(w, x)|_{\mathcal{A}}.$$

Remark
If only semi-global output regulation is sought, this assumption is not needed and the estimate $|\delta_0(w, x)| \leqslant \alpha(|(w, x)|_{\mathcal{A}})$, in which $\alpha(\cdot)$ is linearly bounded in a neighborhood of the origin, suffices.

¶Here $|(w, x)|_{\mathcal{A}}$ denotes the distance of (w, x) from the set $\mathcal{A}$.

3. THE CONTROLLER

3.1. A property of the high-frequency gain matrix

We show in this section that for a system of the class considered in this paper, if the strong minimum phase assumption is fulfilled, the problem of output regulation can be solved globally, by means of a feedback law driven by the redesigned output $\tilde{y}$. The following lemmas, whose proofs can be found in [10], are important in the proof of convergence.

Lemma 1
Suppose $B(w,x)$ is bounded and all its leading principal minors $\Delta_i(w,x)$ satisfy

$$0 < b_0 \leqslant |\Delta_i(w,x)| \tag{12}$$

for some b_0. Then there exist a non-singular diagonal square $m \times m$ matrix E, whose entries are equal either to 1 or to -1, The i-th element on the diagonal of E is equal to $\mathrm{sign}(\Delta_i/\Delta_{i-1})$, with $\Delta_0 = 1$. Note also that E satisfies $E^2 = I$, a property that is often used in the sequel. a symmetric positive definite $m \times m$ matrix $S(w,x)$ and positive numbers a_1, a_2 satisfying

$$a_1|y|^2 \leqslant y^{\mathrm{T}} S(w,x)^{-1} y \leqslant a_2|y|^2\,,$$

and a strictly upper triangular bounded $m \times m$ matrix $U(w,x)$ such that

$$B(w,x) = ES(w,x)(I + U(w,x))\,.$$

Lemma 2
Let $U(w,x)$ be a strictly upper triangular bounded $m \times m$ matrix. Set

$$K = \mathrm{diag}\left(k^m, k^{m-1}, \ldots, k\right). \tag{13}$$

Then, there is a number $k_0 > 0$ such that, if $k > k_0$,

$$[I + U(w,x)]K + K[I + U(w,x)]\mathrm{T} \geqslant kI \qquad \text{for all } (w,x)\,.$$

3.2. The internal model

We proceed now with the design of a control structure possessing an invariant manifold on which the regulated variable vanishes. As known from the literature, this can achieved by including, in the controller, a dynamical system (usually referred to as 'internal model') able to generate, in steady-state, the control input $u^*(t) = \Psi(w(t))$, which is precisely the input required to force the regulated error to remain identically zero.

To the purpose of designing an internal model, inspired by [3], we pick a controllable pair $(F,G) \in \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times 1}$, with F a Hurwitz matrix and $d > 0$, and let $\mathbf{F} \in \mathbb{R}^{md\times md}$ and $\mathbf{G} \in \mathbb{R}^{md\times m}$ be defined as

$$\mathbf{F} = \mathrm{blkdiag}(F, F, \ldots, F)$$
$$\mathbf{G} = \mathrm{blkdiag}(G, G, \ldots, G)\,.$$

The design of the internal model is based on the following fundamental result, proven in [3].

Lemma 3
Let $d > 2n_w + 2$. There exist an $\ell > 0$ and a subset $S \subset \mathbb{C}$ of zero Lebesgue measure such that if the eigenvalues of F [‖] are in $\{\lambda \in \mathbb{C} : \mathrm{Re}[\lambda] \leqslant -\ell\}$, then there exist a differentiable function $\sigma_i : \mathcal{W} \to \mathbb{R}^d$ and a continuous bounded function $\gamma_i : \mathbb{R}^d \to \mathbb{R}$ such that, for all $w \in \mathcal{W}$,

$$\begin{aligned} \frac{\partial \sigma_i}{\partial w} s(w) &= F\sigma_i(w) + G[E\Psi(w)]_i \\ [E\Psi(w)]_i &= \gamma_i(\sigma_i(w)) \end{aligned} \tag{14}$$

‖ Please note that the matrix denoted here by F has nothing to do with the matrix F in (6). We have preferred this mild abuse of notation to the purpose of avoiding unnecessary subscripts.

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in which $[E\Psi(w)]_i$ denotes the i-th entry of $E\Psi(w)$.

Observe that setting

$$\begin{aligned}\sigma(w) &= \mathrm{col}(\sigma_1(w), \sigma_2(w), \ldots, \sigma_m(w)) \\ \gamma(\eta_1, \eta_2, \ldots, \eta_m) &= \mathrm{col}(\gamma_1(\eta_1), \gamma_2(\eta_2), \ldots, \gamma_m(\eta_m)),\end{aligned}$$

the set of all (14) is rewritten in compact form as

$$\begin{aligned}\frac{\partial \sigma}{\partial w} s(w) &= \mathbf{F}\sigma(w) + \mathbf{G}E\Psi(w) \\ E\Psi(w) &= \gamma(\sigma(w)).\end{aligned} \tag{15}$$

Assumption 5
The function $\gamma(\cdot)$ in (15) is at least C^2, is bounded, and has a bounded gradient $\nabla\gamma$.

Remark
We stress that while the existence of a continuous function $\gamma(\cdot)$ is always guaranteed, as proven in [3], the property that such function is C^2 is an extra hypothesis. Because the only domain where the function $\gamma(\cdot)$ matters is a compact set, the second property in the assumption is not restrictive.

With these in mind and proceeding as in [3, 9], we design a controller having the form

$$\begin{aligned}\dot{\eta} &= \mathbf{F}\eta + \mathbf{G}(\gamma(\eta) + \nu) \\ u &= -E(\gamma(\eta) + \nu) \\ \nu &= K\tilde{y},\end{aligned} \tag{16}$$

where K is defined as in (13). This leads to a closed-loop system that can be viewed as a system with input ν and output $\tilde{y}$

$$\begin{aligned}\dot{w} &= s(w) \\ \dot{x} &= f(w,x) - g(w,x)E(\gamma(\eta) + \nu) \\ \dot{\eta} &= \mathbf{F}\eta + \mathbf{G}(\gamma(\eta) + \nu) \\ \tilde{y} &= \tilde{h}(w,x).\end{aligned} \tag{17}$$

controlled by

$$\nu = K\tilde{y}.$$

Note also, in this setup, that the redesigned output $\tilde{y}$ obeys

$$\dot{\tilde{y}} = R(w,x) - B(w,x)E(\gamma(\eta) + \nu).$$

By construction, in view of (15), in composed system (17), if $\nu = 0$ the manifold $(x,\eta) = (\pi(w), \sigma(w))$ is invariant and on this manifold the output $\tilde{y}$ vanishes. Thus, the matter is to show that the control $\nu = K\tilde{y}$ renders this manifold attractive. To prove the desired convergence result, define

$$\mathbf{K} = \mathrm{blkdiag}(k^m I_d, k^{m-1} I_d, \ldots, k I_d)$$

and change variables as

$$\begin{aligned}\eta &\mapsto z := \mathbf{K}^{-1}(\eta - \sigma(w)) \\ \tilde{y} &\mapsto \varsigma := \tilde{y} + K^{-1}[\gamma(\eta) - \gamma(\sigma(w))].\end{aligned} \tag{18}$$

A simple calculation, in which one takes advantage of the fact that $\mathbf{K}^{-1}\mathbf{F}\mathbf{K} = \mathbf{F}$ and that $\mathbf{K}^{-1}\mathbf{G}K = \mathbf{G}$, shows that, in the new coordinates, the closed-loop system reads as

$$\begin{aligned}\dot{w} &= s(w) \\ \dot{x} &= f(w,x) + g(w,x)u \\ \dot{z} &= \mathbf{F}z + \mathbf{G}\varsigma \\ \dot{\varsigma} &= -B(w,x)KE\varsigma + \delta_0(w,x) + \delta_1(w,z,\varsigma)\end{aligned} \tag{19}$$

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in which

$$\begin{aligned}\delta_0(w,x) &= R(w,x) - B(w,x)\Psi(w)\\ \delta_1(w,z,\varsigma) &= K^{-1}[\nabla\gamma(\mathbf{K}z+\sigma(w)) - \nabla\gamma(\sigma(w))][\mathbf{F}\sigma(w)+\mathbf{G}\gamma(\sigma(w))]\\ &\quad + K^{-1}\nabla\gamma(\mathbf{K}z+\sigma(w))\mathbf{K}[\mathbf{F}z+\mathbf{G}\varsigma]\,.\end{aligned}$$

In what follows, it will be shown that system (19), seen as a system with output ς, is uniformly detectable relative to the compact set

$$\mathcal{A}^* = \{(w,x,z) : (w,x)\in\mathcal{A}, z=0\}, \tag{20}$$

with a linear gain function. Moreover, it will be also shown that an estimate of the form

$$|\varsigma(t)| \leqslant \beta(|\varsigma(0)|,t) + \ell\,\|\,|(w,x,z)|_{\mathcal{A}^*}\|_{[0,t)} \tag{21}$$

holds, in which ℓ is a coefficient that can be arbitrarily assigned, by properly choosing the value of the gain parameter k in K. This, in view of small gain theorem, makes it possible to arrive at the desired convergence result.

We begin by establishing appropriate bounds on the terms $\delta_0(w,x)$ and $\delta_1(w,z,\varsigma)$ that appear in the last equation of (19). The term $\delta_0(w,x)$ is bounded as specified in Assumption 4. The relevant properties of $\delta_1(w,z,\varsigma)$ are described in the following lemma.

Lemma 4

There exist positive numbers L_0, c_1, and c_2, all independent of k, such that

$$\begin{aligned}|K^{-1}[\gamma(\mathbf{K}z+\sigma(w)) - \gamma(\sigma(w))]| &\leqslant L_0|z|\\ |\delta_1(w,z,\varsigma)| &\leqslant c_1|\varsigma| + c_2|z|\,,\end{aligned} \tag{22}$$

for all $w\in\mathcal{W}, z\in\mathbb{R}^{md}, \varsigma\in\mathbb{R}^m$.

Proof

As a consequence of Assumption 5, the entries of $\gamma(\eta)$ are globally Lipschitz. Note that $K^{-1}\gamma(\mathbf{K}z+\sigma(w)) - \gamma(\sigma(w))$ is a vector whose i-th element is $k^{i-m-1}[\gamma_i(k^{m+1-i}z_i+\sigma_i(w)) - \gamma_i(\sigma_i(w))]$. This can be bounded as

$$|k^{i-m-1}\left[\gamma_i\left(k^{m+1-i}z_i+\sigma_i(w)\right) - \gamma_i(\sigma_i(w))\right]| \leqslant M_i|z_i|$$

for some positive constant M_i, which yields the first inequality.

As far as the second inequality is concerned, a further insight on the expression of $\nabla\gamma(\eta)$ shows that

$$\nabla\gamma(\eta) = \text{blkdiag}\left(\frac{\partial\gamma_1(\eta_1)}{\partial\eta_1},\ldots,\frac{\partial\gamma_m(\eta_m)}{\partial\eta_m}\right).$$

From this, we can see that

$$K^{-1}\nabla\gamma(\eta)\mathbf{K} = \nabla\gamma(\eta)\,.$$

By Assumption 5, $\nabla\gamma(\eta)$ is bounded. Hence, there exists a positive number L_1 such that

$$|K^{-1}\nabla\gamma(\mathbf{K}z+\sigma(w))\mathbf{K}| \leqslant L_1\,,$$

for all $w\in\mathcal{W}$ and $z\in\mathbb{R}^{md}$. Because $\mathcal{W}$ is compact, there exists a positive number L_2 such that

$$|\mathbf{F}\sigma(w)+\mathbf{G}\gamma(\sigma(w))| \leqslant L_2\,,$$

for all $w\in\mathcal{W}$. Finally, observe that

$$K^{-1}[\nabla\gamma(\mathbf{K}z+\sigma(w)) - \nabla\gamma(\sigma(w))]$$

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is a block-diagonal matrix, with the i-th block given by

$$k^{i-m-1}[\nabla\gamma_i(k^{m+1-i}z_i + \sigma_i(w)) - \nabla\gamma_i(\sigma_i(w))]\,.$$

Assumption 5 implies that also all $\nabla\gamma_i(\eta)$'s are globally Lipschitz. Therefore, there exists a positive number L_3 such that

$$|K^{-1}[\nabla\gamma(\mathbf{K}z + \sigma(w)) - \nabla\gamma(\sigma(w))])| \leqslant L_3|z|\,.$$

All such estimates, in view of the expression of $\delta_1(w, z, \varsigma)$, prove the second estimate in (22). □

Next, we show that system (19) has the indicated detectability property.

Proposition 1
Suppose Assumptions 3, 4, and 5 hold. Then, system (19), viewed as a system with output ς, is uniformly detectable relative to the compact set $\mathcal{A}^*$, with a linear gain function.

Proof
Consider the (w, x) dynamics in system (19). Bearing in mind the estimate (11), which expresses the property that system (9) is strongly minimum phase, the changes of coordinates (18), and the first of (22), we can obtain

$$\begin{aligned}
|(w(t), x(t))|_{\mathcal{A}} &\leqslant \beta_1(|(w(0), x(0))|_{\mathcal{A}}, t) + \ell_1\|\tilde{y}\|_{[0,t)} \\
&= \beta_1(|(w(0), x(0))|_{\mathcal{A}}, t) + \ell_1\left(\|\varsigma + K^{-1}[\gamma(\mathbf{K}z + \sigma(w)) - \gamma(\sigma(w))]\|_{[0,t)}\right) \\
&\leqslant \beta_1(|(w(0), x(0))|_{\mathcal{A}}, t) + \ell_1\left(\|\varsigma\|_{[0,t)} + L_0\|z\|_{[0,t)}\right)\,.
\end{aligned}$$

Therefore, we can conclude that there exists a positive number ℓ_2 such that

$$|(w(t), x(t))|_{\mathcal{A}} \leqslant \beta_1(|(w(0), x(0))|_{\mathcal{A}}, t) + \ell_2\|z\|_{[0,t)} + \ell_1\|\varsigma\|_{[0,t)}\,. \tag{23}$$

Moreover, because $\mathbf{F}$ is a Hurwitz matrix, we also know that there exist positive numbers d_0, d_1, λ_0 such that

$$|z(t)| \leqslant d_0|z(0)|e^{-\lambda_0 t} + d_1\|\varsigma\|_{[0,t)}\,. \tag{24}$$

Letting $\mathcal{A}^*$ denote the set defined in (20) and combining (23) and (24), we obtain an estimate of the form

$$|(w(t), x(t), z(t))|_{\mathcal{A}^*} \leqslant \beta^*(|(w(0), x(0), z(0))|_{\mathcal{A}^*}, t) + \ell^*\|\varsigma\|_{[0,t)} \tag{25}$$

for some class $\mathcal{KL}$ function $\beta^*(\cdot,\cdot)$ and some positive number ℓ^*. □

We have shown that the estimate (25) holds, in which ℓ^* is a fixed number, on which we do not have control. With the small-gain theorem in mind, we establish now for $|\varsigma(t)|$ an estimate of the form (21), in which ℓ is a coefficient that can be arbitrarily assigned, by fixing the value of the gain parameter k in K.

To this end, set $\mu = E\varsigma$, which yields

$$\dot{\mu} = -S(w, x)(I + U(w, x))K\mu + E[\delta_0(w, x) + \delta_1(w, z, \varsigma)]$$

in which the result in Lemma 1 is used. Then take the derivative, along the trajectories of the system, of the positive definite function

$$V(w, x, \mu) = \mu^\top S^{-1}(w, x)\mu\,,$$

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to obtain

$$\begin{aligned}\dot{V}(w,x,\mu) &= 2\mu\mathrm{T}S^{-1}(w,x)E[\delta_0(w,x)+\delta_1(w,z,\varsigma)] - 2\mu\mathrm{T}(I+U(w,x))K\mu + \mu\mathrm{T}\overbrace{S^{-1}(w,x)}^{\cdot}\mu \\ &= 2\mu\mathrm{T}S^{-1}(w,x)E[\delta_0(w,x)+\delta_1(w,z,\varsigma)] - 2\mu\mathrm{T}(I+U(w,x))K\mu \\ &\quad - \mu\mathrm{T}S^{-1}(w,x)[L_fS(w,x)+L_sS(w,x)+\sum_{i=1}^{m}L_{g_i}S(w,x)u_i]S^{-1}(w,x)\mu\,,\end{aligned}$$

The quantity on the right-hand side explicitly depends on u, which is not a good thing if a high-gain feedback is going to be implemented. To avoid this, we invoke an additional hypothesis.

Assumption 6

The matrix $B(w,x)$ is such that $L_{g_i}B(w,x)=0$ for $i=1,\dots,m$. Moreover, there exists a positive number ρ_2, such that, for all $(w,x)\in\mathcal{W}\times\mathbb{R}^n$, the following inequality holds

$$|S^{-1}(w,x)[L_fS(w,x)+L_sS(w,x)]S^{-1}(w,x)|\leqslant\rho_2\,. \tag{26}$$

Remark

The role of the assumption in question is to render the last term in the estimate above *independent* of u. The assumption essentially says that $B(w,x)$ is independent of certain state variables (in the special case of a MIMO system having a vector relative degree $\{r_1,\dots,r_m\}$ and globally defined normal form, this assumption means that the matrix is independent of $\{y_1^{(r_1-1)},\dots,y_m^{(r_m-1)}\}$). A relevant case in which this assumption is guaranteed to hold is the one in which the matrix in question depends only on w (and not on x). This includes the special case in which this matrix is a matrix of constant, but possibly *uncertain*, parameters. In this case, in fact, the entries of such matrix can be seen as constant exogenous variables. Thus, in particular, the results described thereafter covers the relevant case of (multivariable) systems having a constant, but otherwise uncertain, high-frequency gain matrix.

Because the entries of $S(w,x)$ are rational functions of the entries of $B(w,x)$ ([10]; proof of Lemma 3), the first part of this assumption implies $L_{g_i}S(w,x)=0$ for $i=1,\dots,m$. This being the case, bearing in mind the Lemma 2 and 4, we obtain the estimate

$$\begin{aligned}\dot{V}(w,x,\mu)\leqslant{}& -k|\mu|^2+2\mu\mathrm{T}S^{-1}(w,x)E[\delta_0(w,x)+\delta_1(w,z,\varsigma)]\\ &-\mu\mathrm{T}S^{-1}(w,x)[L_fS(w,x)+L_sS(w,x)]S^{-1}(w,x)\mu\,,\end{aligned} \tag{27}$$

from which, using Assumption 4 and (22), we can further obtain

$$\begin{aligned}\dot{V}(w,x,\mu)&\leqslant -k|\mu|^2+2(c_0|(w,x,z)|_{\mathcal{A}\times\{0\}}+c_1|\mu|)|\mu|+\rho_2|\mu|^2\\ &\leqslant -(k-2c_1-\rho_2)|\mu|^2+2c_0|(w,x,z)|_{\mathcal{A}^*}|\mu|\,.\end{aligned} \tag{28}$$

Let ρ_1 be a fixed positive number, and choose a large enough k so that there exist a *positive* number k_2 yielding

$$k\geqslant 2c_1+\rho_2+2\frac{c_0}{\rho_1}+k_2\,.$$

Because $\mu=E\varsigma$, we see that if $|\varsigma|\geqslant\rho_1|(w,x,z)|_{\mathcal{A}\times\{0\}}$, this inequality implies

$$\dot{V}(w,x,E\varsigma)\leqslant -k_2|\varsigma|^2\,.$$

Recall now that $V(w,x,E\varsigma)$ is bounded as

$$a_1|\varsigma|^2\leqslant V(w,x,E\varsigma)\leqslant a_2|\varsigma|^2$$

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Using this estimate, we see that

$$\begin{aligned} & V(w,x,E\varsigma) \geqslant a_2\rho_1^2(|(w,x,z)|_{\mathcal{A}^*})^2 \\ \Longrightarrow\ & a_2|\varsigma(t)|^2 \geqslant a_2\rho_1^2(|(w,x,z)|_{\mathcal{A}^*})^2 \\ \Longrightarrow\ & \dot{V} \leqslant -\tfrac{k_2}{a_2}V \\ \Longrightarrow\ & |\varsigma(t)| \leqslant Me^{-\lambda_1 t}|\varsigma(0)| \end{aligned}$$

for suitable M and $\lambda_1 > 0$. Otherwise

$$\begin{aligned} & V(w,x,E\varsigma) \leqslant a_2\rho_1^2(|(w,x,z)|_{\mathcal{A}^*})^2 \\ \Longrightarrow\ & a_1|\varsigma(t)|^2 \leqslant a_2\rho_1^2(|(w,x,z)|_{\mathcal{A}^*})^2 \\ \Longrightarrow\ & |\varsigma(t)| \leqslant \sqrt{\tfrac{a_2}{a_1}}\rho_1|(w,x,z)|_{\mathcal{A}^*} := \ell_3|(w,x,z)|_{\mathcal{A}^*} \end{aligned}$$

As a result, bearing in mind that ρ_1 is a parameter that can be arbitrarily chosen, all of the above leads to the following result.

Proposition 2
Suppose Assumption 6 holds. For any $\ell > 0$, there is a number k^* such that if the gain parameter k in K satisfies $k > k^*$, there exists a class $\mathcal{KL}$ function β such that for every $\varsigma(0)$, the inequality

$$|\varsigma(t)| \leqslant \beta(|\varsigma(0)|, t) + \ell\|(w,x,z)_{\mathcal{A}^*}\|_{[0,t)} \tag{29}$$

holds along trajectories of system (17).

Remark
If only semi-global output regulation is sought, the (rather restrictive) second part of Assumption 6 could be weakened. If fact, in this case, it suffices that – instead of (26) – an estimate of the form

$$|S^{-1}(w,x)[L_f S(w,x) + L_s S(w,x)]S^{-1}(w,x)| \leqslant \rho_2(|(w,x)|_{\mathcal{A}})$$

in which $\rho_2(\cdot)$ is a strictly non-decreasing function (not necessarily vanishing at 0). Such inequality indeed always holds, because the functions on the left-hand side are continuous.

We are now in the position of presenting the main result of the paper, as detailed in the next proposition. Using the aforementioned proposition, together with Proposition 1, and known results about interconnected ISS stable systems (e.g., [23]), it is readily concluded that the following results hold.

Theorem 1
Suppose Assumptions 1–4 hold. Let $\mathbf{F}$ and $\mathbf{G}$ be chosen as indicated previously and suppose that the function $\gamma(\cdot)$ in the second equation of (15) is such that Assumption 5 holds. Suppose that also Assumption 6 holds . Then there exists a $k^* > 0$ such that for all $k \geqslant k^*$ the controller described in (16) solves the problem of global output regulation.

Remark
The previous result provides conditions under which the problem of output regulation is solved by means of a controller *driven* by the 'redesigned' output $\tilde{y}$. As shown in (4), $\tilde{y}$ is a linear combination of the partial states ξ_j^k, with $j = 1, \ldots, N_k$ and $k = 1, \ldots, k^*$. If such partial states are not directly available for feedback, a dynamical system driven only by the actual regulated output y can be designed that provides appropriate asymptotic estimates of such partial states. The construction of such dynamical system is based on a non-trivial extension of the pioneering results of [20] and [23] on the design of a 'high-gain' estimator of partial states to the purpose of stabilizing a minimum-phase nonlinear system. Details on the structure of such estimator (which in the present MIMO setting is substantially more complicated than the one typically used for SISO systems) can be found in [10]. If such estimator is used, then the problem of output regulation can be solved (only semi-globally, though) by means of a controller driven by the regulated output y. In fact, an observer

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such as (the appropriate MIMO version of) [20] and [23] yields an observation error system, which is ISS stable with respect to the 'input' $R(w,x) + B(w,x)u$, with a linear gain function that can be arbitrarily lowered by increasing the gain parameter of the observer. If the input $\tilde{y}$ in $v = K\tilde{y}$ is replaced by (a saturated version of) its estimate provided by the observer, a system is obtained that is ISS stable with respect the observation error viewed as a input, with a (now fixed) gain function. Thus, in view of the small gain theorem, suffices to conclude that, by increasing the gain parameter of the observer, the regulation error can be made to converge to zero. Details of the analysis are rather standard and need not to be included here.

Remark
We conclude the section with some remarks on the role of the various assumptions upon which the theory is based. We have already observed that the assumption, for the system, of being invertible and input–output linearizable (Assumptions 1 and 2) is quite weaker than the assumption of having a vector relative degree. As a matter of fact, even a linear system that is invertible needs not to have a vector relative degree. The fact that the integers $N_1, \dots, N_{k^*}$ (that, as it is known, characterize the so-called zero structure at infinity [24]) are well-defined and hence independent of x as well as of w is part of the assumption that the system is input–output linearizable. After all, even in the special case that the system has a vector relative degree, the integers that characterize such degree are integers independent of x and w. The matrix $B(w,x)$, as the notation suggests, may depend on x and w, but we assume that it has special properties (Assumption 6). For a SISO system, the relevant property would be that this quantity, a scalar quantity, is bounded away from zero. Again, this implies that the system has a well-defined relative degree. In the MIMO case, the (equivalent) property that the principal minors are bounded away from zero is not enough. One needs to know the appropriate 'signs' by means of which the various components of the output are being fed back (this is reflected in the knowledge of E), but this also is not enough. In the current paper, we have added the assumption $L_{g_i} B(x,w) = 0$, which means that the matrix in question is independent on certain state variables. The assumption is fulfilled in the special, but relevant case, in which such matrix only depends on w and not on x, which includes the important case in which the matrix in question is a matrix of constant, but otherwise uncertain, parameters. We have assumed (Assumption 3) that the system is 'strongly minimum phase', in the sense described in the paper. This assumption is ubiquitous in the theory of nonlinear output regulation. The setting in which our definition is given, following [17], is coordinate free. Finally, we observe that our method reposes also on the assumption (Assumption 5) that the function $\gamma(\cdot)$ in (15) is twice continuously differentiable.

4. A NUMERICAL EXAMPLE

Consider the system

$$\begin{aligned} \dot{z}_1 &= -z_1 + x_1 \\ \dot{x}_1 &= a_1(w,x) + u_1 + \varphi(\mu)u_2 \\ \dot{x}_2 &= x_3 + a_1(w,x) + u_1 + \varphi(\mu)u_2 \\ \dot{x}_3 &= a_2(w,x) + u_2 \\ y_1 &= x_1 \\ y_2 &= x_2 \end{aligned} \tag{30}$$

in which $w \in \mathbb{R}^2$ is an exogenous input generated by the harmonic oscillator

$$\dot{w} = \begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix} w := s(w)\,,$$

whose initial conditions range in the compact set $\{w \in \mathbb{R}^2 : |w| \leqslant R\}$. The argument μ of $\varphi(\mu)$ is an uncertain constant parameter, ranging on a compact set $\mathcal{P}$. Note that, this being the case, we can consider for (30) an 'augmented' exosystem modeled by

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{\mu} &= 0 \end{aligned}$$

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with initial conditions in the compact set $\mathcal{W} = \{w \in \mathbb{R}^2 : |w| \leqslant R\} \times \mathcal{P}$. The functions $a_1(w,x)$ and $a_2(w,x)$ are smooth functions, about which we will be more specific in a moment. The control objective is to have the regulated outputs y_1 and y_2 asymptotically decaying to zero, by means of an output feedback controller of the form (2). Occasionally, for the sake of brevity, we denote x as $\mathrm{col}(z, x_1, x_2, x_3)$.

Following the design procedure detailed in the previous section, defining the new outputs

$$\tilde{y}_1 = y_1\,, \quad \tilde{y}_2 = c y_2 + x_3$$

and taking coordinate transformation $z_2 = y_2 - y_1$, the original controlled system becomes

$$\begin{aligned} \dot{z}_1 &= -z_1 + \tilde{y}_1 \\ \dot{z}_2 &= -c z_2 - c\tilde{y}_1 + \tilde{y}_2 \\ \dot{\tilde{y}} &= R(w,x) + B(\mu)u \end{aligned}$$

in which $c > 0$,

$$R(w,x) = \begin{pmatrix} r_1(w,x) \\ r_2(w,x) \end{pmatrix} = \begin{pmatrix} a_1(w,x) \\ a_2(w,x) + c a_1(w,x) + c x_3 \end{pmatrix},$$

and

$$B(\mu) = \begin{pmatrix} 1 & \varphi(\mu) \\ c & (c\varphi(\mu) + 1) \end{pmatrix}.$$

In this way, the problem is transformed in the problem of designing a controller of the form (8) driven by the redesigned outputs $\tilde{y}_1$ and $\tilde{y}_2$ (that could be estimated by means of a 'high-gain observer' driven by the actual regulated outputs y_1 and y_2). In what follows, we discuss how to arrive at the desired controller having the form (16).

We first seek a solution pair of the regulator equations. An easy calculation shows that necessarily $\pi(w) = 0$, and hence, $\Psi(w)$ is the solution of

$$R(w,0) - B(\mu)\Psi(w) = 0\,,$$

which yields that

$$\Psi(w) = \begin{pmatrix} (1 + c\varphi(\mu))r_1(w,0) - \varphi(\mu) r_2(w,0) \\ -c r_1(w,0) + r_2(w,0) \end{pmatrix}.$$

Observe also that the leading principal minors of $B(\mu)$ are both positive. Hence, the matrix E in Lemma 1 is the identity matrix (see [10] for details).

Suppose now that $a_1(w,0)$ and $a_2(w,0)$ are polynomials of degree 2 in w, that is functions of the form ($i = 1,2$)

$$a_i(w,0) = a_{i,11} w_1^2 + a_{i,12} w_1 w_2 + a_{i,22} w_2^2.$$

As a consequence, setting $\bar{w} = \mathrm{col}(w_1^2,\ w_1 w_2,\ w_2^2)$, it is seen that $\Psi(w)$ can be expressed as

$$\Psi(w) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \bar{w}$$

in which ψ_1 and ψ_2 are 1×3 row vectors, the first one of which depends on the unknown parameter μ. This being the case, a linear internal model can be designed, following t he method of [2]. To this end, observe that $\dot{\bar{w}} = \bar{S}\bar{w}$ in which

$$\bar{S} = \begin{pmatrix} 0 & 2\Omega & 0 \\ -\Omega & 0 & \Omega \\ 0 & -2\Omega & 0 \end{pmatrix}.$$

Then, pick

$$F = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

with a_0, a_1, a_2 positive constants such that F is Hurwitz.

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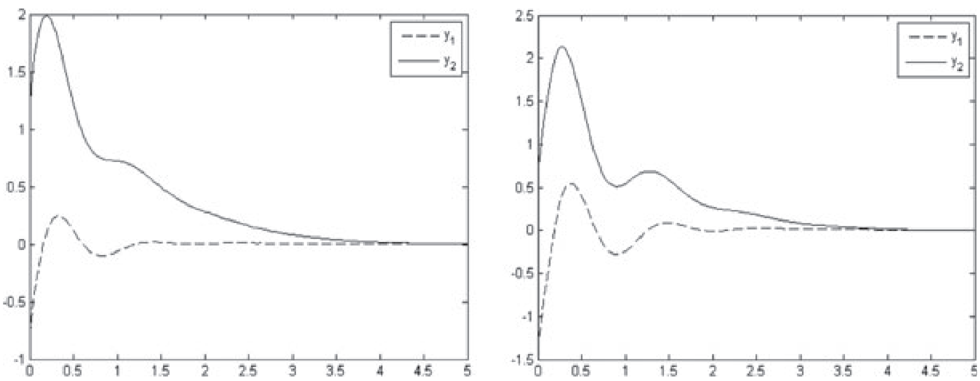

(a) Case $\mu = 2$ (b) Case $\mu = 5$

Figure 1. Trajectories of regulated outputs y_1 and y_2.

Because the pair (F, G) is controllable, there exists Γ such that $(F + G\Gamma)$ has the same characteristic polynomial of $\bar{S}$. Then, it is known that there exists non-singular matrix Σ_i, possibly dependent of the unknown parameter μ, such that

$$\begin{aligned} \Sigma_i \bar{S} &= (F + G\Gamma)\Sigma_i \\ \Psi_i &= \Gamma\Sigma_i\,, \qquad i = 1, 2\,. \end{aligned}$$

As a result, it is easily seen that (14), in which we pick $E = I$, are satisfied with

$$\begin{aligned} \sigma_i(w) &= \Sigma_i \bar{w}\,, \\ \gamma_i(\eta_i) &= \Gamma\eta_i \qquad i = 1, 2\,. \end{aligned}$$

This yields the controller (16)

$$\begin{aligned} \dot{\eta}_1 &= F\eta_1 + G\left(\Gamma\eta_1 + k^2\tilde{y}_1\right) \\ \dot{\eta}_2 &= F\eta_2 + G\left(\Gamma\eta_2 + k\tilde{y}_2\right) \\ u_1 &= -\Gamma\eta_1 - k^2\tilde{y}_1 \\ u_2 &= -\Gamma\eta_2 - k\tilde{y}_2\,. \end{aligned} \tag{31}$$

In what follows, a simulation is presented showing that the controller (31) is such that y_1 and y_2 converge to zero. The simulation is performed with $a_1(w, x) = w_1^2 + \sin(w_2)x_1$, $a_2(w, x) = w_1 w_2 + z_1$, $\varphi(\mu) = \mu$, in which μ varies in the set $[2, 5]$, $\Omega = 1$ and $c = 1$. The matrix F is chosen to have characteristic polynomial $(\lambda + 1)(\lambda + 2)(\lambda + 3)$. By simple calculations, one can obtain $\Gamma = \begin{pmatrix} 6 & 7 & 6 \end{pmatrix}$. Fixing $k = 10$ and all initial conditions $w(0) = \mathrm{col}(1, 0)$, $x(0) = \mathrm{col}(1, -1, 1, 2)$ and $(\eta_1(0), \eta_2(0)) = 0$, we can obtain Figure 1(a) and (b) for $\mu = 2$ and $\mu = 5$, respectively.

5. CONCLUSIONS

In this paper, we have addressed the problem of output regulation for a rather general class of MIMO systems. Specifically, we have considered systems which are invertible, are input–output linearizable, are strongly minimum phase, and have a 'high-frequency gain matrix' whose leading principal minors are bounded away from zero and whose derivative along the vector fields $g_i(x)$ is zero. Exploiting some recent results of ours [10] dealing with the design of feedback stabilizing laws, we have extended to this class of systems the design technique recently proposed in [9] (see also [22]) for the solution of the problem of output regulation in the (more restrictive) case of MIMO systems, which possess a vector relative degree and a globally defined normal form. Under appropriate technical assumptions, we have shown how the problem in question can be solved globally, via partial-state feedback. We have also indicated how such assumptions can be weakened if only semi-global regulation is sought. If the partial-state is not directly available for feedback, semi-global regulation can be achieved by means of a controller driven by the regulated output only.

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