Continuum Mech. Thermodyn. 3 (1991) 155–167

Continuum Mechanics and Thermodynamics

A generalization of the Chapman-Enskog and Grad methods

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By using the method of Chapman-Enskog as a base, we have determined successive approximations of any order to the transport coefficients of shear viscosity and thermal conductivity for a monatomic ideal gas. The expressions for the transport coefficients involve only integrals which can be evaluated once the law of interaction between the spherically symmetrical particles is known. Moreover, we have developed a $(13 + 9N)$-field theory based on the method of Grad and showed that the transition from this theory to a five-field theory leads to the same results as those obtained through the method of Chapman-Enskog.

1 Introduction

The aim of this paper is to investigate the transport coefficients of a monatomic ideal gas, using kinetic theory as a base. In Sec. 2 we begin with the determination of successive approximations of any order to the transport coefficients of shear viscosity and thermal conductivity, basing on the method of Chapman-Enskog [1]. The expressions for the transport coefficients involve only integrals which can be evaluated once the law of interaction between the spherically symmetrical particles is known. As an application we indicate how to get successive approximations to the transport coefficients for gases of hard-sphere particles, as well as for gases whose particles interact according to an interparticle potential function that varies with the inverse power of the distance between the centers of the particles.

In Sec. 3 we follow Refs. [2] and [3] and develop a $(13 + 9N)$-field theory based on the method of moments of Grad [4]. The transition from the $(13 + 9\,N)$-field theory to a five-field theory proceeds by an iterative scheme akin to the Maxwellian procedure [5]. As a consequence the laws of Navier-Stokes and Fourier are obtained, and we show the equivalence between the two methods with respect to the successive approximations to the transport coefficients.

Cartesian notation for tensors is used with angular brackets around two indices denoting the symmetric and traceless part of a tensor.

2 The method of Chapman-Enskog

A. Theory

In the method of Chapman-Enskog a macroscopic state of a monatomic ideal gas is characterized by the five scalar fields of density ϱ, velocity v_i, and temperature T. These fields are defined in terms of the single-particle distribution function $f(\mathbf{x}, \mathbf{c}, t)$ by

$$\varrho(\mathbf{x}, t) = \int mf(\mathbf{x}, \mathbf{c}, t)\, d\mathbf{c}, \tag{1}$$

$$v_i(\mathbf{x}, t) = \frac{1}{\varrho} \int mc_i f(\mathbf{x}, \mathbf{c}, t)\, d\mathbf{c}, \tag{2}$$

$$T(\mathbf{x}, t) = \frac{m}{3k\varrho} \int mC^2 f(\mathbf{x}, \mathbf{c}, t)\, d\mathbf{c}. \tag{3}$$

The corresponding balance equations for the basic fields (1)–(3) are obtained from an equation of transfer derived from the Boltzmann equation, and read

$$\dot{\varrho} + \varrho \frac{\partial v_i}{\partial x_i} = 0, \tag{4}$$

$$\varrho \dot{v}_i + \frac{\partial p_{ij}}{\partial x_j} = 0, \tag{5}$$

$$\frac{3}{2} \varrho \frac{k}{m} \dot{T} + \frac{\partial q_i}{\partial x_i} + p_{ij} \frac{\partial v_i}{\partial x_j} = 0. \tag{6}$$

Equations (4), (5) and (6) represent, respectively, the balance equations of mass, momentum, and specific internal energy, for gases free of external supplies. In the above equations m is the mass of a fluid particle, $C_i = c_i - v_i$ is the peculiar velocity, and the dot denotes the material time derivative. Moreover, the pressure tensor p_{ij}, and the heat flux q_i are defined by

$$p_{ij}(\mathbf{x}, t) = \int mC_iC_j f(\mathbf{x}, \mathbf{c}, t)\, d\mathbf{c}, \tag{7}$$

$$q_i(\mathbf{x}, t) = \tfrac{1}{2} \int mC^2 C_i f(\mathbf{x}, \mathbf{c}, t)\, d\mathbf{c}. \tag{8}$$

The system of balance equations (4)–(6) is closed if one can relate p_{ij} and q_i to the basic fields (1)–(3). This is achieved by integration of Eqs. (7) and (8), once $f(\mathbf{x}, \mathbf{c}, t)$ is determined as a solution of the Boltzmann equation. From the knowledge of the second approximation to the distribution function and from Eqs. (7)

and (8) it follows that

$$p_{ij} = \varrho \frac{k}{m} T \delta_{ij} - 2[\mu]_{(N'+1)} \frac{\partial v_{\langle i}}{\partial x_{j\rangle}}, \tag{9}$$

$$q_i = -[\lambda]_{(N'+1)} \frac{\partial T}{\partial x_i}. \tag{10}$$

Equations (9) and (10) represent the laws of Navier-Stokes and Fourier, while $[\mu]_{(N'+1)}$ and $[\lambda]_{(N'+1)}$ are the coefficients of shear viscosity and thermal conductivity in the $(N'+1)$-order approximation $(N' = 0, 1, 2, \ldots)$, respectively. They are given by

$$[\mu]_{(N'+1)} = \frac{5}{4} \varrho \frac{\mathscr{B}'_{N'+1,N'+1}}{\mathscr{B}_{N'+1,N'+1}}, \tag{11}$$

$$[\lambda]_{(N'+1)} = \frac{75}{16} \varrho \frac{k}{m} \frac{\mathscr{A}'_{N'+1,N'+1}}{\mathscr{A}_{N'+1,N'+1}}, \tag{12}$$

where

$$\mathscr{B}_{N'+1,N'+1} = \sum_{\tau} (\operatorname{sgn} \tau)\, \beta^{(0,\tau_0)} \ldots \beta^{(N',\tau_{N'})}, \tag{13}$$

$$\mathscr{B}'_{N'+1,N'+1} = \delta^{(0,N')} + \sum_{\tau'} (\operatorname{sgn} \tau')\, \beta^{(1,\tau'_1)} \ldots \beta^{(N',\tau'_{N'})}, \tag{14}$$

$$\mathscr{A}'_{N'+1,N'+1} = \sum_{\sigma} (\operatorname{sgn} \sigma)\, \alpha^{(1,\sigma_1)} \ldots \alpha^{(N'+1,\sigma_{N'+1})}, \tag{15}$$

$$\mathscr{A}'_{N'+1,N'+1} = \delta^{(1,N'+1)} + \sum_{\sigma'} (\operatorname{sgn} \sigma')\, \alpha^{(2,\sigma'_2)} \ldots \alpha^{(N'+1,\sigma'_{N'+1})}. \tag{16}$$

In the above equations $\delta^{(N,N')}$ is the Kronecker delta, and τ, τ', σ, and σ' are permutations of $\{0, 1, \ldots, N'\}$, $\{1, 2, \ldots, N'\}$, $\{1, 2, \ldots, N'+1\}$, and $\{2, 3, \ldots, N'+1\}$, respectively. Furthermore, $\beta^{(r,s)}$, and $\alpha^{(r,s)}$ are the following integrals

$$\beta^{(r,s)} = -\frac{\beta^3}{8} \frac{m}{\varrho} \int S^{(r)}_{5/2} C_{\langle i} C_{j\rangle} I[S^{(s)}_{5/2} C_{\langle i} C_{j\rangle}]\, d\mathbf{c}, \quad 0 \leqq r, s \leqq N', \tag{17}$$

$$\alpha^{(r,s)} = -\frac{\beta^2}{4} \frac{m}{\varrho} \int S^{(r)}_{3/2} C_i I[S^{(s)}_{3/2} C_i]\, d\mathbf{c}, \quad 1 \leqq r, s \leqq N'+1, \tag{18}$$

where $S^{(n)}_m$ are Sonine polynomials, and $I[\phi]$ stands for the integral

$$I[\phi] = \int f^{(0)} f^{(0)}_1 [\phi'_1 + \phi' - \phi_1 - \phi]\, gb\, db\, d\varepsilon\, d\mathbf{c}_1. \tag{19}$$

In the latter integral $\mathbf{g} = \mathbf{C}_1 - \mathbf{C}$ is the relative velocity of two particles before collision, b is the impact parameter, ε is the angle between the plane of collision and a reference plane through $\mathbf{g}$, and the primes refer to velocities after collision. Moreover, $f^{(0)}$ is the Maxwellian distribution function

$$f^{(0)} = \frac{\varrho}{m}\left(\frac{\beta}{2\pi}\right)^{3/2} \exp\left(-\frac{\beta C^2}{2}\right), \tag{20}$$

with $\beta = m/kT$, and k denoting the Boltzmann constant.

In order to evaluate, for the general case, the integrals (17) and (18), we introduce the functions

$$F_l^{(r,k)} = \frac{(2r+2l+1)!!}{(2k+2l+1)!!}\,\frac{2^k}{2^r}\,\frac{(-1)^k}{(r-k)!}, \tag{21}$$

which have the following property

$$\sum_{k=s}^{r} F_l^{(r,k)} F_l^{(k,s)} = \delta^{(r,s)}. \tag{22}$$

Hence, we can write the Sonine polynomials in terms of the functions $F_l^{(r,s)}$ as

$$S_{l+1/2}^{(r)} = \sum_{k=0}^{r} F_l^{(r,k)} \frac{\beta^k}{2^k k!} C^{2k}, \tag{23}$$

and the integrals (17) and (18) as

$$\beta^{(r,s)} = \frac{\varrho}{m}\sqrt{\pi\beta} \sum_{k=0}^{r}\sum_{l=0}^{s} F_2^{(r,k)} K^{(k,l)} F_2^{(s,l)}, \tag{24}$$

$$\alpha^{(r,s)} = \frac{\varrho}{m}\sqrt{\pi\beta} \sum_{k=1}^{r}\sum_{l=1}^{s} F_1^{(r,k)} J^{(k,l)} F_1^{(s,l)}, \tag{25}$$

where

$$K^{(k,l)} = -\frac{1}{\pi^{1/2}}\left(\frac{m}{\varrho}\right)^2 \frac{\beta^{k+l+5/2}}{2^{k+l+3}}\,\frac{1}{k!\,l!}\int C^{2k} C_{\langle i}C_{j\rangle} I[C^{2l} C_{\langle i}C_{j\rangle}]\,d\mathbf{c}, \tag{26}$$

$$J^{(k,l)} = -\frac{1}{\pi^{1/2}}\left(\frac{m}{\varrho}\right)^2 \frac{\beta^{k+l+3/2}}{2^{k+l+2}}\,\frac{1}{k!\,l!}\int C^{2k} C_i I[C^{2l} C_i]\,d\mathbf{c}. \tag{27}$$

For the integration of Eqs. (26) and (27) we change the variables of integration by introducing the center of mass velocity $\mathbf{G} = (\mathbf{C} + \mathbf{C}_1)/2$ and the relative velocity $\mathbf{g}$, make use of the binomial expansion for the powers of the velocities and of the following relationship

$$(\mathbf{G}\cdot\mathbf{g})^k\,(\mathbf{G}\cdot\mathbf{g}')^l = \frac{G^{k+l} g^{k+l}}{(k+l+1)!!} \sum_{p=s}^{\min(k,l)} \frac{k!\,l!}{p!\,(k-p)!!\,(l-p)!!} \cos^p\chi. \tag{28}$$

In Eq. (28) one has to take $s = 0$ and p even when k and l are even, and $s = 1$ and p odd when k and l are odd. Moreover, $\chi = \arccos(\mathbf{g}\cdot\mathbf{g}'/g^2)$ is the scattering angle. Now the integration of Eq. (26) in $\mathbf{G}$ and over the directions of $\mathbf{g}$ can be carried out, and it follows after a length calculation

$$K^{(m,n)} = \frac{1}{4^{m+n}\, m!\, n!} \sum_{k=1}^{m/2+1} \sum_{l=1}^{n/2+1} \sum_{q=l+k}^{m'+l+k+4} \sum_{\substack{p=2 \\ p \text{ even}}}^{\min(2k,2l)} \times (2m + 2n - 2q + 5)!!\, \mathscr{L}\mathscr{K}\, \Omega^{(p,q)}, \tag{29}$$

where $m' = m + n - 2l - 2k$, $q' = q - l - k$, and

$$\mathscr{L} = \frac{(2k)!\,(2l)!\,2^{k+l+q+p}}{(2k + 2l + 1)!!\, p!\,(k - p/2)!\,(l - p/2)!}, \tag{30}$$

$$\begin{aligned}\mathscr{K} = {} & \binom{q'}{m'}\binom{2k}{m}\binom{2l}{n} + \binom{q'}{m'+1}\left[\binom{2k}{m}\binom{2l-1}{n} + \binom{2k-1}{m}\binom{2l}{n}\right] \\ & + \binom{q'}{m'+2}\left[\tfrac{1}{4}\binom{2k}{m}\binom{2l-2}{n} + \tfrac{1}{4}\binom{2k-2}{m}\binom{2l}{n}\right. \\ & \left. + \binom{2k-1}{m}\binom{2l-1}{n}\frac{p(2k+2l+1)+4kl}{8kl} - \tfrac{1}{3}\binom{2k}{m+1}\binom{2l}{n+1}\right] \\ & + \binom{q'}{m'+3}\left[\binom{2k-1}{m}\binom{2l-2}{n} + \binom{2k-2}{m}\binom{2l-1}{n}\right]\frac{p(2k+2l+1)}{16kl} \\ & + \binom{q'}{m'+4}\binom{2k-2}{m}\binom{2l-2}{n}\frac{p(p-1)\,[(2k+2l)^2-1]}{64kl(2k-1)(2l-1)}.\end{aligned} \tag{31}$$

Likewise, the integration of Eq. (27) leads to

$$J^{(m,n)} = \frac{1}{4^{m+n-1}\, m!\, n!} \sum_{k=1}^{(m+1)/2} \sum_{l=1}^{(n+1)/2} \sum_{q=l+k}^{m'+l+k+2} \sum_{\substack{p=2 \\ p \text{ even}}}^{\min(2k,2l)} \times (2m + 2n - 2q + 3)!!\, \mathscr{L}\mathscr{J}\, \Omega^{(p,q)}, \tag{32}$$

where

$$\begin{aligned}\mathscr{J} = {} & \binom{q'}{m'}\binom{2k}{m}\binom{2l}{n} + \tfrac{1}{2}\binom{q'}{m'+1}\left[\binom{2k}{m}\binom{2l-1}{n} + \binom{2k-1}{m}\binom{2l}{n}\right] \\ & + \binom{q'}{m'+2}\binom{2k-1}{m}\binom{2l-1}{n}\frac{p(2k+2l+1)}{16kl}.\end{aligned} \tag{33}$$

In the above equations one should take $\binom{k}{n}$ equal to zero whenever $k > n$ or $k < 0$. Moreover, $\Omega^{(l,r)}$ is the following integral

$$\Omega^{(l,r)} = \int_0^\infty \int_0^\infty (1 - \cos^l \chi)\, e^{-y^2}\, y^{2r+3}\, b\, db\, dy. \tag{34}$$

As an example one can get from Eqs. (21), (24), and (29)–(31)

$$\beta^{(4,4)} = \frac{\varrho}{m}\sqrt{\pi\beta}\left[\frac{711\,736\,025}{1\,048\,576}\Omega^{(2,2)} - \frac{50\,053\,575}{65\,536}\Omega^{(2,3)} + \frac{64\,969\,619}{196\,608}\Omega^{(2,4)}\right.$$
$$- \frac{3\,630\,289}{49\,152}\Omega^{(2,5)} + \frac{953\,681}{98\,304}\Omega^{(2,6)} - \frac{29\,393}{36\,864}\Omega^{(2,7)} + \frac{4699}{110\,592}\Omega^{(2,8)}$$
$$- \frac{13}{9216}\Omega^{(2,9)} + \frac{1}{36\,864}\Omega^{(2,10)} + \frac{470\,041}{18\,432}\Omega^{(4,4)} - \frac{28\,223}{2304}\Omega^{(4,5)}$$
$$\left. + \frac{5341}{2304}\Omega^{(4,6)} - \frac{13}{64}\Omega^{(4,7)} + \frac{1}{128}\Omega^{(4,8)} + \frac{1}{96}\Omega^{(6,6)}\right]. \tag{35}$$

Hence, the expressions for the successive approximations to the transport coefficients are functions only of the integrals $\Omega^{(l,r)}$, which can be evaluated once the law of interaction between the spherically symmetrical particles is known.

B. Application

As we have seen the determination of successive approximations to the transport coefficients of shear viscosity and thermal conductivity is reduced to the knowledge of the integrals $\Omega^{(l,r)}$. The simplest case is that of a gas of hard-sphere particles, since these integrals are given by

$$\Omega^{(l,r)} = \frac{a^2}{8}(r+1)!\left[2 - \frac{1+(-1)^l}{l+1}\right], \tag{36}$$

where a is the diameter of the spherically symmetrical particles.

For gases whose particles interact according to an interparticle potential function Φ that varies with the inverse power ν of the distance r between the centers of the particles, i.e.,

$$\Phi = \frac{\varkappa}{(\nu-1)}\frac{1}{r^{(\nu-1)}}, \quad \text{with } \nu > 1 \quad \text{and} \quad \varkappa > 0, \tag{37}$$

the integrals $\Omega^{(l,r)}$ reduce to

$$\Omega^{(l,r)} = \tfrac{1}{2}\left(\frac{\varkappa}{2kT}\right)^{\frac{2}{\nu-1}}\Gamma\left(r+2-\frac{2}{\nu-1}\right)A_l(\nu), \tag{38}$$

where Γ is the gamma function, and $A_l(\nu)$ is a pure number that depend only on l and ν. On the other hand, the pure numbers $A_l(\nu)$ can be evaluated by quadrature from

$$\theta = \int_0^{\frac{\pi}{2}}\left[\frac{2\sin 2\phi}{2\sin 2\phi + (1-\sin 2\phi)(1-\cos^{\nu-1}\psi)/\sin^2\psi}\right]d\psi, \tag{39}$$

$$A_l(\nu) = 2\left(\frac{2}{\nu-1}\right)^{\frac{2}{\nu-1}} \int_0^{\frac{\pi}{4}} [1-(-1)^l \cos^l 2\theta] \left[\frac{1-\frac{\nu-5}{\nu-1}\sin 2\phi}{(1+\sin 2\phi)^{\frac{\nu-5}{\nu-1}} \cos^{\frac{\nu+3}{\nu-1}} 2\phi}\right] d\phi \tag{40}$$

Some values of $A_l(\nu)$ are given in Refs. [1] and [3].

3 The method of Grad

A. The $(13+9N)$*-field theory*

We follow Refs. [2] and [3] and characterize a macroscopic state of a monatomic gas by $13+9N$ scalar fields that correspond to the following moments of the distribution function

$$\varrho(\mathbf{x}, t) = \int mf(\mathbf{x}, \mathbf{c}, t)\, d\mathbf{c}, \tag{41}$$

$$v_i(\mathbf{x}, t) = \frac{1}{\varrho}\int mc_i f(\mathbf{x}, \mathbf{c}, t)\, d\mathbf{c}, \tag{42}$$

$$p_{2N|ij}(\mathbf{x}, t) = \int mC^{2N} C_i C_j f(\mathbf{x}, \mathbf{c}, t)\, d\mathbf{c}, \tag{43}$$

$$q_{2N|i}(\mathbf{x}, t) = \int \frac{m}{2} C^{2(N+1)} C_i f(\mathbf{x}, \mathbf{c}, t)\, d\mathbf{c}. \tag{44}$$

The balance equations for these fields read

$$\dot{\varrho} + \varrho\frac{\partial v_i}{\partial x_i} = 0, \tag{45}$$

$$\varrho\dot{v}_i + \frac{\partial p_{ij}}{\partial x_j} = 0, \tag{46}$$

$$\dot{p}_{2N|ij} + \frac{\partial p_{2N|ijk}}{\partial x_k} + p_{2N|ij}\frac{\partial v_k}{\partial x_k} + p_{2N|ik}\frac{\partial v_j}{\partial x_k} + p_{2N|jk}\frac{\partial v_i}{\partial x_k} + 2Np_{2N-2|ijkl}\frac{\partial v_l}{\partial x_k}$$
$$- 2N\frac{p_{2N-2|ijk}}{\varrho}\frac{\partial p_{kl}}{\partial x_l} - 2\frac{q_{2N-2|j}}{\varrho}\frac{\partial p_{ik}}{\partial x_k} - 2\frac{q_{2N-2|i}}{\varrho}\frac{\partial p_{jk}}{\partial x_k} = P_{2N|ij}, \tag{47}$$

$$\dot{q}_{2N|i} + \tfrac{1}{2}\frac{\partial p_{2N+2|ij}}{\partial x_j} + q_{2N|i}\frac{\partial v_j}{\partial x_j} + q_{2N|j}\frac{\partial v_i}{\partial x_j} + (N+1)\, p_{2N|ijk}\frac{\partial v_k}{\partial x_j}$$
$$- (N+1)\frac{p_{2N|ij}}{\varrho}\frac{\partial p_{jk}}{\partial x_k} - \frac{p_{2N|rr}}{2\varrho}\frac{\partial p_{ij}}{\partial x_j} = Q_{2N|i}. \tag{48}$$

In the above equations $p_{2N|ij}$ and $q_{2N|i}$ are two contracted higher-order tensors. The second-order tensor $p_{0|ij} = p_{ij}$ is identified with the pressure tensor, the vector $q_{0|i} = q_i$ is identified with the heat flux, and the moments for which N is different from zero do not have proper names. Moreover, one should take the term $q_{-2|i}$ equal to zero since it corresponds to the first-order moment of the distribution function. The moments of the distribution function $p_{2N|ijk}$ and $p_{2N|ijkl}$ are related to the moments $p_{2N|ij}$ and $q_{2N|i}$ by

$$p_{2N|ijk} = \tfrac{2}{5}\,[q_{2N|i}\,\delta_{jk} + q_{2N|j}\,\delta_{ik} + q_{2N|k}\,\delta_{ij}], \tag{49}$$

$$\begin{aligned} p_{2N|ijkl} = \tfrac{1}{7}\,[&p_{2N+2|ij}\,\delta_{kl} + p_{2N+2|ik}\,\delta_{jl} + p_{2N+2|il}\,\delta_{jk} + p_{2N+2|kl}\,\delta_{ij} + p_{2N+2|lj}\,\delta_{ik} \\ &+ p_{2N+2|kj}\,\delta_{il} - \tfrac{1}{5}\,p_{2N+2|rr}\,(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})], \end{aligned} \tag{50}$$

while the production terms $P_{2N|ij}$ and $Q_{2N|i}$ are defined by

$$P_{2N|ij} = \int m\,[C_i'C_j'(C')^{2N} - C_iC_jC^{2N}]\,ff_1 gb\;db\;d\varepsilon\;d\mathbf{c}_1\;d\mathbf{c}, \tag{51}$$

$$Q_{2N|i} = \int \frac{m}{2}\,[(C')^{2(N+1)}\,C_i' - C^{2(N+1)}C_i]\,ff_1 gb\;db\;d\varepsilon\;d\mathbf{c}_1\;d\mathbf{c}. \tag{52}$$

In this case the system of balance equations (45)–(48) is closed if one can relate the moment of the distribution function $p_{2N+2|ij}$ and the production terms $P_{2N|ij}$ and $Q_{2N|i}$ to the basic fields. This is attained by integration, if we know the distribution function in terms of the fields ϱ, v_i, $p_{2N|ij}$, and $q_{2N|i}$.

Following Grad we expand the distribution function $f(\mathbf{x}, \mathbf{c}, t)$ in terms of Hermite polynomials about a locally Maxwellian distribution function with the coefficients of the Hermite functions being related to the moments of the distribution function. In terms of the basic fields ϱ, v_i, $p_{2N|ij}$ and $q_{2N|i}$, the distribution function $f(\mathbf{x}, \mathbf{c}, t)$ can be written as

$$\begin{aligned} f = f^{(0)}\Bigg\{1 &+ \frac{\beta^2}{\varrho}\sum_{m=0}^{N'}\sum_{n=0}^{N'} A_{mn}^{(N')}\beta^{(m+n)}C^{2n}p_{2m|\langle ij\rangle}C_iC_j \\ &+ \frac{\beta^2}{\varrho}\sum_{m=0}^{N'}\sum_{n=0}^{N'} \tilde{B}_{mn}^{(N')}\beta^m q_{2m|i}C_i\,[(\beta C^2)^{n+1} - \tfrac{1}{3}\,(2n+5)!!] \\ &+ \frac{\beta^2}{\varrho}\sum_{\substack{m=0\\ N'>0}}^{N'-1}\sum_{n=0}^{N'-1} \tilde{D}_{mn}^{(N')}\beta^m\,p_{2m+2|rr}^{NE} \\ &\times [(\beta C^2)^{n+2} - \tfrac{1}{3}\,(2n+5)!!\,(n+2)\,\beta C^2 + (2n+5)!!\,(n+1)]\Bigg\}. \end{aligned} \tag{53}$$

In Eq. (53) $p_{2N|\langle ij\rangle}$ and $p_{2N|rr}^{NE}$ stand for

$$p_{2N|\langle ij\rangle} = p_{2N|ij} - \tfrac{1}{3}\,p_{2N|rr}\,\delta_{ij}, \tag{54}$$

$$p_{2N|rr}^{NE} = p_{2N|rr} - p_{2N|rr}^{(0)} = p_{2N|rr} - \varrho(2N+3)!!\left(\frac{1}{\beta}\right)^{(N+1)}. \tag{55}$$

The term $p_{2N|\langle ij\rangle}$ denotes the traceless part of $p_{2N|ij}$, while $p_{2N|rr}^{NE}$ and $p_{2N|rr}^{(0)}$ are the non-equilibrium and the equilibrium part of $p_{2N|rr}$, respectively. We call attention to the fact that $p^{NE}_{0|rr}$ is zero since for a monatomic gas the non-equilibrium pressure does not exist. Furthermore, the elements of the matrices $A^{(N')}_{mn}$, $\tilde{B}^{(N')}_{mn}$ and $\tilde{D}^{(N')}_{mn}$ can be obtained from the following recurrence formulae for their inverses

$$(A^{(N')^{-1}})_{mn} = \tfrac{2}{15}(2m + 2n + 5)!!, \tag{56}$$

$$(\tilde{B}^{(N')^{-1}})_{mn} = \tfrac{1}{6}\left[(2m + 2n + 7)!! - \tfrac{1}{3}(2m + 5)!!\,(2n + 5)!!\right], \tag{57}$$

$$(\tilde{D}^{(N')^{-1}})_{mn} = (2m + 2n + 9)!! - \tfrac{1}{3}(2m + 5)!!\,(2n + 5)!!\,(2mn + 4m + 4n + 1). \tag{58}$$

These formulae are valid for all $0 \leqq m, n \leqq N'$.

Once the distribution function is known one can obtain the desired expressions for $p_{2N+2|ij}$, $P_{2N|ij}$, and $Q_{2N|i}$ in terms of the basic fields. Indeed, by insertion of Eq. (53) into their definitions, given by Eqs. (43), (51) and (52), integration and leaving out all nonlinear terms, it follows

$$p_{2N+2|ij} = \frac{(2N+5)!!}{3\beta^{(N+2)}}\varrho\,\delta_{ij} + \tfrac{2}{15}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{(2N+2n+7)!!}{\beta^{(N-m+1)}}A^{(N)}_{mn}\,p_{2m|\langle ij\rangle}$$
$$+ \tfrac{1}{3}\sum_{\substack{m=0\\N>0}}^{N-1}\sum_{n=0}^{N-1}\frac{(2N+2n+5)!!}{\beta^{(N-m)}}\tilde{D}^{(N)}_{mn}\,p^{NE}_{2m+2|rr}\,\delta_{ij}, \tag{59}$$

$$P_{2N|\langle ij\rangle} = -\tfrac{8}{5}\frac{\varrho}{m}\left(\frac{\pi}{\beta}\right)^{1/2}\sum_{m=0}^{N'}\sum_{n=0}^{N'}A^{(N')}_{mn}\frac{2^{N+m}}{\beta^{(N-m)}}N!\,n!\,K^{(N,n)}\,p_{2m|\langle ij\rangle}$$
$$+ \tfrac{4}{3}\frac{\varrho}{m}\sqrt{\pi\beta}\sum_{\substack{m=0\\N'>0}}^{N'-1}\sum_{n=0}^{N'-1}\tilde{D}^{(N')}_{mn}\left(\frac{1}{\beta}\right)^{N-m}L^{(N+1,n+2)}\,p^{NE}_{2m+2|rr}\,\delta_{ij}, \tag{60}$$

$$Q_{2N|i} = -\tfrac{8}{3}\frac{\varrho}{m}\left(\frac{\pi}{\beta}\right)^{1/2}\sum_{m=0}^{N'}\sum_{n=0}^{N'}\tilde{B}^{(N')}_{mn}\frac{2^{N+m}}{\beta^{(N-m)}}(N+1)!\,(n+1)!\,J^{(N+1,n+1)}\,q_{2m|i}. \tag{61}$$

Equations (60) and (61) are valid for all $N = 0, 1, \ldots, N'$. In Eq. (60) $L^{(m,n)}$ is given by

$$L^{(m,n)} = \frac{-1}{2^{m+n-3}}\sum_{k=1}^{m/2}\sum_{l=1}^{n/2}\sum_{q=l+k}^{m'+l+k}\sum_{\substack{p=2\\p\ \text{even}}}^{\min(2k,2l)}(2m + 2n - 2q + 1)!!$$
$$\times\,\mathscr{L}\binom{q'}{m'}\binom{2k}{m}\binom{2l}{n}\,\Omega^{(p,q)}. \tag{62}$$

The system of field equations for the basic fields ϱ, v_i, $p_{2N|ij}$ and q_i can then be obtained by insertion of the constitutive relations (59)–(61) into the balance equations (47) and (48).

B. The five field theory

As we have seen a macroscopic state of a monatomic gas in the five field theory is characterized by the fields of density, velocity and temperature. In this case the pressure deviator $p_{\langle ij\rangle}$ and the heat flux q_i are no longer variables, and must be expressed in terms of the basic fields ϱ, v_i, and T. Here instead of using the method of Chapman-Enskog, for the determination of $p_{\langle ij\rangle}$ and q_i, we shall use the remaining $8 + 9N$ field equations of the $(13 + 9N)$-field theory and a method akin to the Maxwellian iteration procedure. Since we are interested in a linearized theory, the substitution of Eqs. (59)–(61) into Eqs. (47) and (48) and linearization lead to

$$\dot{p}^{NE}_{2N|rr} + 2\frac{\partial q_{2N|i}}{\partial x_i} - \tfrac{2}{3}(N+1)\frac{1}{\beta^N}(2N+3)!!\frac{\partial q_i}{\partial x_i}$$

$$= 4\frac{\varrho}{m}\sqrt{\pi\beta}\sum_{\substack{m=0\\N'>0}}^{N'-1}\sum_{n=0}^{N'-1}\tilde{D}^{(N')}_{mn}\left(\frac{1}{\beta}\right)^{N-m}L^{(N+1,n+2)}\,p^{NE}_{2m+2|rr}, \tag{63}$$

$$\dot{p}_{2N|\langle ij\rangle} + \tfrac{4}{5}\frac{\partial q_{2N|\langle i}}{\partial x_{j\rangle}} + \tfrac{2}{15}\varrho\frac{(2N+5)!!}{\beta^{(N+1)}}\frac{\partial v_{\langle i}}{\partial x_{j\rangle}}$$

$$= -\tfrac{8}{5}\frac{\varrho}{m}\left(\frac{\pi}{\beta}\right)^{1/2}\sum_{m=0}^{N'}\sum_{n=0}^{N'}A^{(N')}_{mn}\frac{2^{N+m}}{\beta^{(N-m)}}N!\,n!\,K^{(N,n)}\,p_{2m|\langle ij\rangle}, \tag{64}$$

$$\dot{q}_{2N|i} + \tfrac{1}{2}\left[\frac{\partial p_{2N+2|\langle ij\rangle}}{\partial x_j} + \tfrac{1}{3}\frac{\partial p^{NE}_{2N+2|rr}}{\partial x_i}\right](1-\delta_{NN'}) - \frac{(2N+5)!!}{6\beta^{(N+1)}}\frac{\partial p_{\langle ij\rangle}}{\partial x_j}$$

$$+ \tfrac{1}{6}\frac{\partial}{\partial x_j}\left[\tfrac{2}{5}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{(2N+2n+7)!!}{\beta^{(N-m+1)}}A^{(N)}_{mn}\,p_{2m|\langle ij\rangle}\right.$$

$$\left.+\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\frac{(2N+2n+5)!!}{\beta^{(N-m)}}\tilde{D}^{(N)}_{mn}p^{NE}_{2m+2|rr}\,\delta_{ij}\right]\delta_{NN'}$$

$$+\frac{(2N+5)!!}{6}\frac{\varrho}{T}\frac{(N+1)}{\beta^{(N+2)}}\frac{\partial T}{\partial x_i} = -\tfrac{8}{3}\frac{\varrho}{m}\left(\frac{\pi}{\beta}\right)^{1/2}\sum_{m=0}^{N'}\sum_{n=0}^{N'}\tilde{B}^{(N')}_{mn}\frac{2^{N+m}}{\beta^{(N-m)}}$$

$$\times (N+1)!\,(n+1)!\,J^{(N+1,n+1)}\,q_{2m|i}. \tag{65}$$

Equation (63) corresponds to the trace of Eq. (47) for $N > 0$ and is valid for all $N = 1, \ldots, N'$. The time derivatives of the density and of the temperature were eliminated from Eq. (63) through the use of the balance equations of mass density and specific internal energy. Equations (64) and (65) are valid for all $N = 0, 1, \ldots, N'$.

For the determination of $p_{2N|\langle ij\rangle}$, $p^{NE}_{2N|rr}$ and $q_{2N|i}$, through the Maxwellian iteration method, we proceed as follows: for the first iteration step we insert the equilibrium values $p_{2N|\langle ij\rangle} = 0$, $q_{2N|i} = 0$, and $p^{NE}_{2N|rr} = 0$ on the left-hand side of Eqs. (63)–(65) and get, on the right-hand side, the first iterated values of $p_{2N|\langle ij\rangle}$,

$q_{2N|i}$, and $p^{NE}_{2N|rr}$, viz.,

$$0 = 4 \frac{\varrho}{m} \sqrt{\pi\beta} \sum_{\substack{m=0 \\ N'>0}}^{N'-1} \sum_{n=0}^{N'-1} \tilde{D}^{(N')}_{mn} \left(\frac{1}{\beta}\right)^{N-m} L^{(N+1,n+2)} p^{NE}_{2m+2|rr}, \tag{66}$$

$$\tfrac{2}{15} \varrho \frac{(2N+5)!!}{\beta^{(N+1)}} \frac{\partial v_{\langle i}}{\partial x_{j\rangle}}$$
$$= -\tfrac{8}{15} \frac{\varrho}{m} \left(\frac{\pi}{\beta}\right)^{1/2} \sum_{m=0}^{N'} \sum_{n=0}^{N'} A^{(N')}_{mn} \frac{2^{N+m}}{\beta^{(N-m)}} N!\, n!\, K^{(N,n)} p_{2m|\langle ij\rangle}, \tag{67}$$

$$\frac{(2N+5)!!}{6} \frac{\varrho}{T} \frac{(N+1)}{\beta^{(N+1)}} \frac{\partial T}{\partial x_i}$$
$$= -\tfrac{8}{3} \frac{\varrho}{m} \left(\frac{\pi}{\beta}\right)^{1/2} \sum_{m=0}^{N'} \sum_{n=0}^{N'} \tilde{B}^{(N')}_{mn} \frac{2^{N+m}}{\beta^{(N-m)}} (N+1)!\, (n+1)!\, J^{(N+1,n+1)} q_{2m|i}. \tag{68}$$

We conclude from Eq. (66) that the first iterated values of $p^{NE}_{2N|rr}$ are zero for all N. Now if we multiply Eq. (67) by $\beta^N F_2^{(r,N)}/(2^N N!)$, sum over all values of N from 0 to r, and use the relationship

$$\sum_{s=0}^{p} \beta^{(r,s)} F_2^{(p,s)} = \frac{\varrho}{m} \sqrt{\pi\beta} \sum_{k=0}^{r} F_2^{(r,k)} K^{(k,p)}, \tag{69}$$

that follows from Eq. (24), we get after some calculation

$$-2\,(\tfrac{5}{4}\varrho)\, \delta^{(0,r)} \frac{\partial v_{\langle i}}{\partial x_{j\rangle}} = \sum_{s=0}^{N'} \beta^{(r,s)} T^{(s)}_{\langle ij\rangle}, \tag{70}$$

where

$$T^{(s)}_{\langle ij\rangle} = 2 \sum_{m=s}^{N'} \sum_{n=0}^{N'} F_2^{(n,s)} \beta^m\, n!\, 2^n A^{(N')}_{mn} p_{2m|\langle ij\rangle}. \tag{71}$$

Moreover, by using the identity

$$\delta^{(p,0)} = \tfrac{2}{15} \sum_{n=0}^{N'} A^{(N')}_{pn} (2n+5)!!, \tag{72}$$

that follows from Eq. (56), it is easy to get from Eq. (71)

$$T^{(0)}_{\langle ij\rangle} = p_{0|\langle ij\rangle} = p_{\langle ij\rangle}. \tag{73}$$

Hence, if we solve Eq. (70) for $p_{\langle ij\rangle}$ we get the same expression given by Eq. (11) for the coefficient of shear viscosity.

Likewise, by multiplication of Eq. (68) by $\beta^N F_1^{(r,N+1)}/(2^N(N+1)!)$, summing over all values of N from 0 to $(r-1)$, and using the relationship that follows from Eq. (25)

$$\sum_{s=1}^{p} \alpha^{(r,s)} F_1^{(p,s)} = \frac{\varrho}{m} \sqrt{\pi\beta} \sum_{k=1}^{p} F_1^{(r,k)} J^{(k,p)}, \tag{74}$$

we get

$$\left(\tfrac{75}{16}\varrho\frac{k}{m}\right)\delta^{(r,1)}\frac{\partial T}{\partial x_i}=\sum_{s=1}^{N'+1}\alpha^{(r,s)}V_i^{(s)}, \tag{75}$$

where

$$V_i^{(s)}=5\sum_{n=s}^{N'+1}\sum_{m=0}^{N'}F_1^{(n,s)}\beta^m 2^{n-1}n!\,\tilde{B}_{m,n-1}^{(N')}q_{2m|i}. \tag{76}$$

Furthermore, by using the identity

$$\delta^{(p,0)}=\tfrac{1}{3}\sum_{n=0}^{N'}\tilde{B}_{pn}^{(N')}(2n+5)!!\,(n+1), \tag{77}$$

which is obtained from Eq. (57), it follows from Eq. (76)

$$V_i^{(1)}=-q_{0|i}=-q_i. \tag{78}$$

Now by solving Eq. (75) for q_i, we get the same expression given by Eq. (12) for the coefficient of thermal conductivity.

C. Remarks

We have shown the equivalence between the methods developed in Secs. 2 and 3 with respect to the successive approximations to the transport coefficients. However, one could ask why in Sec. 3 we have developed a $(13+9N)$-field theory instead of a N-field theory with the full moments $p_{i_1,i_2,\ldots,i_N}$ as basic fields. The answer to this question is based on the fact that a tensor of rank N can be decomposed uniquely (see Weyl [6]) into summands of ranks $N, N-2, N-4, \ldots, 2$ (or 3) for N even (or odd), all of them having its traces equal to zero. Since we are interested in deriving linear laws for the second-order tensor p_{ij} and for the vector q_i (laws of Navier-Stokes and Fourier), we have just taking into account the tensors of lower rank, e.g., the vectors $q_{2N|i}$, and the second-order tensors $p_{2N|ij}$.

Acknowledgement. One of the authors (G. M. K.) gratefully acknowledges the support of the Conselho Nacional de Desenvolvimento Científico e Tecnológico of Brazil.

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Received January 3, 1991