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Exact structure of positive solutions for some p-Laplacian equations ☆

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Abstract

This paper establishes the exact multiplicities and properties of positive solutions for some second order differential equations involving p-Laplacian operator.

Keywords: Positive solution; Exact structure of all solutions; Properties of solutions

1. Introduction

In this paper, we study the exact multiplicities and properties of positive solutions for some second order differential equations involving p-Laplacian operator of the following boundary value problems:

$$\begin{cases} -(\varphi_p(x'))'(t) = \lambda(x^m(t) - x^n(t)), & t \in (0,1), \\ x(t) > 0, & t \in (0,1), \\ x(0) = x(1) = 0, \end{cases} \tag{1.1}$$

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where $\varphi_p(x) = |x|^{p-2}x$ for all $x \in R$ and λ is a positive parameter, $p > 1, m, n$ are constants satisfy $m > n > -1$.

Problem (1.1) comes from a problem raised by Agarwal and O'Regan [1]. Agarwal and O'Regan [1] proved the equation

$$\begin{cases} y''(t) + \delta(y^{-\alpha}(t) + y^{\beta}(t) + 1) = 0, & 0 < t < 1, \\ y(0) = y(1) = 0, & \delta > 0 \text{ a parameter,} \end{cases}$$

with $0 \leqslant \alpha < 1 < \beta$, has a nonnegative solution for all $\delta > 0$ small enough.

Exact multiplicity results are usually difficult to establish; see, e.g., [2]. The exact number of solutions was studied earlier by many authors for both elliptic and ordinary differential equations involving only concave or convex nonlinearities or cubic polynomials (see [3–10]), but detailed properties of the solutions were not considered at all. Liu in [11] and [12] considered the following two point boundary value problem:

$$\begin{cases} -v''(x) = \mu(v^p(x) + v^q(x) + kv(x)), & a \leqslant x \leqslant b, \\ v(x) > 0, & x \in (a, b), \\ v(a) = v(b) = 0, \end{cases} \tag{1.2}$$

where $0 < q < 1 < p$ and $k \geqslant 0$ are fixed given numbers and $\mu > 0$ is parameter. He gave not only the exact number of solutions of (1.2) but also many interesting properties of the solutions. Recently, Cheng in [13] has investigated the following two point boundary value problem:

$$\begin{cases} -y''(t) = \lambda(y^p(t) - y^q(t)), & -1 \leqslant t \leqslant 1, \\ y(t) > 0, & t \in (-1, 1), \\ y(-1) = y(1) = 0, \end{cases} \tag{1.3}$$

where $p > q > -1$ and $\lambda > 0$ is a positive parameter. He only considered the exact number of positive solutions, but do not give the detail properties of positive solutions of (1.3).

Recently, the existence theory of singular boundary value problems for the one-dimension p-Laplacian has been investigated by Jiang in the papers [14–17] with the method of upper and lower solutions, but there only few works on the exact multiplicities and properties of positive solutions for singular p-Laplacian equations (1.1).

Now, in this paper, we first give the exact multiplicity results of solutions of (1.1) and many useful properties of the solutions, so, the main results of this paper are an essential improvement and generalization of paper [13]. Then we will give some several important lemmas. Finally, we will give the proof of the main results.

2. The main result

For convenience, we give some denotes as follows:

$$\beta = \left(\frac{m+1}{n+1}\right)^{1/(m-n)}, \qquad \gamma_p = 2 \cdot \left(\frac{p-1}{p}\right)^{1/p}, \tag{2.1}$$

$$\mu_1 = 2^p \frac{(p-1)(n+1)}{p} \left(\frac{m+1}{n+1}\right)^{(p-n-1)/(m-n)} \times \left(\int_0^1 \left[t^{n+1}\left(1-t^{m-n}\right)\right]^{-1/p} dt\right)^p, \tag{2.2}$$

$$\mu_2 = 2^p m \left(\int_0^1 \left(1-t^p\right)^{-1/p} dt\right)^p, \tag{2.3}$$

$$\mu_3 = 2^p m \left(\int_0^1 \left(t^{1+n}-t^p\right)^{-1/p} dt\right)^p, \tag{2.4}$$

$$Q(p,m,n) = \int_0^1 \frac{p-m-1}{(t^{1+n}(1-t^{m-n}))^{1/p}} dt - \int_0^1 \frac{(m-n)(1-t^{1+n})}{(t^{1+n}(1-t^{m-n}))^{(p+1)/p}} dt. \tag{2.5}$$

Remark. If $p-1 > m > n \geqslant -1/(p+1)$, then

$$\int_0^1 \frac{(m-n)(1-t^{1+n})}{(t^{1+n}(1-t^{m-n}))^{(p+1)/p}} dt = +\infty \quad \text{and}$$
$$\int_0^1 \frac{p-m-1}{(t^{1+n}(1-t^{m-n}))^{1/p}} dt < \infty.$$

This means $Q(p,m,n) = -\infty$, in particular, $Q(p,m,n) < 0$.

Now, we state the main results of this paper as follows.

Theorem 2.1. *If $m > n \geqslant p-1$, then* (1.1) *has exactly one positive solution $x_\lambda(t)$ for any $\lambda > 0$; and the solution $x_\lambda(t)$ satisfies*

$$\lim_{\lambda\to 0^+} \|x_\lambda\| = +\infty, \qquad \lim_{\lambda\to+\infty} \|x_\lambda\| = \beta \quad (\textit{see Fig. } 1\text{a}).$$

Theorem 2.2. *If $m > p-1 > n > -1$, then* (1.1) *has exactly one positive solution $x_\lambda(t)$ for $0 < \lambda < \mu_1$ and none for $\lambda > \mu_1$; moreover, the solution $x_\lambda(t)$ satisfies*

$$\lim_{\lambda\to 0^+} \|x_\lambda\| = +\infty, \qquad \lim_{\lambda\to\mu_1} \|x_\lambda\| = \beta \quad (\textit{see Fig. } 1\text{b}).$$

Theorem 2.3. *If $m = p-1 > n > -1$, then* (1.1) *has exactly one positive solution $x_\lambda(t)$ for $\lambda \in (\mu_2, \mu_3]$ and none for $\lambda \in (0, \mu_2] \cup (\mu_3, +\infty)$; moreover, the solution $x_\lambda(t)$ satisfies*

$$\lim_{\lambda\to\mu_2} \|x_\lambda\| = +\infty, \qquad \lim_{\lambda\to\mu_3} \|x_\lambda\| = \beta \quad (\textit{see Fig. } 1\text{c}).$$

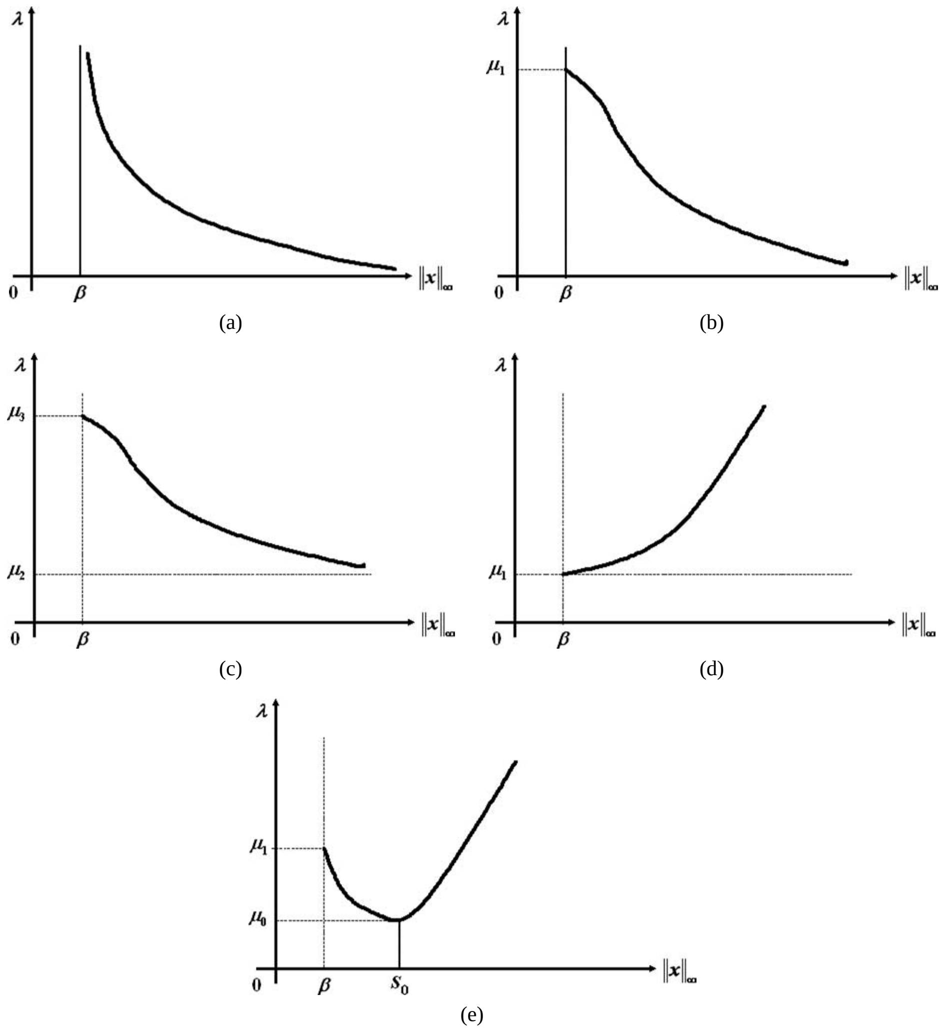

Fig. 1. (a) $m > b \geqslant p-1$, (b) $m > p-1 > n > -1$, (c) $m = p-1 > n > -1$, (d) $p-1 > m > n > -1$, $Q(p,m,n) \geqslant 0$, (e) $p-1 > m > n > -1$, $Q(p,m,n) < 0$.

Theorem 2.4. *Assume that* $p-1 > m > n > -1$.

(1) *If* $Q(p,m,n) \geqslant 0$, *then* (1.1) *has exactly one positive solution* $x_\lambda(t)$ *for* $\lambda \in (\mu_1, +\infty)$ *and none for* $\lambda \in (0, \mu_1)$; *moreover, the solution* $x_\lambda(t)$ *satisfies*

$$\lim_{\lambda \to \mu_1} \|x_\lambda\| = \beta, \qquad \lim_{\lambda \to +\infty} \|x_\lambda\| = +\infty \quad (\textit{see Fig. } 1\text{d}).$$

(2) *If* $Q(p,m,n) < 0$, *then there exists* $\mu_0 \in (0, \mu_1)$ *such that* (1.1) *has exactly two positive solutions* $x_{\lambda,1}, x_{\lambda,2}$ *with* $x_{\lambda,1}(t) < x_{\lambda,2}(t)$ *in* $t \in (0,1)$ *for* $\lambda \in (\mu_0, \mu_1]$, *exactly one for* $\lambda \in (\mu_1, +\infty)$ *or* $\lambda = \mu_0$, *and none for* $\lambda \in (0, \mu_0)$.

Moreover, in the case

(a) $\lambda \in (\mu_0, \mu_1]$: *the two positive solutions* $x_{\lambda,1}(t)$ *and* $x_{\lambda,2}(t)$ *satisfy*

$$\lim_{\lambda\to\mu_1} \|x_{\lambda,1}\| = \beta, \qquad \lim_{\lambda\to\mu_0} \|x_{\lambda,1}\| = \lim_{\lambda\to\mu_0} \|x_{\lambda,2}\| = s_0;$$

(b) $\lambda \in (\mu_1, +\infty)$: *the positive solution* $x_\lambda(t)$ *satisfies*

$$\lim_{\lambda\to+\infty} \|x_\lambda\| = +\infty \quad (\textit{see Fig. } 1\text{e}).$$

3. Several important lemmas

Denote

$$F(t) = \frac{t^{1+m}}{1+m} - \frac{t^{n+1}}{n+1},$$

$$M = M(s,t) = \frac{1}{m+1} s^{m+1}\bigl(1 - t^{m+1}\bigr), \quad t \in (0,1), \tag{3.1}$$

$$N = N(s,t) = \frac{1}{n+1} s^{n+1}\bigl(1 - t^{n+1}\bigr), \quad t \in (0,1). \tag{3.2}$$

Define a function $g : [\beta, +\infty) \to (0, +\infty)$ as

$$g(s) = \gamma_p \int_0^s \bigl[F(s) - F(t)\bigr]^{-1/p}\, dt, \quad s \geqslant \beta,$$

that is

$$g(s) = \gamma_p s \int_0^1 \left[\frac{s^{1+m}(1-t^{1+m})}{1+m} - \frac{s^{n+1}(1-t^{n+1})}{n+1}\right]^{-1/p} dt$$

$$= \gamma_p s \int_0^1 \bigl[M(s,t) - N(s,t)\bigr]^{-1/p}\, dt, \quad s \geqslant \beta. \tag{3.3}$$

Lemma 3.1. *For* $-1 < n < m < +\infty$, $g(s)$ *has continuous derivatives up to the second order on* $[\beta, +\infty)$, *and*

$$g'(s) = \frac{\gamma_p}{p} \int_0^1 \bigl[(p-m-1)M - (p-n-1)N\bigr] \times [M-N]^{-(p+1)/p}\, dt, \tag{3.4}$$

$$
\begin{aligned}
g''(s) &= \frac{\gamma_p}{sp}\int_0^1 \frac{[(p-m-1)(m+1)M-(p-n-1)(n+1)N]}{[M-N]^{(p+1)/p}}\,dt \\
&\quad - \gamma_p\left(\frac{p+1}{sp^2}\right)\int_0^1 \big[(p-m-1)M-(p-n-1)N\big] \\
&\quad \times \big[(m+1)M-(n+1)N\big]\times[M-N]^{-(2p+1)/p}\,dt.
\end{aligned}
\tag{3.5}
$$

Proof. Obviously, $g(s) > 0$, for $s \geqslant \beta$. For $0 \leqslant t \leqslant 1$, we have

$$
\begin{aligned}
&1-t \leqslant 1-t^{n+1} \leqslant (n+1)(1-t), \\
&(1-n)(1-t) \leqslant 1-t^{1-n} \leqslant (1-t), \quad \text{for } n \geqslant 0. \qquad \square
\end{aligned}
$$

It follows that for any $\beta < s_1 < s_2 < +\infty$, there exists a positive constant C depending only on m, n, s_1, and s_2 such that, if $s_1 \leqslant s \leqslant s_2$ and $0 \leqslant t \leqslant 1$, the absolute value of each integrand in (3.3)–(3.5) is less than $C(1-t)^{-1/p}$. This implies that each singular integral in (3.3)–(3.5) converges uniformly with respect to $s_1 \leqslant s \leqslant s_2$. Therefore, $g(s)$ has continuous derivatives up to the second order on $[s_1, s_2]$ and $g'(s)$ and $g''(s)$ have expressions (3.4) and (3.5), respectively. In view of the fact that s_1 and s_2 are arbitrary, we get the result.

Lemma 3.2.

$$
\lim_{s\to\beta} g(s) = \begin{cases} +\infty, & n \geqslant p-1, \\ \mu_1^{1/p}, & n < p-1, \end{cases}
$$

here μ_1 *is given by* (2.2).

Proof. Making use of

$$
\frac{1}{n+1}\beta^{n+1} = \frac{1}{m+1}\beta^{m+1}
$$

and (3.3), we have

$$
\begin{aligned}
\lim_{s\to\beta} g(s) &= \gamma_p(1+n)^{1/p}\beta^{(p-n-1)/p}\int_0^1 \frac{dt}{(t^{1+n}(1-t^{m-n}))^{1/p}} \\
&= \begin{cases} +\infty, & n \geqslant p-1, \\ \mu_1^{1/p}, & n < p-1. \end{cases} \qquad \square
\end{aligned}
$$

Lemma 3.3.

$$
\lim_{s\to\infty} g(s) = \begin{cases} +\infty, & m < p-1, \\ 0, & m > p-1, \\ 2\cdot m^{1/p}\int_0^1 \frac{dt}{(1-t^p)^{1/p}} = \mu_2^{1/p}, & m = p-1. \end{cases}
$$

Proof. By means of (3.1), (3.2) and $m > n$, we have $\forall t \in (0,1)$,

$$\lim_{s\to\infty} \frac{N(s,t)}{M(s,t)} = \lim_{s\to\infty} \frac{(m+1)s^{1+n}(1-t^{1+n})}{(n+1)s^{1+m}(1-t^{1+m})} = 0.$$

Therefore $\forall t \in (0,1)$,

$$\lim_{s\to\infty} \frac{s}{(M(s,t)-N(s,t))^{1/p}} = \lim_{s\to\infty} \frac{s}{(M(s,t))^{1/p}} = \lim_{s\to\infty} \frac{s(1+m)^{1/p}}{(s^{m+1}(1-t^{m+1}))^{1/p}}$$
$$= \begin{cases} +\infty, & m < p-1, \\ 0, & m > p-1, \\ \frac{p^{1/p}}{(1-t^p)^{1/p}}, & m = p-1. \end{cases}$$

(3.3) together with the Lebesgue dominated convergence theorem guarantee this lemma. □

Lemma 3.4. *For $s \geqslant \beta$ and $t \in (0,1)$, we have*

$$\frac{1+n}{1+m} s^{m-n} < \frac{M(s,t)}{N(s,t)} < s^{m-n}.$$

Proof. Denote by $k(t) = (m-n)t^{m+1} - (m+1)t^{m-n} + n + 1$. From $k(0) = n+1$, $k(1) = 0$ and $k'(t) = (m-n)(m+1)t^{m-n-1}(t^{m+1}-1) < 0$, $t \in (0,1)$, we have that $k(t) > 0$ for $t \in (0,1)$. It follows that

$$\frac{\partial}{\partial t} \frac{M(s,t)}{N(s,t)} = \left(\frac{n+1}{m+1}\right) s^{m-n} \frac{t^n k(t)}{(1-t^{1+n})^2} > 0, \quad t \in (0,1).$$

Combining

$$\lim_{t\to 0} \frac{M(s,t)}{N(s,t)} = \frac{n+1}{m+1} s^{m-n} \quad \text{and} \quad \lim_{t\to 1} \frac{M(s,t)}{N(s,t)} = s^{m-n},$$

we get the results of this lemma. □

Lemma 3.5. *For $p-1 > m > n > -1$,*

$$\lim_{s\to\beta} g'(s) = \frac{\gamma_p}{p} \left(\frac{n+1}{\beta^{1+n}}\right)^{1/p} Q(p,m,n),$$

here, $Q(p,m,n)$ is given by (2.5).

Proof.

$$M(\beta,t) - N(\beta,t) = \frac{\beta^{1+n} t^{1+n}(1-t^{m-n})}{1+n}, \qquad N(\beta,t) = \frac{\beta^{n+1}}{n+1}\bigl(1-t^{n+1}\bigr).$$

By Lemma 3.1, we have

$$\begin{aligned}
&\lim_{s\to\beta} g'(s)\\
&= \lim_{s\to\beta} \frac{\gamma_p}{p}\left(\int_0^1 \frac{p-m-1}{(M(s,t)-N(s,t))^{1/p}}\,dt - \int_0^1 \frac{(m-n)N(s,t)}{(M(s,t)-N(s,t))^{(p+1)/p}}\,dt\right)\\
&= \frac{\gamma_p}{p}\left(\int_0^1 \frac{p-m-1}{(M(\beta,t)-N(\beta,t))^{1/p}}\,dt - \int_0^1 \frac{(m-n)N(\beta,t)}{(M(\beta,t)-N(\beta,t))^{(p+1)/p}}\,dt\right)\\
&= \frac{\gamma_p}{p}\left(\frac{n+1}{\beta^{1+n}}\right)^{1/p} Q(p,m,n). \qquad \square
\end{aligned}$$

Lemma 3.6. *For $-1 < n < m < p-1$, $s \in [\beta, +\infty)$, there exists $\delta = \delta(p,m,n) > 0$ such that*

$$g''(s) + \delta g'(s)\frac{m+1}{sp} > \gamma_p \cdot \frac{(p-m-1)(m+1)(1-m^*)(n^*-1)\sqrt[p]{n+1}}{ps^{(2pm-2pn+m+p+1)/p}}, \tag{3.6}$$

where m^*, n^* are given by

$$m^* = \frac{n+1}{m+1}, \qquad n^* = \frac{p-n-1}{p-m-1}. \tag{3.7}$$

Proof. For $-1 < n < m < p-1$, $s \in [\beta, +\infty)$, by (3.4) and (3.5), $g'(s), g''(s)$ can be rewritten by

$$g'(s) = \gamma_p \frac{p-m-1}{p}\int_0^1 G(s,t)H_1\bigl(h(s,t)\bigr)\,dt, \tag{3.8}$$

$$g''(s) = \gamma_p \frac{(p-m-1)(m+1)}{sp^2}\int_0^1 G(s,t)H_2\bigl(h(s,t)\bigr)\,dt, \tag{3.9}$$

where

$$G(s,t) = \bigl(h(s,t)-1\bigr)^{-(2p+1)/p}N^{-1/p}, \tag{3.10}$$

$$h(s,t) = \frac{M(s,t)}{N(s,t)}, \tag{3.11}$$

$$H_1(x) = (x-n^*)(x-1), \tag{3.12}$$

$$H_2(x) = -(p+1)(x-n^*)(x-m^*) + p(x-m^*n^*)(x-1). \tag{3.13}$$

Since

$$H_1\left(\frac{1+n^*}{2}\right) = -\frac{(n^*-1)^2}{4} < 0$$

and

$$\min_{x\in[1,n^*]} H_2(x) = \min\bigl\{H_2(1), H_2(n^*)\bigr\} = (1-m^*)(n^*-1)\{p+1, pn^*\}$$
$$> p(1-m^*)(n^*-1),$$

there exists $\delta = \delta(p, r, s) > 1$ such that

$$\min_{x\in[1,n^*]} \bigl[H_2(x) + \delta H_1(x)\bigr] = p(1-m^*)(n^*-1). \tag{3.14}$$

Let

$$H(x) = H_2(x) + \delta H_1(x), \quad \forall x \in (-\infty, +\infty). \tag{3.15}$$

Then from

$$H(x) = (\delta-1)x^2 + \bigl[(p+1)(m^*+n^*) - p(m^*n^*+1) - \delta(n^*+1)\bigr]x$$
$$- m^*n^* + \delta n^*$$

and

$$\min_{x\in[1,n^*]} H(x) = p(1-m^*)(n^*-1) < (1-m^*)(n^*-1)\min\{p+1, pn^*\}$$
$$= \min\bigl\{H_2(1), H_2(n^*)\bigr\} = \min\bigl\{H(1), H(n^*)\bigr\},$$

we have that $\delta > 1$ and

$$H(x) \geqslant \min_{\xi\in[1,n^*]} H(\xi) = p(1-m^*)(n^*-1), \quad x \in (-\infty, +\infty). \tag{3.16}$$

By (2.3), we have

$$0 < N(s,t) < \frac{s^{n+1}}{n+1}. \tag{3.17}$$

By means of Lemma 2.5 and (3.11), we have

$$0 < m^*s^{m-n} - 1 < h(s,t) - 1 < h(s,t) < s^{m-n}. \tag{3.18}$$

For $s > \beta$ and $t \in (0,1)$, by (3.10), (3.17), and (3.18), we can get

$$G(s,t) > s^{-(2pm-2pn+m+1)/p}(n+1)^{1/p}. \tag{3.19}$$

Making use of (3.8), (3.9), (3.15), (3.16), and (3.19), we obtain

$$g''(s) + \delta g'(s)\frac{m+1}{sp} = \gamma_p \cdot \frac{(p-m-1)(n+1)}{sp^2}\int_0^1 G(s,t)H\bigl(h(s,t)\bigr)\,dt$$
$$> \gamma_p \cdot \frac{(p-m-1)(m+1)(1-m^*)(n^*-1)(n+1)^{1/p}}{ps^{(2pm-2pn+m+p+1)/p}}.$$

This completes the proof of Lemma 3.6.

Combining Lemmas 3.2 and 3.5 gives the results of this lemma. □

4. The proof of main results

Let t_0 be the point at which some solution $x(t)$ of (1.1) takes its maximum $\|x\| = x(t_0)$ and denotes $x(t_0)$ by s. Thus (1.1) may be integrated as follows: multiply the first equality with $x'(t)$ and integrate it from t to t_0, then

$$\frac{p-1}{p}\left|x'(t)\right|^p = \lambda \int\limits_{x(t)}^{s} \left(z^m - z^n\right) dz, \quad t \in (0,1).$$

This implies that $s \geqslant \beta > 1$ and $x'(s) \neq 0$ if $0 \leqslant x(t) < s$. Therefore, $x'(t) > 0$ on $(0, t_0)$, $x'(t) < 0$ on $(t_0, 1)$ and $x'(t_0) = 0$ and

$$\begin{cases} \frac{p-1}{p}(x'(t))^p = \lambda[F(x(t_0)) - F(x(t))], & \text{for } 0 < t \leqslant t_0, \\ \frac{p-1}{p}(-x'(t))^p = \lambda[F(x(t_0)) - F(x(t))], & \text{for } t_0 \leqslant t < 1. \end{cases} \tag{4.1}$$

Take the p-power root of (4.1), and then integrate it from 0 to t or from t to 1, it follows that

$$\begin{cases} \int_0^{x(t)}[F(s) - F(\xi)]^{-1/p}\, d\xi = \left(\frac{p}{p-1}\right)^{1/p}\lambda^{1/p} t, & \text{for } s \geqslant \beta,\ 0 \leqslant t \leqslant t_0, \\ \int_0^{x(t)}[F(s) - F(\xi)]^{-1/p}\, d\xi = \left(\frac{p}{p-1}\right)^{1/p}\lambda^{1/p}(1-t), & \text{for } s \geqslant \beta,\ t_0 \leqslant t \leqslant 1. \end{cases} \tag{4.2}$$

Setting $t = t_0$ in (4.2), we see that $t_0 = 1/2$ and $x(t) = x(1-t)$; that is, any solution $x(t)$ of (1.1) takes its maximum at $1/2$, $x(t)$ is symmetric with respect to $1/2$, $x'(t) > 0$ for $0 < t < 1/2$ and $x'(t) < 0$ for $1/2 < t < 1$. Hence, (1.1) is equivalent to the following problem defined on $[0, 1/2]$:

$$\begin{cases} -(\varphi_p(x'))'(t) = \lambda(x^m(t) - x^n(t)), & 0 < t \leqslant 1/2, \\ x(t) > 0, & 0 < t \leqslant 1/2, \\ x(0) = x'(1/2) = 0. \end{cases} \tag{4.3}$$

Choosing t to be $1/2$ in (4.2), we get that

$$g(s) = 2\left(\frac{p-1}{p}\right)^{1/p} \int\limits_0^s \left[F(s) - F(\xi)\right]^{-1/p} d\xi = \lambda^{1/p}, \quad s \geqslant \beta. \tag{4.4}$$

Conversely, for a given $\lambda > 0$, if s satisfies (4.4), then (4.2) defines a function $x(t)$ on $[0, 1/2]$ satisfying (4.3) with $x(1/2) = s$, and it is easy to see that $x(t)$ is a solution of (4.3). Therefore, we get the important proposition.

Proposition 4.1. *The number of solutions of* (4.3) *is equal to the number of s satisfying* (4.4).

Proof of Theorem 2.1. According to Lemma 3.4, for $s \in [\beta, +\infty)$, $t \in (0,1)$, we have

$$\frac{M(s,t)}{N(s,t)} > \frac{n+1}{m+1} s^{m-n} \geqslant \frac{n+1}{m+1}\beta^{m-n} = 1.$$

Therefore if $m > n \geqslant p-1$, then $0 < (p-n-1)/(p-m-1) < 1$ and

$$(p-m-1)M - (p-n-1)N = (p-m-1)N\left(\frac{M}{N} - \frac{p-n-1}{p-m-1}\right) < 0.$$

By Lemma 3.1, we have $g'(s) < 0$ for $s \in [\beta, +\infty)$. By Lemmas 3.2 and 3.3, if $m > n \geqslant p-1$, then

$$\lim_{s\to\beta} g(s) = +\infty, \qquad \lim_{s\to+\infty} g(s) = 0. \tag{4.5}$$

By the Proposition 4.1, (4.5) and the inverse function theorem, we obtain that if $m > n \geqslant p-1$, then (1.1) has exactly one positive solution $x_\lambda(t)$ for any $\lambda > 0$; and the solution $x_\lambda(t)$ satisfies $\lim_{\lambda\to 0^+}\|x_\lambda\| = +\infty$, $\lim_{\lambda\to+\infty}\|x_\lambda\| = \beta$. This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. For $m > p-1 > n > -1$, then $p-n-1 > 0$, $p-m-1 < 0$ and

$$(p-m-1)M - (p-n-1)N = (p-m-1)N\left(\frac{M}{N} - \frac{p-n-1}{p-m-1}\right) < 0.$$

By Lemma 3.1, we have $g'(s) < 0$ for $s \in [\beta, +\infty)$. By Lemmas 3.2 and 3.3, if $m > n \geqslant p-1$, then

$$\lim_{s\to\beta} g(s) = \mu_1^{1/p}, \qquad \lim_{s\to+\infty} g(s) = 0. \tag{4.6}$$

By Proposition 4.1, (4.6), and the inverse function theorem, we obtain that if $m > p-1 > n > -1$, then (1.1) has exactly one positive solution $x_\lambda(t)$ for $\lambda \in (0, \mu_1]$ and none for $\lambda > \mu_1$; and the solution $x_\lambda(t)$ satisfies $\lim_{\lambda\to 0^+}\|x_\lambda\| = +\infty$, $\lim_{\lambda\to\mu_1}\|x_\lambda\| = \beta$. This completes the proof of Theorem 2.2. □

Proof of Theorem 2.3. According to Lemma 3.4, for $s \in [\beta, +\infty)$, $t \in (0, 1)$, we have

$$\frac{M(s,t)}{N(s,t)} > \frac{n+1}{m+1}s^{m-n} \geqslant \frac{n+1}{m+1}\beta^{m-n} = 1.$$

Therefore if $m = p-1 > n > -1$, then

$$(p-m-1)M - (p-n-1)N = -(p-n-1)N < 0.$$

By Lemma 3.1, we have $g'(s) < 0$ for $s \in [\beta, +\infty)$. By Lemmas 3.2 and 3.3, if $m = p-1 > n > -1$, then

$$\lim_{s\to\beta} g(s) = \mu_3^{1/p}, \qquad \lim_{s\to+\infty} g(s) = \mu_2^{1/p}. \tag{4.7}$$

By Proposition 4.1, (4.7), and the inverse function theorem, we obtain that if $m = p-1 > n > -1$, then (1.1) has exactly one positive solution $x_\lambda(t)$ for $\lambda \in (\mu_2, \mu_3]$ and none for $\lambda \in (0, \mu_2] \cup (\mu_3, +\infty)$; moreover, the solution $x_\lambda(t)$ satisfies $\lim_{\lambda\to\mu_2}\|x_\lambda\| = +\infty$, $\lim_{\lambda\to\mu_3}\|x_\lambda\| = \beta$. The proof is completed. □

Proof of Theorem 2.4. It follows from Lemma 3.6 that if $g'(s) = 0$, then $g''(s) > 0$, and $g'(s)$ has at most one zero in $(\beta, +\infty)$.

Therefore, for case

(i) $Q(p, m, n) \geqslant 0$, then $g'(s) > 0$ for $s \in [\beta, +\infty)$.
By Lemmas 3.2 and 3.3, if $p - 1 > m > n > -1$, then

$$\lim_{s \to \beta} g(s) = \mu_1^{1/p}, \qquad \lim_{s \to +\infty} g(s) = +\infty. \tag{4.8}$$

By Proposition 4.1, (4.8), and the inverse function theorem, we obtain that if $p - 1 > m > n > -1$ and $Q(p, m, n) \geqslant 0$, then (1.1) has exactly one positive solution $x_\lambda(t)$ for $\lambda \in [\mu_1, +\infty)$ and none for $\lambda \in (0, \mu_1)$; moreover, the solution $x_\lambda(t)$ satisfies $\lim_{\lambda \to \mu_1} \|x_\lambda\| = \beta$, $\lim_{\lambda \to +\infty} \|x_\lambda\| = +\infty$.

(ii) $Q(p, m, n) < 0$, then there exists zero point $s_0 \in (\beta, +\infty)$ of $g'(s)$ such that

$$\begin{cases} g'(s) < 0, & \text{for } s \in (\beta, s_0), \\ g'(s) > 0, & \text{for } s \in (s_0, +\infty). \end{cases}$$

Let $\mu_0 = g^p(s_0)$. From Lemmas 3.2 and 3.3, if $p - 1 > m > n > -1$, then

$$\lim_{s \to \beta} g(s) = \mu_1^{1/p}, \qquad \lim_{s \to s_0} g(s) = \mu_0, \qquad \lim_{s \to +\infty} g(s) = +\infty. \tag{4.9}$$

By the Proposition 4.1, (4.9), and the inverse function theorem, we obtain that if $p - 1 > m > n > -1$ and $Q(p, m, n) < 0$, (1.1) has exactly two positive solutions $x_{\lambda,1}, x_{\lambda,2}$ with $\|x_{\lambda,1}\| < \|x_{\lambda,2}\|$ for $\lambda \in (\mu_0, \mu_1]$, exactly one for $\lambda \in (\mu_1, +\infty)$ or $\lambda = \mu_0$, and none for $\lambda \in (0, \mu_0)$.

Moreover, in the case

(a) $\lambda \in (\mu_0, \mu_1]$: the two positive solutions $x_{\lambda,1}(t)$ and $x_{\lambda,2}(t)$ satisfy $\lim_{\lambda \to \mu_1} \|x_{\lambda,1}\| = \beta$; $\lim_{\lambda \to \mu_0} \|x_{\lambda,1}\| = \lim_{\lambda \to \mu_0} \|x_{\lambda,2}\| = s_0$.
(b) $\lambda \in (\mu_1, +\infty)$: the positive solution $x_\lambda(t)$ satisfies $\lim_{\lambda \to +\infty} \|x_\lambda\| = +\infty$.

This completes the proof of Theorem 2.4. □

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