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Discrete boundary finite element schemes for an exterior problem for the time-harmonic Maxwell's equation

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ABSTRACT

A general boundary integral formulation using Galerkin procedure is applied to compute the scattering electric field produced by the diffraction of an incident electromagnetic wave by a perfectly conducting obstacle. This electric field satisfies the three-dimensional time-harmonic Maxwell's equations for which the skin currents and charges are to be approximated using boundary finite element method. With the help of linear and quadratic finite elements of Lagrange type, we introduce an approximate surface on which the discrete formulation is defined, and construct approximate surface currents and charges. In addition, we study the existence and the uniqueness of the solution of the discrete problem and develop some error estimates for the currents and charges. Numerical results are also presented in order to validate our numerical approach.

1. Introduction

Let Ω^i be a bounded domain of $\mathbb{R}^3$ with boundary Γ and Ω^e be its exterior domain. These two domains represent respectively, a perfect conductor and the air. We want to determine the diffracted electric field $\overrightarrow{E}$ satisfying the time-harmonic Maxwell's equation and coming from the diffraction of an electromagnetic wave by the perfectly conducting obstacle Γ. This field $\overrightarrow{E}$ is solution of the problem [5]

$$\begin{cases} \Delta\overrightarrow{E} + k^2\overrightarrow{E} = \overrightarrow{0},\ div\,\overrightarrow{E} = \overrightarrow{0} \text{ in } \Omega^e, \\ \Pi_\Gamma\overrightarrow{E} = -\Pi_\Gamma\overrightarrow{E}^{in} = \overrightarrow{c} \text{ on } \Gamma, \\ \overrightarrow{curl}\,\overrightarrow{E}\wedge\frac{x}{r} - ik\overrightarrow{E} = o\left(\frac{1}{r}\right) \text{ when } r\longrightarrow\infty \end{cases} \tag{1}$$

where $\overrightarrow{E}^{in}$ is the incident electric wave satisfying the Maxwell's equations in air, $k = \omega\sqrt{\mu_0\varepsilon_0} > 0$ is the wave number, $\Pi_\Gamma\overrightarrow{E} = -\overrightarrow{n}\wedge(\overrightarrow{n}\wedge\overrightarrow{E})|_\Gamma$ is the tangential field of $\overrightarrow{E}$ to Γ, $r = |x|$ and $\overrightarrow{n}$ is the outward normal vector to Γ.

Different methods and analysis are developed for the determination of the diffracted field satisfying the time-harmonic Maxwell's equation may be found in many references (see [2,3,7,8,10–12,16,20,23,25,29,33,34]). It is to note that the list of references given above is by no means complete. In [31] MacCamy and Stephan proposed a solution procedure of the problem (1) by boundary integral equations based on the Galerkin method. In this paper, we start by presenting the system of the integral equations of the scattered electric field, showing the necessity of the modification of this system (see [31]) whose proposed variational formulation is not strongly elliptic. The new system verifies a Gårding's inequality (see [46,47]), which is primordial for the numerical analysis. The numerical approximation of the modified variational formulation by a boundary

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element method, requires the introduction of some geometrical approximations for the surface Γ and some finite element spaces. For completeness, we begin by presenting a complete survey of the problem by considering the error resulted from the surface approximation. For this, we first study the error estimates of the potentials forming the solution of the variational formulation and of the electric field in the proposed finite element spaces. Next, we study some error estimates, in the case where the surface approximation is taken into account. Some existence and uniqueness results are also presented for the various variational formulations. Finally, numerical schemes are developed for the numeric implementation for the discrete variational formulation and some numerical results for the calculation of the scattered electric field are also presented.

2. Presentation of the problem

Let V be the simple layer potential defined for a given field u by

$$V(u)(x) = \int_\Gamma u(y)G(x,y)d\Gamma_y, \quad x \in \mathbb{R}^3 \tag{2}$$

with $G(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$.

For $s > 0, H^s(\Gamma)$ denotes a Hilbert space, $H^{-s}(\Gamma)$ its dual space and $\mathcal{H}^s(\Gamma)$ is the space formed by the fields $\vec{u} = \sum_{i=1}^3 u^i \vec{e}_i$, with $u^i \in H^s(\Gamma)$ for an orthonormal basis $\{\vec{e}_i\}_{i=1,2,3}$ of $\mathbb{R}^3$. On these spaces, we consider the following inner products

$$< u, v >= \int_\Gamma u(x)\overline{v(x)}d\Gamma_x \quad \text{and} \quad < \vec{u}, \vec{v} >= \int_\Gamma \vec{u}(x) \cdot \overline{\vec{v}(x)}d\Gamma_x. \tag{3}$$

Next define the spaces $TH^s(\Gamma)$ and $\mathcal{H}^{s-1,s}$ by

$$TH^s(\Gamma) = \left\{\vec{u} \in \mathcal{H}^s(\Gamma) : \ \vec{n} \cdot \vec{u} = 0\right\} \quad \text{and} \quad \mathcal{H}^{s-1,s}(\Gamma) = TH^{s-1}(\Gamma) \times H^s(\Gamma). \tag{4}$$

The solution of the problem (1) has the following representation of Stratton-Shu [43]:

$$\vec{E} = V(\vec{p}) + \overrightarrow{grad}\, V(\lambda) \quad \text{in} \quad \Omega^i \cup \Omega^e \tag{5}$$

where the potentials $\vec{p}$ and λ are respectively the electric surface currents and charges defined by

$$\vec{p} = \left[\vec{n} \wedge \overrightarrow{curl}\, \vec{E}\right]_\Gamma \in TH^{-\frac{1}{2}}(\Gamma) \quad \text{and} \quad \lambda = -\left[\vec{n} \cdot \vec{E}\right]_\Gamma \in H^{\frac{1}{2}}(\Gamma) \tag{6}$$

with $[u]_\Gamma$ as the jump across Γ of a field u.

The main of this work is to solve numerically the following system of boundary integral equations proposed by MacCamy and Stephan [31] involving from (1) with the constraint $div\, \vec{E} = \vec{0}$ on Γ :

$$\begin{cases} \Pi_\Gamma V(\vec{p}) + \overrightarrow{grad}_\Gamma V(\lambda) = \vec{c}, \\ V(div_\Gamma\, \vec{p}) - k^2 V(\lambda) = 0, \end{cases} \tag{7}$$

where $\Pi_\Gamma V(\vec{p})$ is the tangential field of $V(\vec{p})$ to Γ, $\overrightarrow{grad}_\Gamma V(\lambda)$ is the surface gradient of $V(\lambda)$ and $div_\Gamma\, \vec{p}$ is the surface divergence of $\vec{p}$. The variational formulation by Galerkin method of the above system is given by

$$\begin{cases} \text{Find } (\vec{p}, \lambda) \in \mathcal{H}^{r-1,r}(\Gamma) \quad \text{such that for all} \quad (\vec{q}, \mu) \in \mathcal{H}^{r-1,r}(\Gamma) \\ < \Pi_\Gamma V(\vec{p}), \vec{q} > + < \overrightarrow{grad}_\Gamma V(\lambda), \vec{q} >=< \vec{c}, \vec{q} >, \\ < V(div_\Gamma\, \vec{p}), \mu > -k^2 < V(\lambda), \mu >= 0 \end{cases} \tag{8}$$

Since the system (7) is not strongly elliptic, which is a necessary condition for the existence and the uniqueness of the solution, MacCamy and Stephan [31] proposed a modification of these integral equations. They found that there exists a continuous mapping J_Γ defined from $TH^s(\Gamma)$ into $H^{s+1}(\Gamma)$ $(s \in \mathbb{R})$ such that

$$div_\Gamma \Pi_\Gamma V(\vec{p}) = V(div_\Gamma\, \vec{p}) + J_\Gamma(\vec{p}). \tag{9}$$

Therefore we obtain the following new system

$$\begin{cases} \Pi_\Gamma V(\vec{p}) + \overrightarrow{grad}_\Gamma V(\lambda) = \vec{c}, \\ -J_\Gamma(\vec{p}) - (\Delta_\Gamma + k^2)V(\lambda) = -div_\Gamma\, \vec{c}, \end{cases} \tag{10}$$

with $\Delta_\Gamma = div_\Gamma \overrightarrow{grad}_\Gamma$. The advantage of this new system is that it satisfies the Gårding's inequality (see [46,47]).

For the numerical approximation of (10), we propose a variational formulation based on a Galerkin method. This variational formulation has the following form:

$$\begin{cases} \text{Find } (\vec{p}, \lambda) \in \mathcal{H}^{-\frac{1}{2},\frac{1}{2}}(\Gamma) \quad \text{such that for all} \quad (\vec{q}, \mu) \in \mathcal{H}^{-\frac{1}{2},\frac{1}{2}}(\Gamma) \\ < \Pi_\Gamma V(\vec{p}), \vec{q} > + < \overrightarrow{grad}_\Gamma V(\lambda), \vec{q} >=< \vec{c}, \vec{q} >, \\ < -J_\Gamma(\vec{p}), \mu > - < (\Delta_\Gamma + k^2)V(\lambda), \mu >=< -div_\Gamma\, \vec{c}, \mu > . \end{cases} \tag{11}$$

Using Green's formula [2]

$$< div_\Gamma \overrightarrow{p}, \lambda >= - < \overrightarrow{p}, \overrightarrow{grad}_\Gamma \lambda >, \tag{12}$$

the relation (9) and from the fact that the operator Δ_Γ, is self-adjoint, the variational formulation (11) then becomes

$$\begin{cases} \text{Find } (\overrightarrow{p}, \lambda) \in \mathcal{H}^{r-1,r}(\Gamma) \quad \text{such that for all} \quad (\overrightarrow{q}, \mu) \in \mathcal{H}^{r-1,r}(\Gamma) \\ < \Pi_\Gamma V(\overrightarrow{p}), \overrightarrow{q} > - < V(\lambda), div_\Gamma \overrightarrow{q} >=< \overrightarrow{c}, \overrightarrow{q} >, \\ < V(div_\Gamma \overrightarrow{p}), \mu > + < \Pi_\Gamma V(\overrightarrow{p}), \overrightarrow{grad}_\Gamma \mu > - < V(\lambda), (\Delta_\Gamma + k^2)\mu >=< -div_\Gamma \overrightarrow{c}, \mu >, \end{cases} \tag{13}$$

The study of the existence and the uniqueness of the solution of the above continuous problems is developed in [31,32]. In the following we present a complete study of the discrete problem of (11).

3. Approximation of the surface and finite element spaces

Consider the set $\mathcal{T}_a = \{T_1, \ldots, T_N\}$ consisting of plane open triangles with straight edges in $\mathbb{R}^3$. This set is an affine surface mesh for Γ satisfying the following conditions [41]:

– $\mathcal{T}_a$ is a regular boundary element mesh for the affine surface approximation

$$\Gamma_a = \bigcup_{T \in \mathcal{T}_a} \overline{T}. \tag{14}$$

– Γ_a interpolates the surface Γ at the vertices of the triangles.
– The reference mappings $F_T : \hat{T} \to T$ are affine.

Assumption 3.1. There is a neighborhood U_Γ of Γ and a mapping $\Phi : U_\Gamma \to \Gamma$ such that:

(i) $\Gamma_a \subset U_\Gamma$.
(ii) $\Phi|_{\Gamma_a} : \Gamma_a \to \Gamma$ is bi-Lipschitz continuous, i.e., the exists a constant $C_\Phi > 0$ such that

$$C_\Phi \|x - y\| \leqslant \|\Phi(x) - \Phi(y)\| \leqslant C_\Phi^{-1} \|x - y\|, \; \forall x, y \in \Gamma_a. \tag{15}$$

(iii) For any $T \in \mathcal{T}_a$ and $T_\Gamma = \Phi(T)$, the restriction $\Phi|_T : T \to T_\Gamma$ is C^{k-1} diffeomorphism for some $k \geqslant 2$.

Next, we will define the lth order parametric surface approximation by employing the mapping Φ as in Assumption 3.1. Recall that if $\Sigma_{\hat{T}}^l$ is the set of nodal points on the reference element, $(\hat{T}, \mathbb{P}_l, \Sigma_{\hat{T}}^l)$ is a finite element of Lagrange type [4].

Definition 3.2. For given panel $T \in \mathcal{T}_a$ and degree $l \geqslant 1$, the l-parametric reference mapping F_T^l of degree l is the $l - th$ order nodal interpolation of $\Phi \circ F_T$.

For $T \in \mathcal{T}_a$, the image $F_T^l(\hat{T}) = T_h$ is the l-parametric panel of degree l. The set $\mathcal{T}_h = \{T_h : T \in \mathcal{T}_a\}$ is the l-parametric surface mesh of degree l. Then, the l-parametric surface approximation of degree l corresponding to $\mathcal{T}_h$ is given by

$$\Gamma_h = \overline{\bigcup_{T_h \in \mathcal{T}_h} T_h}. \tag{16}$$

Remarks 3.3. The l-parametric reference mapping $F_T^l : \hat{T} \to T_h$ is componentwise a polynomial of degree l and for $l = 1$, the mesh $\mathcal{T}_h = \mathcal{T}_a$.

The analogue to Assumption 3.1 for l-parametric surface approximations of higher order reads as follows [41].

Assumption 3.4. The mapping $\Phi : U_\Gamma \to \Gamma$ satisfies:

(i) $\Gamma_h \subset U_\Gamma$.
(ii) $\Phi|_{\Gamma_h} : \Gamma_h \to \Gamma$ is bi-Lipschitz continuous, i.e., the exists a constant $C_\Phi > 0$ such that

$$C_\Phi \|x - y\| \leqslant \|\Phi(x) - \Phi(y)\| \leqslant C_\Phi^{-1} \|x - y\|, \; \forall x, y \in \Gamma_h. \tag{17}$$

(iii) For any $T_h \in \mathcal{T}_h$ and $T_\Gamma = \Phi(T_h)$, the restriction $\Phi|_{T_h} : T_h \to T_\Gamma$ is C^{k-1} diffeomorphism for some $k \geqslant 2$.

Finally, the mapping Φ which is considered as the orthogonal projection of Γ_h over Γ serves to introduce a triangulation $\mathcal{T} = \{T_\Gamma : T_h \in \mathcal{T}_h\}$ of the exact surface Γ by triangular elements $T_\Gamma = \Phi(T_h) = (\Phi \circ F_T^l)(\hat{T})$ with

$$\Gamma = \overline{\bigcup_{T_\Gamma \in \mathcal{T}} T_\Gamma}. \tag{18}$$

This parametrization of Γ allows us to define a basis of the tangent plane to Γ

$$\vec{e}_\alpha(\xi) = \frac{\partial}{\partial \xi_\alpha}(\Phi \circ F_T^l)(\xi), \quad \text{for } \alpha = 1,2,\ \forall \xi = (\xi_1, \xi_2) \in \hat{T}. \tag{19}$$

We define the tensor matrix $(g_{\alpha\beta})_{\alpha,\beta=1,2}$ such that $g_{\alpha\beta} = \vec{e}_\alpha \cdot \vec{e}_\beta$, and let $(g^{\alpha\beta})_{\alpha,\beta=1,2}$ be its inverse matrix. Recall that in local coordinates the surface measure on Γ is given by $d\Gamma = \sqrt{g}d\xi$ with $g = |\vec{e}_1 \wedge \vec{e}_2|^2 = g_{11}g_{22} - g_{12}g_{21}$ and $d\xi = d\xi_1 d\xi_2$ is the surface measure on $\hat{T}$ [37]. Finally the outward unit normal $\vec{n}$ to Γ is given by

$$\vec{n}(y) = \frac{1}{\sqrt{g}}\left(\vec{e}_1(\xi) \wedge \vec{e}_2(\xi)\right), \quad \text{for} \quad y = (\Phi \circ F_T^l)(\xi) \in T_\Gamma \quad \text{and} \quad \xi \in \hat{T}. \tag{20}$$

Each tangential field $\vec{p}$ to Γ, admits in local coordinates the following form

$$\vec{p}|_{T_\Gamma} = p^1 \vec{e}_1 + p^2 \vec{e}_2. \tag{21}$$

The inner product of the fields $\vec{p}$ and $\overline{\vec{q}}$, defined at the same point, is then defined by $\vec{p} \cdot \overline{\vec{q}} = \sum_{\alpha,\beta=1}^2 g_{\alpha\beta} p^\alpha \bar{q}^\beta$.

For a tangential field $\vec{p}$ the surface divergence is given in local coordinates by

$$div_\Gamma \vec{p}|_{T_\Gamma} = \frac{1}{\sqrt{g}} \sum_{\alpha=1}^2 \frac{\partial}{\partial \xi_\alpha}(\sqrt{g} p^\alpha) \tag{22}$$

and for a scalar field λ we have

$$\overrightarrow{grad}_\Gamma \lambda|_{T_\Gamma} = \sum_{\alpha=1}^2 \left(g^{\alpha 1} \frac{\partial}{\partial \xi_1}\lambda + g^{\alpha 2}\frac{\partial}{\partial \xi_2}\lambda \right)\vec{e}_\alpha. \tag{23}$$

and

$$\Delta_\Gamma \lambda|_{T_\Gamma} = \frac{1}{\sqrt{g}}\sum_{\alpha=1}^2 \frac{\partial}{\partial \xi_\alpha}\left[\sqrt{g}\left(g^{\alpha 1}\frac{\partial}{\partial \xi_1}\lambda + g^{\alpha 2}\frac{\partial}{\partial \xi_2}\lambda\right)\right]. \tag{24}$$

In order to introduce finite element spaces for the numerical approximation we define the space $H^s(\Gamma)$ as in [28]

$$H^s(\Gamma) = \left\{ u : \tilde{u} = u \circ (\Phi \circ F_T^l) \in H^s(\hat{T}),\ T_h \in \mathcal{T}_h \right\}. \tag{25}$$

equipped with the norm

$$\|u\|_{s,\Gamma} = \left(\sum_{T_h \in \mathcal{T}_h} \|\tilde{u}\|^2_{H^s(\hat{T})} \right)^{\frac{1}{2}}. \tag{26}$$

Similarly, the space $TH^s(\Gamma)$ is defined by

$$TH^s(\Gamma) = \left\{ \vec{f} = f^1\vec{e}_1 + f^2\vec{e}_2 : f^1, f^2 \in H^s(\hat{T}),\ T_h \in \mathcal{T}_h \right\} \tag{27}$$

and equipped with the norm

$$\left\|\vec{f}\right\|_{s,\Gamma} = \left(\sum_{T_h \in \mathcal{T}_h} \left[\|f^1\|^2_{s,\hat{T}} + \|f^2\|^2_{s,\hat{T}} \right] \right)^{\frac{1}{2}}. \tag{28}$$

For the norms of negative index, we have, for $u \in H^{-s}(\Gamma)$

$$\|u\|_{-s,\Gamma} = \sup_{v \in H^s(\Gamma)} \frac{|<u, v>|}{\|u\|_{s,\Gamma}} \tag{29}$$

It is to note that the space $H^s(\Gamma)$ can be defined in another manner as

$$H^s(\Gamma) = \left\{ u \in L^2(\Gamma) : \overrightarrow{grad}_\Gamma u \in TH^{s-1}(\Gamma) \right\}. \tag{30}$$

From the partition of Γ, we define a finite element space called (l,k)-space, in the sense of Babuska and Aziz [1].

Definition 3.5. Let Γ be a regular or Lipschitz surface, h be the maximal diameter of the partition of Γ and $h_0 > 0$, such that $0 < h \leqslant h_0$. A space is said to be a finite element (l,k)-space, denoted by $S_h^{l,k}(\Gamma)$ $(l > k \geqslant 0)$, if it verifies:

(i) $S_h^{l,k}(\Gamma) \subset H^k(\Gamma)$,
(ii) $(l-1)$ is the degree of the interpolating polynomial constituting the corresponding finite elements.

In a similar manner, we define the (l,k)-space $\tilde{S}_h^{l,k}(\Gamma)$ of the tangential field functions such that

$$\tilde{S}_h^{l,k}(\Gamma) \subset TH^k(\Gamma). \tag{31}$$

For convenience, we introduce the finite element space $(m \geqslant 1)$

$$\mathcal{S}_h(\Gamma) = \tilde{S}_h^{m,m-1}(\Gamma) \times S_h^{m+1,m}(\Gamma) \tag{32}$$

with

$$S_h^{m+1,m}(\Gamma) = \left\{\lambda \in H^m(\Gamma)/\lambda|_{T_\Gamma} \in \mathbb{P}_m,\ \forall T_\Gamma \in \mathcal{T}\right\}, \tag{33}$$

and

$$\tilde{S}_h^{m,m-1}(\Gamma) = \left\{\overrightarrow{p} \in TH^{m-1}(\Gamma)/\overrightarrow{p}|_{T_\Gamma} = p^1\overrightarrow{e}_1 + p^2\overrightarrow{e}_2 \quad \text{with } p^1,p^2 \in \mathbb{P}_{m-1},\ \forall T_\Gamma \in \mathcal{T}\right\}. \tag{34}$$

We can easily verify that $\mathcal{S}_h(\Gamma) \subset \mathcal{H}^{m-1,m}(\Gamma) \subset \mathcal{H}^{-\frac{1}{2},\frac{1}{2}}(\Gamma)$.

In the same way, we will denote a fixed element on Γ and its corresponding element defined on Γ_h. The approximation of the tangential basis $\{\overrightarrow{e}_\alpha\}_{\alpha=1,2}$ will be given therefore by

$$\overrightarrow{e}^h_\alpha(\xi) = \frac{\partial}{\partial \xi_\alpha} F^l_T(\xi), \quad \text{for } \alpha = 1,2,\ \forall \xi = (\xi_1,\xi_2) \in \hat{T}, \tag{35}$$

which is considered as a basis of the tangent space associated to $T_h = F^l_T(\hat{T}) \in \mathcal{T}_h$. All tangential field $\overrightarrow{p}_h$ to Γ will be defined on Γ_h by

$$\overrightarrow{p}_h|_{T_h} = p_h^1\overrightarrow{e}^h_1 + p_h^2\overrightarrow{e}^h_2. \tag{36}$$

The finite element space $\mathcal{S}_h(\Gamma_h)$ is therefore denoted by

$$\mathcal{S}_h(\Gamma_h) = \tilde{S}_h^{m,m-1}(\Gamma_h) \times S_h^{m+1,m}(\Gamma_h) \tag{37}$$

where

$$S_h^{m+1,m}(\Gamma_h) = \left\{\lambda_h \in H^m(\Gamma_h) : \lambda_h|_{T_h} = \rho_h \quad \text{with } \rho_h \in \mathbb{P}_m,\ \forall T_h \in \mathcal{T}_h\right\} \tag{38}$$

and

$$\tilde{S}_h^{m,m-1}(\Gamma_h) = \left\{\overrightarrow{p}_h \in TH^{m-1}(\Gamma_h) : \overrightarrow{p}_h|_{T_h} = p_h^1\overrightarrow{e}^h_1 + p_h^2\overrightarrow{e}^h_2 \text{ with } p_h^1,p_h^2 \in \mathbb{P}_{m-1},\ \forall T_h \in \mathcal{T}_h\right\}. \tag{39}$$

Note that, for a field u belonging to the above spaces, $u|_{T_\Gamma} = u \circ (\Phi \circ F^l_T)$ for $T_\Gamma \in \mathcal{T}$ and $u|_{T_h} = u \circ F^l_T$ for $T_h \in \mathcal{T}_h$.

4. The discrete problem – case of the exact surface

In this section we will present briefly some results concerning the approximation of the potentials $\overrightarrow{p}$ and λ in the space $\mathcal{S}_h(\Gamma)$ that appears in [31]. These results will be used for the study of the fully discrete formulation of (11).

The approximate solution $(\overrightarrow{p}_h,\lambda_h) \in \mathcal{S}_h(\Gamma)$, by Galerkin method, of the solution $(\overrightarrow{p},\lambda) \in \mathcal{H}^{-\frac{1}{2},\frac{1}{2}}(\Gamma)$ of the problem (11) satisfies the following discretized problem:

$$\begin{cases} \text{Find } (\overrightarrow{p}_h,\lambda_h) \in \mathcal{S}_h(\Gamma) \quad \text{such that for all} \quad (\overrightarrow{q}_h,\mu_h) \in \mathcal{S}_h(\Gamma) \\ < \Pi_\Gamma V(\overrightarrow{p}_h), \overrightarrow{q}_h > + < \overrightarrow{grad}_\Gamma V(\lambda_h), \overrightarrow{q}_h > = < \overrightarrow{c}, \overrightarrow{q}_h >, \\ < -J_\Gamma(\overrightarrow{p}_h), \mu_h > - < (\Delta_\Gamma + k^2)V(\lambda_h), \mu_h > = < -div_\Gamma \overrightarrow{c}, \mu_h > . \end{cases} \tag{40}$$

By taking $U_h = (\overrightarrow{p}_h,\lambda_h)$, $V_h = (\overrightarrow{q}_h,\mu_h)$ and $F = (\overrightarrow{c}, -div_\Gamma \overrightarrow{c})$ this problem can be written as

$$\begin{cases} \text{Find } U_h \in \mathcal{S}_h(\Gamma) \quad \text{such that for all} \quad V_h \in \mathcal{S}_h(\Gamma) \\ \prec \mathcal{A}U_h, V_h \succ = \prec F, V_h \succ . \end{cases} \tag{41}$$

where the operator $\mathcal{A}$ is defined, for $U = (\vec{p}, \lambda)$, by

$$\mathcal{A}U = \begin{pmatrix} \Pi_\Gamma V & \overrightarrow{grad}_\Gamma V \\ -J_\Gamma & -(\Delta_\Gamma + k^2)V \end{pmatrix} \begin{pmatrix} \vec{p} \\ \lambda \end{pmatrix} = \begin{pmatrix} \Pi_\Gamma V(\vec{p}) + \overrightarrow{grad}_\Gamma V(\lambda) \\ -J_\Gamma(\vec{p}) - (\Delta_\Gamma + k^2)V(\lambda) \end{pmatrix}. \tag{42}$$

and $\prec, \succ$ is an inner product on $L^2(\Gamma)^2 \times L^2(\Gamma)^2$ with associated norm defined for $U = (\vec{p}, \lambda) \in \mathcal{H}^{s,t}(\Gamma)$ by

$$\|U\|_{s,t,\Gamma} = \|(\vec{p}, \lambda)\|_{\mathcal{H}^{s,t}(\Gamma)} = \|\vec{p}\|_{s,\Gamma} + \|\lambda\|_{t,\Gamma}. \tag{43}$$

This operator $\mathcal{A}$, which is bijective from $\mathcal{H}^{s-1,s}(\Gamma)$ into $\mathcal{H}^{s,s-1}(\Gamma)$ for any $s \in \mathbb{R}$, verifies the following Gårding's inequality (see [46,47]):

Theorem 4.1. *There exists a real $\delta > 0$ such that*

$$Re \prec \mathcal{A}(\vec{p}, \lambda), (\vec{p}, \lambda) \succ \geqslant \delta\left(\|\vec{p}\|^2_{TH^{-\frac{1}{2}}(\Gamma)} + \|\lambda\|^2_{H^{\frac{1}{2}}(\Gamma)}\right) - |\mathcal{K}((\vec{p}, \lambda), (\vec{p}, \lambda))| \tag{44}$$

for all $(\vec{p}, \lambda) \in \mathcal{H}^{-\frac{1}{2},\frac{1}{2}}(\Gamma)$, where $\mathcal{K}$ is a compact bilinear form on $\mathcal{H}^{-\frac{1}{2},\frac{1}{2}}(\Gamma) \times \mathcal{H}^{-\frac{1}{2},\frac{1}{2}}(\Gamma)$.

Let $\mathcal{P}_h$ be the L^2-orthogonal projection of the space $\mathcal{H}^{s-1,s}(\Gamma)$ over $\mathcal{S}_h(\Gamma)$. Then we will have

$$\mathcal{P}_h\mathcal{A}\mathcal{P}_hU_h = \mathcal{P}_h\mathcal{A}U = \mathcal{P}_hF \tag{45}$$

So that we will be able to define the Galerkin operator $G_\mathcal{A}$ from $\mathcal{H}^{-\frac{1}{2},\frac{1}{2}}(\Gamma)$ into $\mathcal{S}_h(\Gamma)$ by

$$G_\mathcal{A}U = ((\mathcal{P}_h\mathcal{A}\mathcal{P}_h)^{-1}\mathcal{P}_h\mathcal{A})U = U_h, \tag{46}$$

which verifies [31]:

Theorem 4.2. *There exists $h_0 > 0$ such that for all $0 < h \leqslant h_0$, the Galerkin operator $G_\mathcal{A}$ defined from $\mathcal{H}^{-\frac{1}{2},\frac{1}{2}}(\Gamma)$ into $\mathcal{H}^{-\frac{1}{2},\frac{1}{2}}(\Gamma)$ is uniformly bounded independently of h.*

In order to study the existence and the uniqueness of the solution of the problem (41) we recall some general results of Hildebrandt and Wienholtz [15]:

Theorem 4.3. *In a separable Hilbert space $\mathcal{H}$, we consider the following problem*

$$\begin{cases} \text{Find } U \in \mathcal{H} \quad \text{such that for all} \quad V \in \mathcal{H} \\ B(V, U) = L(V) \end{cases} \tag{47}$$

where B is a bilinear and continuous form on $\mathcal{H} \times \mathcal{H}$ and L be a linear and continuous form on $\mathcal{H}$. Let $\mathcal{H}_h$ be a sub-space of $\mathcal{H}$ of finite dimension. Suppose that there are a positive real h_0, a constant $c > 0$ and a family $\{\mathcal{Q}_h\}_{0<h\leqslant h_0}$ of uniformly bounded operators mapping $\mathcal{H}_h$ into $\mathcal{H}_h$ such that

$$c\|U_h\|^2 \leqslant |B(Q_hU_h, U_h)|, \forall U_h \in \mathcal{H}_h. \tag{48}$$

Therefore

(i) *the problem (47) admits at most one solution,*
(ii) *for all $0 < h \leqslant h_0$ the approximate problem of (47)*

$$\begin{cases} \text{Find } U_h \in \mathcal{H}_h \quad \text{such that for all} \quad V_h \in \mathcal{H}_h \\ B(V_h, U_h) = L(V_h) \end{cases} \tag{49}$$

admits a unique solution U_h, that converges to the solution U of the exact problem (47).

According to Gårding's inequality (44) and the uniqueness of the solution of the system (11), the operator $\mathcal{A}$ admits the following decomposition [22]

$$\mathcal{A} = \mathcal{D} + \mathcal{T}, \tag{50}$$

where $\mathcal{A}$ and $\mathcal{D}$ are bounded isomorphisms in $\mathcal{H}^{-\frac{1}{2},\frac{1}{2}}(\Gamma)$ and $\mathcal{T}$ is a compact operator on $\mathcal{H}^{-\frac{1}{2},\frac{1}{2}}(\Gamma)$ defined by

$$\prec \mathcal{T}(\vec{p}, \lambda), (\vec{p}, \lambda) \succ = \mathcal{K}((\vec{p}, \lambda), (\vec{p}, \lambda)), \forall (\vec{p}, \lambda) \in \mathcal{H}^{-\frac{1}{2},\frac{1}{2}}(\Gamma). \tag{51}$$

The operator $\mathcal{D}$ is positive definite, this means that there exists a constant $c > 0$, such that

$$c\|(\vec{p}, \lambda)\|^2_{-\frac{1}{2},\frac{1}{2},\Gamma} \leqslant \prec \mathcal{D}(\vec{p}, \lambda), (\vec{p}, \lambda) \succ, \quad \forall (\vec{p}, \lambda) \in \mathcal{H}^{-\frac{1}{2},\frac{1}{2}}(\Gamma). \tag{52}$$

Theorem 4.4. *For $h_0 > 0$, the problem*

$$\begin{cases} \text{Find } U_h \in \mathcal{S}_h(\Gamma) \quad \text{such that for all} \quad V_h \in \mathcal{S}_h(\Gamma) \\ \prec \mathcal{A}U_h, V_h \succ = \prec \mathcal{A}U, V_h \succ = \prec F, V_h \succ \end{cases} \tag{53}$$

admits a unique solution $U_h \in \mathcal{S}_h(\Gamma),\ 0 < h \leqslant h_0,\ U = (\overrightarrow{p}, \lambda)$ of (11).

Proof. Let's verify that the hypotheses of Theorem 4.3 are satisfied for the case where

$$B(V,U) = \prec V, \mathcal{A}U \succ = \prec \mathcal{A}U, V \succ, \quad \forall U,\ V \in \mathcal{H}^{-\frac{1}{2}\frac{1}{2}}(\Gamma). \tag{54}$$

If $\mathcal{A}$ admits the decomposition (50) with all necessary properties on $\mathcal{D}$ and $\mathcal{T}$, therefore (see [15] for similar result), there exists a constant $c > 0$ and a real h_0 such that

$$c\|U_h\|_{-\frac{1}{2}\frac{1}{2},\Gamma} \leqslant \|\mathcal{P}_h\mathcal{A}U_h\|_{-\frac{1}{2}\frac{1}{2},\Gamma}, \quad \forall U_h \in \mathcal{S}_h(\Gamma) \quad \text{and} \quad 0 < h \leqslant h_0. \tag{55}$$

Let $U_h \in \mathcal{S}_h(\Gamma)$, then using (55), there exists a constant $c > 0$ and a real $h_0 > 0$, such that

$$c\|U_h\|^2_{-\frac{1}{2}\frac{1}{2},\Gamma} \leqslant |\prec \mathcal{P}_h\mathcal{A}U_h, \mathcal{A}U_h \succ|, \quad \forall U_h \in \mathcal{S}_h(\Gamma) \text{ and } 0 < h \leqslant h_0. \tag{56}$$

We can put $\mathcal{Q}_h = \mathcal{P}_h\mathcal{A}$, which is bounded by the bijection of $\mathcal{A}$. Hence, the hypotheses of Theorem 4.3 are satisfied. □

From the decomposition (50), we can introduce an operator $\mathcal{R}$ defined from $\mathcal{H}^{-\frac{1}{2}\frac{1}{2}}(\Gamma)$ into $\mathcal{H}^{-\frac{1}{2}\frac{1}{2}}(\Gamma)$ by

$$\mathcal{R} = \mathcal{I} + \mathcal{D}^{-1}\mathcal{T}. \tag{57}$$

As for (55) the operator $\mathcal{R}$ verifies the following property:

Proposition 4.5. *There exists a constant $c > 0$ such that*

$$c\|U_h\|_{-\frac{1}{2}\frac{1}{2},\Gamma} \leqslant \|\mathcal{R}U_h\|_{-\frac{1}{2}\frac{1}{2},\Gamma}, \quad \forall U_h \in \mathcal{S}_h(\Gamma) \quad \text{and } 0 < h \leqslant h_0. \tag{58}$$

The error estimate analysis for Galerkin method using the finite element of the regular space $\mathcal{S}_h(\Gamma)$ requires the introduction of the following approximation properties and inverse assumptions based on the works of [1,17,18].

Proposition 4.6. *For all $-m-1 \leqslant t \leqslant s \leqslant m+1; t \leqslant m; -m \leqslant s$ and to any $\lambda \in H^s(\Gamma)$, there exists $\lambda_h \in S_h^{m+1,m}(\Gamma)$ such that*

$$\|\lambda - \lambda_h\|_{t,\Gamma} \leqslant c_1 h^{s-t}\|\lambda\|_{s,\Gamma}. \tag{59}$$

Moreover, for all $-m \leqslant t \leqslant s \leqslant m; t \leqslant m-1; -(m-1) \leqslant s$ and to any $\overrightarrow{p} \in TH^s(\Gamma)$, there exists $\overrightarrow{p}_h \in \tilde{S}_h^{m,m-1}(\Gamma)$ such that

$$\|\overrightarrow{p} - \overrightarrow{p}_h\|_{t,\Gamma} \leqslant c_2 h^{s-t}\|\overrightarrow{p}\|_{s,\Gamma}, \tag{60}$$

where c_1 and c_2 are constant that only depend on s and t.

Proposition 4.7. *For all $-m \leqslant t \leqslant s \leqslant m$ and to any $\lambda_h \in S_h^{m+1,m}(\Gamma)$, we have*

$$\|\lambda_h\|_{s,\Gamma} \leqslant c_1 h^{t-s}\|\lambda_h\|_{t,\Gamma}. \tag{61}$$

Moreover, for all $-(m-1) \leqslant t \leqslant s \leqslant m-1$ and to any $\overrightarrow{p}_h \in \tilde{S}_h^{m,m-1}(\Gamma)$, we have

$$\|\overrightarrow{p}_h\|_{s,\Gamma} \leqslant c_2 h^{t-s}\|\overrightarrow{p}_h\|_{t,\Gamma}, \tag{62}$$

where c_1 and c_2 are constant that only depend on s and t.

With the uniform boundedness of the Galerkin operator $G_{\mathcal{A}}$ in $\mathcal{H}^{-\frac{1}{2}\frac{1}{2}}(\Gamma)$ (Theorem 4.2) and the regularity results on the exact solution of $AU = F$ [31,32], we are able to derive error estimates for more regular norms than those of the spaces $TH^{-\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$. Standard techniques from [42] using the results of Propositions 4.6, 4.7 and classical duality arguments using an extension of the lemma of Aubin-Nitsche [6,18] we have the following error estimates [31]:

Theorem 4.8. *Let $U = (\overrightarrow{p}, \lambda)$ be the solution of the problem (11) and $U_h = (\overrightarrow{p}_h, \lambda_h)$ be the one of its approximate problem (41). Then, for h small enough with $h \to 0$, there exists a constant $c > 0$ independent of h and $\overrightarrow{c}$ such that*

$$\|U - U_h\|_{t-1,t,\Gamma} \leqslant c \inf_{V_h \in \mathcal{S}_h(\Gamma)} \|U - V_h\|_{-\frac{1}{2}\frac{1}{2},\Gamma} \leqslant ch^{s-t}\|U\|_{s-1,s,\Gamma} \leqslant ch^{s-t}\|\overrightarrow{c}\|_{s,\Gamma} \tag{63}$$

for $-m-2 \leqslant t \leqslant s \leqslant m+1,\ -m \leqslant -\frac{1}{2} \leqslant m,\ t \leqslant m$ and $-\frac{1}{2} \leqslant s$.

Remarks 4.9. The surface Γ is regular and the incident field $\overrightarrow{E}^{in}$ is also, in general, regular on Γ. Hence (63) shows that we can have an elevated convergence rate by taking sufficiently regular elements.

The best convergence rates, for the case $t = -m-2$ and $s = m+1$, shows that the approximate electric field $\vec{E}_h$ given by

$$\vec{E}_h(x) = V(\vec{p}_h)(x) + \overrightarrow{grad}\,V(\lambda_h)(x), \forall x \in \Omega^e \tag{64}$$

verifies the following theorem [31]:

Theorem 4.10. *For all sub-domain $\widetilde{\Omega} \subset \Omega^e$, there exists a constant $c = c(\widetilde{\Omega})$ such that*

$$\left\|\vec{E} - \vec{E}_h\right\|_{L^\infty(\widetilde{\Omega})} \leqslant ch^{2m+3}\|U\|_{m,m+1,\Gamma} \leqslant ch^{2m+3}\|\vec{c}\|_{m+1,\Gamma}. \tag{65}$$

5. The discrete problem – case of the approximate surface

In this section we present a complete survey for the fully discrete variational formulation of (11), where the surface approximation is taken into account.

The approximation of the surface Γ by Γ_h requires the introduction of an approximate operator $\mathcal{A}_h$ of the operator $\mathcal{A}$ defined by (42). This will be achieved by joining the finite elements defined on Γ to those defined, by transportation, on Γ_h. By taking into account the approximation of the surface Γ, the approximate solution $(\vec{p}_h, \lambda_h) \in \mathcal{S}_h(\Gamma_h)$ by Galerkin procedure of the exact solution $(\vec{p}, \lambda) \in \mathcal{H}^{-\frac{1}{2}\frac{1}{2}}(\Gamma)$ of (11) satisfies the following approximate formulation:

$$\begin{cases} \text{Find } (\vec{p}_h, \lambda_h) \in \mathcal{S}_h(\Gamma_h) \quad \text{such that for all} \quad (\vec{q}_h, \mu_h) \in \mathcal{S}_h(\Gamma_h) \\ < \Pi_{\Gamma_h} V_h(\vec{p}_h), \vec{q}_h >_h + < \overrightarrow{grad}_{\Gamma_h} V_h(\lambda_h), \vec{q}_h >_h = < \vec{c}_h, \vec{q}_h >_h, \\ < -J_{\Gamma_h}(\vec{p}_h), \mu_h >_h - < (\Delta_{\Gamma h} + k^2) V_h(\lambda_h), \mu_h >_h = < -div_{\Gamma_h}\vec{c}_h, \mu_h >_h. \end{cases} \tag{66}$$

where $\vec{c}_h$ is the approximate field of $\vec{c}$. This formulation can be transformed, as for (13), into

$$\begin{cases} \text{Find } (\vec{p}_h, \lambda_h) \in \mathcal{S}_h(\Gamma_h) \quad \text{such that for all} \quad (\vec{q}_h, \mu_h) \in \mathcal{S}_h(\Gamma_h) \\ < \Pi_{\Gamma_h} V_h(\vec{p}_h), \vec{q}_h >_h - < V_h(\lambda_h), div_{\Gamma_h}\vec{q}_h >_h = < \vec{c}_h, \vec{q}_h >_h, \\ < V_h(div_{\Gamma_h}\vec{p}_h), \mu_h >_h + < \Pi_{\Gamma_h} V_h(\vec{p}_h), \overrightarrow{grad}_{\Gamma_h}\mu_h >_h - < V_h(\lambda_h), (\Delta_{\Gamma h} + k^2)\mu_h >_h = < -div_{\Gamma_h}\vec{c}_h, \mu_h >_h. \end{cases} \tag{67}$$

The integral operator V_h in (66) is the operator V defined on Γ_h. We then have for λ_h and $\vec{p}_h$

$$V_h(\lambda_h)(x) = \int_{\Gamma_h} \lambda_h(y_h) G(x, y_h) d\Gamma_{hy}, \quad \forall x \in \Gamma_h \tag{68}$$

and

$$V_h(\vec{p}_h)(x) = \int_{\Gamma_h} \vec{p}_h(y_h) G(x, y_h) d\Gamma_{hy}, \quad \forall x \in \Gamma_h. \tag{69}$$

Referring to these integral operators we can define from (67) the following bilinear forms

$$b_{11}^h(\vec{p}_h, \vec{q}_h) = \int_{\Gamma_h}\int_{\Gamma_h} \vec{p}_h(y_h) \cdot \overline{\vec{q}_h(x_h)} G(x_h, y_h) d\Gamma_{hy} d\Gamma_{hx}, \tag{70}$$

$$b_{12}^h(\lambda_h, \vec{q}_h) = \int_{\Gamma_h}\int_{\Gamma_h} \lambda_h(y_h) \overline{div_{\Gamma_h}\vec{q}_h(x_h)} G(x_h, y_h) d\Gamma_{hy} d\Gamma_{hx}, \tag{71}$$

$$b_{21}^h(\vec{p}_h, \mu_h) = \int_{\Gamma_h}\int_{\Gamma_h} div_{\Gamma_h}\vec{p}_h(y_h) \overline{\mu_h(x_h)} d\Gamma_{hy} d\Gamma_{hx} + \int_{\Gamma_h}\int_{\Gamma_h} \vec{p}_h(y_h) \cdot \overline{\overrightarrow{grad}_{\Gamma_h}\mu_h(x_h)} G(x_h, y_h) d\Gamma_{hy} d\Gamma_{hx}, \tag{72}$$

$$b_{22}^h(\lambda_h, \mu_h) = \int_{\Gamma_h}\int_{\Gamma_h} \lambda_h(y_h)(\Delta_{\Gamma_h} + k^2)\overline{\mu_h(x_h)} G(x_h, y_h) d\Gamma_{hy} d\Gamma_{hx}. \tag{73}$$

By taking $U_h = (\vec{p}_h, \lambda_h), V_h = (\vec{q}_h, \mu_h)$ and $F_h = (\vec{c}_h, -div_{\Gamma_h}\vec{c}_h)$, the formulation (66) is rewritten as

$$\begin{cases} \text{Find } U_h \in \mathcal{S}_h(\Gamma_h) \quad \text{such that for all} \quad V_h \in \mathcal{S}_h(\Gamma_h) \\ \prec \mathcal{A}_h U_h, V_h) \succ_h = \prec F_h, V_h \succ_h \end{cases} \tag{74}$$

where the approximate operator $\mathcal{A}_h$ and the inner product $\prec \cdot, \cdot \succ_h$ are defined in the same way as $\mathcal{A}$ and $\prec \cdot, \cdot \succ$ (see (42)).

In order to use the results of the previous paragraph concerning the study of the error estimates, we will compare the exact solution $U = (\vec{p}, \lambda)$ with an element $\tilde{U}_h = (\tilde{\vec{p}}_h, \tilde{\lambda}_h) \in \mathcal{S}_h(\Gamma)$ evaluated in terms of the approximate solution $U_h = (\vec{p}_h, \lambda_h) \in \mathcal{S}_h(\Gamma_h)$. In fact, to each element $U_h = (\vec{p}_h, \lambda_h) \in \mathcal{S}_h(\Gamma_h)$ with

$$\begin{cases} \vec{p}_h|_{T_h}(y_h) = \sum_{\alpha=1}^{2} p_h^\alpha(\xi)\vec{e}_\alpha^{\,h}(\xi), \\ \lambda_h|_{T_h}(y_h) = \rho_h(\xi), \end{cases} \tag{75}$$

we will associate the element $\tilde{U}_h = (\vec{\tilde{p}}_h, \tilde{\lambda}_h) \in \mathcal{S}_h(\Gamma)$, such that

$$\begin{cases} \vec{\tilde{p}}_h|_{T_\Gamma}(y) = \sum_{\alpha=1}^{2} \tilde{p}_h^\alpha(\xi)\vec{e}_\alpha(\xi) \quad \text{with} \quad \sqrt{g(y)}\tilde{p}_h^\alpha(\xi) = \sqrt{g_h(y_h)}p_h^\alpha(\xi), \\ \tilde{\lambda}_h|_{T_\Gamma}(y) = \tilde{\rho}_h(\xi) \quad \text{with} \quad \sqrt{g(y)}\tilde{\rho}_h(\xi) = \sqrt{g_h(y_h)}\rho_h(\xi), \end{cases} \tag{76}$$

where $y_h = F_T^l(\xi)$, $y = (\Phi \circ F_T^l)(\xi)$, if $\xi \in K$ and $y_h = \Phi^{-1}(y)$ for $y \in \Gamma$. Next note that $u \circ \Phi^{-1}$ is defined on Γ for any function u defined on Γ_h. In this sense, we will transform the integrals in (68) and (69) to integrals on Γ by the mapping Φ defined by Assumption 3.4). For λ_h and $\vec{p}_h$, we have

$$V_h(\lambda_h)(x) = \sum_{T_h \in \mathcal{T}_h} \int_{\hat{T}} \rho_h(\xi)G(x, y_h)\sqrt{g_h(y_h)}d\xi = \int_\Gamma (\lambda_h \circ \Phi^{-1})(y)G(x, y_h)J_h(y)d\Gamma_y \tag{77}$$

and

$$V_h(\vec{p}_h)(x) = \sum_{T_h \in \mathcal{T}_h} \int_{\hat{T}} \sum_{\alpha=1}^{2} p_h^\alpha(\xi)\vec{e}_\alpha^{\,h}(\xi)G(x, y_h)\sqrt{g_h(y_h)}d\xi = \int_\Gamma (\vec{p}_h \circ \Phi^{-1})(y)G(x, y_h)J_h(y)d\Gamma_y. \tag{78}$$

with $J_h(y) = \frac{\sqrt{g_h(y_h)}}{\sqrt{g(y)}}$.

We can easily verify the following lemma permitting to compare the different integral operators of the system (66) applied to $\vec{p}_h$ and λ_h with those applied to $\vec{\tilde{p}}_h$ and $\tilde{\lambda}_h$.

Lemma 5.1. *For all $U_h = (\vec{p}_h, \lambda_h) \in \mathcal{S}_h(\Gamma_h)$ with its associated element $\tilde{U}_h = (\vec{\tilde{p}}_h, \tilde{\lambda}_h) \in \mathcal{S}_h(\Gamma)$ defined by (75) and (76), we have*

$$(i) \qquad \tilde{\lambda}_h|_{T_\Gamma}(y) = J_h(y)\lambda_h|_{T_h}(y_h), \tag{79}$$

$$(ii) \qquad \vec{\tilde{p}}_h|_{T_\Gamma}(y) = J_h(y)\sum_{\alpha=1}^{2} p_h^\alpha(\xi)\left(\vec{e}_\alpha(\xi) - \vec{e}_\alpha^{\,h}(\xi)\right) + J_h(y)\vec{p}_h|_{T_h}(y_h), \tag{80}$$

$$(iii) \qquad div_\Gamma \vec{\tilde{p}}_h|_{T_\Gamma}(y) = J_h(y) div_{\Gamma_h} \vec{p}_h|_{T_h}(y_h), \tag{81}$$

$$\begin{aligned}(iv) \qquad \overrightarrow{grad}_\Gamma \tilde{\lambda}_h|_{T_\Gamma}(y) = {}& \sum_{\alpha,\beta=1}^{2}\left[g^{\alpha\beta}\frac{\partial}{\partial \xi_\beta}\tilde{\rho}_h(\xi) - J_h(y)g_h^{\alpha\beta}\frac{\partial}{\partial \xi_\beta}\rho_h(\xi)\right]\vec{e}_\alpha(\xi) + J_h(y)\sum_{\alpha,\beta=1}^{2}\left(g_h^{\alpha\beta}\frac{\partial}{\partial \xi_\beta}\rho_h(\xi)\right)\left(\vec{e}_\alpha(\xi) - \vec{e}_\alpha^{\,h}(\xi)\right) \\ & + J_h(y)\overrightarrow{grad}_{\Gamma_h}\lambda_h|_{T_h}(y_h),\end{aligned} \tag{82}$$

$$(v) \qquad \Delta_\Gamma \tilde{\lambda}_h|_{T_\Gamma}(y) = \frac{1}{\sqrt{g}}\sum_{\alpha\,\beta=1}^{2}\frac{\partial}{\partial \xi_\alpha}\left[\sqrt{g}g^{\alpha\beta}\frac{\partial}{\partial \xi_\beta}\tilde{\rho}_h(\xi) - \sqrt{g_h}g_h^{\alpha,\beta}\frac{\partial}{\partial \xi_\beta}\rho_h(\xi)\right] + J_h(y)\Delta_{\Gamma_h}\lambda_h|_{T_h}(y_h). \tag{83}$$

5.1. Approximation properties

The analysis of the error estimate for the completely discrete Galerkin method is affected by using the space $\mathcal{S}_h(\Gamma_h)$. The approximate surface Γ_h is defined by finite elements $\left(\hat{T}, \mathbb{P}_l, \Sigma_{\hat{T}}^l\right)$ of Lagrange type with order l. We recall some approximation properties first [36].

First, we take the decomposition of the kernel $G = G_0 + \phi$ with

$$G_0(x, y) = \frac{1}{4\pi|x-y|} \quad \text{and} \quad \phi(x, y) = \frac{e^{ik|x-y|} - 1}{4\pi|x-y|}. \tag{84}$$

We can then define the continuous approximate bilinear forms from (70), for all $\vec{p}_h,\ \vec{q}_h \in S_h^{m,m-1}(\Gamma_h)$

$$a_{11}^h(\vec{p}_h, \vec{q}_h) = \int_{\Gamma_h}\int_{\Gamma_h} \vec{p}_h(y_h) \cdot \overline{\vec{q}_h(x_h)}G_0(x_h, y_h)d\Gamma_{hy}d\Gamma_{hx} \tag{85}$$

and

$$r_{11}^h(\vec{p}_h, \vec{q}_h) = \int_{\Gamma_h}\int_{\Gamma_h} \vec{p}_h(y_h) \cdot \overline{\vec{q}_h(x_h)}\phi(x_h, y_h)d\Gamma_{hy}d\Gamma_{hx}. \tag{86}$$

Similarly, we can define the bilinear forms $a_{12}^h,\ r_{12}^h,\ a_{21}^h,\ r_{21}^h$ and $a_{22}^h,\ r_{22}^h$ from (71), (72) and (73), as well as the corresponding bilinear forms $a_{11},\ r_{11}$, etc.,..., by writing the integrals on Γ. For each triangle K of $\mathcal{T}_h$, the mapping $\Phi \circ F_T^l$ and its derivatives of order $l+1$ are bounded, and $\frac{\partial}{\partial \xi_\alpha}(\Phi \circ F_T^l)$ is a linear mapping defined from $\mathbb{R}^2$ into the tangent plane to Γ with bounded inverse.

Referring to [3,36] we can obtain the following lemma:

Lemma 5.2. *For $y = (\Phi \circ F_T^l)(\xi)$ and $y_h = F_T^l(\xi)$, if $\xi \in K$, we have the following estimates:*

$$(i)\ |y - y_h| \leqslant ch^{l+1}, \tag{87}$$

$$(ii)\ |\vec{e}_\alpha - \vec{e}^{\,h}_\alpha| \leqslant ch^{l+1}, \tag{88}$$

$$(iii)\ \left|\sqrt{g(y)} - \sqrt{g_h(y_h)}\right| \leqslant ch^{l+1} \quad \text{and} \quad |J_h(y) - 1| \leqslant ch^{l+1}, \tag{89}$$

$$(iv)\ |g^{\alpha\beta} - g_h^{\alpha\beta}| \leqslant ch^{l+1}. \tag{90}$$

In order to prove the error consistency for the form a_{11}^h, etc., ..., we propose the following lemma permitting to compare an element $(\vec{p}_h, \lambda_h) \in \mathcal{S}_h(\Gamma_h)$ with its associated element $(\vec{\tilde{p}}_h, \tilde{\lambda}_h) \in \mathcal{S}_h(\Gamma)$.

Lemma 5.3. *With the above notations, we have*

$$(i)\ \left\|\vec{\tilde{p}}_h - J_h(\vec{p}_h \circ \Phi^{-1})\right\|_{0,\Gamma} \leqslant ch^{l+1}\left\|\vec{\tilde{p}}_h\right\|_{0,\Gamma}, \tag{91}$$

$$(ii)\ \left\|\overrightarrow{grad}_\Gamma\,\tilde{\lambda}_h - J_h(\overrightarrow{grad}_{\Gamma_h}\lambda_h \circ \Phi^{-1})\right\|_{0,\Gamma} \leqslant ch^l\|\tilde{\lambda}_h\|_{0,\Gamma}, \tag{92}$$

$$(iii)\ \left\|\Delta_\Gamma\,\tilde{\lambda}_h - J_h(\Delta_{\Gamma_h}\lambda_h \circ \Phi^{-1})\right\|_{0,\Gamma} \leqslant ch^{l-1}\|\tilde{\lambda}_h\|_{0,\Gamma}. \tag{93}$$

Proof. We will first compare the function ρ_h with its corresponding function $\tilde{\rho}_h$ on each triangle $T_h \in \mathcal{T}_h$ (see (75) and (76)), then from the relation

$$|\tilde{\rho}_h(\xi) - \rho_h(\xi)| = |J_h(y)\rho_h(\xi) - \rho_h(\xi)| = |J_h(y) - 1|\,|\rho_h(\xi)|,\ \forall \xi \in \hat{T}, \tag{94}$$

the third property of Lemma 5.2 gives us

$$\|\tilde{\rho}_h - \rho_h\|_{0,\hat{T}} \leqslant ch^{l+1}\|\rho_h\|_{0,\hat{T}} \leqslant ch^{l+1}\|\tilde{\rho}_h\|_{0,\hat{T}}. \tag{95}$$

In the same way, and according to the classical interpolation error [4]

$$\left\|\frac{\partial}{\partial\xi_\beta}\tilde{\rho}_h - \frac{\partial}{\partial\xi_\beta}\rho_h\right\|_{0,\hat{T}} \leqslant ch^{-1}\|\tilde{\rho}_h - \rho_h\|_{0,\hat{T}} \leqslant ch^l\|\tilde{\rho}_h\|_{0,\hat{T}}. \tag{96}$$

(i) Using the property (ii) of Lemma 5.1, we have

$$\left\|\vec{\tilde{p}}_h - J_h(\vec{p}_h \circ \Phi^{-1})\right\|_{0,\Gamma} = \left(\sum_{T_h\in\mathcal{T}_h}\int_{\hat{T}}\left|J_h(y)\sum_{\alpha=1}^{2}P_h^\alpha(\xi)\left(\vec{e}_\alpha(\xi) - \vec{e}^{\,h}_\alpha(\xi)\right)\right|^2\sqrt{g}d\xi\right)^{\frac{1}{2}} \tag{97}$$

and by the estimate (ii) of Lemma 5.2, we obtain

$$\left\|\vec{\tilde{p}}_h - J_h(\vec{p}_h \circ \Phi^{-1})\right\|_{0,\Gamma} \leqslant ch^{l+1}\left(\sum_{T_h\in\mathcal{T}_h}\left[\|\tilde{p}_h^1\|_{0,\hat{T}}^2 + \|\tilde{p}_h^2\|_{0,\hat{T}}^2\right]\right)^{\frac{1}{2}} \leqslant ch^{l+1}\left\|\vec{\tilde{p}}_h\right\|_{0,\Gamma}. \tag{98}$$

(ii) Using the property (iv) of Lemma 5.1, we can write

$$\begin{aligned}\left\|\overrightarrow{grad}_\Gamma\,\tilde{\lambda}_h - J_h(\overrightarrow{grad}_{\Gamma_h}\lambda_h \circ \Phi^{-1})\right\|_{0,\Gamma} \leqslant{}& \left(\sum_{T_h\in\mathcal{T}_h}\int_{\hat{T}}\left|\sum_{\alpha,\beta=1}^{2}\left[g^{\alpha\beta}\frac{\partial}{\partial\xi_\beta}\tilde{\rho}_h(\xi) - J_h(y)g_h^{\alpha\beta}\frac{\partial}{\partial\xi_\beta}\rho_h(\xi)\right]\vec{e}_\alpha(\xi)\right|^2\sqrt{g}d\xi\right)^{\frac{1}{2}}\\ &+\left(\sum_{T_h\in\mathcal{T}_h}\int_{\hat{T}}\left|J_h(y)\sum_{\alpha,\beta=1}^{2}\left(g_h^{\alpha\beta}\frac{\partial}{\partial\xi_\beta}\rho_h(\xi)\right)\left(\vec{e}_\alpha(\xi) - \vec{e}^{\,h}_\alpha(\xi)\right)\right|^2\sqrt{g}d\xi\right)^{\frac{1}{2}}.\end{aligned} \tag{99}$$

Let's take the first term of the right-hand side, that we note I_1. We have

$$\begin{aligned}\left|g^{\alpha\beta}\frac{\partial}{\partial\xi_\beta}\tilde{\rho}_h(\xi) - J_h(y)g_h^{\alpha\beta}\frac{\partial}{\partial\xi_\beta}\rho_h(\xi)\right| \leqslant{}& |g^{\alpha\beta} - g_h^{\alpha\beta}|\left|\frac{\partial}{\partial\xi_\beta}\tilde{\rho}_h(\xi)\right| + |1 - J_h(y)|\left|g_h^{\alpha\beta}\frac{\partial}{\partial\xi_\beta}\tilde{\rho}_h(\xi)\right|\\ &+|J_h(y)g_h^{\alpha\beta}|\left|\frac{\partial}{\partial\xi_\beta}\tilde{\rho}_h(\xi) - \frac{\partial}{\partial\xi_\beta}\rho_h(\xi)\right|.\end{aligned} \tag{100}$$

According to the properties (ii) and (iv) of Lemma 5.2 and the inequality (96), we obtain

$$I_1 \leqslant c\left\{h^{l+1}\left(\sum_{T_h\in\mathcal{T}_h}\left[\left\|\frac{\partial}{\partial\xi_1}\tilde{\rho}_h\right\|^2_{0,\hat{T}}+\left\|\frac{\partial}{\partial\xi_2}\tilde{\rho}_h\right\|^2_{0,\hat{T}}\right]\right)^{\frac{1}{2}}+h^l\|\tilde{\rho}_h\|_{0,\hat{T}}\right\} \leqslant c\{h^{l+1}\|\tilde{\lambda}_h\|^2_{1,\Gamma}+h^l\|\tilde{\lambda}_h\|_{0,\hat{T}}\} \tag{101}$$

and by using the inverse inequality (61), we can deduce that

$$I_1 \leqslant ch^l\|\tilde{\lambda}_h\|_{0,\Gamma}. \tag{102}$$

Similarly, the second term I_2 of the right-hand side of (99) is estimated as

$$I_2 \leqslant ch^l\|\tilde{\lambda}_h\|_{0,\Gamma}. \tag{103}$$

All this gives the estimate (ii).

(iii) The property (v) of Lemma 5.1 gives

$$\left\|\Delta_\Gamma\tilde{\lambda}_h - J_h(\Delta_{\Gamma_h}\lambda_h\circ\Phi^{-1})\right\|_{0,\Gamma} = \left(\sum_{T_h\in\mathcal{T}_h}\int_{\hat{T}}\left|\frac{1}{\sqrt{g}}\sum_{\alpha,\beta=1}^{2}\frac{\partial}{\partial\xi_\alpha}\left[\sqrt{g}g^{\alpha\beta}\frac{\partial}{\partial\xi_\beta}\tilde{\rho}_h(\xi)-\sqrt{g_h}g_h^{\alpha\beta}\frac{\partial}{\partial\xi_\beta}\rho_h(\xi)\right]\right|^2\sqrt{g}d\xi\right)^{\frac{1}{2}}. \tag{104}$$

According to the classical interpolation errors [4]

$$\|\varphi\|_{1,\hat{T}} \leqslant ch^{-1}\|\varphi\|_{0,\hat{T}} \tag{105}$$

for any function φ defined on $\hat{T}$. We then deduce

$$\left\|\Delta_\Gamma\tilde{\lambda}_h - J_h(\Delta_{\Gamma_h}\lambda_h\circ\Phi^{-1})\right\|_{0,\Gamma} \leqslant ch^{-1}I_1 \leqslant ch^{l-1}\|\tilde{\lambda}_h\|_{0,\Gamma}. \tag{106}$$

This completes the proof of the lemma. □

Now let's take two triangles T_h and T'_h of $\mathcal{T}_h$. For $\xi\in\hat{T}$ and $\eta\in\hat{T}'$ we take the points x, y, x_h and y_h defined by

$$\begin{cases} x = (\Phi\circ F^l_{T'})(\eta), \;\; x_h = F^l_{T'}(\eta), \\ y = (\Phi\circ F^l_T)(\xi), \;\; y_h = F^l_T(\xi), \end{cases} \tag{107}$$

which verify the following inequalities of [36]:

$$c|x_h - y_h| \leqslant |x-y| \leqslant c|x_h - y_h|, \tag{108}$$

$$\left||x_h-y_h|^2 - |x-y|^2\right| \leqslant ch^{l+1}|x-y|^2. \tag{109}$$

Thus we obtain the following lemma:

Lemma 5.4. *For the kernels G_0 and ϕ (see (84)), we have the following estimates:*

$$(i)\;\; |G_0(x,y) - G_0(x_h,y_h)| \leqslant ch^{l+1}\frac{1}{|x-y|}, \tag{110}$$

$$(ii)\;\; |\phi(x,y) - \phi(x_h,y_h)| \leqslant ch^{l+1}. \tag{111}$$

5.2. Existence and uniqueness

In this paragraph, we will study the existence and the uniqueness of the solution of the approximate problem (74). First, we will prove some error consistency results based on the works developed in [3,14,36,38].

Theorem 5.5. *The forms a^h_{11} and a^h_{12} verify, for $\vec{p}_h, \vec{q}_h \in \widetilde{S}^{m,m-1}_h(\Gamma_h)$ and $\lambda_h \in S^{m+1,m}_h(\Gamma_h)$:*

$$\left|a_{11}(\vec{\tilde{p}}_h, \vec{\tilde{q}}_h) - a^h_{11}(\vec{p}_h, \vec{q}_h)\right| \leqslant c_1h^l\left\|\vec{p}_h\right\|_{-\frac{1}{2},\Gamma}\left\|\vec{q}_h\right\|_{-\frac{1}{2},\Gamma}, \tag{112}$$

$$\left|a_{12}(\tilde{\lambda}_h, \vec{\tilde{q}}_h) - a^h_{12}(\lambda_h, \vec{q}_h)\right| \leqslant c_2h^{l-\frac{1}{2}}\|\tilde{\lambda}_h\|_{\frac{1}{2},\Gamma}\left\|\vec{q}_h\right\|_{-\frac{1}{2},\Gamma}, \tag{113}$$

where c_1 and c_2 are independent constants of h, $\vec{p}_h$, $\vec{q}_h$ and λ_h.

Proof. By using the mapping Φ and (78), we can write the integrals of the form a_{11}^h over Γ

$$a_{11}^h(\vec{p}_h, \vec{q}_h) = \int_\Gamma \int_\Gamma \vec{p}_h(y_h) \cdot \overline{\vec{q}_h(x_h)} G_0(x_h, y_h) J_h(y) J_h(x) d\Gamma_y d\Gamma_x \tag{114}$$

and therefore we have

$$a_{11}(\vec{\tilde{p}}_h, \vec{\tilde{q}}_h) - a_{11}^h(\vec{p}_h, \vec{q}_h) = \int_\Gamma \int_\Gamma \left[\vec{\tilde{p}}_h(y) \cdot \overline{\vec{\tilde{q}}_h(x)} G_0(x,y) - \vec{p}_h(y_h) \cdot \overline{\vec{q}_h(x_h)} G_0(x_h, y_h) J_h(y) J_h(x)\right] d\Gamma_y d\Gamma_x \tag{115}$$

that we can decompose into two parts

$$I_1 = \int_\Gamma \int_\Gamma \vec{\tilde{p}}_h(y) \cdot \overline{\vec{\tilde{q}}_h(x)} [G_0(x,y) - G_0(x_h, y_h)] d\Gamma_y d\Gamma_x \tag{116}$$

and

$$I_2 = \int_\Gamma \int_\Gamma \left[\vec{\tilde{p}}_h(y) \cdot \overline{\vec{\tilde{q}}_h(x)} - \vec{p}_h(y_h) \cdot \overline{\vec{q}_h(x_h)} J_h(y) J_h(x)\right] G_0(x_h, y_h) d\Gamma_y d\Gamma_x. \tag{117}$$

On the one hand, by using the property (i) of Lemma 5.4, we obtain

$$|I_1| \leqslant ch^{l+1} \int_\Gamma \int_\Gamma \frac{\left|\vec{\tilde{p}}_h(y)\right| \left|\vec{\tilde{q}}_h(x)\right|}{|x-y|} d\Gamma_y d\Gamma_x. \tag{118}$$

Then from [38], the bilinear form of the right-hand side of the inequality (118) defined from $TH^{-\frac{1}{2}}(\Gamma) \times TH^{-\frac{1}{2}}(\Gamma)$ into $(L^2(\Gamma))^2 \times (L^2(\Gamma))^2$ is continuous, and we have

$$|I_1| \leqslant ch^{l+1} \left\|\vec{\tilde{p}}_h\right\|_{0,\Gamma} \left\|\vec{\tilde{q}}_h\right\|_{0,\Gamma}. \tag{119}$$

On the other hand,

$$|I_2| \leqslant c\left\{\int_\Gamma \int_\Gamma \left|\vec{\tilde{p}}_h(y)\right| \left|\vec{\tilde{q}}_h(x) - J_h(x)\vec{q}_h(x_h)\right| G_0(x_h,y_h) d\Gamma_y d\Gamma_x + \int_\Gamma \int_\Gamma \left|\vec{\tilde{p}}_h(y) - J_h(y)\vec{p}_h(y_h)\right| \left|J_h(x)\vec{q}_h(x_h)\right| G_0(x_h,y_h) d\Gamma_y d\Gamma_x\right\}. \tag{120}$$

The kernel $G_0(x_h, y_h)$ can be estimated by $G_0(x,y)$ using the inequality (108), and as for (118), we can prove that

$$|I_2| \leqslant c\left\{\left\|\vec{\tilde{p}}_h\right\|_{0,\Gamma} \left\|\vec{\tilde{q}}_h - J_h(\vec{q}_h \circ \Phi^{-1})\right\|_{0,\Gamma} + \left\|\vec{\tilde{p}}_h - J_h(\vec{p}_h \circ \Phi^{-1})\right\|_{0,\Gamma} \left\|J_h(\vec{q}_h \circ \Phi^{-1})\right\|_{0,\Gamma}\right\}. \tag{121}$$

By combining (121) with the property (i) of Lemma 5.3, we obtain

$$|I_2| \leqslant ch^{l+1} \left\|\vec{\tilde{p}}_h\right\|_{0,\Gamma} \left\|\vec{\tilde{q}}_h\right\|_{0,\Gamma}. \tag{122}$$

From (119) and (122), we can deduce that

$$\left|a_{11}(\vec{\tilde{p}}_h, \vec{\tilde{q}}_h) - a_{11}^h(\vec{p}_h, \vec{q}_h)\right| \leqslant ch^{l+1} \left\|\vec{\tilde{p}}_h\right\|_{0,\Gamma} \left\|\vec{\tilde{q}}_h\right\|_{0,\Gamma}, \tag{123}$$

and the inverse inequality (62) gives (112).

In order to prove (113), the mapping Φ and (77) permit to write the integrals of the form a_{12}^h over Γ as

$$a_{12}^h(\lambda_h, \vec{q}_h) = \int_\Gamma \int_\Gamma \lambda_h(y_h) \overline{div_{\Gamma_h} \vec{q}_h(x_h)} G_0(x_h, y_h) J_h(y) J_h(x) d\Gamma_y d\Gamma_x. \tag{124}$$

Then we have, by using the properties (i) and (iii) of Lemma 5.1,

$$a_{12}(\tilde{\lambda}_h, \vec{\tilde{q}}_h) - a_{12}^h(\lambda_h, \vec{q}_h) = \int_\Gamma \int_\Gamma \tilde{\lambda}_h(y) \overline{div_{\Gamma_h} \vec{\tilde{q}}_h(x)} [G_0(x,y) - G_0(x_h, y_h)] d\Gamma_y d\Gamma_x. \tag{125}$$

From the property (i) of Lemma 5.4 and as for (118), we obtain

$$\left|a_{12}(\tilde{\lambda}_h, \vec{\tilde{q}}_h) - a_{12}^h(\lambda_h, \vec{q}_h)\right| \leqslant ch^{l+1} \int_\Gamma \int_\Gamma \frac{|\tilde{\lambda}_h(y)| \left|div_{\Gamma_h} \vec{\tilde{q}}_h(x)\right|}{|x-y|} d\Gamma_y d\Gamma_x \leqslant ch^{l+1} \|\tilde{\lambda}_h\|_{0,\Gamma} \left\|\vec{\tilde{q}}_h\right\|_{1,\Gamma} \tag{126}$$

and the estimate (113) is then obtained from the inverse property (62). $\square$

Theorem 5.6. *The forms a_{21}^h and a_{22}^h verify, for $\vec{p}_h \in \tilde{S}_h^{m,m-1}(\Gamma_h)$ and $\lambda_h, \mu_h \in S_h^{m+1,m}(\Gamma_h)$:*

$$\left|a_{21}(\vec{\tilde{p}}_h, \tilde{\mu}_h) - a_{21}^h(\vec{p}_h, \mu_h)\right| \leqslant c_1 h^{l-\frac{1}{2}} \left\|\vec{\tilde{p}}_h\right\|_{-\frac{1}{2},\Gamma} \|\tilde{\mu}_h\|_{\frac{1}{2},\Gamma}, \tag{127}$$

$$\left|a_{22}(\tilde{\lambda}_h,\tilde{\mu}_h)-a^h_{22}(\lambda_h,\mu_h)\right|\leqslant c_2 h^{l-1}\|\tilde{\lambda}_h\|_{\frac{1}{2},\Gamma}\|\tilde{\mu}_h\|_{\frac{1}{2},\Gamma},\tag{128}$$

where c_1 and c_2 are independent constants of h, $\vec{p}_h, \lambda_h$ and μ_h.

Proof. A change of variable using the mapping Φ together with (77), (78) and the relations (i) and (iii) of Lemma 5.1, lead to

$$\begin{aligned}a_{21}(\tilde{\vec{p}}_h,\tilde{\mu}_h)-a^h_{21}(\vec{p}_h,\mu_h)=&\int_\Gamma\int_\Gamma div_\Gamma\,\tilde{\vec{p}}_h(y)\overline{\tilde{\mu}_h(x)}[G_0(x,y)-G_0(x_h,y_h)]d\Gamma_y d\Gamma_x+\int_\Gamma\\&\times\int_\Gamma\left[\tilde{\vec{p}}_h(y)\cdot\overline{\overrightarrow{grad}_\Gamma\,\tilde{\mu}_h(x)}G_0(x,y)-\vec{p}_h(y_h)\cdot\overline{\overrightarrow{grad}_{\Gamma_h}\,\mu_h(x_h)}G_0(x_h,y_h)J_h(y)J_h(x)\right]d\Gamma_y d\Gamma_x.\end{aligned}\tag{129}$$

On the one hand, the second term of the right-hand side, that we denote T_2, is estimated in the same way as (115), and we have

$$\begin{aligned}|T_2|\leqslant c\Big\{&h^{l+1}\left\|\tilde{\vec{p}}_h\right\|_{0,\Gamma}\left\|\overrightarrow{grad}_\Gamma\,\tilde{\mu}_h\right\|_{0,\Gamma}+\left\|\tilde{\vec{p}}_h\right\|_{0,\Gamma}\left\|\overrightarrow{grad}_\Gamma\,\tilde{\mu}_h-J_h(\overrightarrow{grad}_{\Gamma_h}\,\mu_h\circ\Phi^{-1})\right\|_{0,\Gamma}\\&+\left\|\tilde{\vec{p}}_h-J_h(\vec{p}_h\circ\Phi^{-1})\right\|_{0,\Gamma}\left\|J_h(\overrightarrow{grad}_{\Gamma_h}\,\mu_h\circ\Phi^{-1})\right\|_{0,\Gamma}\Big\}.\end{aligned}\tag{130}$$

From the properties (i) and (ii) of Lemma 5.3, we can prove that

$$|T_2|\leqslant c\left\{h^{l+1}\left\|\tilde{\vec{p}}_h\right\|_{0,\Gamma}\|\tilde{\mu}_h\|_{1,\Gamma}+h^l\left\|\tilde{\vec{p}}_h\right\|_{0,\Gamma}\|\tilde{\mu}_h\|_{0,\Gamma}+h^{l+1}\left\|\tilde{\vec{p}}_h\right\|_{0,\Gamma}\|\tilde{\mu}_h\|_{1,\Gamma}\right\}\tag{131}$$

and the inverse properties of Proposition 4.7 give

$$|T_2|\leqslant ch^{l-\frac{1}{2}}\left\|\tilde{\vec{p}}_h\right\|_{-\frac{1}{2},\Gamma}\|\tilde{\mu}_h\|_{\frac{1}{2},\Gamma}.\tag{132}$$

On the other hand, the first term T_1 of the right-hand side of (129) can be easily estimated as for (125). Hence

$$|T_1|\leqslant ch^{l+1}\left\|\tilde{\vec{p}}_h\right\|_{1,\Gamma}\|\tilde{\mu}_h\|_{0,\Gamma}\leqslant ch^{l-\frac{1}{2}}\left\|\tilde{\vec{p}}_h\right\|_{-\frac{1}{2},\Gamma}\|\tilde{\mu}_h\|_{\frac{1}{2},\Gamma}.\tag{133}$$

The estimate (127) is then obtained from (132) and (133).

Now to prove (128), the mapping Φ together with (77) and the transposition formula (i) of Lemma 5.1 enable us to write

$$\begin{aligned}a_{22}(\tilde{\lambda}_h,\tilde{\mu}_h)-a^h_{22}(\lambda_h,\mu_h)=&k^2\int_\Gamma\int_\Gamma\tilde{\lambda}_h(y)\overline{\tilde{\mu}_h(x)}[G_0(x,y)-G_0(x_h,y_h)]d\Gamma_y d\Gamma_x+\int_\Gamma\\&\times\int_\Gamma\tilde{\lambda}_h(y)\left[\overline{\Delta_\Gamma\,\tilde{\mu}_h(x)}G_0(x,y)-\overline{\Delta_{\Gamma_h}\,\mu_h(x_h)}G_0(x_h,y_h)J_h(x)\right]d\Gamma_y d\Gamma_x.\end{aligned}\tag{134}$$

First take the second term of the right-hand side that we denote R_2. An analogous argument to the one of the proof of the previous theorem enables us to establish

$$|R_2|\leqslant c\left\{h^{l+1}\|\tilde{\lambda}_h\|_{0,\Gamma}\|\Delta_\Gamma\,\tilde{\mu}_h\|_{0,\Gamma}+\|\tilde{\lambda}_h\|_{0,\Gamma}\left\|\Delta_\Gamma\,\tilde{\mu}_h-J_h(\Delta_{\Gamma_h}\,\mu_h\circ\Phi^{-1})\right\|_{0,\Gamma}\right\}.\tag{135}$$

From the property (iii) of Lemma 5.3 and the inverse inequality (61), we obtain

$$|R_2|\leqslant c\left\{h^{l+1}\|\tilde{\lambda}_h\|_{0,\Gamma}\|\tilde{\mu}_h\|_{2,\Gamma}+h^{l-1}\|\tilde{\lambda}_h\|_{0,\Gamma}\|\tilde{\mu}_h\|_{0,\Gamma}\right\}\leqslant ch^{l-1}\|\tilde{\lambda}_h\|_{\frac{1}{2},\Gamma}\|\tilde{\mu}_h\|_{\frac{1}{2},\Gamma}.\tag{136}$$

For the first term of (134), it can be estimated, using Lemma 5.4 (i), as

$$|R_1|\leqslant ch^{l+1}\|\tilde{\lambda}_h\|_{0,\Gamma}\|\tilde{\mu}_h\|_{0,\Gamma}\leqslant ch^{l+1}\|\tilde{\lambda}_h\|_{\frac{1}{2},\Gamma}\|\tilde{\mu}_h\|_{\frac{1}{2},\Gamma}.\tag{137}$$

This with the estimate (136) lead to (128). □

Similar estimates can be proved for the forms r^h_{11}, r^h_{12}, etc.,. . ., by replacing the kernel G_0 by ϕ and using the property (ii) of Lemma 5.4.

For all tangential field $\vec{c}$ of class at least $\mathcal{C}^1$ given in local coordinates by $\vec{c}=\sum_{\alpha=1}^2 c^\alpha\vec{e}_\alpha$ we can define its approximation as $\vec{c}_h=\sum_{\alpha=1}^2 c^\alpha\vec{e}^{\,h}_\alpha$. A comparison between $\vec{c}_h$ and $\vec{c}$ can be easily obtained. From Lemma 5.2 (ii) we have

$$\left\|\vec{c}-\widetilde{\vec{c}_h}\right\|_{0,\Gamma}\leqslant ch^{l+1}\|\vec{c}\|_{0,\Gamma}\tag{138}$$

with $\widetilde{\vec{c}_h}(x)=\vec{c}_h(x_h)$, for $x=\Phi(x_h)$. For the surface divergence of $\vec{c}_h$, we can easily prove the following estimate:

$$\left\| div_\Gamma \vec{c} - \widetilde{div_{\Gamma_h} \vec{c}_h} \right\|_{0,\Gamma} \leqslant ch^l \|\vec{c}\|_{0,\Gamma}. \tag{139}$$

with $\widetilde{div_{\Gamma_h} \vec{c}_h}(x) = div_{\Gamma_h} \vec{c}_h(x_h)$, for $x = \Phi(x_h)$.

Theorem 5.7. *For $\vec{q}_h \in \tilde{S}_h^{m,m-1}(\Gamma_h)$ and $\mu_h \in S_h^{m+1,m}(\Gamma_h)$:*

$$\left| < \vec{c}, \vec{\tilde{q}}_h > - < \widetilde{\vec{c}_h}, \vec{\tilde{q}}_h > \right| \leqslant c_1 h^{l+1} \|\vec{c}\|_{0,\Gamma} \left\| \vec{\tilde{q}}_h \right\|_{0,\Gamma}, \tag{140}$$

$$\left| < -div_\Gamma \vec{c}, \tilde{\mu}_h > - < -\widetilde{div_{\Gamma_h} \vec{c}_h}, \tilde{\mu}_h > \right| \leqslant c_2 h^l \|\vec{c}\|_{0,\Gamma} \|\tilde{\mu}_h\|_{0,\Gamma} \tag{141}$$

where c_1 and c_2 are independent constants of h, $\vec{c}$, $\vec{c}_h$, $\vec{q}_h$ and μ_h.

Proof. The proof is a consequence of the estimates (138) and (139). □

The approximate bilinear forms a_{11}^h, r_{11}^h, etc.,...(see (85) and (86)), lead to the following decomposition of the operator $\mathcal{A}_h$

$$\mathcal{A}_h = \mathcal{D}_h + \mathcal{T}_h \tag{142}$$

where $\mathcal{D}_h$ and $\mathcal{T}_h$, are considered as approximate operators of $\mathcal{D}$ and $\mathcal{T}$ (see (50)).

Proposition 5.8. *There exists $h_0 > 0$ such that for all $0 < h \leqslant h_0$ and all $(\vec{p}_h, \lambda_h) \in \mathcal{S}_h(\Gamma_h)$,*

$$\left| \prec \mathcal{D}_h(\vec{p}_h, \lambda_h), (\vec{p}_h, \lambda_h) \succ_h \right| \geqslant c \left\| (\vec{p}_h, \lambda_h) \right\|^2_{-\frac{1}{2},\frac{1}{2},\Gamma_h} \tag{143}$$

where c is a strictly positive constant independent of h.

Proof. From the consistency inequalities given by Theorems 5.5 and 5.6 we can deduce that

$$\left| \prec \mathcal{D}(\vec{\tilde{p}}_h, \tilde{\lambda}_h), (\vec{\tilde{p}}_h, \tilde{\lambda}_h)) \succ - \prec \mathcal{D}_h(\vec{p}_h, \lambda_h), (\vec{p}_h, \lambda_h) \succ_h \right| \leqslant ch^{l-1} \left\| (\vec{\tilde{p}}_h, \tilde{\lambda}_h) \right\|^2_{-\frac{1}{2},\frac{1}{2},\Gamma}. \tag{144}$$

Therefore the coerciveness inequality (143) is obtained as a consequence of the one given by (52) for the operator $\mathcal{D}$ and from (144). □

Remarks 5.9. It should be noted here that the norm of $U_h = (\vec{p}_h, \lambda_h) \in \mathcal{S}_h(\Gamma_h)$ defined over Γ_h is equivalent to the norm of $\tilde{U}_h = (\vec{\tilde{p}}_h, \tilde{\lambda}_h) \in \mathcal{S}_h(\Gamma)$ defined over Γ.

Proposition 5.8 assures that the hypotheses of Theorem 4.3 are satisfied. Thus, for all $Z_h \in \mathcal{S}_h(\Gamma_h)$, the problem

$$\begin{cases} \text{Find } W_h \in \mathcal{S}_h(\Gamma_h) \quad \text{such that for all} \quad V_h \in \mathcal{S}_h(\Gamma_h) \\ \prec \mathcal{D}_h W_h, V_h) \succ_h = \prec \mathcal{A}_h Z_h, V_h \succ_h \end{cases} \tag{145}$$

admits a unique solution.

All this leads to define an operator $\mathcal{R}_h$ from $\mathcal{S}_h(\Gamma_h) \times \mathcal{S}_h(\Gamma_h)$ into $\mathcal{S}_h(\Gamma_h) \times \mathcal{S}_h(\Gamma_h)$ which will be an approximate of $\mathcal{R}$ defined by (57). For $Z_h \in \mathcal{S}_h(\Gamma_h)$, we set $\mathcal{R}_h Z_h = W_h$, where W_h is the unique solution of the problem (145).

To $Z_h \in \mathcal{S}_h(\Gamma_h)$ we associate $\tilde{Z}_h \in \mathcal{S}_h(\Gamma)$. It is clear that the problem

$$\begin{cases} \text{Find } W \in \mathcal{S}_h(\Gamma) \quad \text{such that for all} \quad \tilde{V}_h \in \mathcal{S}_h(\Gamma) \\ \prec \mathcal{D}W, \tilde{V}_h) \succ = \prec \mathcal{A}\tilde{Z}_h, \tilde{V}_h \succ \end{cases} \tag{146}$$

has a unique solution, and we have $\mathcal{R}\tilde{Z}_h = W$. By using the decompositions $\mathcal{A} = \mathcal{D} + \mathcal{T}$ and $\mathcal{A}_h = \mathcal{D}_h + \mathcal{T}_h$, we can introduce the problems

$$\begin{cases} \text{Find } U_h \in \mathcal{S}_h(\Gamma_h) \quad \text{such that for all} \quad V_h \in \mathcal{S}_h(\Gamma_h) \\ \prec \mathcal{D}_h U_h, V_h) \succ_h = \prec \mathcal{T}_h Z_h, V_h \succ_h \end{cases} \tag{147}$$

with $U_h = W_h - Z_h$, and

$$\begin{cases} \text{Find } U \in \mathcal{S}_h(\Gamma) \quad \text{such that for all} \quad \tilde{V}_h \in \mathcal{S}_h(\Gamma) \\ \prec \mathcal{D}U, \tilde{V}_h) \succ = \prec \mathcal{T}\tilde{Z}_h, \tilde{V}_h \succ \end{cases} \tag{148}$$

with $U = W - \tilde{Z}_h$. Since $Z_h \in \mathcal{S}_h(\Gamma_h)$ and $\tilde{Z}_h \in \mathcal{S}_h(\Gamma)$, it is obvious that $(W_h - Z_h)$ and $(W - \tilde{Z}_h)$ are the respective solutions of the problems (147) and (148).

Proposition 5.10. *There exists $h_0 > 0$ such that for all $0 < h \leqslant h_0$, we have*

$$\left\|U - \tilde{U}_h\right\|_{-\frac{1}{2},\frac{1}{2},\Gamma} \leqslant c\left(h^2 + h^{l-1}\right)\left\|\tilde{Z}_h\right\|_{-\frac{1}{2},\frac{1}{2},\Gamma}. \tag{149}$$

Proof. According to Proposition 5.8, we can write

$$c\left\|\tilde{U}_h - \tilde{V}_h\right\|^2_{-\frac{1}{2},\frac{1}{2},\Gamma} \leqslant c\|U_h - V_h\|^2_{-\frac{1}{2},\frac{1}{2},\Gamma_h} \leqslant | \prec \mathcal{D}_h U_h, U_h - V_h \succ_h - \prec \mathcal{D}_h V_h, U_h - V_h \succ_h |, \tag{150}$$

for all $V_h \in \mathcal{S}_h(\Gamma_h)$, and by using the equations (147) and (148) this last inequality becomes

$$c\left\|\tilde{U}_h - \tilde{V}_h\right\|^2_{-\frac{1}{2},\frac{1}{2},\Gamma} \leqslant | \prec \mathcal{T}\tilde{Z}_h, \tilde{U}_h - \tilde{V}_h \succ - \prec \mathcal{T}_h Z_h, U_h - V_h \succ_h | + | \prec \mathcal{D}\tilde{V}_h, \tilde{U}_h - \tilde{V}_h \succ - \prec \mathcal{D}_h V_h, U_h - V_h \succ_h | + | \prec \mathcal{D}U, \tilde{U}_h - \tilde{V}_h \succ - \prec \mathcal{D}\tilde{V}_h, \tilde{U}_h - \tilde{V}_h \succ |. \tag{151}$$

The error consistency estimate (144) and its similar one corresponding to $\mathcal{T}$ and $\mathcal{T}_h$, together with the continuity of $\mathcal{D}$, give

$$\left\|\tilde{U}_h - \tilde{V}_h\right\|_{-\frac{1}{2},\frac{1}{2},\Gamma} \leqslant c\left\{h^{l-1}\left(\left\|\tilde{Z}_h\right\|_{-\frac{1}{2},\frac{1}{2},\Gamma} + \left\|\tilde{V}_h\right\|_{-\frac{1}{2},\frac{1}{2},\Gamma}\right) + \left\|U - \tilde{V}_h\right\|_{-\frac{1}{2},\frac{1}{2},\Gamma}\right\}. \tag{152}$$

From this we deduce that

$$\left\|U - \tilde{U}_h\right\|_{-\frac{1}{2},\frac{1}{2},\Gamma} \leqslant \left\|U - \tilde{V}_h\right\|_{-\frac{1}{2},\frac{1}{2},\Gamma} + \left\|\tilde{V}_h - \tilde{U}_h\right\|_{-\frac{1}{2},\frac{1}{2},\Gamma} \leqslant c \inf_{V_h \in \mathcal{S}_h(\Gamma_h)} \left\{\left\|U - \tilde{V}_h\right\|_{-\frac{1}{2},\frac{1}{2},\Gamma} + h^{l-1}\left(\left\|\tilde{Z}_h\right\|_{-\frac{1}{2},\frac{1}{2},\Gamma} + \left\|\tilde{V}_h\right\|_{-\frac{1}{2},\frac{1}{2},\Gamma}\right)\right\}. \tag{153}$$

From the equation $\mathcal{R}\tilde{Z}_h = W$, we can derive that $\mathcal{D}^{-1}\mathcal{T}\tilde{Z}_h = U$. For $\tilde{Z}_h \in \mathcal{H}^{-\frac{1}{2},\frac{1}{2}}(\Gamma)$, clearly $U \in \mathcal{H}^{\frac{3}{2},\frac{5}{2}}(\Gamma)$. Therefore, there exists a constant c independent of h, such that

$$\|U\|_{\frac{3}{2},\frac{5}{2},\Gamma} \leqslant c\left\|\tilde{Z}_h\right\|_{-\frac{1}{2},\frac{1}{2},\Gamma}. \tag{154}$$

If we consider $\tilde{V}_h$ as the projection of U over $\mathcal{S}_h(\Gamma)$, and according to Proposition 4.6 (for $m \geqslant 2$), we can obtain

$$\left\|U - \tilde{V}_h\right\|_{-\frac{1}{2},\frac{1}{2},\Gamma} \leqslant ch^2\|U\|_{\frac{3}{2},\frac{5}{2},\Gamma}. \tag{155}$$

By using the inequalities (154) and (155) we can derive (149). □

Now, by replacing $\tilde{U}_h$ and U by their expressions defined above, the inequality (149) can be rewritten as

$$\left\|\mathcal{R}\tilde{Z}_h - \widetilde{\mathcal{R}_h Z_h}\right\|_{-\frac{1}{2},\frac{1}{2},\Gamma} \leqslant c\left(h^2 + h^{l-1}\right)\left\|\tilde{Z}_h\right\|_{-\frac{1}{2},\frac{1}{2},\Gamma} \leqslant c\left\|\tilde{Z}_h\right\|_{-\frac{1}{2},\frac{1}{2},\Gamma} \tag{156}$$

for all $W \in \mathcal{S}_h(\Gamma_h)$. From this last estimate and by referring on Proposition 4.5, we can establish the following theorem:

Theorem 5.11. *There exists $h_0 > 0$, such that for all $0 < h \leqslant h_0$,*

$$c\|(\overrightarrow{p}_h, \lambda_h)\|_{-\frac{1}{2},\frac{1}{2},\Gamma_h} \leqslant \|\mathcal{R}_h(\overrightarrow{p}_h, \lambda_h)\|_{-\frac{1}{2},\frac{1}{2},\Gamma_h}, \quad \forall (\overrightarrow{p}_h, \lambda_h) \in \mathcal{S}_h(\Gamma_h) \tag{157}$$

where c is a strictly positive constant.

Finally, we deduce that the discrete problem (74) is well-posed.

Theorem 5.12. *The discrete problem (74) admits unique solution.*

Proof. It is sufficient to show that this problem admits at most one solution. Let $U_h = (\overrightarrow{p}_h, \lambda_h) \in \mathcal{S}_h(\Gamma_h)$ be a solution of the homogenous problem of (74). Then the solution of the problem (152) $W_h = 0$, for $Z_h = U_h$. Therefore $\mathcal{R}_h U_h = 0$, and then the inequality (157) gives $U_h = 0$. □

5.3. Error estimates

In this section, we will establish some error estimates for the numerical approximation of quantities as the currents and the charges of the surface Γ developed by the incident wave $\overrightarrow{E}^{in}$ on the obstacle, that are calculated from the solution of the discrete problem. In the proofs, we will use the estimates obtained by Theorem 4.8 and those of errors consistency given by Theorems 5.5 and 5.6, as well as the coerciveness inequality (143).

In what follows $U = (\overrightarrow{p}, \lambda)$ and $U_h = (\overrightarrow{p}_h, \lambda_h)$ designate the respective solutions of the continuous problem

$$\begin{cases} \text{Find } U \in \mathcal{H}^{-\frac{1}{2}\frac{1}{2}}(\Gamma) \quad \text{such that for all} \quad V \in \mathcal{H}^{-\frac{1}{2}\frac{1}{2}}(\Gamma) \\ \prec \mathcal{A}U, V) \succ = \prec F, V \succ \end{cases} \tag{158}$$

and of the discrete problem

$$\begin{cases} \text{Find } U_h \in \mathcal{S}_h(\Gamma_h) \quad \text{such that for all} \quad V_h \in \mathcal{S}_h(\Gamma_h) \\ \prec \mathcal{A}_h U_h, V_h) \succ_h = \prec F_h, V_h \succ_h \end{cases} \tag{159}$$

To $U_h \in \mathcal{S}_h(\Gamma_h)$, we associate the element $\tilde{U}_h \in \mathcal{S}_h(\Gamma)$ (see (75) and (76)) which is the solution of the equation

$$\prec \mathcal{A}\tilde{U}_h, \tilde{V}_h) \succ = \prec \widetilde{F_h}, \tilde{V}_h \succ, \quad V_h \in \mathcal{S}_h(\Gamma_h) \tag{160}$$

with $\widetilde{F_h} = (\widetilde{\vec{c}}_h, -\widetilde{div_{\Gamma_h} \vec{c}_h})$. By following arguments similar to those of Theorem 4.2 (see [31]) and using the Eqs. (158) and (160), we can easily prove that

$$\tilde{U}_h = G_{\mathcal{A}} U - \tilde{A}^{-1}(F - \widetilde{F_h}). \tag{161}$$

Now, let $V_h = (\vec{q}_h, \mu_h)$ be an element of $\mathcal{S}_h(\Gamma_h)$. By using the relation (161) joining U to $\tilde{U}_h$ and the inverse inequalities (61) and (62), we have for $\frac{1}{2} \leqslant t \leqslant m$

$$\begin{aligned} \left\| U - \tilde{U}_h \right\|_{t-1,t,\Gamma} &\leqslant \left\| U - \tilde{V}_h \right\|_{t-1,t,\Gamma} + \left\| G_{\mathcal{A}}(U - \tilde{V}_h) \right\|_{t-1,t,\Gamma} + \left\| \mathcal{A}^{-1}(F - \widetilde{F_h}) \right\|_{t-1,t,\Gamma} \\ &\leqslant \left\| U - \tilde{V}_h \right\|_{t-1,t,\Gamma} + ch^{\frac{1}{2}-t} \left\| G_{\mathcal{A}}(U - \tilde{V}_h) \right\|_{-\frac{1}{2}\frac{1}{2},\Gamma} + \left\| \mathcal{A}^{-1}(F - \widetilde{F_h}) \right\|_{t-1,t,\Gamma}. \end{aligned} \tag{162}$$

This, together with Theorem 4.2 and the continuity of the operator $\mathcal{A}^{-1}$, lead to

$$\left\| U - \tilde{U}_h \right\|_{t-1,t,\Gamma} \leqslant c \left\{ \inf_{V_h \in \in \mathcal{S}_h(\Gamma_h)} \left[\left\| U - \tilde{V}_h \right\|_{t-1,t,\Gamma} + h^{\frac{1}{2}-t} \left\| U - \tilde{V}_h \right\|_{-\frac{1}{2}\frac{1}{2},\Gamma} \right] + \left\| F - \widetilde{F_h} \right\|_{t,t-1,\Gamma} \right\}. \tag{163}$$

The following theorem gives the error estimates for the solution $\tilde{U}_h$ of (160).

Theorem 5.13. *Let $h_0 > 0$. Then for all $0 < h \leqslant h_0$, there exists a constant $c > 0$ independent of h and $\vec{c}$ such that*

$$\left\| U - \tilde{U}_h \right\|_{t-1,t,\Gamma} \leqslant c \left\{ \left(h^{s-t} + h^{l-1} \right) \| U \|_{s-1,s,\Gamma} + h^{l+1-t} \| \vec{c} \|_{0,\Gamma} \right\} \tag{164}$$

for $-m-2 \leqslant t \leqslant s \leqslant m+1,\ -m \leqslant -\frac{1}{2} \leqslant m,\ t \leqslant m$ and $-\frac{1}{2} \leqslant s$.

Proof. First we treat the case $t \geqslant \frac{1}{2}$. Let's fix an element V_h of $\mathcal{S}_h(\Gamma_h)$, we have then by using the inverse inequalities (61) and (62)

$$\left\| U - \tilde{V}_h \right\|_{t-1,t,\Gamma} \leqslant \left\| U - \tilde{W}_h \right\|_{t-1,t,\Gamma} + \left\| \tilde{W}_h - \tilde{V}_h \right\|_{t-1,t,\Gamma} \leqslant \left\| U - \tilde{W}_h \right\|_{t-1,t,\Gamma} + ch^{\frac{1}{2}-t} \left\| \tilde{W}_h - \tilde{V}_h \right\|_{-\frac{1}{2}\frac{1}{2},\Gamma} \tag{165}$$

where W_h is an arbitrary element of $\mathcal{S}_h(\Gamma_h)$. Suppose that V_h is the solution of the system

$$\prec \mathcal{D}_h V_h, W_h) \succ_h = \prec \mathcal{D}U, \tilde{W}_h \succ, \quad \forall W_h \in \mathcal{S}_h(\Gamma_h). \tag{166}$$

In the following, we estimate the second term of the right-hand side of (165). In fact, according to the inequality (143), we can write

$$c \left\| \tilde{V}_h - \tilde{W}_h \right\|^2_{-\frac{1}{2}\frac{1}{2},\Gamma} \leqslant c \| V_h - W_h \|^2_{-\frac{1}{2}\frac{1}{2},\Gamma_h} \leqslant | \prec \mathcal{D}_h(V_h - W_h), V_h - W_h \succ_h |. \tag{167}$$

From the Eq. (166), we can easily obtain

$$c \left\| \tilde{V}_h - \tilde{W}_h \right\|^2_{-\frac{1}{2}\frac{1}{2},\Gamma} \leqslant | \prec \mathcal{D}U, \tilde{V}_h - \tilde{W}_h \succ - \prec \mathcal{D}\tilde{W}_h, \tilde{V}_h - \tilde{W}_h \succ | + | \prec \mathcal{D}\tilde{W}_h, \tilde{V}_h - \tilde{W}_h \succ - \prec \mathcal{D}_h W_h, V_h - W_h \succ_h |. \tag{168}$$

The continuity of the operator $\mathcal{D}$ together with the error consistency estimates (144) give

$$\left\| \tilde{V}_h - \tilde{W}_h \right\|_{-\frac{1}{2}\frac{1}{2},\Gamma} \leqslant c \left\{ \left\| U - \tilde{W}_h \right\|_{-\frac{1}{2}\frac{1}{2},\Gamma} + h^{l-1} \left\| \tilde{W}_h \right\|_{-\frac{1}{2}\frac{1}{2},\Gamma} \right\}. \tag{169}$$

This enable us to rewrite (165) as

$$\left\| U - \tilde{V}_h \right\|_{-\frac{1}{2}\frac{1}{2},\Gamma} \leqslant c \inf_{W_h \in \mathcal{S}_h(\Gamma_h)} \left\{ \left\| U - \tilde{W}_h \right\|_{t-1,t,\Gamma} + h^{\frac{1}{2}-t} \left\| U - \tilde{W}_h \right\|_{-\frac{1}{2}\frac{1}{2},\Gamma} + h^{l-1} \left\| \tilde{W}_h \right\|_{t-1,t,\Gamma} \right\}. \tag{170}$$

Now, by using inequality (163), we are leading to

$$\left\|U-\tilde{U}_h\right\|_{t-1,t,\Gamma} \leqslant c\left\{\inf_{W_h\in\mathcal{S}_h(\Gamma_h)}\left[\left\|U-\tilde{W}_h\right\|_{t-1,t,\Gamma}+h^{\frac{1}{2}-t}\left\|U-\tilde{W}_h\right\|_{-\frac{1}{2},\frac{1}{2},\Gamma}+h^{l-1}\left\|\tilde{W}_h\right\|_{-\frac{1}{2},\frac{1}{2},\Gamma}\right]+\left\|F-\widetilde{F_h}\right\|_{t,t-1,\Gamma}\right\}. \tag{171}$$

For the estimate of $(F-\tilde{F}_h)$, we use the relation

$$\left\|F-\widetilde{F_h}\right\|_{t,t-1,\Gamma} = \sup_{W_h\in\mathcal{S}_h(\Gamma_h)} \frac{|\prec F,\tilde{W}_h\succ - \prec \widetilde{F_h},\tilde{W}_h\succ|}{\left\|\tilde{W}_h\right\|_{-t,-t+1,\Gamma}}, \tag{172}$$

the error estimates given by Theorem 5.7 and the inverse inequalities (61) and (62). We obtain

$$\left\|F-\widetilde{F_h}\right\|_{t,t-1,\Gamma} \leqslant \sup_{W_h\in\mathcal{S}_h\Gamma_h)} \frac{h^l\|\overrightarrow{c}\|_{0,\Gamma}h^{-t+1}\left\|\tilde{W}_h\right\|_{-t,-t+1,\Gamma}}{\left\|\tilde{W}_h\right\|_{-t,-t+1,\Gamma}} \leqslant ch^{l-t+1}\|\overrightarrow{c}\|_{0,\Gamma}. \tag{173}$$

If we take an element $W_h\in\mathcal{S}_h(\Gamma_h)$, such that $\tilde{W}_h$ verifies the estimate (63) then

$$\left\|U-\tilde{W}_h\right\|_{t-1,t,\Gamma} \leqslant ch^{s-t}\|U\|_{s-1,s,\Gamma}, \tag{174}$$

and moreover,

$$\left\|\tilde{W}_h\right\|_{t-1,t,\Gamma} \leqslant c\|U\|_{s-1,s,\Gamma}. \tag{175}$$

Finally, to obtain (164) (for $t\geqslant\frac{1}{2}$), we only have to insert the inequalities (173)–(175) in (171).

For the case $t\leqslant\frac{1}{2}$, we can find the result by using classical duality arguments [6,18]. So the problem (158) and (160) give us

$$\prec \mathcal{A}(U-\tilde{U}_h),\tilde{V}_h\succ = \prec U-\tilde{U}_h,\mathcal{A}^*\tilde{V}_h\succ = \prec F-\widetilde{F_h},\tilde{V}_h\succ, \tag{176}$$

for all $V_h\in\mathcal{S}_h(\Gamma_h)$. Let $-\frac{1}{2}\leqslant\alpha\leqslant m+2$. With the help of (29), we can write

$$\left\|U-\tilde{U}_h\right\|_{-\alpha-1,-\alpha,\Gamma} = \sup_{Z\in H^{\alpha+1,\alpha}(\Gamma)}\frac{|\prec U-\tilde{U}_h,Z\succ|}{\|Z\|_{\alpha+1,\alpha,\Gamma}}. \tag{177}$$

We are then conducted to solve the equation

$$\mathcal{A}^*W=Z, \tag{178}$$

for $Z\in H^{\alpha+1,\alpha}(\Gamma)$, and by the continuity of $\mathcal{A}^{*-1}$ it follows that

$$\|W\|_{\alpha,\alpha+1,\Gamma}\leqslant c\|Z\|_{\alpha+1,\alpha,\Gamma}. \tag{179}$$

Now, by using the Eqs. (176), (178)the inequality (179), we obtain, for all $V_h\in\mathcal{S}_h(\Gamma_h)$

$$\begin{aligned}\left\|U-\tilde{U}_h\right\|_{-\alpha-1,-\alpha,\Gamma} &\leqslant c\sup_{W\in H^{\alpha,\alpha+1}(\Gamma)}\frac{|\prec U-\tilde{U}_h,\mathcal{A}^*W\succ|}{\|W\|_{\alpha,\alpha+1,\Gamma}} \leqslant c\sup_{W\in H^{\alpha,\alpha+1}(\Gamma)}\frac{|\prec U-\tilde{U}_h,\mathcal{A}^*(W-\tilde{V}_h)\succ+\prec F-\widetilde{F_h},\tilde{V}_h\succ|}{\|W\|_{\alpha,\alpha+1,\Gamma}}\\ &\leqslant c\sup_{W\in H^{\alpha,\alpha+1}(\Gamma)}\frac{\left\|U-\tilde{U}_h\right\|_{-\frac{1}{2},\frac{1}{2},\Gamma}\left\|W-\tilde{V}_h\right\|_{-\frac{1}{2},\frac{1}{2},\Gamma}+\left\|F-\widetilde{F_h}\right\|_{\frac{1}{2},-\frac{1}{2},\Gamma}\left\|\tilde{V}_h\right\|_{-\frac{1}{2},\frac{1}{2},\Gamma}}{\|W\|_{\alpha,\alpha+1,\Gamma}}.\end{aligned} \tag{180}$$

By taking $\tilde{V}_h$ as the orthogonal projection of W over $\mathcal{S}_h(\Gamma)$ and referring to the estimates (164), (173) and those of the Proposition 4.6, for $t=\frac{1}{2}$, we can derive (164) with $\alpha=-t$. □

For the solution U_h of the discrete problem (159), we have the following stability result:

Theorem 5.14. *There exists a constant c independent of h and* $\overrightarrow{c}$*, such that*

$$\|U_h\|_{-\frac{1}{2},\frac{1}{2},\Gamma_h}\leqslant c\|\overrightarrow{c}\|_{\frac{1}{2},\Gamma}. \tag{181}$$

Proof. The problem (145) give us with the help of (159), for $Z_h=U_h$, that the equation

$$\prec\mathcal{D}_hW_h,V_h\succ_h = \prec F_h,V_h\succ_h, \quad \forall V_h\in\mathcal{S}_h(\Gamma_h) \tag{182}$$

admits a unique solution $W_h\in\mathcal{S}_h(\Gamma_h)$. According to the coerciveness estimate (143) we obtain

$$\|W_h\|_{-\frac{1}{2},\frac{1}{2},\Gamma_h}\leqslant c\|F_h\|_{\frac{1}{2},-\frac{1}{2},\Gamma_h}\leqslant c\|F\|_{\frac{1}{2},-\frac{1}{2},\Gamma}. \tag{183}$$

The estimate (181) is then a consequence of the equation $\mathcal{R}_hU_h=W_h$ and the inequality (157). □

Once the discrete problem is resolved, we can calculate the approximate solution $\overrightarrow{E}_h$ of the exact solution $\overrightarrow{E}$ of the problem (1), that is defined by

$$\overrightarrow{E}_h(x) = V_h(\overrightarrow{P}_h)(x) + \overrightarrow{grad}_x V_h(\lambda_h)(x), \forall x \in \Omega^e. \tag{184}$$

For the cases $t = -m-2$ and $s = m+1$, we have the following approximation result for the approximate electric field $\overrightarrow{E}_h$.

Theorem 5.15. *For h small enough and for x such that $d(x,\Gamma) \geqslant \delta > 0$, we have*

$$\left|\overrightarrow{E}(x) - \overrightarrow{E}_h(x)\right| \leqslant c\, e(x,\Gamma)\left\{\left(h^{2m+3} + h^{l-1}\right)\|U\|_{m,m+1,\Gamma} + h^{l+m+3}\|\overrightarrow{c}\|_{0,\Gamma} + h^{l+\frac{1}{2}}\|\overrightarrow{c}\|_{\frac{1}{2},\Gamma}\right\} \tag{185}$$

where c is independent of h, U, x and of the function $e(x,\Gamma)$ defined by

$$e(x,\Gamma) = \sum_{j=1}^{m+3} \frac{1}{d^j(x,\Gamma)}. \tag{186}$$

Proof. By change of variable using the mapping Φ and the representations of the exact and approximate solutions of Maxwell's system (5) and (184), we have

$$\overrightarrow{E}(x) - \overrightarrow{E}_h(x) = \int_\Gamma \left[\overrightarrow{p}(y)G(x,y) - J_h(y)\overrightarrow{p}_h(y_h)G(x,y_h)\right]d\Gamma_y + \overrightarrow{grad}_x \int_\Gamma [\lambda(y)G(x,y) - J_h(y)\lambda_h(y_h)G(x,y_h)]d\Gamma_y \tag{187}$$

that we can write as $I_1 + I_2$ with

$$I_1 = \int_\Gamma \left[\overrightarrow{P}(y) - J_h(y)\overrightarrow{p}_h(y_h)\right]G(x,y)d\Gamma_y + \overrightarrow{grad}_x \int_\Gamma \left[\lambda(y) - \tilde{\lambda}_h(y)\right]G(x,y)d\Gamma_y \tag{188}$$

and

$$I_2 = \int_\Gamma J_h(y)\overrightarrow{p}_h(y_h)[G(x,y) - G(x,y_h)]d\Gamma_y + \overrightarrow{grad}_x \int_\Gamma \tilde{\lambda}_h(y)[G(x,y) - G(x,y_h)]d\Gamma_y. \tag{189}$$

On the one hand, we have

$$\begin{aligned} |I_1| &\leqslant c\left\{\left\|\overrightarrow{p} - \vec{\tilde{p}}_h\right\|_{-m-3,\Gamma}\|G(x,y)\|_{m+3,\Gamma} + \left\|\vec{\tilde{p}}_h - J_h(\overrightarrow{p}_h\circ\Phi^{-1})\right\|_{0,\Gamma}\|G(x,y)\|_{0,\Gamma} + \left\|\lambda - \tilde{\lambda}_h\right\|_{-m-2,\Gamma}\left\|\overrightarrow{grad}_x G(x,y)\right\|_{m+2,\Gamma}\right\} \\ &\leqslant c\left\{\left\|U - \tilde{U}_h\right\|_{-m-3,-m-2,\Gamma}\left(\|G(x,y)\|_{m+3,\Gamma} + \left\|\overrightarrow{grad}_x G(x,y)\right\|_{m+2,\Gamma}\right) + \left\|\vec{\tilde{p}}_h - J_h(\overrightarrow{p}_h\circ\Phi^{-1})\right\|_{0,\Gamma}\|G(x,y)\|_{0,\Gamma}\right\}. \end{aligned} \tag{190}$$

Referring to [14,36], we find the following estimates, for all positive integer m

$$\|G(x,y)\|_{m,\Gamma} \leqslant c\left\|\frac{e^{ik|x-y|}}{|x-y|}\right\|_{m,\Gamma} \leqslant c\sum_{j=1}^{m}\frac{1}{d^j(x,\Gamma)} \tag{191}$$

and

$$\left\|\overrightarrow{grad}_x G(x,y)\right\|_{m,\Gamma} \leqslant c\left\|(x-y)\left(\frac{ik}{|x-y|^2} - \frac{1}{|x-y|^3}\right)e^{ik|x-y|}\right\|_{m,\Gamma} \leqslant c\sum_{j=1}^{m+1}\frac{1}{d^j(x,\Gamma)}. \tag{192}$$

All this with the results of Theorem 5.13, the first property of Lemma 5.3, the inverse property (62) and the stability inequality (181) yield

$$|I_1| \leqslant c\left(\sum_{j=1}^{m+3}\frac{1}{d^j(x,\Gamma)}\right)\left\{\left(h^{2m+3} + h^{l-1}\right)\|U\|_{m,m+1,\Gamma} + h^{l+m+3}\|\overrightarrow{c}\|_{0,\Gamma} + h^{l+\frac{1}{2}}\|\overrightarrow{c}\|_{\frac{1}{2},\Gamma}\right\} \tag{193}$$

On the other hand,

$$|I_2| \leqslant c\left\{\left\|J_h(\overrightarrow{p}_h\circ\Phi^{-1})\right\|_{0,\Gamma}\|G(x,y) - G(x,y_h)\|_{0,\Gamma} + \left\|\tilde{\lambda}_h\right\|_{0,\Gamma}\left\|\overrightarrow{grad}_x G(x,y) - \overrightarrow{grad}_x G(x,y_h)\right\|_{0,\Gamma}\right\}. \tag{194}$$

Always referring to [14,36], we have

$$\begin{aligned} \|G(x,y) - G(x,y_h)\|_{0,\Gamma} &\leqslant c\left\|\frac{e^{ik|x-y|}}{|x-y|} - \frac{e^{ik|x-y_h|}}{|x-y_h|}\right\|_{0,\Gamma} \leqslant c\left\{\left\|\frac{e^{ik|x-y|} - e^{ik|x-y_h|}}{|x-y|}\right\|_{0,\Gamma} + \left\|e^{ik|x-y_h|}\left(\frac{1}{|x-y|} - \frac{1}{|x-y_h|}\right)\right\|_{0,\Gamma}\right\} \\ &\leqslant c\frac{h^{l+1}}{d(x,\Gamma)} \end{aligned} \tag{195}$$

and

$$\left\|\overrightarrow{grad}_x G(x,y)-\overrightarrow{grad}_x G(x,y_h)\right\|_{0,\Gamma}\leqslant c\left\{\left\|\left(\frac{ik}{|x-y|^2}-\frac{1}{|x-y|^3}\right)\left((x-y)e^{ik|x-y|}-(x-y_h)e^{ik|x-y_h|}\right)\right\|_{0,\Gamma}\right.$$
$$\left.+\left\|(x-y_h)e^{ik|x-y_h|}\left(\frac{1}{|x-y|^2}-\frac{1}{|x-y_h|^2}\right)\right\|_{0,\Gamma}+\left\|(x-y_h)e^{ik|x-y_h|}\left(\frac{1}{|x-y|^3}-\frac{1}{|x-y_h|^3}\right)\right\|_{0,\Gamma}\right\}\leqslant ch^{l+1}\left(\frac{1}{d^2(x,\Gamma)}+\frac{1}{d^3(x,\Gamma)}\right). \quad (196)$$

By using the inequalities

$$\|\tilde{\lambda}_h\|_{0,\Gamma}\leqslant c\|\tilde{\lambda}_h\|_{\frac{1}{2},\Gamma}\leqslant c\|\lambda_h\|_{\frac{1}{2},\Gamma_h}, \quad (197)$$

$$\left\|J_h(\vec{p}_h\circ\Phi^{-1})\right\|_{0,\Gamma}\leqslant c\left\|\vec{\tilde{p}}_h\right\|_{0,\Gamma}\leqslant ch^{-\frac{1}{2}}\left\|\vec{\tilde{p}}_h\right\|_{-\frac{1}{2},\Gamma}\leqslant ch^{-\frac{1}{2}}\|\vec{p}_h\|_{-\frac{1}{2},\Gamma_h}, \quad (198)$$

Theorem 5.14 and the estimates (195), (196), we obtain

$$|I_2|\leqslant c\left(\sum_{j=1}^{3}\frac{1}{d^j(x,\Gamma)}\right)h^{l+\frac{1}{2}}\|\vec{c}\|_{\frac{1}{2},\Gamma}. \quad (199)$$

To derive (185) we have only to combine (193) and (199). □

6. Particular case

The definition of the space $\mathcal{S}_h(\Gamma_h)$ (see (37)) shows that the scalar function λ_h has an order of differentiability more than the vector function $\vec{p}_h$. According to MacCamy and Stephan [31] the most efficient implementation procedure requires that we are capable to calculate the surface divergence of $\vec{p}_h$ and the surface Laplacian of λ_h. For this, they have proposed to choose $m\geqslant 2$. Also the convergence rate of the error estimates (164) and (185) requires that we take finite elements of order at least 2 ($l\geqslant 2$) for the approximation of the surface. However, we will take into consideration, with justification, the case $m=l=1$.

Using the formula (12) and the relation (9) the second equation of the modified variational formulation (11) becomes

$$< V(div_\Gamma\,\vec{p}),\mu > + < \Pi_\Gamma V(\vec{p}),\overrightarrow{grad}_\Gamma\,\mu > + < \overrightarrow{grad}_\Gamma\,V(\lambda),\overrightarrow{grad}_\Gamma\,\mu > -k^2 < V(\lambda),\mu > = < \vec{c},\overrightarrow{grad}_\Gamma\,\mu > . \quad (200)$$

By adding the above equation to the first one of (11), and using the second equation of (8), we obtain

$$\begin{cases}\text{Find } (\vec{p},\lambda)\in\mathcal{H}^{r-1,r}(\Gamma)\quad\text{such that for all}\quad(\vec{q},\mu)\in\mathcal{H}^{r-1,r}(\Gamma)\\ < \Pi_\Gamma V(\vec{p}),\vec{q}+\overrightarrow{grad}_\Gamma\,\mu > + < \overrightarrow{grad}_\Gamma\,V(\lambda),\vec{q}+\overrightarrow{grad}_\Gamma\,\mu > = < \vec{c},\vec{q}+\overrightarrow{grad}_\Gamma\,\mu >,\\ < V(div_\Gamma\,\vec{p}),\mu > -k^2 < V(\lambda),\mu > = 0.\end{cases} \quad (201)$$

which is the variational formulation (8), with modified test functions.

In the same way, the discrete variational formulation (66) can be transformed into

$$\begin{cases}\text{Find } (\vec{p}_h,\lambda_h)\in\mathcal{S}_h(\Gamma_h)\quad\text{such that for all}\quad(\vec{q},\mu)\in\mathcal{S}_h(\Gamma_h)\\ < \Pi_{\Gamma_h}V(\vec{p}_h),\vec{q}_h+\overrightarrow{grad}_{\Gamma_h}\,\mu_h >_h + < \overrightarrow{grad}_{\Gamma_h}\,V(\lambda_h),\vec{q}_h+\overrightarrow{grad}_{\Gamma_h}\,\mu_h >_h = < \vec{c}_h,\vec{q}_h+\overrightarrow{grad}_{\Gamma_h}\,\mu_h >_h,\\ < V(div_{\Gamma_h}\,\vec{p}_h),\mu_h >_h - k^2 < V(\lambda_h),\mu_h >_h = 0.\end{cases} \quad (202)$$

Proposition 6.1. *Let $(\vec{q}_h,\mu_h)\in\mathcal{S}_h(\Gamma_h)$. Then*

$$(\vec{q}_h+\overrightarrow{grad}_{\Gamma_h}\,\mu_h,\mu_h)\in\mathcal{S}_h(\Gamma_h), \quad (203)$$

for $m=l=1$.

Proof. We have for $\vec{q}_h|_{T_h}(x_h)=\sum_{\alpha=1}^{2}q_h^\alpha(\xi)\,\vec{e}^{\,h}_\alpha(\xi)$ and $\mu_h|_{T_h}(x_h)=\rho_h(\xi)$,

$$\left(\vec{q}_h+\overrightarrow{grad}_{\Gamma_h}\,\mu_h\right)|_{T_h}(x_h)=\sum_{\alpha=1}^{2}\left(q_h^\alpha(\xi)+g_h^{\alpha 1}\frac{\partial}{\partial\xi_1}\rho_h(\xi)+g_h^{\alpha 2}\frac{\partial}{\partial\xi_2}\rho_h(\xi)\right)\vec{e}^{\,h}_\alpha(\xi) \quad (204)$$

with $x_h=F_T^l(\xi)$ and $\xi\in\hat{T}$. For $m=l=1$, $q_h^\alpha\in\mathbb{P}_0$, $\rho_h\in\mathbb{P}_1$ and $g_h^{\alpha\beta}$ are constant, therefore $\left(q_h^\alpha+g_h^{\alpha 1}\frac{\partial}{\partial\xi_1}\rho_h+g_h^{\alpha 2}\frac{\partial}{\partial\xi_2}\rho_h\right)\in\mathbb{P}_0$, which gives us (203). □

This proposition enables us to replace the test function $(\vec{q}_h+\overrightarrow{grad}_{\Gamma_h}\,\mu_h,\mu_h)\in\mathcal{S}_h(\Gamma_h)$ by $(\vec{q}^{\,*}_h,\mu^*_h)\in\mathcal{S}_h(\Gamma_h)$ in the formulation (202). Then we have

$$\begin{cases} \text{Find } (\overrightarrow{p}_h,\lambda_h)\in\mathcal{S}_h(\Gamma_h) \quad \text{such that for all} \quad (\overrightarrow{q}^*_h,\mu^*_h)\in\mathcal{S}_h(\Gamma_h) \\ <\Pi_{\Gamma_h}V_h(\overrightarrow{p}_h),\overrightarrow{q}^*_h>_h + <\overrightarrow{grad}_{\Gamma_h}V_h(\lambda_h),\overrightarrow{q}^*_h>_h = <\overrightarrow{c}_h,\overrightarrow{q}^*_h>_h, \\ <V_h(div_{\Gamma_h}\overrightarrow{p}_h),\mu^*_h>_h - k^2<V_h(\lambda_h),\mu^*_h>_h = 0. \end{cases} \tag{205}$$

The above variational formulation is identical to the discretization of the non-modified system (8). Finally, note that if $m>1$ or $l>1$, the discrete variational formulations of both initial and modified systems are not equivalent.

The error analysis for the solution $U_h=(\overrightarrow{p}_h,\lambda_h)$ of (205) and for the approximate electric field $\overrightarrow{E}_h$ given by (184) can be obtained in the same way as for the general case. Therefore, we have the following similar theorems to Theorem 5.13 and Theorem 5.15:

Theorem 6.2. *Let $h_0>0$. Then for all $0<h\leqslant h_0$, there exists a constant $c>0$ independent of h and $\overrightarrow{c}$ such that*

$$\left\|U-\tilde{U}_h\right\|_{t-1,t,\Gamma}\leqslant c\left\{\left(h^{s-t}+h^{\frac{1}{2}}\right)\|U\|_{s-1,s,\Gamma}+h^{2-t}\|\overrightarrow{c}\|_{0,\Gamma}\right\} \tag{206}$$

for $-3\leqslant t\leqslant s\leqslant 2; t\leqslant 1$ and $-\frac{1}{2}\leqslant s$.

and

Theorem 6.3. *For h small enough and for x such that $d(x,\Gamma)\geqslant\delta>0$, we have*

$$\left|\overrightarrow{E}(x)-\overrightarrow{E}_h(x)\right|\leqslant c\,e(x,\Gamma)\left\{\left(h^5+h^{\frac{1}{2}}\right)\|U\|_{1,2,\Gamma}+h^4\|\overrightarrow{c}\|_{0,\Gamma}+h^{\frac{3}{2}}\|\overrightarrow{c}\|_{\frac{1}{2},\Gamma}\right\} \tag{207}$$

where c is independent of $h,\ U,\ x$ and of the function $e(x,\Gamma)$ defined by

$$e(x,\Gamma)=\sum_{j=1}^{4}\frac{1}{d^j(x,\Gamma)}. \tag{208}$$

7. Numerical results

The numerical approximation schemes that we propose are such that the surface is approximated by planar or curved elements, satisfying the conditions of discretization by finite elements, and that the surface currents $\overrightarrow{p}$ and charges λ are approximated by piecewise polynomial interpolation functions of order $m-1$ and m, respectively, with $m\geqslant 1$ (In our numerical implementation we only consider the cases $m=1$ and $m=2$). This gives a numerical scheme that we will call $\mathbb{D}_{m-1}-\mathbb{P}_m$ scheme representing the space (see (37)) $\mathcal{S}_h(\Gamma_h)=\tilde{S}_h^{m,m-1}(\Gamma_h)\times S_h^{m+1,m}(\Gamma_h)$, where $\tilde{S}_h^{m,m-1}(\Gamma_h)$ and $S_h^{m+1,m}(\Gamma_h)$ are, respectively, the spaces of tangential and scalar piecewise Lagrange polynomial interpolating functions of order $m-1$ and m. In what follows N_f is the number of elements obtained from the discretization of the surface Γ. Note that for the implementation of the $\mathbb{D}_0-\mathbb{P}_1$ scheme with $l=1$ we consider the discrete variational formulation (205).

We consider the case where the obstacle is illuminated by an incident wave $\overrightarrow{E}^{in}$. We recall that the electric scattering field in near-field or Fresnel zone [35,39,44,45] is expressed with the help of the surface currents and charges by

$$\overrightarrow{E}^{sc}(x)=\int_\Gamma\overrightarrow{p}(y)G(x,y)d\Gamma_y+\overrightarrow{grad}_x\int_\Gamma\lambda(y)G(x,y)d\Gamma_y,\ \forall x=(x_1,x_2,x_3)\in\Omega^e. \tag{209}$$

Along a unit direction $\overrightarrow{\omega}$ of $\mathbb{R}^3$ moving away from the obstacle, the scattered field $\overrightarrow{E}^{sc}(r\overrightarrow{\omega})$ seen at a distance r away from the origin, admits the following expansion at infinity

$$\overrightarrow{E}^{sc}(r\overrightarrow{\omega})=\frac{e^{ikr}}{4\pi r}\overrightarrow{\omega}\wedge(\overrightarrow{a}(\overrightarrow{\omega})\wedge\overrightarrow{\omega})+o\left(\frac{1}{r^2}\right) \tag{210}$$

with

$$\overrightarrow{a}(\overrightarrow{\omega})=\int_\Gamma\overrightarrow{p}(y)e^{-ik\overrightarrow{r}(y)\cdot\overrightarrow{\omega}}d\Gamma_y \tag{211}$$

where $\overrightarrow{r}(y)$ is the director vector of y. The Radar Cross Section (RCS) (see [40]) is very important for the industrial and in electromagnetic simulation [9,10,13]. For a reference direction $\overrightarrow{\omega}_0$ of $\mathbb{R}^3$ the RCS is given by the formula

$$\sigma(\overrightarrow{\omega})=\lim_{r\to\infty}4\pi r^2\frac{\left|\overrightarrow{E}^{sc}(r\overrightarrow{\omega})\right|^2}{\left|\overrightarrow{E}^{in}(r_0\overrightarrow{\omega}_0)\right|^2} \tag{212}$$

where the diffracted wave reply in the direction $\overrightarrow{\omega}$ when the incident wave arrives in the direction $\overrightarrow{\omega}_0$. For a unit incident wave, the diffracted electric field in far-field or Fraunhofer zone [35,39,44,45] is therefore characterized by

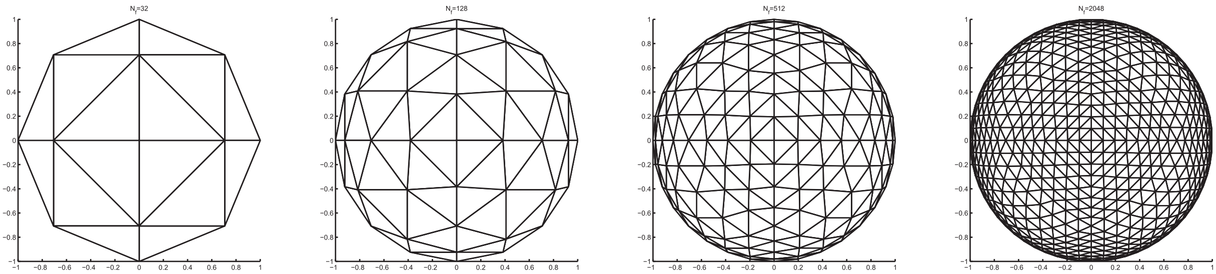

Fig. 1. Discretization and refinement of the unit sphere.

Table 1
Exact and approximate solution for $\mathbb{D}_0 - \mathbb{P}_1$ scheme with $k = \frac{5\pi}{4}$.

d	0.50	0.75	1.00	1.25	1.50	1.75	2.00
Exact	+0.169890	−0.270311	+0.227596	−0.106882	−0.025142	+0.116019	−0.13996
Real part of the exact and approximate component of the solution							
$N_f = 32$	+0.177668	−0.261364	+0.211343	−0.094201	−0.028391	+0.110060	−0.129303
$N_f = 128$	+0.170451	−0.270099	+0.227153	−0.106594	−0.025140	+0.115789	−0.139668
$N_f = 512$	+0.170087	−0.270217	+0.227383	−0.106720	−0.025170	+0.115924	−0.139818
$N_f = 2048$	+0.169981	−0.270233	+0.227452	−0.106773	−0.025163	+0.115958	−0.139868
d	0.50	0.75	1.00	1.25	1.50	1.75	2.00
Exact	+0.330431	−0.109083	−0.075343	+0.173252	−0.175069	+0.104807	−0.005951
Imaginary part of the exact and approximate component of the solution							
$N_f = 32$	+0.312221	−0.092931	−0.079069	+0.165209	−0.161895	+0.094008	−0.002265
$N_f = 128$	+0.329914	−0.108584	−0.075409	+0.172984	−0.174715	+0.104579	−0.005944
$N_f = 512$	+0.330213	−0.108855	−0.075395	+0.173138	−0.174892	+0.104672	−0.005919
$N_f = 2048$	+0.330277	−0.108940	−0.075372	+0.173175	−0.174950	+0.104714	−0.005925

Table 2
Exact and approximate solution for $\mathbb{D}_1 - \mathbb{P}_2$ scheme with $k = \frac{5\pi}{4}$.

d	0.50	0.75	1.00	1.25	1.50	1.75	2.00
Exact	+0.169890	−0.270311	+0.227596	−0.106882	−0.025142	+0.116019	−0.13996
Real part of the exact and approximate component of the solution							
$N_f = 8$	+0.169967	−0.267973	+0.225731	−0.107342	−0.021637	+0.113502	−0.139261
$N_f = 32$	+0.169766	−0.270031	+0.227356	−0.106852	−0.025016	+0.115847	−0.139838
$N_f = 128$	+0.169770	−0.270222	+0.227564	−0.106903	−0.025089	+0.115963	−0.139932
$N_f = 512$	+0.169829	−0.270267	+0.227581	−0.106894	−0.025115	+0.115991	−0.139951
d	0.50	0.75	1.00	1.25	1.50	1.75	2.00
Exact	+0.330431	−0.109083	−0.075343	+0.173252	−0.175069	+0.104807	−0.005951
Imaginary part of the exact and approximate component of the solution							
$N_f = 8$	+0.328791	−0.108778	−0.073275	+0.169516	−0.174791	+0.106602	−0.008643
$N_f = 32$	+0.330182	−0.109046	−0.075170	+0.173000	−0.174903	+0.104775	−0.006026
$N_f = 128$	+0.330402	−0.109130	−0.075266	+0.173183	−0.175035	+0.104813	−0.005988
$N_f = 512$	+0.330419	−0.109108	−0.075304	+0.173218	−0.175053	+0.104812	−0.005970

$$\sigma(\vec{\omega}) = \frac{1}{4\pi}\left|\vec{\omega} \wedge (\vec{a}(\vec{\omega}) \wedge \vec{\omega})\right|^2 = \frac{1}{4\pi}\left|\vec{a}(\vec{\omega}) - (\vec{a}(\vec{\omega}) \cdot \vec{\omega})\vec{\omega}\right|^2. \tag{213}$$

The numerical results in this zone are expressed in terms of the bistatic (in decibels)

$$\sigma_{dB}(\vec{\omega}) = 20\log_{10}\left|\vec{\omega} \wedge (\vec{a}(\vec{\omega}) \wedge \vec{\omega})\right|. \tag{214}$$

This quantity is called monostatic when $\vec{\omega} = -\vec{\omega}_0$.

It is to note that the discretization of (70)–(73) requires the handling of singular and quasi-singular integrals with different order of singularity. The adopted techniques of integration, based on transformations of the domains of integration and

Gaussian quadrature formulas, are presented and developed in [24,26,27,19,30]. For all our examples we take as obstacle the unit sphere (See Fig. 1 for its discretization and refinement).

Example 1. In this first example the field $\overrightarrow{c}$ is constructed using the usual spherical coordinates $r,\ \phi, \theta$ so that the solution of the boundary value problem is known. The field $\overrightarrow{c}$ is given, for $x \in \Gamma$ by

$$\overrightarrow{c}(x) = \frac{1}{r}\left[2h_1(r)cos\theta\overrightarrow{u_r} - \frac{\partial}{\partial r}(rh_1(r))sin\theta\overrightarrow{u_\theta}\right] \tag{215}$$

where $\overrightarrow{u_r},\ \overrightarrow{u_\phi},\ \overrightarrow{u_\theta}$ are the unit vectors of the associated mobile system of the spherical coordinates and $h_1(r)$ is a Bessel spherical function of order 1 in $]0,1[\cup]1,+\infty[$ expressed, using the first kind Hankel function of order 1 $h_1^{(1)}$, by

Table 3
Relative error of the potential for $\mathbb{D}_0 - \mathbb{P}_1$ scheme with $k = \frac{5\pi}{4}$.

$d \setminus h$	$N_f = 32$ 1.000000	*ORD* 1.73	$N_f = 128$ 0.57735	*ORD* 1.91	$N_f = 512$ 0.301511	*ORD* 1.98	$N_f = 2048$ 0.152499
0.50	5.329340	26.0	0.205186	2.59	0.079110	1.64	0.048125
0.75	6.334447	34.1	0.186029	2.20	0.084663	1.52	0.055807
1.00	6.955056	37.3	0.186677	2.05	0.091176	1.49	0.061352
1.25	7.376434	38.2	0.193012	1.99	0.096963	1.48	0.065447
1.50	7.672006	38.3	0.200299	1.97	0.101794	1.48	0.068562
1.75	7.888989	38.1	0.207172	1.96	0.105786	1.49	0.070999
2.00	8.054012	37.8	0.213305	1.96	0.109102	1.50	0.072952
Average	7.087183	35.6	0.198811	2.08	0.095513	1.51	0.063321

Table 4
Relative error of the potential for $\mathbb{D}_1 - \mathbb{P}_2$ scheme with $k = \frac{5\pi}{4}$.

$d \setminus h$	$N_f = 8$ 1.414214	*ORD* 1.41	$N_f = 32$ 1.000000	*ORD* 1.73	$N_f = 128$ 0.57735	*ORD* 1.91	$N_f = 512$ 0.301511
0.50	0.441833	5.91	0.074750	2.24	0.033402	1.99	0.016783
0.75	0.808632	8.35	0.096847	2.81	0.034408	1.99	0.017298
1.00	1.161881	9.41	0.123414	3.54	0.034843	1.98	0.017610
1.25	1.849163	14.8	0.124864	3.54	0.035271	1.98	0.017817
1.50	1.987849	16.9	0.117700	3.30	0.035658	1.99	0.017963
1.75	1.977097	17.7	0.111631	3.10	0.035993	1.99	0.018071
2.00	1.986308	18.6	0.106611	2.94	0.036277	2.00	0.018154
Average	1.458966	13.5	0.107974	3.07	0.035122	1.99	0.017671

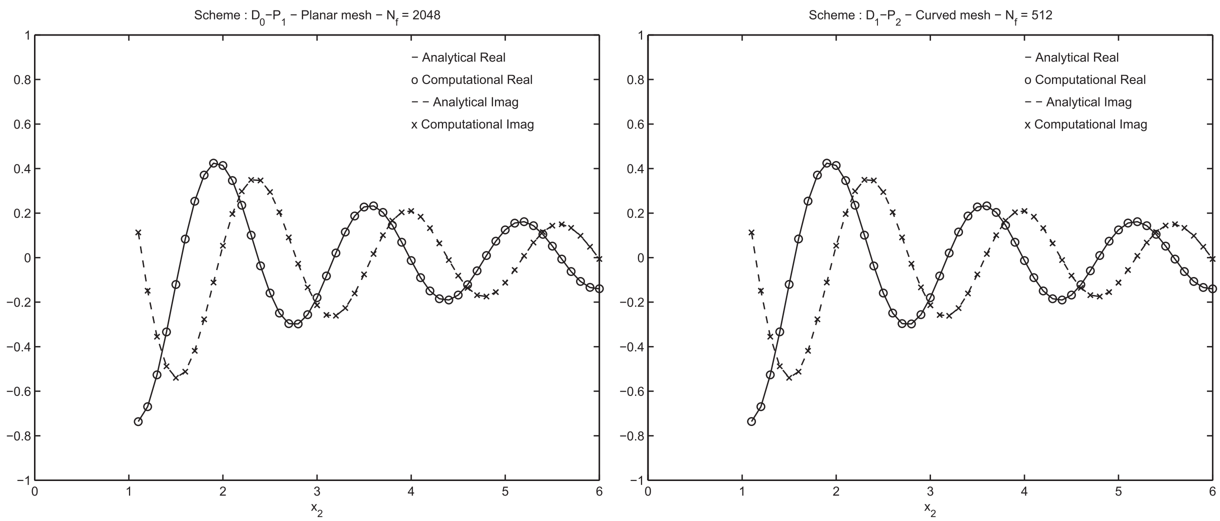

Fig. 2. Real and imaginary parts of the diffracted field for $k = \frac{5\pi}{4}$.

$$h_1(r) = \beta h_1^{(1)} = \beta\left(1 + \frac{i}{kr}\right)\frac{e^{ikr}}{kr}, \quad \text{for } 1 < r < +\infty. \tag{216}$$

The constant $\beta \in \mathbb{C}$ is calculated using the spherical Bessel function of order 1

$$j_1(r) = \frac{sinr}{r^2} - \frac{cosr}{r} \tag{217}$$

such that

$$\beta = \left[j_1(kr) + \frac{\partial}{\partial r} j_1(kr)\right]_{r=1}. \tag{218}$$

Table 5
Exact and approximate bistatic for $\mathbb{D}_0 - \mathbb{P}_1$ scheme with $k = \frac{5\pi}{4}$.

θ	10	20	30	40	50	60	70	80
Exact	−10.68680	−4.799170	−1.500803	0.681146	2.204876	3.270409	3.979513	4.386826
$N_f = 32$	−8.087443	−2.812321	−0.369680	0.942391	1.807793	2.524933	3.129552	3.540887
$N_f = 128$	−10.59357	−4.714888	−1.435902	0.714787	2.202734	3.242507	3.945147	4.359301
$N_f = 512$	−10.68319	−4.791095	−1.489226	0.692046	2.210401	3.268478	3.971500	4.375652
$N_f = 2048$	−10.68872	−4.797982	−1.496463	0.686816	2.209160	3.271145	3.976084	4.380222

Table 6
Exact and approximate bistatic for $\mathbb{D}_1 - \mathbb{P}_2$ scheme with $k = \frac{5\pi}{4}$.

θ	10	20	30	40	50	60	70	80
Exact	−10.68680	−4.799170	−1.500803	0.681146	2.204876	3.270409	3.979513	4.386826
$N_f = 8$	−13.12176	−6.255905	−2.050423	0.647137	2.315416	3.340546	3.990312	4.375836
$N_f = 32$	−10.76015	−4.851318	−1.528907	0.670717	2.202517	3.269003	3.975997	4.380997
$N_f = 128$	−10.68778	−4.800150	−1.501810	0.680024	2.203526	3.268770	3.977596	4.384707
$N_f = 512$	−10.68690	−4.799381	−1.501180	0.680585	2.204145	3.269551	3.978578	4.385857

Table 7
Relative error of the bistatic for $\mathbb{D}_0 - \mathbb{P}_1$ scheme with $k = \frac{5\pi}{4}$.

$d \setminus h$	$N_f = 32$	*ORD*	$N_f = 128$	*ORD*	$N_f = 512$	*ORD*	$N_f = 2048$
	1.000000	1.73	0.57735	1.91	0.301511	1.98	0.152499
10	24.32305	27.9	0.872344	25.9	0.033742	1.88	0.017924
20	41.39985	23.6	1.756175	10.4	0.168265	6.80	0.024760
30	75.36783	17.4	4.324437	5.61	0.771416	2.67	0.289204
40	38.35362	7.77	4.938842	3.09	1.600164	1.92	0.832422
50	18.00931	185	0.097117	0.39	0.250572	1.92	0.194310
60	22.79459	26.7	0.853169	14.5	0.059031	2.26	0.022504
70	21.35840	24.7	0.863571	4.29	0.201353	2.34	0.086165
80	19.28361	30.7	0.627439	2.46	0.254718	1.69	0.150524
Average	32.61128	18.2	1.791637	4.29	0.417408	2.06	0.202227

Table 8
Relative error of the bistatic for $\mathbb{D}_1 - \mathbb{P}_2$ scheme with $k = \frac{5\pi}{4}$.

$\theta \setminus h$	$N_f = 8$	*ORD*	$N_f = 32$	*ORD*	$N_f = 128$	*ORD*	$N_f = 512$
	1.414214	1.41	1.000000	1.73	0.57735	1.91	0.301511
10	22.78474	33.2	0.686345	74.9	0.009160	10.1	0.000903
20	30.35390	27.9	1.086597	53.2	0.020414	4.64	0.004399
30	36.62172	19.6	1.872533	27.9	0.067072	2.67	0.025113
40	4.992888	3.26	1.531164	9.30	0.164721	2.00	0.082452
50	5.013451	46.9	0.106996	1.74	0.061217	1.85	0.033146
60	2.144583	49.9	0.042998	0.86	0.050127	1.91	0.026238
70	0.271375	3.07	0.088358	1.83	0.048156	2.05	0.023481
80	0.250506	1.89	0.132876	2.75	0.048302	2.19	0.022089
Average	12.80414	14.5	0.693483	11.8	0.058646	2.15	0.027227

The analytical solution $\overrightarrow{E}^{sc}(x)$ of the boundary value problem is given by the same formula of $\overrightarrow{c}(x)$ but for $x \in \Omega^e$. The field is calculated along the axis x_2 which it is directed along the axis x_3. The distance to the sphere is expressed in multiples of $\frac{D^2}{2\pi}k$, D being the diameter of the sphere. The Fresnel zone is therefore defined as the set of points having the distance $d\frac{D^2}{2\pi}k$ from the obstacle, with $0.5 \leqslant d \leqslant 2$ [44]. It is to note that in all our numerical results linear finite elements are taken for the surface approximation for the $\mathbb{D}_0 - \mathbb{P}_1$ scheme and quadratic finite elements are taken for the surface approximation for the $\mathbb{D}_1 - \mathbb{P}_2$ scheme. The Tables 1 and 2 give an idea on the obtained results for the solution in Fresnel zone.

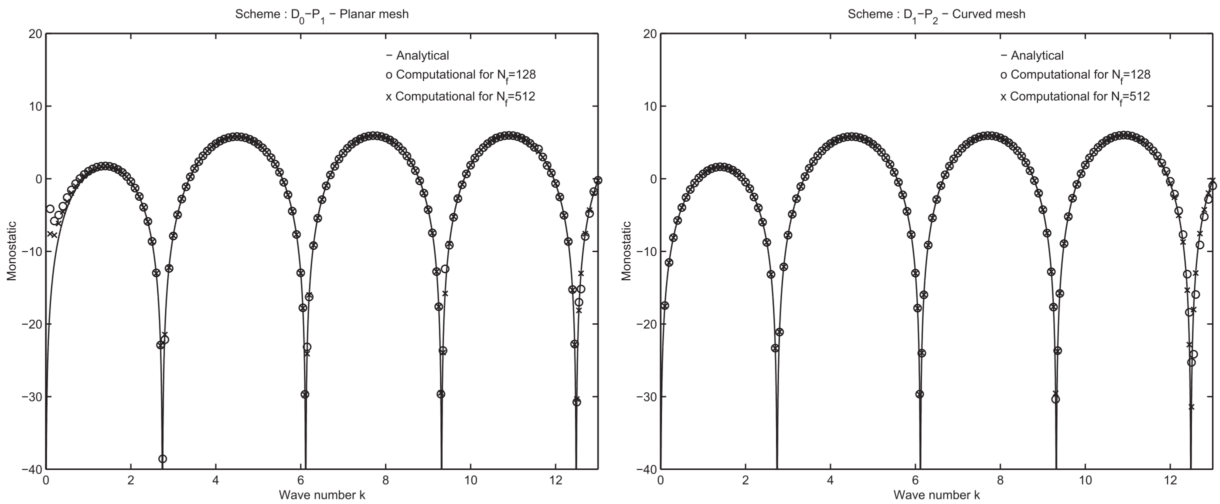

Fig. 3. The monostatic in terms of k.

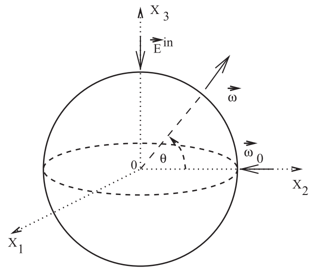

Fig. 4. The obstacle.

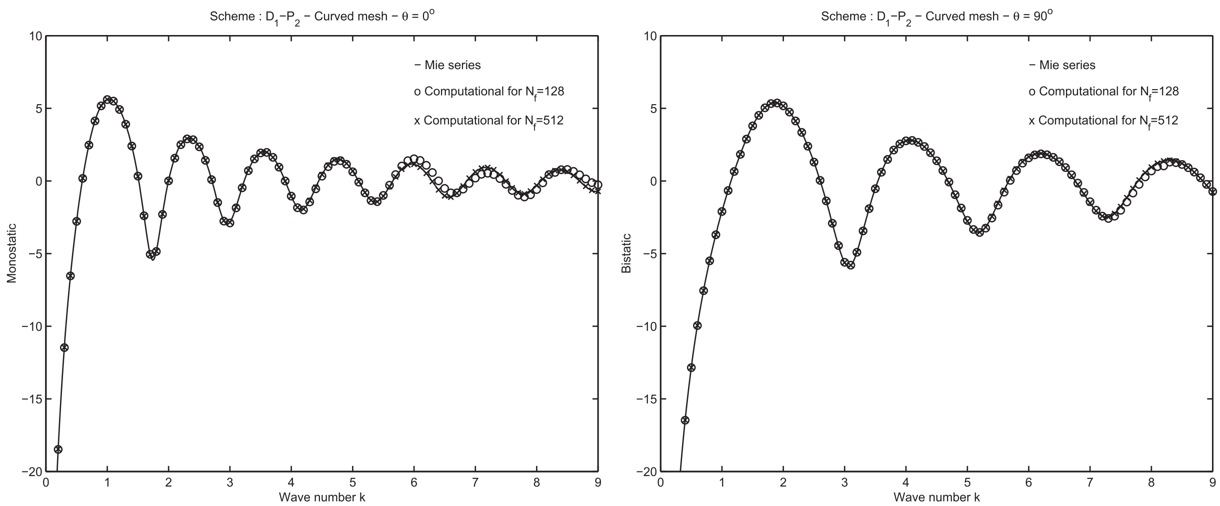

Fig. 5. The monostatic in terms of k.

For the $\mathbb{D}_0 - \mathbb{P}_1$ scheme the Table 3 presents the relative error $RE_{N_f}(x) = 100\frac{|\overrightarrow{E}^{sc}(x) - \overrightarrow{E}_h^{sc}(x)|}{|\overrightarrow{E}^{sc}(x)|}$ of the diffracted field in Fresnel zone for many meshes and for several values of d. This table also gives us an idea on the convergence rate $ORD = \frac{RE_{N_f}}{RE_{4N_f}}$ of our numerical schemes. The convergence rate obtained by this table, notably for h sufficiently small, approves the theoretical one given by Theorem 6.3 for $l = 1$ which is $o(h^{\frac{1}{2}})$. The Table 4 presents relative error of the diffracted field in Fresnel zone for the $\mathbb{D}_1 - \mathbb{P}_2$ scheme. Its clear that the convergence rate is in the same order of theoretical one given by Theorem 5.15 which is $o(h)$. We see will that an improvement of precision is due to the approximation of the surface by quadratic finite elements and to the order of the interpolating polynomial for the approximation of the potentials. The super convergence phenomenon for h no small enough is caused by the approximation of $\overrightarrow{c}$ specially when the surface is approximated by linear finite elements.

The Fig. 2 shows graphical representations of the real and imaginary parts of the diffracted field in Fresnel zone for the $\mathbb{D}_0 - \mathbb{P}_1$ and $\mathbb{D}_1 - \mathbb{P}_2$ schemes.

The bistatic in Fraunhofer zone is calculated by taking $\overrightarrow{\omega}$ in an angular sector θ with respect to the radiation ($\theta = (\overrightarrow{k}, \overrightarrow{\omega})$) in any direction of the plane $x_3 = 0$ such that $0 < \theta < \pi$. Its exact value is given by $20\log_{10}|2\beta sin\theta|$. Tables 5 and 6 give some results for the calculation of the bistatic and Tables 7 and 8 shows the convergence rate of the bistatic for several values of θ and this for the $\mathbb{D}_0 - \mathbb{P}_1$ and $\mathbb{D}_1 - \mathbb{P}_2$ schemes.

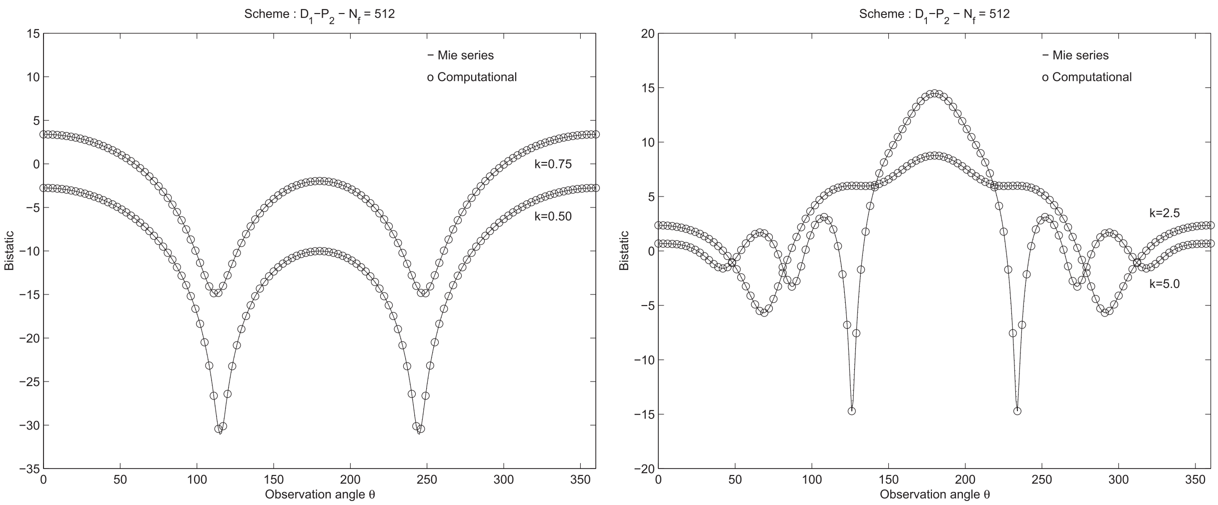

Fig. 6. Bistatic in terms of θ.

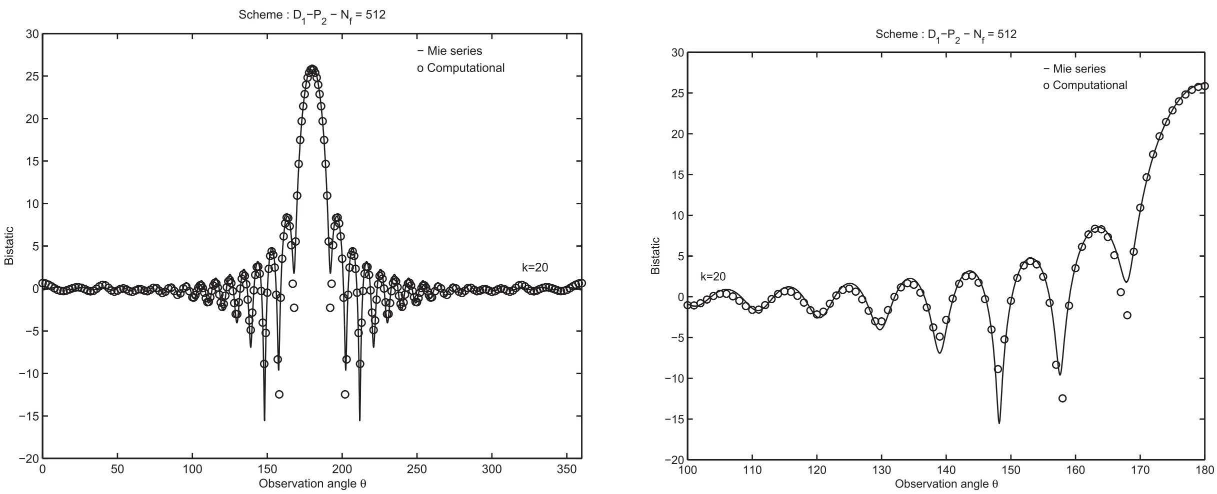

Fig. 7. Bistatic in terms of θ.

The obtained results affirm us the advantage of the $\mathbb{D}_1 - \mathbb{P}_2$ scheme with respect to the $\mathbb{D}_0 - \mathbb{P}_1$ scheme noting that the necessary calculation time for the $\mathbb{D}_0 - \mathbb{P}_1$ scheme to have the same precision as the $\mathbb{D}_1 - \mathbb{P}_2$ scheme is comparatively to much elevated. Similarly for the dimension of the corresponding linear system to solve.

The Fig. 3 gives us graphical representation of the monostatic in terms of the wave number k for the two schemes. We look therefore at what comes back in the direction of the incidence.

Example 2. For the incident wave, we consider the following choice for the boundary condition $\overrightarrow{c}$:

$$\overrightarrow{c}(x) = \overrightarrow{\omega}_0 \wedge (\overrightarrow{\omega}_0 \wedge \overrightarrow{v}) e^{ik(\overrightarrow{\omega}_0 \cdot \overrightarrow{r}(x))}, \tag{219}$$

where $\overrightarrow{v} = (0, 0, 1)$ is the incident wave polarization. As obstacle we take the unit sphere presented in the Fig. 4. The unit vector $\overrightarrow{\omega}$, representing the direction of the observation of the diffracted wave, is taken in the plane (x_2Ox_3) in an angular sector θ with respect to the axis x_2 where the incidence direction is affected. The exact value of the bistatic is given by the Mie series [21].

For the $\mathbb{D}_1 - \mathbb{P}_2$ scheme, we represent in the Fig. 5 the monostatic for $\theta = 0^o$ and the bistatic for $\theta = 90^o$ in terms of the wave number k. On the other hand, the Figs. 6 and 7 shows, for some values of k, the bistatic in terms of θ. It means that we observe the diffraction in all directions around the obstacle in the plane (x_2Ox_3). These figures compare the accuracy of the computed RCS with the analytical Mie-series solution.

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