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ON THE APPLICATION OF CERTAIN INTERVAL ITERATION METHODS TO SOLVING A SYSTEM OF NONLINEAR ALGEBRAIC EQUATIONS

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We consider the problem of finding real solutions of a system of nonlinear algebraic equations using interval analysis. Several versions of Newton and Runge interval iteration methods are presented. The computational aspects of their application are explained.

Many problems in science and technology lead to the solution of nonlinear algebraic systems of equations. For example, truncation methods for the solution of partial differential equations are equivalent to solving special nonlinear algebraic systems with many unknowns [2]. However, the roots of even algebraic equations of degree exceeding four cannot be obtained in the form of radicals. Thus there is enormous value in iteration methods for the solution of such systems.

Classical iteration methods for the solution of nonlinear systems of equations have a number of significant shortcomings, which frequently complicate their use. A "good" initial approximation is necessary for their realization. They are only a way of increasing the accuracy of a unique solution of a system; they do not themselves yield a solution, and they converge locally to a true solution (under rather restrictive conditions). In addition, there must be only one root of the system in a neighborhood of the initial approximation; if there is a multiple root, then in addition one must know the order of the multiplicity of this unknown root. Each real numerical process is accompanied by errors associated with the method, the initial data, as well as round-off (approximation) errors. Jointly all these errors may completely destroy the convergence properties of the method (in particular, when there is instability), and even when a limit exists there is no certainty that one has found a genuine solution. Having solved the approximation problem, it is necessary to find the distance between the obtained solution and the unknown solution of the original problem.

Interval iteration methods, in particular, those considered here, are free of all these shortcomings.

The methods considered here are a generalization within the framework of interval analysis of known iteration methods of Newton and Runge type [1,3] for the solution of functional equations. They possess a number of advantages:

1) every modification (and we examine three of them) of the method finds all real roots of a given system of algebraic equations, including multiple ones;
2) there are no restrictions on the use of this method (other than natural ones);
3) the existence and order of multiplicity of multiple roots need not be known;
4) the methods are globally convergent;
5) the methods take into account the effect of all types of errors and are stable.

These advantages determine the efficiency and effectiveness of the use of interval iteration methods. Computations in terms of intervals are essentially simple and are widely applicable. However, the number of simple computations can become rather large; this is not an essential shortcoming given the availability of high-speed computers.

RESULTS OF INTERVAL ANALYSIS

Our notation is standard, and was used in [4–6]: $x^+ = \sup(x, 0)$; $x^- = \sup(-x, 0) \in R_+{}^n$; $x = x^+ - x^-$; $|x| = \sup(x, -x) = x^+ + x^- \in R_+{}^n$; 0 is the null element in $R_+{}^n$.

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If $\underline{x} \in R$ $(\underline{x} \leq x \leq \bar{x})$, then the set $X = \{x \in R, \underline{x} \leq x \leq \bar{x}\}$ is an interval; for $x = \underline{x} = \bar{x}$, X is a degenerate interval. We denote by $\mathrm{mid}\, X = (1/2)(\bar{x} + \underline{x}) \in R^n$ the arithmetic mean of a point; $\omega(X) = \bar{x} - \underline{x} \in R_+{}^n$ is the width of the interval X.

In the sequel we shall also use $\tilde{x} = \mathrm{mid}\, X$. We note that the definition of the width of an interval differs from the common notation for $n > 1$, so we frequently use the notation $\omega(X) = \|\bar{x} - \underline{x}\|$. Let H(R) denote the set of intervals on R. Analogously for matrices $A = \{a \in R^{m\times n} \mid \underline{a} \leq a \leq \bar{a}\} \in H(R^{m\times n})$ is the set of all interval matrices on $R^{m\times n}$.

Let G denote a set of intervals, $G \in H(R)$. Elementary arithmetic operations on intervals are defined as follows: $I*J = \{x*y \mid x \in I, y \in J\}$ where $I, J \in G$, $* \in \{+,-,\times,/\}$.

For computing width, arithmetic mean, and absolute value we apply the following rules:

Let $A, B \in H(R^n)$, $c \in R^n$. Then

1. $\mathrm{mid}(A \pm B) = \mathrm{mid}\, A \pm \mathrm{mid}\, B$.
2. $\mathrm{mid}(A - \tilde{a}) = 0$.
3. $\mathrm{mid}(cA) = c\, \mathrm{mid}\, A$, and $\mathrm{mid}(Ac) = (\mathrm{mid}\, A)c$.
4. $\omega(A) = 0 \Leftrightarrow A = [a, a]$.
5. $(A \pm B) = \omega(A) + \omega(B)$.
6. $\omega(c,A) = |c|\, \omega(A)$ and $\omega(Ac) = \omega(A)\, |c|$.
7. $|\mathrm{mid}\, A| \leq 1/2\omega(A)$.
8. $|A| = 1/2\omega(A) + |\mathrm{mid}\, A|$.

If $\mathrm{mid}\, B = 0$, then additionally

9. $A\cdot B = |A|\cdot B$, and $B\cdot A = B\cdot |A|$.
10. $\mathrm{mid}(A\cdot B) = 0$ and $\mathrm{mid}(B\cdot A) = 0$, as follows from rules 3 and 9.
11. $\omega(A\cdot B) = |A|\, \omega(B)$, and $\omega(B\cdot A) = \omega(B)\, |A|$, as follows from rules 6 and 9.
12. $A \leq B \Leftrightarrow |\mathrm{mid}\, B - \mathrm{mid}\, A| \leq 1/2(\omega(B) - \omega(A))$.

The interval functions $F{:}H(B) \to H(R^n)$, when $B \subset R^n$, are called the interval closure of f if

$$f(X) \subset F(X), \ \forall X \in H(B), \tag{1}$$

and the interval continuation of f if in addition to (1) we have $F([\underline{x}, \bar{x}]) = f(x)$, $x \in B$.

Let f(x) and $\bar{f}$ be functions defined on $B \subset R^n$, and suppose it is known that $f(x) \leq -\bar{f}$ for all $x \in B'$; then $F_s(x) = [\underline{f}(x), \bar{f}(x)]$, $s \in N$ is called the functional penalty on B. The domain of dependence of the function f on x is denoted $W(f(X)) = \{f(x) \mid x \in X\}$. For example, if $f{:}B \to R$ and $B \subset R$ is a rational expression, then substituting the variable x for X in this expression and representing the arithmetic operations in the form of the corresponding interval operators, we obtain a nondecreasing interval continuation F.

We note that the interval continuation of a function essentially depends on the order of the necessary interval operators. For example, for the function $f(x) = x^3 - 6x^2 + 11x - 6$ we can obtain an interval continuation: $F_1(X) = X^3 - 6x^2 + 11x - 6$; $F_2(X) = ((X - 6)\cdot X + 11)X - 6$; $F_3(X) = (X - 1)(X - 2)(X - 3)$.

Suppose for concreteness that $X = [0, 2]$. Then the domain of the given function $W(f(X)) = [-2/9\sqrt{3}, 2/9\sqrt{3}]$, and the values of the corresponding continuations are $F_1([0, 2]) = [-30, 24]$; $F_2([0, 2]) = [-8, 16]$; $F_3([0, 2]) = [-6, 6]$; $F_1 \supset F_2 \supset F_3 \supset W(f(X))$.

PREREQUISITES FOR THE CONSTRUCTION AND ANALYSIS OF INTERVAL ITERATION METHODS WITH HIGHER ORDERS OF CONVERGENCE

The idea behind the construction of interval iteration methods for the solution of nonlinear algebraic equations consists of the solution of the corresponding linear system of equations, the coefficients of which are not determined, but belong to known intervals. Thus the basic use of interval analysis in the construction of such methods is to obtain analogous theorems about the "mean" value.

Let a representation $f{:}D \subset R^n \to R$ be continuously differentiable at each point of a convex set $D_0 \subset D$. Then for any points $x_1, x_2 \in D_0$ we have the identity $f(x_2) = f(x_1) + f(\xi_1)(x_2 - x_1)$, where $\xi_1 = x_1 + \Theta_1(x_2 - x_1)$, $\Theta_1 \in (0,1)$. If an arbitrary representation f also is a continuously differentiable representation, then analogously we have

$$f'(\xi_1) = f'(x_1) + f''(\xi_2)(\xi_1 - x_1), \tag{2}$$

where $\xi_2 = x_1 + \Theta_2(\xi_1 - x_1)$, $\Theta_2 \in (0, 1)$. Then

$$f(x_2) = f(x_1) + f'(x_1)(x_2 - x_1) + \Theta_1 f''(\xi_2)(x_2 - x_1)^2.$$

Thus in the case of a smooth representation

$$f(x_2) = f(x_1) + f'(x_1)(x_2 - x_1) + \Theta_1 f''(x_1)(x_2 - x_1)^2 + \cdots + \\ + \Theta_1^{n-1}\Theta_2^{n-2} \ldots \Theta_{n-1}^1 f^{(n)}(\xi_n)(x_2 - x_1)^n, \tag{3}$$

where $\Theta_i \in (0, 1)$ $(i = 1, 2, \ldots, n - 1)$, $\xi_n = x_1 + \Theta_n(\xi_{n-1} - x_1)$, and where, as shown in [3], $\Theta_i \to 1/2$ as $x_1 \to x_2$.

It is important to note that the theorem of the mean does not hold for representations $f: R^n \to R^m$, $m > 1$. However, we have an analogous formula (2)

$$F(x_2) = F(x_1) + J(x_1, x_2, \Theta_1)(x_2 - x_1), \tag{4}$$

where

$$J(x_1, x_2, \theta) = \begin{pmatrix} f_1'(x_1 + \Theta_1^{(1)}(x_2 - x_1)) \\ \cdots\cdots\cdots\cdots \\ f_m'(x_1 + \Theta_1^{m}(x_2 - x_1)) \end{pmatrix};$$

$$F = (f_1, \ldots, f_m)^T, \ x_1, \ x_2 \in D_0 \subset D \subset R^n, \ \Theta_1 = (\theta_1^{(1)}, \ldots, \Theta_1^{(m)});$$

$$f_i'(x_1 + \Theta_1^{(i)}(x_2 - x_1)) = ((\partial f_i/\partial x^{(1)}), \ldots, (\partial f_i/\partial x^{(n)}))_{x = x_1 + \Theta_1^{(i)}(x_2 - x_1)}.$$

The matrix $J(x_1, x_2, \Theta_1) \subset L(R^n, R^m)$ cannot be considered an arbitrary Gateaux (Frechet) operator defined at an interstitial point since, generally speaking, all the $\Theta_1^{(i)}$ are different. However, we do have the relation

$$F(x_2) = F(x_1) + F'(x_1)(x_2 - x_1) + H(x_2, x_1, \Theta_2)\,\Theta_1(x_2 - x_1)^2, \tag{5}$$

where

$$[H(x_2, x_1, \Theta_2)hk]^T = (k^T H_1(x_2, x_1, \Theta_2)h, \ldots, k^T H_m(x_2, x_1\Theta_2)h;$$

$$H_i(x_2, x_1, \Theta_2) = f_i''(x_1 + \Theta_2^{(i)}\Theta_1^T(x_2 - x_1));$$

Suppose that we must solve the system of nonlinear algebraic equations

$$f(x) = 0, \tag{6}$$

where $f = (f_1, \ldots, f_n)^T$, $x = (x_1, \ldots, x_n)$. Based on (4), we have

$$f(x) = f(\bar{x}) + J(\xi_1)(x - \bar{x}). \tag{7}$$

We denote by X an interval vector containing x and $\bar{x}$. Then also, $\xi_1 \in X$. Substituting $J(\xi_1)$ for $J(X)$ in (7), we obtain the equation

$$f(\bar{x}) + J(X)(z - \bar{x}) = 0. \tag{8}$$

Obviously the solution of Eq. (6) belongs to the set Z, the elements $z \in Z$ of which satisfy Eq. (8). Thus if $x^* \in X$, then also $x^* \in Z$, if x^* is a solution of Eq. (6).

Using an analogue of Taylor's theorem in terms of a generalized theorem of the mean, for instance in the form (5), we obtain for a definition of the set Z matrix equations of higher order. It is easy to show that even for $n = 1$ the set of solutions obtained in this case under reasonably weak natural restrictions is essentially the set Z already obtained by solving

Eq. (8); this accelerates the search for solutions of (6). However the solution of such matrix equations for $n > 1$ is more difficult. Thus in order to construct interval iteration processes with higher orders of convergence we use linear approximations of such equations.

SEVERAL VERSIONS OF NEWTON'S METHOD

Solving Eq. (8), we obtain various versions of Newton's interval method. They differ in their methodology and in the construction of J(X). Maximal compression of the sequence of intervals is achieved by using the structure of the matrix J(X), applying the theorem of the mean along individual segments, and applying the obtained intermediate results in a sequence of computations. For example, Newton's interval method, starting from the theorem of the mean in (7) applied to a section connecting the points x and $\bar{x}$, or moving along a rib of an n-dimensional parallelepiped containing the unknown solution, may lead to theorems of the mean for each coordinate. In both cases we obtain successively narrower intervals containing the unknown solution, having initially multiplied (8) by the matrix B – approximately the inverse matrix to J_c – the center of the interval matrix J(X). In order to bound the round-off errors of the products Bf(x) and BJ(X), we compute in terms of intervals.

In the general case Newton's interval method consists of the computation of sequences of intervals X_i ($i = 0, 1, \ldots$) according to the formula

$$\begin{aligned} Y_{i+1} &= \bar{x}_i - [J(X_i)]^{-1} f(\bar{x}); \\ x_{i+1} &= Y_{i+1} \cap X_i, \quad i = 0, 1, 2, \ldots \end{aligned} \tag{9}$$

If the initial interval X_0 does not contain the unknown solution, then in a finite number of steps $X_{i+1} = \{\varnothing\}$.

The set Z of solutions to Eq. (8) can be bounded by an n-dimensional parallelepiped with sides parallel to the coordinate axes; this significantly simplifies the computations. This parallelepiped is called a Krawczyk box [4] and is defined as follows:

$$K(X) = \bar{x} - Bf(\bar{x}) + (I - BJ(X))(X - \bar{x}), \tag{10}$$

where I is the identity matrix.

If $x^* \in X$, then $x^* \in K(X)$. As a result we obtain a sequence of intervals $X^{(i)}$ ($i = 0, 1, \ldots$), where

$$X^{(i+1)} = X^{(i)} \cap K(X^{(i)}), \tag{11}$$

which has the same properties as the sequence X_i obtained from formula (9). In Eqs. (10) and (11) the intervals for each coordinate are determined from

$$K_i = \bar{x}_i - g_i + \sum_{j=1}^{i-1} R_{ij}(K'_j - \bar{x}_j) + \sum_{j=i}^{n} R_{ij}(X_j - \bar{x}_j), \tag{12}$$

where $g = Bf(\bar{x})$, $R = I - BJ(X)$, $K_j' = K_j \cap X_j$.

We have the following theorem for a sequence of intervals obtained by Newton's interval method in Krawczyk's interpretation (12).

Theorem 1. Let the initial interval $X^{(0)}$ contain a unique solution x^* of system (6). If for some $k = 0, 1, \ldots$, the condition $t_k < 1$ is satisfied, where $t_k = \sum_{j=1}^{n} |R_{ij}^{(k)}|$, then $X^{(k)} \to x^*$.

Proof. If $x^{(k)}$ is a single point, then $t_k = 0$. The condition placed on t_k is satisfied for a sufficiently small value of $\omega(X_k)$. The set obtained by substituting K_j into the right side of Eq. (12) in place of X_j will contain $K(X^{(k)})$, thus $K(X^{(k)}) \subseteq X^{(k)}$.

Since $\bar{x}^{(k)}$ is the mean point of the interval $X^{(k)}$, then

$$X_j^{(k)} - x_j^{(k)} = 1/2\omega(X_j^{(k)})[-1, 1] \subset 1/2\alpha_k[-1, 1], \tag{13}$$

where $\alpha_k = \omega(X^{(k)})$. Taking (11)–(13) into account, we have $K_i^{(k)} \subset \bar{x}_i^{(k)} - g_i^{(k)} + 1/2\alpha_k \sum_{j=1}^{n} |R_{ij}^{(k)}|[-1, 1] \subset \bar{x}_i^{(k)} - g_i^{(k)} +$

$1/2\alpha_k t_k$ |—1, Thus $\omega(K_i^{(k)}) < \alpha_k t_k$. Thus, using the hypothesis of the theorem, we obtain $\omega(K_i^{(k)}) < \alpha_k$, $\omega(K^{(k)}) < \omega(X^{(k)})$. Taking into account Eq. (11), we finally obtain $\omega(X^{(k+1)}) < \omega(X^{(k)})$.

Consequently, when algorithm (12) is realized we obtain a sequence of closed intervals $X^{(0)} \supset X^{(1)} \supset X^{(2)} \supset X^{(3)} \supset \ldots$.

It follows from the construction of the algorithm (12) that $x^* \in X^{(k)}$ ($k = 0, 1, \ldots$) if $x^* \in X^{(0)}$. Then $X^{(k)} \to x^*$, which is what we had to show.

The box K(X) computed from (12) is not the smallest one containing a solution to system (6). The result is significantly improved if the calculations are performed according to the scheme

$$Y = \tilde{x} - D^{-1}[g + L(X' - x) + U(X - x)]; \quad X' = Y \cap X, \tag{14}$$

where $BJ(X) = P = L + D + U$; L, D, U are, respectively, lower triangular, diagonal, and upper triangular matrices.

We have the following theorem concerning the sequences of intervals obtained from formula (14).

Theorem 2. Let the initial interval X(0) contain a unique solution x^* of system (6). If for some $k = 0, 1, \ldots$ the conditions $\delta_k < 2/\sqrt{3} - 1$, $\rho_k < (1 - \delta)/2$ are satisfied, where $\delta_k = \max | D_{ii}^{(k)} - 1 |$, $\rho_k = \sum_{\substack{j=1 \\ j \neq i}}^{n} | P_{ij}^{(k)} |$ $(i = 1, 2, \ldots, n)$ then $X^{(k)} \to x^*$.

The proof of this theorem is given in [6].

SOLUTION OF A SYSTEM OF NONLINEAR ALGEBRAIC EQUATIONS BY RUNGE'S METHOD

In order to construct interval iteration methods possessing higher order of convergence for the solution of nonlinear algebraic systems of equations we use a Runge–Kutta construction for the solution of the Cauchy problem for ordinary differential equations, by analogy with [3].

We define the function

$$\bar{f}(x) = f(x_n) + \{\alpha_1 f'(x_n) + \alpha_2 f'(x_n + \beta(x - x_n))\}(x - x_n),$$

where α_1, α_2, and β are constants which are chosen so that the solution x^* of Eq. (6) will also satisfy the equation

$$\bar{f}(x) = 0. \tag{15}$$

Using Taylor's theorem,

$$f(x^*) = f(x_n) + f'(x_n)(x^* - x_n) + 1/2!\, f''(x_n + \Theta(x^* - x_n))(x^* - x_n)^2,$$

where $\Theta \in (0, 1)$.

Applying the generalized theorem of the mean in the form (3)–(5), we have

$$f(x^*) = f(x_n) + (\alpha_1 + \alpha_2) f'(x_n)(x - x_n) + \alpha_2 \beta f''(\xi_n) \cdot (x - x_n)^2, \tag{16}$$

where $\xi_n = x_n + \Theta_1 \beta(x^* - x_n)$, $\Theta_1 \in (0, 1)$. Thus if the relations

$$\alpha_1 + \alpha_2 = 1, \quad \alpha_2 \beta = 1/2, \quad \Theta_1 \beta = \Theta, \tag{17}$$

are satisfied, then Eqs. (6) and (15) both have the solution x^*. Since $\Theta_1 \to 1/2$ and $\Theta \to 1/3$ as $x_n \to x^*$, then $\beta \to 2/3$. Analogously

$$f(x_n) + f'(x_n + \Theta_2(x^* - x_n))(x^* - x_n) = 0, \tag{18}$$

where $\Theta_2 \in (0, 1)$.

Let X be an interval vector containing x^* and x_n. Then $x_n + \Theta_2(x^* - x_n) \in X$ as a consequence of (16)–(18), and the set Z whose points are z-solutions of the equation

$$f(x_k') + (1/4f'(x_k') + 3/4f'(\bar{\Omega}_k))(z_k - x_k') = 0, \tag{19}$$

where

$$\bar{\Omega}_k = \Omega_k \cap X_k, \quad x_k' = \underline{x}_k + 1/3(\bar{x}_k + \underline{x}_k);$$

$$\Omega_k = \{y_k : 2/3f(\tilde{x}_k) + f'(X_k)(y_k - \tilde{x}_k) = 0\}.$$

$$\tilde{x}_k = \mathrm{mid}\, X_k, \quad k = 0, 1, \ldots,$$

contains the solution x^*.

Thus the Runge interval method consists in the computation of a sequence of intervals Z_i ($i = 0, 1, \ldots$) according to the formulas

$$Z_{i+1} = x_i' - [1/4f'(x_k') + 3/4f'(\bar{\Omega}_k)]^{-1} f(x_k');$$
$$X_{i+1} = Z_{i+1} \cap X_i,$$

where the values are as in (19).

In order to simplify the computations one can, in addition to determining the intervals, compute a sequence of analogues of the Krawczyk boxes using the formulas

$$K(X_k, b) = x_k' - C_k f(x_k') + (I - C_k J(X_k'))(X_k' - x_k'), \tag{20}$$

where $x_k' = \underline{x}_k + 1/3(\bar{x} - \underline{x}_k)$, $b = 2/3$; $x_k' = K_1(X_k, b) \cap X_k$, $K_1 \times (X_k, b) = \tilde{x}_k - 2/3B_k f(\tilde{x}_k) + (I - 2/3B_k f'(X_k))(X_k - \tilde{x}_k)$; B_k is the approximation of the inversion of the center $f'(X_k)$; C_k is the approximation of the inversion of $[1/4f'(x_k') + 3/4f'(x_k' - 2/3B_k f'(x_k'))]$; $j(X_k') = 1/4f'(x_k') + 3/4f'(X_k')$; I is the identity matrix.

It is possible to reduce the volume of computations connected with the realization of the Runge method. The matrix $A = I - CJ(X')$ is easily determined. However, we require $A(X' - x')$, which can be obtained as follows: $A(X' - x') = X' - x' - CJ(X')(X' - x')$, where $J(X')(X' - x')$ is computed initially. This procedure involves multiplication of a vector by a matrix and thus requires fewer interval operations.

Since x_j ($i = 1, 2, \ldots, n$) is the mean point of X_j', then $X_j' - x_j' = 1/2\omega_j[-1, 1]$, where ω_j is the width of the interval X_j. Thus

$$[J(X')(X' - x')]_i = \sum_{j=1}^{n} [J(X')]_{ij}(X' - x')_j =$$
$$= 1/2[-1, 1] \sum_{l=1}^{n} |[J(X')]_{ij}|\omega_j.$$

Multiplying $J(X')(X' - x')$ by C, we find the k^{th} element

$$[CJ(X')(X' - x')]_k = 1/2[-1, 1] \sum_{j=1}^{n} \left| \sum_{i=1}^{n} c_{ki}[J(X')]_{ij} \right| \omega_j.$$

Having already computed $CJ(X')$, we obtain

$$[CJ(X')(X' - x')]_k = 1/2[-1, 1] \sum_{j=1}^{n} \left| \sum_{l=1}^{n} c_{kl}[J(X')]_{lj} \right| \omega_j.$$

Thus in a Runge method the interval operations can be replaced by interval multiplication of the corresponding numbers by the interval $[-1, 1]$, which significantly reduces the volume of computations.

If the initial interval does not contain a solution of Eq. (6), then in a finite number of steps the Runge method yields the empty set.

In connection with the realization of the interval iteration method it is sometimes necessary to divide an interval by an interval containing zero. In this case we apply extended interval arithmetic[6]. If at a certain step in the realization we find that $X_{i+1} = X_i$, then we divide the box into segments in each of which, as in the new initial interval, the corresponding interval method is performed.

TABLE 1

Iteration	Boundaries of the input intervals	Boundaries of the computed intervals
1	0,100000E+01—0,300000E+01	—0,438466E+00—0,910520E+01
	0,100000E+01—0,300000E+01	—0,438465E+00—0,910520E+01
2	0,100000E+01—0,200000E+01	—0,162607E+02—0,109523E+02
	0,100000E+01—0,300000E+01	—0,100000E+08—0,191786E+01
3	0,100000E+01—0,200000E+01	—0,127907E+01—0,149756E+01
	0,100000E+01—0,191786E+01	—0,115791E+01—0,172113E+01
4	0,100000E+01—0,149756E+01	0,643368E+00—0,134183E+01
	0,100000E+01—0,172113E+01	0,620775E+00—0,129412E+01
5	0,100000E+01—0,134183E+01	0,103244E+01—0,115255E+01
	0,100000E+01—0,129412E+01	0,104456E+01—0,114498E+01
6	0,103244E+01—0,115255E+01	0,110319E+01—0,111562E+01
	0,104456E+01—0,114498E+01	0,110545E+01—0,111262E+01
7	0,110319E+01—0,111562E+01	0,110848E+01—0,110855E+01
	0,110545E+01—0,111262E+01	0,110850E+01—0,110852E+01

TABLE 2

System	Initial interval	Number of solutions in one initial box	Quantity of iterations during realization of the method			
			(12)	(20)	(14)	(21)
(22)	[0,6; 2,9]	1	41	30	6	4
	[0,5; 5]	1	51	22	25	17
	[—5; —0,1]	1	50	21	25	16
(23)	[0; 2]	1	27	10	15	6
	[2; 5]	No root	5	3	3	1
(24)	[1; 3]	1	24	18	7	4
	[0,6; 2,9]	1	18	12	5	3
	[—1; 4]	1	94	56	56	43
(25)	[0,6; 2,9]	1	33	24	5	3
	[3; 10]	No root	15	12	10	5

We examine one of the strategies for this sort of division of an interval X. When dividing X in the coordinate direction x_i we represent X in the form of a union $X^{(1)} \cup X^{(2)}$, where $X_j^{(1)} = X_j^{(2)} = X_j$ for all j = i, and the i^{th} component of the interval X is divided in half, that is, $X_i^{(1)} = [\underline{X}_i, \mathrm{mid}(X_i)]$, $X_i^{(2)} = [\mathrm{mid}\,(X_i), \overline{X}_i]$. Then coordinate directions are sampled by cyclic permutations and the left half of the divided interval is determined for $X^{(1)}$, i.e., for the left half we let $X_i^{(1)} = [\underline{X}_i, \mathrm{mid}\, X_i]$. After pn divisions we have $\omega(X) = 2^{-p}\omega(D)$, where D is an arbitrary n-dimensional parallelepiped.

Proceeding from Eq. (20) and using (14) we obtain

$$Y = x' - D_1^{-1}\,[Cf(x') + L_1(X'' - x') + U_1(X' - x')], \tag{21}$$

where $X'' = Y \cap X'$; $CJ(X) = L_1 + D_1 + U_1$; $X' = \tilde{x} - D_2^{-1}\,[2/3Bf \times (\tilde{x}) + L_2(X''' - \tilde{x}) + U_2(X - \tilde{x})$; $Bf'(X) = L_2 + D_2 + U_2$; $X''' = X' \cap X$.

RESULTS OF THE NUMERICAL EXPERIMENTS

The systems

$$\begin{cases} xy - y - 1 = 0, \\ x^2 - y^2 - 1 = 0; \end{cases} \tag{22}$$

$$\begin{cases} 3x^2 + 3/2y^2 + z^2 - 5 = 0, \\ 6xyz - x + 5y + 3z = 0, \\ 5xz - yz - 1 = 0; \end{cases} \tag{23}$$

$$\begin{cases} 0{,}6x - 2 + 0{,}49x\,(x^2 + y^2) = 0, \\ 0{,}6y - 2 + 0{,}49y\,(x^2 + y^2) = 0; \end{cases} \tag{24}$$

$$\begin{cases} 6x^5 - 25{,}2x^3 + 24x - 6y = 0, \\ 12y - 6x = 0 \end{cases} \tag{25}$$

were solved by the Newton interval iteration method in the form (12), (14) and the Runge interval iteration method in the form (20),(21).

The experimental results confirm the theoretical conclusions that the most effective computational method is method (21). The dynamics of the changes in the intermediate intervals when solving system (24) by method (14) and for fixed $\beta = 2/3$ (accuracy $\varepsilon = 10^{-2}$) are given in Table I.

The results of solving systems (22)−(25) by all the above methods, with accuracy $\varepsilon = 10^{-2}$, with various initial intervals, are given in Table II. For each coordinate we took identical initial intervals, indicated in the table.

If the initial intervals are small, then during realization by methods (20), (21) it is convenient to take a fixed value of $\beta = 2/3$; in the general case β must be changed at each step of the iteration. To this end one must conduct three trial iterations (these are discounted in the computation of the total number of iterations) for $\beta = 1$, $2/3$, $1/100$ and we take β to be the terms of the corresponding sequence converging to 2/3.

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