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The Yamabe Problem for Almost Hermitian Manifolds

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ABSTRACT. The conformal class of a Hermitian metric g on a compact almost complex manifold (M^{2m}, J) consists entirely of metrics that are Hermitian with respect to J. For each one of these metrics, we may define a J-twisted version of the Ricci curvature, the J-Ricci curvature, and its corresponding trace, the J-scalar curvature s^J. We ask if the conformal class of g carries a metric with constant s^J, an almost Hermitian version of the usual Yamabe problem posed for the scalar curvature s. We answer our question in the affirmative. In fact, we show that $(2m-1)s^J - s = 2(2m-1)W(\omega, \omega)$, where W is the Weyl tensor and ω is the fundamental form of g. Using techniques developed for the solution of the problem for s, we construct an almost Hermitian Yamabe functional and its corresponding conformal invariant. This invariant is bounded from above by a constant that only depends on the dimension of M, and when it is strictly less than the universal bound, the problem has a solution that minimizes the almost complex Yamabe functional. By the relation above, we see that when $W(\omega, \omega)$ is negative at least one point, or identically zero, our problem has a solution that minimizes the almost Hermitian Yamabe functional, and the universal bound is reached only in the case of the standard 6-sphere $\mathbb{S}^6$ equipped with a suitable almost complex structure. When $W(\omega, \omega)$ is non-negative and not identically zero, we prove that the conformal invariant is strictly less than the universal bound, thus solving the problem for this type of manifolds as well. We discuss some applications.

1. Introduction

The purpose of this article is to study the analog of the Yamabe problem for almost Hermitian manifolds. We are motivated by our desire to tie properties of almost complex structures to properties of suitable Riemannian tensors, that will lead us to the understanding of the first in terms of the second.

The mere existence of almost complex structures on a compact manifold is still a very difficult subject. Other than an elementary condition on the dimension, it is not easy to see that a given manifold does or does not carry almost complex structures, and when it does, whether any such structure is integrable. For example, it has been a long time since we have known that of all Euclidean spheres, only $\mathbb{S}^2$ and $\mathbb{S}^6$ carry almost complex structures, and we do not yet know if the

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latter carries one that is integrable. The road towards a better understanding of almost complex structures is still quite rough.

When given a compact almost complex manifold (M, J), our general hope would be to study geometric tensors whose behavior, to an extent yet to be determined, is characterized by properties of J. As far as this article is concerned, we start with a given Riemannian metric g that is Hermitian relative to J, and observe that its entire conformal class $[g]$ consists of metrics that are J-Hermitian. Consequently, given a generally defined J-related tensor of a Hermitian metric g, we may ask if the conformal class $[g]$ has a representative for which this tensor has a canonical shape or form.

The simplest example of tensors are scalar ones, and canonical could be chosen well to mean constant in that case. Forgetting momentarily about J, a typical scalar tensor of a Riemannian metric g is its scalar curvature s, and we would then be asking if the conformal class $[g]$ admits a representative with constant scalar curvature. This, of course, is the famous Yamabe problem, completely solved in the mid eighties. Here, we attempt to solve its almost complex version.

For, given any almost Hermitian manifold (M, J, g), we may define a J-twisted version of the Ricci curvature of g, and its corresponding trace, the J-scalar curvature s^J. We study the Yamabe question, this time using the scalar s^J instead of s. In other words, we ask if $[g]$ admits a representative with constant J-scalar curvature, and prove here that this is always the case.

Since in general the tensors s and s^J are quite different from each other, our question is well worthy of consideration. The tensors s and s^J differ not just because the definition of the second requires the presence of an almost complex structure, while the definition of the first does not. These tensors exhibit significant differences from each other when the metric under consideration is not one of Kähler type, even in cases where the almost complex structure J is integrable [5].

Our proof that the almost complex version of the Yamabe problem can always be solved begins with the demonstration of an identity that expresses a suitable linear combination of s and s^J in terms of a function determined by the Weyl tensor of the metric and the fundamental form. We then carefully analyze the problem, according to the sign of this function. The apparent simplicity of the identity we obtain must be tempered by the fact that the Weyl tensor is quite difficult to study, so our task requires some work to get it done.

However, this approach makes it possible to use techniques developed in the solution of the original Yamabe problem, and where possible, we have drawn a dictionary between the two problems that makes it easy to see why these techniques are applicable, or applicable with some modifications. In the case where the function of the previous paragraph is non-negative and not identically zero, we obtained our result only after a substantial improvement of pre-existing techniques. We feel this illustrates the essence of the conformal analysis nearby a point carried out by Lee and Parker [8] (and based on work of R. Graham), and broadens the applicability of that type of idea.

In order to make this article self contained, we include a section on almost Hermitian manifolds, where we recall definitions and set up the notation to be used. We then describe mathematically the problem to be solved, and contrast it with the one associated to the scalar curvature. We proceed to show some relations that hold among the tensors we use, and then prove that the analysis of the almost Hermitian Yamabe problem can be reduced to the study of a conformal invariant that depends upon J. Finally, we study this invariant in detail, proving the solvability of the stated problem, and discussing two applications of the result.

2. Almost Hermitian manifolds

An *almost complex structure* J on a smooth manifold M is a smooth field of automorphisms of the tangent bundle TM satisfying the condition $J^2 = -1\!\!1$. Thus, for each $x \in M$ we have a linear map $J_x : T_xM \to T_xM$ that squares to minus the identity operator, and such that the function $x \to J_x$ is smooth. The manifold M, provided with one such structure, is called an *almost complex manifold.* A Riemannian metric g on M is said to be *Hermitian* relative to J iff $g(JX, JY) = g(X, Y)$ for all vector fields X, Y. The pair (J, g) is called an *almost Hermitian structure* on M. The triple (M, J, g) is referred to as an *almost Hermitian manifold.*

The complexified tangent bundle $\mathbb{C} \otimes TM$ decomposes as $T^{1,0}M \oplus T^{0,1}M$, where the summands are the $+i$ and $-i$ eigenspace of J, respectively. This induces a splitting of the whole complexified tensor bundle into types, for example, $\mathbb{C} \otimes \Lambda^r M = \sum_{p+q=r} \Lambda^p(T^{1,0}M)^* \otimes \Lambda^q(T^{0,1}M)^*$. We denote $\Lambda^p(T^{1,0}M)^* \otimes \Lambda^q(T^{0,1}M)^*$ by $\Lambda^{q,p}_{\mathbb{C}}M$ and refer to its elements as forms of type (q, p). The structure J induces an operator on T^*M defined by

$$J\alpha(X) = -\alpha(JX)\,, \quad \alpha \in T^*M, \ X \in TM\,.$$

This map is extended to the whole of $\Lambda^*_{\mathbb{C}}M$, and we have $J\beta = i^{p-q}\beta$ for any $\beta \in \Lambda^{p,q}_{\mathbb{C}}M$.

The vanishing of the Nijenhuis tensor $N(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]$ is equivalent to the fact that J is induced by a *complex structure* [9]. We say then that J is *integrable.* In that case, the exterior derivative $d\beta$ of a form β of type (p, q) decomposes as the sum of forms of type $(p+1, q)$ and $(p, q+1)$, respectively. This decomposition defines two operators, ∂ and $\overline{\partial}$, such that $d = \partial + \overline{\partial}$, $\partial\overline{\partial} + \overline{\partial}\partial = 0$, $\partial^2 = 0$, and $\overline{\partial}^2 = 0$. For a *non-integrable* almost complex structure J, this decomposition of d *does not* hold.

On an almost Hermitian manifold, we can always define a J-twisted version of its *Ricci tensor*, that we shall call the *J-Ricci tensor.*[1] Indeed, the Ricci tensor $r(X, Y)$ of a Riemannian manifold (M, g) is defined as the trace of the map $L \to R(L, X)Y$, where R is the curvature tensor $R(X, Y)Z = (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})Z$, and ∇ is the Levi-Civita connection. If g is Hermitian relative to J, we define

$$r^J(X, Y) = \text{trace } L \to -J(R(L, X)JY)\,. \tag{2.1}$$

Notice that for a complex manifold (M, J) of Kähler type with Kähler metric g, r^J coincides with r. This follows easily from the Kähler identity $R(X, Y)(JZ) = J(R(X, Y)Z)$, a consequence of the fact that $\nabla J = 0$. In general, though, $r \neq r^J$.

In view of the symmetries of the curvature tensor, (2.1) is essentially the only new tensor we can obtain by computing the trace of a J-twisting of R in two different positions. Indeed, varying the type of trace we take, up to a constant factor or a permutation of the arguments, we obtain the expressions r^J, J^*r^J or J^*r, respectively.

The *scalar curvature* s is the total contraction of the curvature tensor, that is to say, the contraction of the Ricci tensor r. On the other hand, the *J-scalar curvature* s^J is defined as the contraction of r^J. A straightforward calculation shows that in terms of the components of R and J, we have that

$$s = R^{li}{}_{li}\,, \qquad s^J = J^i{}_t R_{ilm}{}^t J^{lm}\,. \tag{2.2}$$

[1] In the existing literature (see, for example, [13]), the J-Ricci tensor is known as the $*$-Ricci tensor, terminology that we avoid to prevent any association with the Hodge $*$ operator. Furthermore, our choice makes it explicit the relationship between J and r^J. For the same reason, we denote by s^J the contraction of r^J, and call it the J-scalar curvature instead of the $*$-scalar curvature.

Associated to an almost Hermitian manifold (M, J, g), we have the *fundamental form*

$$\omega(X, Y) = \omega_g^J(X, Y) = g(JX, Y). \tag{2.3}$$

Interestingly enough, we may also define the alternating tensor

$$\rho^J(X, Y) = -r^J(X, JY),$$

that we call the *J- Ricci form.*

In the presence of an integrable J, the Kähler condition may be stated by simply saying that ω is closed. In that case, ρ^J is closed and coincides with the Ricci form of the Kähler metric. As the example below shows, this property fails for general almost Hermitian manifolds.

Example 2.1. The sphere $\mathbb{S}^6 \subset \mathbb{R}^7$ admits an almost complex structure defined by the vector product $\times$ on $\mathbb{R}^7$ induced by the Cayley numbers. Let g be the Riemannian metric on $\mathbb{S}^6$ as a submanifold of $\mathbb{R}^7$ endowed with the canonical metric $\langle \ , \ \rangle$. Given $x \in \mathbb{S}^6$, the tangent space $T_x\mathbb{S}^6$ at x may be identified with a subspace of $\mathbb{R}^7$. The almost complex structure is defined by

$$\begin{array}{rccc} J_x : T_x\mathbb{S}^6 & \longrightarrow & T_x\mathbb{S}^6 \\ Y & \mapsto & x \times Y \end{array}.$$

This map is an endomorphism of $T_x\mathbb{S}^6$ because $x \times Y$ is in effect, an element of $\mathbb{R}^7$ perpendicular to x. Since the Cayley product is *alternative* (that is to say, $X \cdot (X \cdot Y) = (X \cdot X) \cdot Y$ for all $X, Y \in \mathbb{O}$), we have that $J_x^2 Y = x \times (x \times Y) = x \cdot (x \cdot Y) = (x \cdot x) \cdot Y = -\langle x, x\rangle Y = -Y$. So J defines an almost complex structure on $\mathbb{S}^6$. This structure is not integrable [6]. Furthermore, $g(J_xX, J_xY) = g(X, Y)$. Hence, $(\mathbb{S}^6, J, g)$ is an almost Hermitian manifold.

The sectional curvature of $(\mathbb{S}^6, g)$ is constant and equal to 1. Hence, in an orthonormal frame, we have that $R_{skij} = \delta_{is}\delta_{jk} - \delta_{ik}\delta_{js}$. The metric g is Einstein with Ricci tensor $r = 5g$. The fundamental form ω_g^J is just the almost complex structure J reinterpreted as an alternating two tensor, and a simple calculation shows that $\rho^J = \omega_g^J$. Consequently, the scalar curvature of $(\mathbb{S}^6, g)$ is $s_{\mathbb{S}^6} = 30$, while the J-scalar curvature of $(\mathbb{S}^6, J, g)$ is $s_{\mathbb{S}^6}^J = 6$.

The only spheres $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ that admit almost complex structures are $\mathbb{S}^2$ and $\mathbb{S}^6$ [4]. This result will have some consequences below.

3. The Yamabe problem for almost Hermitian manifolds

The usual Yamabe problem only involves the Riemannian metric g. It was stated in [12], as the following question.

Yamabe problem. *Let (M, g) be a smooth compact Riemannian manifold with scalar curvature s. Is there a metric $\tilde{g}$ in the conformal class of g whose scalar curvature $\tilde{s}_{\tilde{g}}$ is constant ?*

For two-dimensional manifolds, this problem corresponds to the fact that every conformal class of metrics has a representative of constant Gaussian curvature, a rather famous consequence of the uniformization theorem for (real) surfaces. One may attempt to generalize this result to dimensions 3 or greater, searching for special type of metrics in each conformal class. Yamabe posed his problem in pursuit of this, as a way of proving the Poincaré conjecture.

Let n be the dimension of M, and suppose we write the conformal deformation as $\tilde{g} = \varphi^{\frac{4}{n-2}} g$, for some function φ. Then [12] we have the equation

$$4\frac{n-1}{n-2}\Delta_g\varphi + s\,\varphi = \tilde{s}\,\varphi^{\frac{n+2}{n-2}}\,, \tag{3.1}$$

where Δ_g is the Laplace operator $-\nabla^i\nabla_i = -\operatorname{trace}_g \nabla d$. The solution to the Yamabe problem is achieved if we find a positive real-valued smooth function φ that satisfies (3.1) for some constant $\tilde{s}$.

Yamabe attempted to solve this problem by a variational method. Indeed, let $N = 2n/(n-2)$ and set

$$\lambda_N(\varphi) = \frac{\int_M \tilde{s}_{\tilde{g}}\, d\mu_{\tilde{g}}}{\left(\int_M d\mu_{\tilde{g}}\right)^{2/N}} = \frac{4\frac{n-1}{n-2}\int_M |d\varphi|^2\, d\mu_g + \int_M s\varphi^2\, d\mu_g}{\|\varphi\|_N^2}\,. \tag{3.2}$$

The Euler–Lagrange equation of this functional is given by

$$4\frac{n-1}{n-2}\Delta_g\varphi + s\,\varphi = \frac{\lambda_N(\varphi)}{\|\varphi\|_{L^N}^N}\varphi^{N-1}\,,$$

and therefore, solutions to (3.1) can be obtained by finding positive minimizers of $\lambda_N(\varphi)$ in the class of functions in $H^1(M)$ which have L^N-norm equal to one. The constant scalar curvature of the resulting conformally deformed metric is then equal to $\inf_\varphi \lambda_N(\varphi)$. Notice that this transforms Yamabe's original problem into that of finding a metric of constant scalar curvature in the conformal class of g that achieves this infimum.

Since N is the critical exponent in the Sobolev embedding theorem, Yamabe considered the Euler–Lagrange equations for the functionals $\lambda_q(\varphi)$, $2 \le q \le N$, defined as in (3.2) replacing N by q. A function φ is a critical point of $\lambda_q(\varphi)$ if and only if

$$4\frac{n-1}{n-2}\Delta_g\varphi + s\,\varphi = \mu_q\varphi^{q-1}\,, \tag{3.3}$$

where $\mu_q = \lambda_q(\varphi)/\|\varphi\|_q^{q-2}$.

We set

$$\lambda_q(M,[g]) = \inf_\varphi \lambda_q(\varphi)\,,$$

and define

$$\lambda(M,[g]) = \lambda_N(M,[g])\,.$$

(The notation is indicative of the fact that the constant $\lambda(M,[g])$ is a conformal invariant of (M,g).) Since the inclusion $H^1(M) \to L^q(M)$ is compact when $2 \le q < N$, it is relatively easy to find a positive solution φ_q of (3.3), with $\|\varphi_q\|_{L^q} = 1$, which minimizes $\lambda_q(\varphi)$. But the convergence of this sequence as $q \to N$ is a delicate issue, as one cannot prove easily that the limit function is non-zero. Aubin proved that $\lambda(M,[g]) \le \lambda(\mathbb{S}^n,[g_0])$, where $(\mathbb{S}^n, g_0)$ is the n-sphere with its standard Riemannian metric g_0. Moreover, the combined efforts of Yamabe, Trudinger, and Aubin led to the theorem that if $\lambda(M,[g]) < \lambda(\mathbb{S}^n,[g_0])$, then some subsequence of $\{\varphi_q\}$ converges uniformly to a positive smooth function minimizing $\lambda_N(\varphi)$, thus solving the Yamabe problem under such an assumption on $\lambda(M,[g])$.

This shifted the focus of attention to the conformal invariant $\lambda(M,[g])$. In dimensions greater or equal than 6, Aubin proved that if M is not locally conformally flat, then $\lambda(M,[g]) <$

$\lambda(\mathbb{S}^n, [g_0])$. For the remaining cases, R. Schoen proved that this estimate also holds unless M is conformal to the standard sphere. Thus, from the collected efforts of Yamabe, Trudinger, Aubin, and Schoen [12, 11, 1, 10], the following result emerged.

Theorem 3.1. *Let (M^n, g) be a compact Riemannian manifold of dimension $n \geq 3$. The conformal class of g contains metrics which attain the conformal invariant $\lambda(M, [g])$.*

As we saw in Section 2, on an almost Hermitian manifold (M, J, g), the J-scalar curvature s^J ties quite naturally the properties of g with those of J, and it is a natural geometric quantity to study. One then may consider the following version of the problem above.

Yamabe problem for almost Hermitian manifolds. *Let (M, J, g) be a compact almost Hermitian Riemannian manifold with J-scalar curvature s^J. Is there a Hermitian metric $\tilde{g}$ in the conformal class of g whose J-scalar curvature $\tilde{s}^J_{\tilde{g}}$ is constant ?*

The dimension of any almost Hermitian manifold is an even number. Let us then write $n = \dim M = 2m$. The novelty in this new problem lies in the cases where $2m > 2$. Almost Hermitian manifolds of dimension two are all Kähler, and thus, $s^J = s$. In that case, the new problem also reduces to the *classic* consequence of the uniformization theorem mentioned before.

This version of the Yamabe problem is once again an attempt to find a *canonical* metric in the conformal class of g; this time we search for a metric that sits well with the almost complex structure. If g is J invariant, then so will be any of its conformal deformations. The J-scalar curvature is thus well defined, and we may try to find those deformations for which this geometric quantity is a constant. This problem involves no restriction whatsoever on the almost complex structure J, and its formulation is intrinsically outside the realm of Kählerian properties from the outset.

In analyzing this new problem, the first thing to do is to derive an equation relating the J-scalar curvature of the background metric to that of its conformal deformation.

Proposition 3.2. *Let (M, J, g) be a compact almost Hermitian manifold of dimension $n = 2m$. Suppose that $\tilde{g} = \varphi^{\frac{2}{m-1}} g$ is a conformal deformation of g, and let s^J and $\tilde{s}^J$ be the J-scalar curvatures of g and $\tilde{g}$, respectively. Then*

$$\frac{2}{m-1}\Delta_g \varphi + s^J \varphi = \tilde{s}^J \varphi^{\frac{m+1}{m-1}}\,. \tag{3.4}$$

Here Δ_g is the positive Laplacian of the background metric g.

The conformal deformation of g in this statement is described by the same function of φ and the dimension of M as that used in obtaining Equation (3.1). The apparent difference in the exponent arises from the fact that this is expressed here in terms of m and not n: we have $\frac{4}{n-2} = \frac{2}{m-1}$.

Proof. Suppose firstly that $\tilde{g} = e^{2f} g$. Then the $(4, 0)$ Riemannian curvature tensor (see [3], p. 58; their convention for the curvature is the negative of ours) transforms as

$$\tilde{R} = e^{2f}\left(R + g \owedge \left(\nabla df - df \circ df + \frac{1}{2}|df|^2 g\right)\right), \tag{3.5}$$

where $h \owedge k$ is the Kulkarni–Nomizu product of two symmetric 2-tensors.

Given an orthonormal frame $\{\tilde{e}_1, \ldots, \tilde{e}_{2m}\}$ of $\tilde{g}$, we want to calculate

$$\tilde{s}_{\tilde{g}}^J = \sum_{i,j} \tilde{R}\left(\tilde{e}_i, \tilde{e}_j, J\tilde{e}_j, J\tilde{e}_i\right).$$

Let $\{e_1, \ldots, e_{2m}\}$ be an orthonormal frame for g. Then $\{e^{-f}e_1, \ldots, e^{-f}e_{2m}\}$ is an orthonormal frame for $\tilde{g}$. Using this frame and (3.5), we have that

$$\begin{aligned}
\tilde{s}_{\tilde{g}}^J &= e^{-4f} \textstyle\sum_{i,j} \tilde{R}(e_i, e_j, Je_j, Je_i) \\
&= e^{-2f} \left(s_g^J + \textstyle\sum_{i,j} g \owedge \left(\nabla df - df \circ df + \frac{1}{2}|df|^2 g\right)(e_i, e_j, Je_j, Je_i)\right).
\end{aligned}$$

By the definition of the Kulkarni–Nomizu product of two symmetric 2-tensors and the fact that $g(e_i, Je_i) = 0$, given any symmetric tensor h we have that

$$\begin{aligned}
\textstyle\sum_{i,j} g \owedge h(e_i, e_j, Je_j, Je_i) &= \textstyle\sum_{i,j} (g(e_i, Je_j)h(e_j, Je_i) + g(e_j, Je_i)h(e_i, Je_j)) \\
&= 2\textstyle\sum_{i,j} g(e_i, Je_j)h(e_j, Je_i) \\
&= 2\textstyle\sum_{i,j} \sum_{l,k} J^l{}_j J^k{}_i g(e_i, e_l)h(e_j, e_k) \\
&= -2\textstyle\sum_j h(e_j, e_j) = -2\,\mathrm{trace}_g\, h.
\end{aligned}$$

Applying this result to the tensor $h = \nabla df - df \circ df + \frac{1}{2}|df|^2 g$, we obtain that

$$\tilde{s}_{\tilde{g}}^J = e^{-2f}\left(s_g^J + 2\Delta_g f - 2(m-1)|df|^2\right).$$

Setting $e^{2f} = \varphi^{\frac{2}{m-1}}$, we now get

$$\tilde{s}_{\tilde{g}}^J = \varphi^{-\frac{2}{m-1}}\left(s_g^J + \frac{2}{m-1}\frac{\Delta_g \varphi}{\varphi}\right),$$

from which the desired result follows. □

Note that except for the coefficient of the Laplacian in (3.1) and (3.4), these two equations are formally of the same type. The analysis of (3.4) can thus be made with techniques developed in the study of (3.1). In particular, we introduce the functional

$$\lambda_N^J(\varphi) = \frac{\displaystyle\int_M \tilde{s}_{\tilde{g}}^J \, d\mu_{\tilde{g}}}{\left(\displaystyle\int_M d\mu_{\tilde{g}}\right)^{2/N}} = \frac{\displaystyle\frac{2}{m-1}\int_M |d\varphi|^2 \, d\mu_g + \int_M s^J \varphi^2 \, d\mu_g}{\|\varphi\|_N^2}. \tag{3.6}$$

Then we have the following.

Proposition 3.3. *The quantity*

$$\lambda^J(M, [g]) := \inf_{\varphi} \lambda_N^J(\varphi),$$

with $\lambda_N^J(\varphi)$ as in (3.6), is a conformal invariant.

Proof. By (3.4), if $\tilde{g} = \varphi^{\frac{2}{m-1}} g$ we have that $\lambda_N^J(\varphi\psi) = \tilde{\lambda}_N^J(\psi)$. Here, $\tilde{\lambda}_N^J$ is the functional associated with the metric $\tilde{g}$. Therefore, the infima of λ_N^J and $\tilde{\lambda}_N^J$ coincide. (See the analogous proof for the Yamabe problem [2], p, 126.) □

One would hope to solve the Yamabe problem for almost Hermitian manifolds by finding a positive function φ that minimizes $\lambda_N^J(\varphi)$. Indeed, since the Euler–Lagrange equation of $\lambda_N^J(\varphi)$ is given by

$$\frac{2}{m-1}\Delta_g \varphi + s^J\,\varphi = \frac{\lambda_N^J(\varphi)}{\|\varphi\|_{L^N}^N}\varphi^{N-1}\,,$$

a positive minimizer φ with L^N-norm equal to one would solve (3.4) with $\tilde{s}^J = \inf_\varphi \lambda_N^J(\varphi)$, and the metric $\tilde{g} = \varphi^{\frac{2}{m-1}} g$ would have J-scalar curvature equal to the constant $\inf_\varphi \lambda_N^J(\varphi)$.

4. Some global analysis

Given an n-dimensional Riemannian manifold (M^n, g), we have the decomposition ([3], p. 48; once again, we remind the reader about this reference's use of the opposite sign convention for the curvature tensor)

$$R = -\frac{s}{2n(n-1)} g \owedge g - \frac{1}{n-2} g \owedge z - W$$

where s is the scalar curvature, $z = r - \frac{s}{n} g$ is the trace free part of the Ricci tensor r, and W is the Weyl tensor. If we express z in terms of g and r, we obtain

$$R = \frac{s}{2(n-1)(n-2)} g \owedge g - \frac{1}{n-2} g \owedge r - W\,,$$

which written down in the case of a Riemannian manifold (M, g) of dimension $n = 2m$ gives

$$R = \frac{s}{4(m-1)(2m-1)} g \owedge g - \frac{1}{2(m-1)} g \owedge r - W\,. \tag{4.1}$$

We have observed, in the proof of Proposition 3.2, that

$$\begin{aligned} s^J &= \textstyle\sum_{i,j=1}^{2m} R(e_i, e_j, Je_j, Je_i)\,, \\ \textstyle\sum_{i,j=1}^{2m} (g \owedge h)(e_i, e_j, Je_j, Je_i) &= -2\,\mathrm{trace}_g\, h\,. \end{aligned}$$

Hence, if (M^{2m}, g, J) is an almost Hermitian manifold, by (4.1) we get

$$\begin{aligned} s^J &= \frac{(-4m)s}{4(m-1)(2m-1)} - \frac{(-2s)}{2(m-1)} - \sum_{i,j=1}^{2m} W(e_i, e_j, Je_j, Je_i) \\ &= \frac{s}{2m-1} - \sum_{i,j=1}^{2n} W(e_i, e_j, Je_j, Je_i)\,. \end{aligned}$$

Lemma 4.1. *Let $\omega^\#$ be the metric dual of the fundamental form ω of (M, J, g). If $\{e_i\}_{i=1}^{2m}$ is an orthonormal frame for g, then*

$$\sum_{i,j=1}^{2m} W(e_i, e_j, Je_j, Je_i) = -2W\left(\omega^\#, \omega^\#\right)\,.$$

Proof. Let $\{\theta^i\}$ be the basis dual to $\{e_i\}$. By (2.3), we have that $\omega = \sum_{i=1}^m \theta^i \wedge J\theta^i$, and therefore,

$$\omega^\# = \sum_{i=1}^m e_i \wedge Je_i \,. \tag{4.2}$$

By the algebraic Bianchi identity, we have that

$$\sum_{i,j=1}^{2m} W(e_i, e_j, Je_j, Je_i) + \sum_{i,j=1}^{2m} W(e_j, Je_j, e_i, Je_i) + \sum_{i,j=1}^{2m} W(Je_j, e_i, e_j, Je_i) = 0 \,.$$

The first and last summand in the left side of the identity above are equal to each other. Indeed, since the sums are independent of the choice of basis, we can use $\{Je_i\}$ instead of $\{e_i\}$ itself to obtain that $\sum_{i,j=1}^{2m} W(Je_j, e_i, e_j, Je_i) = -\sum_{i,j=1}^{2m} W(Je_j, Je_i, e_j, e_i)$, and by the symmetries of the curvature tensor, we easily see that this last expression is just $\sum_{i,j=1}^{2m} W(e_i, e_j, Je_j, Je_i)$. Hence,

$$\sum_{i,j=1}^{2m} W(e_i, e_j, Je_j, Je_i) = -\frac{1}{2} \sum_{i,j=1}^{2m} W(e_j, Je_j, e_i, Je_i) \,.$$

Once again, the right side of this last expression is independent of the orthonormal basis $\{e_i\}_{i=1}^{2m}$ chosen to compute it. We now select one such basis of the form $\{e_1, \ldots, e_m, Je_1, \ldots, Je_m\}$, and obtain that

$$\sum_{i,j=1}^{2m} W(e_i, e_j, Je_j, Je_i) = -\frac{1}{2} \sum_{i,j=1}^{2m} W(e_j, Je_j, e_i, Je_i) = -2 \sum_{i,j=1}^{m} W(e_j, Je_j, e_i, Je_i) \,.$$

The endomorphism of $\Lambda^2 TM$ given by W is defined by $W(X \wedge Y, Z \wedge W) = W(X, Y, Z, W)$. By (4.2), we then see that the expression above is just $-2W(\omega^\#, \omega^\#)$, as desired. □

We now draw the following important conclusion.

Proposition 4.2. *Let (M^{2m}, J, g) be an almost Hermitian manifold. Then*

$$(2m-1)s^J - s = 2(2m-1)W\left(\omega^\#, \omega^\#\right) \,.$$

To our knowledge, this identity is new. It expresses $(2m-1)s^J$ as a perturbation of s, where the perturbation is fully defined in terms of the Weyl tensor and the fundamental form. In the terminology of [7], this naturally defined perturbation is a function of conformal weight -2. The difficult nature of W forces us to do further analysis of the identity if we are to gain some insight into the behavior of s^J using that of s.

We now study how the J-Ricci tensor r^J transforms under a conformal change in the metric.

Lemma 4.3. *Let $\{e_i\}$ be an orthonormal frame for g. Then*

$$\sum_{i=1}^{2m} (g \owedge h)(X, JY, e_i, Je_i) = 4h^{1,1}(X, Y) \,. \tag{4.3}$$

where $h^{1,1}$ is the J-invariant component of h.

Proof. Given an orthonormal basis $\{e_i\}$, we have that

$$\begin{aligned}
\sum_{i=1}^{2m}(g \owedge h)(X, JY, e_i, Je_i) &= \sum_{i=1}^{2m}(g(X, e_i)h(JY, Je_i) + g(JY, Je_i)h(X, e_i)) \\
&\quad - \sum_{i=1}^{2m}(g(X, Je_i)h(JY, e_i) + g(JY, e_i)h(X, Je_i)) \\
&= 2\sum_{i=1}^{2m}(g(JX, Je_i)h(JY, Je_i) + g(Y, e_i)h(X, e_i)) \\
&= 2(h(JX, JY) + h(X, Y)),
\end{aligned}$$

and the result follows. In obtaining the second equality above, we use the Hermitian property of g, plus the fact that the sums are independent of the choice of basis and can be computed using $\{Je_i\}$ instead. □

Proposition 4.4. *Let (M^{2m}, J, g) be an almost Hermitian manifold. Given a smooth function f on M, consider the Hermitian metric $\tilde{g} = e^{2f}g$. Then*

$$\tilde{r}^J = r^J - 2h_f^{1,1}. \tag{4.4}$$

Here, $h_f^{1,1}$ is the J-invariant component of $h_f(\cdot, \cdot) = \nabla df - df \circ df + \frac{1}{2}|df|^2 g$, and $\tilde{r}^J$ and r^J are the J-Ricci tensors of $\tilde{g}$ and g, respectively.

Proof. Applying the algebraic Bianchi identity to the curvature tensor, we may easily see that $2r^J(X, Y) = -\sum R(X, JY, e_i, Je_i)$, $\{e_i\}$ an orthonormal frame for the metric.

Let $\{e_i\}$ be an orthonormal frame for g. Then, $\{e^{-f}e_i\}$ is an orthonormal frame for $\tilde{g}$. By (3.5) and the remark above, we have that

$$\begin{aligned}
\tilde{r}^J(X, Y) &= -\frac{1}{2}\sum_{i=1}^{2m}\tilde{R}\left(X, JY, e^{-f}e_i, e^{-f}Je_i\right) \\
&= -\frac{1}{2}\sum_{i=1}^{2m}\left(R + g \owedge \left(\nabla df - df \circ df + \frac{1}{2}|df|^2 g\right)\right)(X, JY, e_i, Je_i) \\
&= r^J(X, Y) - \frac{1}{2}\sum_{i=1}^{2m}\left(g \owedge \left(\nabla df - df \circ df + \frac{1}{2}|df|^2 g\right)\right)(X, JY, e_i, Je_i),
\end{aligned}$$

from which the result follows by Lemma 4.3 applied to $h = \nabla df - df \circ df + \frac{1}{2}|df|^2 g$. □

As a consequence of this result, we obtain the following.

Corollary 4.5. *For any almost Hermitian manifold (M, J, g), the skew Hermitian component r^J_{skew} of the J-Ricci tensor is a conformal invariant.*

Proof. By Proposition 4.4, the difference $\tilde{r}^J - r^J$ is a symmetric 2-tensor. The result follows. □

The pure trace part of the J-Ricci tensor of a conformal deformation of a Hermitian metric is perpendicular to this skew Hermitian invariant of the class. This shows the ample room we have to find a conformal deformation of a given metric whose J-scalar curvature is better behaved than the J-scalar curvature of the original one.

By (4.1), we may also show that

$$r^J = -\frac{s}{2(m-1)(2m-1)}g + \frac{2}{(m-1)}r^{1,1} - \sum_i W(e_i, \cdot, J\cdot, Je_i) \tag{4.5}$$

where $r^{1,1}$ is the J-invariant component of the Ricci tensor r. This expression follows by applying Lemma 4.3 twice and using the Hermitian property of g. The expression shows that r^J fails to be symmetric if and only if so does the contribution to it that arises from the Weyl tensor.

5. Analysis of the equation

The problem of conformally deforming the almost Hermitian metric g to a metric of constant J-scalar curvature is equivalent to proving the existence of a constant $\tilde{s}^J$ and a smooth positive function φ satisfying Equation (3.4). One such φ (if more than one exist) can be obtained by minimizing the functional $\lambda_N^J(\varphi)$, in which case $\tilde{s}^J = \lambda^J(M,[g])/\|\varphi\|_{L^N}^{N-2}$.

Yamabe, Trudinger, Aubin, and others have analyzed equations such as (3.4) with a great deal of generality. Of special interest to us are the following two results of Aubin [2], p. 131:

Theorem 5.1 (Aubin). *Let $(M^{n=2m}, g)$ be a compact Riemannian manifold, and consider the equation*

$$2\frac{2m-1}{m-1}\Delta_g\varphi + h(x)\varphi = \lambda f(x)\varphi^{N-1}, \tag{5.1}$$

with h, f smooth functions, $f > 0$, $N = 2n/(n-2) = 2m/(m-1)$, and λ a real number. Let

$$I(\varphi) = \left[2\frac{2m-1}{m-1}\int |d\varphi|^2\, d\mu_g + \int h\varphi^2\, d\mu_g\right]\left(\int f\varphi^N\, d\mu_g\right)^{-2/N},$$

and define $\nu = \inf_\varphi I(\varphi)$, the infimum taken over all non-zero φ's in $H^1(M)$. Then, $\nu \leq 2m(2m-1)\omega_{2m}^{1/m}[\sup f]^{-2/N}$, and if ν is strictly less than this upper bound, then Equation (5.1) has a positive smooth solution φ for $\lambda = \nu$. Here, ω_k is the volume of the k dimensional unit sphere in $\mathbb{R}^{k+1}$.

Proposition 5.2 (Aubin). *Suppose that $m \geq 2$, and let p be a point where f achieves a maximum. If $h(p) - s(p) + (m-2)(\Delta f(p)/f(p)) < 0$, then Equation (5.1) has a positive smooth solution φ for $\lambda = \nu = \inf_\varphi I(\varphi)$. Here s is the scalar curvature of the metric g on the manifold $M^{n=2m}$.*

Let us first multiply (3.4) by $2m-1$ and obtain the equation

$$2\frac{2m-1}{m-1}\Delta_g\varphi + (2m-1)s^J\varphi = (2m-1)\tilde{s}^J\varphi^{\frac{m+1}{m-1}}. \tag{5.2}$$

As a straightforward application of the results above, we obtain the following theorem.

Theorem 5.3. *Let $(M^{n=2m}, J, g)$ be a compact almost Hermitian manifold. Then we have the universal bound*

$$\lambda^J(M,[g]) \leq 2m\,\omega_{2m}^{1/m}.$$

In this expression, ω_{2m} is the volume of the $2m$ dimensional unit sphere in $\mathbb{R}^{2m+1}$. If $\lambda^J(M,[g]) < 2m\,\omega_{2m}^{1/m}$, there exists a smooth positive solution φ_0 of the equation

$$\frac{2}{m-1}\Delta_g\varphi + s^J\varphi = \lambda^J(M,[g])\varphi^{\frac{m+1}{m-1}},$$

with L^N-norm equal to 1, and the metric $\tilde{g} = \varphi_0^{2/(m-1)}g$ has J-scalar curvature equal to $\lambda^J(M,[g])$.

Proof. This is a consequence of Aubin's Theorem 5.1 above, applied with $f \equiv 1$. □

Proposition 5.4. *Let $(M^{n=2m}, J, g)$ be an almost Hermitian manifold. If there exists a point $p \in M$ such that $(2m-1)s^J(p) - s(p) < 0$, then there exists a metric $\tilde{g}$ in the conformal class of g whose J-scalar curvature is equal to $\lambda^J(M,[g])$.*

Proof. Apply Aubin's Proposition 5.2 with $f \equiv 1$ and $h = (2m-1)s^J$. □

For the sake of completeness, and as an introduction to an argument we shall use later on, we sketch the proof of the upper bound $\lambda^J(M,[g]) \leq 2m\,\omega_{2m}^{1/m}$. Recall that the functions

$$u_\alpha(x) = \left(\frac{|x|^2+\alpha^2}{\alpha}\right)^{(2-n)/2} \tag{5.3}$$

realize the optimal Sobolev constant in $\mathbb{R}^n$, that is to say,

$$a\|\nabla u_\alpha\|^2_{L^2(\mathbb{R}^n)} = \Lambda\|u_\alpha\|^2_{L^N(\mathbb{R}^n)}, \tag{5.4}$$

where $a = 4(n-1)/(n-2)$ and $\Lambda = n(n-1)\omega_n^{2/n}$. In the case of an almost Hermitian manifold (M, J, g) of dimension $2m$, we have $a = 2(2m-1)/(m-1)$. We set

$$E(\varphi) = (2m-1)\lambda_N^J(\varphi)\|\varphi\|_N^2 = \int_M \left(a|d\varphi|^2 + (2m-1)s^J\varphi^2\right) d\mu_g .$$

Let K be the maximum of $(2m-1)s^J$ over M, and fix normal coordinates in a neighborhood of a point $p \in M$ where this maximum is achieved. Let us choose a φ of the form $\varphi = \eta u_\alpha$, where η is a smooth radial cutoff function supported in the ball $B_{2\varepsilon}$ centered at p, and identically equal to 1 in B_ε. This local expression is extended by zero to a smooth function on M.

We recall that in normal coordinates we have $d\mu_g = (1+O(r))\,dx$. Hence, by Hölder's inequality and estimating the volume of the Euclidean ball by the n-th power of its radius, we obtain that

$$\int_{B_{2\varepsilon}} (2m-1)s^J\varphi^2\,d\mu_g \leq (1+C\varepsilon)K\int_{B_{2\varepsilon}}\varphi^2\,dx \leq C(1+C\varepsilon)K\varepsilon^2\|\varphi\|^2_{L^N},$$

and so

$$E(\varphi) \leq \int_{B_{2\varepsilon}} a|d\varphi|^2\,d\mu_g + C(1+C\varepsilon)K\varepsilon^2\|\varphi\|^2_{L^N}.$$

By (5.4), the first integral on the right side above can be estimated in terms of Λ, ε and α (see [8], p. 50, for details). Thus, we get

$$E(\varphi) \leq (1+C\varepsilon)\|\varphi\|^2_{L^N}\left(\Lambda + C\alpha + C\varepsilon^2\right).$$

Thus, choosing ε small and then α small, we conclude that

$$(2m-1)\lambda^J(M,[g]) = \inf_{\varphi}(2m-1)\lambda^J(\varphi) = \inf_{\varphi}\frac{E(\varphi)}{\|\varphi\|^2_{L^N}} \leq \Lambda = 2m(2m-1)\omega_{2m}^{1/m},$$

proving the desired estimate for $\lambda^J(M,[g])$. (This is essentially the argument given by Aubin to prove the universal bound for $I(\varphi)$ in Theorem 5.1.)

The result stated in Theorem 5.3 shifts the attention to the analysis of the conformal invariant $\lambda^J(M,[g])$. This is analogous to the way the usual Yamabe problem eventually focused on the invariant $\lambda(M,[g])$. The question to answer now is if there are Hermitian manifolds (M^{2m}, J, g) such that $\lambda^J(M,[g]) = 2m\omega_{2m}^{1/m}$. Those will be the only type of manifolds where the infimum of the functional $\lambda^J(\varphi)$ might not be realizable. If that were the case, the almost complex Yamabe problem might still be solvable, but in those cases the solution would have a rather large constant J-scalar curvature. We analyze this issue next, proving that the infimum is always achieved.

6. The conformal invariant $\lambda^J(M,[g])$

The essential differences between the almost complex Yamabe problem and its usual version are *encoded* in Proposition 4.2. Formally speaking, the functionals (3.2) and (3.6) look very much the same. However, the proposition states that

$$(2m-1)s^J = s + 2(2m-1)W\left(\omega^{\#},\omega^{\#}\right),$$

and this shows how the sign of the conformal term $W(\omega^{\#},\omega^{\#})$ plays an important rôle in the new problem: if non-positive it tends to help in finding the minimizing solution to the problem while, if positive, it produces the opposite effect. For convenience, we shall write $W(\omega,\omega)$ instead of $W(\omega^{\#},\omega^{\#})$.

By Theorem 5.3, if $\lambda^J(M,[g]) \leq 0$ then the almost complex Yamabe problem admits a solution that is a minimizer of the functional $\lambda^J_N(\varphi)$. Moreover, in that case, the minimizing solution is unique. In the sequel, we exploit this remark in combination with the identity above.

Let $\mathcal{A}_H$ be the class of all almost Hermitian manifolds, and define the subclasses W^-, W^0 and W^+ by

$$\begin{aligned}
W^- &= \left\{\left(M^{2m},J,g\right)\in\mathcal{A}_H:\ W(\omega,\omega)(p)<0 \text{ for some } p\in M\right\}\\
W^0 &= \left\{\left(M^{2m},J,g\right)\in\mathcal{A}_H:\ W(\omega,\omega)=0\right\}\\
W^+ &= \left\{\left(M^{2m},J,g\right)\in\mathcal{A}_H:\ W(\omega,\omega)\geq 0,\ W(\omega,\omega)(p)>0 \text{ for some } p\in M\right\},
\end{aligned}$$

respectively. These three subclasses are pairwise disjoint, and $\mathcal{A}_H = W^-\cup W^0\cup W^+$. Furthermore, if (M^{2m},J,g) is in one of these three subclasses, then its entire conformal class is in the same subclass.

6.1. The subclasses W^- and W^0

We begin by considering the subclass W^-.

Theorem 6.1. *For any manifold $(M^{2m},J,g)\in W^-$, the almost Hermitian Yamabe problem can be solved by a minimizer of the functional $\lambda^J_g(\varphi)$. In fact, we have $\lambda^J(M,[g]) < 2m\omega_{2m}^{1/m}$.*

Proof. We apply Proposition 5.4. Indeed, since our manifold belongs to W^-, by Proposition 4.2 there exists a point p where $(2m-1)s^J(p) - s(p) < 0$, and the result follows. We may prove that $\lambda^J(M,[g])$ is strictly less than $2m\omega_{2m}^{\frac{1}{m}}$ using the argument outlined at the end of the previous section, estimating the quotient $E(\varphi)/\|\varphi\|^2_{L^N}$ for a function of the form $\varphi = \eta u_\alpha$ with u_α as in (5.3). We choose the localizing function η to have support in a neighborhood of a point where $W(\omega, \omega)$ is negative and bounded away from 0. □

For almost Hermitian manifolds in W^0, it is quite remarkable that we can solve our problem appealing to the results of R. Schoen [10], somewhat disregarding all together the presence of the almost complex structure. Historically, the conformally flat version of the Yamabe problem was the hardest, and was proven by Schoen in 1984, years after the other cases had been resolved. In our case, the presence of an almost complex structure makes no difference, and Schoen's result applies directly to the solution of the almost Hermitian Yamabe problem in the class W^0.

Observe that our class W^0 contains more than just conformally flat manifolds. It includes, for example, all Kähler Ricci-flat non-flat manifolds, and all almost Hermitian anti-self-dual 4-manifolds.

Theorem 6.2. *For any manifold $(M^{2m}, J, g) \in W^0$, the almost Hermitian Yamabe problem can be solved by a minimizer of $\lambda^J(\varphi)$. In fact, $\lambda^J(M,[g]) \leq 2m\omega_{2m}^{1/m}$, and the equality is achieved if only if (M^{2m}, g) is conformal to the 6 dimensional sphere $\mathbb{S}^6$ with its standard metric. In that case, J is an almost complex structure on $\mathbb{S}^6$ and we have that $s_g^J = 6$.*

Proof. By Proposition 4.2, for these type of manifolds we have that

$$(2m-1)\tilde{s}^J = \tilde{s}\,,$$

where $\tilde{g}$ is an arbitrary Hermitian metric $\tilde{g}$ conformally equivalent to g. Consequently, the functional $\lambda_N(\varphi)$ in (3.2) coincides with $(2n-1)\lambda_N^J(\varphi)$ in (3.6), and the corresponding conformal invariants are thus related:

$$\lambda(M,[g]) = (2m-1)\lambda^J(M,[g])\,.$$

By Schoen's result [10] on $\lambda(M,[g])$, the upper bound for this conformal invariant, namely $2m(2m-1)\omega_{2m}^{\frac{1}{m}}$, is never achieved unless we are in the case of the $2m$ dimensional unit sphere in $\mathbb{R}^{2m+1}$, with its standard metric. It follows that $\lambda^J(M,[g]) < 2m\omega_{2m}^{\frac{1}{m}}$ unless M is the standard metric sphere. The solvability of the almost Hermitian Yamabe problem for this class of manifolds follows applying Theorem 5.3.

If the upper bound is reached, then $M = \mathbb{S}^{2m}$ with its standard metric g. Since M is an almost complex manifold, we must have $(M, g, J) = (\mathbb{S}^6, g, J)$ with J an almost complex structure on $\mathbb{S}^6$ and g Hermitian relative to J. Since the scalar curvature of g is 30, the relationship above between s and s^J implies that $s^J = 6$. □

These results alone have interesting applications.

Corollary 6.3. *Let (M, J, g) be a Kähler manifold whose first Chern class $c_1 = c_1(M,J)$ satisfies the condition $c_1 \cup [\omega]^{m-1} \leq 0$. Here, ω is the Kähler form of g and m is the complex dimension of (M, J). Then there exists a unique representative of the conformal class of g that has non-positive constant J-scalar curvature and minimizes the functional $\lambda_g^J(\varphi)$.*

In particular, this result applies to Kähler manifolds (M, J, g) with $c_1 \leq 0$.

Proof. Since the metric g is Kähler, $s^J = s$ and therefore,

$$2(2m-1)W(\omega,\omega) = (2m-1)s^J - s = (2m-2)s\,.$$

But we have

$$\int s\,d\mu_g = \frac{4\pi}{(m-1)!}c_1 \cup [\omega]^{m-1} \leq 0\,,$$

so s must be negative somewhere, or identically zero, and the manifold (M, J, g) is in W^- or W^0, respectively. In either case, the existence of a solution to the almost complex Yamabe problem follows from the theorems above. The uniqueness follows from the fact that $\lambda^J(M, [g]) \leq 0$. $\square$

In general, conformal deformations of Kähler metrics are not Kähler. Thus, the metric in the corollary above is one where we do not necessarily have $s^J = s$. In fact, if in the conformal class of a Kähler metric the solutions to the usual and almost complex Yamabe problem coincide, then the conformal factor must be a constant and the solution metric is in fact Kähler. This follows by observing that the same function φ must solve (3.1) and (3.4), and therefore, since $s = s^J$, by subtraction we obtain

$$4\Delta\varphi = \left(\lambda_N - \lambda_N^J\right)\varphi^{N-1}\,.$$

Integration of this identity leads to the conclusion that $\lambda_N = \lambda_N^J$, which in turns implies that $\Delta\varphi = 0$, and, therefore, φ is a constant. Consequently, the metric that solves both problems is a rescaling of the original one, thus, Kähler. However, it will rarely be the case that the solutions of these two problems will agree with each other.

Notice how this remark suggests a potential way of proving that a Kähler manifold (M, J, g) with $c_1 < 0$ carries a constant scalar curvature Kähler metric in the same Kähler class as that of g. Indeed, we could consider the space $\mathfrak{M}_g$ of Kähler metrics whose Kähler forms are cohomologous to the Kähler form of g, and prove that for at least one metric in this space, the usual and almost Hermitian Yamabe problem would admit the same solution. For in that case, the solution would be a Kähler metric of constant scalar curvature. We feel there is a good chance that one could complete the details of this idea.

6.2. The subclass W^+

We now consider manifolds in W^+. We prove that for manifolds in this class, $\lambda^J(M, [g])$ never reaches the universal bound.

We do this by working in a special set of normal coordinates, in the same way as these coordinates were used by Lee and Parker [8]. Of course, our selection is made to simplify the conformal analysis of s^J and not that of s, as the above mentioned authors do. We take advantage of the following result, an inhomogeneous version of a theorem of R. Graham (see [8]) that turns out to be essential in our proof.

Theorem 6.4. *Let p be a point in M, $k \geq 0$, and let T be a symmetric $(k+2)$-tensor on T_pM. There exists a unique homogeneous polynomial f of degree $k+2$ in g-normal coordinates such that the metric $\tilde{g} = e^{2f}g$ satisfies*

$$\mathrm{Sym}\left(\tilde{\nabla}^k \tilde{r}_{ij}(p)\right) = T\,.$$

In order to show that $\lambda^J(M, [g])$ never reaches the universal bound, we proceed as we did towards the end of Section 5. We fix normal coordinates in a neighborhood of a point p to be specified later, use (5.3), and consider test functions of the form $\varphi = \eta u_\alpha$, for η a radial cut-off function supported in the ball $B_{2\varepsilon}$, that is identically 1 in B_ε. By (3.6), we have that

$$\begin{aligned}
\lambda_N^J(\varphi)\, \|\varphi\|_N^2 &= \frac{2}{m-1}\int_M |d\varphi|^2\, d\mu_g + \int_M s^J \varphi^2\, d\mu_g \\
&= \frac{2}{m-1}\int_{B_{2\varepsilon}} \left(\eta^2 |du_\alpha|^2 + 2\eta u_\alpha \langle d\eta, du_\alpha\rangle + u_\alpha^2 |d\eta|^2\right) d\mu_g + \int_{B_{2\varepsilon}} s^J \eta^2 u_\alpha^2\, d\mu_g\,.
\end{aligned}$$

We now recall that in normal coordinates we have the expansion

$$\det g_{ij} = 1 - \frac{1}{3} r_{ij} x^i x^j - \frac{1}{6} r_{ij,k} x^i x^j x^k + O\left(|x|^4\right).$$

Hence, by (5.4), we have that

$$\begin{aligned}
\frac{2}{m-1}\int_M \eta^2\, |du_\alpha|^2\, d\mu_g &= \frac{2}{m-1}\int_{B_\varepsilon} |du_\alpha|^2\, d\mu_g + \frac{2}{m-1}\int_{B_{2\varepsilon}-B_\varepsilon} \eta^2\, |du_\alpha|^2\, d\mu_g \\
&\le 2m\omega_{2m}^{\frac{1}{m}} \|\varphi\|_{L^N}^2 - \frac{2}{3(m-1)}\int_{B_\varepsilon} |du_\alpha|^2\, r_{ij} x^i x^j\, dx + E(\varepsilon, \alpha)\,,
\end{aligned}$$

where

$$E(\varepsilon,\alpha) = \frac{2}{(m-1)}\int_{B_\varepsilon} |du_\alpha|^2\, O\left(|x|^3\right) dx + \frac{2}{m-1}\int_{B_{2\varepsilon}-B_\varepsilon} \eta^2\, |du_\alpha|^2\, d\mu_g\,. \tag{6.1}$$

Therefore,

$$\lambda_N^J(\varphi)\, \|\varphi\|_N^2 \le 2m\omega_{2m}^{\frac{1}{m}} \|\varphi\|_{L^N}^2 + S(\varepsilon,\alpha) + \tilde{E}(\varepsilon,\alpha)\,, \tag{6.2}$$

where S is given by

$$S(\varepsilon,\alpha) = -\frac{2}{3(m-1)}\int_{B_\varepsilon} |du_\alpha|^2\, r_{ij} x^i x^j\, dx + \int_{B_\varepsilon} s^J u_\alpha^2\, dx\,, \tag{6.3}$$

while $\tilde{E}$ is given by

$$\tilde{E} = \frac{2}{m-1}\int_{B_{2\varepsilon}-B_\varepsilon} \left(2\eta u_\alpha \langle d\eta, du_\alpha\rangle + u_\alpha^2 |d\eta|^2\right) d\mu_g + \int_{B_{2\varepsilon}-B_\varepsilon} s^J \eta^2 u_\alpha^2\, d\mu_g + E(\varepsilon,\alpha). \tag{6.4}$$

We would like to prove that we can choose ε and α so that $\tilde{E}$ is negligible in comparison with S, and furthermore, use Theorem 6.4 to make a choice of normal coordinates so that the contribution of S in (6.2) is negative. This will imply that the infimum of $\lambda_N^J(\varphi)$ over the unit sphere in $L^N(M)$ is strictly less than the universal bound $2m\omega_{2m}^{\frac{1}{m}}$, that in turn implies the solvability of the almost complex Yamabe problem.

Let us start with the analysis of the term $S(\varepsilon,\alpha)$. We carry out the integration in geodesic polar coordinates centered at the point p. For simplicity of notation, we set $n = 2m$ and $r = |x|$. Then

$$\begin{aligned}
S(\varepsilon,\alpha) &= \omega_{n-1}\int_0^\varepsilon r^{n-1}\left(-\frac{r^2}{3m(m-1)} s\, |du_\alpha|^2 + s^J u_\alpha^2\right) dr \\
&= \omega_{n-1}\int_0^\varepsilon r^{n-1}\left(\frac{r^2+\alpha^2}{\alpha}\right)^{-n}\left[\frac{r^4}{\alpha^2}\left(s^J - \frac{4(m-1)}{3m} s\right) + 2s^J r^2 + s^J \alpha^2\right] dr \\
&= \omega_{n-1}\alpha^2 \int_0^{\frac{\varepsilon}{\alpha}} \sigma^{n-1}(1+\sigma^2)^{-n}\left(\sigma^4\left(s^J - \frac{4(m-1)}{3m} s\right) + 2s^J\sigma^2 + s^J\right) d\sigma\,,
\end{aligned}$$

which follows from (6.3) by simple manipulations plus the fact that $x^i x^j$ integrates over the unit sphere to $\delta^{ij}\omega_{n-1}/n$.

Let $\overline{W}$ be the symmetrization of the tensor $-\sum_i W(e_i, \cdot, J\cdot, Je_i)$. Its trace is equal to the function $W(\omega,\omega)$. We take as p any point of our manifold where this function is strictly positive, for instance, a maximum, and for convenience, we denote its value at the said point by $W(p)$. We then apply Theorem 6.4 to choose normal coordinates and a metric $\tilde{g} = e^{2f}g$ about p, such that

$$\tilde{r}_{ij}(p) = T \stackrel{\text{def}}{=} \ell\overline{W},$$

where ℓ is a positive parameter whose value shall be specified later on. Since f is a homogeneous polynomial of degree 2, we have that $T = \tilde{T}$ at p. From now on, we replace g by $\tilde{g}$, and continue with the argument started above.

By our choice of conformal normal coordinates, the value of the scalar curvature at p is given by $s(p) = \ell W(p)$ and, by Proposition 4.2,

$$s^J(p) = \frac{1}{2m-1}s(p) + 2W(p) = \frac{4m-2+\ell}{2m-1}W(p),$$

so we have that

$$\frac{\sigma^4\left(s^J - \frac{4(m-1)}{3m}s\right) + 2s^J\sigma^2 + s^J}{W(p)} = \frac{4m-2+\ell}{2m-1}\left(1 + 2\sigma^2 + \left(1 - \frac{4(m-1)(2m-1)}{3m(4m-2+\ell)}\ell\right)\sigma^4\right).$$

Given $\varepsilon > 0$, let us define

$$F^\varepsilon(\alpha, l) = \int_0^{\frac{\varepsilon}{\alpha}} \sigma^{n-1}\left(\sigma^2+1\right)^{-n}\left(1 + 2\sigma^2 + \left(1 - \frac{4(m-1)(2m-1)}{3m(4m-2+\ell)}\ell\right)\sigma^4\right)d\sigma\,.$$

Then we have that

$$S(\varepsilon,\alpha) = \omega_{n-1}\,\alpha^2\frac{4m-2+\ell}{2m-1}F^\varepsilon(\alpha,\ell),$$

and its asymptotic behavior is manifestly dependent upon the behavior of the improper integral

$$\int_0^\infty \sigma^{n-1}\left(\sigma^2+1\right)^{-n}\left(1+2\sigma^2+\left(1-\tfrac{4(m-1)(2m-1)}{3m}\right)\sigma^4\right)d\sigma = \lim_{\alpha\to 0^+}\lim_{\ell\to\infty} F^\varepsilon(\alpha,\ell)\,.$$

Lemma 6.5. *Let*

$$I_n(a) = \int_0^a \sigma^{n-1}(\sigma^2+1)^{-n}\left(1+2\sigma^2+\left(1-\tfrac{4(m-1)(2m-1)}{3m}\right)\sigma^4\right)d\sigma\,.$$

Then,

a) *For $n = 2m > 4$, $I_n(a)$ converges to a negative value as $a \to \infty$.*
b) *For $n = 2m = 4$, we have that*

$$\lim_{a\to\infty}\frac{I_4(a)}{\log a} = -1\,.$$

Proof. Assertion (b) is quite obvious. As for (a), observe that

$$\begin{aligned}
\int_0^\infty \sigma^{2m-1}\left(\sigma^2+1\right)^{-2m} d\sigma &= \frac{(m-1)!^2}{2(2m-1)!}, \\
\int_0^\infty \sigma^{2m-1}\left(\sigma^2+1\right)^{-2m} \sigma^2 d\sigma &= \frac{m(m-1)!^2}{2(m-1)(2m-1)!}, \\
\int_0^\infty \sigma^{2m-1}\left(\sigma^2+1\right)^{-2m} \sigma^4 d\sigma &= \frac{m\left(m^2-1\right)(m-2)!^2}{2(m-2)(2m-1)!}.
\end{aligned}$$

Using these results and some elementary manipulations, it then follows that

$$I_n = I_{2m} = -\frac{(m-1)^2(2m-1)^2((m-2)!)^2}{3(m-2)(2m-1)!},$$

expression that is negative in the given range for m. □

We use this Lemma to finish our argument. Since $F^\varepsilon(\alpha, \ell)$ is a continuous function of (α, ℓ), we can choose α sufficiently small in comparison with ε and ℓ large enough so that, for some positive constant $C(m)$, we have that

$$S(\varepsilon, \alpha) \leq \begin{cases} -C(m)W(p)\alpha^2 & \text{if } m > 2 \\ -C(m)W(p)\alpha^2 \log(1/\alpha) & \text{if } m = 2\,. \end{cases}$$

The error term $\tilde{E}(\varepsilon, \alpha)$ in (6.4) may be estimated by the same arguments in [8] (see p. 50–51). In fact, we easily conclude that the first two integrals in the right side of (6.4) are of the order $O(\alpha^{2m-2})$, negligible in comparison with $S(\varepsilon, \alpha)$. This leaves us with the task of estimating the contribution to $\tilde{E}(\varepsilon, \alpha)$ given by $E(\varepsilon, \alpha)$. A rerun of the arguments above show that the integral corresponding to the $O(r^3)$ term in (6.1) is of the order $O(\alpha^3)$, while the integral over the domain $B_{2\varepsilon} - B_\varepsilon$ is of the order $O(\alpha^{2m-2})$ for values of α sufficiently small in comparison with ε.

Combining our estimates, we see that

$$\lambda_N^J(\varphi)\,\|\varphi\|_N^2 \leq \begin{cases} 2m\omega_{2m}^{\frac{1}{m}}\|\varphi\|_{L^N}^2 - CW(p)\alpha^2 + o\left(\alpha^2\right) & \text{if } m > 2 \\ 2m\omega_{2m}^{\frac{1}{m}}\|\varphi\|_{L^N}^2 - CW(p)\alpha^2\log(1/\alpha) + O\left(\alpha^2\right) & \text{if } m = 2 \end{cases}$$

and therefore, by selecting α sufficiently small, we conclude that

$$\lambda^J(M, [g]) = \inf \lambda_N^J(\varphi) < 2m\omega_{2m}^{\frac{1}{m}},$$

as desired.

We summarize our conclusions into the following.

Theorem 6.6. *For any manifold $(M^{2m}, J, g) \in W^+$, we have the strict inequality $\lambda^J(M, [g]) < 2m\omega_{2m}^{1/m}$, and therefore, the almost Hermitian Yamabe problem can be solved by a minimizer of the functional $\lambda_g^J(\varphi)$.*

Example 6.7. Consider the complex projective space $\mathbb{CP}^n$ with the Fubini-Study metric g. Since this metric has constant scalar curvature, it is a critical point of λ_N^J. However, it is not the only critical point, and in fact, its critical value is rather large, far from that associated to the minimum of the functional. For the Fubini-Study metric has scalar curvature $s_g = 4n(n+1)$, and volume $\mu_g = \pi^n/n!$, so the value of the almost complex Yamabe functional for it is $4n(n+1)\pi/(n!)^{\frac{1}{n}}$, value strictly greater than the universal bound $2n\omega_{2n}^{\frac{1}{n}} = 4n\pi\big(2/(2n-1)(2n-3)\cdots 3\cdot 1\big)^{\frac{1}{n}}$. Our result then says that, in the conformal class of the Fubini-Study metric, there exists another metric $\tilde{g}$ whose J-scalar curvature $s_{\tilde{g}}^J$ is strictly less than the value of the almost complex Yamabe functional computed at g. Obviously, this other critical metric is not Kähler.

This result is interesting in its own right, and somewhat surprising. The Fubini-Study metric is not canonical in the sense that it does not realize the infimum of λ_N^J. In fact, if we compute the Hessian of this functional at g seeking directions along which we may decrease its values, we see that this can be accomplished by moving along eigenfunctions of the Laplacian associated with the first non-zero eigenvalue λ_1. However, it is not clear how to keep pushing this argument once λ_N^J decreases all the way to the value λ_1, and we want to go past it.

Besides the interesting fact observed above, this example illustrates the non-uniqueness of critical points in the positive case.

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