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A Newly-Discovered Ancient Value for the Length of the Year

Eliyahu Beller

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1.

Maimonides opens the Ninth Chapter of his treatise *Sanctification of the New Moon*[1] as follows:

> Concerning the solar year, some Sages of Israel say that it consists of 365 days and $\frac{1}{4}$ day, which is 6 h, and some say it is less than $\frac{1}{4}$ day.

In Chap. 10 he writes

> Among the Sages of Israel who say that the solar year is less than [365 and] $\frac{1}{4}$, there are some who say it consists of 365 days 5 h 997 parts[2] (*ḥalaqim*) and 48 *rĕga'im*, where the *rega'* is $\frac{1}{76}$ of a part.

The value of $365\frac{1}{4}$ days for the length of the tropical year was already stated by the third-century talmudic sage Shĕmuel in the Babylonian Talmud.[3] The second value,

$$365^{\mathrm{d}}5^{\mathrm{h}}997^{\mathrm{p}}\tfrac{12}{19} = 365^{\mathrm{d}}5^{\mathrm{h}}55^{\mathrm{m}}25^{\mathrm{s}}\tfrac{25}{57}, \tag{1}$$

is first attested in al-Khwarizmi's treatise on the Jewish Calendar[4] written in the year 823/824.

Maimonides' wording in the above passage from Chap. 10, "... there are some who say ...", implies that he knows of yet another Jewish value for the length of the solar year, which is also less than $365\frac{1}{4}$ days. But in the medieval literature on the Hebrew Calendar we find only the two values quoted above; the third value seems to have disappeared.

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1 A translation into English by S. Gandz was published as *The Code of Maimonides, Sanctification of the New Moon,* Yale Univ. Press, New Haven, 1956. For the quotations in this paper, I found it preferable to do my own translating from the Hebrew.

2 Maimonides had earlier defined a part (*ḥeleq*) to be $\frac{1}{1080}$ of an hour.

3 'Eruvin 56a.

4 See E.S. Kennedy, "Al-Khwārizmī on the Jewish Calendar", *Scripta Mathematica* **27** (1964), 55–59, and T. Langermann, "When was the Hebrew Calendar established? Its antiquity according to al-Khwarizmi's treatise", *Asuppot* Vol. 1, Jerusalem 5747 (1987), 159–168, (in Hebrew).

In this note, we will prove that a statement made in the Talmud Yerushalmi *circa* 300 C.E.[5] presupposes a length of the year which is different from the above two. In particular, we will demonstrate in Sect. 2 that according to the Yerushalmi, the number of months (M) in a 49-year period lies in the following interval:

$$606M \leqq 49y < 606\tfrac{1}{23}M, \tag{2}$$

where M itself, the length of the mean synodic month, satisfies

$$29^{\mathrm{d}}12^{\mathrm{h}}792^{\mathrm{p}} \leqq M \leqq 29^{\mathrm{d}}12^{\mathrm{h}}797^{\mathrm{p}}, \tag{3}$$

where $\mathrm{p} = \frac{1}{1080}$ h. The short interval (3) contains the value

$$M = 29^{\mathrm{d}}12^{\mathrm{h}}793^{\mathrm{p}} \tag{4}$$

which is attested in the Babylonian Talmud,[6] and is the value used in the fixed Jewish Calendar.[7] Using this value of M in (2), we obtain

$$365^{\mathrm{d}}5^{\mathrm{h}}9^{\mathrm{m}}44^{\mathrm{s}}\tfrac{44}{49} \leqq y < 365^{\mathrm{d}}5^{\mathrm{h}}47^{\mathrm{m}}28^{\mathrm{s}}\tfrac{2782}{3381}. \tag{5}$$

The values $365\frac{1}{4}$ and (1) lie outside this interval, so that (5) contains the "lost" Jewish value for the length of the tropical year.

Actually, none of the ancient lengths of the year lie in the interval (5). Hipparchus found the length of the tropical year to be $365^{\mathrm{d}}5^{\mathrm{h}}55^{\mathrm{m}}12^{\mathrm{s}}$, and this was the value used by Ptolemy;[8] the other Greek values were higher.[9] As for the year-lengths used in Babylonian astronomy, Neugebauer[10] showed that they lie in the following range:

$$365^{\mathrm{d}}5^{\mathrm{h}}51^{\mathrm{m}}36^{\mathrm{s}} \leqq y \leqq 365^{\mathrm{d}}6^{\mathrm{h}}27^{\mathrm{m}}12^{\mathrm{s}}.$$

Thus, we have located an otherwise unattested ancient value for the length of the year.

This find was anticipated in 1751 by the talmudist Rabbi Shlomo of Chelm in his monograph *Bĕrekhoth bĕḤeshbon.*[11] Starting with the value (4) of M as an axiom, Rabbi Shlomo showed that the pertinent passage in the Talmud Yerushalmi could not have been working with a year of $365\frac{1}{4}$ days. He went on to demonstrate that the Yerushalmi's statement works out if you let $y = 365^{\mathrm{d}}5^{\mathrm{h}}816^{\mathrm{p}} = 365^{\mathrm{d}}5^{\mathrm{h}}45^{\mathrm{m}}20^{\mathrm{s}}$. This is the value attributed to al-Battani[12] by the 14th-century commentator to Maimonides'

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5 "C.E." stands for Common Era, of which the present year is 1997.

6 Rosh Hashana 25a. See the discussion in Beller, "Ancient Jewish mathematical astronomy", *Archive for History of Exact Sciences* **38** (1988), 51–66, pp. 63, 64.

7 See Beller[6], p. 64, note 66.

8 O. Neugebauer, *A History of Ancient Mathematical Astronomy,* Springer-Verlag, Berlin-Heidelberg-New York, 1975, p. 293.

9 See, for example, Neugebauer[8], pp. 584, 585, 601, 602.

10 Neugebauer[8], p. 529, uses sexagesimal notation: $365;14,39^{\mathrm{d}} \leqq y \leqq 365;16,8^{\mathrm{d}}$.

11 Appended to his book *Mirkeveth ha-Mishneh,* Frankfurt an der Oder, 5511 (1751).

12 In the version of al-Battani's work which came down to us, the value is $365^{\mathrm{d}}5^{\mathrm{h}}46^{\mathrm{m}}24^{\mathrm{s}}$ (see Al-Battani, *Opus Astronomicum*, ed. C.A. Nallino, Part I, Rome 1903, p. 128). Of course, al-Battani's value is not relevant for our purposes, since he lived six centuries after the statement in the Yerushalmi was made.

Sanctification of the New Moon. It was Rabbi Shlomo's verification of selected values which inspired the global investigation conducted in this paper.

In a previous paper,[13] I discussed the evidence that the talmudic calendar council possessed a lunar theory. The value in (5) may have been a parameter in such a theory. If it was anywhere near the right-hand endpoint of interval (5), that would make it the most accurate year-length of antiquity.[14]

2.

A. The Talmudic Passage

At the end of Tractate Sukkah of the Talmud Yerushalmi (the Talmud of the Land of Israel), there is a discussion involving the priestly watches which served in the Temple in Jerusalem. The priests (*kohănim*) were divided into 24 (ordered) watches; each watch served for one week, from Sabbath to Sabbath. The Mishnah described how the watch of Bilgah was permanently penalized for misconduct. The Talmud asked, why wasn't the watch of Bilgah eliminated by dispersing the individual priests of Bilgah among the other 23 watches, so that there would be only 23 watches? The answer was given:

> you could not do that, because . . . a great art was involved [in setting up 24 watches], for no watch is awarded fields of possession twice until each watch is awarded once. Rabbi Avahu (*fl.* 300 C.E.) said: I calculated them, and no watch is awarded fields of possession twice until each watch is awarded once.

The explanation: If a person consecrated his field of possession (that is, a field from his ancestral heritage) and later another man redeemed it by paying its valuation to the Temple treasury, then when Jubilee comes, the field is awarded free of charge to the members of the priestly watch in whose watch the beginning of the Jubilee year falls.[15] The Talmud's statement above can be paraphrased as follows: In a period of 24 consecutive Jubilees, each of the 24 watches will be awarded fields of possession exactly once, with no repeats. If, however, there had been only 23 watches, then there would be at least one repeat in a period of 23 Jubilees, so that at least one watch would be awarded twice, and another would not be awarded at all.

B. The Jubilee Period – 49 Years or 50 Years?

It is universally agreed that 49 years were counted, and that the 50th year was Jubilee. The question is, from which year was the next Jubilee period counted? According to one opinion,[16] the Jubilee year itself was the first year of the next Jubilee period, so

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[13] Beller[6].

[14] According to the formula in the *American Ephemeris,* the length of the tropical year in 300 C.E. was $365^{d}5^{h}48^{m}54.5^{s}$.

[15] If the beginning of Jubilee falls on a Sabbath, the award goes to the outgoing watch.

[16] Babylonian Talmud Rosh Hashanah 9a, Nĕdarim 61a, Arakhin 12b.

that each period was 49 years. Another opinion[16] held that the counting began the year *after* the Jubilee year, so that the Jubilee periods were 50 years each. We do not know a priori which opinion Rabbi Avahu followed; therefore we will test both hypotheses. In Subsection D we will prove that the 50-year hypothesis doesn't work, thus proving that the passage in the Yerushalmi worked with a 49-year period.

C. The Length of the Month (*M*)

We already saw in Sect. 1 that $M = 29^{\mathrm{d}}12^{\mathrm{h}}793^{\mathrm{p}}$ is the only value attested in the Talmud and in the fixed Hebrew Calendar. However, the midrash called *Pirqey dĕRabbi Eli'ezer* (Chap. 7) uses the round value[17]

$$M = 29^{\mathrm{d}}12^{\mathrm{h}}\tfrac{2}{3} = 29^{\mathrm{d}}12^{\mathrm{h}}720^{\mathrm{p}}$$

for the purpose of presenting certain schematic lunar cycles. In order to leave no stone unturned, we will investigate an extended interval which contains the two attested values comfortably in its interior:

$$29^{\mathrm{d}}12^{\mathrm{h}}700^{\mathrm{p}} \leqq M \leqq 29^{\mathrm{d}}12^{\mathrm{h}}820^{\mathrm{p}}. \tag{6}$$

We will prove in Subsection E that the value of M used in the above-quoted passage of the Talmud Yerushalmi lies in interval (3) of Sect. 1, an interval of length 6 parts (20 s) containing $29^{\mathrm{d}}12^{\mathrm{h}}793^{\mathrm{p}}$. Thus, as an added dividend of the present investigation, we will be able to deduce that Rabbi Avahu almost surely used the parameter $M = 29^{\mathrm{d}}12^{\mathrm{h}}793^{\mathrm{p}}$ two centuries after it was stated by Rabban Gamliel, Prince of Israel.[18]

D. Elimination of the 50-Year Hypothesis

A sequence of 24 consecutive Jubilees contains 23 Jubilee periods. In this subsection we will work with the hypothesis that a Jubilee period is 50 years. Our interest is the number of weeks in a 50-year period. But each year, and therefore each Jubilee period, consists of an integral number of lunar months. Thus, we must first clarify the number of months in 50 years. As a first orientation, let us start with the relation from the well-known intercalation cycle: 19 years = 235 months. According to that relation, 50 years equals $50 \times \frac{235}{19} = 618\frac{8}{19}$ months. Thus, in a sequence of 23 consecutive Jubilee periods, some Jubilee periods will consist of 618 months, and others – 619 months.

More generally, let

$$50 \text{ years} = 618 + x \text{ months}, \quad \text{where } 0 \leqq x < 1. \tag{7}$$

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[17] The phrase "and 73 parts" in the printed editions (and some mss.) is definitely a gloss, because it contradicts all the descriptions of the lunar cycles given in Chap. 7. The 73 parts are indeed absent from the medieval citations.

[18] See Beller[6], p. 63.

In order to obtain the sequence of watches in which 24 consecutive Jubilees fall, one must first calculate the exact sequence of 618's and 619's in 23 consecutive Jubilee periods, which we will call the *month sequence.* If a calculator (such as Rabbi Avahu) started with specific value[19] of x in (7), he would naturally round down to an integral number of months, and carry over the remaining fraction of a month to the next Jubilee period. For example, if we have $x = \frac{3}{8}$, the first eight terms in the month sequence would be

$$618, 618, 619, 618, 618, 619, 618, 619,$$

and then the sequence would repeat cyclicly. This resulting month sequence of 23 terms is valid not only for $x = \frac{3}{8}$, but for a certain interval containing the point $x = \frac{3}{8}$.

In order to carry out our investigation, we need to partition the interval $0 \leqq x < 1$ into subintervals such that all the x's in any subinterval lead to the same month sequence of 23 terms. Each endpoint of one of these subintervals is a critical point; as x passes through the critical point, the month sequence changes. A critical point c, then, is characterized by the following property: for some natural number $n \leqq 23$, $[nx] = m - 1$ for x in some left-hand neighborhood of c, while $[nx] = m$ for x in a right-hand neighborhood of c, where $[\cdot]$ denotes the greatest integer function. This implies that $c = \frac{m}{n}$. Note that the point $x = \frac{m}{n}$ itself leads to the same month sequence as the points in the right-hand neighbhorhood. In summary, the endpoints of our subintervals are all fractions of the form $\frac{m}{n}$, $1 \leqq n \leqq 23$, $0 \leqq m < n$, where n and m are integers. If we reduce these fractions to lowest terms and arrange them in ascending order, we obtain the so-called *Farey sequence*[20] of order 23. The subintervals we are looking for are precisely those of the form $[k, k')$, where k, k' are consecutive terms of the Farey sequence. We will call them *Farey intervals.*

We can now start with the first Jubilee year, which begins in the week of watch number 1. For our purposes, the watch's week begins on Sunday.[21] Thus, it would be natural for a calculator (such as Rabbi Avahu) to start with the first day of the first Jubilee year falling on a Sunday.[22] Now we must compute the number of weeks (modulo 24) and days in a Jubilee period of 618 or 619 months; this depends on the value M for the length of the month which was used by the calculator. Our program is to test all values of M in the interval (6). The lowest value in (6) is $M = 29^{\mathrm{d}}12^{\mathrm{h}}700^{\mathrm{p}}$; for that value we have

$$618M = 2,606^{\mathrm{w}}5^{\mathrm{d}}16^{\mathrm{h}}600^{\mathrm{p}} = 14^{\mathrm{w}}5^{\mathrm{d}}16^{\mathrm{h}}600^{\mathrm{p}} \pmod{24^{\mathrm{w}}}, \tag{8}$$

$$619M = 2,611^{\mathrm{w}}0^{\mathrm{d}}5^{\mathrm{h}}220^{\mathrm{p}} = 19^{\mathrm{w}}0^{\mathrm{d}}5^{\mathrm{h}}220^{\mathrm{p}} \pmod{24^{\mathrm{w}}}.$$

In reality, each period of 618 or 619 months consists of an integral number of days. However, there is no need to work with “day sequences”; since we are really interested

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19 which he obtained from his value of the year-length, or from a luni-solar intercalation cycle.

20 See G.H. Hardy & E.M. Wright, *An Introduction to the Theory of Numbers,* 5th ed., Oxford, 1979, p. 23 (where they are called “Farey series”).

21 See note 15.

22 We will return to this question at the end of Subsection E.

in the number of weeks, and the fractional part of a week, we can just as well express the fractional part of a week in days, hours, and parts.

For any value of x in (7), the first Jubilee period consists of 618 months. Thus, if the first Jubilee begins on $1^{\mathrm{w}}0^{\mathrm{d}}0^{\mathrm{h}}0^{\mathrm{p}}$, then using (8) (for $M = 29^{\mathrm{d}}12^{\mathrm{h}}700^{\mathrm{p}}$) we obtain that the second Jubilee would begin on $15^{\mathrm{w}}5^{\mathrm{d}}16^{\mathrm{h}}600^{\mathrm{p}}$, where "w" can be read as "watch", or as "weeks modulo 24" (with the proviso that we write "24^{w}" instead of "0^{w}"). Now for x in any Farey interval, we compute the month sequence, and then determine the watches in which consecutive Jubilees fall, according to (8), until one watch repeats. (See the Appendix for an example of such a table according to the 49-year period).

For the 50-year hypothesis we tested the following intervals:

$$29^{\mathrm{d}}12^{\mathrm{h}}700^{\mathrm{p}} \leqq M < 29^{\mathrm{d}}12^{\mathrm{h}}734^{\mathrm{p}}, \quad 618\tfrac{1}{3}M \leqq 50y < 618\tfrac{2}{3}M,$$

and

$$29^{\mathrm{d}}12^{\mathrm{h}}734^{\mathrm{p}} \leqq M \leqq 29^{\mathrm{d}}12^{\mathrm{h}}820^{\mathrm{p}}, \quad 618\tfrac{1}{3}M \leqq 50y < 618\tfrac{4}{7}M.$$

This covers the following range[23] of values for the length of the year y:

$$365^{\mathrm{d}}4^{\mathrm{h}}59.37^{\mathrm{m}} \leqq y \leqq 365^{\mathrm{d}}7^{\mathrm{h}}22.76^{\mathrm{m}}, \tag{9}$$

which surely includes all reasonable values of y. They all produced repeats. This proves that Rabbi Avahu did not work with a 50-year Jubilee period.

E. The Proof of Inequalities (2) and (3)

We have now established that Rabbi Avahu used a 49-year Jubilee period. To get a first orientation of the number of months in 49 years, we mutliply 49 by the well-known luni-solar ratio $\frac{235}{19}$ to obtain $606\frac{1}{19}$ months. Accordingly, using the methodology and notation of Subsection D, we investigated the intervals

$$29^{\mathrm{d}}12^{\mathrm{h}}700^{\mathrm{p}} \leqq M < 29^{\mathrm{d}}12^{\mathrm{h}}792^{\mathrm{p}}, \quad 605\tfrac{9}{10}M \leqq 49y < 606\tfrac{1}{2}M,$$

and

$$29^{\mathrm{d}}12^{\mathrm{h}}792^{\mathrm{p}} \leqq M \leqq 29^{\mathrm{d}}12^{\mathrm{h}}820^{\mathrm{p}}, \quad 605\tfrac{11}{12}M \leqq 49y < 606\tfrac{4}{23}M,$$

which cover the following range[23] of values for y:

$$365^{\mathrm{d}}4^{\mathrm{h}}15.98^{\mathrm{m}} \leqq y \leqq 365^{\mathrm{d}}7^{\mathrm{h}}39.99^{\mathrm{m}}. \tag{10}$$

It turns out that only the three following cases produce a sequence of watches in 24 consecutive Jubilees with no repeated watches:

$$\begin{array}{lll}
\text{I.} & 29^{\mathrm{d}}12^{\mathrm{h}}792^{\mathrm{p}} \leqq M \leqq 29^{\mathrm{d}}12^{\mathrm{h}}797^{\mathrm{p}}, & 606M \leqq 49y < 606\tfrac{1}{23}M. \\
\text{II.} & 29^{\mathrm{d}}12^{\mathrm{h}}798^{\mathrm{p}} \leqq M \leqq 29^{\mathrm{d}}12^{\mathrm{h}}799^{\mathrm{p}}, & 605\tfrac{22}{23}M \leqq 49y < 606M. \\
\text{III.} & M = 29^{\mathrm{d}}12^{\mathrm{h}}792^{\mathrm{p}}, & 606\tfrac{1}{8}M \leqq 49y < 606\tfrac{3}{23}M.
\end{array}$$

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[23] Of course, we obtain a different y-interval for each value of M. The interval in (9) or (10) is the *intersection* of all of those intervals.

All other cases produce repeats. (Note that the fractional parts of the range of $49y$ in the above three cases are the Farey intervals $[0, \frac{1}{23})$, $[\frac{22}{23}, 1)$, $[\frac{1}{8}, \frac{3}{23})$ of order 23.)

Case II leads to the following range for y:

$$365^{\mathrm{d}}4^{\mathrm{h}}35.45^{\mathrm{m}} \leqq y \leqq 365^{\mathrm{d}}5^{\mathrm{h}}13.87^{\mathrm{m}},$$

which is unreasonably low; the right-hand endpoint is over three quarters of an hour less than the value $365^{\mathrm{d}}6^{\mathrm{h}}$ attested in the Babylonian Talmud. Case III implies

$$365^{\mathrm{d}}6^{\mathrm{h}}57.54^{\mathrm{m}} \leqq y \leqq 365^{\mathrm{d}}7^{\mathrm{h}}2.26^{\mathrm{m}},$$

an unreasonably high range for y. Thus, by elimination we are left with Case I, which consists of the inequalities (2) and (3) discussed in Sect. 1. The Appendix shows the watches in the beginnings of 24 consecutive Jubilees according to I, with $M = 29^{\mathrm{d}}12^{\mathrm{h}}793^{\mathrm{p}}$. Note that for $606M \leqq 49y < 606\frac{1}{23}M$, all 23 Jubilee periods consist of 606 months.

Recall that the passage quoted from the Talmud Yerushalmi stated that if there had been only 23 watches, there would have been a repeat. Indeed, a simple computation for Case I shows that 23 watches lead to a repeat in the first 8 Jubilees. As an extra check, we found that if there were 25 watches, there would be a repeat in the first 13 Jubilees. This verifies that a "great art" was involved in setting up exactly 24 watches.

Until now, we have been working under the assumption that in Rabbi Avahu's calculations, the first Jubilee year began on a Sunday,[24] the first day of the week. This assumption seems reasonable, but one may ask: perhaps Rabbi Avahu meant to say that there would be no repeats, no matter which day of the week the first Jubilee began? The answer is that no values of y and M satisfy this last requirement, and thus our Sunday assumption in justified. Below is a table showing on which days of the week the beginning of the first Jubilee must fall, in order for there to be no repeats in 24 consecutive Jubilees.[25]

	Case	Jubilee must begin
I,	$M = 29^{\mathrm{d}}12^{\mathrm{h}}792^{\mathrm{p}}$	Sunday through Wednesday
I,	$M = 29^{\mathrm{d}}12^{\mathrm{h}}793^{\mathrm{p}}/794^{\mathrm{p}}$	Sunday through Tuesday
I,	$M = 29^{\mathrm{d}}12^{\mathrm{h}}795^{\mathrm{p}}$	Sunday, Monday
I,	$M = 29^{\mathrm{d}}12^{\mathrm{h}}796^{\mathrm{p}}/797^{\mathrm{p}}$	Sunday
II, III		Sunday.

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[24] To be more exact, a day which begins on Saturday night, for the Jewish day begins with the night.

[25] The allowable days for Case I can be deduced from the Appendix.

Appendix

The Watches in 24 Consecutive Jubilees

$$M = 29^{\mathrm{d}}12^{\mathrm{h}}793^{\mathrm{p}}; \quad 606M \leqq 49y < 606\tfrac{1}{23}M.$$

$$606M = 12^{\mathrm{w}}3^{\mathrm{d}}12^{\mathrm{h}}1038^{\mathrm{p}} (\mathrm{mod}\ 24^{\mathrm{w}}).$$

Table 1.

Jubilee	watch	days	hours	parts
1	1	0	0	0
2	13	3	12	1038
3	2	0	1	996
4	14	3	14	954
5	3	0	3	912
6	15	3	16	870
7	4	0	5	828
8	16	3	18	786
9	5	0	7	744
10	17	3	20	702
11	6	0	9	660
12	18	3	22	618
13	7	0	11	576
14	19	4	0	534
15	8	0	13	492
16	20	4	2	450
17	9	0	15	408
18	21	4	4	366
19	10	0	17	324
20	22	4	6	282
21	11	0	19	240
22	23	4	8	198
23	12	0	21	156
24	24	4	10	114

Department of Mathematics
Bar-Ilan University
Ramat Gan, Israel

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