ON THE CONSTRAINT AND SCALING METHODS TO DERIVE THE EQUATIONS OF LINEARLY ELASTIC RODS

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ABSTRACT. The theory of linearly elastic rods may be obtained from three-dimensional elasticity either by the method of internal constraints or by the scaling method. Both methods have been applied to obtain linear plate and shell equations ([1], [2]–[5]); the relationships between the two methods are discussed in [6]. For rods, a version of the constraint method has been developed in [7], whereas a scaling method has been presented in [12]. In this paper a direct comparison is made between the mechanical basis and analytical results of the constraint and the scaling methods, and it is shown how the scaling method yields the same Kirchhoff hypothesis that forms the starting point of the constraint method.

SOMMARIO. La teoria delle travi elastiche lineari puó essere ottenuta a partire dalla teoria tridimensionale dell'elasticitá tanto con il metodo dei vincoli interni che con il metodo di riscalamento. Entrambi i metodi sono giá stati utilizzati per ottenere le equazioni delle piastre e dei gusci lineari ([1], [2]–[5]); in quel contesto, le relazioni tra i due metodi sono state discusse in [6]. Una versione del metodo dei vincoli appropriata al caso delle travi é stata sviluppata in [7], mentre i metodi di riscalamento per le travi si trovano esposti in [12]. Scopo di questo lavoro é compiere un paragone diretto tra i fondamenti meccanici e le risultanze analitiche, rispettivamente, del metodo dei vincoli e di quello di riscalamento, mostrando come il metodo di riscalamento imponga di accogliere proprio quelle ipotesi alla Kirchhoff sulle quali si basa il metodo dei vincoli interni.

KEYWORDS. Rod theories, Internal constraints, Anisotropic bodies, Scaling methods.

1. INTRODUCTION

In this paper I wish to show how the theory of linearly elastic thin rods can be obtained in two alternative ways, namely, with the method of internal constraints and with the scaling method. The first method, introduced in [2] to derive the Germain–Lagrange equation for a thin plate from three-dimensional elasticity, was later applied to sheets [3], laminated plates [4] and shells [5]; for thin rods [7] the method yields, among others, some classical results due to Lamb and Love (see [8]).

It is important to remark that this method has nothing to do with the 'constraint' method proposed by Volterra [9] which consists only in an *a priori* assumption on the displacement field but misses the point of reactive stresses maintaining such displacement. Indeed, the internal constraints I use in this work are closer to the 'material' constraints introduced by Antman and Warner [10] who, however, do not use them to obtain an explicit representation of the admissible displacement field.

The scaling method, introduced in [1], has been applied to a wide class of problems (see [11]), including nonlinearly elastic straight slender rods [12].

In a recent paper, the relationships between the two methods were discussed in the context of the theory of shells [6]. Here I take up the same issue for the theory of rods, and show not only that the two methods yield the same equilibrium equations, but also that the scaling method leads to the same Kirchhoff hypothesis I used as a starting point when I applied the method of internal constraints to thin rods.

2. ROD-LIKE BODIES AND THE METHOD OF INTERNAL CONSTRAINTS

The application of the constraint method to rod-like bodies is presented in full in [7]; here I recall the main results obtained there.

Let $\mathscr{C}$ be a simple, regular curve in the three-dimensional Euclidean space $\mathscr{E}$ (with an associated vector space $\mathscr{V}$), i.e. an invertible, smooth map $\zeta \mapsto \mathbf{x}(\zeta)$ from an interval $]0, L[\subset \mathbb{R}$ into $\mathscr{E}$, with ζ the arc-length; at a typical point $\mathbf{x}$ of $\mathscr{C}$, the orthonormal triad of vectors $\{\mathbf{n}(\mathbf{x}), \mathbf{b}(\mathbf{x}), \mathbf{t}(\mathbf{x})\}$ (i.e. the *normal*, *binormal* and *tangent* unit vector, respectively) form the *Frenet frame.*

For a *rod-like region, modelled on $\mathscr{C}$ and having radius ρ*, I mean a tubular region $\mathscr{T}(\rho)$ with typical point $\mathbf{p}$ such that the mapping

$$\mathbf{p} = \mathbf{p}(z^1, z^2, \zeta) := \mathbf{x}(\zeta) + z^1\mathbf{n}(\mathbf{x}(\zeta)) + z^2\mathbf{b}(\mathbf{x}(\zeta)) \tag{2.1}$$

is a global parametrization of $\mathscr{T}(\rho)$, with $(z^1, z^2) \in \mathscr{D}_0(\rho)$, the disk of $\mathbb{R}^2$ of radius ρ centered at the origin (for the notion of a tubular region, see [13]). At each point $\mathbf{x} \in \mathscr{C}$, the points $\mathbf{p}$ such that

$$(\mathbf{p}(z^1, z^2, \zeta) - \mathbf{x}(\zeta)) \cdot \mathbf{t}(\mathbf{x}(\zeta)) = 0 \tag{2.2}$$

compose a disk of radius ρ centered at $\mathbf{x}$, the *cross-section*

$\mathscr{S}_\rho(\mathbf{x})$ of $\mathscr{T}(\rho)$ at $\mathbf{x}$. At a point $\mathbf{p} \in \mathscr{T}(\rho)$ the natural basis is (here and later I follow the common abuse of notation and denote the functions of ζ and $\mathbf{x}$ by the same symbols)

$$\begin{aligned} &\mathbf{g}_1(\mathbf{p}) = \mathbf{n}(\mathbf{x}), \quad \mathbf{g}_2(\mathbf{p}) = \mathbf{b}(\mathbf{x}), \\ &\mathbf{g}_3(\mathbf{p}) = \mathbf{t}(\mathbf{x}) + z^1\mathbf{n}_{,\zeta}(\mathbf{x}) + z^2\mathbf{b}_{,\zeta}(\mathbf{x}); \end{aligned} \tag{2.3}$$

this basis is related to the dual one through the formula:

$$\mathbf{g}_i(\mathbf{p}) \cdot \mathbf{g}^j(\mathbf{p}) = \delta_i^j \tag{2.4}$$

(here δ_i^j denotes the Kronecker delta, the Latin indices run from 1 to 3, while the Greek ones will run from 1 to 2); for $\mathbf{x} \in \mathscr{C}$ and $(z^1, z^2) \in \mathscr{D}_0(\rho)$ fixed, the shifter

$$\begin{aligned} &\mathbf{A}(\mathbf{p}, \mathbf{x}) = \mathbf{g}_1(\mathbf{p}) \otimes \mathbf{n}(\mathbf{x}) + \mathbf{g}_2(\mathbf{p}) \otimes \mathbf{b}(\mathbf{x}) + \mathbf{g}_3(\mathbf{p}) \otimes \mathbf{t}(\mathbf{x}), \\ &(\mathbf{B}(\mathbf{p}, \mathbf{x}) = \mathbf{g}^1(\mathbf{p}) \otimes \mathbf{n}(\mathbf{x}) + \mathbf{g}^2(\mathbf{p}) \otimes \mathbf{b}(\mathbf{x}) + \mathbf{g}^3(\mathbf{p}) \otimes \mathbf{t}(\mathbf{x})) \end{aligned} \tag{2.5}$$

transforms the Frenet frame at $\mathbf{x}$ into the natural (dual) basis at $\mathbf{p} = \mathbf{x}(\zeta) + z^1\mathbf{n}(\zeta) + z^2\mathbf{b}(\zeta)$.

The area element of the cross-section and the volume element of a rod-like region transform respectively as:

$$\begin{aligned} &\mathbf{m}(\mathbf{p})\, \mathrm{d}s(\mathbf{p}) = \mathbf{g}_1(\mathbf{p})\, \mathrm{d}z^1 \times \mathbf{g}_2(\mathbf{p})\, \mathrm{d}z^2 = \mathbf{t}(\mathbf{x})\, \mathrm{d}s(\mathbf{x}) \\ &\mathrm{d}v(\mathbf{p}) = \mathbf{m}(\mathbf{p})\, \mathrm{d}s(\mathbf{p}) \cdot \mathbf{g}_3(\mathbf{p})\, \mathrm{d}\zeta = \alpha(\mathbf{p}, \mathbf{x})\, \mathrm{d}v(\mathbf{x}), \end{aligned} \tag{2.6}$$

where $\mathbf{m}(\mathbf{p})$ is the normal unit vector to the cross-section at $\mathbf{p}$ and $\alpha := \det \mathbf{A}$; the request that (2.1) be a global parametrization for $\mathscr{T}(\rho)$ implies that $\alpha > 0$. Notice that, as

$$\mathbf{A}(\mathbf{p}, \mathbf{x}) = \mathbf{1}(\mathbf{x}) + (z^1\mathbf{n}_{,\zeta} + z^2\mathbf{b}_{,\zeta}) \otimes \mathbf{t}(\mathbf{x}),$$

with

$$\mathbf{1} = \mathbf{n} \otimes \mathbf{n} + \mathbf{b} \otimes \mathbf{b} + \mathbf{t} \otimes \mathbf{t},$$

one has that

$$\begin{aligned} \alpha &= 1 + (z^1\mathbf{n}_{,\zeta} + z^2\mathbf{b}_{,\zeta}) \cdot \mathbf{t} = 1 - z^1\mathbf{n} \cdot \mathbf{t}_{,\zeta} - z^2\mathbf{b} \cdot \mathbf{t}_{,\zeta} \\ &= 1 - \kappa z^1, \end{aligned}$$

where κ is the *curvature* of $\mathscr{C}$; thus, the local invertibility condition for the mapping (2.1) does not involve the *torsion* τ of $\mathscr{C}$.

For $\varphi = \varphi(\mathbf{p})$ an integrable field over $\mathscr{T}(\rho)$, and for $\mathscr{P}$ a rod-like part of $\mathscr{T}(\rho)$ modelled on the portion $\mathscr{C}_{\mathscr{P}} = \{\mathbf{x} \in \mathscr{C} | \mathbf{x} = \mathbf{x}(\zeta), \zeta \in]\delta, \varepsilon[\}$ of $\mathscr{C}$, one has that

$$\int_{\mathscr{P}} \varphi(\mathbf{p})\, \mathrm{d}v(\mathbf{p}) = \int_\delta^\varepsilon \hat{\varphi}(\mathbf{x}(\zeta))\, \mathrm{d}\zeta,$$

$$\hat{\varphi}(\mathbf{x}) := \int_{\mathscr{S}_\rho(\mathbf{x})} \varphi(\mathbf{p}) \alpha(\mathbf{p}, \mathbf{x})\, \mathrm{d}s(\mathbf{p}). \tag{2.7}$$

For a thin rod the method of internal constraints rests on giving the status of an exact mathematical restriction, effective throughout a rod-like region $\mathscr{T}(\rho)$, to the classical Kirchhoff hypothesis that each cross-section remains rigid and orthogonal to the curve into which the model curve $\mathscr{C}$ is deformed (see [8]). Within the framework of linear elasticity, these restrictions take the form

$$\mathbf{E}(\mathbf{u})\mathbf{g}_\alpha = \mathbf{0} \quad (\alpha = 1, 2) \text{ in } \mathscr{T}(\rho), \qquad \mathbf{E}(\mathbf{u}) = \tfrac{1}{2}(D\mathbf{u} + D\mathbf{u}^{\mathrm{T}}), \tag{2.8}$$

where $D\mathbf{u} = \mathbf{u}_{,\alpha} \otimes \mathbf{g}^\alpha + \mathbf{u}_{,\zeta} \otimes \mathbf{g}^3$ denotes the gradient of $\mathbf{u}$ with respect to $\mathbf{p}$ (here $(\cdot)_{,\alpha}$ denotes partial differentiation with respect to z^α). Without any loss of generality I write $\mathbf{u}$ as

$$\mathbf{u}(\mathbf{p}) = {}^s\mathbf{u}(\mathbf{p}) + u(\mathbf{p})\mathbf{t}(\mathbf{x}), \qquad {}^s\mathbf{u} \cdot \mathbf{t} = 0, \quad u = \mathbf{u} \cdot \mathbf{t}.$$

Then, (2.8) becomes

$$\begin{aligned} &{}^s\mathbf{u}_{,1} + u_{,1}\mathbf{t} + ({}^s\mathbf{u}_{,\beta} \cdot \mathbf{n})\mathbf{g}^\beta + ({}^s\mathbf{u}_{,\zeta} \cdot \mathbf{n} + u\mathbf{t}_{,\zeta} \cdot \mathbf{n})\mathbf{g}^3 = \mathbf{0}, \\ &{}^s\mathbf{u}_{,2} + u_{,2}\mathbf{t} + ({}^s\mathbf{u}_{,\beta} \cdot \mathbf{b})\mathbf{g}^\beta + ({}^s\mathbf{u}_{,\zeta} \cdot \mathbf{b})\mathbf{g}^3 = \mathbf{0}. \end{aligned} \tag{2.9}$$

The scalar system obtained by taking the dot product of (2.9) with $\mathbf{n}$ and $\mathbf{b}$ yields the following expression for ${}^s\mathbf{u}$:

$${}^s\mathbf{u}(\mathbf{p}) = {}^s\mathbf{v}(\mathbf{x}) + \omega(\mathbf{x})(\mathbf{n} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{n})(\mathbf{p} - \mathbf{x}), \quad {}^s\mathbf{v} \cdot \mathbf{t} = 0, \tag{2.10}$$

with ${}^s\mathbf{v} = {}^s\mathbf{v}(\mathbf{x})$, $\omega = \omega(\mathbf{x})$ a vector and a scalar field on $\mathscr{C}$. Taking the inner product of (2.9) and $\mathbf{t}$, and using (2.10), one has the following system for $u = u(\mathbf{p})$:

$$\begin{aligned} &\alpha u_{,1} + \kappa u + {}^s\mathbf{v}_{,\zeta} \cdot \mathbf{n} - z^2\omega_{,\zeta} = 0 \\ &\alpha u_{,2} + {}^s\mathbf{v}_{,\zeta} \cdot \mathbf{b} + z^1\omega_{,\zeta} = 0, \end{aligned} \tag{2.11}$$

whose integrability condition $u_{,12} = u_{,21}$ implies that

$$\omega_{,\zeta} - \kappa u_{,2} = 0, \tag{2.12}$$

and whose solution is

$$u(\mathbf{p}) = \alpha v(\mathbf{x}) - {}^s\mathbf{v}_{,\zeta} \cdot (\mathbf{p} - \mathbf{x}), \tag{2.13}$$

where $v = v(\mathbf{x})$ is a scalar field defined on $\mathscr{C}$. Relations (2.10) and (2.13) yield the following displacement solution of (2.8) in terms of the field $\mathbf{v}(\mathbf{x}) := {}^s\mathbf{v}(\mathbf{x}) + v(\mathbf{x})\mathbf{t}(\mathbf{x})$ (see [7])

$$\mathbf{u}(\mathbf{p}) = \mathbf{v}(\mathbf{x}) + \mathbf{\Omega}(\mathbf{v}(\mathbf{x}))(\mathbf{p} - \mathbf{x}) \tag{2.14}$$

where $\mathbf{\Omega}$, a skew-symmetric tensor, is given by

$$\mathbf{\Omega}(\mathbf{v}(\mathbf{x})) = \mathbf{v}_{,\zeta} \otimes \mathbf{t} - \mathbf{t} \otimes \mathbf{v}_{,\zeta} + \omega(\mathbf{x})(\mathbf{n} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{n}), \tag{2.15}$$

and, as $\mathbf{v}_{,\zeta} = {}^s\mathbf{v}_{,\zeta} + v_{,\zeta}\mathbf{t} + \kappa v\mathbf{n}$,

$$\omega_{,\zeta} - \kappa(\mathbf{v}_{,\zeta} \cdot \mathbf{b}) = \mathbf{\Omega}_{,\zeta}\mathbf{b} \cdot \mathbf{n} = 0. \tag{2.16}$$

For $\mathbf{p} \in \mathscr{T}(\rho)$ fixed, $\mathscr{M}_\mathbf{p} = \operatorname{span}\{\mathbf{t}(\mathbf{x}) \otimes \mathbf{t}(\mathbf{x})\}$ is the subspace of all symmetric tensors satisfying (2.8) at $\mathbf{p}$; notice that

$$\mathscr{M}_\mathbf{p} = \mathscr{M}_\mathbf{x}$$

for all points $\mathbf{p}$ in the cross-section $\mathscr{S}_\rho(\mathbf{x})$. As customary in the presence of internal frictionless constraints such as (2.8), one splits the stress $\mathbf{T}$ into a reactive part $\mathbf{T}^{(\mathrm{R})}$ and an active part $\mathbf{T}^{(\mathrm{A})}$:

$$\begin{aligned} &\mathbf{T} = \mathbf{T}^{(\mathrm{A})} + \mathbf{T}^{(\mathrm{R})}, \\ &\mathbf{T}^{(\mathrm{A})} = \mathbb{C}[\mathbf{E}], \qquad \mathbf{T}^{(\mathrm{A})} \in \mathscr{M}_\mathbf{p}, \\ &\mathbf{T}^{(\mathrm{R})} \in (\mathscr{M}_\mathbf{p})^\perp. \end{aligned} \tag{2.17}$$

One also requires that the linear map $\mathbb{C}: \mathscr{M}_\mathbf{p} \to \mathscr{M}_\mathbf{p}$ reflects the *maximal* material symmetry compatible with the imposed constraints, i.e. (see [14])

$$\mathbb{C} = \mathrm{E}(\mathbf{t}\otimes\mathbf{t})\otimes(\mathbf{t}\otimes\mathbf{t}), \tag{2.18}$$

with $\mathrm{E} > 0$ a material modulus; the constitutive relation (2.18) describes a transversely isotropic, elastic material such as to comply with the kinematical internal constraints expressed by (2.8).

For this material, the stored-energy functional associated with a displacement field $\mathbf{u}$ (2.14) is

$$\Sigma(\mathbf{u}) = \int_{\mathscr{T}(\rho)} \sigma(\mathbf{u})\,\mathrm{d}v(\mathbf{p}),$$

$$\sigma(\mathbf{u}) = \tfrac{1}{2}\mathbf{E}(\mathbf{u})\cdot\mathbb{C}[\mathbf{E}(\mathbf{u})] = \tfrac{1}{2}\mathrm{E}(\mathbf{E}(\mathbf{u})\cdot\mathbf{t}\otimes\mathbf{t})^2; \tag{2.19}$$

in view of (2.7) and (2.14) one has:

$$\Sigma(\mathbf{u}) = \hat{\Sigma}(\mathbf{v}) = \int_0^L \hat{\sigma}(\mathbf{v})\,\mathrm{d}\zeta,$$

$$\hat{\sigma}(\mathbf{v}) := \int_{\mathscr{S}_\rho(\mathbf{x})} \alpha\sigma(\mathbf{v} + \boldsymbol{\Omega}(\mathbf{v})(\mathbf{p}-\mathbf{x}))\,\mathrm{d}s(\mathbf{p}). \tag{2.20}$$

Adding the load potential one obtains the total energy functional for the thin elastic rod; field and boundary equations for the unknown displacement field $\mathbf{v}(\mathbf{x})$ on $\mathscr{C}$ can be obtained using standard variational procedures. Notice that the problem of minimizing the total energy functional over the class of smooth displacement fields $\mathbf{v}(\mathbf{x})$ restricted by (2.16) is equivalent to the problem of minimizing, over all smooth field $\mathbf{v}(\mathbf{x})$ on $\mathscr{C}$, the functional:

$$\Phi(\mathbf{v}) := \bar{\Sigma}(\mathbf{v}) + \int_{\mathscr{T}(\rho)} \mathbf{q}\cdot\mathbf{u}(\mathbf{v})\,\mathrm{d}v(\mathbf{p}) + \int_{\partial_\ell\mathscr{T}(\rho)} \mathbf{s}\cdot\mathbf{u}(\mathbf{v})\,\mathrm{d}s(\mathbf{p}) + {}$$
$$+ \int_{\mathscr{T}(\rho)} \gamma\boldsymbol{\Omega}_{,\zeta}(\mathbf{v})\mathbf{b}\cdot\mathbf{n}\,\mathrm{d}v(\mathbf{p}), \tag{2.21}$$

where $\mathbf{q} = \mathbf{q}(\mathbf{p})$ are the volume loads, $\mathbf{s} = \mathbf{s}(\mathbf{p})$ the surface loads acting on the lateral surface of $\mathscr{T}(\rho)$ defined as:

$$\partial_\ell\mathscr{T}(\rho) := \{\mathbf{p} = \mathbf{x}(\zeta) + z^1\mathbf{n}(\mathbf{x}(\zeta)) + {}$$
$$+ z^2\mathbf{b}(\mathbf{x}(\zeta))\,|\,\zeta\in]0,L[,\ (z^1)^2 + (z^2)^2 = \rho^2\},$$

$\gamma = \gamma(\mathbf{p})$ is a Lagrange multiplier (whose nature will be rendered explicit in the following Remark 1) and $\mathbf{u} = \mathbf{u}(\mathbf{v})$, $\boldsymbol{\Omega} = \boldsymbol{\Omega}(\mathbf{v})$ are given by (2.14)–(2.16). As the strain field satisfying (2.8) is:

$$\mathbf{E} = \alpha^{-1}((\mathbf{v}_{,\zeta} + \boldsymbol{\Omega}_{,\zeta}(\mathbf{p}-\mathbf{x}))\cdot\mathbf{t})\mathbf{t}\otimes\mathbf{t}, \tag{2.22}$$

(2.21), (2.7) and (2.17) yield

$$\Phi(\mathbf{v}) = \int_0^L\int_{\mathscr{S}_\rho(\mathbf{x})} (\tfrac{1}{2}\alpha^{-1}\mathrm{E}(\mathbf{v}_{,\zeta} + \boldsymbol{\Omega}_{,\zeta}(\mathbf{p}-\mathbf{x}))\cdot\mathbf{t})^2\,\mathrm{d}s(\mathbf{x})\,\mathrm{d}\zeta + {}$$
$$+ \int_0^L \Gamma\boldsymbol{\Omega}_{,\zeta}\mathbf{b}\cdot\mathbf{n}\,\mathrm{d}\zeta + {}$$
$$+ \int_0^L\int_{\mathscr{S}_\rho(\mathbf{x})} \alpha\mathbf{q}\cdot(\mathbf{v} + \boldsymbol{\Omega}(\mathbf{p}-\mathbf{x}))\,\mathrm{d}s(\mathbf{x})\,\mathrm{d}\zeta + {}$$
$$+ \int_0^L\int_{\partial\mathscr{S}_\rho(\mathbf{x})} \mathbf{s}\cdot(\mathbf{v} + \boldsymbol{\Omega}(\mathbf{p}-\mathbf{x}))\,\mathrm{d}\ell(\mathbf{x})\,\mathrm{d}\zeta,$$

$$\Gamma := \int_{\mathscr{S}_\rho(\mathbf{x})} \alpha\gamma\,\mathrm{d}s(\mathbf{x}). \tag{2.23}$$

The Euler–Lagrange equations associated with the functional (2.23) are:

$$\{\mathrm{E}[\mathrm{A}(\mathbf{v}_{,\zeta}\cdot\mathbf{t}) + \boldsymbol{\Omega}_{,\zeta}\mathbf{h}\cdot\mathbf{t}]\mathbf{t} + {}$$
$$+ \mathbf{t}\times[(\mathrm{E}\mathbf{t}\times(\boldsymbol{J}\boldsymbol{\Omega}_{,\zeta}\mathbf{t} - \mathbf{h}(\mathbf{v}_{,\zeta}\cdot\mathbf{t})))_{,\zeta} + \mathbf{c}]\}_{,\zeta} + {}$$
$$+ (\Gamma\kappa\mathbf{b})_{,\zeta} + \mathbf{f} = \mathbf{0}, \tag{2.24}$$
$$\Gamma_{,\zeta} + \mathbf{t}\cdot\{[\mathrm{E}\mathbf{t}\times(\boldsymbol{J}\boldsymbol{\Omega}_{,\zeta}\mathbf{t} - \mathbf{h}(\mathbf{v}_{,\zeta}\cdot\mathbf{t}))]_{,\zeta} + \mathbf{c}\} = 0,$$

where

$$\mathbf{f} := \int_{\mathscr{S}_\rho(\mathbf{x})} \alpha\mathbf{q}\,\mathrm{d}s(\mathbf{p}) + \int_{\partial\mathscr{S}_\rho(\mathbf{x})} \mathbf{s}\,\mathrm{d}\ell(\mathbf{p}),$$

$$\mathbf{c} := \int_{\mathscr{S}_\rho(\mathbf{x})} \alpha(\mathbf{p}-\mathbf{x})\times\mathbf{q}\,\mathrm{d}s(\mathbf{p}) + \int_{\partial\mathscr{S}_\rho(\mathbf{x})} (\mathbf{p}-\mathbf{x})\times\mathbf{s}\,\mathrm{d}\ell(\mathbf{p}),$$

$\mathrm{d}\ell(\mathbf{p})$ is the line element on $\partial\mathscr{S}_\rho(\mathbf{x})$, and the *area* A, the vector $\boldsymbol{h}$ and the *Euler tensor* $\boldsymbol{J}$ of the cross-section are:

$$\mathrm{A} := \int_{\mathscr{S}_\rho(\mathbf{x})} \frac{\mathrm{d}s(\mathbf{p})}{\alpha}, \qquad \boldsymbol{h} := \int_{\mathscr{S}_\rho(\mathbf{x})} \frac{(\mathbf{p}-\mathbf{x})\,\mathrm{d}s(\mathbf{p})}{\alpha},$$

$$\boldsymbol{J} := \int_{\mathscr{S}_\rho(\mathbf{x})} \frac{(\mathbf{p}-\mathbf{x})\otimes(\mathbf{p}-\mathbf{x})\,\mathrm{d}s(\mathbf{p})}{\alpha}. \tag{2.25}$$

From equations (2.24) an expression for Γ can be derived, and substituted back in $(2.24)_1$ to obtain the following field equations for $\mathbf{v}$:

$$\mathrm{E}(\kappa\mathbf{b})_{,\zeta}\times\{[(\mathrm{A}(\mathbf{v}_{,\zeta}\cdot\mathbf{t}) - \boldsymbol{h}\cdot(\mathbf{v}_{,\zeta\zeta} - \kappa\omega\mathbf{b} + \kappa\mathbf{n}(\mathbf{v}_{,\zeta}\cdot\mathbf{t})))\mathbf{t}]_{,\zeta} + {}$$
$$+ [\mathbf{t}\times(\boldsymbol{J}\mathbf{v}_{,\zeta\zeta} - \kappa(\mathbf{v}_{,\zeta}\cdot\mathbf{t})\boldsymbol{J}\mathbf{n} + \kappa\omega\boldsymbol{J}\mathbf{b})_{,\zeta} + \boldsymbol{h}(\mathbf{v}_{,\zeta}\cdot\mathbf{t})]_{,\zeta} + {}$$
$$- \kappa\mathbf{b}[\mathbf{t}\cdot(\boldsymbol{J}\mathbf{v}_{,\zeta\zeta} - \kappa(\mathbf{v}_{,\zeta}\cdot\mathbf{t})\boldsymbol{J}\mathbf{n} + \kappa\omega\boldsymbol{J}\mathbf{b})_{,\zeta}$$
$$- \kappa\mathbf{b}\cdot\boldsymbol{h}(\mathbf{v}_{,\zeta}\cdot\mathbf{t})]\} + \boldsymbol{\ell} = \mathbf{0}, \tag{2.26}$$
$$\mathrm{E}\kappa\mathbf{b}\cdot[\boldsymbol{J}\mathbf{v}_{,\zeta\zeta} - \kappa(\mathbf{v}_{,\zeta}\cdot\mathbf{t})\boldsymbol{J}\mathbf{n} + \kappa\omega\boldsymbol{J}\mathbf{b} - \boldsymbol{h}(\mathbf{v}_{,\zeta}\cdot\mathbf{t})] + {}$$
$$+ \mathrm{E}(\kappa\mathbf{b})_{,\zeta}\|(\kappa\mathbf{b})_{,\zeta}\|^{-2}\cdot\{\kappa\mathbf{b}\cdot[(\mathbf{v}_{,\zeta\zeta} - \kappa(\mathbf{v}_{,\zeta}\cdot\mathbf{t})\boldsymbol{J}\mathbf{n} + {}$$
$$+ \kappa\omega\boldsymbol{J}\mathbf{b})\kappa\mathbf{b} - (\mathbf{v}_{,\zeta}\cdot\mathbf{t})(\mathbf{h}_{,\zeta\zeta} + (\boldsymbol{h}\cdot\mathbf{n})\mathbf{n} - (\boldsymbol{h}\cdot\mathbf{b})\mathbf{b})] + {}$$
$$+ \boldsymbol{h}\cdot(\mathbf{v}_{,\zeta\zeta} - \kappa\omega\mathbf{b} + \kappa\mathbf{n}(\mathbf{v}_{,\zeta}\cdot\mathbf{t}))\mathbf{t} - \mathrm{A}(\mathbf{v}_{,\zeta}\cdot\mathbf{t})\mathbf{t} + \mathbf{t}\times{}$$
$$\times[\mathbf{t}\times(\boldsymbol{J}\mathbf{v}_{,\zeta\zeta} - \kappa(\mathbf{v}_{,\zeta}\cdot\mathbf{t})\boldsymbol{J}\mathbf{n} + \kappa\omega\boldsymbol{J}\mathbf{b})]_{,\zeta}\}_{,\zeta\zeta} + \mu = 0;$$

$$\omega_{,\zeta} - \kappa(\mathbf{v}_{,\zeta}\cdot\mathbf{b}) = 0. \tag{2.16}$$

where the load terms are defined to be:

$$\boldsymbol{\ell} = (\kappa\mathbf{b})_{,\zeta}\times(\mathbf{f} - \kappa\mathbf{b}(\mathbf{c}\cdot\mathbf{t}) + (\mathbf{t}\times\mathbf{c})_{,\zeta}),$$

$$\mu = \mathbf{t}\cdot\mathbf{c} + (\kappa\mathbf{b})_{,\zeta}\|(\kappa\mathbf{b})_{,\zeta}\|^{-2}\cdot\{\kappa\mathbf{b}(\mathbf{c}\cdot\mathbf{t}) - \mathbf{f} - (\mathbf{t}\times\mathbf{c})_{,\zeta}\}.$$

REMARK 1. Equations (2.26) can be obtained from the equilibrium equations expressed in terms of the stress and couple resultants, $\mathbf{N}$ and $\mathbf{M}$ (see [15]):

$$\mathbf{N}_{,\zeta} + \mathbf{f} = \mathbf{0}, \qquad \mathbf{M}_{,\zeta} + \mathbf{t} \times \mathbf{N} + \mathbf{c} = \mathbf{0}$$

$$\mathbf{N}(\mathbf{x}) := \int_{\mathscr{S}_\rho(\mathbf{x})} \mathbf{T}(\mathbf{p})\mathbf{t}(\mathbf{x})\, \mathrm{d}s(\mathbf{p}),$$

$$\mathbf{M}(\mathbf{x}) := \int_{\mathscr{S}_\rho(\mathbf{x})} (\mathbf{p} - \mathbf{x}) \times \mathbf{T}(\mathbf{p})\mathbf{t}(\mathbf{x})\, \mathrm{d}s(\mathbf{p}). \tag{2.27}$$

The method of derivation consists of the following steps (see [7] for further details). First, the additive decomposition of $\mathbf{T}$ into an active and a reactive part leads to a similar decomposition for $\mathbf{N}$ and $\mathbf{M}$:

$$\begin{aligned} &\mathbf{N} = \mathbf{N}^{(A)} + \mathbf{N}^{(R)}, \qquad \mathbf{N}^{(A)} \times \mathbf{t} = \mathbf{0}, \qquad \mathbf{N}^{(R)} \cdot \mathbf{t} = 0 \\ &\mathbf{M} = \mathbf{M}^{(A)} + \mathbf{M}^{(R)}, \qquad \mathbf{M}^{(A)} \cdot \mathbf{t} = 0, \qquad \mathbf{M}^{(R)} \times \mathbf{t} = \mathbf{0}. \end{aligned} \tag{2.28}$$

With $(2.28)_1$ the equilibrium equations yield, after some algebra, both the expression of the resultants of the reactions:

$$\begin{aligned} &\mathbf{N}^{(R)} = \mathbf{t} \times (\mathbf{M}^{(A)}_{,\zeta} + \mathbf{c}) + \kappa(\mathbf{M}^{(R)} \cdot \mathbf{t})\mathbf{b} \\ &(\mathbf{M}^{(R)} \cdot \mathbf{t})_{,\zeta} = -\mathbf{t} \cdot (\mathbf{M}^{(A)}_{,\zeta} + \mathbf{c}) \end{aligned} \tag{2.29}$$

and the 'pure' i.e. reaction-free) equilibrium equations:

$$\begin{aligned} &(\kappa\mathbf{b})_{,\zeta} \times [\mathbf{N}^{(A)}_{,\zeta} + (\mathbf{t} \times \mathbf{M}^{(A)}_{,\zeta})_{,\zeta} - \kappa\mathbf{b}(\mathbf{t} \cdot \mathbf{M}^{(A)}_{,\zeta})] + \boldsymbol{\ell} = \mathbf{0}. \\ &[\|(\kappa\mathbf{b})_{,\zeta}\|^{-2}(\kappa\mathbf{b})_{,\zeta} \cdot (\mathbf{N}^{(A)}_{,\zeta} + \mathbf{t} \times \mathbf{M}^{(A)}_{,\zeta\zeta})]_{,\zeta} - \mathbf{t} \cdot \mathbf{M}^{(A)}_{,\zeta} - \mu = 0 \end{aligned} \tag{2.30}$$

Secondly, (2.22), (2.18) and (2.27) allow one to express $\mathbf{N}^{(A)}$ and $\mathbf{M}^{(A)}$ in terms of the displacement field (2.14), (2.15) as:

$$\begin{aligned} &\mathbf{N}^{(A)} = \mathrm{E}(A(\mathbf{v}_{,\zeta} \cdot \mathbf{t}) + \boldsymbol{\Omega}_{,\zeta}\boldsymbol{h} \cdot \mathbf{t})\mathbf{t}, \\ &\mathbf{M}^{(A)} = \mathbf{t} \times (\mathrm{E}\boldsymbol{J}\boldsymbol{\Omega}_{,\zeta}\mathbf{t} - \boldsymbol{h}(\mathbf{v}_{,\zeta} \cdot \mathbf{t})); \end{aligned} \tag{2.31}$$

$(2.26)_1$ is then obtained by substitution of (2.31) into (2.30). Finally, when the constitutive relation $(2.31)_2$ is inserted in $(2.29)_2$ one obtains that Γ, to within an unessential constant, equals the *torque* $\mathbf{M}^{(R)} \cdot \mathbf{t}$.

Notice that, when $\mathscr{C}$ is a *plane* curve with $\tau \equiv 0$, $(2.29)_1$ yields the classical relation between the shear $T = \mathbf{N}^{(R)} \cdot \mathbf{n}$ and the bending moment $M = \mathbf{M}^{(A)} \cdot \mathbf{b}$:

$$T = -M_{,\zeta} - \mathbf{c} \cdot \mathbf{b}. \qquad \square$$

REMARK 2. The boundary equations accompanying (2.24) can be given in terms of $\mathbf{N}$ and $\mathbf{M}$:

$$\mathbf{N} \cdot \mathbf{v} + \mathbf{M} \cdot \boldsymbol{\omega}\big|_0^L = 0, \quad \forall\, \mathbf{v}, \boldsymbol{\omega} \in \mathscr{V}, \tag{2.32}$$

where $\boldsymbol{\omega}$ is the axial vector of $\boldsymbol{\Omega}$. $\square$

3. THE SCALING METHOD FOR THIN RODS

Following [6] I give here a simplified version of the method, consisting of two steps: firstly, both the data and the solution are rescaled; secondly, the energy functional is required to stay bounded above under rescaling.

Let a rod-like region $\mathscr{T}(\rho)$ be made of a linearly elastic, isotropic unconstrained material described by the constitutive law

$$\mathbf{T} = 2\mu\mathbf{E} + \lambda(\operatorname{tr}\mathbf{E})\mathbf{1}, \qquad 2\mu + 3\lambda > 0, \quad \mu > 0 \tag{3.1}$$

where $\mathbf{1}$ is the identity tensor and μ, λ are the Lamé moduli. I assume that

$$\mu(\rho) = \rho^{-4}\bar{\mu}, \qquad \lambda(\rho) = \rho^{-4}\bar{\lambda}, \tag{3.2}$$

and that the region $\mathscr{T}(\rho)$ is mapped one-to-one onto a rod-like region $\bar{\mathscr{T}}$, modelled on a copy of $\mathscr{C}$ and having cross-section of unit radius. The typical point $\bar{\mathbf{p}}$ of $\bar{\mathscr{T}}$ is

$$\bar{\mathbf{p}} = \bar{\mathbf{x}}(\bar{\zeta}) + \bar{z}^1\mathbf{n} + \bar{z}^2\mathbf{b}, \qquad \bar{z}^\alpha \in \mathscr{D}_0(1), \quad \alpha = 1, 2; \tag{3.3}$$

$\bar{\mathscr{T}}$ is parametrized by

$$\bar{z}^\alpha = \rho^{-1}z^\alpha \quad (\alpha = 1, 2), \qquad \bar{\zeta} = \zeta. \tag{3.4}$$

As $\mathbf{x}(\zeta) = \bar{\mathbf{x}}(\bar{\zeta})$, (3.3) and (3.4) yield

$$\mathbf{p} - \mathbf{x} = \rho(\bar{\mathbf{p}} - \bar{\mathbf{x}}); \tag{3.5}$$

moreover, the natural basis transforms as

$$\mathbf{g}_\alpha = \bar{\mathbf{g}}_\alpha \quad (\alpha = 1, 2), \qquad \mathbf{g}_3 = \mathbf{x}_{,\zeta} + \rho(\bar{\mathbf{g}}_3 - \bar{\mathbf{x}}_{,\zeta}), \tag{3.6}$$

and the corresponding relations for the dual basis can be easily obtained from (3.5).

The scaling of data is completed by stipulating that the body force field $\mathbf{q} = \mathbf{q}(\mathbf{p})$ on $\mathscr{T}(\rho)$ and the surface traction field $\mathbf{s} = \mathbf{s}(\mathbf{p})$ on the lateral boundary $\partial_\ell\mathscr{T}$ are

$$\begin{aligned} \mathbf{q}(z^1, z^2, \zeta, \rho) &= \rho^{-2}(\bar{q}_1(\bar{z}^1, \bar{z}^2, \zeta)\mathbf{n} + \\ &\quad + \bar{q}_2(\bar{z}^1, \bar{z}^2, \zeta)\mathbf{b}) + \rho^{-3}\bar{q}_3(\bar{z}^1, \bar{z}^2, \zeta)\mathbf{t}, \\ \mathbf{s}(z^1, z^2, \zeta, \rho) &= \rho^{-1}(\bar{s}_1(\bar{z}^1, \bar{z}^2, \zeta)\mathbf{n} + \bar{s}_2(\bar{z}^1, \bar{z}^2, \zeta)\mathbf{b}) + \\ &\quad + \rho^{-2}\bar{s}_3(\bar{z}^1, \bar{z}^2, \zeta)\mathbf{t}. \end{aligned}$$

As to the solution, I have for the components of $\mathbf{E}$:

$$\begin{aligned} &2E_{\alpha\beta} = u_{\alpha,\beta} + u_{\beta,\alpha} \\ &2E_{13} = u_{3,1} + (1 - \kappa z^1)^{-1}(u_{1,\zeta} - \tau(u_2 + u_{1,2}z^1) + \kappa u_3) \\ &2E_{23} = u_{3,2} + (1 - \kappa z^1)^{-1}(u_{2,\zeta} + \tau(u_1 + u_{2,1}z^2)) \\ &E_{33} = (1 - \kappa z^1)^{-1}(u_{3,\zeta} + \tau(u_{3,1}z^2 - u_{3,2}z^1) + \kappa u_1); \end{aligned} \tag{3.7}$$

I shall confine my attention to situations when the following estimates hold:

$$\begin{aligned} &u_{3,\zeta} + \tau(u_{3,1}z^2 - u_{3,2}z^1) \gg \kappa u_1, \\ &u_{1,\zeta} - \tau(u_2 + u_{1,2}z^1) \gg \kappa u_3; \end{aligned} \tag{3.8}$$

I shall then rescale the displacement field as follows

$$\begin{aligned} &\mathbf{u}(z^1, z^2, \zeta, \rho) \\ &\quad = \bar{u}_1(\bar{z}^1, \bar{z}^2, \zeta)\mathbf{n} + \bar{u}_2(\bar{z}^1, \bar{z}^2, \zeta)\mathbf{b} + \rho\bar{u}_3(\bar{z}^1, \bar{z}^2, \zeta)\mathbf{t}. \end{aligned} \tag{3.9}$$

REMARK 3. Assumptions (3.8) are fulfilled if the position vector depends on the thickness parameter in the following way:

$$\begin{aligned} &\mathbf{p}(\rho) = \mathbf{x}(\rho) + \rho\bar{z}^1\mathbf{n}(\mathbf{x}(\rho)) + \rho\bar{z}^2\mathbf{b}(\mathbf{x}(\rho)), \\ &\mathbf{x}(\rho) = \rho\bar{x}_\alpha\mathbf{i}_\alpha + \zeta\mathbf{i}_3, \end{aligned} \tag{3.10}$$

(here the Cartesian orthogonal coordinates x_α, ζ are relative to the basic vectors $\mathbf{i}_\alpha$, $\mathbf{i}_3$). In this case the Frenet vectors and the curvature and torsion of $\mathscr{C}$ are:

$$\begin{aligned}
&\mathbf{n}(\mathbf{x}(\rho)) = \bar{n}_\alpha \mathbf{i}_\alpha + \rho \bar{n}_3 \mathbf{i}_3 + O(\rho_2),\\
&\mathbf{b}(\mathbf{x}(\rho)) = \bar{b}_\alpha \mathbf{i}_\alpha + \rho \bar{b}_3 \mathbf{i}_3 + O(\rho^2),\\
&\mathbf{t}(\mathbf{x}(\rho)) = \mathbf{i}_3 + \rho \bar{t}_\alpha \mathbf{i}_\alpha + O(\rho^2);\\
&\kappa = \rho\bar{\kappa}, \qquad \tau = \bar{\tau} + O(\rho).
\end{aligned} \tag{3.11}$$

It follows from (3.11) and (3.9) that the left-hand sides of (3.8) are $O(1)$ whereas the right-hand sides are $O(\rho)$. □

Thus, (3.7), (3.8) and (3.9) yield:

$$\begin{aligned}
&E_{\alpha\beta} = \rho^{-1}\bar{E}_{\alpha\beta}, \qquad E_{\alpha 3} = \bar{E}_{\alpha 3} + O(\rho),\\
&E_{33} = \rho \bar{E}_{33} + O(\rho^2)
\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}
&2\bar{E}_{\alpha\beta} = \bar{u}_{\alpha,\beta} + \bar{u}_{\beta,\alpha},\\
&2\bar{E}_{13} = \bar{u}_{3,1} + \bar{u}_{1,\zeta} - \bar{\tau}(\bar{u}_2 + \bar{u}_{1,2}\bar{z}^1),\\
&2\bar{E}_{23} = \bar{u}_{3,2} + \bar{u}_{2,\zeta} + \bar{\tau}(\bar{u}_1 + \bar{u}_{2,1}\bar{z}^2),\\
&\bar{E}_{33} = \bar{u}_{3,\zeta} + \bar{\tau}(\bar{u}_{3,1}\bar{z}^2 - \bar{u}_{3,2}\bar{z}^1).
\end{aligned} \tag{3.13}$$

I shall render now explicit the dependence of the energy functional on ρ. As to the stored energy, since

$$2\sigma(\mathbf{E}) = 2\mu\mathbf{E}\cdot\mathbf{E} + \lambda(\operatorname{tr}\mathbf{E})^2, \tag{3.14}$$

and since $\alpha\, ds(\mathbf{x}) = ds(\bar{\mathbf{x}}) + O(\rho)$, one has that

$$\begin{aligned}
&\int_{\mathscr{S}_\rho(\mathbf{x})} \alpha\sigma(\mathbf{E}(\mathbf{u}))\, ds(\mathbf{x})\\
&= \int_{\mathscr{S}(\bar{\mathbf{x}})} \sigma_0(\bar{\mathbf{E}})\, ds(\bar{\mathbf{x}}) + \rho^{-2}\int_{\mathscr{S}(\bar{\mathbf{x}})} \sigma_1(\bar{\mathbf{E}})\, ds(\bar{\mathbf{x}}) +\\
&\quad + \rho^{-4}\int_{\mathscr{S}(\bar{\mathbf{x}})} \sigma_2(\bar{\mathbf{E}})\, ds(\bar{\mathbf{x}}) + \rho^{-6}\int_{\mathscr{S}(\bar{\mathbf{x}})} \sigma_3(\bar{\mathbf{E}})\, ds(\bar{\mathbf{x}}),
\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}
&2\sigma_0(\bar{\mathbf{E}}) = (2\bar{\mu} + \bar{\lambda})\bar{E}_{33}^2\\
&2\sigma_1(\bar{\mathbf{E}}) = 4\bar{\mu}(\bar{E}_{23}^2 + \bar{E}_{13}^2) + \bar{\lambda}(\bar{E}_{11} + \bar{E}_{22})^2\\
&2\sigma_2(\bar{\mathbf{E}}) = 2\bar{\lambda}\bar{E}_{33}(\bar{E}_{11} + \bar{E}_{22})\\
&2\sigma_3(\bar{\mathbf{E}}) = \bar{\mu}(\bar{E}_{11}^2 + \bar{E}_{22}^2 + 2\bar{E}_{12}^2).
\end{aligned} \tag{3.16}$$

As to the load potential, one obtains that

$$\begin{aligned}
&\pi_v(\mathbf{u}) = \int_{\mathscr{S}_\rho(\mathbf{x})} \alpha\mathbf{q}\cdot\mathbf{u}\, ds(\mathbf{p}) = \int_{\mathscr{S}(\bar{\mathbf{x}})} \bar{b}_i\bar{u}_i\, ds(\bar{\mathbf{x}})\\
&\pi_s(\mathbf{u}) = \int_{\partial\mathscr{S}_\rho(\mathbf{x})} \mathbf{s}\cdot\mathbf{u}\, d\ell(\mathbf{p}) = \int_{\partial\mathscr{S}(\bar{\mathbf{x}})} \bar{s}_i\bar{u}_i\, d\ell(\bar{\mathbf{x}}).
\end{aligned} \tag{3.17}$$

I now assume that the total energy functional is bounded above under rescaling, in the following sense:

$$\lim_{\rho\to 0}\int_0^L (\hat{\sigma}(\mathbf{u}) + \pi_v(\mathbf{u}) + \pi_s(\mathbf{u}))\, d\zeta < +\infty; \tag{3.18}$$

(3.18) implies that $\sigma_1 = \sigma_2 = \sigma_3 = 0$ and therefore, as $\bar{\mu} > 0$ for (3.1) and (3.2):

$$\bar{E}_{\alpha\beta} = 0, \qquad \bar{E}_{\alpha 3} = 0, \tag{3.19}$$

but, in view of (3.8) and (3.12), (3.19) is equivalent to (2.8). In other words, the rescaling method yields precisely those kinematical restrictions that are the starting point of the constraint method.

Moreover, as the displacement field $\bar{\mathbf{u}}$ on $\bar{\mathscr{T}}$ can be obtained from (3.19) and $(3.13)_1$–$(3.13)_3$ the Euler–Lagrange equations associated with the total energy functional (3.18) (as ρ approaches zero) can also be obtained. Indeed, one uses first $(3.19)_1$ and $(3.13)_1$ to have:

$${}^s\bar{\mathbf{u}}(\bar{z}_1, \bar{z}_2, \bar{\zeta}) = {}^s\bar{\mathbf{v}}(\bar{\zeta}) + \bar{\omega}(\bar{\zeta})\mathbf{i}_3 \times (\bar{\mathbf{p}} - \bar{\mathbf{x}}), \tag{3.20}$$

where ${}^s\bar{\mathbf{v}}\cdot\mathbf{i}_3 = 0$, ${}^s\bar{\mathbf{u}}\cdot\mathbf{i}_3 = 0$, and $\bar{\omega} = \bar{\omega}(\bar{\zeta})$ is an unknown function; then, $(3.19)_2$ and $(3.13)_3$ lead to the following system for $\bar{u}_3$:

$$\begin{aligned}
&\bar{u}_{3,1} = \bar{v}_{1,\zeta} - \bar{\tau}\bar{v}_2 + \bar{\omega}_{,\zeta}\bar{z}^1,\\
&\bar{u}_{3,2} = \bar{v}_{2,\zeta} + \bar{\tau}\bar{v}_1 - \bar{\omega}_{,\zeta}\bar{z}^2,
\end{aligned} \tag{3.21}$$

which can be integrated, provided

$$\bar{\omega}_{,\zeta} = 0, \tag{3.22}$$

to give:

$$\bar{u}_3(\bar{z}_1, \bar{z}_2, \bar{\zeta}) = \bar{v}(\zeta) + {}^s\bar{\mathbf{v}}_{,\zeta}\cdot(\bar{\mathbf{p}} - \bar{\mathbf{x}}). \tag{3.23}$$

From (3.20), (3.23) and $(3.13)_4$ one finally obtains the deformation field $\bar{E}_{33}$:

$$\begin{aligned}
\bar{E}_{33} &= \bar{v}_{,\zeta} + ({}^s\bar{\mathbf{v}}_{,\zeta}\cdot(\bar{\mathbf{p}} - \bar{\mathbf{x}}))_{,\zeta} + \mathbf{i}_3\cdot\bar{\tau}\,{}^s\bar{\mathbf{v}}_{,\zeta}\times(\bar{\mathbf{p}} - \bar{\mathbf{x}})\\
&= \bar{v}_{,\zeta} + {}^s\bar{\mathbf{v}}_{,\zeta\zeta}\cdot(\bar{\mathbf{p}} - \bar{\mathbf{x}});
\end{aligned} \tag{3.24}$$

in view of (3.15), $(3.16)_1$, (3.17) and (3.18), the stationary of the total energy functional implies the following Euler–Lagrange equations:

$$\begin{aligned}
&\bar{\mathrm{E}}\bar{A}\bar{v}_{,\zeta\zeta} + \bar{\mathbf{f}}\cdot\mathbf{i}_3 = 0,\\
&\bar{\mathrm{E}}(\bar{J}\,{}^s\bar{\mathbf{v}}_{,\zeta\zeta})_{,\zeta\zeta} + \bar{\mathbf{c}}_{,\zeta} + \bar{\mathbf{f}}\times\mathbf{i}_3 = \mathbf{0},
\end{aligned} \qquad \bar{\mathrm{E}} := 2\bar{\mu} + \bar{\lambda}, \tag{3.25}$$

where $\bar{A}$, $\bar{J}$, $\bar{\mathbf{f}}$ and $\bar{\mathbf{c}}$ are defined just as the corresponding quantities introduced in Section 2 on $\bar{\mathscr{T}}$.

A direct comparison of these results with those obtained by the constraint method can be done when one takes the position vector of the curve $\mathscr{C}$ to be as in $(3.10)_2$. Equations (2.26) and (2.16) become, respectively:

$$\begin{aligned}
&\mathrm{E}Av_{,\zeta\zeta}\mathbf{i}_3 + \mathbf{i}_3\times \mathrm{E}(J\,{}^s\mathbf{v}_{,\zeta\zeta})_{,\zeta\zeta} + \mathbf{i}_3\times\mathbf{c}_{,\zeta} + \mathbf{f} + O(\rho) = \mathbf{0},\\
&\mathbf{i}_3\cdot\mathbf{c} = O(\rho),
\end{aligned} \tag{3.26}$$

$$\omega_{,\zeta} + O(\rho) = 0; \tag{3.27}$$

by (3.9), as ρ approaches zero, (3.26) and (3.27) are equivalent to (3.25) and (3.22), respectively.

Thus, the scaling method yields those kinematical restrictions that are the starting point of the constraint method, and, moreover, the two methods yield the same equilibrium equations.

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