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DYNAMICS OF TOPOLOGICAL DEFECTS IN CRITICAL BINARY FLUIDS, METAMAGNETS AND ^{3}He–^{4}He MIXTURES

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The dynamical theory of topological defects in critical and tricritical systems is presented. Starting with the bulk stochastic equation like the time dependent Ginzburg–Landau model we derive the equation of motion of the topological defects such as interfaces and vortices in a unified way. We are primarily concerned with critical binary fluids, metamagnets and ^{3}He–^{4}He mixtures. The method utilized here is based on the idea originally used by Thiele in his theory of magnetic bubbles.

1. Introduction

Topological defects such as interfaces, vortices and dislocations often play an important role in phase transitions. A typical example is the Kosterlitz–Thouless phase transition[1]) caused by the vortex and dislocation unbindings in superfluid helium and melting, respectively, in two dimensions. The statistical theory of topological defects of the vortex type has been reviewed by Halperin[2]). Some properties of dynamics of first order phase transitions are characterized by the motion of interfaces (domain walls). For instance the scaling law exhibited in the ordering process of quenched thermodynamically unstable systems has been shown to be described by the motion of random interfaces[3]). Thus deriving the equation of motion of topological defects and solving it are of great importance for further understanding of the dynamics of phase transitions.

The dynamical behavior of the systems near the critical point is described by a set of equations for the gross variables. Derivation of the equation of motion of topological defects from these bulk equations is generally a very difficult problem. However, if we assume that the singular domain of the defects specified, for instance, by the thickness of an interface and the core radius of a vortex line is negligibility small compared to other length scales of the problem, we can eliminate the bulk degrees of freedom systematically to extract the motion of the topological defects. So far the equations of motion of interfaces have been derived for the time-dependent Ginzburg–Landau (TDGL) model with a non-conserved[4,5]) and conserved order parameter[6]) (model A and B of ref. 7) and for critical fluids[6]) (model H). The equation of motion of vortex lines has also been obtained starting from model F[8]) and from its modified version[9]).

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In the present paper starting with the bulk equations for the gross variables we develop a systematic method to obtain the equation of motion of the topological defects. Here we employ Thiele's approach[10]) used in the theory of magnetic bubbles. Recently Kawasaki reformulated Thiele's method in a general way and applied it to the study of the motion of discommensurations and dislocations in modulated systems[11]). The basic idea of the Thiele's method is the following. Suppose that there is a topological defect. Existence of the defect violates the translational invariance of the system described by the bulk equations so that a type of Goldstone modes appear. We consider virtual displacement of the topological defect. This causes a distortion of the gross variables due to the Goldstone modes. The energy change by this virtual displacement is represented by the thermodynamic force in the bulk equation of motion. This energy change should be balanced with the work done by the real force which acts on the defect. Once we obtain the Hamiltonian of the topological defect the force is readily derived. In this way the static part is treated separately from the dynamic part, which is an advantage of this method.

The equation of motion which we derived takes the form

$$\boldsymbol{F}(\boldsymbol{a}) = \int \mathrm{d}\boldsymbol{a}' \mathbf{D}(\boldsymbol{a}, \boldsymbol{a}')(\boldsymbol{v}(\boldsymbol{a}') - \boldsymbol{\theta}(\boldsymbol{a}')), \tag{1.1}$$

where $\boldsymbol{F}$ is the force acting at the position $\boldsymbol{a}$ of the defect, $\boldsymbol{v}(\boldsymbol{a})$ the velocity of the defect and $\boldsymbol{\theta}$ the fluctuating random force. The most relevant point of the present theory is the derivation of $\mathbf{D}(\boldsymbol{a}, \boldsymbol{a}')$ in (1.1), which connects $\boldsymbol{F}$ with $\boldsymbol{v}$. The matrix $\mathbf{D}(\boldsymbol{a}, \boldsymbol{a}')$ contains all the dynamic information of the bulk system. Here we describe a systematic formulation to calculate $\mathbf{D}(\boldsymbol{a}, \boldsymbol{a}')$.

We apply the theory to critical fluids and the tricritical systems of metamagnets and ^{3}He–^{4}He mixtures. The dynamics of critical fluids (model H)[7]) is described by the conserved TDGL model accompanied by the transverse part of the momentum density. The dynamics of metamagnets is also described by the non-conserved order parameter with an auxiliary conserved density (model C)[7]). Since the order parameters are both real and scalar in these systems, there occurs an interface which separates two phases in the ordered state. Whereas, in ^{3}He–^{4}He mixtures, there are three gross variables[12]), i.e. the superfluid order parameter, the local concentration difference between ^{3}He and ^{4}He atoms and the local entropy field. Then, since the order parameter is a complex variable, we may generally have both the interfaces and vortex lines in an ordered state. One of the features of our formulation is that two such types of defects can be simultaneously represented by (1.1).

In section 2 we present the general formulation to obtain the equation of motion of the topological defects. $\mathbf{D}(\boldsymbol{a}, \boldsymbol{a}')$ is given by a diagonal representation of the

gross variables. In section 3 we apply this method to critical fluids (model H), metamagnets (model C) and ^{3}He–^{4}He mixtures. We derive the explicit form of the equation of motion of the topological defects, which describes both interfaces and vortex lines. In section 4 we study the dynamical behaviour of an almost planar interface. Analysis of a sufficiently smooth vortex line of ^{3}He–^{4}He mixtures[13]) is made on section 5. Finally we discuss the advantages of this method from various viewpoints in section 6. The technical details are summarized in the appendices.

2. General formulation

In order to derive the equation of motion of the topological defects, we take a set of the gross variables $S_\alpha(\boldsymbol{r}, t)$ ($\alpha = 1, 2, \ldots, N$) which stand for the order parameter, concentration, entropy density, etc., and a set of the bulk stochastic equations of motion for them

$$\frac{\partial}{\partial t} S_\beta^* - \zeta_\beta^* = -\sum_{\alpha=1}^{N} \Lambda_{\beta\alpha} \frac{\delta W}{\delta S_\alpha} \quad (\beta = 1, 2, \ldots, N), \tag{2.1}$$

where S_α^* and ζ_α^* denote the complex conjugate of the gross variable $S_\alpha(\boldsymbol{r}, t)$ and the random force $\zeta_\alpha(\boldsymbol{r}, t)$, respectively. The latter satisfies the fluctuation dissipation relations

$$\langle \zeta_\alpha^*(\boldsymbol{r}, t)\zeta_\beta(\boldsymbol{r}', t')\rangle = k_B T(\Lambda_{\alpha\beta} + \Lambda_{\beta\alpha}^*)\delta(\boldsymbol{r} - \boldsymbol{r}')\delta(t - t') . \tag{2.2}$$

Generally $\Lambda_{\alpha\beta}(\boldsymbol{r})$ contains an operator such as ∇^2 and furthermore may depend on $\{S_\alpha(\boldsymbol{r})\}$. Thus both the reversible and dissipative couplings are contained. Some examples will be given in section 3. W is the free energy functional of $\{S_\alpha\}$. We assume that the stationary solution given by

$$\frac{\delta W}{\delta S_\alpha} = 0 , \quad \alpha \in \{1, 2, \ldots, N\} \tag{2.3}$$

represents the topological defects like an interface or a vortex line.

We wish to derive the stochastic equation of motion for such a defect starting with the above equation (2.1). One of the simplest ways is to employ the method developed by Thiele in the study of domain walls of magnetic bubbles[10]). In order to represent the configuration of a defect we introduce a vector $\boldsymbol{a}$ which parametrizes a position on the defect. For instance in the case of a vortex line $\boldsymbol{a}$ is the contour length and is a scalar. Thus the configuration of a defect is determined by the position vectors $\boldsymbol{r}(\boldsymbol{a})$. Suppose that the system undergoes a virtual

displacement $\delta \boldsymbol{r}(\boldsymbol{a})$ of the defect under a force field $\boldsymbol{F}(\boldsymbol{r})$. The work done by $\boldsymbol{F}(\boldsymbol{r}(\boldsymbol{a}))$ on the defect must balance with the energy change due to the distortion of the field $\{S_\alpha(\boldsymbol{r})\}$:

$$\int \mathrm{d}\boldsymbol{a}\, \delta \boldsymbol{r}(\boldsymbol{a}) \cdot \boldsymbol{F}(\boldsymbol{r}(\boldsymbol{a})) = \int \mathrm{d}\boldsymbol{r}\, \delta \boldsymbol{r}(\boldsymbol{a}) \cdot \sum_{\alpha=1}^{N} (\nabla S_\alpha) \frac{\delta W}{\delta S_\alpha} . \tag{2.4}$$

Generally the force $\boldsymbol{F}(\boldsymbol{r}(\boldsymbol{a}))$ consists of two parts. One is the interaction between the topological defects. The other is the internal driving force. A defect has line or surface tension, and tends to change its configuration in such a way that the energy associated with the defect is lowered. These two forces are specified once we obtain the total energy of the defects. The right-hand side of (2.1) contains their dynamic informations. If we could solve (2.1) for $\delta W/\delta S_\alpha$, substitution of the result into (2.4) gives us the dynamic information of the defect explicitly.

The main task is to derive the inverse operator Λ^{-1}. This, however, is not trivial since there are generally noncommutable operators in the off-diagonal part of Λ.

In order to distinguish the inner product in the space of the gross variables $S_\alpha(\boldsymbol{r})$ ($\alpha = 1, 2, \ldots, N$) from that in the real space we introduce the ket vector $|S(\boldsymbol{r})\rangle$ as

$$|S(\boldsymbol{r})\rangle = (S_1(\boldsymbol{r}), S_2(\boldsymbol{r}), \ldots, S_N(\boldsymbol{r})) . \tag{2.5}$$

The bra vector $\langle S(\boldsymbol{r})|$ is defined as a conjugate to $|S(\boldsymbol{r})\rangle$. Furthermore we define the orthogonal unit vector $|\alpha\rangle$ and $\langle\alpha|$ ($\alpha = 1, 2, \ldots, N$) which span the vector space $|S(\boldsymbol{r})\rangle$:

$$|S(\boldsymbol{r})\rangle = \sum_{\alpha=1}^{N} |\alpha\rangle\langle\alpha \mid S(\boldsymbol{r})\rangle \tag{2.6a}$$

with

$$\langle\alpha|S(\boldsymbol{r})\rangle = S_\alpha(\boldsymbol{r}) . \tag{2.6b}$$

Hereafter we omit $\boldsymbol{r}$ in $S(\boldsymbol{r})$. Thus the summation of the right-hand side of (2.4) is written as

$$\sum_{\alpha=1}^{N} (\nabla S_\alpha) \frac{\delta W}{\delta S_\alpha} = \langle \nabla S| \sum_{\alpha=1}^{N} |\alpha\rangle\left\langle \alpha \,\middle|\, \frac{\delta W}{\delta S}\right\rangle \tag{2.7}$$

with

$$\left\langle \alpha \,\middle|\, \frac{\delta W}{\delta S}\right\rangle = \frac{\delta W}{\delta S_\alpha} . \tag{2.8}$$

Equation (2.1) may also be written as

$$\left\langle \beta \left| \left(-\frac{\partial S^*}{\partial t} + \zeta^* \right) \right\rangle = \sum_{\alpha=1}^{N} \Lambda_{\beta\alpha} \left\langle \alpha \left| \frac{\delta W}{\delta S} \right\rangle \quad (\beta = 1, 2, \ldots, N) . \right. \right. \tag{2.9a}$$

We express this in an abbreviated form as

$$\langle \hat{\beta} | = \sum_{\alpha=1}^{N} \langle \hat{\beta} | \Lambda | \alpha \rangle \langle \alpha | , \tag{2.9b}$$

where

$$\langle \hat{\beta} | \Lambda(\boldsymbol{r}) | \alpha \rangle = \Lambda_{\beta\alpha}(\boldsymbol{r}) . \tag{2.9c}$$

Note here that we have introduced two kinds of bra vectors $\langle \alpha |$ and $\langle \hat{\alpha} |$ ($\alpha = 1, 2, \ldots, N$). The former operates on $|\delta W/\delta S\rangle$ whereas $\langle \hat{\alpha} |$ acts on $|(-\partial S^*/\partial t + \zeta^*)\rangle$.

Equation (2.9b) is to be solved for $\langle \alpha |$. We assume that there exists an inverse operator $\langle \hat{\alpha} | \Lambda | \alpha \rangle^{-1}$ of the diagronal operator $\langle \hat{\alpha} | \Lambda | \alpha \rangle$ for all α. We wish to diagonalize the inverse operator Λ^{-1} and to obtain its eigen vectors. From (2.9b) with $\beta = 1$ we obtain

$$\langle 1 | = \langle \hat{1} | \Lambda | 1 \rangle^{-1} \langle \hat{1} | - \sum_{\alpha=2}^{N} \langle \hat{1} | \Lambda | 1 \rangle^{-1} \langle \hat{1} | \Lambda | \alpha \rangle \langle \alpha | . \tag{2.10}$$

Here and after $\langle \hat{\alpha} | \Lambda(\boldsymbol{r}) | \alpha \rangle^{-1} f(\boldsymbol{r})$ ($\alpha = 1, 2, \ldots, N$) means convolution, i.e.

$$\langle \hat{\alpha} | \Lambda(\boldsymbol{r}) | \alpha \rangle^{-1} f(\boldsymbol{r}) \equiv \int d\boldsymbol{r}' [\langle \hat{\alpha} | \Lambda(\boldsymbol{r}) | \alpha \rangle^{-1} \delta(\boldsymbol{r} - \boldsymbol{r}')] f(\boldsymbol{r}') \tag{2.11}$$

for an arbitrary function $f(\boldsymbol{r})$. Substitution of (2.10) into the right-hand side of (2.9b) yields

$$\langle \hat{\beta} | Q_1 = \sum_{\alpha=2}^{N} \langle \hat{\beta} | \Lambda | \alpha \rangle \langle \alpha | \quad (\beta = 2, 3, \ldots, N) , \tag{2.12}$$

where we have introduced the projection operator Q_1 as

$$Q_1 = 1 - P_1 , \quad P_1 = \Lambda | 1 \rangle \langle \hat{1} | \Lambda | 1 \rangle^{-1} \langle \hat{1} | . \tag{2.13}$$

Equation (2.10) represents $\langle 1 |$ in terms of $\langle \hat{1} |$ and $\langle \alpha |$ ($\alpha = 2, 3, \ldots, N$). The projection Q_1, therefore, removes $\langle 1 |$ from (2.9b). The repeated use of this

elimination of $\langle i|$, $i = 2, 3, \ldots, k-1$, leads to

$$\langle\hat{\beta}|\prod_{n=1}^{k-1} Q_n = \sum_{\alpha=k}^{N} \langle\hat{\beta}|\prod_{n=1}^{k-1} Q_n \Lambda|\alpha\rangle\langle\alpha| \quad (\beta = k, \ldots, N), \tag{2.14}$$

where

$$Q_k = 1 - P_k, \quad P_k = \Lambda|k\rangle\langle\hat{k}|\prod_{n=1}^{k-1} Q_n \Lambda|k\rangle^{-1}\langle\hat{k}|\prod_{n=1}^{k-1} Q_n. \tag{2.15}$$

The product $\Pi_{n=1}^{k-1} Q_n \Lambda$ stands for $Q_1 Q_2 \cdots Q_{k-1}\Lambda$. Here we have assumed the existence of the inverse operator $\langle\hat{k}|\Pi_{n=1}^{k-1} Q_n \Lambda|k\rangle^{-1}$. This will indeed be justified in the systems considered in section 3. The projection Q_k removes $\langle k|$ from the set of equations where $\langle i|$ with $i = 1, 2, \ldots, k-1$ have already been eliminated. The inverse operator $\langle\hat{k}|\Pi_{n=1}^{k-1} Q_n \Lambda|k\rangle^{-1}$ may be represented formally in a continued fraction of rank k. From definition (2.15), the following relations hold for all of the k;

$$\begin{aligned} &\langle\hat{k}|\prod_{n=1}^{k} Q_n = 0, \quad \langle\hat{k}|\left(\prod_{n=1}^{k-1} Q_n\right) P_k = \langle\hat{k}|\prod_{n=1}^{k-1} Q_n, \\ &Q_k \Lambda|k\rangle = 0, \quad P_k \Lambda|k\rangle = \Lambda|k\rangle. \end{aligned} \tag{2.16}$$

From (2.14) with $k = N$, we obtain

$$\langle N| = \langle\hat{N}|\prod_{n=1}^{N-1} Q_n \Lambda|N\rangle^{-1}\langle\hat{N}|\prod_{n=1}^{N-1} Q_n. \tag{2.17}$$

Substitution of (2.17) into (2.14) with $k = \beta$ yields

$$\langle\hat{\beta}|\left(\prod_{n=1}^{\beta-1} Q_n\right) Q_N = \sum_{\alpha=\beta}^{N-1} \langle\hat{\beta}|\prod_{n=1}^{\beta-1} Q_n \Lambda|\alpha\rangle\langle\alpha| \quad (\beta = 1, 2, \ldots, N-1), \tag{2.18}$$

which is similar to (2.14) but $\langle N|$ is now absent. Successive application of this procedure finally yields

$$\langle\alpha| = \langle\hat{\alpha}|\prod_{n=1}^{\alpha-1} Q_n \Lambda|\alpha\rangle^{-1}\langle\hat{\alpha}|\left(\prod_{n=1}^{\alpha-1} Q_n\right)\left(\prod_{m=\alpha+1}^{N} Q_m\right). \tag{2.19}$$

Note that the suffix vector $\langle\hat{\alpha}|$ is now on the right-hand side and $\langle\alpha|$ on the left-hand side. In this way we can express $\delta W/\delta S_\alpha$ in terms of $-(\partial S_\alpha^*/\partial t - \zeta_\alpha^*)$.

Using (2.19) we obtain

$$\Lambda \sum_{\alpha=1}^{N} |\alpha\rangle\langle\alpha| = \sum_{\alpha=1}^{N} P_\alpha \prod_{m=\alpha+1}^{N} Q_m = \sum_{\alpha=2}^{N} P_\alpha \prod_{m=\alpha+1}^{N} Q_m + P_1 \prod_{m=2}^{N} Q_m . \tag{2.20}$$

The second term of (2.20) becomes

$$P_1 \prod_{m=2}^{N} Q_m = P_1 \prod_{m=3}^{N} Q_m - P_1 P_2 \prod_{m=3}^{N} Q_m . \tag{2.21}$$

By regarding (2.21) as a recursion relation we obtain

$$P_1 \prod_{m=2}^{N} Q_m = P_1 - P_1 \sum_{\alpha=2}^{N} P_\alpha \prod_{m=\alpha+1}^{N} Q_m . \tag{2.22}$$

Substituting this into (2.20) yields

$$\sum_{\alpha=1}^{N} P_\alpha \prod_{m=\alpha+1}^{N} Q_m = Q_1 \sum_{\alpha=2}^{N} P_\alpha \prod_{m=\alpha+1}^{N} Q_m + P_1 . \tag{2.23}$$

Using this relation iteratively, (2.20) can be written as

$$\sum_{\alpha=1}^{N} |\alpha\rangle\langle\alpha| = \sum_{\alpha=1}^{N} \prod_{n=1}^{\alpha-1} \hat{Q}_n |\alpha\rangle\langle\hat{\alpha}| \prod_{n=1}^{\alpha-1} Q_n \Lambda |\alpha\rangle^{-1} \langle\hat{\alpha}| \prod_{n=1}^{\alpha-1} Q_n , \tag{2.24}$$

where $\hat{Q}_n$ is defined by

$$\hat{Q}_n = \Lambda^{-1} Q_n \Lambda . \tag{2.25}$$

Equation (2.24) with (2.7) enables us to represent the right-hand side of (2.4) in terms of $-(\partial S_\alpha^*/\partial t - \zeta_\alpha^*)$.

If we retain only the part of $\partial S_\beta^*/\partial t$ and ζ_β^* associated with the defect motion we may write approximately

$$\frac{\partial}{\partial t} S_\beta^*(\boldsymbol{r}) - \zeta_\beta^*(\boldsymbol{r}) = -(\nabla S_\beta^*(\boldsymbol{r})) \cdot (\boldsymbol{v}(\boldsymbol{a}) - \boldsymbol{\theta}(\boldsymbol{a})) , \tag{2.26}$$

where $\boldsymbol{v}$ and $\boldsymbol{\theta}$ are the velocity of the defect and the fluctuating force acting on the defect, respectively. Equation (2.26) is valid when the size of a singular region of a topological defect such as the thickness of an interface or the core

radius of a vortex line is negligibly small compared to other length scales of the problem. In such a situation use of the mean field results for S_α is also justified. We introduce the new variables $(\nabla\tilde{S}_\alpha)$ and $(\nabla\tilde{S}^+_\alpha)$ $(\alpha = 1, 2, \ldots, N)$ defined by

$$(\nabla\tilde{S}_\alpha) = \langle \nabla S | \prod_{n=1}^{\alpha-1} \hat{Q}_n | \alpha \rangle , \tag{2.27}$$

$$(\nabla\tilde{S}^+_\alpha) = \langle \hat{\alpha} | \prod_{n=1}^{\alpha-1} Q_n | \nabla S^* \rangle . \tag{2.28}$$

Then using (2.24) and (2.26), eq. (2.7) can be written as

$$\sum_{\alpha=1}^{N} (\nabla S_\alpha) \frac{\delta W}{\delta S_\alpha} = \int \mathrm{d}\boldsymbol{r}' \sum_{\alpha=1}^{N} (\nabla\tilde{S}_\alpha(\boldsymbol{r})) \mathscr{L}_\alpha(\boldsymbol{r}, \boldsymbol{r}') (\nabla'\tilde{S}^+_\alpha(\boldsymbol{r}')) \cdot (\boldsymbol{v}(\boldsymbol{a}') - \boldsymbol{\theta}(\boldsymbol{a}')) , \tag{2.29}$$

where we have introduced the Green function $\mathscr{L}_\alpha$ defined by

$$\mathscr{L}_\alpha(\boldsymbol{r}, \boldsymbol{r}') \equiv \langle \hat{\alpha} | \prod_{n=1}^{\alpha-1} Q_n \Lambda(\boldsymbol{r}) | \alpha \rangle^{-1} \delta(\boldsymbol{r} - \boldsymbol{r}') . \tag{2.30}$$

The new variables $(\nabla\tilde{S}_\alpha)$ and $(\nabla\tilde{S}^+_\alpha)$ are given by a linear combination of the variables (∇S_β) and (∇S^*_β). However, as can be seen from the definition (2.15) of Q_k the "coefficients" consist of operators. We call $(\nabla\tilde{S}_\alpha)$ and $(\nabla\tilde{S}^+_\alpha)$ the eigenvariable, and $\langle \hat{\alpha} | \prod_{n=1}^{\alpha-1} Q_n \Lambda | \alpha \rangle$ the eigenoperator.

Since $\delta\boldsymbol{r}(\boldsymbol{a})$ in (2.4) is arbitrary we may obtain by substituting (2.26) into it,

$$\boldsymbol{F}(\boldsymbol{r}(\boldsymbol{a})) = \int \mathrm{d}\boldsymbol{a}' \mathbf{D}(\boldsymbol{a}, \boldsymbol{a}') (\boldsymbol{v}(\boldsymbol{a}') - \boldsymbol{\theta}(\boldsymbol{a}')) , \tag{2.31}$$

where

$$\mathbf{D}(\boldsymbol{a}, \boldsymbol{a}') = \int \mathrm{d}\boldsymbol{r}_\perp(\boldsymbol{a}) \int \mathrm{d}\boldsymbol{r}'_\perp(\boldsymbol{a}') \sum_{\alpha=1}^{N} (\nabla\tilde{S}_\alpha(\boldsymbol{r})) \mathscr{L}_\alpha(\boldsymbol{r}, \boldsymbol{r}') (\nabla'\tilde{S}^+_\alpha(\boldsymbol{r}')) . \tag{2.32}$$

Here $\mathrm{d}\boldsymbol{r}_\perp(\boldsymbol{a})$ is the infinitesimal volume element in $d - n$ dimensions with n the dimension of the defect such that

$$\int \mathrm{d}\boldsymbol{r} = \int \mathrm{d}\boldsymbol{a} \int \mathrm{d}\boldsymbol{r}_\perp(\boldsymbol{a}) . \tag{2.33}$$

The force $\boldsymbol{F}(\boldsymbol{r}(\boldsymbol{a}))$ is now the only unknown quantity. Derivation of $\boldsymbol{F}(\boldsymbol{r}(\boldsymbol{a}))$

is, however, merely a static problem. If we obtain the static defect energy W_D starting with the bulk Hamiltonian $W\{S\}$ the force $\boldsymbol{F}(\boldsymbol{r}(\boldsymbol{a}))$ is given by

$$\boldsymbol{F}(\boldsymbol{r}(\boldsymbol{a})) = -\frac{\delta W_D}{\delta \boldsymbol{r}(\boldsymbol{a})}. \tag{2.34}$$

Equating (2.31) with (2.34) we obtain

$$-\frac{\delta W_D}{\delta \boldsymbol{r}(\boldsymbol{a})} = \int d\boldsymbol{a}' \mathbf{D}(\boldsymbol{a}, \boldsymbol{a}')(\boldsymbol{v}(\boldsymbol{a}') - \boldsymbol{\theta}(\boldsymbol{a}')). \tag{2.35}$$

Thus we have completed the whole procedure to derive the equation of motion of the topological defect. When S_α and $\Lambda_{\alpha\beta}$ in (2.1) are specified, following the method described above we may calculate $\tilde{S}_\alpha$, $\tilde{S}^+_\alpha$ and $\mathscr{L}_\alpha(\boldsymbol{r}, \boldsymbol{r}')$ so that we obtain $\mathbf{D}(\boldsymbol{a}, \boldsymbol{a}')$ (see appendix A). The fluctuation dissipation relation for $\boldsymbol{\theta}$ can be derived similarly:

$$\langle \theta_i(\boldsymbol{a}, t)\theta_j(\boldsymbol{a}', t')\rangle = k_B T(D^{-1}_{ij}(\boldsymbol{a}, \boldsymbol{a}') + D^{*-1}_{ji}(\boldsymbol{a}, \boldsymbol{a}'))\delta(t - t'), \tag{2.36}$$

where $\mathbf{D}^{-1}(\boldsymbol{a}, \boldsymbol{a}')$ is defined through

$$\int d\boldsymbol{a}'' \mathbf{D}(\boldsymbol{a}, \boldsymbol{a}'')\mathbf{D}^{-1}(\boldsymbol{a}'', \boldsymbol{a}') = \mathbf{1}\delta(\boldsymbol{a} - \boldsymbol{a}'). \tag{2.37}$$

The correlation is found to be related to the dissipative parts of $D^{-1}_{ij}(\boldsymbol{a}, \boldsymbol{a}')$.[11])

3. Application to various models

Here we apply the above method to critical fluids (model H), metamagnets, and ^{3}He–^{4}He mixtures.

3.1. *Critical fluids*

First we derive the equation of motion for an interface exhibited in model H.[7])

$$\frac{\partial}{\partial t} c - \zeta_c = \lambda_0 \nabla^2 \frac{\delta W}{\delta c} - g_0 \nabla c \cdot \frac{\delta W}{\delta \boldsymbol{j}}, \tag{3.1.1a}$$

$$\frac{\partial}{\partial t}\boldsymbol{j} - \boldsymbol{\zeta}_j = \eta_0 \nabla^2 \frac{\delta W}{\delta \boldsymbol{j}} + g_0 \nabla c \frac{\delta W}{\delta c}, \tag{3.1.1b}$$

where c is the order parameter, $\boldsymbol{j}$ the momentum density, λ_0, η_0 the Onsager coefficients and g_0 the coupling constant. We have omitted the projection operator in (3.1.1b) which selects out the transverse part of $\boldsymbol{j}$. The free energy is given by

$$W = \int \mathrm{d}\boldsymbol{r}\{\tfrac{1}{2}(\nabla c)^2 + P(c, \boldsymbol{j})\}, \tag{3.1.2}$$

where $P(c, \boldsymbol{j})$ is the uniform part of the free energy. It is assumed to exhibit double minima in c below criticality and is bilinear in $\boldsymbol{j}$. Since the order parameter c is real, we have only an interface as a topological defect in the ordered phase. We may derive the static total energy W_s associated with an interface,

$$W_s = \int \mathrm{d}\boldsymbol{r}(\nabla c)^2 = \sigma \int \mathrm{d}\boldsymbol{a}, \tag{3.1.3}$$

where σ is the surface tension given by

$$\sigma = \int \mathrm{d}\boldsymbol{r}_\perp (\nabla c)^2 \tag{3.1.4}$$

and $\mathrm{d}\boldsymbol{a}$ is a surface element of the interface whereas $\mathrm{d}\boldsymbol{r}_\perp$ is a line element normal to the surface. The force $\boldsymbol{F}$ due to this interfacial energy is given by

$$\boldsymbol{F}(\boldsymbol{r}(\boldsymbol{a})) = -\frac{\delta W_s}{\delta \boldsymbol{r}(\boldsymbol{a})} = \sigma K(\boldsymbol{a}) \boldsymbol{n}(\boldsymbol{a}), \tag{3.1.5}$$

where $K(\boldsymbol{a})$ is the mean curvature at the position $\boldsymbol{r}(\boldsymbol{a})$, i.e.

$$K(\boldsymbol{a}) = -\nabla \cdot \boldsymbol{n}(\boldsymbol{a}). \tag{3.1.6}$$

$\boldsymbol{n}(\boldsymbol{a})$ is the unit vector normal to the interface at the position $\boldsymbol{r}(\boldsymbol{a})$.

The matrix $\mathbf{D}$ in (2.31) can be obtained as follows. We put the variable vector $|S\rangle$ as

$$|S\rangle = (S_1, S_2, S_3, S_4) = (j_x, j_y, j_z, c). \tag{3.1.7}$$

Then the operators $\langle\hat{\alpha}|\Lambda|\beta\rangle$ are automatically given from (2.9) and (3.1.1). The nonvanishing components of them become

$$\langle\hat{4}|\Lambda|\alpha\rangle = -\langle\hat{\alpha}|\Lambda|4\rangle = g_0(\nabla_\alpha c) \quad (\alpha = 1, 2, 3), \tag{3.1.8a}$$

$$\langle\hat{\alpha}|\Lambda|\alpha\rangle = -\eta_0\nabla^2, \tag{3.1.8b}$$

$$\langle\hat{4}|\Lambda|4\rangle = -\lambda_0\nabla^2, \tag{3.1.8c}$$

where the ∇_α ($\alpha = 1, 2, 3$) are interpreted as ∂_x, ∂_y and ∂_z, respectively. From (3.1.8) and the definition of Q_n we obtain the "eigen"operators and the corresponding Green functions, i.e. (see appendix A)

$$\langle\hat{\alpha}|\prod_{n=1}^{\alpha-1} Q_n\Lambda|\alpha\rangle = -\eta_0\nabla^2 \quad (\alpha = 1, 2, 3), \tag{3.1.9a}$$

$$-\eta_0\nabla^2\mathscr{L}_\eta(\boldsymbol{r}, \boldsymbol{r}') = \delta(\boldsymbol{r} - \boldsymbol{r}'), \tag{3.1.9b}$$

$$\langle\hat{4}|\prod_{n=1}^{3} Q_n\Lambda|4\rangle = -\lambda_0\nabla^2 + g_0^2(\nabla c(\boldsymbol{r}))\cdot\int d\boldsymbol{r}'\mathscr{L}_\eta(\boldsymbol{r}, \boldsymbol{r}')(\nabla' c(\boldsymbol{r}')), \tag{3.1.10a}$$

$$\langle\hat{4}|\prod_{n=1}^{3} Q_n\Lambda|4\rangle\mathscr{L}_\lambda(\boldsymbol{r}, \boldsymbol{r}') = \delta(\boldsymbol{r} - \boldsymbol{r}'). \tag{3.1.10b}$$

The "eigen"variable $\nabla\tilde{S}_\alpha$ and $\nabla\tilde{S}_\alpha^+$ become (see appendix A)

$$(\nabla\tilde{S}_\alpha) = (\nabla\tilde{S}_\alpha^+) = (\nabla j_\alpha) \quad (\alpha = 1, 2, 3), \tag{3.1.11a}$$

$$(\nabla\tilde{S}_4) = (\nabla c) + g_0(\nabla j)\cdot\int d\boldsymbol{r}'\mathscr{L}_\eta(\boldsymbol{r}, \boldsymbol{r}')(\nabla' c) \equiv (\nabla c_0), \tag{3.1.11b}$$

$$(\nabla\tilde{S}_4^+) = (\nabla c) - g_0(\nabla c)\cdot\int d\boldsymbol{r}'\mathscr{L}_\eta(\boldsymbol{r}, \boldsymbol{r}')(\nabla' j) \equiv (\nabla c_0^+), \tag{3.1.11c}$$

where the inner products in (3.1.11) must be taken between j and $g_0(\nabla c)$. Using formula (2.32) we obtain the equation of motion of the interface for this model;

$$\sigma K(\boldsymbol{a})\boldsymbol{n}(\boldsymbol{a}) = \int d\boldsymbol{a}'\mathbf{D}(\boldsymbol{a}, \boldsymbol{a}')(\boldsymbol{v}(\boldsymbol{a}') - \boldsymbol{\theta}(\boldsymbol{a}')), \tag{3.1.12}$$

where

$$\mathbf{D}(a, a') = \int \mathrm{d}\boldsymbol{r}_{\perp}(a) \int \mathrm{d}\boldsymbol{r}'_{\perp}(a')[(\nabla j(r)) \cdot \mathscr{L}_{\eta}(r, r')(\nabla' j(r')) + (\nabla c_0(r))\mathscr{L}_{\lambda}(r, r')(\nabla' c_0^{+}(r'))] . \tag{3.1.13}$$

In the first term of (3.1.13), the inner products must be taken between j's.

3.2. *Metamagnets and ^{3}He–^{4}He mixtures*

Secondly we consider tricritical dynamics of metamagnets and ^{3}He–^{4}He mixtures. These systems are governed by the following set of stochastic equations[12])

$$\frac{\partial}{\partial t}\psi^* - \eta_{\psi}^* = -2\Gamma^* \frac{\delta W}{\delta \psi} + \mathrm{i}g_2\psi^* \frac{\delta W}{\delta c} + \mathrm{i}g_1\psi^* \frac{\delta W}{\delta q}, \tag{3.2.1a}$$

$$\frac{\partial}{\partial t}c - \eta_c = \mathrm{i}g_2\psi \frac{\delta W}{\delta \psi} - \mathrm{i}g_2\psi^* \frac{\delta W}{\delta \psi^*} + \lambda \nabla^2 \frac{\delta W}{\delta c} + L\nabla^2 \frac{\delta W}{\delta q}, \tag{3.2.1b}$$

$$\frac{\partial}{\partial t}q - \eta_q = \mathrm{i}g_1\psi \frac{\delta W}{\delta \psi} - \mathrm{i}g_1\psi^* \frac{\delta W}{\delta \psi^*} + L\nabla^2 \frac{\delta W}{\delta c} + K\nabla^2 \frac{\delta W}{\delta q}, \tag{3.2.1c}$$

where ψ is the complex order parameter. The relative concentration of ^{3}He and ^{4}He atoms is denoted by c. q is the local entropy density. The random forces η satisfy the fluctuation dissipation relation given by (2.2). The Onsager coefficients λ, K and L are positive. The coupling constants g_1 and g_2 in (3.2.1) are also positive. W is the free energy given by

$$W = \int \mathrm{d}\boldsymbol{r}\{\tfrac{1}{2}|\nabla\psi|^2 + P(|\psi|^2, c, q)\} . \tag{3.2.2}$$

The uniform part of the free energy $P(|\psi|^2, c, q)$ has triple minima in $|\psi|$ below a tricritical point and contains the binary coupling among $|\psi|^2$, c and q. Thus the equilibrium solutions of (3.2.1) are given by

$$-\nabla^2\psi + \xi^{-2}\left(1 - \frac{|\psi|^2}{\psi_s^2}\right)\left(1 - 3\frac{|\psi|^2}{\psi_s^2}\right)\psi = 0 , \tag{3.2.3a}$$

$$c = c_n + (\Delta c)\frac{|\psi|^2}{\psi_s^2}, \tag{3.2.3b}$$

$$q = q_n + (\Delta q)\frac{|\psi|^2}{\psi_s^2}, \tag{3.2.3c}$$

where ξ is the correlation length. ψ_s is the value of the order parameter in the ordered phase, whereas c_n and q_n are the values of c and q in the disordered phase ($\psi = 0$). (Δc) and (Δq) are the differences between two phases, respectively.

Recently Onuki[13]) has studied the dynamics of vortex lines, starting with the same model as (3.2.1). If we put $q_n = \Delta q = 0$, eqs. (3.2.1) reduce to those used by Hohenberg and Nelson[14]). As a particular case eqs. (3.2.1) can describe other systems. For instance, if we ignore the q-field and put $g_1 = L = K = 0$, eqs. (3.2.1) reduce to the model F of Hohenberg and Halperin[7]). Furthermore when ψ is real eqs. (3.2.1a) and (3.2.1b) with $g_2 = 0$ are a dynamic model of metamagnets[15]). Thus eqs. (3.2.1) generally describe the tricritical dynamics.

Equations (3.2.3) contain two types of the topological defects. One is the interface type. The static interfacial energy and force associated with an interface are given by (3.1.3)–(3.1.6) in which $(\nabla c)^2$ is replaced by $|\nabla\psi|^2$. The other topological defect is a vortex line. We may put the order parameter ψ as

$$\psi(\boldsymbol{r}) = |\psi(\boldsymbol{r})|\, e^{i\varphi(\boldsymbol{r})}, \tag{3.2.4}$$

where $\varphi(\boldsymbol{r})$ is the azimuthal angle of $\boldsymbol{r}$ in a local cylindrical coordinate (r, φ, τ) with τ-axis parallel to the vortex line. Substitution of (3.2.4) into (3.2.2) yields the free energy associated with a vortex line. We omit the r-dependence of $|\psi(\boldsymbol{r})|$ which is the constant ψ_s everywhere except for a vortex core region. Thus the vortex line energy is given by

$$W_l \sim \int d\boldsymbol{r}\, \tfrac{1}{2}\psi_s^2 (\nabla\varphi)^2 . \tag{3.2.5}$$

This is valid for the case that the core radius of the vortex line is infinitesimal. The vector $\boldsymbol{a}$ in section 2 is taken to be the contour length τ and is a scalar as mentioned before. Let $\boldsymbol{r}_l(\tau)$ be the position vector of the vortex line parametrized by τ. In terms of $\boldsymbol{r}_l(\tau)$ eq. (3.2.5) may be written as[2])

$$W_l = \kappa_0 \int d\tau \int d\tau' \frac{(\boldsymbol{r}_l'(\tau)\cdot\boldsymbol{r}_l'(\tau'))}{|\boldsymbol{r}_l(\tau) - \boldsymbol{r}_l(\tau')|}, \tag{3.2.6}$$

where

$$\boldsymbol{r}_l'(\tau) = \frac{d}{d\tau}\boldsymbol{r}_l(\tau) \tag{3.2.7}$$

and κ_0 is some constant. Therefore the force $\boldsymbol{F}$ acting on the vortex line is

given by

$$\boldsymbol{F}(\boldsymbol{r}(\tau)) = -\frac{\delta W}{\delta \boldsymbol{r}_1(\tau)} = \boldsymbol{\kappa}(\tau) \times \boldsymbol{v}_s(\boldsymbol{r}_1(\tau)), \tag{3.2.8}$$

where we have introduced the superfluid velocity as

$$\boldsymbol{v}_s(\boldsymbol{r}) = -\kappa_0 \int \mathrm{d}\tau\, \boldsymbol{r}_1'(\tau) \times \boldsymbol{\nabla} \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_1(\tau)|} \tag{3.2.9}$$

and the unit vector $\boldsymbol{\kappa}(\tau)$ parallel to the vortex line at $\boldsymbol{r}_1(\tau)$.

The matrix **D** appeared in (2.31) can be calculated in the following. It should be stressed that **D** for an interface and for a vortex line can be derived simultaneously in a unified way. Let us define the variable $|S\rangle$ as

$$|S\rangle = (S_1, S_2, S_3, S_4) = (\psi, \psi^*, q, c). \tag{3.2.10}$$

Then the operators $\langle\hat{\alpha}|\Lambda|\beta\rangle$ are determined by comparing (2.1) with (3.2.1) and the c.c. of (3.2.1a), though we do not write them down here. The "eigen"operators $\langle\hat{\alpha}|\Pi_{n=1}^{\alpha-1} Q_n \Lambda|\alpha\rangle$ are easily calculated up to $\alpha = 3$ (see appendix A),

$$\langle\hat{1}|\prod_{n=1}^{0} Q_n \Lambda|1\rangle = \frac{1}{2\Gamma^*}, \quad \langle\hat{2}|\prod_{n=1}^{1} Q_n \Lambda|2\rangle = \frac{1}{2\Gamma}, \tag{3.2.11}$$

$$\langle\hat{3}|\prod_{n=1}^{2} Q_n \Lambda|3\rangle = -K\boldsymbol{\nabla}^2 + g_1^2 \frac{\Gamma_1}{|\Gamma|^2}|\psi|^2 \equiv \Lambda_K, \tag{3.2.12}$$

where

$$\Gamma = \Gamma_1 + \mathrm{i}\Gamma_2 \quad (\Gamma_1, \Gamma_2: \text{real}). \tag{3.2.13}$$

In order to calculate $\langle\hat{4}|\,\Pi_{n=1}^{3} Q_n \Lambda|4\rangle$, we need to get the operators

$$\langle\hat{3}|\prod_{n=1}^{2} Q_n \Lambda|4\rangle = \langle\hat{4}|\prod_{n=1}^{2} Q_n \Lambda|3\rangle = -L\boldsymbol{\nabla}^2 + g_1 g_2 \frac{\Gamma_1}{|\Gamma|^2}|\psi|^2 \equiv \Lambda_L, \tag{3.2.14}$$

$$\langle\hat{4}|\prod_{n=1}^{2} Q_n \Lambda|4\rangle = -\lambda\boldsymbol{\nabla}^2 + g_2^2 \frac{\Gamma_1}{|\Gamma|^2}|\psi|^2 \equiv \Lambda_\lambda \tag{3.2.15}$$

and the Green function

$$\Lambda_K \mathscr{L}_K(r, r') = \delta(r - r') . \tag{3.2.16}$$

Then we obtain the eigenoperator and its Green function

$$\langle \hat{4}| \prod_{n=1}^{3} Q_n \Lambda |4\rangle = \Lambda_\lambda(r) - \Lambda_L(r) \int \mathrm{d}r' \mathscr{L}_K(r, r') \Lambda_L(r') , \tag{3.2.17}$$

$$\langle \hat{4}| \prod_{n=1}^{3} Q_n \Lambda |4\rangle \mathscr{L}_c(r, r') = \delta(r, r') . \tag{3.2.18}$$

The eigen variable $\nabla \tilde{S}_\alpha$ and $\nabla \tilde{S}^+_\alpha$ ($\alpha = 1, 2, 3, 4$) are given by (see appendix A)

$$\langle \nabla \tilde{S}| = (\nabla\psi, \nabla\psi^*, \nabla\tilde{q}^-, \nabla\tilde{c}^-)^{\mathrm{T}} , \tag{3.2.19a}$$

$$|\nabla \tilde{S}^+\rangle = (\nabla\psi^*, \nabla\psi, \nabla\tilde{q}^+, \nabla\tilde{c}^+) , \tag{3.2.19b}$$

where

$$\nabla\tilde{q}^\pm \equiv \nabla q + g_1 j^\pm , \tag{3.2.20a}$$

$$j^+ \equiv \frac{\mathrm{i}\psi}{2\Gamma^*}(\nabla\psi^*) - \frac{\mathrm{i}\psi^*}{2\Gamma}(\nabla\psi) , \quad j^- \equiv (\nabla\psi)\frac{\mathrm{i}\psi^*}{2\Gamma^*} - (\nabla\psi^*)\frac{\mathrm{i}\psi}{2\Gamma} , \tag{3.2.20b}$$

$$\nabla\tilde{c}^+ \equiv (\nabla c) + g_2 j^+ - \Lambda_L \int \mathrm{d}r' \mathscr{L}_K(r, r')(\nabla'\tilde{q}^+) , \tag{3.2.20c}$$

$$\nabla\tilde{c}^- \equiv (\nabla c) + g_2 j^- - (\nabla\tilde{q}^-) \int \mathrm{d}r' \mathscr{L}_K(r, r')\Lambda_L . \tag{3.2.20d}$$

$\mathbf{D}(a, a')$ defined by (2.32) is now expressed by the eigenoperators (3.2.11), (3.2.12) and (3.2.17) and by the eigenvariable vectors (3.2.19).

We, however, have an alternative representation for $\mathbf{D}$ which is specific to the model (3.2.1). Note that the operators Λ_K, Λ_L and Λ_λ given by (3.2.12), (3.2.14) and (3.2.15), respectively, are not independent of each other. This originates from the fact that with the linear combination $g_1 c - g_2 q$ the reversible couplings of (3.2.1b) or (3.2.1c) can be eliminated. In fact we can show that there are relations among Λ_K, Λ_L and Λ_λ such that

$$\Lambda_K = \Lambda_{q0} + \frac{L'^2}{L^2}\Lambda_{c0} , \tag{3.2.21a}$$

$$\Lambda_L = \frac{g_2}{g_1}\Lambda_{q0} + \frac{L'}{\lambda'}\left(1 + \frac{g_2 L'}{g_1 \lambda'}\right)\Lambda_{c0}, \tag{3.2.21b}$$

$$\Lambda_\lambda = \frac{g_2^2}{g_1^2}\Lambda_{q0} + \left(1 + \frac{g_2 L'}{g_1 \lambda'}\right)^2 \Lambda_{c0}, \tag{3.2.21c}$$

where

$$\Lambda_{q0} \equiv -K'\nabla^2 + g_1^2 \frac{\Gamma_1}{|\Gamma|^2}|\psi|^2, \quad \Lambda_{c0} \equiv -\lambda'\nabla^2, \tag{3.2.22a}$$

$$L' \equiv L - \frac{g_2}{g_1}K, \quad \lambda' \equiv \lambda - 2\frac{g_2}{g_1}L + \frac{g_2^2}{g_1^2}K, \quad K' \equiv \frac{K\lambda - L^2}{\lambda'}. \tag{3.2.22b}$$

These relations give (see appendix B)

$$\begin{aligned}\mathbf{D}(\boldsymbol{a}, \boldsymbol{a}') = \int \mathrm{d}\boldsymbol{r}_\perp(\boldsymbol{a}) \int \mathrm{d}\boldsymbol{r}'_\perp(\boldsymbol{a}') \Big[& (\nabla\psi(\boldsymbol{r})) \frac{\Gamma\delta(\boldsymbol{r}-\boldsymbol{r}')}{2|\Gamma|^2} (\nabla'\psi^*(\boldsymbol{r}')) \\ & + (\nabla\psi^*(\boldsymbol{r})) \frac{\Gamma^*\delta(\boldsymbol{r}-\boldsymbol{r}')}{2|\Gamma|^2} (\nabla'\psi(\boldsymbol{r}')) + (\nabla q_0^-(\boldsymbol{r}))\mathscr{L}_{q_0}(\boldsymbol{r}-\boldsymbol{r}')(\nabla' q_0^+(\boldsymbol{r}')) \\ & + (\nabla c_0(\boldsymbol{r}))\mathscr{L}_{c_0}(\boldsymbol{r}-\boldsymbol{r}')(\nabla' c_0(\boldsymbol{r}')) \Big],\end{aligned} \tag{3.2.23}$$

where

$$\Lambda_{q0}\mathscr{L}_{q0}(\boldsymbol{r}, \boldsymbol{r}') = \delta(\boldsymbol{r}-\boldsymbol{r}'), \tag{3.2.24a}$$

$$\Lambda_{c0}\mathscr{L}_{c0}(\boldsymbol{r}, \boldsymbol{r}') = \delta(\boldsymbol{r}-\boldsymbol{r}'), \tag{3.2.24b}$$

$$(\nabla c_0) \equiv (\nabla c) - \frac{g_2}{g_1}(\nabla q), \tag{3.2.24c}$$

$$(\nabla q_0^\pm) \equiv (\nabla q) + g_1 \boldsymbol{j}^\pm - \frac{L'}{\lambda'}(\nabla c_0). \tag{3.2.24d}$$

Thus we obtain the equation of motion for two types of topological defects

$$\boldsymbol{F}(\boldsymbol{r}(\boldsymbol{a})) = \int \mathrm{d}\boldsymbol{a}' \mathbf{D}(\boldsymbol{a}, \boldsymbol{a}')(\boldsymbol{v}(\boldsymbol{a}') - \boldsymbol{\theta}(\boldsymbol{a}')), \tag{3.2.25}$$

where the force $\boldsymbol{F}(\boldsymbol{r}(\boldsymbol{a}))$ is given by (3.1.5) for the interface and by (3.2.8) for the vortex line in ^{3}He–^{4}He mixtures.

Finally let us consider the dynamics of metamagnets. It is given by putting $g_1 = L = K = 0$ and $\Gamma_2 = g_2 = 0$ in (3.2.24). Since the order parameter ψ is real and a scalar we have an interface as topological defects. Thus we obtain

$$\sigma K(a)n(a) = \int da' \mathbf{D}(a, a')(v(a') - \theta(a')), \tag{3.2.26}$$

where

$$\mathbf{D}(a, a') = \int dr_\perp(a) \int dr'_\perp(a') \left[(\nabla\psi) \frac{\delta(r - r')}{\Gamma_1} (\nabla'\psi) + (\nabla c)\mathscr{L}_0(r, r')(\nabla' c) \right], \tag{3.2.27}$$

$$-\lambda \nabla^2 \mathscr{L}_0(r, r') = \delta(r - r'). \tag{3.2.28}$$

The quantities σ, $K(a)$ and $n(a)$ have already been defined in critical fluids.

4. Motion of an almost planar interface

In this section we consider the equation of motion of a smoothly curved interface. In model H we assume that the uniform part on the potential $P(c, j)$ is symmetric, i.e. $P(c, j) = P(-c, j)$ so that the equilibrium solution is given by

$$j = 0, \quad c(z) = (\Delta c)\frac{1}{2}\tanh\frac{z}{\xi}, \tag{4.1}$$

where Δc is the difference of the value of the order parameter at the two ordered states and ξ the correlation length. For sufficiently small values of ξ (4.1) expresses a flat interface at $z = 0$. Supposing that the flat interface slightly fluctuates about $z = 0$ with wave number $k_\parallel$ which is parallel to the plane $z = 0$, we may calculate the relaxation rate Ω_k. In this case the equation of motion (3.1.12) is easily solved. The concentration profile can be written as

$$c = c(z - f(r_\parallel, t)), \tag{4.2}$$

where $z = f(r_\parallel, t)$ specifies the configuration of the interface. We put

$$f(r_\parallel, t) = f_0 e^{ik_\parallel r_\parallel - \Omega_k t} \tag{4.3}$$

and take the limit $\xi \to 0$, so that $c(z)$ given by (4.1) is approximated by the step

function. Then we can solve the differential integral equation caused by the operator (3.1.10a). In this way we obtain from (3.1.12)

$$\sigma k_{\parallel}^2 = \left[\frac{2\lambda_0 k_{\parallel}}{(\Delta c)^2} + \frac{g_0^2}{2\eta_0 k_{\parallel}}\right]^{-1} \Omega_k + o(k_{\parallel}\xi)\,. \tag{4.4}$$

Equation (4.4) yields a relaxation rate[6])

$$\Omega_k = \frac{\sigma g_0^2}{2\eta_0} k_{\parallel} + \frac{2\sigma\lambda_0}{(\Delta c)^2} k_{\parallel}^3 \tag{4.5}$$

which is valid for $k_{\parallel}\xi \ll 1$. This method of calculating the relaxation rate of an almost planar interface is also appliciable to metamagnets and ^{3}He–^{4}He mixtures. The mean field profile of the order parameter near a tricritical point is given from (3.1.16a) by[15])

$$\psi(z) = \psi_s \left\{\frac{1}{2}\left(1 + \tanh\frac{z}{\xi}\right)\right\}^{1/2}. \tag{4.6}$$

We have assumed that there is no macroscopic supercurrent i.e. $\nabla\varphi = 0$. In the limit $\xi \to 0$, $\psi(z)$ and the eigenvariables (3.2.15c) and (3.2.15d) become

$$(\nabla\psi) = (\nabla\psi^*) = \psi_s \delta(z)\,, \tag{4.7a}$$

$$(\nabla c_0) = \sigma_1 \delta(z)\,, \tag{4.7b}$$

$$(\nabla q_0) = \sigma_2 \delta(z)\,, \tag{4.7c}$$

where

$$\sigma_1 = \Delta c - \frac{g_2}{g_1}\Delta q\,, \tag{4.8a}$$

$$\sigma_2 = \Delta q - g_1 \frac{\Gamma_2}{2|\Gamma|^2}\psi_s^2 - \frac{L'}{\lambda'}\left(\Delta c - \frac{g_2}{g_1}\Delta q\right). \tag{4.8b}$$

The fluctuations of the interface positions can be represented as described by (4.2) and (4.3). Therefore the equation of motion (3.1.12) becomes

$$\sigma k_{\parallel}^2 = \left[\frac{\Gamma_1\sigma}{|\Gamma|^2} + \frac{\sigma_1^2}{2\lambda' k_{\parallel}} + \frac{\sigma_2^2}{K'(k_{\parallel} + (k_{\parallel}^2 + k_0^2)^{1/2})}\right] \Omega_k\,, \tag{4.9}$$

where

$$k_0^2 = \frac{\Gamma_1}{|\Gamma|^2}\frac{g_1^2\psi_s^2}{K'}. \tag{4.10}$$

We have again solved exactly the eigenvalue problems of the "eigen"operator with standard means. The third term in the bracket of (4.9) appears by solving the Schrödinger equation with a step function-like potential in Λ_{q0} (3.2.12a). Thus the relaxation rate is given by

$$\Omega_k = \left[\frac{\sigma_1^2}{2\lambda'} + \frac{\Gamma_1\sigma}{|\Gamma|^2}k_\parallel + \frac{\sigma_2^2 k_\parallel}{K'(k_\parallel + (k_\parallel^2 + k_0^2)^{1/2})}\right]^{-1}\sigma k_\parallel^3. \tag{4.11}$$

For ^{3}He–^{4}He mixtures an order estimation indicates that the first term in (4.11) is dominant for $k_\parallel\xi \ll 1$. If the normal velocity field, which has been omitted in our model (3.2.1), is taken into account Ω_k would have an additive term like $\sigma k_\parallel/\eta$ with η the shear viscosity just as in the critical fluids[6]), i.e. (4.5), and this additive term dominates the $k_\parallel^3$ term in (4.11). It should be also noted that form (4.4) of the Fourier component of $\mathbf{D}(\boldsymbol{a}, \boldsymbol{a}')$ is quite different from (4.9). This originates from the difference between the differential integral equation which appeared in the derivation of (4.4) and the simple differential one in (4.9).

Finally let us consider metamagnets. The dynamics of metamagnets is obtained by putting $g_1 = L = K = 0$ and $\Gamma_2 = g_2 = 0$, with $\psi^* = \psi$ (real) in (3.2.1). Substituting these conditions into (4.11) we obtain the relaxation rate, i.e.

$$\Omega_k = \frac{\sigma\Gamma_1 k_\parallel^3}{\sigma k_\parallel + (\Gamma_1(\Delta c)^2/2\lambda)}. \tag{4.12}$$

This is consistent with the one obtained by San Miguel et al.[15]).

5. Equation of motion of a vortex line

Here we derive the equation of motion of a smoothly curved vortex line of ^{3}He–^{4}He mixtures. The smoothly curved vortex line may be represented by using a local cylindrical coordinate system (r, φ, τ). τ is the contour length of the vortex line. The mean field solution of the vortex line is given from (3.2.3a) by

$$-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}}{\mathrm{d}r}|\psi| + \frac{1}{r^2}|\psi| + \xi^{-2}\left(1 - \frac{|\psi|^2}{\psi_s^2}\right)\left(1 - 3\frac{|\psi|^2}{\psi_s^2}\right)|\psi| = 0 \tag{5.1}$$

with the boundary condition

$$|\psi(r)| = \begin{cases} 0, & r \to 0, \\ \psi_s, & r \to \infty, \end{cases} \tag{5.2}$$

where we have used (3.2.4).

$$(\nabla\psi) = e^{i\varphi} f' \boldsymbol{\rho} + e^{i\varphi} f \frac{i}{r} \boldsymbol{\nu}, \tag{5.3a}$$

$$(\nabla c_0) = \sigma_1 f f' \boldsymbol{\rho}, \tag{5.3b}$$

$$(\nabla q_0^{\pm}) = \sigma_2 f f' \boldsymbol{\rho} \pm \sigma_3 f^2 \frac{1}{r} \boldsymbol{\nu}, \tag{5.3c}$$

where f and f' denote $|\psi(r)|$ and $df(r)/dr$, and $|\psi(r)|$ is the solution of (5.1) with the boundary condition (5.2). $\boldsymbol{\rho}(\tau)$, $\boldsymbol{\nu}(\tau)$ and $\boldsymbol{\kappa}(\tau)$ are the unit vectors corresponding to the coordinates r, φ and τ, respectively. They may change their directions with varying τ. The coefficient σ_1 and σ_2 are the same as (4.8) and σ_3 is defined by

$$\sigma_3 = \frac{\Gamma_1}{|\Gamma|^2} g_1 \psi_s^2. \tag{5.4}$$

When we substitute (5.3) into (3.2.24) the integral of $\boldsymbol{r}_\perp(\tau)$ is taken over r and φ. Two Green functions involved in the integrand can be approximated by

$$\mathcal{L}_\alpha(\boldsymbol{r}, \boldsymbol{r}') = \delta(\tau - \tau')\mathcal{L}_\alpha(r, \varphi; r', \varphi') \quad (\alpha = c_0, q_0), \tag{5.5}$$

which is justified for a sufficiently smooth vortex line. Thus the integral over φ and φ' on each component of the matrix $\mathbf{D}(\tau, \tau')$ can be readily performed by noting that the unit vector $\boldsymbol{\rho}$ and $\boldsymbol{\nu}$ have a component proportional to the Legendre function $P_1(\cos\varphi)$ or the associated Legendre function $P_1^1(\cos\varphi)$. Consequently the equation of motion is given from (3.2.24) by

$$\boldsymbol{F}(\boldsymbol{r}(\tau)) = -D_1(\boldsymbol{v}(\tau) - \boldsymbol{\theta}(\tau)) + D_2 \boldsymbol{\kappa}(\tau) \times (\boldsymbol{v}(\tau) - \boldsymbol{\theta}(\tau)), \tag{5.6}$$

where

$$D_1 = \pi \int_0^\infty dr\, r \int_0^\infty dr' \left[f'(r) \frac{\Gamma_1 \delta(r - r')}{|\Gamma|} f'(r') + \frac{f(r)}{r} \frac{\Gamma_1 \delta(r - r')}{|\Gamma|^2} \frac{f(r')}{r'} \right.$$

$$+\sigma_1^2 ff'(r)\mathscr{L}_{c_0}(r,r')ff'(r')+\sigma_2^2 ff'(r)\mathscr{L}_{q_0}(r,r')ff'(r')$$

$$-\sigma_3^2\frac{f^2(r)}{r}\mathscr{L}_{q_0}(r,r')\frac{f^2(r')}{r'}\Big], \tag{5.7a}$$

$$D_2=\pi\int_0^\infty \mathrm{d}r\, r\int_0^\infty \mathrm{d}r'\Big[\frac{f(r)}{r}\frac{\Gamma_2\delta(r-r')}{|\Gamma|^2}f'(r')+2\sigma_2\sigma_3\frac{f^2(r)}{r}\mathscr{L}_{q_0}(r,r')ff'(r')\Big]. \tag{5.7b}$$

$\mathscr{L}_{c_0}(r,r')$ and $\mathscr{L}_{q_0}(r,r')$ are now the Green functions corresponding to the operators which depend only on r,

$$\Lambda_{c_{0r}}=-\lambda'\nabla_r^2, \tag{5.8a}$$

$$\Lambda_{q_{0r}}=-K'\nabla_r^2+\frac{\Gamma_1}{|\Gamma|^2}g_1^2|\psi|^2(r), \tag{5.8b}$$

where

$$\nabla_r^2=\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}}{\mathrm{d}r}-\frac{1}{r^2}. \tag{5.9}$$

The prefactor π in (5.7) and the second term of (5.9) come from the integration over φ and φ'. Equating (3.2.8) with (5.6) and taking a vector product with $\boldsymbol{\kappa}$, we obtain finally

$$\boldsymbol{v}=-\frac{D_1}{D_1^2+D_2^2}(\boldsymbol{\kappa}\times\boldsymbol{v}_s)-\frac{D_2}{D_1^2+D_2^2}(\boldsymbol{\kappa}\times(\boldsymbol{\kappa}\times\boldsymbol{v}_s))+\boldsymbol{\theta}, \tag{5.10}$$

where $\boldsymbol{v}$ is the velocity of a vortex line perpendicular to $\boldsymbol{\kappa}$.

Equation (5.10) has the same form as obtained previously for model F. The feature of ^{3}He–^{4}He mixtures enter in the coefficients D_1 and D_2. In fact, if the third term in (5.7a) is absent we recover the result of model F. Finally we note that Onuki[13]) has also derived the equation of motion of a vortex line without the random force for ^{3}He–^{4}He mixtures.

6. Discussions

We have presented a general formulation to derive the equation of motion for the topological defects. One of the advantages of the method is that the

dynamic part can be treated separately from the static one. The force exerting on the defects is obtained by purely static considerations in terms of the defect Hamiltonian. Therefore the main task is to calculate the coefficient $\mathbf{D}(\boldsymbol{a}, \boldsymbol{a}')$ defined by (2.32) which contains the whole dynamic information. This feature of theory enables a systematic and transparent derivation of the defect equation of motion.

In order to evaluate $\mathbf{D}(\boldsymbol{a}, \boldsymbol{a}')$ we need to have the diagonalized representation of the operator Λ^{-1}. By introducing the "eigen"operator $\langle\hat{\alpha}|\prod_{n=1}^{\alpha-1} Q_n \Lambda|\alpha\rangle$ and the "eigen"variables $\tilde{S}_\alpha$ and $\tilde{S}_\alpha^+$ the operator Λ can be diagonalized. The inverse operator Λ^{-1} is represented by the Green function $\mathscr{L}_\alpha$ associated with $\langle\hat{\alpha}|\prod_{n=1}^{\alpha-1} Q_n \Lambda|\alpha\rangle$.

Each model considered here has its own specific characteristic feature. In metamagnets where ψ is real and $g_2 = 0$ we can see from (3.2.1a) and (3.2.1b) that $\langle\hat{\alpha}|\Lambda|\beta\rangle$ is automatically diagonalized. These two diagonal elements result in the denominator of (4.12). In critical fluids the off-diagonal elements can be eliminated by introducing ∇c_0 and ∇c_0^+ given by (3.11b) and (3.11c). Since $j = 0$ the diagonal part is substantially simplified and is given by the differential integral operator (3.1.10a). In ^{3}He–^{4}He mixtures the direct diagonalization produces three operators Λ_K, Λ_L and Λ_λ. However these are not independent. By a suitable choice of the variables c_0 and $q_0^\pm$ as in (3.2.24c) and (3.2.24d) the diagonal elements can be expressed by Λ_{c_0} and Λ_{q_0}.

It is to be noted that our theory can deal with a system having different kinds of defects. For instance it would be an interesting future problem to study the dynamics of a system where vortex lines and interfaces coexist.

Appendix A

Here we briefly describe the derivation of the eigenvariables $(\nabla\tilde{S}_\alpha)$, $(\nabla\tilde{S}_\alpha^+)$ given by (2.27) and (2.28) and the eigenoperator $\langle\hat{\alpha}|\prod_{n=1}^{\alpha-1} Q_n \Lambda|\alpha\rangle$. The essential point is to calculate the product of Q_n, i.e. $\prod_{n=1}^{\alpha-1} Q_n$ $(\alpha = 1, \ldots, N-1)$. From the definition (2.15) of Q_n, we find

$$\prod_{n=1}^{\alpha-1} Q_n = \prod_{n=1}^{\alpha-2} Q_n - \prod_{n=1}^{\alpha-2} Q_n \Lambda|\alpha-1\rangle\langle\hat{\alpha-1}|\prod_{n=1}^{\alpha-1} Q_n \Lambda|\alpha-1\rangle^{-1}\langle\hat{\alpha-1}|\prod_{n=1}^{\alpha-2} Q_n . \tag{A.1}$$

By iteration the right-hand side of (A.1) can be written in the form

$$\prod_{n=1}^{\alpha-1} Q_n = 1 + \sum_{k,m=1}^{\alpha-1} \Lambda|m\rangle\cdots<\hat{k}|, \tag{A.2}$$

where $\cdots$ consists of operators like $\langle \hat{i}|\Lambda|j\rangle$ and $\langle \hat{j}|\prod_{n=1}^{j-1} Q_n \Lambda|j\rangle^{-1}$ $(i, j = 1, \ldots, \alpha - 1)$. Multiplying $\langle \hat{\alpha}|$ with (A.2) from the left the top ket vector $\Lambda|m\rangle$ turns out to be the operator $\langle \hat{\alpha}|\Lambda|m\rangle$. Operating $|\nabla \tilde{S}^*\rangle$ to this from the right gives the αth eigenvariable $(\nabla \tilde{S}_\alpha)$. This is the linear combination of (∇S_k^*) $(k = 1, \ldots, \alpha)$ where the suffix originates from the end dra vector $\langle \hat{k}|$. Similarly $\prod_{n=1}^{\alpha-1} \hat{Q}_n$ may be written as

$$\prod_{n=1}^{\alpha-1} \hat{Q}_n = 1 + \sum_{k,m=1}^{\alpha-1} |m\rangle \cdots \langle \hat{k}|\Lambda\,. \tag{A.3}$$

The multiplication of $|\alpha\rangle$ from the right and $\langle \nabla S^*|$ from the left gives us the corresponding eigenvariable $(\nabla \tilde{S}_\alpha^+)$. The eigenoperator $\langle \hat{\alpha}|\prod_{n=1}^{\alpha-1} Q_n \Lambda|\alpha\rangle$ is obtained by operating $\langle \alpha|$ and $\Lambda|\alpha\rangle$ from both sides. Thus we obtain the diagonal representation of $\mathbf{D}(\boldsymbol{a}, \boldsymbol{a}')$ given by (2.32).

Appendix B

The derivation of (3.2.23) is as follows. From relations (3.2.21) we obtain

$$\Lambda_L = \frac{g_2}{g_1} \Lambda_K + \frac{L'}{\lambda'} \Lambda_{c_0}\,. \tag{B.1}$$

Substitution this into (3.2.17) yields with (2.30)

$$\langle \hat{4}| \prod_{n=1}^{3} Q_n \Lambda|4\rangle = \Lambda_{c_0} \Lambda_K^{-1} \Lambda_{q_0}\,. \tag{B.2}$$

Therefore its inverse operator becomes

$$\langle \hat{4}| \prod_{n=1}^{3} Q_n \Lambda|4\rangle^{-1} = \Lambda_{c_0}^{-1} + \frac{L'^2}{\lambda^2} \Lambda_{q_0}^{-1}\,. \tag{B.3}$$

Using the relations (B.1) and (B.3) the integrand of $\mathbf{D}(\boldsymbol{a}, \boldsymbol{a}')$ can be represented in terms of $\Lambda_{c_0}^{-1}$ and $\Lambda_{q_0}^{-1}$.

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