Int J Game Theory (1998) 27:245–256

International Journal of Game Theory


# Link formation in cooperative situations*

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Received September 1995/Revised version I October 1996/Revised version II April 1997/
Final version September 1997

**Abstract.** In this paper we study the endogenous formation of cooperation structures or communication graphs between players in a superadditive TU game. For each cooperation structure that is formed, the payoffs to the players are determined by an exogenously given solution. We model the process of cooperation structure formation as a game in strategic form. It is shown that several equilibrium refinements predict the formation of the complete cooperation structure or some structure which is payoff-equivalent to the complete structure. These results are obtained for a large class of solutions for cooperative games with cooperation structures.

**Key words:** Link formation, TU game, exogenous solution

## 1. Introduction

The main goal of this paper is to analyse the pattern of cooperation between players in a cooperative game. A full-blown analysis would require simultaneous determination of the coalition structure as well as the payoffs associated with each coalition structure. However, this is an extremely complicated task. Following Hart and Kurz (1983), we address ourselves to the simpler task of analysing the equilibrium pattern of cooperation between players, assuming an *exogenously* given rule or solution which specifies the distribution of payoffs corresponding to each pattern of cooperation.

In contrast to Hart and Kurz (1983), who dealt with coalition structures,

* The authors are grateful to the anonymous referee and Associate Editor for helpful suggestions and comments.

we focus attention on Myerson's (1977) *cooperation structures*[1], rather than coalition structures. A cooperation structure is a graph whose vertices are identified with the players. A link between two players means that these players can carry on meaningful direct negotiations with each other. Notice that a coalition structure is a special kind of cooperation structure where two members $i$ and $j$ are linked if and only if they are in the same coalition.[2] Following Aumann and Myerson (1988), we model situations in which the eventual distribution of payoffs is determined in *two* distinct stages. The first period is devoted to link formation only. During this period, the players cannot enter into binding agreements of any kind, either on the nature of the link formation, or on the subsequent division of payoffs. In the second period, no new links can be formed, but players negotiate over the division of the payoff, *given* the cooperation structure which has formed in the first stage.

We assume that agents' decisions on whether or not to form a link with other agents can be represented as a game in strategic form.[3] In the link formation game, each player announces a set of players with whom he or she wants to form a link. A link is formed between $i$ and $j$ if both players want the link. Given the announcements of the $n$ players, this specification gives the cooperation structure. Suppose there is a rule or solution which determines a distribution of payoffs for each cooperation structure. This, then, also gives the payoff function of the strategic form game. Since this is a well-defined strategic form game, we can use any noncooperative equilibrium concept to analyse the game.

Suppose now that the rule which determines payoffs for each cooperation structure has the property that no agent wants to unilaterally *break* a link with any player. Since no player wants to break a link, and it needs the consent of *two* players to form an additional link, any cooperation structure can be sustained as a Nash equilibrium. We, therefore, use refinements of the Nash equilibrium concept. In particular, we employ undominated Nash equilibrium, coalition-proof Nash equilibrium, and strong Nash equilibrium. Our principal conclusion is that for a wide class of solutions, the first two equilibrium refinements lead to the formation of the full cooperation structure or cooperation structures which are *payoff-equivalent* to this structure. This is also true when the equilibrium concept is that of strong Nash equilibrium, *provided* a strong Nash equilibrium exists. However, we show that there are games in which reasonable solution concepts fail to guarantee the existence of a strong Nash equilibrium.

The plan of this paper is as follows. In section 2 we provide some basic definitions, including those of cooperation structures and solutions for games with cooperation structures. Also, we introduce the properties Component Efficiency, Weak Link Symmetry, and Improvement Property, which we believe to be 'reasonable' properties on such solutions, and we derive some implications of these properties. Section 3 contains the model of link forma-

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[1] See van den Nouweland (1993) for a survey of recent research on games with cooperation structures.

[2] Aumann and Myerson (1988) give examples of negotiation situations which can be modelled by cooperation structures, but not by coalition structures.

[3] This game was originally introduced by Myerson (1991) (p. 448). See also Hart and Kurz (1983), who discuss a similar strategic-form game in the context of the endogenous formation of coalition structures.

tion studied in this paper and the definitions of the equilibrium concepts used to analyze the model. Endogenous cooperation structures corresponding to undominated Nash equilibrium and coalition-proof Nash equilibrium are determined in section 4. We conclude in section 5.

## 2. Cooperation structures and solutions

Let $(N, v)$ be a $TU$ coalitional game, where $N = \{1, 2, \ldots, n\}$ denotes the finite player set and $v$ is a real-valued function on the family $2^N$ of all subsets of $N$ with $v(\emptyset) = 0$. *Throughout this paper, we will assume that v is superadditive*[4].

A *cooperation structure* is a graph $g = (N, L)$ where $N$ is the set of vertices, and $L$ is the *edge set*. An edge will also be called a *link*, and denoted by $l, l'$ etc. For any $S \subseteq N$, we say that players $i, j \in S$ are *connected* in $S$ if there exists a path from $i$ to $j$ that uses only vertices in $S$. The relation 'connected in $N$' is an equivalence relation on $N$. The equivalence classes of this relation are the *connected components* of the graph $g$.

We follow Anmann and Myerson (1988) in interpreting a link between two players as meaning that these players can carry on meaningful direct negotiations with each other. The negotiation to form links takes place in a preliminary period when "for one reason or another, one cannot enter into binding agreements of any kind (such as those relating to subsequent divisions of the payoff..)".[5]

A *solution* is a mapping $\gamma$ which assigns an element in $\mathbb{R}^n$ to each $TU$ game $(N, v)$ and cooperation structure $g = (N, L)$. Since there will be no ambiguity about the underlying game $(N, v)$, we will simply write $\gamma(L)$, $\gamma(L')$, etc., instead of writing $\gamma(N, v, L)$, $\gamma(N, v, L')$, etc.

A solution can for example be generated for any graph $g$ by applying the usual or familiar cooperative solution concepts to the 'graph-restricted game' $(N, v^g)$. This game is defined as follows. Let $S \backslash g$ denote the partition of $S$ into subsets of players that are connected in $S$ by $g$. That is,

$$S \backslash g = \{\{i | j \text{ and } i \text{ are connected in } S \text{ by } g\} | j \in S\} \tag{1}$$

Now, define $v^g : 2^N \to \mathbb{R}$ by

$$v^g(S) = \sum_{T \in S \backslash g} v(T) \tag{2}$$

For instance, for any $g = (N, L)$, the Shapley value of the associated game $(N, v^g)$ is a solution for $(N, v, L)$, and has come to be called the *Myerson value*.[6] Similarly, weighted Myerson values of $(N, v, L)$ are the weighted Shapley values[7] of $(N, v^g)$.

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[4] $v$ is superadditive if for all $S, T \in 2^N$ with $S \cap T = \emptyset$, $v(S) + v(T) \leq v(S \cup T)$.

[5] Aumann and Myerson (1988), page 187. See also Myerson (1977).

[6] Myerson (1977) contains a characterization of the Myerson value. See also Jackson and Wolinsky (1996).

[7] See Kalai and Samet (1988).

A class of solutions which will play a prominent role in this paper is the class satisfying the following 'reasonable' properties on a solution $\gamma$ below.

*Component efficiency (CE):* For all cooperation structures $(N, L)$ and all $S \in 2^N$, if $S$ is a connected component of $(N, L)$, then $\sum_{i \in S} \gamma_i(L) = v(S)$.

*Weak link symmetry (WLS):* For all $i,j \in N$, and all cooperation structures $(N, L)$, if $\gamma_i(L \cup \{i,j\}) > \gamma_i(L)$, then $\gamma_j(L \cup \{i,j\}) > \gamma_j(L)$.

*Improvement property (IP):* For all $i,j \in N$ and all cooperation structures $(N, L)$, if for some $k \in N \setminus \{i,j\}$, $\gamma_k(L \cup \{i,j\}) > \gamma_k(L)$, then $\gamma_i(L \cup \{i,j\}) > \gamma_i(L)$ or $\gamma_j(L \cup \{i,j\}) > \gamma_j(L)$.

These properties all have very simple interpretations. Component efficiency, which was originally used by Myerson (1977), states that the players in a connected component $S$ split the value $v(S)$ amongst themselves. The second property is a very weak form of symmetry. It says that if a new link between players $i$ and $j$ makes $i$ *strictly* better off, then it must also strictly improve the payoff of player $j$. Finally, the improvement property states that if a new link between players $i$ and $j$ strictly improves the payoff of any other player $k$, then the payoff of *either i or j* must also strictly improve.

The class of *weighted Myerson values* satisfies all the properties listed above. There are also others. For instance, if $(N, v)$ is a *convex* game, then the *egalitarian solution* of Dutta and Ray (1989) corresponding to the associated game $(N, v^g)$ also satisfies these properties.

The three properties together imply an interesting fourth property. This is the content of the next lemma.

**Lemma 1.** *Let $\gamma$ be any solution satisfying CE, WLS and IP. Then, for all $i,j \in N$, and all cooperation structures $(N, L)$,*

$$\gamma_i(L \cup \{i,j\}) \geq \gamma_i(L). \tag{3}$$

*Proof.* Suppose for some $i,j \in N$ and $(N, L)$, $\gamma_i(L) > \gamma_i(L \cup \{i,j\})$. Then, by WLS, we must also have $\gamma_j(L) \geq \gamma_j(L \cup \{i,j\})$. But then, since $v$ is superadditive, and $\gamma$ satisfies CE, there must exist $k \notin \{i,j\}$ such that $\gamma_k(L) < \gamma_k(L \cup \{i,j\})$. This shows that $\gamma$ violates IP since $\gamma_i(L) > \gamma_i(L \cup \{i,j\})$ and $\gamma_j(L) \geq \gamma_i(L \cup \{i,j\})$. ■

*Remark 1.* We will denote the property incorporated in equation (3) by *Link Monotonicity*. Note that Link Monotonicity is an appealing property in its own right. It says that a player $i$ should not be worse-off as a result of forming a new link with some player $j$.[8]

*Remark 2.* It is easy to construct examples to show that the Component Efficiency, Weak Link Symmetry, and Improvement properties are independent.

Another consequence of these three properties is derived in the next

[8] Note that the game is superadditive.

lemma. We show that if the formation of a link $\{i,j\}$ affects the payoff of some other player $k$, then it must also affect the payoffs of both players that formed the new link. This property will be used later on in the paper.

**Lemma 2.** *Let $\gamma$ satisfy CE, WLS and IP. Then, for all $i,j \in N$, and all cooperation structures $(N,L)$, if for some $k \in N\backslash\{i,j\}$, $\gamma_k(L \cup \{i,j\}) \neq \gamma_k(L)$, then $\gamma_i(L \cup \{i,j\}) > \gamma_i(L)$ and $\gamma_j(L \cup \{i,j\}) > \gamma_j(L)$.*

*Proof.* Suppose for some $i,j \in N$ and $k \in N\backslash\{i,j\}$, $\gamma_k(L \cup \{i,j\}) \neq \gamma_k(L)$. If $\gamma_k(L \cup \{i,j\}) > \gamma_k(L)$, then from WLS and IP we must have $\gamma_i(L \cup \{i,j\}) > \gamma_i(L)$ and $\gamma_j(L \cup \{i,j\}) > \gamma_j(L)$.

Suppose $\gamma_k(L \cup \{i,j\}) < \gamma_k(L)$. From WLS, either $\gamma_i(L \cup \{i,j\}) > \gamma_i(L)$ and $\gamma_j(L \cup \{i,j\}) > \gamma_j(L)$, or $\gamma_i(L \cup \{i,j\}) \leq \gamma_i(L)$ and $\gamma_j(L \cup \{i,j\}) \leq \gamma_j(L)$. But, in the latter case, CE and superadditivity imply that there exists a $l \notin \{i,j\}$ such that $\gamma_l(L \cup \{i,j\}) > \gamma_l(L)$. This would violate IP, so it must hold that $\gamma_i(L \cup \{i,j\}) > \gamma_i(L)$ and $\gamma_j(L \cup \{i,j\}) > \gamma_j(L)$. This establishes the lemma. ■

While the three properties are all appealing and are satisfied by a large class of solutions, there are other solutions outside this class that seem to be appealing. One such solution is defined below.

For any $i$ and $L$, let $L_i = \{\{i,j\} | j \in N, \{i,j\} \in L\}$, the set of links that are adjacent to $i$, and $l_i = |L_i|$. Let $S_i(L)$ denote the connected component of $L$ containing $i$. Then, the *Proportional Links Solution*, denoted $\gamma^P$, is given by

$$\gamma_i^P(L) = \begin{cases} \dfrac{l_i}{\sum_{j \in S_i(L)} l_j} v(S_i(L)) & \text{if } |S_i(L)| \geq 2 \\ v(\{i\}) & \text{if } S_i(L) = \{i\} \end{cases} \tag{4}$$

for all $L$ and all $i \in N$. The solution $\gamma^P$ captures the notion that the more links a player has with other players, the better are his *relative* prospects in the subsequent negotiations over the division of the payoff. Notice that this makes sense only when the players are equally 'powerful' in the game $(N,v)$. Otherwise, a *big* player may get more than *small* players even if he has fewer links. We leave it to the reader to check that $\gamma^P$ satisfies CE and IP, but not WLS.

## 3. Modelling negotiation processes

As we have remarked before, we model the process of link formation as a game in strategic form.[9] The specific strategic form game that we will construct was first defined by Myerson (1991), and has subsequently been used by Qin (1996). This model is described below.

Let $\gamma$ be a solution. Then, the *linking game* $\Gamma(\gamma)$ associated with $\gamma$ is given by the $(n+2)$-tuple $(N; S_1, \ldots, S_n; f^\gamma)$ where for each $i \in N$, $S_i$ is player $i$'s strategy set with $S_i = 2^{N\backslash\{i\}}$, and the payoff function is the mapping

[9] In contrast, Aumann and Myerson (1988) use an extensive form approach. See Dutta et al. (1995) for a discussion of the two approaches.

$f^{\gamma} : \prod_{i \in N} S_i \to \mathbb{R}^n$ given by

$$f_i^{\gamma}(s) = \gamma_i(L(s)) \tag{5}$$

for all $s \in \prod_{i \in N} S_i$, with

$$L(s) = \{\{i,j\} | j \in s_i, i \in s_j\} \tag{6}$$

The interpretation of (5) and (6) is straightforward. A typical strategy of player $i$ in $\Gamma(\gamma)$ consists of the set of players with whom $i$ wants to form a link. Then (6) states that a link between $i$ and $j$ is formed if and only if they both want to form this link. Thus, each strategy vector $s$ gives rise to a unique co-operation structure $L(s)$. Finally, the payoff to player $i$ associated with $s$ is simply $\gamma_i(L(s))$[10], the payoff that $\gamma$ associates with the cooperation structure $L(s)$.

We will let $\bar{s} = (\bar{s}_1, \ldots, \bar{s}_n)$ denote the strategy vector such that $\bar{s}_i = N \backslash \{i\}$ for all $i \in N$, while $\bar{L} = \{\{i,j\} | i \in N, j \in N\} = L(\bar{s})$ denotes the complete edge set on $N$. A cooperation structure $L$ is *essentially complete for* $\gamma$ if $\gamma(L) = \gamma(\bar{L})$. Hence, if $L$ is essentially complete for $\gamma$, but $L \neq \bar{L}$, then the links which are not formed in $L$ are inessential in the sense that their absence does not change *the payoff vector* from that corresponding to $\bar{L}$. Notice that the property of "essentially complete" is specific to the solution $\gamma$ – a cooperation structure $L$ may be essentially complete for $\gamma$, but not for $\gamma'$.

We now define some equilibrium concepts for any $\Gamma(\gamma)$ that will be used in section 4 below.

The first equilibrium concept that we consider is the undominated Nash equilibrium. For any $i \in N$, $s_i$ *dominates* $s_i'$ iff for all $s_{-i} \in S_{-i}$, $f_i^{\gamma}(s_i, s_{-i}) \geq f_i^{\gamma}(s_i', s_{-i})$ with the inequality being strict for *some* $s_{-i}$. Let $S_i^u(\gamma)$ be the set of undominated strategies for $i$ in $\Gamma(\gamma)$, and $S^u(\gamma) = \prod_{i \in N} S_i^u(\gamma)$. A strategy tuple $s$ is an *undominated Nash equilibrium* of $\Gamma(\gamma)$ if $s$ is a Nash equilibrium and, moreover, $s \in S^u(\gamma)$.

The second equilibrium concept that will be discussed is the Coalition-Proof Nash Equilibrium. In order to define the concept of Coalition-Proof Nash Equilibrium of $\Gamma(\gamma)$, we need some more notation. For any $T \subset N$ and $s_T^* \in S_T := \prod_{i \in T} S_i$, let $\Gamma(\gamma, s_{N \backslash T}^*)$ denote the game induced on subgroup $T$ by the actions $s_{N \backslash T}^*$. So,

$$\Gamma(\gamma, s_{N \backslash T}^*) = \langle T, \{S_i\}_{i \in T}, \tilde{f}^{\gamma} \rangle$$

where for all $j \in T$, $\tilde{f}_j^{\gamma} : \prod_{i \in T} S_i \to \mathbb{R}$ is given by $\tilde{f}_j^{\gamma}((s_i)_{i \in T}) = f_j^{\gamma}((s_i)_{i \in T}, s_{N \backslash T}^*)$ for all $(s_i)_{i \in T} \in S_T$.

The Coalition-Proof Nash Equilibrium is defined inductively as follows: In a single player game, $s^* \in S$ is a *Coalition-Proof Nash Equilibrium* (CPNE) of $\Gamma(\gamma)$ iff $s_i^*$ maximizes $f_i^{\gamma}(s)$ over $S$. Now, let $\Gamma(\gamma)$ be a game with $n$ players, where $n > 1$, and assume that Coalition-Proof Nash Equilibria have been defined for games with less than $n$ players. Then, a strategy tuple $s^* \in S_N := \prod_{i \in N} S_i$ is called *self-enforcing* if for all $T \subsetneq N$, $s_T^*$ is a CPNE in the game

[10] We again remind the reader that we have suppressed the underlying $TU$ game $(N, v)$ in order to simplify the notation.

$\Gamma(\gamma, s^*_{N\setminus T})$. A strategy tuple $s^* \in S_N$ is a CPNE of $\Gamma(\gamma)$ if it is self-enforcing and, moreover, there does not exist another self-enforcing strategy vector $s \in S_N$ such that $f_i^\gamma(s) > f_i^\gamma(s^*)$ for all $i \in N$.

Let CPNE $(\gamma)$ denote the set of CPNE of $\Gamma(\gamma)$.[11] Notice that the notion of CPNE incorporates a kind of 'farsighted' thought process on the part of players since a coalition when contemplating a deviation takes into consideration the possibility of further deviations by subcoalitions.[12]

The third equilibrium concept that we consider is that of strong Nash equilibrium. A strategy tuple $s$ is a *Strong Nash Equilibrium* (SNE) of $\Gamma(\gamma)$ if there is no coalition $T \subseteq N$ and strategies $s'_T \in S_T$ such that

$$f_i^\gamma((s'_T, s_{N\setminus T}) > f_i^\gamma(s) \quad \text{for all } i \in T.$$

We denote the set of SNE of $\Gamma(\gamma)$ SNE $(\gamma)$.

## 4. Equilibrium cooperation structures

In this section, we characterize the sets of equilibrium cooperation structures under the equilibrium concepts defined in the previous section.

We consider refinements of Nash equilibrium because Nash equilibrium itself does not enable us to distinguish between different cooperation structures. If a solution satisfies the properties listed in section 2, then no player wants to unilaterally break a link because of link monotonicity. Further, it needs the consent of *two* players to form a link. Because of these two facts, any cooperation structure can be sustained in a Nash equilibrium.

**Proposition 1.** *Let $\gamma$ be a solution that satisfies CE, WLS, and IP. Then any cooperation structure can be sustained in a Nash equilibrium.*

*Proof.* Let $g = (N, L)$ be a cooperation structure. Define for each player $i \in N$ the strategy $s_i = \{j \in N\setminus\{i\} | \{i,j\} \in L\}$. That is, each player announces that he wants to form links with exactly those players to which he is directly connected in $g$. It is easily seen that $s = (s_i)_{i\in N}$ is a Nash equilibrium of $\Gamma(\gamma)$, because for all $i, j \in N$ it holds that $j \in s_i$ if and only if $i \in s_j$. Further, $L(s) = L$. ■

Our principal objective is to show that the equilibrium concepts of undominated Nash equilibrium and coalition-proof Nash equilibrium both lead to essentially complete cooperation structures for solutions satisfying the properties that are listed in section 2.

**Theorem 1.** *Let $\gamma$ be a solution that satisfies CE, WLS and IP. Then, $\bar{s}$ is an undominated Nash equilibrium of $\Gamma(\gamma)$. Moreover, if $s$ is an undominated Nash equilibrium of $\Gamma(\gamma)$, then $L(s)$ is essentially complete for $\gamma$.*

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[11] See Bernheim, Peleg and Whinston (1987) for discussion of Coalition-Proof Nash Equilibrium.

[12] We mention this because Aumann and Myerson (1988) state that they do not use the 'usual, myopic, here-and-now kind of equilibrium condition', but a 'look ahead' one. Of course, farsightedness can be modelled in many different ways.

*Proof.* First, we show that $\bar{s}_i$ is undominated for all $i \in N$.

So, choose $i \in N$, $s_i \in S_i$ and $s_{-i} \in S_{-i}$ arbitrarily. Let $L = L(\bar{s}_i, s_{-i})$ and $L' = L(s_i, s_{-i})$. Note that, since $s_i \subseteq \bar{s}_i$, $L' \subseteq L$. Also, if $l \in L \backslash L'$, then $i \in l$. So, from repeated application of link monotonicity (see lemma 1),

$$f_i^{\gamma}(\bar{s}_i, s_{-i}) = \gamma_i(L) \geq \gamma_i(L') = f_i^{\gamma}(s_i, s_{-i}) \tag{7}$$

Since $s_i$ and $s_{-i}$ were chosen arbitrarily, this shows that $\bar{s}_i \in S_i^u(\gamma)$. Further, putting $s_{-i} = \bar{s}_{-i}$ in (7), we also get that $\bar{s}$ is a Nash equilibrium of $\Gamma(\gamma)$. So, we may conclude that $\bar{s} \in S^u(\gamma)$.

Now, we show that $L(s)$ is essentially complete for an undominated Nash equilibrium $s$. Choose $s \neq \bar{s}$ arbitrarily. Without loss of generality, let $\{i \in N | s_i \neq \bar{s}_i\} = \{1, 2, \ldots, K\}$. Construct a sequence $\{s^0, s^1, \ldots, s^K\}$ of strategy tuples as follows.

(i) $s^0 = s$

(ii) $s_k^k = \bar{s}_k$ for all $k = 1, 2, \ldots, K$.

(iii) $s_j^k = s_j^{k-1}$ for all $k = 1, 2, \ldots, K$, and all $j \neq k$.

Clearly, $s^K = \bar{s}$. Consider any $s^{k-1}$ and $s^k$. By construction, $s_j^{k-1} = s_j^k$ for all $j \neq k$, while $s_k^k = \bar{s}_k$ and $s_k^{k-1} = s_k$. So, using link monotonicity, we have

$$f_k^{\gamma}(s^k) = \gamma_k(L(s^k)) \geq \gamma_k(L(s^{k-1})) = f_k^{\gamma}(s^{k-1}) \tag{8}$$

Suppose (8) holds with strict inequality. Then, we have demonstrated the existence of strategies $s_{-k}$ such that

$$f_k^{\gamma}(\bar{s}_k, s_{-k}) > f_k^{\gamma}(s_k, s_{-k}) \tag{9}$$

But, (7) and (9) together show that $\bar{s}_k$ *dominates* $s_k$. So, if $s \in S^u(\gamma)$, then (8) must hold with equality. Then it follows from lemma 2 that the payoffs to all players remain unchanged when going from $s^{k-1}$ to $s^k$, so

$$\gamma(L(s^k)) = \gamma(L(s^{k-1})) \tag{10}$$

Since this argument can be repeated for $k = 1, 2, \ldots, K$, we get $\gamma(L(s^0)) = \gamma(L(s^1)) = \cdots = \gamma(L(\bar{s}))$. Hence, if $s \in S^u(\gamma)$, then $L(s)$ is essentially complete. ■

The following example shows that link monotonicity alone does not guarantee the validity of the statements in theorem 1. It is easily seen from the proof of the theorem that $\bar{s}$ is an undominated Nash equilibrium of $\Gamma(\gamma)$ if $\gamma$ is link monotonic, so the first part of the theorem only requires link monotonicity of $\gamma$. However, the second part of the theorem might be violated even if $\gamma$ is link monotonic.

*Example 1.* Consider the TU game $v$ on the player set $\{1, 2, 3\}$ defined by

$$v(S) = \begin{cases} 5 & \text{if } S = N \\ 1 & \text{if } |S| = 2 \\ 0 & \text{otherwise} \end{cases}$$

and the component efficient solution $\gamma$ defined for this game by $\gamma(\{1,2\}) = \gamma(\{2,3\}) = (0,1,0)$, $\gamma(\{1,3\}) = (0,0,1)$, $\gamma(\{1,2\},\{1,3\}) = (2,2,1)$, $\gamma(\{1,2\},\{2,3\}) = (1,4,0)$, and $\gamma(\{1,3\},\{2,3\}) = \gamma(\bar{L}) = (1,3,1)$. It is not hard to see that $\gamma$ satisfies IP and link monotonicity but fails to satisfy WLS. Further, strategy $s_3 = \{1\}$ is an undominated strategy for player 3, and strategies $s_1 = \{2,3\}$ and $s_2 = \{1,3\}$ are undominated strategies for players 1 and 2, respectively. Hence, $s = (s_1, s_2, s_3)$ is an undominated Nash equilibrium of the game $\Gamma(\gamma)$. Note that $L(s)$ is not essentially complete for $\gamma$.

In the following theorem we consider Coalition-Proof Nash Equilibria.

**Theorem 2.** *Let $\gamma$ be a solution satisfying CE, WLS and IP. Then $\bar{s} \in CPNE(\gamma)$. Moreover, if $s \in CPNE(\gamma)$, then $L(s)$ is essentially complete for $\gamma$.*

*Proof.* In fact, we will prove a slightly generalized version of the theorem and show that for each coalition $T \subseteq N$ and all $s_{N\setminus T} \in s_{N\setminus T}$ it holds that $\bar{s}_T \in \text{CPNE}(\gamma, s_{N\setminus T})$ and that for all $s_T^* \in \text{CPNE}(\gamma, s_{N\setminus T})$ it holds that $f^\gamma(s_T^*, s_{N\setminus T}) = f^\gamma(\bar{s}_T, s_{N\setminus T})$. We will follow the definition of Coalition-Proof Nash Equilibrium and proceed by induction on the number of elements of $T$. Throughout the following, we will assume $s_{N\setminus T} \in S_{N\setminus T}$ to be arbitrary.

Let $T = \{i\}$. Then by repeated application of Link Monotonicity we know that $f_i^\gamma(\bar{s}_i, s_{N\setminus\{i\}}) \geq f_i^\gamma(s_i, s_{N\setminus\{i\}})$ for all $s_i \in S_i$. From this it readily follows that $\bar{s}_i \in \text{CPNE}(\gamma, s_{N\setminus\{i\}})$. Now, suppose $s_i^* \in \text{CPNE}(\gamma, s_{N\setminus\{i\}})$. Then, since $f_i^\gamma(s_i^*, s_{N\setminus\{i\}}) \leq f_i^\gamma(\bar{s}_i, s_{N\setminus\{i\}})$, it follows that $f_i^\gamma(s_i^*, s_{N\setminus\{i\}}) = f_i^\gamma(\bar{s}_i, s_{N\setminus\{i\}})$ must hold. Now we use lemma 2 and see that $f^\gamma(s_i^*, s_{N\setminus\{i\}}) = f^\gamma(\bar{s}_i, s_{N\setminus\{i\}})$.

Now, let $|T| > 1$ and assume that we already proved that for all $R$ with $|R| < |T|$ and all $s_{N\setminus R} \in S_{N\setminus R}$ it holds that $\bar{s}_R \in \text{CPNE}(\gamma, s_{N\setminus R})$ and that for all $s_R^* \in \text{CPNE}(\gamma, s_{N\setminus R})$ it holds that $f^\gamma(s_R^*, s_{N\setminus R}) = f^\gamma(\bar{s}_R, s_{N\setminus R})$. Then it readily follows from the first part of the induction hypothesis that $\bar{s}_R \in \text{CPNE}(\gamma, \bar{s}_{T\setminus R}, s_{N\setminus T})$ for all $R \subsetneq T$. This shows that $\bar{s}_T$ is self-enforcing.

Suppose $s_T^* \in S_T$ is also self-enforcing, i.e. $s_R^* \in \text{CPNE}(\gamma, s_{T\setminus R}^*, s_{N\setminus T})$ for all $R \subsetneq T$. We will start by showing that $f_i^\gamma(\bar{s}_T, s_{N\setminus T}) \geq f_i^\gamma(s_T^*, s_{N\setminus T})$ for all $i \in T$, which proves that $\bar{s}_T \in \text{CPNE}(\gamma, s_{N\setminus T})$. So, let $i \in T$ be fixed for the moment. Then repeated application of Link Monotonicity implies that $f_i^\gamma(\bar{s}_T, s_{N\setminus T}) \geq f_i^\gamma(s_i^*, \bar{s}_{T\setminus i}, s_{N\setminus T})$. Further, since $s_{T\setminus\{i\}}^* \in \text{CPNE}(\gamma, s_i^*, s_{N\setminus T})$, it follows from the second part of the induction hypothesis that $f^\gamma(s_i^*, \bar{s}_{T\setminus\{i\}}, s_{N\setminus T}) = f^\gamma(s_T^*, s_{N\setminus T})$. Combining the two last (in)equalities we find that $f_i^\gamma(\bar{s}_T, s_{N\setminus T}) \geq f_i^\gamma(s_T^*, s_{N\setminus T})$.

Note that we will have completed the proof of the theorem if we show that, in addition to $f_i^\gamma(\bar{s}_T, s_{N\setminus T}) \geq f_i^\gamma(s_T^*, s_{N\setminus T})$ for all $i \in T$, it holds that either $f_i^\gamma(\bar{s}_T, s_{N\setminus T}) > f_i^\gamma(s_T^*, s_{N\setminus T})$ for all $i \in T$ (and, consequently, $s_T^* \notin \text{CPNE}(\gamma, s_{N\setminus T})$) or $f_i^\gamma(\bar{s}_T, s_{N\setminus T}) = f_i^\gamma(s_T^*, s_{N\setminus T})$ for all $i \in T$ (and $s_T^* \in \text{CPNE}(\gamma, s_{N\setminus T})$). So, suppose $i \in T$ is such that $f_i^\gamma(\bar{s}_T, s_{N\setminus T}) > f_i^\gamma(s_T^*, s_{N\setminus T})$. Because $s_T^*$ is self-enforcing, we know that $s_{T\setminus\{j\}}^* \in \text{CPNE}(\gamma, s_j^*, s_{N\setminus T})$ for each $j \in T$, and it follows from the induction hypothesis that $f^\gamma(s_T^*, s_{N\setminus T}) = f^\gamma(s_j^*, \bar{s}_{T\setminus j}, s_{N\setminus T})$ for each $j \in T$. Let $j \in T\setminus\{i\}$ be fixed. Then we have just shown that $f_i^\gamma(\bar{s}_T, s_{N\setminus T}) > f_i^\gamma(s_T^*, s_{N\setminus T}) = f_i^\gamma(s_j^*, \bar{s}_{T\setminus j}, s_{N\setminus T})$. We know by repeated application of Link Monotonicity that $f_j^\gamma(\bar{s}_T, s_{N\setminus T}) \geq f_j^\gamma(s_j^*, \bar{s}_{T\setminus j}, s_{N\setminus T})$. However, if this should hold with equality, $f_j^\gamma(\bar{s}_T, s_{N\setminus T}) = f_j^\gamma(s_j^*, \bar{s}_{T\setminus j}, s_{N\setminus T})$, then repeated application of lemma 2 would imply that $f^\gamma(\bar{s}_T, s_{N\setminus T}) =$

$f^{\gamma}(s_j^*, \bar{s}_{T\setminus j}, s_{N\setminus T})$, which contradicts that $f_i^{\gamma}(\bar{s}_T, s_{N\setminus T}) > f_i^{\gamma}(s_j^*, \bar{s}_{T\setminus j}, s_{N\setminus T})$. Hence, we may conclude that $f_j^{\gamma}(\bar{s}_T, s_{N\setminus T}) > f_j^{\gamma}(s_j^*, \bar{s}_{T\setminus j}, s_{N\setminus T})$. Since $f_j^{\gamma}(s_j^*, \bar{s}_{T\setminus j}, s_{N\setminus T}) = f_j^{\gamma}(s_T^*, s_{N\setminus T})$, we now know that $f_j^{\gamma}(\bar{s}_T, s_{N\setminus T}) > f_j^{\gamma}(s_T^*, s_{N\setminus T})$.

This shows that either $f_i^{\gamma}(\bar{s}_T, s_{N\setminus T}) > f_i^{\gamma}(s_T^*, s_{N\setminus T})$ for all $i \in T$ or $f_i^{\gamma}(\bar{s}_T, s_{N\setminus T}) = f_i^{\gamma}(s_T^*, s_{N\setminus T})$ for all $i \in T$. ■

*Remark 3.* We have an example of a solution satisfying CE, WLS and IP, for which CPNE($\gamma$) $\neq \{s | L(s)$ is essentially complete$\}$. In other words, there may be a strategy tuple $s$ which is not in CPNE($\gamma$), though $L(s)$ is essentially complete.

We defined the Proportional Links Solution $\gamma^P$ in section 2, and pointed out that it does not satisfy WLS. It also turns out that the conclusions of theorem 2 are no longer valid in the linking game $\Gamma(\gamma^P)$. While we do not have any general characterization results for $\Gamma(\gamma^P)$, we show below that complete structures will not necessarily be coalition-proof equilibria of $\Gamma(\gamma^P)$ by considering the special case of the 3-player *majority* game.[13]

**Proposition 2.** *Let $N$ be a player set with $|N| = 3$, and let $v$ be the majority game on $|N|$. Then, $s \in CPNE(\gamma^P)$ iff $L(s) = \{\{i,j\}\}$, i.e., only one pair of agents forms a link.*

*Proof.* Suppose only $i$ and $j$ form a link according to $s$. Then, $f_i^{\gamma^P}(s) = f_j^{\gamma^P}(s) = \frac{1}{2}$. Check that if $i$ deviates and forms a link with $k$, then $i$'s payoff remains at $\frac{1}{2}$. Also, clearly $i$ and $j$ together do not have any profitable deviation. Hence, $s$ is coalition-proof.

Suppose $L(s) = \emptyset$. Then, $f_i^{\gamma^P}(s) = 0$ for all $i$. Suppose there are $i$ and $j$ such that $j \in s_i$. Then, $s$ is not a Nash equilibrium since $j$ can profitably deviate to $s_j' = \{i\}$. Note that $L(s_j', s_{-j}) = \{i,j\}$, and $f_j^{\gamma^P}(s_j', s_{-j}) = \frac{1}{2}$.

If $s_i = \emptyset$ for all $i$, then *any* two agents, say $i$ and $j$, can deviate profitably to form the link $\{i,j\}$. Neither $i$ nor $j$ has a further deviation.

Now, suppose that $N$ is a connected set according to $s$. There are two possibilities.

*Case (i)*: $L(s) = \bar{L}$. In that case, $f_i^{\gamma^P} = \frac{1}{3}$ for all $i \in N$. Let $i$ and $j$ deviate and break links with $k$. Then, both $i$ and $j$ get a payoff of $\frac{1}{2}$. Suppose $i$ makes a further deviation. The only deviation which needs to be considered is if $i$ re-establishes a link with $k$. Check that $i$'s payoff *remains* at $\frac{1}{2}$. So, in this case $s$ cannot be a coalition-proof equilibrium.

*Case (ii)*: $L(s) \neq \bar{L}$. Since $N$ is a connected set in $L(s)$, the only possibility is that there exist $i$ and $j$ such that both are connected to $k$, but not to each other. Then, both $i$ and $j$ have a payoff of $\frac{1}{4}$. Let now $i$ and $j$ deviate, break links with $k$ and form a link between each other. Then, their payoff increases to $\frac{1}{2}$. Check that neither player has any further profitable deviation. Again, this shows that $s$ is not coalition-proof. ■

[13] $v$ is a majority game if a majority coalition has worth 1, and all other coalitions have zero worth.

*Remark 4.* The Proportional Links Solution $\gamma^P$ satisfies CE and IP and is link monotonic in the case covered by proposition 2. This observation shows that we cannot replace WLS by link monotonicity in theorem 2.

The last equilibrium concept we discuss is strong Nash equilibrium. Since every strong Nash equilibrium is a coalition-proof Nash equilibrium, it follows immediately from Theorem 2 that for a solution satisfying CE, WLS, and IP it holds that if $s \in \mathrm{SNE}(\gamma)$, then $L(s)$ is essentially complete for $\gamma$. However, strong Nash equilibria might not exist. One might think that for strong Nash equilibria to exist, some condition like *balancedness* of $v$ is needed, but we have examples that show that balancedness of $v$ is not necessary and even convexity of $v$ is not sufficient for nonemptiness of the set of strong Nash equilibria of the linking game.

## Conclusion

In this paper, we have studied the endogenous formation of *cooperation structures* in superadditive TU-games using a strategic game approach. In this strategic game, each player announces the set of players with whom he or she wants to form a link, and a link is formed if and only if both players want to form the link. Given the resulting cooperation structure, the payoffs are determined by some exogenous solution for cooperative games with cooperation structures. We have concentrated on the class of solutions satisfying three appealing properties. We have shown that in this setting both the undominated Nash equilibrium and the Coalition-Proof Nash Equilibrium of this strategic form game predict the formation of the full cooperation structure or some payoff equivalent structure. This is also true for the concept of strong Nash equilibrium, although there are games for which the set of strong Nash equilibria may be empty.[14]

The results obtained in this paper all point in the direction of the formation of the full cooperation structure in a superadditive environment. However, as we have indicated earlier, these results are sensitive to the assumptions on solutions for cooperative games with cooperation structures. Further, the discussion in section 3 of Dutta et al. (1995) shows that in a context where links are formed sequentially rather than simultaneously other predictions may prevail.

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[14] In a separate paper, Slikker et al. (1997), we show that another equilibrium for linking games, the *argmax sets of weighted potentials*, also predicts the formation of the full cooperation structure. See Monderer and Shapley (1996) for various properties of weighted potential games.

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