A COMPLEX ANALYTIC SOLUTION FOR THE ATTITUDE MOTION OF A NEAR-SYMMETRIC RIGID BODY UNDER BODY-FIXED TORQUES

P. TSIOTRAS AND J.M. LONGUSKI

School of Aeronautics and Astronautics, Purdue University, West Lafayette, Indiana 47907, U.S.A.

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Abstract. Although analytic solutions for the attitude motion of a rigid body are available for several special cases, a comprehensive theory does not exist in the literature for the more complicated problems found in spacecraft dynamics. In the present paper, analytic solutions in complex form are derived for the attitude motion of a near-symmetric rigid body under the influence of constant body-fixed torques. The solution is very compact, which enables efficient and rapid machine computation. Numerical simulations reveal that the solution is very accurate when applied to typical spinning spacecraft problems.

Key words: Analytic solutions, rigid body motion, kinematics.

1. Introduction

Mathematicians have been working on the problem of rigid body motion for over two centuries. However, an analytic solution to the general problem of a rigid body under the influence of arbitrary external torques, is far from complete. In fact, most existing analytic theories apply to highly idealized cases, such as torque-free or symmetric bodies. Solutions have been obtained for these and several other special cases by Euler, Jacobi, Poinsot and others, and are reported by Leimanis (1965) and Grammel (1954). Unfortunately these solutions are hardly of practical importance to the more complex problems encountered in spacecraft dynamics and control. In fact, prior to the advent of jet propulsion, the problem of the self-excited rigid body, that is, a body under the influence of body-fixed torques, was mainly of academic interest, and most of the previous analytic theory was merely concerned with the case when the applied torques are dependent on the actual orientation of the body. Furthermore, even for these simple cases, such as the case of torque-free motion of a general rigid body, where an analytic solution involving Jacobian elliptic functions has existed since the late 1800s, most modern authors of classical mechanics texts have elected not to discuss the details of the explicit analytic solution, in favor of the motion analogy to an ellipsoid rolling on an invariable plane, first given by Poinsot (1851).

As a result of the small amount of attention that has been paid to the development of a comprehensive analytic theory which treats the rigid body motion, scientists and engineers have come to rely on numerical methods for the solution of the problem. Even though such numerical solutions are easily

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found by computer simulations, analytic solutions can provide deeper insight into the problem, and can be used in obtaining quick solutions over large intervals of time, in error analyses, and in computer algorithms for on board computations. Recently, new interest has been revived in the area of analytic solutions for the motion of spinning spacecraft. Analytical formulations have been obtained for satellite attitude computations, which significantly extend the classical torque-free and rigid body assumption of Poinsot motion (Cochran, 1972; Kraige and Junkins, 1976; Kraige and Skaar, 1977; Cochran and Shu, 1983). Other authors (Junkins *et al.*, 1973; Morton *et al.*, 1974) have also developed new formulations for Poinsot motion itself. Current interest in the area of analytic solutions for spinning spacecraft is carried on, mainly because they have been found to be extremely useful in control problems and stability analyses associated with this class of vehicles (Likins, 1967; Larson and Likins, 1973; Junkins and Turner, 1980; Golubev and Demidov, 1984; Branets *et al.*, 1984; Winfree and Cochran, 1986). Among the recent developments in this field we can briefly mention especially the work by Larson and Likins (1974), where they obtained a close-form solution for linearized equations in which transverse torques appear, but the spin rate is constant, and also the work by Cochran *et al.* (1982) where an exact solution was obtained for the free motion of a dual-spin spacecraft. For a symmetric rigid body subject to body-fixed torques about its principal axes a solution is given by Bödewadt (1952) and is discussed by Leimanis (1965), but the solution for the orientation of the body in inertial space is incorrect in these references, for reasons explained by Longuski (1984). The case of near symmetry is dealt with by Longuski (1980) which includes an analytic solution for the Eulerian velocities (which reduces to the exact solution of Bödewadt in the case of symmetry) and an approximate analytic solution for the Eulerian angles which provides the orientation of the body in inertial space. The accuracy of the solutions has been tested, and the results are reported by Kia and Longuski (1984). Price (1981), using Longuski's solution as a first order approximation, has developed a semi-analytic solution in the form of power series in one of the applied torques. Although the series converge very rapidly, the method is limited to selected time intervals, so it has short term validity. Van der Ha (1984) gives a perturbation solution for the attitude motion under body-fixed torques, based on the ratio of transverse-to-spin-rotation rate as the small parameter, but his solution is also valid only for short time intervals.

The scope of the present paper is to provide analytic solutions to the problem of the attitude evolution of a near-symmetric rigid body under constant body-fixed torques. The use of complex variables allows the solution to be expressed in a very compact form. The solution of Euler velocities is given in terms of a complex Fresnel integral function, and it is *exact for an axisymmetric body*. For a *near-symmetric body*, the solution is valid when the product of the transverse angular velocities is small relative to the angular acceleration about the spin axis. The solution for Euler angles is more involved, due to the difficulty in evaluating certain integrals in closed form, and it is limited by the assumption that the two angles defining the direction of the spin axis must remain small. As compared to previous related results (Longuski, 1980; Price, 1981), the current solution also has the advantage, that it remains valid even for the case of despinning through the zero spin rate neighborhood. This is a significant extension to the existing theory, as no other analytical solutions for the low spin rate case have been reported in the literature. The accuracy of the solution is tested by numerical simulations and comparison with the solutions of the governing differential equations. Two cases are presented here, the first a spin-up maneuver from 3.15 to 10 rpm, and the second a spin-down maneuver through zero spin rate. Specific parameters were taken from the Galileo spacecraft (in its all-spin mode), and the results reveal an excellent agreement between the 'exact' numerical integration solution, and the analytic solution.

2. Solution for Angular Velocities

Euler's equations for motion of a rigid body with principal axes at the center of mass are

$$M_x = I_x\dot{\omega}_x + (I_z - I_y)\omega_y\omega_z \tag{1a}$$

$$M_y = I_y\dot{\omega}_y + (I_x - I_z)\omega_z\omega_x \tag{1b}$$

$$M_z = I_z\dot{\omega}_z + (I_y - I_x)\omega_x\omega_y \tag{1c}$$

where M_x, M_y and M_z are assumed to be constant body-fixed torques, I_x, I_y and I_z are the moments of inertia about the principal axes and ω_x, ω_y and ω_z are the angular velocity components along the same axes. For a nearly symmetric (about the $\hat{z}$ axis) rigid body $I_x \approx I_y$, one can immediately solve (1c) to obtain

$$\omega_z(t) \approx \frac{M_z}{I_z}\,t + \omega_{z0}\ , \qquad \omega_z(0) = \omega_{z0}\ . \tag{2}$$

The approximation (2) is not only valid for nearly symmetric bodies, but also for the important practical case of spin-stabilized vehicles (such as rockets and spacecraft), since then both Euler velocities ω_x, and ω_y tend to remain small so that their product $\omega_x\omega_y$ can be discarded in a first order approximation. Such a case implies that the spinning is about the principal axis $\hat{z}$ with constant torque M_z whereas M_x and M_y act as disturbance torques lying in the transverse plane. If the magnitude of the disturbance torques M_x and M_y is small compared to the axial torque M_z, as is often the case, then it is safe to discard the product of the transverse angular rates $\omega_x\omega_y$, as it will indeed be much smaller than the axial spin acceleration $\dot{\omega}_z$. In fact, Kia and Longuski (1984) have found through numerical simulations, that whenever $(M_x^2 + M_y^2)^{1/2}/I_z\omega_{z0}^2 < 1$, the approximation (2) tends to remain valid even for asymmetric bodies. This explains why ω_x and ω_y are often referred to as angular velocity errors.

Substituting (2) into (1a) and (1b) provides

$$M_x = I_x\dot{\omega}_x + (I_z - I_y)\omega_y\Big(\frac{M_z}{I_z}\,t + \omega_{z0}\Big) \tag{3a}$$

$$M_y = I_y\dot{\omega}_y + (I_x - I_z)\omega_x\Big(\frac{M_z}{I_z}\,t + \omega_{z0}\Big) \tag{3b}$$

Note that although we have assumed $I_x \approx I_y$ for the solution of ω_z, we have retained the distinction between I_x and I_y in the equations for ω_x and ω_y. This appears to be a trivial extension of the symmetric rigid body case; however, it will be shown that this assumption has significant consequences to the accuracy of the solution.

Rearranging terms, Equations (3) can be written in the following convenient form

$$\dot{\Omega}_x + (At + B)\Omega_y = F_x \tag{4a}$$

$$\dot{\Omega}_y - (At + B)\Omega_x = F_y \tag{4b}$$

where

$$\begin{aligned}
&\Omega_x = \omega_x\sqrt{k_y}\ , \qquad \Omega_y = \omega_y\sqrt{k_x}\ , \\
&F_x = (M_x/I_x)\sqrt{k_y}\ , \qquad F_y = (M_y/I_y)\sqrt{k_x} \\
&k_x = (I_z - I_y)/I_x\ , \qquad k_y = (I_z - I_x)/I_y \\
&A = k\alpha\ , \quad B = k\beta\ , \quad \alpha = M_z/I_z\ , \quad \beta = \omega_{z0}\ , \quad k = \sqrt{k_xk_y}\ .
\end{aligned} \tag{5}$$

The above definitions hold whether the spin axis – here assumed to be $\hat{z}$ – is the one which corresponds to the maximum or the minimum moment of inertia. Without loss of generality, we will assume that the spinning is about the axis of the major principal axis of inertia. The case where spinning is about the intermediate moment of inertia will not be considered here, since it will always result in unstable motion.

Introducing the complex variables

$$\Omega = \Omega_x + i\Omega_y\ , \qquad F = F_x + iF_y \tag{6}$$

we can combine (4a) and (4b) into the following complex scalar equation

$$\dot{\Omega} - i(At + B)\Omega = F\ . \tag{7}$$

This is a linear differential equation with a time-varying coefficient. For the constant spin-rate case it reduces to a linear time-invariant differential equation, which can easily be solved, by standard methods of operational calculus, for several special cases of forcing functions (Kurzhals, 1967). Physically Ω represents the trace of the total transverse rate velocity vector in the *skewed* body-fixed xy plane. The term *skewed* arises from the fact that Ω is not the actual transverse velocity vector – this would be $\omega_x + i\omega_y$ – but it is related to it by (5). That is, when the Ω vector traces a unit circle in the xy plane, the actual velocity vector traces an ellipse with semiaxes k_x and k_y. This difference is the result of the assumed asymmetry, and it vanishes for an axially symmetric body.

It can be easily verified that the solution of (7) is given by

$$\begin{aligned}\Omega(t) &= \Omega_0 \exp\Big[i(\tfrac{1}{2}\,At^2 + Bt + C)\Big] + \\ &\quad + \exp\Big[i(\tfrac{1}{2}\,At^2 + Bt + C)\Big]\,F\int_0^t \exp\Big[-i(\frac{1}{2}\,At^2 + Bt + C)\Big]\,dt \qquad (8)\end{aligned}$$

where the first term of the right-hand side of the equation represents the homogeneous part of the solution, and the second term represents the particular part due to the forcing function F. For reasons that will become obvious later, we choose the constant C to be equal to

$$C = \frac{B^2}{2A}\,. \qquad (9)$$

Then Ω_0 is related to the initial condition on $\Omega(t)$ as follows

$$\Omega(0) = \Omega_0 \exp\Big(i\,\frac{B^2}{2A}\Big)\,. \qquad (10)$$

Thus, given the initial conditions $\omega_x(0)$, $\omega_y(0)$ and $\omega_z(0)$, and the mass properties of the body k_x and k_y, one can use Equation (10), in order to determine Ω_0. Note from (8), that the choice of C affects only the homogeneous part of the solution, so we can always pick the value of the constant C arbitrarily, as long as we define the relation between the constant Ω_0 and the initial condition $\Omega(0)$ in a consistent manner, as done here in (10). With this choice of C, and recalling that F is constant, Equation (8) can be rewritten

$$\begin{aligned}\Omega(t) &= \Omega_0 \exp\Big[i\,\frac{(At+B)^2}{2A}\Big] + \\ &\quad + \exp\Big[i\,\frac{(At+B)^2}{2A}\Big]\,F\int_0^t \exp\Big[-i\,\frac{(A\tau+B)^2}{2A}\Big]\,d\tau\,. \qquad (11)\end{aligned}$$

In order to evaluate the integral involved in (11), consider the following transformation

$$\sigma = \frac{(A\tau + B)^2}{2\,|A|} \tag{12}$$

then

$$(A\tau + B)d\tau = s_3\, d\sigma \quad \text{and} \quad (A\tau + B) = s_2 \sqrt{2\,|A|\,\sigma} \tag{13}$$

where

$$s_3 = \text{sgn}(A) \quad \text{and} \quad s_2 = \text{sgn}(A\tau + B) \tag{14}$$

and sgn(.) is the signum function, defined by

$$\text{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Note that $s_3 = +1$ corresponds to a spin-up maneuver, and $s_3 = -1$ corresponds to a spin-down maneuver. The case $s_3 = 0$ corresponds to constant spin-rate and it will not be considered here. Under the previous definitions, the integral involved in (11) becomes

$$I_\omega = \int_0^t \exp\Big[-i\,\frac{(A\tau + B)^2}{2A}\Big]\, d\tau =$$

$$= \frac{1}{\sqrt{2\,|A|}} \int_{\sigma_0}^{\sigma_1} \text{sgn}\Big(\tau + \frac{B}{A}\Big)\, \frac{\exp(-is_3\sigma)}{\sqrt{\sigma}}\, d\sigma \tag{15}$$

where

$$\sigma_1 = \frac{(At + B)^2}{2\,|A|} \quad \text{and} \quad \sigma_0 = \frac{B^2}{2\,|A|} \tag{16}$$

Integrals of the form

$$C_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\cos(\eta)}{\sqrt{\eta}}\, d\eta\,, \qquad S_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\sin(\eta)}{\sqrt{\eta}}\, d\eta \tag{17}$$

are called Fresnel integrals. They have been extensively studied and their values have been tabulated (Abramowitz and Stegun, 1972). If we now define the complex Fresnel integral function by

$$E(x) = C_2(x) - is_3 S_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\exp(-is_3\sigma)}{\sqrt{\sigma}}\, d\sigma \tag{18}$$

we can readily evaluate I_ω as

$$I_\omega = \sqrt{\frac{\pi}{|A|}}\left\{\mathrm{sgn}\left(t+\frac{B}{A}\right)E\left[\frac{(At+B)^2}{2|A|}\right] - \mathrm{sgn}\left(\frac{B}{A}\right)E\left[\frac{B^2}{2|A|}\right]\right\} =$$

$$= \sqrt{\frac{\pi}{|A|}}\left[s_1 E(\sigma_1) - s_0 E(\sigma_0)\right] \tag{19}$$

where

$$s_1 = \mathrm{sgn}(t + B/A) \quad \text{and} \quad s_0 = \mathrm{sgn}(B/A)\ . \tag{20}$$

3. Evaluation of Fresnel Integrals

Fresnel integrals are notorious for their difficulty in approximating over a large range of their argument. However, an excellent approximation based on the τ-method of Lanczos (1956), and given by Boersma (1960) is satisfactory. According to this method, two approximations are used, one valid for small values of the argument, and the other valid for large values of the argument.

If we define the function

$$f(x) = \int_0^x \frac{e^{-it}}{\sqrt{2\pi t}}\, dt = C_2(x) - iS_2(x) \tag{21}$$

then

$$f(x) = e^{-ix}\sum_{n=0}^{11}(a_n + ib_n)\left(\frac{x}{4}\right)^{n+1/2} \qquad \text{for } 0 \le x \le 4 \tag{22}$$

$$f(x) = \frac{1-i}{2} + e^{-ix}\sum_{n=0}^{11}(c_n + id_n)\left(\frac{4}{x}\right)^{n+1/2} \qquad \text{for } x > 4\ . \tag{23}$$

The numerical values of the coefficients a_n, b_n, c_n and d_n are given by Boersma (1960), and for the reader's convenience, are reproduced here in Table I. The maximum error is 1.6×10^{-9} for the first approximation and 0.5×10^{-9} for the second approximation. The complex functions $E(x)$ and $f(x)$ are related by

$$E(x) = \begin{cases} f(x) & \text{if } s_3 = +1 \\ f^*(x) & \text{if } s_3 = -1 \end{cases} \tag{24}$$

where the asterisk indicates the complex conjugate. The advantage of this approximation is that it provides both Fresnel integrals in complex form, as required by Equation (19). It is then an easy matter to separate the real and imaginary parts, if desired.

Numerical values of coefficients for the Fresnel Integral computation (Boersma, 1960).

$a_0 = +1.595769140$	$b_0 = -0.000000033$	$c_0 = 0$	$d_0 = +0.199471140$
$a_1 = -0.000001702$	$b_1 = +4.255387524$	$c_1 = -0.024933975$	$d_1 = +0.000000023$
$a_2 = -6.808568854$	$b_2 = -0.000092810$	$c_2 = +0.000003936$	$d_2 = -0.009351341$
$a_3 = -0.000576361$	$b_3 = -7.780020400$	$c_3 = +0.005770956$	$d_3 = +0.000023006$
$a_4 = +6.920691902$	$b_4 = -0.009520895$	$c_4 = +0.000689892$	$d_4 = +0.004851466$
$a_5 = -0.016898657$	$b_5 = +5.075161298$	$c_5 = -0.009497136$	$d_5 = +0.001903218$
$a_6 = -3.050485660$	$b_6 = -0.138341947$	$c_6 = +0.011948809$	$d_6 = -0.017122914$
$a_7 = -0.075752419$	$b_7 = -1.363729124$	$c_7 = -0.006748873$	$d_7 = +0.029064067$
$a_8 = +0.850663781$	$b_8 = -0.403349276$	$c_8 = +0.000246420$	$d_8 = -0.027928955$
$a_9 = -0.025639041$	$b_9 = +0.702222016$	$c_9 = +0.002102967$	$d_9 = +0.016497308$
$a_{10} = -0.150230960$	$b_{10} = -0.216195929$	$c_{10} = -0.001217930$	$d_{10} = -0.005598515$
$a_{11} = +0.034404779$	$b_{11} = +0.019547031$	$c_{11} = +0.000233939$	$d_{11} = +0.000838386$

Other approximations for Fresnel integrals and for integrals of Fresnel integrals, using asymptotic and/or series expansions or rational functions can be found in Abramowitz and Stegun (1972).

4. Solution for the Euler Angles

If we use a 3-1-2 Euler angle sequence to describe the orientation of the body-fixed reference frame, with respect to an inertially fixed reference frame, the following kinematic equations hold:

$$\dot{\varphi}_x = \omega_x \cos \varphi_y + \omega_z \sin \varphi_y \tag{25a}$$

$$\dot{\varphi}_y = \omega_y - (\omega_z \cos \varphi_y - \omega_x \sin \varphi_y) \tan \varphi_x \tag{25b}$$

$$\dot{\varphi}_z = (\omega_z \cos \varphi_y - \omega_x \sin \varphi_y) \sec \varphi_x \ . \tag{25c}$$

A small angle approximation for φ_x and φ_y reduces this system of equations to

$$\dot{\varphi}_x = \omega_x + \varphi_y \omega_z \tag{26a}$$

$$\dot{\varphi}_y = \omega_y - \varphi_x \omega_z \tag{26b}$$

$$\dot{\varphi}_z = \omega_z - \varphi_y \omega_x \ . \tag{26c}$$

If we also assume that the product $\varphi_y \omega_x$ is small compared to ω_z, we can immediately solve for φ_z to get

$$\varphi_z(t) = \int_0^t \omega_z(\tau) d\tau \ . \tag{27}$$

Note that the differential Equations (26a) and (26b) are independent of φ_z, hence, the accuracy of the solution of φ_x, and φ_y will not be affected by dropping the term $-\varphi_y \omega_x$ in (26c). If, however, one wishes a more precise solution for φ_z, it may be possible to reinstate the ignored term as a perturbation.

Using the expression for $\omega_z(t)$ from (2), one can readily perform the integration for $\varphi_z(t)$ to obtain

$$\varphi_z(t) = \tfrac{1}{2} \, \frac{M_z}{I_z} \, t^2 + \omega_{z0} t + \varphi_{z0} \; , \qquad \varphi_z(0) = \varphi_{z0} \; . \tag{28}$$

Introducing the complex variables

$$\varphi = \varphi_x + i\varphi_y \quad \text{and} \quad \omega = \omega_x + i\omega_y \tag{29}$$

we can combine (26a) and (26b) into a simple complex scalar equation

$$\dot{\varphi} + i\omega_z \varphi = \omega \; . \tag{30}$$

The solution of (30) has the same form as for the case of the angular velocities, where now the forcing term is $\omega(t)$. The solution is given by

$$\begin{aligned} \varphi(t) \; &= \varphi_0 \exp\Big[-i(\tfrac{1}{2}\,\alpha t^2 + \beta t + \gamma)\Big] + \\ &+ \exp\Big[-i(\tfrac{1}{2}\,\alpha t^2 + \beta t + \gamma)\Big] \int_0^t \omega(\tau) \exp\Big[i(\frac{1}{2}\,\alpha t^2 + \beta t + \gamma)\Big] \, d\tau \; . \end{aligned} \tag{31}$$

Again, the choice of γ affects the homogeneous part in (31) so we can choose the constant γ and the initial conditions such that

$$\gamma = \frac{\beta^2}{2\alpha} \; , \qquad \varphi(0) = \varphi_0 \exp\Big(-i\,\frac{\beta^2}{2\alpha}\Big) \; . \tag{32}$$

Then we can rewrite (31) in the form

$$\begin{aligned} \varphi(t) \; &= \varphi_0 \exp\Big[-i\,\frac{(\alpha t + \beta)^2}{2\alpha}\Big] + \\ &+ \exp\Big[-i\,\frac{(\alpha t + \beta)^2}{2\alpha}\Big] \int_0^t \omega(\tau) \exp\Big[i\,\frac{(\alpha\tau + \beta)^2}{2\alpha}\Big] \, d\tau \; . \end{aligned} \tag{33}$$

Recall that $\omega(t) = \omega_x(t) + i\omega_y(t)$ and $\Omega(t) = \Omega_x(t) + i\Omega_y(t)$. Thus, we can express the angular velocity $\omega(t)$ in terms of $\Omega(t)$ as follows

$$\omega(t) = \frac{\Omega_x(t)}{\sqrt{k_y}} + i\,\frac{\Omega_y(t)}{\sqrt{k_x}} = \frac{\Omega(t) + \Omega^*(t)}{2\sqrt{k_y}} + \frac{\Omega(t) - \Omega^*(t)}{2\sqrt{k_x}} \; . \tag{34}$$

Using this relation the integral in (33) can be rewritten as

$$I_\varphi = \int_0^t \omega(\tau)\exp\Big[i\,\frac{(\alpha\tau+\beta)^2}{2\alpha}\Big]\,d\tau \;=$$

$$= \frac{\sqrt{k_x}+\sqrt{k_y}}{2\sqrt{k_x k_y}}\,I_{\varphi 1} + \frac{\sqrt{k_x}-\sqrt{k_y}}{2\sqrt{k_x k_y}}\,I_{\varphi 2} \tag{35}$$

where

$$\begin{aligned} I_{\varphi_1} &= \int_0^t \Omega(\tau)\exp\Big[i\,\frac{(\alpha\tau+\beta)^2}{2\alpha}\Big]\,d\tau \quad \text{and} \\ I_{\varphi_2} &= \int_0^t \Omega^*(\tau)\exp\Big[i\,\frac{(\alpha\tau+\beta)^2}{2\alpha}\Big]\,d\tau\,. \end{aligned} \tag{36}$$

Let $\lambda = 1/k$, then from (5) $\alpha = \lambda A$ and $\beta = \lambda B$. Using the already known solution for $\Omega(t)$ and the independent variable σ introduced in (12), we can rewrite the integral I_{φ_1} as

$$\begin{aligned} I_{\varphi_1} \;=\; &\Big[\Omega_0 - F\sqrt{\frac{\pi}{|A|}}\,s_0 E(\sigma_0)\Big]\int_0^t \exp\Big[i(\lambda+1)\,\frac{(A\tau+B)^2}{2A}\Big]\,d\tau \,+ \\ &+ F\sqrt{\frac{\pi}{|A|}}\int_0^t \exp\Big[i(\lambda+1)\,\frac{(A\tau+B)^2}{2A}\Big]\,\mathrm{sgn}\Big(\tau+\frac{B}{A}\Big)\,E(\sigma)d\tau\,. \end{aligned} \tag{37}$$

It is not difficult to show that the first integral in the above equation is easily evaluated as in (15), with an obvious change of independent variable, as follows

$$\int_0^t \exp\Big[i(\lambda+1)\,\frac{(A\tau+B)^2}{2A}\Big]\,d\tau \;=$$

$$= \sqrt{\frac{\pi}{|A|\,(\lambda+1)}}\,\Big\{s_1 E^*[(\lambda+1)\sigma_1] - s_0 E^*[(\lambda+1)\sigma_0]\Big\}\,. \tag{38}$$

The evaluation of the second integral in (37) is more involved. Use the transformation (12) to rewrite the integral in the form

$$\int_0^t \exp\Big[i(\lambda+1)\,\frac{(A\tau+B)^2}{2A}\Big]\,\mathrm{sgn}\Big(\tau+\frac{B}{A}\Big)\,E(\sigma)d\tau \;=$$

$$= \frac{1}{\sqrt{2\,|A|}}\,[W_1(\lambda,\sigma_1) - W_1(\lambda,\sigma_0)] \tag{39}$$

where

$$W_1(\lambda, x) = \int_0^x \frac{\exp[is_3(\lambda+1)\eta]\, E(\eta)}{\sqrt{\eta}}\, d\eta \tag{40}$$

is a function to be evaluated later. Then, I_{φ_1} takes the final form

$$\begin{aligned} I_{\varphi_1} &= \left[\Omega_0 - F\sqrt{\frac{\pi}{|A|}}\, s_0 E(\sigma_0)\right] \cdot \\ &\cdot \sqrt{\frac{\pi}{|A|\,(\lambda+1)}} \left\{ s_1 E^*[(\lambda+1)\sigma_1] - s_0 E^*[(\lambda+1)\sigma_0] \right\} + \\ &+ \frac{F}{|A|}\sqrt{\frac{\pi}{2}}\, [W_1(\lambda,\sigma_1) - W_1(\lambda,\sigma_0)]\,. \end{aligned} \tag{41}$$

In a similar way, one can show that the integral I_{φ_2} is given by

$$\begin{aligned} I_{\varphi_2} &= \left[\Omega_0^* - F^*\sqrt{\frac{\pi}{|A|}}\, s_0 E^*(\sigma_0)\right] \cdot \\ &\cdot \sqrt{\frac{\pi}{|A|\,(\lambda-1)}} \left\{ s_1 E^*[(\lambda-1)\sigma_1] - s_0 E^*[(\lambda-1)\sigma_0] \right\} + \\ &+ \frac{F^*}{|A|}\sqrt{\frac{\pi}{2}}\, [W_2(\lambda,\sigma_1) - W_2(\lambda,\sigma_0)] \end{aligned} \tag{42}$$

where

$$W_2(\lambda, x) = \int_0^x \frac{\exp[is_3(\lambda-1)\eta]\, E^*(\eta)}{\sqrt{\eta}}\, d\eta\,. \tag{43}$$

The evaluation of the integrals $W_1(\lambda, x)$ and $W_2(\lambda, x)$ will be discussed next. Without loss of generality, and for the sake of brevity, we will consider only the case when $s_3 = +1$, since for the case $s_3 = -1$ one can simply substitute for $W_1(\lambda, x)$ and $W_2(\lambda, x)$ in (41) and (42) their complex conjugates. First recall that the expression for $E(x)$ involves, according to (24), two different approximations, one valid for $0 \le x \le 4$ and the other valid for $x > 4$. Thus, we can rewrite $W_j(\lambda, x)$ for $j = 1, 2$ as

$$W_j(\lambda, x) = \begin{cases} W_j'(\lambda, x) & 0 \le x \le 4 \\ W_j'(\lambda, 4) + W_j''(\lambda, x) & x > 4 \end{cases} \tag{44}$$

where

$$W_1''(\lambda, x) = \int_4^x \frac{\exp[is_3(\lambda+1)\eta]\, E(\eta)}{\sqrt{\eta}}\, d\eta \tag{45}$$

$$W_2''(\lambda, x) = \int_4^x \frac{\exp[is_3(\lambda - 1)\eta]\, E^*(\eta)}{\sqrt{\eta}}\, d\eta \,. \tag{46}$$

After substitution of (24) into (40) and (43), and integrating term by term, we get for both $W_1(\lambda, x)$ and $W_2(\lambda, x)$ that

$$W_j'(\lambda, x) = \tfrac{1}{2} \sum_{n=0}^{11} \frac{[a_n - i(-1)^j b_n]}{4^n}\, I_n'(x) \qquad \text{for } 0 \le x \le 4 \tag{47}$$

and

$$\begin{aligned} W_j''(\lambda, x) \; &= \frac{1 + i(-1)^j}{2} \left[\frac{2\pi}{\lambda - (-1)^j}\right]^{1/2} \cdot \\ &\cdot \Big\{E^*[(\lambda - (-1)^j)x] - E^*[(\lambda - (-1)^j)4]\Big\} + \\ &+ 2 \sum_{n=0}^{11} [c_n - i(-1)^j d_n]\, 4^n I_n''(x) \qquad \text{for } x > 4 \end{aligned} \tag{48}$$

where $j = 1, 2$, and where

$$I_n'(x) = \int_0^x \exp(i\lambda\eta)\eta^n d\eta \qquad n = 0, 1, 2, ..., 11 \tag{49}$$

$$I_n''(x) = \int_4^x \frac{\exp(i\lambda\eta)}{\eta^{n+1}}\, d\eta \qquad n = 0, 1, 2, ..., 11 \,. \tag{50}$$

These sequences of integrals can be evaluated recursively, using the relationships

$$I_n'(x) = -\frac{ix^n}{\lambda} \exp(i\lambda x) + i\,\frac{n}{\lambda}\, I_{n-1}'(x) \qquad n = 1, 2, ..., 11 \tag{51}$$

$$I_n''(x) = -\left.\frac{\exp(i\lambda x)}{nx^n}\right|_4^x + i\,\frac{\lambda}{n}\, I_{n-1}''(x) \qquad n = 1, 2, ..., 11 \,. \tag{52}$$

The first integrals of the above sequences are

$$I_0'(x) = \int_0^x \exp(i\lambda\eta) d\eta = -\frac{i}{\lambda}\, [\exp(i\lambda x) - 1] \tag{53a}$$

$$I_0''(x) = \int_4^x \frac{\exp(i\lambda\eta)}{\eta}\, d\eta = [Ci(\lambda x) - Ci(\lambda 4)] + i[Si(\lambda x) - Si(\lambda 4)] \,. \tag{53b}$$

Where $Si(x)$ and $Ci(x)$ are the well-known sine and cosine integrals defined by

$$Si(x) = \int_0^x \frac{\sin(t)}{t}\, dt \qquad Ci(x) = \bar{\gamma} + \ln(x) + \int_0^x \frac{\cos(t) - 1}{t}\, dt \tag{54}$$

and $\bar{\gamma}$ = Euler's constant (= 0.57721 ...). The evaluation of the sine and cosine integrals in (54) can be easily performed uniformly, by rational approximations (Abramowitz and Stegun, 1972). The maximum error $\varepsilon(x)$ for these approximations is given by $|\varepsilon(x)| < 5 \times 10^{-7}$.

We should mention in passing, that one should be careful with the definition of the complex function $E(x)$ since its argument should be always positive. It is a well-known fact however, that the following relations hold between the principal moments of inertia of an arbitrary rigid body (excluding the planar case)

$$I_x + I_y > I_z \ , \qquad I_x + I_z > I_y \ , \qquad I_y + I_z > I_x \ . \tag{55}$$

From the first and second equations, along with the definitions for k_x and k_y, we get that

$$-1 < k_x < 1 \quad \text{and} \quad -1 < k_y < 1 \ . \tag{56}$$

As mentioned at the beginning, for a rigid body spinning about one of the two stable principal axes, $k = \sqrt{k_x k_y} > 0$. Hence, the parameter k satisfies the inequality $0 < k < 1$, and since $\lambda = 1/k$, we have

$$1 < \lambda < \infty \ . \tag{57}$$

As a consequence, both $\lambda + 1 > 0$ and $\lambda - 1 > 0$ and the arguments of $E(x)$ are well-defined in Equations (41), (42), and (48).

5. Discussion of the Solution

Taking advantage of the special symmetric structure of the problem, we have used a complex analytic approach to derive analytic solutions for the problem of the attitude motion of a self-excited rigid body. The use of complex variables allowed for the formulation of the solution in a very compact form, which is appealing especially for machine computations. This is very important, since it appears that for future applications, realistic compact analytical expressions modeling the attitude evolution, will become vital in on board attitude control software. Although complications arise from the sign functions in the solution, these functions were included in order to give the solution in complete form, i.e., a solution for both Eulerian rates and angles, valid for both spin-up and spin-down maneuvers, and valid also in the critical neighborhood of zero-spin rate. This is the first time that such a complete solution for this problem is reported in the literature, as far as the authors know.

In complex form the analytic solutions are considerably shorter than previous related results, but they are still quite lengthy for hand computations. Nevertheless, definite conclusions can be drawn about the asymptotic behavior of the solution. By keeping in the solution for example, only those terms that create secular effects, one can capture the essential behavior of the motion, thus gaining invaluable insight into the nature of the problem. In fact, it has been shown in the past that such simplified procedures can be extremely successful in the study of attitude motion and control of modern spacecraft (Longuski, 1989).

6. Numerical Examples

The application of the theory is illustrated by means of practical examples, such as spin-up or spin-down maneuvers of the Galileo spacecraft. Two cases are examined. The first case is a spin-up maneuver from $\omega_z(0) = 3.15$ rpm to $\omega_z(t_f) = 10$ rpm. The second is a spin-down maneuver from $\omega_z(0) = 3.15$ rpm to $\omega_z(t_f) = -3.15$ rpm. For both cases, the following initial conditions are assumed

$$\omega_x(0) = \omega_y(0) = 0 \tag{58}$$

$$\varphi_x(0) = \varphi_y(0) = \varphi_z(0) = 0 \; . \tag{59}$$

Representative values for the mass properties of the Galileo spacecraft (in the all-spin mode) are

$$I_x = 2985 \text{ kg-m}^2 \; , \quad I_y = 2729 \text{ kg-m}^2 \; , \quad I_z = 4183 \text{ kg-m}^2 \; . \tag{60}$$

In general, transverse torques arise during spin-up or spin-down maneuvers, due to error sources such as thruster misalignment or thruster mismatch. The Galileo spacecraft is a rather extreme example of a spacecraft that uses a single thruster for the spin-up and spin-down maneuvers. Moreover, the center of mass does not lie in the plane of the applied thrust. As a result, there are significant torques about all three body-fixed axes. The torques generated about the body axes are given by

$$M_x = -1.253 \text{ Nm} \; , \quad M_y = -1.494 \text{ Nm} \; , \quad M_z = \pm 13.5 \text{ Nm} \tag{61}$$

where the plus sign in M_z corresponds to spin-up, and the minus sign to spin-down.

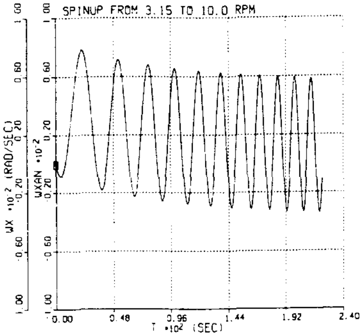

Fig. 1a. Exact and analytic solutions for $\omega_x(t)$.

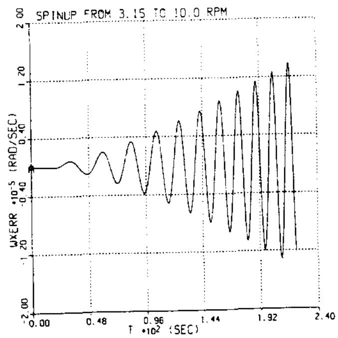

Fig. 1b. Difference between exact and analytic solutions for $\omega_x(t)$.

6.1. Case 1: spin-up from 3.15 to 10 rpm

The analytic solutions for the attitude motion are compared to the 'exact' solutions which are found by numerical integration of Equations (1) and (25). Figure 1 compares the exact solution for $\omega_x(t)$ with the analytic solution. In Figure 1a both exact and analytic solutions are displayed, but they are indistinguishable from one another. Their difference, presented in Figure 1b, has oscillatory behavior, with a linearly increasing envelope. The same plot indicates that the analytic solution for $\omega_x(t)$ deviates from the exact solution by only about 0.1%. Similar results were found also for the solution for $\omega_y(t)$. Figure 2 demonstrates that the linearity assumption (2) for $\omega_z(t)$ is reasonable, since the error indicates a discrepancy of only about 0.01% from the exact solution.

In Figure 3, the exact solution for $\varphi_x(t)$ is compared to the analytic solution. The discrepancy between the exact and analytic solutions is not apparent in Figure 3a, but Figure 3b, which displays their difference reveals an error reaching 0.5%. In Figure 4 the analytic solution for $\varphi_z(t)$ is shown to be within about 0.01% of the exact solution.

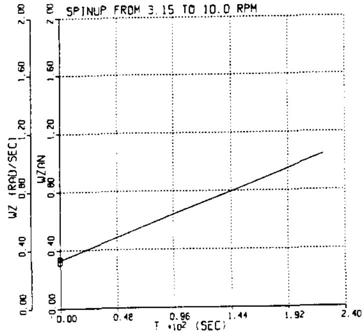

Fig. 2a. Exact and analytic solutions for $\omega_z(t)$.

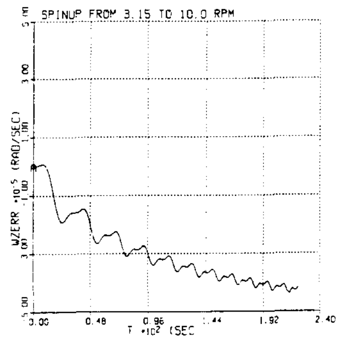

Fig. 2b. Difference between exact and analytic solutions for $\omega_z(t)$.

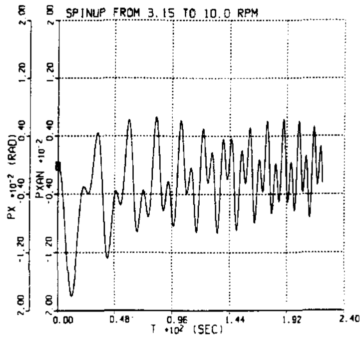

Fig. 3a. Exact and analytic solutions for $\varphi_x(t)$.

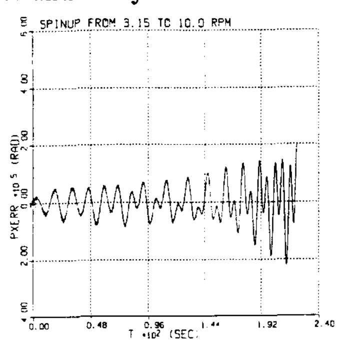

Fig. 3b. Difference between exact and analytic solutions for $\varphi_x(t)$.

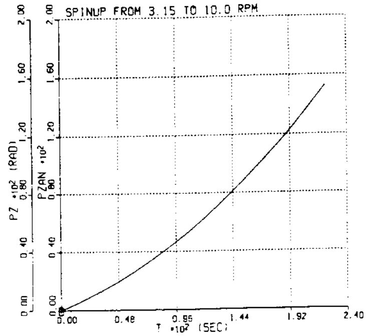

Fig. 4a. Exact and analytic solutions for $\varphi_z(t)$.

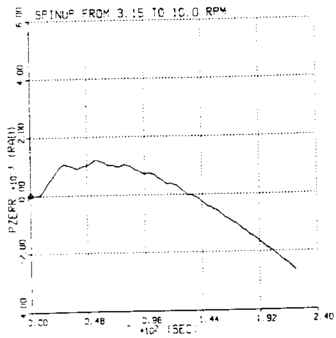

Fig. 4b. Difference between exact and analytic solutions for $\varphi_z(t)$.

6.2. Case 2: spin-down from 3.15 to −3.15 rpm

The second example examines the very important case of despinning, possibly through the region of zero spin rate. Low spin-rate, in conjunction with the nonlinear rate coupling effect inherent in the Euler equations, can have catastrophic consequences, as was vividly demonstrated by the GEOS-1 satellite experiment (Van der Ha, 1984).

Figures 5a and 5b reveal that the assumption $\omega_z(t) = M_z/I_z + \omega_{z0}$ still remains valid, where the error with respect to the exact solution has increased to a maximum of only about 1%. The solutions for $\omega_x(t)$ (shown in Figure 6) and $\omega_y(t)$ are still very accurate, up to the point when $\omega_z(t)$ crosses the critical zero spin rate value. At very low spin rates, however, the transverse torques create large angular displacements, and the small angle approximation for $\varphi_x(t)$ and $\varphi_y(t)$ is no longer valid. The kinematics equations have entered the region of nonlinearity, which is clearly illustrated by the phase shift in Figure 7. The solutions for $\varphi_x(t)$ and $\varphi_y(t)$ are not affected by $\varphi_z(t)$, as mentioned earlier. On the other hand, however, errors introduced in the solutions of $\varphi_x(t)$ and $\varphi_y(t)$ do affect the solution for $\varphi_z(t)$. The perturbative effects of the nonlinearities are clear in Figure 8, where the exact solution for $\varphi_z(t)$ departs from the parabolic solution (28) after crossing zero spin rate. From this point of view, a nonlinear method, such as Poincaré's

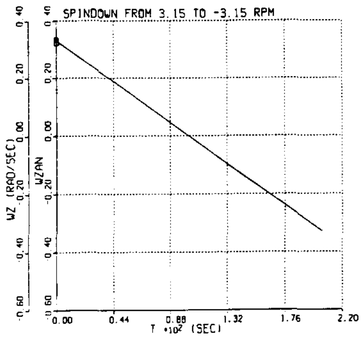

Fig. 5a. Exact and analytic solutions for $\omega_z(t)$.

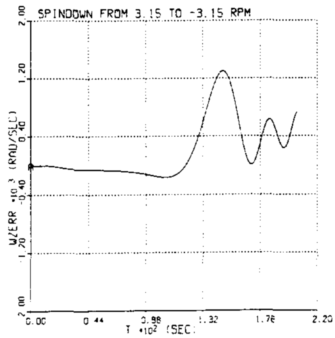

Fig. 5b. Difference between exact and analytic solutions for $\omega_z(t)$.

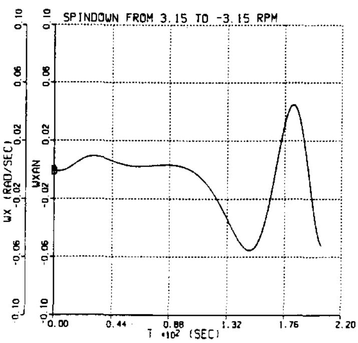

Fig. 6a. Exact and analytic solutions for $\omega_x(t)$.

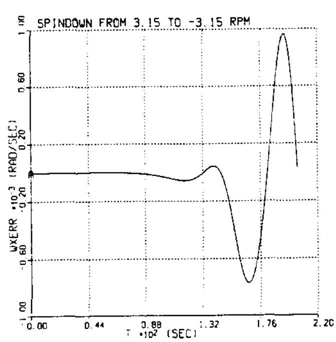

Fig. 6b. Difference between exact and analytic solutions for $\omega_x(t)$.

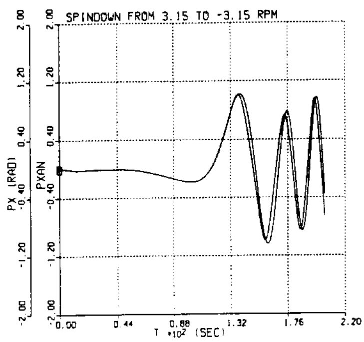

Fig. 7. Exact and analytic solutions for $\varphi_x(t)$.

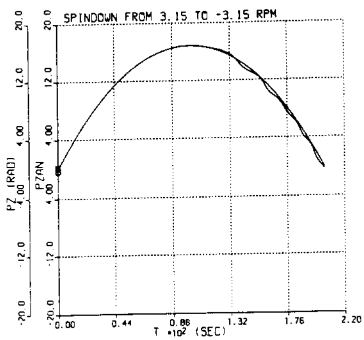

Fig. 8. Exact and analytic solutions for $\varphi_z(t)$.

or Lindstedt's method of small parameters (Blaquière, 1966), could prove to be useful, in a second order approximation of the solution. Regardless of this fact, the solutions are still qualitatively correct, and the analytic solution predicts the time history of the attitude orientation very closely. The degradation of the accuracy of the solution at the low spin rate region should be expected. Further simulations, however, (not presented herein) have shown that low spin-rate by itself is not a matter of concern. Rather the relative magnitude of the transverse to the axial torques, i.e. $(M_x^2 + M_y^2)^{1/2}/M_z$, acting during the time that the body is in the neighborhood of zero spin rate, has proven to be the major factor for the inaccuracy of the analytic solution.

7. Conclusions

Analytic solutions have been derived for the attitude motion of a spinning, self-excited near-symmetric rigid body. The complex representation enables the solution to take a compact form, especially suitable for implementation in maneuver or attitude control software. The solution assumes exact axial symmetry in order to write the solution for the angular velocity about the spinning axis in a linear form, but keeps the distinction of the moments of inertia in the other two equations for the angular velocities. A small angle approximation allows the Euler angles to be given as the solution of a linear, time-varying system with the expression for the angular velocities acting as a forcing function. Numerical simulations reveal that the solutions are very accurate in describing the rotational motion of a typical spacecraft. Current and previous research indicate that such analytic solutions are extremely helpful in capturing the fundamental behavior of the motion and provide insight into the mechanics of the motion, which cannot be derived from numerical solutions.

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