ANNALS OF PHYSICS: 7, 456-465 (1959)

On Weak Time-Symmetric Gravitational Waves

HUZIHIRO ARAKI*

Palmer Physical Laboratory, Princeton, New Jersey

The existence and the nature of the solutions of the initial value condition for the time-symmetric pure gravitational waves are investigated. It is shown that, corresponding to any nonsingular three-dimensional metric which is sufficiently close to the flat metric, there exists one and only one nonsingular solution of the initial value condition which is conformal to that metric. It is further shown that the energy of such gravitational waves is positive definite provided that the original metric is chosen sufficiently close to the flat metric.

1. INTRODUCTION

At the present stage of the general theory of relativity, it is extremely desirable to acquire more knowledge about gravitational waves. For this reason one asks (a) whether there exist everywhere nonsingular pure gravitational waves which are asymptotically flat and, if they exist, (b) whether they have positive total energy.

Foures-Bruhat (*1*) has studied the initial value problem in general relativity and has shown that the solution of the Einstein equation for pure gravitational fields exists for a finite time if a certain condition is satisfied by the initial value on the initial surface. This initial condition takes particularly simple form if the metric is assumed to be time symmetric around the time of the initial surface. In addition to this simplicity, we can tell the total energy of a time symmetric gravitational wave just by comparing its asymptotic behavior on the initial surface for large space-like distances with that of the Schwarzschild metric. Therefore it is interesting to investigate whether there exist any solutions for the initial condition for time symmetric gravitational waves and, if they exist, whether they have positive total energy. Brill (*2*) has already shown that any axially symmetric solution of the initial condition for time symmetric gravitational waves has necessarily a positive total energy.

The aim of the present paper is to give an affirmative answer to the above two questions for the case where the metric on the initial surface is sufficiently close to the flat metric but otherwise arbitrary.

We shall show in Section 2 that, corresponding to any three-dimensional metric

* On leave of absence from Department of Nuclear Engineering, Kyoto University, Kyoto, Japan.

which is everywhere regular, asymptotically flat, and sufficiently close to the flat metric, there exists one and only one solution of the initial conditions for time symmetric gravitational waves which is conformal to the original metric. We shall further show that this solution has a positive total energy if it is sufficiently close to the flat metric. The Appendix proves the theorem used in Section 2.

2. THE INITIAL CONDITION AND THE EXISTENCE OF ITS SOLUTION

The initial value condition for the time symmetric pure gravitational waves (*3*) is given by

$$R = 0 \tag{2.1}$$

where R is the curvature invariant of the three dimensional metric on the initial surface. It is known (*3*) that, given any three dimensional metric g_{lm}, the new metric $\hat{g}_{lm} = \phi^4 g_{lm}$ is a solution of (2.1) if and only if ϕ satisfies the following equation

$$\Delta_g \phi + \tfrac{1}{8} R_g \phi = 0. \tag{2.2}$$

Here Δ_g and R_g are the Laplacian and the curvature invariant in the original metric g_{lm}.

We are interested in a solution of (2.1) which is everywhere nonsingular (twice differentiable), asymptotically flat and everywhere positive definite. Therefore we choose the original metric g_{lm} to be everywhere regular (sufficiently many times differentiable), positive definite, and approaching the flat metric sufficiently rapidly at infinity. Then we require the following condition for ϕ:

$$\begin{aligned} &\text{(a) } \phi \text{ has a continuous second derivative,} \\ &\text{(b) } \phi > 0, \\ &\text{(c) } \phi \to 1 \text{ at infinity.} \end{aligned} \tag{2.3}$$

To make the discussion easier, we assume g_{lm} to be of the form

$$g_{lm} = \delta_{lm} + \epsilon\, \bar{g}_{lm}, \tag{2.4}$$

where $\bar{g}_{lm}$ is an infinitely differentiable function with a compact support and ϵ is a constant which we choose sufficiently small in the later discussion. The restrictions which we imposed here on the behavior at infinity and the differentiability of g_{lm} are by no means minimal. All the results in Section 2 and in Section 3 hold true for any functions g_{lm} for which Theorem 1 below holds and for which $\int | R_g | \, dx$ is finite. On the other hand, our restriction leaves us a comfortably large class of functions for our purpose.

The uniqueness of the solution of Eq. (2.2) is readily seen. Suppose there are two functions ϕ_1 and ϕ_2 of (2.2) satisfying (2.3). Then ϕ_1/ϕ_2 should satisfy (2.2) where we replace g with $\phi_2^4\, g$. Since $R_{(\phi_2^4 g)} = 0$, ϕ_1/ϕ_2 is then a solution of the Laplace equation in the metric $\phi_2^4\, g$ which again satisfies (2.3). It is well known (*4*) that such a solution is identically equal to unity, which proves the uniqueness. As a special case, note that every nonsingular spherically symmetric three dimensional metric is *conformal* to the flat metric. Therefore the uniqueness of the solution of (2.2) implies the well-known result that the only spherically symmetric nonsingular solution of (2.1) *is* the flat metric.

To prove the existence of a solution of (2.2), we make use of the following theorem which we shall prove in the Appendix.

THEOREM 1. Given a three dimensional metric of the form (2.4). There exists the fundamental solution $G_\epsilon(x,x^1)$ (Green's function) of the Laplacian in this metric, which satisfies the following

(1) $G_\epsilon(x,x^1)$ has continuous derivatives of the second order everywhere except for $x = x^1$.

(2) $\Delta_g\, G_\epsilon(x,x^1) = 0$, if $x \neq x^1$.

(3) For $n = 1$ or 2, if $f(x)$ has continuous $(n - 1)$ the derivatives and if $\int |f(x)| / |x|\, dx$ converges at infinity, $\int G_\epsilon\, (x,x^1)\, f(x^1)\, dx^1$ has a continuous nth derivative.

(4) If $f(x)$ has continuous first derivative, and if $\int |f(x)| / |x|\, dx$ converges at infinity, then

$$\Delta_g \int G_\epsilon(x,x^1) f(x^1) g^{1/2}(x^1)\, dx^1 = -f(x), \tag{2.6}$$

where $g(x)$ is the determinant of the matrix g_{lm}.

(5) There exist a positive number N which is independent of x and x^1 such that

$$|G_\epsilon(x,x^1) - G_0(x,x^1)| < N\, |x - x^1|^{-1} \qquad \text{if } x \neq x^1 \tag{2.7}$$

where $|x - x^1| = [\sum (x_i - x_i^1)^2]^{1/2}$ and $G_0\,(x,x^1) = [4\pi\, |x - x^1|\,]^{-1}$. In particular for $|x - x^1| \to \infty$, $|x - x^1|\, G(x - x^1)$ approaches a constant.

(6) For sufficiently small ϵ, N can be chosen independent of ϵ and arbitrarily small.

(7) $G_\epsilon(x,x^1) = G_\epsilon(x^1,x)$. (2.8)

(8) $G_\epsilon(x,x^1) > 0$. (2.9)

The last two equations are irrelevant for the present purpose.

Using $G_\epsilon(x,x^1)$, we can now convert (2.2) into an integral equation

$$\phi(x) = 1 + (1/8) \int G_\epsilon(x,x^1) R_g(x^1) g^{1/2}(x^1) \phi(x^1)\, dx^1. \tag{2.10}$$

To remove the singularity of the kernel, we iterate once

$$\phi(x) = \phi_0(x) + \int K(x,x^1)R_g(x^1)g^{1/2}(x^1)\phi(x^1)\, dx^1, \tag{2.11}$$

where

$$\phi_0(x) = 1 + (1/8) \int G_\epsilon(x,x^1)R_g(x^1)g^{1/2}(x^1)\, dx^1, \tag{2.12}$$

$$K(x,x^1) = (1/64) \int G_\epsilon(x,x'')R_g(x'')G_\epsilon(x'',x^1)g^{1/2}(x'')\, dx''. \tag{2.13}$$

We now examine the boundedness of the kernel $K(x,x^1)$. Since $\bar{g}_{lm}$ is assumed to be an infinitely differentiable function with a compact support, $R_g(x)$ has the same property. By (1) and (3) of Theorem 1, $K(x,x^1)$ is a continuous function of x and x^1 except possibly at $x = x^1$. The boundedness near infinity and near $x = x^1$ can be studied by replacing $R_g(x'')$ by $| R_g(x'') |$ and $G_\epsilon(x,x^1)$ by $| x - x^1 |^{-1}$ because of (5) of Theorem 1. Clearly $K(x,x^1)$ goes to zero if either x and x^1 goes to infinity. Near $x = x^1$ we may replace $| R_g(x'')g^{1/2}(x'') |$ by its upper bound and the integration domain by a sphere of fixed radius around point x^1. The integral can be evaluated in an elementary way and is bounded. Thus we have established the boundedness of $| K(x,x^1) |$.

$$| K(x,x^1) | < K_0 \qquad \text{for all } x \text{ and } x^1. \tag{2.14}$$

Because of (6) of Theorem 1, K_0 can be chosen independent of ϵ for sufficiently small ϵ.

We now proceed to the proof of the existence of the solution of (2.10) and (2.2). For this purpose, we examine the following series

$$\phi(x) = \sum_{n=0}^{\infty} \phi_n(x), \tag{2.15}$$

$$\phi_n(x) = \int K(x, x^1)R_g(x^1)g^{1/2}(x^1)\, dx^1 \int \cdots \int K(x^{(n-1)},x^{(n)})g^{1/2}(x^{(n)})R_g(x^{(n)})\phi_0(x^{(n)})\, dx^{(n)}. \tag{2.16}$$

Replacing $K(x,x^1)$ by K_0 of (2.14) and $\phi_0(x^{(n)})$ by an upper bound ϕ_0, we get the following estimate for ϕ_n,

$$| \phi_n(x) | < \epsilon^n A^n C, \tag{2.17}$$

where A and C are constants independent of x and ϵ for sufficiently small ϵ such that $K_0 \int | R_g(x) | g^{1/2}(x)\, dx < \epsilon A$ and $\phi_0 < C$. This proves the uniform and absolute convergence of the series (2.15). Since $\int G_\epsilon(xx^1) \mid R_g(x^1) \mid g^{1/2}(x^1)\, dx^1$

converges, the uniform convergence of (2.15) guarantees that we can integrate term by term when we substitute (2.15) into (2.10). Then we see that (2.15) is a solution of (2.10). (Notice that two terms of ϕ which correspond to two terms of ϕ_0 separately approach to zero.)

The uniformity of the convergence also implies the continuity of $\phi(x)$, which, in turn, guarantees twice continuous differentiability of $\phi(x)$ because of (2.10) and (3) of Theorem 1. Then by virtue of (4) of Theorem 1, we conclude that $\phi(x)$ satisfies the differential equation (2.2).

The condition (a) of (2.3) has already been proved. (c) is a consequence of (2.10) and (2.7). (b) is clear from (2.15) and (2.17). Summing up we have the following theorem.

THEOREM 2. Given a metric g_{lm} of the form (2.4). For sufficiently small ϵ, there exist a unique solution of (2.2) satisfying (2.3).

3. POSITIVE DEFINITENESS OF TOTAL ENERGY

We are now going to examine the total energy of the gravitational waves by comparing the asymptotic behavior of the three dimensional metric $\hat{g}_{lm}$ of the foregoing section with the Schwarzschild metric.

Suppose the original metric $g_{lm}(x)$ approaches to the flat metric not slower than $|x|^{-2}$. Then $\int R(x^1)g^{1/2}(x^1)\,dx^1$ exists and therefore we see from (2.10) and (5) of Theorem 1, that $\phi \sim 1 + (m^*/2|x|)$ at infinity. By comparing with the Schwarzschild metric we get

$$m^* = G\,M/C^2. \tag{3.1}$$

where G is gravitational constant and M is the total mass of the system.

An expression for m^* can be obtained from Eq. (2.2). Since Δ_g can be expressed as $(g^{-1/2}\partial_l g^{1/2} g^{lm}\partial_m)$, we get by a partial integration

$$m^* = (1/16\pi)\int R(x)\phi(x)g^{1/2}(x)\,dx. \tag{3.2}$$

Another expression can be obtained by similar procedure

$$m^* = (1/2\pi)\int g^{lm}\partial_l(\log\phi)\partial_m(\log\phi)g^{1/2}(x)\,dx + (1/16\pi)\int R(x)g^{1/2}(x)\,dx. \tag{3.3}$$

We shall now prove that m^* is positive definite for sufficiently small ϵ by using (3.2). It turns out that m^* is of order ϵ^2 and hence we calculate m^* up to the order ϵ^2 and show that it is positive definite.

By (2.10) and (6) of Theorem 1,

$$\begin{aligned} 16\pi m^* &= \int R(x)g^{1/2}(x)\,dx \\ &\quad + (1/8)\int R(x)g^{1/2}(x)\,dx\,G_0(xx^1)R(x^1)g^{1/2}(x^1)\,dx^1 + O(\epsilon^3), \end{aligned} \tag{3.4}$$

substituting the explicit expression for $R(x)$ and retaining terms of order ϵ^2 we get

$$
\begin{aligned}
128\pi m^*/\epsilon^2 = \int [&-(\partial_l \bar{g}_{rr})(\partial_l \bar{g}_{ss}) + 2(\partial_l \bar{g}_{rr})(\partial_m \bar{g}_{lm}) \\
&+ 2(\partial_k \bar{g}_{lm})(\partial_k \bar{g}_{lm}) - 4(\partial_m \bar{g}_{lk})(\partial_l \bar{g}_{mk})]\, dx \\
&+ (\tfrac{1}{4}\pi) \int (\partial_l \partial_m \bar{g}_{lm}) \mid x - x^1 \mid^{-1} (\partial_r^1 \partial_s^1 \bar{g}_{rs}^1)\, dx dx^1 + O(\epsilon).
\end{aligned} \tag{3.5}
$$

Here summation over repeated indices should be understood.

We make use of the Fourier transform

$$\bar{g}_{lm}(x) = (2\pi)^{-(3/2)} \int \tilde{g}_{lm}(k) e^{ikx}\, d^3k. \tag{3.6}$$

From reality of $\bar{g}_{lm}(x)$

$$\tilde{g}_{lm}(k)^* = \tilde{g}_{lm}(-k). \tag{3.7}$$

Substituting (3.6) into (3.5) and using (3.7), we get

$$128\pi m^*/\epsilon^2 = \int H(k)\, d^3k + O(\epsilon). \tag{3.8}$$

$$
\begin{aligned}
H(k) = &-k^2 \mid \tilde{g}_{rr} \mid^2 + k_l k_m (\tilde{g}_{rr} \tilde{g}_{lm}^* + \tilde{g}_{rr}^* \tilde{g}_{lm}) \\
&- 4k_l k_m (\tilde{g}_{lr}^* \tilde{g}_{mr}) + 2k^2 (\tilde{g}_{lm}^* \tilde{g}_{lm}) \\
&+ (k_l k_m k_r k_s / k^2)(\tilde{g}_{lm}^* \tilde{g}_{rs}).
\end{aligned} \tag{3.9}
$$

Equation (3.9) can be re-expressed in the following form

$$H(k) = (1/2k^6) \sum_{l,m} \mid F_{lm} \mid^2, \tag{3.10}$$

$$
\begin{aligned}
F_{lm} = &\, 2k^4 \tilde{g}_{lm} - k^4 \delta_{lm} \tilde{g}_{rr} + \delta_{lm} k^2 (k_r \tilde{g}_{rs} k_s) \\
&- 2k^2 (k_m \tilde{g}_{sl} k_s + k_l \tilde{g}_{sm} k_s) \\
&+ k_l k_m k^2 \tilde{g}_{rr} + k_l k_m (k_r \tilde{g}_{rs} k_s).
\end{aligned} \tag{3.11}
$$

We now see that m^* is always positive or zero. If $m^* = 0$, F_{lm} must vanish identically which implies that $\tilde{g}_{lm}$ is of the following form

$$\tilde{g}_{lm} = \tilde{h} \delta_{lm} + i(\tilde{f}_l k_m + \tilde{f}_m k_l),$$

where $\tilde{h}$ and $\tilde{f}_l$ are arbitrary functions of k. In coordinate space, this means that

$$\bar{g}_{lm}(x) = h(x)\delta_{lm} + [\partial_m f_l(x) + \partial_l f_m(x)], \tag{3.12}$$

where $h(x)$ and $f_1(x)$ are related to $\tilde{h}$ and $\tilde{f}_1$ just as the $\bar{g}_{lm}(x)$ are related to $\tilde{g}_{lm}$.

We now see that by a coordinate transformation $x_l' = x_l + \epsilon f_l$ and by conformal transformation by a factor $1 - h$ in the original metric g_{lm}, the same solution of Eq. (2.1) could have been obtained starting from another metric which differs from the flat metric by order ϵ^2. Therefore using above method, the term of order ϵ^4 of m^* is also positive or possibly zero. In the latter case we can repeat the same argument.

We thus conclude (1) any solution of $R = 0$ which is sufficiently close to the flat metric has positive mass energy.

4. CONCLUDING REMARK

We have proved that there are many nontrivial solutions of the initial value problem for the time symmetric gravitational waves, and furthermore that, if the solutions are sufficiently close to the flat metric, they have positive total energy. It will be extremely interesting if one could prove the positive definiteness of the total energy for any value of ϵ. Of course it will be even more interesting if one could prove the positive definiteness of the total energy for any gravitational waves but this is beyond the scope of the approach in this paper.

APPENDIX. PROOF OF THEOREM 1

The existence theorem for the Green function of the second order elliptic equations on a finite domain is known (*4*). We shall make use of it in the form of the following lemma.[1]

Lemma. Given a metric of the form (2.4) and a compact domain D. There exists a Green function $g(x,x^1)$ of the Laplacian in this metric which satisfies

(1) $g(x,x^1)$ is defined for $x,x^1 \in D$, $x \neq x^1$.

(2) $g(x,x^1)$ has continuous second order derivatives for $x \neq x^1$.

(3) $| g(x,x^1) - G_0(x,x^1) | < N | x - x^1 |^{-1}$ with N independent of x and x^1.

(4) If $\phi(x)$ has continuous $(n - 1)$th derivative, then $\int_D g(x,x^1)\phi(x^1)\, dx^1$ has continuous nth derivative for $n = 1$ or 2.

(5) $\Delta_g g(x,x^1) = 0$ for $x \neq x^1$.

(6) $\Delta_g \int_D g(x,x^1)\phi(x^1)g^{1/2}(x^1)\, dx^1 = -\phi(x)$ for $x \in$ interior of D if $\phi'(x)$ is continuous.

(7) N of (3) can be chosen arbitrarily small and independent of ϵ, for sufficiently small ϵ.

(8) $g(x,x^1) - G_0(x,x^1)$ has continuous sixth derivatives if x and x^1 is in the interior of the region where the $\bar{g}_{lm}$ vanish (including $x = x^1$). Furthermore its second derivatives have bounds which go to zero as $\epsilon \to 0$ if x or x^1 stays on a closed set in the interior of the same region.

[1] Properties (7) and (8) of lemma can be proved by constructing $g(x,x^1)$ starting from $G_0(x,x^1) + \rho_1(x,x^1)\,([4\pi s(x,x^1)]^{-1} - G_0(x,x^1))$ where $s(x,x^1)$ is the geodesic distance of x and x^1 and $\rho_1(x,x^1)$ is an infinitely differentiable function which is unity for $| x - x^1 | < \delta$ and zero for $| x - x^1 | > 2\delta$ with suitably chosen δ.

We shall now give a proof of Theorem 1 which is almost the replica of the proof of the existence theorem of the Green function on a finite domain (*4*). In particular, we use the Fredholm theorem (*4*) only for a finite domain of integration.

Suppose $\bar{g}_{lm}$ has a compact support D_1. We choose a sequence of compact sets $D_1 \subset D_2 \subset D_3 \subset D$ such that the boundaries are finite distant apart. We apply the lemma to the domain D. Introducing an infinitely differentiable function $\rho(x,x^1)$ which is unity for x and $x^1 \in D_2$ and which vanishes for x or $x^1 \notin D_3$ we define a function $G_1(x,x^1)$ by

$$G_1(x,x^1) = G_0(x,x^1) + \rho(x,x^1)[g(x,x^1) - G_0(x,x^1)]. \tag{A.1}$$

$G_1(x,x^1)$ is defined for all $x \neq x^1$ and satisfies the properties (2), (3), (4), and (7) of the lemma. Denoting $M(x,x^1) = \Delta_g G_1(x,x^1)$, we have the following properties for M:

(a) $$\Delta_g \int G_1(x,x^1)v(x^1)\, dx^1 g^{1/2}(x^1) = -v(x) + \int M(x,x^1)v(x^1)g^{1/2}(x^1)\, dx^1 \tag{A.2}$$

if $v(x)$ has continuous first derivative and $\int (\,|\,v(x)\,|/|\,x\,|\,)\, dx$ converges at infinity.

(b) $M(x,x^1) = 0$ for $x \notin D$ or x and $x^1 \in D_2$.

(c) $\Delta_{x^1} M(x,x^1) = 0$ for $x^1 \notin D$.

(d) $M(x,x^1)$ has continuous fourth derivative because of (8) of the lemma.

(e) $M(x,x^1)$ has a bound which goes to zero as $\epsilon \to 0$ because of (8) of the lemma.

We now consider the following integral equation

$$\int_D M(x^1,x)z(x^1)g^{1/2}(x^1)\, dx^1 = z(x), \qquad x \in D. \tag{A.3}$$

By the Fredholm theorem, (A.3) has at most a finite number m of linearly independent solutions $z_\mu(x)$; $\mu = 1, \cdots, m$. By Eqs. (A.3) we define an extension of the functions $z_\mu(x)$ on D to functions $\bar{z}_\mu(x)$ on the whole space. We note that, because of (d) above, $\bar{z}_\mu(x)$ has continuous fourth derivative and goes to zero like $|\,x\,|^{-2}$ as $|\,x\,| \to \infty$. Therefore, because of the uniqueness of the solution of the Laplace equation, the $\Delta\bar{z}_\mu(x)$ are linearly independent and the matrix

$$\alpha_{\mu\nu} = \int \Delta_g\bar{z}_\mu(x)\, \Delta_g\bar{z}_\nu(x)g^{1/2}(x)\, dx \tag{A.4}$$

is positive definite. From (c) above and (A.3) we have $\Delta\bar{z}_\mu(x) = 0$ for $x \notin D$.

We set

$$L(x,x^1) = \sum_{\mu\nu} \Delta_g \bar{z}_\mu(x)\bar{z}_\nu(x^1)(\alpha^{-1})_{\mu\nu} \tag{A.5}$$

and

$$f_1(x) = \int M(x,x^1)f(x^1)g^{1/2}(x^1)\,dx^1 - \Delta_g \int L(x,x^1)f(x^1)g^{1/2}(x^1)\,dx^1, \tag{A.6}$$

where $f(x)$ is some once differentiable function for which $\int(f(x)/|x|)\,dx$ converges at infinity. Noting that $f_1(x) = 0$ for $x \notin D$ and $f_1(x)$ is orthogonal to every $z_\mu(x)$ because of (A.3), (A.4), and (A.5), we apply the Fredholm theorem to an integral equation

$$u(x) - \int_D M(x,x^1)u(x^1)g^{1/2}(x^1)\,dx^1 = f_1(x), \qquad x \in D \tag{A.7}$$

to get a solution of the form

$$u(x) = f_1(x) + \int_D \Gamma(x,x^1)g^{1/2}(x^1)\,dx^1\,f_1(x^1), \tag{A.8}$$

where $\Gamma(x,x^1)$ satisfies

$$\begin{aligned}\Gamma(x,x^1) = M(x,x^1) &+ \int M(x,x'')\Gamma(x'',x^1)g^{1/2}(x'')\,dx'' \\ &+ \sum_\mu M(x,a_\mu)z_\mu(x^1)\end{aligned} \tag{A.9}$$

with some constant a_μ.

We extend $\Gamma(x,x^1)$ to $\Gamma(x,x^1)$ by setting $\Gamma(x,x^1) = 0$ if $x \notin D$. Because of (b) and (d) above $\Gamma(x,x^1)$ has continuous second derivatives. Similarly we extend $u(x)$ to $\bar{u}(x)$ which is defined by (A.8) with Γ replaced by $\bar{\Gamma}$. $\bar{u}(x) = 0$ for $x \notin D$.

We now see that $\bar{v}(x)$ defined by

$$\bar{v}(x) = \bar{u}(x) + f(x) \tag{A.10}$$

satisfies

$$\bar{v}(x) - \int M(x,x^1)\bar{v}(x^1)g^{1/2}(x^1)\,dx^1 = f(x) - \Delta_g \int L(x,x^1)f(x^1)g^{1/2}(x^1)\,dx^1. \tag{A.11}$$

Finally putting

$$\phi(x) = \int G_1(x,x^1)g^{1/2}(x^1)\bar{v}(x^1)\,dx^1 - \int L(x,x^1)g^{1/2}(x^1)f(x^1)\,dx^1, \tag{A.12}$$

we get from (A.2) and (A.11)

$$\Delta_g \phi(x) = -f(x), \tag{A.13}$$

which shows in view of (A.6), (A.8), (A.10), and (A.12) that the desired Green function is given by

$$\begin{aligned} G_\epsilon(x,x^1) &= G_1(x,x^1) + \int G_1(x,x'')g^{1/2}(x'')M(x''x^1)\,dx'' \\ &\quad - \int G_1(x,x'')\Delta_g''L(x'',x^1)g^{1/2}(x'')\,dx'' - L(x,x^1) \\ &\quad + \int G_1(x,x'')\,dx''\,g^{1/2}(x'')\int \Gamma(x'',x''')g^{1/2}(x''')\,dx'''(M(x''',x^1) \\ &\quad - \Delta_g'''L(x''',x^1)). \end{aligned} \tag{A.14}$$

Property (1) of Theorem 1 follows from the same properties of L and G_1. Property (2) can be checked for (A.14) using (A.2), definition of M, (A.9), (A.3), (A.5), and (A.4). Property (3) and (5) is a consequence of the corresponding property of G_1. Property (4) has already been shown. Property (6) is a consequence of the same property for G_1 and property (e) above for M because for sufficiently small M, $m = 0$. Property (8) is a consequence of the maximum principle for the solution of the Laplace equation. Property (7) can be obtained by considering the integral

$$\int_{K_\delta(x)+K_\delta(x^1)} \left[G_\epsilon(x'',x)\frac{\partial''}{\partial n_r}G_\epsilon(x'',x^1) - G_\epsilon(x'',x^1)\frac{\partial''}{\partial n_r}G_\epsilon(x'',x)\right] ds_r'',$$

where $K_\delta\ (a)$ denote the surface of a sphere of geodesic radius δ with center at a.

RECEIVED: March 20, 1959

REFERENCES

1. Y. FOURES-BRUHAT, *Acta Math.* **88,** 141 (1952); *J. Rational Mech. Anal.* **4,** 951 (1956).
2. D. R. BRILL, following paper [*Annals of Physics* **7,** 466 (1959).]
3. A. LICHNEROWICZ, "Problèmes globaux en mécanique relativiste." Hermann, Paris, 1939; *J. math. pures appl.* [9] **23,** 37 (1944); *Helv. Phys. Acta* Suppl. IV, p. 176 (1956). See also Ref. *1* and C. W. MISNER AND J. A. WHEELER, *Annals of Physics* **2,** 525 (1957).
4. G. F. D. DUFF, "Partial Differential Equations." Univ. of Toronto Press, Toronto, 1956.