International Journal of Approximate Reasoning 54 (2013) 467–477

Contents lists available at SciVerse ScienceDirect

International Journal of Approximate Reasoning

journal homepage: www.elsevier.com/locate/ijar

Probabilistic stit logic and its decomposition

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ARTICLE INFO

Article history:
Available online 10 September 2012

Keywords:
Agency
Probability
Modal logic
Epistemic attitudes

ABSTRACT

We define an extension of stit logic that encompasses subjective probabilities representing beliefs about simultaneous choice exertion of other agents. This semantics enables us to express that an agent sees to it that a condition obtains under a minimal chance of success. We first define the fragment of XSTIT where choice exertion is not collective. Then we add lower bounds for the probability of effects to the stit syntax, and define the semantics of the newly formed stit operator in terms of subjective probabilities concerning choice exertion of other agents. We show how the resulting probabilistic stit logic faithfully generalizes the non-probabilistic XSTIT fragment. In a second step we analyze the defined probabilistic stit logic by decomposing it into an XSTIT fragment and a purely epistemic fragment. The resulting epistemic logic for grades of believes is a weak modal logic with a neighborhood semantics combining probabilistic and modal logic theory.

1. Introduction

A well established formal theory of agency in philosophy is *stit* theory [1]. *Stit* theory gives an elegant and thoroughly elaborated view on the question of how agents exercise control over the courses of events that constitute our dynamic world. Also *stit* theory provides a view on the fundamentals of cooperation and the limits and possibilities of acting together and/or in interaction. Recently, *stit* theory attracted the attention of computer scientist who are interested in deontic logic and logic for the specification of multi-agent systems [2–4].

One shortcoming of *stit* theory is that its central notion of choice exertion is one that assumes that a choice is always successful. But it is highly unrealistic for formalisms aimed at modeling (group) choice of intelligent agents to assume that action can never fail. This problem cannot be solved by making the connection with Dynamic Logic or the situation calculus, since these formalisms also lack a theory about how actions can be unsuccessful.

This paper assumes we measure success of action against an agent's beliefs about the outcome of its choice. So, the perspective is an internal, subjective one, and the criterion of success is formed by an agent's beliefs about its action. To represent these beliefs we choose here to use probabilities. In particular, we will represent beliefs about simultaneous choice exertion of other agents in a system as subjective probabilities. Several choices have to be made. We will impose that an agent can never be mistaken about its own choice, but that it can be mistaken about choices of others. The actual action performed results from a simultaneous choice exertion of all agents in the system. Then, if an agent can be mistaken about the choices of other agents (including possibly an agent with special properties called 'nature'), the action can be unsuccessful. As a very basic example, consider the opening of a door. An agent exercises its choice to open the door. It cannot be mistaken about that: it knows what it chooses to do. It does this under the belief that there is no other agent on the other side exercising its choice to keep the door closed. So it assigns a low probability to such a choice of any other agent. However, here the agent can be mistaken. And here comes in the notion of unsuccessful action modeled in this paper: as it turns out, in the situation described there actually is an agent at the other side of the door choosing to keep it closed and the agent's opening effort is unsuccessful.

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http://dx.doi.org/10.1016/j.ijar.2012.08.007

To model this, we endow *stit* theory with probabilities in the object language, enabling us to say that an agent exercises a choice for which it believes to have a chance higher than c to see to it that φ results in the next state.

As far as we know, our proposal is the first combining *stit* logic and probability. Possibly unsuccessful actions have been considered in the context of Markov Decision Processes, temporal logic and ATL [5]. Two differences with the present work are that here we start from the richer *stit* theory and that we focus on fundamental properties of the resulting logic in stead of on issues related to planning, policy generation or model checking. An independent motivation for considering action with a chance of success comes from the relation between *stit* theory and game theory. Kooi and Tamminga [6] investigate how to characterize pure strategy equilibria as *stit* formulas. An extension of *stit* logic with probabilistic effects would enable us to also characterize mixed strategy equilibria.

2. The base logic: XSTIT^p

In this section we define the base logic, which is a variant of the logic XSTIT that we call XSTIT^p. The difference with XSTIT is embodied by an axiom schema concerning modality-free propositions p, which explains the name. Another difference with XSTIT is that we do not define the semantics in terms of *relations*, but in terms of functions. We introduce h-relative effectivity functions, which specialize the notion of effectivity function from Coalition Logic [7] by defining choices relative to histories. The function-based semantics explains the formalism better than XSTIT's earlier semantics in terms of relations.

Definition 2.1. Given a countable set of propositions P and $p \in P$, and given a finite set Ags of agent names, and $ag \in Ags$, the formal language $\mathcal{L}_{XSTIT^p}$ is:

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid [ag\ \texttt{xstit}]\varphi \mid X\varphi$$

Besides the usual propositional connectives, the syntax of XSTIT^p comprises three modal operators. The operator $\Box\varphi$ expresses 'historical necessity', and plays the same role as the well-known path quantifiers in logics such as *CTL* and *CTL** [8]. Another way of talking about this operator is to say that it expresses that φ is 'settled'. We abbreviate $\neg\Box\neg\varphi$ by $\Diamond\varphi$. The operator $[ag\ \texttt{xstit}]\varphi$ stands for 'agent ag sees to it that φ in the next state'. We abbreviate $\neg[ag\ \texttt{xstit}]\neg\varphi$ by $\langle ag\ \texttt{xstit}\rangle\varphi$. The third modality is the next operator $X\varphi$. It has a standard interpretation as the transition to a next state.

Definition 2.2. An XSTIT^p-frame is a tuple $\langle S, H, E\rangle$ such that [1] :

1. S is a non-empty set of static states. Elements of S are denoted s, s', etc.
2. H is a non-empty set of possible system histories isomorphic with $\ldots s_{-2}, s_{-1}, s_0, s_1, s_2, \ldots$ with $s_x \in S$ for $x \in \mathbb{Z}$. Elements of H are denoted h, h', etc. We denote that s' succeeds s on the history h by $s' = succ(s, h)$ and by $s = prec(s', h)$. We have the following bundling constraint on the set H:
   a. if $s \in h$ and $s' \in h'$ and $s = s'$ then $prec(s, h) = prec(s', h')$.
3. $E : S \times H \times Ags \mapsto 2^S$ is an h-effectivity function yielding for an agent ag the set of next static states allowed by the choice exerted by the agent relative to a history. We have the following constraints on h-effectivity functions:
   a. if $s \notin h$ then $E(s, h, ag) = \emptyset$.
   b. if $s' \in E(s, h, ag)$ then $\exists h' : s' = succ(s, h')$.
   c. if $s' = succ(s, h')$ and $s' \in h$ then $s' \in E(s, h, ag)$.
   d. $E(s, h, ag_1) \cap E(s, h', ag_2) \neq \emptyset$ for $ag_1 \neq ag_2$.

In Definition 2.2 above, we refer to the states s as 'static states'. This is to distinguish them from 'dynamic states', which are combinations $\langle s, h\rangle$ of static states and histories. Dynamic states function as the elementary units of evaluation of the logic. This means that the basic notion of 'truth' in the semantics of this logic is about dynamic conditions concerning choice exertions. This distinguishes *stit* from logics like Dynamic Logic and Coalition Logic whose central notion of truth concerns static conditions holding for static states.

The name 'h-effectivity functions' for the functions defined in item **3** above is short for 'h-relative effectivity functions'. This name is inspired by similar terminology in Coalition Logic whose semantics is in terms of 'effectivity functions'. Condition **3.a** above states that h-effectivity is empty for history-state combinations that do not form a dynamic state. Condition **3.b** ensures that next state effectivity as seen from a current state s does not contain states s' that are not reachable from the current state through some history. Condition **3.c** expresses the well-known *stit* condition of 'no choice between undivided histories'. Condition **3.d** above states that simultaneous choices of different agents never have an empty intersection. This is the central condition of 'independence of agency'. It reflects that a choice exertion of one agent can never have as a consequence that some other agent is limited in the choices it can exercise simultaneously.

[1] In the meta-language we use the same symbols both as constant names and as variable names, and we assume universal quantification of unbound meta-variables.

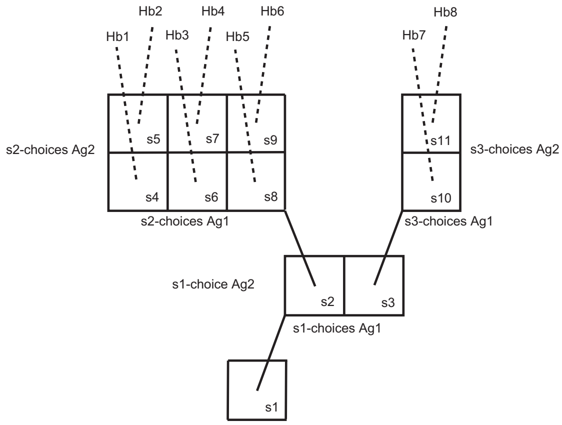

Fig. 1. Visualization of a partial two agent XSTIT^p frame.

The conditions on the frames are not as tight as the conditions in the classical *stit* formalisms of Belnap et al. [1]. Appart from the crucial difference concerning the effect of actions (in XSTIT^p actions take effect in next states), the classical *stit* formalisms assumes a condition that in our meta-language can be represented as:

e. $E(s, h, ag) \neq E(s, h', ag)$ implies $E(s, h, ag) \cap E(s, h', ag) = \emptyset$.

Condition **e.** says that the choices of an agent *ag* are mutually disjoint. Since they result in much tidier pictures, in the example visualization of a frames we consider below, we assume this condition. However, we do not include it in the formal definition of the frames, because it is not modally expressible (e.g., in modal logic we can give axioms characterizing that an intersection is non-empty, but we cannot characterize that an intersection is empty). This means that they will not have an effect on our modal logic of agency whose semantics we will define in terms of the above frames.

Fig. 1 visualizes a frame of the type defined by Definition 2.2. The columns in the games forms linked to each state are the choices of agent ag_1 and the rows are the choices of agent ag_2. Independence of choices is reflected by the fact that the game forms contain no 'holes' in them. Choice exertion in this 'bundled' semantics is thought of as the separation of two bundles of histories: one bundle ensured by the choice exercised and one bundle excluded by that choice.

We now define models by adding a valuation of propositional atoms to the frames of Definition 2.2. We impose that all dynamic state relative to a static state evaluate atomic propositions to the same value. This reflects the intuition that atoms, and modality-free formulas in general do not represent dynamic information. Their truth value should thus not depend on a history but only on the static state. This choice does however make the situation non-standard. It is a constraint on the models, and not on the frames.

Definition 2.3. A frame $\mathcal{F} = \langle S, H, E\rangle$ is extended to a model $\mathcal{M} = \langle S, H, E, V\rangle$ by adding a valuation V of atomic propositions:

- V is a valuation function $V : P \longrightarrow 2^S$ assigning to each atomic proposition the set of static states relative to which they are true.

We evaluate truth with respect to dynamic states built from a dimension of histories and a dimension of static states.

Definition 2.4. Relative to a model $\mathcal{M} = \langle S, H, E, V\rangle$, truth $\langle s, h\rangle \models \varphi$ of a formula φ in a dynamic state $\langle s, h\rangle$, with $s \in h$, is defined as:

$$\begin{array}{lll}
\langle s, h\rangle \models p & \Leftrightarrow & s \in V(p) \\
\langle s, h\rangle \models \neg\varphi & \Leftrightarrow & \text{not } \langle s, h\rangle \models \varphi \\
\langle s, h\rangle \models \varphi \wedge \psi & \Leftrightarrow & \langle s, h\rangle \models \varphi \text{ and } \langle s, h\rangle \models \psi
\end{array}$$

$$\begin{array}{lll}
\langle s, h\rangle \models \Box\varphi & \Leftrightarrow & \forall h' : \text{ if } s \in h' \text{ then } \langle s, h'\rangle \models \varphi \\
\langle s, h\rangle \models X\varphi & \Leftrightarrow & \text{if } s' = succ(s, h) \text{ then } \langle s', h\rangle \models \varphi \\
\langle s, h\rangle \models [ag\ \texttt{xstit}]\varphi & \Leftrightarrow & \forall s', h' : \text{if } s' \in E(s, h, ag) \text{ and} \\
 & & s' \in h' \text{ then } \langle s', h'\rangle \models \varphi
\end{array}$$

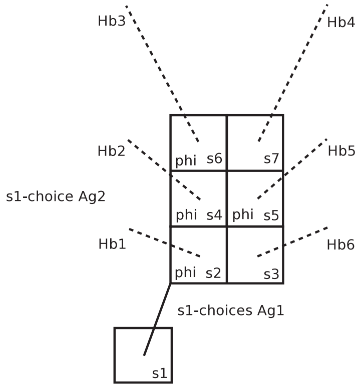

Fig. 2. Visualization of a partial two agent XSTIT^p model.

Satisfiability, validity on a frame and general validity are defined as usual.

Note that the historical necessity operator quantifies over one dimension, and the next operator over the other. The *stit* modality combines both dimensions. Now we proceed with the axiomatization of the base logic.

Fig. 2 gives an example model that we can use to discuss the evaluation of formulas. Relative to static state s_1 and the history h_5 that is part of the bundle of histories Hb_5 we do not have that the choice by agent ag_1 ensures that φ holds, since the other agent has two choices (the bottom one and the top one) for which φ will not be true. So in this model we have that $\langle s_1, h_5\rangle \not\models [ag_1 \texttt{ xstit}]\varphi$. However, relative to, for instance, a history in the bundle Hb_1, the agent ag_1 does guarantee that φ obtains as the result of the choice it excerpts independent of what agent ag_2 chooses simultaneously: for all three choices of the other agent φ is the result. So we have that $\langle s_1, h_1\rangle \models [ag_1 \texttt{ xstit}]\varphi$.

Definition 2.5. The following axiom schemas, in combination with a standard axiomatization for propositional logic, and the standard rules (like necessitation) for the normal modal operators, define a Hilbert system for XSTIT^p:

(p) $p \to \Box p$ for p modality free

S5 for $\Box$

(Lin) $\neg X \neg \varphi \leftrightarrow X\varphi$

(Sett) $\Box X\varphi \to [ag \texttt{ xstit}]\varphi$

(XSett) $[ag \texttt{ xstit}]\varphi \to X\Box\varphi$

(Agg) $[ag \texttt{ xstit}]\varphi \wedge [ag \texttt{ xstit}]\psi \to [ag \texttt{ xstit}](\varphi \wedge \psi)$

(Mon) $[ag \texttt{ xstit}](\varphi \wedge \psi) \to [ag \texttt{ xstit}]\varphi$

(Indep) $\Diamond[ag_1 \texttt{ xstit}]\varphi \wedge \ldots \wedge \Diamond[ag_n \texttt{ xstit}]\psi \to$

$\Diamond([ag_1 \texttt{ xstit}]\varphi \wedge \ldots \wedge [ag_n \texttt{ xstit}]\psi)$

for $Ags = \{ag_1, \ldots, ag_n\}$

Theorem 2.1. *The Hilbert system of Definition 2.5 is complete with respect to the semantics of Definition 2.4.*

The proof strategy is as follows. First we establish completeness of the system *without* the axiom $p \to \Box p$, relative to the frames of Definition 2.2. All remaining axioms are in the Sahlqvist class. This means that all the axioms are expressible as first-order conditions on frames and that together they are complete with respect to the frame classes thus defined, cf. [9]. It is easy to find the first-order conditions corresponding to the axioms, for instance, by using the on-line SQEMA system [10]. So, now we know that every formula consistent in the slightly reduced Hilbert system has a model based on an abstract frame. Left to show is that we can associate such an abstract model to a concrete model based on an XSTIT^p frame as given in Definition 2.2. This takes some effort, since we have to associate worlds in the abstract model to dynamic states in the frames of Definition 2.2 and check all the conditions of Definition 2.2 against the conditions in the abstract model (**3.b** corresponds to (Sett), **3.c** in combination with **2.a** to (XSett), **3.d** to (Indep)). Once we have done this, we have established completeness of the axioms relative to the conditions on the frames. Now the second step is to add the axiom $p \to \Box p$. This axiom does not have a corresponding frame condition. Indeed, the axiom expresses a condition on the models. But then, to show completeness, we only have to show that we can always find a model obtained by the construction just described that satisfies the axiom $p \to \Box p$. But this is straightforward. From all the possible models resulting from the first step, we select the ones where propositional atoms in dynamic states based on the same static state have identical valuations. Since consistent formulas also have to be consistent with the axiom $p \rightarrow \Box p$ for any non-modal formula p, we can always do that. This means that a satisfying model for a consistent formula is always obtainable in this way and that completeness is preserved.

The logic for isolated operators $[ag\ \texttt{xstit}]\varphi$ is KD. The D axiom $\neg[ag\ \texttt{xstit}]\bot$ is derivable from the axioms (Lin) and (XSet) and the standard axioms for normal modal logic. Semantically, the D axiom corresponds with the condition $succ(s, h) \in E(s, h, ag)$ saying that the static state next of some other static state on a history is always in the effectivity set relative to that history state pair for any agent. The condition follows from the condition **3.c** in Definition 2.2 by assuming $h' = h$.

The independence axiom given here is simpler than Xu's axiomatization of independence in his systems for instantaneous deliberative *stit* (see the chapter in [1]). Xu's axiomatization works with sets of axioms for independence, where each axiom in the set deals with a subset of the complete group of agents. This construction is due to the fact that the *deliberative* stit operator of Xu is not normal.

3. Choice exertion with a lower bound on chance of success: XSTIT.Prob

We introduce operators $[ag\ \texttt{xstit}^{\geq c}]\varphi$ with the intended meaning that agent ag exercises a choice for which it believes to have a chance of at least c of bringing about φ. We assume that numbers c are between 1 and 0 and that the set of possible c's is at least countable (that is, a subset of $\mathbb{Q}$). Roughly, the semantics for this new operator is as follows. We start with the multi-agent *stit*-setting of the previous section. Now to the semantic structures we add functions such that in the little game-forms, as visualized by Fig. 1, for each choice of an agent ag we have available the subjective probabilities applying to the simultaneous choices of other agents in the system. For agent ag the sum of the probabilities over the choices of each particular other agent in the system adds up to one. So, the probabilities represent agent ag's beliefs concerning what choices are exerted simultaneously by other agents. In terms of the subjective probability function we define for each choice the sum of the probabilities for each of the choices of all other agents in the system leading to a situation obeying φ.

For the definition of the probabilistic frames, we first define an augmentation function returning the choice an agent has in a given state.

Definition 3.1. The range function $Range : S \times Ags \mapsto 2^{2^S \setminus \emptyset} \setminus \emptyset$ yielding for a state s and an agent ag, the choices this agent has in s is defined as:
$Range(s, ag) = \{Ch \mid \exists h : s \in h \text{ and } Ch = E(s, h, ag)\}$.

A range function is similar to what in Coalition Logic [7] is called an 'effectivity function'. Now we are ready to define the probabilistic *stit* frames.

Definition 3.2. A probabilistic XSTIT^p-frame is a tuple $\langle S, H, E, B\rangle$ such that:

1. $\langle S, H, E\rangle$ is an XSTIT^p-frame.
2. $B : S \times Ags \times Ags \times 2^S \mapsto [0, 1]$ is a subjective probability function such that $B(s, ag_1, ag_2, Ch)$ expresses agent 1's believe that in static state s agent 2 performs a choice resulting in one of the static states in Ch. We apply the following constraints:
   a. $B(s, ag, ag', Ch) = 0$ if $ag \neq ag'$ and $Ch \notin Range(s, ag')$.
   b. $B(s, ag, ag', Ch) > 0$ if $ag \neq ag'$ and $Ch \in Range(s, ag')$.
   c. $\sum_{Ch \in Range(s, ag')} B(s, ag, ag', Ch) = 1$ if $ag \neq ag'$.
   d. $B(s, ag, ag, Ch) = 1$.

Condition **2.a** says that agents only assign non-zero subjective probabilities to choices other agents objectively have. Condition **2.b** says these probabilities are strictly larger than zero. Condition **2.c** says that the sum of the subjective probabilities over the possible choices of other agents add up to 1. Condition **2.d** says that agents always know what choice they exercise themselves. Note that this is not the same as claiming that agents always know what action they perform (which is not the case in our conceptualization). We already explained this difference between choice and action in Section 2. Note that summation of probabilities $B(s, ag, ag, Ch)$ over the choices relative to some state s yields infinity if the number of choices is infinite. This underlines that probabilities are subjective in this theory and that agents are free to choose. If we would see the probabilities in this theory as objective, then the same summation would add up to 1. However, then the choices of the agent whose reasoning we model in our logic are governed by chance, which conflicts with the idea of free will. We assume here that the agent is free to choose, and that whatever it chooses, it is certain about its choice (which, again, is not to say that it is certain about its action). This results in the summation over $B(s, ag, ag, Ch)$ possibly adding up to infinity.

Fig. 3 extends the earlier example model with subjective probabilities for agent ag_1's belief concerning the choice agent ag_2 exerts simultaneously. We see that agent ag_1 believes that the chance that agent ag_2 chooses the top row is 0.6, that the chance for the middle row is 0.3 and the chance for the bottom row is 0.1. It is easy to check that this model satisfies all the conditions discussed above.

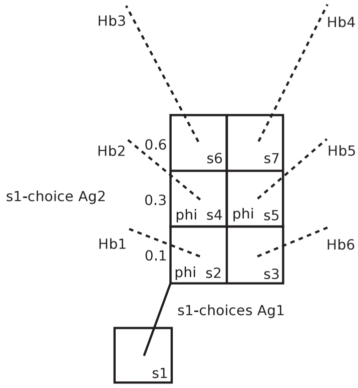

Fig. 3. Visualization of a partial two agent probabilistic XSTIT^p model.

In the sequel we will need an augmentation function yielding for an agent and an arbitrary next static state the chance an agent ascribes to the occurrence of this state (given its belief, i.e., subjective probabilities about simultaneous choice exertion of other agents). For this, we first need the following proposition. To guarantee that the proposition is true, we need the condition **e.** as one of the conditions posed on the frames in Definition 2.2. As we argued, assuming this condition **e.** does not change the logic. The condition ensures that in any state the set of choices of any agent partitions the set of possible next states. From this, the proposition follows immediately.

Proposition 3.1. *For any pair of static states s and s' for which there is an h such that $s' = succ(s, h)$ there is a unique 'choice profile' determining for each agent ag in the system a unique choice $Ch = E(s, h, ag)$ relative to s and s'.*

Now we can define the subjective probabilities agents assign to possible system outcomes. Because of the idea of independence of agency, we can multiply the chances for the choices of the individual agents relative to the system outcome (the resulting static state). Note that this gives a new and extra dimension to the notion of independence that is not available in standard *stit* theories.

Definition 3.3. $BX : S \times Ags \times S \mapsto [0, 1]$ is a subjective probability function concerning possible next static states, defined by

$$BX(s, ag, s') = \prod_{ag' \in Ags} B(s, ag, ag', E(s, h, ag')) \text{ for some } h \text{ such that } s' = succ(s, h).$$

Note that $BX(s, ag, s')$ expresses agent ag's belief in state s that its choice ends up in s' modulo the assumption that ag actually chooses such as to make s' a possible outcome; if ag chooses such that s' is excluded by its choice, the chance for s' is of course 0.

Now before we can define the notion of 'seeing to it under a minimum bound on the probability of success' formally as a truth condition on the frames of Definition 3.2 we need to do more preparations. First we assume that the intersection of the h-effectivity functions of all agents together yields a unique static state. We can safely assume this, because, again, this condition is not modally expressible. In general we can express uniqueness of next states in terms of a D axiom. But note that here the units of evaluation are dynamic states. With a unique next static state corresponds a set of next dynamic states: all the dynamic states that can be built with the static state. We cannot characterize uniqueness of this set using a D axiom. This justifies Definition 3.4 below, that establishes a function characterizing the static states next of a given state that satisfy a formula φ relative to the current choice of an agent.

Definition 3.4. The 'possible next static φ-states' function $PosX : S \times H \times Ags \times \mathcal{L} \mapsto 2^S$ which for a state s, a history h, an agent ag and a formula φ gives the possible next static states obeying φ given the agent's current choice determined by h, is defined by: $PosX(s, h, ag, \varphi) = \{s' \mid s' \in E(s, h, ag)$ and $\langle s', h' \rangle \models \varphi$ for all h' with $s' \in h'\}$.

Now we can formulate the central 'chance of success' (CoS) function that will be used in the truth condition for the new operator. The chance of success relative to a formula φ is the sum of the chances the agent assigns to possible next static states validating φ.

Definition 3.5. The chance of success function $CoS : S \times H \times Ags \times \mathcal{L} \mapsto [0, 1]$ which for a state s and a history h an agent ag and a formula φ gives the chance the agent's choice relative to h is an action resulting in φ is defined by: $CoS(s, h, ag, \varphi) = 0$ if $PosX(s, h, ag, \varphi) = \emptyset$ or else $CoS(s, h, ag, \varphi) = \sum_{s' \in PosX(s,h,ag,\varphi)} BX(s, ag, s')$.

Extending the probabilistic frames of Definition 3.2 to models in the usual way, the truth condition of the new operator is defined as follows.

Definition 3.6. Relative to a model $\mathcal{M} = \langle S, H, E, B, V\rangle$, truth $\langle s, h\rangle \models [ag\ \texttt{xstit}^{\geq c}]\varphi$ of a formula $[ag\ \texttt{xstit}^{\geq c}]\varphi$ in a dynamic state $\langle s, h\rangle$, with $s \in h$, is defined as:

$$\langle s, h\rangle \models [ag\ \texttt{xstit}^{\geq c}]\varphi \Leftrightarrow CoS(s, h, ag, \varphi) \geq c.$$

Using the example model of Fig. 3 we can now discuss truth evaluations on probabilistic *stit* models. As we saw earlier, relative to static state s_1 and the history h_5 that is part of the bundle of histories Hb_5 we do not have that the choice by agent ag_1 ensures that φ holds, since the other agent has two choices (the bottom one and the top one) for which φ will not be true. So in this model we have that $\langle s_1, h_5\rangle \not\models [ag_1\ \texttt{xstit}^{\geq 1}]\varphi$. But we do have that $\langle s_1, h_5\rangle \models [ag_1\ \texttt{xstit}^{\geq 0.3}]\varphi$ since ag_1 believes that with a chance of 0.3 agent ag_2 exerts the choice of the middle row. But, relative to histories in, for instance the bundle $Hb1$, agent ag_1 has better chances to see to it that φ will be true. In particular we have that $\langle s_1, h_1\rangle \models [ag_1\ \texttt{xstit}^{\geq 0.4}]\varphi$, because it can add up the chances of the bottom two rows. Note that this is also true relative to the histories in bundle Hb_3 for which the result is $\neg\varphi$. Here we have a situation where the agent saw to it that φ with a chance of success of at least 0.4, but failed. Also note that situations like these show that it is consistent in the logic to have, for instance, that $[ag_1\ \texttt{xstit}^{\geq\, c}]\varphi \wedge [ag_2\ \texttt{xstit}^{\geq\, c}]\neg\varphi$, that is, if c is not 1.

We now formulate the result that the logic following from Definitions 2.4 and 3.6 naturally extends the base *stit* logic.

Theorem 3.2. *Consider a trivial translation T of probabilistic stit formulas to xstit formulas determined by the mapping: $[ag\ \texttt{xstit}]\varphi = [ag\ \texttt{xstit}^{\geq 1}]\varphi$. Other formulas are mapped to their identical twin. Now, a formula φ is satisfiable in an xstit model according to semantics of Definition 2.4 if and only if $T(\varphi)$ is satisfiable in a probabilistic stit model according to the semantics of Definition 3.6.*

The proof of this theorem follows by careful examination of the probabilistic semantics. If the chance of success must be one, than an agent's beliefs about choice exertion by other agents is irrelevant. The only way in which the agent can be sure is be ensuring φ holds irrespective of what other agents chose. This condition brings the standard xstit semantics back in the probabilistic setting. If we would combine both modalities in one language, we would get the axiom $[ag\ \texttt{xstit}]\varphi \leftrightarrow [ag\ \texttt{xstit}^{\geq 1}]\varphi$. This shows that the probabilistic *stit* operator we gave in Definition 3.6 faithfully generalizes the *stit* operator of our base XSTIT^p system: the objective *stit* operator $[ag\ \texttt{xstit}]\varphi$ discussed in Section 2 comes out as the probabilistic *stit* operator assigning a probability 1 to establishing the effect φ. This is very natural. Where in the standard *stit* setting we can talk about 'ensuring' a condition, in the probabilistic setting we can only talk about establishing an effect with a certain lower bound on the probability of succeeding.

We now define a Hilbert system for the probabilistic stit logic. The system is parametric in the probabilistic variables c and k. This means that the system encodes infinitely many axioms, since there can be infinitely many values for c and k. To obtain a standard Hilbert system we can pose a prior limit to the possible values of probabilities.

Definition 3.7. Relative to the semantics following from Definitions 2.4 and 3.6 we define the following Hilbert system. We assume all the standard derivation rules for the normal modalities X and $\Box$. Furthermore, we assume the standard derivation rules for the weak modality $[ag\ \texttt{xstit}^{\geq c}]\varphi$, like closure under logical equivalence.

(p)	$p \rightarrow \Box p$ for p modality free
	S5 for $\Box$
(D)	$\neg[ag\ \texttt{xstit}^{\geq c}]\bot$ for $c > 0$
(Triv)	$[ag\ \texttt{xstit}^{\geq 0}]\varphi$
(Lin)	$\neg X\neg\varphi \leftrightarrow X\varphi$
(Sett)	$\Box X\varphi \rightarrow [ag\ \texttt{xstit}^{\geq c}]\varphi$
(XSett)	$[ag\ \texttt{xstit}^{\geq 1}]\varphi \rightarrow X\Box\varphi$
(Min)	$[ag\ \texttt{xstit}^{\geq c}]\varphi \rightarrow [ag\ \texttt{xstit}^{\geq k}]\varphi$ for $c \geq k$
(Add)	$[ag\ \texttt{xstit}^{\geq c}]\varphi \wedge [ag\ \texttt{xstit}^{\geq k}]\psi \rightarrow [ag\ \texttt{xstit}^{\geq c+k-1}](\varphi \wedge \psi)$ for $c + k > 1$
(Mon)	$[ag\ \texttt{xstit}^{\geq c}](\varphi \wedge \psi) \rightarrow [ag\ \texttt{xstit}^{\geq c}]\varphi$
(Indep)	$\Diamond[ag_1\ \texttt{xstit}^{\geq c}]\varphi \wedge \ldots \wedge \Diamond[ag_n\ \texttt{xstit}^{\geq k}]\psi \rightarrow$
	$\Diamond([ag_1\ \texttt{xstit}^{\geq c}]\varphi \wedge \ldots \wedge [ag_n\ \texttt{xstit}^{\geq k}]\psi)$ for $Ags = \{ag_1, \ldots, ag_n\}$.

Proposition 3.3. *The Hilbert system is sound relative to the semantics.*

This proposition follows by careful inspection of the semantics.

Proposition 3.4. *The Hilbert system reduces to the complete Hilbert system for xstit after substitution of 1 for the parameter c.*

Note that all axioms for xstit have a natural generalization in the above Hilbert system. The most interesting one is agglomeration that generalizes from the standard normal modal logic axiom (Agg) to the set of weak modal scheme's (Add). From the system we can derive several intuitive properties.

Proposition 3.5. *Derivable properties are the following:*

a. $[ag\ \texttt{xstit}^{\geq c}]\varphi \rightarrow [ag\ \texttt{xstit}^{\geq c}](\varphi \vee \psi)$.
b. $[ag\ \texttt{xstit}^{\geq c}](\varphi \vee \psi) \wedge [ag\ \texttt{xstit}^{\geq 1}]\neg\varphi \rightarrow [ag\ \texttt{xstit}^{\geq c}]\psi$.
c. $[ag\ \texttt{xstit}^{\geq c}]\varphi \wedge [ag\ \texttt{xstit}^{\geq 1}]\psi \rightarrow [ag\ \texttt{xstit}^{\geq c}](\varphi \wedge \psi)$.
d. $\neg([ag\ \texttt{xstit}^{\geq c}]\varphi \wedge [ag\ \texttt{xstit}^{\geq k}]\neg\varphi)$ *for* $c + k > 1$.
e. $[ag\ \texttt{xstit}^{\geq c}]\varphi \rightarrow \langle ag\ \texttt{xstit}^{\geq c}\rangle\varphi$.

For instance, property **d.** tells us that it is not possible, by means of one choice, to have at the same time a chance of c for φ and a chance of k for $\neg\varphi$ if $c + k > 1$.

4. Decomposing XSTIT.Prob

To gain insight in the nature of the $[ag\ \texttt{xstit}^{\geq c}]\varphi$ we will decompose it into the operator for XSTIT from Section 2 and an epistemic component. This will enable us, by making a simulation, to show that the axiomatization of Definition 3.7 is complete. But, maybe more important, it will provide us with a new epistemic logic combining both knowledge and levels of belief, and it will give us insight in the nature of not necessarily successful agency. We start by giving the syntax of the epistemic logic used in the simulation.

Definition 4.1. Given a countable set of propositions P and $p \in P$, and given a finite set Ags of agent names, and $ag \in Ags$, the formal language $\mathcal{L}_{ProbEp}$ is:

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle[ag : \geq c]\rangle\varphi.$$

To emphasize it's weak modal character, we denote the new belief operator with a combination of sharp and square brackets. This alludes to the combination of first order existential and universal quantifications that is present in any first order simulation of a weak modal operator. The reading of the operator $\langle[ag : \geq c]\rangle\varphi$ is "c is a lower bound to agent ag's belief in φ". Another, equivalent reading is "according to agent ag, *there is* at least a chance of c on the truth of φ". A reading closely following the technical definition of the semantics is: "ag believes that at least $c \times 100\%$ of the worlds it considers possible as such, satisfy φ".[2] We will capture this intuition in a neighborhood semantics.

Definition 4.2. A ProbEp-frame is a tuple $\langle S, N\rangle$ such that:

1. S is a set of possible worlds.
2. N is a family of neighborhood functions $N : S \times Ags \times C \mapsto 2^{2^S}$ mapping any combination of a state s, an agent ag and a probability c to a set of neighborhoods of s. We have the following constraints:
   a. $\emptyset \notin N(s, ag, c)$ for $c > 0$.
   b. $N(s, ag, 0) = 2^S$.
   c. $\forall N$ if $N \in N(s, ag, 1)$ then $s \in N$.
   d. $N(s, ag, c) \subseteq N(s, ag, k)$ for $c \geq k$.
   e. if $N \in N(s, ag, c)$ and $N' \in N(s, ag, k)$ then $N \cap N' \in N(s, ag, c + k - 1)$ for $c + k > 1$.
   f. if $N \in N(s, ag, c)$ and $N \subset N'$ then $N' \in N(s, ag, c)$.

The intuition underlying the neighborhood functions is the following. $N(s, ag, c)$ gives for agent ag in state s the clusters of possible worlds who's joint possibility it assigns a degree of belief to of at least c. Since clusters and propositions correspond to each other one-to-one (modulo logic equivalence of the propositions), it will also be convenient to look at the clusters or

[2] Note that this reading may be compatible with both the frequentist and of the Bayesian view on probability.

neighborhoods as propositions and to say that if $N \in N(s, ag, c)$ the agent ag assigns a belief of at least c to the truth of the proposition (modulo logical equivalence) corresponding to N. Now **a.** says that there is no non trivial belief in impossible states of affairs. **b.** says that with a lower bound of chance 0 anything is believed (even impossible states of affairs) **c.** says that certainty about states of affairs includes the truth (analogous to the truth axiom in S5 epistemic modal logic) **d.** says that if c is a lower bound in the belief of a proposition, and $c \geq k$, then k is also a lower bound on the belief of that proposition. **e.** says how to combine lower bounds of beliefs in different propositions (since we cannot assume that propositions are independent we do not have multiplication of lower bounds, as will be explained later) **f.** says that belief is closed under weakening of the propositions believed.

Definition 4.3 (*Truth condition belief operator*). Relative to a model $\mathcal{M} = \langle S, N, V\rangle$, truth $\langle s\rangle \models \langle[ag : \geq c]\rangle\varphi$ of a formula $\langle[ag : \geq c]\rangle\varphi$ in a state $\langle s\rangle$ is defined as ($[\![\varphi]\!]$ is the truth set of φ, that is, the subset of all elements in S satisfying φ):

$$s \models \langle[ag : \geq c]\rangle\varphi \Leftrightarrow [\![\varphi]\!] \in N(s, ag, c).$$

An axiomatization of the probabilistic epistemic logic is obtained by formulating axioms corresponding to the conditions on neighborhood functions.

Definition 4.4 (*Hilbert system belief operator*). Relative to the semantics following from Definitions 4.2 and 4.3 we define the following Hilbert system. We assume the standard derivation rules for the weak modality $\langle[ag : \geq c]\rangle\varphi$, like closure under logical equivalence.

(D) $\neg\langle[ag : \geq c]\rangle\bot$ for $c > 0$

(Triv) $\langle[ag : \geq 0]\rangle\varphi$

(Truth) $\langle[ag : \geq 1]\rangle\varphi \rightarrow \varphi$

(MinB) $\langle[ag : \geq c]\rangle\varphi \rightarrow \langle[ag : \geq k]\varphi$ for $c \geq k$

(Add) $\langle[ag : \geq c]\rangle\varphi \wedge \langle[ag : \geq k]\rangle\psi \rightarrow \langle[ag : \geq c + k - 1]\rangle(\varphi \wedge \psi)$
for $c + k > 1$

(Mon) $\langle[ag : \geq c]\rangle(\varphi \wedge \psi) \rightarrow \langle[ag : \geq c]\rangle\varphi$.

The completeness of this axiomatization relative to the frames of Definition 4.2 follows from general results in neighborhood semantics and monotonic modal logic [11]. We can easily check that the 6 conditions on the frames correspond one-to-one with the axioms in the axiomatization. We have to make one precaution though. To keep the language and the axiomatization finitary, we have to restrict ourselves to finite sets of different possible probabilities. So we assume that numbers c and k are taken from a finite predefined set of probabilities between 0 and 1.

At this point it seems good to clarify and explain the axiom (Add) in more detail. The reader might have expected the axiom to be stronger. In particular the following axiom might have been expected:

$$\langle[ag : \geq c]\rangle\varphi \wedge \langle[ag : \geq k]\rangle\psi \rightarrow \langle[ag : \geq c \times k]\rangle(\varphi \wedge \psi)$$

This axiom would hold if the agent would know that the chances for φ and for ψ would be independent. Take the following example. We have a vase with 10 stones. An agent knows that at least 7 are red and at least 4 have a hole. If the agent knows nothing about the relation between red stones and ones with a hole, when taking one out, the chance that it is a red one with a hole is at least $4/10 \times 7/10 = 28/100$. That these are exactly the chances is modulo the assumption that by taking out a stone we have *no control whatsoever* on whether or not the one we take out will be red or not or will have a hole or not. We might say that we lack any practical knowledge (Anscombe [12]) for taking out a red or one with a hole. Now, in general, we cannot assume that chances are independent. In particular, it might be that the stones with a hole are usually not red and the agent might know about this correlation. In that case the estimation of the chance being at least 28/100 is too high. All the agent knows is that there must be at least one stone in the vase that is red and has a hole, that is, the agent knows that the chance is at least 1/10 (which is the result of the calculation $4/10 + 7/10 - 1$). That is the kind of reasoning expressed by the axiom given in the axiomatization. When modeling reasoning we cannot simply assume that chances are not correlated. In particular this is true for the reasoning about choices that are exerted simultaneously by other agents: we often have information about how different possible effects of choices are correlated.

The logic can be made stronger relative to the same syntax by including some well known properties from epistemic modal logic. One of them is positive introspection. And since this logic talks about knowledge and belief within one framework, we can generalize positive introspection to the following property saying that if we have a certain degree of belief in a proposition then we know we have this degree of belief in that property.

(IntroS) $\langle[ag : \geq c]\rangle\varphi \rightarrow \langle[ag : \geq 1]\rangle\langle[ag : \geq c]\rangle\varphi$

We will not explore these strengthenings here, and proceed by defining the combination of the above epistemic logic with XSTIT.

Definition 4.5 (*XSTIT* $\times$ *ProbEp*)**.** We define the fusion of the normal modal logic XSTIT and the weak modal logic ProbEp as the logic *XSTIT* $\times$ *ProbEp* resulting from freely combining the languages of both logics and combining normal frames and neighborhood frames into mixed frames where in every dynamic state $\langle s, h \rangle$ also the neighborhood function $N(s, ag, c)$ is defined. Note that neighborhood functions do not depend on a history.

A complete axiomatization of the fusion is formed by joining the axiomatizations of the XSTIT fragment and the ProbEp fragment. However, to accomplish this we have to add the axiom $\langle [ag : \geq c] \rangle \varphi \leftrightarrow \Box \langle [ag : \geq c] \rangle \varphi$, to account for the fact that the existence of neighborhoods does not depend on the history, as defined above in Definition 4.5. This is the only interaction between the two fragment logics that we allow.

Theorem 4.1 (Simulation)**.** *The simulation determined by the following identifications is sound and complete:*

$$[ag\ \texttt{xstit}^{\geq c}]\varphi \equiv_{sim} [ag\ \texttt{xstit}]\langle [ag : \geq c\,] \rangle \varphi$$

Other logical operators are simulated by their identical twins.

Proof. We have to prove two directions (soundness and completeness of the simulation). Since we have no completeness for the Hilbert system we gave for XSTIT.Prob in Definition 3.7, we will only look at the semantic characterizations of the logics. We will show then that we can make a mapping between models for formulas with the simulated modality $[ag\ \texttt{xstit}^{\geq c}]\varphi$ and models for formulas with the simulating modality $[ag\ \texttt{xstit}]\langle [ag : \geq c\,] \rangle \varphi$ that preserves satisfiability for formulas in both directions. Our mapping between models of XSTIT.Prob to the models of XSTIT x ProbEp is a follows. The structure of static states, histories and valuations of atomic propositions in both models is identical and the mapping relates them. Furthermore, we associate the neighborhood function $N(s, ag, c)$ in the XSTIT x ProbEp models with functions in the XSTIT.Prob models in the following way:

$$N \in N(succ(s, h), ag, c) \text{ in a model of XSTIT} \times \text{ProbEp}$$
$$\text{if and only if}$$
$$N \in \{X \subseteq E(s, h, ag) \mid (\sum_{s' \in X} B(s, ag, s')) \geq c\} \text{ in the related XSTIT.Prob model}$$

This mapping between the models preserves satisfiability in both directions. This can be proven by induction over the structure of formulas. The main condition to check is whether truth conditions for the operators $[ag\ \texttt{xstit}^{\geq c}]\varphi$ and $[ag\ \texttt{xstit}]\langle [ag : \geq c\,] \rangle \varphi$ in the two settings correspond to each other. But because the mapping between the models defines the neighborhood function in the simulation logic entirely in terms of effectivity functions and the beliefs in next static states in the simulated logic, this is guaranteed. $\Box$

From this we cannot conclude that the system of Definition 3.7 is complete relative to the semantics of Definition 3.6. One way to prove completeness of XSTIT.Prob is to show that the system of Definition 3.7 is at least as strong as the combined systems for XSTIT plus ProbEp plus the single interaction axiom, relative to the language fragment of XSTIT x ProbEp corresponding to XSTIT.Prob. But, at this point this proof is still lacking.

It will be interesting to consider an alternative simulation. The simulation of Theorem 4.1 decomposes the operator $[ag\ \texttt{xstit}^{\geq c}]\varphi$ by specifying what an agent believes after it first has exercised the choice whose outcome it knew to be uncertain. We can also turn this around and decompose the operator into a simulation $\langle [ag : \geq c\,] \rangle [ag\ \texttt{xstit}]\varphi$. Now the uncertainty is, as it were, moved to the start of the agent's choice exertion. In the setup we have given here, the two views are close to equivalent. In particular, in the logic XSTIT x ProbEp we might want to have commutativity between the two operators: $[ag\ \texttt{xstit}]\langle [ag : \geq c\,] \rangle \varphi \leftrightarrow \langle [ag : \geq c\,] \rangle [ag\ \texttt{xstit}]\varphi$. Properties like these are closely related to the 'no learning' and 'no forgetting' properties in temporal epistemic logic and to the reduction axioms for the update of knowledge in dynamic epistemic logics [13]. It will be particularly interesting not to have these commutativity properties and study the situations where agents can learn or forget. The investigation of these issues is left to future research.

5. Conclusion

This paper starts out by defining a base *stit* logic, which is a variant on XSTIT. However, we define the semantics in terms of h-effectivity functions, which does more justice to the nature of the structures interpreting the language. We show completeness relative to this semantics. Then we proceed by generalizing the central *stit* operator of the base language to a probabilistic variant. The original operator comes out as the probabilistic operator assigning a chance 1 to success of a choice. Then we proceeded to decompose the defined probabilistic stit operator into its stit and its belief components. The resulting probabilistic belief logic was given a neighborhood semantics and was briefly analyzed. One of the main routes of future investigation will be to study the updating of probabilistic beliefs on the basis of failed actions or information reaching the agent by other means (communication, announcements, etc.).

There are several more opportunities for future work. An interesting issue is the generalization of the theory in this paper to group choices of agents. If a group makes a choice, we may assume all kinds of conditions on the pooling of information within the group. This means that the chances that agents assign to choices made by agents within the group are generally different than the chances they assign to choices by agents outside the group. How this pooling of information takes form in a setting where beliefs are modeled as subjective probabilities is still an open question to us.

References

[1] N. Belnap, M. Perloff, M. Xu, Facing the Future: Agents and Choices in our Indeterminist World, Oxford, 2001.
[2] J. Broersen, A. Herzig, N. Troquard, Embedding alternating-time temporal logic in strategic STIT logic of agency, J. Logic Comput. 16 (5) (2006) 559–578.
[3] J. Broersen, A. Herzig, N. Troquard, A normal simulation of coalition logic and an epistemic extension, in: D. Samet (Ed.), Proceedings Theoretical Aspects Rationality and Knowledge (TARK XI), ACM Digital Library, Brussels, 2007, pp. 92–101.
[4] P. Balbiani, A. Herzig, N. Troquard, Alternative axiomatics and complexity of deliberative stit theories, J. Philos. Logic, http://dx.doi.org/10.1007/s10992-007-9078-7.
[5] W. Jamroga, A temporal logic for markov chains, in: AAMAS'08: Proceedings of the Seventh International Joint Conference on Autonomous Agents and Multiagent Systems, 2008, pp. 697–704.
[6] B.P. Kooi, A.M. Tamminga, Conflicting obligations in multi-agent deontic logic, in: L. Goble, J.-J. Meyer (Eds.), Proceedings of the Eighth International Workshop on Deontic Logic in Computer Science (DEON'06), Lecture Notes in Computer Science, vol. 4048, Springer, 2006, pp. 175–186.
[7] M. Pauly, A modal logic for coalitional power in games, J. Logic Comput. 12 (1) (2002) 149–166.
[8] E. Emerson, Temporal and modal logic, in: J.v. Leeuwen (Ed.), Handbook of Theoretical Computer Science, Formal Models and Semantics, vol. B, Elsevier Science, 1990, pp. 996–1072.
[9] P. Blackburn, M.d. Rijke, Y. Venema, Modal logic, Cambridge University Press, 2001.
[10] W. Conradie, V. Goranko, D. Vakarelov, Algorithmic correspondence and completeness in modal logic I: the core algorithm SQEMA, Log. Methods Comput. Sci. 2 (1) (2006) 1–26.
[11] H.H. Hansen, Monotonic modal logics, Master's Thesis, ILLC, 2003.
[12] G. Anscombe, Intention, second ed., Cornell University Press, Ithaca, NY, 1963.
[13] H.v. Ditmarsch, W.v.d. Hoek, B. Kooi, Dynamic epistemic logic, in: Synthese Library, vol. 337, Springer, New York, NY, USA, 2007.