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CHARACTERIZATION OF THE ROTATION SET AND EXISTENCE OF PERIODIC POINTS OF ENDOMORPHISMS OF A CIRCLE

W. Li

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1. Introduction

Let us begin with some notation. We identify the circle S^1 with $\mathbb{R}/\mathbb{Z}$. We assume that f is a map of the circle into itself, and then the *lift* $\bar{f}$ of f satisfies

$$\bar{f}(x+1) = \bar{f}(x) + n, \qquad n \in \mathbb{Z}, \qquad x \in \mathbb{R}.$$

The integer n is called the *degree of the map* f. Since every continuous map of the circle into itself, whose degree is not equal to one, has a periodic point, we only consider a continuous map of degree one. By End$\,(S)$ we denote the set of the continuous map $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x+1) = f(x) + 1. \tag{1}$$

On End$\,(S)$ we use the usual C^0 topology. Let $f \in \text{End}\,(S)$ be a C^1 map. We denote by

$$\Sigma(f) = \{x : \; Df(x) = 0\}$$

the *critical set* of f. The point $x \in \Sigma(f)$ is called a *fold* if x is isolated in $\Sigma(f)$ and Df changes sign at x. Let B denote the set of C^1 maps f in End$\,(S)$ such that $\Sigma(f)$ consists of a finite number of folds. It is easy to see that B is a dense subset of End$\,(S)$ (for the proof, see [2]). Let $f \in \text{End}\,(S)$. The point $x \in \mathbb{R}$ is called a *periodic point* of f if there exist an integer p and a natural number q such that $f^q(x) = x + p$, where f^q is the q times iteration of f. The rational number p/q is the *rotation number* of x. Block and Franke proved the following:

Theorem A ([1]). *Consider $f \in B$ with Df of bounded variation and $\Sigma(f)$ consisting of a nonzero number of folds. Then f has a periodic point.*

The above result has an important application to the bifurcation of homoclinic orbits of a nonhyperbolic periodic point [3].

In order to give our results, let us introduce some more designations. If $f \in \text{End}\,(S)$, let

$$f_1(x) = \min_{y \geq x} f(y), \qquad f_2(x) = \max_{y \leq x} f(y). \tag{2}$$

Obviously, f_1 and f_2 are monotone continuous functions satisfying Eq. (1) and

$$f_1(x) \leq f(x) \leq f_2(x). \tag{3}$$

Let

$$A(f) = \{x \in \mathbb{R} : \; f_2(x) > f_1(x)\}.$$

Then $A(f)$ is an empty set if and only if f is monotone. The main result of this paper is the following:

Theorem B. *Let $f \in \text{End}\,(S)$ be a C^1 map, whose first derivative is of bounded variation, satisfying the following conditions:*

(1) $A(f) \neq \varnothing$;

(2) $\Sigma(f) \subset A(f)$.

Then f has a periodic point.

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2. The Proof of Theorem B

We introduce the notion of rotation set for the map of a circle of degree one defined by Newhouse et al. in [3]. For $f \in \mathrm{End}\,(S)$, let

$$r(f,x) = \lim_{n\to+\infty} \sup(f^n(x) - x)/n.$$

Definition ([3]). For $f \in \mathrm{End}\,(S)$ the *rotation set* $r(f)$ of f is the closure of the set $\{r(f,x): x \in \mathbb{R}\}$.

Obviously, if $f, g \in \mathrm{End}\,(S)$ are topologically conjugate, then $r(f) = r(g)$. When $f \in \mathrm{End}\,(S)$ is monotone, $r(f)$ is the usual rotation number [4]. We now quote two lemmas given in [3].

Lemma 1 ([3]). *If $f \in \mathrm{End}\,(S)$ has no periodic points with rotation number p/q, $p \in \mathbb{Z}$, $q \in \mathbb{N}$, then $r(f)$ is contained in the set $\{x \in \mathbb{R}: x < p/q\}$ or in $\{x \in \mathbb{R}: x > p/q\}$.*

Proof. Since $f^q(x) \neq x + p$ for any $x \in \mathbb{R}$, either $f^q(x) - x < p$ for all $x \in \mathbb{R}$ or $f^q(x) - x > p$ for all $x \in \mathbb{R}$. Since $f^q(x) - x$ is periodic in x, we find that there is some $c > 0$ such that $f^q(x) - x < p - c$ or $f^q(x) - x > p + c$ for all $x \in \mathbb{R}$. Therefore, $r(f) \subset \{x \in \mathbb{R}: x \leq (p-c)/q\}$ or $r(f) \subset \{x \in \mathbb{R}: x \geq (p+c)/q\}$.

Corollary. *If $f \in \mathrm{End}\,(S)$, $a, b \in r(f)$, and $a \leq p/q \leq b$ for some rational number p/q, then f has a periodic point with rotation number p/q and, hence, $p/q \in r(f)$.*

From the above corollary we find that $r(f)$ must be either a single point in $\mathbb{R}$ or a closed interval. We denote by $r_1(f)$ and $r_2(f)$ the left and right end of $r(f)$, respectively.

Lemma 2 ([3]). *The functions $f \mapsto r_i(f)$, $i = 1, 2$, are continuous.*

Proof. Notice that for any rational number p/q, we find that $p/q < r_1(f)$ is equivalent to $f^q(x) - x > p$ for all $x \in [0,1]$, which is an "openness condition," i.e., the set of $f \in \mathrm{End}\,(S)$ with $r_1(f) > p/q$ is open. Analogously, the set $\{f \in \mathrm{End}\,(S): r_2(f) < p/q\}$ is open. Finally, $p/q \in (r_1(f), r_2(f))$ if and only if, for some large natural number N, there are $x, y \in [0,1]$ with $f^{Nq}(x) - x > Np + 1$ and $f^{Nq}(y) - y < Np - 1$. Also, this condition is an openness condition, and, hence, $r_1(f)$ and $r_2(f)$ depend continuously on f.

Proposition 1. *Let $g \in \mathrm{End}\,(S)$ be monotone and $r(g)$ be irrational. We denote by $\omega(g,x)$ the set of ω-limit points of x and by $\Omega(g)$ the set of nonwandering points. Then*

(i) *$\omega(g,x)$ is a perfect set;*

(ii) *$\omega(g,x) = \Omega(g)$ for any $x \in S$.*

Proof. We shall first prove (i). It is sufficient to prove that there are no isolated points in $\omega(g,x)$ since $\omega(g,x)$ is a nonempty closed invariant set. If not, let $y \in \omega(g,x)$ be isolated. Suppose that the open intervals $I = (u,y)$ and $J = (y,v)$ are the gaps of $\omega(g,x)$. Let $I_n = g^n(I)$ and $J_n = g^n(J)$. Since $r(g)$ is irrational, $I_n(J_n)$ are pairwise disjoint. Therefore, y is a wandering point, which contradicts $y \in \omega(g,x)$.

Analogously, we can prove that any point of the completion of $\omega(g,x)$ for any x is a wandering one. This implies that the set $\Omega(g)$ is contained in $\omega(g,x)$, which leads to the conclusion of (ii).

Proposition 2. *Let $f \geq g \in \mathrm{End}\,(S)$ be monotone and $r(g)$ be irrational. If there exists a point $u \in \Omega(g)$ such that $f(u) > g(u)$, then $r(f) > r(g)$.*

Proof. Choose a point $x_0 \in \Omega(g)$ near u such that $f(x_0) > g(x_0)$ and x_0 is not an end of a gap of the set $\Omega(g)$. By Proposition 1, the orbit of x_0 is dense in $\Omega(g)$. The choice of x_0 guarantees that this orbit accumulates to x_0 from both sides. Therefore, there exist two sequences $p_i \in \mathbb{Z}$, $q_i \in \mathbb{N}$ tending to infinity as $i \to \infty$ such that

$$g^{q_i}(x_0) - x_0 - p_i = h_i \nearrow 0 \quad \text{as} \quad i \to \infty, \tag{4}$$

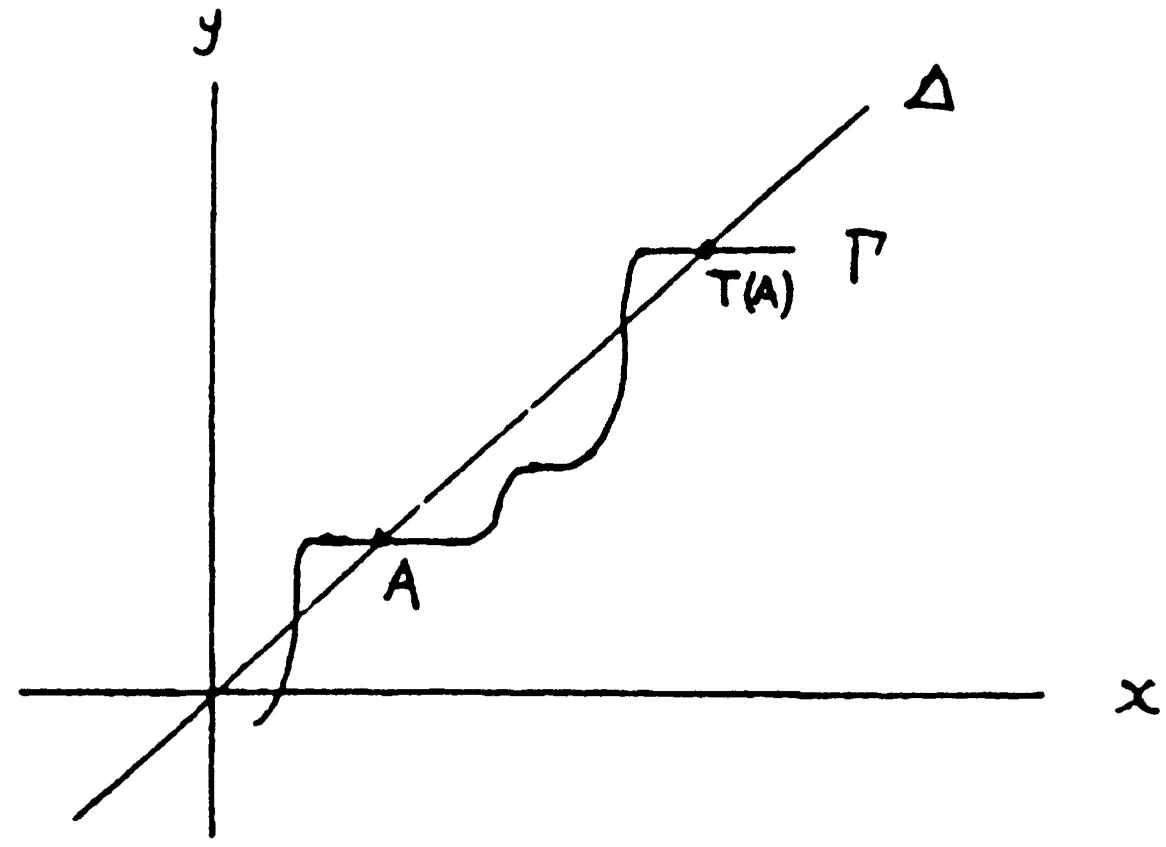

Fig. 1

which, together with the irrationality of $r(g)$, implies

$$p_i/q_i > r(g). \tag{5}$$

On the other hand, setting $y_0 = g(x_0)$, we have

$$f^{q_i}(y_0) \geq f(g^{q_i-1}(y_0)) = f(g^{q_i}(x_0)) = f(x_0 + p_i + h_i) = f(x_0 + h_i) + p_i > g(x_0) + p_i = y_0 + p_i.$$

The last inequality is a consequence of the statement $h_i \to 0$ and holds for $i \gg 1$. This implies $r(f) \geq p_i/q_i > r(g)$.

Proposition 3. *Let* $f \in \mathrm{End}\,(S)$. *Then* $r_i(f) = r(f_i)$, $i = 1, 2$.

Proof. First we will prove the proposition for the map belonging to B. Let $f \in B$. From Eq. (3) we have

$$r(f_1) \leq r_1(f) \leq r_2(f) \leq r(f_2). \tag{6}$$

If $r(f_2) = p/q$ is a rational number, let $C = \{x \in \mathbb{R}\colon f_2(x) > f(x)\}$. We claim that there exists a point $x \in \mathbb{R}$ such that $f_2^q(x) = x + p$ and $x, f_2(x), \ldots, f_2^{q-1}(x)$ do not belong to C.

In fact, suppose that z is a periodic point of f_2 and its orbit intersects C. Then z is an interior point of the set $C_q = \{x \in \mathbb{R}\colon Df_2^q(x) = 0\}$. This can be proved in the following way. Let $y \in \mathrm{Orb}\,(z) \cap C$, $y = f_2^s(z)$, $0 \leq s < q$. Then $Df_2^{s+1} = 0$ in some neighborhood of z. The chain rule implies that the same holds for Df_2^q. Now we will find a periodic point x of f_2 such that x does not belong to $\mathrm{Int}\,(C_q)$, which implies our claim.

Consider the function $f_2^q(x) - p$. Our assumption $r(f_2) = p/q$ implies that its graph Γ has a nonempty intersection with the diagonal $\Delta = \{x = y\}$. The graph Γ has the orientation induced by the natural orientation of the x-axis. It has an infinite series of intersections with the diagonal Δ invariant with respect to the shift T:

$$T:\ (x, y) \mapsto (x + 1, y + 1), \qquad T(\Gamma \cap \Delta) = \Gamma \cap \Delta. \tag{7}$$

If the x-projection of the intersection point $A \in \Gamma \cap \Delta$ is an interior one for the set C_q, then the graph Γ comes from the left half-plane $y > x$ to the right one $y < x$ (see Fig. 1).

If the graph Γ does not return to Δ, then A is the last point of $\Gamma \cap \Delta$, which contradicts Eq. (7). The return point of Γ to Δ from the right half-plane is the point we seek. Its x-projection is the periodic point of f_2, whose orbit does not intersect C.

Let x be a periodic point of f_2, whose orbit does not intersect C. Then we have $f^q(x) = x + p$, which, together with Eq. (6), implies $r_2(f) = r(f_2)$.

If $r(f_2)$ is irrational, let us regard $\mu(c) = r(f_2 + c)$ as a function of c. By Lemma 2, it is continuous. Hence, by Proposition 2, $\mu(0) < \mu(c)$ for $c > 0$. Therefore, we can find a sequence $c_n \to 0$ such that $r(f_2 + c_n)$ are rational. Then we have

$$r_2(f) = \lim_{n\to\infty} r_2(f + c_n) = \lim_{n\to\infty} r(f_2 + c_n) = r(f_2).$$

Analogously, we can prove that $r_1(f) = r(f_1)$.

In general, for $f \in \mathrm{End}\,(S)$, let $f_n \in B$ be a sequence of maps with $f_n \to f$. Then, by Lemma 2, we have

$$r_i(f) = \lim_{n\to\infty} r_i(f_n) = \lim_{n\to\infty} r((f_n)_i) = r(f_i), \qquad i = 1, 2.$$

We now prove Theorem B.

Let $f \in \mathrm{End}\,(S)$ satisfy the conditions of Theorem B. By Lemma 1, the conclusion of Theorem B is equivalent to saying that $r(f)$ cannot consist of only one irrational number. If not, we assume that $r_1(f) = r_2(f)$ is an irrational number. By condition (2), the open set $A(f)$ consists of a finite number of open intervals (mod $\mathbb{Z}$). Hence, we can construct a C^1 diffeomorphism $g \in \mathrm{End}\,(S)$, whose first derivative is of bounded variations, such that $f_1(x) \le g(x) \le f_2(x)$, and there exists a point $x_0 \in A(f)$ satisfying $f_1(x_0) < g(x_0) < f_2(x_0)$. The bounded variation condition for g' is easily satisfied on any interval of $A(f)$. On the completion of $A(f)$, we have $g = f$, and the above condition follows from the assumption of the theorem. Proposition 3, together with our assumption, implies $r(f_1) = r(g) = r(f_2)$. Now $r(g)$ is irrational and g satisfies the condition of the Denjoy theorem. Hence, $\Omega(g) = S$. Now Proposition 2 can be applied to g and f_2. This gives $r(g) < r(f_2)$, a contradiction.

The proof of Theorem B is completed.

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