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EXPRESSIBILITY IN THE ELEMENTARY THEORY OF RECURSIVELY ENUMERABLE SETS WITH REALIZABILITY LOGIC

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Let $\mathcal{E}$ be the lattice of the recursively enumerable (r.e.) subsets of the set N of natural numbers and $\mathcal{A}(\mathcal{E})$ be the Boolean algebra of sets generated by the lattice $\mathcal{E}$. Let $\mathcal{L}$ denote the first-order language in the signature $\langle \varnothing, N; \cup, \cap, ' \rangle$. The elementary theory of r.e. sets with the classical logic has been studied in the literature. In this connection, the theory is sometimes formulated in a signature different from the signature of the language $\mathcal{L}$, but the additional symbols do not extend the class of expressible predicates. We know [7] that important predicates such as hypersimplicity and T-reducibility (and, consequently, all reducibilities) are not expressed in the classical elementary theory of the lattice $\mathcal{E}$. This shows that the language of the set-theoretic signature $\mathcal{L}$ for the classical semantics is ill-adapted for the expression of the predicates of algorithm-theoretic character.

The Gödel numbering $\{\varphi_i\}$ of all partial recursive (p.r.) functions of one variable and the corresponding Post numbering $\{W_i\}$ of all r.e. sets are used for the definition of the elementary theory of the lattice of r.e. sets with realizability logic [6]. We denote the language obtained from $\mathcal{L}$ by the addition of a countable set of constant symbols $\{W_0, W_1, W_2, \dots\}$ by $\mathcal{L}+\mathcal{W}$. It is clear that the terms take values in $\mathcal{A}(\mathcal{E})$ under the interpretation of the new letters as the sets W_i. In all the further statements we will use the letters S, T, S_i , and T_i as metavariables for constant terms of the language $\mathcal{L}+\mathcal{W}$. In this connection, we will often omit the phrase "for all constant terms of the language $\mathcal{L}+\mathcal{W}$." While speaking about the set T, we mean the "standard interpretation" of the term T.

The relation $\mathcal{L}+\mathcal{W}$ ("the natural number e realizes the formula $\mathfrak{A}$ ") for closed formulas of the language $e \circledR \mathfrak{A}$ is defined inductively.

1. $e \circledR (S = T) \Leftrightarrow e = 0$ and the equality of sets S = T is true.

2. The Kleene definition [2] for arithmetical formulas is repeated for propositional connectives.

3. Let $\mathfrak{A}(X)$ be a formula of the extended language with one free variable X . Then

1) $e \circledR \exists X \mathfrak{A}(X) \Leftrightarrow e = 2^a \cdot 3^b \,\&\, a \circledR \mathfrak{A}(W_b)$,

2) $e \circledR \forall X \mathfrak{A}(X) \Leftrightarrow (\forall x)[\varphi_e(x)$ is defined and $\varphi_e(x) \circledR \mathfrak{A}(W_x)\,]$.

The set of all realizable formulas of the language $\mathcal{E}_r$ is called the theory $\mathcal{L}$. A predicate $P(X_1, \dots, X_n)$, defined on $\mathcal{E}$ [on $\mathcal{A}(\mathcal{E})$], is said to be expressible in $\mathcal{E}_r$ if there exists a formula $\mathfrak{A}(X_1, \dots, X_n)$ of the language $\mathcal{L}$ such that

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$$Ⓣ\mathfrak{A}(W_{i_1},\ldots,W_{i_n}) \Leftrightarrow P(W_{i_1},\ldots,W_{i_n}) = t$$

[conforming to $Ⓣ\mathfrak{A}(T_1,\ldots,T_n) \Leftrightarrow P(T_1,\ldots,T_n) = t$].

In the present article we give complete description of the class of the predicates that are expressible in $\mathcal{E}_r$, from which it follows that practically all the predicates of the contensive theory of r.e. sets are expressible. To this end, after recalling certain results from [6], we prove the expressibility of the predicates of the comparison of cardinalities of finite sets $|A| \leq |B|$ and $|A| = |B|$ in Sec. 1. In Sec. 2 we construct an embedding of the arithmetic in $\mathcal{E}_r$, from which it follows that the theory $\mathcal{E}_r$ is undecidable. It is proved in Sec. 3 that the belongingness of a suite of sets to an arbitrary given type of recursive isomorphism is expressible in $\mathcal{E}_r$. Finally, in Sec. 4, by means of the construction of a formula that is universal for the formulas of Sec. 3, we show that each recursively invariant predicate that is arithmetical on the Post indices is expressible in $\mathcal{E}_r$.

By D_u we will denote that a finite set with the canonical index u, $\langle A_1,\ldots,A_n\rangle \equiv \langle B_1,\ldots,B_n\rangle$ means recursive isomorphism of suites of sets, i.e., the existence of a recursive permutation P such that $p(A_1) = B_1,\ldots,p(A_n) = B_n$. In some simple cases, in the formulas of the language $\mathcal{L}+\mathcal{W}$ we write in place of subformulas the predicates expressible by them, e.g., $\mathfrak{X} \neq \mathfrak{Y}$ in place of $\neg(\mathfrak{X} = \mathfrak{Y})$ and $\mathfrak{X} \subseteq \mathfrak{Y}$ in place of $\mathfrak{X} \cap \mathfrak{Y}' = \emptyset$.

1. Finite Sets

At first, we give formulas from [6] for the expression of the predicates "$|A|=1$" and "A is finite." Let us set

$$[|A| = 1] \leftrightharpoons A \neq \emptyset \;\&\; \neg(\exists \mathfrak{Y})[A \cap \mathfrak{Y} \neq \emptyset \;\&\; A \cap \mathfrak{Y}' \neq \emptyset],$$

$$Fin(A) \leftrightharpoons (\forall \mathfrak{Y}_1 \mathfrak{Y}_2)[\mathfrak{Y}_1' = \mathfrak{Y}_2 \supset A \subseteq \mathfrak{Y}_1 \vee \neg(A \subseteq \mathfrak{Y}_1)].$$

The following propositions are valid for these formulas.

Proposition 1.1 [6, Lemma 4.2]. The following statement is valid:

$$Ⓣ[|S| = 1] \Leftrightarrow |S| = 1 \Leftrightarrow 1\,Ⓣ[|S| = 1].$$

Proposition 1.2 [6, Theorem 4.5]. (a) Let S be a constant term of the language $\mathcal{L}+\mathcal{W}$. If the set S is infinite, then $\not{Ⓣ}\, Fin(S)$.

(b) There exists a general recursive (g.r.) function f such that for arbitrary constant term S and natural number u

$$S = D_u \Rightarrow f(u)\,Ⓣ\, Fin(S).$$

(c) There exists a p.r. function g such that for arbitrary constant term S and natural number e

$$e\,Ⓣ\, Fin(S) \Rightarrow D_{g(e)} = S.$$

Proposition 1.1 enables us to model the computations over natural numbers in $\mathcal{E}_r$. For example, it is easily verified [6, Theorem 5.1] that the predicate $A \leq_m B$ is expressed by the formula

$$Red_m(A,B) \leftrightharpoons (\forall \mathfrak{X})[|\mathfrak{X}| = 1 \supset (\exists \mathfrak{Y})[|\mathfrak{Y}| = 1 \;\&\; (\mathfrak{X} \subseteq A \equiv \mathfrak{Y} \subseteq B)]].$$

The "equinformative property" of the canonical index of a finite set A and the realization of the formula $Fin(A)$ enables us to define cannonical indices in $\mathcal{E}_\tau$. For example, the predicate $A \leqslant_c B$ is expressed by the formula

$$Red_c(A,B) = (\forall \mathcal{X})[|\mathcal{X}| = 1 \supset (\exists \mathcal{Y})[Fin(\mathcal{Y}) \& (\mathcal{X} \subseteq A \equiv \mathcal{Y} \subseteq B)]].$$

Let us pass to the predicates of the comparison of cardinalities. Let $[|A| \leqslant |B|]$ denote the formula

$$(\forall \mathcal{X}_1 \mathcal{X}_2)[\mathcal{X}_1 \subseteq \mathcal{X}_2 \subseteq A \supset (\exists \mathcal{Y}_1 \mathcal{Y}_2)[\mathcal{Y}_1 \subseteq \mathcal{Y}_2 \subseteq B \& (\mathcal{X}_2 \setminus \mathcal{X}_1 = \emptyset \equiv \mathcal{Y}_2 \setminus \mathcal{Y}_1 = \emptyset)]].$$

THEOREM 1.1. Let S and T be constant terms of the language $\mathcal{L} + \mathcal{W}$ and the sets S and T be finite. Then

(a) $|S| > |T| \Rightarrow \emptyset [|S| \leqslant |T|]$,

(b) $|S| \leqslant |T| \Rightarrow$. The number realizing the formula $[|S| \leqslant |T|]$ can be effectively found with respect to the canonical indices of the sets S and T.

Proof. (a) Let $S = \{s_1, \dots, s_m\}$ and $|T| = n < m$. We select r.e. sets $U_1 \supset V_1 \supset U_2 \supset V_2 \supset \dots \supset U_m \supset V_m$ such that the set $V = (U_1 \setminus V_1) \cup (U_2 \setminus V_2) \cup \dots \cup (U_m \setminus V_m)$ belongs to the class $\Sigma^{-1}_{2m} \setminus \Pi^{-1}_{2m}$ in Ershov's hierarchy [4].

Defining $s_j \in W_{f_1(i)} \Leftrightarrow i \in V_j$ and $s_j \in W_{f_2(i)} \Leftrightarrow i \in U_j$ for $1 \leqslant j \leqslant m$, with each i $\in$ N we associate the r.e. sets $W_{f_1(i)}, W_{f_2(i)} \subseteq S$.

It is clear that the sets $W_{f_1(i)}$ and $W_{f_2(i)}$ satisfy the condition for $\mathcal{X}_1$ and $\mathcal{X}_2$ in the premise of the implication of the formula $[|S| \leqslant |T|]$ and

$$i \in V \Longleftrightarrow W_{f_2(i)} \setminus W_{f_1(i)} \neq \emptyset.$$

Let us suppose that the formula $[|S| \leqslant |T|]$ is realizable and the functions g_1 and g_2 compute the indices of $\mathcal{X}_1$ and $\mathcal{X}_2$ with respect to the indices of the sets $\mathcal{Y}_1$ and $\mathcal{Y}_2$ in the realization of $[|S| \leqslant |T|]$. Then, substituting $W_{f_1(i)}$ and $W_{f_2(i)}$ for $\mathcal{X}_1$ and $\mathcal{X}_2$, we get

$$i \in V \Leftrightarrow \mathcal{Y}_2^{(i)} \setminus \mathcal{Y}_1^{(i)} \neq \emptyset, \tag{1}$$

where $\mathcal{Y}_k^{(i)}$ means $W_{g_k}(f_1(i), f_2(i))$.

For each p > 0, let us define the r.e. set

$$\begin{aligned} M_p \rightleftharpoons \{i \mid (\exists t_1 \dots t_p)[t_1 < t_2 < \dots < t_p \,\& \\ \& \,\&\{\mathcal{Y}_2^{(i)}(t_q) \setminus \mathcal{Y}_1^{(i)}(t_q) \neq \emptyset \mid q \leqslant p \,\&\, q \text{ is odd}\} \,\& \\ \& \,\&\{\mathcal{Y}_2^{(i)}(t_q) \setminus \mathcal{Y}_1^{(i)}(t_q) = \emptyset \mid q \leqslant p \,\&\, q \text{ is even}\}]\}, \end{aligned}$$

where $\mathcal{Y}_k^{(i)}(t_q)$ is the finite set obtained by carrying out t_q steps in the numbering of the set $\mathcal{Y}_k^{(i)}$. There can be at most n = |T| oscillations of the set $\mathcal{Y}_1^{(i)}, \mathcal{Y}_2^{(i)} \subseteq T$ from empty to nonempty and vice versa in the process of numbering of the sets $\mathcal{Y}_2^{(i)}(t) \setminus \mathcal{Y}_1^{(i)}(t)$. Consequently,

$$\mathcal{Y}_2^{(i)} \setminus \mathcal{Y}_1^{(i)} \neq \emptyset \Leftrightarrow i \in (M_1 \setminus M_2) \cup (M_3 \setminus M_4) \cup \dots \cup (M_{2n-1} \setminus M_{2n}),$$

and, taking (1) into account, we get

$$V=(M_1\setminus M_2)\cup(M_3\setminus M_4)\cup\ldots\cup(M_{2n-1}\setminus M_{2n}).$$

But this means that $V\in\Sigma^{-1}_{2n}$, in contradiction with our choice of the sets U_j and V_j.

(b) Let h be a p.r. function that is a one-to-one mapping of the set S onto a certain subset of T and is not defined outside S. Setting $\mathcal{Y}_\kappa = h(\mathcal{X}_\kappa)$, we get the realization of the formula $[|S|\leq|T|]$ (the realization of the part, contained within the square brackets, of the conclusion of the implication is trivial).

The theorem is proved.

Let $[|A|=|B|]$ denote the formula $[|A|\leq|B|]\,\&\,[|B|\leq|A|]$. It is clear that this formula expresses the predicate $|A|=|B|$. In the sequel, we will often write $[|A|=1]$, $[|A|\leq|B|]$, and $[|A|=|B|]$ without square brackets.

2. Embedding of the Arithmetic in the Theory $\mathcal{E}_\tau$

Let $\mathcal{Y}$ be the language of the arithmetic of first order in the signature $\langle 0,{}',+,\cdot\rangle$. The following theorem shows that the classical arithmetic can be interpreted in the theory $\mathcal{E}_\tau$ in the form of the arithmetic of the cardinalities of finite r.e. sets.

THEOREM 2.1. For each formula $\mathcal{A}(x_1,\ldots,x_n)$ of the language $\mathcal{Y}$ there can effectively be found a formula $\mathcal{L}[\mathcal{A}(x_1,\ldots,x_n)](\mathcal{X}_1,\ldots,\mathcal{X}_n)$ of the language $\mathcal{L}$ such that $\textcircled{\tau}\mathcal{L}[\mathcal{A}(x_1,\ldots,x_n)](T_1,\ldots,T_n)\Leftrightarrow$ the sets $T_1,\ldots,T_n$ are finite and $\mathcal{A}(|T_1|,\ldots,|T_n|)=t$.

Proof. The proof of this theorem is reduced to the proof of Lemma 2.1. We do not give here the standard procedure for translation of the formulas of the language $\mathcal{Y}$ into the language $\mathcal{L}$.

LEMMA 2.1. The predicates $|A|+|B|=|C|$ and $|A|\cdot|B|=|C|$ (for finite sets) are expressible in the theory $\mathcal{E}_\tau$.

Proof. The first predicate is obviously expressed by the formula

$$Fin(A)\,\&\,Fin(B)\,\&\,Fin(C)\,\&\,(\exists Z_1Z_2)[Z_1\cap Z_2=\emptyset\,\&\,Z_1\cup Z_2=C\ \&\ |Z_1|=|A|\ \&\ |Z_2|=|B|].$$

For the second predicate, at first we set

$$\mathcal{L}(A,B,C)\leftrightharpoons Fin(A)\,\&\,Fin(B)\,\&\,Fin(C)\,\&(\forall\mathcal{X}_1\mathcal{X}_2)[\mathcal{X}_1\subseteq\mathcal{X}_2\subseteq A\supset(\exists\mathcal{Y}_1\mathcal{Y}_2)[\mathcal{Y}_1\subseteq\mathcal{Y}_2\subseteq C\,\&$$
$$\&(\mathcal{X}_2\setminus\mathcal{X}_1=\emptyset\supset\mathcal{Y}_2\setminus\mathcal{Y}_1=\emptyset)\&(\mathcal{X}_2\setminus\mathcal{X}_1\neq\emptyset\supset|\mathcal{Y}_2\setminus\mathcal{Y}_1|\geq|B|)]].$$

In the same manner as in the proof of Theorem 1.1, we can show that the formula $\mathcal{L}$ expresses the predicate $|A|\cdot|B|\leq|C|$. In this connection, the sets M_p in the part (a) of the proof should be introduced in the form

$$\begin{aligned}M_p\leftrightharpoons\{i\mid(\exists t_1\ldots t_p)[\,&t_1<t_2<\ldots<t_p\ \&\\ \&\ &\&\{|\mathcal{Y}_2^{(i)}(t_q)\setminus\mathcal{Y}_1^{(i)}(t_q)|\geq|B|\mid q\leq p\ \&\ q \text{ is odd}\}\ \&\\ \&\ &\&\{\mathcal{Y}_2^{(i)}(t_q)\setminus\mathcal{Y}_1^{(i)}(t_q)=\emptyset\mid q\leq p\ \&\ q\text{ is even}\}]\},\end{aligned}$$

and in the part (b), with each element of the set A we should associate a block of $|B|$ elements of the set C. Setting

$$[|A|\cdot|B|=|C|]\leftrightharpoons\mathcal{L}(A,B,C)\,\&\,(\forall Z)[|Z|<|C|\supset\neg\mathcal{L}(A,B,Z)],$$

we regard the proof of the lemma to be complete.

COROLLARY 2.1. The theory $\mathcal{E}_\tau$ is undecidable.

It is clear from Theorem 2.1 that the set of the realizable formulas of the language $\mathcal{L}$ is not even arithmetical.

Remark. In the sequel, we will suppose that all the formulas of $\mathcal{L}[A]$ start with $\neg\neg$ and, consequently, have noninformative realization.

3. Definability of Types of Recursive Isomorphism

By Theorem 4.1 of [6], each predicate that is expressible in $\mathcal{E}_\tau$ is recursive isomorphism has proper subsets that are definable in $\mathcal{E}_\tau$. In this section we will show that each type is definable in $\mathcal{E}_\tau$ and, thus, the types of recursive isomorphism are the smallest classes of the elements of $\mathcal{A}(\mathcal{E})$ that are definable in $\mathcal{E}_\tau$. At first, let us consider the relations $\lambda\mathfrak{X}.\, B \leqslant_1 \mathfrak{X}$ and $\lambda\mathfrak{X}.B \geqslant_1 \mathfrak{X}$, where B is an arithmetical set.

Proposition 3.1. The formulas $B\geqslant_1(\mathfrak{X})$ and $B\leqslant_1(\mathfrak{X})$ of the language $\mathcal{L}$ can be constructed for each arithmetical set B such that (a) $\textcircled{\tau} B \geqslant_1 (T) \Leftrightarrow B \geqslant_1 T$ and (b) $\textcircled{\tau} B \leqslant_1 (T) \Leftrightarrow B \leqslant_1 T$.

Proof. Let B be an arithmetical set. This means that $\mathcal{B}(x) = t \Leftrightarrow x \in B$ for a certain formula $\mathcal{B}(x)$ of the language $\mathcal{Y}$.

Let $p_0, p_1, p_2, \ldots$ be the sequence of all the prime numbers in increasing order. Each nonzero natural number z can be represented uniquely in the form

$$z = p_0^{z_0} \cdot p_1^{z_1} \cdot p_2^{z_2} \cdot \ldots \,. \tag{2}$$

Set

$$(z_i) = \begin{cases} z_i \text{ from (2), if } & z \neq 0; \\ 0, \text{ if } & z = 0. \end{cases}$$

Let C(z) denote the predicate

$$z \neq 0 \;\&\; (\forall i j)[\, i < j \leqslant z \;\&\; (z)_i > 0 \;\&\; (z)_j > 0 \supset (z)_i \neq (z)_j \,].$$

Since predicate C(z) is obviously recursive, it is arithmetical [3, Theorem 14-VIII], and, by Theorem 2.1, there exists a formula $\mathcal{C}(Z)$ of the language $\mathcal{L}$ such that:

$\textcircled{\tau}\ \mathcal{C}(T) \Longleftrightarrow$ the set T is finite and $C(|T|) = t$.

Further, let $\nu_B(z)$ be the number of those i for which $(z)_i \neq 0 \;\&\; (z)_i \dot{-} 1 \in B$, and N(u, z) be the relation $\lambda u z.\, u = \nu_B(z)$, where the function $\nu_B(z)$ is recursive with respect to B, and, consequently, the relation N(u, z) is recursive with respect to B. Therefore, by Theorem 14-VIII of [3], the relation N belongs to Σ_1^B and is expressible by a formula of the language $\mathcal{Y}$ with the additional symbol B(x) for the relation "$x \in B$". But each occurrence of this symbol can be replaced by the formula $\mathcal{B}(x)$ of the language $\mathcal{Y}$. Thus, N(u, z) is an arithmetical predicate. Now, using the operator $\mathcal{L}$ from Theorem 2.1, we can obtain the formula $[|U| = \nu_B(|Z|)]$, which expresses the predicate "the sets U and Z are finite and $|U| = \nu_B(|Z|)$" in $\mathcal{E}_\tau$. We can construct an analogous formula for B' in place of B. Let us set

$$B \geqslant_1 (\mathfrak{X}) \leftrightharpoons (\forall Y)[Fin(Y) \supset (\exists Z)[Fin(Z) \;\&\; \mathcal{C}(Z) \;\&\; |Y \cap \mathfrak{X}| = \nu_B(|Z|) \;\&\; |Y \cap \mathfrak{X}'| = \nu_{B'}(|Z|)]]\,.$$

The last three terms of the conjunction have the following meaning: The cardinality of the set $\mathcal{Z}$ codes the decomposition of $|\mathcal{Y}|$ different [by virtue of $\mathcal{L}(\mathcal{Z})$] natural numbers into prime factors in their nonzero powers, out of which $|\mathcal{Y}\cap\mathcal{X}|$ numbers belong to the set B and $|\mathcal{Y}\cap\mathcal{X}'|$ numbers belong to the set B'. The role of $|\mathcal{Z}|$ in the formula $B\leqslant_1(\mathcal{X})$ is as follows:

$$B\leqslant_1(\mathcal{X})\leftrightharpoons(\forall\mathcal{Z})[Fin(\mathcal{Z})\,\&\,\mathcal{L}(\mathcal{Z})\supset(\exists\mathcal{Y})[Fin(\mathcal{Y})\,\&\,|\mathcal{Y}\cap\mathcal{X}|=\nu_B(|\mathcal{Z}|)\,\&\,|\mathcal{Y}\cap\mathcal{X}'|=\nu_{B'}(|\mathcal{Z}|)]].$$

Let us prove that (a) is fulfilled.

$\Leftarrow$. Let a g.r. function g 1-reduce $T\in\mathcal{A}(\mathcal{E})$ to B. We show how to compute Z with respect to $\mathcal{Y}$.

With respect to the realization of the formula $Fin(\mathcal{Y})$ we can find a list of the elements of the set $\mathcal{Y}$ (Proposition 1.2). Let $\mathcal{Y}=\{y_1,\ldots,y_m\}(m\geqslant 0)$. Let us set

$$\begin{aligned}&z=p_0^{g(y_1)+1}\cdot p_1^{g(y_2)+1}\cdot\ldots\cdot p_{m-1}^{g(y_m)+1}\cdot p_m^0\cdot p_{m+1}^0\cdot\ldots,\\&\mathcal{Z}=\{0,\ldots,z-1\},\qquad |\mathcal{Z}|=z.\end{aligned}$$

It is clear that the list of the elements of the set $\mathcal{Z}$ and the realization of the formula $Fin(\mathcal{Z})$ are computed effectively.

Let us consider the last three terms of the conjunction. By the definition of 1-reducibility, the numbers $g(y_i)$ are pairwise different. Consequently, $C(|\mathcal{Z}|)=t$. Further, we have

$$y_i\in T\Leftrightarrow g(y_i)\in B,\quad i=1,\ldots,m.$$

Consequently,

$$|\mathcal{Y}\cap T|=|\{i\,|\,g(y_i)\in B\}|=|\{i\,|\,(|\mathcal{Z}|)_i\neq 0\,\&\,(|\mathcal{Z}|)_i-1\in B\}|=\nu_B(|\mathcal{Z}|).$$

In the same manner, $|\mathcal{Y}\cap T'|=\nu_{B'}(|\mathcal{Z}|)$. Therefore, the three formulas under consideration are realizable. It is still required to show that their realization can be found effectively with respect to the index of $\mathcal{Y}$ and the realization of $Fin(\mathcal{Y})$. But, by the remark at the end of Sec. 2, we can assume that all of them are realized by the number 0.

$\Rightarrow$. We show how to find a g.r. function g with respect to the realization of the formula $B\geqslant_1(T)$ that 1-reduces T to B. For one-to-one property, let us define g(x) by induction on the argument.

1. In order to compute g(0), we take $\mathcal{Y}=\{0\}$ and find the realization of the formula $Fin(\mathcal{Y})$ with respect to the realization of $Fin(\mathcal{Z})$. We obtain a list of the elements of $\mathcal{Z}$ and $|\mathcal{Z}|$ with respect to the former realization. Since

$$\nu_B(|\mathcal{Z}|)+\nu_{B'}(|\mathcal{Z}|)=|\mathcal{Y}\cap T|+|\mathcal{Y}\cap T'|=|\mathcal{Y}|,\tag{3}$$

it follows that the number $|\mathcal{Z}|$ has exactly one nonzero component $(|\mathcal{Z}|)_i$ and $(|\mathcal{Z}|)_i-1\in B\Leftrightarrow\mathcal{Y}\subseteq T\Leftrightarrow 0\in T$ for this component.

Let us set

$$g(0)=(|\mathcal{Z}|)_{(\mu i)[(|\mathcal{Z}|)_i>0]}-1.$$

It is clear from above that the value of $g(0)$ is defined and is computed effectively.

2. Let $g(0),\dots,g(n)$, be computed such that 1) these numbers are pairwise different and 2) $g(j)\in B \Leftrightarrow j\in T$ for $j\leq n$.

To prove this statement, it is sufficient to show that we can effectively compute a value of $g(n+1)$ such that the conditions 1) and 2) are fulfilled with $n+1$ in place of n.

To this end, we successively compute the finite sets $\mathcal{Y}^{\kappa}$, Z^{κ}, and $V^{\kappa}(\kappa\geq 0)$ by the following algorithm:

$$\mathcal{Y}^{\circ}=\{n+1\};$$

Z^k is the set Z obtained from $B\geqslant_1(T)$ by the realization of the formula $\mathcal{Y}=\mathcal{Y}^{\kappa}$;

$V^{\kappa}\rightleftharpoons\{v\mid(\exists\ell)[v=(|Z^{\kappa}|)_{\ell}-1]\}$, i.e., V^k is the set of those numbers which are encoded in the cardinality of the set Z^{κ};

if $V^{\kappa}\subseteq\{g(0),\dots,g(n)\}$, then $\mathcal{Y}^{\kappa+1}=g^{-1}(V^{\kappa})\cup\{n+1\}$;

if $V^{\kappa}\not\subseteq\{g(0),\dots,g(n)\}$, then the process of computation of sets ends and $g(n+1)=(\mu v)\cdot[v\in V^{\kappa}\setminus\{g(0),\dots,g(n)\}]$.

It is clear that the above-described process is effective. Further, we have

$$|\mathcal{Y}^{\circ}|=1;$$
$$|V^{\kappa}|=|\mathcal{Y}^{\kappa}| \quad (\text{by } (3));$$
$$|\mathcal{Y}^{\kappa+1}|=|V^{\kappa}|+1=|\mathcal{Y}^{\kappa}|+1.$$

This means that $|V^{\kappa}|=\kappa+1$ and a "new" value of the function g appears not later than in the set V^{n+1}. By the same token, it is proved that g is a g.r. function.

The one-to-one property of the function g is ensured since the algorithm is directed to the search of a "new" value for g. Finally,

$$n+1\in T\Leftrightarrow\mathcal{Y}^{\circ}\subseteq T\Leftrightarrow V^{\circ}\subseteq B\Leftrightarrow\mathcal{Y}^{1}=g^{-1}(V^{\circ})\cup\{n+1\}\subseteq T\Leftrightarrow V^{1}\subseteq B\Leftrightarrow\dots\Leftrightarrow V^{\kappa}\subseteq B\Leftrightarrow g(n+1)\in B.$$

The equivalences on the passages from $\mathcal{Y}^{\kappa}$ to V^{κ} follow here from the equality $|\mathcal{Y}\cap T|=\nu_B(|Z|)$ in the formula $B\geqslant_1(T)$, and that on the passages from V^{κ} to $\mathcal{Y}^{\kappa+1}$ follow from the condition 2 in the induction hypothesis.

The statement (a) of the proposition is proved, and the statement (b) is proved in the same manner.

THEOREM 3.1. Let d be a type of recursive isomorphism that contains an arithmetical set. Then the predicate $\lambda\mathcal{X}.\mathcal{X}\in d$ is expressible in $\mathcal{E}_{\tau}$.

Proof. Let B be an arithmetical set of the type d. Let us set $\vartheta_d(\mathcal{X})\rightleftharpoons B\geqslant_1(\mathcal{X})\ \&\ B\leqslant_1(\mathcal{X})$.

Then for each constant term T we have

$$\textcircled{\tau}\,\vartheta_{\underline{d}}(T)\Leftrightarrow\textcircled{\tau}\,B\geqslant_1(T)\ \&\ \textcircled{\tau}\,B\leqslant_1(T)\Leftrightarrow B\geqslant_1 T\ \&\ B\leqslant_1 T\Leftrightarrow B\equiv T\Leftrightarrow T\in\underline{d},$$

where the second passage is valid by virtue of Proposition 3.1 and the third passage is valid by virtue of Myhill's theorem on the coincidence of 1-equivalence and recursive isomorphism [3, Theorem 7-VI].

The theorem is proved.

COROLLARY 3.1. For each set $A \in \mathcal{A}(\mathcal{E})$, the predicate $\lambda \mathfrak{X}. \mathfrak{X} \equiv A$ is expressible in the theory $\mathcal{E}_{\tau}$.

This corollary is valid, since the sets in $\mathcal{A}(\mathcal{E})$ are arithmetical.

Theorem 3.1 is also valid for the types of recursive isomorphism of n-tuples of sets. In this case, the analogue of Proposition 3.1 is formulated in terms of multiple 1-reducibility [1, Sec. 1, Chap. 3]; in this proof, we must write the 2n terms

$$|\mathfrak{Y} \cap \mathfrak{X}| = \nu_B(|\mathfrak{Z}|) \text{ and } |\mathfrak{Y} \cap \mathfrak{X}'| = \nu_{B'}(|\mathfrak{Z}|)$$

in the place of the two conjunctive terms $B \geqslant_1 (\mathfrak{X})$ and $B \leqslant_1 (\mathfrak{X})$:

$$|\mathfrak{Y} \cap \mathfrak{X}_i| = \nu_{B_i}(|\mathfrak{Z}|), \ |\mathfrak{Y} \cap \mathfrak{X}_i'| = \nu_{B_i'}(|\mathfrak{Z}|), \ i = 1, \ldots, n.$$

In the proof of Theorem 3.1, in place of Myhill's theorem we can use a generalization of it [1, Chap. 3, Sec. 1, Corollary 2].

4. Description of Expressible Predicates

For the description of the predicates that are defined on $\mathcal{E}_{\tau}$ and are expressible in $\mathcal{E}$, we establish at first the existence of formulas that are universal over the formulas of Theorem 3.1 with respect to the Post numbering.

Proposition 4.1. For each $n \geqslant 1$ there exists a formula $R\mathfrak{I}s^W(V_1, \ldots, V_n, \mathfrak{X}_1, \ldots, \mathfrak{X}_n)$ of the language $\mathcal{L}$ such that $\circledR R\mathfrak{I}s^W(S_1, \ldots, S_n, T_1, \ldots, T_n) \Leftrightarrow$ the sets $S_1, \ldots, S_n$ are finite and $\langle T_1, \ldots, T_n \rangle \equiv \langle W_{|S_1|}, \ldots, W_{|S_n|} \rangle$.

Proof. We prove the proposition for n = 1. The method of passing to n > 1 is indicated at the end of Sec. 3.

Let $N(u, z, v)$ be the predicate $\lambda u z v. u = \nu_{W_v}(z)$, where $\nu_B(z)$ is the function from the proof of Proposition 3.1. In the same manner as the arguments given there, we can show that $N(u, z, v)$ is an arithmetical predicate. Then, by Theorem 2.1, we can construct the formula $[|U| = \nu_{W_{|V|}}(|\mathfrak{Z}|)]$ with the free variables U, Z, and V, which expresses the predicate $(\lambda U \mathfrak{Z} V)[U, \mathfrak{Z}, V$ are finite and $|U| = \nu_{W_{|V|}}(|\mathfrak{Z}|)]$.

We can construct such a formula for $W'_{|V|}$ also in place of $W_{|V|}$. Let us set

$$WRed_1'(V, \mathfrak{X}) \rightleftharpoons (\forall \mathfrak{Y})[Fin(\mathfrak{Y}) \supset (\exists \mathfrak{Z})[Fin(\mathfrak{Z}) \,\&\, \mathcal{L}(\mathfrak{Z}) \,\&$$
$$\&\, [|\mathfrak{Y} \cap \mathfrak{X}| = \nu_{W_{|V|}}(|\mathfrak{Z}|)] \,\&\, [|\mathfrak{Y} \cap \mathfrak{X}'| = \nu_{W'_{|V|}}(|\mathfrak{Z}|)]]],$$
$$WRed_1''(V, \mathfrak{X}) \rightleftharpoons (\forall \mathfrak{Z})[Fin(\mathfrak{Z}) \,\&\, \mathcal{L}(\mathfrak{Z}) \supset (\exists \mathfrak{Y})[Fin(\mathfrak{Y}) \,\&$$
$$\&\, [|\mathfrak{Y} \cap \mathfrak{X}| = \nu_{W_{|V|}}(|\mathfrak{Z}|)] \,\&\, [|\mathfrak{Y} \cap \mathfrak{X}'| = \nu_{W'_{|V|}}(|\mathfrak{Z}|)]]],$$
$$R\mathfrak{I}s^W(V, \mathfrak{X}) \rightleftharpoons WRed_1'(V, \mathfrak{X}) \,\&\, WRed_1''(V, \mathfrak{X}).$$

The proof the equivalence in the statement of the proposition is a repetition of the proofs of Proposition 3.1 and Theorem 3.1.

We call a predicate $P(\mathfrak{X}_1, \ldots, \mathfrak{X}_n)$ defined on $\mathcal{E}$ arithmetical if the set $\{\langle i_1, \ldots, i_n \rangle \mid P(W_{i_1}, \ldots, W_{i_n}) = t\}$ is arithmetical.

THEOREM 4.1. The following statements are equivalent for each predicate $P(\mathfrak{X}_1, \ldots, \mathfrak{X}_n)$ defined on $\mathcal{E}$: (a) P is expressible in $\mathcal{E}_{\tau}$ and (b) P is a recursively invariant arithmetical predicate.

Proof. (a) $\Leftrightarrow$ (b). The necessity of recursive invariance is shown in [6, Theorem 4.1]. Let us show that P is arithmetical. Let the formula $\mathfrak{A}(x_1,\dots,x_n)$ express the predicate $P(x_1,\dots,x_n)$. By the definition of expressibility, we have

$$\{\langle i_1,\dots,i_n\rangle \mid P(W_{i_1},\dots,W_{i_n})=t\}=\{\langle i_1,\dots,i_n\rangle \mid ⓡ\,\mathfrak{A}(W_{i_1},\dots,W_{i_n})\}.$$

But the definition of realizability for a given formula $\mathfrak{A}$ can be written in a language with the elementary formulas $x=0$, $x=2^y\cdot 3^z$, $\varphi_x(y)=z$ and $y\in W_x$. Since these four relations are arithmetical, the condition of realizability can be written in the form of a formula of the language $\mathcal{G}$ with the free variables $i_1,\dots,i_n$.

(b) $\Rightarrow$ (a). Let $P(x_1,\dots,x_n)$ be a recursively invariant arithmetical predicate. Since it is arithmetical, there exists a formula $\mathcal{P}^W(v_1,\dots,v_n)$ of the language $\mathcal{G}$ such that

$$\mathcal{P}^W(v_1,\dots,v_n)=t \Leftrightarrow P(W_{v_1},\dots,W_{v_n})=t$$

for arbitrary $v_1,\dots,v_n\in N$.

Applying the operator $\mathcal{P}^W$ of Theorem 2.1 to the formula $\mathcal{L}$, we get the formula $\mathcal{L}[\mathcal{P}^W](V_1,\dots,V_n)$, for which

$$ⓡ\,\mathcal{L}[\mathcal{P}^W](S_1,\dots,S_n)\Leftrightarrow S_1,\dots,S_n \text{ are finite and}$$
$$\mathcal{P}^W(|S_1|,\dots,|S_n|)=t\Leftrightarrow S_1,\dots,S_n \text{ are finite and } P(W_{|S_1|},\dots,W_{|S_n|})=t.$$

Let us set

$$\mathfrak{A}_p(x_1,\dots,x_n)\rightleftharpoons(\exists V_1\dots V_n)[Fin(V_1)\,\&\dots\&\,Fin(V_n)\,\&$$
$$\&\,\mathcal{L}[\mathcal{P}^W](V_1,\dots,V_n)\,\&\,RJS^W(V_1,\dots,V_n,x_1,\dots,x_n)].$$

For arbitrary $i_1,\dots,i_n$ we have

$$ⓡ\,\mathfrak{A}_p(W_{i_1},\dots,W_{i_n})\Leftrightarrow(\exists j_1\dots j_n)[ⓡ\,Fin(W_{j_1})\,\&\dots\&\,ⓡ\,Fin(W_{j_n})\,\&$$
$$\&\,ⓡ\,\mathcal{L}[\mathcal{P}^W](W_{j_1},\dots,W_{j_n})\,\&\,ⓡ\,RJS^W(W_{j_1},\dots,W_{j_n},W_{i_1},\dots,W_{i_n})]\Leftrightarrow$$
$$\Leftrightarrow(\exists j_1\dots j_n)[W_{j_1},\dots,W_{j_n} \text{ are finite } \& \,P(W_{|W_{j_1}|},\dots,W_{|W_{j_n}|})=t\,\&$$
$$\&\,\langle W_{|W_{j_1}|},\dots,W_{|W_{j_n}|}\rangle\equiv\langle W_{i_1},\dots,W_{i_n}\rangle]\Leftrightarrow(\exists v_1\dots v_n)[P(W_{v_1},\dots,W_{v_n})=t\,\&$$
$$\&\,\langle W_{v_1},\dots,W_{v_n}\rangle\equiv\langle W_{i_1},\dots,W_{i_n}\rangle]\Leftrightarrow P(W_{i_1},\dots,W_{i_n})=t.$$

The last passage in this chain is valid as a consequence of the recursive invariance of the predicate P.

The theorem is proved.

Using the effective numbering of all the closed terms of the language $\mathcal{L}+\mathcal{W}$, we can prove an analogous theorem for predicates defined on $\mathcal{A}(\mathcal{E})$.

In conclusion, let us observe that the expressibility, described in Theorem 4.1, of all the predicates that are inexpressible "with respect to trivial motives" is not explained only by the introduction of constructive semantics. It has been proved in [5] that in the analogous theory of the Boolean algebra of recursive sets with numbering by characteristic indices even the predicate $|A| = |B|$ is inexpressible. Further examples show that the properties of a theory with realizability logic depend basically on the numbering. In the limiting case of a strongly constructive numbered model, the constructive theory coincides with the classical one.

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ADMISSIBLE RULES FOR PRETABLE MODAL LOGICS

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Recently, the study of admissible rules of inference has been receiving more attention. This interest has been due to extending the possibilities for constructing deductions. The most actively studied admissible rules are those for the intuitionistic logic. Mints [7, 8] and Tsitkin [11, 12] obtained many interesting results in this connection.

Of equal interest is the study of admissible rules for modal logics, both from syntactical reasons and as a result of the close connections between modal logics and extensions of the intuitionistic logic. One of the most important problems in this connection is that of the solubility of the admissibility of rules of inference (this is known for the intuitionistic logic as the Kuznetsov–Friedman problem [14]). It is equivalent to the problem of the solubility of the quasiequational theory of the free algebra of the variety corresponding to this logic.

In this article we consider the problem of the solubility of the admissibility of rules, applied to "strong" modal logics: pretable extensions of S4. It is also undoubtedly interesting to study this problem applied to "weak" logics (especially S4).

In Sec. 1 we give the preliminary information. In particular, we note here that the admissibility problem is soluble for all table logics and for one pretable logic: $PM4 = \sigma(LC)$.

In Sec. 2 we study the structure of the free algebras of the varieties corresponding to the remaining pretable logics.

In Sec. 3, starting with the facts obtained in Sec. 2, we prove that the universal theories of the free algebras of the varieties corresponding to these pretable logics are soluble. It follows from this that the quasiequational theories of the free algebras of all pretable logics are soluble. This, in accordance with the above-mentioned equivalence, shows that the problem of the admissibility of rules is soluble for all pretable logics.

In Sec. 4 we study the bases of quasiidentities of the free algebras of pretable varieties. We prove that the free algebras of varieties corresponding to the three pretable logics have finite bases of quasiidentities. We establish that in the remaining cases the free algebras have no bases of quasiidentities with a finite number of variables.

In Sec. 5 we carry these results over to superintuitionistic logics.

The definitions and notation concerning modal and superintuitionistic logics and their semantics are taken from [4, 10]. The definitions, notation, and results for the general theory of algebraic systems are contained in [5].

1. Preliminary Facts

Let λ be a logic and $\varphi_1(p_1,\ldots,p_n),\ldots,\varphi_m(p_1,\ldots,p_n)$, $\varphi(p_1,\ldots,p_n)$ its formulas. We call the rule $\varphi_1,\ldots,\varphi_m/\varphi$ admissible in λ, if for any $\psi_1,\ldots,\psi_n$, it follows from $\forall j(\varphi_j(\psi_1,\ldots,\psi_n)\in\lambda)$ that

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