THEORY OF THE ELASTIC STABILITY OF COMPRESSIBLE AND INCOMPRESSIBLE COMPOSITE MEDIA

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The present article considers general problems of the theory of the elastic stability of composite media in the presence of finite and small precritical deformations with an arbitrary elastic-potential form. Our investigation was conducted for homogeneous and piecewise-homogeneous anisotropic media. Numerical results were obtained for laminar media. We have elucidated the case in which there is internal loss of stability (in the material structure) for compressible materials with small deformations (plane problem) and for incompressible materials with highly elastic deformations (plane and three-dimensional problems) for the Treloar and Mooney potentials.

In investigating problems of the elastic stability of composite media, as in any other problem of the mechanics of composite media, there are two possible approaches. In the first approach, the medium is considered to be homogeneous and anisotropic, having given characteristics; in the second approach, the medium is treated as being piecewise-homogeneous.

The present article considers general problems of the theory of the elastic stability of compressible and incompressible elastic media with both the first and second approaches, proceeding from three-dimensional linearized equations within the framework of the theory of finite and small precritical deformations. Our investigation is conducted on Lagrangian coordinates, which coincide with the rectangular coordinates before deformation; the quantities characterizing the deformation are the constants ε_{ij} of the Green deformation tensor, with differentiation over the coordinates being indicated by the subscripts after the symbol; summation from 1 to 3 is conducted over repeated subscripts. The perturbations are not indicated by subscripts.

It should be noted that the generalized stresses σ^*_{ij} in the theory of finite deformations [1] and the stresses σ_{ij}[2] in the theory of small deformations are defined in terms of the elastic potential Φ in the following manner:

for compressible materials

$$\sigma^*_{ij}=\frac{1}{2}\left(\frac{\partial}{\partial\varepsilon_{ij}}+\frac{\partial}{\partial\varepsilon_{ji}}\right)\Phi; \quad \sigma_{ij}=\frac{1}{2}\left(\frac{\partial}{\partial\varepsilon_{ij}}+\frac{\partial}{\partial\varepsilon_{ji}}\right)\Phi; \tag{0.1}$$

for incompressible materials

$$\sigma^*_{ij}=\frac{1}{2}\left(\frac{\partial}{\partial\varepsilon_{ij}}+\frac{\partial}{\partial\varepsilon_{ji}}\right)\Phi+pG^{ij}; \quad \sigma_{ij}=\frac{1}{2}\left(\frac{\partial}{\partial\varepsilon_{ij}}+\frac{\partial}{\partial\varepsilon_{ji}}\right)\Phi+p\delta_{ij}. \tag{0.2}$$

Here G^{ij} are the contravariant components of the metric tensor in the deformed state and p is a scalar quantity.

For an anisotropic body, the elastic potential is a function of all components of the Green deformation tensor $\Phi=\Phi(\varepsilon_{ij})$. This function will be assumed to be an arbitrary, continuously twice-differentiable function. We will henceforth use the results obtained previously [3]. The general results are given for compressible materials.

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1. General Relationships and Properties. We will write the basic linearized equations and boundary conditions.

Theory of Finite Precritical Deformations. The basic equations in this case [4] can be written in the form

$$[\sigma^*_{in}(\delta_{mn}+u^0_{m,n})+\sigma^{*0}_{in}u_{m,n}]_{,i}+X^*_m-\rho^*\ddot{u}_m=0;\quad m=1,2,3. \tag{1.1}$$

Here u_m and σ^*_{in} are the perturbations of the displacements and generalized stresses, X^*_m and ρ^* are the perturbations of the mass forces and density of the material per unit volume before deformation.

We write the boundary conditions for part of the surface S_1 with respect to stresses and part of the surface S_2 with respect to displacements [4]:

$$N_i[\sigma^*_{in}(\delta_{mn}+u_{m,n}{}^0)+\sigma^{*0}{}_{in}u_{m,n}]_{S_1}=P^*_m;\quad u_m\big|_{S_2}=0. \tag{1.2}$$

Here N_i are components of the orthonormals to the surface before deformation and P^*_m are the perturbations of the corresponding surface forces acting on the body after deformation, but per unit area before deformation.

We now write the expressions for the deformation-tensor components:

$$2\varepsilon_{ij}{}^0=u_{i,j}{}^0+u_{j,i}{}^0+u_{s,i}{}^0u_{s,j};\quad 2\varepsilon_{ij}=u_{i,j}+u_{j,i}+u_{s,i}{}^0u_{s,j}+u_{s,i}u_{s,j}. \tag{1.3}$$

The volume V and surface $S=S_1+S_2$ refer to the volume and surface of the body before deformation, by virtue of the Lagrangian coordinates.

First Variant of the Theory of Small Precritical Deformations. The basic assumption for this theory is that the elongation and shear (and hence the deformation-tensor components) are negligibly small in comparison with one [2], so that the changes in volume and area are not taken into account. The basic relationships have the form of Eqs. (1.1)-(1.3) if the index is omitted from all quantities.

Second Variant of the Theory of Small Precritical Deformations. In addition to the assumptions of the first variant, it is assumed in this theory that the precritical state can be described geometrically by a linear theory [2]. The basic relationships have the form

$$(\sigma_{im}+\sigma_{in}{}^0u_{m,n})_{,i}+X_m-\rho\ddot{u}_m=0; \tag{1.4}$$

$$N_i(\sigma_{im}+\sigma_{in}{}^0u_{m,n})\big|_{S_1}=P_m;\quad u_m\big|_{S_2}=0; \tag{1.5}$$

$$2\varepsilon_{ij}{}^0=u_{i,j}{}^0+u_{j,i}{}^0;\quad 2\varepsilon_{ij}=u_{i,j}+u_{j,i}. \tag{1.6}$$

Third Variant of the Theory of Small Precritical Deformations. In addition to the assumptions in the first and second variants, the quantities e_{ij} and $e_{ij}{}^0$ are negligibly small in comparison with the quantities [2] ω_j and $\omega_j{}^0$; moreover, we assume $e_{ij}\approx 0$ $(j\neq i)$ in determining ω_j in terms of the displacement vector. The basic relationships have the form

$$[\sigma_{im}+\sigma_{in}{}^0(1-\delta_{mn})u_{m,n}]_{,i}+X_m-\rho\ddot{u}_m=0; \tag{1.7}$$

$$N_i[\sigma_{im}+\sigma_{in}{}^0(1-\delta_{mn})u_{m,n}]_{S_1}=P_m;\quad u_m\big|_{S_2}=0. \tag{1.8}$$

Equation (1.6) remains in force.

We linearize Eq. (0.1). For the theory of finite precritical deformation, we obtain

$$\sigma^*_{ij}=\mu_{ij\alpha\beta}u_{\alpha,\beta}; \tag{1.9}$$

$$\mu_{ij\alpha\beta}=\frac{1}{4}\left(\frac{\partial}{\partial\varepsilon_{\alpha\beta}{}^0}+\frac{\partial}{\partial\varepsilon_{\beta\alpha}{}^0}+u_{\alpha,n}{}^0\frac{\partial}{\partial\varepsilon_{n\beta}{}^0}+u_{\alpha,n}{}^0\frac{\partial}{\partial\varepsilon_{\beta n}{}^0}\right)\left(\frac{\partial}{\partial\varepsilon_{ij}{}^0}+\frac{\partial}{\partial\varepsilon_{ji}{}^0}\right)\Phi^0. \tag{1.10}$$

For the first variant of the theory of small precritical deformations, Eqs. (1.9) and (1.0) remain in force if the index * is omitted.

For the second and third variants of the theory, the linearized relationships are identical and assume the form

$$\sigma_{ij}=\lambda_{ij\alpha\beta}u_{\alpha,\beta};\quad \lambda_{ij\alpha\beta}=\frac{1}{4}\left(\frac{\partial}{\partial\varepsilon_{\alpha\beta}{}^0}+\frac{\partial}{\partial\varepsilon_{\beta\alpha}{}^0}\right)\left(\frac{\partial}{\partial\varepsilon_{ij}{}^0}+\frac{\partial}{\partial\varepsilon_{ji}{}^0}\right)\Phi^0. \tag{1.11}$$

It can be seen from a direct check that the equalities

$$\lambda_{ij\alpha\beta}=\lambda_{ji\alpha\beta}\,; \quad \lambda_{ij\alpha\beta}=\lambda_{ij\beta\alpha}\,; \quad \lambda_{ij\alpha\beta}=\lambda_{\alpha\beta ij} \tag{1.12}$$

are satisfied for Eq. (1.11). It follows from Eq. (1.12) that the linearized relationships (1.11) for the second and third variants of the theory of small precritical deformations have the same form as the relationships for a linear inhomogeneous anisotropic body. The inhomogeneity obtains for inhomogeneous precritical states.

It can be seen from a direct check that the relationships

$$\mu_{ij\alpha\beta}=\mu_{ji\alpha\beta}\,; \quad \mu_{ij\alpha\beta}\neq\mu_{ij\beta\alpha}\,; \quad \mu_{ij\alpha\beta}\neq\mu_{\alpha\beta ij} \tag{1.13}$$

are satisfied for Eq. (1.10). It follows from Eq. (1.13) that the linearized relationships in Eq. (1.9) for the theory of finite precritical deformations and the second variant of the theory of small precritical deformations do not coincide with the relationships for a linear inhomogeneous anisotropic body.

In conformity with our previous study [3], we introduce the quantity

$$\nu_{ij\alpha\beta}=\mu_{in\alpha\beta}(\delta_{jn}+u_{j,n}{}^0)\,. \tag{1.14}$$

It can be seen from a direct check, taking into account Eqs. (1.10), (1.13), and (1.14), that the relationships

$$\nu_{ij\alpha\beta}\neq\nu_{ij\beta\alpha}\,; \quad \nu_{ij\alpha\beta}\neq\nu_{ji\alpha\beta}\,; \quad \nu_{ij\alpha\beta}\neq\nu_{\alpha\beta ij}\,; \quad \nu_{ij\alpha\beta}=\nu_{\beta\alpha ji} \tag{1.15}$$

are satisfied for the quantity in Eq. (1.14).

It follows from Eqs. (1.9), (1.14), and (1.15) that the linearized relationships for the asymmetric Kirchhoff stress tensor $\hat{\sigma}_{ij}=\sigma^*{}_{in}(\delta_{jn}+u_{j,n}{}^0)$ do not coincide with the relationships for a linear inhomogeneous anisotropic body.

Taking the foregoing into account, we write the basic problems with respect to the displacements. For the theory of finite precritical deformations,

$$(\nu_{im\alpha\beta}u_{\alpha,\beta}+\sigma^{*0}{}_{in}u_{m,n})_{,i}+X^*{}_m-\rho^*\ddot{u}_m=0\,; \tag{1.16}$$

$$N_i(\nu_{im\alpha\beta}u_{\alpha,\beta}+\sigma^{*0}{}_{in}u_{m,n})\,|_{S_1}=P^*{}_m\,; \quad u_m|_{S_2}=0\,. \tag{1.17}$$

For the first variant of the theory of small precritical deformations, Eqs. (1.16) and (1.17) are valid if the index * is omitted.

For the second variant of the theory of small precritical deformations,

$$(\lambda_{im\alpha\beta}u_{\alpha,\beta}+\sigma_{in}{}^0u_{m,n})_{,i}+X_m-\rho\ddot{u}_m=0\,; \tag{1.18}$$

$$N_i(\lambda_{im\alpha\beta}u_{\alpha,\beta}+\sigma_{in}{}^0u_{m,n})\,\big|_{S_1}=P_m\,; \quad u_m\big|_{S_2}=0. \tag{1.19}$$

For the third variant of the theory of small precritical deformations,

$$[\lambda_{im\alpha\beta}u_{\alpha,\beta}+\sigma_{in}{}^0(1-\delta_{mn})\,u_{m,n}]_{,i}+X_m-\rho\ddot{u}_m=0\,; \tag{1.20}$$

$$N_i[\lambda_{im\alpha\beta}u_{\alpha,\beta}+\sigma_{in}{}^0(1-\delta_{mn})u_{m,n}]_{S_1}=P_m\,; \quad u_m\big|_{S_2}=0. \tag{1.21}$$

It should be noted that, in the general case, the quantities $\nu_{ij\alpha\beta}$, $\lambda_{ij\alpha\beta}$, $\sigma^{*0}{}_{ij}$ and σ_{ij} depend on x_1, x_2, x_3, and t; we will henceforth assume them to be independent of time.

2. Sufficient Conditions for Use of Euler Method. The quantities in Eqs. (1.16)-(1.21) are represented in the following form:

$$u_m=v_m\exp i\omega t\,; \quad X^*{}_m=Y^*{}_m(x_1,x_2,x_3,\omega)\exp i\omega t\,;$$
$$X_m=Y_m(x_1,x_2,x_3,\omega)\exp i\omega t\,; \quad P^*{}_m=Q^*{}_m(x_1,x_2,x_3,\omega)\exp i\omega t\,;$$
$$P_m=Q_m(x_1,x_2,x_3,\omega)\exp i\omega t\,. \tag{2.1}$$

Substituting Eq. (2.1) into Eqs. (1.16)-(1.21), we obtain the principal equations and boundary conditions. For the theory of finite precritical deformations,

$$(\nu_{im\alpha\beta}v_{\alpha,\beta}+\sigma^{*0}{}_{in}v_{m,n})_{,i}+Y^*{}_m+\rho^*\omega^2v_m=0\,; \tag{2.2}$$

$$N_i(\nu_{im\alpha\beta}v_{\alpha,\beta}+\sigma^{*0}{}_{in}v_{m,n})\,\big|_{S_1}=Q^*{}_m\,; \quad v_m\big|_{S_2}=0. \tag{2.3}$$

Equations (2.2) and (2.3) remain in force for the first variant of the theory of small precritical deformations if we omit the index *.

For the second variant of the theory of small precritical deformations,

$$(\lambda_{im\alpha\beta}v_{\alpha,\beta}+\sigma_{in}{}^0 v_{m,n})_{,i}+Y_m+\rho\omega^2 v_m=0\,; \tag{2.4}$$

$$N_i(\lambda_{im\alpha\beta}v_{\alpha,\beta}+\sigma_{in}{}^0 v_{m,n})\big|_{S_1}=Q_m\,;\qquad v_m\big|_{S_2}=0\,. \tag{2.5}$$

For the third variant of the theory of small precritical deformations,

$$[\lambda_{im\alpha\beta}v_{\alpha,\beta}+\sigma_{in}{}^0(1-\delta_{mn})v_{m,n}]_{,i}+Y_m+\rho\omega^2 v_m=0; \tag{2.6}$$

$$N_i[\lambda_{im\alpha\beta}v_{\alpha,\beta}+\sigma_{in}{}^0(1-\delta_{mn})v_{m,n}]_{S_1}=Q_m\,;\qquad v_m\big|_{S_2}=0. \tag{2.7}$$

We henceforth assume the quantities Y^*_m, Y_m, Q^*_m, and Q_m to be independent of ω.

Retracing the arguments given in the literature [5], we conclude that sufficient conditions for use of the Euler method are that the eigennumbers ω^2 of boundary problems (2.2)-(2.7) be real, which ensures self-consistency of the boundary problems. We will derive these conditions for anisotropic and piecewise-homogeneous materials.

Anisotropic Material. We will first consider the theory of finite precritical deformations. Let $v_i^{(1)}$ and $v_i^{(2)}$ be continuously twice-differentiable functions ($i=1, 2, 3$) satisfying conditions (2.3). The self-consistency condition for the boundary problem in Eqs. (2.2) and (2.3) can then be written in the form [6, 7]

$$\int_V \{v_m{}^{(1)}[(\nu_{im\alpha\beta}v_{\alpha,\beta}{}^{(2)}+\sigma^*{}_{in}{}^0 v_{m,n}{}^{(2)})_{,i}+Y^*{}_m{}^{(2)}+\rho^*\omega^2 v_m{}^{(2)}]-$$
$$-v_m{}^{(2)}[(\nu_{im\alpha\beta}v_{\alpha,\beta}{}^{(1)}+\sigma^*{}_{in}{}^0 v_{m,n}{}^{(1)})_{,i}+Y^*{}_m{}^{(1)}+\rho^*\omega^2 v_m{}^{(1)}]\}\,dV=0\,. \tag{2.8}$$

This condition should be satisfied with any $v_m^{(1)}$ and $v_m^{(2)}$ satisfying boundary conditions (2.3). The indices (1) and (2) designate perturbations of the volume and surface forces corresponding to $v_m^{(1)}$ and $v_m^{(2)}$

We apply the Gauss-Ostrogradskii theorem to Eq. (2.8). After making a series of transformations and taking into account boundary conditions (2.3), we derive the condition

$$\iint_{S_1}(v_m{}^{(1)}Q^*{}_m{}^{(2)}-v_m{}^{(2)}Q^*{}_m{}^{(1)})\,dS+\int_V(v_m{}^{(1)}Y^*{}_m{}^{(2)}-v_m{}^{(2)}Y^*{}_m{}^{(1)})\,dV-$$
$$-\int_V(\nu_{im\alpha\beta}-\nu_{\beta\alpha mi})\,v_{m,i}{}^{(1)}v_{\alpha,\beta}{}^{(2)}\,dV=0\,. \tag{2.9}$$

Taking into account the final equality in Eq. (1.15), we obtain

$$\iint_{S_1}(v_m{}^{(1)}Q^*{}_m{}^{(2)}-v_m{}^{(2)}Q^*{}_m{}^{(1)})\,dS+\int_V(v_m{}^{(1)}Y^*{}_m{}^{(2)}-v_m{}^{(2)}Y^*{}_m{}^{(1)})\,dV=0\,. \tag{2.10}$$

Having made similar transformations, we obtain

$$\iint_{S_1}(v_m{}^{(1)}Q_m{}^{(2)}-v_m{}^{(2)}Q_m{}^{(1)})\,dS+\int_V(v_m{}^{(1)}Y_m{}^{(2)}-v_m{}^{(2)}Y_m{}^{(1)})\,dV=0 \tag{2.11}$$

for all variants of the theory of small precritical deformations.

We have thus determined the sufficient conditions for use of the Euler method to investigate the deformation stability of a nonlinear elastic isotropic body with an arbitrary elastic-potential form for the theory of finite precritical deformations [Eq. (2.10)] and for all variants of the theory of small precritical deformations [Eq. (2.11)]. It must be noted that these conditions are independent of the form of the elastic body, since Eqs. (2.10) and (2.11) do not contain the elastic potential. In the case of the second variant of the theory of small precritical deformations, for a linear-elastic body, Eq. (2.11) was derived previously [5].

Piecewise-Homogeneous Medium. Let us consider the theory of finite precritical deformations. For the k-th component, we can write

$$(\nu_{im\alpha\beta}{}^{(k)}v_{\alpha,\beta}{}^{(k)}+\sigma^{*0(k)}{}_{in}v_{m,n}{}^{(k)})_{,i}+Y^*{}_m{}^{(k)}+\rho^{*(k)}\omega^2 v_m{}^{(k)}=0; \tag{2.12}$$

$$N_i{}^{(k)}(\nu_{im\alpha\beta}{}^{(k)}v_{\alpha,\beta}{}^{(k)}+\sigma^*{}_{in}{}^{0(k)}v_{m,n}{}^{(k)})\big|_{S_1}=Q_m{}^{(k)}\,;\qquad v_m{}^{(k)}\big|_{S_2}=0\,. \tag{2.13}$$

TABLE 1

$h/h^{(1)}$	Parameter	$E^{(1)}/E$			
		200	300	500	800
20	z $\alpha^{(1)}$	14,0 0,160	10,5 0,13	7,2 0,095	—
30	z $\alpha^{(1)}$	9,8 0,160	—	5,1 ,0,115	3,8 0,096
40	z $\alpha^{(1)}$	—	—	3,8 0,120	2,9 0,096
50	z $\alpha^{(1)}$	—	4,4 0,140	3,1 0,120	—

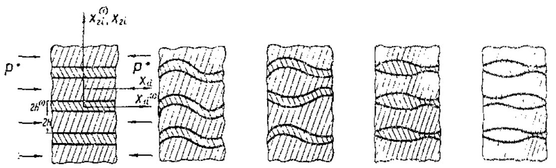

Fig. 1

Here $S = S_1 + S_2$ refers to the external surface. Moreover, the following conditions should be satisfied at the component interfaces:

$$N_i(\nu^+{}_{im\alpha\beta}v^+{}_{\alpha,\beta} + \sigma^{*0+}_{in}v^+{}_{m,n}) = N_i(\nu^-{}_{im\alpha\beta}v^-{}_{\alpha,\beta} + \sigma^{*0-}_{in}v^-{}_{m,n}) ;$$
$$v^+{}_m = v^-{}_m. \tag{2.14}$$

Having written the self-consistency condition, a series of transformations yields Eq. (2.10) when we take into account Eqs. (2.13) and (2.14). We derive Eq. (2.11) in similar fashion for the theory of small precritical deformations. This agreement is to be expected, since Eqs. (2.10) and (2.11) do not contain characteristics of the material.

The sufficient conditions for application of the Euler method to anisotropic and piecewise-homogeneous materials therefore have the form of Eqs. (2.10) and (2.11).

We will now turn to construction of solutions for laminar and fibrous media in homogeneous precritical states.

3. Representation of Solutions for Static Stability. We will assume a piecewise-homogeneous medium to consist of anisotropic components, the elastic potential for each of them having an arbitrary form; the precritical state is assumed to be homogeneous.

Laminaı Medium. We will first consider the case of uniaxial compression along the axis Ox_3, where the component interfaces from the plane $x_1 = \text{const}$. In this case, it is necessary to set $p^*_1 = p^*_2 = 0$ [8]. The function Ψ_i can be written in the form

$$\Psi_1 = (A_1 \operatorname{ch} \eta_1 x_1 + B_1 \operatorname{sh} \eta_1 x_1) \sin \alpha x_2 \sin \beta x_3 ; \quad \eta_i = (\alpha^2 + \zeta_i^2 \beta^2)^{1/2} ;$$
$$\Psi_j = (A_j \operatorname{ch} \eta_j x_1 + B_j \operatorname{sh} \eta_j x_1) \cos \alpha x_2 \cos \beta x_3 ; \quad \alpha = \pi l_1^{-1} ; \tag{3.1}$$
$$\beta = \pi l_2^{-1}, \quad i = 1, 2, 3.$$

The formulas in section 3 are not summed over j. The quantities ζ_i have the form of Eqs. (3.3)-(3.5) in [8].

Let the interfaces form the plane $x_3 = \text{const}$. We will consider diaxial equilibrium compression along the Ox_1 and Ox_2 axes. In this case, it is necessary to set $p^*_3 = 0$ [8]. On a rectangular coordinate system, the function Ψ_i has the form

$$\Psi_1 = (A_1 \operatorname{ch} \zeta_1^{-1} x_3 + B_1 \operatorname{sh} \zeta_1^{-1} x_3) \sin \alpha x_1 \sin \beta x_2 ;$$
$$\Psi_j = (A_j \operatorname{ch} \zeta_j^{-1} x_3 + B_j \operatorname{sh} \zeta_j^{-1} x_3) \cos \alpha x_1 \cos \beta x_2 ; \quad j = 2, 3. \tag{3.2}$$

On the circular cylindrical coordinate system of the axially symmetric problem, the function Ψ_j has the form

$$\Psi_1 = 0 ; \quad \Psi_j = \left(A_j^{(k)} \operatorname{ch} \frac{\varkappa_k x_3}{R\zeta_j} + B_j^{(k)} \operatorname{sh} \frac{\varkappa_k x_3}{R\zeta_j} \right) J_0 \left(r \frac{\varkappa_k}{R} \right). \tag{3.3}$$

Equation (3.3) is not summed for k. The quantities $\varkappa_k$ are roots of the equation $J'_0(\varkappa_k) = 0$.

Fibrous Medium. We will consider the compression along the Ox_3 axis of a unidirectional fibrous medium. It will be assumed that fibers with a circular cross section are directed along the Ox_3 axis. We also make the approximate assumption that a homogeneous uniaxial precritical state is realized in the material under a given load. In this case, it is necessary to set $p^*_1 \equiv p^*_2 = 0$ [8]. We now write the solution for loss of stability without torsion and with torsion of one fiber.

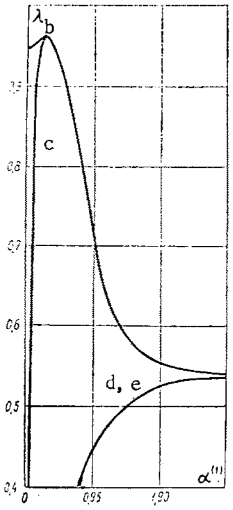

Fig. 2

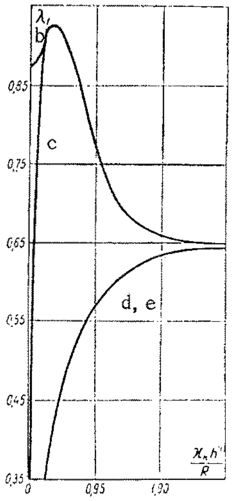

Fig. 3

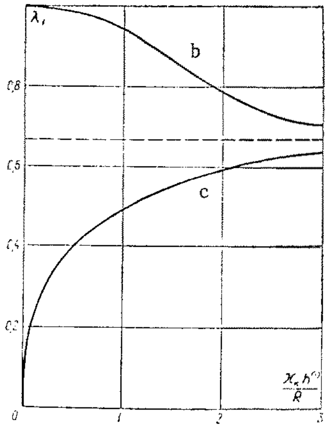

Fig. 4

For loss of stability without fiber torsion,

$$\Psi_1=[A_1^{(p)}K_p(\zeta_1\gamma r)+B_1^{(p)}I_p(\zeta_1\gamma r)]\sin p\theta\sin\gamma x_3;\qquad \gamma=\pi l^{-1};$$

$$\Psi_j=[A_j^{(p)}K_p(\zeta_j\gamma r)+B_j^{(p)}I_p(\zeta_j\gamma r)]\cos p\theta\cos\gamma x_3;\qquad j=2,3. \tag{3.4}$$

Summation for p is not carried out in Eqs. (3.4) and (3.5).

For the case of loss of stability with fiber torsion,

$$\Psi_1=[A_1^{(p)}K_p(\zeta_1\gamma r)+B_1^{(p)}I_p(\zeta_1\gamma r)]\cos(p\theta+\gamma x_3);$$

$$\Psi_j=[A_j^{(p)}K_p(\zeta_j\gamma r)+B_j^{(p)}I_p(\zeta_j\gamma r)]\cos(p\theta+\gamma x_3). \tag{3.5}$$

Taking into account the results of our previous study [8], the solutions to Eqs. (3.1)-(3.5) must be substituted into the boundary conditions in Eq. (2.13) to obtain the characteristic determinants. The solution for dynamic stability can be written in similar fashion, using the general solutions given previously [8]. The solutions for incompressible laminar and fibrous materials can be obtained in a form similar to Eqs. (3.1)-(3.5) [9-11].

4. Numerical Examples. As examples of the utilization of the solutions obtained, we will consider the deformation instability of compressible and incompressible laminar materials.

Plane Problem for Compressible Materials. We will consider a laminar medium under uniaxial compression parallel to the layers (Fig. 1). The medium is assumed to consist of two alternating isotropic components with the characteristics E, ν, h, and $E^{(1)}$, $\nu^{(1)}$, $h^{(1)}$, where 2h and $2h^{(1)}$ are the thicknesses of the binder and filler layers respectively. In the case of a "dead" load,

$$Q_m\equiv0;\quad Y_m\equiv0. \tag{4.1}$$

According to Eqs. (2.11) and (4.1), the Euler method can be used to solve the problem. We will investigate it within the framework of the third variant of the theory of small precritical deformations in Eqs. (1.20) and (1.21). The elastic potential has the form

$$\Phi=\lambda\varepsilon_{ii}|_2+\mu\varepsilon_{ij}\varepsilon_{ij}. \tag{4.2}$$

Taking into account Eq. (3.1) and (3.2) and satisfying the periodicity conditions, we arrive at a characteristic equation for determination of the quantity p_{cr}. Table 1 gives the critical load values ($z=10^4 p_{cr}/E^{(1)}$) and the corresponding values of the parameter $\alpha^{(1)}=\pi h^{(1)}/l$ (for loss of stability) for different combinations of binder and filler layers. Dashes in the table indicate that determinations were not made for the parameters in question. There is no internal loss of stability for the parameter values $h/h^{(1)}=1$, 5, 10, and $E^{(1)}/E=100$, 200, 300, 500, 800, and 1000; the compressive instability in this case is determined by the specimen length.

Plane Problem for Incompressible Materials. Following our previous study [9], we will consider a laminar medium composed of incompressible materials with highly elastic deformations. The medium consists of two alternating isotropic layers of binder and filler parallel to the Ox_1 axis and compression is parallel to the Ox_1 axis with a force P^* (see Fig. 1). The generalized stresses are determined in terms of the elastic potential with the formulas given in the literature [1, 4].

The precritical state and the perturbation components for the i-th layer are defined in the following manner [9]:

$$
\begin{gathered}
u_{1i}{}^0=(\lambda_1-1)x_{1i}; \quad u_{2i}{}^0=(\lambda_2-1)x_{2i}; \quad \lambda_1=\lambda; \quad \lambda_2=\lambda^{-1};\\
\sigma^{*0}{}_{11i}=(\lambda^{-2}-\lambda^2)\left(2\frac{\partial\Phi_i{}^0}{\partial I_{2i}{}^0}+p_i{}^0\right); \quad \sigma^*{}_{11i}=a_1\frac{\partial u_{1i}}{\partial x_{1i}}+p_i\lambda^{-2};\\
\sigma^*{}_{22i}=a_2\frac{\partial u_{2i}}{\partial x_{2i}}+p_i\lambda^2; \quad \sigma^*{}_{12i}=\sigma^*{}_{21i}=a_{12}\frac{\partial u_{1i}}{\partial x_{2i}}+a_{21}\frac{\partial u_{2i}}{\partial x_{1i}};\\
u_{1i}=\lambda\frac{\partial^2}{\partial x_{1i}\partial x_{2i}}\Psi_i; \quad u_{2i}=-\lambda^{-1}\frac{\partial^2}{\partial x_{1i}{}^2}\Psi_i;\\
p_i=-\lambda^2\frac{\partial}{\partial x_{2i}}\left[(\lambda a_1-\lambda^{-1}a_{21}+\sigma^{*0}{}_{11i})\frac{\partial^2}{\partial x_{1i}{}^2}+\lambda a_{12}\frac{\partial^2}{\partial x_{2i}{}^2}\right]\Psi_i;\\
\Psi_i=\Psi_{1i}+\Psi_{2i}; \quad \left(\frac{\partial^2}{\partial x_{2i}{}^2}+\zeta_j{}^2\frac{\partial^2}{\partial x_{1i}{}^2}\right)\Psi_{ji}=0; \qquad j=1,2.
\end{gathered}
\tag{4.3}
$$

Expressions for the quantities a_1, a_2, a_{12}, a_{21}, ζ_1, and ζ_2 were given previously [9]. The calculations are made for the Treloar potential $\Phi=C(I_1-3)$ and the Mooney potential $\Phi=C_1(I_1-3)+C_2(I_2-3)$. Figure 2 shows the root of the characteristic equation λ (λ is the contraction factor along the Ox_1 axis) as a function of the parameter $\alpha^{(1)}=\pi h^{(1)}/l$ for four different forms of loss of stability (see Fig. 1). Internal loss of stability occurs for a material with the parameters $C^{(1)}/C=100$ and $h/h^{(1)}=20$, since the curves corresponding to forms b and c in Fig. 1 have a common maximum lying above that for the other curves. Calculations were made for the parameters $C^{(1)}/C=2, 5, 20, 100$ and $h/h^{(1)}=0.1, 0.5, 10, 20, \infty$. No internal loss of stability was observed for any value of $C^{(1)}/C$ at $h/h^{(1)}=0.1$, 0.5, 5.

Three-Dimensional Problem for Incompressible Materials. We will consider the three-dimensional problem of the loss of stability of an incompressible material uniformly compressed by definite forces P^* with $r=R$. Following our previous study [10] and omitting intermediate results, the procedure is similar to that described in the previous section for the Treloar–Mooney potentials and yields the following characteristic equation for λ_1 (λ_1 is the contraction factor along the coordinate r):

$$
\begin{gathered}
[\beta(\lambda_1^{-3}+\lambda_1^3)-2\lambda_1^{-3}]^2 \operatorname{th}\omega \operatorname{th}\omega^{(1)}\lambda_1^{-3}-4\lambda_1^{-3}(1-\beta)^2 \operatorname{th}\omega \operatorname{th}\omega^{(1)}+\\
+\beta(\lambda_1^{-3}-\lambda_1^3)^2 \operatorname{th}\omega^{(1)} \operatorname{th}\omega^{(1)}\lambda_1^{-3}-\lambda_1^{-3}(\lambda_1^{-3}+\lambda_1^3)^2(1-\beta)^2 \operatorname{th}\omega\lambda_1^{-3} \operatorname{th}\omega^{(1)}\lambda_1^{-3}+\\
+\beta(\lambda_1^{-3}-\lambda_1^3)^2 \operatorname{th}\omega \operatorname{th}\omega\lambda_1^{-3}+[\lambda_1^{-3}(1-2\beta)+\lambda_1^3]^2 \operatorname{th}\omega\lambda_1^{-3} \operatorname{th}\omega^{(1)}=0.
\end{gathered}
\tag{4.4}
$$

Here, $\omega=(\varkappa_k/R)h$; $\omega^{(1)}=(\varkappa_k/R)h^{(1)}$; $J'_0(\varkappa_k)=0$; $\beta=C^{(1)}/C$.

Numerical calculations were made for all four forms of loss of stability (see Fig. 1) with the parameter values $\beta=0$, 2, 5, 20, 100, and $h/h^{(1)}=0.1$, 0.5, 5, 10, 20. Figures 3 and 4 show the parameter λ as a function of the quantity $\omega^{(1)}=(\varkappa_k/R)h^{(1)}$. Internal loss of stability occurs for the material parameters $\beta=20$ and $h/h^{(1)}=20$. The curves in Fig. 4, which correspond to forms b and c in Fig. 1 for a circular plate, approximate to the asymptotic contraction-factor value $\lambda^*{}_1=0.666$ as the parameter ω rises.

CONCLUSIONS

1. A circular plate of arbitrary thickness loses its stability when compressed by about 33%.

2. Loss of stability in the material structure does not occur with all ratios of the geometric and mechanical characteristics of the layers. Internal loss of stability develops when the thickness of the softer layers (binder) is substantially greater than that of the rigid layers (filler).

3. When internal loss of stability occurs, the results are identical for forms of stability loss with the period of the material structure (forms b and d) and with a period twice this value (forms c and e).

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