NONMATRIX VARIETIES

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Throughout this paper the word "algebra" signifies an associative algebra over a fixed F of characteristic zero.

A variety of algebras $\mathfrak{M}$ is called a nonmatrix variety if the algebra of second-order matrices M_2 does not belong to $\mathfrak{M}$. The main example of nonmatrix varieties are the following:

$\mathfrak{N}_k$, the variety of all nilpotent algebras of index $\leqslant k$;

$\mathfrak{A}_0$, the variety of all commutative algebras;

$\mathfrak{A}_1 = \mathrm{Var}\,(G)$, where G is a Grassmann algebra of countable rank;

$\mathfrak{A}_2 = \mathrm{Var}\,(G \otimes G)$.

If $\mathfrak{A}, \mathfrak{B}, \mathfrak{M}$ are varieties and $\mathfrak{A}, \mathfrak{B} \subseteq \mathfrak{M}$, then $\mathfrak{A} \circ_{\mathfrak{M}} \mathfrak{B}$ denotes the $\mathfrak{M}$-product of varieties [1], i.e., the intersection of the usual product $\mathfrak{A} \circ \mathfrak{B}$ with $\mathfrak{M}$.

In the present paper we prove that any nonmatrix variety $\mathfrak{M}$ can be "assembled" from the above-mentioned varieties by means of the operation "$\circ_{\mathfrak{M}}$"; more precisely, we prove the following results:

THEOREM 1. Suppose $\mathfrak{M}$ is a nonmatrix variety, $\mathfrak{M} \not\ni G \otimes G$, and $\mathfrak{A}$ is the largest variety in $\{\mathfrak{O}, \mathfrak{A}_0, \mathfrak{A}_1\}$ ($\mathfrak{O}$ is the zero variety) lying in $\mathfrak{M}$. Then for some k we have

$$\mathfrak{M} = \mathfrak{M}_k \circ_{\mathfrak{M}} \mathfrak{A},$$

where $\mathfrak{M}_k = \mathfrak{M} \cap \mathfrak{N}_k$.

THEOREM 2. Suppose $\mathfrak{M}$ is a nonmatrix variety and $\mathfrak{M} \ni G \otimes G$. Then for certain k, ℓ we have

$$\mathfrak{M} = \mathfrak{M}_k \circ_{\mathfrak{M}} (\mathfrak{A}_2 \circ (\mathfrak{M}_\ell \circ_{\mathfrak{M}} \mathfrak{A}_0)).$$

THEOREM 3. Suppose $\mathfrak{M}$ is a nonmatrix variety, $\mathfrak{A}$ is a finitely based variety, and $\mathfrak{A} \supseteq \mathfrak{A}_2$. Then for some k we have

$$\mathfrak{M} = \mathfrak{M}_k \circ_{\mathfrak{M}} (\mathfrak{A} \cap \mathfrak{M}).$$

It is not known whether the variety $\mathfrak{A}_2$ is finitely based, but if it is, then, by Theorem 3, the conclusion of Theorem 2 can be rewritten in the form $\mathfrak{M} = \mathfrak{M}_k \circ_{\mathfrak{M}} \mathfrak{A}_2$. Note also that the varieties $\mathfrak{N}_2, \mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2$ are indecomposable; hence Theorems 1 and 2 give a decomposition of an arbitrary nonmatrix variety $\mathfrak{M}$ into an $\mathfrak{M}$-product of indecomposable varieties.

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From Theorems 1, 2, and 3 we obtain:

COROLLARY 1. The algebras of an arbitrary nonmatrix variety satisfy the identity

$$[[x_1,y_1],[z_1,t_1],u_1]\cdot\ldots\cdot[[x_n,y_n],[z_n,t_n],u_n]=0$$

for some n.

COROLLARY 2. If $\mathfrak{M}$ is a nonmatrix variety and $\mathfrak{M}\not\ni G\otimes G$, then $\mathfrak{M}$ satisfies the identity

$$[x_1,y_1,z_1]\cdot\ \ldots\ \cdot[x_n,y_n,z_n]=0 \tag{*}$$

for some n.

Latyshev [2] studied the varieties of algebras satisfying an identity of the form

$$[x_1^{(1)},\ldots,x_m^{(1)},[y^{(1)},z^{(1)}]]\cdot\ldots\cdot[x_1^{(k)},\ldots,x_m^{(k)}[y^{(k)},z^{(k)}]]=0. \tag{**}$$

It is easy to show that an identity of the form (**) implies an identity of the form (*); hence, in view of Corollary 2, V. N. Latyshev's class of varieties coincides with the class of nonmatrix varieties that do not contain $G\otimes G$. Therefore in view of the main result of [2], we obtain

COROLLARY 3. If $\mathfrak{M}$ is a nonmatrix variety and $\mathfrak{M}\not\ni G\otimes G$, then $\mathfrak{M}$ is Spechtian.

From Corollary 3 we obtain

COROLLARY 4. The variety of algebras satisfying a nontrivial identity of the form

$$\sum_{i=1}^{n}\alpha_i x^i y x^{n-i}=0,$$

where $\alpha_i\in F$, is Spechtian.

It follows easily from Corollary 1 that any nonmatrix variety satisfies a Lie solvability identity. From this fact and a theorem of Higgins [3] we obtain

COROLLARY 5. If an associative algebra satisfies the identity $[x,\underbrace{y,\ldots,y}_{k}]=0$ for some k, then it satisfies the identity $[x_1,x_2,\ldots,x_m]$ for some m.

The proof of Corollary 5, obtained independently of [3], provides an affirmative answer to a question of Latyshev [4].

From Corollary 3 we also obtain

COROLLARY 6. If $\mathfrak{M}$ is locally weakly Noetherian, then $\mathfrak{M}$ is Spechtian.

From Corollary 5 we obtain for locally weakly Noetherian varieties the following

COROLLARY 7. If $\mathfrak{M}$ is locally weakly Noetherian, then $\mathfrak{M}$ satisfies an identity of the form

$$[x_1,\ldots,x_m]\cdot y_1\cdot\ldots\cdot y_n[z_1,\ldots,z_m]=0.$$

1. PRELIMINARIES

Suppose X is a fixed countable linearly ordered set and $F\langle X\rangle$ is the free associative algebra (without unity) generated by X. Represent X in the form

$$X = Y \cup \left(\bigcup_{i=1}^{\infty} T_i\right),$$

where T_i, Y are countable, pairwise disjoint subsets. Fix the sets T_i, Y. Let

$$T = \bigcup_{i=1}^{\infty} T_i .$$

Consider the ideal $\mathcal{Y}$ of $F\langle X\rangle$ generated by all elements of the form

$$a_1 u a_2 + a_2 u a_1,$$

where $a_1, a_2 \in T_i$, $i = 1, 2, \ldots$; $u \in F\langle X\rangle \cup \{1\}$. The images of the sets T_i, T under the natural homomorphism $F\langle X\rangle \to F\langle X\rangle/\mathcal{Y}$ will be denoted by E_i, E, respectively. Obviously,

$$E = \bigcup_{i=1}^{\infty} E_i .$$

The image of the set Y will gain be denoted by Y. We denote the quotient algebra $F\langle X\rangle/\mathcal{Y}$ by $F_{E,Y}$. Thus, the algebra $F_{E,Y}$ has the following defining relations:

$$a_1 u a_2 = - a_2 u a_1, \tag{1}$$

where $a_1, a_2 \in E_i$, $i = 1, 2, \ldots$; $u \in F_{E,Y} \cup \{1\}$.

Suppose $A \subseteq X$. Consider the linear operator $S_A : F\langle X\rangle \to F\langle X\rangle$ defined as follows:

1. If f is a monomial and $\deg_a f \geq 2$ for some $a \in A$ ($\deg_a f$ is the degree of the polynomial f with respect to the variable a), then $S_A(f) = 0$.

2. If $f = f(a_1, \ldots, a_n, x_1, \ldots, x_m)$ is a monomial, $a_i \in A, x_i \notin A$; $a_1 < a_2 < \ldots < a_n$; $\deg_{a_i} f = 1$; $n \geq 0$, then

$$S_A(f) = \sum_{\sigma \in S(n)} (-1)^{\sigma} f(a_{\sigma(1)}, \ldots, a_{\sigma(n)}, x_1, \ldots, x_m),$$

where $S(n)$ is the symmetric group of degree n. The operator S_A will be called the symmetrizer with respect to the set A.

It is easy to verify the following properties of symmetrizers:

If $A \cap B = \emptyset$, then

$$S_A(S_B(f)) = S_B(S_A(f)). \tag{2}$$

If f is homogeneous and linear with respect to the variables in A, then

$$S_A(S_A(f)) = n!\, S_A(f), \tag{3}$$

where n is the number of variables in A on which f depends.

The operator ${}_T S : F\langle X\rangle \to F\langle X\rangle$ is defined as follows: if the polynomial f does not depend on the variables in T_i for $i > k$, then ${}_T S(f) = S_{T_1} \ldots S_{T_k}(f)$.

Put ${}_T S(F\langle X\rangle) = M_{T,Y}$. The natural homomorphism $F\langle X\rangle \to F_{E,Y}$ induces a homomorphism of spaces $\varphi : M_{T,Y} \to F_{E,Y}$.

LEMMA 1. φ is an isomorphism.

Proof. Suppose $f \in \operatorname{Ker}\varphi = M_{T,Y} \cap \mathcal{J}$. Since the spaces $M_{T,Y}$, $\mathcal{J}$ are homogeneous and $f \in M_{T,Y}$, we may assume that $f = {}_T S(g)$, where g is a polynomial that is multilinear with respect to the variables in T . Then in view of (2) and (3) we have

$$ {}_T S(f) = {}_T S({}_T S(g)) = S_{T_1} S_{T_1} \dots S_{T_K} S_{T_K}(g) = n_1! \cdot \ldots \cdot n_K! f, \tag{4} $$

where n_i is the number of variables in T_i on which g depends ($n_i = 0$ for $i > K$). On the other hand, it is easy to see that ${}_T S(\mathcal{J}) = 0$. Therefore, $f \in \mathcal{J}$ implies ${}_T S(f) = 0$. It follows from this and (4) that $f = 0$.

It remains to show that $M_{T,Y} + \mathcal{J} = F\langle X\rangle$. Suppose f is an arbitrary monomial. If $\deg_t f \geqslant 2$ for some $t \in T$, then, by definition of the ideal $\mathcal{J}$, we have $f \in \mathcal{J}$. In the opposite case, it is easy to see, using (1), that ${}_T S(f) = n! \cdot \ldots \cdot n_K! f \pmod{\mathcal{J}}$, where n_i is the number of variables in T_i occurring in f .

The lemma is proved.

If Γ is an arbitrary T-ideal of the algebra $F\langle X\rangle$ and A is an arbitrary algebra, we denote by $\Gamma(A)$ the ideal of values of the polynomials of Γ on A , i.e., the ideal generated by the elements $f(a_1, \dots, a_n)$, where $f \in \Gamma$, $a_i \in A$.

LEMMA 2. Suppose Γ is an arbitrary T-ideal of $F\langle X\rangle$. Then

$$ \varphi(\Gamma \cap M_{T,Y}) = \Gamma(F_{E,Y}). $$

Proof. Suppose $f = f(x_1, \dots, x_n) \in \Gamma \cap M_{T,Y}$. Then $\varphi(f) = f(\varphi(x_1), \dots, \varphi(x_n)) \in \Gamma(F_{E,Y})$. Conversely, suppose $g \in \Gamma(F_{E,Y})$. We may assume that g is homogeneous; but then Γ contains a homogeneous polynomial $f = f(t_1^{(1)}, \dots, t_{n_1}^{(1)}, \dots, t_1^{(K)}, \dots, t_{n_K}^{(K)}, y_1, \dots, y_s)$, $t_i^{(j)} \in T_j$, $y_i \in Y$, such that $g = f(e_1^{(1)}, \dots, e_{n_1}^{(1)}, \dots, e_1^{(K)}, \dots, e_{n_K}^{(K)}, y_1, \dots, y_s)$, $e_i^{(j)} \in E_j$, $y_i \in Y$. Put

$$ h = \frac{1}{n_1! \cdot \ldots \cdot n_K!} \, {}_T S(f). $$

Obviously, $h \in \Gamma \cap M_{T,Y}$. Using (1), it is easy to show that $\varphi(h) = g$.

The lemma is proved.

Definition 1. Let Z be the subspace of $F\langle X\rangle$ generated by the set X . An ideal I of the algebra $F\langle X\rangle$ will be called an S-ideal if for any $f(x_1, \dots, x_n) \in I$ and any $z_1, \dots, z_n \in Z$ we have $f(z_1, \dots, z_n) \in I$, i.e., if I is stable under linear substitutions.

It is clear that any S-ideal is stable under linearizations, hence is generated and defined by a set of multilinear polynomials lying in I.

Definition 2. Suppose A is an algebra and Z is a subspace of A generating A as a ring. An ideal of $F\langle X\rangle$ of the form $T[A,Z] = \{f(x_1, \dots, x_n) \in F\langle X\rangle \mid \forall z_1, \dots, z_n \in Z \; f(z_1, \dots, z_n) = 0\}$ will be called the ideal of identities of the pair (A, Z) .

It is clear that $T[A,Z]$ is an S-ideal and contains the ideal of identities of the algebra A.

The subalgebra of $F_{E,Y}$ generated by the set

$$\bigcup_{i=1}^{t} E_i \cup \{y_1, \dots, y_r\}$$

will be denoted by $F_{t,r}$.

Proposition 1. For any T-ideal Γ there exist natural numbers t, r such that $\Gamma = T[A,Z]$, where $A = F_{t,r}/\Gamma(F_{t,r})$ and Z is the space generated by the classes of elements

$$\bigcup_{i=1}^{t} E_i \cup \{y_1, \dots, y_r\} .$$

Proof. We will make use of the representation theory of the symmetric group $S(n)$. For the details of this theory see [5-7]. The facts we need are all given in [8].

Let P_n be the space of multilinear polynomials of degree n in the variables $x_1, \dots, x_n \in X$. Put $\Gamma_n = \Gamma \cap P_n, d_n = \dim_F P_n/\Gamma_n$. It is known [9] that for some natural number $a > 1$ we have

$$d_n \leqslant a^n \tag{5}$$

for any n.

The dimension over F of a minimal left $FS(n)$-submodule of the left $FS(n)$-module P_n corresponding to the Young table D will be denoted by δ_D. A formula for δ_D can be found in [5, 7, 8].

If p, q are natural numbers, we denote by $D_{p,q}$ the Young table of degree pq consisting of p equal columns.

Put $t = a^2 + 1$ and choose r so that

$$\delta_{D_{t,r+1}} > a^{2t(r+1)}. \tag{6}$$

Such a choice is possible, since $\lim\limits_{q \to \infty} \dfrac{\delta_{D_{p,q}}}{(p-1)^{pq}} = \infty$ (this can be proved by using the formula for δ_D and Stirling's formula).

We will prove that t, r are the desired natural numbers.

Let $B_n = T[A,Z] \cap P_n$. Since $T[A,Z], \Gamma$ are S-ideals, it follows that B_n, Γ_n are left $FS(n)$-submodules of the module P_n and the S-ideals $T[A,Z], \Gamma$ are defined by these submodules. Therefore, it suffices to prove that $B_n = \Gamma_n$ for any n. But since P_n is a completely reducible $FS(n)$-module, it suffices to show that for any n and any table D of degree n we have

$$\Gamma_n \cap U_D = B_n \cap U_D, \tag{7}$$

where U_D is the homogeneous component of P_n corresponding to the table D.

We begin the proof of (7). Suppose D is an arbitrary table. There are only two possible cases.

Case 1. The table D contains the table $D_{t,r+1}$. We will show that $\Gamma_n \cap U_D = U_D$. Since $\Gamma \subseteq T[A,Z]$, equality (7) will follow.

Suppose first that $n \leq 2t(r+1)$. The formula (see [7]) $b_D = \sum_{D'} b_{D'}$, where the sum extends over all tables D' of degree $n-1$ that are contained in D, implies $b_D \geq b_{D_{t,r+1}}$. Therefore, if $U_D \not\subseteq \Gamma_n$, it follows from (6) that

$$d_n = \dim_F P_n/\Gamma_n \geq \dim_F U_D/U_D \cap \Gamma_n \geq b_D \geq b_{D_{t,r+1}} > a^{2t(r+1)} \geq a^n,$$

which contradicts (5). Therefore $U_D \subseteq \Gamma_n$.

Suppose $f(x_1,\ldots,x_{t(r+1)}) = \sum_{\sigma \in S(t(r+1))} \alpha_\sigma x_{\sigma(1)} \cdot\ldots\cdot x_{\sigma(t(r+1))}$ is an arbitrary polynomial in $U_{D_{t,r+1}}$. We will show that any polynomial of the form

$$g(x_1,\ldots,x_n) = \sum_{\sigma \in S(t(r+1))} \alpha_\sigma x_{\sigma(1)} \cdot y_1 x_{\sigma(2)} y_2 \cdot\ldots\cdot x_{\sigma(t(r+1))} y_{t(r+1)}, \tag{8}$$

where $y_i \in \{x_{t(r+1)+1},\ldots,x_n\}$, $y_i \neq y_j$, belongs to Γ. Indeed, for some permutation $\tau \in S(n)$ we have $g(x_1,\ldots,x_n) = (f(x_1,\ldots,x_{t(r+1)})x_{t(r+1)+1} \cdot\ldots\cdot x_n)\tau$ (the action of $\tau \in S$ on P_n from the right is defined by $(x_{j_1} x_{j_2} \cdot\ldots\cdot x_{j_n})\tau = x_{j_{\tau(1)}} \cdot\ldots\cdot x_{j_{\tau(n)}}$). It follows from Proposition 1 of [8] that

$$g \in \bigoplus_{D' \supset D_{t,r+1}} U_{D'}\tau = \bigoplus_{D' \supset D_{t,r+1}} U_{D'} \subseteq \Gamma.$$

If we make the substitution $x_i = u_i$ in (8), where $i > t(r+1)$, $u_i \in F\langle X\rangle \cup \{1\}$, we obtain

$$\sum_{\sigma \in S(t(r+1))} \alpha_\sigma x_{\sigma(1)} u_1 x_{\sigma(2)} u_2 \cdot\ldots\cdot x_{\sigma(t(r+1))} u_{t(r+1)} \in \Gamma. \tag{9}$$

Now suppose that n is an arbitrary number and h is any polynomial in U_D generating a minimal submodule. It follows from Remarks 1 and 2 of [8] that h can be represented as a linear combination of polynomials of the form (8), where $y_i \in F\langle X\rangle \cup \{1\}$,

$$\sum_{\sigma \in S(t(r+1))} \alpha_\sigma x_{\sigma(1)} \cdot\ldots\cdot x_{\sigma(t(r+1))} \in U_{D_{t,r+1}},$$

which, in view of (9), lie in Γ. Therefore, h, hence also U_D, is contained in Γ. Equality (7) is proved.

Case 2. The table D does not contain the table $D_{t,r+1}$. Since $\Gamma \subseteq T[A,Z]$, it suffices to prove that

$$U_D \cap B_n \subseteq U_D \cap \Gamma_n. \tag{10}$$

Suppose D^* is an arbitrary diagram of the table D (i.e., some filling in of the cells of D by the integers from 1 to n), and P_{D^*} (respectively, Q_{D^*}) is the subgroup of $S(n)$ consisting of all permutations fixing all rows (respectivley, columns) of the diagram D^*. Put

$$e_{D^*} = \sum_{p\in P_{D^*}} p\cdot \sum_{q\in Q_{D^*}} (-1)^q q .$$

It follows from the representation theory of the group $S(n)$ that the minimal ideal V_D of $FS(n)$, corresponding to the table D, is generated as left ideal by the elements e_{D^*}. Therefore, the module $U_D\cap B_n$ is generated by the elements $e_{D^*}f$, where $f\in U_D\cap B_n$.

Let $\mathcal{L}_i$ (respectively, N_i) be the set of numbers appearing in the i-th column (respectively, row) of the diagram D^*. Since D does not contain the table $D_{t,z+1}$, we can choose integers k, ℓ such that

$$0 < k \leq t, \quad 0 \leq \ell \leq z, \tag{11}$$

$$\bigcup_{i=1}^{k} \mathcal{L}_i \bigcup_{i=1}^{\ell} N_i = \{1,\dots,n\}. \tag{12}$$

Consider the subgroups $P^0_{D^*} = \{p\in P_{D^*} \mid p(j)=j \quad \forall j\in \bigcup_{i=1}^{k}\mathcal{L}_i\}$, $Q^0_{D^*} = \{q\in Q_{D^*} \mid q(j)=j \quad \forall j\notin \bigcup_{i=1}^{k}\mathcal{L}_i\}$.

We have

$$e_{D^*} = \sum_{p'\in P'} p'\cdot \sum_{p\in P^0_{D^*}} p\cdot \sum_{q\in Q^0_{D^*}} (-1)^q q \cdot \sum_{q'\in Q'} \alpha_{q'} q',$$

where P' (respectively, Q') is a set of representatives of the left (respectively, right) cosets of P_{D^*} (Q_{D^*}) modulo the subgroup $P^0_{D^*}$ ($Q^0_{D^*}$); $\alpha_{q'} = \pm 1$. Therefore, the module $U_D\cap B_n$ is generated by the elements

$$h_{D^*,f} = \Big(\sum_{p\in P^0_{D^*}} p\cdot \sum_{q\in Q^0_{D^*}} (-1)^q\cdot q\Big) f,$$

where $f\in U_D\cap B_n$.

Suppose $f=f(x_{i_{1,1}},\dots,x_{i_{1,n_1}},\dots,x_{i_{k,1}},\dots,x_{i_{k,n_k}},\dots,x_{j_{1,1}},\dots,x_{j_{1,m_1}},\dots,x_{j_{\ell,1}},\dots,x_{j_{\ell,m_\ell}})$, where $\{i_{s,1},\dots,i_{s,n_s}\}=\mathcal{L}_s$, $\{j_{s,1},\dots,j_{s,m_s}\}=N_s\setminus\bigcup_{i=1}^{k}\mathcal{L}_i$. In view of (12), such a representation is possible. It is easy to see that the polynomial $h_{D^*,f}$ is, to within sign and a renaming of the variables, a linearization of the polynomial

$$g = S_{T_1} S_{T_2}\dots S_{T_k}(f(t_1^{(1)},\dots,t_{n_1}^{(1)},\dots,t_1^{(k)},\dots,t_{n_k}^{(k)},\underbrace{y_1,\dots,y_1}_{m_1},\dots,\underbrace{y_\ell,\dots,y_\ell}_{m_\ell})).$$

Since $T[A,Z]$ is an S-ideal, we have $g\in T[A,Z]$; but then, in view of the definition of $T[A,Z]$ and (11), we obtain

$$h = g\big|_{t_i^{(j)}=e_i^{(j)},\ j=1,\dots,k;\ i=1,\dots,n_j} \in \Gamma(F_{t,z}),$$

where $e_i^{(j)}=\varphi(t_i^{(j)})$ (φ is the canonical isomorphism $M_{T,Y}\to F_{E,Y}$). Therefore, by Lemma 2, $\varphi^{-1}(h)\in\Gamma\cap M_{T,Y}\subseteq\Gamma$. Since $g\in M_{T,Y}$, $\varphi(g)=h$, and φ is an isomorphism, it follows that $g\in\Gamma\cap M_{T,Y}\subseteq\Gamma$. Therefore, $h_{D^*,f}\in\Gamma_n$. This proves (10), hence also (11).

The proposition is proved.

2. ALGEBRA G_2

Let I denote the S-ideal generated by the polynomials

$$[x_1, x_2, x_3], \tag{13}$$

$$\sum_{\sigma\in S(3)} [x_{\sigma(1)}, y_1][x_{\sigma(2)}, y_2][x_{\sigma(3)}, y_3], \tag{14}$$

where $x_i, y_j \in X$. Put $G_2 = F\langle X\rangle / I$. In view of the definition, this algebra is defined by the relations

$$[x_1, x_2, x_3] = 0, \tag{15}$$

$$\sum_{\sigma\in S(3)} [x_{\sigma(1)}, y_1][x_{\sigma(2)}, y_2][x_{\sigma(3)}, y_3] = 0, \tag{16}$$

where x_i, y_j are arbitrary generators.

In this section we will show that the algebra G_2 is an analog in some sense of the algebra $G\otimes G$.

LEMMA 3. Suppose G^* is a Grassmann algebra with 1. Then the algebra $G^*\otimes G^*$ is a homomorphic image of G_2 .

Proof. Suppose e_i are generators of the algebra G^* . We will identify the element $1\otimes 1$ with 1, and $e_i\otimes 1$ with e_i . We denote the element $1\otimes e_i$ by f_i . The algebra $G^*\otimes G^*$ is generated by the countable set $\{e_1, e_2, \dots\} \cup \{f_1, f_2, \dots\} \cup \{1\}$, and we need only verify relations (15) and (16) for the generators; this verification is immediate.

The lemma is proved.

A polynomial $f\in F\langle X\rangle$ will be called a commutator polynomial if f is homogeneous in all variables and can be represented as a linear combination of polynomials of the form $[y_1, y_2]\cdot\ldots\cdot[y_{2k-1}, y_{2k}]$, where $y_i\in X$.

LEMMA 4. Suppose A is an S -ideal and $A\supseteq I$. If A contains a commutator polynomial g that is not in I , then A contains a commutator polynomial f such that

1. $f\notin I$;

2. $f = \sum_{\sigma\in S(k)} (-1)^\sigma h(x_{\sigma(1)}, \dots, x_{\sigma(k)}, y_1, \dots, y_l)$, where h is a commutator polynomial, and x_i, y_j are pairwise distinct elements of X ;

3. $\deg_{x_i} f = 1$, $\deg_{y_i} f = 2$.

Proof. Let P_n be the left $FS(n)$ -module of multilinear polynomials in the variables $x_1, \dots, x_n$. Since A, I are S -ideals, it follows that $A_n = A\cap P_n$ and $I_n = I\cap P_n$ are submodules. We may assume that $g\in P_n$ and generates a minimal submodule K . Let U_D be the homogeneous component of P_n containing K . We will show that the table D contains at most two columns.

Suppose this is not so. As we observed in the proof of Proposition 1, K is generated by the polynomials $e_{D^*}g$, where D^* is an arbitrary diagram of D and

$$e_{D^*} = \sum_{p \in P_{D^*}} p \cdot \sum_{q \in Q_{D^*}} (-1)^q q.$$

Let P' be the subgroup of P_{D^*} consisting of the permutations fixing any number except for the first three (for definiteness we assume these three are 1, 2, 3) appearing in the first row. Then

$$e_{D^*} = \sum_{p \in P'} p \cdot \sum_{\sigma \in S(n)} \alpha_\sigma \sigma$$

for certain $\alpha_\sigma \in F$. Put

$$g_1 = \Big(\sum_{\sigma \in S(n)} \alpha_\sigma \sigma \Big) g.$$

It is easy to see that g_1 is a commutator polynomial, $g_1 \in K$, and

$$e_{D^*} g = \Big(\sum_{p \in P'} p \Big) g_1 (x_1, \dots, x_n) = \frac{|P'|}{3} \sum_{\sigma \in S(3)} g_1 (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_4, \dots, x_n).$$

Also, using relations (16) and (15), it is easy to see that the right-hand side is equal to 0 in G_2, i.e., $e_{D^*} g \in I$. Therefore, $K \subseteq I$, which contradicts the choice of g.

It follows from what has been proved and from Remarks 1 and 2 of [8] that K is generated by a polynomial that is a linearization of one of the form

$$h = [S_{k+l}(y_1, \dots, y_l, x_1, \dots, x_k) S_l (y_1, \dots, y_l)] \Big(\sum_{\sigma \in S(k+2l)} \alpha_\sigma \sigma \Big),$$

where $S_n(z_1, \dots, z_n)$ is a standard polynomial and $\alpha_\sigma \in F$. Putting

$$f = \sum_{\sigma \in S(k)} (-1)^\sigma h (x_{\sigma(1)}, \dots, x_{\sigma(k)}, y_1, \dots, y_l),$$

we have $f = k! \, h$. Therefore, a linearization of f generates K, hence $f \notin I$. Since K is generated by a commutator polynomial, it follows that all polynomials in K, and also h and f, are commutator polynomials.

The lemma is proved.

LEMMA 5. Suppose A is an S-ideal and $A \supseteq I$. If A contains a commutator polynomial f that is not in I, then A contains a polynomial of the form $[z_1, t_1]^2 \cdot \ldots \cdot [z_p, t_p]^2$, where z_i, t_j are pairwise distinct variables in X.

Proof. In view of Lemma 4, we may assume that f possesses properties 1-3 of Lemma 4.

The proof is by induction on the degree of f. The basis of the induction, $\deg f = 2$, is obvious.

Suppose

$$\deg f > 2, \quad f = \sum_{\sigma \in S(k)} (-1)^\sigma h (x_{\sigma(1)}, \dots, x_{\sigma(k)}, y_1, \dots, y_l).$$

There are only three possibilities.

Case 1. $l \geq 2$. We will show that

$$f \equiv [y_{\ell-1}, y_\ell]^2 f_1(x_1, \ldots, x_k, y_1, \ldots, y_{\ell-2}) (\mathrm{mod}\, I), \tag{17}$$

where f_1 is a commutator polynomial. Indeed, since f is a commutator polynomial, it follows from (15) that f can be represented as a linear combination of polynomials of the form

$$[z_1, z_2][z_3, z_4][z_5, z_6][z_7, z_8]\, g_z, \tag{18}$$

where $z_i \in \{x_1, \ldots, x_k, y_1, \ldots, y_\ell\}$, and g_z is a commutator polynomial not depending on $y_{\ell-1}, y_\ell$. We may assume that $z_1 = z_3 = y_{\ell-1}$ (otherwise we use (15)). We may also assume that $z_2 = y_\ell$. For if $z_2 \neq y_\ell$, but $z_4 = y_\ell$, then we can use (15). If $z_2 \neq y_\ell$ and $z_4 \neq y_\ell$, then we can assume $z_6 = z_8 = y_\ell$. Using (15) and (16), we obtain

$$[y_{\ell-1}, z_2][y_{\ell-1}, z_4][z_5, y_\ell][z_7, y_\ell] \overset{(16)}{\equiv} -[y_{\ell-1}, z_2][y_{\ell-1}, y_\ell][z_5, z_4][z_7, y_\ell] -$$

$$-[y_{\ell-1}, z_2][y_{\ell-1}, y_\ell][z_5, y_\ell][z_7, z_4] \overset{(15)}{\equiv} -[y_{\ell-1}, y_\ell][y_{\ell-1}, z_2][z_5, z_4][z_7, y_\ell] -$$

$$-[y_{\ell-1}, y_\ell][y_{\ell-1}, z_2][z_5, y_\ell][z_7, z_4] (\mathrm{mod}\, I),$$

from which it follows that the polynomial (18) can be represented modulo I as a linear combination of polynomials of the form (18) in which $z_1 = z_3 = y_{\ell-1}, z_2 = y_\ell$. If $z_4 \neq y_\ell$, then, as above, we may assume $z_6 = y_\ell$. Modulo I we have the congruences

$$[y_{\ell-1}, y_\ell][y_{\ell-1}, z_4][z_5, y_\ell] \overset{(16)}{\equiv} -[y_{\ell-1}, z_4][y_{\ell-1}, y_\ell][z_5, y_\ell] - [y_{\ell-1}, y_\ell]^2 [z_5, z_4] \overset{(15)}{\equiv}$$

$$\overset{(15)}{\equiv} -[y_{\ell-1}, y_\ell][y_{\ell-1}, z_4][z_5, y_\ell] - [y_{\ell-1}, y_\ell]^2 [z_5, z_4] (\mathrm{mod}\, I).$$

Therefore

$$[y_{\ell-1}, y_\ell][y_{\ell-1}, z_4][z_5, y_\ell] \equiv -\frac{1}{2}[y_{\ell-1}, y_\ell]^2 [z_5, z_4] (\mathrm{mod}\, I).$$

Congruence (17) is proved.

Let A_1 be the S-ideal generated by the set $\{f_1\} \cup I$. Using (15) and (17), it is easy to see that $[y_{\ell-1}, y_\ell]^8 A_1 \subseteq A$. Since $f_1 \notin I$ and $\deg f_1 < \deg f$, it follows from the induction assumption that the assertion of the lemma holds for A_1, hence also, in view of what was said above, for A.

Case 2. $\ell = 1$. We will show that this case is impossible. Indeed, for some $\alpha \in F$ we have

$$f \equiv \alpha \sum_{\sigma \in S(k)} (-1)^\sigma [x_{\sigma(1)}, y_1][x_{\sigma(2)}, y_1][x_{\sigma(3)}, x_{\sigma(4)}] \cdot \ldots \cdot [x_{\sigma(k-1)}, x_{\sigma(k)}] =$$

$$= \frac{\alpha}{2} \sum_{\sigma \in S(k)} (-1)^\sigma [[x_{\sigma(1)}, y_1], [x_{\sigma(2)}, y_1]][x_{\sigma(3)}, x_{\sigma(4)}] \cdot \ldots \cdot [x_{\sigma(k-1)}, x_{\sigma(k)}] \equiv 0 (\mathrm{mod}\, I).$$

Case 3. $\ell = 0$. Here we may assume that $f = S_{2q}(x_1, \ldots, x_{2q})$. Since A is an S-ideal we have $g = S_{2q}(y_1, y_2, x_3, \ldots, x_{2q})[y_1, y_2] \in A$, where $x_i, y_j \in X$. If $g \notin I$, then g satisfies the conditions of the first case and $g \equiv [y_1, y_2]^2 f_1(x_3, \ldots, x_{2q}) \pmod{I}$, where $f_1 \notin I$. If A_1 is the S-ideal generated by the set $\{f_1\} \cup I$, then, using (15), it is easy to show that $[y_1, y_2]^2 A_1 \subseteq A$. Since $\deg f_1 < \deg f$, the assertion of the lemma follows from the induction assumption.

Thus, it remains to prove that $g \notin I$. In view of Lemma 3, it suffices to show that the algebra $G^* \otimes G^*$ satisfies the relation

$$\sum_{\sigma, \tau \in S(2)} S_{2q}(y_{\tau(1)}, z_{\sigma(1)}, x_3, \ldots, x_{2q})[y_{\tau(2)}, z_{\sigma(2)}] \neq 0 \tag{19}$$

for certain $y_i, z_j, x_s \in \{e_1, e_2, \ldots\} \cup \{f_1, f_2, \ldots\}$ (the left-hand side of (19) is a linearization of the polynomial g). Put $y_1 = e_1$, $y_2 = f_1$, $z_1 = e_2$, $z_2 = f_2$, $x_i = e_i$. Then the left-hand side of (19) is equal to

$$\begin{aligned} S_{2q}(e_1, \ldots, e_{2q})[f_1, f_2] + S_{2q}(f_1, f_2, e_3, \ldots, e_{2q})[e_1, e_2] = 2(2q)!\, e_1 \cdot \ldots \cdot e_{2q} f_1 f_2 + \\ + 4(2q-2)! \Big(\sum_{j=0}^{2q-2} \sum_{i=0}^{j} (-1)^{i+j} e_1 \cdot \ldots \cdot e_i f_1 e_{i+1} \cdot \ldots \cdot e_j f_2 e_{j+1} \cdot \ldots \cdot e_{2q-2}\Big) e_{2q-1} e_{2q} = \\ = [2(2q)! + 4q(2q-2)!]\, e_1 \cdot \ldots \cdot e_{2q} f_1 f_2 \neq 0. \end{aligned}$$

The lemma is proved.

If A is an algebra, we denote by $T[A]$ the ideal of identities of A. Let $\Gamma_2 = T[G \otimes G]$, where G is a Grassmann algebra of countable rank, and let $\Gamma_1 = T[G]$.

Proposition 2. $T[G_2] = \Gamma_2$.

Proof. It is well known that $\Gamma_1 = \{[x, y, z]\}^T$. Therefore, since G^* (a Grassmann algebra with 1) satisfies the identity $[x, y, z] = 0$, it follows that $T[G^*] = \Gamma_1$. It is now easy to see that $T[G^* \otimes G^*] = \Gamma_2$. Therefore, it suffices to prove that $T[G^* \otimes G^*] = T[G_2]$.

Let $\mathcal{L}$ be the subspace of G^* generated by the elements $1 = 1 \otimes 1$, $e_i = e_i \otimes 1$, $f_i = 1 \otimes e_i$. Consider the S-ideal of identities $I_1 = T[G^* \otimes G^*, \mathcal{L}]$ of the pair $(G^* \otimes G^*, \mathcal{L})$ (see Definition 2). Obviously, $T[G^* \otimes G^*]$ is the largest T-ideal contained in I_1 and $T[G_2]$ is the largest T-ideal contained in I. Therefore, it suffices to prove that $I_1 = I$.

It follows from Lemma 3 that $I_1 \supseteq I$. Suppose $f \in I_1$, $f \notin I$. Since I_1, I are S-ideals, we may assume that f is a multilinear polynomial in the variables $x_1, \ldots, x_n$.

If $A = \{x_{i_1}, \ldots, x_{i_k}\} \subseteq \{x_1, \ldots, x_n\}$, where $i_1 < i_2 < \ldots < i_k$, then we denote by x_A the monomial $x_{i_1} \cdot \ldots \cdot x_{i_k}$.

Modulo I the polynomial f can be represented in the form

$$f = \sum_A x_A f_A,$$

where f_A is a commutator polynomial in the variables in $\{x_1, \ldots, x_n\} \setminus A$. Let A be a set of maximal cardinality such that $f_A \notin I$. We may assume without loss of generality that $A = \{x_1, \ldots, x_k\}$. Consider the polynomial $g(x_{k+1}, \ldots, x_n) = f(x_1, \ldots, x_n)|_{x_1 = x_2 = \ldots = x_k = 1}$. Since $f \in I_1$, we have $g \in I_1$. Obviously, $g \equiv f_A \pmod{I}$. Therefore, $f_A \subseteq I_1$. Then, by Lemma 5, I_1 contains the polynomial $h=[z_1,t_1]^2\cdot\ldots\cdot[z_p,t_p]$ for some p, where z_i, y_j are pairwise distinct variables in X. It remains to show that $h\notin I_1$. Put $z_i=e_{2i-1}+f_{2i-1}$, $t_i=e_{2i}+f_{2i}$. Then we easily see that $[z_1,t_1]^2\cdot\ldots\cdot[z_p,t_p]^2=4^p e_1\cdot\ldots\cdot e_{2p}f_1\cdot\ldots\cdot f_{2p}$, i.e., $h\notin I_1$.

The proposition is proved.

Proposition 3. Suppose Γ is a T-ideal such that $\Gamma\supseteq\Gamma_2$, $\Gamma\neq\Gamma_2$. Then for some natural number q we have $\Gamma\supseteq\Gamma_1^q$.

Proof. Consider the S-ideal $S=\Gamma+I$. Since Γ_2 is the largest T-ideal contained in I and since $\Gamma\neq\Gamma_2$, it follows that $S\neq I$. Suppose $f\in S$, $f\notin I$. Since S, I are S-ideals, we may assume that f is a multilinear polynomial in the variables $x_1,\ldots,x_n$. As in Proposition 2, we represent f in the form

$$f=\sum_A x_A f_A,$$

where f_A is a commutator polynomial in the variables in $\{x_1,\ldots,x_n\}\setminus A$. Let A be a set of maximal cardinality such that $f_A\notin I$. We may assume without loss of generality that $A=\{x_1,\ldots,x_k\}$. Put

$$g(y_1,\ldots,y_k,z_1,\ldots,z_k,x_{k+1},\ldots,x_n)=f(x_1,\ldots,x_n)\Big|_{\substack{x_1=[y_1,z_1]\\ \vdots \\ x_k=[y_k,z_k]}}, \tag{20}$$

where y_i, z_j, x_s are pairwise distinct variables in X. Obviously,

$$g\equiv[y_1,z_1]\cdot\ldots\cdot[y_k,z_k]f_A(x_{k+1},\ldots,x_n)\ (\mathrm{mod}\ I). \tag{21}$$

We will show that $g\in S$. Since $f\in S$, we have $f=f_1+f_2$, where $f_1\in\Gamma$, $f_2\in I$. We define polynomials g_1, g_2 by means of (20) (instead of f use f_1 and f_2). Then $g=g_1+g_2$. Since Γ is T-ideal $g_1\in\Gamma$. The polynomial f_2 can be represented in the form $f_2=\sum_i u_i h_i v_i$, where $u_i, v_i\in F\langle X\rangle\cup\{1\}$, and the h_i are polynomials of the form (13) or (14). If $x_1,\ldots,x_k$ do not occur in h_i, then

$$u_i h_i v_i\Big|_{\substack{x_1=[y_1,z_1]\\ \vdots \\ x_k=[y_k,z_k]}}=\sum u' h_i v'. \tag{22}$$

In the opposite case, the left-hand side of (22) lies in the S-ideal generated by the polynomials of the form (13). Therefore, $g_2\in I$. Then it follows from (21) that $[y_1,z_1]\cdot\ldots\cdot[y_k,z_k]f_A\in I$, where f_A is a computer polynomial, $f_A\notin I$. Using (15), it is easy to see that

$$[y_1,z_1]\cdot\ldots\cdot[y_k,z_k]E\subseteq S, \tag{23}$$

where E is the S-ideal generated by $\{f_A\}\cup I$. By Lemma 5, $E\ni[y_{k+1},z_{k+1}]^2\cdot\ldots\cdot[y_p,z_p]^2$ for some p. Therefore, in view of (23) and (15), we obtain $[y_1,z_1]^2\cdot\ldots\cdot[y_p,z_p]^2\in S$.

Put

$$h(x_1,x_2,t_1,t_2)=\sum_{\sigma,\tau\in S(2)}[x_{\sigma(1)},t_{\tau(1)}]\cdot[x_{\sigma(2)},t_{\tau(2)}],$$

where $x_i, t_j \in X$ (h is a linearization of the polynomial $[y,z]^2$). Since S is an S-ideal, it follows from what was said above that

$$u(x_1,\ldots,x_{2p},t_1,\ldots,t_{2p})=h(x_1,x_2,t_1,t_2)\cdot\ldots\cdot h(x_{2p-1},x_{2p},t_{2p-1},t_{2p})\in S,$$

where x_i, t_j are pairwise distinct variables in X.

Put $v(x_1,\ldots,x_{4p},y_1,\ldots,y_{4p})=u(c_1,\ldots,c_{4p})$, where $c_i=[x_i,y_i]$ and v is multilinear. We will show that $v\in\Gamma$. Indeed, since $u\in S=\Gamma+I$, we have $v=v_1+v_2$, where $v_1\in\Gamma$ and v_2 lies in the S-ideal generated by the polynomials $[c_i,c_j,c_k]$,

$$\sum_{\sigma\in S(3)}[c_{\sigma(1)},c_4][c_{\sigma(2)},c_5][c_{\sigma(3)},c_6],$$

where $c_i=[x_i,y_i]$. Therefore, it suffices to show that these polynomials lie in Γ_2, or, equivalently, that the algebra $G\otimes G$ (or $G^*\otimes G^*$) satisfies the identities

$$[c_1,c_2,c_3]=0, \tag{24}$$

$$\sum_{\sigma\in S(3)}[c_{\sigma(1)},c_4][c_{\sigma(2)},c_5][c_{\sigma(3)},c_6]=0, \tag{25}$$

where $c_i=[x_i,y_i]$.

The algebra $G^*\otimes G^*$ can be represented in the form

$$G^*\otimes G^*=Z+\sum_i e_iZ+\sum_i f_iZ+\sum_{i,j}e_if_jZ,$$

where Z is the center of $G^*\otimes G^*$. Using these decompositions, we see easily that for any $x_i, y_i\in G^*\otimes G^*$ we have

$$c_i=[x_i,y_i]\in Z+\sum_i e_iZ+\sum_i f_iZ.$$

After this observation it is trivial to verify (24) and (25).

It is well known and easy to prove that if $g=f_1\cdot\ldots\cdot f_n\in\Gamma$, where Γ is a T-ideal and the polynomials $g,f_1,\ldots,f_n$ are multilinear, then $T_1\cdots T_n\subseteq\Gamma$, where $T_i=\{f_i\}^T$. In view of this observation and what was proved above, it follows that $T^p\subseteq\Gamma$, where T is the T-ideal generated by the set $\{h(c_1,c_2,c_3,c_4)\}\cup\Gamma_2$, where $c_i=[x_i,y_i]$. It remains to prove that $T\supseteq\Gamma_1^z$ for some z. We isolate this fact in a separate lemma.

<u>LEMMA 6.</u> $T\supseteq\Gamma_1^z$.

<u>Proof.</u> Since identity (24) holds modulo Γ_2 and since $T\supseteq\Gamma_2$, we have modulo T the identities

$$\frac{1}{2}h(c_1,c_2,c_3,c_4)\overset{(24)}{\equiv}[c_1,c_2][c_3,c_4]+[c_1,c_4][c_3,c_2]\equiv 0, \tag{26}$$

$$[c_1,c_2][c_3,c_4]=-[c_1,c_4][c_3,c_2]. \tag{27}$$

It is easy to see that in the algebra $G\otimes G$, hence also modulo T, we have the identity

$$[[x_1,x_2],[x_3,x_4],x_5] = 0. \tag{28}$$

In (26) we make the replacement $x_1 = x_6 c_5$, where $c_5 = [x_5, y_5]$. Using (27) and (28), we obtain the following congruence modulo T:

$$0 \equiv [x_6, y_1]([c_5,c_2][c_3,c_4] + [c_5,c_4][c_3,c_2]) + c_5([[x_6,y_1],c_2][c_3,c_4] +$$
$$+ [[x_6,y_1],c_4][c_3,c_2]) + x_6([[c_5,y_1],c_2][c_3,c_4] + [[c_5,y_1],c_4][c_3,c_2]) + ([x_6,c_2][c_3,c_4] + [x_6,c_4][c_3,c_2])[c_5,y_1].$$

Then

$$([x_6,c_2][c_3,c_4] + [x_6,c_4][c_3,c_2])[c_5,y_1] \equiv 0 \ (\mathrm{mod}\ T). \tag{29}$$

Making the substitution $x_6 = x_6 c_7$, where $c_7 = [x_7, y_7]$, we obtain the congruence

$$([x_6,c_2]c_7[c_3,c_4] + [x_6,c_4]c_7[c_3,c_2])[c_5,y_1] \equiv 0 \ (\mathrm{mod}\ T),$$

from which it follows that

$$([x_6,c_2,c_7][c_3,c_4] + [x_6,c_4,c_7][c_3,c_2])[c_5,y_1] \equiv 0 \ (\mathrm{mod}\ T). \tag{30}$$

Using (27) and (28), we obtain

$$[x_6,c_2,c_7][c_3,c_4] \overset{(27)}{\equiv} -[x_6,c_2,c_4][c_3,c_7] \overset{(28)}{\equiv} -[x_6,c_4,c_2][c_3,c_7] \overset{(27)}{\equiv} [x_6,c_4,c_7][c_3,c_2] \ (\mathrm{mod}\ T).$$

In view of (30),

$$[x_6,c_4,c_7][c_3,c_2][c_5,y_1] \equiv 0 \ (\mathrm{mod}\ T). \tag{31}$$

Let B be the T-ideal generated by the polynomial $[x_1,x_2,x_3,[x_4,x_5]]$, where $x_i \in X$. It follows from (31) that $T \supseteq B^2 \cdot T_1$. It remains to show that $B \supseteq T_1^3$.

In the identity (modulo B) $[x_1,x_2,x_3,[x_4,x_5]] = 0$ we make the substitution $x_4 = x_4 \cdot [x_6,x_7]$ and obtain the congruence

$$0 \equiv [x_1,x_2,x_3,[x_4\cdot[x_6,x_7],x_5]] \equiv [x_1,x_2,x_3,x_4\cdot[x_6,x_7,x_5]] \equiv [x_1,x_2,x_3,x_4][x_6,x_7,x_5] \equiv 0 \ (\mathrm{mod}\ B). \tag{32}$$

Then making the substitution $x_3 = x_3 \cdot [x_8, x_9]$ in (32) we obtain

$$0 \equiv [x_1,x_2,x_3\cdot[x_8,x_9],x_4][x_6,x_7,x_5] \equiv [[x_1,x_2,x_3]\cdot[x_8,x_9],x_4] \times$$
$$\times [x_6,x_7,x_5] \equiv [x_1,x_2,x_3][x_8,x_9,x_4][x_6,x_7,x_5] \ (\mathrm{mod}\ B).$$

The lemma and the proposition are proved.

3. NONMATRIX VARIETIES

In this section, $\mathfrak{M}$ is a fixed nonmatrix variety and Γ is the ideal of identities of $\mathfrak{M}$. Suppose p, q are arbitrary nonnegative integers. Put $\bar{F}_{p,q} = F_{p,q}/\Gamma(F_{p,q})$. The images of the sets E_i, $Y_q = \{y_1, \ldots, y_q\}$ under the natural homomorphism $F_{p,q} \to \bar{F}_{p,q}$ will be denoted by $\bar{E}_i$, $\bar{Y}_q = \{z_1, \ldots, z_q\}$ respectively.

Let $\mathcal{U}_{p,q}$ be the ideal of $\bar{F}_{p,q}$ generated by all elements of the form $[z_i, z_j]$, $[e,f,z_i]$, $[ef, gh]$, $[e,f,g]$, where

$$z_i, z_j \in \bar{Y}_q, e, f, g, h \in \bigcup_{i=1}^{p} \bar{E}_i.$$

LEMMA 7. If p, q are arbitrary, then $\mathcal{U}_{p,q}$ is a nilpotent ideal of the algebra $\bar{F}_{p,q}$.

Proof. Let $V^{(1)}_{i,j}$, where $1 \le i, j \le q$, be the ideal generated by the element $[z_i, z_j]$, $z_i, z_j \in \bar{Y}_q$; $V^{(2)}_{\kappa,\ell,i}$, where $1 \le \kappa, \ell \le p$, $1 \le i \le q$, the ideal generated by all elements $[e,f,z_i]$, where $e \in \bar{E}_\kappa, f \in \bar{E}_\ell, z_i \in \bar{Y}_q$; $V^{(3)}_{\kappa,\ell,\mu,\nu}$ is the ideal generated by the elements $[ef, gh]$, where $e \in \bar{E}_\kappa$, $f \in \bar{E}_\ell$, $g \in \bar{E}_\mu$, $h \in \bar{E}_\nu$, $1 \le \kappa, \ell, \mu, \nu \le p$; $V^{(4)}_{\kappa,\ell,\mu}$, where $1 \le \kappa, \ell, \mu \le p$, the ideal generated by all elements $[e,f,g]$, where $e \in \bar{E}_\kappa, f \in \bar{E}_\ell, g \in \bar{E}_\mu$. Since $\mathcal{U}_{p,q}$ is the sum of the ideals $V^{(1)}_{i,j}$, $V^{(2)}_{\kappa,\ell,i}$, $V^{(3)}_{\kappa,\ell,\mu,\nu}$, $V^{(4)}_{\kappa,\ell,\mu}$, the number of which is finite, it suffices to prove the nilpotency of which of these four ideals for fixed $i, j, \kappa, \ell, \mu, \nu$.

Since $\mathfrak{M}$ is a nonmatrix variety, the commutator ideal of the algebra $A = F\langle X\rangle/\Gamma$ is a nil ideal. Therefore, A satisfies the identity $([x,t]y)^m = 0$ for some m. This means that for any x, t the right ideal $E = [x,t]A$ satisfies the identity $x^m = 0$, where $m = m(\Gamma)$. By the Nagata–Higman theorem [10], E is nilpotent. But then the ideal generated by $[x,t]$ is nilpotent. Therefore, the algebra A, hence any algebra in $\mathfrak{M}$, satisfies the identity

$$[x,t]\, y_1 [x,t]\, y_2 \cdot \ldots \cdot y_n [x,t] = 0 \tag{33}$$

for an arbitrary $n = n(m)$, where some of the variables y_r can be absent. It follows at once from (33) that $V^{(1)}_{i,j}$ is nilpotent.

Linearizing (33), we obtain an equivalent identity

$$\sum_{\sigma,\tau \in S(n+1)} [x_{\sigma(1)}, t_{\tau(1)}]\, y_1 \cdot \ldots \cdot y_n\, [x_{\sigma(n+1)}, t_{\tau(n+1)}] = 0. \tag{34}$$

We make the substitutions $x_s = [e_s, f_s]$ in (34), where $e_s \in \bar{E}_\kappa, f_s \in \bar{E}_\ell$; $t_s = z_i$, where $z_i \in \bar{Y}_q$; $y_s = a_s$, where $a_s \in \bar{F}_{p,q} \cup \{1\}$. In the algebra $\bar{F}_{p,q}$ we obtain

$$\begin{aligned} 0 &= (n+1)! \sum_{\sigma \in S(n+1)} [e_{\sigma(1)}, f_{\sigma(1)}, z_i]\, a_1 \cdot \ldots \cdot a_n\, [e_{\sigma(n+1)}, f_{\sigma(n+1)}, z_i] \overset{(1)}{=} \\ &= [(n+1)!]^2 \sum_{\sigma \in S(n+1)} (-1)^\sigma [e_1, f_{\sigma(1)}, z_i]\, a_1 \cdot \ldots \cdot a_n\, [e_{n+1}, f_{\sigma(n+1)}, z_i] \overset{(1)}{=} \\ &= [(n+1)!]^3 [e_1, f_1, z_i]\, a_1 \cdot \ldots \cdot a_n\, [e_{n+1}, f_{n+1}, z_i], \end{aligned}$$

from which it follows at once that $V^{(2)}_{\kappa,\ell,i}$ is nilpotent.

In exactly the same way we can prove that $V^{(3)}_{\kappa,\ell,\mu,\nu}$ is nilpotent. To do this we make the replacement $x_s = e_s f_s$, $t_s = g_s h_s$, $y_s = a_s$ in (34), where

$$e_s \in \bar{E}_\kappa, f_s \in \bar{E}_\ell, g_s \in \bar{E}_\mu, h_s \in \bar{E}_\nu, a_s \in \bar{F}_{p,q} \cup \{1\}.$$

It remains to show that the ideal $V^{(4)}_{\kappa,\ell,\mu}$ is nilpotent. Put

$$V^{(3)} = \sum_{1 \le \kappa,\ell,\mu,\nu \le p} V^{(3)}_{\kappa,\ell,\mu,\nu}.$$

Since the ideals $V^{(3)}_{\kappa,\ell,\mu,\nu}$ are nilpotent, so is $V^{(3)}$.

Suppose $e_s \in \bar{E}'_\kappa$, $g_s \in \bar{E}_\mu$, $f \in \bar{E}_\nu$. Modulo $V^{(3)}$ we have the congruences

$$[e_1 e_2, g_1] f [e_3 e_4, g_2] = e_1 e_2 g_1 f [e_e e_4, g_2] - g_1 e_1 e_2 f [e_3 e_4, g_2] \equiv$$
$$\equiv g_1 f e_1 e_2 [e_3 e_4, g_2] - e_2 f g_1 e_1 [e_3 e_4, g_2] = g_1 f e_1 [e_3 e_4, e_2 g_2] - e_2 f g_1 [e_3 e_4, e_1 g_2] \equiv 0,$$

i.e., $[e_1 e_2, g_1] f [e_3 e_4, g_2] \equiv 0 \pmod{V^{(3)}}$.

Therefore, since $V^{(3)}$ is nilpotent, $\Gamma(F_{p,q})$ (hence also $\Gamma(F_{E,Y})$) contains, for some m, all elements of the form

$$[e_1 e_2, g_1] f_1 [e_3 e_4, g_2] f_2 \cdot \ldots \cdot f_{m-1} [e_{2m-1} e_{2m}, g_m] f_m,$$

where $e_s \in E_\kappa, g_s \in E_\mu, f_s \in E_\nu$. It follows from this and Lemmas 1 and 2 that the algebra $A = F\langle X\rangle/\Gamma$ (and also any algebra in $\mathfrak{M}$) satisfies the identity

$$\sum_{\substack{\sigma \in S(2m)\\ \tau,\pi \in S(m)}} (-1)^\sigma (-1)^\tau (-1)^\pi [x_{\sigma(1)} x_{\sigma(2)}, y_{\tau(1)}] t_{\pi(1)} \cdot \ldots \cdot [x_{\sigma(2m-1)} x_{\sigma(2m)}, y_{\tau(m)}] t_{\pi(m)} = 0. \tag{35}$$

If in (35) we put $x_\tau = e_\tau$, $y_\tau = g_\tau$, $t_\tau = a_\tau$, where $e_\tau \in \bar{E}_\kappa, g_\tau \in \bar{E}_\mu, a_\tau \in \bar{F}_{E,Y}$ $(\bar{F}_{E,Y} = F_{E,Y}/\Gamma(F_{E,Y}))$ we obtain in the algebra $\bar{F}_{E,Y}$ the equality

$$0 = \sum_{\sigma \in S(m)} (-1)^\sigma [e_1 e_2, g_1] a_{\sigma(1)} \cdot \ldots \cdot [e_{2m-1} e_{2m}, g_m] a_{\sigma(m)} \overset{(1)}{=}$$

$$\overset{(1)}{=} h_m(b_1, \ldots, b_m) = 0, \text{ where } h_m(x_1, \ldots, x_m) = \sum_{\sigma \in S(m)} x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(m)};\ b_s = [e_{2s-1} e_{2s}, g_s] a_s.$$

Therefore, the right ideal $\mathcal{L}_{\kappa,\mu}$ of $\bar{F}_{E,Y}$ generated by all elements $[e_1 e_2, g] a$, where $e_s \in \bar{E}_\kappa, g \in \bar{E}_\mu, a \in \bar{F}_{E,Y}$, satisfies the identity $h_m(x_1, \ldots, x_m) = 0$. By the Nagata–Higman theorem [10], we see that $\mathcal{L}_{\kappa,\mu}$, hence also the ideal $W_{\kappa,\mu}$ generated by the elements $[e_1 e_2, g]$, is nilpotent, i.e., for some $\tau = \tau(m)$ the ideal $\Gamma(F_{E,Y})$ contains all elements of the form

$$[e_1 e_2, g_1] a_1 \cdot \ldots \cdot a_{\tau-1} [e_{2\tau-1} e_{2\tau}, g_\tau],$$

where $e_s \in E_\kappa, g_s \in E_\mu, a_s \in F_{E,Y} \cup \{1\}$. Again applying Lemmas 1 and 2, we see that any algebra in $\mathfrak{M}$ satisfies the identity

$$\sum_{\substack{\sigma \in S(2\tau)\\ \tau \in S(\tau)}} (-1)^\sigma (-1)^\tau [x_{\sigma(1)} x_{\sigma(2)}, y_{\tau(1)}] t_1 \cdot \ldots \cdot t_{\tau-1} [x_{\sigma(2\tau-1)} x_{\sigma(2\tau)}, y_{\tau(\tau)}] = 0, \tag{36}$$

where some of the variables t_s can be absent.

Put $W'_{\kappa,\mu} = W_{\kappa,\mu} \cap F_{p,q}$, $W = \sum_{1 \le \kappa, \mu \le p} W'_{\kappa,\mu}$. In view of what was said above, W is nilpotent.

If in (36) we make the substitution $x_{2s-1} = e_s, x_{2s} = f_s, y_s = g_s, t_s = a_s$, where $e_s \in \bar{E}_\kappa, f_s \in \bar{E}_\ell$, $g_s \in \bar{E}_\mu$, $a_s \in \bar{F}_{p,q} \cup \{1\}$ we obtain modulo W a congruence in the algebra $\bar{F}_{p,q}$:

$$0=\sum_{\substack{\sigma\in S(2\tau)\\ \tau\in S(\tau)}}(-1)^{\sigma}(-1)^{\tau}[x_{\sigma(1)}x_{\sigma(2)},y_{\tau(1)}]t_1\cdot\ldots\cdot t_{\tau-1}[x_{\sigma(2\tau-1)}x_{\sigma(2\tau)},y_{\tau(\tau)}]\Bigg|_{\substack{x_{2s-1}=e_s\\ x_{2s}=f_s\\ y_s=g_s\\ t_s=a_s}}=$$

$$=\sum_{\sigma,\tau,\pi\in S(\tau)}(-1)^{\sigma}(-1)^{\tau}(-1)^{\pi}[e_{\sigma(1)},f_{\tau(1)},g_{\pi(1)}]a_1\cdot\ldots\cdot a_{\tau-1}[e_{\sigma(\tau)},f_{\tau(\tau)},g_{\pi(\tau)}],$$

hence, in view of (1),

$$[e_1,f_1,g_1]a_1[e_2,f_2,g_2]a_2\cdot\ldots\cdot a_{\tau-1}[e_\tau,f_\tau,g_\tau]\equiv 0\ (\mathrm{mod}\ W).$$

This implies that the ideal $V^{(4)}_{\kappa,\ell,\mu}$ is nilpotent.

The lemma is proved.

Proposition 4. For any p there exist $\alpha=\alpha(p)$ such that $T[\bar{F}_{p,0}]\supseteq\Gamma_2^{\alpha}$ ($T[\bar{F}_{p,0}]$ is the ideal of identities of the algebra $\bar{F}_{p,0}$).

Proof. In view of Lemma 7, it suffices to show that $T[\bar{F}_{p,0}/\mathcal{U}_{p,0}]\supseteq\Gamma_2$. In view of Proposition 2, it is enough to prove that in the algebra $\bar{F}_{p,0}/\mathcal{U}_{p,0}$ the generators in $\bigcup_{i=1}^{p}E_i$ satisfy (15) and (16).

In this case the ideal $\mathcal{U}_{p,0}$ is generated by the elements $[e,f,g]$, $[ef,gh]$, where $e,f,g,h\in\bigcup_{i=1}^{p}E_i$. Relation (15) follows trivially from the definition of $\mathcal{U}_{p,0}$. We will show that in the algebra $\bar{F}_{p,0}$ the generators in $\bigcup_{i=1}^{p}E_i$ satisfy modulo $\mathcal{U}_{p,0}$ the relation

$$\sum_{\sigma\in S(3)}[x_{\sigma(1)},y_1]x_{\sigma(2)}x_{\sigma(3)}=0. \tag{37}$$

Indeed,

$$\sum_{\sigma\in S(3)}[x_{\sigma(1)},y_1]x_{\sigma(2)}x_{\sigma(3)}\equiv\sum_{\sigma\in S(3)}x_{\sigma(1)}[x_{\sigma(2)},y_1]x_{\sigma(3)}=\sum_{\sigma\in S(3)}x_{\sigma(1)}x_{\sigma(2)}y_1x_{\sigma(3)}-$$

$$-\sum_{\sigma\in S(3)}x_{\sigma(1)}y_1x_{\sigma(2)}x_{\sigma(3)}\equiv\sum_{\sigma\in S(3)}y_1x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}-\sum_{\sigma\in S(3)}x_{\sigma(1)}y_1x_{\sigma(2)}x_{\sigma(3)}=\sum_{\sigma\in S(3)}[x_{\sigma(1)},y_1]x_{\sigma(2)}x_{\sigma(3)},$$

which implies (37). From (37) we obtain

$$0\equiv\sum_{\sigma\in S(3)}[[x_{\sigma(1)},y_1]x_{\sigma(2)}x_{\sigma(3)},y_2,y_3]=2\sum_{\sigma\in S(3)}[[x_{\sigma(1)},y_1][x_{\sigma(2)},y_2]x_{\sigma(3)},y_3]\equiv 2\sum_{\sigma\in S(3)}[x_{\sigma(1)},y_1][x_{\sigma(2)},y_2][x_{\sigma(3)},y_3].$$

Relation (16) is proved.

The proposition is proved.

From now on, t,τ are fixed natural numbers such that the T-ideal Γ satisfies the conclusion of Proposition 1, and $p=t+t\tau$.

LEMMA 8. Suppose $f=f(x_1,\ldots,x_n)$ is a multilinear polynomial and $f\in T[\bar{F}_{p,0}]$. Let T_1 denote the T -ideal generated by the multilinear polynomial

$$g=f(x_1,\ldots,x_n)S_{2(\tau+n)}(y_1,\ldots,y_{2(\tau+n)}),$$

where $x_i, y_j \in X$, and S_k is a standard polynomial, and let T_2 denote the T-ideal generated by all multilinear polynomials

$$h_W = f\left(w_1 S_{2(r+1)}(y_1^{(1)},\ldots,y_{2(r+1)}^{(1)}),\ldots,w_n S_{2(r+1)}(y_1^{(n)},\ldots,y_{2(r+1)}^{(n)})\right),$$

where $w_i, y_j^{(i)} \in X$, some of the variables w_i can be absent, and W is the set of variables w_i occurring in h_W. Then $T_1^\beta, T_2^\beta \subseteq T$ for some natural number β.

Proof. By Proposition 1, $T = T[A,Z]$, where $A = \bar{F}_{t,r}$, and Z is the subspace of A generated by the set $\bigcup_{i=1}^{t} \bar{E}_i \cup \bar{Y}_r$. By Lemma 7, the ideal $\mathcal{U}_{t,r}$ is nilpotent. Let β be the index of nilpotency of $\mathcal{U}_{t,r}$. In view of what was said above, it suffices to show that $g, h_W \in \mathcal{U}_{t,r}$ for any $x_i, y_i, y_j^{(i)}, w_i \in \bigcup_{i=1}^{t} \bar{E}_i \cup \bar{Y}_r$.

Consider the subalgebra D of $\bar{F}_{t,r}$ generated by the set $\bigcup_{i=1}^{t} \bar{E}_i \cup (\bigcup_{j=1}^{r} \bigcup_{i=1}^{t} z_j \bar{E}_i)$, where $z_j \in \bar{Y}_r$. Since the elements of $z_j \bar{E}_i$ satisfy (1), the algebra D is a homomorphic image of $\bar{F}_{p,0}$, where $p = t + tr$.

If more than r elements among $y_1,\ldots,y_{2(r+k)}$ belong to $\bar{Y}_r$, then at least two of these elements are equal to and we have $g = 0$. Similarly, if more than r elements among $y_1^{(i)}, \ldots, y_{2(r+1)}^{(i)}$ belong to $\bar{Y}_r$ for some i, then $h_W = 0$. In the opposite case, since

$$S_{2k}(x_1,\ldots,x_{2k}) = 1/2^k \sum_{\sigma \in S(2k)} (-1)^\sigma [x_{\sigma(1)}, x_{\sigma(2)}] \cdot \ldots \cdot [x_{\sigma(2k-1)}, x_{\sigma(2k)}],$$

it follows that $S_{2(r+n)}(y_1,\ldots,y_{2(r+n)})$ can be represented modulo $\mathcal{U}_{t,r}$ as a linear combination of elements of the form $c_1 \cdot \ldots \cdot c_n d$, where $c_i = [a_i, b_i]$,

$$a_i, b_i \in \bigcup_{i=1}^{t} \bar{E}_i,\ d \in D,$$

and $S_{2(r+1)}(y_1^{(i)},\ldots,y_{2(r+1)}^{(i)})$ can be represented as a linear combination of elements of the form cd, where $c = [a,b]$, $a, b \in \bigcup_{i=1}^{t} \bar{E}_i$, $d \in D$. Therefore, since modulo $\mathcal{U}_{t,r}$ the elements c_i, c lie in the center, it follows that g and h_W can be represented modulo $\mathcal{U}_{t,r}$ as a linear combination of elements of the form $f(d_1,\ldots,d_n) d_{n+1}$, where $d_i \in D$. Since $f \in T[\bar{F}_{p,0}] \subseteq T[D]$, we have $f(d_1,\ldots,d_n) = 0$. This means that $g, h_W \in \mathcal{U}_{t,r}$.

The lemma is proved.

To prove the main results we will need

LEMMA 9. If a nonmatrix variety $\mathfrak{L}$ does not contain the Grassmann algebra G, then $\mathfrak{L}$ satisfies the identity $[x_1, x_2] \cdot \ldots \cdot [x_{2n-1}, x_{2n}] = 0$ for some n.

Proof. Since $G \notin \mathfrak{L}$, it follows that $\mathfrak{L}$ satisfies the Capelli identities of some order k, as shown in [11]. Then, as is well known [13], $\mathfrak{L}$ is generated by a (k − 1)-generator algebra. Since $\mathfrak{L}$ is a nonmatrix variety, the commutator ideal of any finitely generated algebra in $\mathfrak{L}$ is nilpotent.

The lemma is proved.

We can now turn to the proof of the main results.

THEOREM 1. Suppose $\mathfrak{M}$ is a nonmatrix variety $\mathfrak{M} \not\ni G \otimes G$, and $\mathfrak{A}$ is the largest variety in $\{\mathfrak{O}, \mathfrak{A}_0, \mathfrak{A}_1\}$ contained in $\mathfrak{M}$. Then for some κ we have $\mathfrak{M} = \mathfrak{M}_\kappa \circ_{\mathfrak{M}} \mathfrak{A}$, where $\mathfrak{M}_\kappa = \mathfrak{M} \cap \mathfrak{N}_\kappa$.

Proof. If $\mathfrak{A} = \mathfrak{O}$, the conclusion of the theorem follows from the Nagata–Higman theorem, and if $\mathfrak{A} = \mathfrak{A}_0$, from Lemma 9. Suppose $\mathfrak{A} = \mathfrak{A}_1$, t, τ are natural numbers such that the ideal of identities Γ of the variety $\mathfrak{M}$ satisfies the conclusion of Proposition 1, and $p = t + t\tau$.

By Proposition 4, the algebra $\bar{F}_{p,0}$ contains a nilpotent ideal $\mathcal{U}$ such that $T[\bar{F}_{p,0}/\mathcal{U}] \supseteq \Gamma_2 + \Gamma$. Since $\mathfrak{M} \not\ni G \otimes G$, we have $\Gamma + \Gamma_2 \neq \Gamma_2$. Applying Proposition 3 to the T-ideal $T[\bar{F}_{p,0}/\mathcal{U}]$, we obtain $T[\bar{F}_{p,0}] \supseteq \Gamma_1^{\gamma}$ for some γ. Since $\Gamma_1 = \{[x, y, z]\}^T$, we obtain, applying Lemma 8 to the polynomial $f = [x_1, y_1, z_1] \cdot \ldots \cdot [x_\gamma, y_\gamma, z_\gamma]$, that $(\Gamma_1^{\gamma} T)^{\beta} \subseteq \Gamma$ for some β, where T is the T-ideal generated by $\{S_{2(\tau+3\gamma)}(y_1, \ldots, y_{2(\tau+3\gamma)})\} \cup \Gamma$. Since the identity $S_t(x_1, \ldots, x_t) = 0$ is not satisfied on G, it follows from Lemma 9 that $T \supseteq \Gamma_0^n$ for some n, where $\Gamma_0 = \{[x, y]\}^T$. In view of what was said above, $\Gamma \supseteq (\Gamma_1^{\gamma} T)^{\beta} \supseteq (\Gamma_1^{\gamma} \Gamma_0^n)^{\beta} \supseteq \Gamma_1^{\beta(\gamma+n)}$.

The theorem is proved.

THEOREM 2. Suppose $\mathfrak{M}$ is a nonmatrix variety and $\mathfrak{M} \ni G \otimes G$. Then $\mathfrak{M} = \mathfrak{M}_\kappa \circ_{\mathfrak{M}} (\mathfrak{A}_2 \circ_{\mathfrak{M}} (\mathfrak{M}_\ell \circ_{\mathfrak{M}} \mathfrak{A}_0))$ for certain κ, ℓ.

Proof. Suppose t, τ are natural numbers such that the ideal of identities Γ of the variety $\mathfrak{M}$ satisfies the conclusion of Proposition 1, and let $p = t + t\tau$. Put $M = F\langle X\rangle/\Gamma$. By Lemma 8, the algebra M contains ideals $\mathcal{J}_0, \mathcal{J}_2, \mathcal{J}_0 \subseteq \mathcal{J}_2$, such that $\mathcal{J}_0$ is nilpotent, $T[\mathcal{J}_2/\mathcal{J}_0] \supseteq T[\bar{F}_{p,0}]$, and algebra $M/\mathcal{J}_2$ satisfies the standard identity of degree $\tau + 1$. By Proposition 4, M contains an ideal $\mathcal{J}_1, \mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \mathcal{J}_2$, such that $\mathcal{J}_1$ is nilpotent and $T[\mathcal{J}_2/\mathcal{J}_1] \supseteq \Gamma_2$. By Lemma 8, M contains an ideal $\mathcal{J}_3$, $\mathcal{J}_2 \subseteq \mathcal{J}_3$, such that the algebra $\mathcal{J}_3/\mathcal{J}_2$ is nilpotent and $M/\mathcal{J}_3$ is commutative.

The theorem is proved.

THEOREM 3. Suppose $\mathfrak{M}$ is a nonmatrix variety, $\mathfrak{A}$ is a finitely based variety, and $\mathfrak{A} \supseteq \mathfrak{A}_2$. Then $\mathfrak{M} = \mathfrak{M}_\kappa \circ_{\mathfrak{M}} (\mathfrak{A} \cap \mathfrak{M})$ for some κ.

Proof. This theorem is equivalent to the following assertion: If V is a finitely based T-ideal and $V \subseteq \Gamma_2$, then V is nilpotent modulo $\Gamma = T[\mathfrak{M}]$. We may assume that V is generated by a single multilinear polynomial $f_0(x_1, \ldots, x_m)$.

Suppose t, τ are natural numbers such that the T-ideal Γ satisfies the conclusion of Proposition 1, and let $p = t + t\tau$. By Proposition 4, $T[\bar{F}_{p,0}] \supseteq V^{\alpha}$ for some α. Applying Lemma 8 to the polynomial $f = f_0(x_1^{(1)}, \ldots, x_m^{(1)}) \cdot \ldots \cdot f_0(x_1^{(\alpha)}, \ldots, x_m^{(\alpha)})$, we obtain $(V^{\alpha} T)^{\beta} \subseteq \Gamma$ for some β, where $T = \{S_{2(\tau+\alpha m)}(x_1, \ldots, x_{2(\tau+\alpha m)})\}^T + \Gamma$. By Lemma 9, $T \supseteq \Gamma_0^n$ for some n, where $\Gamma_0 = \{[x, y]\}^T$. In view of what was said above, $\Gamma \supseteq (V^{\alpha} T)^{\beta} \supseteq (V^{\alpha} \Gamma_0^n)^{\beta} \supseteq V^{\beta(\alpha+n)}$.

The theorem is proved.

We turn to the corollaries.

Corollary 1 follows from Theorem 3, since the variety of algebras satisfying the identity $[[x,y],[z,t],h]=0$ contains the variety $\mathfrak{A}_2$.

Corollary 2 follows from Theorem 1, since $T[\mathfrak{A}_1]=\{[x,y,z]\}^T$.

Corollary 3 follows from Corollary 2 and a theorem of Latyshev [2].

Corollary 4 follows from Corollary 3, since an identity of the form

$$\sum_i \alpha_i x^i y x^{n-i}=0$$

is not satisfied by the algebras M_2 and $G\otimes G$.

Indeed, since $T[G\otimes G]=T[G^*\otimes G^*]$, where G^* is a Grassmann algebra with unity, it suffices to show that this identity is not satisfied by the algebras M_2 and $G^*\otimes G^*$. Suppose the algebra M_2 (or $G^*\otimes G^*$) satisfies the identity $\sum_i \alpha_i x^i y x^{n-i}=0$. Since our algebra has a unity, this identity implies the identity $[y,\underbrace{x,\dots,x}_{n}]=0$ (we need only make the substitution $x\longrightarrow x+1$ and take the homogeneous component of smallest degree). Now to see that the identity $[y,x,\dots,x]=0$ is not satisfied by M_2 and $G^*\otimes G^*$, it suffices to make the following substitutions: in the first case, $x=e_{11}$, $y=e_{12}+e_{21}$ and in the second $x=e_1\otimes 1$, $y=\sum_{i=2}^{N} e_i\otimes f_i$.

Proof of Corollary 5. Put

$$W_m=\{[x_1,\dots,x_m]\}^T,\quad \mathcal{U}_{q,n}=\{[x_1,\dots,x_q,\underbrace{y,\dots,y}_{n}]\}^T.$$

It suffices to show that for any $q,n\geq 1$ there exists $p=p(q,n)$ such that $\mathcal{U}_{q,n}\supseteq W_{p(q,n)}$. The proof is by induction on n . For $n=1$ the assertion is trivial. Fix n . Suppose for any q there exists $p(q,n)$ such that $\mathcal{U}_{q,n}\supseteq W_{p(q,n)}$ (induction assumption). We fix q and will prove that there exists $s=p(q,n+1)$ such that

$$\mathcal{U}_{q,n+1}\supseteq W_s .$$

It is easy to see that the identity $[x_1,\dots,x_q,y,\dots,y]=0$ is not satisfied by $G\otimes G$. Therefore, it follows from Theorem 1 that $\mathcal{U}_{q,n+1}\supseteq\Gamma_1^{\alpha}$. Thus, it suffices to show that for any $\beta\geq 1$ there exists $t=t(\beta)$ such that

$$\mathcal{U}_{q,n+1}+\Gamma_1^{\beta}\supseteq W_{t(\beta)}. \tag{38}$$

The proof is by induction on β . For $\beta=1$ the assertion is trivial. Fix β . Suppose (38) holds (induction assumption). We will show that there exists $\tau=t(\beta+1)$ such that $\mathcal{U}_{q,n+1}+\Gamma_1^{\beta+1}\supseteq W_\tau$.

It follows from (38) that

$$\mathcal{U}_{q,n+1}+\Gamma_1^{\beta+1}\supseteq(\mathcal{U}_{q,n+1}+\Gamma_1^{\beta})\Gamma_1\supseteq W_{t(\beta)}{}' .$$

Analogously,

$$\mathcal{U}_{q,n+1} + \Gamma_1^{\beta+1} \supseteq \Gamma_1 W_{t(\beta)}.$$

Therefore,

$$\mathcal{U}_{q,n+1} + \Gamma_1^{\beta+1} \supseteq W_{t(\beta)}\Gamma_1 + \Gamma_1 W_{t(\beta)} + \mathcal{U}_{q,n+1}. \tag{39}$$

Put $\gamma = max\,(q, t(\beta))$. Then, modulo $\mathcal{U}_{q,n+1}$ and $\mathcal{U}_{q,n+1} + \Gamma_1^{\beta+1}$, we have the identity

$$[x_1,\ldots,x_\gamma,\underbrace{y,\ldots,y}_{n+1}] = 0. \tag{40}$$

Making a partial linearization in (40), we obtain

$$0 = \sum_{i=1}^{n+1} [x_1,\ldots,x_\gamma,\underbrace{y,\ldots,y}_{i-1},z,\underbrace{y,\ldots,y}_{n-i+1}] = n[x_1,\ldots,x_\gamma,\underbrace{y,\ldots,y}_{n},z] + \sum_{i=1}^{n}\alpha_i[x_1,\ldots,x_\gamma,[z,\underbrace{y,\ldots,y}_{i}],\underbrace{y,\ldots,y}_{n-i}] \tag{41}$$

for certain $\alpha_i \in F$.

Making the substitution $z = [u,v]$ in (41), by virtue of (39) we obtain modulo $\mathcal{U}_{q,n+1} + \Gamma_1^{\beta+1}$ the identity

$$[x_1,\ldots,x_\gamma,\underbrace{y,\ldots,y}_{n},[u,v]] = 0. \tag{42}$$

From (42) modulo $\mathcal{U}_{q,n+1} + \Gamma_1^{\beta+1}$ we obtain the identity

$$[[x_1,\ldots,x_\gamma,\underbrace{y,\ldots,y}_{n}]t,\ [u,v]] = 0. \tag{43}$$

It follows from (42) and (43) that $[\mathcal{U}_{\gamma,n},[u,v]] \subseteq \mathcal{U}_{q,n+1} + \Gamma_1^{\beta+1}$. By the induction assumption, $\mathcal{U}_{\gamma,n} \supseteq W_{p(\gamma,n)}$. Therefore, modulo $\mathcal{U}_{q,n+1} + \Gamma_1^{\beta+1}$ we have the identity

$$[x_1,\ldots,x_\mu,[u,v]] = 0,\quad \mu = p(\gamma,n),\quad \mu \geq \gamma. \tag{44}$$

From (40) modulo $\mathcal{U}_{q,n+1} + \Gamma_1^{\beta+1}$ we obtain the identity

$$\sum_{\sigma\in S(n+1)} [x_1,\ldots,x_\mu,y_{\sigma(1)},\ldots,y_{\sigma(n+1)}] = 0. \tag{45}$$

It follows from (44) and (45) that

$$0 = \sum_{\sigma\in S(n+1)} [x_1,\ldots,x_\mu,y_{\sigma(1)},\ldots,y_{\sigma(n+1)}] = (n+1)!\,[x_1,\ldots,x_\mu,y_1,\ldots,y_{n+1}]$$

as required.

Corollary 6 follows from Corollary 3. Actually, it is known [12] that since $\mathfrak{M}$ is locally weakly Noetherian, the algebras in $\mathfrak{M}$ satisfy an identity of the form

$$[x,y,\ldots,y]z^k[u,t,\ldots,t] = 0, \tag{46}$$

but, as is easily seen, (46) is not satisfied by the algebras M_2 and $G\otimes G$.

Corollary 7 follows immediately from (46) and Corollary 5.

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LITERATURE CITED

1. A. I. Mal'tsev, Algebraic Systems [in Russian], Nauka, Moscow (1970).
2. V. N. Latyshev, "The finite basis property for the identities of certain rings," Usp. Mat. Nauk, 32, No. 4, 259-266 (1977).
3. P. J. Higgins, "Lie rings satisfying the Engel condition," Proc. Cambridge Phil. Soc., 50, No. 1, 8-15 (1954).
4. V. N. Latyshev, "Nonmatrix varieties of associative algebras," Doctoral Dissertation, Moscow State Univ., Moscow (1978).
5. G. de B. Robinson, Representation Theory of the Symmetric Group, Univ. of Toronto Press (1961).
6. C. Curtis and I. Reiner, Representation Theory of Finite Groups and Associative Algebras, Interscience, New York (1962).
7. H. Boerner, Representations of Groups, North-Holland, Amsterdam (1963).
8. A. R. Kemer, "The Specht property of T-ideals with power growth of codimensions," Sib. Mat. Zh., 19, No. 1, 54-69 (1978).
9. A. Regev, "Existence of identities in $A \otimes B$," Israel J. Math., 11, No. 2, 131-152 (1972).
10. G. Higman, "On a conjecture of Nagata," Proc. Cambridge Phill. Soc., 52, No. 1, 1-4 (1956).
11. A. R. Kemer, "Remark on the standard identity," Mat. Zametki, 23, No. 5, 753-757 (1978).
12. A. Z. Anan'in, "Locally finitely approximable and locally representable varieties of algberas," Algebra Logika, 16, No. 1, 3-23 (1977).
13. Yu. P. Razmyslov, "The Jacobson radical in PI-algebras," Algebra Logika, 13, No. 3, 337-360 (1974).