WAVE PACKET FORMATION IN AN INCOMPRESSIBLE BOUNDARY LAYER UNDER THE INFLUENCE OF IMPULSIVE VIBRATION OF THE WALL

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The formation of a Tollmien—Schlichting wave packet excited by the impulsive motion of a localized area of the boundary surface is investigated. A solution of the mixed problem for the linearized system of Navier—Stokes equations is constructed by the Fourier method for a parallel main flow. The initial stage of the disturbed motion is analyzed.

Surface vibrations can have a considerable (in some cases decisive) effect on laminar-turbulent boundary layer transition. According to modern ideas [1], the breakdown of laminar flow under the influence of wall vibration involves the generation and subsequent downstream growth of natural oscillations of the boundary layer — Tollmien—Schlichting waves. The first theoretical studies of this question [2, 3] were concerned with the excitation of a Tollmien—Schlichting wave by a vibrator — a narrow strip of the surface in harmonic vibration. Subsequently, in [4, 5] the wave packet generated when the vibrator starts up was investigated; in these studies the Reynolds number was assumed to be infinitely large and the characteristic frequency of the vibrations corresponded to the neighborhood of the lower branch of the neutral curve. As shown by analysis [6, 7], transferring the results of asymptotic stability theory to the case of finite Reynolds numbers gives only a qualitative picture of the disturbed motion.

1. Formulation of the Problem

We will investigate the unsteady perturbations of the laminar boundary layer on a semi-infinite flat plate located in a uniform viscous incompressible flow at zero angle of attack. We introduce the Cartesian coordinate system with origin at the point O a distance L from the tip of the plate, x axis parallel to the free-stream velocity vector, and y axis perpendicular to the surface of the plate (Fig. 1). In what follows, all the quantities are assumed to be dimensionless, the fluid density ρ, the free-stream velocity U_∞, and the characteristic dimension $\ell = (\nu L/U_\infty)^{1/2}/\tau$ (ν is the kinematic viscosity coefficient, $\tau = 0.332...$) being taken as the basic units. We denote by $U(y)$ the velocity profile of the undisturbed flow at the point O; the Reynolds number $R = U_\infty \ell/\nu$.

We assume that at times $t < 0$ the flow is undisturbed, and that when $t > 0$ perturbations are generated by the vibration of the wall, points on which are displaced in the vertical direction in accordance with the law

$$y_w = \varepsilon f(t, x), \quad \varepsilon \ll 1 \tag{1.1}$$

In solving the perturbation problem we assume a parallel main flow; accordingly, in what follows the function f is assumed to be significantly different from zero only when $|x| \gg L$ (although in this case the flow region considered contains points $|x| \gg 1$).

We denote by $\varepsilon q(t, x, y)$ the perturbations of the velocity vector components and the pressure ($q = v_x, v_y, p$). The functions q satisfy the linear system of equations

$$\frac{\partial v_x}{\partial t} + U\frac{\partial v_x}{\partial x} + \frac{dU}{dy}v_y + \frac{\partial p}{\partial x} = \frac{1}{R}\frac{\partial^2 v_x}{\partial y^2}, \quad \frac{\partial v_y}{\partial t} + U\frac{\partial v_y}{\partial x} + \frac{\partial p}{\partial y} = 0, \quad \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{1.2}$$

As compared with the complete system of linearized Navier—Stokes equations, in system (1.2) we have omitted the terms associated with the nonparallelism of the main flow and, moreover, terms lower than the latter [8]; in this connection, in (1.2) we have retained only the leading dissipative term. The boundary conditions at y = 0 follow from

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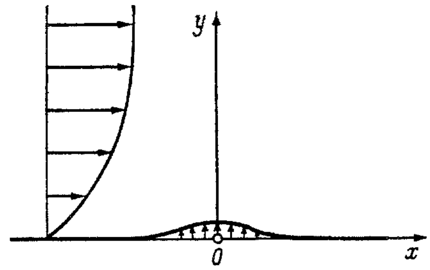

Fig. 1

the no-slip conditions written for the wall (1.1):

$$v_x(t,x,0)=-f(t,x), \quad v_y(t,x,0)=\frac{\partial f}{\partial t}(t,x) \tag{1.3}$$

As $x \to \pm\infty$ and as $y \to \infty$ we impose on the solution of system (1.2) the damping conditions $q \to 0$.

The solution of the mixed problem thus formulated will be sought in the form of a Fourier–Laplace transformation in the variables x, t

$$\begin{gathered} q=\int_{-\infty+i\sigma}^{+\infty+i\sigma} d\omega \int_{-\infty}^{+\infty} f^*(\omega,k)\, q^*(y;\omega,k)\exp(ikx-i\omega t)\,dk \\ f^*=(2\pi)^{-2}\int_0^{\infty}\int_{-\infty}^{+\infty} f(t,x)\exp(-ikx+i\omega t)\,dt\,dx \end{gathered} \tag{1.4}$$

For convenience, the transforms of the unknown functions have been normalized on f^*; the value of the real constant σ is so chosen that the contour of integration with respect to ω lies above all the poles of the integrand expression. By means of transformation (1.4) the system of partial differential equations (1.2) for the perturbations q is reduced to a system of ordinary differential equations for the transforms q^*. We introduce the new unknown function $\varphi(y;\omega,k)$, setting $v_y^*=-ik\varphi$. As a result, the problem of calculating the functions q^* is reduced to the inhomogeneous boundary-value problem for an equation of the Orr–Sommerfeld type

$$\left(U-\frac{\omega}{k}\right)\left(\frac{d^2\varphi}{dy^2}-k^2\varphi\right)-\frac{d^2U}{dy^2}\varphi=\frac{1}{ik\mathrm{R}}\frac{d^4\varphi}{dy^4}, \quad \varphi(0)=\frac{\omega}{k}, \quad \frac{d\varphi}{dy}(0)=-1, \quad \varphi(\infty)=0 \tag{1.5}$$

The first two boundary conditions (1.5) follow from conditions (1.3); the last of conditions (1.5) is a consequence of the damping of the perturbations as $y \to \infty$: it eliminates from the solution two exponentially increasing, linearly independent integrals. The damping of the flow perturbations as $|x| \to \infty$ is ensured by the application of transformation (1.4) to the solution of the mixed problem formulated.

Let us consider whether the problem is correctly posed. As in the problem of a harmonic vibrator in an incompressible boundary layer [9], the function p^* has a singularity of the type $|k|^{-1}$ as $k \to 0$; accordingly, generally speaking, the Fourier integral for the pressure perturbation diverges and as $x \to \pm\infty$ the function p behaves as $\ln|x|$ (i.e., does not satisfy the damping condition). This is a consequence of the assumption that the fluid is incompressible. In fact, the vertical motion of part of the surface leads to the displacement of fluid, as a result of which as $r = (x^2 + y^2)^{1/2} \to \infty$ the expansion of the disturbed potential Φ (outside the boundary layer zone) will contain a term of the source type. In the first approximation from the Cauchy–Lagrange integral we have $p \sim -\partial\Phi/\partial t \sim -\pi^{-1} \ln r\,\partial Q/\partial t$, where Q(t) is the volume flow rate of the source. Calculating Q for the given wall motion (1.1), we finally obtain

$$p \sim -\frac{1}{\pi}\frac{\partial^2}{\partial t^2}\left[\int_{-\infty}^{+\infty} f(t,x)\,dx\right]\ln\sqrt{x^2+y^2}$$

As $x \to \pm\infty$ this expansion will also be valid within the boundary layer zone, in particular at $y = 0$, since at these x the disturbed external pressure varies only weakly along the plate (on distances of the order of the boundary layer thickness).

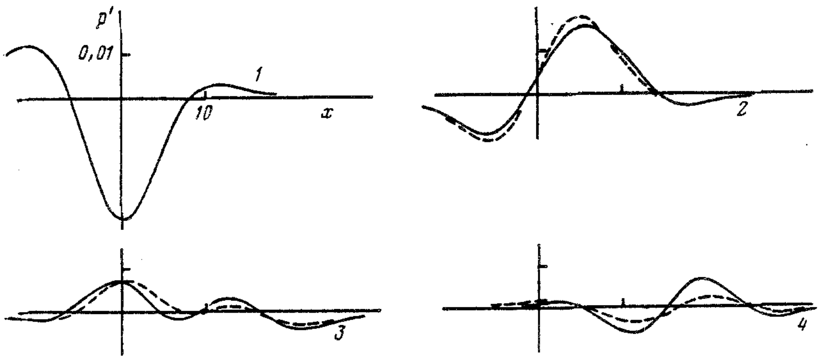

Fig. 2

As for the functions v_x, v_y, $\partial p/\partial x$, and $\partial p/\partial y$, they satisfy the damping conditions as $|x| \to \infty$ and can be determined from relations (1.4) and (1.5). We note that the problem will also be properly posed with respect to the function p if at each moment of time there is no net displacement of fluid by the moving part of the plate, i.e.,

$$\int_{-\infty}^{+\infty} f(t, x)\, dx \equiv 0$$

In this case $f^*(\omega, 0) \equiv 0$; accordingly, the Fourier integral for the function p converges and therefore $p \to 0$ as $|x| \to \infty$.

2. Calculation Method and Results

We will analyze the disturbed motion with reference to the function $p'(t, x) = \partial p(t, x, 0)/\partial x$ for wall motion of the type $f = f_0 = (ct/3)^3 \exp(-ct - ax^2 + 3)$; $c > 0$, $a > 0$. This choice of function is dictated by considerations of continuity with respect to t of the boundary conditions for v_x and v_y at $y = 0$ ($f_0 = \partial f_0/\partial t = 0$ when $t = 0$). It is also convenient because of the smoothness of the function f_0, which determines the fairly rapid damping of its transform f_0^* as $|k|$, $|\omega| \to \infty$ and ultimately leads to the good convergence of the integrals (1.4). The class of wall vibrations in question satisfies the normalization condition $\max f_0 = 1$.

In order to reduce the volume of computations, the integral (1.4), written for the function p', was expressed in the form:

$$p' = \exp(\sigma t)\, \mathrm{Re}\left[\int_0^{\infty}\int_{-\infty}^{+\infty} P'(\omega, k) \exp(ikx - i\omega t)\, d\omega\, dk\right]$$
$$P' = \frac{c^3}{9\pi\sqrt{\pi a}\mathrm{R}} [\omega + i(c+\sigma)]^{-4} \exp\left(-\frac{k^2}{4a} + 3\right) \frac{d^3\varphi}{dy^3}(0; \omega + i\sigma, k) \qquad (2.1)$$

The values of $P'(\omega, k)$ were tabulated at the nodes of a nonuniform rectangular grid, which was made denser in the areas of sharp variation of P' (in the examples considered below the number of nodes was approximately 4000). At each node the function was evaluated as follows. From the given parameters R, k, and $\omega' = \omega + i\sigma$ by the method of orthogonalization [10] we calculated two linearly independent integrals φ_1, φ_2 of Eq. (1.5) which decrease as $y \to \infty$. The general solution of the equation satisfying the last of conditions (1.5) was represented in the form $\varphi = c_1\varphi_1 + c_2\varphi_2$. The coefficients c_1 and c_2 were uniquely determined from the first two of conditions (1.5). At the same time we calculated the quantity $d^3\varphi/dy^3$ entering into the last of Eqs. (2.1) (it is one of the unknown functions when Eq. (1.5) is integrated by the Runge–Kutta method) and, together with it, the unknown quantity P'.

In order to calculate the function $p'(t, x)$ we replaced the integral (2.1) by a sum of integrals over the subdivision rectangles; in each rectangle the function $P'(\omega, k)$ was approximated by the surface of a hyperbolic paraboloid $\alpha + \beta\omega + \gamma k + \delta\omega k$ (generalization of the piecewise-linear approximation for the one-dimensional case [9]). The values of the complex constants α, β, γ, and δ were determined from the condition of coincidence of P′ and the approximating function at the vertices of the rectangle. This approximation made it possible to calculate each of the integrals entering into the sum in accordance with an exact formula. The accuracy of the calculations (~1%) was checked by doubling the cell dimensions and varying the dimensions of the region of integration.

It should be noted that in evaluating the integral (2.1) fixing the subdivisions imposes limitations on the region of variation of the variables t, x in which the integral can be calculated with given accuracy. In order to calculate the values of the integral (2.1) for ever larger values of the parameters t, x it would be necessary to subdivide the integration domain ever more finely, which would lead to considerable computational difficulties. However, there is no need for this: the assumption that the main flow is parallel itself limits the region of variation of the variable x; therefore in the formulation in question it is possible to investigate only the initial stage of formation of the wave packet; its further development downstream can be studied only by methods that describe the evolution of waves in weakly inhomogeneous flows.

In Fig. 2 we have plotted the results of calculating the longitudinal gradient for times t = 5, 20, 30, and 50 (curves 1—4, respectively). The broken curves correspond to the stable (R = 500) and the continuous curves to the unstable (R = 1500) boundary layer: in the chosen units the Reynolds number corresponding to loss of stability is approximately 900. The parameters c = 0.2 and a = 0.02 were chosen on the basis of the condition of coincidence of the characteristic time and length scales for vibration (1.1) and the Tollmien—Schlichting wave; in this case the maximum deviation of the wall is reached at t = 15.

As may be seen from the curves, at the beginning of the wall motion the disturbed flow depends weakly on R. Wave packet formation is completed by t = 50, when the surface scarcely differs from a flat plate. At this moment the difference between the cases of stable and unstable flow is already clearly visible.

In conclusion, we recall that the results of the calculations can be treated as the solution of the well-posed problem of excitation of boundary layer oscillations by wall motion of the type $f = \partial f_0/\partial x$. In this case p′ should be understood as the function p(t, x, 0).

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