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Complex Analysis and Operator Theory

A Note on the Integrability of Maximal Functions Related to Differentiation Bases

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Abstract In this article we study maximal operators related to differentiation bases. In particular, we find a sufficient condition for the integrability of the maximal function. This complements earlier results of Hardy and Littlewood, de Guzmán and Kita. We also give a geometric application of the result.

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1 Introduction

The Hardy–Littlewood maximal function of a locally integrable function $f : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y)|\, dy,$$

where the supremum is taken over all balls centered at x. When Hardy and Littlewood introduced the operator they showed that if f belongs to the class $L \log L$, then Mf is locally integrable (see Theorem 12 in [7]). This can be seen as a consequence of the

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weak type inequality of the maximal operator. For further results for the integrability of Hardy–Littlewood maximal function we refer to [5,9,12] and Chap. 4 in [10], see also Sect. 12.3 in [6].

When the maximal operator is changed so that the supremum is taken not over balls but over some other sets, then the functions in L^1 usually are no longer mapped to weak L^1. Instead, the corresponding maximal operator $M_{\mathcal{B}}$ satisfies a weak type ϕ inequality with ϕ usually different from identity. These weak type inequalities occur very naturally since the weak type ϕ inequality of the maximal operator $M_{\mathcal{B}}$ is equivalent to the basis $\mathcal{B}$ to differentiate the class $\phi(L)$ (for details, see Theorem 6.4.11 of Rubio in [4], see also [13]). It is shown by de Guzmán (see p. 64 in [3]) that in the case of rectangles parallel to the coordinate axes, it is sufficient that $f \in L(1+\log^+ L)^n$ in order to $M_{\mathcal{B}} f$ be integrable. The crucial ingredient there is the weak type inequality for functions in $L(1 + \log^+ L)^{n-1}$.

The main contribution of this work is to show that the integrability of a maximal function can be deduced by elementary methods for other bases than merely balls or cubes. Our approach also includes the theorems of Hardy and Littlewood and de Guzmán as special cases. At the end we apply the theorem to obtain a covering lemma that might be of independent interest.

2 Maximal Functions and Differentiation Bases

We will consider any collection of open subsets of $\mathbb{R}^n$ a differentiation basis, denoted by $\mathcal{B}$. We assume also that $\mathcal{B}$ is homothecy invariant, i.e. whenever $B \in \mathcal{B}$, $x \in \mathbb{R}^n$ and $\alpha > 0$, then $x + \alpha B \in \mathcal{B}$. A result by Rubio shows that such a basis $\mathcal{B}$ differentiates $\phi(L)$ if and only if the maximal operator is of weak type ϕ, i.e.

$$|\{x \in B : M_{\mathcal{B}} f(x) > t\}| \le c \int_B \phi(f(x)/t)\, dx$$

for all $f \in \phi(L)$ where the Orlicz space $\phi(L)$ consists of functions for which

$$\int_{\mathbb{R}^n} \phi(\alpha |f(x)|)\, dx$$

is finite for some α; for more details on Orlicz spaces see e.g. [11]. Importantly, given an Orlicz spaces $\phi(L)$ (satisfying a so called Δ_2 condition which is not very restrictive) it is possible to construct a differentiation basis for which the weak type inequality ϕ is optimal for the related maximal operator, see [14] for details.

Next we present the main theorem.

Theorem 2.1 *Let $B \subset \mathbb{R}^n$ be a ball and let ϕ be a strictly convex function with $\phi(0) = 0$. Suppose μ is a positive measure with $\mu(B)$ finite. Define*

$$\psi(t) = \int_0^t \phi'(s)(1 + \log^+ s)\, ds$$

Suppose the maximal operator related to the basis $\mathcal{B}$ is of weak type ϕ. Then the maximal operator satisfies

$$\int_B M_{\mathcal{B}} f \, d\mu \le c\mu(B) + c \int_B \psi(f)\, d\mu$$

with constant c independent of the function f.

Proof Without loss of generality we may assume that f is positive. We have the following Cavalieri principle for measurable functions:

$$\int_{\{x \in B : f(x) > t\}} \phi(f)\, d\mu = \phi(t)\mu(\{x \in B : f(x) > t\}) + \int_t^\infty \phi'(s)\mu(\{x \in B : f(x) > s\})\, ds.$$

The maximal operator is bounded in L^∞ and sublinear, so

$$\int_B M_{\mathcal{B}} f \, d\mu \le 2\mu(B) + c \int_2^\infty \int_{\{x \in B : f(x) > t/2\}} \phi(f/t)\, d\mu\, dt$$

and further by Cavalieri principle and a change of variables

$$\int_B M_{\mathcal{B}} f \, d\mu \le 2\mu(B) + c \int_2^\infty \phi(1/2)\mu(\{x \in B : f(x) > t/2\})\, dt + c \int_2^\infty \int_{t/2}^\infty \phi'(s/t)\mu(\{x \in B : f(x) > s\}) \frac{ds}{t}\, dt.$$

The first integral here is bounded since f is integrable. For the second we integrate by parts and use the convexity of ϕ to obtain

$$\begin{aligned}
&\int_2^M \int_{t/2}^\infty \phi'(s/t)\mu(\{x \in B : f(x) > s\}) \frac{ds}{t}\, dt \\
&\quad = \int_{M/2}^\infty \phi'(2s/M)\mu(\{x \in B : f(x) > s\}) \log M\, ds \\
&\qquad + \int_2^M \phi'(1/2)\mu(\{x \in B : f(x) > t/2\}) \log t\, dt. \\
&\quad \le \int_1^\infty \phi'(s)(1 + log^+ s)\mu(\{x \in B : f(x) > s\})\, ds
\end{aligned}$$

which gives the statement of the theorem by Cavalieri principle. □

Remark 2.2 The same argument can be modified to give a similar condition for the maximal function to belong to $\theta(L)$. Indeed, let the maximal operator be of weak type ϕ and set

$$\psi(t) = \int_1^t \phi'(s) \int_1^s \frac{\theta'(p)}{p}\, dp\, ds,$$

when $t > 1$ and $\psi(t) = 0$ for $0 < t \le 1$. Then the maximal operator maps $\psi(L)$to $\theta(L)$. In particular, the maximal operator related to the basis of rectangles in the coordinate directions maps $L(1 + \log^+ L)^{n+\alpha}$ to the class $L(1 + \log^+ L)^\alpha$ with $\alpha > 0$.

The following geometric consequence is mentioned in [8], Exercise 2.11, for balls, but the origin of the statement is not known to the authors (see also [2]).

Corollary 2.3 *Let $\mathcal{B}$ be a differentiation basis and assume that the corresponding maximal operator $M_{\mathcal{B}}$ is of weak type ϕ. Fix $\lambda > 0$ and assume there is a constant $c > 0$ such that for any $B \in \mathcal{B}$ we have*

$$\mu(\lambda B) \le c\mu(B).$$

Let Q_0 be a cube in $\mathbb{R}^n$ and B_i a family of pairwise disjoint sets from the differentiation basis $\mathcal{B}$ with $B_i \subset Q_0$ for all i. Then the overlap function

$$f(x) = \sum_i \chi_{\lambda B_i}(x)$$

belongs to the dual of the space $\psi(L)$, where ψ is defined as in the Theorem 2.1.

Proof We show that

$$\int_{Q_0} fg\,d\mu \leq c\mu(Q_0) + c\int_{Q_0} \psi(g)\,d\mu$$

is finite for all $g \in \psi(L)$, which gives the result when combined with Landau's resonance theorem (see Lemma 2.6 in [1]). Indeed for any B_i we have

$$\fint_{B_i} M_{\mathcal{B}} g\,d\mu \geq \inf_{\lambda B_i} M_{\mathcal{B}} g \geq \fint_{\lambda Q_i} g\,d\mu.$$

By the Theorem 2.1, the maximal operator maps $\psi(L)$ boundedly to L^1 and since the sets B_i are disjoint we have

$$\begin{aligned}\int_{Q_0} fg d\mu &= \sum_i \mu(\lambda B_i) \fint_{\lambda B_i} g\,d\mu \\ &\leq \sum_i \mu(\lambda B_i) \fint_{B_i} M_{\mathcal{B}} g\,d\mu \leq c\int_{Q_0} M_{\mathcal{B}} g\,d\mu \\ &\leq c\mu(Q_0) + c\int_{Q_0} \psi(g)\,d\mu\end{aligned}$$

as wished.

The corollary can be illustrated with help of the ractangles in the direction of the coordinate axes, since then the overlap function is in the class $\exp(L^{1/n})$. This is a larger class of functions than only $\exp(L)$ where the overlap function belongs to as soon as there is a uniform bound for the eccentricity of the rectangles.

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