Discontinuous Inverse Eigenvalue Problems

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Introduction

In this paper we consider inverse Sturm–Liouville problems in which the eigenfunctions have a discontinuity in an interior point. We shall show that if the potential is known over half the interval and if one boundary condition is given, then the potential and the other boundary condition are uniquely determined by the eigenvalues. The position of the discontinuity and the jump in the eigenfunctions are also uniquely determined. This generalizes a theorem by Hochstadt and Lieberman [13].

Our work has been motivated by the inverse problem for the torsional modes of the Earth. Here the main discontinuity is caused by reflection of the shear waves at the base of the crust. We shall show that if the density is known in the lower mantle and the velocity distribution of the shear waves is given throughout the mantle and in the crust, then the density is uniquely determined by the eigenfrequencies of the torsional modes with a fixed angular order. The velocity of the shear waves in the upper mantle and in the crust can be replaced by an additional torsional spectrum. This generalizes two results by Hald [10].

The proof of Hochstadt and Lieberman [13] is based on an integral representation of the eigenfunctions of a regular Sturm–Liouville problem and the Cauchy integral technique. By using the theory of translation operators Hald [10] has proved a slightly more general theorem. I have been unable to extend this technique to eigenvalue problems with discontinuities. Instead the proof in this paper follows the original arguments by Hochstadt and Lieberman. Our main technical tool is a generalization of the Povzner–Levitan integral representation of the eigenfunctions and an adaptation of the Cauchy integral technique to problems with discontinuities. The difficulty here is that the eigenvalues of such problems can behave quite randomly and do not have an asymptotic expansion.

It has been pointed out by Anderssen and Chandler [3] that the density may not be uniquely determined even if all torsional overtones are given. Hald [11] has presented an explicit example of this phenomenon. It is therefore the information on the density in the lower mantle that leads to uniqueness. If the density is not known, then additional information must be provided to ensure uniqueness. Thus Sabatier [24] has shown that the density and the velocity of the shear waves are uniquely determined by the eigenvalues and the normalizing constants of two torsional spectra. All these arguments are for earth models without interior

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discontinuities. I have not yet found any example of non-uniqueness for problems with one interface.

It may seem peculiar to assume that the density is given in the lower mantle. However, if certain physical assumptions are satisfied, then the density can be determined from the velocities of the compression waves and the shear waves provided the mass of the core and the density at the core-mantle boundary are given. These physical assumptions are indeed satisfied in the lower mantle, but not in the upper mantle, in many of the earth models which fit the available data.

It has been observed by Anderssen and Cleary [4] that the eigenfrequencies of realistic earth models do not have an asymptotic expansion due to internal discontinuities. The phenomenon is called the solotone effect. McNabb, Anderssen and Lapwood [22] have studied it from a theoretical point of view and found the leading term in the Wronskian. This expression can be used to approximate the eigenfrequencies provided the discontinuities in the mantle are small and explains the observed variations in the computed eigenfrequencies (see Anderssen [2] and Lapwood and Usami [16], page 220).

I suspect that the theory presented here can be extended to other inverse problems with discontinuities, e.g. the determination of a symmetric potential from one spectrum or the determination of a general potential from two spectra or from one spectrum plus the normalizing constants. It should also be possible to prove uniqueness for problems with several discontinuities. In this paper we do not consider the question of existence. However, our calculations show that a slight change in the position of the discontinuity, say from a rational number to an irrational number, may change the higher eigenvalues drastically. This indicates that the position of the discontinuity cannot be changed if the eigenvalues are given and then slightly perturbed. Finally we mention that Krueger [16] and Symes [25] have considered inverse problems for one-dimensional elastic media with discontinuities and discuss both existence and uniqueness questions. The problems are studied in the time domain and the reflection data is used to determine the characteristic impedance of the elastic medium.

1. Main Result

In this section we shall state our main result. The proof depends on the asymptotic behavior of the eigenvalues and the eigenfunctions of Sturm–Liouville problems with one discontinuity. We must therefore extend the usual theory. By using a new approach we actually improve some estimates due to K. Jörgens [14] and Titchmarsh [28].

THEOREM 1. *Consider the eigenvalue problem*

$$-u''+q(x)u=\lambda u \tag{1.1}$$

on the interval $0 < x < \pi$ *and with the boundary conditions*

$$u'(0) - hu(0) = u'(\pi) + Hu(\pi) = 0 \tag{1.2}$$

and with jump conditions

$$u(d+) = au(d-), \qquad u'(d+) = a^{-1}u'(d-) + bu(d-), \tag{1.3}$$

where q *is an integrable function,* $0 < d < \frac{1}{2}\pi$, $a > 0$ *and* $|a-1| + |b| > 0$. *Let* $\lambda_0, \lambda_1, \cdots$ *be the eigenvalues. Consider the eigenvalue problem with a, b, d, h, H,* λ *and q replaced by* $\tilde{a}, \tilde{b}, \tilde{d}, \tilde{H}, \tilde{\lambda}$ *and* $\tilde{q}$. *If* $\lambda_j = \tilde{\lambda}_j$ *for* $j \geqq 0$, $H = \tilde{H}$ *and* $q = \tilde{q}$ *almost everywhere in* $(\frac{1}{2}\pi, \pi)$, *then* $a = \tilde{a}$, $b = \tilde{b}$, $d = \tilde{d}$, $h = \tilde{h}$ *and* $q = \tilde{q}$ *almost everywhere.*

Remark. Thus if the eigenvalues and one of the boundary conditions are fixed and the potential q is given in half of the interval, then the potential and the other boundary condition are uniquely determined even if the differential equation has an interior discontinuity. This result is a generalization of a theorem due to Hochstadt and Lieberman [13]. They assume that $a = 1$, $b = 0$ and $h = \tilde{h}$. The last restriction is unnecessary (see Hald [10]). The proof presented here follows Hochstadt and Lieberman's proof in outline, but is much longer.

Beginning of the proof: Let $u(x, \lambda)$ be the solution of the differential equation (1.1) that satisfies the initial condition $u = 1$ and $u' = h$ at $x = 0$ and the jump condition (1.3). Note that u is not defined at $x = d$ (see Figure 1). It is well known that a solution of a Sturm–Liouville problem satisfies a Volterra integral equation of the second kind. By using this integral equation we can estimate the solution and its derivative and show that u is an entire function in λ of order $\frac{1}{2}$. Next we consider the Wronskian $\omega(\lambda) = -u'(\pi) - Hu(\pi)$. Its roots are real and simple and by using the estimates of u and u' we can give crude upper and lower bounds for the eigenvalues of the differential equation. Finally, we present an example which shows that in general the eigenvalues do not have an asymptotic expansion.

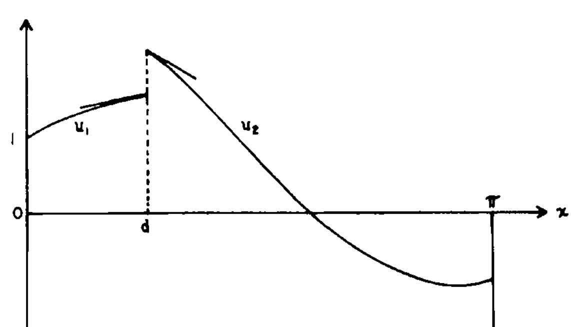

Figure 1. An eigenfunction u of a Sturm–Liouville problem with a discontinuity at $x = d$.

To study the solution u of equation (1.1) we introduce two functions u_1 and u_2 such that $u_1 = u$ for $x < d$ and $u_2 = u$ for $x > d$. Let $k = \sqrt{\lambda}$ with $k > 0$ when $\lambda > 0$ and assume that $x < d$. Then u_1 satisfies the Volterra integral equation

$$u_1(x) = \cos kx + \frac{h}{k} \sin kx + \frac{1}{k} \int_0^x \sin k(x-t) q(t) u_1(t)\, dt, \tag{1.4}$$

see Titchmarsh [28], page 9. On the other hand, the solution of this integral equation satisfies equation (1.1) and has the correct initial values. Equation (1.4) was introduced by Liouville [20], page 24. He used it to find the first two terms in the asymptotic expansion of the eigenvalues. The integral equation for u_2 is more complicated. Let $x > d$. If we multiply both sides of the equation $u'' + k^2 u = q(t)u$ by $G(x, t)$, integrate with respect to t from 0 to x, integrate by parts twice and use equation (1.3), we are led to a differential equation for G with discontinuity conditions for G and G_t at $t = d$. By solving the equation for G with $G(x, x) = 0$ and $G_t(x, x) = -1$ and inserting the result we get the Volterra integral equation

$$\begin{aligned}
u_2(x) = {} & a \cos k(x-d) \cos kd - a^{-1} \sin k(x-d) \sin kd + \frac{b}{k} \sin k(x-d) \cos kd \\
& + \frac{h}{k} \left[a^{-1} \sin k(x-d) \cos kd + a \cos k(x-d) \sin kd \right. \\
& \qquad \left. + \frac{b}{k} \sin k(x-d) \sin kd \right] \\
& + \frac{1}{k} \int_0^d \left[a^{-1} \sin k(x-d) \cos k(d-t) + a \cos k(x-d) \sin k(d-t) \right. \\
& \qquad \left. + \frac{b}{k} \sin k(x-d) \sin k(d-t) \right] q(t) u_1(t)\, dt \\
& + \frac{1}{k} \int_d^x \sin k(x-t) q(t) u_2(t)\, dt.
\end{aligned} \tag{1.5}$$

It follows from equation (1.4) and (1.5) that the solution u satisfies a Volterra integral equation for $x \neq d$. The kernel has a discontinuity in one of the variables and is integrable in the other and the inhomogeneous term is discontinuous at $x = d$. This problem is not included in Bôcher's theory of integral equations with regularly distributed discontinuities (see [5], pages 13–16), but it is partially covered by Evans' generalization. Thus the integral equation has a unique piecewise continuous solution if the potential is piecewise continuous (see Evans [8]). If the potential is integrable in Lebesgue's sense and bounded, we can use a result by Young [30]. In our case the potential is just integrable and we prefer to analyze the two equations separately. To estimate u_1 and u_2 we use

LEMMA 1. *Consider the integral equation*

$$u(x)-\int_a^x K(x,t)q(t)u(t)\,dt=f(x),$$

where f and K are continuous and q is integrable. This equation has a unique solution u which is continuous and satisfies

$$|u(x)|\leqq M(x)\,e^{L(x)\rho(x)},$$

where

$$M(x)=\max_{a\leqq t\leqq x}|f(t)|,\quad L(x)=\max_{a\leqq t\leqq x}|K(x,t)|\quad \textit{and}\quad \rho(x)=\int_a^x|q(t)|\,dt.$$

Remark. If f and K are entire functions of a parameter λ, then u is also an entire function. If f and K are twice differentiable with respect to x and $K(x,x)=0$, then u is also twice differentiable. Our proof is similar to the proof by K. Jörgens [14], page 4.4, but simpler.

Proof: We use the method of successive approximations. Let $u_0=f$ and define

$$u_{n+1}(x)=\int_a^x K(x,t)q(t)u_n(t)\,dt. \tag{1.6}$$

We shall show that the infinite series $u=\sum u_n$ converges and is a solution of the integral equation. Let $M_n(x)=\max_{a\leqq t\leqq x}|u_n(t)|$, and note that $M=M_0$. I claim that

$$M_n\leqq M\frac{(L\rho)^n}{n!} \tag{1.7}$$

for all x. We observe first that $M_1\leqq ML\rho$. Assume now that (1.7) is valid for the integer n. By using equation (1.6) we see that

$$\begin{aligned}|u_{n+1}(x)|&\leqq\int_a^x L(x)|q(t)|M(t)\frac{(L(t)\rho(t))^n}{n!}\,dt\\&\leqq M(x)\frac{(L(x)\rho(x))^{n+1}}{(n+1)!}\end{aligned}$$

since $\rho'=|q|$ a.e. This completes the proof of the inequality (1.7). Thus the series $\sum u_n$ converges uniformly and we get the estimate in the lemma. By using equation (1.6) we obtain

$$u-f-\int_a^x Kqu\,dt=u-\sum_0^{N+1}u_n-\int_a^x Kq\left(u-\sum_0^N u_n\right)dt.$$

The right-hand side tends to zero uniformly provided x is bounded and u is therefore a solution of the integral equation. To prove the uniqueness we assume that there are two continuous solutions u and v. Thus $u-v$ satisfies the homogeneous integral equation. The estimate in the existence proof then implies that $\max|u-v|$ is less than $\max|u-v|(L\rho)^n/n!$. By letting n tend to infinity we see that $u=v$. This completes the proof.

The existence and the uniqueness of a solution of the differential equation (1.1) with the jump condition (1.3) can be established either by using a standard theorem for each of the subintervals (see K. Jörgens [14], page 4.3) or by combining Lemma 1 with equations (1.4) and (1.5). In our next lemma we shall show how the solution u depends upon the parameter λ.

LEMMA 2. *Let u_1 and u_2 be the solutions of equations* (1.4) *and* (1.5). *Let $\sqrt{\lambda}=\sigma+i\tau$ and $c=\max(|b|, |h|, |H|, \int_0^\pi |q(t)|\,dt)$. Then u_1 and u_2 are entire functions of λ of order $\frac{1}{2}$ and*

$$|u_1(x,\lambda)|\leqq(1+\pi c)\,e^{cx+|\tau|x}, \qquad 0\leqq x\leqq d,$$

$$|u_2(x,\lambda)|\leqq(1+\pi c)^3\,e^{cx+|\tau|x}(a+a^{-1}), \qquad d\leqq x\leqq \pi.$$

Remark. Titchmarsh [28], pages 7 and 10, derives two bounds for the eigenfunctions. The first is valid for all λ, but $c+|\tau|$ is replaced by $(\max|q|+|\lambda|)^{1/2}$. The second is similar to ours, but only valid for λ sufficiently large. In K. Jörgens notes [14], page 4.3, $cx+|\tau|x$ is replaced by $(1+|\lambda|)^{1/2}\int_0^x \max(1,|q|)\,dt$, but his estimate is derived for more general equations.

Proof: Let $k=\sqrt{\lambda}$ and assume that $\nu\geqq|\tau|$. To estimate u_1 we rewrite equation (1.4) as

$$e^{-\nu z}u_1(x)=\left[\cos kx+hx\frac{\sin kx}{kx}\right]e^{-\nu x}$$
$$+\int_0^x \frac{\sin k(x-t)}{k(x-t)}\,e^{-\nu(x-t)}(x-t)q(t)\,e^{-\nu t}u_1(t)\,dt.$$

We observe now that

$$|\cos kx|, |\sin kx|, \left|\frac{\sin kx}{kx}\right|\leqq e^{\nu x}, \tag{1.8}$$

if $x\geqq 0$ and $\nu\geqq|\mathscr{I}m\,k|$. By using these bounds and the notation of Lemma 1 we find that M and L are less than $1+|h|x$ and x. Since f and K are analytic functions of λ we see that all the iterates in the method of successive approximations are analytic. Thus the solution u_1 is analytic in the strip $|\mathscr{I}m\sqrt{\lambda}|\leqq\nu$ as the infinite series converges uniformly. Since ν is arbitrary we conclude that u_1 is an entire function. By using Lemma 1 we obtain

$$|u_1(x, \lambda)| \leqq (1+|h|x)\, e^{x\rho(x)}\, e^{\nu x},$$

where $\rho(x)=\int_0^x |q|\, dt$ and $0 \leqq x \leqq d$. We get the sharpest bound if $\nu = |\tau|$. Thus u_1 is an entire function of λ of order $\frac{1}{2}$. To estimate u_2 we multiply both sides of equation (1.5) by $e^{-\nu x}$ which we rewrite as $e^{-\nu(x-d)}\, e^{-\nu d}$. Using equation (1.8), our estimate for $e^{-\nu x} u_1$ and the inequality $1 < a + a^{-1}$ we find that, in the notation of Lemma 1,

$$\begin{aligned}|f(x)| &\leqq a+a^{-1}+|b|(x-d)+|h|[a^{-1}(x-d)+ad+|b|(x-d)d] \\ &\quad + \int_0^d [a^{-1}(x-d)+ad+|b|(x-d)d]|q(t)|(1+\pi c)\, e^{d\rho(d)}\, dt \\ &\leqq (1+c\pi)^3\, e^{d\rho(d)}(a+a^{-1}).\end{aligned}$$

Thus $M(x)$ is less than $(1+c\pi)^3 e^{x\rho}(a+a^{-1})$ and $L < x$. We can now apply Lemma 1 to the integral equation for $e^{-\nu x}u_2$ and derive the bound for u_2. This completes the proof.

To derive the integral representation of the eigenfunctions and estimate the higher eigenvalues we must study the behavior of u_1 and u_2 for large λ. We get the leading terms in the asymptotic expansion by setting $b=h=0$ and $q \equiv 0$ in equation (1.4) and (1.5).

LEMMA 3. *Let u_1 and u_2 be the solutions of equations* (1.4) *and* (1.5). *Let $k=\sqrt{\lambda}=\sigma+i\tau$ and $c=\max(|b|, |h|, |H|, \int_0^\pi |q(t)|\, dt)$. If $|k| \geqq 3c$, then*

$$|u_1(x)| \leqq 2\, e^{|\tau|x}, \qquad 0 \leqq x \leqq d,$$

$$|u_1(x)-\cos kx| \leqq \frac{3c}{|k|}\, e^{|\tau|x},$$

$$|u_2(x)| \leqq 3\, e^{|\tau|x}(a+a^{-1}), \qquad d \leqq x \leqq \pi,$$

$$|u_2(x)-a\cos k(x-d)\cos kd+a^{-1}\sin k(x-d)\sin kd| \leqq \frac{5c}{|k|}\, e^{|\tau|x}(a+a^{-1}),$$

$$|u_2'(x)+ka\sin k(x-d)\cos kd+ka^{-1}\cos k(x-d)\sin kd| \leqq 5c\, e^{|\tau|x}(a+a^{-1}).$$

Remark. If $c=0$, then we set $c/k=0$ for all k. Our proof is based on the Volterra integral equation for the eigenfunctions. The approach was used by Liouville [20], page 27, and has been extended by Hobson [12] and Borg [6].

Proof: Let $\nu=|\tau|$. To estimate u_1 we rewrite equation (1.4) as

$$e^{-\nu x}\left[u_1-\cos kx-\frac{h}{k}\sin kx\right]$$
$$=\frac{1}{k}\int_0^x \sin k(x-t)\, e^{-\nu(x-t)}q(t)\, e^{-\nu t}\left[\cos kt+\frac{h}{k}\sin kt\right] dt$$
$$+\frac{1}{k}\int_0^x \sin k(x-t)\, e^{-\nu(x-t)}q(t)\, e^{-\nu t}\left[u_1-\cos kt-\frac{h}{k}\sin kt\right] dt.$$

Since $|\cos kt|$ and $|\sin kt|$ are bounded by $e^{\nu t}$ and $|h/k|\leqq\frac{1}{3}$ we find in the notation of Lemma 1 that $M\leqq\frac{4}{3}c/|k|$, $L\leqq 1/|k|$ and $\rho(d)\leqq c$. Using Lemma 1 we get

$$e^{-\nu x}|u_1-\cos kx-\frac{h}{k}\sin kx|\leqq\frac{4}{3}\frac{c}{|k|}e^{c/|k|}.$$

Since $|h/k|\leqq c/|k|\leqq\frac{1}{3}$ we have proved the first two inequalities in Lemma 3.

To estimate u_2 we use the same approach. Let ψ denote the first six terms on the right side of equation (1.5). We get an integral equation for $e^{-\nu x}(u_2-\psi)$ by multiplying both sides of equation (1.5) by $e^{-\nu x}$ and reordering the terms. Since $|b/k|$ and $|h/k|$ are less than $\frac{1}{3}$ we find in the notation of Lemma 1 that

$$|f(x)|\leqq\frac{1}{|k|}\int_0^d [a^{-1}+a+\tfrac{1}{3}]|q|\cdot 2dt+\frac{1}{|k|}\int_d^x |q|([a+a^{-1}+\tfrac{1}{3}]+\tfrac{1}{3}[a^{-1}+a+\tfrac{1}{3}])\,dt,$$

where we have used the fact that $|e^{-\nu x}u_1|\leqq 2$ and equation (1.8). Thus M and L are less than $2(a+a^{-1}+\frac{1}{3})c/|k|$ and $1/|k|$. Using Lemma 1 and the inequality $2\leqq a+a^{-1}$ we get

$$e^{-\nu x}|u_2-\psi|\leqq\frac{7}{3}\frac{c}{|k|}e^{c/|k|}(a+a^{-1})\leqq\frac{10}{3}\frac{c}{|k|}(a+a^{-1}).$$

Since $|\psi|\leqq\frac{14}{9}(a+a^{-1})\,e^{\nu x}$ and the last four terms in ψ are no greater than $\frac{10}{6}(a+a^{-1})c/|k|\,e^{\nu x}$, we have proved the next two inequalities in Lemma 3. By differentiating both sides of equation (1.5) we see after reordering that

$$u_2'(x)+ka\sin k(x-d)\cos kd+ka^{-1}\cos k(x-d)\sin kd$$
$$=b\cos k(x-d)\cos kd+h\Big[a^{-1}\cos k(x-d)\cos kd-a\sin k(x-d)\sin kd$$
$$\tag{1.9} +\frac{b}{k}\cos k(x-d)\sin kd\Big]+\int_0^d\Big[a^{-1}\cos k(x-d)\cos k(d-t)$$
$$-a\sin k(x-d)\sin k(d-t)+\frac{b}{k}\cos k(x-d)\sin k(d-t)\Big]q(t)u_1(t)\,dt$$
$$+\int_d^x\cos k(x-t)q(t)u_2(t)\,dt.$$

More precisely, there exists an absolutely continuous function which agrees with u_2' almost everywhere. In the following investigations we denote this function by u_2'. To get the last inequality in Lemma 3 we multiply both sides of equation (1.9) by $e^{-\nu x}$, use equation (1.8) and the first and the third inequality in Lemma 3. This completes the proof.

Next we study the distribution of the eigenvalues of problem (1.1)–(1.3). Here λ is an eigenvalue if the solution with the initial data $u=1$ and $u'=h$ at $x=0$ satisfies the boundary condition $u'+Hu=0$ at $x=\pi$. We shall prove the existence of the eigenvalues by the Cauchy integral technique. We set $\sqrt{\lambda}=k$ and let R_n be the rectangle in the k-plane with vertices at $\pm\nu+i0$ and $\pm\nu+i\nu$, where $\nu=n-\frac{1}{2}$ (see Titchmarsh [28], page 13). Finally we let Γ_n be the contour in the λ-plane that corresponds to the points of R_n for which $\mathscr{Im}\, k>0$.

LEMMA 4. *Let u be the solution of equations* (1.1)–(1.3) *with $u=1$ and $u'=h$ at $x=0$ and let $\omega(\lambda)=-u'(\pi)-Hu(\pi)$. Then ω is an entire function of λ of order $\frac{1}{2}$. Its roots $\lambda_0<\lambda_1<\cdots$ are real and simple. Let $m=\max(a,a^{-1})$ and assume that $n\geqq m+26c(1+m^2)$, where $c=\max(|b|,|h|,|H|,\int_0^\pi|q|\,dt)$. Set $\sqrt{\lambda}=\sigma+i\tau$. Then*

$$|\omega(\lambda)|\geqq\frac{\sqrt{|\lambda|}e^{|\tau|\pi}}{6m}$$

for all points λ on the contour Γ_n and $|\sqrt{\lambda_n}-n|<\frac{1}{2}$.

Remark. If $a\approx 1$, then our estimate differs from the best possible estimate for the smooth case by a factor of 3. The trouble in our case is that the leading term in the Wronskian cannot be obtained by inspection. In fact the roots can behave randomly and can lie very close to the contour Γ_n if a is either very large or very small. Thus it becomes difficult to estimate the Wronskian from below. Similar bounds can be obtained for eigenvalue problems with several discontinuities provided these are small. If the discontinuities are large then I cannot estimate the eigenvalues in any useful manner and do not know how to choose the contours Γ_n.

Proof: Let ω_0 be the leading term in the Wronskian ω. We shall show that $|\omega-\omega_0|<|\omega_0|$ on the contour Γ_n if n is sufficiently large. Since $\omega=\omega_0+\omega-\omega_0$ it follows from Rouche's theorem that ω_0 and ω have the same number of zeroes inside Γ_n. To get the leading term of ω we imagine that $b=h=H=0$ and that $q\equiv 0$. This corresponds to $c=0$ and thus

$$\omega_0=ka\sin k(\pi-d)\cos kd+ka^{-1}\cos k(\pi-d)\sin kd.$$

Since $\omega-\omega_0$ is equal to $-(u_2'(\pi)+\omega_0)-Hu_2(\pi)$, we conclude from Lemma 3 that

$$|\omega-\omega_0|\leqq 8c\,e^{|\tau|\pi}(a+a^{-1}) \tag{1.10}$$

for $|k| \geqq 3c$. We shall now derive lower bounds for ω_0 and ω. Using the definition of ω_0 and elementary trigonometric formulas we see that

$$\omega_0 = k(a+a^{-1})\tfrac{1}{2}[\sin k\pi + \alpha \sin k\pi\beta], \tag{1.11}$$

where $\alpha = (a-a^{-1})/(a+a^{-1})$ and $\beta = 1-2d/\pi$. Note that α is always between -1 and 1. Since $0 < d < \frac{1}{2}\pi$, we find that $0 < \beta < 1$. Let $k = \pm\nu + i\tau$ be a point on the rectangle R_n, i.e., $\nu = n - \frac{1}{2}$. We can then estimate

$$|\sin k\pi + \alpha \sin k\pi\beta| \geqq \cosh \tau\pi - |\alpha| \cosh \tau\pi\beta \geqq \tfrac{1}{2} e^{|\tau|\pi}[1-|\alpha|].$$

It is more complicated to estimate ω_0 when $k = \sigma + i\nu$. We observe first that

$$|\sin k\pi + \alpha \sin k\pi\beta| \geqq \sinh \nu\pi - |\alpha| \cosh \nu\pi\beta \geqq \tfrac{1}{2} e^{\nu\pi}(1-|\alpha|)\left[1 - \frac{1+|\alpha|}{1-|\alpha|}e^{-2\nu\pi}\right],$$

since $|\sin(x+iy)| > \sinh |y|$. We observe next that $(1+|\alpha|)/(1-|\alpha|)$ is equal to m^2 where $m = \max(a, a^{-1})$. Since $n \geqq m \geqq 1$ we conclude that $m^2 e^{-2\nu\pi}$ is less than $e^{-\pi}$. Combining the estimates for the three sides of R_n we see from equation (1.11) that

$$|\omega_0| \geqq |k|(a+a^{-1})\tfrac{1}{2}\tfrac{1}{2} e^{|\tau|\pi}(1-|\alpha|)[1-e^{-\pi}]$$

for all points λ on the contour Γ_n. We split the lower bound for $|\omega|$ in two parts. By combining the lower bound for $|\omega_0|$ with the upper bound for $|\omega - \omega_0|$, using the fact that $(a+a^{-1})(1-|\alpha|) = 2/m$ and that $1 - e^{-\pi} > \frac{1}{3} + \frac{8}{13}$, we find that

$$|\omega| \geqq |\omega_0| - |\omega - \omega_0| > |k| 2/m \tfrac{1}{2}\, \tfrac{1}{2} e^{|\tau|\pi} \tfrac{1}{3} + (a+a^{-1})\, e^{|\tau|\pi} \tfrac{1}{2}\, \tfrac{8}{13}[|k|\tfrac{1}{2}(1-|\alpha|) - 26c].$$

Since $|k| \geqq \nu > n - m$ and $\frac{1}{2}(1-|\alpha|) = (1+m^2)^{-1}$, we conclude that the expression $[\cdots]$ is positive provided $n \geqq m + 26c(1+m^2)$. We have therefore derived the bound for ω stated in the lemma. Since $|\omega - \omega_0| < |\omega_0|$ on the contour Γ_n it follows from Rouche's theorem that ω_0 and $\omega_0 + \omega - \omega_0$ have the same number of zeroes inside Γ_n.

We shall first study the zeroes of ω_0. Note that $\omega_0 = 0$ at $\lambda = 0$. To show that the remaining roots of $\omega_0(\lambda) = 0$ are positive, we split the equation $\sin k\pi + \alpha \sin k\pi\beta = 0$ into its real and imaginary parts and find after some calculations that

$$(\sin^2 \sigma\pi - \cosh^2 \tau\pi\beta)(\sinh^2 \tau\pi - \sinh^2 \tau\pi\beta) = (1-\alpha^2) \sinh^2 \tau\pi\beta \cosh^2 \tau\pi\beta.$$

Since $|\alpha|$ and $|\beta|$ are less than 1 we conclude that $\tau = 0$ and $\lambda = k^2 > 0$. We shall now prove that the roots are simple. This is certainly the case at $\lambda = 0$ since $\omega_0'(0) = (a+a^{-1})\frac{1}{2}\pi(1+\alpha\beta)$. If $\lambda > 0$ is a root of ω_0, then

$$\begin{aligned} |\omega_0'(\lambda)| &= (a+a^{-1})\tfrac{1}{4}\pi|\cos k\pi + \alpha\beta \cos k\pi\beta| \\ &\geqq (a+a^{-1})\tfrac{1}{4}\pi[(1-\alpha^2 \sin^2 k\pi\beta)^{1/2} - |\alpha\beta|(1-\sin^2 k\pi\beta)^{1/2}]. \end{aligned}$$

The smallest value of $[\cdots]$ is $1-|\alpha\beta|$ if $|\beta|\geqq|\alpha|$ and $(1-\alpha^2)^{1/2}(1-\beta^2)^{1/2}$ otherwise. Hence

$$|\omega_0'(\lambda)|\geqq(a+a^{-1})\tfrac{1}{4}\pi(1-\alpha^2)^{1/2}(1-\beta^2)^{1/2}=\tfrac{1}{2}\pi(1-\beta^2)^{1/2},$$

where we have used the definition of α. Thus the roots of $\omega_0(\lambda)=0$ are simple. It can be shown by examples that the lower bound for ω_0' is the best possible, e.g. $\alpha=.70689$, $\beta=.57697$ and $\sqrt{\lambda}=\frac{13}{6}$.

To count the number of zeroes of ω_0 we consider Figure 2. Since the sign of $\sin k\pi+\alpha\sin k\pi\beta$ is $(-1)^n$ at $k=n+\frac{1}{2}$ we can conclude that each interval $[n-\pi^{-1}\arcsin|\alpha|,\ n+\pi^{-1}\arcsin|\alpha|]$ contains at least one value k such that $\omega(k^2)=0$. Thus ω_0 has at least n zeroes inside Γ_n. Since $|\alpha\sin k\pi\beta|<|\sin k\pi|$ on the three sides of R_n it follows from Rouche's theorem that $\frac{1}{2}(a+a^{-1})\sqrt{\lambda}\sin\sqrt{\lambda}\,\pi$ and ω_0 have the same number of zeroes inside Γ_n. Thus ω_0 and therefore also ω have exactly n zeroes inside Γ_n. By modifying Titchmarsh's proof in [28], pages 11–12, we find that the roots $\lambda_0,\cdots,\lambda_{n-1}$ of $\omega(\lambda)=0$ are real and simple. Finally we observe that $|\sqrt{\lambda_n}-n|<\frac{1}{2}$ since the signs of $\omega((n-\frac{1}{2})^2)$ and $\omega((n+\frac{1}{2})^2)$ are opposite. This completes the proof.

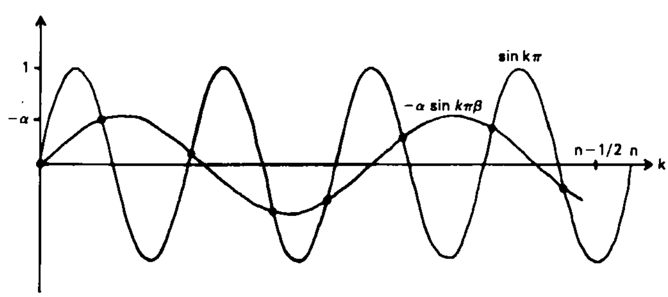

Figure 2. Graphical determination of the roots of $\sin k\pi+\alpha\sin k\pi\beta=0$.

We shall now show that the eigenvalues of a problem with a discontinuity may not have an asymptotic expansion. Consider a vibrating string with fixed ends. Thus $-y''=\lambda\rho y$ and $y(0)=y(\pi)=0$, where $y(x)$ is the displacement at the point x, ρ is the density per unit length and the length is π. We set $\rho=\frac{1}{6}(4-\sqrt{2})$ for $0\leqq x<d$ and $\rho=\frac{1}{2}(4-\sqrt{2})$ for $d<x\leqq\pi$, where $d=\frac{1}{14}(9-3\sqrt{2})\pi$, and assume that y and y' are continuous. A short calculation shows that the eigenvalues λ_n satisfy the equation

$$\sin\sqrt{\lambda}\,\pi-\frac{1}{2}\sin\sqrt{\lambda}\,\frac{\pi}{\sqrt{2}}=0,$$

(see Figure 2). Let $\sqrt{\lambda_n}=n+\delta_n$, and note that $\pi^{-1}\arcsin\frac{1}{2}=\frac{1}{6}$. We have calculated the first 1000 roots of this equation and present the result in Figure 3 as a histogram with mesh width $\frac{1}{60}$. Clearly the eigenvalues do not have an asymptotic

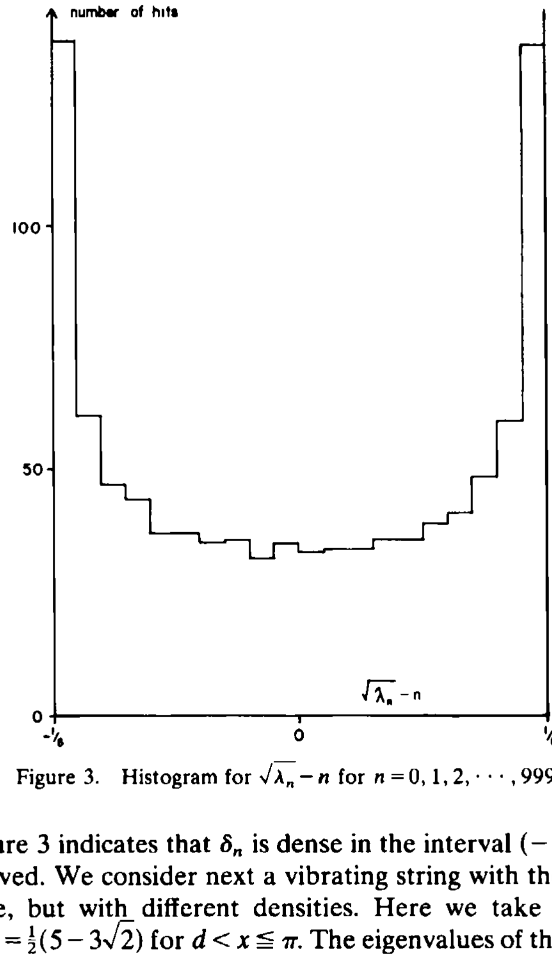

Figure 3. Histogram for $\sqrt{\lambda_n} - n$ for $n = 0, 1, 2, \cdots, 999$.

expansion. Figure 3 indicates that δ_n is dense in the interval $(-\frac{1}{6}, \frac{1}{6})$, and this can actually be proved. We consider next a vibrating string with the discontinuity at the same place, but with different densities. Here we take $\rho = \frac{1}{2}(3+\sqrt{2})$ for $0 \leqq x < d$ and $\rho = \frac{1}{2}(5-3\sqrt{2})$ for $d < x \leqq \pi$. The eigenvalues of this problem satisfy the equation

$$\sin\sqrt{\lambda}\,\pi - \frac{1}{\sqrt{2}}\sin\sqrt{\lambda}\,\frac{\pi}{2} = 0.$$

The roots of this equation can be given by inspection. They are $\sqrt{\lambda_{2n}} = 2n$ and $\sqrt{\lambda_{2n-1}} = 2n - 1 + (-1)^n(2/\pi)\arcsin(1/2\sqrt{2})$. Thus there is no asymptotic expansion, but the eigenvalues fall in three distinct classes, each of which has an asymptotic expansion.

2. Integral Representation of the Eigenfunctions

In this section we shall study the eigenfunctions of a Sturm–Liouville problem with an interior discontinuity. We shall show that the solution of the differential equation (1.1) can be written as a trigonometric function plus an integral. The kernel in this integral depends upon the potential $q(x)$ and the constants a, b, d and h, but is independent of the parameter λ. Our calculations show that the kernel is not continuous in general, but given by a different formula in each of the five regions indicated in Figure 4.

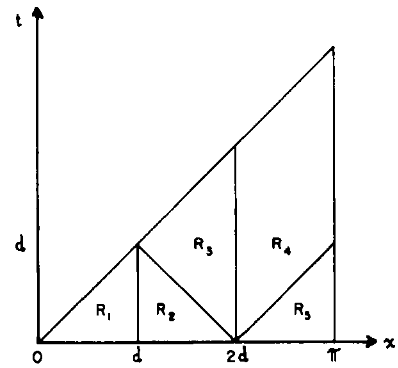

Figure 4. The kernel $K(x, t)$ is defined for $0 \leqq t \leqq x \leqq \pi$.

LEMMA 5. *Let u be the solution of equations* (1.1)–(1.3) *with $u = 1$ and $u' = h$ at $x = 0$ and with $\lambda = k^2$. Set*

$$\varphi(x, k^2) = \cos kx, \qquad 0 \leqq x < d,$$
$$= a \cos k(x-d) \cos kd - a^{-1} \sin k(x-d) \sin kd, \qquad d < x \leqq 2\pi.$$

There exists a bounded function $K(x, t)$ such that

$$u(x, k^2) = \varphi(x, k^2) + \int_0^x K(x, t) \cos kt \, dt$$

for all k and $x \neq d$. Set $K(x, t) = 0$ if $t < 0$ or $t > x$. The kernel K is continuous in the regions R_1, R_2, $R_3 \cup R_4$ and R_5, where

$$R_1: 0 \leqq x \leqq d, \quad 0 \leqq t \leqq x,$$
$$R_2: d < x < 2d, \quad 0 \leqq t \leqq 2d - x,$$
$$R_3: d < x < 2d, \quad 2d - x < t \leqq x,$$
$$R_4: 2d \leqq x \leqq \pi, \quad x - 2d \leqq t \leqq x,$$
$$R_5: 2d < x \leqq \pi, \quad 0 \leqq t < x - 2d.$$

The restriction of K to any one of the domains R_1, R_2, $R_3 \cup R_4$ or R_5 can be extended as a continuous function on the closure of the set.

Remark. This lemma is our main technical result. It is a generalization of the Povzner–Levitan representation of the eigenfunctions of a Sturm–Liouville problem. The usual representation can be derived either by solving a Goursat problem (cf. Gel'fand and Levitan [9]) or by using Riemann's method of integrating hyperbolic equations (cf. Povzner [23], Levitan [18], Levin [17], page 424) or by using the Paley–Wiener theorem (cf. Zikov [31], Chadan and Sabatier [7], Sabatier [25]). We have extended the last technique to problems with interior discontinuities. Note that φ is the solution of equations (1.1)–(1.3) with $h=b=0$ and $q(x)\equiv 0$. Thus it carries part of the discontinuity, but not all.

Proof: We remember first that $u=u_1$ for $x<d$ and $u=u_2$ for $x>d$. Thus it follows from Lemma 2 and Lemma 3 that $u-\varphi$ satisfies the condition in the Paley–Wiener theorem (see Paley and Wiener [22]). Since $u-\varphi$ is an even function of k and real if k is real, there exists for all values of $x\neq d$ a square integrable function K such that

$$u(x,k^2)-\varphi(x,k^2)=\int_0^x K(x,t)\cos kt\,dt \tag{2.1}$$

for all $x\neq d$. Using the Fourier transform we find that

$$K(x,t)=\frac{2}{\pi}\int_0^\infty \cos kt[u(x,k^2)-\varphi(x,k^2)]\,dk \tag{2.2}$$

almost everywhere. Here the integral is interpreted as the limit in the mean in the L^2 sense. Since we can change the kernel in equation (2.1) on a set of measure zero without changing the integral, we use equation (2.2) in the interior of the regions $R_1,\cdots,R_5$ and define the values of K at the boundaries such that the kernel is continuous in each region.

We shall now study the function $u=u_1$ in the interval $0\leqq x<d$ and determine the kernel $K=K_1$ in the region R_1. Equation (1.4) implies that

$$u-\varphi=\frac{h}{k}\sin kx+\frac{1}{k}\int_0^x \sin k(x-t)q(t)\cos kt\,dt+E_1, \tag{2.3}$$

where

$$E_1(x,k^2)=\frac{1}{k}\int_0^x \sin k(x-t)q(t)[u_1(t,k^2)-\cos kt]\,dt. \tag{2.4}$$

Since $k^{-1}\sin kx$ is equal to $\int_0^x \cos k\tau\,d\tau$, we see that the contribution from the first term in equation (2.3) to the kernel K_1 is h if $0<t<x$. We consider next the second term in the equation (2.3). Using the trigonometric addition formulas

and a change of variables it can be rewritten as

$$\frac{1}{2k}\int_0^x \sin kx\, q(t)\, dt+\frac{1}{2k}\int_0^x \sin k(x-2t)q(t)\, dt))\, dy.$$

$$=\int_0^x \cos k\tau \frac{1}{2}\int_0^x q(t)\, dt\, d\tau+\frac{1}{4k}\int_{-x}^x \sin ky\, q(\tfrac{1}{2}(x-y))\, dy.$$

By reflecting around $y=0$ we see that the last term is equal to

$$\frac{1}{4k}\int_0^x \sin ky[q(\tfrac{1}{2}(x-y))-q(\tfrac{1}{2}(x+y))]\, dy$$

$$=\int_0^x \cos k\tau \frac{1}{4}\int_\tau^x [q(\tfrac{1}{2}(x-y))-q(\tfrac{1}{2}(x+y))]\, dy\, d\tau,$$

where we have used Fubini's theorem to interchange the order of integration. Thus it follows from the Fourier integral theorem and an additional change of variables that the contribution from the second term in equation (2.3) to the kernel K_1 is

$$\frac{1}{2}\int_0^x q(s)\, ds+\frac{1}{2}\int_0^{(x-t)/2} q(s)\, ds-\frac{1}{2}\int_{(x+t)/2}^x q(s)\, ds. \tag{2.5}$$

Finally, we consider the last term in equation (2.3). By combining equation (2.4) with Lemma 2 and Lemma 3 we see that E_1 is a continuous function of x and k and that $|E_1|\leqq 3c^2/k^2$ for $k\geqq 3c$. This implies that the integral

$$K_{01}(x,t)=\frac{2}{\pi}\int_0^\infty \cos kt\, E_1(x,k^2)\, dk$$

converges uniformly with respect to k and K_{01} is therefore a continuous function of x and t (see Titchmarsh [27], page 25). Thus we can express the kernel K_1 as

$$K_1(x,t)=h+\frac{1}{2}\int_0^{(x+t)/2} q(s)\, ds+\frac{1}{2}\int_0^{(x-t)/2} q(s)\, ds+K_{01}(x,t), \tag{2.6}$$

and this formula can be extended to the boundary of the region R_1. We note that the first three terms in equation (2.6) are absolutely continuous with respect to x and t. Inserting the expression for $u_1-\varphi$ into equation (2.4), evaluating the terms with coefficients k^{-2} and observing that the remainder—which decays like k^{-3}—is continuously differentiable with respect to x, we can show that K_{01} is continuously differentiable with respect to x and t and not just continuous. We do not need this refinement and will not present the details.

We shall now study the function $u=u_2$ in the intervals $d<x<2d$ and $2d<x<\pi$ and determine the kernels K_2, K_3, K_4 and K_5 in the regions R_2, R_3, R_4 and R_5. As our technique is similar to the previous case we present only the main steps. From equation (1.5) it follows that

$$u_2-\varphi=\left(b+\frac{h}{a}\right)\frac{1}{k}\sin k(x-d)\cos kd+\frac{ha}{k}\cos k(x-d)\sin kd$$
$$+\frac{1}{k}\int_0^d [a^{-1}\sin k(x-d)\cos k(d-t)$$
$$+a\cos k(x-d)\sin k(d-t)]q(t)\cos kt\,dt$$
$$+\frac{1}{k}\int_d^x \sin k(x-t)q(t)[a\cos k(t-d)\cos kd$$
$$-a^{-1}\sin k(t-d)\sin kd]\,dt+E_2$$

for $x>d$, where

$$E_2(x,k^2)=\frac{hb}{k^2}\sin k(x-d)\sin kd+\frac{b}{k^2}\int_0^d \sin k(x-d)\sin k(d-t)q(t)u_1(t)\,dt$$
$$+\frac{1}{k}\int_0^d [a^{-1}\sin k(x-d)\cos k(d-t)$$
$$+a\cos k(x-d)\sin k(d-t)]q(t)[u_1(t,k^2)-\cos kt]\,dt$$
$$+\frac{1}{k}\int_d^x \sin k(x-t)q(t)[u_2(t,k^2)-\varphi(t,k^2)]\,dt.$$

Using the trigonometric addition formulas and several changes of variables we find, after some reordering, that

$$u_2-\varphi=[b+h(a+a^{-1})]\frac{\sin kx}{2k}+[b+h(a^{-1}-a)]\frac{\sin k(x-2d)}{2k}$$
$$+\frac{a+a^{-1}}{2k}\int_0^x \sin k(x-t)q(t)\cos kt\,dt$$
$$+(a^{-1}-a)\left[\int_0^d q\,dt-\int_d^x q\,dt\right]\frac{\sin k(x-2d)}{4k} \tag{2.7}$$
$$+\frac{a^{-1}-a}{4k}\int_0^d \sin k(x-2d+2t)q(t)\,dt$$
$$+\frac{a^{-1}-a}{4k}\int_d^x \sin k(2t-x-2d)q(t)\,dt+E_2.$$

Using equation (2.5) we can write down the contribution from the first term and the third term in equation (2.7) to the kernels K_2, K_3, K_4 and K_5, namely

$$\tfrac{1}{2}b+\tfrac{1}{2}(a+a^{-1})\left[h+\frac{1}{2}\int_0^{(x+t)/2} q(s)\,ds+\frac{1}{2}\int_0^{(x-t)/2} q(s)\,ds\right].$$

Since $\sin k(x-2d)$ is equal to $\operatorname{sign}(x-2d)\sin k|x-2d|$, we see that the contribution from the second term and the fourth term in equation (2.7) to the kernels K_2 and K_5 is equal to

$$\operatorname{sign}(x-2d)\left\{\tfrac{1}{2}b+\tfrac{1}{2}(a^{-1}-a)\left[h+\frac{1}{2}\int_0^d q(s)\,ds-\int_d^x q(s)\,ds\right]\right\},$$

while there is no contribution to K_3 and K_4 in the interior of the regions R_3 and R_4 because here $|x-2d|<t$. We consider now the remaining two integrals in equation (2.7) and assume that $d<x<2d$. Since $\sin ky = k\int_0^y \cos k\tau\,d\tau$ we find after substituting $2t\pm x-2d$ by y, splitting the first integral in two parts and using Fubini's theorem that the fifth term and the sixth term in equation (2.7) can be written as

$$\begin{aligned}
&\int_0^x \cos k\tau\,d\tau \int_\tau^x \tfrac{1}{2}q(\tfrac{1}{2}(y-x+2d))\\
&-\int_0^{2d-x} \cos k\tau\,d\tau \int_\tau^{2d-x} \tfrac{1}{2}q(\tfrac{1}{2}(-y-x+2d))\,dy\\
&-\int_0^{2d-x} \cos k\tau\,d\tau \int_{2d-x}^{x} \tfrac{1}{2}q(\tfrac{1}{2}(-y+x+2d))\,dy\\
&-\int_{2d-x}^{x} \cos k\tau\,d\tau \int_\tau^{x} \tfrac{1}{2}q(\tfrac{1}{2}(-y+x+2d))\,dy,
\end{aligned}$$

except for a common factor of $\frac{1}{4}(a^{-1}-a)$. By using the Fourier integral theorem we conclude that the contribution from the fifth term and the sixth term in equation (2.7) to the kernel K_2 is

$$\tfrac{1}{4}(a^{-1}-a)\left[\int_{(2d-x+t)/2}^{d} q(s)\,ds-\int_0^{(2d-x-t)/2} q(s)\,ds-\int_d^x q(s)\,ds-0\right]$$

while the contribution to the kernel K_3 is equal to

$$\tfrac{1}{4}(a^{-1}-a)\left[\int_{(2d-x+t)/2}^{d} q(s)\,ds-0-0-\int_d^{(2d+x-t)/2} q(s)\,ds\right]. \tag{2.8}$$

Assume now that $2d<x<\pi$. Then similar arguments show that the contribution from the fifth and the sixth term in equation (2.7) to the kernel K_4 is given by equation (2.8) and that the contribution to the kernel K_5 is equal to

$$\tfrac{1}{4}(a^{-1}-a)\left[\int_0^d q(s)\,ds+\int_{(2d+x+t)/2}^{x} q(s)\,ds-\int_d^{(2d+x-t)/2} q(s)\,ds\right].$$

Finally, let $K_{02}(x,t)$ be defined as the kernel $K_{01}(x,t)$ but corresponding to the term E_2 in equation (2.7). Since $E_2(x,k^2)$ is continuous in x and k and $|E_2|\leqq 6(a+a^{-1})c^2/k^2$ for $k\geqq 3c$, we see that K_{02} is a continuous function of x

and t. By combining our results we conclude that each of the kernels K_2, K_3, K_4 and K_5 is continuous in the interior of its domain of definition and that it can be extended as a continuous function to the boundary of the region. Moreover, we observe that

$$K_3(x,t)=\tfrac{1}{2}b+\tfrac{1}{2}(a+a^{-1})\left[h+\frac{1}{2}\int_0^{(x+t)/2} q(s)\,ds+\frac{1}{2}\int_0^{(x-t)/2} q(s)\,ds\right]$$
$$+\tfrac{1}{4}(a-a^{-1})\left[\int_d^{d+(x-t)/2} q(s)\,ds-\int_{d-(x-t)/2}^{d} q(s)\,ds\right]+K_{02}(x,t),$$

and that the kernel K_4 is given by the same formula. Since K_3 and K_4 are the restrictions of the kernel K to the regions R_3 and R_4 this implies that $K(x,t)$ is a continuous function in $R_3 \cup R_4$. This completes the proof.

The kernel in Lemma 5 has an unexpected discontinuity at $x=d$. The integral representation may not be well suited for generalizations, but it is sufficient for our purpose. From the definition of φ follows that $\varphi(d+)=a\varphi(d-)$. Combining this result with equation (1.3) we find that

$$\int_0^d [K(d+,t)-aK(d-,t)]\cos kt\,dt=0$$

for all k. Since the expression $[\cdots]$ is a continuous function of t, we conclude by using the Fourier integral theorem that $K_2=aK_1$ for $x=d$ and $0<t<d$. Thus the kernel has a discontinuity at $x=d$ if $a\neq 1$ (cf. Krueger [15]).

3. Further Study of the Wronskian

So far we have studied the eigenvalues and the eigenfunctions of a particular eigenvalue problem. In this section we shall assume that two different eigenvalue problems have the same eigenvalues and show that the two discontinuities must occur at the same point. The jumps in the eigenfunctions are also the same, but we cannot yet prove that the jumps in their derivatives are identical.

LEMMA 6. *The constants a and d in the eigenvalue problem* (1.1)–(1.3) *are uniquely determined by the eigenvalues provided* $0<d<\frac{1}{2}\pi$ *and* $|a-1|+|b|>0$.

Remark. If $a=1$ and $b=0$, then u and u' are continuous at $x=d$. Hence there is no discontinuity and d can be anywhere. Thus we insist that $|a-1|+|b|>0$. An additional reason is that the eigenvalues of a problem with a discontinuity and of one without cannot agree.

Proof: We have proved in Lemma 4 that the Wronskian $\omega(\lambda)$ is an entire function of λ of order $\frac{1}{2}$ and that its roots are simple. Thus it follows from

Hadamards theorem (see [27], page 250) that ω is uniquely determined up to a multiplicative constant by the canonical product of genus zero formed by the eigenvalues. Let ω and $\tilde{\omega}$ be the Wronskians for two different eigenvalue problems that have the same eigenvalues. Then $\omega = C\tilde{\omega}$ for all λ, where $C \neq 0$. We shall show that $C = 1$. Let ω_0 be the leading term in the Wronskian ω. Since $\omega_0 - C\tilde{\omega}_0$ is equal to $C(\tilde{\omega} - \tilde{\omega}_0) - (\omega - \omega_0)$, it follows from equation (1.11) that

$$\begin{aligned} \tfrac{1}{2}[a + a^{-1} - C(\tilde{a} + \tilde{a}^{-1})]k \sin k\pi + \tfrac{1}{2}(a + a^{-1})\alpha k \sin k\pi\beta \\ - \tfrac{1}{2}C(\tilde{a} + \tilde{a}^{-1})\tilde{\alpha} k \sin k\pi\tilde{\beta} = C(\tilde{\omega} - \tilde{\omega}_0) - (\omega - \omega_0), \end{aligned} \tag{3.1}$$

where $\beta = 1 - 2d/\pi$ and $\tilde{\beta} = 1 - 2\tilde{d}/\pi$. Let $\bar{c}$ be the maximum of c and $\tilde{c}$; see Lemma 2. We multiply now both sides of equation (3.1) by $T^{-2} \sin k\pi$ and integrate with respect to k from $3\bar{c}$ to T. Since $\pi\beta$ and $\pi\tilde{\beta}$ are different from π, we find that

$$\tfrac{1}{2}[a + a^{-1} - C(\tilde{a} + \tilde{a}^{-1})]\left[\frac{1}{4} + O\left(\frac{1}{T}\right)\right] - \alpha O\left(\frac{1}{T}\right) - \tilde{\alpha} O\left(\frac{1}{T}\right) = O\left(\frac{1}{T}\right),$$

where we have used equation (1.10) to estimate the right-hand side of equation (3.1). By letting T tend to infinity we conclude that

$$a + a^{-1} = C(\tilde{a} + \tilde{a}^{-1}). \tag{3.2}$$

Thus the first term in equation (3.1) vanishes. To progress we must distinguish two cases, namely $d = \tilde{d}$ and $d \neq \tilde{d}$. Let us suppose that $d = \tilde{d}$ or equivalently that $\beta = \tilde{\beta}$. We can then combine the remaining two terms on the left-hand side of equation (3.1). Multiplying both sides of this equation by $T^{-2} \sin k\pi\beta$, integrating with respect to k from $3\bar{c}$ to T, and using equation (1.10), we see that

$$\tfrac{1}{2}(a + a^{-1})[\alpha - \tilde{\alpha}]\left[\frac{1}{4} + O\left(\frac{1}{T}\right)\right] = O\left(\frac{1}{T}\right).$$

Here we have used the fact that the discontinuity does not occur at $\frac{1}{2}\pi$, i.e., $\beta \neq 0$. By letting T tend to infinity we see that $\alpha = \tilde{\alpha}$. Thus it follows from equation (3.2) that $a = C\tilde{a}$ and $a^{-1} = C\tilde{a}^{-1}$. This implies that $C^2 = 1$, but since a and $\tilde{a}$ are positive we conclude that $C = 1$. We have therefore proved that $a = \tilde{a}$ if $d = \tilde{d}$.

We consider now the case $d \neq \tilde{d}$ (or $\beta \neq \tilde{\beta}$) and will show that it leads to a contradiction. This part of the proof lies much deeper than the previous one. We observe first that equation (3.2) is still valid. Multiplying equation (3.1) by $T^{-2} \sin k\pi\beta$, integrating with respect to k from $3\bar{c}$ to T and using equation (1.10) we obtain

$$\tfrac{1}{2}(a + a^{-1})\alpha\left[\frac{1}{4} + O\left(\frac{1}{T}\right)\right] - \tilde{\alpha} O\left(\frac{1}{T}\right) = O\left(\frac{1}{T}\right).$$

At this point we have used that both discontinuities lie in the interval $(0, \frac{1}{2}\pi)$, i.e., that β and $\tilde{\beta}$ are positive. Indeed the last equation is wrong if $d + \tilde{d} = \pi$. We can now let T tend to infinity and conclude that $\alpha=0$ and similar arguments show that $\tilde{\alpha}=0$. Thus $a=\tilde{a}=1$ and it follows from equation (3.2) that $C=1$ and $\omega=\tilde{\omega}$. To continue we must study the Wronskian in some detail. Since $\omega=-u_2'(\pi)-Hu_2(\pi)$, we find after setting $a=1$ in equation (1.9) and using the addition formulas that

$$\omega=k\sin k\pi-A\cos k\pi-\tfrac{1}{2}b\cos k\pi\beta-\int_0^\pi \cos k(\pi-2t)q(t)\,dt-E_3, \tag{3.3}$$

where $A=\frac{1}{2}b+h+H+\frac{1}{2}\int_0^\pi q(t)\,dt$ and

$$\begin{aligned}E_3(k^2)=H(u_2(\pi)-\cos k\pi)&+\frac{hb}{k}\cos k(\pi-d)\sin kd\\&+\int_0^d \cos k(\pi-t)q(t)[u_1-\cos kt]\,dt\\&+\int_d^\pi \cos k(\pi-t)q(t)[u_2-\cos kt]\,dt\\&+\frac{b}{k}\int_0^d \cos k(\pi-d)\sin k(d-t)q(t)u_1(t)\,dt.\end{aligned}$$

It follows from Lemma 2 and Lemma 3 that E_3 satisfies the conditions in the Paley–Wiener theorem. Since E_3 is an even function of k and real if k is real, there exists a square integrable function V_3 such that

$$E_3(k^2)=\int_0^\pi V_3(t)\cos kt\,dt.$$

By using a change of variables we can rewrite the fourth term in equation (3.3) as

$$\int_0^\pi \tfrac{1}{2}[q(\tfrac{1}{2}(\pi+t))+q(\tfrac{1}{2}(\pi-t))]\cos kt\,dt=\int_0^\pi V_2(t)\cos kt\,dt.$$

Let V be the function V_2+V_3. Since $\omega=\tilde{\omega}$ for all λ, we combine equation (3.3) with the corresponding expression for $\tilde{\omega}$ with A, b and V replaced by $\tilde{A}$, $\tilde{b}$ and $\tilde{V}$ and obtain

$$(A-\tilde{A})\cos k\pi+\tfrac{1}{2}b\cos k\pi\beta-\tfrac{1}{2}\tilde{b}\cos k\pi\tilde{\beta}=\int_0^\pi (\tilde{V}-V)\cos kt\,dt$$

for all k. Now, our assumption is that $0<\beta\neq\tilde{\beta}<1$. Thus we multiply both sides of the last equation by $T^{-1}\cos k\pi\beta$ and integrate with respect to k from $3\bar{c}$ to T and get

$$\begin{aligned}&(A-\tilde{A})O\left(\frac{1}{T}\right)+\frac{1}{2}b\left[\frac{1}{2}+O\left(\frac{1}{T}\right)\right]-\tilde{b}O\left(\frac{1}{T}\right)\\&\quad=\int_0^\pi (\tilde{V}-V)\frac{1}{T}\int_0^T \cos k\pi\beta\cos kt\,dk\,dt,\end{aligned} \tag{3.4}$$

where we have used Fubini's theorem to interchange the order of integration. Let us express the right-hand side of equation (3.4) as $\int (\tilde{V}-V)f_T\,dt$. Since $|f_T|\leqq 1$ and f_T tend to zero at all points $t\neq \pi\beta$, it follows from Lebesgue's theorem on dominated convergence that the right-hand side of equation (3.4) tends to zero as T tends to infinity. This implies that $b=0$ and a similar argument shows that $\tilde{b}=0$. Since $|a-1|+|b|>0$, we have arrived at a contradiction. Hence $d=\tilde{d}$ and this completes the proof.

We mention that the conclusion in Lemma 6 is valid even if a finite number of eigenvalues are not known. To prove this result we let ω and $\tilde{\omega}$ be the Wronskians for the two different eigenvalue problems. If $\lambda_j=\tilde{\lambda}_j$ for all j except for a finite number, then it follows from Hadamard's theorem that

$$\omega - C\tilde{\omega} = C\left(\prod_{j\in\Lambda}\left[1+\frac{\tilde{\lambda}_j-\lambda_j}{\lambda-\tilde{\lambda}_j}\right]-1\right)\tilde{\omega} \tag{3.5}$$

for all λ, where $C\neq 0$ and the index set Λ consists of those j for which $\lambda_j\neq\tilde{\lambda}_j$. Let $M=\max_\Lambda(|\lambda_j|,|\tilde{\lambda}_j|)$. Writing the product in terms of exponentials and using the fact that $|\log(1+x)|\leqq 2|x|$ for all $|x|\leqq\frac{1}{2}$ we see that

$$\prod_{j\in\Lambda}\left[1+\frac{\tilde{\lambda}_j-\lambda_j}{\lambda-\tilde{\lambda}_j}\right]=\exp\left\{2\vartheta\cdot\sum_{\Lambda}\frac{|\tilde{\lambda}_j-\lambda_j|}{\lambda-|\lambda_j|}\right\}=\exp\left\{\frac{2\vartheta' 2M}{\frac{4}{5}\lambda}\right\}$$

for all $\lambda\geqq 5M$. Here ϑ and ϑ' are less than 1 in absolute value. Since $|e^x-1|\leqq xe^x$ for $x\geqq 0$, we can estimate the factor $\{\cdots\}$ in equation (3.5) by $5e/\lambda$. By combining equations (1.10) and (1.11) we see that $|\tilde{\omega}|\leqq 4k(\tilde{a}+\tilde{a}^{-1})$ for $k\geqq 3\tilde{c}$. The right-hand side of equation (3.5) is therefore less than $20CeM(\tilde{a}+\tilde{a}^{-1})/k$ if k is greater than $3\tilde{c}$ and $(5M)^{1/2}$. We observe now that if a finite number of eigenvalues differ, then we get an extra term in equation (3.1), namely the right-hand side of equation (3.5). By multiplying this extra term by either sin or cos divided by T^2 or T and integrating with respect to k from the largest of $3\tilde{c}$ and $(5M)^{1/2}$ to T we get an extra contribution of order $\log(T)/T$. Since this contribution tends to zero as T tends to infinity, we conclude that the arguments in the previous proof are valid, even if a finite number of eigenvalues are different.

We shall now show that a problem with a discontinuity in the first half of the interval cannot have the same eigenvalues as a problem without any discontinuities. Suppose that the two problems do have the same eigenvalues and let $\tilde{\omega}$ be the Wronskian for the smooth problem. Then $\tilde{a}=1$ and $\tilde{b}=0$. This implies that $\tilde{\alpha}=0$ and thus the third term in equation (3.1) vanishes. By following the proof of Lemma 6 we see that equation (3.2) is still valid. We can then multiply equation (3.1) by $T^{-2}\sin k\pi\beta$ and integrate with respect to k from $3\tilde{c}$ to T and conclude that $\alpha=0$ and $a=1$. Hence it follows from equation (3.2) that $C=1$. By using the integral representation of the Wronskian and copying the second part of the proof we find that $b=0$. We have therefore shown that neither eigenvalue problem has a discontinuity and this contradicts our assumption.

The above arguments fail if the discontinuity is in the middle of the interval. In this case a problem with a discontinuity can have exactly the same eigenvalues as a problem without. Consider for example the two eigenvalue problems

$$-u''=\lambda u, \qquad\qquad -u''=\lambda u,$$
$$u'(0)-hu(0)=u'(\pi)=0, \qquad\qquad u'(0)-\tilde{h}u(0)=u'(\pi)+\tilde{H}u(\pi)=0,$$
$$u(\tfrac{1}{2}\pi+)=au(\tfrac{1}{2}\pi-),$$
$$u'(\tfrac{1}{2}\pi+)=a^{-1}u'(\tfrac{1}{2}\pi-)+bu(\tfrac{1}{2}\pi-),$$

where $b=h(a-a^{-1})$. Using equation (1.9) we see that the eigenvalues of the first problem are the roots of the equation

$$\frac{\tan(\sqrt{\lambda}\,\pi)}{\sqrt{\lambda}}=\frac{h(1+\alpha)}{\lambda-h\alpha},$$

where $\alpha=(a-a^{-1})/(a+a^{-1})$. On the other hand, the eigenvalues of the second problem satisfy the same equation with $h(1+\alpha)$ and $h\alpha$ replaced by $\tilde{h}+\tilde{H}$ and $\tilde{h}\tilde{H}$. If $\tilde{h}$ and $\tilde{H}$ are positive and $2\tilde{h}\tilde{H}<\tilde{h}+\tilde{H}$, then there exist three positive constants a, b and h such that the two problems have the same eigenvalues. Thus, by using only the eigenvalues we cannot in general determine whether or not there is a discontinuity in the interior of the interval.

4. An Integral Equation

In this section we shall show that the difference between two potentials satisfies a homogeneous integral equation if the eigenvalues of the two problems are equal. By using this integral equation we can prove that the two potentials must be equal (see Section 6).

LEMMA 7. *Let $u=u(x,\lambda)$ be the solution of equation* (1.1) *that satisfies the initial condition $u=1$, $u'=h$ at $x=0$ and the jump condition* (1.3). *Let $\tilde{u}$ be defined similarly with $a, b, d, h, H,$ and q replaced by $\tilde{a}, \tilde{b}, \tilde{d}, \tilde{h}, \tilde{H}$ and $\tilde{q}$. Set $u_-=u(d-)$ and $\tilde{u}_-=\tilde{u}(\tilde{d}-)$. If $\lambda_j=\tilde{\lambda}_j$ for $j\geqq 0$, $H=\tilde{H}$ and $q=\tilde{q}$ almost everywhere in $(\tfrac{1}{2}\pi,\pi)$, then $a=\tilde{a}$, $d=\tilde{d}$ and*

$$b-\tilde{b}=-\frac{a^2-a^{-2}}{2a}\int_d^{\pi/2}(q-\tilde{q})\,dt, \tag{4.1}$$

$$h-\tilde{h}=-\frac{1}{2}\int_0^d(q-\tilde{q})\,dt-\frac{1}{2a^2}\int_d^{\pi/2}(q-\tilde{q})\,dt, \tag{4.2}$$

$$\int_0^d(q-\tilde{q})(u\tilde{u}-\tfrac{1}{2})\,dt+\int_d^{\pi/2}(q-\tilde{q})\left(u\tilde{u}-\frac{1}{2a^2}-\tfrac{1}{2}(a^2-a^{-2})u_-\tilde{u}_-\right)dt=0, \tag{4.3}$$

where the last equation holds for all values of λ.

Remark. If we can show that the two potentials are equal in the first half of the interval, then it follows from equations (4.1) and (4.2) that $b=\tilde{b}$ and $h=\tilde{h}$ and the proof of Theorem 1 is complete. This step will be taken in Section 6. If $a=1$ and $b=\tilde{b}=0$, then equation (4.3) reduces to the integral equation considered by Hochstadt and Lieberman [13] and Hald [10]. In this case, equation (4.1) is trivially satisfied and equation (4.2) follows from the asymptotic expansion of the eigenvalues. Although the eigenvalues may not have an asymptotic expansion if $|a-1|+|b|>0$, we can derive equations (4.1) and (4.2) from a detailed study of the Wronskian; for a special case see equation (3.3). Here we present another approach.

Proof: It follows from the differential equations for u and $\tilde{u}$ that

$$(u\tilde{u}'-\tilde{u}u')'+(q-\tilde{q})u\tilde{u}=0.$$

Since the two eigenvalue problems have the same eigenvalues we conclude from Lemma 6 that $a=\tilde{a}$ and $d=\tilde{d}$. Integrating both sides of the above equation from 0 to d and from d to π, using the boundary conditions and the jump conditions we obtain

$$h-\tilde{h}+a(b-\tilde{b})u_-\tilde{u}_-+\int_0^{\pi/2}(q-\tilde{q})u\tilde{u}\,dt=0 \tag{4.4}$$

if $\lambda=\lambda_j$. Here we have used the fact that $H=\tilde{H}$ and that the two potentials agree in the last half of the interval. Denote the left-hand side of equation (4.4) by $\Phi(\lambda)$. Thus $\Phi=0$ if $\lambda=\lambda_j$. We shall show that $\Phi\equiv 0$. Let $k=\sqrt{\lambda}=\sigma+i\tau$. It follows from Lemma 2 and Lemma 3 that Φ is an entire function of λ of order $\frac{1}{2}$ and that

$$|\Phi(\lambda)|\leqq 18\bar{c}m\,e^{|\tau|\pi}$$

if $|k|\geqq 3\bar{c}$, where $\bar{c}=\max(c,\tilde{c})$ and $m=\max(a,a^{-1})$. Our next argument is in essence due to Hochstadt and Lieberman [13]. We introduce the function $\Psi=\Phi/\omega$, where ω is the Wronskian of equation (1.1)–(1.3). Since Φ and ω vanish at $\lambda=\lambda_j$ and all the eigenvalues are simple, we see that Ψ is an entire function in λ. We shall now estimate Ψ at the points λ on the contour Γ_n, where $n\geqq m+26\bar{c}(1+m^2)$. By combining our estimates for Φ with Lemma 4 we find that

$$|\Psi|\leqq\frac{108\bar{c}m^2}{\sqrt{|\lambda|}}$$

for all points along the contour. Since $\sqrt{|\lambda|}\geqq n-\frac{1}{2}$, we conclude that $\Psi=0$ by letting n tend to infinity and using the maximum principle. This implies that $\Phi\equiv 0$ and equation (4.4) is therefore valid for all values of λ or k.

We shall derive equations (4.1), (4.2) and (4.3) from equation (4.4) and our arguments will be similar to the arguments in Section 3. First we express equation

(4.4) as

$$h-\tilde{h}+a(b-\tilde{b})\cos^2 kd+\int_0^{\pi/2}(q-\tilde{q})\varphi^2\,dt+E_4=0, \tag{4.5}$$

where $\varphi=\varphi(t,k^2)$ is defined in Lemma 5 and

$$E_4(k^2)=a(b-\tilde{b})[(u_--\cos kd)\tilde{u}_-+\cos kd(\tilde{u}_--\cos kd)]$$
$$+\int_0^{\pi/2}(q-\tilde{q})[(u-\varphi)\tilde{u}+\varphi(\tilde{u}-\varphi)]\,dt.$$

Let $\lambda=k^2$ be real and positive. Since $2a\leqq(a+a^{-1})^2$ and $|\varphi|\leqq a+a^{-1}$, it follows from Lemma 3 that

$$|E_4|\leqq\frac{49\bar{c}^2(a+a^{-1})^2}{k}$$

if $k\geqq 3\bar{c}$. Note that E_4 vanishes identically if $\bar{c}=0$. We consider next the third term in equation (4.5). From the definition of φ follows that it is equal to

$$\int_0^d(q-\tilde{q})\cos^2 kt\,dt+\int_d^{\pi/2}(q-\tilde{q})(\tfrac{1}{2}(a+a^{-1}))^2[\cos kt+\alpha\cos k(t-2d)]^2\,dt,$$

where $\alpha=(a-a^{-1})/(a+a^{-1})$. By using some trigonometric identities and several changes of variables we see that the second integral is equal to

$$(\tfrac{1}{2}(a+a^{-1}))^2\left[(\tfrac{1}{2}+\tfrac{1}{2}a^2)\int_d^{\pi/2}(q-\tilde{q})\,dt+\alpha\cos 2kd\int_d^{\pi/2}(q-\tilde{q})\,dt\right]+E_5,$$

where

$$E_5(k^2)=(\tfrac{1}{2}(a+\tilde{a}^{-1}))^2\left[\frac{1}{2}\int_d^{\pi/2}(q-\tilde{q})\cos 2kt\,dt+\tfrac{1}{2}\alpha^2\int_0^d(q-\tilde{q})(2d-t)\cos 2kt\,dt\right.$$
$$+\tfrac{1}{2}\alpha^2\int_0^{\pi/2-d}(q-\tilde{q})(t+2d)\cos 2kt\,dt$$
$$\left.+\alpha\int_0^{\pi/2-d}(q-\tilde{q})(t+d)\cos 2kt\,dt\right]$$
$$=\int_0^{\pi/2}V_5(t)\cos 2kt\,dt.$$

Here V_5 is an integrable function and its definition depends upon whether $d\leqq\frac{1}{4}\pi$ or $d>\frac{1}{4}\pi$. By combining our results we find that equation (4.5) can be rewritten as

$$A+B\cos 2kd+\int_0^{\pi/2}V(t)\cos 2kt\,dt+E_4=0, \tag{4.6}$$

where V is equal to $\frac{1}{2}(q-\tilde{q})+V_5$ if $t \leqq d$ and equal to V_5 otherwise. In addition,

$$A=h-\tilde{h}+\tfrac{1}{2}a(b-\tilde{b})+\frac{1}{2}\int_0^d (q-\tilde{q})\,dt+\tfrac{1}{4}(a^2+a^{-2})\int_d^{\pi/2}(q-\tilde{q})\,dt,$$

$$B=\tfrac{1}{2}a(b-\tilde{b})+\tfrac{1}{4}(a^2-a^{-2})\int_d^{\pi/2}(q-\tilde{q})\,dt.$$

We shall show that both A and B are zero. We can therefore solve the last two equations for $b-\tilde{b}$ and $h-\tilde{h}$ and arrive at the results given in equations (4.1) and (4.2). To prove that $B=0$ we multiply both sides of equation (4.6) by $T^{-1}\cos 2kd$ and integrate with respect to k from $3\tilde{c}$ to T. Fubini's theorem and our estimates for E_4 yield

$$AO\left(\frac{1}{T}\right)+B\left[\frac{1}{2}+O\left(\frac{1}{T}\right)\right]+\int_0^{\pi/2}V(t)\frac{1}{T}\int_{3\tilde{c}}^{T}\cos 2kd\cos 2kt\,dk\,dt+O\left(\frac{\log T}{T}\right)=0.$$

Since the integral $T^{-1}\int\cos 2kd\cos 2kt\,dk$ is bounded by 1 and tends to zero if $t\neq d$ and T tends to infinity we conclude from Lebesgue's theorem of dominated convergence that the third term in the last equation tends to zero as T tends to infinity. Thus $B=0$. We shall now show that $A=0$. By using the Riemann–Lebesgue lemma we see that the third term in equation (4.6) tends to zero as k tends to infinity. Since $E_4=O(k^{-1})$, it follows from equation (4.6) that A must be equal to zero. Finally we derive equation (4.3) by combining equation (4.1) and equation (4.2) with equation (4.3) and reorder the terms. This completes the proof.

5. Integral Representation of Products

In this section we shall study the products of the eigenfunctions of two Sturm–Liouville problems; see equation (4.3). The result will be used to derive an integral equation for the function $q-\tilde{q}$, which does not involve the parameter λ.

LEMMA 8. *Let u, $\tilde{u}$, u_- and $\tilde{u}_-$ be defined as in Lemma 7 and assume that $d=\tilde{d}$ and $a=\tilde{a}$. Let $k=\sqrt{\lambda}$. Then there exists a bounded function $\bar{K}(x,t)$ such that*

$$u\tilde{u}-\tfrac{1}{2}=\tfrac{1}{2}\cos 2kx+\frac{1}{2}\int_0^x \bar{K}(x,t)\cos 2kt\,dt, \qquad 0\leqq x<d,$$

$$u\tilde{u}-\tfrac{1}{2}a^{-2}-\tfrac{1}{2}(a^2-a^{-2})u_-\tilde{u}_-=\frac{1}{2}[A\cos 2kx+B\cos 2k(x-d)+C\cos 2k(x-2d)]$$
$$+\frac{1}{2}\int_0^x \bar{K}(x,t)\cos 2kt\,dt, \qquad d<x\leqq\pi,$$

for all k, where $4A=(a+a^{-1})^2$, $2B=a^2-a^{-2}$ and $4C=(a-a^{-1})^2$.

Remark. The representation for $0 \leqq x < d$ is well known (see Levitan [19], page 75) and the expression for $d < x \leqq \pi$ is the natural generalization. Note that B and C can also be written as $2A\alpha$ and $A\alpha^2$ and the second expression thus reduces to the first if $a = 1$ (see also Sabatier [25]). Our derivation is similar to Hochstadt and Lieberman's [13]. We use the integral representation of the eigenfunctions (see Lemma 5), while Levitan's proof is based on the theory of translation operators.

Proof: Let $0 \leqq x < d$. It follows from Lemma 5 that

$$
\begin{aligned}
u\tilde{u} - \frac{1}{2} = \frac{1}{2}\cos 2kx + \int_0^x \{K + \tilde{K}\}(x, t) \cos kx \cos kt \, dt \\
+ \int_0^x \int_0^x K(x, s)\tilde{K}(x, t) \cos ks \cos kt \, ds \, dt.
\end{aligned} \tag{5.1}
$$

We denote the last two terms in this equation by E_6 and E_7. By using the trigonometric addition formulas and a change of variables we find that

$$
\begin{aligned}
E_6 &= \int_0^x \frac{1}{2}\{K + \tilde{K}\}(x, x-y) \cos ky \, dy + \int_x^{2x} \frac{1}{2}\{K + \tilde{K}\}(x, y-x) \cos ky \, dy \\
&= \int_0^{2x} \cos ky \frac{1}{2}\{K + \tilde{K}\}(x, |x-y|) \, dy.
\end{aligned}
$$

Let $V(s, t) = \frac{1}{2}K(x, s)\tilde{K}(x, t)$. We can then rewrite the last term in equation (5.1) as

$$
E_7(x) = \int_0^x ds \int_0^x V(s, t) \cos k(s+t) \, dt + \int_0^x ds \int_0^x V(s, t) \cos k(s-t) \, dt. \tag{5.2}
$$

We shall analyze the first term in this equation. Let $y = s + t$ and integrate with respect to t. Interchanging the order of integration we see that the first term in equation (5.2) is equal to

$$
\int_0^x \cos ky \, dy \int_0^y V(s, y-s) \, ds + \int_x^{2x} \cos ky \, dy \int_{y-x}^x V(s, y-s) \, ds.
$$

Since $\int_0^y V(s, y-s) \, ds = \int_0^y V(y-s, s) \, ds$ and a similar result holds for $\int_{y-x}^x V \, ds$, we can express the first term in equation (5.2) as

$$
\begin{aligned}
&\int_0^x \cos ky \, dy \int_0^y \frac{1}{2}[V(s, y-s) + V(y-s, s)] \, ds \\
&+ \int_x^{2x} \cos ky \, dy \int_{y-x}^x \frac{1}{2}[V(s, y-s) + V(y-s, s)] \, ds \\
&= \int_0^{2x} \cos ky \, dy \int_0^y \frac{1}{2}[V(s, y-s) + V(y-s, s)] \, ds,
\end{aligned}
$$

where we have used the fact that $V(s,t)$ vanishes if s or t are less than zero or if s or t are greater than x. The second term in equation (5.2) can be analyzed in a similar manner and we conclude that

$$E_7(x)=\int_0^{2x}\cos ky\,dy\int_0^y \frac{1}{4}[K(x,s)\tilde{K}(x,y-s)+K(x,y-s)\tilde{K}(x,s)]\,ds$$
$$+\int_0^x \cos ky\,dy\int_0^{x-y}\frac{1}{2}[K(x,s)\tilde{K}(x,y+s)+K(x,y+s)\tilde{K}(x,s)]\,ds.$$

By inserting the expressions for E_6 and E_7 in equation (5.1) and changing the variable y to $t=\frac{1}{2}y$ we arrive at the integral representation of $u\tilde{u}-\frac{1}{2}$ for $0\leqq x<d$. The details of the formula for $\tilde{K}$ are not important.

We turn now to the second integral representation in Lemma 8. Let $d<x\leqq\pi$. By using Lemma 5 we see that

$$\begin{aligned}(5.3)\quad & u\tilde{u}-\tfrac{1}{2}a^2-\tfrac{1}{2}(a^2-a^{-2})u_-\tilde{u}_-\\ &=\varphi^2-\tfrac{1}{2}a^{-2}-\tfrac{1}{2}(a^2-a^{-2})\cos^2 kd+\varphi\int_0^x\{K+\tilde{K}\}(x,t)\cos kt\,dt\\ &\quad+\int_0^x\int_0^x K(x,s)\tilde{K}(x,t)\cos ks\cos kt\,ds\,dt\\ &\quad-\tfrac{1}{2}(a^2-a^{-2})\Bigg[\int_0^d\{K+\tilde{K}\}(d-,t)\cos kd\cos kt\,dt\\ &\qquad+\int_0^d\int_0^d K(d-,s)\tilde{K}(d-,t)\cos ks\cos kt\,ds\,dt\Bigg].\end{aligned}$$

It follows from the double angle formula and the definition of φ that

$$\begin{aligned}&\varphi^2-\tfrac{1}{2}a^2-\tfrac{1}{2}(a^2-a^{-2})\cos^2 kd\\ &\quad=\tfrac{1}{2}[(\tfrac{1}{2}(a+a^{-1}))^2\cos 2kx+\tfrac{1}{2}(a^2-a^{-2})\cos 2k(x-d)\\ &\qquad+(\tfrac{1}{2}(a-a^{-1}))^2\cos 2k(x-2d)].\end{aligned}$$

Thus we have found the leading terms in the integral representation of $u\tilde{u}-\cdots$ for $x>d$. By using the definition of φ we get

$$\begin{aligned}(5.4)\quad &\varphi\int_0^x\{K+\tilde{K}\}(x,t)\cos kt\,dt=\tfrac{1}{2}(a+a^{-1})\int_0^x\{K+\tilde{K}\}(x,t)\cos kx\cos kt\,dt\\ &\qquad+\tfrac{1}{2}(a-a^{-1})\int_0^x\{K+\tilde{K}\}(x,t)\cos k(x-2d)\cos kt\,dt.\end{aligned}$$

The first term on the right-hand side of this equation is simply $(a+a^{-1})E_6(x)$, and we note that the formula for E_6 is valid for all $x \geqq 0$. Let E_8 be the last integral in equation (5.4) and set $K+\tilde{K}=2V$. Using the trigonometric addition formulas and a change of variables we obtain

$$E_8(x)=\int_{x-2d}^{2x-2d} V(2d-x+y)\cos ky\,dy+\int_{2d-x}^{2d} V(x-2d+y)\cos ky\,dy.$$

We consider the two cases $x-2d<0$ and $x-2d \geqq 0$ separately. Assume that $x<2d$. Splitting the first integral into two parts, $\{y<0\}$ and $\{y>0\}$, and combining the results we see that

$$E_8(x)=\int_0^{2x-2d} V(2d-x+y)\cos ky\,dy+\int_0^{2d} V(|x-2d+y|)\cos ky\,dy.$$

We observe now that $V(t)\equiv 0$ if t is less than zero or if t is greater than x. The region of integration can therefore be extended to the interval $(0,2x)$ and we obtain

$$E_8(x)=\int_0^{2x}\cos ky\,\tfrac{1}{2}[\{K+\tilde{K}\}(x,|2d-x+y|)+\{K+\tilde{K}\}(x,|2d-x-y|)]\,dy.$$

It turns out that this formula also covers the case $x \geqq d$. By inserting this expression into equation (5.4) and combining our results we conclude from equation (5.3) that

$$\begin{aligned} u\tilde{u}-\tfrac{1}{2}a^{-2}-\tfrac{1}{2}(a^2-a^{-2})u_-\tilde{u}_- &= \tfrac{1}{2}[A\cos 2kx+B\cos 2k(x-d)+c\cos 2k(x-2d)] \\ &\quad+\tfrac{1}{2}(a+a^{-1})E_6(x)+\tfrac{1}{2}(a-a^{-1})E_8(x)+E_7(x) \\ &\quad-\tfrac{1}{2}(a^2-a^{-2})[E_6(d-)+E_7(d-)]. \end{aligned}$$

Since the integration on $E_6(d-)$ and $E_7(d-)$ can be extended from the interval $(0,2d)$ to $(0,2x)$ we arrive at the integral representation for $u\tilde{u}-\cdots$ for $d<x\leqq \pi$ by changing the variable y in E_6, E_7 and E_8 to $t=\tfrac{1}{2}y$. This completes the proof of Lemma 8.

6. Completion of the Proof

In this section we shall complete the proof of Theorem 1. The basic idea is to translate the homogeneous integral equation (4.3) into four inhomogeneous integral equations that are independent of λ and then show, step by step, that the inhomogeneous terms must vanish.

Let $Q = q - \tilde{q}$. We shall show that $Q = 0$ almost everywhere (a.e.). By combining equation (4.3) with Lemma 8 we get

$$
\begin{aligned}
0 = &\int_0^d Q(x)\left[\cos 2kx + \int_0^x \bar{K}(x,t)\cos 2kt\,dt\right]dx \\
(6.1)\qquad &+ \int_d^{\pi/2} Q(x)\left[A\cos 2kx + B\cos 2k(x-d) + C\cos 2k(x-2d)\right. \\
&\qquad\qquad \left. + \int_0^x \bar{K}(x,t)\cos 2kt\,dt\right]dx
\end{aligned}
$$

for all k. We consider first the two terms with $\bar{K}$. Since Q is integrable and $\bar{K}$ bounded, it follows from Fubini's theorem that

$$
\begin{aligned}
&\int_0^d Q(x)\int_0^x \bar{K}(x,t)\cos 2kt\,dt\,dx + \int_d^{\pi/2} Q(x)\int_0^x \bar{K}(x,t)\cos 2kt\,dt\,dx \\
&\quad = \int_0^{\pi/2} \cos 2kt \int_t^{\pi/2} \bar{K}(x,t)Q(x)\,dx\,dt.
\end{aligned}
$$

Let E_9 denote the term in equation (6.1) that involves the constant C. Replacing $x-2d$ by t we see that

$$E_9(k^2) = \int_{-d}^{\pi/2-2d} Q(2d+t)\cdot C\cdot \cos 2kt\,dt.$$

We treat the two cases $\frac{1}{2}\pi - 2d > 0$ and $\frac{1}{2}\pi - 2d \leqq 0$ separately. Assume that $d < \frac{1}{4}\pi$. By splitting the integral at $t = 0$ we find after a change of variables that

$$(6.2)\qquad E_9(k^2) = \int_0^d \cos 2kt\cdot C\cdot Q(2d-t)\,dt + \int_0^{\pi/2-2d} \cos 2kt\cdot C\cdot Q(2d+t)\,dt.$$

On the other hand, if $d \geqq \frac{1}{4}\pi$ we replace t by $-t$ and get

$$(6.3)\qquad E_9(k^2) = \int_{2d-\pi/2}^{d} \cos 2kt\cdot C\cdot Q(2d-t)\,dt.$$

Combining our results and using a change of variables we can rewrite equation (6.1) as

$$
\begin{aligned}
&\int_0^d \cos 2kt\,Q(t)\,dt + \int_d^{\pi/2} \cos 2kt\cdot A\cdot Q(t)\,dt \\
(6.4)\qquad &\quad + \int_0^{\pi/2-d} \cos 2kt\cdot B\cdot Q(d+t)\,dt \\
&\quad + E_9(k^2) + \int_0^{\pi/2}\cos 2kt\int_t^{\pi/2}\bar{K}(x,t)Q(x)\,dx\,dt = 0,
\end{aligned}
$$

where E_9 is given by equation (6.2) if $d < \frac{1}{4}\pi$ and otherwise by equation (6.3).

Equation (6.4) involves integration over two, three or four intervals on the t-axis depending upon the relative position of $d, \frac{1}{2}\pi - d$ and $|\frac{1}{2}\pi - 2d|$. We have four different possibilities:

$$\text{I: } 0 < d < \tfrac{1}{6}\pi; \quad 0 < d < \tfrac{1}{2}\pi - 2d < \tfrac{1}{2}\pi - d < \tfrac{1}{2}\pi,$$
$$\text{II: } \tfrac{1}{6}\pi \leqq d < \tfrac{1}{4}\pi; \quad 0 < \tfrac{1}{2}\pi - 2d \leqq d < \tfrac{1}{2}\pi - d < \tfrac{1}{2}\pi,$$
$$\text{III: } \tfrac{1}{4}\pi \leqq d < \tfrac{1}{3}\pi; \quad 0 \leqq 2d - \tfrac{1}{2}\pi < \tfrac{1}{2}\pi - d \leqq d < \tfrac{1}{2}\pi,$$
$$\text{IV: } \tfrac{1}{3}\pi \leqq d < \tfrac{1}{2}\pi; \quad 0 < \tfrac{1}{2}\pi - d \leqq 2d - \tfrac{1}{2}\pi < d < \tfrac{1}{2}\pi.$$

Take case I. From equations (6.4) and (6.2) it follows that

$$\begin{aligned}
&\int_0^d \cos 2kt\Big[Q(t) + B\cdot Q(d+t) + C\cdot Q(2d-t) + C\cdot Q(2d+t) \\
&\qquad\qquad + \int_t^{\pi/2} \bar{K}(x,t)Q(x)\,dx\Big]\,dt \\
&\quad + \int_d^{\pi/2-2d} \cos 2kt\Big[A\cdot Q(t) + B\cdot Q(d+t) + C\cdot Q(2d+t) \\
&\qquad\qquad + \int_t^{\pi/2} \bar{K}(x,t)Q(x)\,dx\Big]\,dt \\
&\quad + \int_{\pi/2-2d}^{\pi/2-d} \cos 2kt\Big[A\cdot Q(t) + B\cdot Q(d+t) + \int_t^{\pi/2} \bar{K}(x,t)Q(x)\,dx\Big]\,dt \\
&\quad + \int_{\pi/2-d}^{\pi/2} \cos 2kt\Big[A\cdot Q(t) + \int_t^{\pi/2} \bar{K}(x,t)Q(x)\,dx\Big]\,dt = 0
\end{aligned} \tag{6.5}$$

for all k. Since the trigonometric system $\{\cos 2nx\}$ is complete in $L^1(0, \frac{1}{2}\pi)$, we conclude that all four terms $[\cdots]$ must vanish a.e. We shall show that $Q = 0$ a.e. in each of the four intervals. Let $\frac{1}{2}\pi - d < t < \frac{1}{2}\pi$. Then

$$A\cdot Q(t) + \int_t^{\pi/2} \bar{K}(x,t)Q(x)\,dx = 0$$

a.e. But this is a homogeneous Volterra integral equation and its solution is identically zero a.e. We assume next that $\frac{1}{2}\pi - 2d < t < \frac{1}{2}\pi - d$. Then $d + t > \frac{1}{2}\pi - d$ and this implies that $Q(d+t) = 0$ a.e. Since $Q(x) = 0$ for $x > \frac{1}{2}\pi - d$, it follows from equation (6.5) that

$$A\cdot Q(t) + \int_t^{\pi/2-d} \bar{K}(x,t)Q(x)\,dx = 0$$

a.e. This is a homogeneous Volterra equation and Q is therefore zero a.e. in the interval $(\frac{1}{2}\pi - 2d, \frac{1}{2}\pi - d)$. We consider next the interval $d < t < \frac{1}{2}\pi - 2d$ which we break up into several parts. Let us determine the integer m such that $\frac{1}{2}\pi-(m+1)d \leqq d < \frac{1}{2}\pi - md$. Then

$$(d, \tfrac{1}{2}\pi - 2d] = (d, \tfrac{1}{2}\pi - md] \bigcup_{j=2}^{m-1} (\tfrac{1}{2}\pi - (j+1)d, \tfrac{1}{2}\pi - jd].$$

We must study the subintervals one by one. Assume that $m > 2$ and let $\frac{1}{2}\pi - 3d < t < \frac{1}{2}\pi - 2d$. Then $2d+t > d+t > \frac{1}{2}\pi - 2d$ and thus $Q(2d+t)$ and $Q(d+t)$ must vanish a.e. Hence it follows from equation (6.5) that

$$A \cdot Q(t) + \int_t^{\pi/2-2d} \bar{K}(x,t) Q(x)\, dx = 0$$

a.e. Q is therefore zero a.e. for $t > \frac{1}{2}\pi - 3d$. By repeating this argument for each subinterval and using the fact that $d+t > \frac{1}{2}\pi - md$ in the last subinterval, we conclude that $Q=0$ a.e. for $t > d$. Finally, we let $0 < t < d$. Since $d+t$, $2d-t$ and $2d+t$ are greater than d, it follows from equation (6.5) that

$$Q(t) + \int_t^d \bar{K}(x,t) Q(x)\, dx = 0$$

a.e. This is again a Volterra equation and its solution is identically zero. We have therefore proved that $Q=0$ a.e. in the interval $(0, \frac{1}{2}\pi)$ or equivalently that $q = \tilde{q}$. Using equations (4.1) and (4.2), we conclude that $b = \tilde{b}$ and $h = \tilde{h}$ and this completes the proof of Theorem 1 in case I.

The proofs of the remaining cases are similar and only the essential points will be presented. First we note that the solution of the integral equation for the top interval must vanish a.e. Then we observe that the inhomogeneous terms in the integral equation for the next interval are evaluated at points where Q vanishes. Thus $Q=0$ a.e. in the next interval and we can continue the argument.

We consider now case II. It follows from equations (6.4) and (6.2) that

$$\int_0^{\pi/2-2d} \cos 2kt \left[Q(t) + B \cdot Q(d+t) + C \cdot Q(2d-t) + C \cdot Q(2d+t) + \int_t^{\pi/2} \bar{K}(x,t) Q(x)\, dx \right] dt$$
$$+ \int_{\pi/2-2d}^{d} \cos 2kt \left[Q(t) + B \cdot Q(d+t) + C \cdot Q(2d-t) + \int_t^{\pi/2} \bar{K}(x,t) Q(x)\, dx \right] dt$$

$$+\int_d^{\pi/2-d} \cos 2kt\left[A\cdot Q(t)+B\cdot Q(d+t)+\int_t^{\pi/2} \bar{K}(x,t)Q(x)\,dx\right]dt$$

$$+\int_{\pi/2-d}^{\pi/2} \cos 2kt\left[A\cdot Q(t)+\int_t^{\pi/2} \bar{K}(x,t)Q(x)\,dx\right]dt=0$$

for all k. Since $\{\cos 2nx\}$ is complete in $(0,\frac{1}{2}\pi)$, all four terms $[\cdots]$ must vanish. Thus $Q=0$ a.e. in $(\frac{1}{2}\pi-d,\frac{1}{2}\pi)$. Assume next that $d<t<\frac{1}{2}\pi-d$. Then $d+t>2d\geqq\frac{1}{2}\pi-d$ and $Q(d+t)=0$ a.e. Thus $Q=0$ a.e. in $(d,\frac{1}{2}\pi-d)$. Let now $\frac{1}{2}\pi-2d<t<d$. Then $d+t>\frac{1}{2}\pi-d$ and $2d-t>d$ and this implies that $Q(d+t)=Q(2d-t)=0$ a.e. Hence $Q=0$ a.e. in $(\frac{1}{2}\pi-2d,d)$. Suppose finally that $0<t<\frac{1}{2}\pi-2d$. Then $d+t>d$, $2d-t>6d-\pi+\frac{1}{2}\pi-2d$ and $2d+t>2d\geqq\frac{1}{2}\pi-d$ and this implies that $Q(d+t)=Q(2d-t)=Q(2d+t)=0$ a.e. Thus $Q=0$ a.e. in $(0,\frac{1}{2}\pi)$ and this completes the proof of Theorem 1 in case II.

We consider now case III. From equations (6.4) and (6.3) it follows that

$$\int_0^{2d-\pi/2} \cos 2kt\left[Q(t)+B\cdot Q(d+t)+\int_t^{\pi/2} \bar{K}(x,t)Q(x)\,dx\right]dt$$

$$+\int_{2d-\pi/2}^{\pi/2-d} \cos 2kt\left[Q(t)+B\cdot Q(d+t)+C\cdot Q(2d-t)\right.$$
$$\left.+\int_t^{\pi/2} \bar{K}(x,t)Q(x)\,dx\right]dt$$

$$+\int_{\pi/2-d}^{d} \cos 2kt\left[Q(t)+C\cdot Q(2d-t)+\int_t^{\pi/2} \bar{K}(x,t)Q(x)\,dx\right]dt$$

$$+\int_d^{\pi/2} \cos 2kt\left[A\cdot Q(t)+\int_t^{\pi/2} \bar{K}(x,t)Q(x)\,dx\right]dt=0$$

for all k. We observe first that $Q=0$ a.e. in $(d,\frac{1}{2}\pi)$. Assume next that $\frac{1}{2}\pi-d<t<d$. Then $2d-t>d$ and $Q(2d-t)=0$ a.e. Thus $Q=0$ a.e. in $(\frac{1}{2}\pi-d,d)$. Let now $2d-\frac{1}{2}\pi<t<\frac{1}{2}\pi-d$. Then $d+t$ and $2d-t$ are greater than $4d-\pi+\frac{1}{2}\pi-d$ and this implies that $Q(d+t)=Q(2d-t)=0$ a.e. Hence $Q=0$ a.e. in $(2d-\frac{1}{2}\pi,\frac{1}{2}\pi-d)$. Suppose finally that $0<t<2d-\frac{1}{2}\pi$. Then $d+t>d$ and $Q(d+t)=0$ a.e. Consequently, $Q=0$ a.e. in $(0,\frac{1}{2}\pi)$ and this completes the proof of Theorem 1 in case III.

Finally we consider case IV. It follows from equations (6.4) and (6.3) that

$$\int_0^{\pi/2-d} \cos 2kt\left[Q(t)+B\cdot Q(d+t)+\int_t^{\pi/2} \bar{K}(x,t)Q(x)\,dx\right]dt$$

$$+\int_{\pi/2-d}^{2d-\pi/2} \cos 2kt\left[Q(t)+\int_t^{\pi/2} \bar{K}(x,t)Q(x)\,dx\right]dt$$

$$+\int_{2d-\pi/2}^{d}\cos 2kt\left[Q(t)+C\cdot Q(2d-t)+\int_t^{\pi/2}\bar{K}(x,t)Q(x)\,dx\right]dt$$
$$+\int_d^{\pi/2}\left[A\cdot Q(t)+\int_t^{\pi/2}\bar{K}(x,t)Q(x)\,dx\right]dt=0$$

for all k. This case is easy. We observe first that $Q=0$ a.e. in $(d,\frac{1}{2}\pi)$. Assume next that $2d-\frac{1}{2}\pi<t<d$. Then $2d-t>d$ and $Q(2d-t)=0$ a.e. Thus $Q=0$ a.e. in $(2d-\frac{1}{2}\pi, d)$ and the same conclusion holds for the interval $(\frac{1}{2}\pi-d, 2d-\frac{1}{2}\pi)$. Suppose finally that $0<t<\frac{1}{2}\pi-d$. Then $d+t>d$ and this implies that $Q(d+t)=0$ a.e. Consequently, $Q=0$ a.e. in the whole interval $(0,\frac{1}{2}\pi)$ and this completes the proof of Theorem 1.

7. Inverse Problems for the Earth

In this section we shall study the inverse eigenvalue problem for the Earth. We shall show that if the density is given in the lower mantle, then the density is uniquely determined in the upper mantle and in the crust by one torsional spectrum provided the velocity of the S-waves is given throughout the mantle and in the crust. In particular, the density jump at the crust is uniquely determined. The velocity of the S-waves in the upper mantle and in the crust can be replaced by an additional torsional spectrum.

The inverse problem for the Earth amounts to determining the material properties inside the Earth. To analyze this problem we must agree on a simple model for the Earth. Here we assume that the Earth is spherically symmetric and consists of a central core, the mantle and the crust. The core itself consists of an inner core which is solid, and the outer core which is fluid. The mantle and the crust are both solid and their interface is called the Mohorovičić discontinuity. It lies about 10 km below the sea level in the oceanic areas and under the continents at a depth of approximately 40 km. We also assume that the Earth consists of a perfect elastic isotropic material. Now, solid elastic material can transmit two kinds of waves, namely compression waves and shear waves denoted by P and S. Only compression waves can pass through a fluid. No shear waves have been observed to pass through the outer core and this indicates that it is molten.

To describe the elastic material we can use the density, the incompressibility and the rigidity. We can also use the density and the velocities of the P- and S-waves. This provides an equivalent description. Under certain assumptions the velocities α and β of the P- and S-waves can be computed from travel time observations, i.e., the time it takes a disturbance to travel from its source to the seismographic station. This is an inverse problem and amounts to solving an integral equation. By using this technique Jeffreys and, independently, Gutenberg and Richter derived, around 1940, average velocity distributions through the Earth.

To determine the density we need either additional physical assumptions or additional data. In his early work on earth model, Bullen delineated regions in the Earth in which he assumed that the material was chemically homogeneous and devoid of phase changes and that the temperature gradient was adiabatic. Under these assumptions the increase in density is solely due to the increase in pressure of the above lying material. This is the content of the Adams–Williamson's equation. Using this equation, Bullen was able to determine the density ρ throughout the Earth and arrived at his model A.

In the last 20 years a new kind of data has been observed, namely the periods of the free oscillations of the Earth, or equivalently the eigenfrequencies of the Earth. There are two types of oscillations, spheroidal oscillations and torsional oscillations. The spheroidal oscillations involve the whole Earth whereas the torsional oscillations are restricted to the mantle and the crust. The periods can be determined by a Fourier analysis of the seismograms from a single seismographic station. By comparing the observed periods with the periods computed for various earth models one can reject old models and derive new ones. This approach raises the following question: what data will determine the density inside the Earth uniquely? We shall present a partial answer to this question.

THEOREM 2. *Let $R_c < R_m < R$ be given and assume that $\rho(r)$ and $\beta(r)$ are positive and twice continuously differentiable for $R_c \leqq r \leqq R_m$ and $R_m \leqq r \leqq R$. Consider the eigenvalue problem*

$$-(r^4\rho\beta^2\dot{u})^{\cdot}+(l+2)(l-1)r^2\rho\beta^2 u=\omega^2 r^4\rho u, \qquad \dot{u}(R_c)=\dot{u}(R)=0, \tag{7.1}$$

with the continuity conditions

$$u_+=u_-, \qquad r^4\rho_+\beta_+^2\dot{u}_+=r^4\rho_-\beta_-^2\dot{u}_- \tag{7.2}$$

at $r=R_m$. Here $\dot{u}=du/dr$ and $u_+=\lim_{r\to R_m+} u(r)$. Let $\beta(r)$ be given and determine the constant r_0 in the interval $R_c<r<R$ such that

$$\int_{R_c}^{r_0}\beta^{-1}\,dr=\int_{r_0}^{R}\beta^{-1}\,dr.$$

Assume that ρ is known for $R_c\leqq r\leqq r_0$ and that $r_0<R_m$. Then one spectrum $\{\omega_n^2\}$ determines $\rho(r)$ uniquely.

Remark. Equation (7.1) is obtained by substituting $U=ru$ in the equation for the torsional modes for a spherically symmetric, non-rotating Earth (see Alterman, Jarosch and Pekeris [1]). The equations in (7.2) come from the continuity of the displacement and the stress at the interface. The constant l is called the angular order. It is a positive integer and has its origin in the separation of variables in the equation of motions. In addition, R_c is the radius of the core,

R_m is the distance from the center of the Earth to the Mohorovičić discontinuity and R is the radius of the Earth. In practice $R_c \approx 3473$ km, $R_m \approx 6338$ km and $R = 6371$ km. The value of r_0 corresponds to a depth of approximately 1300 km. The domain in which we assume that the density is known is therefore included in the region in which Bullen used Adams–Williamson's equation to determine the density. The proof of Theorem 2 is based on Theorem 1. In essence it is a paraphrase of the proof for the inverse problem without discontinuities (see Hald [10]).

Proof: Using the Liouville transformation we can transform equation (7.1) into Liouville normal form with an interior discontinuity. Let $r = R - z$. We introduce the new independent variable

$$x = \psi(z) = \frac{1}{K}\int_0^z \frac{1}{\beta(R-\xi)}\,d\xi, \tag{7.3}$$

$$K = \frac{1}{\pi}\int_{R_c}^{R} \frac{1}{\beta(r)}\,dr. \tag{7.4}$$

Since β is positive, ψ has an inverse function which we denote by $z = \varphi(x)$. Because β is given, we see that the constant K and the functions ψ and φ are uniquely determined by the data. Let $d = \psi(R - R_m)$. If β has a jump discontinuity at $r = R_m$, then φ is continuous but not differentiable at $x = d$. Instead of u we introduce the dependent variable

$$y(x) = f(x)u(r),$$
$$f(x) = r^2(\rho(r)\beta(r))^{1/2},$$

where $r = R - \varphi(x)$. Note that the function y may have a discontinuity at $x = d$ even though u is continuous at $r = R_m$. Now equation (7.3) implies that $x_r = -(K\beta(r))^{-1}$. Thus it follows from the definition of y that

$$fy' - f'y = f^2 r_x u_r = -K(r^4\rho\beta^2\dot{u}), \tag{7.5}$$

where $y' = dy/dx$ and $x \neq d$. We can now complete the Liouville transformation. By differentiating both sides of this equation with respect to x, we obtain after some reordering the differential equation

$$-y'' + \left[\frac{f''(x)}{f(x)} + K^2(l+2)(l-1)\frac{\beta^2}{r^2}\right]y = \omega^2K^2y, \tag{7.6}$$

for the intervals $0 < x < d$ and $d < x < \pi$. Since $\dot{u}$ vanishes at $r = R_c$ and $r = R$, it follows from equation (7.5) that the function y satisfies the boundary conditions

$$y'(0) - hy(0) = y'(\pi) + Hy(\pi) = 0.$$

Here $h = f'(0)/f(0)$ and $H = -f'(\pi)/f(\pi)$. Using the definition of $y(x)$ and equation (7.5) we see that the continuity conditions in (7.2) for u and $\dot{u}$ are transformed into

$$\frac{y_+}{f_+}=\frac{y_-}{f_-}, \qquad f_+y'_+-f'_+y_+=f_-y'_--f'_-y_-$$

at $x=d$. We can rewrite these equations in the form $y_+=ay_-$ and $y'_+=a^{-1}y'_-+by_-$, where $a=f_+/f_-$ and $b=f'_+/f_--f'_-/f_+$. Thus the differential equation for the torsional modes of the Earth with an interior discontinuity can be transformed into the eigenvalue problem considered in Theorem 1.

Let $q(x)$ be the potential $[\cdots]$ in equation (7.6). We shall show that q is uniquely determined in $(0, \pi)$ and so are the constants a and b. We observe first that the function β^2/r^2 is evaluated at $r=R-\varphi(x)$ and thus known for $0\leqq x\leqq \pi$. The point r_0 has been chosen such that it corresponds to $x=\frac{1}{2}\pi$. Indeed, by using the definition of r_0 and equations (7.3) and (7.4) we see that

$$x-\tfrac{1}{2}\pi=\tfrac{1}{2}\pi\int_r^{r_0}\beta^{-1}(r)\,dr\Big/\int_{R_c}^{r_0}\beta^{-1}(r)\,dr.$$

This shows that the interval $R_c\leqq r\leqq r_0$ is mapped onto the interval $\frac{1}{2}\pi\leqq x\leqq\pi$ in a one-to-one manner. Moreover, our assumption $r_0<R_m<R$ implies that $0<d<\frac{1}{2}\pi$. Since $f(x)=r^2\sqrt{\rho\beta}$ with $r=R-\varphi(x)$ and $\rho(r)$ is known for $R_c\leqq r\leqq r_0$, we conclude that the function $f(x)$ and consequently also the potential $q(x)$ are uniquely determined in the interval $\frac{1}{2}\pi\leqq x\leqq\pi$. Finally the constant H in the boundary condition at $x=\pi$ is equal to $f'(\pi)/f(\pi)$ and thus uniquely determined.

We consider now another earth model and the corresponding Liouville normal form with $\rho, \beta, \omega, a, b, d, f, h, H, K$ and q replaced by $\tilde\rho, \tilde\beta, \tilde\omega, \tilde a, \tilde b, \tilde d, \tilde f, \tilde h, \tilde H, \tilde K$ and $\tilde q$. Since $\beta=\tilde\beta$, we see that $d=\tilde d$ and $K=\tilde K$ and we have just shown that $f=\tilde f$ and $q=\tilde q$ for $\frac{1}{2}\pi\leqq x\leqq\pi$ and that $H=\tilde H$.

Assume $\omega_n^2=\tilde\omega_n^2$ for $n=0,1,2,\cdots$, i.e., the two problems have the same eigenvalues. In Section 3 we observed that a problem with a discontinuity in the first half of the interval cannot have the same eigenvalues as a problem without any discontinuities. Thus there are two cases. Either $|a-1|+|b|$ and $|\tilde a-1|+|\tilde b|$ are both positive or they are both equal to zero. In the first case it follows from Theorem 1 that $a=\tilde a$, $b=\tilde b$, $h=\tilde h$ and $q=\tilde q$ for all $x\neq d$. Here we have used the fact that ρ and β are twice continuously differentiable for $r\neq R_m$. Thus both f and $\tilde f$ satisfy the linear differential equation

$$f''(x)=\left[q(x)-K^2(l+2)(l-1)\frac{\beta^2(R-\varphi(x))}{(R-\varphi(x))^2}\right]f(x) \tag{7.7}$$

for $d<x<\frac{1}{2}\pi$ and $0<x<d$. Since $f=\tilde f$ and $f'=\tilde f'$ at $x=\frac{1}{2}\pi$, we see that $f=\tilde f$ for $d<x<\frac{1}{2}\pi$. Thus $f_+=\tilde f_+$ and $f'_+=\tilde f'_+$. But $a=\tilde a$ and $b=\tilde b$ and this implies that $f_-=\tilde f_-$ and $f'_-=\tilde f'_-$. We can therefore solve equation (7.7) on the other side of the discontinuity and conclude that $f=\tilde f$ for $0<x<d$. Since $f(x)=r^2\sqrt{\rho\beta}$ it

follows that

$$\rho(r)=\frac{f^2(\psi(R-r))}{r^4\beta(r)}$$

for $r_0<r<R_m$ and $R_m<r<R$. Thus $\rho=\tilde{\rho}$ for all $r\neq R_m$.

The proof for the second case is simpler. Here $a=\tilde{a}=1$ and $b=\tilde{b}=0$. Thus f and f' are continuous at $x=d$ and the same is true for $\tilde{f}$. This does not mean that ρ and β are continuous at $r=R_m$. But it shows that the transformed equation cannot have an interior discontinuity in y or y'. We can therefore use Lemma 1 of Hald [10] and conclude that $h=\tilde{h}$ and $q=\tilde{q}$ for all $x\neq d$. The remaining arguments in case 2 are identical to the arguments in case 1 and this completes the proof.

The proof of Theorem 2 does not use the full strength of Theorem 1. The position of the discontinuity is not determined by the eigenvalues, but is fixed by the velocity distribution which we assume is known. We shall now show that if the density and the velocity is given in the lower mantle, then the depth of the Mohorovičić discontinuity and the density and the velocity of the S-waves are uniquely determined in the upper mantle and in the crust by two torsional spectra.

THEOREM 3. *Let $R_c<R_m<R$ and assume that ρ and β are positive and twice continuously differentiable for $R_c\leqq r\leqq R_m$ and $R_m\leqq r\leqq R$. Consider the eigenvalue problem* (7.1) *with the continuity condition* (7.2). *Let $K=\lim(n/\omega_n)$, where $\{\omega_n^2(l)\}$ is the spectrum for a fixed value of l. Assume that ρ and β are given for $R_c\leqq r\leqq r_0$, where r_0 is determined by*

$$\int_{R_c}^{r_0}\beta^{-1}(r)\,dr=\tfrac{1}{2}\pi K.$$

Assume that $r_0<R_m$. If $\rho\cdot\beta$ has a jump discontinuity at $r=R_m$, then R_m, $\rho(r)$ and $\beta(r)$ are uniquely determined by two torsional spectra $\{\omega_n^2(l_1)\}$ and $\{\omega_n^2(l_2)\}$.

The result can be proved by combining the technique from the previous proof with the arguments for the smooth problem (see Hald [10]). We shall not present the details here. Note however that the assumption on $\rho(r)\cdot\beta(r)$ ensures that the Liouville transformation leads to an eigenvalue problem with an interior discontinuity. The assumption is satisfied in all the earth models I know.

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