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Reduction procedures for calculating the determinant of the adjacency matrix of some graphs and the singularity of square planar grids

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Abstract

Let G be a graph without loops and multiple edges. If $V(G) = \{v_1, v_2, \ldots, v_n\}$, we define the *adjacency matrix* of G to be the $n \times n$ (0, 1)-matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if $v_i v_j \in E(G)$ and $a_{ij} = 0$ otherwise. G is said to be *singular* if the matrix $A(G)$ is singular. Reduction procedures which will decrease the amount of computation needed to obtain the determinant of the adjacency matrices of some graphs are introduced. One of these reduction procedures is used in proving the singularity of square planar grid $P_n \times P_n$.

1. Introduction

By a *graph* G we mean a pair $(V(G), E(G))$, where $V(G)$ is a finite nonempty set of elements called *vertices* and $E(G)$ is a finite set of unordered pairs of distinct elements of $V(G)$ called *edges*. The *order* of the graph is the number of vertices. If uv is an edge in G, then u and v are said to be *adjacent* vertices or we also say that u is a *neighbor* of v. We may also denote the edge uv by $[v, u]$ and the set of neighbors of v by $N(v)$.

The *product* of two graphs $G_1 = (X_1, E_1)$ and $G_2 = (X_2, E_2)$ is the graph $G_1 \times G_2$ where $V(G_1 \times G_2) = X_1 \times X_2$ and $[(x_1, x_2), (y_1, y_2)] \in E(G_1 \times G_2)$ iff one of the following conditions is satisfied:

(1) $x_1 = y_1$ and $[x_2, y_2] \in E_2$;

(2) $x_2 = y_2$ and $[x_1, y_1] \in E_1$.

The product $P_m \times P_n$ is called a *planar grid* (see Fig. 1).

The study of singular graphs is of particular importance in organic chemistry where conjugated hydrocarbons are associated with graphs whose vertices correspond to the carbon atoms and whose edges correspond to chemical bonds between these atoms. Other relationships between chemistry and singular graphs are outlined in [3].

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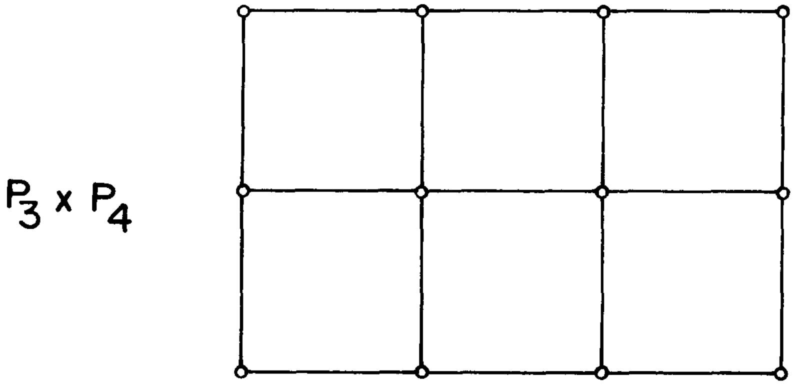

Fig. 1. $P_3 \times P_4$.

The computation of the determinant of any $n \times n$ matrix for large n is generally difficult; likewise the characterization of singular graphs. Thus, this paper presents some reduction formulas which in some cases enable the determinant of the adjacency matrix of some graphs to be determined by simple numerical calculation. This paper gives also the proof of the singularity of square planar grid using one of the reduction formulas.

We shall prove a theorem which will allow us to assume only connected graphs throughout this paper.

Theorem A. *G is singular if and only if at least one of its components is singular.*

Proof. Let $G_1, G_2, \ldots, G_n$ be components of G. Then we can rearrange the vertices of G in such a way that its adjacency matrix is of the form

$$A(G) = \begin{bmatrix} A(G_1) & & & 0 \\ & A(G_2) & & \\ & & \ddots & \\ 0 & & & A(G_n) \end{bmatrix},$$

where $A(G_i)$ is the adjacency matrix of component G_i. Hence, $\det A(G) = \det A(G_1) \det A(G_2) \cdots \det A(G_n)$. Therefore G is singular if and only if at least one of the G_i's is singular. $\square$

The proof of the above theorem simply says that the determinant of the adjacency matrix of a disconnected graph is a product of the determinant of the adjacency matrices of its components.

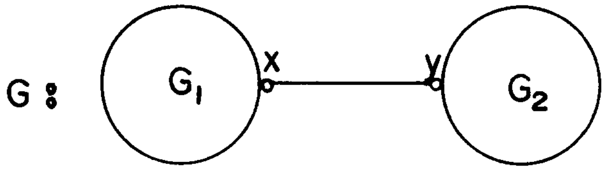

Fig. 2.

2. Reduction formulas

Theorem 1. *Let G be the graph obtained by joining the vertex x of the graph G_1 to the vertex y of the graph G_2 by an edge (see Fig. 2). Then* $\det A(G) = \det A(G_1) \det A(G_2) - \det A(G_1 - x) \det A(G_2 - y)$.

Proof. Let $v_1, v_2, \ldots, v_{n_1}$ be the vertices of G_1 and $v_{n_1+1}, v_{n_1+2}, \ldots, v_n$ be the vertices of G_2. Without loss of generality assume $v_{n_1} = x$ and $v_{n_1+1} = y$. Then the adjacency matrix of G has the following form:

$$A(G) = \left[\begin{array}{ccc|ccc} & & & 0 & & \\ & A(G_1) & & \vdots & & 0 \\ & & & 0 & & \\ & & & 1 & 0 \quad \cdots & 0 \\ \hline 0 & \cdots & 1 & & & \\ & & 0 & & A(G_2) & \\ & 0 & \vdots & & & \\ & & 0 & & & \end{array}\right].$$

Applying the Laplacian development [2] to the determinant of the matrix $A(G)$, we obtain

$$\det A(G) = (-1)^{2s_{n_1}} \det A(G_1) \det A(G_2) + (-1)^{s_{n_1} + s_{n_1-1} + n_1 + 1} \det A(G_1 - x) \det A(G_2 - y),$$

where $s_n = 1 + 2 + 3 + \cdots + n$. By algebraic simplification the theorem is proved. □

Theorem 2. *Let G be a graph of order $n > 2$ and let x_1, x_2 be distinct vertices in G such that $N(x_1) \subseteq N(x_2)$. Let G' be the graph obtained from G by removing all the edges $x_2 y$, where $y \in N(x_1)$. Then* $\det A(G) = \det A(G')$.

Proof. Let $1, 2, \ldots, n$ denote the vertices of G and, without loss of generality, assume $x_i = i$ for $i = 1, 2$ and $N(x_1) = \{3, 4, \ldots, m + 2\}$ (see Fig. 3). The adjacency matrix $A(G)$

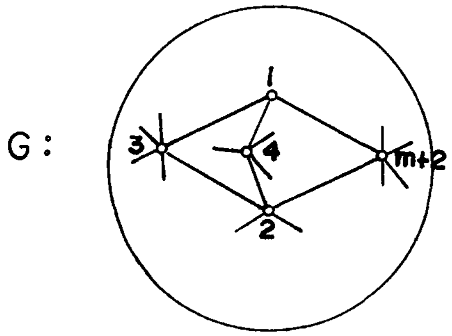

Fig. 3.

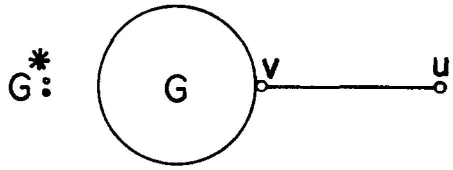

Fig. 4.

of G has the following form:

$$A(G) = \left[\begin{array}{cc|cccccccc}
0 & 0 & 1 & 1 & \dots & 1 & 0 & \dots & 0 \\
0 & 0 & 1 & 1 & \dots & 1 & a_{2,m+3} & \dots & a_{2,n} \\
\hline
1 & 1 & & & & & & & \\
1 & 1 & & & & & & & \\
\vdots & \vdots & & & & & & & \\
1 & 1 & & & & A(G - \{x_1, x_2\}) & & & \\
0 & a_{2,m+3} & & & & & & & \\
\vdots & \vdots & & & & & & & \\
0 & a_{2,n} & & & & & & &
\end{array}\right]$$

where $a_{2,k} = 0$ or 1, $k = m + 3, \dots, n$. If we subtract row 1 from row 2 and column 1 from column 2 of matrix $A(G)$, we see that the resulting matrix has determinant equal to $\det A(G')$. Therefore $\det A(G) = \det A(G')$. □

Corollary 1. *Let G be a graph and v be any vertex of G. If G^* is the graph obtained from G by joining v to a new vertex u, then* $\det A(G^*) = -\det A(G - v)$. *See Fig.* 4.

Proof. Clearly the set of neighbors $N(v)$ of v is disjoint from $\{u\}$ and $N(u) \subseteq N(i)$ $\forall i \in N(v)$. Applying Theorem 2 to the neighbors of v repeatedly we obtain a disjoint graph whose components are the path P_2 and $G - v$. By the proof of Theorem A,

$$\det A(G^*) = \det A(P_2) \det A(G - v).$$

But it is easy to see that $\det A(P_2) = -1$. Therefore $\det A(G^*) = -\det A(G - v)$. □

Corollary 2. *Let $C_4 = [1, 2, 3, 4, 1]$ be a subgraph of G where* $\deg_G(1) = 2$. *If G' is the graph obtained from G by removing the edges* $[2, 3]$ *and* $[3, 4]$, *then*

$$\det A(G') = \det A(G).$$

Proof. Apply Theorem 2 with $|N(x_1)| = 2$. $\square$

An *induced subgraph* H of G is a subgroup of G where for every vertex x, y of H, $[x, y] \in E(H)$ if $[x, y] \in E(G)$.

Theorem 2. *Let* $P_6 = [1,2,3,4,5,6]$ *be an induced subgraph of* G *with* $\deg_G(2) = \deg_G(3) = \deg_G(4) = \deg_G(5) = 2$. *If* H *is the graph formed from* $G - \{2,3,4,5\}$ *by joining vertices* 1 *and* 6 *with an edge, then* $\det A(G) = \det A(H)$. *See Fig.* 5.

Proof. Let $\{1,2,3,\dots,n\}$ be the set of vertices of G. Then $A(G)$ is of the form

$$A(G)=\left[\begin{array}{ccc|c}
 & & & a_{17}\quad \cdots\cdots \quad a_{1n} \\ \cline{4-4}
 & A(P_6) & & 0 \\ \cline{4-4}
 & & & b_{67}\quad \cdots\cdots \quad b_{6n} \\ \hline
a_{17} & & b_{67} & \\
\vdots & 0 & \vdots & A(G-P_6) \\
a_{1n} & & b_{6n} &
\end{array}\right],$$

where a_{1i} and b_{6i}, $i = 7,8,\dots,n$, are either 0 or 1. Apply the following operations to the matrix $A(G)$. Subtract row 2 from row 4 and column 2 from column 4. Then add row 5 to row 1 and column 5 to column 1. The resulting matrix is the adjacency matrix of the graph shown in Fig. 6. By Corollary 1, $\det A(G) = \det A(H)$. $\square$

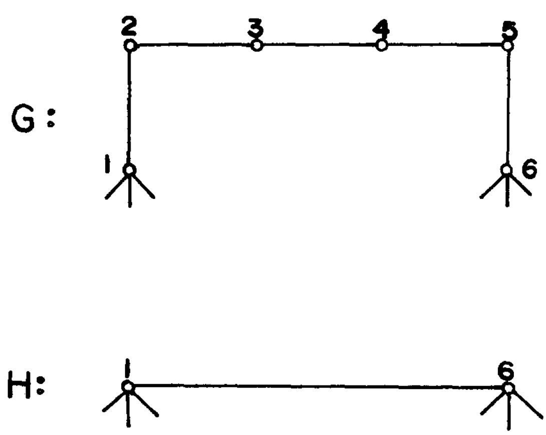

Fig. 5.

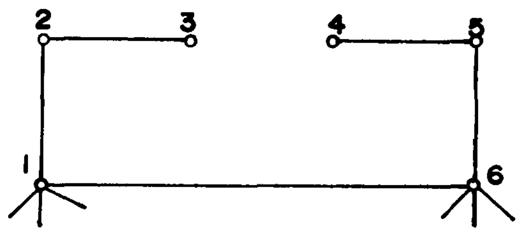

Fig. 6.

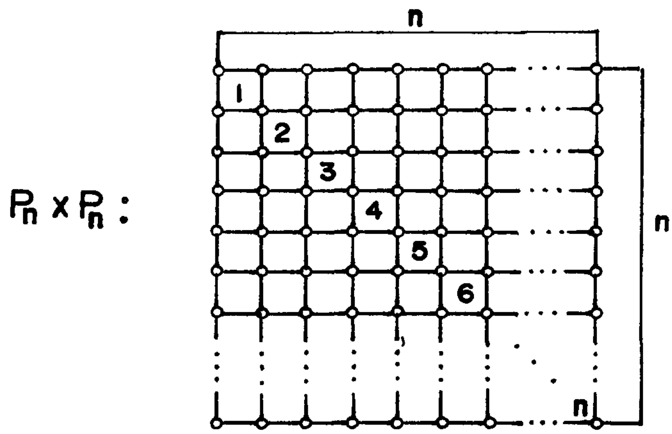

Fig. 7. $P_n \times P_n$.

Theorem 4 (Gervacio [1]). *Let x and y be the two adjacent vertices of G such that* $N(x) - \{y\} = N(x) - \{x\}$. *Then*

$$\det A(G) = -2 \det A(G - x) - \det A(G - \{x, y\}).$$

3. Singularity of square planar grid

Theorem 5. *Every square planar grid* $P_n \times P_n$ *is singular.*

Proof. Let $P_n \times P_n$ be the graph shown in Fig. 7. The diagonal squares are numbered from 1 to n. Applying Corollary 2 repeatedly on the diagonal squares of $P_n \times P_n$, we obtain a disconnected graph where one of its components is an isolated vertex. Therefore by Theorem A, $\det A(P_n \times P_n) = 0$ and $P_n \times P_n$ is singular. □

References

[1] S.V. Gervacio, A study of singular bipartite graphs, A Research Project Funded by the National Research Council of the Philippines, Metro Manila, Philippines, 1991.
[2] H.M. Rara, Singular and non-singular graphs, Ph.D. Thesis, Ateneo de Manila University, Metro Manila, Philippines 1991.
[3] R.J. Wilson, Singular graphs, Proc. 1st Japan Conf. in Graph Theory and its Applications, Hakone, Japan, 1986.