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ScienceDirect
Discrete Mathematics 308 (2008) 180–183
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

The size of m-irreducible blocking sets and of the sets of class $[0, n_1, \ldots, n_l]$

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Received 30 October 2004; received in revised form 3 August 2005; accepted 27 November 2006
Available online 2 June 2007

Abstract

We provide an upper bound of the size of an m-irreducible blocking set in a linear space. This upper bound is a generalization of the Bruen–Thas bound in π_q and improves it if $m > (q^2 + q - q\sqrt{q})/(q\sqrt{q} + 1)$. We prove that in a finite affine plane α_q of order q, two blocking sets mutually complementary are both irreducible, if and only if $q = 4$. Moreover, we determine bounds of the size of a set of class $[0, n_1, \ldots, n_l]$ in π_q, $n_i \equiv 1 \bmod d$, $i = 1, \ldots, l$, $2 \leqslant d < q$.

Keywords: Blocking sets; Baer subplanes

1. Introduction

A *blocking set* B in π_q is a set of points such that every line has a point in B and a point outside B. A blocking set is *irreducible* if it does not properly contain a blocking set. Equivalently, through any point of B there is a tangent line to B (that is a line meeting B at a unique point). A blocking set B_m is *m-irreducible*, if through any point of B_m there are at least m tangents, $m \geqslant 1$. We get an upper bound of $|B_m|$ which generalizes the Bruen–Thas bound [3] and improves it, whenever

$$m > \frac{q^2 + q - q\sqrt{q}}{q\sqrt{q} + 1}.$$

Moreover, we prove that in α_q the complement of an irreducible blocking set is irreducible, if and only if $q = 4$, while in π_q the complement of an irreducible blocking set is reducible [5]. At last, if I denotes a set of class $[0, n_1, \ldots, n_l]$, $n_i \equiv 1 \bmod d$, $i = 1, \ldots, l$, $2 \leqslant d < q$, we get suitable bounds for $|I|$.

2. Upper bound of the size of a set T_m in a linear space

A linear space is a pair $(\mathscr{S}, \mathscr{L})$, where $\mathscr{S}$ is a non-empty set whose elements we call *points* and $\mathscr{L}$ a non-empty set of subsets of $\mathscr{S}$ called *lines*, such that there is a unique line through two distinct points and each line has at least two points. We denote by T_m a set of $(\mathscr{S}, \mathscr{L})$ such that through every point there are at least m, $m \geqslant 1$, tangent lines to T_m.

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doi:10.1016/j.disc.2006.11.030

Let $b=|\mathscr{L}|$ and let $b \geqslant 2$, e the number of the external lines to T_m, t the number of the tangents to T_m, s the number of the secants to T_m, that is the lines meeting T_m at more than one point. We get

$$e+s+t=b, \tag{1}$$

$$t \geqslant |T_m| \cdot m. \tag{2}$$

Let $\mathscr{L}(T_m)$ be the set of the intersections of T_m with the secant lines of T_m. The pair $(T_m, \mathscr{L}(T_m))$ is a linear space. By the de Bruijn–Erdös Theorem [4], we get

$$s \geqslant |T_m|. \tag{3}$$

By (1) it follows

$$t+s=b-e. \tag{4}$$

By (2) and (3) it follows $t+s \geqslant (m+1)|T_m|$. Hence $|T_m| \leqslant b/(m+1)$.

So the following theorem holds:

Theorem 1. *The size of a set T_m in a linear space satisfies the inequality*

$$|T_m| \leqslant \frac{b}{m+1}.$$

3. Upper bound of the size of an m-irreducible blocking set in π_q

Let $(\mathscr{S}, \mathscr{L})$ be a finite projective plane π_q and let T_m be an m-irreducible blocking set B_m in π_q. By Theorem1 we get

$$|B_m| \leqslant \frac{q^2+q+1}{m+1}. \tag{5}$$

The Bruen–Thas upper bound for $|B_m|$ [3] is

$$|B_m| \leqslant q\sqrt{q}+1. \tag{6}$$

It is easy to prove that the bound (5) improves the bound (6) if

$$m > \frac{q^2+q-q\sqrt{q}}{q\sqrt{q}+1}.$$

If q is a square, a Baer subplane of π_q is a m-irreducible blocking set, with $m=q-\sqrt{q}$ and of size $q+\sqrt{q}+1$. In this case in (5) the equality holds. More precisely we prove the following:

Theorem 2. *In π_q, with q a square, a set B is a Baer subplane, if and only if, B is an m-irreducible blocking set with $m=q-\sqrt{q}$.*

Proof. If B is a Baer subplane of π_q, B is an m-irreducible blocking set, with $m=q-\sqrt{q}$. For, a Baer subplane is a $(q+\sqrt{q}+1)$-set of type $(1, \sqrt{q}+1)$ and through every point there are $q-\sqrt{q}$ tangent lines to B.

Conversely, let B_m be an m-irreducible blocking set of π_q, with $m=q-\sqrt{q}$. By (5) it follows

$$|B_m| \leqslant \frac{q^2+q+1}{q-\sqrt{q}+1}=q+\sqrt{q}+1. \tag{7}$$

In [2] Bruen proved that

$$|B_m| \geqslant q+\sqrt{q}+1 \tag{8}$$

and the equality holds, if and only if, B_m is a Baer subplane. By (7) and (8) we get $|B_m|=q+\sqrt{q}+1$ and therefore B_m is a Baer subplane. □

If $(\mathscr{S}, \mathscr{L})$ is a finite affine plane α_q, by Theorem 1 the following theorem follows

Theorem 3. *If B_m is an m-irreducible blocking set in α_q, then*

$$|B_m| \leqslant \frac{q^2+q}{m+1}.$$

4. Irreducible blocking sets in α_q and their complements in α_q

The following is known [5]:

Theorem 4. *In π_q the complement of an irreducible blocking set is reducible.*

In α_q the following theorem holds:

Theorem 5. *In α_q two blocking sets B and B' mutually complementary are both irreducible, if and only if, $q=4$. It follows that in α_q, $q>4$, the complement of an irreducible blocking set is reducible.*

Proof. Let B be an irreducible blocking set in α_q such that its complement B' is irreducible. We get $|B'|>q+1$. For, B' has at least two distinct points X, Y. The line XY contains a point Z not in B'. Every line through Z distinct from XY contains at least a point of B' and therefore $|B'| \geqslant q+2 > q+1$. Since B' is irreducible, through every point of B' there is at least a tangent to B'. It follows that there are at least $q+2$ distinct lines $(q-1)$-secant to B, that is lines having $q-1$ points in B. The directions of such lines cannot be distinct, since in α_q there are $q+1$ directions. Therefore there are two distinct lines, namely z and t, both $(q-1)$-secant to B and having the same direction δ. Let Z and T be the points of z and t not in B. Let $\ell_1, \ldots, \ell_{q-2}$ be the lines of α_q with direction δ distinct from z and t. Every line ℓ_j, $j=1,\ldots,q-2$, contains at least a point L_j in B. Let

$$U = (z\backslash\{Z\}) \cup (t\backslash\{T\}) \cup \{L_1, \ldots, L_{q-2}\}.$$

Obviously $B \supseteq U$. Let us prove that $B \subseteq U$. For, if X is a point of $B\backslash U$, it is $X \neq T$, $X \neq Z$, since Z and T are not in B. It follows that there is an index j, $1 \leqslant j \leqslant q-2$, such that $X \in \ell_j\backslash\{L_j\}$. Since B is irreducible, through L_j there is at least a line u tangent to B at L_j. The line u is distinct from ℓ_j, since ℓ_j is not tangent to B. Therefore the direction of u is distinct from δ. It follows that u meets z and t at Z and T, respectively, therefore $u = ZT$ and $L_j = ZT \cap \ell_j$. Similarly, substituting L_j by X, we get $X = ZT \cap \ell_j$, hence $X = L_j$: a contradiction since $X \neq L_j$. The contradiction proves that a point $X \in B\backslash U$ does not exist. Therefore $B \subseteq U$ and hence $B = U$. It follows that $|B| = |U| = q-1+q-1+q-2 = 3q-4$. Similarly we prove that $|B'| = 3q-4$. Since B and B' are mutually complementary, we get $|B|+|B'| = q^2$, hence $q^2-6q+8=0$, whose solutions are 2 and 4. The solution 2 is impossible, since in $AG(2,2)$ there are no blocking sets. Therefore $q=4$. Conversely, if $q=4$, two mutually complementary blocking sets are both irreducible, since it is known [5] that in $AG(2,4)$ there is a unique blocking set of size 8, up to an affinity. □

5. Bounds of the size of a set of class $[0, n_1, \ldots, n_l]$, $n_i \equiv 1 \bmod d$, $i=1,\ldots,l$, $2 \leqslant d < q$ in π_q

Let S be a set of points in π_q such that

(i) For every line ℓ meeting S, $|\ell \cap S| \equiv 1 \bmod d$, $\quad 2 \leqslant d < q$;
(ii) $\triangle = [1+d(q+2)]^2 - 4d(q^2+q+1) > 0$.

The following theorem holds:

Theorem 6. *Let S be a set of π_q satisfying* (i) *and* (ii). *Then either*

$$|S| \leqslant \frac{1+d(q+2)-\sqrt{\triangle}}{2} \tag{9}$$

or

$$|S| \geqslant \frac{1 + d(q+2) + \sqrt{\triangle}}{2}. \tag{10}$$

Proof. Let

$$\frac{1 + d(q+2) - \sqrt{\triangle}}{2} < |S| < \frac{1 + d(q+2) + \sqrt{\triangle}}{2}. \tag{11}$$

Let P be a point of S. Let n be the number of tangent lines to S at P. The number of lines through P not tangent to S is $q+1-n$ and each of them meets S in at least d points distinct from P. It follows $n \geqslant q+1-(|S|-1)/d$. Ranging the points of S over the lines through P and by i), we get $|S| \equiv 1 \bmod d$ and therefore $(|S|-1)/d$ is an integer. Let $m = q+1-(|S|-1)/d$. We prove that m is positive. For, the following conditions hold:

$$d < q, \tag{12}$$

$$|S| < \frac{1 + d(q+2) + \sqrt{\triangle}}{2}, \tag{13}$$

$$\frac{1 + d(q+2) + \sqrt{\triangle}}{2} < d(q+1) + 1 \quad \Leftrightarrow \quad d < q. \tag{14}$$

The conditions (12) and (13) hold by hypothesis. The condition (14) can be proved by easy calculations. By (12) and (14) we get

$$\frac{1 + d(q+2) + \sqrt{\triangle}}{2} < d(q+1) + 1.$$

By the above condition and (13), it follows

$$|S| < d(q+1) + 1,$$

which is equivalent to

$$\frac{|S|-1}{d} < q+1.$$

Since $m = q+1-(|S|-1)/d$, the previous condition provides $m > 0$.

By (5) we get

$$|S|^2 - |S|[1 + d(q+2)] + d(q^2+q+1) \geqslant 0. \tag{15}$$

By (ii) the roots $|S|_1$ and $|S|_2$ ($|S|_1 < |S|_2$) of the left hand side of (15) are both real and distinct and we get either $|S| \leqslant |S|_1$, or $|S| \geqslant |S|_2$, which contradicts (11). $\square$

If q is a square and $q \geqslant 9$, the right hand side of (9) by easy calculations becomes $q + \sqrt{q} + 1$, while, if $q = 4$, its value is 6, which is different from $4 + \sqrt{4} + 1 = 7$. It follows that the Baer subplanes, for $q \geqslant 9$, are examples such that in (9) the equality holds. The hermitian arcs, whose size is $q\sqrt{q} + 1$, are examples satisfying (10) for any square q.

In [1] (Proposition 7) an upper bound for the size of a set S in $PG(2, q)$, $q = p^h$, and satisfying (i) with $d = p$, is shown. Theorem 6 provides an upper bound for $|S|$ in any projective plane π_q.

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