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Perturbations from a kind of quartic Hamiltonians under general cubic polynomials

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Abstract In this paper we investigate the perturbations from a kind of quartic Hamiltonians under general cubic polynomials. It is proved that the number of isolated zeros of the related abelian integrals around only one center is not more than 12 except the case of global center. It is also proved that there exists a cubic polynomial such that the disturbed vector field has at least 3 limit cycles while the corresponding vector field without perturbations belongs to the saddle loop case.

Keywords: abelian integral, elliptic Hamiltonian, homoclinic bifurcation

MSC(2000): 58F14, 58F21, 58F30, 34C05

1 Introduction and main results

It is known that the families of planar differential equations are often applied to describing mathematically natural phenomena. However, relatively simple nonlinear systems defeat a complete study since no general methods are known to investigate the properties of solutions such as the number and location of limit cycles. The Liénard equations

$$\dot{x} = y, \quad \dot{y} = P(x) + yQ(x) \tag{1.1}$$

are among the commonest examples in nonlinear systems in the mechanical and electrical engineering because they take the familiar Van der Pol's equation

$$\ddot{x} + \alpha\phi(x)\dot{x} + x = 0 \tag{1.2}$$

and the Duffing's equation

$$\ddot{x}(t) + \delta\dot{x}(t) - x + x^3 = 0 \tag{1.3}$$

as their typical models. As for this limited class, Hilbert's 16th problem is still unsolved. Also they are unavoidable in the study of local bifurcations by means of rescaling techniques. A Liénard equation is said to be type (m, n) if $\deg P = m$ and $\deg Q = n$ in (1.1).

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In [1–4], Dumortier and Li studied systematically the Liénard equations of type $(3,2)$, that are Hamilton vector fields $\dot{x}=y$, $\dot{y}=P(x)$ under perturbations given by $\delta y(x^2+\beta x+\alpha)\frac{\partial}{\partial y}$ for small $\delta>0$ and arbitrary $(\delta,\beta,\alpha)\in\mathbb{R}^3$. After linear rescaling, the Hamiltonians $\frac{1}{2}y^2+\int_0^x P(s)ds$ are given by

$$H(x,y)=\frac{1}{2}y^2+\frac{a}{4}x^4+\frac{b}{3}x^3+\frac{c}{2}x^2,\quad (a,b,c)\in\mathbb{R}^3,\quad a\neq 0. \tag{1.4}$$

In other words, they studied the bifurcations of the vector fields

$$\dot{x}=y,\ \dot{y}=-(ax^3+bx^2+cx)+(\delta x^2y+\delta\beta xy+\delta\alpha y), \tag{1.5}$$

for arbitrary $(a,b,c,\delta,\beta,\alpha)\in\mathbb{R}^6$ with $a\neq 0$ and $0<\delta\ll 1$.

Motivated by [1–4], in this paper we intend to investigate the perturbations of the same vector fields as in [1–4], that is,

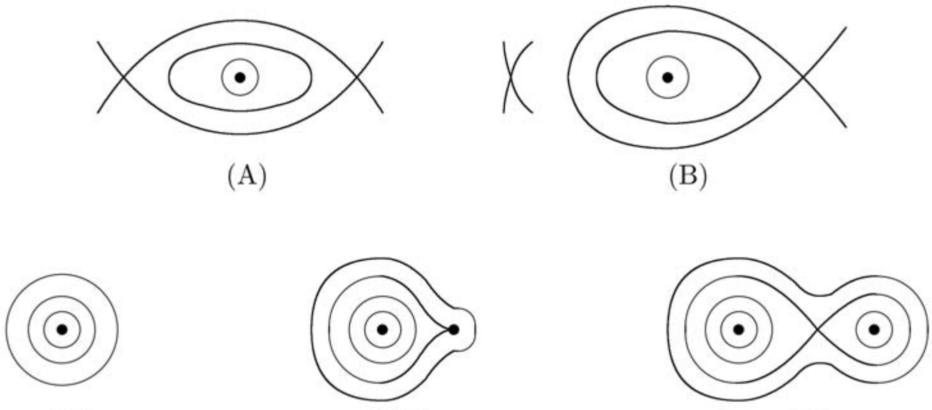
$$\dot{x}=y,\ \dot{y}=-(ax^3+bx^2+cx),\quad \forall(a,b,c)\in\mathbb{R}^3,\ a\neq 0, \tag{X_0}$$

under a general form of cubic perturbations. More precisely, we will study the bifurcations of the vector fields

$$\begin{cases}\dot{x}=y+\varepsilon\displaystyle\sum_{0\leqslant i+j\leqslant 3}p_{ij}x^iy^j=:H_y+\varepsilon P_3(x,y),\\ \dot{y}=-(ax^3+bx^2+cx)+\varepsilon\displaystyle\sum_{0\leqslant i+j\leqslant 3}q_{ij}x^iy^j=:-H_x+\varepsilon Q_3(x,y),\end{cases} \tag{$\tilde{X}_\varepsilon$}$$

where p_{ij}, q_{ij} and ε are arbitrary constants with $0<\varepsilon\ll 1$ for i,j satisfying $0\leqslant i+j\leqslant 3$.

For arbitrary $(a,b,c)\in\mathbb{R}^3$ with $a\neq 0$, vector fields (X_0) can be classified into five different types according to the phase portraits. They are respectively called the cases of two saddle cycle, saddle loop, global center, cuspidal loop and figure eight-loop, shown in Figure 1 (A)-(E).



(A) (B) (C) (D) (E)

Figure 1

In the rest of the paper we essentially intend to prove the following theorem.

Theorem 1.1. *Consider the family of vector fields $(\tilde{X}_\varepsilon)$ that are vector fields (X_0) under a general form of cubic perturbations. Suppose that a,b and c are arbitrary constants with $a\neq 0$ such that every vector field in (X_0) has at least one center. Then the following statements hold*:

(i) $(\tilde{X}_\varepsilon)$ *is equivalent to*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -(ax^3 + bx^2 + cx) + \varepsilon(q_{01}y + q_{11}xy + q_{21}x^2y + q_{03}y^3) \end{cases} \qquad (X_\varepsilon)$$

in the study of abelian integrals.

(ii) *The number of the isolated zeros of the related abelian integrals of* (X_ε) *around only one center is not more than* 12 *for the cases of* (A), (B), (D) *and* (E).

(iii) *There exist* $a^*, b^*, c^*, q_{01}^*, q_{11}^*, , q_{21}^*, q_{03}^*$ *and* $\varepsilon^* > 0$ *such that the corresponding vector field* (X_ε^*) *has at least* 3 *limit cycles while the corresponding vector field* (X_0) *belongs to the case* (B).

Remark 1.1. (i) Since we consider the perturbations of vector fields (X_0) under a general form of cubic polynomials, we cannot use the techniques developed in [1–4]. Also it is difficult to obtain the sharp upper bounds of the number of isolated zeros of the related abelian integrals and limit cycles.

(ii) Theorem 1.1 will be proved by investigating the number of the intersection points of two curves by means of general Rolle's theorem (in the spirit of Khovanskii's method, see Theorem B in Section 3).

(iii) It follows from Theorem 1 in [1] that 2 is the sharp upper bound of limit cycles for vector fields (1.5) if the unperturbed vector field (X_0) belongs to the case (B). Combining Theorem 1 in [1] with Theorem 1.1, it is clear that the general form of cubic perturbations do change the number of limit cycles bifurcated from (X_0).

(iv) Zhao and Zhang[5] proved that the number of isolated zeros of Abelian integrals determined by vector fields (X_0) under perturbations of polynomials of degree n is not more than $7n + 5$ for any $n \in N$. Liu[6] studied the upper bound of the number of zeros of the related abelian integral for the case (E).

(v) For Hamiltonian (1.4) with certain symmetric properties, such as $H(x, y) = \frac{1}{2}y^2 + x^4 - x^2$, $H(x, y) = \frac{1}{2}y^2 + x^2 - x^4$, the number of zeros of the related abelian integrals has been estimated in [7–11], and most of the results were proved by Petrov's method.

The paper is organized as follows. In Section 2, we first prove the first conclusion in Theorem 1.1 and then obtain the expression of abelian integrals $I(h)$. Next, we introduce some results on $I_i(h)$ with $i = 1, 2, 3$. In Section 3, we obtain the upper bounds of the number of zeros of the related abelian integrals. In Section 4, we complete the proof of Theorem 1.1.

2 Preliminaries

Denote by Γ_h the planar ovals defined by $H(x, y) = h$ for $h \in \Sigma$, where Σ is the maximal domain of the level curves.

2.1 Proof of Theorem 1.1 (i) and properties related to $I_i(h)$

Lemma 2.1. *The family of vector fields* $(\tilde{X}_\varepsilon)$ *is equivalent to* (X_ε) *in the study of abelian integrals.*

Proof. By [7], the displacement function (which is also called Mel'nikov function) of $(\tilde{X}_\varepsilon)$ can be expressed as

$$M(h, \varepsilon) = \varepsilon(I(h) + o(\varepsilon)), \quad h \in \Sigma, \tag{2.1}$$

where

$$I(h)=\oint_{\Gamma_h}P_3(x,y)dy-Q_3(x,y)dx \tag{2.2}$$

is the abelian integrals. By Green's formula,

$$I(h)=\int\int_{\mathrm{int}\Gamma_h}\left(\frac{\partial P_3}{\partial x}+\frac{\partial Q_3}{\partial y}\right)dxdy,$$

where

$$\begin{aligned}\frac{\partial P_3}{\partial x}+\frac{\partial Q_3}{\partial y}=&(p_{10}+q_{01})+(2p_{20}+q_{11})x+(p_{11}+2q_{02})y\\&+(3p_{30}+q_{21})x^2+2(p_{21}+q_{12})xy+(p_{12}+3q_{03})y^2.\end{aligned}$$

Suppose that $x_l(h)$ and $x_r(h)$ ($x_l(h)<0<x_r(h)$) are abscissa of the intersection points of Γ_h to the x-axis. Then for $x\in(x_l(h),x_r(h))$, Γ_h can be expressed as

$$\begin{aligned}&-y_+(x,h)\leqslant y(x,h)\leqslant y_+(x,h),\\&y_+(x,h)=\sqrt{2\left[h-\left(\frac{a}{4}x^4+\frac{b}{3}x^3+\frac{c}{2}x^2\right)\right]},\quad h\in\Sigma.\end{aligned}$$

And hence

$$I(h)=\int_{x_l(h)}^{x_r(h)}dx\int_{-y_+(x,h)}^{y_+(x,h)}F(x,y)dy,$$

where

$$F(x,y)=(p_{10}+q_{01})+(2p_{20}+q_{11})x+(3p_{30}+q_{21})x^2+(p_{12}+3q_{03})y^2.$$

Noting that p_{ij} and q_{ij} are arbitrary constants, without loss of generality, we can suppose that $p_{ij}\equiv 0$. So,

$$I(h)=\int\int_{\mathrm{int}\Gamma_h}[q_{01}+q_{11}x+q_{21}x^2+3q_{03}y^2]dxdy.$$

The result follows from the fact that the family of vector fields (X_ε) has the same abelian integrals as $(\tilde{X}_\varepsilon)$. We complete this proof.

Let us introduce some notations.

$$I_i(h)=\oint_{\Gamma_h}x^iydx,\quad h\in\Sigma,\quad i=0,1,2,\dots, \tag{2.3}$$

$$Q(h)=\frac{I_2(h)}{I_0(h)},\quad h\in\Sigma, \tag{2.4}$$

$$\omega(h)=\frac{I_1''(h)}{I_0''(h)},\quad \text{for } h\in\Sigma \ \text{ such that } \ I_0''(h)\neq 0. \tag{2.5}$$

Lemma 2.2. *For (X_ε), the abelian integrals $I(h)$ can be expressed as*

$$I(h)=(\alpha_1h+\alpha_2)I_0(h)+\beta I_1(h)+\gamma I_2(h), \tag{2.6}$$

where

$$\alpha_1=\frac{12}{5}q_{03},\quad \alpha_2=q_{01},\quad \beta=q_{11}+\frac{bc}{5a}q_{03},\quad \gamma=q_{21}+\frac{b^2-3ac}{5a}q_{03}.$$

Proof. By (2.2),

$$I(h) = q_{01}I_0(h) + q_{11}I_1(h) + q_{21}I_2(h) + \oint_{\Gamma_h} q_{03}y^3dx. \tag{2.7}$$

Along Γ_h, we have $y^2 = 2h - \frac{a}{2}x^4 - \frac{2b}{3}x^3 - cx^2$, then

$$\oint_{\Gamma_h} y^3dx = 2hI_0(h) - \frac{a}{2}I_4(h) - \frac{2b}{3}I_3(h) - cI_2(h). \tag{2.8}$$

In what follows, we will calculate $I_3(h)$ and $I_4(h)$. Along Γ_h, we also have

$$ydy + (ax^3 + bx^2 + cx)dx = 0. \tag{2.9}$$

Noting $\oint_{\Gamma_h} y^2dy = 0$, we get

$$aI_3(h) + bI_2(h) + cI_1(h) = 0. \tag{2.10}$$

By (2.9),

$$xy^2dy + (ax^4 + bx^3 + cx^2)ydx = 0. \tag{2.11}$$

Noting

$$\begin{aligned}\oint_{\Gamma_h} xy^2dy &= \int\int_{\mathrm{int}\Gamma_h} y^2dxdy = \oint_{\Gamma_h} \frac{y^3}{3}dx = \oint_{\Gamma_h} \frac{2}{3}y\left[h - \frac{a}{4}x^4 - \frac{b}{3}x^3 - \frac{c}{2}x^2\right]dx \\ &= \frac{2}{3}\left\{hI_0(h) - \frac{a}{4}I_4(h) - \frac{b}{3}I_3(h) - \frac{c}{2}I_2(h)\right\},\end{aligned}$$

together with (2.11), we can get

$$\frac{2h}{3}I_0(h) + \frac{5a}{6}I_4(h) + \frac{7b}{9}I_3(h) + \frac{2c}{3}I_2(h) = 0. \tag{2.12}$$

Substituting (2.10) and (2.12) into (2.8), we get

$$\begin{aligned}\oint_{\Gamma_h} y^3dx &= \frac{12h}{5}I_0(h) - \frac{b}{5}I_3(h) - \frac{3c}{5}I_2(h) \\ &= \frac{12h}{5}I_0(h) + \frac{b}{5}\cdot\frac{1}{a}(bI_2(h) + cI_1(h)) - \frac{3c}{5}I_2(h) \\ &= \frac{12h}{5}I_0(h) + \frac{b^2 - 3ac}{5a}I_2(h) + \frac{bc}{5a}I_1(h).\end{aligned}$$

Then by (2.7), we have

$$\begin{aligned}I(h) &= \left[q_{01} + \frac{12}{5}q_{03}h\right]I_0(h) + \left[q_{11} + \frac{bc}{5a}q_{03}\right]I_1(h) + \left[q_{21} + \frac{b^2 - 3ac}{5a}q_{03}\right]I_2(h) \\ &= (\alpha_1h + \alpha_2)I_0(h) + \beta I_1(h) + \gamma I_2(h).\end{aligned}$$

We complete this proof.

2.2 The properties of $I_i(h)$, $Q(h) = \frac{I_2(h)}{I_0(h)}$ and $\omega(h) = \frac{I_1''(h)}{I_0''(h)}$

The following results were proved in [1].

Theorem A.

(i)

$$N\begin{pmatrix} I_0 \\ I_1 \\ I_2 \end{pmatrix} = (4hE + S)\begin{pmatrix} I_0' \\ I_1' \\ I_2' \end{pmatrix},$$

where E is the identity matrix, and

$$N=\begin{pmatrix} 3 & 0 & 0\\ \frac{b}{3a} & 4 & 0\\ \frac{3ac-b^2}{3a^2} & \frac{2b}{3a} & 5\end{pmatrix},\quad S=\begin{pmatrix} 0 & \frac{bc}{3a} & \frac{b^2-3ac}{3a}\\ 0 & \frac{3ac^2-cb^2}{3a^2} & \frac{b(4ac-b^2)}{3a^2}\\ 0 & \frac{-bc(4ac-b^2)}{3a^3} & \frac{3a^2c^2-5ab^2c+b^4}{3a^3}\end{pmatrix}.$$

(ii) *If $b(2b^2-9ac)\neq 0$, then*

$$I_2''=\frac{12a}{2b^2-9ac}hI_0''+\frac{1}{b}\left(\frac{36a^2}{2b^2-9ac}h-c\right)I_1''. \tag{2.13}$$

(iii) *If $I_1(h)\equiv 0$, then $Q(h)$ satisfies the following equation*:

$$G(h)Q'(h)=a_{20}+a_{22}Q-Q(a_{00}+a_{02}Q), \tag{2.14}$$

where

$$\begin{aligned}
G(h)&=12h[144a^3h^2+12(b^4-6ab^2c+6a^2c^2)h-c^3(2b^2-9ac)],\\
a_{00}&=12[108a^3h^2+(10b^4-61ab^2c+63a^2c^2)h-c^3(2b^2-9ac)],\\
a_{02}&=-15a[12a(b^2-3ac)h+c^2(2b^2-9ac)],\\
a_{20}&=-12[12a(b^2-3ac)h+c^2(2b^2-9ac)]h,\\
a_{22}&=180a[12a^2h-c(b^2-3ac)]h.
\end{aligned}$$

(iv) *If $b(2b^2-9ac)\neq 0$, then $\omega(h)$ satisfies the following Riccati equation*:

$$G(h)\omega'(h)=-b_{01}\omega^2-(b_{00}-b_{11})\omega+b_{10}, \tag{2.15}$$

where

$$\begin{aligned}
b_{00}&=\frac{432a^3(36ac-7b^2)}{2b^2-9ac}h^2-(864a^2c^2-996ab^2c+168b^4)h+12(2b^2-9ac)c^3,\\
b_{01}&=\frac{3888a^4(b^2-3ac)}{b(2b^2-9ac)}h^2+\frac{12a(54a^2c^2+9ab^2c-2b^4)}{b}h+\frac{(2b^2-9ac)(10b^2-9ac)c^2}{b},\\
b_{10}&=-\frac{144a^2b(7b^2-27ac)}{2b^2-9ac}h^2+12bc(2b^2-9ac)h,\\
b_{11}&=\frac{432a^3(72ac-17b^2)}{2b^2-9ac}h^2-12(2b^2-9ac)(5b^2-8ac)h.
\end{aligned}$$

Remark 2.1. $G(h)\neq 0$ for $h\in\Sigma$.

3 Proof of Theorem 1.1 (ii)

The following results are from [8–10].

Theorem B. *Let $F(x,y)$ and $G(x,y)$ be two functions continuous on $\overline{\mathcal{D}}$ and smooth in $\mathcal{D}$, where $\mathcal{D}=(x_1,x_2)\times(y_1,y_2)$. If $F_x'(x,y)$ and $F_y'(x,y)\neq 0$, then in the region $\mathcal{D}$ the number of solutions of the system $\begin{cases}F(x,y)=0,\\ G(x,y)=0,\end{cases}$ is not more than 1 plus the number of solutions of the system*

$$\begin{cases}F(x,y)=0,\\ F_y'(x,y)G_x'(x,y)-F_x'(x,y)G_y'(x,y)=0.\end{cases}$$

Denote by $\sharp\{f(h)=0\ (h\in\Sigma)\}$ the number of isolated zeros of $f(h)$ for $h\in\Sigma$.

Lemma 3.1. *Suppose that $b=0$. Then*

$$\sharp\{I(h)=0\ (h\in\Sigma)\}\leqslant \max(4,q), \tag{3.1}$$

where q is the number of monotone intervals of $Q(h)$ on Σ.

Proof. For $b=0$, $H(x,y)=\frac{1}{2}y^2+\frac{a}{4}x^4+\frac{c}{2}x^2,\ (a,b,c)\in\mathbb{R}^3, a\neq 0$, which yields that the level curves $H(x,y)=h$ are symmetric to both the x-axis and the y-axis, and hence $I_1(h)\equiv 0$. By (2.6), we have

$$I(h)=(\alpha_1 h+\alpha_2)I_0(h)+\gamma I_2(h),\quad h\in\Sigma, \tag{3.2}$$

where α_1, α_2 and γ are arbitrary constants. If $\gamma=0$, then $I(h)=(\alpha_1 h+\alpha_2)I_0(h)$, which implies that $I(h)$ has at most 1 isolated zero in Σ since $I_0(h)\neq 0$. For $\gamma\neq 0$, since

$$I(h)=\gamma I_0(h)\left[\frac{\alpha_1 h+\alpha_2}{\gamma}+\frac{I_2(h)}{I_0(h)}\right]=:\gamma[(\widetilde{\alpha}_1 h+\widetilde{\alpha}_2)+Q(h)]I_0(h),$$

the equation $I(h)=0$ has the same solutions as the equation

$$(\widetilde{\alpha}_1 h+\widetilde{\alpha}_2)+Q(h)=0,\ h\in\Sigma, \tag{3.3}$$

where $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ are arbitrary constants. The solutions of (3.3) are the solutions of the system

$$\begin{cases} G(h,y):=y-Q(h)=0, & h\in\Sigma,\\ F(h,y):=y+(\widetilde{\alpha}_1 h+\widetilde{\alpha}_2)=0, & h\in\Sigma. \end{cases} \tag{3.4}$$

Let us discuss the case of $\tilde{\alpha}_1=0$. Then the system (3.4) is equivalent to

$$\begin{cases} y=Q(h), & h\in\Sigma,\\ y=-\widetilde{\alpha}_2, & h\in\Sigma. \end{cases} \tag{3.5}$$

The solutions of (3.5) can be found by investigating the intersections between the curve $y=Q(h)$ with arbitrary line $y=-\tilde{\alpha}_2$. Since the number of monotone intervals of $Q(h)$ is q, the line $y=-\tilde{\alpha}_2$ can intersect to the graph of $Q(h)$ at most q times for arbitrary $\tilde{\alpha}_2$, that is, $\sharp\{I(h)=0\ (h\in\Sigma)\}\leqslant q$ if $\tilde{\alpha}_1=0$.

Now let us consider the case of $\widetilde{\alpha}_1\neq 0$. It follows from Theorem B that the number of solutions of the system (3.4) is not more than 1 plus the number of solutions of the system

$$\begin{cases} Q'(h)=-\widetilde{\alpha}_1, & h\in\Sigma,\\ Q(h)=-(\widetilde{\alpha}_1 h+\widetilde{\alpha}_2), & h\in\Sigma. \end{cases} \tag{3.6}$$

Noting $G(h)\neq 0$, we get that the system (3.6) is equivalent to

$$\begin{cases} G(h)Q'(h)=-\widetilde{\alpha}_1 G(h), & h\in\Sigma,\\ Q(h)=-(\widetilde{\alpha}_1 h+\widetilde{\alpha}_2), & h\in\Sigma. \end{cases} \tag{3.7}$$

Let $f(h)=G(h)Q'(h)+\widetilde{\alpha}_1 G(h)$. Then, by (2.14),

$$f(h)=[a_{20}+a_{22}Q-Q(a_{00}+a_{02}Q)]+\widetilde{\alpha}_1 G(h). \tag{3.8}$$

Substituting the second formula of system (3.7) into (3.8), we obtain

$$f(h) = [a_{20} + a_{22}Q - Q(a_{00} + a_{02}Q)]|_{Q(h)=-(\widetilde{\alpha}_1 h+\widetilde{\alpha}_2)} + \widetilde{\alpha}_1 G(h). \tag{3.9}$$

By Theorem A, $\deg a_{02} = 1$, $\deg a_{20}$, $\deg a_{22}$, $\deg a_{00} = 2$ and $\deg G(h) = 3$, hence formula (3.9) shows that $f(h)$ is a cubic polynomial in h. Since $f(h)$ has at most 3 isolated zeros for $h \in \mathbb{R}$, we know that the system (3.6) has at most 3 isolated zeros for $h \in \mathbb{R}$, and then the system (3.4) has at most 4 isolated zeros for $h \in \Sigma$. To sum up, we have proved

$$\sharp\{I(h) = 0\ (h \in \Sigma)\} \leqslant \begin{cases} 1, & \text{if } \gamma = 0, \\ q, & \text{if } \gamma \neq 0 \text{ and } \tilde{\alpha}_1 = 0, \\ 4, & \text{if } \gamma \neq 0 \text{ and } \tilde{\alpha}_1 \neq 0. \end{cases}$$

This completes the proof.

We will obtain the zeros of $I(h)$ by studying the isolated zeros of $I''(h)$.

Lemma 3.2. *If $2b^2 - 9ac \neq 0$, then*

$$I''(h) = (\sigma_1 h + \sigma_2)I_0''(h) + (\sigma_3 h + \sigma_4)I_1''(h), \quad h \in \Sigma, \tag{3.10}$$

where σ_i $(i = 1, 2, 3, 4)$ are arbitrary constants.

Proof. Indeed, by (2.6), we have

$$I'(h) = (\alpha_1 h + \alpha_2)I_0'(h) + \beta I_1'(h) + \gamma I_2'(h) + \alpha_1 I_0(h). \tag{3.11}$$

Noting that q_{01}, q_{11}, q_{21} and q_{03} are arbitrary, we know that $\alpha_1, \alpha_2, \beta$ and γ are arbitrary too. By Theorem A(i), we get

$$3I_0 = 4hI_0' + \frac{bc}{3a}I_1' + \frac{b^2 - 3ac}{3a}I_2'. \tag{3.12}$$

Substituting (3.12) into (3.11), we have

$$I'(h) = (\rho_1 h + \rho_2)I_0'(h) + \rho_3 I_1'(h) + \rho_4 I_2'(h), \tag{3.13}$$

where ρ_i $(i = 1, 2, 3, 4)$ are arbitrary constants. Then

$$I''(h) = \rho_1 I_0'(h) + (\rho_1 h + \rho_2)I_0''(h) + \rho_3 I_1''(h) + \rho_4 I_2''(h). \tag{3.14}$$

It follows from Theorem A(i) that

$$(N - 4E)\begin{pmatrix} I_0' \\ I_1' \\ I_2' \end{pmatrix} = (4hE + S)\begin{pmatrix} I_0'' \\ I_1'' \\ I_2'' \end{pmatrix}. \tag{3.15}$$

The first line of (3.15) yields

$$-I_0'(h) = 4hI_0'' + \frac{bc}{3a}I_1'' + \frac{b^2 - 3ac}{3a}I_2''. \tag{3.16}$$

Substituting (3.16) and (2.13) into (3.14), we can obtain (3.10). We complete this proof.

Lemma 3.3. *If $b(2b^2 - 9ac) \neq 0$ and $I_0''(h) \neq 0$ for $h \in \Sigma$, then the following results hold*:

(i) *If* $(\sigma_1 h+\sigma_2)$ *and* $(\sigma_3 h+\sigma_4)$ *have a common factor, then*

$$\sharp\{I(h)=0\ (h\in\Sigma)\}\leqslant l+2, \tag{3.17}$$

where l *is the number of the monotone intervals of* $\omega(h)$ *for* $h\in\Sigma$.

(ii) *If* $\sigma_1\sigma_4=\sigma_2\sigma_3$, *then* $\sharp\{I(h)=0\ (h\in\Sigma)\}\leqslant l+2$.

(iii) *If* $\sigma_1\sigma_4\neq\sigma_2\sigma_3$, *then*

$$\sharp\{I(h)=0\ (h\in\Sigma)\}\leqslant \max(6,l+2). \tag{3.18}$$

In one word,

$$\sharp\{I(h)=0\ (h\in\Sigma)\}\leqslant \max(6,l+2). \tag{3.19}$$

Proof. (i) If $(\sigma_1 h+\sigma_2)$ and $(\sigma_3 h+\sigma_4)$ have a common factor, then there exists $k\neq 0$ such that $\sigma_1 h+\sigma_2=k(\sigma_3 h+\sigma_4),\ h\in\mathbb{R}$. Hence

$$I''(h)=(\sigma_3 h+\sigma_4)[kI_0''(h)+I_1''(h)]=(\sigma_3 h+\sigma_4)I_0''(h)[k+\omega(h)],$$

and then

$$\sharp\{I''(h)=0\ (h\in\Sigma)\}\leqslant 1+\sharp\{\omega(h)=-k(h\in\Sigma)\}.$$

Since $\omega(h)$ has l monotone intervals on Σ, we can get

$$\sharp\{I''(h)=0\ (h\in\Sigma)\}\leqslant 1+l.$$

Therefore $I(h)$ has at most $l+3$ isolated zeros for $h\in\Sigma\cup\partial\Sigma$ by Rolle's theorem. Since $I(h)$ is always equal to zero at one point of $\partial\Sigma$, we can obtain (3.17).

(ii) If $\sigma_1\sigma_4=\sigma_2\sigma_3$, $I''(h)$ has the following expressions:

$$\sigma_1\sigma_4=\sigma_2\sigma_3\begin{cases}\text{(a) } \sigma_2,\sigma_4\neq 0,\ \text{then } I''(h)=(\sigma_3 h+\sigma_4)I_0''[k+\omega(h)] \text{ for some } k;\\ \text{(b) } \sigma_2=0,\sigma_4\neq 0,\ \text{then } I''(h)=(\sigma_3 h+\sigma_4)I_0''\omega(h)=(\sigma_3 h+\sigma_4)I_0''(0+\omega(h));\\ \text{(c) } \sigma_2=\sigma_4=0,\ \text{then } I''(h)=\sigma_1 hI_0''+\sigma_3 hI_1''=h(\sigma_1 I_0''+\sigma_3 I_1'');\\ \text{(d) } \sigma_4=0,\sigma_2\neq 0,\ \text{then } I''(h)=(\sigma_1 h+\sigma_2)I_0''.\end{cases}$$

So cases (a)–(c) belong to (i), and $I''(h)$ has at most 1 isolated zero for (d).

(iii) First we assume that $\sigma_3 h+\sigma_4\neq 0$ for $h\in\Sigma$. Then by (3.10),

$$I''(h)=0\Longleftrightarrow\frac{\sigma_1 h+\sigma_2}{\sigma_3 h+\sigma_4}=-\omega(h)\Longleftrightarrow\begin{cases}y=-\omega(h),\\ y=\dfrac{\sigma_1 h+\sigma_2}{\sigma_3 h+\sigma_4}.\end{cases} \tag{3.20}$$

By Theorem B, the number of solutions of the system (3.20) is not more than 1 plus the number of solutions of the following system:

$$\begin{cases}\omega'(h)=-\dfrac{\sigma_1\sigma_4-\sigma_2\sigma_3}{(\sigma_3 h+\sigma_4)^2},\quad h\in\Sigma,\\ \omega(h)=\dfrac{\sigma_1 h+\sigma_2}{\sigma_3 h+\sigma_4},\quad h\in\Sigma.\end{cases} \tag{3.21}$$

The system (3.21) is equivalent to

$$\begin{cases} G(h)\omega'(h) = G(h)\Big[-\dfrac{\sigma_1\sigma_4-\sigma_2\sigma_3}{(\sigma_3 h+\sigma_4)^2}\Big], & h\in\Sigma,\\ \omega(h)=\dfrac{\sigma_1 h+\sigma_2}{\sigma_3 h+\sigma_4}, & h\in\Sigma. \end{cases} \tag{3.22}$$

Let

$$f(h) = G(h)\omega'(h) + \frac{\sigma_1\sigma_4-\sigma_2\sigma_3}{(\sigma_3 h+\sigma_4)^2}G(h). \tag{3.23}$$

Substituting (2.15) and the second formula of (3.22) into (3.23), we get

$$\begin{aligned} f(h) = {} & \frac{1}{(\sigma_3 h+\sigma_4)^2}\{-b_{01}(\sigma_1 h+\sigma_2)^2-(b_{00}-b_{11})(\sigma_1 h+\sigma_2)(\sigma_3 h+\sigma_4)\\ & +b_{10}(\sigma_3 h+\sigma_4)^2+(\sigma_1\sigma_4-\sigma_2\sigma_3)G(h)\}\\ := {} & \frac{1}{(\sigma_3 h+\sigma_4)^2}f_1(h). \end{aligned}$$

Noting that b_{ij} and $G(h)$ are polynomials in h of degrees 2 and 3 respectively, we know that $f_1(h)$ is a polynomial in h of degree not more than 4, which has at most 4 isolated zeros for $h\in R$. So we can obtain that $I''(h)$ has at most 5 isolated zeros for $h\in\Sigma\cup\partial\Sigma$. Since $I(h)$ is always equal to zero at one point of $\partial\Sigma$, we conclude that $I(h)$ has at most 6 zeros for $h\in\Sigma$, that is,

$$\sharp\{I(h)=0\ (h\in\Sigma)\}\leqslant 6. \tag{3.24}$$

Secondly, we assume that there exists $\widetilde{h}\in\Sigma$ such that $\sigma_3\widetilde{h}+\sigma_4=0$. If $I''(\widetilde{h})\neq 0$, let $\tilde{\Sigma}=\{(-\infty,\widetilde{h})\cup(\widetilde{h},\infty)\}\cap\Sigma$. Then $\sigma_3 h+\sigma_4\neq 0$ for $h\in\tilde{\Sigma}$ and $I(h)$ has at most 6 isolated zeros for $h\in\tilde{\Sigma}$. Noting $I''(\widetilde{h})\neq 0$, we can conclude that $I(h)$ has at most 6 isolated zeros for $h\in\Sigma$. If $I''(\widetilde{h})=0$, we can get that $\sigma_1\widetilde{h}+\sigma_2=0$ since $I_0''(h)\neq 0$ for $h\in\Sigma$. Hence $\sigma_3\widetilde{h}+\sigma_4=0, \sigma_1\widetilde{h}+\sigma_2=0$, which implies that $\sigma_3 h+\sigma_4$ and $\sigma_1 h+\sigma_2$ have a common factor. According to (i) of this lemma, we know that $\sharp\{I(h)=0(h\in\Sigma)\}\leqslant l+2$. This completes the proof.

Proof of Theorem 1.1(ii). For case (A), we take the same normal forms as in [1]: $H(x,y)=\frac{y^2}{2}-\frac{1}{4}x^4+\frac{1}{2}x^2$. By [7], $Q'(h)>0$ for $h\in\Sigma=(0,1/4)$. It follows from Lemma 3.1 that $\sharp\{I(h)=0(h\in\Sigma)\}\leqslant 4$.

For case (B), we take the same normal form as in [1]:

$$H(x,y)=\frac{y^2}{2}-\frac{1}{4}x^4-\frac{\lambda-1}{3}x^3+\frac{\lambda}{2}x^2,\quad \lambda>1,\quad \Sigma=\Big(0,\frac{2\lambda+1}{12}\Big),$$

where $\Gamma_h\downarrow$ the center $(0,0)$ as $h\downarrow 0$. It follows from [1] that the number of monotone intervals of $\omega(h)$ is at most 2. Noting that $2b^2-9ac\neq 0$ and $I_0''(h)\neq 0$ for $h\in\Sigma$, by Lemma 3.3, we obtain that $\sharp\{I(h)=0(h\in\Sigma)\}\leqslant 6$.

For case (D), we take the same normal form as in [2]:

$$H(x,y)=\frac{y^2}{2}+\frac{1}{4}x^4+\frac{1}{3}x^3,\quad \Sigma=\Big(-\frac{1}{12},0\Big)\cup(0,+\infty),$$

where $\Gamma_h\downarrow$ the center $(-1,0)$ as $h\downarrow-\frac{1}{12}$. It follows from [2] that the number of monotone intervals of $\omega(h)$ is at most 3. Noting that $2b^2-9ac\neq 0$ and $I_0''(h)\neq 0$ for $h\in\Sigma$, by Lemma 3.3, we obtain that $\sharp\{I(h)=0(h\in\Sigma)\}\leqslant 6$.

For case (E), we take the same normal form as in [4]:

$$H(x,y)=\frac{y^2}{2}+\frac{1}{4}x^4+\frac{1-\lambda}{3}x^3-\frac{\lambda}{2}x^2,\quad \lambda\in(0,1]. \tag{3.25}$$

The Hamiltonian (3.25) has 3 critical values h_1, h_2 and 0, where

$$h_1=-\frac{1}{12}(2\lambda+1),\quad h_2=-\frac{1}{12}\lambda^3(\lambda+2),\quad h_1\leqslant h_2.$$

We denote the compact components of the level curves of H by, respectively,

$$\begin{aligned}\Gamma_h^L&=\{(x,y)|H(x,y)=h,\ h\in(h_1,0),x<0\},\\ \Gamma_h^R&=\{(x,y)|H(x,y)=h,\ h\in(h_2,0),x>0\},\\ \Gamma_h^{L,R}&=\{(x,y)|H(x,y)=h,\ h\in(0,+\infty)\}.\end{aligned}$$

Then $\Gamma_h^L\downarrow$ the center $(-1,0)$ as $h\downarrow h_1$, $\Gamma_h^R\downarrow$ the center $(\lambda,0)$ as $h\downarrow h_2$, and $\Gamma_h^{L,R}\downarrow\Gamma_0$ as $h\downarrow 0$, where Γ_0 is the double homoclinic loop through the saddle $(0,0)$ and surrounding $(-1,0)$ and $(\lambda,0)$.

If $\lambda=1$, then $b=0$. It follows from [7] that the number of monotone intervals of $Q(h)$ on $(-\frac{1}{4},0)\cup(0,+\infty)$ is 2. Noting that $2b^2-9ac\neq 0$ and $I_0''(h)\neq 0$ for $h\in\Sigma$, by Lemma 3.1, we obtain that $\sharp\{I(h)=0,h\in(h_1,0)\}\leqslant 8$.

For the case of $\lambda\in(0,1)$, it was proved in [4] that $\omega(h)$ is strictly increasing for $h\in(h_1,0)$. By [11], $I_0''(h)\neq 0$ for $h>0$. Noting that $2b^2-9ac\neq 0$ and we consider the related abelian integrals around just one center, by Lemma 3.3, we obtain that $\sharp\{I(h)=0,h\in(h_1,0)\}\leqslant 12$. This completes the proof.

4 Proof of Theorem 1.1 (iii)

The purpose of this section is to prove the following results.

Theorem 4.1. *Consider the perturbations of case* (B). *Then*

(i) *there exists $\lambda_0>1$ such that for $\lambda\geqslant\lambda_0$, (X_ε) has at most* 3 *limit cycles for arbitrary q_{ij} and $\varepsilon>0$ near the original homoclinic loop*;

(ii) *there exist $\lambda=\lambda^*$, $\varepsilon^*>0(|\ \varepsilon^*|\ll 1)$ and $q_{ij}^*\in\mathbb{R}$ such that the corresponding vector field (X_{ε^*}) has at least* 3 *limit cycles.*

We will use the same normal form as in [1], which is the case of $a=-1$, $b=1-\lambda$ and $c=\lambda$ with $\lambda>1$ in (1.4). Then system (X_ε) is

$$\begin{cases}\dot{x}=y,\\ \dot{y}=x(x+\lambda)(x-1)+\varepsilon(q_{01}+q_{11}x+q_{21}x^2+q_{03}y^2)y,\end{cases} \tag{4.1}$$

and

$$\Gamma_h=\{(x,y)|H(x,y)=h,0<h<h_1\},\ h_1=\frac{2\lambda+1}{12}.$$

The homoclinic loop Γ is determined by $H(x,y)=h_1$. It follows from [16] that for h near h_1, $I(h)$ can be expressed as

$$I(h)=c_1+c_2|h-h_1|\ln|h-h_1|+c_3|h-h_1|+c_4|h-h_1|^2\ln|h-h_1|+\cdots, \tag{4.2}$$

where

$$c_1 = I(h_1), \quad c_2 = \left(\frac{\partial P_3}{\partial x} + \frac{\partial Q_3}{\partial y}\right)\bigg|_{(1,0)}, \quad \text{if } c_1 = 0;$$
$$c_3 = \int_{-\infty}^{+\infty} \left(\frac{\partial P_3}{\partial x} + \frac{\partial Q_3}{\partial y}\right)\bigg|_{\Gamma} (x(t), y(t))dt, \quad \text{if } c_1 = c_2 = 0;$$
$$c_4 = V_1 \quad \text{if } c_1 = c_2 = c_3 = 0,$$

and c_4 is the first saddle quantity.

Theorem C[12]**.** *If $c_i = 0$ for $i = 1, 2, \ldots, k-1$ and $c_k \neq 0$, then (X_ε) has at most $k-1$ limit cycles near Γ.*

4.1 The calculations of c_1 and c_2

Noting that

$$I_i(h) = 2\int_{x_l(h)}^{x_r(h)} x^i y_+(x,h)\, dx, \quad h \in (0, h_1],\ i = 0, 1, 2,$$
$$x_l(h_1) = x^* = -\frac{2}{3}\lambda - \frac{1}{3} + \frac{\sqrt{2}}{3}\sqrt{(\lambda-1)(2\lambda+1)}, \quad y_+(x, h_1) = \frac{1}{\sqrt{6}}\sqrt{f(x)}, \tag{4.3}$$
$$f(x) = 3(x-1)^2(x-x^*)(x-\tilde{x}), \quad \tilde{x} = -\frac{1}{3}\left[2\lambda + 1 + \sqrt{2(\lambda-1)(2\lambda+1)}\right],$$

by straightforward calculations, we get $I_i(h_1) = a_{i1}v_1 + a_{i2}v_2 + a_{i3}v_3 + a_{i0},\ i = 0, 1, 2$, where

$$v_1 = \sqrt{2\lambda+2}, \quad v_2 = \ln(2\lambda + 4 + 3\sqrt{2\lambda+2}), \tag{4.4}$$
$$v_3 = \ln((\lambda-1)(2\lambda+1)), \tag{4.5}$$

$$a_{21} = \frac{\sqrt{2}}{405}(70\lambda^4 + 65\lambda^3 - 54\lambda^2 - 43\lambda + 16),$$
$$a_{22} = \frac{\sqrt{2}}{1701}(7\lambda + 5 + 3\sqrt{2})(-7\lambda - 5 + 3\sqrt{2})(\lambda-1)(2\lambda+1)^2,$$
$$a_{23} = -\frac{1}{2}a_{22}; \quad a_{20} = \frac{\sqrt{2}\ln 2}{486}(\lambda-1)(7\lambda^2 + 10\lambda + 1)(2\lambda+1)^2,$$
$$a_{11} = -\frac{\sqrt{2}}{54}(10\lambda^2 + 19\lambda + 10)(\lambda-1),$$
$$a_{12} = \frac{\sqrt{2}}{162}(5\lambda+7)(\lambda-1)(2\lambda+1)^2,$$
$$a_{13} = -\frac{1}{2}a_{12},$$
$$a_{10} = -\frac{\sqrt{2}\ln 2}{324}(5\lambda+7)(\lambda-1)(2\lambda+1)^2,$$
$$a_{01} = \frac{2\sqrt{2}}{9}(\lambda^2 + \lambda + 1),$$
$$a_{02} = -\frac{2\sqrt{2}}{27}(\lambda+2)(2\lambda+1)(\lambda-1),$$
$$a_{03} = -\frac{1}{2}a_{02},$$
$$a_{00} = \frac{\sqrt{2}\ln 2}{27}(\lambda-1)(2\lambda+1)(\lambda+2).$$

By (2.6), we get

$$\begin{aligned} c_1 =& I_0(h_1)q_{01} + I_2(h_1)q_{21} + \frac{1}{5}[(I_1(h_1) - I_2(h_1))\lambda^2 \\ &+ (2I_0(h_1) - I_1(h_1) - I_2(h_1))\lambda + (I_0(h_1) - I_2(h_1))]q_{03} + I_1(h_1)q_{11}. \end{aligned} \tag{4.6}$$

It is easy to see that

$$c_2 = q_{01} + q_{11} + q_{21}. \tag{4.7}$$

4.2 The calculations of c_3

It is difficult to express Γ as the function in variable t. We will calculate c_3 by the following way. By (4.2), if $c_1 = c_2 = 0$, we have

$$I'(h) = c_3 + 2c_4(h - h_1)\ln(h - h_1) + c_4(h - h_1) + \cdots ,$$

which yields that $c_3 = I'(h_1)$. Along Γ_0, we have $\frac{y^2}{2} - \frac{1}{4}x^4 - \frac{\lambda-1}{3}x^3 + \frac{\lambda}{2}x^2 = h_1$, which yields that $y \cdot \frac{\partial y}{\partial h_1} = 1$. Hence by (4.3),

$$I_i'(h_1) = 2\int_{x_l(h_1)}^{x_r(h_1)} \frac{x^i}{y_+(x,h_1)}dx = 2\sqrt{2}\int_{x^*}^{1} \frac{x^i}{(1-x)\sqrt{(x-x^*)(x-\tilde{x})}}dx, \tag{4.8}$$

where $x_r(h_1) = 1,\ x_l(h_1) = x^*$ are abscissa of the intersection points of Γ to the x-axis. Noting that

$$I'(h_1) = \alpha_1 I_0(h_1) + (\alpha_1 h_1 + \alpha_2)I_0'(h_1) + \beta I_1'(h_1) + \gamma I_2'(h_1),$$

and $I_0(h_1)$ is known, we should calculate

$$\Psi = (\alpha_1 h_1 + \alpha_2)I_0'(h_1) + \beta I_1' h_1) + \gamma I_2'(h_1). \tag{4.9}$$

Substituting (4.8) into (4.9), we get

$$\Psi = 2\sqrt{2}\int_{x^*}^{1} \frac{\alpha_1 h_1 + \alpha_2 + \beta x + \gamma x^2}{(1-x)\sqrt{(x-x^*)(x-\tilde{x})}}dx. \tag{4.10}$$

By Lemma 2.2,

$$\alpha_1 h_1 + \alpha_2 + \beta + \gamma = \left[\frac{bc}{5a} + \frac{b^2 - 3ac}{5a} + \frac{12}{5}h_1\right]q_{03}.$$

Noting that $a = -1$, $b = 1 - \lambda$, $c = \lambda$ and $c_2 = 0$, we get

$$\frac{bc}{5a} + \frac{b^2 - 3ac}{5a} + \frac{12}{5} \cdot \frac{2\lambda + 1}{12} \equiv 0$$

and hence

$$\alpha_1 h_1 + \alpha_2 + \beta x + \gamma x^2 = (x - 1)[\gamma x - (\alpha_1 h_1 + \alpha_2)].$$

So, (4.10) is a normal integral and

$$\begin{aligned} \Psi = \frac{\sqrt{2}}{15}\Big[& -30\sqrt{2\lambda+2}q_{21} + 10q_{21}\ln(4 + 2\lambda + 3\sqrt{2\lambda+2}) + 4q_{03}\ln(4 + 2\lambda + 3\sqrt{2\lambda+2}) \\ & -5\ln((\lambda-1)(2\lambda+1))q_{21} + 6\sqrt{2\lambda+2}q_{03} - 5\ln(2)q_{21} + 30q_{01}\ln(4 + 2\lambda + 3\sqrt{2\lambda+2}) \\ & -2q_{03}\ln((\lambda-1)(2\lambda+1)) - 2q_{03}\ln(2) - 15q_{01}\ln((\lambda-1)(2\lambda+1)) + 3\lambda^2\ln((\lambda-1) \\ & \cdot(2\lambda+1))q_{03} + 2\lambda^3\ln((\lambda-1)(2\lambda+1))q_{03} - 10\lambda\ln((\lambda-1)(2\lambda+1))q_{21} + 6\sqrt{2\lambda+2}q_{03}\lambda \end{aligned}$$

$$
\begin{aligned}
&+6\sqrt{2\lambda+2}q_{03}\lambda^2 - 3q_{03}\ln(2)\lambda - 3q_{03}\ln((\lambda-1)(2\lambda+1))\lambda + 3\lambda^2\ln(2)q_{03} \\
&+2\lambda^3\ln(2)q_{03} - 10\lambda\ln(2)q_{21} - 15q_{01}\ln(2) - 6q_{03}\lambda^2\ln(4+2\lambda+3\sqrt{2\lambda+2}) \\
&+20\lambda q_{21}\ln(4+2\lambda+3\sqrt{2\lambda+2}) - 4q_{03}\lambda^3\ln(4+2\lambda+3\sqrt{2\lambda+2}) \\
&+6q_{03}\lambda\ln(4+2\lambda+3\sqrt{2\lambda+2})\Big].
\end{aligned}
$$

For simplicity, let

$$b_1 = q_{01},\ b_2 = q_{21},\ b_3 = q_{03},\ b_4 = q_{11}, \tag{4.11}$$

then c_3 can be expressed as

$$
\begin{aligned}
&c_3 = I'(h_1) = 2\sqrt{6}(z_1b_1 + z_2b_2 + z_3b_3) - \frac{12}{5}I_0b_3, \qquad (4.12)\\
&z_1 = -\frac{\sqrt{3}}{6}(-2v_2 + v_3 + \ln 2);\\
&z_2 = -\frac{\sqrt{3}}{18}(6v_1 - (2+4\lambda)v_2 + (1+2\lambda)v_3 + (1+2\lambda)\ln 2);\\
&z_3 = \frac{\sqrt{3}}{15}\left[(\lambda^2+\lambda+1)v_1 - \frac{1}{3}(\lambda+2)(2\lambda+1)(\lambda-1)v_2 + \frac{1}{6}(\lambda+2)(2\lambda+1)(\lambda-1)v_3\right].
\end{aligned}
$$

4.3 The calculations of c_4

Lemma 4.1[13]. *For the vector field $\dot{x} = y + f(x,y),\ \dot{y} = x + g(x,y)$ with $f(0,0) = g(0,0)$, the first saddle quantity is given by*

$$
\begin{aligned}
V_1 =& [f_{xxx} - f_{xyy} + g_{xxy} - g_{yyy} + f_{xy}(f_{yy} - f_{xx}) \\
&+ g_{xy}(g_{yy} - g_{xx}) - f_{xx}g_{xx} + f_{yy}g_{yy}]|_{(0,0)}.
\end{aligned} \tag{4.13}
$$

Let $u = x - 1, v = y$, then (4.1) is reduced to

$$
\begin{cases}
\dot{u} = v,\\
\dot{v} = (u+1)(u+1+\lambda)u + \varepsilon[b_1 + b_4(u+1) + b_2(u+1)^2]v + \varepsilon b_3 v^3.
\end{cases} \tag{4.14}
$$

Since $c_2 = 0$, (4.14) can be rewritten as

$$
\begin{cases}
\dot{u} = v,\\
\dot{v} = (\lambda+1)u + (2+\lambda)u^2 + u^3 + \varepsilon(b_4u + 2b_2u + b_2u^2)v + \varepsilon b_3 v^3.
\end{cases} \tag{4.15}
$$

Let $\xi = u, \eta = \frac{1}{\sqrt{\lambda+1}}v$ and $\tau = \sqrt{\lambda+1}\,t$, then (4.15) is reduced to

$$\frac{d\xi}{d\tau} = \eta, \quad \frac{d\eta}{d\tau} = \xi + \frac{1}{\lambda+1}g(\xi, \sqrt{\lambda+1}\,\eta), \tag{4.16}$$

where

$$g(\xi, \sqrt{\lambda+1}\,\eta) = (2+\lambda)\xi^2 + \xi^3 + \varepsilon(b_4\xi + 2b_2\xi + b_2\xi^2)\sqrt{\lambda+1}\,\eta + \varepsilon b_3(\lambda+1)\sqrt{\lambda+1}\,\eta^3.$$

By (4.13), noting that $\varepsilon > 0$ and $\lambda > 1$, we get

$$c_4 = -2\frac{\lambda+2}{\lambda+1}b_4 + \left[2 - \frac{4(\lambda+2)}{\lambda+1}\right]b_2 + (-6\lambda - 6)b_3. \tag{4.17}$$

4.4 Proof of Theorem 4.1

(i) We can see from (4.6), (4.7), (4.12) and (4.17) that $c_i (i = 1, 2, 3, 4)$ are linearly dependent on b_1, b_2, b_3 and b_4. Let us consider a system of linear simultaneous equations

$$c_1 = c_2 = c_3 = c_4 = 0 \tag{4.18}$$

with variables b_1, b_2, b_3 and b_4. Denote by $\det(\lambda)$ the determinant of coefficient. By straightforward calculations, we get

$$\begin{aligned}
&4920750(s+1)^4 \det(\lambda)\\
&\quad = 33600(v_3 - 2v_2 + \ln 2)^2\lambda^{11} + 67200(v_3 - 2v_2 + \ln 2)(4\ln 2 + 3v_1 - 8v_2 + 4v_3)\lambda^{10}\\
&\qquad +[4166400v_2^2 - 4166400v_2v_3 - 3024000v_2v_1 + 1041600v_3^2 + 1512000v_3v_1 + 302400v_1^2\\
&\qquad -4166400\ln 2v_2 + 2083200\ln 2v_3 + 1512000\ln 2v_1 + 1041600(\ln 2)^2]\lambda^9\\
&\qquad +[8668800v_2^2 - 8668800v_2v_3 - 7570080v_2v_1 + 2167200v_3^2 + 3785040v_3v_1 + 2116800v_1^2\\
&\qquad -8668800\ln 2v_2 + 4334400\ln 2v_3 + 3785040\ln 2v_1 + 2167200(\ln 2)^2]\lambda^8\\
&\qquad +[6476400v_2^2 - 6476400v_2v_3 - 381600v_2v_1 + 1619100v_3^2 + 190800v_3v_1 + 2056320v_1^2\\
&\qquad -6476400\ln 2v_2 + 3238200\ln 2v_3 + 190800\ln 2v_1 + 1619100(\ln 2)^2]\lambda^7\\
&\qquad +[-8668800v_2^2 + 8668800v_2v_3 + 35751960v_2v_1 - 2167200v_3^2 - 17875980v_3v_1\\
&\qquad -15840360v_1^2 + 8668800\ln 2v_2 - 4334400\ln 2v_3 - 17875980\ln 2v_1 - 2167200(\ln 2)^2]\lambda^6\\
&\qquad +[-21294000v_2^2 + 21294000v_2v_3 + 77717520v_2v_1 - 5323500v_3^2 - 38858760v_3v_1\\
&\qquad -54152280v_1^2 + 21294000\ln 2v_2 - 10647000\ln 2v_3 - 38858760\ln 2v_1 - 5323500(\ln 2)^2]\lambda^5\\
&\qquad +[-12247200v_2^2 + 12247200v_2v_3 + 74007720v_2v_1 - 3061800v_3^2 - 37003860v_3v_1\\
&\qquad -75250080v_1^2 + 12247200\ln 2v_2 - 6123600\ln 2v_3 - 37003860\ln 2v_1 - 3061800(\ln 2)^2]\lambda^4\\
&\qquad +[5821200v_2^2 - 5821200v_2v_3 + 29966400v_2v_1 + 1455300v_3^2 - 14983200v_3v_1\\
&\qquad -55721520v_1^2 - 5821200\ln 2v_2 + 2910600\ln 2v_3 - 14983200\ln 2v_1 + 1455300(\ln 2)^2]\lambda^3\\
&\qquad +[10449600v_2^2 - 10449600v_2v_3 - 425880v_2v_1 + 2612400v_3^2 + 212940v_3v_1\\
&\qquad -24480360v_1^2 - 10449600\ln 2v_2 + 5224800\ln 2v_3 + 212940\ln 2v_1 + 2612400(\ln 2)^2]\lambda^2\\
&\qquad +[4695600v_2^2 - 4695600v_2v_3 - 3501360v_2v_1 + 1173900v_3^2 + 1750680v_3v_1\\
&\qquad -7258680v_1^2 - 4695600\ln 2v_2 + 2347800\ln 2v_3 + 1750680\ln 2v_1 + 1173900(\ln 2)^2]\lambda\\
&\qquad +722400v_2^2 - 722400v_2v_3 - 583560v_2v_1 + 180600v_3^2 + 291780v_3v_1 - 1319760v_1^2\\
&\qquad -722400\ln 2v_2 + 361200\ln 2v_3 + 291780\ln 2v_1 + 180600(\ln 2)^2.
\end{aligned}$$

Let

$$4920750(s+1)^4 \det(\lambda) := \sum_{i=0}^{11} \eta_i\lambda^i = \lambda^{11}(\eta_{11} + \sum_{i=0}^{10} \frac{\eta_i\lambda^i}{\lambda^{11}}),$$

where $\eta_{11} = 33600(v_3 - 2v_2 + \ln 2)^2$. By (4.4) and (4.5), we can get that $\lim_{\lambda\to+\infty}(v_3 - 2v_2) = 0$, and

$$\lim_{\lambda\to+\infty} \eta_{11} = 33600(\ln 2)^2, \qquad \lim_{\lambda\to+\infty} \frac{\eta_i\lambda^i}{\lambda^{11}} = 0 \ (i = 0, 1, \dots, 10).$$

Hence, there exists $\lambda_0 > 1$ such that $\det(\lambda) > 0$ for $\lambda > \lambda_0$. Noting that $(b_1, b_2, b_3, b_4) \neq 0$, we can conclude that for $\lambda > \lambda_0$, (X_ε) has at most 3 limit cycles near the homoclinic loop Γ.

(ii) Let $\lambda^* = 7$. Then we can get

$$\begin{aligned}
c_1 &= -32.892b_1 - 1389.4b_2 + 17526b_3 + 211.96b_4,\\
c_2 &= -41.24127447b_2 + 532.8347318b_3 + 7.444120151b_4, \text{ if } c_1 = 0,\\
c_3 &= 53.676934b_3 - 3.01069714b_4, \text{ if } c_1 = c_2 = 0,\\
c_4 &= -7.205209670b_4, \text{ if } c_1 = c_2 = c_3 = 0.
\end{aligned}$$

There are exactly 3 limit cycles bifurcated from Γ if

$$c_4 < 0,\ c_3 > 0,\ c_2 < 0, c_1 < 0,\ |c_1| \ll |c_2| \ll |c_3| \ll |c_4|. \tag{4.19}$$

It is easy to prove that there are b_i^*, $i = 1, 2, 3, 4$, such that (4.19) holds. So (X_ε) with $(b_1, b_2, b_3, b_4) = (b_1^*, b_2^*, b_3^*, b_4^*)$ has at least 3 limit cycles near Γ for $0 < \varepsilon \ll 1$. This completes the proof.

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