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Canonical formulation of spin in general relativity*

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The present article aims at an extension of the canonical formalism of Arnowitt, Deser, and Misner from self-gravitating point-masses to objects with spin. This would allow interesting applications, e.g., within the post-Newtonian (PN) approximation. The extension succeeded via an action approach to linear order in the single spins of the objects without restriction to any further approximation. An order-by-order construction within the PN approximation is possible and performed to the formal 3.5PN order as a verification. In principle both approaches are applicable to higher orders in spin. The PN next-to-leading order spin(1)-spin(1) level was tackled, modeling the spin-induced quadrupole deformation by a single parameter. All spin-dependent Hamiltonians for rapidly rotating bodies up to and including 3PN are calculated.

Contents

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1 Introduction

Though general relativity has seen and passed many experimental tests, one of its most fascinating predictions, namely gravitational waves, has not been observed *directly*. However, observations of certain binary pulsar signals are in good agreement with the energy loss predicted by general relativity due to gravitational waves, see, e.g., [1]. This indirect observation of gravitational waves originates from Hulse and Taylor (first found for the binary pulsar PSR B1913+16) and was awarded the Nobel Prize in 1993. Nowadays there is less doubt that gravitational waves exist, and one aims at a direct observation with assiduous efforts, both by experiments on Earth, e.g., LIGO, VIRGO, GEO 600, and by the future space mission LISA [2]. The direct measurement of gravitational waves is not only interesting, but would furthermore open up an entirely new spectrum for astronomical observations. Such gravitational wave astronomy is expected to have great impact on astrophysics and fundamental physics [2], possibly starting a new era in these fields.

Beside the experimental challenge of measuring extraordinarily small relative changes in length ($\lesssim 10^{-21}$ detectable by now) there are important problems to be solved on the theoretical side in order to successfully establish the new field of gravitational wave astronomy. The theoretical challenge lies within the area of data analysis, for both the noise dominated [3] and signal dominated [4] cases. An accurate understanding and knowledge of the expected gravitational wave signals is a key ingredient to allow faithful astronomical or astrophysical statements from the data analysis process. An appealing source for gravitational waves is the inspiral and merger of two compact objects, like black holes and neutron stars. The advantage of this kind of source is its quite periodic behavior, which can be studied over long periods of time. However, minute changes in frequency and amplitude of the gravitational waves need to be predicted in an accurate way. While fully numerical methods are ideal to study the very late inspiral (or plunge) and merger phases of compact objects, the post-Newtonian approximation to general relativity provides a good analytic handle on the inspiral phase and can give accurate predictions over many orbits. The post-Newtonian approximation was pushed to high orders for nonspinning objects, see, e.g., [5], and it is desirable to catch up to these orders for the spinning case.

A successful and efficient way to calculate the *conservative* part of the dynamics of two compact objects within the post-Newtonian approximation is based on the canonical formalism of Arnowitt, Deser, and Misner (ADM). However, this formalism has been coupled so far to nonspinning point-like objects only. The main goal of the present work is to extend this coupling to spinning objects. Not only this is useful for subsequent applications, but an interesting problem as such (though rather mathematical). To linear order in spin the problem is solved using an action approach, similar to a treatment of spin-$\frac{1}{2}$ Dirac fields coupled to gravity given by Kibble [6]. Further, an order-by-order construction of the canonical formalism with spin is given as a check. This construction is based on consistency conditions on the formalism. In particular it is sufficient to rely on a certain form of total linear and angular momentum expressed in terms of canonical variables in order to reproduce the result of the action approach to next-to-next-to-leading order in the post-Newtonian approximation. The assumed form of total linear and angular momentum, i.e., the generators of translations and rotations, guarantees that a great part of the global Poincaré algebra is fulfilled. The connection to the action approach is given by Noether's theorem on conserved quantities.

Higher orders in spin correspond to quadrupole and even higher multipole corrections. Both the action approach and an order-by-order construction are in principle applicable to canonical formulations at higher orders in spin. However, only the next-to-leading order spin(1)-spin(1) level will be tackled here. This requires a modeling of the spin-induced quadrupole deformation, described by a single parameter for each object. This parameter is not only distinct for black holes and neutron stars, for the latter kind of object it also depends on the assumed equation of state or on other details of a particular theoretical neutron star model. If gravitational wave astronomy becomes available with a high enough precision in the future, one may hope to *measure* this (and maybe other) neutron star parameter.

The results obtained here within the post-Newtonian approximation cover the next-to-leading order spin(1)-spin(2) and spin(1)-spin(1) conservative Hamiltonians. The conservative next-to-leading order spin-orbit Hamiltonian was reproduced. For maximally rotating bodies all Hamiltonians up to and including the third post-Newtonian order are now known. A maximal rotating body is defined to have a dimensionless spin (i.e., rescaled by the mass of the object and identical to the dimensionless Kerr parameter for black holes) of value one, corresponding to an extremal Kerr black hole. Notice that millisecond pulsars (or neutron stars) and black holes can easily have dimensionless spins bigger than $\frac{1}{10}$ (a rough approximation for the sun yields $\frac{1}{5}$ [7]). Thus spins close to maximal ones are expected to be astrophysically relevant. In this case the next-to-leading order spin Hamiltonians obtained here are needed for an accurate description of the dynamics during the inspiral phase. It was found recently in [8] that spin effects as such and in particular the orientations of the spins have a big impact on the event rates expected in detectors, especially when spins are close to maximum.

If the fourth post-Newtonian order Hamiltonian for nonspinning objects could be obtained in the future, the spin Hamiltonians calculated here would be applicable to an even larger class of binaries (with smaller spins). Notice that the effective one-body approach for nonspinning objects, see, e.g., [9], is able to cover such higher post-Newtonian orders by calibration to numerical relativity and further provides predictions for the full waveforms, including merger and ringdown phases. An extension of the effective one-body approach to spinning objects is possible [10]. Subsequent implementation of higher order spin Hamiltonians seems to be interesting, and was already performed for the next-to-leading order spin-orbit Hamiltonian [11].

Now the organization of the present article is given, with references to relevant published work of the author for certain sections (for a short review see also [12]). In Sect. 2 spinning objects in special and general relativity are reviewed. Further, an overview of canonical formulations of general relativity is given, with emphasis on the ADM formalism and coupling to nonspinning objects. In Sect. 3 the action approach to the canonical formulation of self-gravitating spinning objects to linear order in spin is performed [13]. An order-by-order construction based on consistency considerations is performed to next-to-next-to-leading order in Sect. 4 [14, 15] as a check. In Sect. 5 first general quadrupole corrections to the equations of motion and the stress-energy tensor are given [16] and then used to extend the canonical formalism to spin-induced quadrupole deformation at next-to-leading order [17–19]. As an application of the formalism, conservative Hamiltonians at next-to-leading order are derived in Sect. 6. These are the spin-orbit [14] (derived earlier by Damour, Jaranowski, and Schäfer), spin(1)-spin(2) [20], and spin(1)-spin(1) Hamiltonians, the latter was first derived for black holes [17, 18] and later for compact objects in general (including neutron stars) [19]. Finally, conclusions and outlook are given in Sect. 7.

Lower case Latin indices from the beginning of the alphabet (a, b, $\dots$) label the individual spinning objects and then consequently take on values from one to the number of objects. Three different frames are utilized in this article, denoted by different indices. Greek indices (α, μ, $\dots$) refer to the coordinate frame, upper case Latin indices from the middle of the alphabet (I, J, $\dots$) belong to a local Lorentz frame, and upper case Latin indices from the beginning of the alphabet (A, B, $\dots$) the denote the so called body-fixed Lorentz frame. Lower case Latin indices from the middle of the alphabet (i, j, $\dots$) are used for the spatial part of the mentioned frames and are running through $i = 1, 2, 3$. In order to distinguish the three frames when splitting them into spatial and time part, we write $a = (0), (i)$ for Lorentz indices (or $a = (0), (1), (2), (3)$ in more detail), $A = [0], [i]$ for the body-fixed frame, and $\mu = 0, i$ for the coordinate frame. Indices appearing twice in a product are implicitly summed over its index range, except for label indices of the objects. Round and square brackets are also used for index symmetrization and antisymmetrization, respectively, e.g., $A^{(\mu\nu)} \equiv \frac{1}{2}(A^{\mu\nu} + A^{\nu\mu})$. Partial derivatives are denoted by ∂_μ or by a comma as an index $_{,\mu}$. Similarly, the 4-dimensional covariant derivative is written as $_{||\mu}$ and the induced 3-dimensional one as $_{;i}$. A 3-dimensional vector is also written in boldface, e.g., $\mathbf{x}$. The signature of spacetime is taken to be $+2$. Units are such that the speed of light c and the gravitational constant G are equal to one. Other symbols are defined in this article on their first occurrence. For convenience also a summary of defined symbols is given in the appendix.

2 Preliminaries

This section gives a short review of the achievements regarding spin in the theory of relativity as well as the canonical formulation of general relativity. Emphasis is put on the problems to be solved if one aims at a canonical formulation of self-gravitating spinning objects in the pole-dipole approximation.

2.1 Spin in special relativity

Spin already has very interesting properties in special relativity. Its canonical structure is obtained here as a consequence of the Poincaré algebra by introducing the spin as a specific part of the total angular momentum.

2.1.1 Center, spin, and mass dipole

The 4-dimensional total linear momentum P^μ and total angular momentum $J^{\mu\nu} = -J^{\nu\mu}$ of a physical system are conserved quantities due to Poincaré invariance. The 4-dimensional total spin tensor $S^{\mu\nu}$ can then be defined by

$$J^{\mu\nu} = Z^\mu P^\nu - P^\mu Z^\nu + S^{\mu\nu} . \tag{2.1}$$

That is, spin is the difference of total angular momentum and its orbital part. However, a different choice for the yet arbitrary center Z^μ of the system will result in a different spin $S^{\mu\nu}$ (with $J^{\mu\nu}$ being unchanged). This just expresses the dependence of angular momenta on the choice of a reference point. Separating time and space components

$$J^{ij} = Z^i P^j - P^i Z^j + S^{ij} , \qquad J^{i0} = Z^i E - P^i t + S^{i0} , \tag{2.2}$$

one infers that the spin transforms as

$$S^{ij} \to S^{ij} + \delta Z^i P^j - P^i \delta Z^j , \qquad S^{i0} \to S^{i0} + \delta Z^i E , \tag{2.3}$$

under a change of the center $Z^i \to Z^i - \delta Z^i$. $E \equiv P^0$ is the total energy and $t \equiv Z^0$ the time coordinate. Notice that J^{i0} is the total mass dipole of the system at $t = 0$ relative to the coordinate origin, so (2.2) tells us that S^{i0} is the mass dipole relative to the center Z^i. This explains the transformation property (2.3). One may also describe the 3-dimensional spin S^{ij} as the flow dipole and $S^{\mu\nu}$ as the 4-dimensional dipole moment of the system relative to the center Z^i.

By its definition (2.1), $S^{\mu\nu}$ transforms as a tensor under Lorentz boosts, with interesting consequences. In classical mechanics the center of mass, i.e., the center for which the mass dipole vanishes, is independent of the reference frame. In special relativity such a center can in general not be found. Under a Lorentz boost all components of $S^{\mu\nu}$ transform, so if the mass dipole S^{i0} vanishes in one reference frame, it will only be zero in all others if the system has no spin, $S^{\mu\nu} = 0$. A nice graphic interpretation is given by Fig. 1. Notice that a spinning system in special relativity has a minimal extension of the order S/M orthogonal to the axis of rotation [21, 22]. Here S is the spin length, $2S^2 = S^{\mu\nu}S_{\mu\nu}$, and M is the rest mass of the system, $M^2 = -P^\mu P_\mu$. In general relativity, S/M is the radius coordinate of the ring singularity of Kerr spacetime [23].

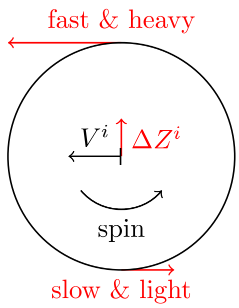

Fig. 1 (online colour at: www.ann-phys.org) If a spinning spherical symmetric object moves with a velocity V^i to the left, its upper hemisphere moves faster with respect to the reference system than its lower hemisphere. Thus the upper hemisphere has a higher relativistic mass than the lower one – the object acquires a mass dipole $E\Delta Z^i$ [21].

However, by virtue of (2.3) one can always choose the center Z^i such that the mass dipole S^{i0} vanishes in one specific reference frame characterized by a timelike vector f_μ. That is, the center is then the center of mass as observed in this frame. It holds

$$S^{\mu\nu} f_\nu = 0 , \tag{2.4}$$

which is the so called spin supplementary condition. This condition fixes the center and ensures that the spin tensor $S^{\mu\nu}$ has three independent components only. Basically three important such conditions can be found in the literature [21, 24],

$$f_\mu = P_\mu\,, \quad \text{or} \quad S^{\mu\nu}P_\nu = 0\,, \tag{2.5}$$

$$f_\mu = -\delta^0_\mu\,, \quad \text{or} \quad \tilde{S}^{\mu 0} = 0\,, \tag{2.6}$$

$$f_\mu = P_\mu - M\delta^0_\mu\,, \quad \text{or} \quad \hat{S}^{\mu\nu}P_\nu - M\hat{S}^{\mu 0} = 0\,. \tag{2.7}$$

In the following, we will indicate center and spin belonging to the second condition [22, 24] by a tilde, $\tilde{Z}^i$ and $\tilde{S}^{\mu\nu}$, a hat relates to the third condition [24, 25], $\hat{Z}^i$ and $\hat{S}^{\mu\nu}$, while center and spin of the first condition [26] are just denoted by Z^i and $S^{\mu\nu}$. We call Z^i the center of inertia, $\tilde{Z}^i$ the center of mass, and $\hat{Z}^i$ the center of spin [21]. Notice that the first condition is manifestly covariant, and is called covariant spin supplementary condition here. A different covariant condition is discussed in Sect. 2.2. The third condition is called canonical spin supplementary condition, which will be explained in the following.

2.1.2 Poincaré algebra

The Poincaré group is one of the most important groups in physics. Its generators P^μ and $J^{\mu\nu}$ obey the Poisson bracket realization of the well-known Poincaré algebra

$$\{P^\mu, P^\nu\} = 0\,, \qquad \{P^\mu, J^{\rho\sigma}\} = -\eta^{\mu\rho}P^\sigma + \eta^{\mu\sigma}P^\rho\,, \tag{2.8}$$

$$\{J^{\mu\nu}, J^{\rho\sigma}\} = -\eta^{\nu\rho}J^{\mu\sigma} + \eta^{\mu\rho}J^{\nu\sigma} + \eta^{\sigma\mu}J^{\rho\nu} - \eta^{\sigma\nu}J^{\rho\mu}\,, \tag{2.9}$$

where $\eta^{\mu\nu}$ is the Minkowski metric. Splitting space and time one gets, see, e.g, [27],

$$\{P_i, P_j\} = 0, \quad \{P_i, E\} = 0, \quad \{J_i, E\} = 0, \quad \{G_i, P_j\} = E\delta_{ij}, \quad \{G_i, E\} = P_i\,, \tag{2.10}$$

$$\{J_i, P_j\} = \epsilon_{ijk}P_k, \quad \{J_i, J_j\} = \epsilon_{ijk}J_k, \quad \{J_i, G_j\} = \epsilon_{ijk}G_k, \quad \{G_i, G_j\} = -\epsilon_{ijk}J_k, \tag{2.11}$$

with the total angular momentum vector $J_i = \frac{1}{2}\epsilon_{ijk}J^{jk}$ and the 3-dimensional Levi-Civita symbol ϵ_{ijk}. The boost vector J^{i0} has an explicit dependence on time t, which was split off as

$$J^{i0} = G^i - P^i t\,. \tag{2.12}$$

This defines the vector G^i, which is related to the spin supplementary condition $\tilde{S}^{\mu 0} = 0$ with center of mass $\tilde{Z}^i$ by $G^i = \tilde{Z}^i E$, cf. (2.2).

Notice that in general relativity total linear and angular momentum can be defined for asymptotically flat spacetimes as global quantities by certain surface integrals. In this case all considerations of this and the following section remain valid in full general relativity, see Sect. 4.1.3.

2.1.3 Canonical structure

Using $\tilde{Z}^i = G^i/E$, $\tilde{S}_{ij} = J_{ij} - \tilde{Z}^i P_j + P_i \tilde{Z}^j$ and the Poincaré algebra (2.10, 2.11), the Poisson brackets between P_i, $\tilde{Z}^i$, and $\tilde{S}_{ij}$ follow as

$$\{\tilde{Z}^i, P_j\} = \delta_{ij}\,, \qquad \{\tilde{Z}^i, \tilde{Z}^j\} = -\frac{\tilde{S}_{ij}}{E^2}\,, \qquad \{\tilde{S}_{ij}, \tilde{Z}^k\} = \frac{P_i\tilde{S}_{kj}}{E^2} + \frac{P_j\tilde{S}_{ik}}{E^2}\,, \tag{2.13}$$

$$\{\tilde{S}_{ij}, \tilde{S}_{kl}\} = \mathcal{P}_{ki}\tilde{S}_{jl} - \mathcal{P}_{kj}\tilde{S}_{il} - \mathcal{P}_{li}\tilde{S}_{jk} + \mathcal{P}_{lj}\tilde{S}_{ik}\,, \tag{2.14}$$

all other zero, where

$$\mathcal{P}_{ij} = \delta_{ij} - \frac{P_i P_j}{E^2} \,, \qquad \mathcal{P}^{-1}_{ij} = \delta_{ij} + \frac{P_i P_j}{M^2} \,, \tag{2.15}$$

and δ_{ij} is the Kronecker symbol.

Now we proceed to the canonical spin supplementary condition (2.7), which can be written as $(E + M)\hat{S}^{i0} = \hat{S}^{ij} P_j$. From (2.3) and $\tilde{S}^{i0} = 0$ we get

$$\hat{Z}^i - \tilde{Z}^i = \delta Z^i = -\frac{\hat{S}^{i0}}{E} = \frac{P_k \hat{S}_{ki}}{E(E+M)} \,. \tag{2.16}$$

Having δZ^i, Eq. (2.3) relates $\hat{S}^{ij}$ and $\tilde{S}^{ij}$ by

$$\tilde{S}_{ij} = \hat{S}_{ij} + \frac{P_j P_k \hat{S}_{ki}}{E(E+M)} - \frac{P_i P_k \hat{S}_{kj}}{E(E+M)} \,. \tag{2.17}$$

Contraction with P_i leads to $E P_i \tilde{S}_{ij} = M P_i \hat{S}_{ij}$. Finally, in terms of $\tilde{S}_{ij}$ one has

$$\hat{Z}^i = \tilde{Z}^i + \frac{P_k \tilde{S}_{ki}}{M(E+M)} \,, \qquad \hat{S}_{ij} = \tilde{S}_{ij} + \frac{P_i P_k \tilde{S}_{kj}}{M(E+M)} - \frac{P_j P_k \tilde{S}_{ki}}{M(E+M)} \,. \tag{2.18}$$

The Poisson brackets (2.13, 2.14) transform into[1]

$$\{\hat{Z}^i, P_j\} = \delta_{ij} \,, \qquad \{\hat{S}_{ij}, \hat{S}_{kl}\} = \delta_{ik}\hat{S}_{jl} - \delta_{jk}\hat{S}_{il} - \delta_{il}\hat{S}_{jk} + \delta_{jl}\hat{S}_{ik} \,, \tag{2.19}$$

all other zero. Thus $\hat{Z}^i$, P_j, and $\hat{S}_{ij}$ are *canonical variables*. This realization is due to Pryce [24, 28]. Newton and Wigner further showed that $\hat{Z}^i$ is the only center with this property [25].

Similarly, we can proceed to the covariant spin supplementary condition (2.5) by

$$Z^i = \hat{Z}^i + \frac{P_k \hat{S}_{ki}}{M(E+M)} \,, \qquad S_{ij} = \hat{S}_{ij} + \frac{P_i P_k \hat{S}_{kj}}{M(E+M)} - \frac{P_j P_k \hat{S}_{ki}}{M(E+M)} \,, \tag{2.20}$$

and find the Poisson brackets,

$$\{Z^i, Z^j\} = \mathcal{P}_{ik}\mathcal{P}_{jl}\frac{S_{kl}}{M^2} \,, \qquad \{S_{ij}, Z^k\} = \frac{\mathcal{P}_{im}\mathcal{P}_{jn}}{M^2}(P_m S_{nk} + P_n S_{km}) \,, \tag{2.21}$$

$$\{Z^i, P_j\} = \delta_{ij} \,, \qquad \{S_{ij}, S_{kl}\} = \mathcal{P}^{-1}_{ki} S_{jl} - \mathcal{P}^{-1}_{kj} S_{il} - \mathcal{P}^{-1}_{li} S_{jk} + \mathcal{P}^{-1}_{lj} S_{ik} \,, \tag{2.22}$$

all other zero.

To conclude, there are several possibilities for spin supplementary conditions and centers, however, only (2.7) leads to canonical variables, (2.19). This is an important fact for a canonical formulation of spin in general relativity.

2.2 Spin in general relativity

It is well-known that spin in general relativity leads to certain gravitomagnetic effects, see, e.g., [29]. In this section the pole-dipole approximation for compact objects is introduced, providing an analytic description of spin in general relativity.

[1] Notice that Poisson brackets with M were calculated according to its definition $M^2 = E^2 - P_i P_i$.

2.2.1 Gravitational skeleton

In electrostatics, the multipole approximation of a charge density ρ,

$$\rho(\mathbf{x}) = \left(q - q^i \partial_i + \frac{1}{2!} q^{ij} \partial_i \partial_j - \dots \right) \delta(\mathbf{x}) \,, \tag{2.23}$$

can be obtained from a Taylor series of its Fourier transform in the form

$$\rho(\mathbf{k}) = \left(q + i q^i k_i + \frac{1}{2!} i^2 q^{ij} k_i k_j + \dots \right) (2\pi)^{-3/2} \,, \tag{2.24}$$

by the well-known transition formulas for the Dirac delta distribution $\delta(\mathbf{x}) \leftrightarrow (2\pi)^{-3/2}$ and partial coordinate derivative $\partial_i \leftrightarrow -ik_i$. Here $\mathbf{x} = (x^i)$ are the spatial coordinates and $\mathbf{k} = (k_i)$ the corresponding ones in Fourier space. The quantities q, q^i, and q^{ij} are the electric monopole, dipole, and quadrupole. The potential ϕ follows as

$$\phi = -4\pi \Delta^{-1} \rho = \left(q - q^i \partial_i + \frac{1}{2!} q^{ij} \partial_i \partial_j - \dots \right) \frac{1}{|\mathbf{x}|} \,, \tag{2.25}$$

where $\Delta = \partial_i \partial_i$ is the Laplacian and Δ^{-1} its inverse operator (with the usual boundary conditions). In most textbooks, the multipole approximation is derived directly for the potential or the field. Notice that the multipole approximation breaks down at high values of $\mathbf{k}$, i.e., in the ultraviolet, or at small values of $\mathbf{x}$ in the potential. This is the reason for the divergent self-energy of the approximated charge density (2.23).

Now the multipole approximation is applied to the stress-energy tensor $T^{\mu\nu}$. As it is desirable to have a manifestly covariant approximation scheme, we write

$$\sqrt{-g} T^{\mu\nu} = \int \mathrm{d}\tau \left[t^{\mu\nu} \delta_{(4)} - (t^{\mu\nu\alpha} \delta_{(4)})_{||\alpha} + \frac{1}{2!} (t^{\mu\nu\alpha\beta} \delta_{(4)})_{||(\alpha\beta)} - \dots \right] . \tag{2.26}$$

Here τ is the proper time of a representative worldline $z^\rho(\tau)$, g the determinant of the 4-dimensional metric $g_{\mu\nu}$, $\delta_{(4)} = \delta(x^\rho - z^\rho(\tau))$, and $t^{\mu\nu\cdots}$ are 4-dimensional covariant multipole moments. If one performs the τ integration in (2.26) by eliminating the time part of $\delta_{(4)}$ and writes the covariant derivatives as partial derivatives and Christoffel symbols, then (2.26) indeed takes on the form of (2.23). Equation (2.26) in substance is Mathisson's *gravitational skeleton* [30], but in the form given by W. M. Tulczyjew [31]. Interestingly enough Mathisson unknowably used a test-function formulation of the delta distribution, years before this formulation was used by Laurent Schwartz for his mathematically rigorous *Théorie des Distributions* [32].

The divergent self-interactions already present in electrostatics become more severe if the field equations are nonlinear. If the distributional stress-energy tensor (2.26) is used as a source for a nonlinear field equation, products of distributions will appear, which lack a mathematical definition. However, this problem can be overcome, as in quantum field theory, by a regularization and renormalization program. In particular, dimensional regularization [33] is most useful for theories involving gauge freedoms, like general relativity. Dimensional regularization has been employed successfully in post-Newtonian calculations [34–36] to a high order of nonlinearity. However, many treatments of multipole approximations in general relativity avoid these problems by considering (2.26) for test bodies only, which by definition are neglected as a source of the gravitational field.

The relation between source multipoles related to $T^{\mu\nu}$ used here and field multipoles [37] was considered in [38]. Only for linear theories like electrostatics this relation is straightforward.

2.2.2 Pole-dipole approximation

The stress-energy tensor (2.26) must fulfill

$$T^{\mu\nu}{}_{||\nu} = 0 \,. \tag{2.27}$$

This corresponds to Mathisson's *variational equations of mechanics* [30] and imposes certain conditions on the multipole moments. In the pole-dipole approximation only monopole $t^{\mu\nu}$ and dipole $t^{\mu\nu\alpha}$ are kept in (2.26). Evaluating (2.27) one sees that $t^{\mu\nu}$ and $t^{\mu\nu\alpha}$ can be expressed in terms of a vector p^μ and an antisymmetric tensor $S^{\mu\nu}$, which have to fulfill the dynamic equations

$$\frac{\mathrm{D}S^{\mu\nu}}{\mathrm{d}\tau} = 2p^{[\mu}u^{\nu]}\,, \qquad \frac{\mathrm{D}p_\mu}{\mathrm{d}\tau} = -\frac{1}{2}R^{(4)}_{\mu\rho\beta\alpha}u^\rho S^{\beta\alpha}\,, \tag{2.28}$$

with $u^\mu = \frac{\mathrm{d}z^\mu}{\mathrm{d}\tau}$, D the 4-dimensional covariant differential, and $R^{(4)}_{\mu\rho\beta\alpha}$ the 4-dimensional Riemann tensor defined by

$$a_{\mu||\alpha\beta} - a_{\mu||\beta\alpha} = R^{(4)}_{\nu\mu\alpha\beta}a^\nu\,, \tag{2.29}$$

for an arbitrary a_μ. The stress-energy tensor can be written as

$$\sqrt{-g}T^{\mu\nu} = \int \mathrm{d}\tau \left[u^{(\mu}p^{\nu)}\delta_{(4)} - \left(S^{\alpha(\mu}u^{\nu)}\delta_{(4)}\right)_{||\alpha}\right]. \tag{2.30}$$

p^μ and $S^{\mu\nu}$ are the linear momentum and spin of the object and now play the role of monopole and dipole moment. Their equations of motion were already derived by Mathisson [30] within his manifestly covariant formalism, albeit restricted to a specific spin supplementary condition. In the general form (2.28) they were first given by Papapetrou [39], however, his method was not manifestly covariant. W. M. Tulczyjew gave a derivation of (2.28) as well as of the stress-energy tensor (2.30) in a manifestly covariant way [31], using essentially Mathisson's method. Further important rederivations have been performed in [40, 41]. Higher multipole corrections will be discussed in Sect. 5.1.

Obviously a spinning object in general relativity does not follow a geodesic. For test bodies this effect can be studied numerically, see, e.g., [42, 43]. Further, without giving a relation between p^μ and u^μ, the system of Eqs. (2.28) is not closed.

2.2.3 Spin supplementary condition

A spin supplementary condition (2.4) must be preserved in time. Using (2.28) this leads to a relation between p^μ and u^μ [44],

$$p^\mu = \frac{1}{-f_\alpha u^\alpha}\left(-f_\nu p^\nu u^\mu + S^{\mu\nu}\frac{\mathrm{D}f_\nu}{\mathrm{d}\tau}\right)\,, \tag{2.31}$$

and thus, for a suitable f_ν, closes the system of Eqs. (2.28). A good spin supplementary condition is the covariant one,

$$S^{\mu\nu}p_\nu = 0\,, \tag{2.32}$$

or $f_\mu = p_\mu$, which has been suggested in the context of general relativity in [31]. Indeed, this condition guarantees existence and uniqueness of a corresponding worldline $z^\rho(\tau)$ [45]. The mass quantity m, $p_\mu p^\mu = -m^2$, and the spin length S, $2S^2 = S_{\mu\nu}S^{\mu\nu}$, are conserved for this condition. A covariant condition has the advantage that the relation between p^μ and u^μ (2.31) is manifestly covariant. However, also the noncovariant condition $\tilde{S}^{\mu 0} = 0$ was applied in general relativity [46].

A different covariant condition is given by

$$S^{\mu\nu}u_\nu = 0\,, \tag{2.33}$$

or $f_\mu = u_\mu$, which was used in both special [47] and general relativity [30, 48]. While there are no serious objections to use this condition, as it closes the system of Eqs. (2.28), it has some features which are usually not wanted. The condition (2.33) does not uniquely specify a worldline. Instead, the worldline depends on the choice of initial conditions and in general performs a kind of classical *Zitterbewegung* around the worldline defined by (2.32), see [31, 43]. As quadrupole corrections are needed to describe a black hole at the quadratic level in spin [37], we will only consider the pole-dipole approximation at linear order in spin here. Then the conditions (2.32) and (2.33) are fully equivalent and it holds $p_\mu = m u_\mu$.

As a generalization of the canonical spin supplementary condition (2.7) to general relativity one could take

$$\hat{S}^{\mu\nu} p_\nu + m \hat{S}^{\mu\nu} n_\nu = 0 \,, \tag{2.34}$$

with some timelike unit vector n_ν. However, it needs to be proven if or under which conditions (2.34) leads to canonical variables.

Finally, contraction of the first relation in (2.28) with u_ν leads to the well-known formula

$$p^\mu = -u_\nu p^\nu u^\mu - \frac{\mathrm{D}(S^{\mu\nu})}{\mathrm{d}\tau} u_\nu \,. \tag{2.35}$$

This relation, however, does not close the system of Eqs. (2.28), it just is a component of (2.28).

2.3 Canonical formulation of general relativity

In this section the canonical formalism of ADM [49, 50] is introduced. Possible couplings to matter are reviewed, and point-masses are treated in detail. Finally alternatives to the ADM approach are discussed.

2.3.1 The ADM formalism

The Einstein-Hilbert action of general relativity W_G is given by a spacetime integral over the Lagrangian density $\mathcal{L}_G$ as

$$W_G[g_{\mu\nu}] = \int \mathrm{d}^4x \, \mathcal{L}_G \,, \qquad \mathcal{L}_G = \frac{1}{16\pi} \sqrt{-g} R^{(4)} \,, \tag{2.36}$$

where $R^{(4)}$ is the 4-dimensional Ricci scalar. Alternatively the action can be varied with respect to the tetrad field $e_{I\mu}$ instead of $g_{\mu\nu}$, see Sect. 3.1.3. In order to find a canonical form of this action it is convenient to perform a splitting of spacetime into a stack of 3-dimensional hypersurfaces with constant time coordinate t. In these coordinates the unit normal vector n^μ, $n_\mu n^\mu = -1$, of the hypersurfaces has the components

$$n_\mu = (-N, 0, 0, 0) \,, \quad \text{or} \quad n^\mu = \frac{1}{N}(1, -N^i) \,, \tag{2.37}$$

where N is the lapse function and N^i the shift vector. With the help of the projector[2]

$$\gamma^{\mu\nu} = g^{\mu\nu} + n^\mu n^\nu = \begin{pmatrix} 0 & 0 \\ 0 & \gamma^{ij} \end{pmatrix} \,, \tag{2.38}$$

this splitting can be constructed in a geometrical way, see, e.g., [51]. The 3-dimensional hypersurfaces have an induced metric $g_{ij} = \gamma_{ij}$, with $\gamma_{ik}\gamma^{kj} = \delta_{ij}$, a Riemann tensor R_{ijkl}, a Ricci tensor R_{ij}, and a Ricci scalar R. These quantities are intrinsic geometric objects of the hypersurfaces, whereas the extrinsic curvature

$$K_{ij} \equiv -n_{(i||j)} = \frac{1}{2N}\left(-\gamma_{ij,0} + 2N^k{}_{;(i}\gamma_{j)k}\right) \,, \tag{2.39}$$

[2] Notice that $0 = n_\mu \gamma^{\mu\nu} = -N\gamma^{0\nu}$ and thus $\gamma^{0\nu} = 0$ for our choice of the time coordinate.

depends on their embedding in spacetime.

Applying this splitting of spacetime to the Lagrangian density $\mathcal{L}_G$ leads to

$$\mathcal{L}_G = \frac{1}{16\pi} N\sqrt{\gamma}\left[R + K_{ij}K^{ij} - (\gamma_{ij}K^{ij})^2\right] + (\mathrm{td})\,, \tag{2.40}$$

where (td) denotes a total divergence, which is neglected for now. Instead of varying with respect to the ten independent components of $g_{\mu\nu}$, we now use γ_{ij}, N, and N^i. Notice that no time derivatives of N and N^i appear. In order to obtain a canonical formulation we have to introduce the field momentum

$$\pi^{ij} = 16\pi\frac{\partial\mathcal{L}_G}{\partial\gamma_{ij,0}} = \sqrt{\gamma}(\gamma^{ij}\gamma^{kl} - \gamma^{ik}\gamma^{jl})K_{kl}\,, \tag{2.41}$$

where (2.39) was used. This can be inverted as

$$K_{ij} = \frac{1}{2\sqrt{\gamma}}(\gamma_{ij}\gamma_{kl} - 2\gamma_{ik}\gamma_{jl})\pi^{kl}\,. \tag{2.42}$$

The Legendre transformed Lagrangian density then reads

$$\mathcal{L}_G = \frac{1}{16\pi}\pi^{ij}\gamma_{ij,0} - N\mathcal{H}^{\text{field}} + N^i\mathcal{H}_i^{\text{field}} + (\mathrm{td})\,, \tag{2.43}$$

$$\mathcal{H}^{\text{field}} = -\frac{1}{16\pi\sqrt{\gamma}}\left[\gamma R - \gamma_{ij}\gamma_{kl}\pi^{ik}\pi^{jl} + \frac{1}{2}\left(\gamma_{ij}\pi^{ij}\right)^2\right]\,, \qquad \mathcal{H}_i^{\text{field}} = \frac{1}{8\pi}\gamma_{ij}\pi^{jk}_{\ ;k}\,, \tag{2.44}$$

and the action is additionally varied with respect to π^{ij} now. Notice that N and N^i play the role of Lagrange multipliers after Legendre transformation, the corresponding constraints are the vanishing of $\mathcal{H}^{\text{field}}$ and $\mathcal{H}_i^{\text{field}}$.

A subsequent gauge fixing is subtle as it requires a fine-tuning of the action, see, e.g., [52]. As shown in [50, 53, 54] by different methods, see also [55], one must replace the total divergence in (2.43) by $-\frac{1}{16\pi}\mathcal{E}_{i,i}$ for asymptotically flat spacetimes, where $\mathcal{E}_i = \gamma_{ij,j} - \gamma_{jj,i}$. This is related to the total energy E of asymptotically flat spacetimes by

$$E = \frac{1}{16\pi}\oint \mathrm{d}^2 s_i \mathcal{E}_i\,, \tag{2.45}$$

where $\oint \mathrm{d}^2 s_i$ denotes an integral over the asymptotic boundary of a spatial hypersurfaces at fixed time. This ADM energy will turn out to be the generator of time evolution after gauge fixing. For further discussion of boundary terms in the action of general relativity, also for the case of not asymptotically flat spacetimes, see, e.g., [56]. However, for asymptotically flat spacetimes the gravitational Hamiltonian may be written as

$$H_G = \int \mathrm{d}^3x\,(N\mathcal{H}^{\text{field}} - N^i\mathcal{H}_i^{\text{field}}) + E[\gamma_{ij}]\,. \tag{2.46}$$

Indeed, the action has the canonical structure momentum π^{ij} times velocity $\gamma_{ij,0}$ minus Hamiltonian H_G. Variation thus results in Hamilton's equations

$$\frac{\partial\pi^{ij}}{\partial t} = -16\pi\frac{\delta H_G}{\delta\gamma_{ij}} \equiv \{\pi^{ij}, H_G\}\,, \qquad \frac{\partial\gamma_{ij}}{\partial t} = 16\pi\frac{\delta H_G}{\delta\pi^{ij}} \equiv \{\gamma_{ij}, H_G\}\,, \tag{2.47}$$

where δ denotes the variational derivative here and the equal-time Poisson brackets are given by

$$\{\gamma_{ij}(\mathbf{x}), \pi^{kl}(\mathbf{x}')\} = 16\pi\delta_{k(i}\delta_{j)l}\delta(\mathbf{x} - \mathbf{x}')\,. \tag{2.48}$$

Before gauge fixing, the surface term E has no impact on these field equations, which could be obtained from local variations[3]. As further explained in Sect. 2.3.3, the gauge fixing is accompanied with solving the constraints $\mathcal{H}^{\text{field}} = 0$ and $\mathcal{H}_i^{\text{field}} = 0$, so H_G then turns into the ADM energy E. To make this more concrete, we choose the ADM transverse-traceless gauge conditions

$$\partial_j(\gamma_{ij} - \tfrac{1}{3}\gamma_{kk}\delta_{ij}) = 0\,, \qquad \pi^{ii} = 0\,, \tag{2.49}$$

in which the transverse-traceless decomposition of γ_{ij} and π^{ij} may be written as

$$\gamma_{ij} = \left(1 + \frac{\phi}{8}\right)^4 \delta_{ij} + h_{ij}^{\text{TT}}\,, \tag{2.50}$$

$$\pi^{ij} = \tilde{\pi}^{ij} + \pi^{ij\text{TT}}\,, \tag{2.51}$$

where h_{ij}^{TT} and $\pi^{ij\text{TT}}$ are transverse-traceless, e.g, $h_{ii}^{\text{TT}} = h_{ij,j}^{\text{TT}} = 0$, and the longitudinal $\tilde{\pi}^{ij}$ is related to a vector potential $\tilde{\pi}^i = \Delta^{-1}\pi^{ij}{}_{,j}$ by

$$\tilde{\pi}^{ij} = \tilde{\pi}^i{}_{,j} + \tilde{\pi}^j{}_{,i} - \frac{1}{2}\delta_{ij}\tilde{\pi}^k{}_{,k} - \frac{1}{2}\Delta^{-1}\tilde{\pi}^k{}_{,ijk}\,. \tag{2.52}$$

The advantage of this gauge is that in (2.50) there is a trace term but no longitudinal part related to a vector potential, while in (2.51) it is the other way around. Because of the orthogonality of the individual parts of the transverse-traceless decomposition, the kinetic term $\pi^{ij}\gamma_{ij,0}$ in the action turns into $\pi^{ij\text{TT}}h_{ij,0}^{\text{TT}}$. Then only the transverse-traceless parts remain dynamical variables. Now the four field constraints can be solved for the four nondynamical variables ϕ and $\tilde{\pi}^i$ in terms of h_{ij}^{TT} and $\pi^{ij\text{TT}}$. An analytic solution for ϕ and $\tilde{\pi}^i$, however, can in general only be given in some approximation scheme. Notice that ADM introduced two slightly different gauges [50], the one used here was actually seldom used by ADM themselfes. However, the gauge used here is better for applications, as the form of the trace term in (2.50) is adapted to the Schwarzschild metric in isotropic coordinates (with obvious advantages for perturbative expansions). The action turns into

$$W_G[h_{ij}^{\text{TT}}, \pi^{ij\text{TT}}] = \frac{1}{16\pi}\int \mathrm{d}^4x\, \pi^{ij\text{TT}} h_{ij,0}^{\text{TT}} - \int \mathrm{d}t\, H_{\text{ADM}}\,, \tag{2.53}$$

where the ADM Hamiltonian H_{ADM} is just the ADM energy E expressed in terms of the gauge-reduced canonical variables h_{ij}^{TT} and $\pi^{ij\text{TT}}$,

$$H_{\text{ADM}} = E[h_{ij}^{\text{TT}}, \pi^{ij\text{TT}}] = -\frac{1}{16\pi}\int \mathrm{d}^3x\, \Delta\phi[h_{ij}^{\text{TT}}, \pi^{ij\text{TT}}]\,. \tag{2.54}$$

Notice that the surface integral (2.45) was written as a volume integral now and the asymptotic behavior of ϕ was used. The action must be varied only with respect to the independent components of h_{ij}^{TT} and $\pi^{ij\text{TT}}$, which is ensured with the help of the transverse-traceless projector

$$\begin{aligned}\delta_{ij}^{\text{TT}kl} = \tfrac{1}{2}[&(\delta_{ik} - \Delta^{-1}\partial_i\partial_k)(\delta_{jl} - \Delta^{-1}\partial_j\partial_l) + (\delta_{il} - \Delta^{-1}\partial_i\partial_l)(\delta_{jk} - \Delta^{-1}\partial_j\partial_k) \\ &- (\delta_{kl} - \Delta^{-1}\partial_k\partial_l)(\delta_{ij} - \Delta^{-1}\partial_i\partial_j)]\,.\end{aligned} \tag{2.55}$$

The Poisson brackets after gauge fixing correspondingly read

$$\{h_{ij}^{\text{TT}}(\mathbf{x}), \pi^{kl\text{TT}}(\mathbf{x}')\} = 16\pi\delta_{ij}^{\text{TT}kl}\delta(\mathbf{x} - \mathbf{x}')\,. \tag{2.56}$$

[3] However, one should not constrain to local variations for asymptotically flat spacetimes [54].

2.3.2 Matter couplings

Point-masses are the simplest kind of matter that can be coupled to general relativity. Its contribution to the action is just

$$W_M[g_{\mu\nu}, z^\mu] = \int \mathrm{d}\tau\, L_M\,, \qquad L_M = -m\sqrt{-g_{\mu\nu}(z^\rho)u^\mu u^\nu}\,. \tag{2.57}$$

This action is invariant under a change of the parameter τ, which simplifies the variation as no constraint of the form $u_\mu u^\mu = -1$ is needed. m is assumed to be a constant. Variation of the action leads to the equations of motion

$$\frac{\mathrm{D}}{\mathrm{d}\tau}\left[\frac{u^\mu}{\sqrt{-u_\rho u^\rho}}\right] = 0\,. \tag{2.58}$$

These equations only have a unique solution if a gauge for τ is chosen. The Einstein field equations now have a source $T^{\mu\nu}$,

$$R^{\mu\nu}_{(4)} - \frac{1}{2}g^{\mu\nu}R_{(4)} = 8\pi T^{\mu\nu}\,, \qquad \text{with } \sqrt{-g}T^{\mu\nu} \equiv 2\frac{\delta W_M}{\delta g_{\mu\nu}}\,, \tag{2.59}$$

and $R^{\mu\nu}_{(4)}$ the 4-dimensional Ricci tensor. The singular stress-energy tensor density reads explicitly

$$\sqrt{-g}T^{\mu\nu} = \int \mathrm{d}\tau \frac{m u^{(\mu}u^{\nu)}}{\sqrt{-u_\rho u^\rho}}\delta_{(4)}\,. \tag{2.60}$$

The 4-dimensional momentum is introduced as

$$p_\mu = \frac{\partial L_M}{\partial u^\mu} = m\frac{u_\mu}{\sqrt{-u_\rho u^\rho}}\,. \tag{2.61}$$

It obviously holds[4]

$$L_M = \frac{\partial L_M}{\partial u^\mu}u^\mu = p_\mu u^\mu\,. \tag{2.62}$$

Thus a Legendre transformation leads to a vanishing canonical (i.e., defined as usual) Hamiltonian. Its place is taken by the mass-shell constraint

$$p_\mu p^\mu + m^2 = 0\,, \tag{2.63}$$

which has to be added to the action via a Lagrange multiplier $\lambda(\tau)$, as further explained in the next section. This constraint is a consequence of the inability to express u_μ uniquely in terms of p_μ, which in turn is due to invariance under reparametrization, or gauging, of τ. Indeed, it is a common feature of reparametrization invariant actions that the canonical Hamiltonian vanishes and the time evolution is instead generated by certain constraints. As seen in the last section this also holds for general relativity, whose action is invariant under reparametrizations of spacetime, or general coordinate transformations.

Up to now the matter action was transformed into[5]

$$W_M[g_{\mu\nu}, z^\mu, p_\mu, \lambda] = \int \mathrm{d}\tau\,(p_\mu u^\mu - H_{M\tau})\,, \qquad H_{M\tau} = \lambda(g^{\mu\nu}p_\mu p_\nu + m^2)\,. \tag{2.64}$$

[4] Due to Euler's theorem, this actually holds for any Lagrangian which is a homogeneous function of degree one in the velocity u^μ. This in turn is required by reparametrization invariance.

[5] Fields within the matter action are always taken at the position z^μ from now on.

Notice that the Hamiltonian $H_{M\tau}$ generates an evolution with respect to the arbitrary parameter τ. Further the variation δp_μ leads to $u_\mu = 2\lambda p_\mu$ and from the mass-shell constraint (2.63) one thus has $\lambda = \frac{1}{2m}\sqrt{-u_\mu u^\mu}$. It may be checked that the equations of motion for p_μ and the stress-energy tensor are equivalent to the ones above, which justifies the Legendre transformation in the presence of constraints. More on constrained Hamiltonian dynamics is discussed in the next section. By solving the constraint and applying the gauge choice $\tau = z^0 \equiv t$, or $u^0 = 1$, the action is expressed in terms of the independent variables p_i and z^i. It holds

$$p_0 = ({\gamma_0}^\mu - n_0 n^\mu) p_\mu = g_{0i}\gamma^{ij} p_j + N np \,, \tag{2.65}$$

where $np \equiv n^\mu p_\mu$. From the constraint we get

$$(\gamma^{\mu\nu} - n^\mu n^\nu) p_\mu p_\nu + m^2 = 0 \qquad \Rightarrow \qquad np = -\sqrt{m^2 + \gamma^{ij} p_i p_j} \,. \tag{2.66}$$

Further we have

$$0 = n_i = n^\mu g_{\mu i} = \frac{1}{N}(g_{0i} - N^j g_{ji}) \qquad \Rightarrow \qquad g_{0i} = \gamma_{ij} N^j \,. \tag{2.67}$$

Putting all together we arrive at

$$W_M[\gamma_{ij}, N, N^i, z^i, p_i] = \int \mathrm{d}t \, (p_i \dot{z}^i - H_M) \,, \qquad H_M = -p_0 = -N np - N^i p_i \,, \tag{2.68}$$

where a dot $\dot{}$ denotes the total time derivative $\frac{\mathrm{d}}{\mathrm{d}t}$. The original Hamiltonian $H_{M\tau}$ vanishes by virtue of the constraint. Variation of the matter variables z^i and p_i results in Hamilton's equations with H_M as the matter part of the Hamiltonian. Thus z^i and p_i have the Poisson brackets $\{z^i, p_j\} = \delta_{ij}$, all other zero. As in the last section, the variables γ_{ij}, π^{ij}, N, and N^i are now used for the gravitational field.

The gauge fixing procedure is analogous to the last section, there are just certain matter corrections to the field constraints following from the N- and N^i-variations,

$$\mathcal{H} \equiv \mathcal{H}^{\text{field}} + \mathcal{H}^{\text{matter}} = 0 \,, \qquad \mathcal{H}_i \equiv \mathcal{H}_i^{\text{field}} + \mathcal{H}_i^{\text{matter}} = 0 \,, \tag{2.69}$$

where

$$\mathcal{H}^{\text{matter}} = -np\,\delta = \sqrt{m^2 + \gamma^{ij} p_i p_j}\,\delta \,, \qquad \mathcal{H}_i^{\text{matter}} = p_i \delta \,, \tag{2.70}$$

with $\delta = \delta(\mathbf{x} - \mathbf{z})$. The first relation in (2.69) is called the Hamilton constraint, while the second one is the momentum constraint. The ADM Hamiltonian H_{ADM} still results from the ADM energy by solving the field constraints using the gauge conditions (2.50, 2.51), but now also depends on the matter variables z^i and p_i, which have entered via source terms of the constraints. All field *and matter* interaction terms in the action,

$$W = \frac{1}{16\pi} \int \mathrm{d}^4 x \, \pi^{ij\mathrm{TT}} h^{\mathrm{TT}}_{ij,0} + \int \mathrm{d}t \Big[p_i \dot{z}^i - H_{\text{ADM}} \Big] \,, \tag{2.71}$$

are contained in the ADM Hamiltonian, or the ADM energy. This is a unique feature of general relativity, and still holds for couplings to other matter and even other *fields* [53].

Finally we review the most important couplings of matter and fields to gravity that have received a canonical formulation, see also [57]. Besides for point-masses [50, 58], such canonical formulations were found for fluids [59], massive scalar fields [55, 60], spin-$\frac{1}{2}$ Dirac fields [6, 61–64], and gauge spin-1 fields, including Maxwell [60, 65] and Yang-Mills [66]. Problematic from a canonical point of view are derivative-coupled theories [57], like Dirac fields and also pole-dipole objects. It is thus fortunate that the sought-for canonical formulation of pole-dipole objects will be seen to resemble to Dirac fields coupled to gravity, for which a canonical formulation was found. Though the classical spin of pole-dipole objects is not restricted in its size, we consider pole-dipole objects only at linear order in spin here. This means the spin is treated as an infinitesimal quantity and thus formally takes on the smallest (nonzero) classical value, which seems to give rise to similarities to the minimal (nonzero) quantized spin $\frac{1}{2}$ of Dirac fields. Thus the achievements on canonical formulations of Dirac fields coupled to gravity served as a very useful guide here, in particular the paper of Kibble [6]. However, an additional problem to be solved for spinning objects in general relativity concerns the canonical spin supplementary condition. So far the canonical formulation of spinning objects was found for test-bodies in an external gravitational field [67], see also the very recent work in [44].

2.3.3 Other formalisms and constrained Hamiltonian dynamics

Before gauge fixing, general relativity possesses a canonical formulation in the presence of the constraints $\mathcal{H} = 0$ and $\mathcal{H}_i = 0$. There exists a general framework to handle such a constrained Hamiltonian dynamics, which was developed most notably by Dirac as a general route to canonical quantization [68], see also [52, 69, 70]. Further important work was done by Bergmann and his collaborators, but focused on general relativity and its canonical quantization [71]. Though Dirac also considered the canonical formulation of general relativity [72], his approach is formulated in a very general way. Early work on this subject can even be traced back to Rosenfeld [73]; for a historical review, see, e.g, [74, 75]. A particular important achievement of ADM for canonical general relativity was the identification of the ADM Energy as the Hamiltonian after gauge fixing [49, 76]. Yet another canonical treatment of general relativity was given by Schwinger [55]. This formulation is similar to the ADM one, essentially only different variables were used and many more such reformulations are possible. A further very appealing formulation was given by Ashtekar [77], in whose variables the gravitational constraints considerably simplify, and which forms the basis of loop quantum gravity, see, e.g., [78].

We will now summarize some of the results of Rosenfeld, Dirac, and Bergmann on constrained Hamiltonian dynamics. In the last section the mass-shell constraint (2.63) manifests the inability to uniquely express the velocity u^μ in terms of the corresponding momentum p_μ. The standard route to a Hamiltonian seems to be impassable in such a situation as the Legendre transformation can not be applied in its usual way. The solution, however, is simple. The Legendre transformation may formally be performed as usual if one adds the emerging constraints via Lagrange multipliers to the action. The additional degrees of freedom introduced by these multipliers correctly parametrize the ambiguity present in the relation between velocities and momenta. Further, it can be shown that the dynamics of the transformed action is equivalent to the dynamics of the original action. The constraints arising at this stage are entitled as *primary* and the Hamiltonian is called the *total Hamiltonian* (or Dirac Hamiltonian), as it includes the primary constraints via Legendre multipliers.

The next step in the analysis of constrained Hamiltonian dynamics consists of evaluating the consistency requirement that all primary constraints must be preserved under the time evolution given by the total Hamiltonian. Of course, some of the resulting consistency conditions can be identically fulfilled or lead to contradictions (then the dynamics must be considered as inconsistent). Moreover some conditions are restrictions for the Lagrange multipliers appearing in the total Hamiltonian. Due to linearity of the total Hamiltonian in the Lagrange multipliers, these restrictions are actually *linear* equations. Further, one might also obtain new (independent) constraints from the consistency conditions. Such new constraints are called *secondary* constraints. For these new constraints the same consistency requirement applies, and one is eventually lead to further conditions on the Lagrange multipliers and/or to further secondary constraints and so on. Finally, one ends up with a complete set of constraints and linear equations for the Lagrange multipliers.

The linear equations for the Lagrange multipliers can be used to eliminate certain linear combinations of these multipliers from the equations of motion. The usual situation known from courses on classical mechanics is that all multipliers are uniquely fixed. However, in the general case some combinations of

Lagrange multipliers could remain unfixed and thus remain as arbitrary degrees of freedom in the equations of motion. The interpretation is that these degrees of freedom are *physically irrelevant* and correspond to a *gauge freedom* of the theory. That is, the corresponding independent Lagrange multipliers can be chosen at will, interpreted as choosing a gauge. Hamiltonian formulations of gauge theories will inevitably involve constraints.

The Lagrange multipliers enter the total Hamiltonian together with the primary constraints. Instead of characterizing the gauge freedom of a theory by undetermined combinations of Lagrange multipliers, one can give a description in terms of corresponding primary constraints. For this purpose it is useful to introduce the notion of *first class* and *second class* constraints. First class constraints are defined to have vanishing Poisson brackets with all other constraints. A constraint that is not first class is called second class. In addition to being first or second class, the constraints can still be primary or secondary, and one thus has four categories of constraints now. An important fact is that the number of independent primary first class constraints is equal to the number of unfixed Lagrange multipliers in the equations of motion and thus to the number of gauge degrees of freedom.

Not only the primary first class constraints but also all secondary first class constraints are related to gauge symmetries [79, 80] (at least under certain reasonable conditions), see also [75]. To be more precise, all first class constraints, primary as well as secondary, appear in the generators of gauge symmetries on phase space. The algebra of first class constraints is therefore related to the algebra of gauge symmetry generators of the theory. For general relativity, the algebra of first class constraints reads [53, 55] (at least for the vacuum case and for coupling to point-masses)

$$\{\mathcal{H}(\mathbf{x}),\mathcal{H}(\mathbf{x}')\} = -\left[\mathcal{H}_i(\mathbf{x})\gamma^{ij}(\mathbf{x}) + \mathcal{H}_i(\mathbf{x}')\gamma^{ij}(\mathbf{x}')\right]\partial_j\delta(\mathbf{x}-\mathbf{x}')\,, \tag{2.72}$$

$$\{\mathcal{H}_i(\mathbf{x}),\mathcal{H}(\mathbf{x}')\} = -\,\mathcal{H}(\mathbf{x})\,\partial_i\delta(\mathbf{x}-\mathbf{x}')\,, \tag{2.73}$$

$$\{\mathcal{H}_i(\mathbf{x}),\mathcal{H}_j(\mathbf{x}')\} = -\,\mathcal{H}_j(\mathbf{x})\,\partial_i\delta(\mathbf{x}-\mathbf{x}') - \mathcal{H}_i(\mathbf{x}')\,\partial_j\delta(\mathbf{x}-\mathbf{x}')\,. \tag{2.74}$$

If one goes to the constraint surface by $\mathcal{H} = 0 = \mathcal{H}_i$, then the right-hand sides vanish. Thus $\mathcal{H}$ and $\mathcal{H}_i$ are indeed first class. In order to relate this algebra to 4-dimensional diffeomorphism invariance one should include lapse and shift as well as corresponding momenta into phase space [79]. It should be noted that though the total Hamiltonian of general relativity (2.46) is composed of the first class constraints and looks quite similar to the generator of gauge transformations, the time evolution given by this Hamiltonian is not just a gauge effect, see, e.g, [81].

One can elaborate more on the distinction between first and second class constraints. Obviously one can recombine the whole set of constraints into some equivalent set. We consider the case that such a recombination brings as many constraints as possible from the second class into the first class. One can then show by a reductio ad absurdum that the matrix $c_{ab} = \{\psi_a, \psi_b\}$, where ψ_a are the constraints that remain second class after recombination[6], is invertible, $\det(c_{ab}) \neq 0$. The Dirac bracket $\{A, B\}^*$ between two phase space functions A and B is then defined by

$$\{A, B\}^* = \{A, B\} - \{A, \psi_a\}c^{-1}_{ab}\{\psi_b, B\}\,. \tag{2.75}$$

This bracket satisfies the laws known from the Poisson bracket. Further, it leads to the correct equations of motion together with the total Hamiltonian. The Dirac bracket can thus be used as a substitute for the Poisson bracket. However, whereas one may use the constraints only after all Poisson brackets were calculated[7], the second class constraints $\psi_a = 0$ can be used *before* an application of the Dirac bracket without changing the result (e.g., one has $\{A, \psi_a\}^* = 0$ for all A and ψ_a). If one restricts to use the Dirac bracket instead of the Poisson bracket, one can use the second class constraints $\psi_a = 0$ to solve for certain phase space variables and eliminate them from all quantities. Then one has performed an actual reduction

6 The indices a and b label constraints in this section.

7 For a more detailed exposition it is useful to introduce the concepts of weak and strong equality.

of the degrees of freedom. Within this formalism, gauge conditions are constraints added by hand that bring all (or just some) first class constraints into the second class. The reduction of degrees of freedom via gauge fixing then follows with the help of the Dirac bracket in a straightforward way.

3 Action approach

In this section an extension of the ADM formalism for point-masses to the pole-dipole approximation is obtained linear in spin. The derivation is based on a corresponding extension of the point-mass action.

3.1 Action of the spherical top

It is remarkable that equations of motion (2.28) and stress-energy tensor (2.30) in the pole-dipole approximation are independent of the specific object, i.e., are the same for black holes and neutron stars. It is thus expected that *any* specific action for a spinning object coupled to general relativity will contain (2.28) and (2.30) to some approximation (e.g., linear in spin). The pole-dipole action found here will be based on the simplest spinning object imaginable – the spherical top.

3.1.1 Newtonian case

The spherical top is well-known in classical mechanics. However, we review it here in a way that allows an easy transition to the special relativistic treatment in [69, 82]. We consider in this section a top with its center of mass resting at the coordinate origin. The center of mass motion can be added easily. The top can be described as a rigid body consisting of many point-masses labeled by an index a, with positions z^i_a and masses m_a. In terms of body-fixed (constant) coordinates $z^{[i]}_a$ it holds $z^i_a(t) = \Lambda^{[j]i}(t) z^{[j]}_a$, with a time-dependent rotation matrix $\Lambda^{[j]i}$, $\Lambda^{[k]i}\Lambda^{[k]j} = \delta_{ij}$. Here and in the following we will indicate 3-dimensional indices in the body-fixed coordinate system by square brackets. The rotation matrix can be expressed in terms of three independent angle variables, $\Lambda^{[i]j} = \Lambda^{[i]j}(\varphi_1, \varphi_2, \varphi_3)$, e.g., the Euler angles. The antisymmetric[8] angular velocity tensor is given by $\Omega^{ij} = \Lambda^{[k]i}\dot{\Lambda}^{[k]j}$. A spherical top is completely characterized by *one* moment of inertia I, it holds $2\sum_a m_a z^{[i]}_a z^{[j]}_a = I\delta_{ij}$. The Lagrangian of the free spherical top then reads

$$L(\Lambda^{[i]j}, \Omega^{ij}) = \frac{1}{2}\sum_a m_a \dot{z}^i_a \dot{z}^i_a = \frac{1}{4} I \Omega^{ij}\Omega^{ij} = \frac{1}{2} I \Omega^i \Omega^i \,, \tag{3.1}$$

where $\Omega^i = \frac{1}{2}\epsilon_{ijk}\Omega^{jk}$ is the usual angular velocity vector. The spin is the generalized momentum of the angular velocities of the form

$$S_{ij} = 2\frac{\partial L}{\partial \Omega^{ij}} = I\Omega_{ij} \,. \tag{3.2}$$

Legendre transformation leads to

$$L = \frac{1}{2} S_{ij}\Omega^{ij} - H \,, \tag{3.3}$$

with the Hamiltonian $H(\Lambda^{[i]j}, S_{ij}) = \frac{1}{4I} S_{ij} S_{ij}$. This specific Hamiltonian is actually independent of $\Lambda^{[i]j}$.

In order to derive the general Euler-Lagrange equations of the Lagrangian (3.3) we are not varying the independent angle variables, but instead use $\delta\theta^{ij} = \Lambda^{[k]i}\delta\Lambda^{[k]j}$ as independent variations. Notice that $\delta\theta^{ij}$

[8] The antisymmetry immediately follows from the time derivative of $\Lambda^{[k]i}\Lambda^{[k]j} = \delta_{ij}$.

is antisymmetric, and thus indeed corresponds to three independent variations of the angle variables. The result is

$$\Omega^{ij} = \Lambda^{[k]i}\dot{\Lambda}^{[k]j} = 2\frac{\partial H}{\partial S_{ij}}\,, \qquad \dot{S}_{ij} = 2S_{k[i}\Omega_{j]k} - \Lambda^{[k]i}\frac{\partial H}{\partial \Lambda^{[k]j}} + \Lambda^{[k]j}\frac{\partial H}{\partial \Lambda^{[k]i}}\,. \tag{3.4}$$

These are Hamilton's equations for $\Lambda^{[k]j}$ and S_{ij}. The Poisson brackets fulfill

$$\dot{A} = \{A, H\} + \frac{\partial A}{\partial t}\,, \tag{3.5}$$

for a general quantity A. Comparing with (3.4) we can read off

$$\{\Lambda^{[i]j}, \Lambda^{[k]l}\} = 0\,, \qquad \{\Lambda^{[i]j}, S_{kl}\} = \Lambda^{[i]k}\delta_{lj} - \Lambda^{[i]l}\delta_{kj}\,, \tag{3.6}$$

$$\{S_{ij}, S_{kl}\} = \delta_{ik}S_{jl} - \delta_{jk}S_{il} - \delta_{il}S_{jk} + \delta_{jl}S_{ik}\,. \tag{3.7}$$

Alternatively one could use canonical variables based on the angle variables,

$$\{\varphi_i, p_j^\varphi\} = \delta_{ij}, \qquad \text{with } p_i^\varphi = \frac{\partial L(\varphi_j, \dot{\varphi}_k)}{\partial \dot{\varphi}_i}\,, \tag{3.8}$$

as in most textbooks. Further, if the Hamiltonian is independent of $\Lambda^{[i]j}$, which will always be the case in the following, the spin length is a constant and it is possible to describe each spin by only two independent canonical variables instead of six contained in S_{ij} and $\Lambda^{[i]j}$, see, e.g., [83, 84]. However, we prefer the variables S_{ij} and $\Lambda^{[i]j}$ here.

3.1.2 Special relativistic case

In the relativistic case there are no rigid bodies. However, one can define a top in a purely mathematical way [69, 82, 85] as a worldline with a Lorentz matrix $\Lambda_{A\mu}$, $\eta^{AB}\Lambda_{A\mu}\Lambda_{B\nu} = \eta_{\mu\nu}$, such that $\Lambda_{A\mu}$ is a pure rotation,

$$\Lambda_{A\mu} = \begin{pmatrix} -1 & 0 \\ 0 & \Lambda_{[i]j} \end{pmatrix}\,, \tag{3.9}$$

in some frame defined by f_μ. (Upper case Latin indices from the beginning of the alphabet refer to the body-fixed frame and have the values $A = [0], [i]$.) This can be formulated as

$$\Lambda_{[0]\mu} = \frac{f_\mu}{\sqrt{-f_\nu f^\nu}}\,, \quad \text{or} \quad \eta^{[0]A} = -\frac{\Lambda^{A\mu}f_\mu}{\sqrt{-f_\nu f^\nu}}\,. \tag{3.10}$$

It holds

$$\Omega^{\mu\nu} = \Lambda_A{}^\mu\frac{\mathrm{d}\Lambda^{A\nu}}{\mathrm{d}\tau}\,, \qquad S_{\mu\nu} = 2\frac{\partial L(u^\mu, \Omega^{\mu\nu})}{\partial \Omega^{\mu\nu}}\,, \qquad p_\mu = \frac{\partial L(u^\mu, \Omega^{\mu\nu})}{\partial u^\mu}\,, \tag{3.11}$$

see, e.g, [69] or the next section. The spin supplementary condition belonging to (3.10) reads

$$S_{\mu\nu}f^\mu = 0\,. \tag{3.12}$$

It will be seen in Sect. 5.2.2 in which sense this belongs to (3.10). Notice that only three relations of (3.10) are independent, e.g., one could equivalently require $\Lambda^{[i]\mu}f_\mu = 0$ only. The same holds for (3.12).

There are many ways to implement the conditions (3.10) and (3.12) in an action approach, see, e.g., [47, 69, 82]. We require here that (3.10) and (3.12) are preserved under the time evolution given by the action and try to directly construct such an action. An alternative, rather indirect, approach would be to add the supplementary conditions to some action with the help of Lagrange multipliers. As well known from classical mechanics, this modifies the dynamics by constraint forces, which ensure that the supplementary conditions are preserved in time. However, one should carefully check the consistency, in particular one should be able to find a solution for the Lagrange multipliers. Also no further (secondary) constraints should appear, which would be physically unacceptable (we want to have exactly three independent rotational degrees of freedom). Finally, the Lagrange multipliers can be eliminated from the action, leading to a dynamics which preserves the constraints and thus to the action we try to find directly here.

3.1.3 Minimal coupling to gravity

The next logical step is a minimal coupling of the special relativistic spherical top defined in the last section to gravity. Such a coupling was already treated in [86] based on the developments in [82]. In [87] even nonminimal couplings leading to higher multipole corrections were considered[9]. Notice that [88] is not a further development of [87], but is a completely different action approach. More recently yet another approach was given in [89] with focus on an application to the post-Newtonian approximation.

The matter variables $\Lambda_{A\mu}$ have the problem that they fulfill

$$\Lambda_{A\mu}\Lambda^{A}{}_{\nu} = g_{\mu\nu}\,. \tag{3.13}$$

That is, $\Lambda_{A\mu}$ is not independent under variation of the metric. For the Dirac field, one has a similar problem with the gamma matrices γ_μ, as it holds $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$. This problem can be overcome by writing $\Lambda^{A}{}_{\mu} = \Lambda^{AI}e_{I\mu}$ and treating Λ^{AI} and the tetrad field $e_{I\mu}$ as independent variables. From

$$\Lambda_{AI}\Lambda^{A}{}_{J} = \eta_{IJ}\,, \quad \text{or} \quad \gamma_I\gamma_J + \gamma_J\gamma_I = 2\eta_{IJ}\,, \tag{3.14}$$

it is obviously now consistent that Λ_{AI} and γ_I are constant under variations of the tetrad field $e_{I\mu}$. We have three bases involved, a body-fixed basis, a local Lorentz basis (denoted by upper case Latin indices from the middle of the alphabet), and a coordinate basis. The field equations are obtained by an unconstrained variation of $e_{I\mu}$. The metric $g_{\mu\nu} = e_{I\mu}e^{I}{}_{\nu}$ as well as the connection are not varied independently. For the variation of Λ_{AI} one has to take into account the condition (3.14).

In [86, 87, 89] matter and field degrees of freedom are not clearly separated in the action. For example, in [87] the equations of motion for the matter variables were obtained by adding (3.13) as a constraint to the action with the help of Lagrange multipliers, whereas the field equations were obtained from an unconstrained variation of $\Lambda_{A\mu}$. However, we need to separate matter and field degrees of freedom here, which is essential for the canonical reduction in the next sections.

The covariant angular velocity in the local Lorentz basis can be defined as

$$\Omega^{IJ} = \Lambda_{A}{}^{I}\frac{\mathrm{D}\Lambda^{AJ}}{\mathrm{d}\tau} = \Lambda_{A}{}^{I}\left[\frac{\mathrm{d}\Lambda^{AJ}}{\mathrm{d}\tau} - \Lambda^{A}{}_{K}\omega_{\mu}{}^{KJ}(z^\rho)u^\mu\right]\,. \tag{3.15}$$

Here $\omega_\mu{}^{IJ}$ are the Ricci rotation coefficients, $e_{I\alpha}e_{J\beta}\omega_\mu{}^{IJ} = -\Gamma^{(4)}_{\beta\alpha\mu} + e^{K}{}_{\alpha,\mu}e_{K\beta}$, and $\Gamma^{(4)}_{\alpha\mu\nu} = \frac{1}{2}(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha})$ is the 4-dimensional Christoffel symbol of first kind. Notice that the covariant derivative does not act on indices referring to the body-fixed frame. The matter action shall be of the general form

$$W_M[e_{I\mu}, z^\mu, \Lambda^{AI}] = \int \mathrm{d}\tau\, L_M(u^\mu, \Omega^{\mu\nu}, g_{\mu\nu}(z^\rho))\,. \tag{3.16}$$

The Lagrangian L_M is restricted to depend on the velocities u^μ and $\Omega^{\mu\nu} = e_I{}^{\mu}e_J{}^{\nu}\Omega^{IJ}$ only, and not on the "coordinates" z^μ and Λ^{AI} directly. This ensures the covariance of the action. The action shall be invariant

[9] This obviously goes beyond a spherical top, however, the formalism stays the same.

under reparametrizations, so u^μ is not constrained. If we let the Lagrangian depend on the curvature tensor, we would include quadrupole corrections, see Sect. 5.2. An important relation is given by (2c) in [82] or (9) in [87], which reads here

$$0 = \frac{\partial L_M}{\partial u^\alpha} u^\beta + 2 \frac{\partial L_M}{\partial \Omega^{\alpha\nu}} \Omega^{\beta\nu} - 2 \frac{\partial L_M}{\partial g_{\beta\nu}} g_{\alpha\nu} \,, \tag{3.17}$$

and is a consequence of L_M being a scalar[10]. Similar to Sect. 3.1.1, the Euler-Lagrange equations are obtained with the help of the antisymmetric variations $\delta\theta^{IJ} = \Lambda_A{}^I \delta\Lambda^{AJ}$, e.g.,

$$\delta\Omega^{IJ} = \frac{\mathrm{D}\delta\theta^{IJ}}{\mathrm{d}\tau} + 2\Omega_K{}^{[I}\delta\theta^{J]K} - R^{(4)IJ}_{\mu\nu} u^\nu \delta z^\mu - \frac{\mathrm{D}}{\mathrm{d}\tau}\left(\omega_\mu{}^{IJ}\delta z^\mu\right) - u^\mu \delta\omega_\mu{}^{IJ} \,. \tag{3.18}$$

The $\delta\theta^{IJ}$-variation then leads to

$$\frac{\mathrm{D}}{\mathrm{d}\tau}\left[\frac{\partial L_M}{\partial \Omega^{\mu\nu}}\right] = \frac{\partial L_M}{\partial \Omega^{\mu\rho}} \Omega^\rho{}_\nu - \frac{\partial L_M}{\partial \Omega^{\nu\rho}} \Omega^\rho{}_\mu \,. \tag{3.19}$$

The δz^μ-variation is subtle as it is not manifestly covariant, see, e.g., the second last term in (3.18). This is due to the fact that Λ^{AI} is held constant for the variation of the worldline δz^μ, which is not a covariant process (e.g., in contrast to a parallel transport of Λ^{AI} to the new worldline). However, using the equations of motion for Λ^{AI}, (3.19), and the covariance of L_M, (3.17), the result of the δz^μ-variation reads

$$\frac{\mathrm{D}}{\mathrm{d}\tau}\left[\frac{\partial L_M}{\partial u^\mu}\right] = -R^{(4)\alpha\beta}_{\mu\nu} u^\nu \frac{\partial L_M}{\partial \Omega^{\alpha\beta}} \,, \tag{3.20}$$

and is manifestly covariant now. Further, by virtue of (3.17) we can write (3.19) as

$$2\frac{\mathrm{D}}{\mathrm{d}\tau}\left[\frac{\partial L_M}{\partial \Omega^{\mu\nu}}\right] = \frac{\partial L_M}{\partial u^\mu} u_\nu - \frac{\partial L_M}{\partial u^\nu} u_\mu \,. \tag{3.21}$$

At last, the field equations follow from the $\delta e_{I\mu}$-variation as

$$R^{\mu\nu}_{(4)} - \frac{1}{2} g^{\mu\nu} R_{(4)} = 8\pi T^{\mu\nu} \,, \qquad \text{with } \sqrt{-g} T^{\mu\nu} \equiv e_I{}^\mu \frac{\delta W_M}{\delta e_{I\nu}} \,, \tag{3.22}$$

where the left-hand side results from the Einstein-Hilbert part (2.36) and the stress-energy tensor density $\sqrt{-g}T^{\mu\nu}$ reads explicitly

$$\sqrt{-g} T^{\mu\nu} = \int \mathrm{d}\tau \left[u^{(\mu} g^{\nu)\alpha} \frac{\partial L_M}{\partial u^\alpha} \delta_{(4)} - \left(2 \frac{\partial L_M}{\partial \Omega^{\alpha\beta}} g^{\alpha\rho} g^{\beta(\mu} u^{\nu)} \delta_{(4)} \right)_{||\rho} \right] . \tag{3.23}$$

Here the important relation (3.17) was used again and the antisymmetric part

$$\sqrt{-g} T^{[\mu\nu]} = \int \mathrm{d}\tau \left[-\frac{\mathrm{D}}{\mathrm{d}\tau}\left(\frac{\partial L_M}{\partial \Omega^{\alpha\beta}}\right) + 2\frac{\partial L_M}{\partial \Omega^{\alpha\rho}} \Omega^\rho{}_\beta \right] g^{\alpha[\mu} g^{\nu]\beta} \delta_{(4)} = 0 \,, \tag{3.24}$$

vanishes, see (3.19). Indeed, (3.19) is equivalent to $T^{[\mu\nu]} = 0$.

Comparing (2.28) and (2.30) with (3.20), (3.21), and (3.23) we get

$$S_{\mu\nu} = 2\frac{\partial L_M}{\partial \Omega^{\mu\nu}} \,, \qquad p_\mu = \frac{\partial L_M}{\partial u^\mu} \,, \tag{3.25}$$

[10] Loosely speaking, one can read (3.17) as “the number of upper indices minus the number of lower indices in L_M is zero.” This is derived in [87] from an infinitesimal coordinate transformation.

as in the special relativistic case. It should be noted that the given derivation basically follows along the lines of Bailey and Israel [87], but the used variables are similar to Porto [89], which resembles to [69]. However, the variables that are varied here differ from both [87] and [89]. The relation between p_μ and u_μ is fixed by (3.25), which means that the action already implements a specific spin supplementary condition, cf. Eq. (2.31). If this would not be the case, then (3.25) should be of the form (2.35), which is impossible due to the assumed absence of accelerations in the action. The approach in [82, 86] includes accelerations of the worldline coordinate. Further, a noncovariant supplementary condition (e.g., the canonical one) will result in a not manifestly covariant relation between p_μ and u_μ, (3.25), and thus one needs a not manifestly covariant action. An approach via Lagrange multipliers as discussed in Sect. 3.1.2 seems to be better when using such conditions from the start. Here we will start with the covariant supplementary conditions and go over to the canonical ones later by a change of variables.

Now we have to find a suitable reparametrization-invariant Lagrangian. An intuitive guess is (see Sect. 5.2 for more elaborated considerations)

$$L_M = \frac{1}{\sqrt{-u_\rho u^\rho}} \left[m_0 u_\mu u^\mu + \frac{I}{4} \Omega_{\mu\nu} \Omega^{\mu\nu} \right] , \tag{3.26}$$

where m_0 and I shall be constants. Then it holds

$$S_{\mu\nu} = \frac{I \Omega_{\mu\nu}}{\sqrt{-u_\rho u^\rho}} , \qquad p_\mu = \left(m_0 + \frac{1}{4I} S_{\mu\nu} S^{\mu\nu} \right) \frac{u_\mu}{\sqrt{-u_\rho u^\rho}} , \tag{3.27}$$

and the dynamical mass $m = \sqrt{-p_\mu p^\mu}$ is given by

$$m = m_0 + \frac{1}{4I} S_{\alpha\beta} S^{\alpha\beta} , \tag{3.28}$$

or $m = m_0$ to linear order in spin. Then (3.27) agrees with (2.31) for $f_\mu = p_\mu$ at linear order in spin, which implies that the corresponding spin supplementary condition (3.12) is preserved in time. (3.10) only needs to be preserved to zeroth order in spin, which is also the case (see also Sect. 5.2.2). Due to reparametrization invariance, L_M must be a homogeneous function of degree one in the velocities and Euler's theorem leads to

$$L_M = \frac{\partial L_M}{\partial u^\mu} u^\mu + \frac{\partial L_M}{\partial \Omega^{\mu\nu}} \Omega^{\mu\nu} = p_\mu u^\mu + \frac{1}{2} S_{\mu\nu} \Omega^{\mu\nu} . \tag{3.29}$$

A Legendre transformation in u^μ and $\Omega^{\mu\nu}$ thus leads to a vanishing result. Further, the mass-shell constraint (2.63) follows from (3.27), but no constraint on $S_{\mu\nu}$ arises from (3.27) as opposed to [69]. (Indeed, in [69] the action was constructed such that the constraint (3.12) arises directly from the action in this way.) Similar to Sect. 2.3.2 we finally have

$$W_M[e_{I\mu}, z^\mu, p_\mu, S_{\mu\nu}, \Lambda^{AI}, \lambda] = \int \mathrm{d}\tau \left[p_\mu u^\mu + \frac{1}{2} S_{\mu\nu} \Omega^{\mu\nu} - H_{M\tau} \right] , \tag{3.30}$$

with the function[11] $H_{M\tau}$ containing the mass-shell constraint only, $H_{M\tau} = \lambda(g^{\mu\nu} p_\mu p_\nu + m^2)$. This is the extension of (2.64) to the pole-dipole approximation at linear order in spin. We could also add the supplementary conditions,

$$S_{i\nu} p^\nu = 0 , \qquad \Lambda^{[i]J} p_J = 0 , \tag{3.31}$$

to the action with the help of Lagrange multipliers. However, this will not change the dynamics as these (independent) conditions are already preserved in time and their Lagrange multipliers therefore vanish. Reference [13] immediately started with the action in the form of Eq. (3.30) without the detailed derivation given in this section.

[11] Notice that $H_{M\tau}$ is not a Hamiltonian as $\frac{1}{2} S_{\mu\nu} \Omega^{\mu\nu}$ in (3.30) also contains interaction terms.

3.2 Reduction of the matter variables

Next a fully reduced canonical formalism is derived. For this the action is put on the constraint surface. That is, all supplementary conditions, constraints, and gauge conditions are solved in terms of certain truly independent variables that parametrize the constraint surface. The equations of motion for this reduced number of variables could then be obtained by varying the action with respect to these variables. However, we will transform the action to a new set of reduced variables such that the equations of motion can easily be seen to resemble to Hamilton's equations. This allows for an easy identification of the Hamiltonian and corresponding Poisson brackets. Thus a fully reduced canonical formalism for spinning objects coupled to general relativity is found [13]. Remember that the necessity for a variable transformation to obtain standard canonical Poisson brackets is already present in the flat space case, see (2.20). A treatment using Dirac brackets (2.75) seems to be more complicated, as one has to consider the brackets for each pair of variables then, whereas here we only have to handle the action (a single scalar).

The derivation sketched above is very similar to the treatment of Dirac fields coupled to gravity by Kibble [6]. In this section we concentrate on the matter part of the action only.

3.2.1 Reduced matter action

Similar to Sect. 2.3.2 we solve the matter constraints (now including the supplementary conditions (3.31)) as

$$np \equiv n^\mu p_\mu = -\sqrt{m^2 + \gamma^{ij} p_i p_j}\,, \tag{3.32}$$

$$nS_i \equiv n^\mu S_{\mu i} = \frac{p_k \gamma^{kj} S_{ji}}{np} = \gamma_{ij} nS^j\,, \qquad \Lambda^{[j](0)} = \Lambda^{[j](i)} \frac{p_{(i)}}{p^{(0)}}\,, \qquad \Lambda^{[0]I} = -\frac{p^I}{m}\,, \tag{3.33}$$

in terms of the independent variables p_i, S_{ij}, and $\Lambda^{[i](k)}$. On the constraint surface it holds $H_{M\tau} = 0$.

For simplicity, we will immediately constrain ourselves to the Schwinger time gauge [55],

$$e^{(0)\mu} = -n^\mu\,, \tag{3.34}$$

see also [6, 61, 64], as lapse and shift then turn into Lagrange multipliers in the matter action [13], like in the ADM formalism for nonspinning objects. This gauge condition effectively reduces the tetrad $e^{I\mu}$ to a triad $e^{(i)j}$, it holds

$$e^{(0)}{}_i = 0 = e_{(i)}{}^0\,, \quad e^{(0)}{}_0 = N = 1/e_{(0)}{}^0\,, \quad e^{(i)}{}_0 = N^j e^{(i)}{}_j\,, \tag{3.35}$$

$$N^i = -N e_{(0)}{}^i\,, \quad \gamma_{ij} = e^{(m)}{}_i e_{(m)j}\,, \quad \gamma^{ij} = e_{(m)}{}^i e^{(m)j}\,. \tag{3.36}$$

A further convenient gauge choice is $\tau = z^0 = t$ for the yet arbitrary parameter τ. In terms of the independent variables the matter Lagrangian (3.29) reads explicitly

$$\begin{aligned} L_M = &\left[p_i + K_{ij} nS^j + A^{kl} e_{(j)k} e^{(j)}{}_{l,i} - \left(\frac{1}{2} S_{kj} + \frac{p_{(k} nS_{j)}}{np}\right) \Gamma^{kj}{}_i\right] \dot{z}^i + \frac{nS^i}{2np} \dot{p}_i + A^{ij} e_{(k)i} e^{(k)}{}_{j,0} \\ &+ \left[S_{(i)(j)} + \frac{nS_{(i)} p_{(j)} - nS_{(j)} p_{(i)}}{np}\right] \frac{\Lambda_{[k]}{}^{(i)} \dot{\Lambda}^{[k](j)}}{2} - \int \mathrm{d}^3x \left(N \mathcal{H}^{\text{matter}} - N^i \mathcal{H}_i^{\text{matter}}\right), \end{aligned} \tag{3.37}$$

with the 3-dimensional Christoffel symbols Γ_{kji}, the abbreviation A^{ij} defined by

$$\gamma_{ik} \gamma_{jl} A^{kl} = \frac{1}{2} S_{ij} + \frac{nS_i p_j}{2np}\,, \tag{3.38}$$

and the matter parts of the gravitational constraints given by

$$\mathcal{H}^{\text{matter}} = -np\,\delta - K^{ij}\,\frac{p_i nS_j}{np}\,\delta - (nS^k\delta)_{;k}\,, \tag{3.39}$$

$$\mathcal{H}_i^{\text{matter}} = (p_i + K_{ij}nS^j)\delta + \left(\frac{1}{2}\gamma^{jk}S_{ik}\delta + \gamma^{jk}\,\frac{p_{(i}nS_{k)}}{np}\,\delta\right)_{;j}\,. \tag{3.40}$$

These coincide with the densitized projections

$$\mathcal{H}^{\text{matter}} = \sqrt{\gamma}T_{\mu\nu}n^\mu n^\nu\,, \qquad \mathcal{H}_i^{\text{matter}} = -\sqrt{\gamma}T_{i\nu}n^\nu\,, \tag{3.41}$$

of the stress-energy tensor (2.30) at linear order in spin. For consistency this must of course be the case, as the gravitational constraints can also be obtained by such projections of the Einstein equations directly, instead of by varying the action with respect to N and N^i. However, in spite of the simplifying premature (but only partial) gauge fixing (3.34) of the tetrad, the matter Lagrangian (3.37) is still complicated compared to the nonspinning case (2.68). In particular, the canonical structure is not immediately visible in the used variables.

3.2.2 Canonical matter Variables

One already knows from special relativity that the variables in the covariant spin supplementary condition have quite complicated Poisson brackets. Thus the complicated structure of the matter action in these variables found in the last section is not surprising. We will now try to simplify the structure of the matter Lagrangian by introducing new variables, which will turn out to possess standard canonical Poisson brackets. These new variables are indicated by a hat. An intuitive guess from the special relativistic case (2.20) is

$$z^i = \hat{z}^i - \frac{nS^i}{m-np}\,, \qquad nS_i = -\frac{p_k\gamma^{kj}\hat{S}_{ji}}{m}\,, \qquad S_{ij} = \hat{S}_{ij} - \frac{p_i nS_j}{m-np} + \frac{p_j nS_i}{m-np}\,, \tag{3.42}$$

belonging to the condition (2.34), as well as

$$\Lambda^{[i](j)} = \hat{\Lambda}^{[i](k)}\left(\delta_{kj} + \frac{p_{(k)}p^{(j)}}{m(m-np)}\right), \tag{3.43}$$

see (3.60c) in [69]. These redefinitions replace A^{kl} in (3.37) by the quantity $\hat{A}^{ij}$ given by

$$\gamma_{ik}\gamma_{jl}\hat{A}^{kl} = \frac{1}{2}\hat{S}_{ij} + \frac{mp_{(i}nS_{j)}}{np(m-np)}\,. \tag{3.44}$$

Then the first line of (3.37) suggests to introduce a new linear momentum for the matter as

$$\hat{p}_i = p_i + K_{ij}nS^j + \hat{A}^{kl}e_{(j)k}e^{(j)}{}_{l,i} - \left(\frac{1}{2}S_{kj} + \frac{p_{(k}nS_{j)}}{np}\right)\Gamma^{kj}{}_i\,, \tag{3.45}$$

which reduces to $\hat{p}_i = p_i$ in the special relativistic case. The matter Lagrangian now turns into (still approximating linear in spin)

$$L_M = \hat{p}_i\dot{\hat{z}}^i + \frac{1}{2}\hat{S}_{(i)(j)}\hat{\Omega}^{(i)(j)} - H_M\,, \tag{3.46}$$

where $\hat{\Omega}^{(i)(j)} = \hat{\Lambda}_{[k]}{}^{(i)}\dot{\hat{\Lambda}}^{[k](j)}$ and

$$H_M = -\hat{A}^{ij}e_{(k)i}e^{(k)}{}_{j,0} + \int \mathrm{d}^3x\,(N\mathcal{H}^{\text{matter}} - N^i\mathcal{H}_i^{\text{matter}})\,. \tag{3.47}$$

Notice that $\hat{\Lambda}^{[i](k)}$ is a 3-dimensional rotation matrix, $\hat{\Lambda}_{[k]}{}^{(i)}\hat{\Lambda}^{[k](j)} = \delta_{ij}$. Therefore $\hat{\Omega}^{(i)(j)}$ is antisymmetric and should be interpreted as an angular velocity tensor. The action thus has the canonical structure momenta times velocities minus Hamiltonian H_M. The Poisson brackets for the matter part read

$$\{\hat{z}^i, \hat{p}_j\} = \delta_{ij}\,, \qquad \{\hat{\Lambda}^{[i](j)}, \hat{S}_{(k)(l)}\} = \hat{\Lambda}^{[i](k)}\delta_{lj} - \hat{\Lambda}^{[i](l)}\delta_{kj}\,, \tag{3.48}$$

$$\{\hat{S}_{(i)(j)}, \hat{S}_{(k)(l)}\} = \delta_{ik}\hat{S}_{(j)(l)} - \delta_{jk}\hat{S}_{(i)(l)} - \delta_{il}\hat{S}_{(j)(k)} + \delta_{jl}\hat{S}_{(i)(k)}\,, \tag{3.49}$$

all other zero, similar to (3.6, 3.7). It is important that all extrinsic curvature terms are eliminated from (3.37, 3.39, 3.40) by the redefinition of the linear momentum (3.45). Terms of this type are the reason for potential problems with derivative-coupled theories [57], so it is good that they disappear. This is similar to the Dirac field case, which can be made a nonderivative-coupled theory by a redefinition of the Dirac field. Further the $\dot{p}_i$-term in (3.37) was removed by the redefinition of the position (3.42).

If we consider test spinning bodies in an external field, then one immediately gets the fully reduced Hamiltonian in the time gauge by inserting the metric (i.e., γ_{ij}, N, and N^i) as well as a suitable triad $e_{(k)i}$ (subject only to $e_{(k)i}e^{(k)}{}_j = \gamma_{ij}$) into (3.47). Canonical formulations of test spinning bodies were already obtained in [67] by a direct construction of the symplectic structure and also very recently in [44] using a Dirac bracket approach. In the latter paper the Hamiltonian was explicitly obtained for the Kerr metric. In the next section we will also be able to put the field part into canonical form.

Given the fact that, at least in the time gauge (3.34), the supplementary condition (2.34) leads to a canonical spin and position variable, it seems to be simpler to immediately start with an action implementing (2.34), thus skipping the need for variable redefinitions. However, one can not be sure in advance that (2.34) leads to canonical variables. Further, it should be noted that only the *structure* of the action was simplified by above redefinitions. The redefinitions still have to be applied to (3.39, 3.40), making these expressions more complicated, see (6.33–6.35). Thus one has a conservation of trouble here and starting directly with (2.34) does not seem to simplify the calculation. In fact, it could be subtle to correctly implement the noncovariant condition (2.34) into the action. However, this succeeded for test spinning objects in [44].

3.3 Full gauge reduction

The discussion of the field part is not as simple as for nonspinning objects. First, we need the tetrad form of the ADM formalism as derived in [62]. Second, the matter action depends on the partial time derivative of the tetrad, which necessitates matter corrections to the canonical field momentum. Indeed, the canonical momentum conjugate to $e_{(k)j}$ is given by

$$\bar{\pi}^{(k)j} = 8\pi \frac{\partial(\mathcal{L}_G + \mathcal{L}_M)}{\partial e_{(k)j,0}} = e^{(k)}{}_i\,\pi^{ij} + 8\pi e^{(k)}{}_i\,\hat{A}^{ij}\,\hat{\delta}\,, \tag{3.50}$$

where $\mathcal{L}_M$ is the density version of (3.46), obtained by introducing $\hat{\delta} = \delta(x^i - \hat{z}^i)$ in certain terms, and π^{ij} is still given by (2.41). Remember that (3.39, 3.40) do not contain the extrinsic curvature after redefining the matter variables. Legendre transformation leads to

$$W = \frac{1}{8\pi}\int \mathrm{d}^4x\,\bar{\pi}^{(k)j}e_{(k)j,0} + \int \mathrm{d}t\left[\hat{p}_i\dot{\hat{z}}^i + \frac{1}{2}\hat{S}_{(i)(j)}\hat{\Omega}^{(i)(j)} - H\right], \tag{3.51}$$

$$H = \int \mathrm{d}^3x\left(N\mathcal{H} - N^i\mathcal{H}_i + \lambda_{ij}\pi^{[ij]}\right) + E[\gamma_{ij}]\,, \tag{3.52}$$

where $\mathcal{H} \equiv \mathcal{H}^{\text{field}} + \mathcal{H}^{\text{matter}}$ and $\mathcal{H}_i \equiv \mathcal{H}_i^{\text{field}} + \mathcal{H}_i^{\text{matter}}$ with (2.44) and (3.39, 3.40). In tetrad gravity one has the additional constraint $\pi^{[ij]} = 0$, or $\bar{\pi}^{[ij]} = 8\pi\hat{A}^{[ij]}\hat{\delta}$, which was added to the Hamiltonian H via a Lagrange multiplier $\lambda_{ij} = -\lambda_{ji}$.

3.3.1 Spatial symmetric gauge

The constraint $\pi^{[ij]} = 0$ is eliminated by a further partial gauge fixing now. The spatial symmetric gauge for the triad $e_{(i)j} = e_{ij} = e_{ji}$ is imposed, which was suggested by Kibble for a canonical formulation of the Dirac field coupled to gravity [6] (however, Kibble was using the Schwinger canonical formalism [55]). In this gauge, the triad is the symmetric matrix square-root of the positive definite induced metric, $e_{ij}e_{jk} = \gamma_{ik}$, or

$$(e_{ij}) = \sqrt{(\gamma_{ij})} \,. \tag{3.53}$$

Thus the triad is fully given in terms of the metric, which is now the variable to be varied. We may therefore define an object B^{kl}_{ij} as

$$2B^{kl}_{ij} = e_{mi}\frac{\partial e_{mj}}{\partial \gamma_{kl}} - e_{mj}\frac{\partial e_{mi}}{\partial \gamma_{kl}} \,, \tag{3.54}$$

which enables us to write

$$e^{(k)}{}_i e_{(k)j,\mu} = B^{kl}_{ij}\gamma_{kl,\mu} + \frac{1}{2}\gamma_{ij,\mu} \,. \tag{3.55}$$

The action obviously takes on the form

$$W = \frac{1}{16\pi}\int \mathrm{d}^4x\, \hat{\pi}^{ij}\gamma_{ij,0} + \int \mathrm{d}t\left[\hat{p}_i\dot{\hat{z}}^i + \frac{1}{2}\hat{S}_{(i)(j)}\hat{\Omega}^{(i)(j)} - H\right], \tag{3.56}$$

$$H = \int \mathrm{d}^3x\,\left(N\mathcal{H} - N^i\mathcal{H}_i\right) + E[\gamma_{ij}] \,, \tag{3.57}$$

with the new canonical field momentum conjugate to γ_{ij} given by

$$\hat{\pi}^{ij} = \pi^{ij} + 8\pi\hat{A}^{(ij)}\hat{\delta} + 16\pi B^{ij}_{kl}\hat{A}^{[kl]}\hat{\delta} \,. \tag{3.58}$$

We have thus reduced the tetrad form of the ADM formalism to its metric form, still coupled to spinning objects.

3.3.2 ADM transverse-traceless gauge

Finally, the gauge fixing for the induced metric follows along the same lines as for nonspinning objects in Sect. 2.3.2. We apply the gauge conditions

$$\partial_j(\gamma_{ij} - \tfrac{1}{3}\gamma_{kk}\delta_{ij}) = 0 \,, \qquad \hat{\pi}^{ii} = 0 \,. \tag{3.59}$$

However, notice that the ADM transverse-traceless condition for the canonical field momentum $\hat{\pi}^{ii} = 0$ differs from the original one, $\pi^{ii} = 0$. Correspondingly we now have the decomposition

$$\hat{\pi}^{ij} = \tilde{\hat{\pi}}^{ij} + \hat{\pi}^{ij\mathrm{TT}} \,, \qquad \tilde{\hat{\pi}}^{ij} = \tilde{\hat{\pi}}^{i}{}_{,j} + \tilde{\hat{\pi}}^{j}{}_{,i} - \frac{1}{2}\delta_{ij}\tilde{\hat{\pi}}^{k}{}_{,k} - \frac{1}{2}\Delta^{-1}\tilde{\hat{\pi}}^{k}{}_{,ijk} \,, \tag{3.60}$$

instead of (2.51). The decomposition for the metric (2.50) is still valid. The ADM Hamiltonian then results from solving the field constraints $\mathcal{H} = 0 = \mathcal{H}_i$ together with the gauge conditions as

$$H_{\mathrm{ADM}} = E[\hat{z}^i, \hat{p}_i, \hat{S}_{(i)(j)}, h^{\mathrm{TT}}_{ij}, \hat{\pi}^{ij\mathrm{TT}}] = -\frac{1}{16\pi}\int \mathrm{d}^3x\,\Delta\phi \,, \tag{3.61}$$

and the fully reduced Poisson brackets of the field read

$$\{h_{ij}^{\mathrm{TT}}(\mathbf{x}), \hat{\pi}^{kl\mathrm{TT}}(\mathbf{x}')\} = 16\pi\delta_{ij}^{\mathrm{TT}kl}\delta(\mathbf{x}-\mathbf{x}')\,, \tag{3.62}$$

all other zero. The Poisson brackets (3.48, 3.49) of course still hold. The fully reduced action finally reads

$$W = \frac{1}{16\pi}\int \mathrm{d}^4x\,\hat{\pi}^{ij\mathrm{TT}}h_{ij,0}^{\mathrm{TT}} + \int \mathrm{d}t\left[\hat{p}_i\dot{\hat{z}}^i + \frac{1}{2}\hat{S}_{(i)(j)}\hat{\Omega}^{(i)(j)} - H_{\mathrm{ADM}}\right]. \tag{3.63}$$

This is the extension of the nonspinning case in (2.71). The new spin interactions enter via the ADM Hamiltonian H_{ADM} after solving the constraints, which now have spin corrections in its source terms, (3.39, 3.40).

4 Symmetry generator approach

As we have seen in Sect. 2.3 and also in the last section, after all constraints as well as supplementary and gauge conditions have been eliminated, the Hamiltonian is given by the ADM energy depending on the fully reduced canonical variables. However, while it is not problematic to calculate the ADM energy at least to some order in a perturbative way, it will then depend on the variables appearing in the stress-energy tensor (2.30) and equation of motion (2.28), for which the canonical structure is not known. If one could someway find the transformation between these variables and fully reduced canonical variables with usual Poisson brackets, then the ADM energy can be expressed in terms of canonical variables and turns into the ADM Hamiltonian. In this section we try to construct this variable transformation order-by-order in some perturbation scheme by looking at certain consistency conditions. It is expected that if one proceeds to higher and higher orders, then one also needs to devise more and more consistency conditions. However, in the post-Newtonian approximation one may reach an order high enough for all currently relevant applications by just relying on a specific form of total linear and angular momentum expressed in terms of canonical variables [14,15]. Notice that this approach is not as powerful as the action approach [13] discussed in the last section, however, it succeeded earlier and is still valuable at higher orders in spin as well as for a check of the action approach at linear order in spin.

4.1 Symmetries and conserved quantities

Now the symmetries and corresponding conserved quantities for asymptotically flat spacetimes are reviewed. These conserved quantities generate their symmetries on phase space. For total linear and angular momentum this leads to a very specific form when expressed in terms of canonical variables.

4.1.1 Global rotations and translations

It is intuitively clear that an asymptotically flat spacetime can be transformed into a physically equivalent one by a 3-dimensional rotation and/or translation of each 3-dimensional hypersurface, i.e., of the whole spacetime. This means that asymptotically flat spacetimes posses a global symmetry[12] under rotations and translations, i.e., under the 3-dimensional Euclidean group. In fact, one even has a global symmetry under the Poincaré group, which will be discussed in Sect. 4.1.3. How the symmetry under the Euclidean group is represented on the coordinates crucially depends on the chosen coordinate system even in flat space. If the coordinate system resembles to a Cartesian one in the asymptotics, then a good guess for the symmetry transformation is $x^i \to \Lambda_{ij}(x^j + a^j)$, with x^i the coordinates of the 3-dimensional hypersurfaces, a^i a constant vector describing a translation, and a rotation matrix Λ_{ij}. The rotation matrix is parametrized by a

[12] A global symmetry depends on parameters which may not vary over spacetime.

constant antisymmetric matrix $\omega^{ij} = -\omega^{ji}$ in the form $\Lambda = e^{\omega}$. A field, e.g., the induced metric γ_{ij}, then transforms as

$$\gamma_{ij}(\mathbf{x}) \to \Lambda_{ik}\Lambda_{jl}\gamma_{kl}(\Lambda^{-1}\mathbf{x} - \mathbf{a})\,, \tag{4.1}$$

where the vector $\mathbf{a}$ has components a^i. However, for this transformation to be a global symmetry and not just a particular gauge transformation, the gauge conditions must be invariant under this representation of the Euclidean group. This is indeed fulfilled for the ADM gauge conditions (2.49) or (3.59) (remember that a^i and Λ_{ij} are *constant*). Further, the local basis shall rotate the same way as the coordinate basis, i.e., $e_{(i)j}(\mathbf{x}) \to \Lambda_{ik}\Lambda_{jl}e_{(k)l}(\Lambda^{-1}\mathbf{x} - \mathbf{a})$. We assume here that the tetrad was reduced to a triad with the help of the time gauge (3.34), as in the action approach. The triad gauge shall be compatible with this transformation property, which is the case for (3.53).

Looking at infinitesimal transformations, i.e., a^i and ω^{ij} shall be small, it holds

$$x^i \to x^i + a^i + \omega^{ij}x^j\,, \tag{4.2}$$

or for a tensor field (4.1)

$$\gamma_{ij} \to \gamma_{ij} - a^k\partial_k\gamma_{ij} - \omega^{kl}x^l\partial_k\gamma_{ij} + \omega^{ik}\gamma_{kj} + \omega^{jk}\gamma_{ik}\,. \tag{4.3}$$

This is just the Lie-shift given by the infinitesimal coordinate transformation (4.2), i.e., $\gamma_{ij} \to \gamma_{ij} - \mathcal{L}_{\delta x^k}\gamma_{ij}$. Similarly, the canonical variables transform as

$$\hat{z}^i_a \to \hat{z}^i_a + a^i + \omega^{ij}\hat{z}^j_a\,, \qquad \hat{p}_{ai} \to \hat{p}_{ai} + \omega^{ij}\hat{p}_{aj}\,, \tag{4.4}$$

$$\hat{\Lambda}^{[i](j)}_a \to \hat{\Lambda}^{[i](j)}_a + \omega^{jk}\hat{\Lambda}^{[i](k)}_a\,, \qquad \hat{S}_{a(i)(j)} \to \hat{S}_{a(i)(j)} + \omega^{im}\hat{S}_{a(m)(j)} + \omega^{jm}\hat{S}_{a(i)(m)}\,, \tag{4.5}$$

$$h^{\mathrm{TT}}_{ij} \to h^{\mathrm{TT}}_{ij} - a^k\partial_k h^{\mathrm{TT}}_{ij} - \omega^{kl}x^l\partial_k h^{\mathrm{TT}}_{ij} + \omega^{ik}h^{\mathrm{TT}}_{kj} + \omega^{jk}h^{\mathrm{TT}}_{ik}\,, \tag{4.6}$$

$$\hat{\pi}^{ij\mathrm{TT}} \to \hat{\pi}^{ij\mathrm{TT}} - a^k\partial_k\hat{\pi}^{ij\mathrm{TT}} - \omega^{kl}x^l\partial_k\hat{\pi}^{ij\mathrm{TT}} + \omega^{ik}\hat{\pi}^{kj\mathrm{TT}} + \omega^{jk}\hat{\pi}^{ik\mathrm{TT}}\,. \tag{4.7}$$

A label index was attached to the matter variables now. In (4.5) the transformation property of the local basis was used. Notice that the body-fixed basis in (4.5) stays unchanged. For (4.6) and (4.7) the transverse-traceless projection was commuted with the infinitesimal coordinate change.

4.1.2 Symmetry generators

Now we try to construct the generators of infinitesimal rotations and translations, P_i and J_{ji}. These are of course nothing else than 3-dimensional total linear and angular momentum. With the help of these generators the transformation rule for an arbitrary phase space function A must read

$$A \to A + \tfrac{1}{2}\omega^{ij}\{A, J_{ji}\} + a^i\{A, P_i\}\,. \tag{4.8}$$

It is sufficient to guarantee this transformation rule for all canonical variables. Comparing (4.8) with (4.4–4.7), using the standard Poisson brackets (3.48, 3.49) for each object as well as (3.62), one can indeed construct P_i and J_{ij}. It is immediately clear that P_i and J_{ij} are a sum of matter and field parts,

$$P_i = P^{\mathrm{matter}}_i + P^{\mathrm{field}}_i\,, \qquad J_{ij} = J^{\mathrm{matter}}_{ij} + J^{\mathrm{field}}_{ij}\,. \tag{4.9}$$

In order to get P_i, one sets $\omega^{ij} = 0$ and a^i is taken to be arbitrary. Then among the matter variables only $\hat{z}^i_a$ is transformed. Comparing (4.8) with (4.4) one obtains $\delta_{ij} = \{\hat{z}^i_a, P_j\} = \frac{\partial P_j}{\partial \hat{p}_{ai}}$ for each particle, and thus

$$P^{\mathrm{matter}}_i = \sum_a \hat{p}_{ai}\,. \tag{4.10}$$

Similarly, for the field part one gets $-\partial_k h_{ij}^{\mathrm{TT}} = \{h_{ij}^{\mathrm{TT}}, P_k\}$ as well as $-\partial_k \hat{\pi}^{ij\mathrm{TT}} = \{\hat{\pi}^{ij\mathrm{TT}}, P_k\}$, which leads to

$$P_i^{\text{field}} = -\frac{1}{16\pi}\int \mathrm{d}^3x\, \hat{\pi}^{kl\mathrm{TT}} h_{kl,i}^{\mathrm{TT}}\,. \tag{4.11}$$

The derivation of J_{ij} is analogous, with the result

$$J_{ij}^{\text{matter}} = \sum_a (\hat{z}_a^i \hat{p}_{aj} - \hat{z}_a^j \hat{p}_{ai}) + \sum_a \hat{S}_{a(i)(j)}\,, \tag{4.12}$$

$$J_{ij}^{\text{field}} = -\frac{1}{16\pi}\int \mathrm{d}^3x\, (x^i \hat{\pi}^{kl\mathrm{TT}} h_{kl,j}^{\mathrm{TT}} - x^j \hat{\pi}^{kl\mathrm{TT}} h_{kl,i}^{\mathrm{TT}}) - \frac{1}{16\pi}\int \mathrm{d}^3x\, 2(\hat{\pi}^{ik\mathrm{TT}} h_{kj}^{\mathrm{TT}} - \hat{\pi}^{jk\mathrm{TT}} h_{ki}^{\mathrm{TT}})\,. \tag{4.13}$$

The ADM Hamiltonian H_{ADM} is by construction manifestly invariant under global rotations and translations (at least in the considered gauges). Comparing $H_{\text{ADM}} \to H_{\text{ADM}}$ with (4.8) one sees that total linear and angular momentum have vanishing Poisson brackets with the ADM Hamiltonian and are thus conserved.

Yet another symmetry specific to objects with spin is given by constant rotations of the body-fixed frame,

$$\hat{\Lambda}_a^{[i](j)} \to \hat{\Lambda}_a^{[i](j)} + \omega_a^{[i][k]} \hat{\Lambda}_a^{[k](j)}\,, \tag{4.14}$$

parametrized by a constant antisymmetric matrix $\omega_a^{[i][j]} = -\omega_a^{[j][i]}$ for each object. The corresponding generators read $J_{a[i][j]}^{\text{body}} = \hat{\Lambda}_a^{[i](k)} \hat{\Lambda}_a^{[j](l)} \hat{S}_{a(k)(l)}$ and are also conserved quantities, $J_{a[i][j]}^{\text{body}} = \text{const}$. A corollary of this is that

$$J_{a[i][j]}^{\text{body}} J_{a[i][j]}^{\text{body}} = \hat{S}_{a(i)(j)} \hat{S}_{a(i)(j)} = \text{const}\,. \tag{4.15}$$

As the ADM Hamiltonian H_{ADM} is invariant under the transformations (4.4–4.7) and (4.14), these transformations are also a symmetry of the action (3.63). The corresponding conserved quantities P_i, J_{ij}, and $J_{a[i][j]}^{\text{body}}$ can then be obtained by standard Noether arguments [90] and come out identical to above results.

4.1.3 Global Poincaré invariance

The global symmetry under the Euclidean group discussed in the last section is only a part of the bigger global symmetry under the Poincaré group. Besides total linear and angular momentum, also the boost vector J^{i0} and the energy $E = H_{\text{ADM}}$ of the system generate a symmetry of the action and are conserved quantities for asymptotically flat spacetimes. However, the infinitesimal transformations generated by H_{ADM} and J^{i0}, similar to (4.8), are in general highly nonlinear in the considered gauges and may not be written down immediately, as opposed to (4.4–4.7). Further, J^{i0} explicitly depends on time, see (2.12). H_{ADM} and J^{i0} can be calculated by surface integrals at spatial infinity, see, e.g., [54, 91]. For the total energy $E = H_{\text{ADM}}$ this was already found in (2.45) and the boost vector J^{i0} is given by (2.12) with

$$G^i = \frac{1}{16\pi}\oint \mathrm{d}^2 s_k \left[x^i(\gamma_{kl,l} - \gamma_{ll,k}) - \gamma_{ik} + \delta_{ik}\gamma_{ll}\right]\,. \tag{4.16}$$

Similarly, for 3-dimensional total linear and angular momentum it holds

$$P_i = -\frac{1}{8\pi}\oint \mathrm{d}^2 s_k \pi^{ik}\,, \qquad J_{ij} = -\frac{1}{8\pi}\oint \mathrm{d}^2 s_k (x^i \pi^{jk} - x^j \pi^{ik})\,. \tag{4.17}$$

When these quantities are expressed in terms of canonical variables (after gauge fixing), they fulfill the Poincaré algebra (2.10, 2.11). Notice that all Poisson brackets in (2.10, 2.11) involving P_i and J_{ij} just reflect the transformation property (4.8). Similar to the special relativistic case in Sect. 2.1, one can define different total spins and centers for a gravitating system in asymptotically flat spacetimes. In particular, a center and total spin of the system with standard Poisson brackets can be constructed (this was exploited recently in [92]).

4.1.4 Symmetry generators from integral formulas

For simplicity we assume that γ_{ij} does not need a redefinition in order to receive a canonical meaning. However, this might be necessary at higher orders in spin. For the canonical field momentum $\hat{\pi}^{ij}$ we allow spin corrections by the ansatz

$$\hat{\pi}^{ij} = \pi^{ij} + 16\pi \sum_a \pi_a^{ij} \hat{\delta}_a \,, \tag{4.18}$$

where π_a^{ij} contains the yet undetermined spin corrections. The gauge condition $\hat{\pi}^{ii} = 0$ with the subsequent decomposition (3.60) is assumed to hold. The surface integrals from the last section can be transformed into volume integrals using the Gauss theorem. With the decomposition (2.50) it follows

$$E = -\frac{1}{16\pi} \int \mathrm{d}^3x \, \Delta\phi \,, \qquad G^i = -\frac{1}{16\pi} \int \mathrm{d}^3x \, x^i \Delta\phi \,. \tag{4.19}$$

However, it is not possible to express E and G^i in terms of the canonical variables without solving the nonlinear constraint equations for ϕ. Similarly one gets

$$P_i = -\frac{1}{8\pi} \int \mathrm{d}^3x \, \hat{\pi}^{ik}{}_{,k} \,, \qquad J_{ij} = -\frac{1}{8\pi} \int \mathrm{d}^3x \, (x^i \hat{\pi}^{jk}{}_{,k} - x^j \hat{\pi}^{ik}{}_{,k}) \,. \tag{4.20}$$

Here one can exploit the momentum constraint $\mathcal{H}_i \equiv \mathcal{H}_i^{\text{field}} + \mathcal{H}_i^{\text{matter}} = 0$ to further evaluate P_i and J_{ij} without needing to actually solve the constraints. Using (2.44, 2.50, 3.60) the momentum constraint can *exactly* be written as

$$\hat{\pi}^{ik}{}_{,k} = -8\pi(\mathcal{H}_i^{\text{matter}} + \mathcal{H}_i^{\pi\text{matter}}) + \frac{1}{2}\hat{\pi}^{jk\text{TT}} h^{\text{TT}}_{jk,i} - (\hat{\pi}^{jk\text{TT}} h^{\text{TT}}_{ki})_{,j} - \Delta\left(\hat{V}^k h^{\text{TT}}_{ki}\right) + \hat{B}^{ij}{}_{,j} \,, \tag{4.21}$$

with the definitions

$$\mathcal{H}_i^{\pi\text{matter}} = \sum_a \left[\pi_a^{jk} \gamma_{jk,i} \hat{\delta}_a - 2(\gamma_{ik} \pi_a^{kj} \hat{\delta}_a)_{,j}\right] , \tag{4.22}$$

$$\hat{B}^{ij} = \left[1 - \left(1 + \tfrac{1}{8}\phi\right)^4\right] (\hat{\tilde{\pi}}^{ij} + \hat{\pi}^{ij\text{TT}}) + \hat{V}^k (h^{\text{TT}}_{ki,j} + h^{\text{TT}}_{kj,i} - h^{\text{TT}}_{ij,k}) - \frac{1}{3}\hat{V}^k{}_{,k} h^{\text{TT}}_{ij} \,, \tag{4.23}$$

and the alternative vector potential

$$\hat{V}^i = \left(\delta_{ij} - \frac{1}{4}\partial_i \partial_j \Delta^{-1}\right) \hat{\tilde{\pi}}^j \,, \tag{4.24}$$

for which it holds

$$\hat{\tilde{\pi}}^{ij} = \hat{V}^i{}_{,j} + \hat{V}^j{}_{,i} - \frac{2}{3}\delta_{ij} \hat{V}^k{}_{,k} \,. \tag{4.25}$$

One can calculate $\mathcal{H}_i^{\text{matter}}$ using (3.41). Notice that $\hat{B}^{ij} = \hat{B}^{ji}$ and $\hat{B}^{ii} = 0$. Further the last two terms in (4.21) do not contribute to (4.20). Obviously, (4.20) are a sum of matter and field parts, (4.9). The field parts are identical to (4.11) and (4.13). However, the matter parts now read

$$P_i^{\text{matter}} = \int \mathrm{d}^3x \, (\mathcal{H}_i^{\text{matter}} + \mathcal{H}_i^{\pi\text{matter}}) \,, \tag{4.26}$$

$$J_{ij}^{\text{matter}} = \int \mathrm{d}^3x \, (x^i \mathcal{H}_j^{\text{matter}} + x^i \mathcal{H}_j^{\pi\text{matter}} - x^j \mathcal{H}_i^{\text{matter}} - x^j \mathcal{H}_i^{\pi\text{matter}}) \,. \tag{4.27}$$

For consistency, these must turn into (4.10) and (4.12) when expressed in terms of canonical variables.

4.2 Construction of canonical variables

In this section we will formulate the important consistency conditions and apply them order-by-order in the post-Newtonian approximation to find canonical variables.

4.2.1 Consistency conditions

In Sect. 2.2.3 it was seen that the spin length S_a given by $2S_a^2 = S_{a\mu\nu}S_a^{\mu\nu}$ is a conserved quantity in the covariant spin supplementary condition. This can also be derived from the action (3.30) using the symmetry under constant 4-dimensional Lorentz transformations of the body-fixed frame, see also [69]. This conserved quantity must be identical to the one in (4.15), as both were derived from the same symmetry (though only the 3-dimensional rotation part is relevant after the supplementary conditions were eliminated). Thus it must hold

$$S_{a\mu\nu}S_a^{\mu\nu} = \hat{S}_{a(i)(j)}\hat{S}_{a(i)(j)} \,, \tag{4.28}$$

providing a relation between covariant spin $S_{a\mu\nu}$ and canonical spin $\hat{S}_{a(i)(j)}$. This is one important consistency condition we will impose.

Further, one can calculate $\mathcal{H}_i^{\text{matter}}$ and thus (4.26, 4.27) in terms of (noncanonical) variables in the covariant supplementary condition with the help of (3.41) and (2.30),

$$\mathcal{H}_i^{\text{matter}} = \sum_a \left[(p_{ai} + K_{ij} n S_a^j)\delta_a + \left(\frac{1}{2}\gamma^{jk} S_{aik}\delta_a + \gamma^{jk} \frac{p_{a(i} n S_{ak)}}{n p_a} \delta_a \right)_{;j} \right] , \tag{4.29}$$

see (3.40). Then (4.26, 4.27) must coincide with (4.10, 4.12), leading to conditions on the transformation between canonical variables and variables in the covariant supplementary condition. We write this as a condition on $\mathcal{H}_i^{\text{matter}}$ in the form

$$\mathcal{H}_i^{\text{matter}} = \sum_a \left[(\hat{p}_{ai} - \pi_a^{jk}\gamma_{jk,i})\hat{\delta}_a + \frac{1}{2}(s_a^{ij}\hat{\delta}_a)_{,j} \right] , \tag{4.30}$$

where the symmetric part of s_a^{ij} is not constrained, but it has to hold

$$s_a^{[ij]} = \hat{S}_{a(i)(j)} + 2\pi_a^{jk} h_{ki}^{\text{TT}} - 2\pi_a^{ik} h_{kj}^{\text{TT}} \,. \tag{4.31}$$

This condition on $\mathcal{H}_i^{\text{matter}}$ is the most general one[13] that guarantees that (4.26, 4.27) coincide with (4.10, 4.12).

Above conditions are sufficient for the post-Newtonian order considered here. Another condition that could be useful at even higher orders (in particular also higher orders in spin) would be the fulfillment of the Poincaré algebra. However, all Poisson brackets in (2.10, 2.11) involving P_i and J_{ij} are fulfilled by construction due to the transformation property (4.8) if above conditions hold, thus giving nothing new. In [14] it was considered whether the construction of the constraint algebra (2.72–2.74), which is related to diffeomorphism invariance and thus more fundamental than global Poincaré invariance, could be used to construct canonical variables. However, this approach seems to be unmanageable.

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[13] In the pole-dipole approximation at most one partial derivative can appear in $\mathcal{H}_i^{\text{matter}}$. Further it was assumed that the variables from different objects do not mix (e.g., as $\hat{p}_1\hat{\delta}_2$) at this stage.

4.2.2 Canonical variables

First we evaluate the condition on the spin length given by (4.28). We will first construct a specific transformation between S_{aij} and $\hat{S}_{a(i)(j)}$ and then discuss its uniqueness. Inspired by the flat space case (2.20), we first apply the transformation

$$S_{aij} = \hat{S}_{aij} - \frac{p_{ai} nS_{aj}}{m_a - np_a} + \frac{p_{aj} nS_{ai}}{m_a - np_a}\,, \qquad nS_{ai} = -\frac{p_{ak}\gamma^{kj}\hat{S}_{aji}}{m_a}\,, \tag{4.32}$$

to the conserved quantity $S_{a\mu\nu}S_a^{\mu\nu} = \gamma^{ki}\gamma^{lj}S_{akl}S_{aij} - 2\gamma^{ij}nS_{ai}nS_{aj}$, with the result $S_{a\mu\nu}S_a^{\mu\nu} = \gamma^{ki}\gamma^{lj}\hat{S}_{akl}\hat{S}_{aij}$. With the help of an arbitrary triad $e_{(i)j}$ this can be written in a local basis as $S_{a\mu\nu}S_a^{\mu\nu} = \hat{S}_{a(i)(j)}\hat{S}_{a(i)(j)}$, so we have found a possible transformation allowed by (4.28). The ambiguities that are left can best be discussed in terms of the spin vector $\hat{S}_{a(i)}$. As we are still considering the linear order in spin, any further transformation of $\hat{S}_{a(i)}$ must be linear in spin and must leave the expression $\hat{S}_{a(i)}\hat{S}_{a(i)}$ *invariant* (notice $\hat{S}_{a(i)(j)}\hat{S}_{a(i)(j)} = 2\hat{S}_{a(i)}\hat{S}_{a(i)}$). Therefore only a rotation of the spin vector as a further transformation is possible, which can be absorbed into the yet arbitrary triad $e_{(i)j}$.

A comparison of (4.29) with (4.30) leads to

$$\hat{p}_{ai} = p_{ai} + K_{ij} nS_a^j + \pi_a^{jk}\gamma_{jk,i} - \left(\frac{1}{2}S_{akj} + \frac{p_{a(k}nS_{aj)}}{np_a}\right)\Gamma^{kj}{}_i\,, \tag{4.33}$$

without any ambiguity. Now (4.29) is of the form (4.30), so (4.31) is the only condition that is left. In order to evaluate (4.31) we first need to read off s_a^{ij}. For the redefinition of the position variable we use

$$z_a^i = \hat{z}_a^i - \frac{nS_a^i}{m_a - np_a} + z_{\Delta a}^i\,, \tag{4.34}$$

where $z_{\Delta a}^i$ is a yet unknown correction to the flat space case (2.20). Comparing (4.29) expressed in terms of the new variables with (4.30) leads to

$$s_a^{ij} = \gamma^{jk}\hat{S}_{aik} + \gamma^{jk}\gamma^{lp}\frac{2\hat{p}_{al}\hat{p}_{a(i}\hat{S}_{ak)p}}{n\hat{p}_a(m_a - n\hat{p}_a)} - 2\hat{p}_{ai}z_{\Delta a}^j\,, \tag{4.35}$$

with the definition $n\hat{p}_a = -\sqrt{m_a^2 + \gamma^{ij}\hat{p}_{ai}\hat{p}_{aj}}$. The only ambiguities in the transition to canonical variables are now given by π_a^{ij}, $z_{\Delta a}^i$, and the triad $e_{(i)j}$. We try to fix these ambiguities by considering (4.31) with (4.35) order-by-order in the post-Newtonian approximation, which is introduced in the next section.

From the action approach we know that the ambiguity of $e_{(i)j}$ should just be a gauge freedom. Thus different choices for $e_{(i)j}$ should be canonically equivalent. Indeed, it was shown in [84] that a spin rotation is just a canonical transformation at linear order in spin. However, a canonical transformation may change all variables, but $\hat{p}_{ai}$ as well as h_{ij}^{TT} can not be changed any more. Thus the canonical representation was already partly fixed and we must therefore still keep $e_{(i)j}$ as general as allowed by the restriction on the triad gauge made in Sect. 4.1.1.

4.2.3 Post-Newtonian approximation

The idea behind the post-Newtonian approximation is that for slowly moving bodies and weak gravitational forces the Newtonian physics is recovered as a first approximation. For two objects this means that their relative velocity v shall be small compared to the speed of light c. In Newtonian physics the time average of kinetic and potential energy is of the same order if the virial theorem applies, which is the case for bound systems. Then one has

$$\frac{v^2}{c^2} \sim \frac{GM}{c^2 r} \ll 1\,, \tag{4.36}$$

where M is the total mass of the system and r the typical distance of the objects. An expansion in the dimensionless quantities (4.36) obviously is also an expansion in c^{-2}. We will therefore think of the post-Newtonian expansion as an expansion in c^{-2}. However, this is a rather formal point of view as it depends on the choice of units whether c^{-2} is actually a small number (e.g., in our units it is equal to one). As seen later, there may be half post-Newtonian orders corresponding to c^{-1}.

As post-Newtonian orders are formally counted in terms of the velocity of light c originally present in the equations, i.e., before setting $c = 1 = G$, one should introduce G and c back into all expressions. However, this would undo the advantages achieved by setting $c = 1 = G$. Instead, we will assign an order in powers of c^{-1} directly to our variables. When setting $c = 1 = G$ only one unit is needed, which we choose to be the unit of spatial distances, e.g., meters. Then the values of all masses m_a must be given in meters, which is obtained by multiplying their values in kilograms by G/c^2. Therefore we just count the masses to be of the order c^{-2}, as this is the power of the speed of light that would be introduced into the expressions if we restore the original units. Similar arguments apply to the other matter variables and we have the counting rules

$$\hat{z}_a = \mathcal{O}\left(c^0\right), \qquad m_a = \mathcal{O}\left(c^{-2}\right), \qquad \hat{p}_a = \mathcal{O}\left(c^{-3}\right). \tag{4.37}$$

Notice that an energy receives a counting of c^{-4}, which gives the absolute order of the Newtonian Hamiltonian (being an energy) within these counting rules. However, one obtains *different* counting rules for the matter variables if one uses kilograms instead of meters to replace all units when setting $c = 1 = G$. This convention is also often used and leads to different *absolute* orders in c^{-1}, e.g., a mass now receives a counting of c^0 and the Newtonian Hamiltonian is at the absolute order c^{-2}. But relative orders are always the same, so only a counting relative to the Newtonian order (or to the leading order if the Newtonian order vanishes) finally makes sense when using such counting rules. The correct absolute Newtonian order is c^0, as it must prevail when $c^{-1} \to 0$.

The formal counting may nicely be applied to more complicated situations, e.g., when spins are present. For dimensional reasons only we are thus counting the spins of the order c^{-3}. This has some computational advantages, e.g., similarities to calculations for nonspinning objects are more manifest, see Sect. 6.2.2. Here post-Newtonian orders should always be understood in the formal sense if not otherwise stated. However, the spin of a (Kerr) black hole is given by Gm^2a/c, where m is the mass of the black hole and $a = 0 \ldots 1$ is the dimensionless Kerr parameter. The maximal spin of an object is defined as Gm^2/c (which *is* the maximal spin of a black hole, $a = 1$), and additionally has to be counted as c^{-1}. If the spins are maximal, one therefore has to add half a post-Newtonian order relative to the formal counting for each spin variable appearing in a specific expression.

If the spins are not maximal, one has to be careful when classifying spin effects into post-Newtonian orders. For example, if the spin is $\frac{1}{100}$ of the maximal one and the orbital velocity is $\frac{1}{100}$ of the speed of light, then each spin variable corresponds to one extra order in v/c relative to the maximal spin case, or half a post-Newtonian order. At a later time during the inspiral the spin length has not changed much[14], however, the orbital velocity might have increased, e.g., to $\frac{1}{10}$ of the speed of light. Then each spin variable even corresponds to two additional orders in the velocity or one post-Newtonian order relative to the maximal spin case. To conclude, while the spin length does essentially stay constant during the inspiral, the orbital velocity will increase and one expects that spin effects slightly shift to higher post-Newtonian orders during inspiral. Therefore, assigning a post-Newtonian order to spin contributions in the Hamiltonian seems to make no sense in general, except for maximal spins or within the formal counting. However, this discussion is only superficial, the relevance of spin effects also crucially depends on the orientation of the spins and the mass ratio of the objects. Due to these problems we will prefer to classify spin effects by leading order, next-to-leading order, etc. when possible.

Counting rules for other quantities may be derived from (4.37). For example, ϕ results from solving the constraints, and one may easily see that its leading order must be identical to the leading order of the matter

[14] In the approximation considered here the spin length is even exactly constant.

source of the Hamilton constraint $\mathcal{H}^{\text{matter}}$, which is c^{-2} (this will become obvious in Sect. 6.1.2). Similarly one gets counting rules for the other field variables by considering the matter source of the field equations. Without going into detail, we state here that

$$\phi = \mathcal{O}\left(c^{-2}\right), \qquad h^{\mathrm{TT}}_{ij} = \mathcal{O}\left(c^{-4}\right), \qquad \tilde{\pi}^{ij} = \mathcal{O}\left(c^{-3}\right), \qquad \pi^{ij\mathrm{TT}} = \mathcal{O}\left(c^{-5}\right), \tag{4.38}$$

are the correct counting rules for the fields. In general the fields include different post-Newtonian orders, (4.38) only gives the leading orders. The Taylor expansion of the fields in terms of c^{-1} is written as, e.g.,

$$\phi = \phi_{(2)} + \phi_{(4)} + \phi_{(6)} + \mathcal{O}\left(c^{-7}\right), \tag{4.39}$$

where a number in round brackets denotes the absolute order in c^{-1} within the counting given by (4.37) (this should not be confused with indices in the local basis). The vanishing of the odd orders $\phi_{(3)}$ and $\phi_{(5)}$ is explained by the vanishing of the corresponding orders in the source terms.

4.2.4 Final fixation of the canonical variables

First we try to find a way to parametrize the ambiguity in the triad when the induced metric is kept fixed. If one considers the perturbative expansion of $e^{i(k)}e^{j(k)} = \gamma^{ij}$ under the assumption that the leading order is given by $e^{i(k)}_{(0)} = \delta_{ik}$, then one sees that the symmetric part of $e^{i(k)}$ is uniquely fixed at each order, while the antisymmetric part $\hat{e}^{ij} \equiv \frac{1}{2}(e^{i(j)} - e^{j(i)})$ is arbitrary. Therefore $\hat{e}^{ij}$ parametrizes the rotational degrees of freedom left in the definition of the local basis and thus the ambiguity of the canonical spin variable. In particular, the leading post-Newtonian orders read

$$e^{i(j)}_{(2)} = \hat{e}^{ij}_{(2)} - \frac{1}{4}\delta_{ij}\phi_{(2)}, \quad e^{i(j)}_{(4)} = \hat{e}^{ij}_{(4)} - \frac{1}{2}\hat{e}^{ik}_{(2)}\hat{e}^{jk}_{(2)} - \frac{1}{4}\delta_{ij}\phi_{(4)} + \frac{3}{64}\delta_{ij}\phi^2_{(2)} - \frac{1}{2}h^{\mathrm{TT}}_{ij}. \tag{4.40}$$

Notice that $\hat{e}^{ij}$ is needed only on the worldlines. In the following we use the abbreviation $\hat{e}^{ij}_a \equiv \hat{e}^{ij}(\hat{z}^k_a)$.

Next we make an ansatz for π^{ij}_a, $z^i_{\Delta a}$, and $\hat{e}^{ij}_a$ at each post-Newtonian order. For this purpose it is important that π^{ij}_a has the dimension length squared, $z^i_{\Delta a}$ the dimension length, and $\hat{e}^{ij}_a$ is dimensionless. Further, π^{ij}_a and $z^i_{\Delta a}$ must be linear in spin, while $\hat{e}^{ij}_a$ must be independent of the spins. The fields h^{TT}_{ij} and $\hat{\pi}^{ij\mathrm{TT}}$ are always taken at the position $\hat{z}^i_a$ in such an ansatz and $\hat{z}^i_a$ should not appear directly. Of course one also has to take into account that π^{ij}_a must be symmetric and $\hat{e}^{ij}_a$ antisymmetric. Considering possible ansätze under these restrictions we infer that the leading orders are at least $\pi^{ij}_a = \mathcal{O}\left(c^{-5}\right)$, $z^i_{\Delta a} = \mathcal{O}\left(c^{-2}\right)$, and $\hat{e}^{ij}_a = \mathcal{O}\left(c^{-6}\right)$. From (4.35) the first orders of $s^{[ij]}_a$ then follow as

$$s^{[ij]}_{a(3)} = \hat{S}_{a(i)(j)}, \quad s^{[ij]}_{a(5)} = \hat{p}_{aj}z^i_{\Delta a(2)} - \hat{p}_{ai}z^j_{\Delta a(2)}, \quad s^{[ij]}_{a(7)} = \hat{p}_{aj}z^i_{\Delta a(4)} - \hat{p}_{ai}z^j_{\Delta a(4)}. \tag{4.41}$$

Evaluating (4.31) one concludes that $z^i_{\Delta a(2)} = 0$ and $z^i_{\Delta a(4)} = 0$. Thus we have anticipated the correct redefinition of the position (4.34) to this order.

For $s^{[ij]}_{a(9)}$ one has

$$s^{[ij]}_{a(9)} = \hat{e}^{ik}_{a(6)}\hat{S}_{a(k)(j)} - \hat{p}_{ai}z^j_{\Delta a(6)} + \frac{1}{4m_a^2}\hat{p}_{ak}h^{\mathrm{TT}}_{lj}(\hat{p}_{ai}\hat{S}_{a(l)(k)} + \hat{p}_{al}\hat{S}_{a(i)(k)}) - (i \leftrightarrow j), \tag{4.42}$$

where $(i \leftrightarrow j)$ denotes an exchange of the indices i and j. The most general solution of (4.31) under above restrictions is

$$\pi^{ij}_{a(5)} = \frac{1-C}{8m_a^2}(\hat{p}_{ai}\hat{p}_{ak}\hat{S}_{a(k)(j)} + \hat{p}_{aj}\hat{p}_{ak}\hat{S}_{a(k)(i)}), \tag{4.43}$$

$$\hat{e}^{ij}_{a(6)} = \frac{C}{4m_a^2}\hat{p}_{ak}(\hat{p}_{ai}h^{\mathrm{TT}}_{jk} - \hat{p}_{aj}h^{\mathrm{TT}}_{ik}), \quad z^i_{\Delta a(6)} = \frac{C}{4m_a^2}\hat{p}_{aj}(\hat{S}_{a(k)(i)}h^{\mathrm{TT}}_{jk} + \hat{S}_{a(k)(j)}h^{\mathrm{TT}}_{ik}), \tag{4.44}$$

at this order and now depends on an arbitrary constant C.

However, we can remove the ambiguity C by a canonical transformation with an infinitesimal generator

$$g = \frac{C}{4m_a^2}\hat{p}_{ai}\hat{p}_{ak}\hat{S}_{a(k)(j)}\int d^3x\, h_{ij}^{\mathrm{TT}}\hat{\delta}_a\,. \tag{4.45}$$

An arbitrary phase space function A then transforms as $A \to A + \{A, g\}$ to the required order. Applied to the canonical variables one obtains

$$h_{ij}^{\mathrm{TT}} \to h_{ij}^{\mathrm{TT}}\,, \qquad \hat{\pi}^{ij\mathrm{TT}} \to \hat{\pi}^{ij\mathrm{TT}} - \delta_{kl}^{\mathrm{TT}ij}\sum_a \frac{4\pi C}{m_a^2}\hat{p}_{ak}\hat{p}_{am}\hat{S}_{a(m)(l)}\hat{\delta}_a\,, \tag{4.46}$$

$$\hat{S}_{a(i)(j)} \to \hat{S}_{a(i)(j)} - \hat{e}_{a(6)}^{ik}\hat{S}_{a(k)(j)} - \hat{e}_{a(6)}^{jk}\hat{S}_{a(i)(k)}\,, \tag{4.47}$$

$$\hat{z}_a^i \to \hat{z}_a^i - z_{\Delta a(6)}^i\,, \qquad \hat{p}_{ai} \to \hat{p}_{ai} - \frac{C}{4m_a^2}\hat{p}_{al}\hat{p}_{aj}\hat{S}_{a(j)(k)}h_{kl,i}^{\mathrm{TT}}\,. \tag{4.48}$$

This indeed removes all terms depending on C from the source expressions $\mathcal{H}^{\mathrm{matter}}$ and $\mathcal{H}_i^{\mathrm{matter}}$ at the considered order. We can therefore choose $C = 0$, which leads to agreement with the action approach. The triad then is in the spatial symmetric gauge $\hat{e}^{ij} = 0$ and all variable transformations are the same as in the action approach at the considered post-Newtonian order. In particular, using (3.55) and (3.58) in (3.45) leads to (4.33). Further, in the action approach we found that $z_{\Delta a}^i = 0$ and

$$\pi_a^{ij} = \frac{1}{2}\hat{A}_a^{(ij)} + B_{kl}^{ij}\hat{A}_a^{[kl]}\,, \tag{4.49}$$

or more explicitly using (3.44)

$$\pi_a^{ij} = \gamma^{ik}\gamma^{jl}\frac{m_a\hat{p}_{a(k}nS_{al)}}{2n\hat{p}_a(m_a - n\hat{p}_a)} + \frac{1}{2}B_{kl}^{ij}\gamma^{km}\gamma^{ln}\hat{S}_{amn}\,. \tag{4.50}$$

Using $B_{kl}^{ij} = \mathcal{O}(c^{-4})$, cf. Eq. (3.37) in [15], the post-Newtonian expansion of (4.50) agrees with the findings in this section. The check of the action approach given here is valid to the formal 3.5 post-Newtonian order.

5 Higher orders in spin

Higher orders in spin require higher multipole moments, e.g., a black hole has a nonzero quadrupole at the quadratic level in spin [37]. We will constrain to quadrupole and quadratic order in spin in this section. Besides spin-induced quadrupole deformations discussed here, also tidal deformations induced through the gravitational field of other objects have been treated in the literature, see, e.g., [93, 94].

5.1 Quadrupole approximation

The extension of the pole-dipole approximation to higher multipoles was already essentially completed some time ago [41, 95–97], see also [98, 99], most notably by Dixon. It should be stressed that Dixon's method incorporates Mathisson's pioneering ideas [100].

5.1.1 Quadrupole approximation from Tulczyjew's method

A more direct application of Mathisson's ideas to the quadrupole order was given in [16] with the help of W. M. Tulczyjew's method [31], see also [101]. There the quadrupole moment $t^{\mu\nu\alpha\beta}$ was kept in (2.26) when evaluating (2.27). In addition to p^μ and $S^{\mu\nu}$ now various quadrupole moments appear. It is suitable to introduce a reduced quadrupole moment $J^{\mu\nu\alpha\beta}$ with symmetries

$$J^{\nu\rho\beta\alpha} = J^{[\nu\rho][\beta\alpha]} = J^{\beta\alpha\nu\rho}\,, \qquad J^{\nu[\rho\beta\alpha]} = 0 \quad \Leftrightarrow \quad J^{\nu\rho\beta\alpha} + J^{\nu\beta\alpha\rho} + J^{\nu\alpha\rho\beta} = 0\,. \tag{5.1}$$

Thus $J^{\rho\beta\alpha\nu}$ has the same (algebraic) symmetries as the Riemann tensor. This quadrupole moment is able to incorporate all quadrupole contributions from $t^{\mu\nu\alpha\beta}$ that remain after (2.27) was evaluated. The equations of motion then take on the simple form

$$\frac{\mathrm{D}S^{\mu\nu}}{\mathrm{d}\tau} = 2p^{[\mu}u^{\nu]} + \frac{4}{3}R^{(4)}_{\alpha\beta\rho}{}^{[\mu}J^{\nu]\rho\beta\alpha}\,, \quad \frac{\mathrm{D}p_\mu}{\mathrm{d}\tau} = -\frac{1}{2}R^{(4)}_{\mu\rho\beta\alpha}u^\rho S^{\beta\alpha} - \frac{1}{6}R^{(4)}_{\nu\rho\beta\alpha||\mu}J^{\nu\rho\beta\alpha}\,, \tag{5.2}$$

and agree with Dixon [97]. The reduced moment $J^{\mu\nu\alpha\beta}$ is also optimal to give a simplified expression for the stress-energy tensor, reading

$$\begin{aligned}&\sqrt{-g}T^{\mu\nu}\\ &= \int \mathrm{d}\tau\bigg[u^{(\mu}p^{\nu)}\delta_{(4)} + \frac{1}{3}R^{(4)}_{\alpha\beta\rho}{}^{(\mu}J^{\nu)\rho\beta\alpha}\delta_{(4)} + \Big(u^{(\mu}S^{\nu)\alpha}\delta_{(4)}\Big)_{||\alpha} - \frac{2}{3}\Big(J^{\mu\alpha\beta\nu}\delta_{(4)}\Big)_{||(\alpha\beta)}\bigg].\end{aligned} \tag{5.3}$$

In this form the stress-energy tensor was first given in [16]. This stress-energy tensor, together with the ansatz for $J^{\mu\nu\alpha\beta}$ at the quadratic level in spin given below, can be applied to the derivation of the next-to-leading order radiation field, see [102] for the spin-orbit case (the leading order is given in [103]). Besides this formula for the stress-energy tensor, a further interesting result in [16] is the relation between the $t^{\mu\nu\cdots}$ moments and Dixon's reduced moments p^μ, $S^{\mu\nu}$, and $J^{\mu\nu\alpha\beta}$. This relation could be used to study alternatives to Dixon's integral formulas for the multipole moments or to discuss the relation between moments belonging to different representative worldlines (for the latter see section VII.C in [16]).

The spin supplementary condition $S^{\mu\nu}f_\nu = 0$ is preserved in time if

$$p^\mu = \frac{1}{-f_\alpha u^\alpha}\left(-f_\nu p^\nu u^\mu + S^{\mu\nu}\frac{\mathrm{D}f_\nu}{\mathrm{d}\tau} + \frac{4}{3}f_\nu R^{(4)}_{\alpha\beta\rho}{}^{[\mu}J^{\nu]\rho\beta\alpha}\right)\,, \tag{5.4}$$

which should give a relation between p^μ and u^μ. This extends (2.31) to the quadrupole approximation. The extension of (2.35) reads

$$p^\mu = -u_\nu p^\nu u^\mu - \frac{\mathrm{D}(S^{\mu\nu})}{\mathrm{d}\tau}u_\nu + \frac{4}{3}u_\nu R^{(4)}_{\alpha\beta\rho}{}^{[\mu}J^{\nu]\rho\beta\alpha}\,. \tag{5.5}$$

5.1.2 Decomposition of the quadrupole

In order to parametrize the quadrupole deformation due to spin we try to find the most general covariant ansatz for $J^{\mu\nu\alpha\beta}$ quadratic in the spin tensor that is relevant for the post-Newtonian order in question. It is suitable to consider the orthogonal decomposition of $J^{\mu\nu\alpha\beta}$ with respect to the vector f_μ to which the spin is orthogonal, $S^{\mu\nu}f_\nu = 0$. This decomposition reads

$$J^{\nu\rho\beta\alpha} = Q^{\nu\rho\beta\alpha} - \frac{1}{\sqrt{-f_\nu f^\nu}}(f^{[\nu}Q^{\rho]\beta\alpha} + f^{[\alpha}Q^{\beta]\rho\nu}) - \frac{3}{-f_\nu f^\nu}f^{[\nu}Q^{\rho][\beta}f^{\alpha]}\,, \tag{5.6}$$

where $Q^{\nu\rho\beta\alpha}$, $Q^{\rho\beta\alpha}$, and $Q^{\rho\beta}$ are called stress, flow, and mass quadrupole here and are orthogonal to f_μ in each index, see also [104]. They further have the symmetries

$$Q^{\nu\rho\beta\alpha} = Q^{[\nu\rho][\beta\alpha]} = Q^{\beta\alpha\nu\rho}\,, \quad Q^{\nu[\rho\beta\alpha]} = 0 \quad \Leftrightarrow \quad Q^{\nu\rho\beta\alpha} + Q^{\nu\beta\alpha\rho} + Q^{\nu\alpha\rho\beta} = 0\,, \tag{5.7}$$

$$Q^{\rho\beta\alpha} = Q^{\rho[\beta\alpha]}\,, \quad Q^{[\rho\beta\alpha]} = 0 \quad \Leftrightarrow \quad Q^{\rho\beta\alpha} + Q^{\beta\alpha\rho} + Q^{\alpha\rho\beta} = 0\,, \quad Q^{\rho\beta} = Q^{(\rho\beta)}\,. \tag{5.8}$$

In a local basis with f^μ giving the time direction these moments only have spatial components (due to the orthogonality of these moments to f^μ). One may therefore decompose these moments further in the local basis into parts transforming under irreducible representations of the *3-dimensional* rotation group SO(3). For the mass quadrupole this SO(3)-decomposition reads, written in the coordinate frame,

$$Q_{\mu\nu} = Q^{\mathrm{STF}}_{\mu\nu} + \frac{1}{3}P_{\mu\nu}Q^\rho{}_\rho\,, \tag{5.9}$$

with the orthogonal projector $P^{\mu\nu} = g^{\mu\nu} - \frac{1}{f_\rho f^\rho} f^\mu f^\nu$ (notice $P^{\mu\nu} g_{\mu\nu} = 3$). Here $Q^{\rm STF}_{\mu\nu}$ is symmetric and trace-free (STF) in the local frame. In the coordinate frame the trace-free property reads $Q^{\rm STF}_{\mu\nu} g^{\mu\nu} = 0$ and of course it holds $Q^{\rm STF}_{\mu\nu} f^\nu = 0$. Obviously $Q^{\mu\nu}$ has six independent components, five contained in the symmetric trace-free part and one in the scalar part $Q^\rho{}_\rho$. The same holds for $Q^{\mu\nu\alpha\beta}$ with a more complicated decomposition into symmetric trace-free and scalar parts, whereas $Q^{\mu\nu\alpha}$ has even eight independent components corresponding to a symmetric trace-free and a vector part [104].

5.1.3 Ansatz for the mass quadrupole

We will now constrain to $f_\mu = p_\mu$ and to the Newtonian limit, the latter to identify the dominant contributions. Then only the mass multipoles are important for the dynamics. Though flow and stress multipoles do in general not vanish in the Newtonian limit [96], they give no contribution to the gravitational field and can be neglected. Therefore the decomposition of the quadrupole moment (5.6) just reads

$$J^{\nu\rho\beta\alpha} = -\frac{3}{m_p^2} p^{[\nu} Q^{\rho][\beta} p^{\alpha]} \,, \tag{5.10}$$

where the dynamical mass defined by $p_\mu p^\mu = -m_p^2$ is now denoted as m_p. Also the trace part of the mass quadrupole gives no contribution to the gravitational field outside the body and one can thus assume $Q^\rho{}_\rho = 0$. The mass quadrupole induced by spin is then given by the ansatz

$$Q_{\mu\nu} = Q^{\rm STF}_{\mu\nu} = \frac{C_Q}{m_p} \left(S_{\mu\rho} S_\nu{}^\rho - \frac{1}{3} P_{\mu\nu} S^{\rho\sigma} S_{\rho\sigma} \right) , \tag{5.11}$$

and is parametrized only by C_Q in the Newtonian limit and quadratic level in spin, see also [105]. For black holes one has $C_Q = 1$ [37] while for neutron star models C_Q depends on the equation of state [106].

Though the ansatz for the quadrupole (5.10, 5.11) was given in the Newtonian limit only, it was written in a manifestly covariant way and we will now consider its implications in full general relativity. However, we stay at the quadratic level in spin. It will be shown in the next section by relying on investigations in [107] that this ansatz indeed holds to next-to-leading order in the post-Newtonian approximation. It is easy to see from (5.2) that the spin length S given by $2S^2 = S^{\mu\nu} S_{\mu\nu}$ is conserved for our quadrupole ansatz and spin supplementary condition $S^{\mu\nu} p_\mu = 0$. But the mass m_p is not conserved. However, the new mass-like parameter m defined by

$$m = m_p - \frac{1}{6} R^{(4)}_{\nu\rho\beta\alpha} J^{\nu\rho\beta\alpha} \,, \tag{5.12}$$

is conserved for our ansatz quadratic in spin. Finally (5.4) can be written as

$$p^\mu = m u^\mu - \frac{1}{2m} R^{(4)}_{\rho\nu\alpha\beta} S^{\mu\rho} S^{\alpha\beta} u^\nu + \frac{1}{2} R^{(4)}_{\delta\alpha\beta\nu} Q^{\alpha\beta} u^\nu (2 g^{\delta\mu} + u^\delta u^\mu) \,, \tag{5.13}$$

and gives a relation between p^μ and u^μ.

5.2 Action approach

It is shown in this section that allowing nonminimal couplings in the action approach from Sect. 3.1.3 corresponds to certain higher multipole corrections, see also [87]. The couplings in the action needed for spin-induced quadrupole deformations at next-to-leading order in the post-Newtonian approximation can be found in [107].

5.2.1 Nonminimal couplings

We now generalize the ansatz (3.16) for the action to nonminimal couplings. More precisely, the Lagrangian is allowed to contain the Riemann curvature tensor,

$$W_M[e_{I\mu}, z^\mu, \Lambda^{AI}] = \int \mathrm{d}\tau \, L_M(u^\mu, \Omega^{\mu\nu}, g^{\mu\nu}(z^\rho), g_{\mu\nu}(z^\rho), R^{(4)}_{\mu\nu\alpha\beta}(z^\rho)) \,. \tag{5.14}$$

The Euler-Lagrange equations of this action follow as in Sect. 3.1.3 in a straightforward way. It is easy to see that (3.19) stays unchanged,

$$\frac{\mathrm{D}}{\mathrm{d}\tau}\left[\frac{\partial L_M}{\partial \Omega^{\mu\nu}}\right] = \frac{\partial L_M}{\partial \Omega^{\mu\rho}}\Omega^\rho{}_\nu - \frac{\partial L_M}{\partial \Omega^{\nu\rho}}\Omega^\rho{}_\mu \,. \tag{5.15}$$

However, the important relation (3.17) now reads

$$0 = \frac{\partial L_M}{\partial u^\alpha}u^\beta + 2\frac{\partial L_M}{\partial \Omega^{\alpha\nu}}\Omega^{\beta\nu} + 2\frac{\partial L_M}{\partial g^{\alpha\nu}}g^{\beta\nu} - 2\frac{\partial L_M}{\partial g_{\beta\nu}}g_{\alpha\nu} - 4\frac{\partial L_M}{\partial R^{(4)}_{\beta\nu\rho\delta}}R^{(4)}_{\alpha\nu\rho\delta} \,. \tag{5.16}$$

Using this identity and the definitions

$$p_\mu = \frac{\partial L_M}{\partial u^\mu} \,, \qquad S_{\mu\nu} = 2\frac{\partial L_M}{\partial \Omega^{\mu\nu}} \,, \qquad J^{\mu\nu\alpha\beta} = -6\frac{\partial L_M}{\partial R^{(4)}_{\mu\nu\alpha\beta}} \,, \tag{5.17}$$

the Euler-Lagrange equations for the matter variables turn into (5.2), whereas for the field variables one obtains the Einstein equations with the stress-energy tensor (5.3). Higher multipoles are covered by including symmetrized covariant derivatives of the curvature tensor in the Lagrangian [87].

An action invariant under general coordinate transformations always leads to a stress-energy tensor fulfilling (2.27). As well known, this can be shown from the Noether identity [90] following from general covariance, see also Eq. (18.23) in [108]. However, this does not mean that an action approach as envisaged here always leads to the most general equations of motion allowed by (2.27). Dixon's derivation essentially only evaluated (2.27) and thus covers a much more general situation, though the Euler-Lagrange equations obtained in this section are identical to the equations of motion found by Dixon (but it is not a priori clear that this will be the case). In particular, the multipole moments will always be implicitly fixed by the other variables on which the Lagrangian was chosen to depend on, cf. (5.17). This means that the quadrupole is not a dynamical variable within our action approach and our action does not cover, e.g., quadrupole oscillation modes or tidal resonances, see, e.g., [109]. Notice that (2.27) puts no constraints on equations of motion related to dynamical quadrupole degrees of freedom. However, a dynamical quadrupole requires further dynamical variables in the action principle. For a good effective description of extended objects via an action one thus needs some intuition on the relevant degrees of freedom that should enter into an ansatz for the effective action. The action (5.14) includes translational and rotational degrees of freedom of the object, which is expected to be a good choice for the inspiral phase.

An advantage of the action approach is that it is easier to find conserved quantities or constant parameters. In particular, the conservation of the spin length immediately follows from the symmetry under Lorentz transformations of the body-fixed frame. Further all parameters in the action, e.g., a mass-like parameter, are constant simply by assumption. It is much more difficult to find such constant quantities if one only considers Dixon's results together with a specific ansatz for the quadrupole moment, see, e.g., Eq. (5.12) or the discussion in reference [16].

It is important that one may eliminate the Ricci tensor (and scalar) from the matter Lagrangian L_M by a suitable redefinition of the metric [110]. This was already found in [111] within a slightly different situation and is based on the observation that in a perturbative context the use of lower order equations of motion in the perturbation part of the action corresponds to a redefinition of variables, see [112]. If we take some additions to the point-mass Lagrangian as a perturbation, then we may eliminate the Ricci tensor by using the Einstein field equations with the point-mass stress-energy tensor as a source, corresponding to an irrelevant redefinition of the metric. However, the point-mass stress-energy tensor then gives rise to singular self-interactions in the matter Lagrangian L_M, which are formally neglected [111]. The conclusion is that one may use the *vacuum* field equations $R^{(4)}_{\mu\nu} = 0$ in the matter Lagrangian L_M. (This will also be used in a slightly modified way in Sect. 5.2.3.) The matter Lagrangian can therefore be restricted to depend on the completely trace-free Weyl tensor $C^{(4)}_{\mu\nu\alpha\beta}$,

$$C^{(4)}_{\mu\alpha\nu\beta} = R^{(4)}_{\mu\alpha\nu\beta} + g_{\alpha[\nu} R^{(4)}_{\beta]\mu} - g_{\mu[\nu} R^{(4)}_{\beta]\alpha} + \frac{1}{3} g_{\mu[\nu} g_{\beta]\alpha} R^{(4)} \,, \tag{5.18}$$

instead of $R^{(4)}_{\mu\nu\alpha\beta}$. This would give rise to corresponding modified multipole moments defined analogous to (5.17). Further, the Weyl tensor can be split into electric $E^{(4)}_{\mu\nu}$ and magnetic $B^{(4)}_{\mu\nu}$ parts,

$$E^{(4)}_{\mu\nu} = C^{(4)}_{\mu\alpha\nu\beta} u^\alpha u^\beta \,, \qquad B^{(4)}_{\mu\nu} = \frac{1}{2} \epsilon^{(4)}_{\mu\rho\alpha\beta} C^{(4)\,\alpha\beta}_{\nu\sigma} u^\rho u^\sigma \,, \tag{5.19}$$

with $\epsilon^{(4)}_{\mu\alpha\beta\rho}$ the 4-dimensional Levi-Civita symbol, leading to definitions for corresponding electric and magnetic multipoles as partial derivatives of L_M. It could be interesting to consider the impact on the equations of motion and the stress-energy tensor from letting L_M depend on $C^{(4)}_{\mu\nu\alpha\beta}$ or $E^{(4)}_{\mu\nu}$ and $B^{(4)}_{\mu\nu}$ instead of $R^{(4)}_{\mu\nu\alpha\beta}$. However, this will not be necessary here.

5.2.2 Legendre transforms and supplementary conditions

Up to now the Lagrangian L_M is completely arbitrary and the equations of motion fully agree with Dixon at the quadrupole level. The question is which supplementary conditions (3.10) and (3.12) belong to L_M, or how L_M must be chosen to fit with specific supplementary conditions. We only require here that (3.10) and (3.12) are preserved in time, which leads to

$$\frac{\mathrm{D}\Lambda^{AI}}{\mathrm{d}\tau} f_I + \Lambda^{AI} \left(\eta_{IJ} - \frac{f_I f_J}{f_K f^K} \right) \frac{\mathrm{D} f^J}{\mathrm{d}\tau} = 0 \,, \qquad S^\mu{}_\rho \Omega^{\rho\nu} f_\nu + S^{\mu\nu} \frac{\mathrm{D} f_\nu}{\mathrm{d}\tau} = 0 \,, \tag{5.20}$$

where (5.15) with (5.17) was used. Both conditions are fulfilled if we have

$$\Omega^{\mu\nu} f_\nu + P^{\mu\rho} \frac{\mathrm{D} f_\rho}{\mathrm{d}\tau} = 0 \,. \tag{5.21}$$

In this sense the conditions (3.10) and (3.12) belong together (however, there may be exceptions). This is the condition our action shall fulfill here. Notice that (5.4) only guarantees that (3.12) is preserved in time so that the second relation in (5.20) holds, but this does not imply (5.21). However, comparing (5.17) with (5.4) can still be useful. Further, at the quadratic level in spin we need to fulfill (5.21) only to linear order in spin.

It is suitable to define a new function $R_M(u^\mu, S_{\mu\nu}, g^{\mu\nu}, g_{\mu\nu}, R^{(4)}_{\mu\nu\alpha\beta})$ via Legendre transformation, $R_M = L_M - \frac{1}{2} S_{\mu\nu} \Omega^{\mu\nu}$. It holds

$$p_\mu = \frac{\partial R_M}{\partial u^\mu} \,, \qquad \Omega^{\mu\nu} = -2 \frac{\partial R_M}{\partial S_{\mu\nu}} \,, \qquad J^{\mu\nu\alpha\beta} = -6 \frac{\partial R_M}{\partial R^{(4)}_{\mu\nu\alpha\beta}} \,. \tag{5.22}$$

An ansatz for R_M then has to fulfill the condition (5.21) with (5.22) inserted. This gives a partial differential equation for R_M. It holds $R_M = p_\mu u^\mu$, which is a consequence of the reparametrization invariance of the matter action. Notice that R_M is similar to the Routhian used in [107,113,114]. For the Routhian the Ricci rotation part in the term $\frac{1}{2} S_{\mu\nu} \Omega^{\mu\nu}$, cf. Eq. (3.15), is not subtracted from the Lagrangian L_M. Therefore the Routhian is not a covariant function, whereas R_M introduced here is covariant.

Due to reparametrization invariance a full Legendre transformation in u^μ and $\Omega^{\mu\nu}$ leads to a vanishing result. However, as in Sect. 2.3.2 we define a function $H_{M\tau}$ which contains the mass-shell constraint, suitably generalized to the quadratic-in-spin level, together with a Lagrange multiplier λ. It holds

$$u^\mu = \frac{\partial H_{M\tau}}{\partial p_\mu} \,, \qquad \Omega^{\mu\nu} = 2 \frac{\partial H_{M\tau}}{\partial S_{\mu\nu}} \,, \qquad J^{\mu\nu\alpha\beta} = 6 \frac{\partial H_{M\tau}}{\partial R^{(4)}_{\mu\nu\alpha\beta}} \,. \tag{5.23}$$

It is also possible to give an ansatz for the mass-shell constraint and thus for $H_{M\tau}$ directly. This ansatz must be chosen such that the condition (5.21) with (5.23) inserted is fulfilled. Further, one may simplify the quadratic-in-spin corrections to $H_{M\tau}$ by using the leading order constraint $p^\mu p_\mu = -m^2$, corresponding to a redefinition of the Lagrange multiplier [112].

5.2.3 Leading order

The coupling terms found in [107] adapted to our notation and conventions read

$$R_M = \frac{1}{\sqrt{-u_\sigma u^\sigma}} \left(m u_\mu u^\mu - \frac{1}{2m} R^{(4)}_{\mu\nu\alpha\beta} S^{\rho\mu} S^{\alpha\beta} u^\nu u_\rho + \frac{C_{ES^2}}{2m} E^{(4)}_{\mu\nu} S^\mu{}_\rho S^{\rho\nu} \right) . \tag{5.24}$$

It was found in [107] that these coupling terms are the most general ones at quadratic level in spin sufficient for the next-to-leading order in the post-Newtonian approximation. Besides these terms corresponding to quadrupole deformation due to spin, one could also treat tidal deformations, see, e.g., [93], using nonminimal couplings in the action given in [94, 115]. With the equivalence of Riemann and Weyl tensors within the matter action, see Sect. 5.2.1, we can write R_M as

$$R_M = \frac{1}{\sqrt{-u_\sigma u^\sigma}} \left(m u_\mu u^\mu - \frac{1}{2m} R^{(4)}_{\mu\nu\alpha\beta} S^{\rho\mu} S^{\alpha\beta} u^\nu u_\rho - \frac{1}{2} R^{(4)}_{\alpha\mu\beta\nu} Q^{\alpha\beta} u^\mu u^\nu \right) , \tag{5.25}$$

where $Q_{\mu\nu}$ is given by (5.11) and $C_{ES^2} = C_Q$. If we set $f_\mu = p_\mu$ and thus

$$S^{\mu\nu} p_\nu = S^{\mu\nu} \frac{\partial R_M}{\partial u^\nu} = 0 \, , \tag{5.26}$$

we obviously reproduce (5.10) and (5.13) within the gauge $u_\sigma u^\sigma = -1$ by plugging (5.25) into (5.22). Further (5.21) is fulfilled[15] to the considered order in spin by using (5.2) and (5.22), i.e.,

$$0 = -2 \frac{\partial R_M}{\partial S_{\mu\nu}} \frac{\partial R_M}{\partial u^\nu} - \frac{1}{2} P^{\mu\sigma} R^{(4)}_{\sigma\rho\beta\alpha} u^\rho S^{\beta\alpha} + P^{\mu\sigma} R^{(4)}_{\nu\rho\beta\alpha||\sigma} \frac{\partial R_M}{\partial R^{(4)}_{\nu\rho\beta\alpha}} \, . \tag{5.27}$$

The last term is of higher order here as this condition must be fulfilled to linear order in spin only. Notice that m depends on spin according to $m = m_0 + \frac{1}{4I} S_{\alpha\beta} S^{\alpha\beta}$, see also (3.28). (Otherwise the Legendre transformation between L_M and R_M would not be possible.)

An equivalent description in terms of $H_{M\tau}$ reads

$$H_{M\tau} = \lambda \left(m^2 + p_\mu p^\mu + \frac{1}{m^2} R^{(4)}_{\mu\nu\alpha\beta} S^{\rho\mu} S^{\alpha\beta} p^\nu p_\rho - \frac{C_Q}{m^2} R^{(4)}_{\alpha\mu\beta\nu} S^\alpha{}_\rho S^{\rho\beta} p^\mu p^\nu \right) , \tag{5.28}$$

with the action still given by (3.30). The derivation of a canonical formalism now follows along the same lines as in Sect. 3. First the matter constraints are solved. The only difference to the linear-in-spin case arises in the mass-shell constraint, which follows from the variation of λ. The solution of this constraint reads

$$np \equiv n^\mu p_\mu = -\sqrt{m^2 + \gamma^{ij} p_i p_j} + \frac{C_Q}{2m^2 \sqrt{m^2 + \gamma^{ij} p_i p_j}} R^{(4)}_{\alpha\mu\beta\nu} S^\alpha{}_\rho S^{\rho\beta} p^\mu p^\nu \, . \tag{5.29}$$

The last term was not yet split into time and space parts. This splitting leads to quite many terms, so one should restrict to some post-Newtonian order. Though all formulas are sufficient for the next-to-leading order, we will for simplicity only treat the leading order in this section. Then we have

$$np = -\sqrt{m^2 + \gamma^{ij} p_i p_j} - \frac{C_Q}{2mN} \gamma^{kl} \gamma^{im} \gamma^{jn} S_{ik} S_{jl} (K_{mn,0} + N_{;mn}) \, . \tag{5.30}$$

[15] One could also consider the most general ansatz for R_M and ask for which choice of f_μ the condition (5.21) is fulfilled. This would allow one to study the impact of the supplementary conditions on the dynamics.

The only contribution to the action quadratic in spin then arises from the term Nnp in the matter Lagrangian, see (2.68). Problematic is the partial time derivative of the extrinsic curvature. In consideration of the definition $2NK_{ij} = -\gamma_{ij,0} + 2N_{(i;j)}$ we see that the $K_{mn,0}$-term produces time-derivatives of lapse and shift, as well as a double time-derivative of γ_{ij}. This does not fit well to the derivation of the canonical formalism as given in Sect. 3. In order to overcome these problems, we eliminate $K_{mn,0}$ with the help of the vacuum field equations, cf. the discussion in Sect. 5.2.1. Finally one ends up with just a quadratic-in-spin correction $\mathcal{H}^{\text{matter}}_{\text{S}^2}$ to the source of the Hamilton constraint $\mathcal{H}^{\text{matter}}$ of the form

$$\mathcal{H}^{\text{matter}}_{\text{S}^2} = \frac{C_Q}{2m}\gamma^{kl}R^{ij}S_{ik}S_{jl}\delta\,. \tag{5.31}$$

This source term is quite unusual in the sense that it is not a specific projection of the stress-energy tensor (5.3, 5.10, 5.11), i.e., $\mathcal{H}^{\text{matter}} \neq \sqrt{\gamma}T_{\mu\nu}n^\mu n^\nu$. This is due to the implicit redefinition of variables performed by using the vacuum field equations in the matter action. However, the leading order Hamiltonian resulting from this source term is identical to the well-known one obtained in Sect. 6.2.2. The variable redefinitions from Sect. 3.2.2 are still correct at the leading order. (There are no additional terms that need to be cancelled in the action and all quadratic spin contributions from the redefinitions in Sect. 3.2.2 are of higher order.)

At the next-to-leading order the calculation gets much more involved. In particular there are more time derivatives of the extrinsic curvature that must be eliminated and the variable redefinitions from Sect. 3.2.2 need corrections quadratic in spin. It is also relevant whether the field variables in the variable transformations are taken at the new or at the old particle position. Further corrections to the canonical field momentum seem to be necessary, too. We will therefore study an alternative derivation oriented at the symmetry generator approach from Sect. 4 in the following.

5.3 Symmetry generator approach

We now sketch the derivation of the canonical formalism at quadratic level in spin via the approach from Sect. 4. However, essentially only the calculation of the source terms of the constraints as certain projections of the stress-energy tensor is used here, the determination of canonical variables by looking at the symmetry generators will only be touched lightly.

5.3.1 Leading order

First we calculate the source of the field constraints as certain projections of the stress-energy tensor (5.3, 5.10, 5.11), e.g., $\mathcal{H}^{\text{matter}} = \sqrt{\gamma}T_{\mu\nu}n^\mu n^\nu$. To leading order we have

$$\mathcal{H}^{\text{matter}}_{\text{S}^2} = \sum_a \left(\frac{1}{2}\gamma^{ki}\gamma^{lj}Q_{aij}\delta_a\right)_{;kl}\,, \tag{5.32}$$

and no corrections appear in $\mathcal{H}^{\text{matter}}_i$. The variable redefinitions found at the linear order in spin are therefore sufficient here, as they are followed from $\mathcal{H}^{\text{matter}}_i$ in the symmetry generator approach. For $C_{Qa} = 1$ this source term is in agreement with the source of the Kerr metric in approximate ADM coordinates found in [116]. It further gives the correct Hamiltonian, see Sect. 6.2.2.

Obviously the derivation of the leading order in this section is much simpler than the one via the action approach. But this does not need to be true at the next-to-leading order. The problem is that it is not guaranteed that the variable redefinitions can be uniquely fixed by just the conditions (4.30) and (4.31). The action approach is much more systematic and should therefore be preferred at the next-to-leading order. However, in the next section a shortcut to the next-to-leading order Hamiltonian is described, which combines the approach of the present section with the Poincaré algebra approach in [117].

5.3.2 Next-to-leading order static source terms

In [117] Hergt and Schäfer constructed the part of the next-to-leading order Hamiltonian that depends on $\hat{p}_i$ (i.e., the nonstatic part) from an ansatz for this Hamiltonian (together with a suitable ansatz for the source of the constraints). The coefficients in this ansatz could be uniquely fixed up to a canonical transformation by considering the Poincaré algebra (2.10, 2.11). The degrees of freedom corresponding to the ambiguity in the canonical representation are given by the coefficients that enter via an ansatz for the center of mass vector G_i. However, the static (i.e., $\hat{p}_i = 0$) part of the Hamiltonian is left completely undetermined by the Poincaré algebra approach in [117].

In order to get the complete next-to-leading order Hamiltonian only the static part of the Hamiltonian is missing, as well as the corresponding center of mass vector G_i. The latter is needed to consistently fix the canonical representation of the nonstatic part of the Hamiltonian given in [117]. Fortunately the center of mass vector does not depend on $\hat{p}_i$ at the considered order. Therefore both the static part of the Hamiltonian and the center of mass vector are determined if we only know the static part of the source of the constraints. For $p_i = 0$ we get from the stress-energy tensor (5.3, 5.10, 5.11)

$$\mathcal{H}^{\text{matter}}_{\mathrm{S}^2,\, p_i=0} = \sum_a \left(\frac{1}{2} \gamma^{ki} \gamma^{lj} Q_{aij} \delta_a \right)_{;kl} , \tag{5.33}$$

but no further contributions to $\mathcal{H}^{\text{matter}}_i$ arise. Though there is no difference to (5.32), this source term is now valid to next-to-leading order for the case $p_i = 0$.

However, we need the source terms for the case $\hat{p}_i = 0$ and not for $p_i = 0$. Also position and spin variables are not yet the canonical ones and we must discuss whether the variable redefinitions will have an impact on the source terms in the static case. As there are no contributions to $\mathcal{H}^{\text{matter}}_i$ at quadratic level in spin for $p_i = 0$, no further *static* contributions to the redefinition of spin and momentum variables can arise from the conditions (4.30) and (4.31). Though static contributions to $z^i_{\Delta a(4)}$ could be necessary, they can be removed by a canonical transformation with generator $-\hat{p}_i z^i_{\Delta a(4)}$. (Notice that in the case $\hat{p}_i = 0$ this transformation only changes the position variable.) Finally only the redefinitions found at the linear order in spin are relevant and only (4.33) gives contributions in the static case. The result for the static source finally reads

$$\begin{aligned}\mathcal{H}^{\text{matter}}_{\mathrm{S}^2,\, \hat{p}_i=0} = \sum_a \Bigg[& \left(\frac{1}{2} \gamma^{ki} \gamma^{lj} \hat{Q}_{aij} \hat{\delta}_a \right)_{;kl} + \frac{1}{8m_a} \gamma_{mn} \gamma^{pj} \gamma^{ql} \gamma^{mi}{}_{,p} \gamma^{nk}{}_{,q} \hat{S}_{1ij} \hat{S}_{1kl} \hat{\delta}_a \\ & + \frac{1}{4m_a} \left(\gamma^{ij} \gamma^{mn} \gamma^{kl}{}_{,m} \hat{S}_{aln} \hat{S}_{ajk} \hat{\delta}_a \right)_{,i} \Bigg] ,\end{aligned} \tag{5.34}$$

where

$$\hat{Q}_{aij} = \frac{C_{Qa}}{m} \left(\gamma^{kl} \hat{S}_{aik} \hat{S}_{ajl} - \frac{1}{3} \gamma_{ij} \gamma^{kl} \gamma^{mn} \hat{S}_{akm} \hat{S}_{aln} \right) . \tag{5.35}$$

Equation (5.34) was found for the black hole case $C_{Qa} = 1$ in [17] from a 3-dimensional covariant ansatz for $\mathcal{H}^{\text{matter}}_{\mathrm{S}^2,\, p_i=0}$ containing four coefficients. Two of these coefficients were fixed by matching to the Kerr metric, but the other two gave no contribution to the Hamiltonian or to the center of mass vector. One of the latter two coefficients would also arise here if we would have kept the trace part of the mass quadrupole, $Q^{\rho}{}_{\rho}$. The ansatz in [17] was generalized to arbitrary C_{Qa} in [19].

The derivation given in this section is quite involved and it would thus be desirable to give a more coherent one with the help of the action approach in the future. This would also facilitate further investigations of quadrupole or higher multipole effects with the help of canonical methods.

6 Results for Hamiltonians

In this section the obtained canonical formalism is applied to calculations within the post-Newtonian approximation. In particular, the next-to-leading order spin corrections to the conservative Hamiltonian are derived. The Hamiltonians are checked with the help of the global Poincaré algebra.

In this section we make use of xTensor [118], a free package for Mathematica [119], especially of its fast index canonicalizer based on the package xPerm [120].

6.1 Post-Newtonian expansion

The post-Newtonian expansion of the ADM Hamiltonian has been well studied for nonspinning objects, for the second post-Newtonian level see [121], and up to and including the 3.5 post-Newtonian order see [34, 122, 123]. From this expanded Hamiltonian the approximate equations of motion can be derived in a straightforward way. In this section we derive general formulas for the ADM Hamiltonian up to and including the formal second post-Newtonian order, which will then be applied to calculate spin corrections to the Hamiltonian in Sect. 6.2. Another interesting application would be to obtain spin corrections to the post-Minkowskian Hamiltonian, see, e.g., [125] for the nonspinning case.

Besides the ADM formalism, there are various other methods available for post-Newtonian calculations. The equations of motion at the first post-Newtonian order are due to Einstein, Infeld, and Hoffmann [126], obtained with the help of a surface integral approach. This method got further developed and applied up to and including the third post-Newtonian level, see, e.g., [127]. A further important method uses point-masses in harmonic gauge, which also succeeded to derive the third post-Newtonian order equations of motion; for a review see [5]. This method has advantages for flux and waveform calculations, which succeeded up to the third post-Newtonian order [128] (corresponding to the knowledge of the equations of motion at the 5.5 post-Newtonian level, which seem to be impossible to obtain directly). Another approach in the harmonic gauge is the direct integration of the relaxed Einstein equations, see, e.g., [129]. More recently also methods inspired by quantum field theory were developed, see, e.g, [110, 130]. An advantage of these methods is that some of the very sophisticated and systematic techniques for perturbative calculations used in high energy physics can be applied in a straightforward way.

6.1.1 Review of the formalism

We now give a short summary of the calculation of the ADM Hamiltonian. First the field constraints

$$\frac{1}{16\pi\sqrt{\gamma}}\left[\gamma R+\frac{1}{2}\left(\gamma_{ij}\pi^{ij}\right)^2-\gamma_{ij}\gamma_{kl}\pi^{ik}\pi^{jl}\right]=\mathcal{H}^{\text{matter}}\,,\qquad -\frac{1}{8\pi}\gamma_{ij}\pi^{jk}_{\ ;k}=\mathcal{H}_i^{\text{matter}}\,, \tag{6.1}$$

have to be solved within the ADM transverse traceless gauge, which for the metric leads to the decomposition

$$\gamma_{ij}=\left(1+\frac{\phi}{8}\right)^4\delta_{ij}+h^{\text{TT}}_{ij}\,, \tag{6.2}$$

at least to linear order in spin. Such a solution can in general only be found in some approximation scheme and we consider the post-Newtonian one here. Having the decomposition (6.2) one can solve the Hamilton constraint for ϕ (this will become obvious in the next section). Then we can calculate the ADM Hamiltonian

$$H_{\text{ADM}}=-\frac{1}{16\pi}\int \mathrm{d}^3x\,\Delta\phi\,, \tag{6.3}$$

which must be expressed in terms of the canonical variables. It is suitable to already express the source terms $\mathcal{H}^{\text{matter}}$ and $\mathcal{H}_i^{\text{matter}}$ in terms of the canonical matter variables, which is done in Sect. 6.2.1. Then no further redefinition of the matter variables is necessary.

However, it seems to be simpler to perform the redefinition of the field momentum after solving the constraints. As the gauge condition at linear order in spin now reads $\hat{\pi}^{ii} = 0$, or, with (4.18), (4.50), and $B^{kl}_{ij}\delta_{kl} = 0$,

$$\pi^{ii} = -16\pi \sum_a \pi^{ii}_a \hat{\delta}_a = -16\pi \sum_a \delta_{ij}\gamma^{ik}\gamma^{jl} \frac{m_a \hat{p}_{ak} n S_{al}}{2n\hat{p}_a(m_a - n\hat{p}_a)} \hat{\delta}_a \,, \tag{6.4}$$

the decomposition (2.51) is not valid any more. But we can still use the general decomposition

$$\pi^{ij} = \pi^{ij\mathrm{TT}} + \tilde{\pi}^{ij} + \breve{\pi}^{ij} \,, \tag{6.5}$$

with

$$\pi^{ij\mathrm{TT}} = \delta^{\mathrm{TT}ij}_{kl} \pi^{kl} \,, \qquad \breve{\pi}^{ij} = \frac{1}{2}\left(\delta_{ij} - \partial_i\partial_j\Delta^{-1}\right)\pi^{kk} \,, \tag{6.6}$$

$$\tilde{\pi}^{ij} = \tilde{\pi}^i{}_{,j} + \tilde{\pi}^j{}_{,i} - \frac{1}{2}\delta_{ij}\tilde{\pi}^k{}_{,k} - \frac{1}{2}\Delta^{-1}\tilde{\pi}^k{}_{,ijk} \,, \tag{6.7}$$

and the vector potential is still $\tilde{\pi}^i = \Delta^{-1}\pi^{ij}{}_{,j}$. This can be shown by inserting (6.6, 6.7) and (2.55) into (6.5), which then turns into an identity. The new part $\breve{\pi}^{ij}$ can immediately be obtained using (6.4). After the constraints have been solved using this decomposition, we go over to the canonical field momentum $\hat{\pi}^{ij\mathrm{TT}}$ by

$$\pi^{ij\mathrm{TT}} = \hat{\pi}^{ij\mathrm{TT}} - 16\pi \sum_a \delta^{\mathrm{TT}ij}_{kl} \pi^{kl}_a \hat{\delta}_a \,. \tag{6.8}$$

No redefinition of h^{TT}_{ij} is needed at the linear order in spin.

6.1.2 Expansion of the constraints

Now we expand the constraints according to the formal post-Newtonian counting rules introduced in Sect. 4.2.3. Notice that only the field parts ϕ, $\tilde{\pi}^{ij}$, and $\breve{\pi}^{ij}$ are expanded, but not h^{TT}_{ij} and $\pi^{ij\mathrm{TT}}$. The latter are still dynamical variables in the ADM Hamiltonian and can be expanded only after their equations of motion were obtained and solved. For the Hamilton constraint we get

$$-\frac{1}{16\pi}\Delta\phi_{(2)} = \mathcal{H}^{\text{matter}}_{(2)} \,, \qquad -\frac{1}{16\pi}\Delta\phi_{(4)} = \mathcal{H}^{\text{matter}}_{(4)} - \frac{1}{8}\mathcal{H}^{\text{matter}}_{(2)}\phi_{(2)} \,, \tag{6.9}$$

$$\begin{aligned} -\frac{1}{16\pi}\Delta\phi_{(6)} &= \mathcal{H}^{\text{matter}}_{(6)} - \frac{1}{8}\left(\mathcal{H}^{\text{matter}}_{(4)}\phi_{(2)} + \mathcal{H}^{\text{matter}}_{(2)}\phi_{(4)}\right) + \frac{1}{64}\mathcal{H}^{\text{matter}}_{(2)}\phi^2_{(2)} \\ &\quad + \frac{1}{16\pi}\left[\left(\tilde{\pi}^{ij}_{(3)}\right)^2 - \frac{1}{2}\left(\phi_{(2)}h^{\mathrm{TT}}_{ij}\right)_{,ij}\right] \,, \end{aligned} \tag{6.10}$$

$$\begin{aligned} -\frac{1}{16\pi}\Delta\phi_{(8)} &= \frac{1}{16\pi}\left[\frac{1}{8}\phi_{(2)}\left(\tilde{\pi}^{ij}_{(3)}\right)^2 + 2\tilde{\pi}^{ij}_{(3)}\tilde{\pi}^{ij}_{(5)} - \frac{1}{16}\phi_{(2),i}\phi_{(2),j}h^{\mathrm{TT}}_{ij} + \frac{1}{4}\left(h^{\mathrm{TT}}_{ij,k}\right)^2\right] \\ &\quad + \mathcal{H}^{\text{matter}}_{(8)} - \frac{1}{8}\left(\mathcal{H}^{\text{matter}}_{(6)}\phi_{(2)} + \mathcal{H}^{\text{matter}}_{(4)}\phi_{(4)} + \mathcal{H}^{\text{matter}}_{(2)}\phi_{(6)}\right) \\ &\quad + \frac{1}{64}\left(\mathcal{H}^{\text{matter}}_{(4)}\phi^2_{(2)} + 2\mathcal{H}^{\text{matter}}_{(2)}\phi_{(2)}\phi_{(4)}\right) - \frac{1}{512}\mathcal{H}^{\text{matter}}_{(2)}\phi^3_{(2)} + (\mathrm{td}) \,, \end{aligned} \tag{6.11}$$

up to and including the formal second post-Newtonian order. These equations can be solved iteratively for ϕ by applying an inverse Laplacian to them. The ADM Hamiltonian (6.3) results from an integration over the right-hand sides of these equations. It was used that $\breve{\pi}^{ij} = \mathcal{O}\left(c^{-9}\right)$ at linear order in spin.

However, we also have to solve the momentum constraint as π^{ij} appears on the right-hand side of the Hamilton constraint. The expansion of the momentum constraint immediately follows from the exact formula

$$\tilde{\pi}^{ij}{}_{,j} = -8\pi\mathcal{H}_i^{\text{matter}} + B^{ij}{}_{,j} + C^i - \Delta\left(V^k h_{ki}^{\text{TT}}\right) + \frac{1}{2}\pi^{jk\text{TT}} h_{jk,i}^{\text{TT}} - (\pi^{jk\text{TT}} h_{ki}^{\text{TT}})_{,j} \,, \tag{6.12}$$

with

$$B^{ij} = \left[1 - \left(1 + \tfrac{1}{8}\phi\right)^4\right](\tilde{\pi}^{ij} + \pi^{ij\text{TT}}) + V^k(h_{ki,j}^{\text{TT}} + h_{kj,i}^{\text{TT}} - h_{ij,k}^{\text{TT}}) - \frac{1}{3}V^k{}_{,k} h_{ij}^{\text{TT}} \,, \tag{6.13}$$

$$C^i = \frac{1}{2}\breve{\pi}^{jk}\gamma_{jk,i} - \breve{\pi}^{jk}\gamma_{ij,k} \,, \tag{6.14}$$

which is analogous to (4.21). Here we introduced the alternative vector potential

$$V^i = \left(\delta_{ij} - \frac{1}{4}\partial_i\partial_j\Delta^{-1}\right)\tilde{\pi}^j \,, \tag{6.15}$$

for which it holds

$$\tilde{\pi}^{ij} = V^i{}_{,j} + V^j{}_{,i} - \frac{2}{3}\delta_{ij}V^k{}_{,k} \,. \tag{6.16}$$

To the considered order we thus have

$$\tilde{\pi}^{ij}_{(3),j} = -8\pi\mathcal{H}^{\text{matter}}_{(3)i} \,, \qquad \tilde{\pi}^{ij}_{(5),j} = -8\pi\mathcal{H}^{\text{matter}}_{(5)i} - \frac{1}{2}\left(\phi_{(2)}\tilde{\pi}^{ij}_{(3)}\right)_{,j} \,. \tag{6.17}$$

With the help of $\tilde{\pi}^i = \Delta^{-1}\tilde{\pi}^{ij}{}_{,j}$, the expanded momentum constraint can be solved iteratively for $\tilde{\pi}^i$ by applying an inverse Laplacian to it. $\tilde{\pi}^{ij}$ and V^i then follow from (6.7) and (6.15).

6.1.3 Formulas for Hamiltonians

The first contribution to the ADM Hamiltonian (6.3) results from an integration over the first relation in (6.9) as

$$H_0 = \int \mathrm{d}^3x\, \mathcal{H}^{\text{matter}}_{(2)} \,. \tag{6.18}$$

Notice that $\mathcal{H}^{\text{matter}}_{(2)}$ is just the Newtonian mass density, so H_0 is the constant energy belonging to the total Newtonian mass. Similarly, from the second relation in (6.9) we obtain the Newtonian Hamiltonian

$$H_{\text{N}} = \int \mathrm{d}^3x \left[\mathcal{H}^{\text{matter}}_{(4)} - \frac{1}{8}\phi_{(2)}\mathcal{H}^{\text{matter}}_{(2)}\right] . \tag{6.19}$$

$\phi_{(2)}$ results from (6.9) as

$$\phi_{(2)} = -16\pi\Delta^{-1}\mathcal{H}^{\text{matter}}_{(2)} \,, \tag{6.20}$$

and agrees up to a factor with the Newtonian gravitational potential of the mass distribution $\mathcal{H}^{\text{matter}}_{(2)}$. $\mathcal{H}^{\text{matter}}_{(4)}$ is the Newtonian kinetic energy density.

Next we proceed to the Hamiltonian at the first post-Newtonian order. However, we first apply the partial integration formulas

$$\frac{1}{8}\phi_{(4)}\mathcal{H}^{\text{matter}}_{(2)} = \frac{1}{8}\left(\mathcal{H}^{\text{matter}}_{(4)} - \frac{1}{8}\phi_{(2)}\mathcal{H}^{\text{matter}}_{(2)}\right)\phi_{(2)} + (\text{td}) \,, \quad \frac{1}{16\pi}\left(\tilde{\pi}^{ij}_{(3)}\right)^2 = V^i_{(3)}\mathcal{H}^{\text{matter}}_{(3)i} + (\text{td}) \,, \tag{6.21}$$

to the right-hand side of (6.10), following from the formal solution

$$\phi_{(4)} = -16\pi\Delta^{-1}\left[\mathcal{H}^{\text{matter}}_{(4)} - \frac{1}{8}\phi_{(2)}\mathcal{H}^{\text{matter}}_{(2)}\right], \tag{6.22}$$

of the Hamilton constraint and from (6.16, 6.17). $V^i_{(3)}$ is determined by (6.15) and

$$\tilde{\pi}^i_{(3)} = -8\pi\Delta^{-1}\mathcal{H}^{\text{matter}}_{(3)i}\,. \tag{6.23}$$

Finally we get for the first post-Newtonian (PN) order Hamiltonian

$$H_{\text{1PN}} = \int \mathrm{d}^3x\left[\mathcal{H}^{\text{matter}}_{(6)} - \frac{1}{4}\phi_{(2)}\mathcal{H}^{\text{matter}}_{(4)} + \frac{1}{32}\phi^2_{(2)}\mathcal{H}^{\text{matter}}_{(2)} + V^i_{(3)}\mathcal{H}^{\text{matter}}_{(3)i}\right]. \tag{6.24}$$

Notice that all terms in the Hamiltonian involve matter source terms $\mathcal{H}^{\text{matter}}$ or $\mathcal{H}^{\text{matter}}_i$ and are thus integrations over delta distributions only. Further only the Newtonian potential $\phi_{(2)}$ and the leading order vector potential $V^i_{(3)}$ need to be determined (lapse N and shift N^i are not even needed at any higher order). This shows the efficiency of the ADM formalism in calculating the conservative post-Newtonian dynamics.

In the same way one can obtain a formula for the second post-Newtonian Hamiltonian

$$\begin{aligned} H^{\text{ADM}}_{\text{2PN}} = \int \mathrm{d}^3x\Bigg[&\mathcal{H}^{\text{matter}}_{(8)} - \frac{1}{8}\left(2\phi_{(2)}\mathcal{H}^{\text{matter}}_{(6)} + \phi_{(4)}\mathcal{H}^{\text{matter}}_{(4)}\right) - \frac{1}{256}\phi^3_{(2)}\mathcal{H}^{\text{matter}}_{(2)} \\ &+ \frac{1}{64}\left(2\phi^2_{(2)}\mathcal{H}^{\text{matter}}_{(4)} + 3\phi_{(2)}\phi_{(4)}\mathcal{H}^{\text{matter}}_{(2)}\right) + 2V^i_{(3)}\mathcal{H}^{\text{matter}}_{(5)i} \\ &+ \frac{1}{16\pi}\left(-\phi_{(2)}\left(\tilde{\pi}^{ij}_{(3)}\right)^2 - \frac{1}{8}\phi_{(2),i}\phi_{(2),j}h^{\text{TT}}_{ij} + \frac{1}{4}\left(h^{\text{TT}}_{ij,k}\right)^2\right)\Bigg], \end{aligned} \tag{6.25}$$

where the partial integrations

$$\begin{aligned} \phi_{(6)}\mathcal{H}^{\text{matter}}_{(2)} = \phi_{(2)}\mathcal{H}^{\text{matter}}_{(6)} &- \frac{1}{8}\left(\phi^2_{(2)}\mathcal{H}^{\text{matter}}_{(4)} + \phi_{(2)}\phi_{(4)}\mathcal{H}^{\text{matter}}_{(2)}\right) + \frac{1}{64}\phi^3_{(2)}\mathcal{H}^{\text{matter}}_{(2)} \\ &+ \frac{1}{16\pi}\left[\phi_{(2)}\left(\tilde{\pi}^{ij}_{(3)}\right)^2 + \frac{1}{2}\phi_{(2),i}\phi_{(2),j}h^{\text{TT}}_{ij}\right] + (\text{td})\,, \end{aligned} \tag{6.26}$$

$$\tilde{\pi}^{ij}_{(3)}\tilde{\pi}^{ij}_{(5)} = 16\pi V^i_{(3)}\mathcal{H}^{\text{matter}}_{(5)i} - \frac{1}{2}\phi_{(2)}\left(\tilde{\pi}^{ij}_{(3)}\right)^2 + (\text{td})\,, \tag{6.27}$$

were used. Notice that $\phi_{(6)}$ and $\tilde{\pi}^{ij}_{(5)}$ were eliminated from the Hamiltonian by these partial integrations. Therefore no solutions to the constraints besides (6.20), (6.22), and (6.23) have to be determined explicitly. The Hamiltonian $H^{\text{ADM}}_{\text{2PN}}$ has the additional label ADM as it still depends on the dynamical field variable h^{TT}_{ij}. The elimination of h^{TT}_{ij} from $H^{\text{ADM}}_{\text{2PN}}$ leads to the matter-only Hamiltonian H_{2PN} and is discussed in the next section. Further $\pi^{ij\text{TT}}$ first appears at the formal third post-Newtonian level.

Notice that the obtained formulas are valid for quite general source expressions $\mathcal{H}^{\text{matter}}$ and $\mathcal{H}^{\text{matter}}_i$, not only to the ones linear in spin.

6.1.4 Matter-only Hamiltonian

In the last section the ADM Hamiltonian H_{ADM} was expanded as

$$H_{\text{ADM}} = H_0 + H_{\text{N}} + H_{\text{1PN}} + H^{\text{ADM}}_{\text{2PN}} + \cdots\,. \tag{6.28}$$

The conservative matter-only Hamiltonian results from plugging the solution for h^{TT}_{ij} and $\hat{\pi}^{ij\mathrm{TT}}$ into the *action*, Eq. (3.63) (and a subsequent elimination of emerging higher order time derivatives of the matter variables), see [123]. As $\hat{\pi}^{ij\mathrm{TT}}$ is neglected at the considered order, the first term in (3.63) does not contribute here. Therefore $H^{\mathrm{ADM}}_{\mathrm{2PN}}$ turns into the matter-only Hamiltonian H_{2PN} by simply inserting the solution for h^{TT}_{ij} into $H^{\mathrm{ADM}}_{\mathrm{2PN}}$.

The field evolution can be obtained from the ADM Hamiltonian by

$$\frac{\partial h^{\mathrm{TT}}_{ij}}{\partial t} = \{h^{\mathrm{TT}}_{ij}, H_{\mathrm{ADM}}\} = 16\pi\delta^{\mathrm{TT}ij}_{kl}\frac{\delta H_{\mathrm{ADM}}}{\delta\hat{\pi}^{kl\mathrm{TT}}}\,, \qquad \frac{\partial\hat{\pi}^{ij\mathrm{TT}}}{\partial t} = \{\hat{\pi}^{ij\mathrm{TT}}, H_{\mathrm{ADM}}\} = -16\pi\delta^{\mathrm{TT}ij}_{kl}\frac{\delta H_{\mathrm{ADM}}}{\delta h^{\mathrm{TT}}_{kl}}\,. \tag{6.29}$$

However, as the $\hat{\pi}^{ij\mathrm{TT}}$-contributions are of higher order here, we formally just have

$$0 = \delta^{\mathrm{TT}ij}_{kl}\frac{\delta H^{\mathrm{ADM}}_{\mathrm{2PN}}}{\delta h^{\mathrm{TT}}_{kl}}\,, \tag{6.30}$$

or explicitly, given that $\mathcal{H}^{\mathrm{matter}}_{(8)}$ has contributions linear in h^{TT}_{ij},

$$\Delta h^{\mathrm{TT}}_{ij} = 2\delta^{\mathrm{TT}kl}_{ij} f_{(4)kl}\,, \qquad \text{with } f_{(4)ij} = 16\pi\frac{\delta\left(\int \mathrm{d}^3x\,\mathcal{H}^{\mathrm{matter}}_{(8)}\right)}{\delta h^{\mathrm{TT}}_{ij}} - \frac{1}{8}\phi_{(2),i}\phi_{(2),j}\,. \tag{6.31}$$

Using the formal solution $h^{\mathrm{TT}}_{ij} = 2\delta^{\mathrm{TT}kl}_{ij}\Delta^{-1}f_{(4)kl}$, *all* contributions of h^{TT}_{ij} to H_{2PN} can be collected as

$$+\frac{1}{16\pi}\int \mathrm{d}^3x\,\frac{1}{4}h^{\mathrm{TT}}_{ij}\Delta h^{\mathrm{TT}}_{ij} = +\frac{1}{16\pi}\int \mathrm{d}^3x\,\frac{1}{2}h^{\mathrm{TT}}_{ij}f_{(4)ij}\,. \tag{6.32}$$

If one is interested in the spin contribution of this integral only, one can obviously perform a partial integration in (6.32) in such a way that only the spin part of h^{TT}_{ij} is needed, see also [14]. This is desirable as the spin-dependent part of h^{TT}_{ij} is much simpler than the spin-independent part.

Though the discussion of h^{TT}_{ij} was straightforward here, it is quite subtle to obtain the post-Newtonian expansion of (6.29) at higher orders. Indeed, it is not easy to correctly implement the boundary conditions into the solution of the first order equations (6.29). At higher orders (6.29) can be converted into a (second order) wave equation for h^{TT}_{ij}, with source terms expanded according to the post-Newtonian counting rules. In [122, 123] this wave equation is then solved order by order using a near zone expansion of the retarded solution up to the 3.5 post-Newtonian order, corresponding to the boundary condition of no incoming gravitational waves; see also, e.g., [5] for other aspects like tails. Equation (6.31) is indeed the leading order near zone approximation of the wave equation for h^{TT}_{ij}. The solution for h^{TT}_{ij} at higher orders is responsible for the half post-Newtonian orders in the matter-only Hamiltonian, starting at the 2.5 post-Newtonian order.

6.2 Spin corrections to the Hamiltonian

By now there are a lot of results regarding spin effects at the conservative orders in the post-Newtonian approximation. The main goal of this sections is to derive the next-to-leading order spin effects within the developed formalism, which were tackled only recently. Even higher post-Newtonian orders linear in spin were derived recently in [44] for test spinning objects in the Kerr metric. Also Hamiltonians of cubic and higher order in spin were obtained for binary black holes [44,116,117]. The calculation of the leading order dissipative spin-orbit and spin(1)-spin(2) Hamiltonians was prepared in [15]. The corresponding equations of motion were already obtained [131]; see also the considerations in terms of orbital elements in [132].

More work needs to be done for an application of the Hamiltonians derived in this section to gravitational wave astronomy. In particular the spin contributions to the next-to-leading order radiation field are only known for the spin-orbit case [102], but not yet for the spin(1)-spin(2) and spin(1)-spin(1) cases (for the latter case the stress-energy tensor derived in Sect. 5 is needed). Further, it would be useful to find a parametrization of the orbits by solving the equations of motion, i.e., extending the solutions from [133, 134] at least to some of the new Hamiltonians. Finally, one should consider to incorporate the new Hamiltonians into the very successful effective one-body approach [9], which already succeeded for the leading order spin Hamiltonians [10] as well as for the next-to-leading order spin-orbit Hamiltonian [11].

Formulas and regularization procedures for the integrals that need to be solved in this section are given in, e.g., [122, 123, 135]. Some parts of the calculations were also checked using Riesz kernels in arbitrary dimension, see, e.g., [35].

6.2.1 Field constraints in canonical variables

Before starting the calculation of the Hamiltonians, it is suitable to express the source terms of the constraints $\mathcal{H}^{\text{matter}}$ and $\mathcal{H}_i^{\text{matter}}$ in terms of the canonical matter variables. Then the formulas provided in Sect. 6.1.3 automatically give the Hamiltonian (the redefinition of $\pi^{ij\text{TT}}$ is not necessary here). Applying the variable redefinitions from Sect. 3.2.2 to (3.39, 3.40) leads to

$$\begin{aligned}\mathcal{H}^{\text{matter}} = \sum_a \Bigg[& -n\hat{p}_a\hat{\delta}_a - \frac{1}{2}\left(\frac{\hat{S}_{ali}\hat{p}_{aj}}{n\hat{p}_a} + \gamma^{mn}\frac{\hat{S}_{ami}\hat{p}_{aj}\hat{p}_{an}\hat{p}_{al}}{(n\hat{p}_a)^2(m_a - n\hat{p}_a)}\right)\gamma^{kl}\gamma^{ij}_{\ ,k}\hat{\delta}_a \\ & + \frac{\hat{p}_{aj}\gamma^{ji}}{n\hat{p}_a}\hat{A}^{kl}_a e_{(m)k}e^{(m)}{}_{l,i}\hat{\delta}_a - \left(\frac{\hat{p}_{al}}{m_a - n\hat{p}_a}\gamma^{ij}\gamma^{kl}\hat{S}_{ajk}\hat{\delta}_a\right)_{,i}\Bigg] ,\end{aligned} \tag{6.33}$$

$$\mathcal{H}_i^{\text{matter}} = \sum_a \left[\hat{p}_{ai}\hat{\delta}_a - \hat{A}^{kl}_a e_{(m)k}e^{(m)}{}_{l,i}\hat{\delta}_a + \frac{1}{2}\left(s^{ij}_a\hat{\delta}_a\right)_{,j}\right] , \tag{6.34}$$

where

$$s^{ij}_a = \gamma^{jk}\hat{S}_{aik} + \gamma^{jk}\gamma^{lp}\frac{2\hat{p}_{al}\hat{p}_{a(i}\hat{S}_{ak)p}}{n\hat{p}_a(m_a - n\hat{p}_a)} , \tag{6.35}$$

and $\hat{A}^{kl}$ given by (3.44). These source expressions are valid in general, also within the test-spin Hamiltonian (3.47). In the spatial symmetric gauge it holds $\hat{A}^{kl}_a e_{(m)k}e^{(m)}{}_{l,\mu} = \pi^{kl}_a\gamma_{kl,\mu}$, where (3.55) and (4.49) were used and π^{kl}_a is given by (4.50). Notice that the variable redefinitions from the action approach leading to these expressions have been checked up to and including the formal 3.5 post-Newtonian order by the symmetry generator approach [15]. This includes the formal third post-Newtonian or next-to-next-to-leading order linear in spin, which for maximal spin is at the 3.5 post-Newtonian order in the spin-orbit case and at the fourth post-Newtonian order in the spin(1)-spin(2) case. It was shown in [15] as a further check to the same level of approximation that the wave equation for h^{TT}_{ij} following from the ADM Hamiltonian agrees with the Einstein equations, which again verifies that the used variables are canonical.

The expansion of $\mathcal{H}^{\text{matter}}$ sufficient for the formal second post-Newtonian Hamiltonian reads

$$\mathcal{H}^{\text{matter}}_{(2)} = \sum_a m_a\hat{\delta}_a , \qquad \mathcal{H}^{\text{matter}}_{(4)} = \sum_a \left[\frac{\hat{\mathbf{p}}^2_a}{2m_a}\hat{\delta}_a + \frac{1}{2m_a}\hat{p}_{ai}\hat{S}_{a(i)(j)}\hat{\delta}_{a,j}\right] , \tag{6.36}$$

$$\begin{aligned}\mathcal{H}^{\text{matter}}_{(6)} = \sum_a \Bigg[& -\frac{(\hat{\mathbf{p}}^2_a)^2}{8m_a^3}\hat{\delta}_a - \frac{\hat{\mathbf{p}}^2_a}{4m_a}\phi_{(2)}\hat{\delta}_a + \frac{1}{4m_a}\hat{p}_{ai}\hat{S}_{a(i)(j)}\phi_{(2),j}\hat{\delta}_a \\ & - \frac{\hat{\mathbf{p}}^2_a}{8m_a^3}\hat{p}_{ai}\hat{S}_{a(i)(j)}\hat{\delta}_{a,j} - \frac{1}{4m_a}\hat{p}_{ai}\hat{S}_{a(i)(j)}(\phi_{(2)}\hat{\delta}_a)_{,j}\Bigg] ,\end{aligned} \tag{6.37}$$

$$\mathcal{H}^{\text{matter}}_{(8)} = \sum_a \left[\frac{(\hat{\mathbf{p}}_a^2)^3}{16m_a^5}\hat{\delta}_a + \frac{(\hat{\mathbf{p}}_a^2)^2}{8m_a^3}\phi_{(2)}\hat{\delta}_a + \frac{5\hat{\mathbf{p}}_a^2}{64m_a}\phi_{(2)}^2\hat{\delta}_a - \frac{\hat{\mathbf{p}}_a^2}{4m_a}\phi_{(4)}\hat{\delta}_a - \frac{1}{2m_a}\hat{p}_{ai}\hat{p}_{aj}h^{\text{TT}}_{ij}\hat{\delta}_a \right.$$
$$- \frac{\hat{\mathbf{p}}_a^2}{8m_a^3}\hat{p}_{ai}\hat{S}_{a(i)(j)}\phi_{(2),j}\hat{\delta}_a - \frac{5}{32m_a}\hat{p}_{ai}\hat{S}_{a(i)(j)}\phi_{(2)}\phi_{(2),j}\hat{\delta}_a$$
$$\left. + \frac{1}{4m_a}\hat{p}_{ai}\hat{S}_{a(i)(j)}\phi_{(4),j}\hat{\delta}_a + \frac{1}{2m_a}\hat{p}_{ai}\hat{S}_{a(j)(k)}h^{\text{TT}}_{ij,k}\hat{\delta}_a \right] + (\text{td})\,, \tag{6.38}$$

where $\hat{\mathbf{p}}_a = (\hat{p}_{ai})$. The expansion of the source $\mathcal{H}^{\text{matter}}_i$ is given by (6.34) and

$$s^{ij}_{a(3)} = \hat{S}_{a(i)(j)}\,, \qquad s^{ij}_{a(5)} = -\frac{1}{2m_a^2}\hat{p}_{ak}(\hat{p}_{ai}\hat{S}_{a(j)(k)} + \hat{p}_{aj}\hat{S}_{a(i)(k)})\,. \tag{6.39}$$

Notice that the triad terms in (6.34) do not contribute at the considered order.

The expansion of the static source terms needed at the spin(1)-spin(1) order follow from (5.34) as

$$\mathcal{H}^{\text{matter}}_{(4)\,\mathrm{S}^2,\,\hat{p}_i=0} = \sum_a \frac{1}{2}\hat{Q}_{a(i)(j)}\hat{\delta}_{a,ij}\,, \tag{6.40}$$

$$\mathcal{H}^{\text{matter}}_{(6)\,\mathrm{S}^2,\,\hat{p}_i=0} = \sum_a \left[\frac{1}{4}\hat{Q}_{a(i)(j)}(\phi_{(2),i}\hat{\delta}_a)_{,j} - \frac{1}{4}\hat{Q}_{a(i)(j)}(\phi_{(2)}\hat{\delta}_a)_{,ij} \right.$$
$$\left. + \frac{1}{8m_a}\hat{S}_{a(i)(k)}\hat{S}_{a(j)(k)}(\phi_{(2),i}\hat{\delta}_a)_{,j} \right], \tag{6.41}$$

$$\mathcal{H}^{\text{matter}}_{(8)\,\mathrm{S}^2,\,\hat{p}_i=0} = -\sum_a \frac{1}{32m_a}\hat{S}_{a(i)(k)}\hat{S}_{a(k)(j)}\phi_{(2),i}\phi_{(2),j}\hat{\delta}_a + (\text{td})\,, \tag{6.42}$$

and it holds

$$\hat{Q}_{a(i)(j)} = \frac{C_{Qa}}{m_a}\left(\hat{S}_{a(i)(k)}\hat{S}_{a(j)(k)} - \frac{2}{3}\delta_{ij}\hat{\mathbf{S}}_a^2\right). \tag{6.43}$$

Here $\hat{\mathbf{S}}_a = (\hat{S}_{a(i)})$ and $\hat{S}_{a(i)} = \frac{1}{2}\epsilon_{ijk}\hat{S}_{a(j)(k)}$. No further contributions to $\mathcal{H}^{\text{matter}}_i$ arise in the spin(1)-spin(1) case.

6.2.2 Leading-order

The leading order spin effects are at the formal first post-Newtonian order and their Hamiltonian can be obtained from (6.24), which of course gives the first post-Newtonian Hamiltonian in the nonspinning case. The needed solutions of the constraints read

$$\phi_{(2)} = 4\sum_a \frac{m_a}{\hat{r}_a}\,, \qquad \tilde{\pi}^i_{(3)} = \sum_a \left[2\frac{\hat{p}_{ai}}{\hat{r}_a} + \hat{S}_{a(i)(j)}\left(\frac{1}{\hat{r}_a}\right)_{,j} \right], \tag{6.44}$$

$$V^i_{(3)} = \sum_a \left[2\frac{\hat{p}_{ai}}{\hat{r}_a} - \frac{1}{4}\hat{p}_{aj}\hat{r}_{a,ij} + \hat{S}_{a(i)(j)}\left(\frac{1}{\hat{r}_a}\right)_{,j} \right], \tag{6.45}$$

where $\hat{r}_a = |\mathbf{x} - \hat{\mathbf{z}}_a|$ and $\hat{\mathbf{z}}_a = (\hat{z}^i_a)$. The leading order (LO) spin-orbit (SO) Hamiltonian follows as

$$H_{\mathrm{SO}}^{\mathrm{LO}} = \sum_a \sum_{b \neq a} \frac{1}{\hat{r}_{ab}^2} (\hat{\mathbf{S}}_a \times \hat{\mathbf{n}}_{ab}) \cdot \left[\frac{3m_b}{2m_a} \hat{\mathbf{p}}_a - 2\hat{\mathbf{p}}_b \right] , \tag{6.46}$$

where $\hat{r}_{ab} = |\hat{\mathbf{z}}_a - \hat{\mathbf{z}}_b|$ and $\hat{\mathbf{n}}_{ab} = (\hat{\mathbf{z}}_a - \hat{\mathbf{z}}_b)/\hat{r}_{ab}$. This Hamiltonian is at the 1.5 post-Newtonian order for maximal spins. Further, the leading order spin(a)-spin(b), or $\mathrm{S}_a\mathrm{S}_b$, Hamiltonian results as

$$H_{\mathrm{S}_a\mathrm{S}_b}^{\mathrm{LO}} = \sum_a \sum_{b \neq a} \frac{1}{2\hat{r}_{ab}^3} \left[3(\hat{\mathbf{S}}_a \cdot \hat{\mathbf{n}}_{ab})(\hat{\mathbf{S}}_b \cdot \hat{\mathbf{n}}_{ab}) - (\hat{\mathbf{S}}_a \cdot \hat{\mathbf{S}}_b) \right] . \tag{6.47}$$

For maximal spins this Hamiltonian is at the second post-Newtonian level. Finally, the leading order spin(a)-spin(a), or S_a^2, Hamiltonian is given by

$$H_{\mathrm{S}_a^2}^{\mathrm{LO}} = \sum_a \sum_{b \neq a} \frac{C_{Qa} m_b}{2m_a \hat{r}_{ab}^3} \left[3(\hat{\mathbf{S}}_a \cdot \hat{\mathbf{n}}_{ab})^2 - \hat{\mathbf{S}}_a^2 \right] , \tag{6.48}$$

which is also at the second post-Newtonian order for maximal spins. All Hamiltonians in this section are valid for arbitrary many spinning objects. The Poisson brackets are the standard canonical ones, i.e.,

$$\{\hat{z}_a^i, \hat{p}_{aj}\} = \delta_{ij} , \qquad \{\hat{S}_{a(i)}, \hat{S}_{a(j)}\} = \epsilon_{ijk} \hat{S}_{a(k)} , \tag{6.49}$$

zero otherwise.

The leading order spin effects derived here are well-known for black holes ($C_Q = 1$), see, e.g., [136, 137]. For the leading order C_Q-dependence see [105, 137].

6.2.3 Next-to-leading order

Now we proceed to the formal second post-Newtonian Hamiltonian (6.25), which includes the next-to-leading order spin effects. There we also need the functions

$$\phi_{(4)} = \sum_a \left[\frac{2\hat{\mathbf{p}}_a^2}{m_a \hat{r}_a} - \sum_{b \neq a} \frac{2m_a m_b}{\hat{r}_{ab} \hat{r}_a} + \frac{2\hat{p}_{ai} \hat{S}_{a(i)(j)}}{m_a} \left(\frac{1}{\hat{r}_a} \right)_{,j} + 2\hat{Q}_{a(i)(j)} \left(\frac{1}{\hat{r}_a} \right)_{,ij} \right] , \tag{6.50}$$

$$\begin{aligned} \tilde{\pi}_{(3)}^{ij} = \sum_a \Bigg[& 2\hat{p}_{ai} \left(\frac{1}{\hat{r}_a} \right)_{,j} + 2\hat{p}_{aj} \left(\frac{1}{\hat{r}_a} \right)_{,i} - \delta_{ij} \hat{p}_{ak} \left(\frac{1}{\hat{r}_a} \right)_{,k} - \frac{1}{2} \hat{p}_{ak} \hat{r}_{a,ijk} \\ & - \hat{S}_{a(k)(i)} \left(\frac{1}{\hat{r}_a} \right)_{,kj} - \hat{S}_{a(k)(j)} \left(\frac{1}{\hat{r}_a} \right)_{,ki} \Bigg] . \end{aligned} \tag{6.51}$$

Notice that there are C_{Qa}-contributions in (6.50). We restrict to two spinning objects in this section. The results provided here complete the knowledge of spin corrections to the Hamiltonian up to and including the third post-Newtonian order for maximal spins.

Next-to-leading order spin-orbit Following the method developed here, the next-to-leading order (NLO) spin-orbit Hamiltonian results as [14]

$$\begin{aligned} H_{\mathrm{SO}}^{\mathrm{NLO}} = & -\frac{((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1) \cdot \hat{\mathbf{n}}_{12})}{\hat{r}_{12}^3} \left[\frac{11m_2}{2} + \frac{5m_2^2}{m_1} \right] + \frac{((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_1) \cdot \hat{\mathbf{n}}_{12})}{\hat{r}_{12}^3} \left[6m_1 + \frac{15m_2}{2} \right] \\ & - \frac{((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1) \cdot \hat{\mathbf{n}}_{12})}{\hat{r}_{12}^2} \left[\frac{5m_2 \hat{\mathbf{p}}_1^2}{8m_1^3} + \frac{3(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2)}{4m_1^2} - \frac{3\hat{\mathbf{p}}_2^2}{4m_1 m_2} + \frac{3(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})}{4m_1^2} \right. \end{aligned}$$

$$
+ \frac{3(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})^2}{2m_1 m_2}\Bigg] + \frac{((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_1) \cdot \hat{\mathbf{n}}_{12})}{\hat{r}_{12}^2} \left[\frac{(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2)}{m_1 m_2} + \frac{3(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})}{m_1 m_2} \right]
$$

$$
+ \frac{((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1) \cdot \hat{\mathbf{p}}_2)}{\hat{r}_{12}^2} \left[\frac{2(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})}{m_1 m_2} - \frac{3(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})}{4m_1^2} \right] + (1 \leftrightarrow 2)\,, \tag{6.52}
$$

where $(1 \leftrightarrow 2)$ indicates an exchange of particle labels, and is identical to the one derived earlier in [84]. The next-to-leading order spin-orbit case was first tackled on the level of the equations of motion in [138] and was later rederived and improved in [139] (both in the harmonic gauge). Within the ADM canonical formalism the Hamiltonian $H^{\rm NLO}_{\rm SO}$ corresponding to these equations of motion was obtained in [84] from the spin equation of motion (2.28). The linear-in-G part of $H^{\rm NLO}_{\rm SO}$ was also derived in [116] from corresponding source terms of the constraints, similar to the approach used here (however, in [116] the source terms were obtained from the approximate Kerr metric in the ADM transverse traceless gauge). Very recently derivations within the effective field theory approach also succeeded [140].

Next-to-leading order spin(1)-spin(2) The spin(1)-spin(2), or $\mathrm{S_1S_2}$, Hamiltonian reads [20]

$$
\begin{aligned}
H^{\rm NLO}_{\rm S_1S_2} = {} & \frac{1}{2m_1 m_2 \hat{r}_{12}^3}\big[\tfrac{3}{2}((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1) \cdot \hat{\mathbf{n}}_{12})((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_2) \cdot \hat{\mathbf{n}}_{12}) + \tfrac{1}{2}(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2)(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2) \\
& + 6((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_1) \cdot \hat{\mathbf{n}}_{12})((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_2) \cdot \hat{\mathbf{n}}_{12}) - \tfrac{1}{2}(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_2)(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{p}}_1) \\
& - 15(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12}) + (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1)(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{p}}_2) \\
& - 3(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2) + 3(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_2)(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12}) \\
& + 3(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{p}}_1)(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12}) + 3(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1)(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12}) \\
& + 3(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{p}}_2)(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12}) - 3(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2)(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})\big] \\
& + \frac{3}{2m_1^2 \hat{r}_{12}^3}\big[-((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1) \cdot \hat{\mathbf{n}}_{12})((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_2) \cdot \hat{\mathbf{n}}_{12}) + (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2)(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \\
& - (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{p}}_1)(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})\big] + \frac{3}{2m_2^2 \hat{r}_{12}^3}\big[(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2)(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})^2 \\
& - ((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_2) \cdot \hat{\mathbf{n}}_{12})((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_1) \cdot \hat{\mathbf{n}}_{12}) - (\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_2)(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})\big] \\
& + \frac{6(m_1 + m_2)}{\hat{r}_{12}^4}\big[(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2) - 2(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{n}}_{12})\big]\,,
\end{aligned} \tag{6.53}
$$

and was confirmed by [114,141]. Notice that no agreement with the result in [142] could be found, see [20]. Indeed, the result in [142] turned out to be incomplete [20, 114].

Next-to-leading order spin(1)-spin(1) A nonreduced potential (i.e., with the spin supplementary condition not eliminated on the level of the potential) for the next-to-leading order spin(1)-spin(1), or S_1^2, dynamics is given in [107, 143]. Within the method described in Sect. 5.3.2 an equivalent Hamiltonian $H^{\rm NLO}_{\rm S_1^2}$ will be derived here. This Hamiltonian was first given only for the black hole case ($C_{Q1} = 1$) in [17] and then generalized to arbitrary C_{Q1} later [19]. However, the comparison with [107, 143] was quite cumbersome. First agreement with [107] could not even be found in the spin precession equation [17], however, this finally succeeded after identifying a sign typo in [107], see [18] (all for the case $C_{Q1} = 1$). After a further correction [143] full agreement was finally found in [19], now also for arbitrary C_{Q1}. For this comparison the potential from [107, 143] was first transformed into a fully reduced Hamiltonian in [19] by a Legendre transformation and an elimination of the spin supplementary condition using Dirac brackets (2.75). Then a canonical transformation leading to our result in [19] was searched for and found.

The result for general compact objects (including neutron stars) is [19]

$$
\begin{aligned}
H^{\mathrm{NLO}}_{\mathrm{S}_1^2} = \frac{m_2}{m_1^3 \hat{r}_{12}^3} \Bigg[& \left(-\frac{5}{4} + \frac{3}{2} C_{Q1}\right) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1)^2 + \left(-\frac{21}{8} + \frac{9}{4} C_{Q1}\right) \hat{\mathbf{p}}_1^2 (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \\
& + \left(\frac{15}{4} - \frac{9}{2} C_{Q1}\right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1) + \left(-\frac{9}{8} + \frac{3}{2} C_{Q1}\right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \hat{\mathbf{S}}_1^2 \\
& + \left(\frac{5}{4} - \frac{5}{4} C_{Q1}\right) \hat{\mathbf{p}}_1^2 \hat{\mathbf{S}}_1^2 \Bigg] + \frac{1}{m_1^2 \hat{r}_{12}^3} \Bigg[-\frac{15}{4} C_{Q1} (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \\
& + \left(-\frac{3}{2} + \frac{9}{2} C_{Q1}\right) (\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1) + \left(-\frac{3}{2} + \frac{9}{4} C_{Q1}\right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2) \hat{\mathbf{S}}_1^2 \\
& + \left(-3 + \frac{3}{2} C_{Q1}\right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_2) + \left(\frac{3}{2} - \frac{3}{2} C_{Q1}\right) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_2) \\
& + \left(3 - \frac{21}{4} C_{Q1}\right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 + \left(\frac{3}{2} - \frac{3}{4} C_{Q1}\right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12}) (\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12}) \hat{\mathbf{S}}_1^2 \Bigg] \\
& + \frac{C_{Q1}}{m_1 m_2 \hat{r}_{12}^3} \left[\frac{9}{4} \hat{\mathbf{p}}_2^2 (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 - \frac{3}{4} \hat{\mathbf{p}}_2^2 \hat{\mathbf{S}}_1^2 \right] + \frac{m_2}{\hat{r}_{12}^4} \Bigg[\left(-3 - \frac{3}{2} C_{Q1}\right) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \\
& + \left(2 + \frac{1}{2} C_{Q1}\right) \hat{\mathbf{S}}_1^2 \Bigg] + \frac{m_2^2}{m_1 \hat{r}_{12}^4} \left[(1 + 2 C_{Q1}) \hat{\mathbf{S}}_1^2 + (-1 - 6 C_{Q1}) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \right] . \qquad (6.54)
\end{aligned}
$$

The corresponding spin(2)-spin(2) Hamiltonian $H^{\mathrm{NLO}}_{\mathrm{S}_2^2}$ simply results from an exchange of particle labels. According to Sect. 5.3.2 the linear-in-G part was derived with the help of the Poincaré algebra method from [117], while the G^2 part (the last two lines) results from the source expressions (6.40–6.42) derived in the present article.

Notice that for black holes ($C_{Q1} = 1$) this Hamiltonian was already found in [17], for the first time including the correct center of mass motion. Further, the earlier result for the general case in [107, 143] is not a fully reduced Hamiltonian. The Hamiltonian presented here is on a higher level of sophistication with advantages for applications, e.g., the spin vectors appearing in our Hamiltonian have a constant length and it is easier to obtain all equations of motion in terms of these "good" spin variables.

6.2.4 Center of mass and Poincaré algebra

The post-Newtonian expansion of the center of mass vector

$$
G^i = -\frac{1}{16\pi} \int \mathrm{d}^3 x \, x^i \Delta\phi = G^i_{\mathrm{N}} + G^i_{\mathrm{1PN}} + G^i_{\mathrm{2PN}} + \cdots , \qquad (6.55)
$$

can be obtained from the expanded Hamilton constraint (6.9, 6.10). To the formal second post-Newtonian order this leads to

$$
G^i_{\mathrm{N}} = \int \mathrm{d}^3 x \, x^i \, \mathcal{H}^{\mathrm{matter}}_{(2)} , \qquad G^i_{\mathrm{1PN}} = \int \mathrm{d}^3 x \, x^i \left[\mathcal{H}^{\mathrm{matter}}_{(4)} - \frac{1}{8} \mathcal{H}^{\mathrm{matter}}_{(2)} \phi_{(2)} \right] , \qquad (6.56)
$$

$$G^i_{\text{2PN}} = \int \mathrm{d}^3x \left[x^i \left(\mathcal{H}^{\text{matter}}_{(6)} - \frac{1}{8} \left(\mathcal{H}^{\text{matter}}_{(4)} \phi_{(2)} + \mathcal{H}^{\text{matter}}_{(2)} \phi_{(4)} \right) \right.\right.$$
$$\left.\left. + \frac{1}{64} \mathcal{H}^{\text{matter}}_{(2)} \phi^2_{(2)} + V^i_{(3)} \mathcal{H}^{\text{matter}}_{(3)i} \right) + \frac{1}{16\pi} \frac{5}{2} V^i_{(3)} \tilde{\pi}^k_{(3),k} \right]. \tag{6.57}$$

For the formula for G^i_{2PN} partial integrations were applied, similar as in Sect. 6.1.3. Notice that $\tilde{\pi}^k_{(3),k}$ is spin-independent. For results in the nonspinning case see [124].

The contributions to the center of mass vector corresponding to the leading order spin Hamiltonians follow from $\mathbf{G}_{\text{1PN}} = (G^i_{\text{1PN}})$ as

$$\mathbf{G}^{\text{LO}}_{\text{SO}} = \sum_a \frac{1}{2m_a} (\hat{\mathbf{p}}_a \times \hat{\mathbf{S}}_a) , \qquad \mathbf{G}^{\text{LO}}_{\text{S}_1\text{S}_2} = 0 , \qquad \mathbf{G}^{\text{LO}}_{\text{S}^2_1} = 0 . \tag{6.58}$$

From $\mathbf{G}_{\text{2PN}} = (G^i_{\text{2PN}})$ the next-to-leading order parts result as

$$\mathbf{G}^{\text{NLO}}_{\text{SO}} = \sum_a \sum_{b \neq a} \frac{m_b}{4 m_a \hat{r}_{ab}} \left[((\hat{\mathbf{p}}_a \times \hat{\mathbf{S}}_a) \cdot \hat{\mathbf{n}}_{ab}) \frac{5\hat{\mathbf{z}}_a + \hat{\mathbf{z}}_b}{\hat{r}_{ab}} - 5(\hat{\mathbf{p}}_a \times \hat{\mathbf{S}}_a) \right] - \sum_a \frac{\hat{\mathbf{p}}^2_a}{8m^3_a} (\hat{\mathbf{p}}_a \times \hat{\mathbf{S}}_a)$$
$$+ \sum_a \sum_{b \neq a} \frac{1}{\hat{r}_{ab}} \left[\frac{3}{2} (\hat{\mathbf{p}}_b \times \hat{\mathbf{S}}_a) - \frac{1}{2} (\hat{\mathbf{n}}_{ab} \times \hat{\mathbf{S}}_a)(\hat{\mathbf{p}}_b \cdot \hat{\mathbf{n}}_{ab}) - ((\hat{\mathbf{p}}_b \times \hat{\mathbf{S}}_a) \cdot \hat{\mathbf{n}}_{ab}) \frac{\hat{\mathbf{z}}_a + \hat{\mathbf{z}}_b}{\hat{r}_{ab}} \right], \tag{6.59}$$

$$\mathbf{G}^{\text{NLO}}_{\text{S}_1\text{S}_2} = \frac{1}{2} \sum_a \sum_{b \neq a} \left[\left(3(\hat{\mathbf{S}}_a \cdot \hat{\mathbf{n}}_{ab})(\hat{\mathbf{S}}_b \cdot \hat{\mathbf{n}}_{ab}) - (\hat{\mathbf{S}}_a \cdot \hat{\mathbf{S}}_b) \right) \frac{\hat{\mathbf{z}}_a}{\hat{r}^3_{ab}} + (\hat{\mathbf{S}}_b \cdot \hat{\mathbf{n}}_{ab}) \frac{\hat{\mathbf{S}}_a}{\hat{r}^2_{ab}} \right], \tag{6.60}$$

$$\mathbf{G}^{\text{NLO}}_{\text{S}^2_1} = \frac{m_2}{m_1} \left[C_{Q1} \left(3(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 - \hat{\mathbf{S}}^2_1 \right) \frac{\hat{\mathbf{z}}_1 + \hat{\mathbf{z}}_2}{4\hat{r}^3_{12}} + (1 + C_{Q1}) \hat{\mathbf{S}}^2_1 \frac{\hat{\mathbf{n}}_{12}}{2\hat{r}^3_{12}} \right.$$
$$\left. - (1 + 3C_{Q1})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12}) \frac{\hat{\mathbf{S}}_1}{2\hat{r}^2_{12}} \right]. \tag{6.61}$$

Notice that (6.59) and (6.60) are valid for arbitrary many spinning objects, while (6.61) holds for two objects only. For two objects $\mathbf{G}^{\text{NLO}}_{\text{SO}}$ was already found in [84]. Further, $\mathbf{G}^{\text{NLO}}_{\text{S}^2_2}$ simply results from an exchange of particle labels in (6.61)

Now one can check whether the Poincaré algebra (2.10, 2.11) is fulfilled, which is indeed the case (the Hamiltonian plays of course the role of the energy E). At the spin-orbit level this was already shown in [84]. At the spin(1)-spin(1) level this holds by construction, as most terms of the Hamiltonian $H^{\text{NLO}}_{\text{S}^2_1}$ were obtained from the Poincaré algebra via an ansatz in [117]. However, the fulfillment of the Poincaré algebra provides a thorough check of $H^{\text{NLO}}_{\text{SO}}$ and $H^{\text{NLO}}_{\text{S}_1\text{S}_2}$.

7 Conclusions and outlook

The first main goal of this work, the extension of the ADM canonical formalism from nonspinning to spinning objects, succeeded to linear order in spin via an action approach. The result was verified by an independent order-by-order derivation. Even the extension to higher orders in spin is well understood now, but somewhat more complicated and requires further approximations, like the post-Newtonian one. The second main goal of this work, the calculation of conservative Hamiltonians for inspiralling binaries relevant for gravitational wave astronomy, was then straightforward. The effort of first deriving the canonical formalism was payed off by its efficiency in the calculation of these Hamiltonians. New results are the next-to-leading order spin(1)-spin(2) and spin(1)-spin(1) Hamiltonians, and the spin-orbit Hamiltonian derived earlier by Damour, Jaranowski, and Schäfer was confirmed. All Hamiltonians through the third post-Newtonian order for maximal spin are known.

The next most interesting Hamiltonian which could be calculated is the conservative next-to-next-to-leading order spin-orbit one, which is at the 3.5 post-Newtonian level for maximally rotating objects. Notice that the verification of the canonical formalism given in this article via the order-by-order construction already covered this case. Leading order *dissipative* Hamiltonians are also envisaged and its calculation was already prepared in [15]. For maximal spins these Hamiltonians are even at the fourth post-Newtonian order in the spin-orbit case and at the 4.5 post-Newtonian order in the spin(1)-spin(2) case. The extension of a recent result within the *post-Minkowskian* approximation [125] to spinning objects would also be desirable, as it could be applied to the gravitational scattering of spinning bodies moving at relativistic speed.

Further, more work needs to be done for an application of the new Hamiltonians presented in this article to gravitational wave astronomy. In particular the spin contributions to the next-to-leading order radiation field are only known for the spin-orbit case [102]. This result should be extended to the spin(1)-spin(2) case, as well as to the spin(1)-spin(1) case. For the latter the spin(1)-spin(1) contributions to the stress-energy tensor given in this article are crucial. Also an implementation of the new results given here into the very successful effective one-body approach would be appealing.

Another though rather mathematical development for the future would be to consider the full constraint algebra, gravitational field and supplementary conditions, at different stages of gauge fixing, as well as a treatment using Dirac brackets; see also [64] for the case of Dirac fields.

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Another significant contribution to this thesis was given by Steven Hergt, as he often was a coauthor of mine and I am grateful for the prolific collaboration with him. Regarding the content of this article, he calculated the spin(1)-spin(1) center of mass vector (6.61) as well as the linear-in-G part of the spin(1)-spin(1) Hamiltonian (6.54) and provided the calculations regarding the Poincaré algebra in Sect. 6.2.4. He also did most of the calculations for the full comparison (i.e., including the center of mass motion) of the spin(1)-spin(1) Hamiltonian (6.54) with the result in [107, 143], see [19]. He further often checked my calculations, in particular the ones for the spin-orbit and spin(1)-spin(2) Hamiltonians (6.52, 6.53) as well as the G^2 part of the spin(1)-spin(1) Hamiltonian (6.54).

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Appendix A Symbols

$\hat{A}^{ij}$	defined by (3.44)
B^{ij}_{kl}	defined by (3.54)
c	speed of light, usually $c = 1$ here
C_Q	mass-quadrupole parameter, see (5.11)
δ	defined as $\delta = \delta(x^i - z^i)$
$\hat{\delta}$	defined as $\hat{\delta} = \delta(x^i - \hat{z}^i)$
$\delta_{(4)}$	defined as $\delta_{(4)} = \delta(x^\mu - z^\mu)$
δ_{ij}	Kronecker delta, $(\delta_{ij}) = \mathrm{diag}(1,1,1)$
$\delta^{\mathrm{TT}kl}_{ij}$	transverse traceless projector, see (2.55)
Δ	Laplace operator, $\Delta = \partial_i \partial_i$
Δ^{-1}	inverse of Δ for usual boundary conditions
$e_{I\mu}$	tetrad field, $g_{\mu\nu} = e_{I\mu} e^I{}_\nu$
e_{ij}	triad in the symmetric gauge, see (3.53)
$\hat{e}^{ij}$	defined as $\hat{e}^{ij} \equiv \frac{1}{2}(e^{i(j)} - e^{j(i)})$
$\hat{e}^{ij}_a$	defined as $\hat{e}^{ij}_a \equiv \hat{e}^{ij}(\hat{z}^k_a)$
ϵ_{ijk}	3-dimensional Levi-Civita symbol
E	energy of the system, see (4.19)
$E^{(4)}_{\mu\nu}$	electric part of $C^{(4)}_{\mu\nu\alpha\beta}$, see (5.19, 5.18)
f^μ	timelike vector in conditions (3.12, 3.10)
ϕ	trace part of the induced metric, see (6.2)
g	defined as $g = \det(g_{\mu\nu})$
$g_{\mu\nu}$	4-dimensional metric
γ	defined as $\gamma = \det(\gamma_{ij})$
$\gamma_{\mu\nu}$	projector (2.38), contains induced metric
G	gravitational constant, usually $G = 1$ here
G^i	center of mass vector, see (4.19)
Γ_{kij}	3-dim. Christoffel symbol of first kind
$\Gamma^{(4)}_{\alpha\mu\nu}$	4-dim. Christoffel symbol of first kind
h^{TT}_{ij}	transverse traceless part of γ_{ij}, see (6.2)
H	general symbol for a Hamiltonian
H_{ADM}	ADM Hamiltonian, see (3.61)
$\mathcal{H}$	Hamilton constraint, see (6.1)
$\mathcal{H}_i$	momentum constraint, see (6.1)
$\mathcal{H}^{\mathrm{field}}$	field part of $\mathcal{H}$, see (2.44)
$\mathcal{H}^{\mathrm{field}}_i$	field part of $\mathcal{H}_i$, see (2.44)
$\mathcal{H}^{\mathrm{matter}}$	matter part of $\mathcal{H}$, see Sect. 6.2.1
$\mathcal{H}^{\mathrm{matter}}_i$	matter part of $\mathcal{H}_i$, see Sect. 6.2.1
$\mathcal{H}^{\pi\mathrm{matter}}_i$	defined by (4.22)
I	moment of inertia of a spherical top
$J^{\mu\nu}$	total angular momentum, see (2.12, 4.9)
$J^{\mu\nu\alpha\beta}$	Dixon's quadrupole moment
K_{ij}	extrinsic curvature, see (2.39)
λ	Lagrange multiplier
L_M	matter Lagrangian
$\mathcal{L}_G$	field Lagrangian density (2.36, 2.40, 2.43)
$\mathcal{L}_M$	matter Lagrangian density
Λ^{AI}	Lorentz matrix, see section 3.1.3
$\hat{\Lambda}^{[i](j)}$	canonical rotation matrix defined by (3.43)
m	constant mass-like parameter, see (3.28)
m_0	constant mass-like parameter, see (3.26)
m_p	dynamical mass, $m_p^2 = -p_\mu p^\mu$
M	mass of the system, $M^2 = -P_\mu P^\mu$
n^μ	normal vector for (3+1)-split, see (2.37)
$\hat{\mathbf{n}}_{ab}$	defined by $\hat{\mathbf{n}}_{ab} = (\hat{\mathbf{z}}_a - \hat{\mathbf{z}}_b)/\hat{r}_{ab}$
np	defined by $np = n^\mu p_\mu$, see (3.32) or (5.30)
$n\hat{p}$	defined by $n\hat{p} = -\sqrt{m^2 + \gamma^{ij}\hat{p}_i\hat{p}_j}$
nS_i	defined by $nS_i = n^\mu S_{\mu i}$, see (3.33, 3.42)
N	lapse function, see (2.37)
N^i	shift vector, see (2.37)
$\omega_\mu{}^{IJ}$	Ricci rotation coefficients
$\Omega^{\mu\nu}$	angular velocity tensor, see (3.15)
$\hat{\Omega}^{(i)(j)}$	angular velocity $\hat{\Omega}^{(i)(j)} = \hat{\Lambda}_{[k]}{}^{(i)}\dot{\hat{\Lambda}}^{[k](j)}$
p_μ	linear momentum, see (2.31, 5.4)
$\hat{p}_i$	canonical momentum conjugate to $\hat{z}^i$
P_μ	total linear momentum, $P_0 = -E$, (4.9)
$P_{\mu\nu}$	the projector $P_{\mu\nu} = g_{\mu\nu} - \frac{1}{f_\rho f^\rho} f^\mu f^\nu$
π^{ij}	defined by (2.41)
$\pi^{ij\mathrm{TT}}$	transverse traceless part of π^{ij}, (6.5, 6.6)
π^{ij}_a	spin correction to $\hat{\pi}^{ij\mathrm{TT}}$, see (4.18)
$\tilde{\pi}^i$	vector potential for $\tilde{\pi}^{ij}$, see (6.7)
$\tilde{\pi}^{ij}$	vector potential part of π^{ij}, see (6.5, 6.7)
$\breve{\pi}^{ij}$	trace part of π^{ij}, see (6.5, 6.6)
$\hat{\pi}^{ij}$	canonical field momentum, see (3.58)
$\hat{\pi}^{ij\mathrm{TT}}$	transverse traceless part of $\hat{\pi}^{ij}$, see (3.60)
$\hat{\tilde{\pi}}^i$	vector potential for $\hat{\tilde{\pi}}^{ij}$, see (3.60)
$\hat{\tilde{\pi}}^{ij}$	longitudinal part of $\hat{\pi}^{ij}$, see (3.60)
$Q^{\mu\nu}$	mass quadrupole part of $J^{\mu\nu\alpha\beta}$, see (5.6)
$\hat{Q}_{ij}$	defined by (5.35)
$\hat{r}_a$	defined by $\hat{r}_a = \lvert\mathbf{x} - \hat{\mathbf{z}}_a\rvert$
$\hat{r}_{ab}$	defined by $\hat{r}_{ab} = \lvert\hat{\mathbf{z}}_a - \hat{\mathbf{z}}_b\rvert$
R	3-dimensional Ricci scalar, $R = \gamma^{ij} R_{ij}$
$R^{(4)}$	4-dimensional Ricci scalar, $R = g^{\mu\nu} R^{(4)}_{\mu\nu}$
R_{ij}	3-dim. Ricci tensor, $R_{ij} = \gamma^{kl} R_{ikjl}$
$R^{(4)}_{\mu\nu}$	4-dim. Ricci tensor, $R^{(4)}_{\mu\nu} = g^{\alpha\beta} R^{(4)}_{\mu\alpha\nu\beta}$
R_{ijkl}	3-dim. Riemann tensor, sign as in (2.29)
$R^{(4)}_{\mu\nu\alpha\beta}$	4-dim. Riemann tensor, see (2.29)
s^{ij}_a	defined by (4.30), see also (6.35)
S	spin length, $2S^2 = S^{\mu\nu} S_{\mu\nu}$
$S^{\mu\nu}$	spin tensor, usually restricted to (2.32)
$\hat{S}^{\mu\nu}$	canonical spin tensor, see (3.49)
$\hat{S}_{(i)}$	canonical spin vector, $\hat{S}_{(i)} = \frac{1}{2}\epsilon_{ijk}\hat{S}_{(j)(k)}$
$\hat{\mathbf{S}}$	canonical spin vector, $\hat{\mathbf{S}} = (\hat{S}_{(i)})$
t	time coordinate, $x^0 \equiv t$ or $z^0 \equiv t$

τ	worldline parameter
$t^{\mu\nu\cdots}$	multipole moments, see (2.26)
$\delta\theta^{IJ}$	variation for Λ^{AI}, $\delta\theta^{IJ} = \Lambda_A{}^I \delta\Lambda^{AJ}$
$T^{\mu\nu}$	stress-energy tensor
(td)	denotes a total divergence
u^μ	4-velocity, $u^\mu = \frac{dz^\mu}{d\tau}$
V^i	vector potential for $\tilde{\pi}^{ij}$, see (6.15, 6.16)
$\hat{V}^i$	vector potential for $\hat{\tilde{\pi}}^{ij}$, see (4.24, 4.25)
W	full action, $W = W_G + W_M$
W_G	Einstein-Hilbert action, see (2.36)
W_M	matter part of the action W
x^μ	spacetime coordinates, $x^0 \equiv t$
z^μ	worldline function, $z^0 \equiv t$
$\hat{z}^i$	canonical position variable
z^i_Δ	possible correction to $\hat{z}^i$, see (4.34)

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