Optim Lett (2014) 8:2135–2143
DOI 10.1007/s11590-013-0720-8

ORIGINAL PAPER

A simple SSD-efficiency test

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Received: 26 June 2013 / Accepted: 24 December 2013 / Published online: 29 January 2014

Abstract A linear programming SSD-efficiency test capable of identifying a dominating portfolio is proposed. It has $T + n$ variables and at most $2T + 1$ constraints, whereas the existing SSD-efficiency tests are either unable to identify a dominating portfolio, or require solving a linear program with at least $O(T^2 + n)$ variables and/or constraints.

Keywords Stochastic dominance · Portfolio analysis · Linear programming

1 Introduction

The concept of second-order stochastic dominance (SSD), introduced in economics by Hadar and Russell [7] and Rothschild and Stiglitz [18], has become one of the central concepts in risk modelling. We say that portfolio rate of return X, modelled as a random variable (r.v.) on some probability space, dominates r.v. Y by SSD, and write $X \succ_2 Y$, if X is preferred to Y for any risk-averse expected utility maximizer (that is, for any agent with increasing and concave utility function, see [13]). Thus, the notion of SSD allows one to compare some of the investment opportunities without knowing exact utility function of a particular agent. This is particularly important, because identifying a utility function is a difficult task, which resorts to an extensive questionnaire procedure.

Given a (convex) set $\mathcal{V}$ of admissible portfolio rates of return, r.v. $Y \in \mathcal{V}$ is called SSD-dominated within $\mathcal{V}$, if $X \succ_2 Y$ for some $X \in \mathcal{V}$ (in this case, X will be called a dominating portfolio), and SSD-efficient otherwise. The SSD definition implies that an optimal investment for a risk-averse expected-utility maximizer belongs to the

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set of SSD-efficient portfolios. However, this fact goes beyond the expected utility theory: only SSD-efficient portfolios may be optimal for an agent who maximizes convex Yaari dual utility function (see [19, Theorem 2]), minimizes a law-invariant convex risk measure (see [4, Corollary 4.59]), or uses the mean-deviation model [5]. SSD-efficiency is also central to solving the inverse portfolio problem for identifying investor's risk preferences, see [6]. This motivates the following question: determine whether the given portfolio Y is SSD-efficient within a given set $\mathcal{V}$, and if not, find a dominating portfolio.

Assuming that the underlying probability space is a finite T-element set $\Omega = \{\omega_1, \ldots, \omega_T\}$, and $\mathcal{V}$ consists of linear combinations of rates of return of n assets, Post [15] developed a linear program with $O(T+n)$ variables and constraints[1], which tests whether a given $Y \in \mathcal{V}$ is SSD-efficient subject to the additional assumption

$$Y(\omega_i) \neq Y(\omega_j), \quad i \neq j, \tag{1}$$

i.e. that ties do not occur in the distribution of Y. This assumption holds with probability 1 if the distribution of Y is an approximation of a continuous one using T Monte-Carlo simulations. However, assumption (1) is rarely encountered in practical applications. As explained in [15, Section II-C], tied returns may occur, for example, when analysing bootstrap pseudo-samples or evaluating a riskless alternative. Even if (1) holds for base assets, it might fail for some mixtures of them, or for derivative securities. Kopa and Post [9] show how the analysis can be generalized using a weakly increasing ranking to account for ties.[2]

Based on his test, Post [15, Section IV] made a surprising conclusion that some of the popular financial indices are SSD-dominated and hence cannot be optimal investments for a risk-averse agent. Post's test, however, fails to identify a dominating portfolio X if Y is SSD-dominated. In other words, for an agent holding a portfolio with rate of return Y, this test may show the *existence* of better investment opportunities within set $\mathcal{V}$, but does not identify them.

Assuming (1), Post [16] shows that a given portfolio is dominated by its mixture with the dual solution portfolio of Post [15], provided that the mixture lies in the local neighborhood with the same strictly increasing ranking as the evaluated portfolio. For a general case, several SSD-efficiency linear programming tests capable to identify a dominating portfolio have been developed, see e.g. [8–10]. Moreover, methods developed by Dentcheva and Ruszczyński [1,2] and Kopa and Chovanec [8] can be used to find an *optimal* dominating portfolio under various definitions of optimality. However, all those tests use $O(T^2+n)$ variables and constraints, which can be computationally intense, because the typical values of T are above 10^2 or even 10^3. Fabian et al. [3] introduced a technique for solving optimization problems with SSD constraints which uses $O(n)$ variables, but with cuts from an exponential number of inequali-

[1] Throughout the paper, we do not count the bounds on decision variables (such as $x_j \geq 0$) as constraints, because the linear programming algorithms can efficiently deal with such bounds. This does not make any significant difference for our linear program, because it has at most $T+n$ such bounds.

[2] A method for treatment of ties has been outlined already in [15, Section II-C]. However, no SSD-test for the case with ties has been explicitly formulated in that paper.

ties added algorithmically. In contrast, Luedtke [12] suggested a test with $O(T+n)$ constraints but $O(T^2+n)$ variables. The existence of a linear programming SSD-efficiency test with $O(T+n)$ variables and constraints, and to be capable of finding a dominating portfolio, was an open question. Such a test is the main result of this work.

We present a linear programming test, with $T+n$ variables and $2T+1$ constraints, which, given any portfolio return $Y \in \mathcal{V}$, possibly with ties, tests whether Y is SSD-efficient within $\mathcal{V}$, and if not, finds a dominating portfolio X. A possible limitation is that our solution portfolio X may itself be inefficient, that is, dominated by a third portfolio Z. If this limitation is of concern, an additional test may be needed, e.g., the full Kopa and Post [9] test.

2 The SSD-efficiency test

Let $\Omega = \{\omega_1, \ldots, \omega_T\}$ be a finite probability space, with probability measure $\mathbb{P}$ such that $\mathbb{P}[\omega_i] = p_i,\ i = 1, \ldots, T$. A random variable (r.v.) is any function $X : \Omega \to \mathbb{R}$. $F_X(x) = \mathbb{P}[X \leq x]$ and $q_X(\alpha) = \inf\{x | F_X(x) > \alpha\}$ will denote the cumulative distribution function (CDF) and quantile function of an r.v. X, respectively. We say, that r.v. X dominates r.v. Y by SSD, and write $X \succ_2 Y$, if $Eu(X) \geq Eu(Y)$ for every increasing concave function $u : \mathbb{R} \to \mathbb{R}$, with inequality being strict for some u. Equivalently, $X \succ_2 Y$ if and only if

$$\int_0^\alpha q_X(\beta)\, d\beta \geq \int_0^\alpha q_Y(\beta)\, d\beta, \quad \alpha \in (0, 1], \tag{2}$$

with inequality being strict for some $\alpha \in (0, 1]$, see Theorem 2.58 in [4].

Let $\mathcal{V}$ be a convex set of r.v.s, and $Y \in \mathcal{V}$ be fixed. This paper presents a test which determines if Y is SSD-efficient within $\mathcal{V}$, and if not, finds a dominating portfolio.

Because Y is fixed and given, we can assume without loss in generality that

$$Y(\omega_i) \leq Y(\omega_j), \quad i < j. \tag{3}$$

Given any r.v. U, we denote $(s(1), \ldots, s(T))$ a permutation of set $(1, \ldots, T)$ such that $Y(\omega_{s(i)}) \leq Y(\omega_{s(j)}),\ i < j$, and $U(\omega_{s(i)}) \leq U(\omega_{s(j)})$ whenever $Y(\omega_{s(i)}) = Y(\omega_{s(j)}),\ i < j$.

Proposition 1 *Y is SSD-dominated if and only if there exists a non-zero r.v. $U \in (\mathcal{V} - Y)$ such that $\sum_{i=1}^{t} u_{s(i)} p_{s(i)} \geq 0,\ t = 1, \ldots, T$. Moreover, in this case $X = Y + \varepsilon U$ is a dominating portfolio for any $\varepsilon \in (0, \min\{1, \min_{i \in J} \frac{y_{s(i+1)} - y_{s(i)}}{u_{s(i)} - u_{s(i+1)}}\}]$ where $J \subset \{1, \ldots, T\}$ is the set of indices[3] for which $y_{s(i+1)} - y_{s(i)} > 0$ and $u_{s(i)} - u_{s(i+1)} > 0$.*

[3] The set of such indices may be an empty set. Throughout the paper, we will use the convention that the minimum over an empty set is equal to $+\infty$.

Proof If Y is SSD-dominated, $X \succ_2 Y$ for some $X \in \mathcal{V}$. Take $U = X - Y$. Then $\sum_{i=1}^{t} X(\omega_{s(i)})p_{s(i)} \geq \int_0^{\alpha_t} q_X(\beta)d\beta \geq \int_0^{\alpha_t} q_Y(\beta)d\beta = \sum_{i=1}^{t} y_{s(i)}p_{s(i)}$, where $y_i = Y(\omega_i)$, $i = 1, \ldots, T$, and $\alpha_t = \sum_{i=1}^{t} p_{s(i)}$, $t = 1, \ldots, T$. Hence $\sum_{i=1}^{t} u_{s(i)}p_{s(i)} \geq 0$, $t = 1, \ldots, T$.

Conversely, let U and ε be as described. Then $X(\omega_{s(1)}) \leq \cdots \leq X(\omega_{s(T)})$, where $X = Y + \varepsilon U$. Indeed, $X(\omega_{s(i)}) \leq X(\omega_{s(i+1)})$ is equivalent to $\varepsilon(u_{s(i)} - u_{s(i+1)}) \leq y_{s(i+1)} - y_{s(i)}$, which holds due to definition of ε if $i \in J$, and due to $\varepsilon(u_{s(i)} - u_{s(i+1)}) \leq 0 \leq y_{s(i+1)} - y_{s(i)}$ if $i \notin J$. Thus,

$$\int_0^{\alpha_t} q_X(\beta)\, d\beta = \sum_{i=1}^{t} X(\omega_{s(i)})p_{s(i)} \geq \sum_{i=1}^{t} y_{s(i)}p_{s(i)} = \int_0^{\alpha_t} q_Y(\beta)\, d\beta, \quad t = 1, \ldots, T. \tag{4}$$

Because functions $f_X(\alpha) = \int_0^{\alpha} q_X(\beta)\, d\beta$ and $f_Y(\alpha) = \int_0^{\alpha} q_Y(\beta)\, d\beta$ are piecewise linear with "vertices" at $\alpha = \alpha_t$, $t = 1, \ldots, T$, (4) implies (2). Let $t = t_0$ be the smallest index such that $u_{s(t)} \neq 0$. Then $0 \leq \sum_{i=1}^{t_0} u_{s(i)}p_{s(i)} = u_{s(t_0)}p_{s(t_0)}$, hence $u_{s(t_0)} > 0$, and strict inequality holds in (4) for $t = t_0$. Thus, $X \succ_2 Y$.

Finally, because r.v.s Y and $Y + U$ belong to $\mathcal{V}$, and $\varepsilon \in (0, 1]$, $X = Y + \varepsilon U \in \mathcal{V}$ due to convexity of $\mathcal{V}$. □

Proposition 1 cannot be applied directly for constructing a linear programming SSD-efficiency test, because permutation s depends on U and hence unknown in advance.

Let $I_k \subset \{1, \ldots, T\}$, $k = 1, \ldots, l$ be the sets of indices (of cardinality at least 2) such that $Y(\omega_i) = Y(\omega_j)$ if and only if $i, j \in I_k$ for some k. Let also $J_k = \{i \in \{1, \ldots, T\} \,|\, i < j, \forall j \in I_k\}$, $k = 1, \ldots, l$, and $I = \{i \in \{1, \ldots, T-1\} \,|\, Y(\omega_i) < Y(\omega_{i+1})\} \cup \{T\}$. Then $\{1, \ldots, T\} = I \cup I_1 \cup \cdots \cup I_l$.

Proposition 2 *For an r.v. U, the following statements are equivalent*

(a) $\sum_{i=1}^{t} u_{s(i)}p_{s(i)} \geq 0$, $t = 1, \ldots, T$;
(b) $\sum_{i=1}^{t} u_i p_i \geq 0$, $t \in I$ *and* $\sum_{i \in J_k} u_i p_i + \sum_{i \in I_k} u_i^- p_i \geq 0$, $k = 1, \ldots, l$, *where* $x^- = \min(x, 0)$.

Proof (a) ⇒ (b): Because

$$\sum_{i=1}^{t} u_{s(i)}p_{s(i)} = \sum_{i=1}^{t} u_i p_i, \quad t \in I, \tag{5}$$

(a) implies first statement of (b). For $k \in \{1, \ldots, l\}$, let t_0 be the largest index $t \in I_k$ with $u_{s(t)} \leq 0$, and let t_0 be the largest index in J_k if $u_{s(t)} > 0$, $\forall t \in I_k$. Then

$$0 \leq \sum_{i=1}^{t_0} u_{s(i)}p_{s(i)} = \sum_{i \in J_k} u_i p_i + \sum_{i \in I_k,\, s^{-1}(i) \leq s^{-1}(t_0)} u_i p_i = \sum_{i \in J_k} u_i p_i + \sum_{i \in I_k} u_i^- p_i,$$

and (b) follows.

(b) $\Rightarrow$ (a): If $t \in I$, (a) follows from the first statement of (b) and (5). If $t \in I_k$ for some $k \in \{1, \ldots, l\}$,

$$\begin{aligned}\sum_{i=1}^{t} u_{s(i)} p_{s(i)} &= \sum_{i \in J_k} u_i p_i + \sum_{i \in I_k,\, s^{-1}(i) \le s^{-1}(t)} u_i p_i \\ &\ge \sum_{i \in J_k} u_i p_i + \sum_{i \in I_k,\, s^{-1}(i) \le s^{-1}(t)} u_i^- p_i \ge \sum_{i \in J_k} u_i p_i + \sum_{i \in I_k} u_i^- p_i \ge 0.\end{aligned}$$

□

It follows from Propositions 1 and 2 that the program

$$\begin{aligned} \max \quad & \sum_{t \in I} \left(\sum_{i=1}^{t} u_i p_i \right) + \sum_{k=1}^{l} \left(\sum_{i \in J_k} u_i p_i + \sum_{i \in I_k} u_i^- p_i \right), \\ \text{s.t.} \quad & \sum_{i=1}^{t} u_i p_i \ge 0,\ t \in I, \quad \sum_{i \in J_k} u_i p_i + \sum_{i \in I_k} u_i^- p_i \ge 0,\ k = 1, \ldots, l \\ & U = (u_1, \ldots, u_T) \in \mathcal{V} - Y \end{aligned} \tag{6}$$

has a positive optimal objective value if and only if Y is SSD-dominated, and in this case a dominating portfolio is $X = Y + \varepsilon U$ with $\varepsilon = \min\{1, \min_{i \in J} \frac{y_{s(i+1)} - y_{s(i)}}{u_{s(i)} - u_{s(i+1)}}\}$ as in Proposition 1. The program (6) is not linear because of the presence of u_i^- but can be linearised in a standard way by introducing variables v_i together with constraints $v_i \le u_i$ and $v_i \le 0$.

Let

$$\mathcal{V} = \left\{ X \;\middle|\; X = \sum_{j=1}^{n} r_j x_j, \ \ \sum_{j=1}^{n} x_j = 1, \ \ x_j \ge 0, \ \ j = 1, \ldots, n \right\}, \tag{7}$$

where $r_1, \ldots, r_n$ are the rates of return of n assets, x_j is the fraction of capital invested into asset j, $\sum_{j=1}^{n} x_j = 1$ is the budget constraint, and $x_j \ge 0,\ j = 1, \ldots, n$ are optional no-short-selling constraints. Let $r_{ij} = r_j(\omega_i),\ i = 1, \ldots, T,\ j = 1, \ldots, n$ be the return of asset j under scenario ω_i. Then condition $U \in \mathcal{V} - Y$ in (6) becomes $u_i = \sum_{j=1}^{n} r_{ij} x_j - y_i,\ i = 1, \ldots, T$. Because $Y \in \mathcal{V}$, $y_i = \sum_{j=1}^{n} r_{ij} x_j^0\ i = 1, \ldots, T$, for some $x_1^0, \ldots, x_n^0$, and the condition $U \in \mathcal{V} - Y$ becomes $u_i = \sum_{j=1}^{n} r_{ij}(x_j - x_j^0),\ i = 1, \ldots, T$.

Hence, for $\mathcal{V}$ given by (7), program (6) can be written as

$$\begin{aligned} \max_{x_j, u_i, v_i} \quad & \sum_{t \in I} \left(\sum_{i=1}^{t} p_i u_i \right) + \sum_{k=1}^{l} \left(\sum_{i \in J_k} p_i u_i + \sum_{i \in I_k} v_i p_i \right), \\ \text{s.t.} \quad & \sum_{i=1}^{t} p_i u_i \ge 0,\ t \in I, \quad \sum_{i \in J_k} p_i u_i + \sum_{i \in I_k} v_i p_i \ge 0,\ k = 1, \ldots, l \end{aligned}$$

$$
\begin{aligned}
& u_i = \sum_{j=1}^{n} r_{ij}(x_j - x_j^0), \quad i = 1, \dots, T, \\
& v_i \le 0, \quad v_i \le u_i, \quad i \in (I_1 \cup \dots \cup I_l), \\
& \sum_{j=1}^{n} x_j = 1, \quad x_j \ge 0, \quad j = 1, \dots, n,
\end{aligned} \tag{8}
$$

or, after excluding u_i,

$$
\begin{aligned}
\max_{x_j, v_i} \quad & \sum_{t \in I} \left(\sum_{i=1}^{t} p_i \sum_{j=1}^{n} r_{ij}(x_j - x_j^0) \right) + \sum_{k=1}^{l} \left(\sum_{i \in J_k} p_i \sum_{j=1}^{n} r_{ij}(x_j - x_j^0) \sum_{i \in I_k} v_i p_i \right), \\
\text{s.t.} \quad & \sum_{i=1}^{t} p_i \sum_{j=1}^{n} r_{ij}(x_j - x_j^0) \ge 0, \ t \in I, \quad \sum_{i \in J_k} p_i \sum_{j=1}^{n} r_{ij}(x_j - x_j^0) \\
& \quad + \sum_{i \in I_k} v_i p_i \ge 0, \ k = 1, \dots, l \\
& v_i \le 0, \quad v_i \le \sum_{j=1}^{n} r_{ij}(x_j - x_j^0), \quad i \in (I_1 \cup \dots \cup I_l), \\
& \sum_{j=1}^{n} x_j = 1, \quad x_j \ge 0, \quad j = 1, \dots, n.
\end{aligned} \tag{9}
$$

The resulting linear programming test has $T + n$ variables and at most $2T + 1$ constraints.

Example 1 Assume that there are $T = 3$ equiprobable scenarios, $n = 2$ assets with returns $r_1 = (0.24, 0, 0.06)$ and $r_2 = (0.04, 0.12, 0.12)$, and the benchmark portfolio has weights $(0, 1)$. In this case, $Y = (0.04, 0.12, 0.12)$, and assumption (1) does not hold. In the notation introduced before Proposition 2, $l = 1$, $I_1 = \{2, 3\}$, $J_1 = \{1\}$, $I = \{1, 3\}$, and the linear program (9) takes the form

$$
\begin{aligned}
\max_{x_1, x_2, v_2, v_3} \quad & 3\frac{1}{3}(0.24(x_1 - 0) + 0.04(x_2 - 1)) + \frac{1}{3}0.12(x_2 - 1) \\
& + \frac{1}{3}(0.06(x_1 - 0) + 0.12(x_2 - 1)) + \frac{1}{3}v_2 + \frac{1}{3}v_3, \\
\text{s.t.} \quad & \frac{1}{3}(0.24(x_1 - 0) + 0.04(x_2 - 1)) \ge 0, \\
& \frac{1}{3}(0.24(x_1 - 0) + 0.04(x_2 - 1)) + \frac{1}{3}0.12(x_2 - 1) + \frac{1}{3}(0.06(x_1 - 0) \\
& + 0.12(x_2 - 1)) \ge 0, \\
& \frac{1}{3}(0.24(x_1 - 0) + 0.04(x_2 - 1)) + \frac{1}{3}v_2 + \frac{1}{3}v_3 \ge 0, \\
& v_2 \le 0.12(x_2 - 1),
\end{aligned}
$$

$$\begin{aligned}
&v_3 \le 0.06(x_1 - 0) + 0.12(x_2 - 1),\\
&v_2 \le 0, \quad v_3 \le 0,\\
&x_1 + x_2 = 1, \quad x_1 \ge 0, \quad x_2 \ge 0,
\end{aligned} \tag{10}$$

which simplifies to

$$\begin{aligned}
\max_{x_1,x_2,v_2,v_3} \quad & 0.26x_1 + 0.12(x_2 - 1) + \frac{1}{3}v_2 + \frac{1}{3}v_3,\\
\text{s.t.} \quad & 6x_1 + (x_2 - 1) \ge 0,\\
& 15x_1 + 14(x_2 - 1) \ge 0,\\
& 0.24x_1 + 0.04(x_2 - 1)) + v_2 + v_3 \ge 0,\\
& v_2 \le 0.12(x_2 - 1),\\
& v_3 \le 0.06x_1 + 0.12(x_2 - 1),\\
& v_2 \le 0, \quad v_3 \le 0,\\
& x_1 + x_2 = 1, \quad x_1 \ge 0, \quad x_2 \ge 0.
\end{aligned} \tag{11}$$

The optimal solution $x_1 = 1$, $x_2 = 0$, $v_2 = -0.12$, $v_3 = -0.06$, with the corresponding objective value $0.26x_1 + 0.12(x_2 - 1) + \frac{1}{3}v_2 + \frac{1}{3}v_3 = 0.08 > 0$, hence the benchmark portfolio is not SSD-efficient. Next, $U = (0.2, -0.12, -0.06)$, $s(i) = i$, $i = 1, 2, 3$, $J = \{1\}$, $\varepsilon = \min\{1, \frac{y_2-y_1}{u_1-u_2}\} = 0.25$, and a dominating portfolio $X = Y + 0.25U$ has weights $(0, 1) + 0.25(1, -1) = (0.25, 0.75)$.

Example 1 is a slight modification of Example 4 in [9], which illustrates that their dual test returns the same dominating portfolio and has 14 variables and 10 constraints. In contrast, test (11) has just 4 variables and 6 constraints.

Kopa and Post [9] compared the size of different SSD-efficiency tests in the case $T = 480$, $n = 12$. Table 1 presents their data for Post [15] dual test, Kuosmanen [10] test, Kopa [8] test, Kopa and Post [9] dual test, and Kopa and Post [9] reduced dual test, together with the corresponding data for Post [16] test, Luedtke [12] test

Table 1 SSD-tests comparison

Test	Size (constraints × variables)	$T = 480, n = 12$	AT	DP	EP
Post [15] dual test, Eq. (12)	$(T + 1) \times (T + n - 1)$	481×491	No	No	No
Kuosmanen [10] test, Th. 6	$(T^2 + T + 1) \times (3T^2 + n)$	$230{,}881 \times 691{,}212$	No	Yes	Yes
Kopa [8], Eq. (16)	$(T^2 + T + 1) \times (T^2 + 2T + n)$	$230{,}881 \times 231{,}372$	No	Yes	Yes
Post [16] test, Eq. (5)	$(T + 1) \times (T + n - 1)$	481×491	No	Yes	No
Kopa and Post [9] reduced test	$(T + 1) \times (T + n)$	481×492	Yes	No	No
Kopa and Post [9] full test	$(T^2 + 1) \times (T^2 + T + n)$	$230{,}401 \times 230{,}892$	Yes	Yes	Yes
Luedtke [12] test, Eq. (cSSD1)	$(3T + n + 1) \times (T^2 + n)$	$1{,}453 \times 230{,}412$	Yes	Yes	Yes
Proposed test	$(2T + 1) \times (T + n)$	961×492	Yes	Yes	No

and our proposed test.[4] It shows that the proposed test has substantially smaller size than the existing tests which allow ties and are able to identify a dominating portfolio.

3 Conclusions and future research

We have constructed a linear program (8)–(9) with $O(T+n)$ variables and constraints, such that its objective value is strictly positive if and only if the evaluated portfolio Y is SSD-dominated within admissible set $\mathcal{V}$ given by (7). If Y is SSD-dominated, the output of the program can be used to construct a dominating portfolio.

The suggested SSD test is relevant for portfolio management: a dominating portfolio may be suggested as an alternative for an investor who is currently holding the benchmark portfolio Y. One may argue that a dominating portfolio may in general be SSD-inefficient, and, even if efficient, it is generally not optimal for the investor who holds the portfolio Y. Indeed, if the exact utility function of the investor is known, then it may be optimized to find an *optimal* portfolio with respect to his/her preferences. However, in practice, an investor rarely knows his/her utility function. In this case, determining the optimal portfolio is impossible, and a risk-averse investor may be advised to buy a dominating portfolio, which, in general, is not optimal, but anyway is better than the portfolio he/she currently holds, no matter what his/her utility function is.

An obvious question for future research is whether there exists a linear programming SSD-efficiency test with $O(T+n)$ variables and constraints returning a dominating portfolio which is in addition *SSD-efficient*. Another interesting research direction would be generalizing the results of this paper to higher order stochastic dominance. We say, that r.v. X dominates r.v. Y by N-th order stochastic dominance, or NSD, and write $X \succ_N Y$, if $Eu(X) \geq Eu(Y)$ for every function $u \in U_N$, with inequality being strict for some u, where U_N is the set of N times differentiable functions $u : \mathbb{R} \to \mathbb{R}$ such that $(-1)^{n-1}u^{(n)}(x) \geq 0$, $\forall x \in \mathbb{R}$, $n = 1, \ldots, N$. An r.v. $Y \in \mathcal{V}$ is called NSD-efficient within set $\mathcal{V}$, if there are no $X \in \mathcal{V}$ such that $X \succ_N Y$. It would be interesting to obtain a linear programming NSD-efficiency test capable of identifying a dominating portfolio. However, this is not entirely straightforward. Our SSD-efficiency test relies on the quantile characterization (2) of SSD. Theorem 4 in [11] claims (without proof) that a similar representation holds at least for $N = 3$, namely, $X \succ_3 Y$ if and only if

$$\int_0^\alpha \left(\int_0^\beta q_X(\gamma)\, d\gamma \right) d\beta \geq \int_0^\alpha \left(\int_0^\beta q_Y(\gamma)\, d\gamma \right) d\beta, \quad \alpha \in (0, 1]. \tag{12}$$

However, Ng [14] provides a counterexample to this statement. To the best of our knowledge, no convenient representation of NSD in terms of quantile functions is

[4] The columns AT, DP, and EP indicate whether the test allows ties in the return distribution, whether the results of the test can be used to identify a dominating portfolio, and whether the solution portfolio is SSD-efficient, correspondingly.

known for $N \geq 3$. Recently, Post and Kopa [17] derived a representation of the NSD criterion in terms of piecewise polynomials and co-lower partial moments, and used it to develop an efficient linear programming NSD-efficiency test for any N. However, their test cannot identify a dominating portfolio. This issue calls for new ideas.

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