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CHAOS IN THE SOFTENING DUFFING SYSTEM UNDER MULTI-FREQUENCY PERIODIC FORCES *

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Abstract: *The chaotic dynamics of the softening-spring Duffing system with multi-frequency external periodic forces is studied. It is found that the mechanism for chaos is the transverse heteroclinic tori. The Poincaré map, the stable and the unstable manifolds of the system under two incommensurate periodic forces were set up on a two-dimensional torus. Utilizing a global perturbation technique of Melnikov the criterion for the transverse interaction of the stable and the unstable manifolds was given. The system under more but finite incommensurate periodic forces was also studied. The Melnikov's global perturbation technique was therefore generalized to higher dimensional systems. The region in parameter space where chaotic dynamics may occur was given. It was also demonstrated that increasing the number of forcing frequencies will increase the area in parameter space where chaotic behavior can occur.*

Key words: multi-frequency excitation; softening Duffing system; chaos; heteroclinic torus

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Introduction

Since 1980s, the possibility of chaotic dynamics in single-degree-of-freedom nonlinear oscillator under sinusoidal excitation has become a firmly-established fact. The analysis of such systems is facilitated by studying an associated two-dimensional Poincaré map. Construction of the Poincaré map involves the elimination of at least one of the variables of the problem resulting in a lower dimensional problem to be studied. The Poincaré map provides an insightful and striking display of a rich underlying global structure of the system. Many concepts that are somewhat cumbersome to state for ordinary differential equations may often be succinctly stated for the associated Poincaré map. In many systems the underlying structure of the Poincaré map which is responsible for the chaotic dynamics is the transverse intersection of the stable and the unstable manifolds of a hyperbolic periodic orbit, such as the lateral vibration of the buckled beam

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under periodic force. Such intersections yield transverse homoclinic orbits (or heteroclinic orbits) resulting in chaotic dynamics of the Smale horseshoe type. For a large class of periodically forced single-degree-of-freedom systems Melnikov has developed a computable measurement of the distance between the stable and the unstable manifolds of hyperbolic periodic orbits which can be used to establish the presence of transverse homoclinic orbits (or heteroclinic orbits) in specific systems and hence, can also provide a criterion for the existence of chaotic dynamics in specific systems[1].

Although there has been great progress in the study of the nature of chaotic dynamics in periodically forced single-degree-of-freedom nonlinear systems, little effort has been put into the study of quasiperiodically forced systems despite that recently the existence of chaotic dynamics for systems under multi-frequency excitation has been established through experiment and numerical simulation. This lack of progress can probably be traced to the fact that there is no natural reduction to a two-dimensional Poincaré map for quasiperiodically forced systems. The double Poincaré section method promotes the application of the Melnikov method to high-dimensional nonlinear systems. Moon and Holmes[2] experimentally observed the chaotic vibration in the Holmes-type Duffing system forced with two incommensurate harmonic signals and revealed the fractal nature of this strange attractor in four-dimensional phase space by using a double Poincaré section. Wiggins[3~4] studied the chaotic dynamics of the quasiperiodically forced Holmes-type Duffing oscillator and found that the mechanism for chaos is the transverse homoclinic tori. A criterion for the existence of chaos was given utilizing a generalization of a global perturbation technique of Melnikov and the effect of the number of forcing frequencies on the region of chaos in parameter space was demonstrated. Kayo and Wiggins[5] studied the bifurcation to homoclinic tori in the quasiperiodically forced Holmes-type Duffing oscillator, showed how homoclinic tori gave rise to chaotic dynamics for single-degree-of-freedom quasiperiodically forced oscillators in much the same way as Smale horseshoes in the periodically forced case, and gave a complete description of the bifurcation set in the five-dimensional parameter space for the two-frequency forced Holmes-type Duffing oscillator.

Furthermore, Heagy and Ditto[6] investigated the transition from two-frequency quasiperiodicity to chaotic behavior in the two-frequency parametrically driven Holmes-type Duffing oscillator. Lu Qi-shao *et al*.[7~8] studied the local bifurcation and Hopf bifurcation of the two-frequency parametrically forced nonlinear oscillator. Yagasaki[9~11] dived himself into nonlinear system subjected to combined parametric and external excitation. Chen Yu-shu and Wang De-shi[12] investigated the transition from quasiperiodicity to chaotic dynamics in the nonlinear system subjected to combined parametric and external excitation by computing the Melnikov function. Kapitaniak[13] studied the combined bifurcation of the Duffing oscillator without the linear term under multi-frequency excitation and the dynamics of this system under random noise by F-P-K equation[14~15]. Bi Qin-sheng *et al*.[16] studied all kinds of bifurcation modes and their boundaries of the hard-spring Duffing system with two periodic excitations as well as the possible ways to chaos by introducing nonlinear frequency, using Floquet theory, and referring to the characteristics of the solution when it passes through the transition boundaries which were presented by Leung[17].

This paper generalizes the work of Wiggins. A criterion for the existence of chaos is given by utilizing the global perturbation technique of Melnikov. The effect of the number of forcing frequencies on the region of chaos in parameter space is also analyzed.

1 Softening-Spring Duffing System Under Two Periodic Forces

There exist periodic motion, quasiperiodic motion and chaos in the softening-spring Duffing system with two periodic forces. Its differential equation is

$$\begin{cases} \dot{x} = y, \ \dot{y} = -x + x^3 + \varepsilon(f\cos\theta_1 + f\cos\theta_2 - \gamma y), \\ \dot{\theta}_1 = \omega_1, \ \dot{\theta}_2 = \omega_2, \end{cases} \tag{1}$$

where ε is presumed small, γ is the damping factor, f is the amplitude of the excitation, and ε, γ, and f are all positive. ω_1 and ω_2 are the frequencies of the excitation and they are positive real numbers. x and y are the displacement and velocity of the forced mass, respectively. Eq.(1) is the quasiperiodically forced softening-spring Duffing oscillator having the augmented phase space given by $R^2 \times S^1 \times S^1$, where $S^1 \times S^1$ is a two-dimensional torus given by $\omega_1 t$ and $\omega_2 t$. Hence, system (1) is a differential dynamical system on the torus. The topological characteristics of the system changes with the ratio between ω_1 and ω_2. Assuming ω_1 and ω_2 satisfy

$$n_1\omega_1 + n_2\omega_2 = 0, \tag{2}$$

ω_1 and ω_2 are commensurate or the ratio ω_1/ω_2 is a rational number when n_1 and n_2 are nonzero integers. Contrarily, ω_1 and ω_2 are incommensurate or the ratio ω_1/ω_2 is an irrational number when n_1 and n_2 are both zeros.

When ω_1 and ω_2 are commensurate, the solution trajectory is a closed curves on the two-dimensional torus, as shown in Fig.1. When ω_1 and ω_2 are incommensurate, however, the system (1) has more complicated dynamics. There exists not only the frequency of $(n_1/n_2)\omega_1$ and $(n_1/n_2)\omega_2$, but also the frequency of $|(n_1/n_2)(\omega_1 \pm \omega_2)|$ in the response[17]. The solution of the system is therefore quasiperiodic[18]. Chaos occurs when all the quasiperiodic solutions lose their stability[19].

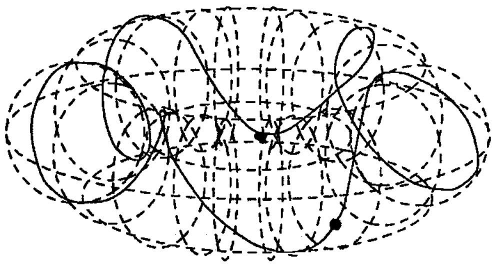

Fig.1 Closed curve on the two-dimensional torus

The study of the system (1) can be reduced to the study of an associated three-dimensional Poincaré map obtained by defining a three-dimensional cross-section to the four-dimensional phase space $R^2 \times S^1 \times S^1$ by fixing the phase of one of the angular variables and allowing the remaining three variables that start on the cross-section to evolve in time under the flow generated by the system (1) until they return to the cross-section. To be more precise, the cross-section, Σ, is given by

$$\Sigma = \{(x, y, \theta_1, \theta_2) \in R^2 \times S^1 \times S^1 \mid \theta_2 = \theta_{20}\}, \tag{3}$$

where, for definiteness, the phase of θ_2 is fixed and the Poincaré map $P_\varepsilon : \Sigma \to \Sigma$ is defined as

$$(x(0), y(0), \theta_1(0) \equiv \theta_{10}) \to \left(x\left(\frac{2\pi}{\omega_2}\right), y\left(\frac{2\pi}{\omega_2}\right), \theta_1\left(\frac{2\pi}{\omega_2}\right) = \frac{2\pi\omega_1}{\omega_2} + \theta_{10}\right). \tag{4}$$

For $\varepsilon = 0$, the system (1) is a completely integrable hamiltonian system having a two-dimensional normally hyperbolic invariant torus given by

$$\{(x, y, \theta_1, \theta_2) \in R^2 \times S^1 \times S^1 \mid x = y = 0, \theta_1, \theta_2 \in [0, 2\pi]\}, \tag{5}$$

with trajectories on the torus given by

$$(x(t), y(t), \theta_1(t), \theta_2(t)) = (0, 0, \omega_1 t + \theta_{10}, \omega_2 t + \theta_{20}). \tag{6}$$

As we all-know, the free vibration of the softening-spring Duffing system has two saddle points, $(1,0)$ and $(-1,0)$, and a heteroclinic loop given by two heteroclinic trajectories connecting the two saddle points $(\pm 1, 0)$ on the two-dimensional phase plane. The parametric equation of the two pieces of heteroclinic trajectories is[1]

$$x_{\pm}^{0}(t) = \pm \operatorname{th}\left(\frac{\sqrt{2}}{2} t\right), \quad y_{\pm}^{0}(t) = \pm \frac{\sqrt{2}}{2} \operatorname{sech}^2\left(\frac{\sqrt{2}}{2} t\right). \tag{7}$$

However, the two saddle points $(\pm 1, 0)$ on the two-dimensional phase plane become two one-dimensional normally hyperbolic invariant tori (namely, circles) τ_{10} and τ_{20}, and the stable and the unstable manifolds of the tori are given by

$$(x(t), y(t), \theta_1(t), \theta_2(t)) =$$
$$\left(\pm \operatorname{th}\left(\frac{\sqrt{2}}{2} t\right), \pm \frac{\sqrt{2}}{2} \operatorname{sech}^2\left(\frac{\sqrt{2}}{2} t\right), \omega_1 t + \theta_{10}, \omega_2 t + \theta_{20}\right). \tag{8}$$

Utilizing this information a complete picture of the global integrable dynamics of the unperturbed Poincaré map, P_0, can be obtained. In particular, P_0 has two one-dimensional normally hyperbolic invariant tori, τ_{10} and τ_{20}. They have a pair of two-dimensional stable and unstable manifolds, $W^s(\tau_{10})$ and $W^u(\tau_{10})$, $W^s(\tau_{20})$ and $W^u(\tau_{20})$, respectively, see Fig.2. For $\varepsilon = 0$, there is no chaotic dynamics since P_0 is integrable.

For $\varepsilon \neq 0$ and small, the perturbed Poincaré map, P_ε, still possesses two one-dimensional normally hyperbolic invariant tori, $\tau_{1\varepsilon}$ and $\tau_{2\varepsilon}$, having two-dimensional stable manifolds and unstable manifolds, $W^s(\tau_{1\varepsilon})$ and $W^u(\tau_{1\varepsilon})$, $W^s(\tau_{2\varepsilon})$ and $W^u(\tau_{2\varepsilon})$, respectively. Now, $W^s(\tau_{1\varepsilon})$ and $W^u(\tau_{2\varepsilon})$, $W^u(\tau_{1\varepsilon})$ and $W^s(\tau_{2\varepsilon})$ may intersect transversely yielding transverse heteroclinic orbits. The behavior of $W^s(\tau_{1\varepsilon})$ and $W^u(\tau_{2\varepsilon})$, $W^u(\tau_{1\varepsilon})$ and $W^s(\tau_{2\varepsilon})$ can be determined by computing the Melnikov function for quasiperiodically forced systems. The Melnikov function is the $O(\varepsilon)$ term in a power series expansion in ε for the distance between $W^s(\tau_{1\varepsilon})$ and $W^u(\tau_{2\varepsilon})$ (or $W^u(\tau_{1\varepsilon})$ and $W^s(\tau_{2\varepsilon})$). Thus if the Melnikov function has simple zeros, then $W^s(\tau_{1\varepsilon})$ and $W^u(\tau_{2\varepsilon})$ (or $W^u(\tau_{1\varepsilon})$ and $W^s(\tau_{2\varepsilon})$) intersect transversely. In this case the Melnikov function for each respective branch of the stable and the unstable manifolds is given by

$$M_{\pm}(\theta_1, \theta_2) = \int_{-\infty}^{+\infty} [-\gamma y_{\pm}^{0}(t) + f\cos\omega_1(t + t_0) + f\cos\omega_2(t + t_0)] y_{\pm}^{0}(t) \mathrm{d}t =$$
$$\int_{-\infty}^{+\infty} \left[\pm \frac{\sqrt{2}}{2} \operatorname{sech}^2\left(\frac{\sqrt{2}}{2} t\right)\right]\left[\pm \frac{\sqrt{2}}{2}(-\gamma) \operatorname{sech}^2\left(\frac{\sqrt{2}}{2} t\right) +\right.$$
$$\left. f\cos\omega_1(t + t_0) + f\cos\omega_2(t + t_0)\right] \mathrm{d}t =$$
$$-\frac{2\sqrt{2}}{3}\gamma + \sqrt{2}\pi\omega_1 \operatorname{csch}\left(\frac{\sqrt{2}}{2}\pi\omega_1\right) f\cos\omega_1 t_0 + \sqrt{2}\pi\omega_2 \operatorname{csch}\left(\frac{\sqrt{2}}{2}\pi\omega_2\right) f\cos\omega_2 t_0 =$$
$$-I_0\gamma \pm I_1 f\cos\theta_{10} \pm I_2 f\cos\theta_{20}, \tag{9}$$

where $I_0 = \frac{2\sqrt{2}}{3}$, $I_1 = \sqrt{2}\pi\omega_1 \operatorname{csch}\left(\frac{\sqrt{2}}{2}\pi\omega_1\right)$, $I_2 = \sqrt{2}\pi\omega_2 \operatorname{csch}\left(\frac{\sqrt{2}}{2}\pi\omega_2\right)$, $\theta_{10} = \omega_1 t_0$, and $\theta_{20} = \omega_2 t_0$.

Upon examination of the Melnikov function, when

$$\frac{f}{\gamma} > \frac{I_0}{I_1 + I_2}, \tag{10}$$

$W^s(\tau_{1\varepsilon})$ and $W^u(\tau_{2\varepsilon})$ (or $W^u(\tau_{1\varepsilon})$ and $W^s(\tau_{2\varepsilon})$) intersect transversely in a set of points that is topologically a one-dimensional torus which is called a transverse heteroclinic torus, see Fig.3. It is well-known that there must be a countable infinity of transverse heteroclinic points as long as there exists a transverse heteroclinic point in the periodically forced single-degree-of-freedom Duffing system. Similarly, in this case there must be a countable infinity of such transverse heteroclinic tori so long as there exists a transverse heteroclinic torus. And hence occurs the chaotic dynamics in the sense of Smale horseshoe.

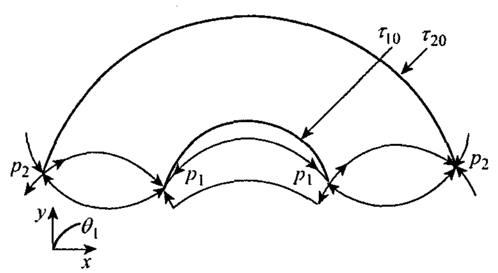

Fig.2 Heteroclinic geometry of the phase of P_0, cut away half view

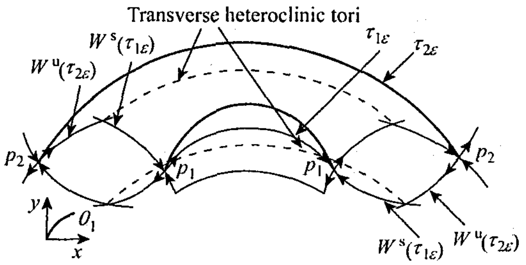

Fig.3 Transverse heteroclinic tori of P_ε

2 Multi-Frequency Excitation

Add more frequencies of excitation to the system (1), and it becomes

$$\begin{cases} \dot{x} = y, \ \dot{y} = -x + x^3 + \varepsilon(f\cos\theta_1 + f\cos\theta_2 + \cdots + f\cos\theta_n - \gamma y), \\ \dot{\theta}_1 = \omega_1, \cdots, \dot{\theta}_n = \omega_n. \end{cases} \tag{11}$$

The study of Eq. (11) can be reduced to the study of an associated $(n-1)$-dimensional Poincaré map having two $(n-2)$-dimensional normally hyperbolic invariant tori with $(n-1)$-dimensional stable and unstable manifolds. Intersection of the stable and the unstable manifolds is determined by calculating the Melnikov function. In Fig.4 the lines $f = m_1\gamma$, $f = m_2\gamma$, and $f = m_n\gamma$ represent lines above which transverse heteroclinic tori occur for the softening-spring Duffing system forced at 1, 2, and n frequencies, respectively. m_1, m_2 and m_n are obtained from the Melnikov function and are given by

$$m_1 = \frac{I_0}{I_1}, \ m_2 = \frac{I_0}{I_1 + I_2}, \ \cdots, \ m_n = \frac{I_0}{I_1 + I_2 + \cdots + I_n}, \tag{12}$$

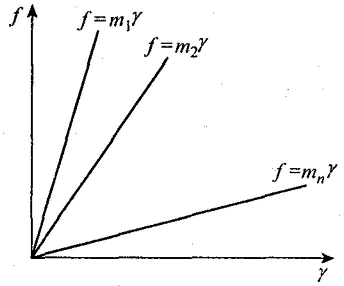

Fig.4 Regions of chaos in $f-\gamma$ space as a function of the number of forcing frequencies

where $I_i = \sqrt{2}\pi\omega_i \operatorname{csch}\left(\frac{\sqrt{2}}{2}\pi\omega_i\right), \quad i = 1,2,\cdots,n.$

From Fig.4 we can see that increasing the number of forcing frequencies will extend the area in parameter space where chaotic behavior can occur, and hence increase the likelihood of chaotic dynamics. This result is in agreement with that of Wiggins[3] about the Holmes-type Duffing system under quasiperiodic excitation.

3 Conclusions

1) When the softening-spring Duffing system is forced with n(n is arbitrary but finite) frequencies, its associated (n - 1)-dimensional Poincaré map has two (n - 2)-dimensional normally hyperbolic invariant tori with (n - 1)-dimensional stable and unstable manifolds. Transverse intersections of the stable and the unstable manifolds give rise to transverse heteroclinic tori. This is the mechanism for chaos. Moreover, the more the frequencies in excitation, the more likely chaos is to occur.

2) When the excitation amplitudes corresponding to different frequencies is nonidentical, the Melnikov function (9) becomes

$$M_{\pm}(\theta_1, \theta_2) = -I_0\gamma \pm I_1 f_1 \cos\theta_{10} \pm I_2 f_2 \cos\theta_{20}. \tag{13}$$

So, when γ and f_i(i = 1,2) satisfy

$$\gamma < (I_1 f_1 + I_2 f_2)/I_0, \tag{14}$$

the Poincaré map of the system (1) has transverse heteroclinic tori, that is, chaos in the sense of Smale horseshoe occurs.

3) If θ_1 and θ_2 in the system (1) are slowly varying, the normally hyperbolic invariant tori of the system may be subject to phase locking. Here the system (1) can be written as

$$\begin{cases} \dot{x} = y, \ \dot{y} = -x + x^3 + \varepsilon(f\cos\theta_1 + f\cos\theta_2 - \gamma y), \\ \dot{\theta}_1 = \omega_1 + \varepsilon\alpha\sin(\theta_1 - \theta_2), \ \dot{\theta}_2 = \omega_2 + \varepsilon\alpha\sin(\theta_2 - \theta_1), \end{cases} \tag{15}$$

where α is a constant. The system (1) is the special case of the system (15) in which the two frequencies are constant (i.e., $\alpha = 0$).

For $\varepsilon > 0$ is sufficiently small, the system (15) has invariant tori which might be subject to phase locking. Yagasaki[20] studied phase locking in the Holmes-type Duffing system and found that when the invariant torus is subject to phase locking, chaotic dynamics resulting from transverse intersection between its stable and unstable manifolds may be interrupted and two periodic orbits are created in a saddle-node bifurcation. So likewise, the softening-spring Duffing system under multi-frequency excitation which is slowly varying and its phase locking are worth studying.

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