Three-Body Bound-State Calculation in Momentum Space.

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The determination of 3-body bound and scattering states for « realistic » two-body and three-body interactions, which are in general not separable, is still a formidable task. One would like to have a clear and economic algorithm to allow an easy test of various nuclear-interaction models. To that aim we have looked into the method proposed by KIM [1] for the calculation of bound states. One is faced with a problem in two continuous variables $p_l = \frac{1}{2}|\boldsymbol{k}_m - \boldsymbol{k}_n|$, $q_l = \frac{2}{3}|\boldsymbol{k}_l - \frac{1}{2}(\boldsymbol{k}_m + \boldsymbol{k}_n)|$, where $\boldsymbol{k}_l$, $l = 1, 2, 3$ are the momenta of the 3 particles. One gains a drastic reduction in the dimension of the problem if the p-dependence can be exhausted by a few suitably chosen basis functions $\varphi_n(p)$, taken from a complete set. We want to propose such a set, which is tightly adapted to the problem under consideration. With an eye towards meson-theoretical nucleon-nucleon interactions the method is developped in momentum space, however, demonstrated only for three identical bosons interacting in relative s-states.

The Faddeev wave function amplitude then obeys the equation

$$(1) \qquad (p_1^2 + \tfrac{3}{4}q_1^2 - E)u(p_1, q_1) + \int_0^\infty \mathrm{d}p_1' v(p_1, p_1') u(p_1', q_1) + \int_0^\infty \mathrm{d}q_2' \int_{-1}^{1} \mathrm{d}t \frac{q_1 q_2'}{p_1' p_2} v(p_1, p_1') u(p_2', q_2') = 0 ,$$

where

$$p_1' = \sqrt{q_2'^2 + \tfrac{1}{4}q_1^2 + q_2' q_1 t}, \qquad p_2' = \sqrt{\tfrac{1}{4}q_2'^2 + q_1^2 + q_2' q_1 t},$$

and, for instance, for a local interaction $v(p, p')$ is given by

$$v(p, p') = \frac{2}{\pi} \int_0^\infty \mathrm{d}r \sin pr V(r) \sin p' r .$$

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[1] Y. E. KIM *Journ. Math. Phys.*, **10**, 1491 (1969); Y. E. KIM and A. TUBIS: *Phys. Rev. C*, **1**, 1627 (1970).

We choose an orthonormal set $\{\varphi_n(p)\}$, expand $u(p, q)$:

$$u(p, q) = \sum_n \varphi_n(p) b_n(q) , \tag{2}$$

and insert this expansion into (1). Then after projecting (1) onto φ_n we are left with a coupled homogeneous system of integral equations for the unknown functions $b_n(q)$

$$(\tfrac{3}{4}q^2 - E) + \sum_{n'} c_{nn'} b_{n'}(q) + \sum_{n'} \int_0^\infty \mathrm{d}q' K_{nn'}(q, q') b_n(q') = 0 , \tag{3}$$

where

$$c_{nn'} = \int_0^\infty \mathrm{d}p\, \varphi_n(p) p^2 \varphi_{n'}(p) + \int_0^\infty \mathrm{d}p \int_0^\infty \mathrm{d}p'\, \varphi_n(p) v(p, p') \varphi_{n'}(p') , \tag{4a}$$

$$K_{nn'}(q, q') = \int_0^\infty \mathrm{d}p \int_{-1}^{1} \mathrm{d}t\, \varphi_n(p) \frac{v(p, p_1') \varphi_n(p_2')}{p_1' p_2'} qq' . \tag{4b}$$

Equation (1) clearly shows that the p-dependence of $u(p, q)$ is strongly influenced by the two-body force $v(p, p')$. Thus for instance $u(r, R)$ (r, R are the conjugate co-ordinates to p, q) will be suppressed for small r if $\langle r|V|r'\rangle$ is repulsive at small distances. It is essential to incorporate that feature into the basis functions $\varphi_n(p)$. Therefore we looked for a discrete basis adapted to v by confining the two-body subsystem essentially to a finite region in space. Since we want to determine $\varphi_n(p)$ directly in momentum space, it is convenient to take a harmonic-oscillator potential in addition to v. Then the functions $\varphi_n(p)$ are found as the square-integrable solutions to

$$\left(p^2 + \alpha \frac{\mathrm{d}^2}{\mathrm{d}p^2} - \varepsilon_n\right) \varphi_n(p) + \int_0^\infty \mathrm{d}p'\, v(p, p') \varphi_n(p') = 0 . \tag{5}$$

The basis functions $\varphi_n(p)$ can be optimized by adjusting the width parameter α, see below. The same set, however, in co-ordinate space, had been used by BARRETT, HEWITT and MCCARTHY [2] to solve the Bethe-Goldstone equation. This integro-differential equation is solved in two steps:

a) Use a rough discretisation with a unique step length and the trapez rule to approximate the integral. By this way one gets a symmetric matrix and the eigenvalues can be found by the method of Householder-Givens transformation and bisection [3].

b) Now take a fine discretisation and variable step lengths as well as the Simpson rule and a five-point formula to represent the derivative. The (lowest) eigenvalues of the resulting, nonsymmetrical matrix are calculated by the method of inverse iteration [3] using the approximate values of step *a*) as a starting point. We doublet the step widths starting with $\Delta p = 0.075\ \mathrm{fm}^{-1}$ for the smallest p values ($(0 \div 3)\ \mathrm{fm}^{-1}$) to $\Delta p = 1.2\ \mathrm{fm}^{-1}$ for the largest ones ($(12 \div 26)\ \mathrm{fm}^{-1}$).

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[2] B. R. BARRETT, R. G. L. HEWITT and R. J. MCCARTHY: *Phys. Rev. C*, **3**, 1137 (1971).
[3] H. J. WILKINSON: *The Algebraic Eigenvalue Problem* (Oxford, 1965).

The first few eigenfunctions for $\alpha = 1$ fm^{-4} and the Malfliet-Tjon potential V ([4]) (we used the same parameters as in ref. ([5])) are plotted in fig. 1. The difference from pure harmonic-oscillator states is showing up in the slow decrease and in an additional node due to the short-range repulsion of v. For later purposes the two-body bound-state wave function (binding energy $\varepsilon_b = -0.361$ MeV) is also presented in fig. 1. The matrix elements (4*a*), (4*b*) have been evaluated by GAUSS-LEGENDRE integration, taking 40 points for the p-integration and at most 24 points for the t-integration. For the q'-integration in eq. (3) 10 G points turned out to be sufficient. Thus the dimension of the problem (1) is reduced to $10 \times N_p$ where N_p is the number of necessary basis functions $\varphi_n(p)$. Again the eigenvalue problem (3) is easily solved by the method of inverse iteration. One may require that method since the integral operator is not Hermitian. However, the procedure is nothing else than the method of inverse iteration applied to the Schrödinger equation and rewritten in the form of Faddeev equations.

We show in table I the 3-body binding energy depending on N_p for various values of α and the Malfliet-Tjon potential V. We like to stress that the converged wave function $u(p, q)$ is almost independent of α in the studied domain. The best convergence occurs for $\alpha = 1$ fm^{-4}. In fig. 2 we compare the expansion (2) truncated at various N_p with $u(p, q)$ for the optimal choice of $\alpha = 1$ fm^{-4} and a fixed q value corresponding roughly to the maximum of u. One also learns from fig. 2 that the two-body bound-state wave function, shown in fig. 1, would not be a good candidate to represent $u(p, q)$. The two-body basis function optimizing the overlap with $u(p, q)$ has its maximum at larger p values, because the two particles in the 3-body system feel an additional attraction via the third particle.

TABLE I. – *3-particle binding energies for the Malfliet-Tjon potential* V *depending on the oscillator parameter* α *and on the truncation of expansion* (2).

N_p	α/fm^{-4}							
	0.2	0.5	0.8	1.0	1.2	2.5	5.0	10.0
1	−6.65	−7.54	−7.83	−7.90	−7.90	−7.26	−5.02	+0.08
2	−7.12	−7.50	−7.56	−7.56	−7.55	−7.36	−6.92	−4.90
3	−7.47	−7.72	−7.77	−7.78	−7.77	−7.67	−7.36	−6.58
4	−7.61	−7.74	−7.75	−7.75	−7.75	−7.73	−7.67	−7.49
5	−7.67	−7.76	−7.76	−7.76	−7.76	−7.73	−7.63	−7.37

For the present state of nuclear physics the accuracy of roughly 2%, which is reached taking only 3 basis functions, seems to be sufficient. Other numerical errors (accuracy of integration etc.) show up in the same order of magnitude.

In order to test the method further on, we have also calculated the binding energies for the Mongan II ([6]) and Reid ([7]) singlet s-wave potentials. We found −2.01 MeV and −1.13 MeV, respectively, which differ slightly from the results −2.06 and −1.02,

([4]) R. A. MALFLIET and J. A. TJON: *Nucl. Phys.*, **127** A, 161 (1969).
([5]) B. L. G. BAKKER: *Zeit. f. Phys.*, **272** A, 335 (1975).
([6]) T. R. MONGAN: *Phys. Rev.*, **178**, 1597 (1969).
([7]) R. V. REID: *Ann. of Phys.*, **50**, 411 (1968).

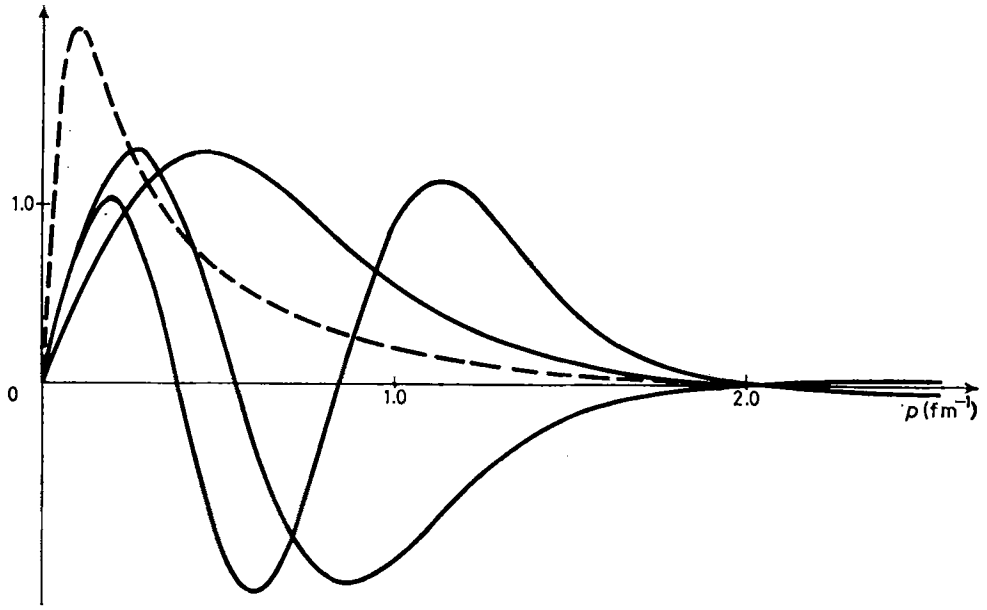

Fig. 1. – The first three eigenfunctions $\varphi_n(p)$ of eq. (5) for the Malfliet-Tjon potential V and a width parameter $\alpha = 1$ fm^{-1}. The dashed curve denotes the two-body bound-state wave function ($\alpha = 0$)

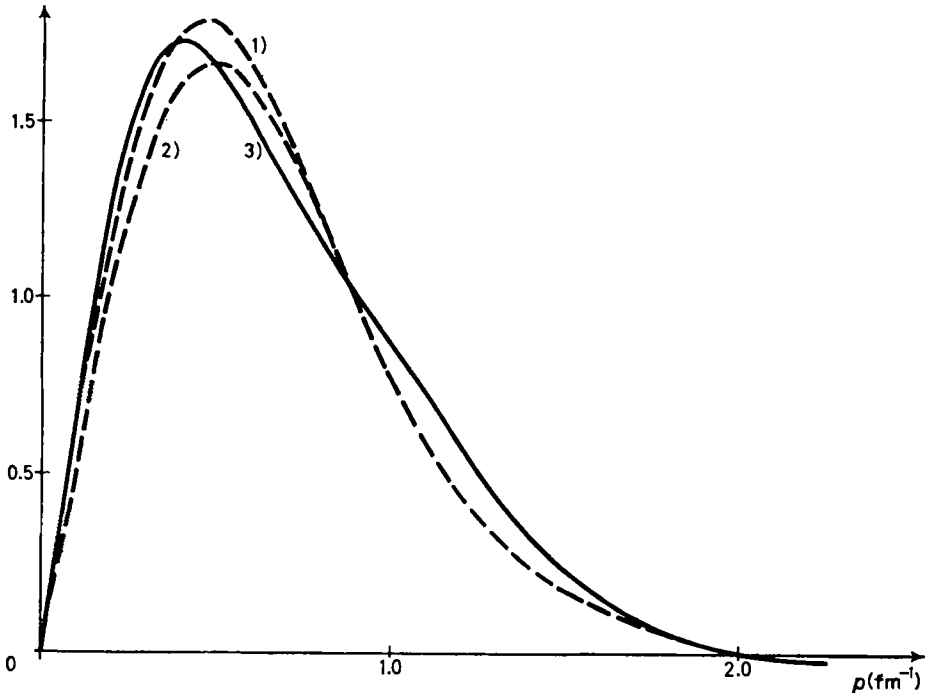

Fig. 2. – Comparison of expansion (2) truncated at various $N_p(1, 2, \ldots)$. The solid curve denotes the converged wave function $N_p = 5$, which cannot be distinguished in the figure from the wave function corresponding to $N_p = 3$, $q = 0.274$.

determined by an UPE expansion ([8]). Although the convergence is not as good as for the stronger Malfliet-Tjon V potential, 3 basis functions are sufficient for an accuracy of a few percent. Furthermore, because of the lower binding energy smaller α values of $\alpha \approx 0.5\,\mathrm{fm}^{-4}$ and $\alpha \approx 0.2\,\mathrm{fm}^{-4}$ turn out to be optimal.

The numerical method described so far is easy to handle and (for $N_p = 5$ and 10 G points in q) takes about 4 ½ minutes on a Telefunken TR 440. A numerical investigation of 3 nucleons along the same line is under way.

([8]) E. HARMS: *Phys. Rev. C*, **1**, 1667 (1970).