Vibrations of Rotating Beams with Tip Mass

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1. Introduction

The problem of determining the influence of a tip mass placed at the free end of a uniform bar rotating at a constant speed about an axis at the other end and vibrating transversely to the plane of rotation (see Figure 1) has been considered previously [1][2]). It was shown, through the use of Rayleigh-Ritz and Southwell methods, that the presence of the tip mass depressed the lowest frequency. For this case, then, the effect of the tip mass on the kinetic energy outweighs the contribution to the potential energy produced by centrifugal force. Dr. Kurt Hohenemser, of McDonnell Aircraft Co., pointed out in an informal oral discussion that this result agreed with calculations made by his group for the lowest frequency. It appeared to them, however, that the second and higher frequencies behaved in the opposite fashion with an increase in tip mass raising these frequencies. Furthermore, he suggested that frequencies higher than the first are of more direct concern to the designer of helicopter blades.

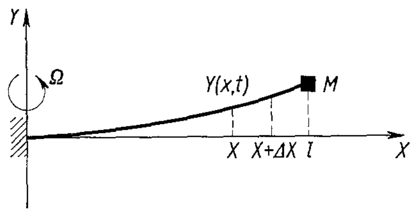

Figure 1
Rotating beam with tip mass.

The purpose of this investigation is an attempt to analyze the effect of a tip mass on frequencies higher than the first. It will be shown that asymptotic or singular perturbation methods can be employed to give information even in the case of variable flexural rigidity and variable mass distribution for high speeds of rotation. The uniform beam can be studied more explicitly through these methods as well as by ordinary perturbation techniques, upper and lower bound estimates based on the Rayleigh-Ritz approach, and an extended

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[2]) Numbers in brackets refer to references page 391.

Southwell procedure [2]. The general beam will be considered first and the detailed analysis for the beam of uniform section will be discussed later. It will be seen that these methods allow one partially to verify Dr. Hohenemser's remarks.

The asymptotic or singular perturbation method is partly an extension and partly a special case of a general theory developed earlier by Moser [3]. An attempt to provide a broader extension of Moser's work on general linear differential equations is now being carried out by Professor Joseph B. Keller of New York University and one of the authors of the present paper. The chief difference between the problem considered here and the general one of [3] lies in the presence of the eigenvalue and expansion parameter in one of the boundary conditions. This boundary condition is due to the tip mass.

2. The equations of motion

The deflections of the beam are assumed to be sufficiently small that the ordinary, linear, Euler-Bernoulli theory can be applied; in addition, shear deformation and rotary inertia are neglected. Let us consider a coordinate system which rotates with the beam at the constant angular velocity Ω (see Figure 1). The origin 0 is at the clamped end of the beam, the x-axis sweeps out the plane of rotation, and $y(x, t)$ measures the deflection out of the plane of rotation. Let EI represent the flexural rigidity of the beam, ϱ the mass per unit volume, A the cross-sectional area, and M the mass of the bob. All but the last can be functions of position x in $0 \leqq x \leqq l$, where l is the length of the beam.

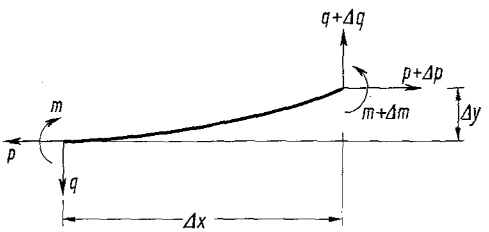

Figure 2
Stress resultants on a typical beam section.

The equations of motion can be obtained by considering an elementary section as shown in Figure 2 running from x to $x + \Delta x$. Let $p(x, t)$ represent the tensile force at x, $q(x, t)$ the shear force, and $m(x, t)$ the bending moment. The equation of motion in the y-direction thus becomes

$$\frac{\partial q}{\partial x} = \varrho \, A \, \frac{\partial^2 y}{\partial t^2}, \tag{2.1}$$

and the moment equation about one end of the element is

$$\frac{\partial m}{\partial x} + q - p\,\frac{\partial y}{\partial x} = 0\,. \tag{2.2}$$

Differentiation of Equation (2.2) and substitution of Equation (2.1) yields

$$\frac{\partial^2 m}{\partial x^2} - \frac{\partial}{\partial x}\left(p\,\frac{\partial y}{\partial x}\right) = -\,\varrho\,A\,\frac{\partial^2 y}{\partial t^2}\,.$$

From the Euler-Bernoulli law, $m = EI\,\partial^2 y/\partial x^2$; the last equation becomes

$$\frac{\partial^2}{\partial x^2}\left(EI\,\frac{\partial^2 y}{\partial x^2}\right) - \frac{\partial}{\partial x}\left(p\,\frac{\partial y}{\partial x}\right) = -\,\varrho\,A\,\frac{\partial^2 y}{\partial t^2}\,. \tag{2.3}$$

The total tensile force at the section x is

$$p(x) = \int\limits_x^l \varrho(\xi)\,A(\xi)\,\Omega^2\,\xi\,d\xi + M\,l\,\Omega^2\,. \tag{2.4}$$

The beam is clamped at $x = 0$ and hence satisfies the boundary conditions

$$y(0,t) = \frac{\partial y}{\partial x}\,(0,t) = 0\,.$$

At the opposite end, the bending moment must vanish, or

$$EI\,\frac{\partial^2 y}{\partial x^2} = 0\,.$$

The final boundary condition is found from the equation of motion of the mass M in the vertical direction; namely

$$-\,q = M\,\frac{\partial^2 y}{\partial t^2} \quad \text{at} \quad x = l\,.$$

However, q may be eliminated in favor of m by means of Equation (2.2) with the result

$$\frac{\partial m}{\partial x} - p\,\frac{\partial y}{\partial x} = M\,\frac{\partial^2 y}{\partial t^2} \quad \text{at} \quad x = l\,.$$

The Euler-Bernoulli law and the fact that $p(l,t) = M\,l\,\Omega^2$ reduce this equation to the boundary condition

$$\frac{\partial}{\partial x}\left(EI\,\frac{\partial^2 y}{\partial x^2}\right) - M\,l\,\Omega^2\,\frac{\partial y}{\partial x} = M\,\frac{\partial^2 y}{\partial t^2} \quad \text{at} \quad x = l\,.$$

These equations can be simplified considerably by introducing harmonic oscillations $y(x,t) = w(x)\,\exp(i\,\lambda\,t)$ and the new non-dimensional variables $s = x/l$ and $v(s) = w(x)/l$. In addition, non-dimensional coefficients and parameters can be defined by

$$e(s) = E(l\,s)\,\frac{I(l\,s)}{(EI)_{max}}\,,$$

$$r(s) = \frac{\varrho(l\,s)\,A(l\,s)}{\frac{1}{l}\int\limits_0^l \varrho(x)\,A(x)\,dx}\,,$$

$$\alpha^2 = l^4\,\Omega^2\,\frac{1}{l}\int\limits_0^l \frac{\varrho(x)\,A(x)\,dx}{(EI)_{max}}\,,$$

$$\beta^2 = l^4\,\lambda^2\,\frac{1}{l}\int\limits_0^l \frac{\varrho(x)\,A(x)\,dx}{(EI)_{max}}\,,$$

$$\gamma^2 = \frac{M}{\int\limits_0^l \varrho(x)\,A(x)\,dx}\,,$$

where the maximum is taken over the range $0 \leqq x \leqq l$. It should be noted that $0 \leqq e(s) \leqq 1$ on $0 \leqq s \leqq 1$. The quantity β^2 is the non-dimensional eigenvalue which is being sought, α^2 is the non-dimensional speed of rotation, and γ^2 is the mass-ratio whose influence is to be examined.

The final eigenvalue problem can now be stated as

$$(e\,v'')'' - \alpha^2\left(v'\left\{\int\limits_s^1 r(\sigma)\,\sigma\,d\sigma + \gamma^2\right\}\right)' = \beta^2\,r(s)\,v\,, \qquad 0 < s < 1\,, \tag{2.5}$$

$$v(0) = v'(0) = 0\,, \tag{2.6}$$

$$v''(1) = 0\,, \tag{2.7}$$

$$\{e(s)\,v''\}' - \gamma^2\,\alpha^2\,v' = -\gamma^2\,\beta^2\,v\,, \quad \text{at } s = 1\,, \tag{2.8}$$

where primes denote differentiation with respect to s. The problem is thus to find the eigenvalues β^2 and, in particular, to examine their dependence on the parameters α^2 and γ^2. A closed form solution is, in general, almost impossible to find. For the general case, then, an asymptotic or singular perturbation method, appropriate for large values of α^2, that is, high speeds of rotation, will be used.

3. Application of a singular perturbation method

It is clear that a straight-forward application of a perturbation procedure in terms of $1/\alpha^2$ is not possible, for the lowest order term arising from Equation (2.5) must then satisfy a differential equation of the form

$$-\alpha^2 \left(v' \left\{ \int\limits_s^1 r(\sigma)\, \sigma\, d\sigma + \gamma^2 \right\} \right)' = \beta^2\, r(s)\, v\,,$$

which is of second order in the dependent variable v. On the other hand, since all four boundary conditions remain, the problem is over prescribed, and has no solution. Similar questions have been discussed rather generally by MOSER [3] using an asymptotic solution. His analysis does not include the present case in which the eigenvalue and expansion parameter appear in the boundary conditions. Nevertheless, his method can be extended to the case in question at the expense of a little more algebraic manipulation.

Let us first set $\varepsilon = 1/\alpha^2$ and $\mu = \beta^2/\alpha^2$; Equation (2.5) becomes

$$\varepsilon\, (e\, v'')'' - \left(v' \left\{ \int\limits_s^1 r(\sigma)\, \sigma\, d\sigma + \gamma^2 \right\} \right)' = \mu\, r(s)\, v\,.$$

The term *reduced equation* will refer to the second order equation which results when ε is set equal to zero. The appropriate parameter for expansion is not ε itself, but one which depends on the difference in the orders of the original differential equation and the reduced equation; namely, $\eta = \varepsilon^{1/2}$.

The basic differential equation becomes

$$\eta^2\, (e\, v'')'' - \left(v' \left\{ \int\limits_s^1 r(\sigma)\, \sigma\, d\sigma + \gamma^2 \right\} \right)' = \mu\, r(s)\, v\,; \tag{3.1}$$

boundary conditions (2.6) and (2.7) remain unchanged; whereas, Equation (2.8) is transformed into

$$\eta^2\, (e\, v'')' - \gamma^2\, v' = -\gamma^2\, \mu\, v \quad \text{at} \quad s = 1\,. \tag{3.2}$$

One seeks a solution of the form

$$\left.\begin{aligned} v(s) &= B\,(s;\,\eta)\, \exp\{\eta^{-1}\, h(s)\} \\ &= \sum_{n=0} B_n(s)\, \eta^n \exp\{\eta^{-1}\, h(s)\}\,, \qquad B_0(s) \neq 0\,. \end{aligned}\right\} \tag{3.3}$$

Substitution of this function into Equation (3.1) and division by $\exp\{\eta^{-1} h(s)\}$ yields the following equation arranged in terms of ascending powers of η:

$$\left.\begin{aligned}
&\frac{1}{\eta^2}\left[e\,h'^4\,B - h'^2\left(\int_s^1 r(\sigma)\,\sigma\,d\sigma + \gamma^2\right) B\right]\\
&+\frac{1}{\eta}\left[6\,e\,h'^2\,h''\,B + 4\,e\,h'^3\,B' + 2\,e'\,h'^3\,B + r\,s\,h'\,B\right.\\
&\qquad\qquad\left. - \left(\int_s^1 r(\sigma)\,\sigma\,d\sigma + \gamma^2\right)(h''\,B + 2\,h'\,B')\right]\\
&+\left[4\,e\,h'\,h'''\,B + 3\,e\,h''^2\,B + 12\,e\,h'\,h''\,B' + 6\,e\,h'^2\,B'' + 6\,e'\,h'\,h''\,B\right.\\
&\qquad\left. + 6\,e'\,h'^2\,B' + e''\,h'^2\,B + r\,s\,B' - \left(\int_s^1 r(\sigma)\,\sigma\,d\sigma + \gamma^2\right) B''\right]\\
&+\eta\,[e\,h''''\,B + 4\,e\,h'''\,B' + 6\,e\,h''\,B'' + 4\,e\,h'\,B''' + 2\,e'\,h'''\,B\\
&\qquad\qquad + 6\,e'\,h''\,B' + 6\,e'\,h'\,B'' + 2\,e''\,h'\,B' + e''\,h''\,B]\\
&+\eta^2\,[e\,B'''' + 2\,e'\,B''' + e''\,B''] - \mu\,r\,B = 0\,.
\end{aligned}\right\}\tag{3.4}$$

Since the coefficient of each power of η must vanish independently, and $B_0(s) \neq 0$, it follows from the first term that

$$e\,h'^4 - h'^2\left(\int_s^1 r(\sigma)\,\sigma\,d\sigma + \gamma^2\right) = 0\,.$$

Thus there are four solutions for h'; namely,

$$h_1' = 0\,,\quad h_2' = 0\,,\quad h_3' = -\left[\frac{1}{e}\left(\int_s^1 r(\sigma)\,\sigma\,d\sigma + \gamma^2\right)\right]^{1/2},$$

$$h_4' = \left[\frac{1}{e}\left(\int_s^1 r(\sigma)\,\sigma\,d\sigma + \gamma^2\right)\right]^{1/2},$$

which can be integrated directly when the mass distribution is known. It is important to observe that $h_3'(s) \leqq 0$ and $h_4'(s) \geqq 0$ for $0 \leqq s \leqq 1$.

Corresponding to each value of h is a solution $v(s)$ which will be written in the form

$$\left.\begin{aligned} U_1 &\equiv u_{10} + \eta\, u_{11} + \eta^2\, u_{12} + \cdots + \eta^\sigma\, R_{1\sigma} && (\sim h_1)\,,\\ U_2 &\equiv u_{20} + \eta\, u_{21} + \eta^2\, u_{22} + \cdots + \eta^\sigma\, R_{2\sigma} && (\sim h_2)\,,\\ V \exp(\eta^{-1} h_3) &\equiv \exp(\eta^{-1} h_3)\,(v_0 + \eta\, v_1 + \cdots + \eta^\sigma\, S_\sigma) && (\sim h_3)\,,\\ W \exp(-\,\eta^{-1} h_3) &\equiv \exp(-\,\eta^{-1} h_3)\,(w_0 + \eta\, w_1 + \cdots + \eta^\sigma\, T_\sigma) && (\sim h_4)\,. \end{aligned}\right\} \tag{3.5}$$

MOSER [3] has shown that U_1, U_2, V, W will form a fundamental system of solutions of (3.1). Furthermore, the remainder terms $R_{1\sigma}$, $R_{2\sigma}$, S_σ, and T_σ are functions of s, η, and μ which can be estimated by

$$\left|\frac{\partial^l}{\partial s^l} R_{1\sigma}\right| + \left|\frac{\partial^l}{\partial s^l} R_{2\sigma}\right| + \left|\frac{\partial^l}{\partial s^l} S_\sigma\right| + \left|\frac{\partial^l}{\partial s^l} T_\sigma\right| < C_\sigma$$

for $0 \leqq s \leqq 1$ and $l = 0, \ldots, 3$. This holds if η is sufficiently small and μ is in the interval $|\,\mu\,| \leqq \mu^*$ where $\mu^* > 1$ is any fixed number. The constant C_σ is independent of s, η, and μ and the fundamental system is infinitely differentiable with respect to s in $0 \leqq s \leqq 1$ and analytic in μ for $|\,\mu\,| < \mu^*$.

The equations satisfied by u_{ij}, v_i, w_i can be found by substituting into Equation (3.4) and equating coefficients of like powers of η. It is readily seen that for $k = 1, 2$, $m = 0, 1$, and $n \geqq 2$

$$-\left\{\left(\int_s^1 r(\sigma)\,\sigma\,d\sigma + \gamma^2\right) u'_{km}\right\}' = \mu\, r\, u_{km}\,,$$

$$-\left\{\left(\int_s^1 r(\sigma)\,\sigma\,d\sigma + \gamma^2\right) u'_{kn}\right\}' + (e\, u''_{k,n-2})'' = \mu\, r\, u_{kn}\,.$$

Equations satisfied by v_k and w_k can be found in similar fashion.

The general solution is

$$v = A U_1 + B U_2 + C V \exp(\eta^{-1} h_3) + D W \exp(-\,\eta^{-1} h_3)\,, \tag{3.6}$$

where A, B, C, D are constants to be determined from the boundary conditions. Inserting Equation (3.6) into the boundary conditions (2.6), (2.7), and (3.2) leads to a set of four linear, homogeneous, algebraic equations in the unknown coefficients A, B, C, and D. The subsequent fourth order determinant guaranteeing non-trivial solutions yields the secular equation from which the eigenvalues can be found.

In order to simplify the structure of this determinant, for the present, at least, let us set

$$[V \exp(\eta^{-1} h_3)]^{(n)} = V_n \exp(\eta^{-1} h_3)\,,$$

$$[W \exp(-\,\eta^{-1} h_3)]^{(n)} = W_n \exp(-\,\eta^{-1} h_3)\,, \quad n = 1, 2, 3$$

where

$$\left.\begin{aligned}
V_1 &= \eta^{-1} h_3' V + V', \quad V_2 = \eta^{-2} h_3'^2 V + \frac{1}{\eta} (h_3'' V + 2 h_3' V') + V'', \\
V_3 &= \eta^{-3} h_3'^2 V + \eta^{-2} (3 h_3' h_3'' V + 3 h_3'^2 V') \\
&\qquad + \eta^{-1} (h_3''' V + 3 h_3'' V' + 3 h_3' V'') + V''', \\
W_1 &= - \eta^{-1} h_3' W + W', \\
W_2 &= \eta^{-2} h_3'^2 W - \eta^{-1} (h_3'' W + 2 h_3' W') + W'', \\
W_3 &= - \eta^{-3} h_3'^3 W + \eta^{-2} (3 h_3' h_3'' W + 3 h_3'^2 W') \\
&\qquad - \eta^{-1} (3 h_3' W'' + 3 h_3'' W' + h_3''' W) + W'''.
\end{aligned}\right\} \tag{3.7}$$

The determinantal equation thus becomes

$$\Delta = \begin{vmatrix}
U_1(0) & U_1'(0) & \eta^2 [e(1) U_1''(1)]' - \gamma^2 U_1'(1) + \gamma^2 \mu U_1(1) & U_1''(1) \\
U_2(0) & U_2'(0) & \eta^2 [e(1) U_2''(1)]' - \gamma^2 U_2'(1) + \gamma^2 \mu U_2(1) & U_2''(1) \\
V(0) \exp\{\eta^{-1} h_3(0)\} & V_1(0) \exp\{\eta^{-1} h_3(0)\} & T_1(1) & V_2(1) \exp\{\eta^{-1} h_3(1)\} \\
W(0) \exp\{-\eta^{-1} h_3(0)\} & W_1(0) \exp\{-\eta^{-1} h_3(0)\} & T_2(1) & W_2(1) \exp\{-\eta^{-1} h_3(1)\}
\end{vmatrix} = 0$$

where

$$T_1(1) = [\eta^2 \{e(1) V_3(1) + e'(1) V_2(1)\} - \gamma^2 V_1(1) + \gamma^2 \mu V(1)] \exp\{\eta^{-1} h_3(1)\},$$

and

$$T_2(1) = [\eta^2 \{e(1) W_3(1) + e'(1) W_2(1)\} - \gamma^2 W_1(1) + \gamma^2 \mu W(1)] \exp\{-\eta^{-1} h_3(1)\}.$$

The factor $\exp \eta^{-1} \{h_3(0) - h_3(1)\}$ can be taken outside this determinant by multiplying the third row by $\exp\{-\eta^{-1} h_3(0)\}$ and the fourth row by $\exp\{\eta^{-1} h_3(1)\}$. The first two elements in the third row no longer contain an exponential factor; whereas the third and fourth elements contain the factor $\exp \eta^{-1} \{h_3(1) - h_3(0)\}$. Similarly, the first two elements of the fourth row contain this exponential factor; whereas, the last two elements have no exponential factor. Since $h_3(s)$ is a decreasing function, the terms containing $\exp \eta^{-1} \{h_3(1) - h_3(0)\}$ will vanish exponentially while the other terms will either grow or decay algebraically. Consequently, the terms with exponential factors may be dropped and the determinant $\varDelta$ replaced by $\varDelta'$ containing zero in each of these entries.

The determinant $\varDelta'$ can be most conveniently expanded by the third row yielding

$$\left.\begin{aligned}
\varDelta' = {} & V(0)\exp\eta^{-1}\{h_3(0)-h_3(1)\}\times \\
& \begin{vmatrix}
U_1'(0) & \eta^2\,[e(1)\,U_1''(1)]' - \gamma^2\,U_1'(1) + \gamma^2\,\mu\,U_1(1) & U_1''(1) \\
U_2'(0) & \eta^2\,[e(1)\,U_2''(1)]' - \gamma^2\,U_2'(1) + \gamma^2\,\mu\,U_2(1) & U_2''(1) \\
0 & \eta^2\,[e(1)\,W_3(1) + e'(1)\,W_2(1)] - \gamma^2\,W_1(1) + \gamma^2\,\mu\,W(1) & W_2(1)
\end{vmatrix} \\
& - [V'(0) + \eta^{-1}\,h_3'(0)\,V(0)]\exp\eta^{-1}\{h_3(0)-h_3(1)\}\times \\
& \begin{vmatrix}
U_1(0) & \eta^2\,[e(1)\,U_1''(1)]' - \gamma^2\,U_1'(1) + \gamma^2\,\mu\,U_1(1) & U_1''(1) \\
U_2(0) & \eta^2\,[e(1)\,U_2''(1)]' - \gamma^2\,U_2'(1) + \gamma^2\,\mu\,U_2(1) & U_2''(1) \\
0 & \eta^2\,[e(1)\,W_3(1) + e'(1)\,W_2(1)] - \gamma^2\,W_1(1) + \gamma^2\,\mu\,W(1) & W_2(1)
\end{vmatrix} = 0\,.
\end{aligned}\right\} \quad (3.8)$$

Upon substituting Equations (3.5), and (3.7), one observes that the dominant term in $\varDelta'$ is of the order η^{-3}. Hence multiplication of $\varDelta'$ by η^3 will leave each of the determinants with positive powers of η only.

If the common factor $\exp\eta^{-1}\{h_3(0)-h_3(1)\}$ is dropped, the determinants may be expanded to give the following eigenvalue equation

$$F_0(\mu) + F_1(\mu)\,\eta + \cdots + F_\sigma(\mu)\,\eta^\sigma + \eta^{\sigma+1}\,R_\sigma\,(\eta,\mu) = 0\,, \qquad (3.9)$$

where one sees from MOSER's result on the behavior of the fundamental system of solutions that there exists a constant C_σ such that $|\,R_\sigma(\eta,\mu)\,| < C_\sigma$ for all $\eta > 0$ sufficiently small and $|\,\mu\,| < \mu^*$. Furthermore, each of the functions $F_0(\mu)\,,\ldots,F_\sigma(\mu)$ is an analytic function of η in this region.

4. Evaluation of the zero and first order terms

Despite the generality of the coefficients, it is possible to make some explicit statements concerning the eigenvalues, at least through zero and first order terms in the asymptotic expansion. Since the functions $F_0,\ldots,F_\sigma$ are analytic in μ and, as will be seen, F_0 has only simple zeros, Equation (3.8) can be inverted in the form

$$\mu = \mu_0 + \mu_1\,\eta + \cdots + \mu_\sigma\,\eta^\sigma + \eta^{\sigma+1}\,r_\sigma\,,$$

where, again, the remainder term is bounded. It will be shown that sufficient information can be obtained about μ_0 and μ_1 to shed some light on Dr. HOHENEMSER's statement concerning the frequency behavior.

The differential equations satisfied by the functions u_{km} have already been stated. In addition, the initial values of the functions and their first derivatives can be prescribed provided only that these conditions yield independent functions. For this purpose, one may set

$$u_{10}(0) = 0\,, \quad u'_{10}(0) = 1\,; \quad u_{20}(0) = 1\,, \quad u'_{20}(0) = 0\,,$$

$$u_{11}(0) = 0\,, \quad u'_{11}(0) = 1\,; \quad u_{21}(0) = 1\,, \quad u'_{21}(0) = 0\,.$$

One then finds, upon expanding Equation (3.8), that

$$F_0(\mu) = h'_3(0)\, V_0(0)\, h'^2_3(1)\, W_0(1)\, \{-\gamma^2\, u'_{10}(1) + \gamma^2\, \mu\, u_{10}(1)\}\,.$$

Since $h'_3(0) \neq 0$, $h'_3(1) \neq 0$ and $V_0(0)$ and $W_0(1)$ can be chosen so as not to vanish, $F_0(\mu_0) = 0$ becomes

$$-u'_{10}(1) + \mu\, u_{10}(1) = 0\,. \tag{4.1}$$

In addition,

$$-\left\{\left(\int_s^1 r(\sigma)\,\sigma\,d\sigma + \gamma^2\right)u'_{10}\right\}' = \mu\, r\, u_{10} \quad 0 < s < 1\,, \tag{4.2}$$

and

$$u_{10}(0) = 0\,. \tag{4.3}$$

Equations (4.1), (4.2), and (4.3) form the eigenvalue problem which describes the motion of a heavy flexible string, carrying a mass at one end, rotating about the other end in a horizontal plane, and vibrating transversely to the plane of rotation. The differential equation is precisely the reduced equation, and the boundary conditions are those which would arise from retaining the first of Equations (2.6) and (3.2) with $\eta = 0$. Thus the singular perturbation method has selected from among the original boundary conditions those which are appropriate to the zero-order problem.

The n^{th} eigenvalue of the problem defined by Equations (4.1), (4.2), and (4.3) will be denoted by $\mu_0^{(n)}$. Inspection shows that $u_{10} = s$ is one of the eigenfunctions of this problem corresponding to the eigenvalue $\mu = 1$. This is indeed the lowest eigenvalue since $u_{10} = s$ vanishes only at the boundary $s = 1$ (see [4]). Therefore

$$\mu_0^{(1)} = 1 \text{ corresponding to } u_{10} = s\,.$$

Unfortunately, the zero order term for the lowest mode yields no information about the dependence on γ since it is independent of this parameter. It is therefore necessary to examine $\mu_1^{(1)}$.

The solutions for higher modes are unfortunately not as obvious as the first; however, it is possible to obtain the behavior of $\mu_0^{(n)}$ for large order n.

Since eigenvalues ν of a differential equation of the form

$$(p\, u')' + \nu\, \varrho\, u = 0\,, \quad \sigma < s < \pi$$

satisfy [5]

$$\lim_{n\to\infty} \frac{n^2}{\nu_n} = \frac{1}{\pi^2}\left(\int\limits_0^{\pi}\left(\frac{\varrho}{p}\right)^{1/2} ds\right)^2,$$

then for the problem in question

$$\mu_0^{(n)} \sim \frac{n^2\,\pi^2}{\int\limits_0^1 [\gamma(s)]^{1/2}\left[\int\limits_s^1 \gamma(\sigma)\, d\sigma + \gamma^2\right]^{-1/2} ds} \tag{4.4}$$

Equation (4.4) shows that as γ^2 increases, $\mu_0^{(n)}$ will also increase. Therefore, for sufficiently high speeds of rotation and sufficiently high modes of vibration, the natural frequencies will be increased if the ratio of the tip mass to the beam mass is increased.

Let us now return to the computation of the first order term μ_1. Carrying through the necessary algebraic calculations in Equation (3.8), the function $F_1(\mu)$ can be found. The work can be considerably simplified, however, by observing that a solution of the form $\mu = \mu_0 + \mu_1$ is desired and the functions F_0 and F_1 are analytic in μ. Thus a solution of

$$F_0\,(\mu_0 + \eta\,\mu_1) + \eta\, F_1\,(\mu_0 + \eta\,\mu_1) = 0$$

is to be found. Applying a Taylor-series expansion and keeping only the first two terms yields

$$F_0(\mu_0) + \eta\,\mu_1 \frac{\partial F_0(\mu_0)}{\partial \mu} + \eta\, F_1(\mu_0) = 0\,.$$

Since $F_0(\mu_0) = 0$,

$$\mu_1 = -\frac{F_1(\mu_0)}{\frac{\partial F_0(\mu_0)}{\partial \mu}}\,.$$

In order to find μ_1, one need only evaluate

$$\left.\begin{aligned} F_1(\mu_0) &= V_0(0)\, h_3'^2(1)\, W_0(1)\, [-\gamma^2\, u_{20}'\,(1, \mu_0) + \gamma^2\,\mu_0\, u_{20}\,(1, \mu_0)] \\ \frac{\partial F_0(\mu_0)}{\partial \mu} &= h_3'(0)\, h_3'^2(1)\, V_0(0)\, W_0(1) \\ &\qquad \left[-\gamma^2 \frac{\partial u_{10}'}{\partial \mu}\,(1, \mu_0) + \gamma^2\, u_{10}\,(1, \mu_0) + \gamma^2\,\mu_0 \frac{\partial u_{10}}{\partial \mu}\,(1, \mu_0)\right]. \end{aligned}\right\} \tag{4.5}$$

The function $F_1(\mu_0)$ has been found by expanding Δ' and making use of Equations (4.1) and (4.2).

It is not necessary to determine the functions u_{20} and $\partial u_{10}/\partial\mu$ explicitly since Equation (4.5) can be rewritten in terms of u_{10} alone. The derivatives with respect to the eigenvalue μ can be found by considering a perturbation procedure. If one writes

$$\mu = \mu_0 + \delta + \cdots,$$

then

$$u_{10}(s, \mu) = u_{10}(s, \mu_0) + \delta u(s),$$

and

$$u(s) = \frac{\partial u_{10}(s, \mu_0)}{\partial\mu}.$$

Applying the usual perturbation expansion to Equations (4.2) and (4.3), one obtains

$$\left.\begin{aligned} &\left[\left(\int_s^1 r\,\sigma\,d\sigma + \gamma^2\right) u_{10}'(s, \mu_0)\right]' + \mu_0\, r\, u_{10}(s, \mu_0) = 0, \quad u_{10}(0, \mu_0) = 0, \\ &\left[\left(\int_s^1 r\,\sigma\,d\sigma + \gamma^2\right) u'(s)\right]' + \mu_0\, r\, u(s) + r\, u_{10}(s, \mu_0) = 0, \quad u(0) = 0. \end{aligned}\right\} \tag{4.6}$$

Multiplying the second of Equations (4.6) by $u_{10}(s, \mu_0)$, integrating from 0 to 1, applying integration by parts, and the first of Equations (4.6), we find that

$$u_{10}(1, \mu_0)\,\gamma^2\,u'(1) - u_{10}'(1, \mu_0)\,\gamma^2\,u(1) + \int_0^1 r\,u_{10}^2(s, \mu_0)\,ds = 0.$$

Now $u_{10}(s, \mu_0)$ also satisfies the boundary condition

$$u_{10}'(1, \mu_0) = \mu_0\, u_{10}(1, \mu_0)$$

Thus

$$u_{10}(1, \mu_0)\,\gamma^2\,[u'(1) - \mu_0\, u(1)] + \int_0^1 r\,u_{10}^2(s, \mu_0)\,ds = 0$$

and hence

$$\frac{\partial F}{\partial\mu}(\mu_0) = h_3'(0)\,V_0(0)\,h_3'^2(1)\,W_0(1)\left[\left\{\int_0^1 \frac{r\,u_{10}^2(s, \mu_0)\,ds}{u_{10}(1, \mu_0)}\right\} + \gamma^2\,u_{10}(1, \mu_0)\right].$$

Similarly the terms in $F_1(\mu_0)$ involving $u_{20}(1, \mu_0)$ can also be written in terms of $u_{10}(s, \mu_0)$, for u_{20} satisfies

$$\left[\left(\int_s^1 r\,\sigma\,d\sigma + \gamma^2\right) u_{20}'(s, \mu_0)\right]' + \mu_0\, r\, u_{20}(s, \mu_0) = 0.$$

Multiplying by $u_{10}(s, \mu_0)$ and integrating from 0 to 1, one finds

$$\int_0^1 u_{10}(s, \mu_0) \left[\left(\int_s^1 r\, \sigma\, d\sigma + \gamma^2 \right) u'_{20}(s, \mu_0) \right]' ds + \mu_0 \int_0^1 r\, u_{10}(s, \mu_0)\, u_{20}(s, \mu_0)\, ds = 0\,.$$

If this expression is integrated twice by parts and the boundary conditions $u'_{10}(0, \mu_0) = u_{20}(0, \mu_0) = 1$ applied, the desired expression for u_{20} is found to be

$$-\gamma^2\, u'_{20}(1, \mu_0) + \mu_0\, \gamma^2\, u_{20}(1, \mu_0) = \frac{\int_0^1 r\, \sigma\, d\sigma + \gamma^2}{u_{10}(1, \mu_0)}\,.$$

Now

$$h'_3(0) = -\,[e(0)]^{-1/2} \left[\int_0^1 r\, \sigma\, d\sigma + \gamma^2 \right]^{1/2}.$$

Therefore

$$\mu_1 = \frac{\left[e(0) \left\{ \int_0^1 r\, \sigma\, d\sigma + \gamma^2 \right\} \right]^{1/2}}{\int_0^1 r\, u_{10}^2(s, \mu_0)\, ds + \gamma^2\, u_{10}^2(1, \mu_0)}\,. \tag{4.7}$$

Thus the linear term can be expressed entirely in terms of the eigenfunction of the reduced problem which in itself was required for the zero order problem. It is interesting to note that the denominator in Equation (4.7) is essentially to total kinetic energy of the string and added mass which is the physical counter-part of the reduced problem.

For the lowest eigenvalue

$$\mu_1 = \frac{\left[e(0) \left\{ \int_0^1 r\, \sigma\, d\sigma + \gamma^2 \right\} \right]^{1/2}}{\int_0^1 r\, s^2\, ds + \gamma^2}\,. \tag{4.8}$$

Observing that

$$\int_0^1 r\, \sigma\, d\sigma \geqq \int_0^1 r\, s^2\, ds$$

it is easily seen that $d\mu_1/d\gamma^2 \leqq 0$ and hence μ_1 is a decreasing function of γ^2.

From the two term asymptotic expansion for μ it follows that for sufficiently high speeds of rotation the lowest natural frequency decreases as the tip mass is increased. Furthermore, the limits obtained in Equations (4.4) and (4.7) when γ^2 approaches zero are finite. Thus the problem for the beam without tip mass can be studied as an interior limit problem. It cannot be handled directly by the singular perturbation method since the underlying theorem requires that the coefficient of the leading term of the reduced equation shall not vanish in the closed interval $0 \leqq s \leqq 1$.

Finally, the Southwell lower bound technique [6] yields as a lower bound for the lowest eigenvalue,

$$\mu \geqq 1 + \nu ,$$

where ν is the lowest eigenvalue of the problem

$$\eta^2 \, (e \, v'')'' = \nu \, r \, v \, , \qquad 0 < s < 1 \, ,$$

$$v(0) = v'(0) = 0 \, , \quad v''(1) = 0 \, ,$$

$$\eta^2 \, (e \, v'')' - \gamma^2 \, v' = - \gamma^2 \, \nu \, v \ \text{ at } \ s = 1 \, .$$

Thus a one term asymptotic expansion will be a lower bound for the eigenvalue. On the other hand, the second term in the asymptotic expansion contains a factor which is linear in $1/\eta$ whereas the Southwell bound does not. Hence for small η, the asymptotic result will exceed this bound.

5. The uniform beam. Singular perturbation methods

More specific results can be obtained for the case of the uniform beam since the reduced problem is then capable of explicit solution. In this case $e(s) = r(s) = 1$ and the boundary value problem becomes

$$\left.\begin{aligned} \eta^2 \, v'''' - \left[v' \left\{ \frac{1}{2} \, (1 - s^2) + \gamma^2 \right\} \right]' = \mu \, v \, , \quad 0 < s < 1 \, , \\ v(0) = v'(0) = v''(1) = 0 \, , \\ \eta^2 \, v''' - \gamma^2 \, v' = - \gamma^2 \, \mu \, v \ \text{ at } \ s = 1 \, . \end{aligned}\right\} \tag{5.1}$$

The reduced problem now takes the form

$$\left.\begin{aligned} - \left[\left\{ \frac{1}{2} \, (1 - s^2) + \gamma^2 \right\} u_{10}' \right]' = \mu \, u_{10} \, , \quad 0 < s < 1 \, , \\ u_{10}(0) = 0 = - \, u_{10}'(1) + \mu \, u_{10}(1) \, . \end{aligned}\right\} \tag{5.2}$$

In order to solve the differential equation in problem (5.2), let us rewrite it in the form

$$\{(1 - s^2 + 2 \, \gamma^2) \, u_{10}'\}' + 2 \, \mu \, u_{10} = 0$$

and set $x = s/\sqrt{1 + 2\gamma^2}$. Under this transformation the differential equation becomes

$$\frac{d}{dx}\left\{(1 - x^2)\frac{du_{10}}{dx}\right\} + 2\mu u_{10} = 0, \quad 0 < x < \frac{1}{\sqrt{1 + 2\gamma^2}} \tag{5.2a}$$

or LEGENDRE's equation. Setting

$$k(k + 1) = 2\mu \tag{5.2b}$$

the solution satisfying $u_{10}(0) = 0$, $du_{10}(0)/dx = 1$ may be written in terms of the Legendre function $P_k(x)$ and the associated Legendre function $Q_k(x)$ as

$$u_{10} = \frac{1}{2}\sin\frac{k\pi}{2}\,\frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{k}{2}+1\right)}\,P_k\left(\frac{s}{\sqrt{1+2\gamma^2}}\right) + \cos\frac{k\pi}{2}\,\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}+1\right)\Gamma\left(\frac{1}{2}\right)}\,Q_k\left(\frac{s}{\sqrt{1+2\gamma^2}}\right). \tag{5.3}$$

The eigenvalues μ_0 can now be found by applying the second boundary condition in Equations (5.2). It should be noted that even this is somewhat awkward since the Legendre functions are not of integral order unless $\gamma^2 = 0$. There are, however, several techniques which lead to very close approximations.

Upper bounds for the eigenvalues can be found from the Rayleigh-Ritz technique. The standard methods of proof [7] will show that the lowest eigenvalue is given by

$$2\mu = \min_w \left[\frac{\int_0^1 (1 - s^2 + 2\gamma^2)\, w'^2\, ds}{\int_0^1 w^2\, ds + \gamma^2 w^2(1)}\right]$$

where the only requirements on w are piecewise continuity of the first derivatives and $w(0) = 0$. The second boundary condition in Equation (5.2) will appear as a natural boundary condition. In fact, the resulting natural boundary condition is actually

$$\lim_{s\to 1}\,(1 - s^2)\, w' + 2\gamma^2 [w'(1) - \mu\, w(1)] = 0\,. \tag{5.4}$$

The limit term is of no importance except in the case $\gamma^2 = 0$. Furthermore, if one derives the basic equations of the original rotating beam problem from a minimum principle, a similar limiting condition, guaranteeing no logarithmic singularity at $s = 1$, appears in the natural boundary conditions.

Since the solution of the eigenvalue problem given by the differential equation and first boundary condition in (5.2) with $\gamma^2 = 0$ and the second boundary condition given by Equation (5.4) is an odd Legendre polynomial, a trial function for the minimum principle has been taken as

$$w = a\, P_1(s) + b\, P_3(s) + c\, P_5(s)\,.$$

Here P_1, P_3, P_5 are Legendre polynomials and the coefficients a, b, c are to be found from the minimizing condition. The lowest root of the resulting cubic equation is the lowest eigenvalue. The other two roots are upper bounds for the second and third eigenvalues. The upper bound for the second root as a function of γ^2 is labeled Rayleigh-Ritz in Figure 3.

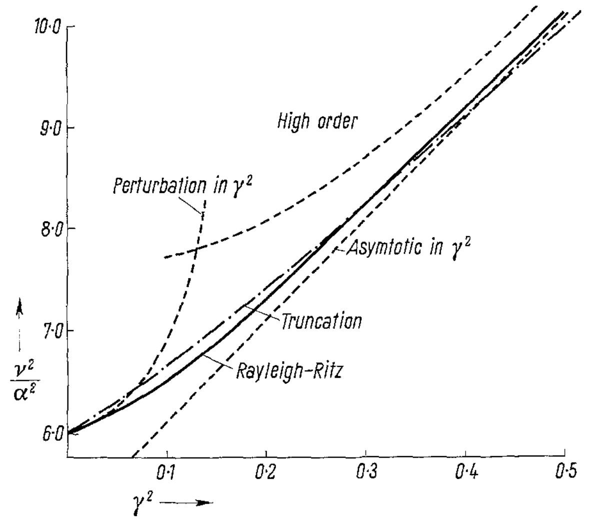

Figure 3

Approximative eigenvalues of the reduced equation as a function of tip mass.

A second method of approximation makes use of a truncated power series. It is more convenient to transform the equation into the form (5.2a) with k defined by (5.2b). The solution which satisfies $u_{10}(0) = 0$ and $u'_{10}(0) = 1$ can be written as the hypergeometric series

$$u_{10} = x + \sum_{m=1}^{\infty} (-1)^m \frac{(k+2)\cdots(k+2\,m)\,(k-1)\cdots(k-2\,m+1)}{(2\,m+1)!}\, x^{2m+1}\,.$$

Differentiating this series and substituting the results into the second boundary condition yields an equation involving k and even powers of the quantity δ

defined by
$$\delta = \frac{1}{(1 + 2\,\delta^2)^{1/2}}\,.$$

Terms of order δ^6 have been neglected and the resulting equation solved for δ^2 as a function of k. The inverse function can be found from the graph of the solution. The final results are marked Truncation in Figure 3 where it should be noted that the exact solution for $\gamma^2 = 0$ is 6.0.

A third method is appropriate for small values of the added mass ratio and consists of a perturbation scheme in γ^2. For this case, expansions of the form

$$u_{10} = \sum_{n=0}^{\infty} w_n (2\,\gamma)^{2n}\,, \quad \mu = \sum_{n=0}^{\infty} \nu_n\,(2\,\gamma)^{2n}$$

are taken. In order to take the case of the lowest order term, corresponding to $\gamma = 0$, it is necessary to consider the boundary condition at $s = 1$ in the form given by Equation (5.4). Save for this remark, the procedure is completely standard and will therefore not be discussed in detail.

It is found that the coefficient of the γ^2 term vanishes and the solution for the second eigenvalue up to terms of order γ^4 is

$$\mu = [6 + 420\,\gamma^4 + 0(\gamma^6)]\,.$$

As indicated in Figure 3, the results are only good for small values of γ^2. The same procedure has been applied to the n^{th} eigenvalue where it has again been found that the γ^2 term in missing, and the coefficient of γ^4 is positive provided that $n \geqq 2$. Consequently, for small values of γ^2 the n^{th} eigenvalue μ_0 is an increasing function of γ^2 for $n \geqq 2$. Recalling that μ_0 is the first term of the asymptotic expansion, we find a somewhat stronger result than that found for the beam with variable section; namely, for high speeds of rotation and small ratios of tip mass to beam mass, the second and higher frequencies increase with the tip mass.

Asymptotic solutions for large γ^2 can also be obtained by setting

$$\varepsilon^2 = \frac{1}{2\,\gamma^2}\,, \quad \lambda^2 = \frac{\mu}{\gamma^2}$$

and seeking solution in the form

$$u_{10} = \sum_{n=0} w_n\,\varepsilon^{2n}\,, \quad \lambda^2 = \sum_{n=0} \lambda_n^2\,\varepsilon^{2n}\,,$$

valid for small values of ε. The resulting differential equations have constant coefficients and the first two terms in the perturbation series can be readily found. The n^{th} eigenvalue ($n \geqq 2$) is given by

$$\mu_0 = 2\,(n-1)^2\,\pi^2\,\gamma^2 + \left[\frac{1}{4} + \frac{1}{3}\,(n-1)^2\,\pi^2\right] + 0\left(\frac{1}{\gamma^2}\right).$$

It should be noted once again that for $n \geqq 2$, the eigenvalue is an increasing function of γ^2. Thus our conclusion previously found to hold for small ratios of tip mass to beam mass is now seen to be valid for very large ratios. The results for $n = 2$ are also plotted in Figure 3 and are seen to be reasonably good even for moderate values of γ^2.

Finally, for the sake of completeness, Equation (4.4) for high order modes has been evaluated for this special case for $n = 2$. One should not expect good agreement for the approximation in this case, but Figure 3 indicates that the results are not completely unreasonable.

Once μ_0 has been determined, the function u_{10} is fixed by Equation (5.3), and the second coefficient μ_1 in the asymptotic expansion can be found by evaluating Equation (4.7). This calculation leads to the results shown in Table 1 on this page.

Table 1

First mode				Second mode			
γ^2	α	β_{ASY}	β_{MV}	γ^2	α	β_{ASY}	β_{MV}
0	2	2·87	4·10	0	5	14·34	25·34
	4	4·95	5.45		10	26·67	33·32
	6	6·98	7·16		15	38·95	43·34
	7	7·99	8·10	0·25	5	17·21	22·72
0·25	2	2·64	3·25		10	31·33	33·59
	4	4·68	4·85		15	45·35	46·23
	6	6·70	6·70	0·50	5	20·40	23·49
0·50	2	2·53	2·93		10	36·50	36·74
	4	4·56	4·60		15	52·43	51·56
	6	6·57	6·46	1·0	5	26·44	25·46
0·75	2	2·46	2·65		10	46·32	42·92
	4	4·49	4·40		15	65·87	61·63
	6	6·50	6·30				

In this table β_{ASY} is calculated from the two-term asymptotic series, and

$$\beta_{MV} = \left\{\frac{1}{2}\left(\beta_l^2 + \beta_u^2\right)\right\}^{1/2}$$

where β_l^2 and β_u^2 are lower and upper bounds, respectively, for β^2, found by the methods of Section 6. An inspection of the table reveals that the asymptotic results are valuable even for fairly small values of α. In general, the larger the tip mass, and the lower the mode of vibration, the better will be the asymptotic approximation.

6. The uniform beam. Upper and lower bounds on the second eigenvalue

In the case of the uniform beam, a method previously used in [2] can also be applied to yield upper and lower bounds for the second eigenvalue for this case in which there is a tip mass. Since the technique has been explained in detail in the earlier reference, a brief outline only will be given here.

The Rayleigh quotient for the eigenvalue problem set by Equations (2.5) to (2.8) for the case of the uniform beam is

$$R(v) = \frac{D(v)}{H(v)} = \frac{D(v, v)}{H(v, v)}$$

where

$$D(u, v) = \int_0^1 v'' u'' \, ds + \frac{1}{2} \alpha^2 \int_0^1 v' u' (1 - s^2 + 2 \gamma^2) \, ds ,$$

$$H(u, v) = \int_0^1 v u \, ds + \gamma^2 v(1) u(1) .$$

The following minimum principle can be proved for the lowest eigenvalue; namely,

$$\beta^2 = \min_u R(u)$$

where u has a piecewise continuous second derivative and satisfies boundary conditions (2.6). The minimizing function is the first eigenfunction and satisfies Equations (2.7) and (2.8) as natural boundary conditions. Corresponding principles can be proved for the n^{th} eigenvalue; and, on the basis of these, the Rayleigh-Ritz method can be applied.

Thus, using the admissible function

$$u = a s^2 + b s^3 + c s^4$$

in the minimizing principle with a, b, c as free parameters leads to a cubic equation in the approximate frequencies. Each of these roots, in turn, is an upper bound to the three lowest eigenvalues.

Lower bounds can be found by a generalization of Southwell's method based on a minimum principle for the sum of the first n eigenvalues [8] which states that if the eigenvalues are ordered according to increasing magnitude, $\beta_1^2 \leqq \beta_2^2 \leqq \cdots \beta_n^2$, then

$$\sum_{i=1}^{n} \beta_i^2 = \min_{\phi_1, \ldots, \phi_n} [D(\phi_1, \phi_1) + \cdots + D(\phi_n, \phi_n)]$$

where the ϕ_i are admissible functions for the minimum principle also satisfying

$$H(\phi_i, \phi_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

Consequently,

$$\sum_{i=1}^{n} \beta_i^2 \geqq \min_{\phi_1, \ldots, \phi_n} \sum_{i=1}^{n} D_A(\phi_i, \phi_i) + \min_{\psi_1, \ldots, \psi_n} \sum_{i=1}^{n} D_B(\psi_i, \psi_i) \tag{6.1}$$

where

$$D_A(u, u) = \int_0^1 u''^2 \, ds \,, \quad D_B(u, u) = \frac{1}{2} \alpha^2 \int_0^1 u'^2 \, (1 - s^2 + 2\gamma^2) \, ds$$

and the ψ_i satisfy only the single boundary condition $\psi_i(0) = 0$, but are otherwise admissible functions fulfilling $H(\psi_i, \psi_j) = \delta_{ij}$.

It is easily shown that

$$\min_{\phi_i} \sum_{i=1}^{n} D_A(\phi_i, \phi_i) = \sum_{i=1}^{n} \mu_i^2$$

where μ_i^2 are the eigenvalues of the system

$$w'''' = \mu^2 w \,, \quad 0 < s < 1 \,,$$

$$w(0) = w'(0) = 0 \,,$$

$$w''(1) = 0 \,, \quad w'''(1) + \mu^2 \gamma^2 w(1) = 0 \,.$$

This problem corresponds to the non-rotating beam with tip mass; and, having constant coefficients, can be solved by direct methods. Clearly the eigenvalues μ_i^2 depend on γ but not on α. On the other hand, it can be shown that

$$\min_{\psi_i} \sum_{i=1}^{n} D_B(\psi_i, \psi_i) = \sum_{i=1}^{n} \nu_i^2$$

where ν_i^2 are the eigenvalues of the reduced problem, Equation (5.2), written in the form

$$[(1 - s^2 + 2\gamma^2) \, w']' + 2 \frac{\nu^2}{\alpha^2} w = 0 \,,$$

$$w(0) = 0 \,, \quad \lim_{s \to 1} (1 - s^2 + 2\gamma^2) \, w' - 2\gamma^2 \frac{\nu^2}{\alpha^2} w(1) = 0 \,.$$

Equation (6.1) then yields the lower bounds

$$\beta_1^2 \geqq \mu_1^2 + \nu_1^2 \,, \quad \beta_1^2 + \beta_2^2 \geqq \mu_1^2 + \mu_2^2 + \nu_1^2 + \nu_2^2$$

or

$$\beta_2^2 \geqq \mu_1^2 + \mu_2^2 + \nu_1^2 + \nu_2^2 - \bar{\beta}_1^2$$

where $\bar{\beta}_1^2$ represents an upper bound to β_1^2.

The results of evaluating these bounds for the second eigenvalue are shown in Figure 4. It should be noted that the bounds are rather close. Indeed, the ratio of the difference between the upper and the lower bounds for β_2 to the mean value is less than 6% for all values of α and γ shown. The dependence of the eigenvalue on γ^2 is clearly exhibited in these bounds. For the non-rotating beam, $\alpha^2 = 0$, the higher the tip mass the lower the frequency. As α^2 is increased, however, the situation is reversed. This cross-over in behavior takes place first for the higher values of γ^2.

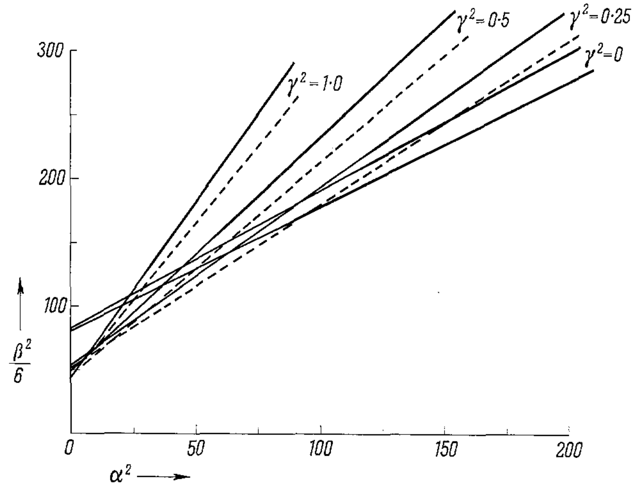

Figure 4
Frequency versus speed of rotation for several values of typ mass.

It should be noted that for $\gamma^2 > 0$, there is a value of α^2 beyond which the lower bound curve rises above the upper bound curve for $\gamma^2 = 0$. Hence this calculation proves that for sufficiently high rotational speeds the second eigenvalue is an increasing function of the tip mass to beam mass ratio.

7. Additional effects

Certain additional effects can be traced without too much difficulty. If, for example, the tip mass is of finite size, its rotary inertia can be analyzed in the following manner. Let $\varkappa$ be the radius of gyration of the tip mass about its own center of mass. Then the boundary condition which leads to Equation

(2.7) must be replaced by

$$EI\frac{\partial^2 y}{\partial x^2} + M\,\varkappa^2\frac{\partial^2}{\partial t^2}\left(\frac{\partial y}{\partial x}\right) = 0\,, \quad x = l\,.$$

If the same substitutions are used as before with the additional parameter $k = \varkappa/l$, Equation (2.7) should be replaced by

$$\eta^2\, e(1)\, v''(1) - \gamma^2\, k^2\, \mu\, v'(1) = 0\,. \tag{7.1}$$

The problem which is then to be examined by an asymptotic expansion consists of the differential equation (3.1) and boundary conditions (2.6), (3.2), and (7.1). The change in boundary conditions effects the analysis only in the last column of the eigenvalue determinant. The last column in the two determinants appearing in the expansion of Δ', Equation (3.8) becomes

$$\eta^2\sum \eta^n\, u''_{1n}(1)\, e(1) - \gamma^2\, k^2\, \mu \sum \eta^n\, u'_{1n}(1)\,,$$

$$\eta^2\sum \eta^n\, u''_{2n}(1)\, e(1) - \gamma^2\, k^2\, \mu \sum \eta^n\, u'_{2n}(1)\,,$$

$$e(1)\, h_3'^2(1)\, W(1) - e(1)\,\eta\,\{h_3''(1)\, W(1) + 2\, h_3'(1)\, W'(1)\} + \eta^2\, e(1)\, W''(1)$$
$$- \gamma^2\, k^2\, \mu \left\{-\frac{1}{\eta}\, h_3'(1)\, W(1) + W'(1)\right\}.$$

The dominant term in the determinants is now of order η^{-2} rather than η^{-3} as in the previous case. Consequently multiplication of Δ' by η^2 will change the determinant to one in which the entries contain non-negative powers of η. The first two entries of the third column remain the same, whereas the last element becomes

$$\eta\, e(1)\, h_3'^2(1)\, W(1) - e(1)\,\eta^2\,\{h_3''(1)\, W(1) + 2\, h_3'(1)\, W'(1)\} + \eta^3\, e(1)\, W''(1)$$
$$+ \gamma^2\, k^2\, \mu\, h_3'(1)\, W(1) - \eta\,\gamma^2\, k^2\, \mu\, W'(1)\,.$$

In the previous expansion of Δ' through the first two terms, the only contribution from the third column came from the term $h_3'^2(1)\, W_0(1)$ appearing as a factor. This is still true in the modified expansion except that this factor is replaced by $\gamma^2\, k^2\, \mu\, h_3'(1)\, W_0(1)$. Thus the functions $F_0(\mu)$ and $F_1(\mu)$ in Equation (3.9) become

$$F_0(\mu) = h_3'(0)\, V_0(0)\,\gamma^2\, k^2\, \mu\, h_3'(1)\, W_0(1)\,\{-\gamma^2\, u'_{10}(1,\mu) + \gamma^2\, \mu\, u_{10}(1,\mu)\}\,,$$

$$F_1(\mu) = V_0(0)\,\gamma^2\, k^2\, \mu\, h_3'(1)\, W_0(1)\,\{-\gamma^2\, u'_{20}(1,\mu_0) + \gamma^2\, \mu_0\, u_{20}(1,\mu_0)\}\,.$$

Since $\mu = 0$ is a trivial solution only, as can be seen from the Rayleigh quotient for example, μ_0 is unchanged. Furthermore the presence of μ in the multiplicative factor in $F_0(\mu)$ does not add an additional term to $\partial F_0(\mu_0)/\partial\mu$ since

$\{-\gamma^2 u'_{10}(1, \mu_0) + \gamma^2 \mu_0 u_{10}(1, \mu_0)\} = 0$. Consequently,

$$\mu_1 = -\frac{F_1(\mu_0)}{\left\{\frac{\partial F_0(\mu_0)}{\partial \mu}\right\}}$$

will also be unchanged.

The foregoing analysis thus shows that the rotational inertia of the tip mass does not affect the first two terms in the asymptotic expansion and consequently will not change the conclusions regarding general behavior drawn from these two terms.

8. Conclusion

Several methods for obtaining approximate solutions to the eigenvalue problem posed by the transverse vibrations of a rotating beam carrying a tip mass have been considered. These include asymptotic representations in terms of the rotational speed and upper and lower bound methods based on minimum principles. In the first case, explicit formula are given for the first two terms and the lowest eigenvalue is determined to within quadratures which depend only on the section properties. Higher frequencies have been explicitly calculated for the uniform beam as well as upper and lower bounds on the second frequency.

These results show the following behavior with regard to the question raised by Dr. KURT HOHENEMSER on the influence of tip mass. For high rotational speeds and general section properties, the lowest frequency decreases as the tip mass is increased; whereas, for sufficiently high modes, the frequency increases with the tip mass. In the case of the uniform beam, upper and lower bound analyses show that the latter statement is true even for the second mode.

9. Acknowledgment

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Zusammenfassung

Es werden die zur Rotationsebene normalen Biegeschwingungen eines rotierenden Stabes mit Endmasse studiert; insbesondere wird im Anschluss an eine frühere Arbeit, die sich auf die Grundschwingung beschränkte, der Einfluss der Endmasse bei grösseren Winkelgeschwindigkeiten auf die höheren Eigenschwingungen untersucht. Dabei werden Störungsmethoden und asymptotische Näherungen in Verbindung mit den klassischen Verfahren von RAYLEIGH-RITZ und SOUTHWELL angewendet. Für Stäbe veränderlichen Querschnittes wird nachgewiesen, dass für Eigenschwingungen genügend hoher Ordnung und hinreichend grosse Winkelgeschwindigkeiten die Eigenfrequenzen durch die Endmasse erhöht werden. Für Stäbe konstanten Querschnittes werden numerische Schranken angegeben, welche dieses Resultat bereits für die erste Oberschwingung beweisen. Dieses Verhalten steht im Gegensatz zu demjenigen der Grundschwingung, deren Eigenfrequenz durch eine Endmasse stets erniedrigt wird.

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Méthodes de calcul des milieux hétérogènes multiplicateurs de neutrons. – Application à un système de 5 sources de neutrons au plutonium-béryllium dans un modérateur de graphite[1])

Par PAUL GAVIN, CHARLES MANDRIN et BERNARD VITTOZ, Lausanne[2])

1. Introduction

L'assimilation d'un milieu hétérogène à un milieu homogène ne peut se faire que sous les conditions suivantes:

a) Le milieu hétérogène est finement divisé en cellules identiques.
b) Le libre parcours moyen des neutrons rapides et thermiques est au moins du même ordre de grandeur que les dimensions de la cellule.

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