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Intrinsic four-point properties

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Abstract. Many characterizations of euclidean spaces (real inner product spaces) among metric spaces have been based on euclidean four point embeddability properties. Related "intrinsic" four point properties have also been used to characterize euclidean or hyperbolic spaces among a suitable class of metric spaces. The present paper provides new characterizations of euclidean or hyperbolic spaces based on intrinsic four point properties which are related to known four point embedding properties.

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Introduction

The question of whether the (metric) embeddability of finite sets of points of a metric space into euclidean space will imply the embeddability of the entire metric space into euclidean space has a long history. In the late 1920s and early 1930s "distance" spaces were studied in which to every ordered pair of elements of the space, a non-negative real number was assigned. The researchers may have observed that an equivalent definition of a metric space to that emanating from the work of Fréchet would be to require that for each triple of labels of points of such a "distance" space there exists a one to one distance-preserving correspondence between that triple and some Euclidean triple, i.e., an equivalent definition of a metric space would be to require that triples of a distance space be congruently embeddable in Euclidean space. Similarly, a semi-metric space could be defined as a distance space in which all pairs of points are congruently embeddable in Euclidean space.

Hence it was reasonable that in 1932, Wilson [15] showed that, defining a space M to be (metrically) convex provided for all p, q in M, $p \neq q$, there exists an x in M such that $px + xq = pq$, $p \neq x \neq q$ (where juxtaposition of the letters denotes the distance of the points); and defining a space M to be externally convex provided for all p, q in M, $p \neq q$, there exists an r in M such that $pq + qr = pr$, $p \neq r \neq q$, then a complete, convex, externally convex metric space is congruent to a generalized euclidean space if and only if each quadruple of its points is congruently embeddable in euclidean space.

Thus the question arose, can a complete, convex, externally convex metric space be shown to be embeddable in euclidean space if a restricted set of quadruples are assumed embeddable. A result by Jordan and von Neumann [12] in 1935 gave an answer to that question in the narrower class of normed linear spaces. The metric space equivalent of what they showed was that in a metric space in which every quadruple of points that contains a linear triple, one point of which is the midpoint of the remaining two, was embeddable in a Euclidean space, then the metric space was a generalized euclidean space, i.e., a real inner product space.

In 1953 Blumenthal [5] presented a proof that in the environment of complete, convex, externally convex metric spaces, the embeddability of quadruples containing a linear triple was sufficient to imply that the space is congruent with a generalized euclidean space. Restated, using terminology that became common at that time, Blumenthal showed that the weak euclidean four point property was sufficient to show that a complete, convex, externally convex metric space was generalized euclidean. Using this terminology, we say that Jordan and von Neuman used the feeble euclidean four point property to show that normed linear spaces with that property are euclidean. It was later shown that the feeble euclidean four point property was also sufficient to imply that a complete, convex, externally convex metric space was generalized euclidean.

Some further investigations restricted those quadruples assumed to be embeddable in the euclidean space to quadruples containing an isosceles triple. Some of the definitions and results from [3,8] and earlier papers are collected here.

A metric quadruple p, q, r, s is said to be of type

$\mathbf{T_1}$ if and only if it contains a linear triple, i.e., a triple in which one distance is the sum of the other two. *(weak)*

$\mathbf{T_2}$ if and only if r is between q and s and $qr = rs$. *(feeble)*

$\mathbf{T_3}$ if and only if q, r, s are linear and $pq = ps$. *(isosceles weak)*

$\mathbf{T_4}$ if and only if r is between q and s and $qr = rs$, $pq = ps$. *(isosceles feeble)*

$\mathbf{T_5}$ if and only if r is between q and s while $qr = 2rs$, $pq = pr$. *(external isosceles feeble)*

$\mathbf{T_6}$ if and only if $qr = pr = pq$ and q, r, s are linear. *(equilateral weak)*

$\mathbf{T_7}$ if and only if $qr = pr = pq$ and s is between q and r and $qs = rs$. *(equilateral feeble)*

A metric space is said to have, respectively, the weak, feeble, isosceles weak, isosceles feeble, external isosceles feeble, equilateral weak, equilateral feeble euclidean four point property according as every quadruple of type $\mathbf{T_1}$, $\mathbf{T_2}$, $\mathbf{T_3}$,

$\mathbf{T_4}$, $\mathbf{T_5}$, $\mathbf{T_6}$, $\mathbf{T_7}$ is congruently embeddable in the (two dimensional) euclidean space.

Over a period of years it has been shown that in a normed linear space, the embeddability of quadruples of types $\mathbf{T_1}, \mathbf{T_2}, \mathbf{T_3}, \mathbf{T_4}, \mathbf{T_5}$ or $\mathbf{T_6}$ is sufficient to show that the space is generalized euclidean, whereas for complete, convex, externally convex metric spaces, embeddability of quadruples of types $\mathbf{T_1}, \mathbf{T_2}, \mathbf{T_3}$, or $\mathbf{T_5}$ is sufficient.

A counterexample by Kelly in [6] showed that, even in the environment of normed linear spaces, assuming the embeddability of quadruples of type $\mathbf{T_7}$ was not sufficient to imply that the normed linear space was generalized Euclidean and thus not sufficient in metric spaces.

In 1983 it was shown in [3] that the equilateral weak euclidean four point property was sufficient to imply that the space was generalized euclidean. That is, in a complete, convex, externally convex metric space, if every quadruple consisting of an equilateral triple and a fourth point linear with some two points of the triple is embeddable in Euclidean space, then the metric space is also embeddable, where three points are said to be *linear* provided one of the distances is the sum of the other two distances.

In 1982, papers [9] and [11] presented characterizations of generalized euclidean spaces among metric spaces and normed linear spaces, respectively, using quadruples that contain a linear triple but adding the restriction that some three of the six distances between the four points are equal. The particular properties that were considered in those papers are given below. Note that Property $\mathbf{P_2}$ is the equilateral weak euclidean four point property.

$\mathbf{P_1}$ If p, q, r, s are elements of M with $pq = pr = rs$ and if betweenness qrs holds, then p, q, r, s is congruently embeddable in E_2 (the euclidean plane).

$\mathbf{P_2}$ If p, q, r, s are elements of M with $pq = pr = qr$ and q, r, s are linear, then p, q, r, s is congruently embeddable in E_2.

$\mathbf{P_3}$ If p, q, r, s are elements of M with $pr = qr = sr$ and if betweenness qrs holds, then p, q, r, s is congruently embeddable in E_2.

$\mathbf{P_4}$ If p, q, r, s are elements of M with $pq = pr = sr$ and if rqs or rsq holds, then p, q, r, s is congruently embeddable in E_2.

Three results from [9] and [11] are:

> A complete, convex, externally convex metric space is a generalized euclidean space if and only if it satisfies one of the following three conditions:
>
> $\mathbf{P_1}$ and $\mathbf{P_2}$ or $\mathbf{P_1}$ and $\mathbf{P_3}$ or $\mathbf{P_1}$ and $\mathbf{P_4}$.

1. Intrinsic properties

While over the years, as we see above, many characterizations of real inner product spaces among metric spaces have been based on euclidean four point embedding properties [1] and [5], in 1941 Busemann [7] showed that a finitely compact metric space with a unique metric line joining each pair of distinct points was euclidean or hyperbolic if and only if the equidistant locus of each pair of points contained the metric segment joining the pair of points. As a consequence of this characterization, so-called intrinsic four point properties have been found to characterize euclidean and hyperbolic spaces among an appropriate class of metric spaces. In general a metric space is said to satisfy an intrinsic four point property provided that whenever two triples of points of the space are congruent (isometric), say (p, q, r) congruent to (p', q', r'), (denoted $p, q, r \cong p', q', r'$), this congruence (or isometry) can be extended to a congruence of quadruples (p, q, r, s) and (p', q', r', s') for some suitable choice of points s, s' in the space. Thus, for example, as points collinear with q and r, q' and r', respectively, as in [14], or as metric midpoints of the pairs q and r, q$'$ and r$'$ respectively as in [13]. In each of these instances, the intrinsic four point property under consideration (the congruence extension postulate in [14] or the intrinsic feeble four point property in [13]) was shown to characterize hyperbolic and euclidean spaces among finitely compact, convex, externally convex metric spaces in which the metric line joining two distinct points was unique. In [2], Andalafte generalized the results in [14] by restricting the triples (p, q, r) and (p', q', r') to isosceles triples.

In [3] a partial answer is given to some of the questions raised in [2]. In one characterization, s and s' are assumed to be the feet of p and p' on the metric lines determined by the other two points of each triple, i.e., points on those lines which are closest to p and p' respectively. In a second characterization in [3] it is shown that the congruent triples p, q, r and p', q', r' may be restricted to equilateral triples, with s on the line determined by q, r, and s' on the line determined by q', r', an analogy to the equilateral weak four point embedding property. A third characterization generalizes the external isosceles feeble four point property of [8].

The purpose of this paper is to find other intrinsic four point properties which will combine to establish that a finitely compact, convex, externally convex metric space with unique metric lines is euclidean or hyperbolic. Some of the intrinsic four point properties we shall discuss and investigate are:

Intrinsic Property $\mathbf{IP_1}$ If $p \notin L$, $q, r, s \in L$, $p' \notin L'$, $q', r', s' \in L'$, with $pq = pr = rs$, $p'q' = p'r' = r's'$ and with qrs and $q'r's'$ holding, if $p, q, r \cong p', q', r'$, (and hence $rs = r's'$,) then the congruence $p, q, r \cong p', q', r'$ can be extended to $p, q, r, s \cong p', q', r', s'$.

Intrinsic Property $\mathbf{IP_2}$ (also called the intrinsic equilateral weak four point property) If p, q, r and p', q', r' are congruent equilateral triples of points and if $s \in L(q, r)$ and $s' \in L'(q', r')$ satisfy $qs = q's'$ and $rs = r's'$ then the congruence $p, q, r \cong p', q', r'$ can be extended to a congruence $p, q, r, s \cong p', q', r', s'$.

It was shown in [4] that this characterizes euclidean or hyperbolic spaces.

Intrinsic Property $\mathbf{IP_3}$ If p, q, r and p', q', r' are congruent equilateral triples and $q, s, r \cong q', s', r'$ with qsr holding, then the congruence $p, q, r \cong p', q', r'$ can be extended to $p, q, r, s \cong p', q', r', s'$.

Intrinsic Property $\mathbf{IP_4}$ If $p \notin L$, $q, r, s \in L$, $p' \notin L'$, $q', r', s' \in L'$, with $pq = pr = rs$, $p'q' = p'r' = r's'$, and with $\sim qrs$ and $\sim q'r's'$ holding, if $p, q, r \cong p', q', r'$ (so that $rs = r's'$ and $qs = q's'$), then the congruence $p, q, r \cong p', q', r'$ can be extended to $p, q, r, s \cong p', q', r', s'$.

Note that each of these properties contains the restriction that three of the six distances of the quadruple must be equal. The purpose of this paper is to show how the intrinsic four point properties $\mathbf{IP_1}$, $\mathbf{IP_2}$, $\mathbf{IP_3}$, $\mathbf{IP_4}$ combine to imply that a finitely compact, convex, externally convex metric space with unique metric lines is euclidean or hyperbolic.

2. Intrinsic four point property $\mathbf{IP_1}$

In this section we shall explore the consequences of the Intrinsic Four Point Property $\mathbf{IP_1}$ in a complete, convex, externally convex metric space M in which every two points lie on exactly one metric line.

Theorem 2.1. *Given a metric line L, $p \notin L$, f_p a foot of p on L, $q, r \in L$ such that $pf_p = qf_p = rf_p$ and qf_pr, then $pq = pr$.*

Proof. If f_p is an endpoint of a segment of feet of p on L, let s_n, t_n be points of this segment of feet such that s_n approaches f_p and t_n approaches f_p with $s_n \neq t_n$ for any n. Apply $\mathbf{IP_1}$ to p, s_n, t_n, q_n and p, t_n, s_n, r_n, where r_n and q_n are chosen near r and q, respectively, so that $\mathbf{IP_1}$ applies. This leads to $pq_n = pr_n$ for each n and since $s_n, t_n \to f_p$ it follows that $q_n \to q$ and $r_n \to r$ and the result follows from the continuity of the metric.

In the event that f_p is not an endpoint of a segment of feet, then there exist sequences $\{s_n\}$, $\{t_n\}$ of L such that $s_nf_pt_n$ holds with $ps_n = pt_n$ for all n, with both sequences converging to f_p. Introducing q_n, r_n as above, one can again use $\mathbf{IP_1}$ and the continuity of the metric to prove $pq = pr$. □

Theorem 2.2. *Given L a metric line of M, $p \notin L$, with f_1, f_2 distinct feet of p on L, then letting q, r, denote the two points of L with distance pf_1 from f_1 and s, t denote the two points of L with distance pf_2 from f_2, then the points q, r, s, t all have the same distance from p.*

Proof. By Theorem 2.1, $pq = pr$ and $ps = pt$. Furthermore since $p, f_1, f_2 \cong p, f_2, f_1$ is isosceles, applying $\mathbf{IP_1}$ yields $pq = pt$ which proves the theorem. □

Theorem 2.3. *In a space M, if $p \notin L$ and there exist two feet f_1, f_2 of p on L, then $f_1f_2 \leq pf_1 = pf_2$.*

Proof. Supposing the contrary, there exist two feet f_1, f_2 of p such that $f_1f_2 > pf_1$. The set of feet of p on L is bounded, so without loss of generality, we may assume that these two feet have the largest distance of any two feet. Then the function $f_2t - pt$, for $t \in S(f_1, f_2)$ is positive at $t = f_1$ and negative at $t = f_2$. Therefore, there exists an $x \in S(f_1, f_2)$ such that $px = xf_2$. If $px = pf_1$, then let y denote the point of L such that yf_1x and $yf_1 = xf_2$. Therefore by $\mathbf{IP_1}$ since $p, x, f_1 \cong p, f_1, x$ with $xf_2 = yf_1$ on L, it follows that $py = pf_2$ and therefore y is a foot of p with $yf_2 > f_1f_2$, contrary to f_1, f_2 being feet of maximum distance. If $px > pf_1$, then, since the function $pt - px$ is a continuous function for $t \in L$ which is negative at $t = f_1$ and positive for $tx > 2px$ and tf_1f_2, it follows that there exists a $y \in L$ such that yf_1x, $py = px$ and $px = xf_2$. Therefore $p, y, x \cong p, x, y$ and hence by $\mathbf{IP_1}$, letting z be a point of L such that zyx and $zy = xf_2$, it follows that z is a foot of p, a contradiction, which proves the theorem. □

Theorem 2.4. *In a space M, the foot of a point p on a line L is unique.*

Proof. Supposing the contrary, let f_1, f_2 be points of L that are distinct feet of p. Then by Theorem 2.2, there exists q_1, r_1, s_1, t_1 of L such that $q_1s_1 = q_1f_2 - s_1f_2 = f_1f_2 = r_1t_1$ and $pq_1 = pr_1 = ps_1 = pt_1$. Since $f_1f_2 \leq pf_1$ by Theorem 2.3, $f_1r_1t_1$ and $f_1f_2r_1$ hold. Since $p, q_1, r_1 \cong p, r_1, q_1$ and choosing $q_2 \in L$ such that $q_2q_1f_1$ and $q_2q_1 = pq_1$, it follows by applying $\mathbf{IP_1}$ to p, q_1, r_1, q_2, there exists an $r_2 \in L$ such that $p, q_2, r_2 \cong p, r_2, q_2$. Similarly, $p, s_1, t_1 \cong p, t_1, s_1$ implies there exist $s_2, t_2 \in L$ such that $s_2s_1 = ps_1 = t_1t_2$. The labeling may be assumed so that the points occur on L in the order $q_2s_2q_1s_1f_1f_2r_1t_1r_2t_2$, and, as in Theorem 2.2, $pq_2 = pr_2 = ps_2 = pt_2$ with $r_1t_1 = r_2t_2 = q_1s_1 = q_2s_2$.

In general for arbitrary $n \geq 2$, there exists q_n, r_n, s_n, t_n points of L following the above pattern, where q_n and r_n are the points such that $q_nq_{n-1} = q_{n-1}p = r_nr_{n-1} = r_{n-1}p$ with $q_nq_{n-1}f_1$ and $f_2r_{n-1}r_n$ holding, and similarly for s_n, t_n. Thus it follows from $\mathbf{IP_1}$ that for each n, $pq_n = pr_n = ps_n = pt_n$, and that $t_{n-1}r_n = t_nt_{n-1} - f_1f_2$. Note also that $t_ir_i = f_1f_2$, for all $i \geq 2$. Then $pt_n - pr_1 = pt_n - pr_n + pr_n - pt_{n-1} + pt_{n-1} - pr_{n-1} + pr_{n-1} - pt_{n-2} + pt_{n-2} - pr_{n-2} + pr_{n-2} - pt_{n-3} + pt_{n-3} - pr_{n-3} + \cdots + pr_3 - pt_2 + pt_2 - pr_2 + pr_2 - pt_1$. Note the equality holds since $pr_1 = pt_1$. Hence $pt_n - pr_1 = 0 + pr_n - pt_{n-1} + 0 + pr_{n-1} - pt_{n-2} + 0 + pr_{n-2} - pt_{n-3} + 0 + \cdots + pr_3 - pt_2 + 0 + pr_2 - pt_1 < r_nt_{n-1} + r_{n-1}t_{n-2} + r_{n-2}t_{n-3} + \cdots + r_3t_2 + r_2t_1 = t_nt_{n-1} - f_1f_2 + t_{n-1}t_{n-2} - f_1f_2 + t_{n-2}t_{n-3} - f_1f_2 + \cdots + t_3t_2 - f_1f_2 + t_2t_1 - f_1f_2 = t_1t_n - (n-1)f_1f_2 = r_1t_n - (n-1)f_1f_2 - f_1f_2 = r_1t_n - (n)f_1f_2$. Since $t_nr_1 - pr_1 < pt_n$, then $t_nr_1 - 2pr_1 < pt_n - pr_1 < r_1t_n - nf_1f_2$. This implies $-2pr_1 < -(n)f_1f_2$, which fails for $n > \dfrac{2pr_1}{f_1f_2}$, resulting in a contradiction. □

Theorem 2.5. *In a space M, if p is a point not on a line L, then the distance px between p and a point $x \in L$ is monotone increasing as x recedes from a foot f of p on L. (monotone property)*

Proof. If the contrary is assumed, there exist points $q, r, s \in L$ which satisfy the betweenness qrs and for which $pq = pr = ps$. Observe that the continuity of the metric and the uniqueness of the foot of a point on a line (Theorem 2.4) imply that the foot of $x \in S(q,p) \cup S(p,s)$ varies continuously with x. Moreover, the feet of q and s, respectively, are q, s and, therefore, it follows that there is a point t between p and q or between p and s whose foot on L is r. But then, assuming betweenness ptq, we would have $pr < pt + tr < pt + tq = pq$, a contradiction to $pq = pr$. A similar contradiction results if pts holds, completing the proof. □

Theorem 2.6. *If L is a line of M, $p \notin L$, f the foot of p on L, then if fx_1x_2 holds, then $x_2y(x_2) < x_2y(x_1)$, where $y(x)$ is the point of L with distance px from x in the direction of f from x.*

Proof. By the monotone property $px_2 > px_1$ but $px_2 - px_1 < x_1x_2$. Therefore $x_2y(x_2) = px_2 < px_1 + x_1x_2 = x_1y(x_1) + x_1x_2 = x_2y(x_1)$, as was to be shown. □

Theorem 2.7. *If L is a line of M, $p \notin L$, then there exists a pair of points $s, t \in L$ such that the triple p, s, t is equilateral.*

Proof. Let f denote the foot of p on L and let q, r denote the points of L such that betweenness qfr holds and $qf = rf = pf$. Hence by Theorem 2.1, $pq = pr$. Then, letting q^*, r^* be points of L such that frr^* and q^*qf with $qq^* = rr^* = pq = pr$, by $\mathbf{IP_1}$, $pq^* = pr^*$. For $x \in S(f, r^*)$ denote by $g(x)$ the point of L lying in the direction of f from x such that $g(x)x = xp$. Consider the continuous function of x given by $g(x)p - xp$. At $x = f$, since $g(f) = q$, the function is positive. At $x = r^*$, we determine that $g(r^*) \in S(r^*, q)$ since $pr^* < pr + rr^* < pf + fr + rr^* = qr^*$ and hence by the monotone property $pg(r^*) < pr^*$, and hence the function $g(x)p - xp$ is negative at $x = r^*$. Therefore there exist points s and $t (= g(s))$ for which p, s, t is an equilateral triple, as required. □

In the next section, properties $\mathbf{IP_3}$ and $\mathbf{IP_4}$ will be introduced in addition to $\mathbf{IP_1}$ to establish further results leading to new characterization results for euclidean or hyperbolic spaces.

3. Intrinsic four point properties $\mathbf{IP_1}$, $\mathbf{IP_3}$, $\mathbf{IP_4}$

Throughout this section, let M^* denote a finitely compact, convex, externally convex metric space with unique metric lines.

Theorem 3.1. *In a space M^*, given an equilateral triple p, q, r with L the line determined by q, r, then the midpoint of q, r is the foot of p on L.*

Proof. Let x be any point in $S(q, r)$ other than the midpoint of q, r. Then since $p, q, r \cong p, r, q$ then by $\mathbf{IP_3}$, that congruence can be extended to a congruence $p, q, r, x \cong p, r, q, x'$ where $qx' = rx$, $qx = rx'$, $px = px'$. (The point x' is the reflection of x about the midpoint of q, r). It follows from the monotone property that the base xx' of every such isosceles triple pxx' must contain the foot f. The only point contained in every such segment is the midpoint of q, r. □

Theorem 3.2. *If L is a line of M^*, $p \notin L$, then there exists a pair of points $s, t \in L$, unique except for order, such that the triple p, s, t is equilateral.*

Proof. Given L a line and $p \notin L$, then by Theorem 2.7, there exists a pair of points $s, t \in L$ such that the triple p, s, t is equilateral. To show uniqueness, suppose p, q', r' is another equilateral triple. Theorem 3.1 implies that the foot f of p on L must be both the midpoint of q, r and the midpoint of q', r'. Without loss of generality we may assume betweenness $r'q'q$ (so that $q'r'r$ holds), which leads to $pq < pq' + q'q = qq' + q'r' = qr' < qr$, a contradiction. □

Theorem 3.3. *Given an equilateral triple p, q, r, if $x, x' \in S(q, r)$ such that $xf = x'f$, where f is the foot of p on $L(q, r)$, then $px = px'$.*

Proof. Since f is the midpoint of q, r by Theorem 3.1, then by appropriate labeling $qx = qr - rx = rx' = qr - qx'$. Since $x, x' \in S(q, r)$, then by $\mathbf{IP_3}$ the congruence $p, q, r \cong p, r, q$ can be extended so that $p, q, r, x \cong p, r, q, x'$, which proves the theorem. □

Theorem 3.4. *In a space M^* the Intrinsic property $\mathbf{IP_2}$ holds.*

Proof. The proof will be broken down into a number of steps.

1. Let p, q, r and p', q', r' be congruent equilateral triples of points and $s \in L(q, r)$, $s' \in L'(q', r')$ satisfying $qs = q's'$ and $rs = r's'$. It must be shown that the congruence $p, q, r \cong p', q', r'$ can be extended to a congruence $p, q, r, s \cong p', q', r', s'$, that is, $ps = p's'$.

Let f denote the foot of p on $L(q, r)$ and f' denote the foot of p' on $L'(q', r')$. Recall that, by Theorem 3.1, f is the midpoint of q, r and f' is the midpoint of q', r'. Define points u, u' in L, L', respectively, to be in the same direction from f as s, s' with $fu = f'u' = pf$. Note that $pf = p'f'$ follows immediately from an application of $\mathbf{IP_3}$ to equilateral triples p, q, r and p', q', r' together with f, f'. Let v, v' be the reflections of u, u' about f, f'. Similarly, let t, t' be the reflections of s, s' about f, f'. This basic setup will be used throughout the remainder of the proof.

In the event that qsr holds, $ps = p's'$ follows immediately by application of $\mathbf{IP_3}$. It is, therefore, sufficient to show that the extension is possible when qrs holds, which will be accomplished by examining the cases where rsu holds, $s = r$, and rus holds.

2. It will first be shown that the congruence may be extended in the event that rsu (and thus $r's'u'$) holds. Consider the function $zs - pz$ for $z \in S(q, r)$ and observe that this function is positive at $z = q$ (because $qs > qr = pq$) and negative at $z = r$ (because $pr > fp = fu > ru > rs$). Thus, there exists $x \in S(q, r)$ such that $xs = px$. The fact that $xs = px > pf = fu > fs$ implies the betweenness qxf. Choose $y \in S(f, r)$ so that Theorem 3.3 applies, which yields $py = px$. Since $p, x, y \cong p, y, x$ and $xs = yt$, the application of $\mathbf{IP_4}$ leads to $ps = pt$. Now, let $x', y' \in L'$ correspond to $x, y \in L$. It follows directly from $\mathbf{IP_3}$ that $p'x' = px = py = p'y'$, but x was chosen so that $xs = px$ (and $x's' = p'x'$), so that $xs = x's'$. Finally, as $px = p'x' = py = p'y' = xs = x's'$, application of $\mathbf{IP_4}$ yields $ps = p's'$, as desired.

3. If $s = u$ (so that $s' = u'$), choose a sequence of points $\{s_n\}$ such that rs_nu holds and $s_n \to u$. Letting $\{s'_n\}$ be the corresponding sequence in L' (so that $s'_n \to u'$), it follows from the continuity of the metric and the preceding argument that $ps = pt$ and $ps = p's'$.

4. In the remaining case both fus and $f'u's'$ hold. Relabel $s_1 = s$ and $s'_1 = s'$ and observe that there must exist a point x such that fxs_1 holds with $px = xs_1$. (The function $px - xs_1$ is positive at $x = s_1$ and negative at $x = f$.) Denote this point by s_2. If $s_2f > uf$ then by continuity of the metric, there exists a point x such that fxs_2 holds with $px = xs_2$. Denote this point s_3. This process can be continued, determining unique points $s_1, s_2, s_3, s_4, \ldots, s_n$ as long as $s_nf > uf$. Notice that $s_{n+1}s_n = ps_{n+1} > pf > 0$, so the process must terminate in a finite number of steps. Let s_{n+1} be the first member of the sequence such that $s_{n+1}f \leq uf = pf$. Note that s_{n+1} cannot be f for then s_n must be u and hence its distance from f is not greater than uf. Given the points $s_1, s_2, s_3, s_4, \ldots, s_n, u, s_{n+1}, f$ in the half-line of L determined by f, s_1, let $t_1, t_2, t_3, t_4, \ldots, t_n, v, t_{n+1}, f$ be a congruent set of points in the other half-line of L, i.e. $s_1, s_2, s_3, s_4, \ldots, s_n, u, s_{n+1}, f \cong t_1, t_2, t_3, t_4, \ldots, t_n, v, t_{n+1}, f$. Since $qs = q's'$ and $rs = r's'$ by hypothesis, and since by Theorem 3.1, f is the midpoint of q, r and f' is the midpoint of q', r', it follows that the congruence of q, f, r with q', f', r' can be extended to a congruence of L with L' with "primed" points of L' corresponding to "unprimed" points of L.

The point s_{n+1} must satisfy $fs_{n+1}r$, $s_{n+1} = r$, $rs_{n+1}u$, or $s_{n+1} = u$. These subcases will be treated below to complete the proof of the theorem.

a. Suppose $fs_{n+1}r$ holds. Property $\mathbf{IP_3}$ implies $ps_{n+1} = p's'_{n+1}$ and by Theorem 3.3, $pt_{n+1} = ps_{n+1}$. Now, by construction, $ps_{n+1} = s_{n+1}s_n$ and since $pt_{n+1} = ps_{n+1} = t_nt_{n+1}$, it follows from $\mathbf{IP_1}$ that $pt_n = ps_n$. Again, with $pt_n = ps_n = t_nt_{n-1} = s_ns_{n-1}$, it follows that $pt_{n-1} = ps_{n-1}$. Continuing this process eventually yields $pt_i = ps_i$ for all $i = 1, 2, \ldots, n, n+1$.

Since s_{n+1} and t_{n+1} are elements of $S(q, r)$, several applications of $\mathbf{IP_3}$ show that $p, q, r, s_{n+1}, t_{n+1} \cong p', q', r', s'_{n+1}, t'_{n+1}$. Then from $p's'_{n+1} = p't'_{n+1} = t'_nt'_{n+1} = s'_ns'_{n+1}$, it follows by $\mathbf{IP_1}$ that $ps_n = p't'_n = p's'_n$. Again from $t_{n-1}t_n = pt_n = ps_n = s_{n-1}s_n$ and $t'_{n-1}t'_n = p't'_n = p's'_n = s'_{n-1}s'_n$, it follows from $\mathbf{IP_1}$ that $p's'_{n-1} = ps_{n-1}$. Repeating this process leads to $ps_1 = p's'_1$, as desired.

b. Suppose $s_{n+1} = r$. We have the congruence $p, s_{n+1}, t_{n+1} \cong p', s'_{n+1}, t'_{n+1}$ since $s_{n+1} = r$ and $t_{n+1} = q$. The remainder of the argument follows as in the previous scenario.

c. Suppose $rs_{n+1}u$ holds. The identities $ps_{n+1} = p's'_{n+1}$ and $pt_{n+1} = ps_{n+1}$ follow from the argument given in Step 2. As before, $t_nt_{n+1} = s_ns_{n+1}$, so $\mathbf{IP_1}$ can be used to prove $ps_n = pt_n$ and, repeating the argument, that $ps_i = pt_i$ for all $i = 1, 2, \ldots, n-1$.

Incorporating the corresponding points on L', from $s'_{n+1}s'_n = s_{n+1}s_n = t_{n+1}$ $t_n = t'_{n+1}t'_n$ it follows by $\mathbf{IP_1}$ that $ps_n = p's'_n$. Using $p', s'_{n+1}, t'_{n+1} \cong p', t'_{n+1},$

s'_{n+1} and $s'_{n+1}s'_n = t'_{n+1}t'_n$ we conclude from $\mathbf{IP_1}$ that $pt_n = p't'_n$. Continuing this process we obtain $p's'_i = ps_i$, for $i = 1, 2, \ldots, n-1$ and thus $ps_1 = p's'_1$.

d. Suppose $s_{n+1} = u$. The identities $ps_{n+1} = p's'_{n+1}$ and $pt_{n+1} = ps_{n+1}$ follow from the argument given in Step 3. The remainder of the argument follows as in the previous scenario. □

4. Characterization theorems

Since it was shown in [4] that in a space M the Intrinsic property $\mathbf{IP_2}$ characterizes euclidean or hyperbolic space, Theorem 3.4 leads to the following result.

Theorem 4.1. *In a complete, convex, externally convex metric space with unique metric lines, Intrinsic Properties* $\mathbf{IP_1}$, $\mathbf{IP_3}$, $\mathbf{IP_4}$ *together characterize euclidean or hyperbolic space.*

We can define a stronger version of $\mathbf{IP_4}$ by replacing the restriction $rs = pq = pr$, by $rs \leq pq = pr$, namely: Intrinsic Property $\mathbf{IP_4^*}$: If $p \notin L, q, r, s \in L, p' \notin L', q', r', s' \in L'$, with $rs \leq pq = pr, p'q' = p'r'$, and with $\sim qrs$ and $\sim q'r's'$ holding, if $p, q, r \cong p', q', r'$ and both $qs = q's'$ and $rs = r's'$, then the congruence $p, q, r \cong p', q', r'$ can be extended to $p, q, r, s \cong p', q', r', s'$.

Then it is easily shown that a space M^*, that possesses property $\mathbf{IP_4^*}$ also possesses property $\mathbf{IP_3}$. Thus we have the additional characterization theorem below.

Theorem 4.2. *In a complete, convex, externally convex metric space with unique metric lines, properties* $\mathbf{IP_1}$ *and* $\mathbf{IP_4^*}$ *characterize euclidean or hyperbolic space.*

If $\mathbf{IP_4^*}$ is further strengthened by eliminating the restriction concerning qrs or $\sim qrs$ on L, we obtain Intrinsic Property $\mathbf{IP_4^{**}}$: If $p \notin L, q, r, s \in L, p' \notin L', q', r', s' \in L'$, with $rs \leq pq = pr, p'q' = p'r'$, if $p, q, r \cong p', q', r'$ and both $qs = q's'$ and $rs = r's'$ hold, then the congruence $p, q, r \cong p', q', r'$ can be extended to $p, q, r, s \cong p', q', r', s'$.

Then we observe that intrinsic property $\mathbf{IP_4^{**}}$ actually implies $\mathbf{IP_1}$ and we thus obtain our final theorem.

Theorem 4.3. *In a complete, convex, externally convex metric space with unique metric lines, intrinsic property* $\mathbf{IP_4^{**}}$ *characterizes euclidean or hyperbolic space.*

References

[1] Amir, D.: *Characterizations of inner product spaces*. BirkhŠuser Verlag, Basel (1986)

[2] Andalafte, E.Z.: *An intrinsic isosceles four point property which characterizes hyperbolic and euclidean spaces*. Demonstratio Math. **XV**, 507–514 (1982)

[3] Andalafte, E.Z., Freese, R.W.: *An equilateral four point property which characterizes generalized euclidean spaces.* J. Geom. **20**, 151–154 (1983)

[4] Andalafte, E.Z., Freese, R.W.: *New intrinsic four point properties which characterize hyperbolic and euclidean spaces.* J. Geom. **73**, 39–48 (2002)

[5] Blumenthal, L.M.: *Theory and applications of distance geometry.* Oxford University Press, Oxford (1953) (Reprinted New York (1970))

[6] Blumenthal, L.M.: Four point properties and norm postulates. In: Kelly, L.M. (eds.) The Geometry of Metric and Linear Spaces, Lecture Notes in Mathematics, vol. 490. Springer Verlag, Berlin (1974)

[7] Busemann, H.: *On Leibnitz's definition of planes.* Amer. J. Math. **63**, 101–111 (1941)

[8] Freese, R.W.: *Criteria for inner product spaces.* Proc. Amer. Math. Soc. **19**, 953–958 (1968)

[9] Freese, R.W., Andalafte, E.Z.: *A new class of four point properties which characterize euclidean spaces.* J. Geom. **18**, 43-53 (1982)

[10] Freese, R.W., Andalafte, E.Z.: *Metrization of orthogonality and characterizations of inner product spaces.* J. Geom. **39**, 28–37 (1990)

[11] Freese, R.W., Andalafte, E.Z., Diminnie, C.: *New four point properties which characterize inner product spaces.* Math. Japonica **27**, 253–261 (1982)

[12] Jordan, P., von Neumann, J.: *On inner products in linear metric spaces.* Ann. Math. **36**, 719–723 (1935)

[13] Valentine, J.E., Andalafte, E.Z.: *Intrinsic four point properties which characterize hyperbolic and euclidean spaces.* Bull. Acad. Polonaise **XXI**, 1103–1106 (1973)

[14] Valentine, J.E., Wayment, S.G.: *Metric characterizations of hyperbolic and euclidean spaces.* Colloq. Math. **XXV**, 259–264 (1972)

[15] Wilson, W.A.: *A relation between metric and euclidean spaces.* Amer. J. Math. **54**, 505–517 (1932)

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