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Reaction–advection random motions in inhomogeneous media

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Abstract

A probabilistic approach to reaction–advection systems with a finite propagation speed is developed. The convergence to traveling-wave solutions in inhomogeneous media is studied. The results are heavily based on the author's previous paper [Branching random motions, nonlinear hyperbolic systems and travelling waves, Preprint of Simón Bolívar University, Caracas, Venezuela, 2003, 32 pp., submitted for publication].

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1. Reaction–diffusion and reaction–advection systems: traveling waves in homogeneous media

1.1. Reaction–diffusion and reaction–advection systems: McKean's representation

The problem of wave front propagation for reaction–diffusion and reaction–advection models has attracted a widespread attention in recent years. This attention is increased by the multiple applications of these models in physics, chemical kinetics and biology (see, e.g. the reports by Fort and Mendez [4], Hadeler [7], Hillen and Othmer [8] and references therein). The simplest model uses the Fisher–Kolmogorov–Petrovskii–Piskunov equation [3,11]:

$$\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + \lambda(v^2 - v), \tag{1}$$

where λ is the reaction rate parameter. This reaction–diffusion equation is widely used to describe the spread of genes in a population [3,4,11], population growth, the spread of an epidemic [14], combustion waves [19], etc.

Beginning with pioneering work [11] it is known that Eq. (1) has monotone traveling-wave solutions $v = w(x-\alpha t)$, $0 \le w \le 1$ with velocities $\alpha \ge \alpha_* = \sqrt{2\lambda}$. Moreover a solution to (1) with an initial step-function converges to the traveling wave with minimal velocity α_*.

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The shortcoming of this model are evident and well known. Information in Eq. (1) propagates at infinite speed, which leads to lack of inertia, independence of increments of the underlying random motion and other nonphysical features. More realistic models are based on cutoff tricks [1,10], on integro-differential/integro-difference equations instead of Eq. (1) or on hyperbolic scaling (equivalently, on random walks with finite velocity, instead of diffusion process respecting to Eq. (1)) [4,6].

On the other hand there is a large literature on traveling fronts and scaling limits originated in kinetic models of the Boltzmann equation (see [2,9,20]). The other area where similar models arise is bioremediation of porous media [5,15,21,22] (in the case of several space dimensions see [18]).

Our treatment is concentrated on the so-called Cattaneo reaction–diffusion hyperbolic system (which is also called reaction–advection system, cf., e.g. [4,7,21]):

$$
\begin{aligned}
-\frac{\partial v_+}{\partial \tau} - \frac{\partial v_+}{\partial x} &= \mu_+(v_- - v_+) - \lambda_+ v_+ + \lambda_+ F_+(v_+, v_-),\\
-\frac{\partial v_-}{\partial \tau} + \frac{\partial v_-}{\partial x} &= \mu_-(v_+ - v_-) - \lambda_- v_- + \lambda_- F_-(v_+, v_-), \quad t > \tau
\end{aligned} \tag{2}
$$

with the terminal conditions:

$$
v_+|_{\tau\uparrow t} = v_-|_{\tau\uparrow t} = \theta(x). \tag{3}
$$

Here v_+ and v_- are the concentrations of reacting particles, the nonlinearities

$$
F_+(v_+, v_-) = \sum_{k+l\geq 2, k,l\geq 0} \beta^+_{kl} v_+^k v_-^l, \qquad F_-(v_+, v_-) = \sum_{k+l\geq 2, k,l\geq 0} \beta^-_{kl} v_+^k v_-^l
$$

represent reaction terms.

System (2) can be interpreted in the framework of branching random motions. First consider a particle, initially (at time τ) situated at a point $x \in (-\infty, \infty)$. At time τ the particle chooses an initial direction with equal probability and moves on a line $(-\infty, \infty)$ with unit velocity. Then it repeatedly takes an opposite direction at the random instants $T_1, T_2, \ldots$ The interarrival times $T_{n+1} - T_n, n = 1, 2, \ldots$ are independent and exponentially distributed (with rate $\mu_+ > 0$ for a forward moving particle, and with rate $\mu_- > 0$ for a backward moving one). Thus

$$
X(x, t) = x + \xi \int_0^t (-1)^{N(s)}\, \mathrm{d}s, \tag{4}
$$

where $\xi = \pm 1$ indicates the initial velocity and $N = N(t), t \leq 0$ is the Poisson process (with parameters $\mu_\pm$) independent of ξ. The state of the process at time t is defined by $(X(x, t), \sigma(t))$, where $X(x, t)$ is the particle's current position and $\sigma(t)$ the sign of its current velocity (positive or negative according with the number of turns that occurred).

The process $X = X(x, t)$ is called a telegraph process. Its distribution density $p(y - x, \tau, t)$ in the case $\mu_+ = \mu_- = \mu$ satisfies the telegraph equation:

$$
\frac{\partial^2 p}{\partial t^2} + 2\mu \frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2}. \tag{5}
$$

To introduce a reaction we consider a particle commencing the random motion X for an exponentially distributed holding time S independent of X. At S, the particle splits into a random number of pieces (offsprings). These new particles continue along independent paths of this random motion with the same parameters $\mu_\pm$ starting at $X(x, S)$, and are subject to the same splitting rule as the original particle. After an elapsed time $t - \tau$ we have $n = n(t - \tau)$ particles located at $X_1(x, t), \ldots, X_n(x, t)$, where $n(t - \tau)$ is stochastic.

In this context the reaction terms $F_+(v_+, v_-)$ and $F_-(v_+, v_-)$ are the probability generating functions of breeding rule; β^+_{kl} (β^-_{kl}) denote the probability of k forward and l backward moving offsprings of a particle, which has forward (backward) direction at a splitting time, and λ_+ and λ_- are the breeding rates of the forward and backward moving particle, respectively. System (2) can be derived by a standard renewal argument (see, e.g. [6] or [16]). Precisely speaking, for any function $g = g(x, \sigma, t), x \in (-\infty, \infty), \sigma = \pm, t \geq \tau$ the conditional expectations under the condition of a certain initial (at time τ) direction:

$$v_\pm(x, \tau, t) = \mathbb{E}_{\pm,\tau} g(X_1(x, t), \sigma_1(t), t), \dots, g(X_n(x, t), \sigma_n(t), t) \tag{6}$$

give a solution to a terminal-value problem for system (2) with terminal conditions $v_\pm|_{\tau\uparrow t} = g_\pm(x, t)$. Here and below $\sigma_k = \sigma_k(t), k = 1, \dots, n$ denote the directions of particles' velocities, $\mathbb{E}_{\pm,\tau}$ is the conditional expectation with respect to a certain initial velocity (+ for a forward and − for a backward initial direction at time τ), $g_\pm \equiv g(\cdot, \pm, \cdot)$.

Thus the conditional probabilities:

$$v_+(x, \tau, t) = \mathbb{P}_{+,\tau}(X_1(x, t) > 0, \dots, X_n(x, t) > 0), \tag{7}$$

$$v_-(x, \tau, t) = \mathbb{P}_{-,\tau}(X_1(x, t) > 0, \dots, X_n(x, t) > 0), \tag{8}$$

solve system (2) with the Heaviside terminal conditions

$$g_\pm = \theta(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Eqs. (7) and (8) give the so-called McKean's representation of the solution to terminal-value problem (2) and (3) (cf. [13]).

In this paper we will explore the backward equations of the type of Eq. (2). Certainly, it is possible to write down and research forward equations as well.

1.2. Rankine–Hugoniot jump condition and propagation of discontinuities

Taking aside the problem of uniqueness of solutions to terminal-value problem (2) and (3) (the existence follows from representation (7) and (8)) we concentrate now on the propagation of discontinuities of McKean's solution arising from the jump of the terminal condition given in Eq. (3). It is common for a system like Eq. (2), that its solution can only have discontinuities across the characteristics (Rankine–Hugoniot jump condition).

This property follows from the McKean's representation (7) and (8). Indeed, since the left-most particle must be in the interval $[x-(t_0-\tau), x+(t_0-\tau)]$, if the initial location at time $\tau < t_0$ is at point x then the McKean's solution to (2) and (3) is identically zero for $x \leq -(t_0 - \tau)$ and is identically one for $x > t_0 - \tau$. It is clear that the solution preserves the continuity of the terminal data. Also it must be strictly increasing in x inside the characteristic cone $-(t_0 - \tau) < x < t_0 - \tau$.

The values of jumps across the characteristics can be evaluated as follows. It is clear that $v_-(-t_0 + \tau + 0, \tau) = 0$ and $v_+(t_0 - \tau - 0, \tau) = 1$. To estimate $v_+(-t_0 + \tau + 0, \tau)$ consider the particle starting forwards at position $-(t_0 - \tau)$ at time τ. The only way that the left-most particle can be at position 0 at time t_0 is if there have been no turns of the particle or any of its descendants so far. In the particular case of splitting in two parts (i.e. the generating function F_+ is a quadratic form) it is clear that the probability of this event is

$$\begin{aligned} v_+(-t_0 + \tau + 0, \tau) &= \mathbb{P}_{+,(x-(t_0-\tau)+0,\tau)}(X_1(t_0) > 0, \dots, X_n(t_0) > 0) \\ &= \frac{\lambda_+ + \mu_+}{((1 - \beta^+_{20})\lambda_+ + \mu_+)\, \mathrm{e}^{(\lambda_+ + \mu_+)(t_0 - \tau)} + \beta^+_{20}\lambda_+}, \end{aligned}$$

where $\beta_{20}^{+} = F_+(1, 0)$ is the probability that both descendants of the forward moving particle move forwards as well.

To evaluate the jump of v_- at characteristics $x = t_0 - \tau$ we can consider the embedded birth and death process on this line (with $\delta = \mu_- + \lambda_-\beta_{20}^{-}$ as the intensity of death and with $\beta = \lambda_-\beta_{02}^{-}$ as the intensity of birth, again for a "splitting-in-two" case). In the other words the particle dies when it turns forward or both descendants at splitting time move forward. The only way that the left-most particle can be at position 0 at time t_0 is if the initial particle starts backwards and this embedded process has not become extinct. Thus,

$$v_-(t_0 - \tau - 0, \tau) = \frac{\delta(1 - \mathrm{e}^{(\beta-\delta)(t_0-\tau)})}{\delta - \beta\,\mathrm{e}^{(\beta-\delta)(t_0-\tau)}},$$

(cf. [12]).

1.3. Traveling waves in homogeneous media: convergence results

A traveling-wave solution to system (2) is defined by $v_\pm(x, \tau, t) = w_\pm(x - \alpha(t - \tau))$, where $\lim_{x\to-\infty} w_\pm = 0$, and $\lim_{x\to+\infty} w_\pm = 1$. It follows from Eq. (2) that $w_\pm$ form a solution to

$$\begin{aligned} -(1+\alpha)w'_+ &= \mu_+(w_- - w_+) - \lambda_+ w_+ + \lambda_+ F_+(w_+, w_-),\\ (1-\alpha)w'_- &= \mu_-(w_+ - w_-) - \lambda_- w_- + \lambda_- F_-(w_+, w_-). \end{aligned} \tag{9}$$

Our main objective is to study traveling-wave solutions to system (2) (and its generalizations) and prove convergence results. We apply the plan of McKean [13] to the hyperbolic system (2). This plan originally realized for the parabolic model (1) consists of the following three steps:

(1) by means of a Feynman–Kac formula, prove the convergence of suitably centered solution to a traveling front;
(2) study the stability properties of traveling fronts with respect to the velocity value;
(3) analytically identify the limits in step 1 as a traveling-wave solution.

Note that system (2) (as well as Eq. (9)) has two stationary solutions: $v_+ = v_- \equiv 0$ and $v_+ = v_- \equiv 1$. We assume the following.

(C1) There are no other stationary solutions of system (9), i.e. the algebraic system

$$\mu_+(y - x) + \lambda_+(F_+(x, y) - x) = 0, \qquad \mu_-(x - y) + \lambda_-(F_-(x, y) - y) = 0$$

has no solutions x, y, such that $0 \le x, y \le 1$, except $\{0, 0\}$ and $\{1, 1\}$.

To describe the second assumption we define the following expected numbers of particles born in each splitting instant:

$$J_{11} = \sum k\beta_{kl}^{+}, \qquad J_{12} = \sum l\beta_{kl}^{+}, \qquad J_{21} = \sum k\beta_{kl}^{-}, \qquad J_{22} = \sum l\beta_{kl}^{-}.$$

Note that the matrix

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial F_+}{\partial u_+} & \dfrac{\partial F_+}{\partial u_-} \\[2ex] \dfrac{\partial F_-}{\partial u_+} & \dfrac{\partial F_-}{\partial u_-} \end{pmatrix}_{|u_+=u_-=1}$$

represents the Jacobian of the nonlinearity $\{F_+(u_+, u_-), F_-(u_+, u_-)\}$ at $\{1, 1\}$.

Let

$$b_{11} = \mu_+ + \lambda_+(1 - J_{11}), \qquad b_{22} = \mu_- + \lambda_-(1 - J_{22}), \qquad b_{12} = \mu_+ + \lambda_+ J_{12},$$
$$b_{21} = \mu_- + \lambda_- J_{21}. \tag{10}$$

We assume that b_{ij}, $i, j = 1, 2$ satisfy the following condition.

(C2) $b_{11} + b_{22} < 2\sqrt{b_{12}b_{21}}$, $b_{22} > 0$.

Define

$$\alpha_* = \frac{b_{11}^2 - b_{22}^2 + 4\sqrt{b_{12}b_{21}(b_{12}b_{21} - b_{11}b_{22})}}{(b_{11} - b_{22})^2 + 4b_{12}b_{21}}.$$

From (C2) it follows that $0 < \alpha_* < 1$ [16].

Theorem 1. *If conditions C1 and C2 hold and $\alpha \geq \alpha_*$, then there exists one and, modulo translation, only one wave solution traveling with speed α.*

To prove this theorem it suffices to analyze the linearizations of system (2) at the stationary points $\{0, 0\}$ and $\{1, 1\}$. The first one is always unstable, while the second is a stable node if condition (C2) holds (see [16]).

It is interesting to understand how the steady-state front $w_\pm$ develops starting from a typical terminal condition $g_\pm$. Mathematically speaking, the question is in the asymptotic behavior of $v_\pm = v_\pm(x, \tau, t)$ as $\tau \downarrow -\infty$. The following theorem contains on the convergence of $v_\pm$ under suitable centering [16].

Theorem 2. *If C1 holds, then*

$$v_+(x + m_+, \tau, t) \to w_+(x), \qquad v_-(x + m_-, \tau, t) \to w_-(x) \tag{11}$$

as $\tau \downarrow -\infty$, $x, t \in (-\infty, \infty)$ with some centering terms $m_\pm = m_\pm(t - \tau)$, which are defined so as to satisfy

$$v_+(m_+, \tau, t) = v_-(m_-, \tau, t) = \tfrac{1}{2}. \tag{12}$$

Note that in this theorem we obtain the existence of limits (11) only. Regarding $w_\pm$ as a traveling wave is a separate problem (see Theorem 4).

Following McKean's steps we prove this theorem exploiting the Feynman–Kac representation for solutions of linear hyperbolic systems, which is interesting in itself.

Consider the terminal-value problem:

$$-\frac{\partial \bar{v}_+}{\partial \tau} - \frac{\partial \bar{v}_+}{\partial x} = \bar{\mu}_+(x, \tau)(\bar{v}_- - \bar{v}_+) + \bar{k}_+(x, \tau)\bar{v}_+,$$
$$-\frac{\partial \bar{v}_-}{\partial \tau} + \frac{\partial \bar{v}_-}{\partial x} = \bar{\mu}_-(x, \tau)(\bar{v}_+ - \bar{v}_-) + \bar{k}_-(x, \tau)\bar{v}_-, \qquad \tau < t,$$
$$\bar{v}_+|_{\tau\uparrow t} = \bar{g}_+(x, t), \qquad \bar{v}_-|_{\tau\uparrow t} = \bar{g}_-(x, t). \tag{13}$$

Here $\bar{k}_\pm = \bar{k}_\pm(x, \tau)$, $\bar{\mu}_\pm = \bar{\mu}_\pm(x, \tau)$, $\tau \leq t$, $x \in (-\infty, \infty)$ are functions with possible discontinuities concentrated on the characteristics $x = \pm(t - \tau)$, where $\bar{g}_\pm = \bar{g}_\pm(x, t)$ are bounded left-continuous in x and continuous in t, $t \geq \tau$ functions.

Let $\bar{X} = \bar{X}(t)$, $t \geq \tau$ be a random motion (4) with respect to parameters $\bar{\mu}_\pm$.

Theorem 3 (Feynman–Kac connection). *Let v_+, v_- form a solution to system (13), $\mathfrak{t}$, $\tau < \mathfrak{t} < t$ be a stopping time for $\bar{X}$. Then $\bar{v}_+$, $\bar{v}_-$ have the representation*:

$$\bar{v}_+(x,\tau) = \mathbb{E}_{+,(x,\tau)}\bar{v}(\bar{X}(\mathfrak{t}), \bar{\sigma}(\mathfrak{t}), \mathfrak{t}) \exp\left(\int_\tau^{\mathfrak{t}} \bar{k}(\bar{X}(s), \bar{\sigma}(s), s)\,\mathrm{d}s\right), \tag{14}$$

$$\bar{v}_-(x,\tau) = \mathbb{E}_{-,(x,\tau)}\bar{v}(\bar{X}(\mathfrak{t}), \bar{\sigma}(\mathfrak{t}), \mathfrak{t}) \exp\left(\int_\tau^{\mathfrak{t}} \bar{k}(\bar{X}(s), \bar{\sigma}(s), s)\,\mathrm{d}s\right). \tag{15}$$

As before we repeatedly unite by $h(\cdot, \sigma, \cdot)$, $\sigma = \pm$ both h_+ and h_- for all functions h of this type.

In the last step we must identify the limits in Eq. (11) as a traveling front. The following theorem supplies a sufficient condition [16].

Theorem 4. *If the limit*

$$\lim_{\tau\downarrow-\infty}(m_-(\tau) - m_+(\tau)) = \beta \tag{16}$$

exists, then

$$\lim_{\tau\downarrow-\infty}(-\dot{m}_+(\tau)) = \lim_{\tau\downarrow-\infty}(-\dot{m}_-(\tau)) = \alpha_*.$$

Furthermore, the limits in Eq. (11) *form a (biased) wave front. More precisely,* $\{w_+ = w_+(x), w_-^* = w_-(x-\beta)\}$ *(and* $\{w_+^* = w_+(x+\beta), w_- = w_-(x)\}$*) is a (modulo translation unique) wave solution traveling with the velocity* α_*.

Here $\dot{m}_+$ and $\dot{m}_-$ denote derivatives with respect to τ.

To prove this theorem we fix $t \in (-\infty, \infty)$ and consider functions $V_\pm$ and $V_\pm^*$ of the following form:

$$\begin{aligned} V_+(x,\tau) &= v_+(x+m_+, \tau, t), \qquad & V_-(x,\tau) &= v_-(x+m_-, \tau, t),\\ V_+^*(x,\tau) &= v_+(x+m_-, \tau, t), \qquad & V_-^*(x,\tau) &= v_-(x+m_+, \tau, t). \end{aligned}$$

Clearly, $V_+(0,\tau) = V_-(0,\tau) = 1/2$.

In these notations system (2) leads to

$$\begin{aligned} -\frac{\partial V_+}{\partial \tau} - (c - \dot{m}_+)\frac{\partial V_+}{\partial x} &= \mu_+(V_-^* - V_+) + \lambda_+(F_+(V_+, V_-^*) - V_+),\\ -\frac{\partial V_-}{\partial \tau} + (c + \dot{m}_-)\frac{\partial V_-}{\partial x} &= \mu_-(V_+^* - V_-) + \lambda_-(F_-(V_+^*, V_-) - V_-),\\ -\frac{\partial V_+^*}{\partial \tau} - (c - \dot{m}_-)\frac{\partial V_+^*}{\partial x} &= \mu_+(V_- - V_+^*) + \lambda_+(F_+(V_+^*, V_-) - V_+^*),\\ -\frac{\partial V_-^*}{\partial \tau} + (c + \dot{m}_+)\frac{\partial V_-^*}{\partial x} &= \mu_-(V_+ - V_-^*) + \lambda_-(F_-(V_+, V_-^*) - V_-^*). \end{aligned} \tag{17}$$

First note that by Eqs. (11) and (16) the following limits exist

$$\begin{aligned} &\lim_{\tau\downarrow-\infty} V_+(x,\tau) = w_+(x), \qquad &&\lim_{\tau\downarrow-\infty} V_-(x,\tau) = w_-(x),\\ &\lim_{\tau\downarrow-\infty} V_+^*(x,\tau) = w_+(x+\beta), \qquad &&\lim_{\tau\downarrow-\infty} V_-^*(x,\tau) = w_-(x-\beta). \end{aligned}$$

Integrating the first two equations of (17) in τ from $\tau-1$ to τ and in x from 0 to x and passing to the limit as $\tau\downarrow -\infty$ we obtain

$$-\left(c+a_+)(w_+(x)-\frac{1}{2}\right)=\int_0^x[\mu_+(w_-(x'-\beta)-w_+(x'))+\lambda_+(F_+(w_+(x'),w_-(x'-\beta))-w_+(x'))]\,\mathrm{d}x',$$
$$\left(c-a_-)(w_-(x)-\frac{1}{2}\right)=\int_0^x[\mu_-(w_+(x'+\beta)-w_-(x'))+\lambda_-(F_-(w_+(x'+\beta),w_-(x'))-w_-(x'))]\,\mathrm{d}x',$$

where $a_+=\lim_{\tau\downarrow-\infty}(-\dot{m}_+(\tau))$ and $a_-=\lim_{\tau\downarrow-\infty}(-\dot{m}_-(\tau))$. Similarly, from the last two equations of (17) it follows

$$\begin{aligned}&-(c+a_-)(w_+(x+\beta)-w_+(x_0+\beta))\\&\quad=\int_{x_0}^x[\mu_+(w_-(x')-w_+(x'+\beta))+\lambda_+(F_+(w_+(x'+\beta),w_-(x'))-w_+(x'+\beta))]\,\mathrm{d}x',\\&(c-a_+)(w_-(x-\beta)-w_-(x-\beta))\\&\quad=\int_{x_0}^x[\mu_-(w_+(x')-w_-(x'-\beta))+\lambda_-(F_-(w_+(x'),w_-(x'-\beta))-w_-(x'-\beta))]\,\mathrm{d}x'.\end{aligned}$$

Differentiating these two pairs of coupled equations we conclude that the pair $\{w_+(x),w_-(x-\beta)\}$ forms a traveling-wave solution with velocity a_+, and the pair $\{w_+(x+\beta),w_-(x)\}$ is a traveling wave with velocity a_-.

From Theorem 1 it follows

$$a_\pm\geq\alpha_*c.$$

On the other hand by comparison arguments and the exact formulas for solutions of linear version of Eq. (2) we can prove

$$a_\pm\leq\alpha_*c.$$

Therefore, $a_+=a_-=\alpha_*c$ and the theorem is proved.

Corollary 5 . *If conditions of* Theorems 1–3 *hold, then* $v_\pm(x,\tau,t)\to 0$ *as* $\tau\downarrow-\infty$ *uniformly in* x *from any bounded set.*

In general, the question whether the limit (16) really exists is still open. Nevertheless it is easy to check Eq. (16) at least for the isotropic case.

Proposition 6. *Let* $J_{11}=J_{22}=J_{12}=J_{21}$ *and* $\mu_+=\mu_-=\mu$, $\lambda_+=\lambda_-=\lambda$. *Then*

$$\lim_{\tau\downarrow-\infty}\psi(\tau)=\frac{2c}{2\mu+\lambda}. \tag{18}$$

Proof. Let T and S be the first turning and the first branching time, respectively. Denote $\xi=\min(T,S)$, $\psi(\tau)=m_-(\tau)-m_+(\tau)$. Note, that if $S<T$, then the system “loses its memory”. Conditioning we can obtain

$$\psi(\tau-\xi)=\mathbb{P}(S<T)\mathbb{E}(2c\xi|S<T)+\mathbb{P}(S>T)\mathbb{E}(2c\xi-\psi(\tau)|S<T)=2c\mathbb{E}\xi-\mathbb{P}(S>T)\psi(\tau).$$

Thus,

$$\int_0^\infty\psi(\tau-t)(\mu+\lambda)\,\mathrm{e}^{-(\mu+\lambda)t}\,\mathrm{d}t=\frac{2c}{\mu+\lambda}-\psi(\tau)\frac{\mu}{\mu+\lambda}.$$

The general solution of this equation is

$$\psi(\tau) = C\,\mathrm{e}^{\lambda(\mu+\lambda)\tau/\mu} + \frac{2c}{2\mu+\lambda},$$

which leads directly to Eq. (18). □

2. Traveling waves in inhomogeneous media

In this section we spread these results to inhomogeneous media. We consider the system (2) with an amplitude modulation:

$$\begin{aligned} -\frac{\partial u_+}{\partial \tau} - c(x)\frac{\partial u_+}{\partial x} &= \mu_+(u_- - u_+) - \lambda_+ u_+ + \lambda_+ F_+(u_+, u_-), \\ -\frac{\partial u_-}{\partial \tau} + c(x)\frac{\partial u_-}{\partial x} &= \mu_-(u_+ - u_-) - \lambda_- u_- + \lambda_- F_-(u_+, u_-), \quad t > \tau \end{aligned} \tag{19}$$

and with the same terminal conditions as before

$$u_+|_{\tau\uparrow t} = u_-|_{\tau\uparrow t} = \theta(x). \tag{20}$$

Here $c = c(x)$ is some positive continuous function.

Consider the particles which move with variable velocity $c = c(x)$, turning and splitting according with the previous rules. As before the solution to Eqs. (19) and (20) can be interpreted in the McKean's form

$$u_\pm(x, \tau, t) = \mathbb{P}_{\pm,\tau}(X_1(x, t) > 0, \dots, X_n(x, t) > 0).$$

Theorem 7. *Assume the conditions C1, C2 and Eq. (16) hold. Then*

(1) *If*

$$\int_0^\infty \frac{\mathrm{d}z}{c(z)} = \infty, \tag{21}$$

then

$$u_\pm(\sigma^{-1}(\sigma(x) + m_\pm), \tau, t) \to w_\pm(\sigma(x)) \tag{22}$$

as $\tau \downarrow -\infty$. Here $\sigma(x) = \int_0^x (\mathrm{d}z/c(z))$, σ^{-1} is the inverse function to σ, $m_\pm = m_\pm(t-\tau)$ are defined in Eq. (12) *and $w_\pm$ form a traveling front defined by* Eq. (11);

(2) *If*

$$\int_0^\infty \frac{\mathrm{d}z}{c(z)} < \infty \tag{23}$$

then

$$u_\pm(m(x, \tau, t), \tau, t) \to 0 \tag{24}$$

for any $m(x, \tau, t) \to +\infty$ as $\tau \downarrow -\infty$.

Proof. First note that if condition (21) holds, then the inverse σ^{-1} of function σ exists. It is clear that $u_\pm(x, \tau, t) = v_\pm(\sigma(x), \tau, t)$, where v_+, v_- resolve system (2). Thus $u_\pm(\sigma^{-1}(\sigma(x) + m_\pm), \tau, t) = v_\pm(\sigma(x) + m_\pm, \tau, t)$ and the convergence (22) follows from Theorems 2 and 4.

In the case Eq. (23), function $\sigma = \sigma(x)$ is bounded and hence by Corollary 5 $u_\pm(m(x, \tau, t), \tau, t) = v_\pm(\sigma(m(x, \tau, t)), \tau, t) \to 0$ as $\tau \downarrow -\infty$. □

Remark 8. Define the characteristics $\lambda(x, t)$ as the solution to the Cauchy problem:

$$\frac{d\lambda}{dt} = c(\lambda), \qquad \lambda|_{t=0} = x. \tag{25}$$

Note that $\lambda(x, t) = \sigma^{-1}(\sigma(x) + t)$. Therefore, the convergence (22) is equivalent to the following:

$$u_\pm(\lambda(x, m_\pm), \tau, t) \to w_\pm(\sigma(x)) \tag{26}$$

as $\tau \downarrow -\infty$.

Notice that inside the characteristic cone $\lambda(x, \tau - t) < y < \lambda(x, t - \tau)$ the functions $u_\pm = u_\pm(y, \tau, t)$ are continuous.

The following theorem gives some details to the particular case of the first part of Theorem 7.

Theorem 9. *Let $c(x) \to c_0 > 0$ as $x \to \infty$, such that*

$$\int_R^\infty \left| \frac{1}{c(x)} - \frac{1}{c_0} \right| dx \to 0 \tag{27}$$

as $R \to \infty$. Then

$$u_\pm(\omega(x) + c_0 m_\pm(t - \tau), \tau, t) \to w_\pm(\sigma(x)) \tag{28}$$

as $\tau \downarrow -\infty$ for some continuous function $\omega = \omega(x)$.

Proof (Cook's method [17]). We first prove that the following limit exists

$$\lim_{t\to\infty} (\lambda(x, t) - c_0 t) = \omega(x) \tag{29}$$

and it is continuous in x. By Eq. (25) we have

$$\lambda(x, t) - c_0 t = \int_0^t [c(\lambda(x, s)) - c_0]\, ds + x.$$

Thus the limit (29) exists, if

$$\lim_{T\to\infty} \int_T^\infty [c(\lambda(x, s)) - c_0]\, ds = 0.$$

This is true, because the latter integral equals to

$$\int_{\lambda(x,T)}^\infty |c(y) - c_0| \frac{dy}{c(y)} = c_0 \int_{\lambda(x,T)}^\infty \left| \frac{1}{c(y)} - \frac{1}{c_0} \right| dy,$$

which by (27) tends to 0 as $T \to \infty$ together with $\lambda(x, T) \to \infty$.

Thus the theorem follows from the continuity of functions $u_\pm$ inside the characteristic cone. □

Remark 10. In the terms of scattering theory $\omega(x) \equiv \Omega_-^{-1}x$, where Ω_- is a wave operator [17]. For example, in the case of local perturbation, i.e. $c(x) \equiv c_0$ for $|x| > R$, it is easy to calculate the size of shift $\omega(x)$. It has the form

$$\omega(x) = x + \begin{cases} R - x - c_0 \int_x^R \dfrac{\mathrm{d}z}{c(z)}, & |x| < R, \\ 0, & x > R, \\ 2R - c_0 \int_{-R}^R \dfrac{\mathrm{d}z}{c(z)}, & x < -R. \end{cases}$$

It is interesting to observe the pattern formation in the structured media, i.e. to study the asymptotics of solutions to system (19) for a suitably scaled velocity fields.

Definition 11. We call the velocity field $c = c(x)$ slow varying, if

$$\lim_{\varepsilon\downarrow 0} \sup_{-\infty<x<\infty} \left| \varepsilon \int_0^{x/\varepsilon} \frac{\mathrm{d}z}{c(z)} - \bar{\sigma}x \right| = 0. \tag{30}$$

Theorem 12. *Assume that* $c = c(x)$, $x \in (-\infty, \infty)$ *satisfies* Eq. (30). *Let* $u^\varepsilon = u^\varepsilon(x, \tau, t)$ *be a solution to system* (19) *with* $c^\varepsilon = c(x/\varepsilon)$ *instead of* c. *Then*

$$u_\pm^\varepsilon \left(x + \frac{m_\pm(t-\tau)}{\bar{\sigma}}, \tau, t \right) \to w_\pm(\bar{\sigma}x) \tag{31}$$

as $\varepsilon \downarrow 0$, $\tau \downarrow -\infty$.

Proof. First note that by Eq. (30)

$$\sigma^\varepsilon(x) \equiv \int_0^x \frac{\mathrm{d}z}{c^\varepsilon(z)} = \varepsilon \int_0^{x/\varepsilon} \frac{\mathrm{d}z}{c(z)} \to \bar{\sigma}x \tag{32}$$

uniformly in $x \in (-\infty, \infty)$ as $\varepsilon \downarrow 0$. Thus $\sigma^\varepsilon(x) + m_\pm(t-\tau)$ and $\sigma^\varepsilon(x + m_\pm(t-\tau)/\bar{\sigma})$ are inside the cone

$$K_\delta = \{(x, \tau) : -(\alpha_* + \delta)(t-\tau) < x < (\alpha_* + \delta)(t-\tau)$$

for some $\delta > 0$, $\alpha_* + \delta < 1$ (and for sufficiently large $|\tau|$).

It follows from Theorem 3 that inside this cone $v_\pm = v_\pm(x, \tau, t)$ are uniformly Lipschitz continuous:

$$|v_\pm(x, \tau, t) - v_\pm(x', \tau, t)| \le L|x - x'|$$

for any $x, x' \in K_\delta$ and $|\tau| \ge T$. Therefore:

$$\begin{aligned} &\left| u_\pm^\varepsilon(\lambda^\varepsilon(x, m_\pm(t-\tau)), \tau, t) - u_\pm^\varepsilon\left(x + \frac{m_\pm(t-\tau)}{\bar{\sigma}}, \tau, t \right) \right| \\ &\quad = \left| v_\pm(\sigma^\varepsilon(x) + m_\pm(t-\tau), \tau, t) - v_\pm\left(\sigma^\varepsilon\left(x + \frac{m_\pm(t-\tau)}{\bar{\sigma}} \right), \tau, t \right) \right| \\ &\quad \le L \left| \sigma^\varepsilon(x) + m_\pm(t-\tau) - \sigma^\varepsilon\left(x + \frac{m_\pm(t-\tau)}{\bar{\sigma}} \right) \right|. \end{aligned} \tag{33}$$

The right hand side goes to 0 as $\varepsilon \downarrow 0$ uniformly in x and τ.

By Eqs. (12) and (32):

$$u_{\pm}^{\varepsilon}(\lambda^{\varepsilon}(x, m_{\pm}(t-\tau)), \tau, t) - v_{\pm}(\sigma^{\varepsilon}(x) + m_{\pm}(t-\tau), \tau, t) \to w_{\pm}(\sigma^{\varepsilon}(x)) \quad (34)$$

as $\tau \downarrow -\infty$ uniformly in ε.

Finally, by continuity of $w_{\pm}$:

$$|w_{\pm}(\sigma^{\varepsilon}(x)) - w_{\pm}(\bar{\sigma}x)| \to 0 \quad (35)$$

as $\varepsilon \downarrow 0$. Combining (33)–(35) we have the assertion of the theorem. □

Remark 13. Note that the parameter $\bar{\sigma}\alpha_*^{-1}$ has the sense of effective wave speed. In the particular case of 1-periodic function $c = c(x)$, condition (30) holds with $\bar{\sigma} = \int_0^1 \mathrm{d}z/c(z)$. In the biological context the convergence in Eq. (31) describes the shape and the velocity of a traveling front in a series of cells.

References

[1] E. Brunet, B. Derrida, Shift in the velocity of a front due to a cutoff, Phys. Rev. E 56 (3) (1997) 2597–2604.
[2] R.E. Caflisch, G.C. Papanicolaou, The fluid dynamical limit of a nonlinear model Boltzmann equation, Commun. Pure Appl. Math. 32 (5) (1979) 589–616.
[3] R.A. Fisher, The advance of advantageous genes, Ann. Eugenics 7 (1937) 335–369.
[4] J. Fort, V. Mendez, Wavefronts in time-delayed reaction–diffusion systems. Theory and comparison to experiment, Rep. Prog. Phys. 65 (2002) 895–954.
[5] C.S. Simmons, T.R. Ginn, B.D. Wood, Stochastic-convective transport with nonlinear reaction: mathematical framework, Water Resour. Res. 31 (11) (1995) 2675–2688;
T.R. Ginn, C.S. Simmons, B.D. Wood, Stochastic-convective transport with nonlinear reaction: biodegradation with microbial growth, Water Resour. Res. 31 (11) (1995) 2689–2700.
[6] K.P. Hadeler, Nonlinear propagation in reaction transport systems, in: Differential Equations with Applications to Biology, Halifax, NS, 1997, pp. 251–257;
K.P. Hadeler, Fields Inst. Commun., vol. 21, Amer. Math. Soc., Providence, RI, 1999.
[7] K.P. Hadeler, Reaction transport systems in biological modelling in mathematics inspiring by biology, in: Lecture Notes in Mathematics 1714, Springer, Berlin, 1999, pp. 95–150.
[8] T. Hillen, H.G. Othmer, The diffusion limit of transport equations derived from velocity-jump processes, SIAM J. Appl. Math. 61 (3) (2000) 751–775;
H.G. Othmer, T. Hillen, The diffusion limit of transport equations. II. Chemotaxis equations, SIAM J. Appl. Math. 62 (4) (2002) 1222–1250.
[9] S. Kawashima, A. Matsumura, Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion, Commun. Math. Phys. 101 (1) (1985) 97–127.
[10] D.A. Kessler, Z. Ner, L.M. Sander, Front propagation: precursors, cutoffs, and structural stability, Phys. Rev. E 58 (1) (1998) 107–114.
[11] A. Kolmogorov, I. Petrovskii, N. Piskunov, Étude de l'équation de la diffusion avec croissance de la quantité de la matère et son apllication a un problème biologique, Moscow University, Bull. Math. 1 (1937) 1–25.
[12] O.D. Lyne, Travelling waves for a certain first-order coupled PDE system, Electron. J. Prob. 5 (14) (2000) 1–40.
[13] H.P. McKean, Application of Brownian motion to the equation of Kolmogorov–Petrovskii–Piskunov, Commun. Pure Appl. Math. 28 (1975) 323–331.
[14] J.D. Murray, Mathematical Biology, Springer, Berlin, 1989.
[15] S. Oya, A. Valocchi, Characterization of traveling waves and analytical estimation of pollutant removal in one-dimensional subsurface bioremediation modeling, Water Resour. Res. 33 (5) (1997) 1117–1127.
[16] N. Ratanov, Branching random motions, nonlinear hyperbolic systems and travelling waves, Preprint of Simón Bolívar University, Caracas, Venezuela, 2003, 32 pp., submitted for publication. http://www.cesma.usb.ve.
[17] M. Reed, B. Simon, Methods of Modern Mathematical Physics, III, Scattering Theory, Academic Press, New York, 1979.
[18] H. Schwetlick, Travelling fronts for multidimensional nonlinear transport equations, Ann. Inst. Henri Poincaré, Analyse Nonlinéaire 17 (4) (2000) 523–550.
[19] A.I. Volpert, V.A. Volpert, Vl.A. Volpert, Traveling wave solutions of parabolic systems, Translated from the Russian Manuscript by J.F. Heyda, Translations of Mathematical Monographs, 140, American Mathematical Society, Providence, RI, 1994, xii+448 pp.
[20] Z.P. Xin, The fluid-dynamic limit of the Broadwell model of the nonlinear Boltzmann equation in the presence of shocks, Commun. Pure Appl. Math. 44 (6) (1991) 679–713.
[21] J.X. Xin, J.M. Hyman, Stability, relaxation and oscillation of biodegradation fronts, SIAM J. Appl. Math. 61 (2) (2000) 472–505.
[22] J. Xin, D. Zhang, Stochastic analysis of biodegradation fronts in one-dimensional heterogeneous porous media, Adv. Water Res. 22 (2) (1998) 103–116.