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Chaos, Solitons and Fractals 22 (2004) 1039–1046

CHAOS
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Intuitionistic fuzzy metric spaces

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Accepted 24 February 2004

Abstract

Using the idea of intuitionistic fuzzy set due to Atanassov [Intuitionistic fuzzy sets. in: Sgurev V. (Ed.), VII ITKR's Session, Sofia June, 1983; Fuzzy Sets Syst. 20 (1986) 87], we define the notion of intuitionistic fuzzy metric spaces as a natural generalization of fuzzy metric spaces due to George and Veeramani [Fuzzy Sets Syst. 64 (1994) 395] and prove some known results of metric spaces including Baire's theorem and the Uniform limit theorem for intuitionistic fuzzy metric spaces.

1. Introduction

The concept of fuzzy topology may have very important applications in quantum particle physics particularly in connections with both string and $\epsilon^{(\infty)}$ theory which were given and studied by Elnaschie [8,9].

One of the most important problems in fuzzy topology is to obtain an appropriate concept of fuzzy metric space. This problem has been investigated by many authors [5,10,12,15,17] from different points of views. In particular, George and Veeramani [12] have introduced and studied a notion of fuzzy metric space with the help of continuous t-norms, which constitutes a slight but appealing modification of the one due to Kramosil and Michalek [17]. On the other hand, Atanassov [1] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets [24], and later there has been much progress in the study of intuitionistic fuzzy sets by many authors [1–7,13,15].

In this paper, using the idea of intuitionistic fuzzy sets, we define the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to George and Veeramani [12]. In Section 3, we define a Hausdorff topology on this intuitionistic fuzzy metric space and show that every metric induces an intuitionistic fuzzy metric. Further we introduce the notion of Cauchy sequences in an intuitionistic fuzzy metric space and prove the Baire's theorem [21] for intuitionistic fuzzy metric spaces. In Section 4, we are finding a necessary and sufficient condition for an intuitionistic fuzzy metric space to be complete and show that every separable intuitionistic fuzzy metric space is second countable and that every subspace of an intuitionistic fuzzy metric space is separable. Finally, we prove the Uniform limit theorem [20] for intuitionistic fuzzy metric spaces.

2. Intuitionistic fuzzy metric spaces

Definition 2.1 [22]. A binary operation $* : [0.1] \times [0, 1] \to [0, 1]$ is continuous t-norm if $*$ is satisfying the following conditions:

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doi:10.1016/j.chaos.2004.02.051

(a) $*$ is commutative and associative;
(b) $*$ is continuous;
(c) $a * 1 = a$ for all $a \in [0, 1]$;
(d) $a * b \leqslant c * d$ whenever $a \leqslant c$ and $c \leqslant d$, and $a, b, c, d \in [0, 1]$.

Definition 2.2 [22]. A binary operation $\diamondsuit : [0.1] \times [0, 1] \to [0, 1]$ is continuous t-conorm if $\diamondsuit$ is satisfying the following conditions:

(a) $\diamondsuit$ is commutative and associative;
(b) $\diamondsuit$ is continuous;
(c) $a \diamondsuit 0 = a$ for all $a \in [0, 1]$;
(d) $a \diamondsuit b \leqslant c \diamondsuit d$ whenever $a \leqslant c$ and $c \leqslant d$, and $a, b, c, d \in [0, 1]$.

Note 2.3. The concepts of triangular norms (t-norms) and triangular conorms (t-conorms) are known as the axiomatic skeletons that we use for characterizing fuzzy intersections and unions, respectively. These concepts were originally introduced by Menger [19] in his study of statistical metric spaces. Several examples for these concepts were proposed by many authors (see [6,7,11,14,16,22,23]).

Remark 2.4
(a) For any $r_1, r_2 \in (0, 1)$ with $r_1 > r_2$, there exist $r_3, r_4 \in (0, 1)$ such that $r_1 * r_3 \geqslant r_2$ and $r_1 \geqslant r_4 \diamondsuit r_2$.
(b) For any $r_5 \in (0, 1)$, there exist $r_6, r_7 \in (0, 1)$ such that $r_6 * r_6 \geqslant r_5$ and $r_7 \diamondsuit r_7 \leqslant r_5$.

Definition 2.5. A 5-tuple $(X, M, N, *, \diamondsuit)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\diamondsuit$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$, $s, t > 0$,

(a) $M(x, y, t) + N(x, y, t) \leqslant 1$;
(b) $M(x, y, t) > 0$;
(c) $M(x, y, t) = 1$ if and only if $x = y$;
(d) $M(x, y, t) = M(y, x, t)$;
(e) $M(x, y, t) * M(y, z, s) \leqslant M(x, z, t + s)$;
(f) $M(x, y, \cdot) : (0, \infty) \to (0, 1]$ is continuous;
(g) $N(x, y, t) > 0$;
(h) $N(x, y, t) = 0$ if and only if $x = y$;
(i) $N(x, y, t) = N(y, x, t)$;
(j) $N(x, y, t) \diamondsuit N(y, z, s) \geqslant N(x, z, t + s)$;
(k) $N(x, y, \cdot) : (0, \infty) \to (0, 1]$ is continuous.

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Remark 2.6. Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1 - M, *, \diamondsuit)$ such that t-norm $*$ and t-conorm $\diamondsuit$ are associated [18], i.e. $x \diamondsuit y = 1 - ((1 - x) * (1 - y))$ for any $x, y \in X$.

Remark 2.7. In intuitionistic fuzzy metric space X, $M(x, y, \cdot)$ is non-decreasing and $N(x, y, \cdot)$ is non-increasing for all $x, y \in X$.

Example 2.8 (*Induced intuitionistic fuzzy metric*). Let (X, d) be a metric space. Denote $a * b = ab$ and $a \diamondsuit b = \min\{1, a + b\}$ for all $a, b \in [0, 1]$ and let M_d and N_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$M_d(x, y, t) = \frac{ht^n}{ht^n + md(x, y)}, \quad N_d(x, y, t) = \frac{d(x, y)}{kt^n + md(x, y)}$$

for all $h, k, m, n \in \mathbf{R}^+$. Then $(X, M_d, N_d, *, \diamondsuit)$ is an intuitionistic fuzzy metric space.

Remark 2.9. Note the above example holds even with the t-norm $a * b = \min\{a, b\}$ and the t-conorm $a \diamondsuit b = \max\{a, b\}$ and hence (M, N) is an intuitionistic fuzzy metric with respect to any continuous t-norm and continuous t-conorm. In the above example by taking $h = k = m = n = 1$, we get

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}, \quad N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.$$

We call this intuitionistic fuzzy metric induced by a metric d the standard intuitionistic fuzzy metric.

Example 2.10. Let $X = \mathbf{N}$. Define $a * b = \max\{0, a + b - 1\}$ and $a \diamondsuit b = a + b - ab$ for all $a, b \in [0, 1]$ and let M and N be fuzzy sets on $X^2 \times (0, \infty)$ as follows:

$$M(x, y, t) = \begin{cases} \frac{x}{y} & \text{if } x \leqslant y, \\ \frac{y}{x} & \text{if } y \leqslant x, \end{cases} \quad N(x, y, t) = \begin{cases} \frac{y - x}{y} & \text{if } x \leqslant y, \\ \frac{x - y}{x} & \text{if } y \leqslant x \end{cases}$$

for all $x, y \in X$ and $t > 0$. Then $(X, M, N, *, \diamondsuit)$ is an intuitionistic fuzzy metric space.

Remark 2.11. Note that, in the above example, t-norm $*$ and t-conorm $\diamondsuit$ are not associated. And there exists no metric d on X satisfying

$$M(x, y, t) = \frac{t}{t + d(x, y)}, \quad N(x, y, t) = \frac{d(x, y)}{t + d(x, y)},$$

where $M(x, y, t)$ and $N(x, y, t)$ are as defined in above example. Also note that the above functions (M, N) is not an intuitionistic fuzzy metric with the t-norm and t-conorm defined as $a * b = \min\{a, b\}$ and $a \diamondsuit b = \max\{a, b\}$.

3. Topology induced by an intuitionistic fuzzy metric

Definition 3.1. Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space, and let $r \in (0, 1)$, $t > 0$ and $x \in X$. The set $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r, N(x, y, t) < r\}$ is called the open ball with center x and radius r with respect to t.

Theorem 3.2. *Every open ball $B(x, r, t)$ is an open set.*

Proof. Let $B(x, r, t)$ be an open ball with center x and radius r with respect to t. Let $y \in B(x, r, t)$. Then $M(x, y, t) > 1 - r$ and $N(x, y, t) < r$. Since $M(x, y, t) > 1 - r$, there exists $t_0 \in (0, t)$ such that $M(x, y, t_0) > 1 - r$ and $N(x, y, t_0) < r$. Put $r_0 = M(x, y, t_0)$. Since $r_0 > 1 - r$, there exists $s \in (0, 1)$ such that $r_0 > 1 - s > 1 - r$. Now for given r_0 and s such that $r_0 > 1 - s$, there exist $r_1, r_2 \in (0, 1)$ such that $r_0 * r_1 > 1 - s$ and $(1 - r_0) \diamondsuit (1 - r_2) \leqslant s$. Put $r_3 = \max\{r_1, r_2\}$ and consider the open ball $B(y, 1 - r_3, t - t_0)$. We claim $B(y, 1 - r_3, t - t_0) \subset B(x, r, t)$. Now, let $z \in B(y, 1 - r_3, t - t_0)$. Then $M(y, z, t - t_0) > r_3$ and $N(y, z, t - t_0) < r_3$. Therefore

$$M(x, z, t) \geqslant M(x, y, t_0) * M(y, z, t - t_0) \geqslant r_0 * r_3 \geqslant r_0 * r_1 \geqslant 1 - s > 1 - r$$

and

$$N(x, z, t) \leqslant N(x, y, t_0) \diamondsuit N(y, z, t - t_0) \leqslant (1 - r_0) \diamondsuit (1 - r_3) \leqslant (1 - r_0) \diamondsuit (1 - r_2) \leqslant s < r.$$

Thus $z \in B(x, r, t)$ and hence $B(y, 1 - r_3, t - t_0) \subset B(x, r, t)$. $\square$

Remark 3.3. Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. Define $\tau_{(M,N)} = \{A \subset X\colon$ for each $x \in A$, there exist $t > 0$ and $r \in (0, 1)$ such that $B(x, y, t) \subset A\}$. Then $\tau_{(M,N)}$ is a topology on X.

Remark 3.4

(a) From Theorem 3.2 and Remark 3.3, every intuitionistic fuzzy metric (N, M) on X generates a topology $\tau_{(M,N)}$ on X which has as a base the family of open sets of the form $\{B(x, r, t) : x \in X,\ r \in (0, 1),\ t > 0\}$.

(b) Since $\{B(x, \frac{1}{n}, \frac{1}{n}) : n = 1, 2, \ldots\}$ is a local base at x, the topology $\tau_{(M,N)}$ is first countable.

Theorem 3.5. *Every intuitionistic fuzzy metric space is Hausdorff.*

Proof. Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. Let x and y be two distinct points in X. Then $0 < M(x,y,t) < 1$ and $0 < N(x,y,t) < 1$. Put $r_1 = M(x,y,t)$, $r_2 = N(x,y,t)$ and $r = \max\{r_1, 1 - r_2\}$. For each $r_0 \in (r, 1)$, there exist r_3 and r_4 such that $r_3 * r_3 \geqslant r_0$ and $(1 - r_4)\diamondsuit(1 - r_4) \leqslant 1 - r_0$. Put $r_5 = \max\{r_3, r_4\}$ and consider the open balls $B(x, 1 - r_5, \frac{t}{2})$ and $B(y, 1 - r_5, \frac{t}{2})$. Then clearly $B(x, 1 - r_5, \frac{t}{2}) \cap B(y, 1 - r_5, \frac{t}{2}) = \phi$. For if there exists $z \in B(x, 1 - r_5, \frac{t}{2}) \cap B(y, 1 - r_5, \frac{t}{2})$, then

$$r_1 = M(x,y,t) \geqslant M\left(x,z,\frac{t}{2}\right) * M\left(z,y,\frac{t}{2}\right) \geqslant r_5 * r_5 \geqslant r_3 * r_3 \geqslant r_0 > r_1$$

and

$$r_2 = N(x,y,t) \leqslant N\left(x,z,\frac{t}{2}\right)\diamondsuit N\left(z,y,\frac{t}{2}\right) \leqslant (1 - r_5)\diamondsuit(1 - r_5) \leqslant (1 - r_4)\diamondsuit(1 - r_4) \leqslant 1 - r_0 < r_2,$$

which is a contradiction. Hence $(X, M, N, *, \diamondsuit)$ is Hausdorff. □

Remark 3.6. Let (X, d) be a metric space. Let

$$M(x,y,t) = \frac{t}{t + d(x,y)}, \quad N(x,y,t) = \frac{d(x,y)}{kt + d(x,y)}, \quad k \in \mathbf{R}^+$$

be the intuitionistic fuzzy metric defined on X. Then the topology τ_d induced by the metric d and the topology $\tau_{(M,N)}$ induced by the intuitionistic fuzzy metric (M, N) are the same.

Definition 3.7. Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. A subset A of X is said to be **IF**-bounded if there exist $t > 0$ and $r \in (0, 1)$ such that $M(x,y,t) > 1 - r$ and $N(x,y,t) < r$ for all $x, y \in A$.

Remark 3.8. Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space induced by a metric d on X. Then $A \subset X$ is IF-bounded if and only if it is bounded.

Theorem 3.9. *Every compact subset A of an intuitionistic fuzzy metric space $(X, M, N, *, \diamondsuit)$ is IF-bounded.*

Proof. Let A be a compact subset of an intuitionistic fuzzy metric space X. Fix $t > 0$ and $0 < r < 1$. Consider an open cover $\{B(x, r, t) : x \in A\}$ of A. Since A is compact, there exist $x_1, x_2, \ldots, x_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n} B(x_i, r, t)$. Let $x, y \in A$. Then $x \in B(x_i, r, t)$ and $y \in B(x_j, r, t)$ for some i, j. Thus we have $M(x, x_i, t) > 1 - r$, $N(x, x_i, t) < r$, $M(y, x_j, t) > 1 - r$ and $N(y, x_j, t) < r$. Now, let $\alpha = \min\{M(x_i, x_j, t) : 1 \leqslant i, j \leqslant n\}$ and $\beta = \max\{N(x_i, x_j, t) : i \leqslant i,\ j \leqslant n\}$. Then $\alpha > 0$ and $\beta > 0$. Now, we have

$$M(x,y,3t) \geqslant M(x,x_i,t) \ * \ M(x_i,x_j,t) \ * \ M(x_j,y,t) \geqslant (1 - r) \ * \ (1 - r) \ * \ \alpha > 1 - s_1 \quad \text{for some } 0 < s_1 < 1$$

and

$$N(x,y,3t) \leqslant N(x,x_i,t)\diamondsuit N(x_i,x_j,t)\diamondsuit N(x_j,y,t) \leqslant r\diamondsuit r\diamondsuit\beta < s_2 \quad \text{for some } 0 < s_2 < 1.$$

Taking $s = \max\{s_1, s_2\}$ and $t' = 3t$, we have $M(x,y,t') > 1 - s$ and $N(x,y,t') < s$ for all $x, y \in A$. Hence A is **IF**-bounded. □

From Remark 3.8 and Theorems 3.5 and 3.9 we have the following:

Remark 3.10. In an intuitionistic fuzzy metric space every compact set is closed and bounded.

Theorem 3.11. *Let $(X, M, N, *, \diamondsuit)$ be a fuzzy metric space and $\tau_{(M,N)}$ be the topology on X induced by the fuzzy metric. Then for a sequence $\{x_n\}$ in X, $x_n \to x$ if and only if $M(x_n, x, t) \to 1$ and $N(x_n, x, t) \to 0$ as $n \to \infty$.*

Proof. Fix $t > 0$. Suppose $x_n \to x$. Then for $r \in (0, 1)$, there exists $n_0 \in \mathbf{N}$ such that $x_n \in B(x, r, t)$ for all $n \geqslant n_0$. Then $1 - M(x_n, x, t) < r$ and $N(x_n, x, t) < r$ and hence $M(x_n, x, t) \to 1$ and $N(x_n, x, t) \to 0$ as $n \to \infty$.

Conversely, if for each $t > 0$, $M(x_n, x, t) \to 1$ and $N(x_n, x, t) \to 0$ as $n \to \infty$, then for $r \in (0,1)$, there exists $n_0 \in \mathbf{N}$ such that $1 - M(x_n, x, t) < r$ and $N(x_n, x, t) < r$ for all $n \geqslant n_0$. It follows that $M(x_n, x, t) > 1 - r$ and $N(x_n, x, t) < r$ for all $n \geqslant n_0$. Thus $x_n \in B(x, r, t)$ for all $n \geqslant n_0$ and hence $x_n \to x$. □

Definition 3.12. Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in X is said to be Cauchy if for each $\epsilon > 0$ and each $t > 0$, there exists $n_0 \in \mathbf{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ and $N(x_n, x_m, t) < \epsilon$ for all $n,\ m \geqslant n_0$.
(b) $(X, M, N, *, \diamondsuit)$ is called complete if every Cauchy sequence is convergent with respect to $\tau_{(M,N)}$.

Theorem 3.13. *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space such that every Cauchy sequence in X has a convergent subsequence. Then $(X, M, N, *, \diamondsuit)$ is complete.*

Proof. Let $\{x_n\}$ be a Cauchy sequence and let $\{x_{i_n}\}$ be a subsequence of $\{x_n\}$ that converges to x. We prove that $x_n \to x$. Let $t > 0$ and $\epsilon \in (0,1)$. Choose $r \in (0,1)$ such that $(1-r) * (1-r) \geqslant 1 - \epsilon$ and $r \diamondsuit r \leqslant \epsilon$. Since $\{x_n\}$ is Cauchy sequence, there is $n_0 \in \mathbf{N}$ such that $M(x_m, x_n, \frac{t}{2}) > 1 - r$ and $N(x_m, x_n, \frac{t}{2}) < r$ for all $m,\ n \geqslant n_0$. Since $x_{n_i} \to x$, there is positive integer i_p such that $i_p > n_0$, $M(x_{i_p}, x, \frac{t}{2}) > 1 - r$ and $N(x_{i_p}, x, \frac{t}{2}) < r$. Then, if $n \geqslant n_0$,

$$M(x_n, x, t) \geqslant M\left(x_n, x_{i_p}, \frac{t}{2}\right) * M\left(x_{i_p}, x, \frac{t}{2}\right) > (1-r) \ * \ (1-r) \geqslant 1 - \epsilon$$

and

$$N(x_n, x, t) \leqslant N\left(x_n, x_{i_p}, \frac{t}{2}\right) \diamondsuit N\left(x_{i_p}, x, \frac{t}{2}\right) < r \diamondsuit r \leqslant \epsilon.$$

Therefore $x_n \to x$ and hence $(X, M, N, *, \diamondsuit)$ is complete. □

Theorem 3.14. *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space and let A be a subset of X with the subspace intuitionistic fuzzy metric $(M_A, N_A) = (M|_{A^2 \times (0,\infty)}, N|_{A^2 \times (0,\infty)})$. Then $(A, M_A, N_A, *, \diamondsuit)$ is complete if and only if A is a closed subset of X.*

Proof. Suppose that A is a closed subset of X and let $\{x_n\}$ be a Cauchy sequence in $(A, M_A, N_A, *, \diamondsuit)$. Then $\{x_n\}$ is a Cauchy sequence in X and hence there is a point X in X such that $x_n \to x$. Then $x \in \overline{A} = A$ and thus $\{x_n\}$ converges in A. Hence $(A, M_A, N_A, *, \diamondsuit)$ is complete.

Conversely, suppose that $(A, M_A, N_A, *, \diamondsuit)$ is complete and A is not closed. Let $x \in \overline{A} \setminus A$. Then there is a sequence $\{x_n\}$ of points in A that converges to x and thus $\{x_n\}$ is a Cauchy sequence. Thus for each $0 < \epsilon < 1$ and each $t > 0$, there is $n_0 \in \mathbf{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ and $N(x_n, x_m, t) < \epsilon$ for all $n,\ m \geqslant n_0$. Since $\{x_n\}$ is a sequence in A, $M(x_n, x_m, t) = M_A(x_n, x_m, t)$ and $N(x_n, x_m, t) = N_A(x_n, x_m, t)$. Therefore $\{x_n\}$ is a Cauchy sequence in A. Since $(A, M_A, N_A, *, \diamondsuit)$ is complete, there is a $y \in A$ such that $x_n \to y$. That is, for each $0 < \epsilon < 1$ and each $t > 0$, there is $n_0 \in \mathbf{N}$ such that $M_A(y, x_n, t) > 1 - \epsilon$ and $N_A(y, x_n, t) < \epsilon$ for all $n \geqslant n_0$. But since $\{x_n\}$ is a sequence in A and $y \in A$, $M(y, x_n, t) = M_A(y, x_n, t)$ and $N(y, x_n, t) = N_A(y, x_n, t)$. Hence $\{x_n\}$ converges in $(X, M, N, *, \diamondsuit)$ to both x and y. Since $x \notin A$ and $y \in A$, $x \neq y$. This is a contradiction. □

Lemma 3.15. *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. If $t > 0$ and $r, s \in (0,1)$ such that $(1-s) * (1-s) \geqslant (1-r)$ and $s \diamondsuit s \leqslant r$, then $\overline{B(x, s, \frac{t}{2})} \subset B(x, r, t)$.*

Proof. Let $y \in \overline{B(x, s, \frac{t}{2})}$ and let $B(y, s, \frac{t}{2})$ be an open ball with center x and radius s. Since $B(y, s, \frac{t}{2}) \cap B(x, s, \frac{t}{2}) \neq \phi$, there is a $z \in B(y, s, \frac{t}{2}) \cap B(x, s, \frac{t}{2})$. Then we have

$$M(x, y, t) \geqslant M\left(x, z, \frac{t}{2}\right) * M\left(y, z, \frac{t}{2}\right) > (1-s) * \ (1-s) \geqslant 1 - r$$

and

$$N(x, y, t) \leqslant N\left(x, z, \frac{t}{2}\right) \diamondsuit N\left(y, z, \frac{t}{2}\right) < s \diamondsuit s \leqslant r.$$

Hence $z \in B(x, r, t)$ and thus $\overline{B(x, s, \frac{t}{2})} \subset B(x, r, t)$. □

Theorem 3.16. *A subset A of an intuitionistic fuzzy metric space $(X, M, N, *, \diamondsuit)$ is nowhere dense if and only if every nonempty open set in X contains an open ball whose closure is disjoint from A.*

Proof. Let U be a nonempty open subset of X. Then there exists a nonempty open set V such that $V \subset U$ and $V \cap \overline{A} \neq \phi$. Let $x \in V$. Then there exist $r \in (0,1)$ and $t > 0$ such that $B(x,r,t) \subset V$. Choose $s \in (0,1)$ such that $(1-s) * (1-s) \geqslant 1-r$ and $s \diamondsuit s \leqslant r$. By Lemma 3.15, $\overline{B(x,s,\frac{t}{2})} \subset B(x,r,t)$. Thus $B(x,s,\frac{t}{2}) \subset U$ and $\overline{B(x,s,\frac{t}{2})} \cap A = \phi$.

Conversely, suppose A is not nowhere dense. Then $\mathrm{int}(\overline{A}) \neq \phi$, so there exists a nonempty open set U such that $U \subset \overline{A}$. Let $B(x,r,t)$ be an open ball such that $B(x,r,t) \subset U$. Then $\overline{B(x,r,t)} \cap A \neq \phi$. This is a contradiction. □

Now, we shall prove Baire's theorem for intuitionistic fuzzy metric space.

Theorem 3.17 (Baire's theorem). *Let $\{U_n : n \in \mathbf{N}\}$ be a sequence of dense open subsets of a complete intuitionistic fuzzy metric space $(X, M, N, *, \diamondsuit)$. Then $\bigcap_{n\in\mathbf{N}} U_n$ is also dense in X.*

Proof. Let V be a nonempty open set of X. Since U_1 is dense in X, $V \cap U_1 \neq \phi$. Let $x_1 \in V \cap U_1$. Since $V \cap U_1$ is open, there exist $r_1 \in (0,1)$ and $t_1 > 0$ such that $B(x_1, r_1, t_1) \subset V \cap U_1$. Choose $r_1' < r_1$ and $t_1' = \min(t_1, 1)$ such that $\overline{B(x_1, r_1', t_1')} \subset V \cap U_1$. Since U_2 is dense in X, $B(x_1, r_1', t_1') \cap U_2 \neq \phi$. Let $x_2 \in B(x_1, r_1', t_1') \cap U_2$. Since $B(x_1, r_1', t_1') \cap U_2$ is open, there exist $r_2 \in (0, \frac{1}{2})$ and $t_2 > 0$ such that $B(x_2, r_2, t_2) \subset B(x_1, r_1', t_1') \cap U_2$. Choose $r_2' < r_2$ and $t_2' = \min\{t_2, \frac{1}{2}\}$ such that $\overline{B(x_2, r_2', t_2')} \subset B(x_1, r_1', t_1') \cap U_2$. Continuing in this manner, we obtain a sequence $\{x_n\}$ in X and a sequence $\{t_n'\}$ such that $0 < t_n' < \frac{1}{n}$ and

$$\overline{B(x_n, r_n', t_n')} \subset B(x_{n-1}, r_{n-1}', t_{n-1}') \cap U_n.$$

Now we claim that $\{x_n\}$ is a Cauchy sequence. For a given $t > 0$ and $\epsilon > 0$, choose $n_0 \in \mathbf{N}$ such that $\frac{1}{n_0} < t$ and $\frac{1}{n_0} < \epsilon$. Then for $n \geqslant n_0$ and $m \geqslant n$,

$$M(x_n, x_m, t) \geqslant M\left(x_n, x_m, \frac{1}{n}\right) \geqslant 1 - \frac{1}{n} > 1 - \epsilon, \quad N(x_n, x_m, t) \leqslant N\left(x_n, x_m, \frac{1}{n}\right) \leqslant \frac{1}{n} < \epsilon.$$

Therefore $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x \in X$ such that $x_n \to x$. Since $x_k \in B(x_n, r_n', t_n')$ for $k \geqslant n$, we obtain $x \in \overline{B(x_n, r_n', t_n')}$. Hence $x \in \overline{B(x_n, r_n', t_n')} \subset B(x_{n-1}, r_{n-1}', t_{n-1}') \cap U_n$ for all n. Therefore $V \cap (\bigcap_{n\in\mathbf{N}} U_n) \neq \phi$. Hence $\bigcap_{n\in\mathbf{N}} U_n$ is dense in X. □

Note 3.18. Since any complete intuitionistic fuzzy metric space cannot be represented as the union of a sequence of nowhere dense sets, it is not of the first category. Hence every complete intuitionistic fuzzy metric space is of the second category.

Remark 3.19. Since every metric induces an intuitionistic fuzzy metric and intuitionistic fuzzy metric is a generalization of fuzzy metric, Baire's theorem for complete metric space [21] and Baire's theorem for complete fuzzy metric space [12] are particular cases of the above theorem.

4. Some properties of complete intuitionistic fuzzy metric spaces

Definition 4.1. Let $(X, N, M, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. A collection $\{F_n\}_{n\in\mathbf{N}}$ is said to have intuitionistic fuzzy diameter zero if for each $r \in (0,1)$ and each $t > 0$, there exists $n_0 \in \mathbf{N}$ such that $M(x,y,t) > 1-r$ and $N(x,y,t) < r$ for all $x, y \in F_{n_0}$.

Remark 4.2. A nonempty subset F of an intuitionistic fuzzy metric space X has intuitionistic fuzzy diameter zero if and only if F is a singleton set.

Theorem 4.3. *An intuitionistic fuzzy metric space $(X, M, N, *, \diamondsuit)$ is complete if and only if every nested sequence $\{F_n\}_{n\in\mathbf{N}}$ of nonempty closed sets with intuitionistic fuzzy diameter zero have nonempty intersection.*

Proof. First suppose that the given condition is satisfied. We claim that $(X, M, N, *, \diamondsuit)$ is complete. Let $\{x_n\}$ be a Cauchy sequence in X. Set $B_n = \{x_k : k \geqslant n\}$ and $F_n = \overline{B_n}$, then we claim that $\{F_n\}$ has intuitionistic fuzzy diameter zero. For given $s \in (0,1)$ and $t > 0$, we choose $r \in (0,1)$ such that $(1-r) * (1-r) * (1-r) > 1-s$ and $r \diamondsuit r \diamondsuit r < s$. Since $\{x_n\}$ is Cauchy sequence, there exists $n_0 \in \mathbf{N}$ such that $M\big(x_n, x_m, \frac{1}{3}t\big) > 1-r$ and $N\big(x_n, x_m, \frac{1}{3}t\big) < r$ for all m, $n \geqslant n_0$. Therefore, $M\big(x, y, \frac{1}{3}t\big) > 1-r$ and $N\big(x, y, \frac{1}{3}t\big) < r$ for all $x, y \in B_{n_0}$.

Let $x, y \in F_{n_0}$. Then there exist sequences $\{x'_n\}$ and $\{y'_n\}$ in B_{n_0} such that $x'_n \to x$ and $y'_n \to y$. Hence $x'_n \in B(x, r, \frac{t}{3})$ and $y'_n \in B(y, r, \frac{t}{3})$ for sufficiently large n. Now we have

$$M(x,y,t) \geqslant M\left(x, x'_n, \frac{t}{3}\right) * M\left(x'_n, y'_n, \frac{t}{3}\right) * M\left(y'_n, y, \frac{t}{3}\right) > (1-r) * (1-r) * (1-r) > 1-s$$

and

$$N(x,y,t) \leqslant N\left(x, x'_n, \frac{t}{3}\right) \diamondsuit N\left(x'_n, y'_n, \frac{t}{3}\right) \diamondsuit N\left(y'_n, y, \frac{t}{3}\right) < r \diamondsuit r \diamondsuit r < s.$$

Therefore, $M(x,y,t) > 1-s$ and $N(x,y,t) < s$ for all $x, y \in F_{n_0}$. Thus $\{F_n\}$ has intuitionistic fuzzy diameter zero and hence by hypothesis $\bigcap_{n \in \mathbf{N}} F_n$ is nonempty.

Take $x \in \bigcap_{n \in \mathbf{N}} F_n$. We show that $x_n \to x$. Then, for $r \in (0,1)$ and $t > 0$, there exists $n_1 \in \mathbf{N}$ such that $M(x_n, x, t) > 1 - r$ and $N(x_n, x, t) < r$ for all $n \geqslant n_1$. Therefore, for each $t > 0$, $M(x_n, x, t) \to 1$ and $N(x_n, x, t) \to 0$ as $n \to \infty$ and hence $x_n \to x$. Therefore, $(X, M, N, *, \diamondsuit)$ is complete.

Conversely, suppose that $(X, M, N, *, \diamondsuit)$ is complete and $\{F_n\}_{n \in \mathbf{N}}$ is nested sequence of nonempty closed sets with intuitionistic fuzzy diameter zero. For each $n \in \mathbf{N}$, choose a point $x_n \in F_n$. We claim that $\{x_n\}$ is a Cauchy sequence. Since $\{F_n\}$ has intuitionistic fuzzy diameter zero, for $t > 0$ and $r \in (0,1)$, there exists $n_0 \in \mathbf{N}$ such that $M(x,y,t) > 1-r$ and $N(x,y,t) < r$ for all $x, y \in F_{n_0}$. Since $\{F_n\}$ is nested sequence, $M(x_n, x_m, t) > 1-r$ and $N(x_n, x_m, t) < r$ for all $n,\ m \geqslant n_0$. Hence $\{x_n\}$ is a Cauchy sequence. Since $(X, M, N, *, \diamondsuit)$ is complete, $x_n \to x$ for some $x \in X$. Therefore, $x \in \overline{F_n} = F_n$ for every n, and hence $x \in \bigcap_{n \in \mathbf{N}} F_n$. This completes our proof. □

Remark 4.4. The element $x \in \bigcap_{n \in \mathbf{N}} F_n$ is unique. For if there are two elements $x, y \in \bigcap_{n \in \mathbf{N}} F_n$, since $\{F_n\}$ has intuitionistic fuzzy diameter zero, for each fixed $t > 0$, $M(x,y,t) > 1 - \frac{1}{n}$ and $N(x,y,t) < \frac{1}{n}$ for each $n \in \mathbf{N}$. This implies $M(x,y,t) = 1$ and $N(x,y,t) = 0$ and hence $x = y$.

Note that the topologies induced by the standard intuitionistic fuzzy metric and the corresponding metric are the same. So we have the following:

Corollary 4.5. *A metric space (X, d) is complete if and only if every nested sequence $\{F_n\}_{n \in \mathbf{N}}$ of nonempty closed sets with diameter tending to zero have nonempty intersection.*

Theorem 4.6. *Every separable intuitionistic fuzzy metric space is second countable.*

Proof. Let $(X, M, N, *, \diamondsuit)$ be the given separable intuitionistic fuzzy metric space. Let $A = \{x_n : n \in \mathbf{N}\}$ be a countable dense subset of X. Consider the family $\mathscr{B} = \{B(x_j, \frac{1}{k}, \frac{1}{k}) : j, k \in \mathbf{N}\}$. Then $\mathscr{B}$ is countable. We claim that $\mathscr{B}$ is a base for the family of all open sets in X. Let U be any open set in X and let $x \in U$. Then there exist $t > 0$ and $r \in (0,1)$ such that $B(x, r, t) \subset U$. Since $r \in (0,1)$, we can choose a $s \in (0,1)$ such that $(1-s) * (1-s) > 1-r$ and $s \diamondsuit s < r$. Take $m \in \mathbf{N}$ such that $\frac{1}{m} < \min\{s, \frac{t}{2}\}$. Since A is dense in X, there exists $x_j \in A$ such that $x_j \in B(x, \frac{1}{m}, \frac{1}{m})$. Now, if $y \in B(x_j, \frac{1}{m}, \frac{1}{m})$, then

$$\begin{aligned} M(x,y,t) &\geqslant M\left(x, x_j, \frac{t}{2}\right) * M\left(y, x_j, \frac{t}{2}\right) \geqslant M\left(x, x_j, \frac{1}{m}\right) * M\left(y, x_j, \frac{1}{m}\right) \geqslant \left(1 - \frac{1}{m}\right) * \left(1 - \frac{1}{m}\right) \\ &\geqslant (1-s) * (1-s) > 1-r \end{aligned}$$

and

$$N(x,y,t) \leqslant N\left(x, x_j, \frac{t}{2}\right) \diamondsuit N\left(y, x_j, \frac{t}{2}\right) \leqslant N\left(x, x_j, \frac{1}{m}\right) \diamondsuit N\left(y, x_j, \frac{1}{m}\right) \leqslant \frac{1}{m} \diamondsuit \frac{1}{m} \leqslant s \diamondsuit s < r.$$

Thus, $y \in B(x, r, t) \subset U$ and hence $\mathscr{B}$ is a base. □

Remark 4.7. Since second countability is hereditary property and second countability implies separability, we obtain the following: Every subspace of a separable intuitionistic fuzzy metric space is separable.

Definition 4.8. Let X be any nonempty set and $(Y, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. Then a sequence $\{f_n\}$ of functions from X to Y is said to converge uniformly to a function f from X to Y if given $t > 0$ and $r \in (0,1)$, there exists $n_0 \in \mathbf{N}$ such that $M(f_n(x), f(x), t) > 1 - r$ and $N(f_n(x), f(x), t) < r$ for all $n \geqslant n_0$ and for all $x \in X$.

Theorem 4.9 (Uniform limit theorem). *Let $f_n : X \to Y$ be a sequence of continuous functions from a topological space X to an intuitionistic fuzzy metric space $(Y, M, N, *, \diamondsuit)$. If $\{f_n\}$ converges uniformly to $f : X \to Y$, then f is continuous.*

Proof. Let V be open set of Y and let $x_0 \in f^{-1}(V)$. We wish to find a neighborhood U of x_0 such that $f(U) \subset V$. Since V is open, there exist $t > 0$ and $r \in (0,1)$ such that $B(f(x_0), r, t) \subset V$. Since $r \in (0,1)$, we choose a $s \in (0,1)$ such that $(1-s) * (1-s) * (1-s) > 1-r$ and $s \diamondsuit s \diamondsuit s < r$. Since $\{f_n\}$ converges uniformly to f, given $t > 0$ and $s \in (0,1)$, there exists $n_0 \in \mathbf{N}$ such that $M\left(f_n(x), f(x), \frac{t}{3}\right) > 1-s$ and $N\left(f_n(x), f(x), \frac{t}{3}\right) < s$ for all $n \geqslant n_0$ and for all $x \in X$. Since f_n continuous for all $n \in \mathbf{N}$, there exists a neighborhood U of x_0 such that $f_n(U) \subset B(f_n(x_0), s, \frac{t}{3})$. Hence $M\left(f_n(x), f_n(x_0), \frac{t}{3}\right) > 1-s$ and $N\left(f_n(x), f_n(x_0), \frac{t}{3}\right) < s$ for all $x \in U$. Now

$$M(f(x), f(x_0), t) \geqslant M\left(f(x), f_n(x), \frac{t}{3}\right) * M\left(f_n(x), f_n(x_0), \frac{t}{3}\right) * M\left(f_n(x_0), f(x_0), \frac{t}{3}\right)$$
$$\geqslant (1-s) * (1-s) * (1-s) > 1-r$$

and

$$N(f(x), f(x_0), t) \leqslant N\left(f(x), f_n(x), \frac{t}{3}\right) \diamondsuit N\left(f_n(x), f_n(x_0), \frac{t}{3}\right) \diamondsuit N\left(f_n(x_0), f(x_0), \frac{t}{3}\right) \leqslant s \diamondsuit s \diamondsuit s < r.$$

Thus, $f(x) \in B(f(x_0), r, t) \subset V$ for all $x \in U$. Hence $f(U) \subset V$ and so f is continuous. □

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