Analytical Methods for Calculating Continuous Symmetry Measures and the Chirality Measure

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Abstract: We provide analytical solutions of the Continuous Symmetry Measure (CSM) equation for several symmetry point-groups, and for the associated Continuous Chirality Measure (CCM), which are quantitative estimates of the degree of a symmetry-point group or chirality in a structure, respectively. We do it by solving analytically the problem of finding the minimal distance between the original structure and the result obtained by operating on it all of the operations of a specific $\boldsymbol{G}$ symmetry point group. Specifically, we provide solutions for the symmetry measures of all of the improper rotations point group symmetries, $\boldsymbol{S_n}$, including the mirror (S_1, C_S), inversion (S_2, C_i) as well as the higher $\boldsymbol{S_n}$s ($n > 2$ is even) point group symmetries, for the rotational $\boldsymbol{C_2}$ point group symmetry, for the higher rotational $\boldsymbol{C_n}$ symmetries ($n > 2$), and finally for the $\boldsymbol{C_{nh}}$ symmetry point group. The chirality measure is the minimal of all $\boldsymbol{S_n}$ measures.

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Introduction: The Approach

Measuring symmetry and chirality on a continuous scale is a concept that has found already many applications in practically all domains of chemistry and related areas.[1–4] Several algorithms and computational tools have been proposed to approach this type of structural analysis. Particularly well-developed have been the general Continuous Symmetry Measure (CSM) [5] and the closely related Continuous Chirality Measure (CCM) [6], which allow quantification of the content of any symmetry point group [1,3–5], of any specific (polyhedral) shape [7,8], and of chirality [1,3,4,6], all stemming from the same basic approach: Estimation of the minimal distance to symmetry, that is, to the (searched) nearest symmetrical structure. An important on-going challenge has been to provide analytical solutions of the appropriate distance functions. In some instances this has been achieved,[5,9] but a systematic analytical approach to the 3D symmetry point groups has remained as an open challenge.

A first step towards that goal was recently published in this journal,[10] where a methodology for finding the minimal distance between a given structure and the structure which is obtained after applying a selected symmetry operation on it, was developed (the Symmetry Operation Measure). A symmetry measure, however, should evaluate the distance of a structure from all of the symmetry operations that comprise the symmetry group $\boldsymbol{G}$ as a whole. Here we report the successful achievement of this goal by developing a general methodology and applying it on the following symmetry point groups: All of the improper rotational point group symmetries, $\boldsymbol{S_n}$, including mirror symmetry (S_1,C_S), inversion (S_2,C_i), the higher $\boldsymbol{S_n}$s ($n > 2$ is even), the rotational $\boldsymbol{C_2}$ point group symmetry, the rotational $\boldsymbol{C_n}$ symmetries $n > 2$, and finally the $\boldsymbol{C_{nh}}$ symmetry point group. The chirality measure follows directly: It is the minimal of all $\boldsymbol{S_n}$ measures. Particularly, pleasing is the result that the analytical solutions thus obtained coincide with the CSM values which were used so far in the literature, and which have been obtained by the "folding-unfolding (F-U) algorithm" (described in Section "The $S(\boldsymbol{G})$ measure and the folding-unfolding algorithm").

Here is an outline of our new approach (see Fig. 1): The symmetry measure, $S(\boldsymbol{G})$, estimates the distance between the studied structure, $\mathbf{Q}_k = \begin{pmatrix} x_k \\ y_k \\ z_k \end{pmatrix}$, $k = 1 \ldots N$, and the resulting structure obtained by operating on it all of the n operations (g_i, $i = 1 \ldots n$) of a specific $\boldsymbol{G}$-symmetry point group. It is a minimal distance in the sense that the parameters of the symmetry

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transform (axes orientation angles and planes locations) minimize the measure. Operating all of the n symmetry operations of $\boldsymbol{G}$ on the original structure, $\mathbf{Q}_k$, n objects are obtained, $\mathbf{V}_{k,i}$, $k = 1\ldots N$, $i = 1\ldots n$ (Fig. 1c). The collection of all of the n objects, namely an overall object with nN points must be $\boldsymbol{G}$-symmetric because of the way this collection was built (see Fig. 1 for a graphic explanation). That overall $\boldsymbol{G}$-symmetric object of nN points can be represented as $\mathbf{V}_{k,i} = \mathbf{R}_i\ \mathbf{Q}_k$, where $\mathbf{R}_i$ is a transform matrix associated with the symmetry operation.

Note that the N vertices of $\mathbf{Q}_k$ can be labeled in various ways, each of which leads to a different distance between $\mathbf{Q}_k$ and $\mathbf{V}_{k,i}$. One should be very careful to notice that not all permutations between the labels are allowed. In general, the allowed permutations are only those for which the nN-points object has the desired $\boldsymbol{G}$ symmetry. (These allowed permutations are those where the number of points which are interchanged by the permutation is the order of the $\boldsymbol{G}$ symmetry group or a whole divider of it. For example, if $\boldsymbol{G}$ is the $\boldsymbol{C_3}$ symmetry then the allowed permutation can interchange only between three different points or between each point with itself; see also Appendix A). The specific label-permutated object is then $\mathbf{V'}_{k,i} = \mathbf{R}_i\ \mathbf{P}_{k,i}\ \mathbf{Q}_k = \mathbf{R}_i\ \mathbf{Q'}_{k,i}$ where the matrix $\mathbf{P}_{k,i}$ determines the permutation. Note also that $\mathbf{Q}_k$ and $\mathbf{V}_{k,i}$ are of the same size because the symmetry operation does not change it. Our task then is to locate the positions of all of the g_i symmetry elements such that the following distance function will be minimal:

$$\boldsymbol{S}(\boldsymbol{G}) = \min\frac{1}{2Nnd^2}\sum_{k=1}^{N}\sum_{i=1}^{n}\left|\mathbf{Q}_k - \mathbf{V'}_{k,i}\right|^2 = \min\frac{1}{2Nnd^2} \times \sum_{k=1}^{N}\sum_{i=1}^{n}\left|\mathbf{Q}_k - \mathbf{R}_i\mathbf{Q'}_{k,i}\right|^2 = 1 - \frac{1}{Nnd^2}\max\sum_{k=1}^{N}\sum_{i=1}^{n}\mathbf{Q}_k^{\mathrm{T}}\mathbf{R}_i\mathbf{Q'}_{k,i} \quad (1)$$

where d is the root mean square (r.m.s.) size of the original structure, $\mathbf{Q}_k$. We can see now that the distance function is between two sets of nN points: $n\mathbf{Q}_k$ and $\mathbf{V'}_{k,i}$; and that in practice, finding the minimal distance is a maximization problem [because of the right-hand form of eq. (1)]. The minimal value of $S(\boldsymbol{G})$ is 0 (the object is $\boldsymbol{G}$-symmetric) and the maximal value is 1 (obtained, for instance, if inversion is evaluated for a biatomic molecule with two different atoms, where only this permutation is allowed). The factor of 2 in the denominator is introduced in order to compensate for counting each distance between points, twice. (For practical reasons, see Section "The $S(\boldsymbol{G})$ measure and the folding-unfolding algorithm", we expanded in previous reports the range to 0–100).

Finally, returning to the permutations problem, this can be a forbidding task for large molecules, because the number of permutations grows with the number of vertices as $N!$. Excluding the unallowed permutations (Appendix A) reduces the number of tested permutations, and additional reduction is possible by taking into account the (selected) equivalency or nonequivalency of vertices (Appendix A).

The $S(G)$ Symmetry Measure

The Inversion Symmetry Group, C_i

The case of inversion symmetry is the simplest, because one does not need to find any axis. In this case $n = 2$, $\mathbf{R_1} = -\mathbf{I}$, and $\mathbf{R_2} = \mathbf{I}$, where $\mathbf{I}$ is the unit matrix. $S(\boldsymbol{C}_i)$ is easily obtained from (1):

$$S(\boldsymbol{C}_i) = \frac{1}{2}\left(1 + \frac{1}{Nd^2}\sum_{k=1}^{N}\mathbf{Q}_k^{\mathrm{T}}\mathbf{Q'}_k\right). \quad (2)$$

The Mirror Symmetry Group, C_S

Next we move to the mirror symmetry group, $\boldsymbol{C_S} = \{E,\sigma\}$, $n = 2$. Here one needs to find the optimal parameters of the mirror plane, minimizing $S(\boldsymbol{C_s})$, which, as explained above, is a maximization problem of eq. (1). As was shown in ref. 10, the problem leads to finding of minimal eigenvalue, $\lambda_{\min}$ of 3×3 matrix, and the eigenvector $\mathbf{m}_{\min}$ associated with this eigenvalue that gives the direction of optimal rotation axis. Let us define the perpendicular to the mirror plane vector of the direction cosines, $\mathbf{m}$, which satisfies the condition:

$$\mathbf{m}^{\mathrm{T}}\mathbf{m} = 1. \quad (3)$$

The transform matrix can be represented as $\mathbf{R_1} = \mathbf{I} - 2\mathbf{m}\mathbf{m}^{\mathrm{T}}$, $\mathbf{R_2} = \mathbf{I}$, and eq. (1) is then reduced to:

$$S(\boldsymbol{C}_s) = \frac{1}{2}\left(1 - \frac{1}{Nd^2}\sum_{k=1}^{N}\mathbf{Q}_k^{\mathrm{T}}\mathbf{Q'}_k\right) + \frac{1}{Nd^2}\min\sum_{k=1}^{N}\mathbf{Q}_k^{\mathrm{T}}\mathbf{m}\mathbf{m}^{\mathrm{T}}\mathbf{Q'}_k \quad (4)$$

As was shown in ref. 10, the minimum of the quadratic form in (4) is equal to $\frac{1}{2}\lambda_{\min}$, where $\lambda_{\min}$ is the minimal eigenvalue of symmetrical 3×3 matrix $\mathbf{A}$, which is defined as

$$\mathbf{A} = \sum_{k=1}^{N}(\mathbf{Q'}_k\mathbf{Q}_k^{\mathrm{T}} + \mathbf{Q}_k\mathbf{Q'}_k^{\mathrm{T}}), \quad (5)$$

which has three real eigenvalues. The eigenvector $\mathbf{m}_{\min}$ which is associated with this eigenvalue gives the normal to the optimal reflection plane. From eq. (4), one obtains:

$$\boldsymbol{S}(\boldsymbol{C}_s) = \frac{1}{2}\left\{1 + \frac{1}{Nd^2}\left(\lambda_{\min} - \sum_{k=1}^{N}\mathbf{Q}_k^{\mathrm{T}}\mathbf{Q'}_k\right)\right\} \quad (6)$$

The Twofold Rotational Symmetry Group, C_2

For the case of $\boldsymbol{C}_2 = \{E,C_2\}$, one needs to find the optimal direction of the rotation axis, which is given by the direction cosine vector, $\mathbf{m}$. As in the case of $\boldsymbol{C_s}$, the rotation axis to be found passes through the center of the mass of the structure.[5] The transform matrix in this case is $\mathbf{R_1} = 2\mathbf{m}\mathbf{m}^{\mathrm{T}} - \mathbf{I}$, $\mathbf{R_2} = \mathbf{I}$. As was shown in ref. 10, the problem is of finding the maximal

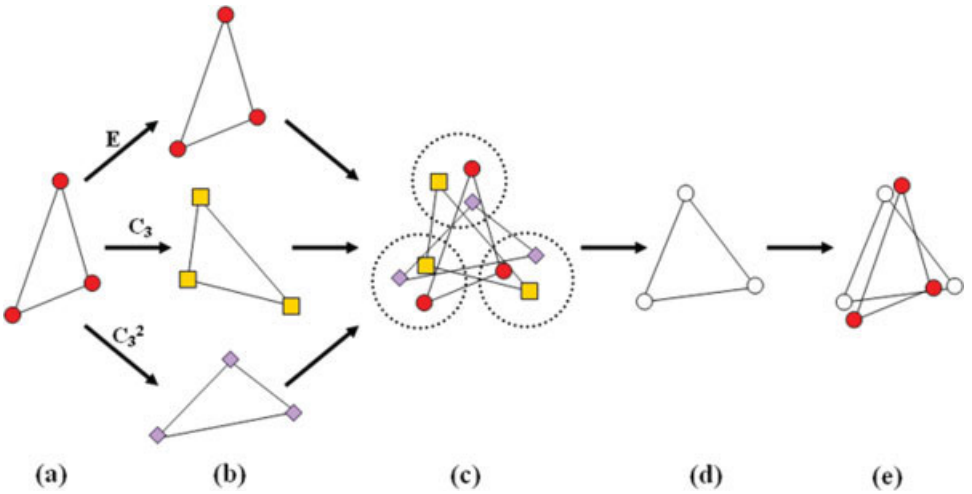

Figure 1. Illustration of the method: Suppose one wishes to determine the symmetry measure of the point-group $\boldsymbol{C}_3$, $S(\boldsymbol{C}_3)$, of the triangle on the left ((a) red), $\mathbf{Q}_k$. One operates on it the three symmetry operations of C_3 (b) the superposition of which (c) $\mathbf{V}_{k,i}$, is a C_3-symmetric object of three ($n = 3$) clusters of 3 ($N = 3$) points each (a total of $nN = 9$ points structure). The symmetry measure $S(\boldsymbol{C}_3)$ is the distance between the nine points of $\mathbf{V}_{k,i}$ and the three points of $\mathbf{Q}_k$ taken 3 times. The nearest object to $\mathbf{Q}_k$ which has the desired C_3 symmetry is the averaged $\mathbf{V}_{k,i}$, namely, the C_3-symmetric open circles (d) ($\mathbf{Q}'_k$)—see Section "Getting the nearest symmetric structure"). Equivalently, $S(\boldsymbol{C}_3)$ is the minimal distance between $\mathbf{Q}_k$ and $\boldsymbol{Q}'_k$, as shown in (e)—see also Section "The $S(\boldsymbol{G})$ measure and the folding-unfolding algorithm."

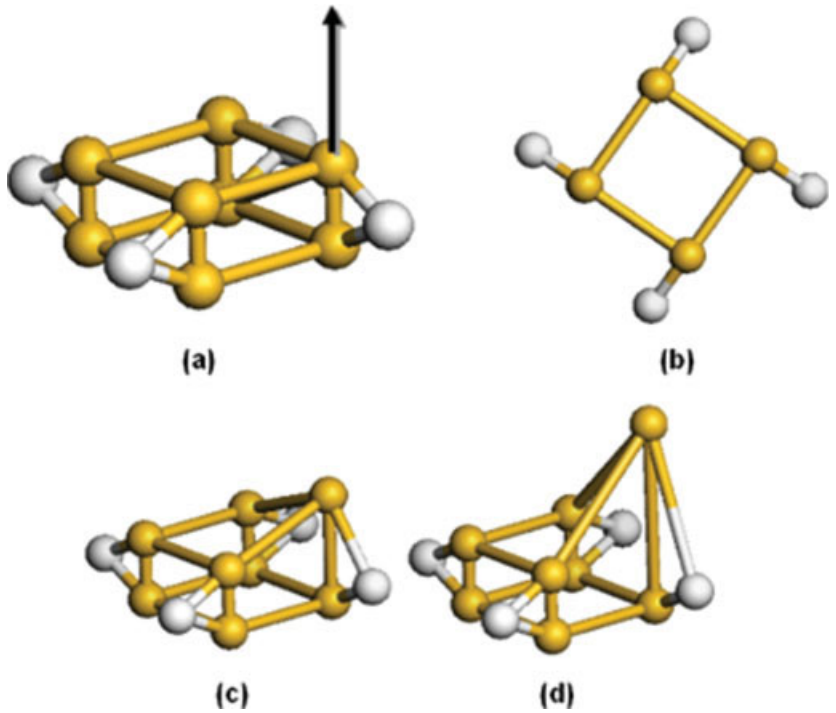

Figure 2. The analyzed 12 atoms structure. (a) The starting fully symmetric structure. (b) Top view of the structure. The distortion is carried out by elongating the edged shown by the arrow in the z-direction. Structures (c) and (d) are two examples of the distorted structures. The deformation parameter is defined as the ratio between the z-coordinates of the vertex which carries the arrow, and the z-coordinates of the same vertex after elongation.

eigenvalue, $\lambda_{\max}$ of the matrix **A** and the associated eigenvector $\mathbf{m}_{\max}$. The final equation then follows from eq. (1) with $n = 2$:

$$S(C_2) = \frac{1}{2}\left\{1 + \frac{1}{Nd^2}\left(\sum_{k=1}^{N}\mathbf{Q}_k^{\mathrm{T}}\mathbf{Q}'_k - \lambda_{\max}\right)\right\}. \quad (7)$$

The n-Fold Rotational Symmetry Groups, C_n

As for $S(C_2)$, here too, the maximization of (1) should be carried out with respect to the axis parameters. The axis passes through the center of the mass of the structure.[5] A C_n symmetry group consists of n symmetry operations $\{C_n^i,\ i = 1\ldots n\}$, which correspond to n symmetric transforms

$$\mathbf{R_i} = \cos\theta_i\mathbf{I} + (1 - \cos\theta_i)\mathbf{m}\mathbf{m}^{\mathrm{T}} + \sin\theta_i\Re(\mathbf{m}), \quad (8)$$

where $\theta_i = \frac{2\pi}{n}i$ is a rotation angle and $\Re(\mathbf{m}) = \begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix}$ is an antisymmetric matrix. Using the results of our previous study,[10] the products in the sum of eq. (1) can be written as

$$\mathbf{Q}_k^{\mathrm{T}}\mathbf{R}_i\mathbf{Q}'_{k,i} = \cos\theta_i\mathbf{Q}_k^T\mathbf{Q}'_{k,i} + (1 - \cos\theta_i)\mathbf{Q}_k^{\mathrm{T}}\mathbf{m}\mathbf{m}^T\mathbf{Q}'_{k,i} + \sin\theta_i[\mathbf{m}\mathbf{Q}'_{k,i}\mathbf{Q}_k] \quad (9)$$

where the brackets in last term denote scalar triple product. One can see that eq. (9) does not depend on the direction of rotation (clockwise or anticlockwise); the last term is odd with respect to angle θ_i, but changing the rotation direction leads to changing the place of the vectors $\mathbf{Q}'_{k,i}$ and $\mathbf{Q}_k$ and therefore to change in the sign of triple product.

Maximization of (1) with respect to the direction cosines in this case is simplified into maximization of the difference of the quadratic and linear forms

$$\mathbf{m}^{\mathrm{T}}\mathbf{A}\mathbf{m} - \mathbf{m}^{\mathrm{T}}\mathbf{B} \quad (10)$$

where the symmetric 3×3 matrix **A** and vector **B** are defined as

$$\mathbf{A} = \sum_{\mathrm{i}=1}^{\mathrm{n}}(1 - \cos\theta_i)\sum_{k=1}^{N}(\mathbf{Q}'_{k,i}\mathbf{Q}_k^{\mathrm{T}} + \mathbf{Q}_k\mathbf{Q}'^{\mathrm{T}}_{k,i}) \quad (11)$$

$$\mathbf{B} = \sum_{i=1}^{n}\sin\theta_i\sum_{k=1}^{N}\mathbf{Q}_k\times\mathbf{Q}'_{k,i} \quad (12)$$

Using the same method as in ref. 10, one first obtains the parameter $\lambda_{\max}$ as a real root of the 6th order polynomial equation

$$\sum_{i=1}^{3}\left(\frac{\mathbf{m}_i^{\mathrm{T}}\mathbf{B}}{\lambda_i - \lambda_{\max}}\right)^2 = 1 \quad (13)$$

and then the directional cosines vector for optimal rotation axes, $\mathbf{m_{max}}$:

$$\mathbf{m_{max}} = \sum_{i=1}^{3}\frac{\mathbf{m}_i^{\mathrm{T}}\mathbf{B}}{\lambda_i - \lambda_{\max}}\mathbf{m}_i \quad (14)$$

In (13) and (14), $\mathbf{m}_i$ and λ_i, $i = 1,2,3$ are the eigenvectors and eigenvalues of the **A**-matrix, respectively. Finally, the expression for $S(C_n)$ becomes:

$$S(C_{\mathrm{n}}) = 1 - \frac{1}{Nnd^2}\left(\sum_{i=1}^{n}\cos\theta_i\sum_{k=1}^{N}\mathbf{Q}_{\mathrm{k}}^{\mathrm{T}}\mathbf{Q}'_{k,i} + \lambda_{\max} - \frac{1}{2}\mathbf{m}_{\max}^{\mathrm{T}}\mathbf{B}\right) \quad (15)$$

In the case of $n = 2$ (C_2-symmetry group), eq. (15) provides eq. (7).

The n-Fold Improper-Rotation Symmetry Groups, S_n

The S_n symmetry point groups involve, we recall, rotation by an angle of $\theta_i = 2\pi i/n$, $i = 1,\ldots n$ and reflection with respect to a plane, which is perpendicular to the rotation axis. The S_n groups are defined for even n only, since it does not satisfy the requirement $S_n^n = E$ for odd n values.[11] An S_n symmetry group consists of n elements $\{S_n^1, C_n^2, S_n^3 \ldots S_n^{n-1}, E\}$, which correspond to the following symmetric transforms

$$\mathbf{R}_{\mathrm{i}} = \cos\theta_i\mathbf{I} + ((-1)^i - \cos\theta_i)\mathbf{m}\mathbf{m}^{\mathrm{T}} + \sin\theta_i\Re(\mathbf{m}), \quad (16)$$

where $i = 1\ldots n$. The S-measure is evaluated analogously to the C_n case. The difference is in matrix **A** only [instead of eq. (11)]

$$\mathbf{A} = \sum_{i=1}^{n}((-1)^i - \cos\theta_i)\sum_{k=1}^{N}(\mathbf{Q}'_{k,i}\mathbf{Q}_k^{\mathrm{T}} + \mathbf{Q}_k\mathbf{Q}'^{\mathrm{T}}_{k,i}) \quad (17)$$

In this case, all final eqs. (13–15) are valid for $S(S_n)$ as well.

The special case of inversion symmetry, C_i, can be obtained from the general solution (15). In this case $n = 2$, $\theta = 180^\circ$ and eq. (15) becomes eq. (2). The other specific case is mirror symmetry, C_S. Here $n = 2$, $\theta = 0$, and eq. (15) becomes eq. (6).

The C_{nh} Symmetry Point-Groups

Our final example demonstrates that the general approach developed here, is applicable not only to groups generated by a single symmetry element (all previous examples), but also to groups generated by more than one symmetry elements. Specifically, we treat here the C_{nh} groups, and others will be described in subsequent reports. We recall that a C_{nh} group is composed of $C_n\cdot\sigma_h$ where the mirror plane is perpendicular to the rotation axis. A C_{nh} group is a direct product of the C_n and C_i point groups for even values of n or of C_s group for odd n values.[11] In contrast with the symmetry groups treated above, a C_{nh} group includes $2n$ elements, resulting in the following definition of $S(C_{nh})$:

$$S(\boldsymbol{C_{nh}}) = \min \frac{1}{4Nnd^2} \left\{ \sum_{k=1}^{N} \sum_{i=1}^{n} \left| \mathbf{Q}_k - \mathbf{R}_{\mathbf{C_n}i} \mathbf{Q'}_{k,i} \right|^2 + \sum_{j=1}^{N} \sum_{i=1}^{n} \left| \mathbf{Q}_j - \mathbf{R}_{2i} \mathbf{Q'}_{j,i} \right|^2 \right\}. \quad (18)$$

The first sum in (18) corresponds to $\boldsymbol{C_n}$ symmetry and the second either to $\boldsymbol{C_i}$ symmetry (for even values of n), or to $\boldsymbol{C_s}$ symmetry (for odd values of n):

For even n:

$$\mathbf{R_{2i}} = -\cos\theta_i \mathbf{I} - (1 - \cos\theta_i)\mathbf{m}\mathbf{m}^{\mathrm{T}} - \sin\theta_i \Re(\mathbf{m}), \quad (19a)$$

and for odd n:

$$\mathbf{R_{2i}} = \cos\theta_i \mathbf{I} - (1 + \cos\theta_i)\mathbf{m}\mathbf{m}^{\mathrm{T}} + \sin\theta_i \Re(\mathbf{m}). \quad (19b)$$

In (18), we assume that all symmetry elements have the same weight. As in the previous cases, both the reflection plane and the rotation axis contain the center of mass of the structure, and therefore, a single vector of directional cosines characterizes the plane and axis. In contrast with the other symmetry groups, considered in this study, the total number of summations is $2Nn$ (instead of Nn). The sums in eq. (18) are dependent because their values depend on the same axis and its direction. We arrive then to a similar problem as in the case of $\boldsymbol{C_n}$ symmetry, but now matrix **A** and vector **B** are defined as:

For even value of n:

$$\mathbf{A} = \sum_{i=1}^{n} (1 - \cos\theta_i) \left[\sum_{k=1}^{N} (\mathbf{Q'}_{k,i}\mathbf{Q}_k^{\mathrm{T}} + \mathbf{Q}_k\mathbf{Q'}_{k,i}^{\mathrm{T}}) - \sum_{j=1}^{N} (\mathbf{Q'}_{j,i}\mathbf{Q}_j^{\mathrm{T}} + \mathbf{Q}_j\mathbf{Q'}_{j,i}^{\mathrm{T}}) \right] \quad (20a)$$

$$\mathbf{B} = \sum_{i=1}^{n} \sin\theta_i \left[\sum_{k=1}^{N} \mathbf{Q}_k \times \mathbf{Q'}_{k,i} - \sum_{k=1}^{N} \mathbf{Q}_j \times \mathbf{Q'}_{j,i} \right] \quad (21a)$$

For odd value of n:

$$\mathbf{A} = \sum_{i=1}^{n} \left[\sum_{k=1}^{N} (1 - \cos\theta_i)(\mathbf{Q'}_{k,i}\mathbf{Q}_k^{\mathrm{T}} + \mathbf{Q}_k\mathbf{Q'}_{k,i}^{\mathrm{T}}) - (1 + \cos\theta_i) \sum_{j=1}^{N} (\mathbf{Q'}_{j,i}\mathbf{Q}_j^{\mathrm{T}} + \mathbf{Q}_j\mathbf{Q'}_{j,i}^{\mathrm{T}}) \right] \quad (20b)$$

$$\mathbf{B} = \sum_{i=1}^{n} \sin\theta_i \left[\sum_{k=1}^{N} \mathbf{Q}_k \times \mathbf{Q'}_{k,i} + \sum_{k=1}^{N} \mathbf{Q}_j \times \mathbf{Q'}_{j,i} \right] \quad (21b)$$

Using the procedure described above (Section "The n-fold rotational symmetry groups, $\boldsymbol{C_n}$"), the $S(\boldsymbol{C_{nh}})$ measure is calculated using eq. (15).

The Chirality Measure

Because chirality implies lack of improper symmetry, a chirality measure, namely a quantitative estimation of the degree of that structural property, can be based on estimating how close is the (chiral) object from having improper symmetry. We therefore have the chirality measure at hand: It will be the minimal of all $S(\boldsymbol{S_n})$ values (n = 1,2,4...). In most cases it will be $S(chir) = S(\boldsymbol{C_s}) = S(\boldsymbol{S_1})$, in others $S(chir) = S(\boldsymbol{C_i}) = S(\boldsymbol{S_2})$ in fewer cases $S(chir) = S(\boldsymbol{S_4})$ and so on. Since in most cases inspection of the studied structure is enough in order to guess which is the nearest $\boldsymbol{S_n}$, a practical solution is to calculate that $S(\boldsymbol{S_n})$, or in case of doubt, few $S(\boldsymbol{S_n})$ values, and pick the smallest.

We mention here that another method for measuring chirality, is to estimate the minimal distance between one enantiomer and the other which amounts to finding the maximal overlap between the enantiomers.[12] The conversion factor between the two measures is 4: *S(enantiomers overlap) = 4S(chir)* (in our web-site for measuring symmetry and chirality, we currently use this approach[13]). The origin of this factor is explained in Appendix B. It should be noted however that the latter method requires heavy computation because it is based on numerical minimization procedures, and that it does not provide information about the nearest achiral structure (Section "Getting the nearest symmetric structure").

Illustration of the *S(G)* Estimation Method

For illustration of the method described in Sections "The $S(G)$ symmetry measure" and "The chirality measure," we calculate CSM and CCM of the structure shown in Figure 2. The structure consists of 12 vertices and when fully $\boldsymbol{C_{4h}}$-symmetric, has inversion, mirror symmetry, $\boldsymbol{C_2}$, $\boldsymbol{C_4}$, $\boldsymbol{S_4}$, $\boldsymbol{C_{2h}}$. The structure is then deformed in the z-direction (as shown in the figure by the arrow), thus leading to nonzero $S(\boldsymbol{G})$ symmetry measures for all of the original types of symmetry. The deformation parameter is defined as the ratio between the z-coordinates of nondeformed and deformed structures (Fig. 2a).

Figure 3 demonstrates dependence of the various $S(\boldsymbol{G})$ measures on the magnitude of the deformation parameter. It is seen that the order of the lines (the sensitivity of the measure to distortion) follows intuition and expectations (note that by definition, the measures are comparable to each other). Thus, $S(\boldsymbol{C_s})$, is the smallest, because having only a mirror plane is the least demanding symmetry. More demanding is to have a C_4 element, and indeed, $\boldsymbol{S(C_4)}$, $\boldsymbol{S(S_4)}$, and $\boldsymbol{S(C_{4h})}$ are the upper lines in that figure while $\boldsymbol{S(C_2)}$, $\boldsymbol{S(C_i)}$ ($= \boldsymbol{S(S_2)}$) and $\boldsymbol{S(C_{2h})}$ are in between. Finally, since $S(\boldsymbol{C_s}) < S(\boldsymbol{C_i}) < \boldsymbol{S(S_4)}$, $S(\boldsymbol{C_s})$ is the chirality measure, *S(Chir)*.

Further Comments on the $S(G)$ Symmetry Measure

Getting the Nearest Symmetric Structure

An important outcome of our methodology is that one can get the actual shape of the nearest structure, $\hat{\mathbf{Q}}_k$, which has the desired $\boldsymbol{G}$ symmetry. That structure is the average within each

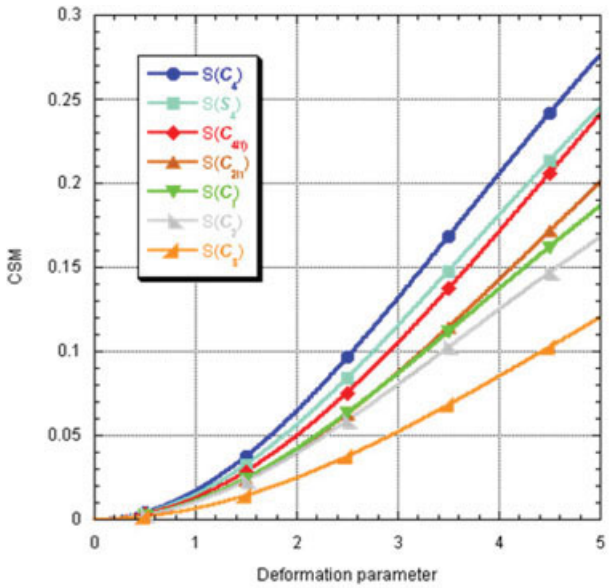

Figure 3. Dependence of $S(\boldsymbol{G})$ measures for different symmetry point groups on the deformation parameter for the distortion of the structure of Figure 2. In this case, the chirality measure is $S(\boldsymbol{C}_s)$.

of the n clusters that compose $\mathbf{V}_{k,i}$ (Fig. 1d):

$$\hat{\mathbf{Q}}_k = \frac{1}{n}\sum_{i=1}^{n}\mathbf{V}'_{i,k} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{R}_i\mathbf{Q}'_{i,k} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{R}_i\mathbf{P}_{k,i}\mathbf{Q}_k \qquad (22)$$

where $k = 1\ldots N$ is the cluster number and i is the point index inside the cluster. The number of points inside of any cluster is equal to the symmetry order.

The S(G) Measure and the Folding-Unfolding Algorithm

In all of the previous studies a different algorithm, the Folding-Unfolding (F-U) algorithm was used. We now show that $\boldsymbol{S}(\boldsymbol{G}) = \boldsymbol{S}_{\mathrm{F-U}}(\boldsymbol{G})$. The definition of $\boldsymbol{S}_{\mathrm{F-U}}(\boldsymbol{G})$ is based on the minimal distance between two structures: the original one, $\mathbf{Q}_k$ (Fig. 1a) and the nearest $\boldsymbol{G}$-symmetric structure, $\hat{\mathbf{Q}}_k$ (Fig. 1d; the distance is shown in Fig. 1e). The measure, $\boldsymbol{S}_{\mathrm{F-U}}(\boldsymbol{G})$, is defined as[5]

$$S_{\mathrm{F-U}}(\boldsymbol{G}) = \min\frac{1}{Nd^2}\sum_{k=1}^{N}\left|\mathbf{Q}_k - \hat{\mathbf{Q}}_k\right|^2, \qquad (23)$$

If the structure is of the desired $\boldsymbol{G}$-symmetry, then $\boldsymbol{S}_{\mathrm{F-U}}(\boldsymbol{G}) = 0$; and the maximal value that $\boldsymbol{S}_{\mathrm{F-U}}(\boldsymbol{G})$ can take is 1.[5] (In practice, since most of the interesting physical questions associated with symmetry, relate to small deviations, and therefore we found it convenient to expand the scale by 100 in previous reports.[1,3,4]) Equation (23) has been solved by a number of algorithms,[9] particularly by the "folding-unfolding" algorithm (demonstrated in Fig. 4) according to which one operates on each point of $\mathbf{Q}_k$ (Fig. 4a) one of the different symmetry operations of $\boldsymbol{G}$ (g_i, $i = 1\ldots n$)). The result of this "folding" step—Figure 4b—is a cluster of points (Fig. 4c). One takes then the center of mass of the minimal possible cluster (open circle in Fig. 4c) and "unfolds" it (Fig. 4d) by operating on this point all the g_i^{-1}, $i = 1\ldots n$. An object with the desired $\boldsymbol{G}$-symmetry—the nearest one—is obtained (Fig. 4d), $\hat{\mathbf{Q}}_k$. The symmetry measure is based on the distance between $\mathbf{Q}_k$ and $\hat{\mathbf{Q}}_k$ (Fig. 4e).

It is seen that the algorithms used for $\boldsymbol{S}(\boldsymbol{G})$ (see Fig. 1) and for $\boldsymbol{S}_{\mathrm{F-U}}(\boldsymbol{G})$ (see Fig. 4) are quite similar, except for the averaging step. We now show the equivalency of the two methods:

Let us define $\mathbf{b}_k = \mathbf{Q}_k - \hat{\mathbf{Q}}_k|_{\min}$, $k = 1\ldots n$, where $\hat{\mathbf{Q}}_k$ is obtained by the F-U procedure (see Fig. 4), and let us define $\mathbf{a}_{k,i} = \mathbf{Q}_k - \mathbf{V}'_{k,i}|_{\min}, k, i = 1\ldots n$, where $\mathbf{V}'_{k,i}$ is obtained after the minimization procedure described above (see Fig. 1). Using these definitions, the folding-unfolding $\boldsymbol{S}_{\mathrm{F-U}}(\boldsymbol{G})$ and the analytical $\boldsymbol{S}(\boldsymbol{G})$ measures can be written as:

$$S_{\mathrm{F-U}}(\boldsymbol{G}) = \frac{1}{nd^2}\sum_{k=1}^{N}|\mathbf{b}_k|^2;\ \sum_{k=1}^{n}\mathbf{b}_k = 0, \qquad (24)$$

and:

$$\boldsymbol{S}(\boldsymbol{G}) = \frac{1}{2n^2d^2}\sum_{i=1}^{n}\sum_{k=1}^{n}|\mathbf{a}_{k,i}|^2 \qquad (25)$$

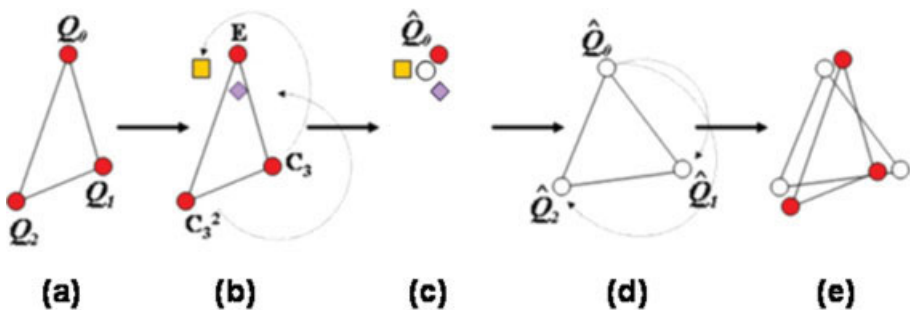

Figure 4. The Folding-Unfolding method: (a) One wishes to determine $S_{F-U}(\boldsymbol{C}_3)$ of the triangle on the left in red, $\mathbf{Q}_k$. (b) The folding step: One operates on each of the vertices one of the three operations on $\mathbf{Q}_0$, $\mathbf{C}_3^1$ on $\mathbf{Q}_1$, and $\mathbf{C}_3^2$ on $\mathbf{Q}_2$. A cluster forms (c), which is mass-averaged into the open circle, Q'_0. (d) The unfolding step: The blue dot is unfolded into the $\boldsymbol{C}_3$-symmetric object, $\mathbf{Q}'_k$, with the three operations. The symmetry measure is based on the distance between $\mathbf{Q}_k$ and $\mathbf{Q}'_k$ (e).

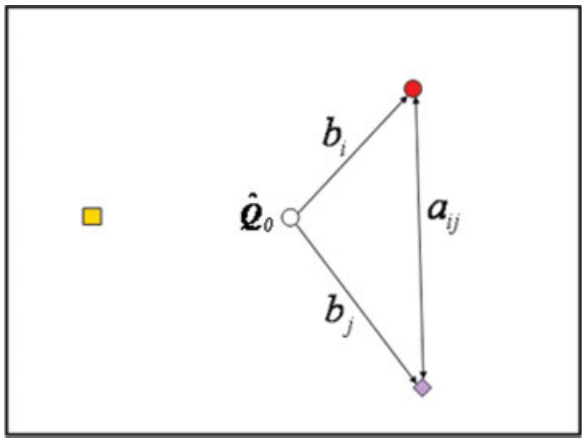

Figure 5. The distances, $a_{i,j}$, between different points of the cluster in Fig. 4c (one distance is shown), and the distances of these points (b_i and b_j) to the center of mass of the cluster (open circle). [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

respectively. Figure 5 shows one cluster from which it is clear that $\mathbf{a}_{k,i} = \mathbf{b}_k - \mathbf{b}_i$; then, from eqs. (24) and (25) one obtains

$$S(G) = \frac{1}{2n^2d^2}\sum_{i=1}^{n}\sum_{k=1}^{n}|\mathbf{a}_{k,i}|^2 = \frac{1}{2n^2d^2}\sum_{i=1}^{n}\sum_{k=1}^{n}|\mathbf{b}_k - \mathbf{b}_i|^2$$
$$= \frac{1}{2n^2d^2}\sum_{i=1}^{n}\sum_{k=1}^{n}(|\mathbf{b}_k|^2 + |\mathbf{b}_i|^2 - 2\mathbf{b}_k\mathbf{b}_i) = \frac{1}{nd^2}\sum_{i=1}^{n}|\mathbf{b}_i|^2$$
$$- \frac{1}{n^2d^2}\sum_{i=1}^{n}\mathbf{b}_i\sum_{k=1}^{n}\mathbf{b}_k = \frac{1}{nd^2}\sum_{i=1}^{n}|\mathbf{b}_i|^2. \quad (26)$$

From (26), the relationship between the two estimations is

$$S(G) = S_{\mathrm{F-U}}(G). \quad (27)$$

The Relation of the Shape Measure to S(G)

In earlier studies, it was found useful to treat separately the special case where $\hat{\mathbf{Q}}_\mathbf{k}$ which minimizes eq. (23) is a predetermined shape.[7,8] *S*(*Shape*) coincides with *S*(***G***) when in the predetermined shape there are no degrees of freedom in the specific shape of the nearest symmetrical object, namely for all regular polygons and for the Platonic polyhedra. For example, if $\hat{\mathbf{Q}}_\mathbf{k}$ is a perfect 6-vertices hexagon, then *S*($\boldsymbol{C_6}$) and *S*(*six vertices hexagon*), all coincide. However, consider an object constructed from 3 vertices, for which one wishes to measure *S*($\boldsymbol{D_{3h}}$) symmetry. In this case, there are two possible objects that are constructed from 3 vertices and have $\boldsymbol{D_{3h}}$ symmetry: a perfect triangle and a line. *S*($\boldsymbol{D_{3h}}$) will provide the minimal of the two possibilities. However, if the shape is closer to a line but one wishes to determine *S*(*triangle*), then $S(\boldsymbol{D}_{3h}) < S(\text{triangle})$.

Note that the Shape measure can be used as a chirality overlap-measure (Section "Illustration of the *S*(***G***) estimation method"): The best (maximal) overlap between the enantiomers is equivalent to finding the Shape content of one enantiomer in its counter enantiomer.

Acknowledgments

The authors gratefully acknowledge useful discussions with Prof. Santiago Alvarez, a long-term collaborator on symmetry, chirality, and shape measures.

Appendix A: Orbits, Permutations, and Equivalency

Orbit

We recall that an orbit comprises of a (sub-) group of vertices all of which are interchanged by all of the operations of the symmetry group ***G***. A structure can have one orbit which includes all of its N vertices (such as when operating the group $\boldsymbol{C}_3$ on a triangle, Fig. A1); or several orbits (such as when operating $\boldsymbol{C}_4$ on an octahedron, Fig. A2, Table A2) In the later case there are several ways of dividing into obits, one of which is shown.

Pairing

A point in the original orbit, and a point in the symmetry operated orbit, comprise a pair. The process of evaluating and minimizing *S*(***G***) requires finding the best of all possible pairings for all vertices. See Tables A1 and A2 for examples of possible pairings.

Permutations

A second way for representing this information is by using permutations. Here, for each symmetry operation all the possible pairings are presented. For example, for the C_3 operation on the triangle in Figure A1 the permutation is

$$1 \rightarrow 2$$
$$2 \rightarrow 3\,.$$
$$3 \rightarrow 1$$

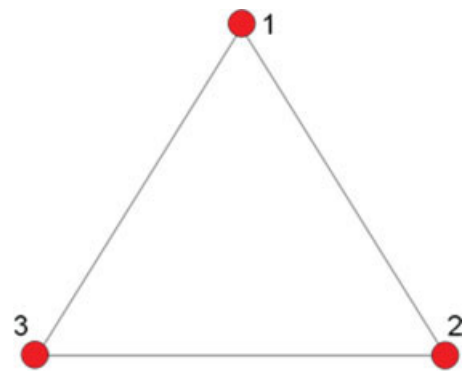

Figure A1. Triangle with its vertices numbered, for use in Table A1. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

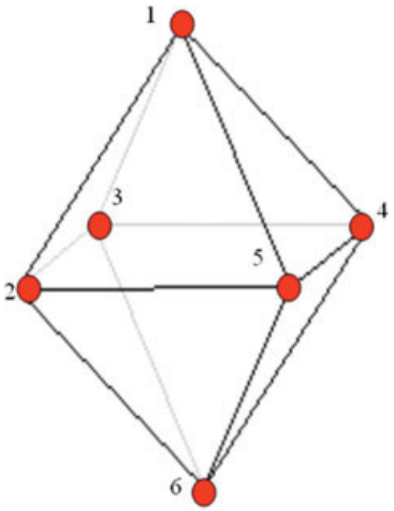

Figure A2. Octahedron with its vertices numbered for use in Table A2. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

The permutations provide the pairs from which one can construct the orbit. Thus, the orbit (1 2 3) is equivalent of saying that the permutation for E is

$$1 \rightarrow 1$$
$$2 \rightarrow 2\,,$$
$$3 \rightarrow 3$$

the permutation for C_3 is

$$1 \rightarrow 2$$
$$2 \rightarrow 3$$
$$3 \rightarrow 1$$

and the permutation for C_3^2 is

$$1 \rightarrow 3$$
$$2 \rightarrow 1\,.$$
$$3 \rightarrow 2$$

In what follows we shall use the permutations notation, keeping in mind that the statements are also true for orbits. We wish then to find the permutations that minimize $S(\boldsymbol{G})$. When comparing two structures, each built of N atoms, $N!$ possible permutations are possible. Thus, if N is, typically, 15 there are 1.3×10^{12} possible permutations for each symmetry operation, making the computation impractical. We must therefore find ways to reduce this number drastically.

Table A1. The Orbit and the Pairings for Calculating $S(\boldsymbol{C}_3)$ of a Triangle.

Orbit	1-2-3		
Symmetry operations	E	C_3^1	$C_3^2 = C_3^{-1}$
Pairs	1-1	1-2	1-3
	2-2	2-3	2-1
	3-3	3-1	3-2

Table A2. The Orbits and the Pairings for Calculating $S(\boldsymbol{C}_4)$ of an Octahedron.

Orbits	1-1-1-1				2-3-4-5				6-6-6-6			
Symmetry operations	E	C_4^1	C_4^2	C_4^3	E	C_4^1	C_4^2	C_4^3	E	C_4^1	C_4^2	C_4^3
Pairs	1-1	1-1	1-1	1-1	2-2	2-3	2-4	2-5	6-6	6-6	6-6	6-6
	1-1	1-1	1-1	1-1	3-3	3-4	4-2	5-4	6-6	6-6	6-6	6-6
	1-1	1-1	1-1	1-1	4-4	4-5	3-5	4-3	6-6	6-6	6-6	6-6
	1-1	1-1	1-1	1-1	5-5	5-2	5-3	3-2	6-6	6-6	6-6	6-6

Equivalency

One way to reduce the number of possible permutations is based on the selected equivalency criterion for the vertices of the structure. The definition of equivalency is open to the user: Equivalent points are points of the same element, same chemical nature, same nearest neighbors, same complete graph of connectivity, or in fact, any selected labeling which may be geometric, chemical or physical in nature. Symmetry operation can relate only points that are defined as equivalent. By this labeling method the number of symmetry allowed permutations drops significantly.

G-Symmetry Preserving Permutations

Once the equivalency criterion has been selected, the allowed permutations for the symmetry measure are restricted to only those which preserve the $\boldsymbol{G}$ symmetry on the nN-sized object when all operations are carried out on the original structure; that is, that the resulting object is not of a different $\boldsymbol{G}'$ symmetry. In general, these allowed permutations are those where the number of points in a given structure which are interchanged by the permutation (the number of points in each permutation) is the order of the $\boldsymbol{G}$ symmetry group or a whole divider of it. An example for allowed and disallowed permutations (leading in the latter case to a $\boldsymbol{G}'$-symmetric structure with $S(\boldsymbol{G}') < S(\boldsymbol{G})$) is the following: Suppose we wish to evaluate the degree of inversion symmetry, $S(\boldsymbol{C}_i)$, of an equilateral triangle (Fig. A3 left), a situation of two operations, E and i, and three points. Let us perform these operations on the triangle: E provides the original red triangle, and i the inversed triangle (Fig. A3 right).

Next we match pairs of vertices; there are $3! = 6$ possible pairings, permutations:

$$A \leftrightarrow A' \quad A \leftrightarrow A' \quad A \leftrightarrow B' \quad A \leftrightarrow C' \quad A \leftrightarrow B' \quad A \leftrightarrow B' \quad A \leftrightarrow C'$$
$$B \leftrightarrow B'\,,\ B \leftrightarrow C'\,,\ B \leftrightarrow A'\,,\ B \leftrightarrow A'\,,\ B \leftrightarrow A'\,,\ B \leftrightarrow C'\,,\ B \leftrightarrow B'$$
$$C \leftrightarrow C' \quad C \leftrightarrow B' \quad C \leftrightarrow C' \quad C \leftrightarrow B' \quad C \leftrightarrow B' \quad C \leftrightarrow A' \quad C \leftrightarrow A'$$

The first permutation (left column) where every point is interchanged with itself (Fig. A4) is allowed because the number of points interchanged, 1 (all 3 orbits contain 1 atom), is a whole divider of the order of the symmetry group, 2. In this case, the nearest object with C_i symmetry is the collapse of all vertices to the center of mass, and one gets $\boldsymbol{S}'(\boldsymbol{C}_i) = 1$, namely the maximal possible value.

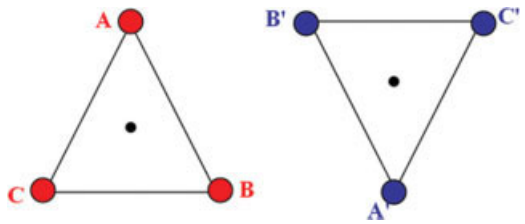

Figure A3. Operating the identity (left) and the inversion (right) on a triangle. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

The 4th-column permutation (Fig. A5) is an example for a disallowed one: all three points are interchanged between themselves, and this number is not a whole divider of the order of the symmetry group (two). In this case the result is another triangle, which does not have the required G-symmetry (C_i), but G' symmetry (C_s in 2D), with $S'(C_i'') = 0.25$. In fact this is the lowest $S'(C_i)$ value of all permutations, but is not a measure of C_i symmetry. The 5th permutation leads to a similar result.

The correct evaluation of $S(C_i)$ is provided by the second, third and sixth permutations (Fig. A6 shows the second-column permutation: one point is interchanged with itself, while the other two points interchange between themselves), leading to a nearest object with C_i symmetry with $S(C_i) = 0.5$, which is the smallest legitimate $S'(C_i)$ value. Notably, this value is larger than obtained from the disallowed 4th permutation case. This is also true for the third and 6th permutations.

Let us generalize: We have a group of m equivalent points and a symmetry group of the order n (n can be the order of the symmetry group or a whole divider of it). By definition every orbit contains n points (which may be n different points, or the same point taken n times). Now there are a number of possibilities, leading to the following permutations:

1. We assume that every point interchanges with itself (every point is conjugated only to itself), leading to m orbits, where every orbit containing one atom duplicated n times. There is only one permutation of this kind.
2. We assume that there is only one orbit of n points and that all of the other $m-n$ points are conjugated only to themselves. The total number of possible permutations in this case is $\binom{m}{n} = \frac{m!}{n!(m-n)!}$
3. We continue similarly and assume that there are 2, 3, 4... orbits of n points and that the rest of the points are conjugated only to themselves. The number of permutations is $\prod_{j=1}^{k} \frac{1}{j} \cdot \binom{m-(j-1)\cdot n}{n} = \prod_{j=1}^{k} \frac{1}{j} \cdot \frac{[m-(j-1)\cdot n]!}{n!\cdot(m-j\cdot n)!}$ where k is the number of sets of n points.

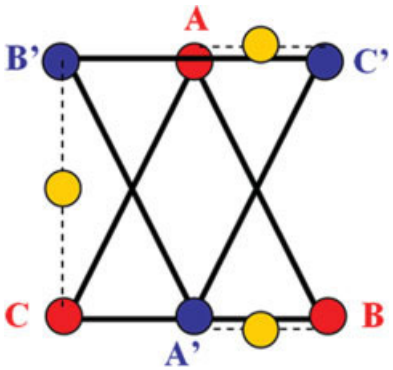

Figure A5. The nearest object with C_i symmetry obtained by the 4th permutation. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

For example, consider a 5-points structure - a,b,c,d,e - for which we wish to evaluate the degree of the C_s symmetry point group. Here $m = 5$ and $n = 2$:

1. Every point is conjugated only to itself, leading to 5 orbits: (a,a) (b,b) (c,c) (d,d) (e,e).
2. There is one group of 2 points and the other 3 points are conjugated only to themselves. There are 10 permutations: ((a,b) (c,c) (d,d) (e,e); (a,c) (b,b) (d,d) (e,e); (b,d) (a,a) (c,c) (e,e); etc.)
3. There are two group of 2 points conjugated to each other, and each of the remaining points is conjugated only to itself. There are 15 possible permutations: ((a,b) (c,d) (e,e); (b,e) (d,c) (a,a); (a,b) (d,e) (c,c); etc.)

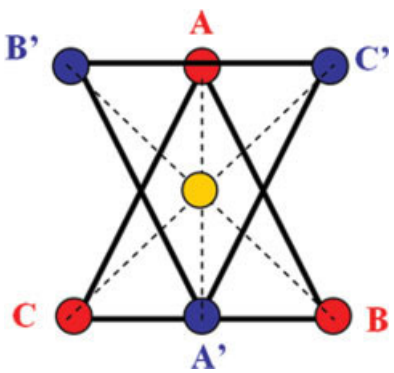

Figure A4. The nearest object with C_i symmetry obtained by the first permutation. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

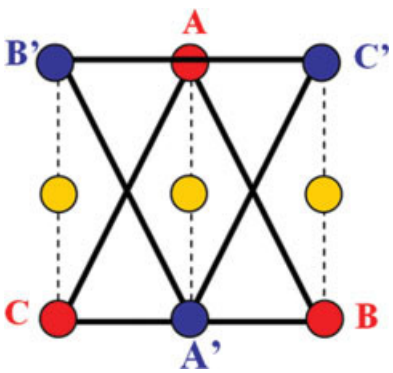

Figure A6. The nearest object with C_i symmetry obtained by the second permutation. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

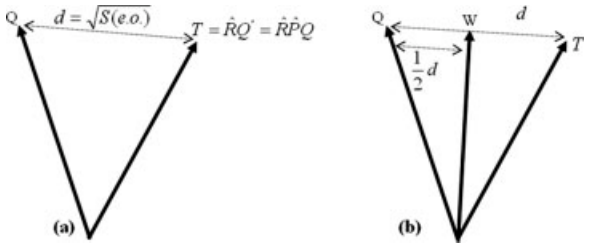

Figure B1. Graphical representation for the origin of the factor 4 between *S(enantiomers overlap)* and *S(chiral)*.

These are all the possible permutations (we cannot build, e.g., 3 groups of 2 different points). Thus, for 5 points there are 5! = 120 possible permutations but only 26 are ***G***-symmetry allowed. This analysis saves significant computations time and it is very critical when dealing with large number of points, and guards against obtaining irrelevant symmetry measure values.

Appendix B: The Relation Between *S*(enantiomers overlap) = 4*S*(chiral)

To understand the origin of the factor 4 between *S*(enantiomers overlap) (*S*(*e.o.*)) and the *S*(*chir*), we define the enantiomeric image, T, of Q as $T = \hat{\sigma}Q' = \hat{\sigma}\hat{P}Q$, where $\hat{P}$ is a permutation operator that takes the best pairing between the two structures T and Q' (where Q' is a structure identical to Q but with a different labeling of vertices). Graphically, we can describe *S*(*e.o.*) on a 3N hypersphere[14] as in Fig. B1(a). In terms of the CSM we recall that by definition it is the square of the minimal distance between the original object, Q, to the closest object that is achiral, W. As is seen in Fig. B1(b) in terms of Q and T, W is: $W = \frac{1}{2}(Q+T) = \frac{1}{2}(Q + \hat{\sigma}\hat{P}Q)$, and therefore $S(Chirc) = \left(\frac{d}{2}\right)^2 = \frac{d^2}{4} = \frac{S(e.o.)}{4}$.

Note added in proof

Upon completion of this report we have found a way to further reduce the number of needed permutations (see Appendix A): Since all of the symmetry groups in this report are cyclic (that is, all of the symmetry operations of the group are different powers of the same basic symmetry operation, $g_x = g_0^x,\ x = 1., 2, ..n$), the permutations must follow the same cyclic way. This means that all the permutations are different powers of the same basic permutation, $P_x = P_0^x,\ x = 1., 2, ..n$. In other words, since the permutations for the different operations are linked, for sweeping over the all permutations of all operations, one should find the best permutation for only one operation, and the rest follow simply as its powers. We thank Amir Zayit for computational verification.

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