Nonlinear Analysis: Real World Applications 11 (2010) 1190–1197

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

Short communication

New results of global robust exponential stability of neural networks with delays

Weirui Zhao *, Qing Zhu

Department of Mathematics, Wuhan University of Technology, Wuhan, 430070, China

ARTICLE INFO

Article history:
Received 18 March 2008
Accepted 12 January 2009

Keywords:
Neural networks
Delays
Global robust stability

ABSTRACT

In this paper, the global robust exponential stability of interval neural networks with delays is investigated. Employing homeomorphism techniques and Lyapunov functions, we establish some sufficient conditions for the existence, uniqueness, and global robust exponential stability of the equilibrium point for delayed neural networks. It is shown that the obtained results improve and generalize the previously published results.

1. Introduction

Recently, the dynamical behaviors of delayed neural networks have attracted increasing attention due to their applicability in some optimization problems and pattern recognition problems. In some of these applications, it is required to ensure that the equilibrium point of the designed neural network is unique and globally stable. Thus, the stability of neural networks with delays, such as Hopfield neural networks, cellular neural networks, bidirectional associative memory neural networks, has been extensively investigated and various sufficient conditions of stability have been presented for delayed neural networks (see, for example, [1–9]). However, stability may be destroyed by some unavoidable uncertainty caused by the existence of modeling errors, external disturbance and parameter fluctuation, which would lead to complex dynamical behaviors. Thus, it is important to investigate the robustness of neural networks against such uncertainties and deviations. Many researchers have studied robust stability properties of neural networks with delays and presented various sufficient conditions for the uniqueness and global robust stability of the equilibrium point of neural networks with delays (see, for example, [10–21]). The authors of [10,15] analyzed the global asymptotic robust stability of delayed neural networks. In addition, the authors of [3] analyzed the locally asymptotic robust stability of stochastic time delay differential systems. To the best of our knowledge, few authors have considered the global robust exponential stability of delayed neural networks. In this paper, we focus on the global robust exponential stability of the delayed neural networks. By applying homeomorphism techniques and constructing a suitable Lyapunov functional, we obtain some new criteria for the global robust exponential stability of delayed neural networks with unbounded activation functions. By comparisons with some previous results, it is shown that our results generalize and improve some results in [10,12,15,19].

The organization of this paper is as follows. In the next section, some problem formulations and preliminaries are given. The existence and uniqueness of the equilibrium point are addressed by applying homeomorphism techniques, and then we present a few new sufficient conditions ensuring the global robust exponential stability of delayed neural networks in Section 3. Then, in Section 4, comparisons between our results and the previous results in the literature are given. In addition, an example is also given to illustrate the effectiveness of our results. Finally, some conclusions are drawn in Section 5.

* Corresponding author.
E-mail address: wrzhao@whut.edu.cn (W. Zhao).

1468-1218/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.nonrwa.2009.01.008

2. Problem formulations and preliminaries

In this paper, we address the following delayed neural network model defined by the following equations:

$$\dot{x}(t) = -Cx(t) + Af(x(t)) + Bf(x(t-\tau)) + U, \tag{1}$$

where $C = \mathrm{diag}(c_1, c_2, \dots, c_n) > 0$, $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^{\mathrm{T}}$ is the state vector associated with the neurons, $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ are the connection weight and the delayed connection weight matrices, respectively; $U = (u_1, u_2, \dots, u_n)^{\mathrm{T}} \in R^n$ is a constant external input vector, τ is a transmission delay, and f_j is activation functions, $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \dots, f_n(x_n(t)))^{\mathrm{T}} \in R^n, f(x(t-\tau)) = (f_1(x_1(t-\tau_1)), f_2(x_2(t-\tau_2)), \dots, f_n(x_n(t-\tau_n)))^{\mathrm{T}} \in R^n$.

In order to establish the stability conditions for neural network (1), we have the usual assumption on the functions.

There exist some positive constants G_i such that

$$0 \le \frac{f_i(x) - f_i(y)}{x - y} \le G_i, \quad \forall x, y \in R, x \ne y, i = 1, 2, \dots, n,$$

where the positive number G_i denotes a Lipschitz constant. This class of functions will be denoted by $f \in \Im$.

The quantities c_i, a_{ij}, and b_{ij} may be considered as intervalized as follows

$$\begin{cases} C_I := [\underline{C}, \overline{C}] = \{C = \mathrm{diag}(c_i) : \underline{C} \le C \le \overline{C}, \text{ i.e., } \underline{c}_i \le c_i \le \overline{c}_i, i = 1, 2, \dots, n\}, \\ A_I := [\underline{A}, \overline{A}] = \{A = (a_{ij})_{n\times n} : \underline{A} \le A \le \overline{A}, \text{ i.e., } \underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}, i, j = 1, 2, \dots, n\}, \\ B_I := [\underline{B}, \overline{B}] = \{B = (b_{ij})_{n\times n} : \underline{B} \le B \le \overline{B}, \text{ i.e., } \underline{b}_{ij} \le b_{ij} \le \overline{b}_{ij}, i, j = 1, 2, \dots, n\}. \end{cases} \tag{2}$$

In order to derive sufficient conditions for the global robust exponential stability of the neural network (1), we will need the following definition and lemmas:

Definition 2.1. The neural network model given by (1) with the parameter ranges defined by (2) is globally robust exponentially stable if the unique equilibrium point $u^* = (u_1^*, u_2^*, \dots, u_n^*)^{\mathrm{T}}$ of the model is globally exponentially stable for all $A \in A_I$, $B \in B_I$, $C \in C_I$.

Lemma 2.2 ([7]). *It is given any real matrices X, Y, C of appropriate dimensions and a scalar $\epsilon_0 > 0$, where $C > 0$. Then the following inequality holds:*

$$X^{\mathrm{T}}Y + Y^{\mathrm{T}}X \le \epsilon_0 X^{\mathrm{T}}CX + \frac{1}{\epsilon_0} Y^{\mathrm{T}}C^{-1}Y.$$

In particular, if X and Y are vectors, $X^{\mathrm{T}}Y \le (X^{\mathrm{T}}X + Y^{\mathrm{T}}Y)/2$.

Lemma 2.3 ([6]). *Continuous map $H(x) : R^n \to R^n$ is homeomorphic if:*

1. *$H(x)$ is injective;*
2. *$\lim_{\|x\|\to\infty} \|H(x)\| = \infty$.*

Lemma 2.4 ([12]). *For $\forall A \in [\underline{A}, \overline{A}]$, $\forall B \in [\underline{B}, \overline{B}]$, we have*

$$\|A\|_2 \le \|A^*\|_2 + \|A_*\|_2, \qquad \|B\|_2 \le \|B^*\|_2 + \|B_*\|_2,$$

where $A^* = \frac{\overline{A}+\underline{A}}{2}$, $A_* = \frac{\overline{A}-\underline{A}}{2}$, $B^* = \frac{\overline{B}+\underline{B}}{2}$, $B_* = \frac{\overline{B}-\underline{B}}{2}$.

3. Main results and their proof

In this section, by using homeomorphism techniques and the Lyapunov functional, several conditions on the existence, uniqueness, and global robust exponential stability of the equilibrium point of neural network (1) are presented.

Theorem 3.1. *Let $f \in \Im$. The neural network (1) has a unique equilibrium point if there exists a positive definite diagonal matrix $P = \mathrm{diag}(p_1, p_2, \dots, p_n)$, such that*

$$\Omega = -2P\underline{C}G^{-1} - S + 2\|P\|(\|B^*\|_2 + \|B_*\|_2)I < 0,$$

where I is the identity matrix, $S = (s_{ij})_{n\times n}$ with $s_{ii} = -2p_i\overline{a}_{ii}$, $s_{ij} = -\max(|p_i\overline{a}_{ij} + p_j\overline{a}_{ji}|, |p_i\underline{a}_{ij} + p_j\underline{a}_{ji}|)$ for $i \ne j$.

Proof. *Step* 1. Let $x^* = (x_1^*, x_2^*, \dots, x_n^*)^{\mathrm{T}}$ denote an equilibrium point of neural network (1). We have:

$$-c_i x_i^* + \sum_{j=1}^{n} (a_{ij} + b_{ij}) f_j(x_j^*) + u_i = 0, \quad i = 1, 2, \dots, n. \tag{3}$$

Define the map $H(x)$ as follows:

$$H(x) = (H_1(x), H_2(x), \dots, H_n(x))^{\mathrm{T}},$$

where $H_i(x) = -c_i x_i + \sum_{j=1}^{n}(a_{ij} + b_{ij}) f_j(x_j) + u_i$.

Obviously, (3) has a unique solution if and only if $H(x)$ is a homeomorphism on R^n. Under the assumptions on the activation functions, $x \neq y$ will imply two cases:

Case (i): $x \neq y$ and $f(x) - f(y) = 0$. We have

$$H(x) - H(y) = -C(x - y) \neq 0,$$

which gives $H(x) \neq H(y)$.

Case (ii): $x \neq y$ and $f(x) - f(y) \neq 0$. In this case, by Lemma 2.4, we have

$$\begin{aligned}
\Delta &= 2(f(x) - f(y))^{\mathrm{T}} P(H(x) - H(y)) = -2\sum_{i=1}^{n}(f_i(x_i) - f_i(y_i)) p_i (c_i x_i - c_i y_i) \\
&\quad + (f(x) - f(y))^{\mathrm{T}}(PA + A^{\mathrm{T}}P)(f(x) - f(y)) + 2(f(x) - f(y))^{\mathrm{T}} PB(f(x) - f(y)) \\
&\leq -2\sum_{i=1}^{n}(f_i(x_i) - f_i(y_i)) \frac{c_i p_i}{G_i}(f_i(x_i) - f_i(y_i)) \\
&\quad + (f(x) - f(y))^{\mathrm{T}}(PA + A^{\mathrm{T}}P)(f(x) - f(y)) + 2(f(x) - f(y))^{\mathrm{T}} PB(f(x) - f(y)) \\
&\leq -2|f(x) - f(y)|^{\mathrm{T}} P\underline{C}G^{-1}|f(x) - f(y)| - |f(x) - f(y)|^{\mathrm{T}} S|f(x) - f(y)| \\
&\quad + 2\|P\|\|B\|_2 (f(x) - f(y))^{\mathrm{T}}(f(x) - f(y)) \\
&\leq -2|f(x) - f(y)|^{\mathrm{T}} P\underline{C}G^{-1}|f(x) - f(y)| - |f(x) - f(y)|^{\mathrm{T}} S|f(x) - f(y)| \\
&\quad + 2\|P\|(\|B^*\|_2 + \|B_*\|_2)(f(x) - f(y))^{\mathrm{T}}(f(x) - f(y)) \\
&\leq |f(x) - f(y)|^{\mathrm{T}} \Omega |f(x) - f(y)| < 0.
\end{aligned} \tag{4}$$

Hence, we have proved that $H(x) \neq H(y)$ when $x \neq y$.

Step 2. We will show that $\|H(x)\| \to \infty$ when $\|x\| \to \infty$. In (4), let $y = 0$, which yields

$$2(f(x) - f(0))^{\mathrm{T}} P(H(x) - H(0)) \leq -\lambda(f(x) - f(0))^{\mathrm{T}}(f(x) - f(0)), \tag{5}$$

where $\lambda > 0$ is the minimum eigenvalue of the matrix $-\Omega$. It follows from (5) that

$$\lambda\|f(x) - f(0)\|_2^2 \leq 2\|f(x) - f(0)\|_\infty \sum_{i=1}^{n}|H_i(x) - H_i(0)| p_i.$$

By the fact

$$\|f(x)\|_\infty - \|f(0)\|_\infty \leq \|f(x) - f(0)\|_\infty \leq \|f(x) - f(0)\|_2,$$

we obtain

$$\lambda\|f(x)\|_\infty - \lambda\|f(0)\|_\infty \leq \lambda\|f(x) - f(0)\|_2 \leq 2\max_{1\leq i\leq n} p_i \|H(x) - H(0)\|_1 \leq 2\max_{1\leq i\leq n} p_i (\|H(x)\|_1 + \|H(0)\|_1).$$

Thus,

$$\|H(x)\|_1 \geq \frac{1}{2\max\limits_{1\leq i\leq n} p_i}(\lambda\|f(x)\|_\infty - \lambda\|f(0)\|_\infty) - \|H(0)\|_1,$$

from which it can be easily concluded that $\|H(x)\| \to \infty$ as $\|f(x)\| \to \infty$, or equivalently $\|H(x)\| \to \infty$ as $\|x\| \to \infty$. Hence, by Lemma 2.3, $H(x)$ must be a homeomorphism on R^n. That gives that there is a unique point $x = x^*$ such that $H(x^*) = 0$. Therefore, the neural network (1) has a unique equilibrium point. (It should be pointed out here that, in the above proof, we have assumed that the activation functions are unbounded. The same results can be shown to hold for bounded activation functions.) $\square$

For convenience, we will shift the equilibrium point x^* of the neural network (1) to the origin by using the transformation

$$y_i(t) = x_i(t) - x_i^*, g_i(y_i(t)) = f_i(y_i(t) + x_i^*) - f_i(x_i^*), \quad i = 1, 2, \dots, n.$$

The neural network (1) can be transformed into the following matrix–vector form:

$$\dot{y}(t) = -Cy(t) + Ag(y(t)) + Bg(y(t - \tau)). \tag{6}$$

In order to gain the main results, we give the following lemma.

Lemma 3.2. *Assume that all conditions of Theorem 3.1 hold, then there exist a positive number ϵ_1 and a positive definite diagonal matrix $W = \mathrm{diag}(w_1, w_2, \ldots, w_n)$ such that $\frac{\mathrm{d}V(t)}{\mathrm{d}t}|_{(6)} \le -\frac{\epsilon_1}{2} y^{\mathrm{T}}(t)\underline{C}y(t)$, where*

$$V(y(t), t) = \frac{\epsilon_1}{2} y^{\mathrm{T}}(t)y(t) + 2\sum_{i=1}^{n} p_i \int_0^{y_i(t)} g_i(s)\mathrm{d}s + \sum_{i=1}^{n} \int_{t-\tau_i}^{t} g_i^2(y_i(\eta))w_i \mathrm{d}\eta. \tag{7}$$

Proof. Let $V_1(y(t), t) = \frac{1}{2} y^{\mathrm{T}}(t)y(t)$, $V_2(y(t), t) = 2\sum_{i=1}^{n} p_i \int_0^{y_i(t)} g_i(s)\mathrm{d}s + \sum_{i=1}^{n} \int_{t-\tau_i}^{t} g_i^2(y_i(\eta))w_i \mathrm{d}\eta$ for some positive definite diagonal matrix W. Define the Lyapunov functional as follows

$$V(y(t), t) = \epsilon_1 V_1(y(t), t) + V_2(y(t), t),$$

where the scalar $\epsilon_1 > 0$ and the matrix W will be determined later. Then we have

$$\dot{V}(y(t), t) = \epsilon_1 \dot{V}_1(y(t), t) + \dot{V}_2(y(t), t), \tag{8}$$

where

$$\begin{aligned} \dot{V}_1(y(t), t) &= -y^{\mathrm{T}}(t)Cy(t) + y^{\mathrm{T}}(t)Ag(y(t)) + y^{\mathrm{T}}(t)Bg(y(t-\tau)), \\ \dot{V}_2(y(t), t) &= -2g^{\mathrm{T}}(y(t))PCy(t) + 2g^{\mathrm{T}}(y(t))PAg(y(t)) + 2g^{\mathrm{T}}(y(t))PBg(y(t-\tau)) \\ &\quad + g^{\mathrm{T}}(y(t))Wg(y(t)) - g^{\mathrm{T}}(y(t-\tau))Wg(y(t-\tau)). \end{aligned} \tag{9}$$

We can rewrite $\dot{V}_1$ as the following format

$$\dot{V}_1(y(t), t) = -y^{\mathrm{T}}(t)Cy(t) + \left(y^{\mathrm{T}}(t)\frac{C^{\frac{1}{2}}}{\sqrt{2}}\right)\sqrt{2}C^{-\frac{1}{2}}Ag(y(t)) + \left(y^{\mathrm{T}}(t)\frac{C^{\frac{1}{2}}}{\sqrt{2}}\right)\sqrt{2}C^{-\frac{1}{2}}Bg(y(t-\tau)),$$

by Lemmas 2.2 and 2.4, we gain

$$\begin{aligned} \dot{V}_1(y(t), t) &\le -\frac{1}{2} y^{\mathrm{T}}(t)Cy(t) + g^{\mathrm{T}}(y(t))A^{\mathrm{T}}C^{-1}Ag(y(t)) + g^{\mathrm{T}}(y(t-\tau))B^{\mathrm{T}}C^{-1}Bg(y(t-\tau)) \\ &\le -\frac{1}{2} y^{\mathrm{T}}(t)Cy(t) + \|C^{-1}\|\|A\|_2^2 g^{\mathrm{T}}(y(t))g(y(t)) + \|C^{-1}\|\|B\|_2^2 g^{\mathrm{T}}(y(t-\tau))g(y(t-\tau)) \\ &\le -\frac{1}{2} y^{\mathrm{T}}(t)Cy(t) + \|C^{-1}\|(\|A^*\|_2 + \|A_*\|_2)^2 g^{\mathrm{T}}(y(t))g(y(t)) \\ &\quad + \|C^{-1}\|(\|B^*\|_2 + \|B_*\|_2)^2 g^{\mathrm{T}}(y(t-\tau))g(y(t-\tau)) \\ &\le -\frac{1}{2} y^{\mathrm{T}}(t)\underline{C}y(t) + Mg^{\mathrm{T}}(y(t))g(y(t)) + Mg^{\mathrm{T}}(y(t-\tau))g(y(t-\tau)), \end{aligned} \tag{10}$$

where $M = \max(\|\underline{C}^{-1}\|(\|A^*\|_2 + \|A_*\|_2)^2, \|\underline{C}^{-1}\|(\|B^*\|_2 + \|B_*\|_2)^2) \ge 0$.

Since $x_i g_i(x_i) \ge G_i^{-1}(g_i(x_i))^2$, we have

$$-2g^{\mathrm{T}}(y(t))PCy(t) \le -2g^{\mathrm{T}}(y(t))PCG^{-1}g(y(t)).$$

Thus, we can gain

$$\begin{aligned} \dot{V}_2(y(t), t) &\le -2g^{\mathrm{T}}(y(t))PCG^{-1}g(y(t)) + 2g^{\mathrm{T}}(y(t))PAg(y(t)) + 2g^{\mathrm{T}}(y(t))PBg(y(t-\tau)) \\ &\quad + g^{\mathrm{T}}(y(t))Wg(y(t)) - g^{\mathrm{T}}(y(t-\tau))Wg(y(t-\tau)) \\ &\le -g^{\mathrm{T}}(y(t))(2P\underline{C}G^{-1})g(y(t)) - |g^{\mathrm{T}}(y(t))|S|g(y(t))| \\ &\quad + 2\|P\|\|B\|_2 g^{\mathrm{T}}(y(t))g(y(t-\tau)) + g^{\mathrm{T}}(y(t))Wg(y(t)) - g^{\mathrm{T}}(y(t-\tau))Wg(y(t-\tau)). \end{aligned} \tag{11}$$

Moreover, we have

$$\begin{aligned} 2\|P\|\|B\|_2 g^{\mathrm{T}}(y(t))g(y(t-\tau)) &\le \|P\|\|B\|_2(g^{\mathrm{T}}(y(t))g(y(t)) + g^{\mathrm{T}}(y(t-\tau))g(y(t-\tau))) \\ &\le \|P\|(\|B^*\|_2 + \|B_*\|_2)(g^{\mathrm{T}}(y(t))g(y(t)) + g^{\mathrm{T}}(y(t-\tau))g(y(t-\tau))). \end{aligned} \tag{12}$$

Therefore, it follows from (11) and (12) that

$$\begin{aligned} \dot{V}_2(y(t), t) &\le |g^{\mathrm{T}}(y(t))|(-2P\underline{C}G^{-1} - S + W + \|P\|(\|B^*\|_2 + \|B_*\|_2))|g(y(t))| \\ &\quad - g^{\mathrm{T}}(y(t-\tau))(W - \|P\|(\|B^*\|_2 + \|B_*\|_2))g(y(t-\tau))). \end{aligned} \tag{13}$$

Since $\Omega < 0$, there exists $\epsilon_2 > 0$ such that $\Omega + 2\epsilon_2 I < 0$. If we define $W = \|P\|(\|B^*\|_2 + \|B_*\|_2)I + \epsilon_2 I$, which is a positive definite diagonal matrix, we can have

$$\dot{V}_2(y(t),t) \le -\epsilon_2 g^{\mathrm{T}}(y(t))g(y(t)) - \epsilon_2 g^{\mathrm{T}}(y(t-\tau))g(y(t-\tau)).$$

If we choose

$$\epsilon_1 = \begin{cases} \dfrac{\epsilon_2}{2M} & M > 0, \\ 1 & M = 0, \end{cases}$$

then $\dot{V}(x(t),t) \le -\frac{\epsilon_1}{2} y^{\mathrm{T}}(t)\underline{C}y(t)$. □

Furthermore, we have the following theorem.

Theorem 3.3. *Assume that all conditions of Theorem 3.1 hold, then the neural network (1) is globally robust exponentially stable.*

Proof. Let $c = \min_{1\le i\le n} \underline{c}_i$, $\overline{\tau} = \max_{1\le i\le n} \tau_i$. Obviously, $V(y(t),t)$ which is defined in Lemma 3.2, is a positive definite and radially unbounded Lyapunov functional.

We can choose a positive number $\epsilon > 0$ satisfying the following condition:

$$\epsilon\epsilon_1 - \epsilon_1 c + 2\epsilon\|GP\| + 2\epsilon\overline{\tau}e^{\epsilon\overline{\tau}}\|G^2 W\| < 0. \tag{14}$$

By Lemma 3.2, we can get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(e^{\epsilon t}V(y(t),t)) &= \epsilon e^{\epsilon t}V(y(t),t) + e^{\epsilon t}\dot{V}(y(t),t) \\ &\le \epsilon e^{\epsilon t}\left(\frac{\epsilon_1}{2}y^{\mathrm{T}}(t)y(t) + 2\sum_{i=1}^{n} p_i \int_0^{y_i(t)} g_i(s)\mathrm{d}s + \sum_{i=1}^{n}\int_{t-\tau_i}^{t} g_i^2(y_i(\eta))w_i\mathrm{d}\eta\right) - \frac{\epsilon_1 e^{\epsilon t}}{2}y^{\mathrm{T}}(t)\underline{C}y(t) \\ &\le \frac{1}{2}e^{\epsilon t}\left(\epsilon\epsilon_1 y^{\mathrm{T}}(t)y(t) - \epsilon_1 y^{\mathrm{T}}(t)\underline{C}y(t) + 4\epsilon\sum_{i=1}^{n} p_i\int_0^{y_i(t)} g_i(s)\mathrm{d}s\right) \\ &\quad + \epsilon e^{\epsilon t}\sum_{i=1}^{n}\int_{t-\tau_i}^{t} g_i^2(y_i(\eta))w_i\mathrm{d}\eta. \end{aligned} \tag{15}$$

Since $\sum_{i=1}^{n} p_i \int_0^{y_i(t)} g_i(s)\mathrm{d}s \le \sum_{i=1}^{n} p_i \int_0^{y_i(t)} G_i s\mathrm{d}s \le \frac{1}{2}y^{\mathrm{T}}(t)GPy(t)$, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(e^{\epsilon t}V(y(t),t)) &\le \frac{1}{2}e^{\epsilon t}(\epsilon\epsilon_1 y^{\mathrm{T}}(t)y(t) - \epsilon_1 y^{\mathrm{T}}(t)\underline{C}y(t) + 2\epsilon y^{\mathrm{T}}(t)GPy(t)) + \epsilon e^{\epsilon t}\sum_{i=1}^{n}\int_{t-\tau_i}^{t} g_i^2(y_i(\eta))w_i\mathrm{d}\eta \\ &\le \frac{1}{2}e^{\epsilon t}(\epsilon\epsilon_1 - \epsilon_1 c + 2\epsilon\|GP\|)y^{\mathrm{T}}(t)y(t) + \epsilon e^{\epsilon t}\sum_{i=1}^{n}\int_{t-\tau_i}^{t} g_i^2(y_i(\eta))w_i\mathrm{d}\eta. \end{aligned} \tag{16}$$

Integrating both sides of (16) from 0 to an arbitrary positive number s, we can obtain

$$e^{\epsilon s}V(y(s),s) - V(y(0),0) \le \frac{1}{2}(\epsilon\epsilon_1 - \epsilon_1 c + 2\epsilon\|GP\|)\int_0^s e^{\epsilon t}y^{\mathrm{T}}(t)y(t)\mathrm{d}t + \epsilon\int_0^s e^{\epsilon t}\sum_{i=1}^{n}\int_{t-\tau_i}^{t} g_i^2(y_i(\eta))w_i\mathrm{d}\eta\mathrm{d}t. \tag{17}$$

Estimating the second term on the right-hand side of (17) by changing the integrals, we have

$$\begin{aligned} \epsilon\int_0^s e^{\epsilon t}\sum_{i=1}^{n}\int_{t-\tau_i}^{t} g_i^2(y_i(\eta))w_i\mathrm{d}\eta\mathrm{d}t &= \epsilon\sum_{i=1}^{n}\int_0^s e^{\epsilon t}\int_{t-\tau_i}^{t} g_i^2(y_i(\eta))w_i\mathrm{d}\eta\mathrm{d}t \\ &\le \epsilon\sum_{i=1}^{n}\int_{-\overline{\tau}}^{s}\int_{\max[\eta,0]}^{\min[\eta+\overline{\tau},s]} e^{\epsilon t}\mathrm{d}t g_i^2(y_i(\eta))w_i\mathrm{d}\eta \\ &\le \epsilon\sum_{i=1}^{n}\int_{-\overline{\tau}}^{s}\left(\int_{\eta}^{\eta+\overline{\tau}} e^{\epsilon t}\mathrm{d}t\right)g_i^2(y_i(\eta))w_i\mathrm{d}\eta \\ &\le \epsilon\sum_{i=1}^{n}\int_{-\overline{\tau}}^{s}\overline{\tau}e^{\epsilon(\eta+\overline{\tau})}g_i^2(y_i(\eta))w_i\mathrm{d}\eta \\ &\le \epsilon\sum_{i=1}^{n}\int_{-\overline{\tau}}^{s}\overline{\tau}e^{\epsilon(\eta+\overline{\tau})}y_i^2(\eta)G_i^2 w_i\mathrm{d}\eta \end{aligned}$$

$$\leq \epsilon\bar{\tau}\mathrm{e}^{\epsilon\bar{\tau}}\|G^2W\|\int_{-\bar{\tau}}^{s}\mathrm{e}^{\epsilon\eta}y^{\mathrm{T}}(\eta)y(\eta)\mathrm{d}\eta$$

$$\leq \epsilon\bar{\tau}\mathrm{e}^{\epsilon\bar{\tau}}\|G^2W\|\left(\int_{-\bar{\tau}}^{0}\mathrm{e}^{\epsilon\eta}y^{\mathrm{T}}(\eta)y(\eta)\mathrm{d}\eta+\int_{0}^{s}\mathrm{e}^{\epsilon\eta}y^{\mathrm{T}}(\eta)y(\eta)\mathrm{d}\eta\right). \tag{18}$$

It follows from (14), (17) and (18) that

$$\begin{aligned}\mathrm{e}^{\epsilon s}V(y(s),s)-V(y(0),0) &\leq \frac{1}{2}(\epsilon\epsilon_1-\epsilon_1 c+2\epsilon\|GP\|+2\epsilon\bar{\tau}\mathrm{e}^{\epsilon\bar{\tau}}\|G^2W\|)\int_0^s\mathrm{e}^{\epsilon t}y^{\mathrm{T}}(t)y(t)\mathrm{d}t\\ &\quad+\epsilon\bar{\tau}\mathrm{e}^{\epsilon\bar{\tau}}\|G^2W\|\int_{-\bar{\tau}}^{0}\mathrm{e}^{\epsilon t}y^{\mathrm{T}}(t)y(t)\mathrm{d}t\\ &\leq\left(\epsilon\bar{\tau}\mathrm{e}^{\epsilon\bar{\tau}}\|G^2W\|\int_{-\bar{\tau}}^{0}\mathrm{e}^{\epsilon t}\mathrm{d}t\right)\|\phi\|^2\equiv M_1\|\phi\|^2.\end{aligned} \tag{19}$$

Thus, we have

$$V(y(t),t)\leq(V(y(0),0)+M_1\|\phi\|^2)\mathrm{e}^{-\epsilon t},\quad \forall t>0, \tag{20}$$

where

$$\begin{aligned}V(y(0),0)&=\frac{\epsilon_1}{2}y^{\mathrm{T}}(0)y(0)+2\sum_{i=1}^{n}p_i\int_0^{y_i(0)}g_i(s)\mathrm{d}s+\sum_{i=1}^{n}\int_{-\tau_i}^{0}g_i^2(y_i(\eta))w_i\mathrm{d}\eta\\ &\leq\frac{1}{2}(\epsilon_2+2\|PG\|+2\bar{\tau}\|G^2W\|)\|\phi\|^2\equiv M_2\|\phi\|^2.\end{aligned} \tag{21}$$

By (7) and (20), we can obtain

$$\frac{\epsilon_1}{2}y^{\mathrm{T}}(t)y(t)\leq V(y(t),t)\leq(M_1+M_2)\|\phi\|^2\mathrm{e}^{-\epsilon t},\quad\forall t>0.$$

That is

$$\|y(t)\|\leq\sqrt{\frac{2}{\epsilon_1}(M_1+M_2)}\|\phi\|\mathrm{e}^{-\frac{\epsilon}{2}t},$$

which implies the neural network (1) is globally robust exponentially stable. This completes the proof. □

By Theorem 3.3, we have the following corollary.

Corollary 3.4. *Let $f\in\Im$. The neural network (1) is globally robust exponentially stable if there exist a positive definite diagonal matrix $P=\mathrm{diag}(p_1,p_2,\ldots,p_n)$ and a positive definite symmetric matrix H, such that*

$$\Omega_1=-2P\underline{C}G^{-1}-S+H+\|P\|^2\|H^{-1}\|(\|B^*\|_2+\|B_*\|_2)^2I<0,$$

where $S=(s_{ij})_{n\times n}$ with $s_{ii}=-2p_i\bar{a}_{ii}$, $s_{ij}=-\max(|p_i\bar{a}_{ij}+p_j\bar{a}_{ji}|,|p_i\underline{a}_{ij}+p_j\underline{a}_{ji}|)$ for $i\neq j$.

Proof. By Lemma 2.2, we have

$$\frac{1}{\|P\|(\|B^*\|_2+\|B_*\|_2)}H+\|P\|(\|B^*\|_2+\|B_*\|_2)H^{-1}\geq 2I,$$

which implies that

$$H+\|P\|^2\|H^{-1}\|(\|B^*\|_2+\|B_*\|_2)^2I\geq H+\|P\|^2(\|B^*\|_2+\|B_*\|_2)^2H^{-1}\geq 2\|P\|(\|B^*\|_2+\|B_*\|_2)I.$$

It follows from the above inequality that

$$-2P\underline{C}G^{-1}-S+2\|P\|(\|B^*\|_2+\|B_*\|_2)I\leq-2P\underline{C}G^{-1}-S+H+\|P\|^2\|H^{-1}\|(\|B^*\|_2+\|B_*\|_2)^2I<0.$$

This result stems from Theorem 3.3. This proof is completed. □

Let $P=I,H=I$ in Corollary 3.4, we have the following corollary.

Corollary 3.5. *Let $f\in\Im$. The neural network (1) is globally robust exponentially stable if:*

(1) *the symmetric matrix $S=(s_{ij})_{n\times n}$ is positive definite,*
(2) $\|B^*\|_2+\|B_*\|_2\leq\sqrt{2r-1}$,

where $r=\min_{1\leq i\leq n}\left(\frac{\underline{c}_i}{G_i}\right)$, and $S=(s_{ij})_{n\times n}$ with $s_{ii}=-2\bar{a}_{ii}$, $s_{ij}=-\max(|\bar{a}_{ij}+\bar{a}_{ji}|,|\underline{a}_{ij}+\underline{a}_{ji}|)$ for $i\neq j$.

4. Comparison and example

In this section, we will compare our results with those in [10,12,15,19], and show the effectiveness of our results by an example.

Firstly, in [19], the authors gave some globally robust exponential stability criteria for the neural network defined by (1) with bounded activation functions. However, in Corollary 3.4, we do not require the activation functions to be bounded. Therefore, results in Corollary 3.4 are less restrictive than those given in [19]. On the other hand, in Corollary 3.5, we have derived that the neural network defined by (1) is globally robust exponentially stable. However, the authors of [12] have only proved that the neural network defined by (1) is globally asymptotically robust stable under the same conditions and bounded activation functions. Therefore, our results have generalized those results in [12].

Secondly, the following globally asymptotically robust stability conditions are presented in [10,15].

Theorem 4.1 (*[10]*). *Let $f \in \Im$. The neural network (1) is globally asymptotically robust stable if:*

(1) *the symmetric matrix $S = (s_{ij})_{n\times n}$ is positive definite,*
(2) $\|B^*\|_2 + \|B_*\|_2 \le r$,

where $r = \min_{1\le i\le n}\left(\frac{c_i}{G_i}\right)$, and $S = (s_{ij})_{n\times n}$ with $s_{ii} = -2\overline{a}_{ii}$, $s_{ij} = -\max(|\overline{a}_{ij} + \overline{a}_{ji}|, |\underline{a}_{ij} + \underline{a}_{ji}|)$ for $i \ne j$.

Theorem 4.2 (*[15]*). *Let $f \in \Im$. The neural network (1) is globally asymptotically robust stable if there exists a positive definite diagonal matrix $P = \mathrm{diag}(p_1, p_2, \ldots, p_n)$, such that*

$$\Omega = -2rI - S + 2\|P\|(\|B^*\|_2 + \|B_*\|_2)I < 0,$$

where $r = \min_{1\le i\le n}\left(\frac{p_i\underline{c}_i}{G_i}\right)$, $S = (s_{ij})_{n\times n}$ with $s_{ii} = -2p_i\overline{a}_{ii}$, $s_{ij} = -\max(|p_i\overline{a}_{ij} + p_j\overline{a}_{ji}|, |p_i\underline{a}_{ij} + p_j\underline{a}_{ji}|)$ for $i \ne j$.

Remark 4.3. The following inequality holds

$$-2\min_{1\le i\le n}\left(\frac{p_i\underline{c}_i}{G_i}\right)I - S + 2\|P\|(\|B^*\|_2 + \|B_*\|_2)I \ge -2P\underline{C}G^{-1} - S + 2\|P\|(\|B^*\|_2 + \|B_*\|_2)I,$$

which implies that the conditions in [10,15] are stronger than those in our Theorem 3.3. In addition, the authors of [10,15] have only proved that the neural network defined by (1) is globally asymptotically robust stable. However, in Theorem 3.3, we have proved that the neural network is globally robust exponentially stable. Therefore, our results improve those results in [10,15].

In addition, in order to show that our condition provides a new sufficient criterion for determining the equilibrium and stability properties of system (1), we consider the following example.

Example 4.4. Consider a second-order neural network with

$$\begin{aligned} &\overline{C} = \underline{C} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \overline{A} = \begin{pmatrix} -2 & 0.5 \\ 0.5 & -5 \end{pmatrix}, \qquad \underline{A} = \begin{pmatrix} -3 & 0.5 \\ 0.5 & -6 \end{pmatrix}, \\ &\overline{B} = \underline{B} = \begin{pmatrix} 0.75 & 0.75 \\ 0.75 & 0.75 \end{pmatrix}, \qquad f_1(x) = \frac{1}{2}(x + \sin x), \qquad f_2(x) = \frac{1}{2}(x + \cos x). \end{aligned} \tag{22}$$

Firstly, there exists a positive definite diagonal matrix $P = \mathrm{diag}\{1, 0.5\}$, such that

$$S = \begin{pmatrix} 4 & -0.75 \\ -0.75 & 5 \end{pmatrix}, \qquad \Omega = -2P\underline{C}G^{-1} - S + 2\|P\|(\|B^*\|_2 + \|B_*\|_2)I = \begin{pmatrix} -2 & 0.75 \\ 0.75 & -1 \end{pmatrix} < 0.$$

So all conditions given in Theorem 3.3 hold, and the neural network (22) is globally robust exponentially stable.

However, $\|B^*\|_2 + \|B_*\|_2 > r$, which implies that those results obtained in [10] cannot be applied for system (22). Further, those results obtained in [12,19] cannot be applied for system (22) because $f_i(x)$ $(i = 1, 2)$ are unbounded.

5. Conclusions

In this paper, we study the global robust exponential stability of interval neural networks with delays, and derive general sufficient conditions for the global robust exponential stability of delayed neural networks based on the homeomorphic techniques and the Lyapunov method. Comparisons between our results and the previous results have been made. They show that our results improve and generalize the previous ones derived in the literature. An example is also given to illustrate the effectiveness of our results.

References

[1] Y. Hou, T. Liao, J. Yan, Global asymptotic stability for a class of nonlinear neural networks with multiple delays, Nonlinear Anal. 67 (2007) 3037–3040.
[2] X. Li, L. Huang, H. Zhu, Global stability of cellular neural networks with constant and variable delays, Nonlinear Anal. 53 (2003) 319–333.
[3] W. Zhang, B.S. Chen, Q. Li, Feedback stabilization of nonlinear stochastic time-delay systems with state and control-dependent noise, in: Proceedings of the 2004 American Control Conference (ACC), Boston, USA, June 30–July 2 2004, pp. 3689–3692.
[4] W. Zhao, H. Zhang, Globally exponential stability of neural network with constant and variable delays, Phys. Lett. A 352 (2006) 350–357.
[5] W. Zhao, Global exponential stability analysis of Cohen–Grossberg neural network with delays, Commun. Nonlinear Sci. Numer. Simul. 13 (2008) 847–856.
[6] W. Zhao, A. Yan, Stability analysis of neural networks with both variable and unbounded delays, Chaos Solitons Fractals (2007) doi:10.1016/j.chaos.2007.08.017.
[7] W. Zhao, Y. Tan, Harmless delays for global exponential stability of Cohen-Grossberg neural networks, Math. Comput. Simulation 74 (2007) 47–57.
[8] W. Zhao, Dynamics of Cohen-Grossberg neural network with variable coefficients and time-varying delays, Nonlinear Anal. RWA 9 (2008) 1024–1037.
[9] Q. Zhou, Global exponential stability of BAM neural networks with distributed delays and impulses, Nonlinear Anal. RWA (2007) doi:10.1016/j.nonrwa.2007.08.019.
[10] S. Arik, Global robust stability analysis of neural networks with discrete time delays, Chaos Solitons Fractals 26 (2005) 1407–1414.
[11] S. Arik, Global robust stability of delayed neural networks, IEEE Trans. Circuits Syst. I 50 (1) (2003) 156–160.
[12] J. Cao, D. Huang, Y. Qu, Global robust stability of delayed recurrent neural networks, Chaos Solitons Fractals 23 (2005) 221–229.
[13] J. Cao, D. Ho, X. Huang, LMI-based criteria for global robust stability of bidirectional associative memory networks with time delay, Nonlinear Anal. 66 (2007) 1558–1572.
[14] C. Li, X. Liao, R. Zhang, A. Prasad, Global robust exponential stability analysis for interval neural networks with time-varying delays, Chaos Solitons Fractals 25 (2005) 751–757.
[15] N. Ozcan, S. Arik, Global robust stability analysis of neural networks with multiple time delays, IEEE Trans. Circuits Syst. I 53 (1) (2006) 166–176.
[16] L. Rong, LMI-based criteria for robust stability of Cohen-Grossberg neural networks with delay, Phys. Lett. A 339 (2005) 63–73.
[17] V. Singh, A novel global robust stability criterion for neural networks with delay, Phys. Lett. A 337 (2005) 369–373.
[18] V. Singh, Novel global robust stability criterion for neural networks with delay, Chaos Solitons Fractals (2008) doi:10.1016/j.chaos.2008.01.001.
[19] W. Wu, B. Cui, Global robust exponential stability of delayed neural networks, Chaos Solitons Fractals 35 (2008) 747–754.
[20] W. Xiong, B. Xu, Some criteria for robust stability of Cohen-Grossberg neural networks with delays, Chaos Solitons Fractals 36 (2008) 1357–1365.
[21] D. Yang, New delay-dependent global asymptotic stability criteria of delayed Hopfield neural networks, Nonlinear Anal. RWA (2007) doi:10.1016/j.nonrwa.2007.06.008.