Arch. Math. 81 (2003) 255–257
0003–889X/03/030255–03
DOI 10.1007/s00013-003-0823-z

Archiv der Mathematik

On pure subgroups of locally compact abelian groups

By

PETER LOTH

Abstract. In this note, we construct an example of a locally compact abelian group $G = C \times D$ (where C is a compact group and D is a discrete group) and a closed pure subgroup of G having nonpure annihilator in the Pontrjagin dual $\hat{G}$, answering a question raised by Hartman and Hulanicki. A simple proof of the following result is given: Suppose $\mathfrak{K}$ is a class of locally compact abelian groups such that $G \in \mathfrak{K}$ implies that $\hat{G} \in \mathfrak{K}$ and nG is closed in G for each positive integer n. If H is a closed subgroup of a group $G \in \mathfrak{K}$, then H is topologically pure in G exactly if the annihilator of H is topologically pure in $\hat{G}$. This result extends a theorem of Hartman and Hulanicki.

All considered groups are Hausdorff abelian topological groups and will be written additively. For a group G and a positive integer n we shall write $nG = \{ng : g \in G\}$ and $G[n] = \{g \in G : ng = 0\}$. If H is a subgroup of G, the subgroup $\{g \in G : ng \in H\}$ will be written $n^{-1}H$. The Pontrjagin dual of a locally compact abelian (LCA) group G is denoted by $\hat{G}$, and $(\hat{G}, H)$ denotes the annihilator of H in the dual $\hat{G}$. By $\mathbf{Z}$ we mean the additive group of integers, $\mathbf{Z}(n)$ is the cyclic group of order n and $\mathbf{R}$ denotes the additive group of real numbers with the usual topology.

A subgroup H of a group G is called *pure* (*in* G) if for every positive integer n and element $h \in H$, the equation $nx = h$ is solvable in H whenever it is solvable in G. In other words, H is pure in G exactly if $nG \cap H = nH$, or equivalently, if $G[n] + H = n^{-1}H$ for each positive integer n. The literature shows the importance of this concept (see e.g. [2]).

Lemma 1. *Consider the following conditions for an* LCA *group G, a closed subgroup H and a positive integer n:*

(i) $nG \cap H = nH$;
(ii) $\overline{nG \cap H} = \overline{nH}$;
(iii) $\overline{\hat{G}[n] + (\hat{G}, H)} = n^{-1}(\hat{G}, H)$.

Then we have: (i) $\not\Rightarrow$ (ii) $\not\Rightarrow$ (i) *and* (ii) $\Leftrightarrow$ (iii).

Mathematics Subject Classification (1991): Primary 20K27, 22B05; Secondary 20K45, 22D35.

Proof. The first assertion follows from [6] Example 2.4. The annihilator of $\overline{nG} \cap H$ is equal to $\overline{\hat{G}[n] + (\hat{G}, H)}$ by [5] Theorems 23.29(b), 24.10 and 24.22. For $\chi \in \hat{G}$, we have $n\chi \in (\hat{G}, H)$ if and only if $\chi \in (\hat{G}, nH) = (\hat{G}, \overline{nH})$. Therefore, (ii) implies (iii). Since closed subgroups of G are identical if they have the same annihilator (see [5] Theorem 24.10), (iii) implies (ii). $\square$

If G is discrete or compact and H is a closed subgroup of G, then in the above lemma, (i) and (ii) are equivalent and $\hat{G}[n] + (\hat{G}, H)$ is a closed subgroup of $\hat{G}$. Therefore, in this case the subgroup H is pure in G if and only if its annihilator $(\hat{G}, H)$ is pure in $\hat{G}$. Let us say that an LCA group G *has correct purity* if a closed subgroup of G is pure exactly if its annihilator is pure.

In [4], Hartman and Hulanicki established a characterization of closed subgroups of LCA groups whose annihilator is pure and determined conditions ensuring correct purity. Recall that a topological group is said to have *no small subgroups* if there is a neighborhood of 0 which does not contain any nontrivial subgroups. Moskowitz [7] showed that LCA groups without small subgroups have the form $\mathbf{R}^n \times (\mathbf{R}/\mathbf{Z})^m \times D$ where m and n are nonnegative integers and D is a discrete group, and that their dual groups are the compactly generated LCA groups. The following theorem is the main result of Hartman and Hulanicki's paper [4]. We would like to remark that a short proof of this result can be found in [6].

Theorem 2 [4, Hartman and Hulanicki]. *Let $\mathfrak{C}$ denote the class of* LCA *groups which are either compactly generated or have no small subgroups. Suppose G is a group in $\mathfrak{C}$. Then G has correct purity.*

In their paper [4], Hartman and Hulanicki asked whether Theorem 2 is still valid if $\mathfrak{C}$ is replaced by the class $\mathfrak{C}'$ of all LCA groups of the form $C \times D$ where $C \in \mathfrak{C}$ and D is a discrete group (see also [1] p. 95). The following example shows that the answer to this question is no.

Example 3. We construct a group $G \in \mathfrak{C}'$ and a closed pure subgroup H of G having nonpure annihilator. Let p be a prime and n a positive integer. Then we set $G = pB \times D$ where B is the topological direct product of $\aleph_0$ copies of the group $\mathbf{Z}(p^{n+1})$ and D is the subgroup $\bigoplus_{\aleph_0} \mathbf{Z}(p^{n+1})$ of B, taken discrete. Notice that $H = \{(px, px) : x \in D\}$ is a discrete, hence closed subgroup of G. It is clear that $p^k G \cap H = p^k H$ for each positive integer k, therefore H is pure in G. Now assume the annihilator $X = (\hat{G}, H)$ is pure in $\hat{G}$. Then $X + \hat{G}[p^n]$ is closed in $\hat{G}$ and therefore $(X + \hat{G}[p^n])/\hat{G}[p^n]$ is a closed subgroup of the compact group $\hat{G}/\hat{G}[p^n]$. Since $\hat{G}[p^n]$ is σ-compact, $(X + \hat{G}[p^n])/\hat{G}[p^n]$ is topologically isomorphic to $X/X[p^n]$ by [3] Theorem 3.3. We have a topological isomorphism $(X/X[p^n])\hat{} \to (\hat{X}, X[p^n]) = \overline{(p^n \hat{X})}$ (see [5] Theorems 23.25 and 24.22), so the group $p^n \hat{X}$ is discrete by [5] Theorem 23.17. On the other hand, $p^n(G/H)$ is topologically isomorphic to $p^n \hat{X}$ (cf. [5] Theorem 24.3). It follows that $p^n G + H$ is a discrete subgroup of G. But then $p^n G + p^{n-1} H = \bigoplus_{\aleph_0} p^n \mathbf{Z}(p^{n+1}) \times p^n D$ is a discrete subgroup of G, which is impossible. Therefore the annihilator of H is not pure in $\hat{G}$.

Notice that if $G \in \mathfrak{C}'$, then $\hat{G} \in \mathfrak{C}'$ and nG is closed in G for each positive integer n because the groups in $\mathfrak{C}'$ have the form $\mathbf{R}^n \times C \times D$ where C is a compact group and D is a discrete group. A subgroup H of a group G is called *topologically pure* (*in* G) if

$$\overline{nG} \cap H = \overline{nH}$$

for each positive integer n. We say that an LCA group G *has correct topological purity* if a closed subgroup of G is topologically pure exactly if its annihilator is topologically pure. "Purity" and "topological purity" are equivalent concepts in $\mathfrak{C}$ because closed subgroups of groups in $\mathfrak{C}$ are again in $\mathfrak{C}$ (cf. [7] Theorem 2.6), hence the following result extends Theorem 2:

Theorem 4. *Let $\mathfrak{K}$ denote a class of* LCA *groups having the following property: If $G \in \mathfrak{K}$, then $\hat{G} \in \mathfrak{K}$ and nG is closed in G for each positive integer n. Suppose G is a group in $\mathfrak{K}$. Then G has correct topological purity.*

Proof. Suppose H is a topologically pure subgroup of $G \in \mathfrak{K}$. Let n be a positive integer, $\chi \in \hat{G}$ and assume $n\chi \in (\hat{G}, H)$. By Lemma 1, $\chi \in \overline{\hat{G}[n] + (\hat{G}, H)}$, so there exists a net $\{\chi_i\}_{i\in\Lambda}$ in $\hat{G}[n] + (\hat{G}, H)$ which converges to χ. But then $\{n\chi_i\}_{i\in\Lambda}$ converges to $n\chi$, hence $n\chi \in \overline{n(\hat{G}, H)}$. Consequently we have

$$\overline{n\hat{G}} \cap (\hat{G}, H) = n\hat{G} \cap (\hat{G}, H) = \overline{n(\hat{G}, H)},$$

therefore $(\hat{G}, H)$ is topologically pure in $\hat{G}$. The converse follows from duality. $\square$

Acknowledgment. The author would like to thank the referee for the helpful comments and suggestions.

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Received: 4 April 2002

Peter Loth
Department of Mathematics
Sacred Heart University
5151 Park Avenue
Fairfield, Connecticut 06825-1000
USA
lothp@sacredheart.edu