DEF 26, 1 – 17 (2003)

Decisions in Economics and Finance

An efficient binomial method for pricing American options

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Received: 16 July 2001 / Accepted: 19 April 2002

Abstract. We present a new method for obtaining fast and accurate estimates of the price of an American put option by binomial trees. The method is based on the interpolation of suitable values obtained by modifying the contractual strike. A time-saving procedure allows us to derive all the interpolating data from a unique standard backward procedure.

Mathematics Subject Classification (2000): 91B28

Journal of Economic Literature Classification: C63

1. Introduction

In this paper we treat the classical problem of high precision pricing of standard American put options. In the absence of an explicit formula, several methods have been developed to obtain efficient estimates of the price. The main techniques used to this purpose are: Monte Carlo, trees, differential equations and related approximation methods. Here we are interested in binomial trees, for their ability to couple speed and precision.

Besides the well-known backward binomial procedure introduced in the seminal work of Cox, Ross and Rubinstein (1979) and the successive trinomial extension due to Kamrad and Ritchken (1991), several other different methods based on backward procedures have been developed. Among those of special relevance are the "accelerated binomial method" of Breen (1991) and the "BBSR" (Black-Scholes modification of the binomial algorithm with

The authors would like to thank an anonymous referee for helpful comments. We also thank Antonino Zanette for his help in the refinements of the numerical procedures.

Richardson extrapolation) of Broadie and Detemple (1996). According to relevant expert opinion the last seems to be the most efficient combination of speed and precision.

The BBSR applies two-point Richardson extrapolation to adjusted binomial evaluations for a convenient pair, typically $n/2$, n, of even numbers describing the length (number of steps) of binomial trees. The adjustment comes from inserting at every node of the last but one step the Black-Scholes European values of the option in place of those given by the usual binomial continuation values.

We present here another binomial based approach based on a proper interpolation procedure of binomial values. We too apply two-point Richardson extrapolation to adjusted binomial values, but our adjustment follows a strategy inspired by the desire to escape from the evaluation bias induced by the strike specification error. Concerning this point we recall that the idea of recognizing and neutralizing, with proper procedures, sources of error in binomial pricing goes back to Boyle–Lau (1994) in their path-breaking paper on pricing of single barrier options. Later Derman, Kani, Ergener and Bardhan (1995) explicitly distinguished between various sources of error in binomial procedures. Following their line of reasoning we attack directly the strike specification error embedded in the pure binomial values of fixed strike not at the money options. To reach this goal we compute, given n, a set of $2s$ (s being greater than or equal to 3) pure binomial values of $2s$ different options having computational strikes lying exactly on the nodes of the tree around the contractual strike. Intuitively these evaluations are unaffected by strike specification errors and may be used to interpolate at the contractual strike to obtain an adjusted binomial value in turn unaffected by an error of this type. Finally a two-point Richardson extrapolation is applied to a pair of interpolated values. We will use the acronym BIR (binomial interpolation with Richardson extrapolation) for such a procedure.

It is interesting to note that there is no need to run backward procedures for $2s$ different binomial trees, as we are able to obtain the entire set of $2s$ prices by managing just one enlarged tree, or equivalently, and even more simply, one standard binomial tree of $n + 2s$ steps. This in turn gives us a computational speed of BIR at an adequate level as well as a simple implementation.

Before proceeding to American options we tested the precision and speed efficiency of BIR on a random sample of 5000 European options extracted from a distribution whose parameters mimic those applied in the milestone study of Broadie and Detemple. The results of the test confirmed the great superiority of BBSR among the existing tree methods and revealed that BIR is in turn more efficient than BBSR. But, when we pass to American put options we face further sources of error: the loss of opportunity to exercise early in any interval between two discrete times of the tree and the early exercise boundary specification error. These new sources of error drastically reduce the precision of both BBSR and BIR American evaluations in comparison with the analogous European ones. However, while BIR and BBRS remain unequivocally better than competitors, a ranking of the two is no longer clear. Indeed while BIR seems to retain more efficiency if the precision is measured through the mean (absolute) relative error (MRE), the ranking becomes doubtful when we evaluate precision through the root mean squared relative error (RMSRE). Nevertheless a careful look at the data reveals an interesting characteristic of the errors which is common to both methods but deeper for BIR: very few critical cases out of the 5000 of the sample are responsible for the largest part of the overall mean quadratic error in the American case. More precisely the precision of any single evaluation depends on the relevance of the American quality (that is, of the early exercise opportunity) of the option. Roughly speaking, for weak American options (that is, options in which the early exercise opportunity is not relevant) BIR and BBSR behave as in the European case, that is, exhibit high efficiency; on the contrary, for strong American options (with the exception of those where immediate exercise is optimal), the scenario is completely modified and we found (both for BIR and BBSR) big and unstable (with respect to n) errors in comparison with the analogous European ones. This empirical evidence suggested to us the idea that a uniform treatment of any option is not an efficient strategy. It should be intuitively better to separate the critical options from the normal ones and conveniently increase (e.g., multiplying by a given constant β) the number of steps of the tree only for the critical options.

The success of this strategy depends of course on the ability to detect with surgical precision a small enough number of critical options to which the costly (about β^2 times for any critical option) additional effort should be devoted.

An intuitive selection criteria is to choose as critical the not immediately exercisable options having a ratio between the estimated American value and the European one which is greater than a constant α. Without any claim to define optimal rules, but rather searching for a good rule, we choose in this paper $\beta = 4$, $\alpha = 1.5$. Accordingly, less than 100 options out of the 5000 of the sample turned out to be critical, so that the additional burden was only about 30% of the computational time, but with a relevant increase in precision.

With reference to our sample and the given pair of parameters, we measured the MRE and RMSRE efficiency of BIRS (BIR with selection) as well as of BBSRS (BBSR with selection). We found that BIRS is meaningfully more efficient than BIR while BBSR seems to be more efficient than BBSR only for RMSRE while of about the same efficiency for MRE. Finally BIRS is more efficient than BBSRS.

The plan of the paper is the following: Sect. 2 introduces the interpolation procedure to adjust binomial values on which Richardson extrapolation is based; Sect. 3 discusses the technique decisive to grant the computational speed of BIR. Numerical results are presented and discussed in Sect. 4.

2. Basic considerations and the interpolation procedure

We consider a standard at-the-money American put option, with initial value A of the underlying, strike K, time to maturity T, volatility σ of the underlying, priced on an arbitrage free market with risk free interest rate r.

If for such an option the sequence of binomial prices $P(n)$ as a function of the number n of steps of a standard binomial tree, which is the counterpart of the continuous model, displays a regular behavior (as shown in Fig. 1, where we indicate, separately, the odd and even prices of two at-the-money options), the application of an extrapolation procedure usually gives good precision and a high speed estimate of the continuous price.

Often a two-point Richardson extrapolation of the form

$$P = \frac{n_2 P(n_2) - n_1 P(n_1)}{n_2 - n_1}$$

(here n_1, n_2 are distinct natural numbers of the same parity, typically $n_2 = 2n_1$) is applied. It could be said that the regularity of the quantization errors of the sequence of even binomial prices is exploited through the extrapolation to obtain reliable estimates of the price.

The situation changes completely if we consider not-at-the-money (American put) options. Typically, as shown in Fig. 2, the subsequences of odd and even binomial evaluations lose any monotone character.

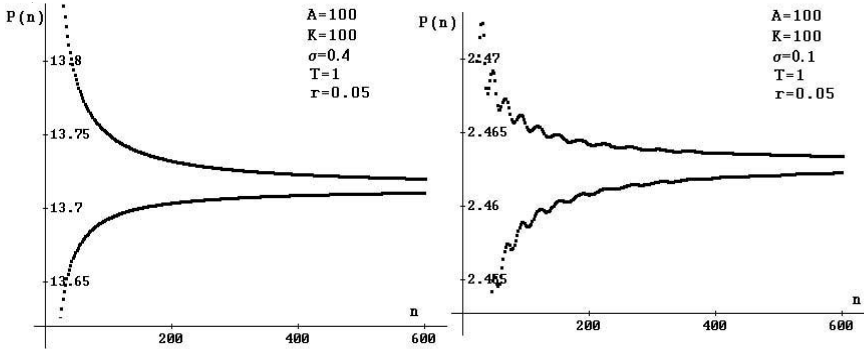

Fig. 1.

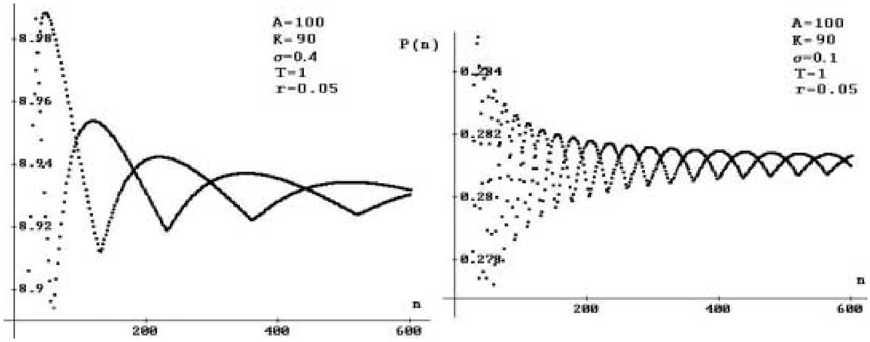

Fig. 2.

Now the naive Richardson extrapolation is no longer applicable. A successful attempt to overcome the problem has been proposed by Broadie and Detemple (1996) with their BBSR model. In brief, they obtain a regularization of the binomial sequence through a convenient adjustment of the pure binomial values. The adjustment comes from inserting at every node of the last but one step the Black-Scholes European values of the option in spite of the continuation values applied in the usual backward procedure. According to the relevant literature this approach seems to be the most efficient in pricing American put options. Here we offer an alternative method to obtain a regular pattern of adjusted binomial values to which Richardson extrapolation may be efficiently applied.

The starting point of our idea is that at least part of the non-regular behavior comes from the so-called strike specification error and that, to overcome the problem, Richardson extrapolation should be applied to a pair of critical values of n (of the same parity) for which the strike specification error is neutralized. This could be done by choosing as critical numbers those values of n for which a row of nodes of the binomial tree lies as close as possible to the level of the contractual strike. It is easy to check that the critical even [odd] values of n may be found via the rule:

$$n_i = \text{even}\left(\frac{(2i)^2\sigma^2 T}{ln^2(\frac{K}{A})}\right) \quad \left[\text{odd}\left(\frac{(2i+1)^2\sigma^2 T}{ln^2(\frac{K}{A})}\right)\right], \qquad i \geq 1$$

where even(x) [odd(x)] denotes the even [odd] integer closest to x.

We remark that this formula is analogous to that used by Boyle–Lau (1994) to derive the sequence of critical numbers to be used in precise pricing of single barrier options. Now, while this method seems to give high precision estimates of the price (see Gaudenzi–Pressacco (2000)), it is computationally efficient only for sufficiently deep in-the-money or out-of-the-money options, while for near at-the-money options the critical pairs may well be too high so as to destroy computational efficiency. Then we are forced to follow another road to reach our goal.

Specifically, consider a binomial tree of n steps (multiple of 4) and a positive integer s. Denote by $K_m = Ae^{m\sigma\sqrt{\frac{T}{n}}}$ $(m = -n, -n+2, \dots, n-2, n)$ the value of the underlying at the nodes of the tree at the time of expiration, and by P_m the binomial prices computed for the option with strike K_m.

We select a grid of a suitable even number of "computational strikes" K_m around the contractual strike K. To this end we consider the largest even integer m^* such that $K_{m^*} \leq K$ (a simple computation shows that m^* is the largest even integer less than or equal to $\frac{\ln\frac{K}{A}}{\sigma\sqrt{\frac{T}{n}}}$), and take K_{m_i} for $m_i = m^*+2(i-s), i = 1, \dots, 2s$. We give an adjusted binomial evaluation $P_{\text{interp}}(n)$ of the price of the option by just interpolating at the contractual strike K the $2s$ selected points $(K_{m_i}, P_{m_i}), i = 1, \dots, 2s$.

We use, as interpolation method, the polynomial interpolation in Lagrange form, i.e., for the l points (x_i, y_i), $i = 1, \dots, l$, with $x_1 < x_2 < \dots < x_l$, their interpolation evaluated at x is

$$f(x) = \sum_{i=1}^{l} y_i \left[\prod_{j\in\{1,\dots,l\}\setminus\{i\}} \frac{(x - x_j)}{(x_i - x_j)} \right].$$

Sequences of adjusted binomial values through the described interpolation procedure seem now to display the same regularity which characterizes a sequence of binomial prices of an at-the-money option as shown in Fig. 3, where to the odd and even values of the binomial prices of Fig. 2 we add the even and odd interpolations evaluated with 10 interpolating points ($s = 5$).

The explanation of the regularity is easy: generated by binomial values free of strike specification error, the interpolated price should inherit the same quality. Then we propose to apply Richardson extrapolation to the adjusted values $P_{\text{interp}}(n/2)$, $P_{\text{interp}}(n)$. It is natural to call this procedure BIR (binomial interpolation with Richardson extrapolation), so we denote by BIR(n) the evaluation obtained this way.

An important modification of the described procedure may yet be necessary. Indeed it is well-known that, for an American put option, there is a critical value $\underline{K}$ of the strike such that for $K > \underline{K}$ the put should be exercised immediately. The price of the option for $K > \underline{K}$ is then $K - A$. In this case obviously not only is the interpolation unnecessary but it also introduces errors in the estimate. However the binomial method allows us to localize $\underline{K}$. After computing the $2s$ prices $P_{m_1}, \dots, P_{m_{2s}}$ as before, we check if $P_{m_s} = P_{m^*} = K_{m^*} - A$. In this case $K \geq \underline{K}$, and hence the price must be $K - A$. Otherwise we look for the largest index s_0 such that $P_{m_{s_0}} > K_{m_{s_0}} - A$, $P_{m_{s_0+1}} = K_{m_{s_0+1}} - A$. As $P_{m_s} > K_{m_s} - A$ one has t

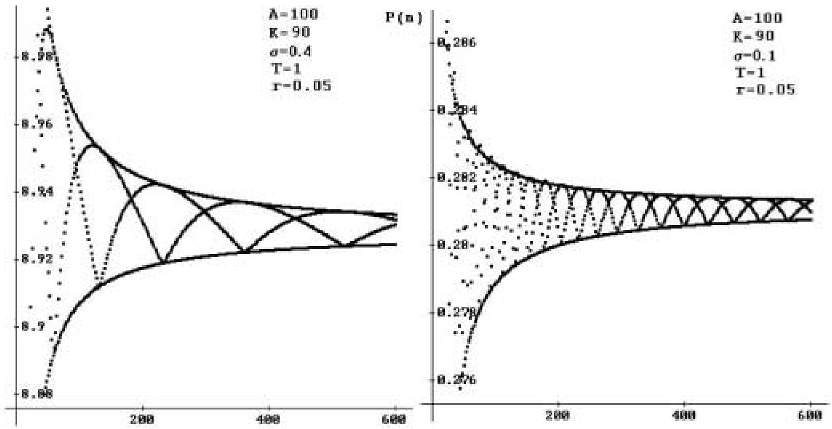

Fig. 3.

$s \leq s_0 \leq 2s$ ($s_0 = 2s$ means $P_{2s} > K_{2s} - A$). If $s_0 > s$ then we interpolate only the first s_0 points $(K_{m_1}, P_{m_1}), \ldots, (K_{m_{s_0}}, P_{m_{s_0}})$ at K. If $s_0 = s$ (hence $P_{m_s} > K_{m_s} - A$, $P_{m_{s+1}} = K_{m_{s+1}} - A$) $\underline{K}$ lies between K_{m_s} and $K_{m_{s+1}}$. Using the interpolating data $(K_{m_1}, P_{m_1}), \ldots, (K_{m_s}, P_{m_s})$ we can obtain an approximation K' of $\underline{K}$ (e.g., applying Newton's method to evaluate the intersection between the function interpolating such points and the straight line $P = K - A$). Finally if $K > K'$ then the price will be $K - A$; otherwise we interpolate the s points $(K_{m_1}, P_{m_1}), \ldots, (K_{m_s}, P_{m_s})$ at K.

3. Saving computational time

We now discuss a problem linked to the computational efficiency of the interpolation procedure. Indeed, to interpolate we need, for each critical value of n, $2s$ (in order to have high precision s must be greater than or equal to 3) binomial pure values of the option, one for each different value of the computational strike. At first sight this requires running the backward procedure for $2s$ different trees and this in turn would imply multiplying the computational time by $2s$, a bad source of inefficiency.

We present here a procedure to escape from this trap which allows us to work only with a tree of $n + 2s$ steps, so that the computational time needed is only slightly greater than that implied by the standard binomial procedure.

We start by considering a binomial tree with an even number n of steps and denote by $N_{i,j}$, $i = 0, \ldots, n$, $j = 0, \ldots, i$, the nodes of the tree.

Consider now the standard “backward” procedure for the evaluation of the American put. With $u = \exp\left(\sigma\sqrt{\frac{T}{n}}\right)$, the prices at the expiration time, in correspondence of the nodes $N_{n,j}$, are $P(n, j) = \max(K - Au^{2j-n}, 0)$. After computing, by backward induction, all the values $P(i, j)$, $P(0, 0)$ gives the price of the option with n steps, strike K and initial value A of the underlying.

We now enlarge the original tree adding, at every step, a further s nodes at the top and a further s nodes at the bottom of the tree (see Fig. 4). The values of the underlying at step i of the enlarged tree are Au^{2j-i}, $j = -s, \ldots, -1, 0, 1, \ldots, i+s$. We apply the standard backward induction to the enlarged tree, computing prices along the whole enlarged tree (and in particular at the $2s+1$ initial nodes), and again denote by $P(i, j)$, the value of the option at step i with j varying between $-s$ and $i+s$.

Consider now the initial value $A_1 = Au^2$ and the tree of vertices

$$N'_{0,0} = N_{0,1}, \quad N'_{n,0} = N_{n,1}, \quad N'_{n,n} = N_{n,n+1}.$$

Note that

$$P'(n, j) = \max(K - Au^{2j-2-n}, 0) = \max(K - A_1u^{2j-n}, 0),$$

$j = 0, \ldots, n$; therefore $P(0, 1) = P'(0, 0)$ gives the price of the option with initial value A_1 and strike K. We can conclude that $u^{-2}P(0, 1)$ gives the price of the option with initial value A and strike Ku^{-2}; in fact if $P(n, A, K)$ denotes the binomial price of an option with initial value of the underlying A and strike K, then $P(n, \alpha A, \alpha K) = \alpha P(n, A, K)$ for all positive constants α.

In the same way we see that $u^{-2j}P(0, j)$ provides the price of the option with initial value A and strike Ku^{-2j} for all $j = -s, \ldots, s$. If we choose as strike K, the pivotal computational strike K_{m_s} (see previous section), then evaluating the prices in the enlarged tree we can find all the prices needed for the interpolation; in fact

$$P_{m_i} = u^{-2(s-i)}P(0, s-i), \quad i = 1, \ldots, 2s. \tag{1}$$

As regards computational problems we remark finally that the construction of the enlarged tree is not necessary if the classical Cox–Ross–Rubinstein procedure has been already implemented. In fact for the standard binomial tree with $n^* = n + 2s$ steps and expiration time $T^* = \frac{n+2s}{n}T$ (so that $T^*/n^* = T/n$), by applying the backward induction just for n times along the tree and denoting by $P^*(i, j)$ the corresponding values, one has $P^*(2s, j+s) = P(0, j)$, $j = -s, \ldots, s$, so that $P(0, s-j) = P^*(2s, 2s-j)$, $j = 0, \ldots, 2s$. Therefore by (1)

$$P_{m_i} = P^*(2s, 2s-i)u^{2i-2s}, \quad i = 0, \ldots, 2s.$$

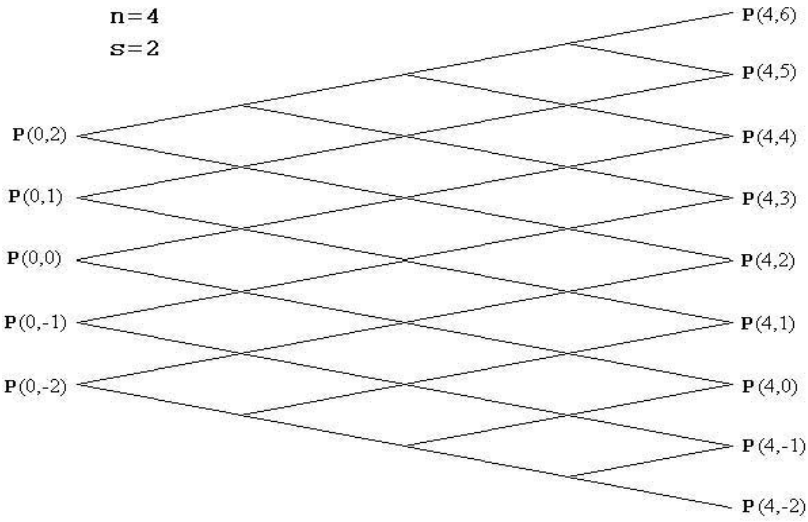

Fig. 4.

The computational time needed to evaluate $P_{\text{interp}}(n)$ can be easily determined. It corresponds to the time needed by the standard backward method for a binomial tree of $n+2s$ steps. More precisely we would have to subtract the time of the last $2s$ steps of the tree and to add the time needed for the interpolation, but these times are very close to each other (in any case for n sufficiently large, say $n \geq 100$, and $s \leq 5$, both are negligible with respect to the time needed by the Cox–Ross–Rubinstein method).

4. Comparison with other methods

In this section we compare the BIR with the BBSR of Broadie-Detemple (which is known as the best tree method for pricing American put options). Data are also provided about other classical, but surely less efficient, tree methods such as: Cox–Ross–Rubinstein binomial (CRR), average (AVE) (i.e., $\frac{P(n+1)+P(n)}{2}$) and trinomial (TRIN).

To this end we extract a random sample of 5000 options from a distribution characterized by the following parameters: volatility σ is distributed uniformly between 0.1 and 0.6. Time to maturity T is, with probability 0.75, uniform between 0.1 and 1.0 years and, with probability 0.25, uniform between 1.0 and 5.0 years. We fix the initial asset price $A = 100$ and take the strike price K to be uniform between 70 and 130. The risk free rate of interest r is distributed uniformly between 0.0 and 0.1.

4.1. European options

For an initial test of the precision of the methods, we apply them to the case of European put options, whose exact price is already known. We emphasize that in this case the BIR requires only simple interpolation, without the additional procedures described at the end of Sect. 2. For each of the 4573 options of the sample, whose price is greater than 0.5, we compute the absolute value of the relative error with respect to the continuous Black-Scholes benchmark, and then the MRE (mean relative error) and the (RMSRE) (root mean squared relative error). We also offer a bidimensional picture of the errors with computational time in the vertical axis and RMSRE in the hor-

Table 1. European options: MRE

n	CRR	AVER	TRIN	BBSR	BIR
100	$1.97\ 10^{-3}$	$8.19\ 10^{-4}$	$8.80\ 10^{-4}$	$3.66\ 10^{-5}$	$9.54\ 10^{-6}$
200	$9.73\ 10^{-4}$	$4.10\ 10^{-4}$	$4.32\ 10^{-4}$	$1.72\ 10^{-5}$	$2.36\ 10^{-6}$
400	$4.78\ 10^{-4}$	$2.06\ 10^{-4}$	$2.19\ 10^{-4}$	$8.15\ 10^{-6}$	$5.9\ 10^{-7}$
600	$3.18\ 10^{-4}$	$1.36\ 10^{-4}$	$1.44\ 10^{-4}$	$5.33\ 10^{-6}$	$2.6\ 10^{-7}$
800	$2.38\ 10^{-4}$	$1.02\ 10^{-4}$	$1.08\ 10^{-4}$	$4.01\ 10^{-6}$	$1.5\ 10^{-7}$

Table 2. European options: RMSRE

n	CRR	AVER	TRIN	BBSR	BIR
100	$2.80\ 10^{-3}$	$1.35\ 10^{-3}$	$1.40\ 10^{-3}$	$5.65\ 10^{-5}$	$1.90\ 10^{-5}$
200	$1.41\ 10^{-3}$	$6.83\ 10^{-4}$	$6.73\ 10^{-4}$	$2.62\ 10^{-5}$	$4.67\ 10^{-6}$
400	$6.80\ 10^{-4}$	$3.41\ 10^{-4}$	$3.49\ 10^{-4}$	$1.24\ 10^{-5}$	$1.16\ 10^{-6}$
600	$4.52\ 10^{-4}$	$2.27\ 10^{-4}$	$2.30\ 10^{-4}$	$8.09\ 10^{-6}$	$5.2\ 10^{-7}$
800	$3.42\ 10^{-4}$	$1.71\ 10^{-4}$	$1.72\ 10^{-4}$	$6.09\ 10^{-6}$	$3.0\ 10^{-7}$

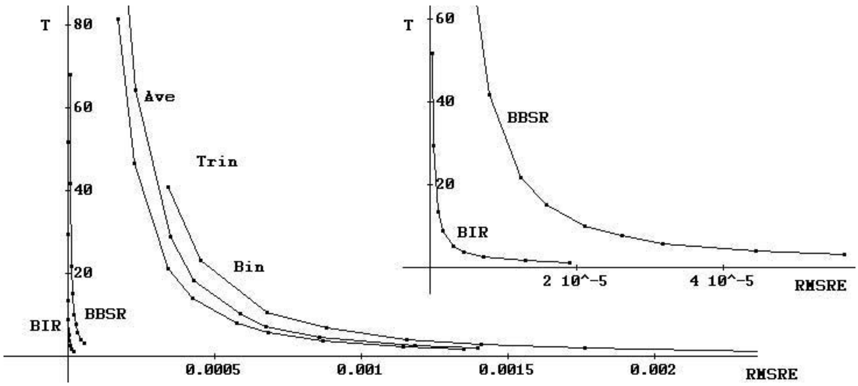

Fig. 5.

izontal axis. As previously stated the data confirm than BBSR is strongly more efficient than the other classical tree methods but BIR appears to be more efficient than BBSR.

4.2. *American options*

We pass now to the core of the paper: American put options. Here other sources of error must be considered: the loss of opportunity to exercise early in any interval between two discrete times of the tree and the early exercise boundary specification error, coming from the failure of the boundary curve to follow properly the levels of the nodes of the tree. To have a clear idea of the impact of these sources of error we use the same 5000 options of the sample and, after having evaluated the benchmark American price, we again compute (for each of the 4588 options whose price is greater that 0.5) the MRE and RMSRE referred to the benchmark.

The benchmark was obtained as follows: we evaluate the BBSR and the BIR prices at canonical values of n steps, e.g., 10000, 15000, 20000, stopping the process as soon as the relative difference between the two prices is less that 0.000002. Then we keep as benchmark the average of the two prices. In the few cases when the relative difference at 20000 steps is still larger than 0.000002 the benchmark is the BBSR value at 50000 steps.

We present tables summarizing the precision (MRE and RMSRE respectively) and (in parentheses) the total computational time in seconds required by a Pentium III processor at 700 MH for a standard number of steps, along with bidimensional pictures of the precision speed efficiency. A table with the number of cases for which each method offers the best estimate of the

Table 3. American options: MRE

n	CRR	AVER	TRIN	BBSR	BIR
100	$1.61{\cdot}10^{-3}$	$8.85{\cdot}10^{-4}$	$7.90{\cdot}10^{-4}$	$2.14{\cdot}10^{-4}$	$1.37{\cdot}10^{-4}$
	(1.67)	(3.38)	(3.26)	(3.30)	(2.65)
200	$7.95{\cdot}10^{-4}$	$4.53{\cdot}10^{-4}$	$3.93{\cdot}10^{-4}$	$8.98{\cdot}10^{-5}$	$5.36{\cdot}10^{-5}$
	(6.36)	(12.78)	(12.89)	(10.19)	(9.00)
400	$3.91{\cdot}10^{-4}$	$2.33{\cdot}10^{-4}$	$1.99{\cdot}10^{-4}$	$3.70{\cdot}10^{-5}$	$2.23{\cdot}10^{-5}$
	(24.83)	(49.77)	(51.06)	(34.82)	(33.07)
600	$2.61{\cdot}10^{-4}$	$1.56{\cdot}10^{-4}$	$1.32{\cdot}10^{-4}$	$2.49{\cdot}10^{-5}$	$1.35{\cdot}10^{-5}$
	(55.81)	(111.80)	(114.88)	(73.73)	(72.75)
800	$1.93{\cdot}10^{-5}$	$1.16{\cdot}10^{-5}$	$9.87{\cdot}10^{-5}$	$1.69{\cdot}10^{-5}$	$8.61{\cdot}10^{-6}$
	(99.36)	(198.94)	(206.37)	(128.14)	(127.46)

price is also provided. In the implementations care was taken to properly tune the methods; in particular, to compute the values of the cumulative normal appearing in BBSR, B. Moro's formula (as suggested in Broadie–Detemple (1996)) was used.

Table 4. American options: RMSRE

n	CRR	AVER	TRIN	BBSR	BIR
100	$2.36{\cdot}10^{-3}$	$1.23{\cdot}10^{-3}$	$1.42{\cdot}10^{-3}$	$6.13{\cdot}10^{-4}$	$5.71{\cdot}10^{-4}$
200	$1.15{\cdot}10^{-3}$	$6.44{\cdot}10^{-4}$	$6.36{\cdot}10^{-4}$	$2.79{\cdot}10^{-4}$	$2.63{\cdot}10^{-4}$
400	$5.75{\cdot}10^{-4}$	$3.24{\cdot}10^{-4}$	$3.19{\cdot}10^{-4}$	$9.76{\cdot}10^{-5}$	$1.29{\cdot}10^{-4}$
600	$3.80{\cdot}10^{-4}$	$2.17{\cdot}10^{-4}$	$2.19{\cdot}10^{-4}$	$1.15{\cdot}10^{-4}$	$8.57{\cdot}10^{-5}$
800	$2.82{\cdot}10^{-4}$	$1.62{\cdot}10^{-4}$	$1.62{\cdot}10^{-4}$	$6.60{\cdot}10^{-5}$	$5.13{\cdot}10^{-5}$

Table 5. American options: the best estimate

n	CRR	AVER	TRIN	BBSR	BIR
100	3.0% (139)	4.6% (215)	4.4% (207)	13.7% (637)	74.3% (3455)
200	2.3% (105)	3.7% (170)	3.9% (181)	11.6% (541)	78.6% (3656)
400	2.0% (92)	2.7% (126)	3.1% (142)	10.3% (477)	82.0% (3816)
800	2.0% (94)	2.2% (101)	2.3% (109)	8.5% (396)	85.0% (3953)

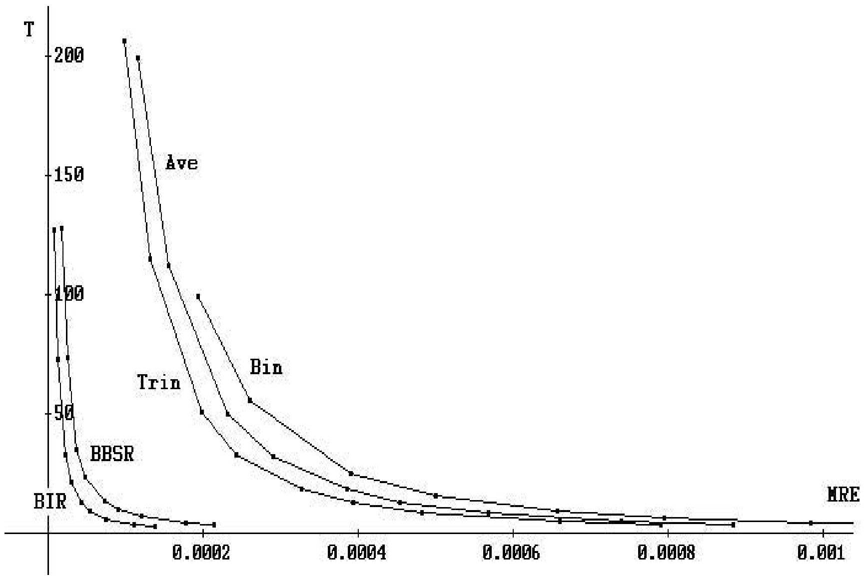

Fig. 6.

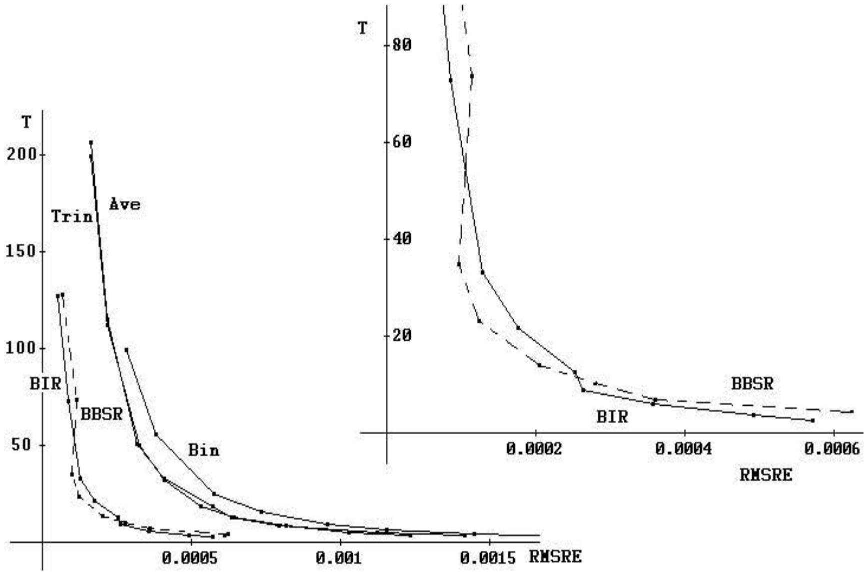

Fig. 7.

4.3. Selection procedure

A comment on the previous tables is in order. While BIR and BBSR remain unequivocally more efficient than the other methods, BIR loses a relevant part of its superiority in comparison with BBSR. Nevertheless it still gives higher efficiency when precision is measured by MRE, while the ranking is no longer clear in the case of RMSRE (see Fig. 7). Indeed, with respect to the European case, the ratios between RMSRE and MRE values increase meaningfully for BIR and BBSR, but this effect is more relevant for BIR, as can be seen by a glance at Table 6.

The previous data, as well as the data concerning the percentage of best estimates, seem to imply that for the BIR method (and partially for BBSR) there are only a few options which are responsible for large errors. This re-

Table 6. Ratio RMSRE/MRE: American and European cases

n	CRR	AVER	TRIN	BBSR	BIR
	Amer – Eur	Amer – Eur	Amer – Eur	Amer – Eur	Amer – Eur
100	1.46 – 1.42	1.39 – 1.65	1.79 – 1.59	2.87 – 1.54	4.16 – 1.99
200	1.45 – 1.45	1.42 – 1.67	1.62 – 1.56	3.10 – 1.52	4.90 – 1.98
400	1.47 – 1.42	1.39 – 1.66	1.61 – 1.59	2.64 – 1.52	5.79 – 1.97
800	1.46 – 1.44	1.39 – 1.68	1.64 – 1.58	3.91 – 1.52	5.96 – 2.00

mark suggests abandoning the uniform strategy which applies the same tree length to any option, and adopting on the contrary a strategy of segmentation of the set of options, increasing the basic number of steps for the critical options. Quite likely these are the options for which the American quality (i.e., the possibility of an early but non-immediate exercise) is relevant. A good candidate for a parametric segmentation of the market could be the ratio between the estimated American and the Black-Scholes European values.

More precisely we suggest selecting as critical all the options for which the estimated American value, $\mathrm{BIR}(n)$, is greater than the value of immediate exercise and larger than 1.5 times the Black-Scholes price of the twin European option. For our set of 5000 options such criteria are satisfied by less that 100 options. For these selected cases we evaluate $\mathrm{BIR}(4n)$ instead of $\mathrm{BIR}(n)$. The same selection strategy is applied to BBSR. It is natural to use the acronym BIRS for BIR with selection and BBSRS for BBSR with selection.

Our tables and graphs offer comparisons between the described methods which confirm the efficiency gain obtained through segmentation.

Table 7. American options: MRE

n	BBSR	BIR	BBSRS	BIRS
100	$2.14 \cdot 10^{-4}$ (3.30)	$1.37 \cdot 10^{-4}$ (2.65)	$1.69 \cdot 10^{-4}$ (4.00)	$9.32 \cdot 10^{-5}$ (3.31)
200	$8.97 \cdot 10^{-5}$ (10.19)	$5.36 \cdot 10^{-5}$ (9.00)	$7.19 \cdot 10^{-5}$ (12.75)	$3.55 \cdot 10^{-5}$ (11.55)
400	$3.70 \cdot 10^{-5}$ (34.82)	$2.23 \cdot 10^{-5}$ (33.07)	$3.07 \cdot 10^{-5}$ (45.10)	$1.41 \cdot 10^{-5}$ (43.17)
600	$2.49 \cdot 10^{-5}$(73.73)	$1.35 \cdot 10^{-5}$ (72.75)	$1.87 \cdot 10^{-5}$ (96.85)	$8.31 \cdot 10^{-6}$ (95.39)
800	$1.69 \cdot 10^{-5}$ (128.1)	$8.61 \cdot 10^{-6}$ (127.5)	$1.31 \cdot 10^{-5}$ (169.3)	$5.32 \cdot 10^{-6}$ (167.6)

Table 8. American options: RMSRE

n	BBSR	BIR	BBSRS	BIRS
100	$6.13 \cdot 10^{-4}$	$5.71 \cdot 10^{-4}$	$2.70 \cdot 10^{-4}$	$2.19 \cdot 10^{-4}$
200	$2.79 \cdot 10^{-4}$	$2.63 \cdot 10^{-4}$	$1.22 \cdot 10^{-4}$	$8.66 \cdot 10^{-5}$
400	$9.76 \cdot 10^{-5}$	$1.29 \cdot 10^{-4}$	$5.18 \cdot 10^{-5}$	$3.84 \cdot 10^{-5}$
600	$1.15 \cdot 10^{-4}$	$8.57 \cdot 10^{-5}$	$3.18 \cdot 10^{-5}$	$2.86 \cdot 10^{-5}$
800	$6.60 \cdot 10^{-5}$	$5.13 \cdot 10^{-5}$	$2.28 \cdot 10^{-5}$	$1.86 \cdot 10^{-5}$

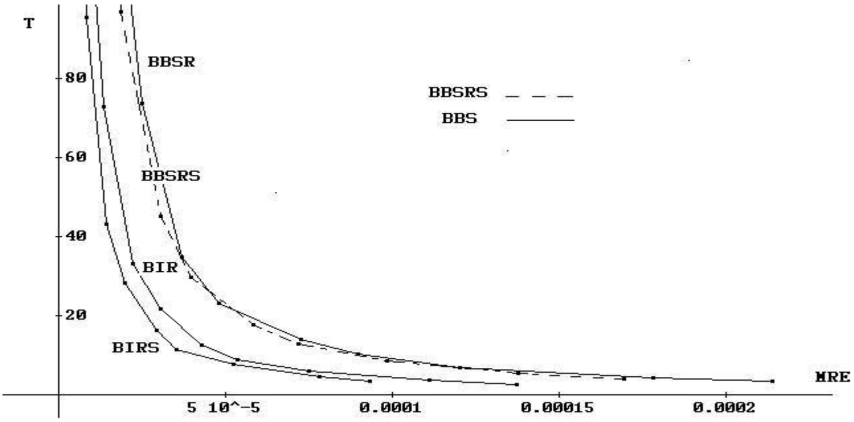

Fig. 8.

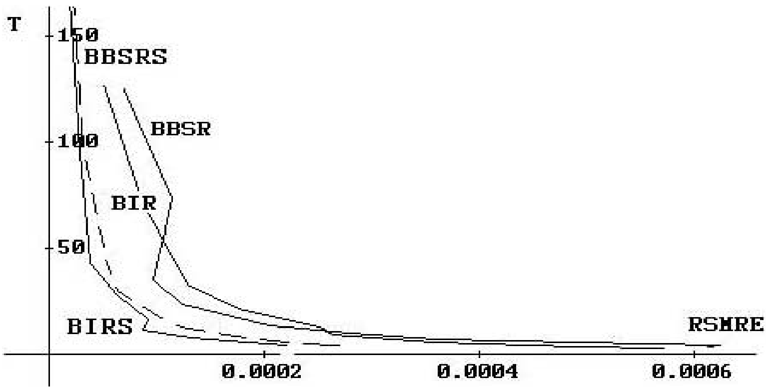

Fig. 9.

4.4. Final remarks

It is interesting to note that, when precision is measured by MRE, BIRS is slightly more efficient than BIR, while the efficiency of BBSR and BBSRS do not differ (see Fig. 8). As regards efficiency, the selection is almost neutral for the MRE level. Things seem to change at the RMSRE level: here, as indicated by Fig. 9, BIRS is better than BIR and BBSRS is better than BBSR, so that the selection strategy is more efficient than the uniform one, but the improvement seems to be greater for BIR.

The efficiency measures given by MRE and RSMRE may differ because of the shape of the distribution of errors which is (at least under our parameters) not symmetric but rather Paretian with a thin and long tail of big and somewhat random errors. Then while the MRE is driven by the large percentage of small errors of weak American options, the RSMRE is more sensitive to the large errors coming from the critical options (for example, in the case $n = 200$, 14 of the 4588 options of the sample are responsible for 50% (BIR) and 38% (BBSR) of RSMRE). Moreover the random character of the large errors explains the possible lack of monotonicity of BBSR and BIR (see Fig. 7).

The American quality of the options as captured by the ratio between the estimated American value and the Black-Scholes (BS) value of the European twin displayed a high correlation with the magnitude of errors: in the case $n = 200$, 39 of the 40 options of the sample with the largest and comparatively large errors (which are responsible for the 65% of the quadratic error) were critical options with a ratio greater than 1.5 (in the case $n = 100$, all 40 options with the largest errors are critical).

Then the suggestion of devoting more attention (computational time) to the critical options proves to be efficient for a dealer whether he applies BBSR or BIR. Indeed despite their differences both methods share common characteristics: they offer very quick and precise estimates for weak American options, while on the other hand estimates for critical options are affected by large and somewhat erratic (with n) errors. The improvement in precision obtained through an adequate increase in computational time is negligible for non-critical options, while it is relevant for the critical ones. Hence the efficiency (for both methods) of a segmentation strategy based on the American/BS ratios, which is almost free of cost. In addition to the American estimation, which should be computed in any case, it requires only the computation of a BS value and of a ratio between the two (a negligible complication in comparison with the base computational burden).

In conclusion the BIR(S) methods seem to be efficient in pricing American put options. In substance this comes from the clear superior efficiency of BIR in pricing European puts, an efficiency which extends to non-critical American options and then to the complete sample by having recourse to the selection strategy (BIRS).

These results seem to be quite robust: of course a different choice of the distribution of parameters of volatility, risk free rate, strike and time to maturity could alter the proportion of critical options to be selected. But this would be a problem only if an absolutely unusual combination of inflationary interest rates and long maturities were present with high probability. In any other scenario the number of critical options should be very small in comparison with the sample number and this in turn surely establishes the efficiency of BIRS.

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