Rheol. Acta 12, 311–318 (1973)

From the Civil Engineering Department College of Engineering University of Utah Salt Lake City (U.S.A.)

On the general theory of Steklov-Aging materials

By J. Edmund Fitzgerald

With 14 figures

(Received October 27, 1972)

1. Introduction

The application of the linear and nonlinear theories of viscoelasticity has had wide application in the characterization and stress analysis of solid propellants, highly solids loaded polymers, and asphalt pavement during the past two decades.

So long as the characterization and verification of the selected constitutive equation was confined to the usual loading or straining conditions such as a constant strain rate, ramp relaxation, or constant stress creep the comparison of theory with experiment was generally satisfactory. This result was to be expected since the theory and experiment are often a curve fitting exercise over a narrow range of load types.

When viscoelastic theory is applied to the prediction of repeated loads such as a sawtooth input or multiple ramp input, the predictive results of the theory are often quite unsatisfactory. The above statement applies to linear viscoelasticity as well as to the several forms of nonlinear viscoelasticity utilizing the multiple integral approach.

Details of the previous statements are presented in *Farris* and *Fitzgerald* (1970), *Fitzgerald* and *Farris* (1970), and Chapter XI of *Fitzgerald* and *Hufferd* (1971). A most excellent comparative review is given in *Stafford* (1969).

Fig. 1 demonstrates the above problem based upon some of *Farris*' experiments, using a filled polyurethane propellant.

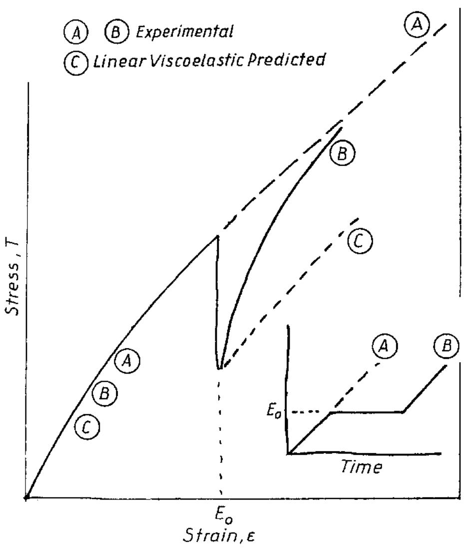

Fig. 1. Stress for an interrupted constant strain rate test

The use of a repeated sawtooth strain history is typified by the curves of fig. 2 for a highly solids loaded polybutadiene acrilonitrile propellant (*Bennett*, 1971).

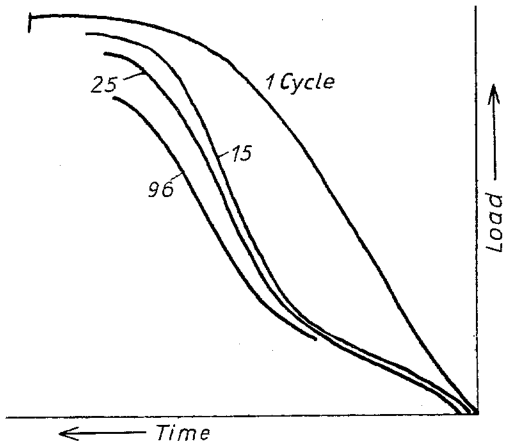

Fig. 2. Uniaxial constant rate cycling of PBAN propellant

Previous publications by the author and his co-workers employed the use of homogeneous functions of the strain history made specific through *Lebesgue* norms. The *Lebesgue* norm, $\|\varepsilon\|_{L_P}$ is essentially the weighted P-summable integral of the infinitesimal strain history, ε wherein

$$\|\varepsilon\|_{L_P} = \Big[\int_{\text{meas } s} |\varepsilon|^P \, h_P(s) \, ds \Big]^{1/P}. \qquad [1]$$

The term $h_P(s)$ is a positive decreasing function of s for fading memory theories where P is taken as unity. Certain boundedness restrictions apply to $h_P(s)$.

For $P \to \infty$, the *Chebyshev* norm results wherein

$$\|\varepsilon\|_{L_\infty} = \text{ess. sup} \, |\varepsilon| \, . \qquad [2]$$

Using a rather simple expression (*Farris*, 1970) for stress, T, versus strain ε, wherein

$$T_{11}(t) = 435 \left[\frac{\|\varepsilon_{11}\|_\infty}{\|\varepsilon_{11}\|_{21}} \right]^{2.25} \varepsilon_{11}(t) \qquad [3]$$

the results of fig. 3 were obtained (*Farris*, 1970).

It should be noted that the general form of eq. [3], which is the L_∞ norm divided by the L_P norm to the nth power (with $P \geqslant 1$), when multiplied by ε_{11} yields the following results:

For a constant strain rate test, $\varepsilon_{11} = Rt$

$$\|\varepsilon_{11}\|_\infty = Rt; \quad \|\varepsilon_{11}\|_P = Rt - \frac{t^{1/P}}{(P+1)^{n/P}}$$

$$T_{11} = A(P+1)^{n/P} R t^{1-n/P}$$

or

$$T_{11} = A(P+1)^{n/P} \varepsilon_{11} t^{-n/P}. \qquad [4]$$

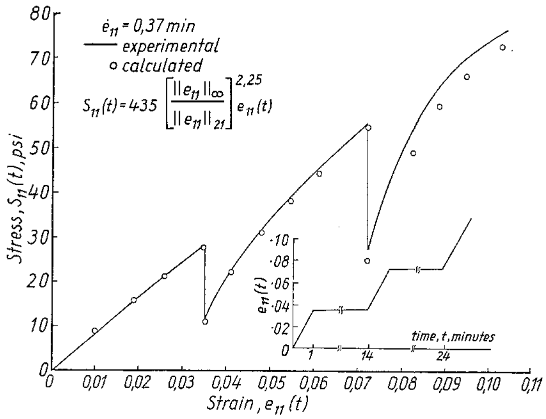

Fig. 3. Comparison of calculated and observed stress-strain output for an interrupted ramp strain input

For a step relaxation test, $\varepsilon_{11} = \varepsilon_0$

$$\|\varepsilon_{11}\|_\infty = \varepsilon_0; \quad \|\varepsilon_{11}\|_P = \varepsilon_0 t^{1/P}$$

$$T_{11} = A\varepsilon_0 t^{-n/P}. \qquad [5]$$

The last result predicts an inverse power law for the relaxation modulus, $At^{-n/P}$, so that the value of A and the slope, n/P, can be determined from a step relaxation test since $G(t)_{\text{relax.}} = At^{-n/P}$.

The constant strain rate test yields the secant modulus

$$G(t)_{\text{secant}} = (P+1)^{n/P} G(t)_{\text{relax.}} \qquad [6]$$

from which $P + 1$, hence P and n may be determined.

The predictions of fig. 3 were made using data obtained as described. A comparison of the above norm expression with linear viscoelasticity for a polyurethane propellant is given in fig. 4 (*Farris*, 1970).

A more general expansion of *Farris*' earlier expressions has been given by *Vakily* and *Fitzgerald* (1972) wherein the stress function involves a sum of *Lebesgue* norm ratios and the naturally occuring inverse, power law kernel in the viscoelastic integral as follows:

$$T_{11}(t) = \sum_{i=0}^{N} \sum_{J=0}^{N} A_{ij} \left(\frac{\|f\|\, q_i}{\|f\|_{P_i}}\right)^{n_i} \times \int_0^t (t-\tau)^{-m_j} \dot{e}_{11}(\tau)\, d\tau. \qquad [7]$$

The first term of the above expansion, $i = j = 0$ with $q_0 = \infty$ yields

$$T_{11}(t) = A_{00} \left[\frac{\|\varepsilon_{11}(t)\|_\infty}{\|\varepsilon_{11}(t)\|_{P_0}}\right]^{n_0} \varepsilon_{11}(t) \qquad [8]$$

which is the previous *Farris* expression, eq. [3].

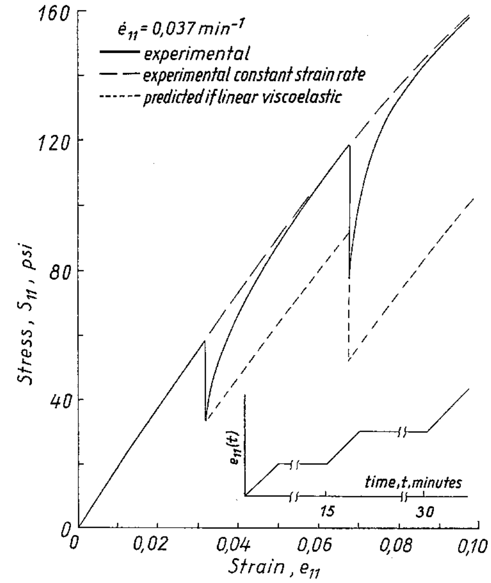

Fig. 4. Stress output for interrupted constant strain rate test

Including the current value of the strain, $|\varepsilon_{11}|$ and simplifying a three term expansion of eq. [7] results in

$$T_{11}(t) = A_1 \left[\frac{|\varepsilon_{11}|}{\|\varepsilon_{11}\|_{P_0}}\right]^{n_0} \varepsilon_{11}(t) + A_2 \left[1 - \frac{|\varepsilon_{11}|}{\|\varepsilon_{11}\|_\infty}\right]^{n_1} \times \int_0^t (t-\tau)^{-m_1} \dot{\varepsilon}_{11}(\tau)\, d\tau. \qquad [9]$$

where we have defined $\|\varepsilon\|_0 = |\varepsilon|$.

Various other specific forms of the above *Lebesgue* norms may be formulated.

Applying eq. [12] to a series of compression tests on a sand-asphalt mixture produced the following expression (*Vakily* and *Fitzgerald*, 1972) where the constants were evaluated from constant strain rate "step" relaxation tests:

$$T(t) = 310 \left\{ \left(\frac{|\varepsilon|}{\|\varepsilon\|_9}\right)^{4.32} \varepsilon(t) + \left[1 - \frac{|\varepsilon|}{\|\varepsilon\|_\infty}\right] \int_0^t (t-\tau)^{-0.8} \dot{\varepsilon}(\tau)\, d\tau \right\}. \qquad [10]$$

The moduli for a strain of 0.37 and 0.5% are shown in figs. 5 and 6.

The constant strain rate tests for two different rates are given in fig. 7.

The ramp relaxation test results are given in fig. 8.

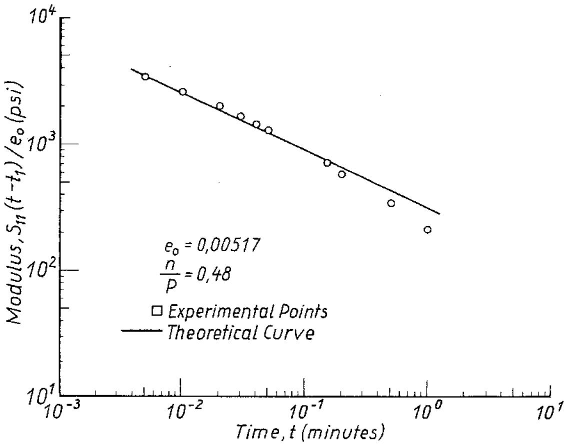

Fig. 5. Relaxation of sand-asphalt for first test

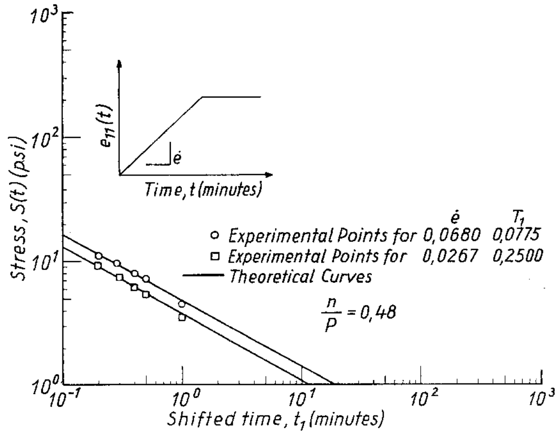

Fig. 8. Stress output for two different ramp relaxation tests

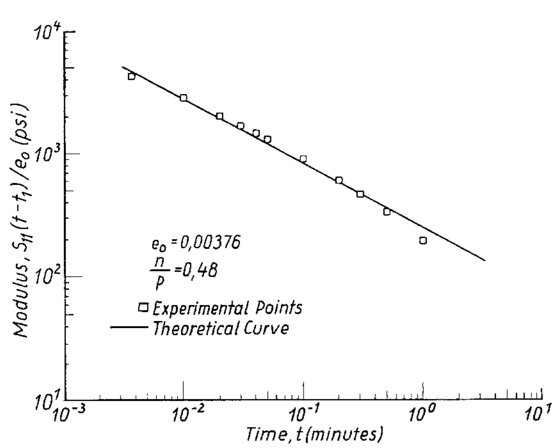

Fig. 6. Relaxation modulus of sand-asphalt for second test

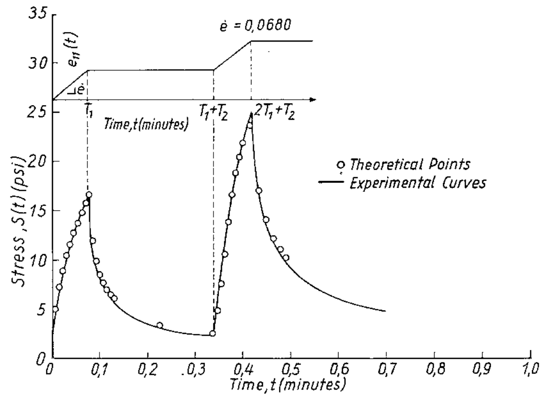

Fig. 9. Stress output for interrupted ramp strain input

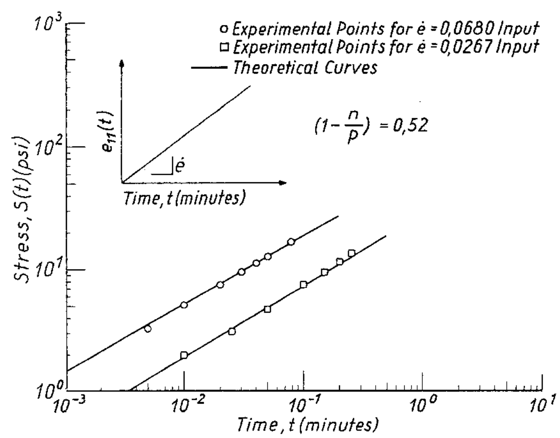

Fig. 7. Stress output for two different constant strain rate tests

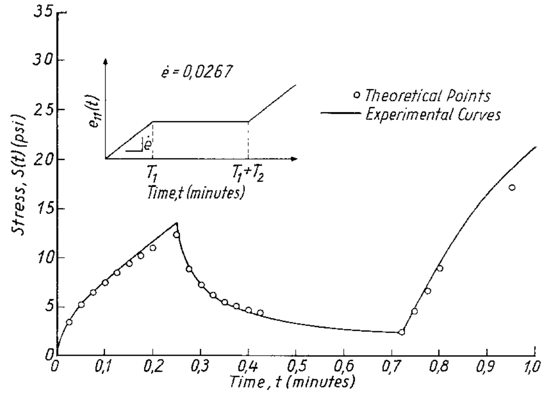

Fig. 10. Stress output for interrupted ramp strain input

Using the expression, eq. [10], derived from the above data, predictions shown as open circles and experiments, shown as solid lines, were conducted for:

1. An interrupted ramp strain input at two different strain rates, shown in figs. 9 and 10. A comparison of the above predictions with those of linear viscoelasticity is given in fig. 11.

2. A single constant strain rate "sawtooth" strain, shown in fig. 12.

3. A repeated constant strain rate test cycled between a set maximum strain and zero load, fig. 13.

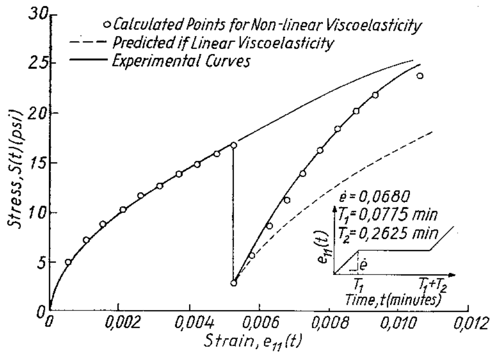

Fig. 11. Comparison of calculated and observed stress-strain output for an interrupted ramp strain input

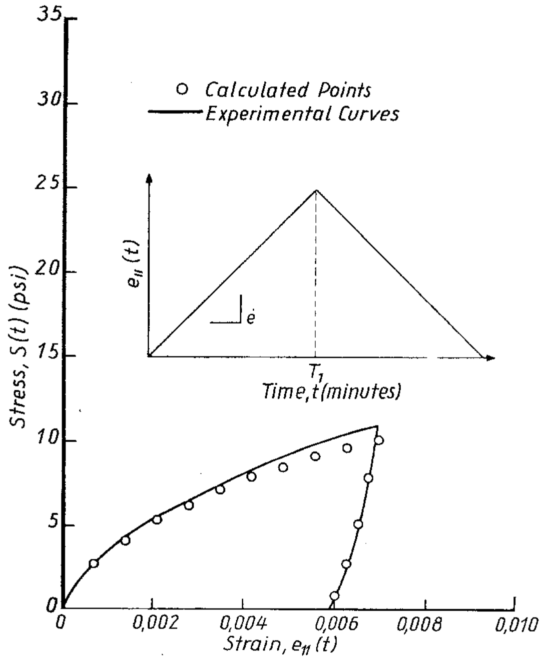

Fig. 12. Comparison of calculated and observed stress-strain behavior of sand-asphalt

Again, it is clear that the sand-asphalt material exhibits a degree of permanent memory similar to the filled polyurethane propellant and the PBAN propellant. Figs. 5 through 13 are from *Vakily* and *Fitzgerald* (1972).

An expression for the stress such as

$$T(t) = f\,[U(t),\ \|U(t)\|_{L_\infty}] \qquad [11]$$

produces a stress-strain relation with no relaxation but with the *Mullins* effect predominant (*Mullins*, 1947).

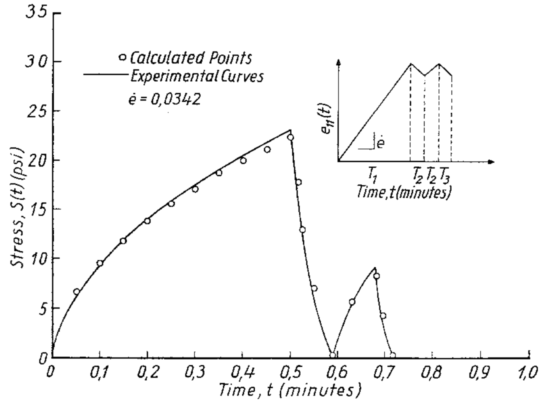

Fig. 13. Comparison of calculated and observed stress output for a reversed ramp strain test

A mathematical justification for the above norms, and their restrictions, is given in *Hufferd* and *Fitzgerald* (1972) including thermodynamic implications.

In general, one may express the *Cauchy* stress tensor, T, as a function of the norms, for example

$$T = f\,[U(t),\ \|U\|_{L_\infty},\ \|U\|_{L_P}] \qquad [12]$$

where U is the positive square root of the right *Cauchy-Green* (finite strain) tensor. The resulting polynomial function following the well known *Rivlin-Spencer* expansion for initially isotropic materials will produce an expression up to the second power in the U and its norms with coefficients that are polynomials, or preferably here, rational fractions in the joint invariants

$$\mathrm{tr}\,U, \quad \mathrm{tr}\,U^2, \quad --, \quad \mathrm{tr}\,\|U\|_{L_P}, \quad \text{etc.}$$

Again, a three-dimensional expression similar to *Farris'* is derived for a single term, homogeneous form for T, namely

$$T(t) = A\left[\frac{\mathrm{tr}\,\|U\|_\infty}{\mathrm{tr}\,\|U\|_P}\right]^n (U(t) - 1). \qquad [13]$$

Application of the above tensorial equation to a uniaxial test produce results similar to those previously shown.

2. Aging Effects and the Steklov Average

The *Lebesgue* norms previously described are such that they produce the integral of the strain history on a P-summable basis. The use of a non-unit weighting function, $h_P(s)$, will provide for fading memory effects. The *Lebesgue* norms may have a basis in microscopic theory as presented by *Farris* in his doctoral thesis.

Nevertheless, the norms as used herein are simply continuum postulates whose use is justified by the results shown herein and in the various quoted references.

With the integrals as used, however, no aging effects are included.

Consider now a class of materials whose response is governed by certain weighted averages of the past strain history as well as by the present value of the strain and its several time derivatives.

A norm on such a space may be constructed as follows for a generalized input A

$$\|A^t(s)\|_{S_P^N} = \sum_{i=0}^{N} |A^{(i)}(o)|\, h_i(o) + \sum_{P=1}^{\infty}\left[\frac{1}{\operatorname{mes} D}\int_D |A_r^t(s)|^P\, h_P^P(s)\, ds\right]^{1/P}$$

with

$$\left[\frac{1}{\operatorname{mes} D}\int_D h_P^P(s)\, ds\right]^{1/P} < \infty \quad \text{and} \quad h_P^P(s) \geq 0 \qquad [14]$$

and $A^{(i)}(o)$ the present i-th rates of change. The above is actually a semi-norm unless meas D is finite.

Considering especially the second summand in eq. [14], we shall call it a *Steklov Average* since it is a generalization of the *Steklov-Lebesgue* average given in *Kantorovich* and *Akilov* (1964).

A slight variation in the above, which will be termed the *modified Steklov Average* and which satisfies all the requirements of a norm (or semi-norm) is given by

$$\|A\|_{\bar{S}_P} = \left[\frac{1}{(1 + k \operatorname{meas} D)}\int_D |A_r^t(s)|^P\, h_P^P(s)\, ds\right]^{1/P}$$

with $k \geq 0$. [15]

Dropping the fading memory factor $h_P(s)$, for simplicity, results then in

$$\|A\|_{\bar{S}_P} = \left[\frac{1}{(1 + kt)}\int_{s=0}^{t} |A|^P\, ds\right]^{1/P} \qquad [16]$$

where we have taken meas $D = t$ for the usual rather smooth physical inputs, A.

Looking at the *Lebesgue* norm ratios of eqs. [7] or [8] for example and the definitions eqs. [1] and [16] yields the following relation between the L_P norms and the S_P norms

$$\left[\frac{\|A\|_{\bar{S}_Q}}{\|A\|_{\bar{S}_P}}\right]^n = \left[\frac{\|A\|_{L_Q}}{\|A\|_{L_P}}\right]^n (1 + kt)^{\left(\frac{Q-P}{QP}\right)n} \qquad [17]$$

and for $Q \to \infty$,

$$\left[\frac{\|A\|_{\bar{S}_\infty}}{\|A\|_{\bar{S}_P}}\right]^n = \left[\frac{\|A\|_{L_\infty}}{\|A\|_{L_P}}\right]^n (1 + kt)^{n/P}. \qquad [18]$$

It is readily shown that the ratio

$$\frac{\|A\|_{\bar{S}_Q}}{\|A\|_{\bar{S}_P}} \geq 1 \;\forall\, Q > P, \quad P \geq 1 \qquad [19]$$

since the dominance of L_Q over L_P is well known when $Q > P$ (*Kantorovich* and *Akilov*, 1964) and for $k \geq 0$, the muliplier is also ≥ 1.

Defining T^L as the stress obtained from *Lebesgue* norms as in the previous section 1.0, the stress obtained by substituting modified *Steklov* norms is T^S where

$$T^S = T^L(1 + kt)^{\frac{Q-P}{QP}n} \qquad [20]$$

or with $Q \to \infty$

$$T^S = T^L(1 + kt)^{n/P}. \qquad [21]$$

Consider for example a material governed by a simple permanent memory norm relation such as eq. (3)

$$T_{11}^L = A\left[\frac{\|\varepsilon_{11}\|_\infty}{\|\varepsilon_{11}\|_{L_{21}}}\right]^n \varepsilon_{11}(t). \qquad [22]$$

For a step relaxation test with strain magnitude ε_0 [22] yields

$$T_{11}^L = A\varepsilon_0 t^{-n/P}. \qquad [23]$$

Applying instead the ratio of the modified *Steklov* norms S_P produces

$$T_{11}^s = A\varepsilon_0 t^{-n/P}(1 + kt^*)^{n/P}. \qquad [24]$$

For a material which is strained immediately upon being formed, $t = t^*$. Otherwise however, t is the time relative to the beginning of the step strain whereas t^* is the actual time from the initial creation of the material to the present. The difference in behavior of the L_P and S_P norms is shown in Fig. 14. A step load-unload input is used with the L_1, $\bar{S}_1$ and $L_\infty = S_\infty = \varepsilon$ max. curves plotted for an example. After the load occurs, L_1 is constant as is L_∞. However, the S_1 norm reduces with time since it is averaging over the life of the specimen.

The relation between the L_P and S_S norms is, from [1] and [16]

$$\|A\|_{\bar{S}_P} = \|A\|_{L_P}(1 + kt)^{-1/P}. \qquad [25]$$

It will also be noticed from eq. [21] that the exponent of the "aging" term $1 + kt^*$ is equal to the positive numerical slope of the inverse power law relaxation modulus, n/P.

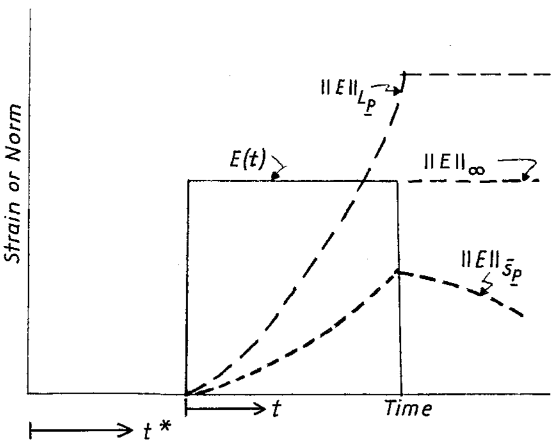

Fig. 14. Time vs. norms

Consider a typical solid propellant with $n/P = 0.2$, then

$$T_{11} = A\varepsilon_0 t^{-0.2}(1 + kt^*)^{0.2}. \qquad [26]$$

If the relaxation modulus increases by 20% over a one-year ambient aging then $k \cong 3 \times 10^{-8}$, $\sec^{-1}$.

A typical CTPB propellant increases its modulus by 20% in 100 days. With a slope $n/P = 0.2$, hence, the value of k is $15 \times 10^{-8} \sec^{-1}$.

It is also to be noted that for large values of kt^*, the aging is described by a straight line on a log-log scale. Further for $n/P = 0$, no relaxation and no strain rate effects, the aging is nonexistent.

Because of the small values of k, the use of the L_P rather than the S_P norms for short time loads on newly formed materials is fully equivalent. Further, for short loading times relative to the life of the material, $(1 + kt^*)^{n/P}$ is essentially a constant and *Steklov* aging reduces to the socalled homothetic aging process (*De Arriaga*, 1969).

For finite strain, the essential results also hold. Consider an incompressible material subject to a simple step elongation with a stretch ratio of λ where λ is the stretched length divided by the initial length. Using a simple constitutive equation of the form of [13] but with modified *Steklov* norms instead of *Lebesgue* norms as in [18] results in

$$T(t) = A\left[\frac{\lambda + 2}{\lambda + 2\lambda^{-1/2}}\right]^n t^{-n/P}(1 + kt^*)^{n/P} \times (\lambda - 1). \qquad [27]$$

For a material stretched when newly formed and then held at the stretch ratio, λ, with $t = t^*$

$$T(t) = A\left[\frac{\lambda + 2}{\lambda + 2\lambda^{-1/2}}\right]^n (t^{-1} + k)^{n/P}(\lambda - 1). \qquad [28]$$

For long times then as $t \to \infty$,

$$T(t) = A\left[\frac{\lambda + 2}{\lambda + 2\lambda^{-1/2}}\right]^n k^{n/P}(\lambda - 1). \qquad [29]$$

One could obviously extend the expansion to higher order terms in U, but the essential point to be made is that the material does not fully relax as $t \to \infty$. For the typical material previously mentioned, if the relaxation modulus at one second were 1000 psi, the final modulus would reduce to 60 psi. In addition, there can be considered an elastic component with no loss of generality.

3. Conclusions

It has been proposed that the characterization of relaxing, rate sensitive materials be based upon certain weighted averages of the past history which are semi-norms called modified *Steklov* averages.

The rationale is based upon previously mentioned guidance from molecular theory but is primarily based upon the postulate that the present response of a material is governed by its present deformation gradient and a selected weighted P-summable *Steklov* average [14].

Since in engineering practice, one seldom knows the detailed past of a structure's thermal and deformation history but usually knows the average and the maximums, no serious loss of applicability should result.

The justification for the use of the proposed norms is based upon the several examples given herein where the predictions using norms is shown to be superior in accuracy to the predictions based upon linear viscoelasticity. The use of the $\bar{S}_\infty$ or L_∞ maximum value norms implies that the material also is sensitive to the maximum strain value it has ever been subjected to. The possible extension of the concept to viscoplastic materials is under study.

Farris has shown that for a selected class of solid propellants the use of the conventional time-temperature superposition integral is valid. The present author suggests that the time temperature shift integral

$$\xi = \int_{\tau=0}^{t} \frac{\tau}{\alpha_T} d\tau \qquad [30]$$

for reduced time ξ with the temperature shift factor α_T be used in the modified *Steklov* norm for aging effects. It is, of course, equally possible to use an absolute reaction rate correction for the kt term in [16].

The use of a rather simple homogeneous fraction involving only the max. norm and the P-norm of the modified *Steklov* average [22] produces four specific results for infinitesimal deformations

1. the relaxation modulus is described by an inverse power law, $t^{-n/P}$;
2. the constant strain rate curves are described by a power law whose exponent is unity plus the relaxation modulus exponent, $t^{1-n/P}$;
3. the relaxation and secant moduli are subject to an age hardening process n/P described by the factor $(1 + kt)$;
4. for very long term stretching and aging, the material stress response is nonvanishing as shown in [29].

It has been also shown that for short aging times the form of the equations reduces to ratios of *Lebesgue* norms only. For short term loading relative to the age of the material, a homothetic aging process is produced.

There does not exist a unique inverse for strain as a function of stress, i. e., a general creep inverse. Indeed, an inverse does not generally exist unless a nonconstant weighting function, $h_P(s)$ is used. Even then, there results a nonlinear integral equation of the type

$$\varphi(s)^m = \int |\varphi(s)|\, h_P(s)\, ds; \qquad \varphi(s) = \Lambda^P(s) \qquad [31]$$

with the exponent m a rational fraction whose value will generally be near unity.

Since the present value of stress is governed by certain averages of the past strain history, uniqueness in the inverse is not to be expected. It is therefore suggested that a creep law be formulated in the same fashion as [22], for example

$$\varepsilon(t) = f[\|T\|_{S_P}, \|T\|_{S_\infty}, T(t)] \qquad [32]$$

made specific as

$$\varepsilon(t) = B\left[\frac{\|T\|_{S_P}}{\|T\|_{S_\infty}}\right]^m T(t) \qquad [33]$$

which for a constant stress, T_0, results in

$$\varepsilon(t) = BT_0 t^{m/P}(1 + kt)^{-m/P}. \qquad [34]$$

The calculation time necessary to use the various norm equations given herein is much shorter for general inputs than the time needed for viscoelasticity. This computer time saving results from the fact that the norms are only a number at the present time whose value changes by the modified average at each step. If, however, nonconstant weighting functions, $h_P(s)$, are used, no computational advantage results.

Acknowledgements

The author wishes to acknowledge the past and continuing efforts of his colleagues *R. Farris, W. Hufferd, J. Vakily*, and *M. Quinlan* in pursuing and contributing to the ideas herein presented.

This work was sponsored in part by the Office of Naval Research under Contract No. 00014-67-A-0325-0001 and the Air Force Office of Scientific Research under Contract No. 72-2332.

Summary

A broad class of materials possessing both instantaneous nonlinear elasticity and dissipation in addition to fading memory with aging effects is described. The measure of the generalized input function, Λ, which is a multiplet in F, the deformation gradient; θ, the temperature; $g = \operatorname{grad}\theta$, as well as various chemical affinities, A_k; is given by a semi-norm over a *Banach* space. With the definition of the history $\Lambda^t = \Lambda^t(s) \equiv \Lambda^t(t-s)$; $s\varepsilon(0,\infty)$ and with the restriction of Λ^t to past history given by $\Lambda^t_r = \Lambda^t_r(s) = \Lambda^t_r(t-s)$; $s\varepsilon(0,\infty)$ the semi-norm is:

$$\|\Lambda^t(s)\|_{S^N_P} = \sum_{i=0}^{N} |\Lambda^{(i)}(o)|\, h_i(o) + \sum_{P=1}^{\infty}\left[\frac{1}{1 + k \operatorname{meas} D}\int_D |\Lambda^t_r(s)|^p\, h^p_p(s)\, ds\right]^{1/p}$$

$$\text{with } \left[\frac{1}{1 + k \operatorname{meas} D}\int_D h^P_P(s)\, ds\right]^{1/P} < \infty$$

$$\text{and } h^P_P(s) \geq 0. \qquad [35]$$

and $\Lambda^i(0)$ the present ith time rate of change.

The second summand is a modification of the *Steklov Average* (*Kantorovich*, 1964) to P – integrable *Lebesgue* functions.

It is assumed that the generalized response $\Omega(t)$ is a nonlinear function(al) of the present input $\Lambda(t)$ and a material property-history kernel determined by the *Steklov-Lebesgue* norm $\|\Lambda^t(s)\|_{S^N_P}$ such that

$$\Omega(t) = F[\|\Lambda^t(s)\|_{S^N_P}]\, \Lambda(t). \qquad [36]$$

Eq. [35] shows that as time increases and that the influence of the past history on present response decreases. *For a given finite duration input, its influence decreases both the longer in the past it occurred and the older the material is.* This latter effect is termed *Steklov-aging*. As the age of the material becomes very large the past history effects are obliterated and from eq. [35]

$$\lim_{\substack{t\to\infty \\ \text{mes}\,D\to\infty}} \|A^t(s)\|_{S_P^N} = \sum_{i=0}^{N} |A^{(i)}(o)|\, h_i(o). \qquad [37]$$

Thus a *Steklov-aging material under long term aging approaches the behavior of a nonlinear viscoelastic or Markoffian type material.* For very small inputs, $\Lambda(0)$ and small time rates of change of inputs, $\Lambda^{(i)}(0)$, the material, after long term aging, becomes linearly viscoelastic.

Examples of the theory as applied to solid propellants and a sand-asphalt concrete are given.

Zusammenfassung

Es wird eine breite Klasse von Materialien beschrieben, die zusätzlich zu einem schwindenden Rückerinnerungsvermögen infolge Alterungserscheinungen sowohl ein nicht-lineares Elastizitätsverhalten als auch Dissipation aufweisen. Als Maß für die verallgemeinerte Eingangsfunktion, Λ, die ein Multiplet in F is, des Deformationsgradienten; der Temperatur, θ, $g = \text{grad}\,\theta$ und verschiedener chemischer Affinitäten, A_k, wird eine Halb-Norm über einen *Banach*-Raum verwendet. Mit der Definition der Beanspruchungsgeschichte $\Lambda^t = \Lambda^t(s) = \Lambda^t(t-s)\, s\,\varepsilon\,[0,\infty]$ und mit der Einschränkung, daß die Gegenwart ausgeschlossen ist, d. h. $\Lambda_r^t = \Lambda_r^t(s) = \Lambda_r^t(t-s)$; $s\,\varepsilon(0,\infty)$ gibt:

$$\|\Lambda^t(s)\|_{S_P^N} = \sum_{i=0}^{N} |\Lambda^{(i)}(o)|\, h_i(o) + \sum_{P=1}^{\infty} \left[\frac{1}{1 + k\,\text{meas}\,D} \int_D |\Lambda_r^t(s)|^p\, h_p^p(s)\, ds\right]^{1/p}$$

$$\text{with} \left[\frac{1}{1 + k\,\text{meas}\,D} \int_D h_P^P(s)\, ds\right]^{1/P} < \infty$$

$$\text{and} \quad h_P^P(s) \geq 0. \qquad [35]$$

$\Lambda^i(0)$ die i-te Ableitung nach der Zeit (Gegenwart).

Der zweite Summand ist die Modifikation des *Steklov-Durchschnittswertes* (*Kantorovich*, 1964) zu P-integrierbaren *Lebesgue-Funktionen.*

Es wird angenommen, daß die verallgemeinerte Abhängigkeit $\Omega(t)$ von $\Lambda(t)$ eine nicht-lineare Funktion (Funktional) der gegenwärtigen Eingangsgrößen und eines Stoffeigenschaften-Vorgeschichte Kernel ist, das durch die *Steklov-Lebesgue*-Norm $\|\Lambda^t(s)\|_{S_P^N}$ vorgegeben ist. Somit ergibt sich:

$$\Omega(t) = F\,[\|\Lambda^t(s)\|_{S_P^N}]\,\Lambda(t). \qquad [36]$$

Gl. [35] besagt, daß der Einfluß der Vorgeschichte auf das Gegenwartsverhalten mit zunehmender Zeit geringer wird. *Der Einfluß eines Inputs endlicher Zeitdauer nimmt ab, je mehr Zeit verstreicht und je älter das Material ist.* Diese letztere Beobachtung wird als *Stecklov-Alterung* bezeichnet. Mit hohem Materialalter wird der Einfluß der Beanspruchungsvorgeschichte völlig ausgelöscht und Gl. [35] lautet dann

$$\lim_{\substack{t\to\infty \\ \text{mes}\,D\to\infty}} \|\Lambda^t(s)\|_{S_P^N} = \sum_{i=0}^{N} |\Lambda^{(i)}(o)|\, h_i(o). \qquad [37]$$

Damit *nähert sich das Verhalten eines Steklov-alternden Materials mit zunehmendem Alter mehr und mehr dem eines nicht-linearen viscoelastischen Materials oder „Markoff"-Materials an.* Wenn die Eingangsgrößen und die zeitlichen Änderungen der Eingangsgrößen klein sind, dann wird das Material nach langer Alterung linear viscoelastisch.

Es werden Beispiele der Theorie in ihrer Anwendung auf Festkörper Treibstoffe und Sand-Asphalt-Beton beschrieben.

References

1) *Bennett, J.*, Private Communication, Thiokol Chemical Co. (1971).

2) *De Arriaga, F.*, Homothetic Aging Viscoelastic Materials. Int. Conf. on Structure and Materials, Southampton 1969.

3) *Farris, R. J.*, Homogeneous Constitutive Equations for Materials with Permanent Memory. UTEC TH 70-083, Project Themis Report AFOSR 70-1962 TR, University of Utah, June 1970.

4) *Farris, R. J.* and *J. E. Fitzgerald*, Deficiencies of Viscoelastic Theories as Applied to Solid Propellants. Bulletin JANNAF Mechanical Behavior Working Group, 8th Meeting 1969, CPIA Publication No. **193**, Vol. **1**, March 1970.

5) *Fitzgerald, J. E.* and *R. J. Farris*, Characterization and Analysis Methods for Nonlinear Viscoelastic Materials. Project Themis Report, UTEC TH 70-204, University of Utah, November 1970.

6) *Fitzgerald, J. E.* and *W. L. Hufferd*, Engineering Structural Analysis of Solid Propellants. CPIA Publication No. **214**, The Johns Hopkins University, July 1971.

7) *Hufferd, W. L.* and *J. E. Fitzgerald*, A Thermodynamic Theory of Materials with Permanent Memory. Proc. 1971 Int. Conf. on Mechanical Behavior of Materials, Vol. III, pp. 530–540, Soc. Mtr'l. Sci., Japan, 1972.

8) *Kantorovich, L. V.* and *G. P. Akilov*, Functional Analysis in Normed Spaces (New York 1964; translated from the 1959 Russian original).

9) *Mullins, L. J.*, J. Rubber Res. **16**, 275–289 (1947).

10) *Stafford, R. O.*, J. Mech. Phys. Solids **17**, 339–358 (1969).

11) *Fitzgerald, J. E.* and *J. Vakily*, Nonlinear Characterization of Sand Asphalt Concrete by Means of Permanent Memory Norms. UTEC CE 72-079, Univ. Utah, 1972. Soc. Exper. Stress Anat. papor No. 2105.

Author's address:

Prof. Dr. *J. Edmund Fitzgerald*

Civil Engineering Department, College of Engineering, University of Utah, Salt Lake City, Utah 84112 (U.S.A.)