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ORIGINAL PAPER

The number of *K*-determination of topological spaces

B. Cascales · M. Muñoz · J. Orihuela

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Abstract We introduce a cardinal function that assigns to each topological space Y a cardinal number $\ell\Sigma(Y)$ that *measures* how the space is determined by its compact subsets via upper semicontinuous compact valued maps defined on metric spaces. By doing so we extend and take to a different dimension the study of the so-called countably K-determined spaces (or Lindelöf Σ-spaces) and their associates Gul'ko compacta. We study the behaviour of $\ell\Sigma(\cdot)$ with respect to the usual operations for topological spaces as well as some of the standard operations within the category of Banach spaces. We study the relationship of $\ell\Sigma(\cdot)$ with regard to other cardinal functions like for instance the weight $w(\cdot)$ of spaces, for which we observe that although for any compact space K we always have $\ell\Sigma(C(K), \tau_p) \leq w(C(K), \tau_p)$ there is a space $\mathbb{Y}$ such that $w(\mathbb{Y}) < \ell\Sigma(\mathbb{Y})$: the example $\mathbb{Y}$ is a subspace of $\beta\mathbb{N}$ of cardinality $2^{2^{\omega}}$ whose compact subsets are finite. We also study some weakening of G_δ-conditions for diagonal of compact spaces that still imply metrizability of the underlying space and that have numerous applications in functional analysis. We close the paper establishing the relationship between $\ell\Sigma(\cdot)$, the Σ-degree introduced by Hödel and the class of strong Σ-spaces studied by Nagami and others.

We dedicate this paper to our friend and colleague Gabriel Vera who retired this year.

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B. Cascales (✉) · J. Orihuela
Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo,
30100 Espinardo, Murcia, Spain
e-mail: beca@um.es

J. Orihuela
e-mail: joseori@um.es

M. Muñoz
Departamento de Matemática Aplicada y Estadística, Universidad Politécnica de Cartagena,
Campus Muralla del Mar, 30202 Cartagena, Murcia, Spain
e-mail: maria.mg@upct.es

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1 Introduction

In this paper we exploit once again the idea, quite useful in topology and its applications to analysis, of using cardinal functions defined in the category of Hausdorff topological spaces Y's to study the impact of these invariants on properties of the subjacent spaces, properties of its compact subsets, properties of the space of continuous functions $C(Y)$, etc. Good examples of these cardinal functions are: the cardinality $|Y|$, the weight $w(Y)$, the tightness $t(Y)$, the Lindelöf number $\ell(Y)$, the density $d(Y)$, the character $\chi(Y)$ and the network number $nw(Y)$. For the definition of these cardinal functions, the relationship between them and related results we refer the reader to the references [1–3] and [13]. Here is a good and very useful example, that we will use later, of how these cardinal functions interact:

Theorem 1 (Arkhangel'ski–Pytkeev [3,27]) *Let Y be a topological space and $\mathfrak{m}$ a cardinal number. Then, $t\big(C_p(Y)\big) \leq \mathfrak{m}$ if, and only if, $\ell(Y^n) \leq \mathfrak{m}$ for every $n \in \mathbb{N}$.*

A good deal of research along these lines has been done in past decades as well as in recent years: many papers could be referenced as source of inspiration for the ideas presented here but those with the strongest influence are [6,29]. Beside the latter, the interested reader could find the book by Kąkol et al. [18] and the references therein as a good source of comprehensive information in the area this paper deals with.

Throughout this paper all cardinal numbers we shall use are infinite. A cardinal number $\mathfrak{m}$ is identified with the set of all ordinals less than $\mathfrak{m}$; in particular $\mathfrak{m}$ is a set of cardinality $\mathfrak{m}$ that is also considered as a topological space endowed with the discrete topology; if $\mathfrak{n}$ is another cardinal number we will consider the product space $\mathfrak{m}^{\mathfrak{n}}$ endowed with the product topology (with respect to the discrete topology on each factor); also, $\mathfrak{m}^{\mathfrak{n}}$ will denote the cardinality of the set $\mathfrak{m}^{\mathfrak{n}}$. Finally, 2^A stands for the family of all subsets of A: when a cardinal number $\mathfrak{m}$ is considered as a set, $2^{\mathfrak{m}}$ is also a set of cardinality $2^{\mathfrak{m}}$. ω denotes the smallest infinite cardinal.

The next definitions, that were introduced in [24], establish the cardinal functions announced in the abstract that we thoroughly study in this paper.

Definition 2 Let Y be a topological space.

(i) The number $\ell\Sigma(Y)$ of *K-determination* of Y is defined as the smallest cardinal number $\mathfrak{m}$ for which there are a metric space (M, d) of weight $\mathfrak{m}$ and a usco map $\phi : M \to 2^Y$ such that $Y = \bigcup\{\phi(x) : x \in M\}$.

(ii) The number $Nag(Y)$ of *Nagami* of Y is defined as the smallest cardinal number $\mathfrak{m}$ for which there are a topological space X of weight $\mathfrak{m}$ and a usco map $\phi : X \to 2^Y$ such that $Y = \bigcup\{\phi(x) : x \in X\}$.

Recall that if X and Y are topological spaces, a multi-valued map $\phi : X \to 2^Y$ is said to be *usco* if it is compact valued and upper semicontinuous, *i.e.*, for every $x \in X$ the set $\phi(x)$ is compact, non-empty and for every open set $V \subset Y$ with $\phi(x) \subset V$ there is an open neighborhood U of x such that $\phi(U) = \bigcup_{y \in U} \phi(y) \subset V$.

Let us note that the class of spaces Y's with $\ell\Sigma(Y) = \omega$ is the class that authors in functional analysis refer to as countably K-determined spaces, [17, Sec. 5.1]; it is worth mentioning that in topology, this class of spaces is referred to as Lindelöf Σ-spaces.

Here is a brief description of the contents of the paper. We start in Sect. 2 by giving a characterization of the existence of usco maps $\phi : X \to 2^Y$ whose range covers Y in terms of families of closed sets in βY that determine Y. In Sect. 3 we study the behaviour of $\ell\Sigma(\cdot)$ and $Nag(\cdot)$ on subspaces, unions, products, usco images, when taking generated subspaces and closures in Banach spaces, etc. In particular we prove that for a compact space K we always have that $\ell\Sigma(C(K), \tau_p) \leq w(C(K), \tau_p)$ and we give an example of a space $\mathbb{Y}$ such that $w(\mathbb{Y}) < \ell\Sigma(\mathbb{Y})$ that also exhibits that $Nag(\mathbb{Y}) < \ell\Sigma(\mathbb{Y})$. In Sect. 4 we offer some applications to spaces of continuous functions and we use $\ell\Sigma(\cdot)$ to extend the results of the so-called Gul'ko compact spaces (or compact spaces of type $\mathcal{E}_2$), see [29], to the general setting of compact spaces K homeomorphic to pointwise compact sets of some $C(Y)$, where $\ell\Sigma(Y) = \mathfrak{m}$ is an arbitrary cardinal number. In Sect. 5 we study some weakening of G_δ-conditions for diagonal of compact spaces that still imply metrizability of the underlying space and that have numerous applications in functional analysis; these results strengthen and extend classical ones as well as a result recently published in [9]. Finally, in Sect. 6 we study the relationship between the Σ-degree $\Sigma(Y)$ introduced by Hödel and the number of Nagami, obtaining that $Nag(Y) = \max\{\Sigma(Y), \ell(Y)\}$ for any completely regular space Y.

Our notation and terminology are standard. We take the books by Engelking [13], Kelley [19] and Köthe [21], as our references for topology, Banach and topological vector spaces. Our topological spaces are Hausdorff and usually referred to by letters $M, X, Y, Z, \ldots$; compact spaces are denoted by $K, L, \ldots$ Given a topological space Z we denote by $C(Z)$ the space of real continuous functions defined on Z; $\tau_p(Z)$ (or just τ_p if no misunderstanding arises) is the topology in $C(Z)$ of pointwise convergence on Z; when $Z = K$ is compact $\|\cdot\|_\infty$ denotes the supremum norm on $C(K)$. Given a space Z we denote by $\mathcal{K}(Z)$ the family of all compact subsets of Z. The diagonal of Z is the subset of $Z \times Z$ given by $\Delta := \{(x, x) : x \in Z\}$. Finally, when $(Y, \|\cdot\|)$ is a Banach space, B_Y denotes its closed unit ball, Y^* is the (topological) dual space and Y^{**} is the bidual; the *weak* topology of Y is denoted by w and w* is the *weak** topology in the dual. If Z is a subset of Y we denote by span{Z} the vector subspace generated by Z.

2 A basic result

Our goal in this section is to characterize, see Theorem 5, when a topological space is the usco image of a subspace Σ of some $\mathfrak{m}^{\mathfrak{n}}$.

We use the notions of filter and filter base as introduced in [13, p. 52]. The notions of net and subnet used here can be found in [13, p. 49] and [19, p. 65]. We say that the filter $\mathcal{F}$ *subconverges* to a subset L in a topological space Y, if given any open subset $V \subset Y$ with $L \subset V$ there exists $F \in \mathcal{F}$ such that $F \subset V$. We say that a filter base $\mathcal{B}$ *subconverges* to L when the filter $\mathcal{F}$ generated by $\mathcal{B}$ subconverges to L. A good deal of information about subconvergent filters and their relative compactoid filters can be found in [5,12,22] and the references therein.

Nice examples of subconvergent filters are produced by usco maps. Indeed, a multi-valued map $\phi : X \to 2^Y$ is a usco map when for every $x \in X$ the set $\phi(x)$ is non empty and compact and the filter base $\mathcal{B} = \{\phi(N) : N \in \mathcal{N}_x\}$ subconverges to $\phi(x)$, where $\mathcal{N}_x$ is any basis of open neighborhoods of x.

Lemma 3 *Let T be a topological space, $Y \subset T$ a subspace and $\mathcal{B}$ a filter base in Y. If $\mathcal{B}$ subconverges to a compact subset L of Y, then*

$$\bigcap_{B \in \mathcal{B}} \overline{B}^Y = \bigcap_{B \in \mathcal{B}} \overline{B}^T \subset Y.$$

Proof The inclusion $\bigcap_{B\in\mathcal{B}} \overline{B}^Y \subset \bigcap_{B\in\mathcal{B}} \overline{B}^T$ is obvious. To finish we prove that $\bigcap_{B\in\mathcal{B}} \overline{B}^T \subset L$. Note that $\mathcal{B}$ is also a filter base in T that subconverges to $L \subset T$. If $x \notin L$ we can take U_x and V_x open sets in T with $x \in U_x$, $L \subset V_x$ and $U_x \cap V_x = \emptyset$. Using that $\mathcal{B}$ subconverges to L we can choose $B \in \mathcal{B}$ such that $B \subset V_x \subset T\backslash U_x$. Since $T\backslash U_x$ is closed in T we conclude that $\overline{B}^T \subset T\backslash U_x$ and therefore $x \notin \overline{B}^T$. This finishes the proof.

The proof of the result that follows is included for the sake of completeness.

Lemma 4 *For any topological space X with $\mathfrak{m} = w(X)$ and $\mathfrak{n} = \chi(X)$, there are a subspace $\Sigma \subset \mathfrak{m}^{\mathfrak{n}}$ and a continuous onto map $f : \Sigma \to X$.*

Proof Let $\mathcal{O} = \{O_i : i \in \mathfrak{m}\}$ be a base for the topology of X. We define $\Sigma \subset \mathfrak{m}^{\mathfrak{n}}$ as follows

$$\Sigma := \{(i_j)_{j\in\mathfrak{n}} \in \mathfrak{m}^{\mathfrak{n}} : (O_{i_j})_{j\in\mathfrak{n}} \text{ is a base of open neighborhoods for some } x \in X\}.$$

The map $f : \Sigma \to X$ given by $f((i_j)_j) := \bigcap_{j\in\mathfrak{n}} O_{i_j}$ satisfies the conditions that we are looking for. The map f is well-defined and onto by the definition of Σ. We will prove that f is continuous. Let $(i_j)_{j\in\mathfrak{n}}$ be an element of Σ and $f((i_j)_{j\in\mathfrak{n}}) = x$. It means that $(O_{i_j})_{j\in\mathfrak{n}}$ is a base of open neighborhoods of x. Let U be an open neighborhood of x. Then there exists $j_0 \in \mathfrak{n}$ such that $x \in O_{i_{j_0}} \subset U$. We define the open set in Σ given by

$$G := \{(s_j)_{j\in\mathfrak{n}} \in \Sigma : s_{j_0} = i_{j_0}\}.$$

Then $(i_j)_{j\in\mathfrak{n}} \in G$ and every element $(s_j)_{j\in\mathfrak{n}} \in G$ satisfies that

$$f(s_j) = \bigcap_{j\in\mathfrak{n}} O_{s_j} \subset O_{s_{j_0}} = O_{i_{j_0}} \subset U.$$

Hence $x \in f(G) \subset U$ and the continuity of f is concluded.

Now we have all the elements to prove the main result of this section.

Theorem 5 *Let Y be a topological space and $\mathfrak{n} \leq \mathfrak{m}$ two cardinal numbers. The following statements are equivalent:*

(i) *There are a topological space X with $w(X) \leq \mathfrak{m}$ and $\chi(X) \leq \mathfrak{n}$ and a usco map $\phi : X \to 2^Y$ such that $Y = \bigcup\{\phi(x) : x \in X\}$;*
(ii) *There are a topological subspace $\Sigma \subset \mathfrak{m}^{\mathfrak{n}}$ and a usco map $\Phi : \Sigma \to 2^Y$ such that $Y = \bigcup\{\Phi(\alpha) : \alpha \in \Sigma\}$;*

Furthermore, if Y is completely regular the above two conditions are equivalent to the following one.

(iii) *There is a family of closed subsets $\mathcal{A} = \{A_i : i \in \mathfrak{m}\}$ in βY, with the property that for every $y \in Y$ there is a subset $L \subset \mathfrak{m}$ with $|L| \leq \mathfrak{n}$ such that*

$$y \in \bigcap_{l\in L} A_l \subset Y.$$

Proof (i)⇒(ii) If we we assume (i), Lemma 4 provides us with a subspace $\Sigma \subset \mathfrak{m}^{\mathfrak{n}}$ and a continuous onto map $f : \Sigma \to X$; the composition $\Phi := \phi \circ f : \Sigma \to 2^Y$ satisfies (ii). The implication (ii)⇒(i) is clear.

For the implication (iii)⇒(ii) we consider the family $\Sigma \subset \mathfrak{m}^{\mathfrak{n}}$ defined as follows

$$\Sigma := \{(i_j)_{j\in\mathfrak{n}} \in \mathfrak{m}^{\mathfrak{n}} : \emptyset \neq \bigcap_{j\in\mathfrak{n}} A_{i_j} \subset Y\}.$$

Now we prove that the map $\Phi : \Sigma \to 2^Y$ given by $\Phi((i_j)_{j\in\mathfrak{n}}) = \bigcap_{j\in\mathfrak{n}} A_{i_j}$ is usco. It is clear that $\Phi((i_j)_{j\in\mathfrak{n}})$ is compact and non-empty in Y. To prove that Φ is upper semicontinuous, we consider $(i_j)_{j\in\mathfrak{n}} \in \Sigma$ and an open set $O \subset Y$ such that $\Phi((i_j)_{j\in\mathfrak{n}}) = \bigcap_{j\in\mathfrak{n}} A_{i_j} \subset O$. Let $O_{\beta Y} \subset \beta Y$ be an open set such that $O_{\beta Y} \cap Y = O$. Since βY is compact and the sets $(A_{i_j})_{j\in\mathfrak{n}}$ are closed in βY, we have that there exists a finite set $J_0 \subset \mathfrak{n}$ such that $\bigcap_{j\in J_0} A_{i_j} \subset O_{\beta Y}$. We consider the set

$$V = \{(s_j)_{j\in\mathfrak{n}} \in \Sigma : s_j = i_j,\ j \in J_0\},$$

which is clearly an open neighborhood of $(i_j)_{j\in\mathfrak{n}}$ in Σ. We claim that $\Phi(V) \subset O$. Indeed, for every $(s_j)_{j\in\mathfrak{n}} \in V$ we have that

$$\Phi((s_j)_{j\in\mathfrak{n}}) = \bigcap_{j\in\mathfrak{n}} A_{s_j} \subset \bigcap_{j\in J_0} A_{s_j} = \bigcap_{j\in J_0} A_{i_j} \subset O_{\beta Y}.$$

On the other hand $\Phi((s_j)_{j\in\mathfrak{n}}) \subset Y$, so $\Phi((s_j)_{j\in\mathfrak{n}}) \subset O_{\beta Y} \cap Y = O$ and this proves that Φ is usco.

To finish the proof we establish that $Y = \bigcup\{\Phi((i_j)_{j\in\mathfrak{n}}) : (i_j)_{j\in\mathfrak{n}} \in \Sigma\}$. Given $y \in Y$ the assumptions in (iii) ensure us that there is $L \subset \mathfrak{m}$ with $|L| \le \mathfrak{n}$ and $y \in \bigcap_{l\in L} A_l \subset Y$. Fix $l_0 \in L$ and $g : L \to \mathfrak{n}$ any injective map. If we define $(i_j)_{j\in\mathfrak{n}}$ by $i_j = l$ if $j = g(l)$ and $i_j = l_0$ otherwise, then $y \in \Phi((i_j)_{j\in\mathfrak{n}}) = \bigcap_{l\in L} A_l$ and the proof is over.

(i)⇒(iii). We assume that (i) holds and we fix $\mathcal{O} = \{O_i \subset X : i \in \mathfrak{m}\}$ a basis of the topology of X. We prove now that the family $\mathcal{A} = \{\overline{\phi(O_i)}^{\beta Y} : i \in \mathfrak{m}\}$ satisfies the conditions required in (iii). Fix $y \in Y$ and take $x \in X$ such that $y \in \phi(x)$. We observe that there is a family $\mathcal{N}_x$ with $|\mathcal{N}_x| \le \mathfrak{n}$ with

$$\mathcal{N}_x \subset \{O_i \in \mathcal{O} : x \in O_i\}$$

and such that $\mathcal{N}_x$ is a basis of neighborhoods of the point x in X. Therefore $\phi(\mathcal{N}_x)$ is a filter base in Y that subconverges to $\phi(x)$ and so Lemma 3 applies to allow us to conclude

$$y \in \phi(x) \subset \bigcap\{\overline{\phi(O_i)}^{\beta Y} : O_i \in \mathcal{N}_x\} \subset Y,$$

that finishes the proof.

3 The number of *K*-determination of a space

This section is devoted to studying the behaviour of $\ell\Sigma(\cdot)$ and $Nag(\cdot)$ on subspaces, unions, products, usco images, linear operations, etc. We also study the relationship of $\ell\Sigma(\cdot)$ and $Nag(\cdot)$ with other classical cardinal functions like the weight $w(\cdot)$ or density character $d(\cdot)$ as well as some counterexamples providing limitation to the established relationship.

We start by noting that given a topological space Y the number $Nag(Y)$ as introduced in Definition 2 is well defined. On the other hand $\ell\Sigma(Y)$ is also well defined since every topological space is the continuous image of itself with the discrete topology, that is a metric space. It is worth noticing that after Lemma 4, for the definition of $\ell\Sigma(Y)$ we can use the family of first-countable spaces instead of the family of metric spaces.

As a first consequence of Theorem 5 we have,

Proposition 6 *Let Y be a completely regular topological space and $\mathfrak{m}$ a cardinal number. The following statements are equivalent:*

(i) $Nag(Y) \le \mathfrak{m}$ (*resp.* $\ell\Sigma(Y) \le \mathfrak{m}$);

(ii) *There is a family of closed sets* $\{A_i : i \in \mathfrak{m}\}$ *in* βY*, such that for every* $y \in Y$ *there is a set* $J \subset \mathfrak{m}$ *(resp. with* $|J| \leq \omega$*) such that* $y \in \bigcap_{i\in J} A_i \subset Y$.

The proposition that follows summarizes some properties of $\ell\Sigma(\cdot)$.

Proposition 7 *For topological spaces* Y, $(Y_j)_{j\in J}$ *and* Z *the following properties hold:*

(i) $Nag(Y) \leq w(Y)$ *and* $\ell(Y) \leq Nag(Y) \leq \ell\Sigma(Y) \leq |Y|$*; when* Y *is a metric space then* $\ell(Y) = Nag(Y) = \ell\Sigma(Y)$*;*
(ii) *if* J *is a finite or countable set then*

$$\ell\Sigma\left(\prod_{j\in J} Y_j\right) \leq \sup_{j\in J} \ell\Sigma(Y_j);$$

(iii) *let* $Z \subset Y$ *be a closed subspace, then* $\ell\Sigma(Z) \leq \ell\Sigma(Y)$*;*
(iv) *if* $\phi : Y \to 2^Z$ *is a usco map such that* $Z = \bigcup\{\phi(y) : y \in Y\}$*, then* $\ell\Sigma(Z) \leq \ell\Sigma(Y)$*; this applies in particular to continuous single-valued onto maps* $\phi : Y \to Z$*;*
(v) *let* (Y, τ) *be a topological space and* $\mathcal{G}$ *a topology in* Y *coarser than* τ*; then*

$$d(Y,\tau) \leq \max\{\ell\Sigma(Y,\tau), nw(Y,\mathcal{G})\};$$

(vi) *if* $(Y_j)_{j\in J}$ *is a family of subspaces of* Y *and* $|J| \leq \mathfrak{m}$ *then*

$$\ell\Sigma\left(\bigcup_{j\in J} Y_j\right) \leq \sup\left\{\mathfrak{m}, \sup_{j\in J} \ell\Sigma(Y_j)\right\};$$

(vii) *if* Y *be a topological vector space and* $Z \subset Y$ *then*

$$\ell\Sigma(\mathrm{span(Z)}) \leq \ell\Sigma(\mathrm{Z}).$$

Proof (i) Most of the statements are easy and their proofs are left to the reader. We will just establish $\ell(Y) \leq Nag(Y)$. Let $\phi : X \to 2^Y$ be a usco map from some topological space X such that $Y = \bigcup\{\phi(x) : x \in X\}$. Let $\mathcal{O} = \{O_i : i \in I\}$ be an open cover of Y. Since $\phi(x)$ is compact for each $x \in X$, there exists a finite set of indexes $i_1^x, i_2^x, \ldots, i_{n_x}^x \in I$ such that $\phi(x) \subset O_{i_1^x} \cup O_{i_2^x} \cup \cdots \cup O_{i_{n_x}^x}$. We define now the open set $O(x) = \bigcup_{j=1}^{n_x} O_{i_j^x}$. Since ϕ is a usco map, there exists an open neighborhood $U(x)$ of x such that $\phi(U(x)) \subset O(x)$. Since, $\ell(X) \leq w(X)$, see [13, Theorem 3.8.12], and $\mathcal{U} = \{U(x) : x \in X\}$ is an open cover of X, there exists a subset $S \subset X$ with $|S| \leq w(X)$ such that $X = \bigcup_{x\in S} U(x)$. Hence, we have

$$Y = \bigcup_{x\in S} \phi(U(x)) \subset \bigcup_{x\in S} O(x) = \bigcup_{x\in S}\bigcup_{j=1}^{n_x} O_{i_j^x}.$$

Thus we have $\ell(Y) \leq w(X)$ and consequently, $\ell(Y) \leq Nag(Y)$.

(ii) For each $j \in J$ there exists a metric space M_j with $w(M_j) = \ell\Sigma(Y_j)$ and a usco map $\phi_j : M_j \to 2^{Y_j}$ such that $Y_j = \bigcup\{\phi_j(x) : x \in M_j\}$. The metric space $M = \prod_{j\in J} M_j$ with the product topology satisfies that $w(M) \leq \sup_{j\in J} w(M_j) = \sup_{j\in J} \ell\Sigma(Y_j)$ and the map $\phi : M \to 2^{\prod_{j\in J} Y_j}$ defined by $\phi(x) := \prod_{j\in J} \phi_j(x_j)$, $x = (x_j)_{j\in J}$, is a usco map, see [13, Theorem 3.2.10], satisfying $\prod_{j\in J} Y_j = \bigcup\{\phi(x) : x \in M\}$. Hence, $\ell\Sigma(\prod_{j\in J} Y_j) \leq \sup_{j\in J} \ell\Sigma(Y_j)$.

(iii) and (iv) straightforwardly follow from the definitions.

(v) Let M be a metric space with $w(M) = \ell\Sigma(Y, \tau)$ and $\phi : M \to 2^Y$ a τ-usco map such that $Y = \bigcup\{\phi(x) : x \in M\}$. Let us define $T := M \times (Y, \mathcal{G})$. An appeal to [13, Exercise 3.1.J] ensures us that $nw(T) \leq \max\{nw(M), nw(Y, \mathcal{G})\}$. Since $nw(M) \leq w(M)$ we have that

$$nw(T) \leq \max\{\ell\Sigma(Y, \tau), nw(Y, \mathcal{G})\}. \tag{1}$$

Consider the subspace of T defined by $W := \{(x, y) \in T : y \in \phi(x)\}$. The inequality $nw(W) \leq nw(T)$ and (1) imply

$$nw(W) \leq \max\{\ell\Sigma(Y, \tau), nw(Y, \mathcal{G})\}. \tag{2}$$

Let $p : W \to (Y, \mathcal{G})$ be the continuous projection map given by $p(x, y) := y$. We claim that $p : W \to (Y, \tau)$ is also continuous. Observe that if $(x_j, y_j)_{j\in D}$ is a convergent net to (x, y) in W then $(y_j)_{j\in D}$ $\mathcal{G}$-converges to y; now we show that $(y_j)_{j\in D}$ actually τ-converges to y. Indeed, since the map ϕ is τ-usco, every subnet of $(y_j)_{j\in D}$ has a τ-cluster point in $\phi(x)$; all these cluster points are the same, namely y, because $(y_j)_{j\in D}$ $\mathcal{G}$-converges to y. Therefore $(y_j)_{j\in D}$ τ-converges to y and our claim is proved. Now, since $p : W \to (Y, \tau)$ is onto and continuous we have $d(Y, \tau) \leq d(W) \leq nw(W)$. Finally, the latter and inequality (2) imply

$$d(Y, \tau) \leq \max\{\ell\Sigma(Y, \tau), nw(Y, \mathcal{G})\},$$

and (v) is proved.

Properties (vi) and (vii) are easy to check.

Remark 8 We note that properties (ii)–(vii) in Proposition 7 hold for $Nag(\cdot)$ with similar proofs to those for $\ell\Sigma(\cdot)$. We stress that (ii) can be strengthen as follows:

(ii)'

$$Nag\Big(\prod_{j\in J} Y_j\Big) \leq \max\left\{|J|, \sup_{j\in J} Nag(Y_j)\right\}.$$

Take notice that all inequalities in property (i) in Proposition 7 can be strict. It is easy to provide an example of a space Y with $Nag(Y) < w(Y)$: if X is a reflexive Banach space of infinite dimension then the space $Y = (X, \mathrm{W})$ satisfies $Nag(Y) = \ell\Sigma(Y) = \omega < w(Y)$. If Y is the Songerfrey line then $\omega = \ell(Y) < Nag(Y)$ because otherwise $\omega = \ell(Y) = Nag(Y)$ and therefore, $\ell(Y^2) \leq Nag(Y^2) = Nag(Y) = \omega$ that cannot be because Y^2 is not Lindelöf, see [13, Example 3.8.15]. To have the inequality $\ell\Sigma(Y) < |Y|$ suffices to take $Y = \mathbb{R}$.

To provide an example of space Y with $Nag(Y) < \ell\Sigma(Y)$ is a bit more delicate. Nonetheless, we shall indeed provide a sharp example $\mathbb{Y}$ for which we have $Nag(\mathbb{Y}) \leq w(\mathbb{Y}) < \ell\Sigma(\mathbb{Y})$. The following properties of $\beta\mathbb{N}$ are used in the constructions that follow: $w(\beta\mathbb{N}) = 2^\omega$ and $|\beta\mathbb{N}| = 2^{2^\omega}$ [13, Corollary 3.6.12]. Every infinite closed set $F \subset \beta\mathbb{N}$ contains a subspace homeomorphic to $\beta\mathbb{N}$; in particular $w(F) = 2^\omega$ and $|F| = 2^{2^\omega}$ [13, Theorem 3.6.14]. If E and O are subsets of $\mathbb{N}$ such that $E \cap O = \emptyset$, then $\overline{E} \cap \overline{O} = \emptyset$ being the closures taken in $\beta\mathbb{N}$.

The wording of example 9 below is due to J. Pelant who sent us these details in a private communication long ago. We refer the reader to the paper by Tkachuk [30], where a stronger result can be found.

Example 9 There is a subspace $\mathbb{Y}$ of $\beta\mathbb{N}$ for which $|\mathbb{Y}| = 2^{2^\omega}$ and its compact subsets are finite.

Proof (Construction) We prove first that $\beta\mathbb{N}$ contains 2^{2^ω} copies of itself. Let us write

$$\mathcal{C} := \{K \subset \beta\mathbb{N} : K \text{ is homeomorphic to } \beta\mathbb{N}\},$$

and let us convince ourselves that $|\mathcal{C}| \leq 2^{2^\omega}$ and $|\mathcal{C}| \geq 2^{2^\omega}$. To prove the first inequality we simply observe that every $K \in \mathcal{C}$ is separable and consequently $K = \overline{D(K)}$ for some countable subset $D(K) \subset \beta\mathbb{N}$; hence $\mathcal{C}$ has at most the same cardinality than the family of the countable subsets of $\beta\mathbb{N}$, that is $|\mathcal{C}| \leq \left(2^{2^\omega}\right)^\omega = 2^{2^\omega}$. The other way around. We prove that $\mathcal{C}$ has at least 2^{2^ω} elements. Indeed, if $E \subset \mathbb{N}$ stands for the set of even numbers and $O \subset \mathbb{N}$ for the set of odd numbers, then their closures $\overline{E}$ and $\overline{O}$ in $\beta\mathbb{N}$ are disjoint copies of $\beta\mathbb{N}$. Hence $|\overline{O}| = 2^{2^\omega}$ and thus the family

$$\{\overline{E} \cup \{o\} : o \in \overline{O}\}$$

is made up of 2^{2^ω} different copies of $\beta\mathbb{N}$.

The construction of $\mathbb{Y}$ mimics the construction of Bernstein's sets in the real line, see [11, Claim 8.8.1 in Ch. VI]. We start by writing down $\mathcal{C} = \{K_\alpha : 0 \leq \alpha < 2^{2^\omega}\}$. Since $|K_\alpha| = 2^{2^\omega}$ for every $\alpha < 2^{2^\omega}$, by transfinite induction we can choose different points x_α and y_α in $K_\alpha \setminus \left(\{x_\gamma : 0 \leq \gamma < \alpha\} \cup \{y_\gamma : 0 \leq \gamma < \alpha\}\right)$. We claim that both subspaces $A := \{x_\gamma : 0 \leq \gamma < 2^{2^\omega}\}$ and $B := \{y_\gamma : 0 \leq \gamma < 2^{2^\omega}\}$ are suitable to be $\mathbb{Y}$. Indeed, observe that $|A| = |B| = 2^{2^\omega}$ and that $A \cap B = \emptyset$. Take $\mathbb{Y} := A$ and let us prove that its compact subsets are finite. If we assume that there is an infinite compact set $K \subset \mathbb{Y}$, then K contains a copy of $\beta\mathbb{N}$, meaning that there is $0 \leq \alpha < 2^{2^\omega}$ such that $K_\alpha \subset K \subset A = \mathbb{Y}$. This implies that $x_\alpha, y_\alpha \in A$ and we reach a contradiction with the fact that $y_\alpha \in B$ and $A \cap B = \emptyset$. The proof is over.

Proposition 10 *The space $\mathbb{Y}$ constructed in Example 9 satisfies*

$$Nag(\mathbb{Y}) \leq w(\mathbb{Y}) < \ell\Sigma(\mathbb{Y}).$$

Proof We only have to prove that $w(\mathbb{Y}) < \ell\Sigma(\mathbb{Y})$. To do that, we begin by observing that if (M, d) is a metric space of weight at most 2^ω then $|M| \leq 2^\omega$, see [16, Theorem 4.1]. On the other hand the equality $w(\beta\mathbb{N}) = 2^\omega$ implies that $w(\mathbb{Y}) \leq 2^\omega$. We prove now that the inequality $\ell\Sigma(\mathbb{Y}) \leq w(\mathbb{Y})$ cannot hold. Indeed, if this were the case there would be a metric space (M, d) with weight at most 2^ω and a usco map $\phi : M \to 2^{\mathbb{Y}}$ such that $\mathbb{Y} = \bigcup\{\phi(x) : x \in M\}$. Bearing in mind that $|M| \leq 2^\omega$ and that each $\phi(x)$ is finite we obtain that $|\mathbb{Y}| \leq 2^\omega$ that contradicts the fact that $|\mathbb{Y}| = 2^{2^\omega}$ and the proof is finished.

Observe that in [4, Theorem 15] is produced and an example of the kind $Y = (C(K), \tau_p)$ with $Nag(Y) < \ell\Sigma(Y)$; according to inequality (4) in Proposition 12 below, this example satisfies $Nag(Y) < \ell\Sigma(Y) \leq w(Y)$ as opposed to $Nag(\mathbb{Y}) \leq w(\mathbb{Y}) < \ell\Sigma(\mathbb{Y})$ as established in Proposition 10.

Next lemma is needed several times in the paper. The rest of the section is devoted to some preliminary study of $\ell\Sigma(\cdot)$ in $C(K)$ and Banach spaces that will be useful in Sect. 4.

Lemma 11 *Let τ_1 and τ_2 be two comparable topologies on Y with the same compact sets then*

$$\ell\Sigma(Y, \tau_1) = \ell\Sigma(Y, \tau_2).$$

Proof Assume that τ_2 is finer than τ_1. On one hand $\ell\Sigma(Y, \tau_1) \leq \ell\Sigma(Y, \tau_2)$ after property (iv) in Proposition 7. On the other hand, if M is a metric space with $w(M) = \ell\Sigma(Y, \tau_1)$ and $\phi : M \to 2^Y$ a τ_1-usco map with $Y = \bigcup\{\phi(x) : x \in M\}$, then by [5, Proposition 2.3] $\phi : M \to 2^Y$ is τ_2-usco, and therefore $\ell\Sigma(Y, \tau_2) \leq \ell\Sigma(Y, \tau_1)$. The proof is over.

Proposition 12 *Let K be a compact space and $Y \subset C(K)$ a linear subspace then*

$$\ell\Sigma(Y, \tau_p) = \ell\Sigma(Y, \mathrm{w}); \tag{3}$$

In particular

$$\begin{aligned}\ell\Sigma(C(K), \tau_p) &= \ell\Sigma(C(K), \mathrm{w}) \leq w(C(K), \|\cdot\|_\infty)\\ &= d(C(K), \mathrm{w}) = d(C(K), \tau_p) \leq w(C(K), \tau_p).\end{aligned} \tag{4}$$

Proof To prove equality (3) we will prove that

$$\ell\Sigma(Y \cap B_{C(K)}, \tau_p) \overset{(a)}{=} \ell\Sigma(Y \cap B_{C(K)}, \mathrm{w})$$

and

$$\ell\Sigma(Y \cap B_{C(K)}, \tau_p) \overset{(b)}{=} \ell\Sigma(Y, \tau_p), \quad \ell\Sigma(Y \cap B_{C(K)}, \mathrm{w}) \overset{(c)}{=} \ell\Sigma(Y, \mathrm{w}).$$

Equality (a) follows from Lemma 11 bearing in mind that $(Y \cap B_{C(K)}, \tau_p)$ and $(Y \cap B_{C(K)}, \mathrm{w})$ have the same compact subsets [14, Theorem 4.2] and τ_p is coarser than w. We prove now equality (b). Note that $Y \cap B_{C(K)}$ is closed in (Y, τ_p), that for any $n \in \mathbb{N}$ the spaces $(Y \cap B_{C(K)}, \tau_p)$ and $(n(Y \cap B_{C(K)}), \tau_p)$ are homeomorphic and that

$$\bigcup_{n\in\mathbb{N}} n(Y \cap B_{C(K)}) = \bigcup_{n\in\mathbb{N}} Y \cap nB_{C(K)} = Y.$$

All the above and properties (iii), (iv) and (vi) in Proposition 7 allow us to conclude that

$$\ell\Sigma(Y \cap B_{C(K)}, \tau_p) \leq \ell\Sigma((Y, \tau_p)) \leq \sup_{n\in\mathbb{N}} \ell\Sigma(n(Y \cap B_{C(K)}), \tau_p) = \ell\Sigma(Y \cap B_{C(K)}, \tau_p),$$

and therefore equality (b) is proved. Likewise, equality (c) can be proved and consequently (3) follows from $(a) + (b) + (c)$.

To finish, note that $\ell\Sigma(C(K), \tau_p) = \ell\Sigma(C(K), \mathrm{w})$ in (4) follows from (3) with $Y = C(K)$. Since the identity map $i : (C(K), \|\cdot\|_\infty) \to (C(K), \mathrm{w})$ is continuous we have that $\ell\Sigma(C(K), \mathrm{w}) \leq w(C(K), \|\cdot\|_\infty)$. The rest of statements in (4) are well known. The proof is over.

Proposition 13 *Let Y be a Banach space and let $Z \subset Y$ be a dense subspace. Then*

$$\ell\Sigma(Y, \mathrm{w}) \leq \ell\Sigma(Z, \mathrm{w}).$$

Proof Here we consider Y canonically embedded in Y^{**}. If this is so, note that the weak topology w on Y is the topology induced by the weak* topology w* of Y^{**}.

Let (M, d) be a metric space with $w(M) = \ell\Sigma(Z, \mathrm{w})$ and let $\varphi : M \to 2^Z$ be a w-usco map such that $Z = \bigcup\{\varphi(x) : x \in M\}$. For every $n \in \mathbb{N}$ we define the usco map extension of φ with values in (Y^{**}, w^*) by $\varphi_n(x) := \varphi(x) + 2^{-n}B_{Y^{**}}$ for every $x \in M$. Now, for any given $k \in \mathbb{N}$ let us define the map

$$\phi_k(x_1, x_2, \ldots, x_k) := \varphi_1(x_1) \cap \varphi_2(x_2) \cap \cdots \cap \varphi_k(x_k)$$

for $(x_1, x_2, \ldots, x_k) \in M^k$, with values in (Y^{**}, w^*) too. With domain the subspace $\mathcal{M}_k := \{x \in M^k : \phi_k(x) \neq \emptyset\}$ the map ϕ_k is usco. Indeed, take a sequence $(x_1^n, x_2^n, \ldots, x_k^n) \in \mathcal{M}_k$ that converges to $(x_1, x_2, \ldots, x_k)$ together with $y_n \in \phi_k(x_1^n, x_2^n, \ldots, x_k^n)$, for $n = 1, 2, \ldots$ Since each φ_i is w*-upper semicontinuous there is a w*-cluster point y of the sequence $(y_n)_n$ with

$$y \in \varphi_1(x_1) \cap \cdots \cap \varphi_k(x_k) = \phi_k(x_1, x_2, \ldots, x_k).$$

To finish the proof we consider

$$\phi((x_n)_n) := \bigcap_{n=1}^{\infty} \varphi_n(x_n)$$

defined on $\mathcal{M} := \{(x_n)_n \in M^{\mathbb{N}} : \phi((x_n)_n) \neq \emptyset\}$. Observe that we have

$$\phi((x_n)_n) \subset \bigcap_{n=1}^{\infty} \left\{ Z + \frac{1}{2^n} B_{Y^{**}} \right\} = \overline{Z}^{\|\cdot\|} = Y$$

and that ϕ is compact valued in (Y, w). Since Z is dense in Y and $\varphi(M) = Z$ we also have that $\phi(\mathcal{M}) = Y$. We claim that ϕ is w-upper semicontinuous. Indeed, let us consider $(x_n)_n \in \mathcal{M}$ and a weak open set W in Y with $\phi((x_n)_n) \subset W$. Take a w*-open subset V of Y^{**} such that $W = V \cap Y$. Since $\bigcap_{n=1}^{\infty} \varphi_n(x_n)$ is non-empty and w-compact, there is an integer m such that we have $\phi_m(x_1, \ldots, x_m) \subset V$. The upper semicontinuity of ϕ_m provides us with an open neighborhood U of $(x_1, \ldots, x_m) \in M^m$ such that $\phi_m(U \cap \mathcal{M}_m) \subset V$. Then we have the inclusion $\phi\Big((U \times M \times \cdots \times M \cdots) \cap \mathcal{M}\Big) \subset V \cap Y = W$, that proves that ϕ is w-usco.

Summing up, $\ell\Sigma(Y, \mathrm{w}) \leq w(\mathcal{M}) \leq w(M) = \ell\Sigma(Z, \mathrm{w})$. The proof is over.

Proposition 14 *Let K be a compact space and let Z be a subset of $C(K)$. If Z separates points of K, then*

$$\ell\Sigma(C(K), \tau_p) \leq \ell\Sigma(Z, \tau_p).$$

Proof Let $\mathcal{A}$ be the subalgebra in $C(K)$ generated by Z and the constant functions $\mathrm{span}\mathbb{1}$. We prove first that $\ell\Sigma(\mathcal{A}, \tau_p) \leq \ell\Sigma(Z, \tau_p)$. Observe that if we define $Y := Z \cup \{\mathrm{span}\mathbb{1}\}$ and for every $n \in \mathbb{N}$

$$Z_n := \{f_1 \cdot f_2 \cdot \ldots \cdot f_n : \; f_i \in Y, \; i = 1, \ldots, n\}$$

then $\mathcal{A} = \mathrm{span}\Big(\bigcup_{n=1}^{\infty} Z_n\Big)$. Being for every $n \in \mathbb{N}$ the multiplication

$$\begin{aligned} (Y, \tau_p)^n &\longrightarrow (Z_n, \tau_p) \\ (f_1, f_2, \ldots, f_n) &\longrightarrow f_1 \cdot f_2 \cdot \ldots \cdot f_n, \end{aligned}$$

continuous and onto, we can apply properties (ii), (iv), (vi) and (vii) of Proposition 7 to conclude that

$$\begin{aligned} \ell\Sigma(\mathcal{A}, \tau_p) &\leq \ell\Sigma\left(\bigcup_{n\in\mathbb{N}} (Z_n, \tau_p)\right) \leq \sup_n \ell\Sigma(Z_n, \tau_p) \leq \sup_n \ell\Sigma\big((Y, \tau_p)^n\big) \\ &= \ell\Sigma(Y, \tau_p) \leq \ell\Sigma(Z, \tau_p). \end{aligned}$$

Since Z separates the points of K, the Stone-Weierstrass's theorem, see [19, p. 244], implies that the subalgebra $\mathcal{A}$ is dense in $(C(K), \|\cdot\|_\infty)$ and Proposition 13 ensures us that $\ell\Sigma(C(K), \mathrm{w}) \leq \ell\Sigma(\mathcal{A}, \mathrm{w})$. On the other hand equality (3) in Proposition 12 allows us to conclude that $\ell\Sigma(C(K), \tau_p) = \ell\Sigma(C(K), \mathrm{w})$ and $\ell\Sigma(\mathcal{A}, \mathrm{w}) = \ell\Sigma(\mathcal{A}, \tau_p)$. Combining all the above, we finally conclude that $\ell\Sigma(C(K), \tau_p) \leq \ell\Sigma(Z, \tau_p)$ and the proof is over.

4 Applications to $C(Y)$ spaces

In this section we take advantage of our previous study for $\ell\Sigma(\cdot)$ to obtain some results in C_p-theory. Next result is an extension of [29, Theorem 3.4].

Proposition 15 *For a compact space K and a cardinal number $\mathfrak{m}$, the following statements are equivalent:*

(i) *there is a set $Z \subset C(K)$ that separates the points of K and such that $\ell\Sigma(Z, \tau_p) \leq \mathfrak{m}$;*
(ii) *$\ell\Sigma(C(K), \tau_p) \leq \mathfrak{m}$;*
(iii) *there is a topological space Y with $\ell\Sigma(Y) \leq \mathfrak{m}$ such that K is homeomorphic to a pointwise compact subset of $C(Y)$.*

Proof The implication (i)⇒(ii) follows from Proposition 14. For the implication (ii)⇒(iii) it suffices to take $Y = (C(K), \tau_p)$. Finally, to prove the implication (iii)⇒(i) we proceed as follows: assuming that $K \subset C(Y)$ we define the continuous map $j : Y \to (C(K), \tau_p)$ given by $j(y)(f) := f(y)$, for $y \in Y$ and $f \in K$; note that $j(Y)$ separates points of K and property (iv) of Proposition 7 tells us that $\ell\Sigma(j(Y), \tau_p) \leq \ell\Sigma(Y)$; hence (i) holds for $Z := j(Y)$ and the proof is over.

Proposition 16 *For any topological space Y the following inequalities hold*

$$t(C(Y), \tau_p) \leq Nag(Y) \leq \ell\Sigma(Y).$$

Proof Using properties (i) and (ii) of Proposition 7 we have that

$$\sup_{n\in\mathbb{N}} \ell(Y^n) \leq \sup_{n\in\mathbb{N}} Nag(Y^n) = Nag(Y)$$

Arkhangel'skiĭ's Theorem [3, Theorem II.1.1, p. 45] implies that $t(C(Y), \tau_p) \leq Nag(Y)$ and the proof is over.

Given a cardinal number $\mathfrak{m}$, a topological space Y is called *strongly $\mathfrak{m}$-monolithic* if for every set $A \subset Y$ with $|A| \leq \mathfrak{m}$ we have $w(\overline{A}) \leq \mathfrak{m}$, see [3].

Proposition 17 *If Y is any topological space, then every compact subset K of $(C(Y), \tau_p)$ is strongly $\ell\Sigma(Y)$-monolithic.*

Proof Take A a subset of K with $\omega \leq |A| \leq \ell\Sigma(Y)$. Define $\delta : Y \to (C(\overline{A}), \tau_p(\overline{A}))$ by $\delta(y)(h) := h(y)$ for $y \in Y$ and $h \in \overline{A}$. The map δ is well-defined and continuous. Since $\delta(Y)$ separates the points of $\overline{A}$, Proposition 14 implies that $\ell\Sigma(C(\overline{A}), \tau_p(\overline{A})) \leq \ell\Sigma(\delta(Y), \tau_p(\overline{A}))$. On the other hand, since δ is continuous we can use property (iv) of Proposition 7 to deduce that $\ell\Sigma(\delta(Y), \tau_p(\overline{A})) \leq \ell\Sigma(Y)$. Combining the two inequalities established as of now we obtain that

$$\ell\Sigma(C(\overline{A}), \tau_p(\overline{A})) \leq \ell\Sigma(Y). \tag{5}$$

Note also that by the very definitions involved we have the inequalities

$$nw(C(\overline{A}), \tau_p(A)) \leq w(C(\overline{A}), \tau_p(A)) \leq |A| \leq \ell\Sigma(Y). \tag{6}$$

Combining the inequalities (5) and (6) and bearing in mind property (v) of Proposition 7 we conclude that $d(C(\overline{A}), \tau_p(\overline{A})) \leq \ell\Sigma(Y)$. Now since $w(\overline{A}) = d(C(\overline{A}), \tau_p(\overline{A}))$, [3, Theorem I.1.5], we conclude that $w(\overline{A}) \leq \ell\Sigma(Y)$ and the proof is over.

A topological space Y is called *angelic* (Fremlin) if every relatively countably compact subset A of Y is relatively compact and its closure $\overline{A}$ is made up of the limits of sequences from A; if moreover the separable compact subsets of Y are metrizable then Y is said to be *superangelic*. All Banach spaces with their weak topologies are superangelic spaces. A good classical reference for angelic spaces is [14]. A good deal of results about angelic spaces can be found in [6,7,26] and the recent book [18].

If we specialize the previous results in this section for countable cardinals we obtain another proof of the superangelic character of $(C(Y), \tau_p)$ when Y is a countably K-determined space, result that was originally established in [26] with a very much different approach and techniques.

Recall that a topological space Y is said to be a *k-space* if for any set $A \subset Y$, A is closed if, and only if, the intersection of A with any compact subspace K of Z is closed in K. If (Y, τ) is a topological space then the family τ^k of subsets of Y with open intersections with all compact subsets of (Y, τ), is a topology on Y with properties:

(a) τ is coarser than τ^k;
(b) τ and τ^k have the same compact sets;
(c) (Y, τ^k) is a k-space;
(d) $\tau = \tau^k$ if, and only if, (Y, τ) is a k-space.

Corollary 18 ([26]) *If Y is countably K-determined then $(C(Y), \tau_p)$ is superangelic.*

Proof Being Y countably K-determined means that $\ell\Sigma(Y) = \omega$. Lemma 11 implies that $\ell\Sigma(Y, \tau^k) = \omega$ too. Note that $(C(Y, \tau), \tau_p)$ is a subspace of $(C(Y, \tau^k), \tau_p)$, and then angelic lemma [14, p. 28], ensures us that to prove that $(C(Y, \tau), \tau_p)$ is angelic (superangelic) it is enough to prove that $(C(Y, \tau^k), \tau_p)$ is angelic (superangelic). Therefore for the rest of the proof we can and do assume that $(Y, \tau) = (Y, \tau^k)$ is a k-space with $\ell\Sigma(Y) = \omega$. Since Y is a k-space we can use [14, Ex. 1.21 b)] to obtain that every relatively countably compact set $A \subset (C(Y), \tau_p)$ is relatively compact. On the other hand separable compact subsets of $(C(Y), \tau_p)$ are metrizable after Proposition 17.

To finally establish that $(C(Y), \tau_p)$ is superangelic we prove that if A is a relatively compact subset of $(C(Y), \tau_p)$ and $f \in \overline{A}$ then there is a sequence in A that converges to f. Indeed, since $f \in \overline{A}$ and $t(C(Y), \tau_p) \leq \omega$ after Proposition 16, there exists a countable set $D \subset A$ such that $f \in \overline{D}$. Proposition 17 comes into play again to imply that $\overline{D}$ is τ_p-metrizable. Hence, there exists a sequence $(f_n)_n$ in $D \subset A$ which converges to f in the pointwise convergence topology and the proof finishes.

5 Compact spaces with G_δ-diagonal and their relatives

Following the terminology of [9,31], given topological spaces M and Y, an M-ordered compact cover of a space Y is a family $\mathcal{F} = \{F_K : K \in \mathcal{K}(M)\} \subset \mathcal{K}(Y)$ such that

$$\bigcup \mathcal{F} = Y \quad \text{and} \quad K \subset L \text{ implies } F_K \subset F_L \quad \text{for any } K, L \in \mathcal{K}(M).$$

Y is said to be *strongly dominated by the space M* if there exists an M-ordered compact cover $\mathcal{F}$ of the space Y such that the family $\mathcal{F}$ swallows all compact subsets of Y in the sense that for any compact $C \subset Y$ there is $F \in \mathcal{F}$ such that $C \subset F$. Pioneer results about strongly dominated spaces by second countable spaces are the following:

Theorem 19 (Christensen [10], Th. 3.3) *A second countable topological space is strongly $\mathbb{N}^{\mathbb{N}}$-dominated if and only if it is completely metrizable.*

Theorem 20 (Cascales–Orihuela [6], Th. 1) *If K is a compact space such that $(K \times K)\setminus\Delta$ is strongly $\mathbb{N}^{\mathbb{N}}$-dominated then K is metrizable.*

Theorem 20 has had a good number of applications over the years and the techniques developed there inspired the work by others: a good account of applications of this result and its techniques can be found in [18]. Recently, in [9] it have been proved that the hypothesis strongly $\mathbb{N}^{\mathbb{N}}$-dominated in Theorem 20 can be weakened to strongly dominated by some second countable space. Our aim here is to extend these results about metrizability of compact sets as well as to offer a newer more general and clarifying approach to them.

Theorem 21 *Let K be a compact space and $\mathfrak{m}$ a cardinal number. The following statements are equivalent:*

(i) $w(K) \leq \mathfrak{m}$;
(ii) *There exists a metric space M with $w(M) \leq \mathfrak{m}$ and a family $\mathcal{O} = \{O_L : L \in \mathcal{K}(M)\}$ of open sets in $K \times K$ that is basis of the neighborhoods of Δ such that $O_{L_1} \subset O_{L_2}$ whenever $L_2 \subset L_1$ in $\mathcal{K}(M)$;*
(iii) $(K \times K)\setminus\Delta$ *is strongly dominated by a metric M with $w(M) \leq \mathfrak{m}$.*

Proof The implication (i)⇒(ii) goes as follows. Assuming that (i) holds, we have that $d(C(K), \|\cdot\|_\infty) = w(K) \leq \mathfrak{m}$, see [3, Theorem I.1.5]. Take a family $\{f_i : i \in A\} \subset C(K)$ with $|A| \leq \mathfrak{m}$ that is $\|\cdot\|_\infty$-dense. We define $M := A$ endowed with the discrete topology and for every compact (hence finite) set $L \subset M$ we consider

$$O_L := \bigcap_{i \in L} \left\{(x, y) \in K \times K : |f_i(x) - f_i(y)| < \frac{1}{|L|}\right\}$$

Each O_L is open in $K \times K$ and it is easily proved that $O_{L_1} \subset O_{L_2}$ whenever $L_2 \subset L_1$ in $\mathcal{K}(M)$. On the other hand, since

$$\overline{O_L} \subset \bigcap_{i \in L} \left\{(x, y) \in K \times K : |f_i(x) - f_i(y)| \leq \frac{1}{|L|}\right\}$$

the density of $\{f_i : i \in A\}$ in $(C(K), \|\cdot\|_\infty)$ imply that $\Delta = \bigcap\{\overline{O_L} : L \in \mathcal{K}(M)\}$; this last equality and a standard compactness argument allow us to conclude that $\{O_L : L \in \mathcal{K}(M)\}$ is a basis for the open neighborhoods of Δ in $K \times K$, and therefore (ii) is satisfied.

The equivalence (ii)⇔(iii) is easily established by taking complements and defining $F_L := (K \times K)\setminus O_L$ when the O_L's are given and $O_L := (K \times K)\setminus F_L$ when the F_L's are given.

To finish we prove that (ii)⇒(i). Let us assume that (ii) holds and given $m \in \mathbb{N}$ and a sequence $(L_1, L_2, \ldots)$ in $\mathcal{K}(M)$ we define

$$\varphi(m, L_1, L_2, \ldots) := \bigcap_{n \in \mathbb{N}} \{f \in mB_{C(K)} : |f(x) - f(y)| \leq \frac{1}{n}, \text{ for all } (x, y) \in O_{L_n}\}. \tag{7}$$

Note that each $\varphi(m, L_1, L_2, \ldots)$ is $\|\cdot\|_\infty$-bounded, closed and equicontinuous as a family of functions defined on K. Therefore, Ascoli's theorem, see [19, p. 234], implies that $\varphi(m, L_1, L_2, \ldots)$ is compact in $(C(K), \|\cdot\|_\infty)$. If $(\mathcal{K}(M), h)$ is the lattice of compact subsets of M with the Hausdorff distance, then $w(\mathcal{K}(M), h) = w(M)$ [28, Proposition 2.4.14]. Therefore the product $M' := \mathbb{N} \times \prod_{n=1}^{\infty}(\mathcal{K}(M), h)$ of countably many copies of $(\mathcal{K}(M), h)$ and $\mathbb{N}$ is still a metric space with $w(M') = w(M)$. Note that the formula (7) defines a multi-map $\varphi : M' \to \mathcal{K}(C(K), \|\cdot\|_\infty)$. Being $\mathcal{O}$ a basis of neighborhoods of Δ implies that $C(K) = \bigcup\{\varphi(x) : x \in M'\}$. On the other hand φ has the following property:

[P] If a sequence $(x_k)_k$ *converges in* M'*, then* $\bigcup\{\varphi(x_k) : k \in \mathbb{N}\}$ *is relatively compact in* $(C(K), \|\cdot\|_\infty)$.

Indeed, if $x_k := (m_k, L_1^k, L_2^k, \dots)$ converges when $k \to \infty$ to $x = (m, L_1, L_2, \dots)$ in M' then there is $l \in \mathbb{N}$ with $m_k \leq l$ for every $k \in \mathbb{N}$ and $S_n := \bigcup_k L_n^k \cup L_n$ is compact in M for every $n \in \mathbb{N}$ after, [23, Lemma 1.11.2]. The decreasing order in $\mathcal{O}$ easily implies that

$$\bigcup\{\varphi(x_k) : k \in \mathbb{N}\} \subset \varphi(l, S_1, S_2, \dots),$$

and therefore property *[P]* is proved. An appeal to [5, Theorem 3.1] (see also [8, Corollary 3.1]) provides us with an usco map $\psi : M' \to \mathcal{K}(C(K), \|\cdot\|_\infty)$ such that $\varphi(x) \subset \psi(x)$ for every $x \in M'$. Thus $C(K) = \bigcup\{\varphi(x) : x \in M'\} = \bigcup\{\phi(x) : x \in M'\}$. Summarizing, we have proved that $\ell\Sigma(C(K), \|\cdot\|_\infty) \leq w(M') = \mathfrak{m}$. Since $(C(K), \|\cdot\|_\infty)$ is metric we have that $\mathfrak{m} \geq \ell\Sigma(C(K), \|\cdot\|_\infty) = \ell(C(K), \|\cdot\|_\infty) = d(C(K), \|\cdot\|_\infty) = w(K)$ and the proof is over.

Specializing the above result for $\mathfrak{m} = \omega$ we obtain:

Corollary 22 *For a compact space* K *the following statements are equivalent:*

(i) K *is metrizable;*
(ii) Δ *is* G_δ *in* $K \times K$*;*
(iii) $\Delta = \bigcap_{n=1}^\infty G_n$ *where* $\{G_n : n \in \mathbb{N}\}$ *is basis of open neighborhoods of* Δ*;*
(iv) $(K \times K)\backslash\Delta = \bigcup_{n=1}^\infty F_n$*, with* $\{F_n : n \in \mathbb{N}\}$ *increasing family of compact sets that swallows all the compact subsets in* $(K \times K)\backslash\Delta$*;*
(v) $(K \times K)\backslash\Delta = \bigcup_{\alpha\in\mathbb{N}^\mathbb{N}} F_\alpha$*, with* $\{F_\alpha : \alpha \in \mathbb{N}\}$ *increasing family of compact sets (*i.e. $F_\alpha \subset F_\beta$ *whenever* $\alpha \leq \beta$ *in the coordinatewise order of* $\mathbb{N}^\mathbb{N}$*) that swallows all the compact subsets in* $(K \times K)\backslash\Delta$*;*
(vi) $(K \times K)\backslash\Delta$ *is strongly dominated by a Polish space;*
(vii) $(K \times K)\backslash\Delta$ *is strongly dominated by a separable metric space;*
(viii) $(K \times K)\backslash\Delta$ *is Lindelöf.*

Proof The equivalence (i)⇔(ii) is a classical result due to Šneĭder that can be found in [13, Exercise 4.2.B]. The implication (i)⇔(viii) is also classical: (i)⇒(viii) is obvious and (viii)⇒(ii) is a nice exercise. The implication (ii)⇒(iii) is proved by showing that if Δ is a G_δ set then there is a decreasing sequence $\{G_n : n \in \mathbb{N}\}$ of open neighborhoods of Δ with the property $\Delta = \bigcap_{n=1}^\infty \overline{G_n}$: this last equality implies that $\{G_n : n \in \mathbb{N}\}$ is basis of open neighborhoods of Δ. For the implication (iii)⇒(iv) we define $F_n := (K \times K)\backslash G_n$. For the proof of (iv)⇒(v) if $\{F_n : n \in \mathbb{N}\}$ are given we simply take $\{F_\alpha : \alpha \in \mathbb{N}\}$ defined by $F_\alpha := F_{\pi_1(\alpha)}$ where $\pi_1 : \mathbb{N}^\mathbb{N} \to \mathbb{N}$ is the projection in the first coordinate. Similar arguments work for (v)⇒(vi). Assume that (v) holds. Given a compact set $L \subset \mathbb{N}^\mathbb{N}$ we define $\alpha(L) \in \mathbb{N}^\mathbb{N}$ by the formula

$$\alpha(L) := (\sup \pi_1(L), \sup \pi_2(L), \dots, \sup \pi_n(L), \dots)$$

and $F_L := F_{\alpha(L)}$, where $\pi_n : \mathbb{N}^\mathbb{N} \to \mathbb{N}$ is the nth-projection, for every $n \in \mathbb{N}$. The family $\{F_L : L \in \mathcal{K}(\mathbb{N}^\mathbb{N})\}$ strongly dominates $(K \times K)\backslash\Delta$ and therefore (vi) holds. The implication (vi)⇒(vii) is obvious and the implication (vii)⇒(i) follows from Theorem 21. The proof is over.

Let us mention that Theorem 20 was originally stated with the hypothesis (v) in the Corollary above instead of the hypothesis of $\mathbb{N}^\mathbb{N}$-domination. We refer to [9] for more information about spaces (strongly) dominated by Polish and second countable spaces.

6 The number of Nagami and the Σ-degree of a topological space

In this last section we study the relationship between $Nag(\cdot)$ and the Σ-degree of a topological space that was introduced by Hödel [15] influenced by Nagami's ideas [25].

To define the Σ-degree of a topological space we have to fix some notation first. If Y is a set, $\mathcal{F}$ is a cover of Y and p belongs to Y we write

$$C(p,\mathcal{F}) := \bigcap\{F \in \mathcal{F} :\ p \in F\}.$$

Observe that when Y is a topological space and $\mathcal{F}$ is a locally finite cover, then for every $p \in Y$ the family $\{F \in \mathcal{F} :\ p \in F\}$ is finite.

Definition 23 Let Y be a regular topological space.

(i) A *strong* Σ*-net* for a topological space Y is a collection of locally finite covers $\{\mathcal{F}_\alpha :\ \alpha \in A\}$ whose elements are closed subsets in Y verifying the following conditions:

  (a) $C(p) = \bigcap\{C(p,\mathcal{F}_\alpha) :\ \alpha \in A\}$ is compact, for every $p \in Y$;
  (b) $\{C(p,\mathcal{F}_\alpha) :\ \alpha \in A\}$ is a basis for $C(p)$ in the sense that for each open set U such that $C(p) \subset U$, there exists $\alpha \in A$ satisfying $C(p,\mathcal{F}_\alpha) \subset U$.

(ii) The Σ*-degree* of Y, $\Sigma(Y)$, is the smallest cardinal number $\mathfrak{m}$ such that Y has a strong Σ-net $\{\mathcal{F}_\alpha :\ \alpha \in A\}$ with $|A| = \mathfrak{m}$.
(iii) Y is said to be a *strong* Σ*-space* (see [25]) when $\Sigma(Y) = \omega$.

Next observation will be used in the theorem that follows. A short proof for the sake of completeness is included.

Lemma 24 *If Y is a topological space and $\mathcal{F}$ a locally finite cover of Y, then $|\mathcal{F}| \le \ell(Y)$.*

Proof For every $y \in Y$, let U_y be an open neighborhood of y which meets finitely many elements of $\mathcal{F}$. Then there exists $Z \subset Y$ such that $|Z| = \ell(Y)$ verifying $Y = \bigcup\{U_y :\ y \in Z\}$. Clearly, we have that

$$\mathcal{F} = \bigcup_{y\in Z}\{F \in \mathcal{F} :\ F \cap U_y \ne \emptyset\},$$

and each $\{F \in \mathcal{F} :\ F \cap U_y \ne \emptyset\}$ is finite. Thus, we conclude that $|\mathcal{F}| \le |Z| = \ell(Y)$.

Theorem 25 *Let Y be a completely regular topological space, then*

$$Nag(Y) = \max\{\Sigma(Y), \ell(Y)\}.$$

Proof First, we will prove that $Nag(Y) \ge \Sigma(Y)$. Let $\mathcal{F} = \{A_j :\ j \in J\}$ be a family of closed sets in βY with $|J| = Nag(Y)$ as described in Proposition 6. Let $\mathcal{L}$ be the family of finite subsets of J with the property that for each $F \in \mathcal{L}$ we have that

$$A_F := \bigcap_{j\in F} A_j \cap Y \ne \emptyset.$$

For every $F \in \mathcal{L}$ we define the locally finite cover $\mathcal{F}_F := \{A_F, Y\}$. The family of covers $\{\mathcal{F}_F :\ F \in \mathcal{L}\}$ is a strong Σ-net in Y. Indeed, for each $p \in Y$ we have that

$$C(p) = \bigcap\{A_F :\ p \in A_F,\ F \in \mathcal{L}\} = \bigcap\{A_j :\ j \in J,\ p \in A_j\} \subset Y.$$

Thus, $C(p)$ is compact. On the other hand, if we take an open set O in Y such that $C(p) \subset O$ we can find an open set $O_{\beta Y}$ of βY such that $O_{\beta Y} \cap Y \subset O$. Now because of the compactness of βY there exists $F \in \mathcal{L}$ such that

$$\bigcap\{A_j : j \in F, \ p \in A_j\} \subset O_{\beta Y}.$$

Thus, $A_F \subset O_{\beta Y} \cap Y$ and the proof that $\{\mathcal{F}_F : F \in \mathcal{L}\}$ is a strong Σ-net in Y is finished. Bearing in mind Proposition 7 we have $Nag(Y) \geq \ell(Y)$, and therefore we conclude that

$$Nag(Y) \geq \max\{\Sigma(Y), \ell(Y)\}.$$

We will prove now the converse inequality. Assume that $\{\mathcal{F}_\alpha : \alpha \in A\}$ is a strong Σ-net in Y. After Lemma 24, we have that $|\mathcal{F}_\alpha| \leq \ell(Y)$ for every $\alpha \in A$. Consider the family $\mathcal{T} := \bigcup_{\alpha \in A} \mathcal{F}_\alpha$ and let $\mathcal{F}$ be the family of finite non empty intersections of members of $\mathcal{T}$. It is clear that $|\mathcal{F}| \leq \sup\{|A|, \ell(Y)\}$. Now we prove that the family $\{\overline{F}^{\beta Y} : F \in \mathcal{F}\}$ satisfies property (ii) in Proposition 6. Fix $p \in Y$ and consider $\mathcal{B}(p) = \{F \in \mathcal{F} : p \in F\}$. Note that $\mathcal{B}(p)$ is a filter base in Y. On the other hand $\mathcal{B}(p)$ subconverges to the non empty compact set $\bigcap_{\alpha \in A} C(p, \mathcal{F}_\alpha) = C(p) \subset Y$. Indeed, by property (b) in the definition of strong Σ-net above, if $U \subset Y$ is open and $C(p) \subset U$ then there is $\alpha \in A$ such that

$$C(p, \mathcal{F}_\alpha) = \bigcap\{F \in \mathcal{F}_\alpha : p \in F\} \subset U.$$

Observe that $C(p, \mathcal{F}_\alpha) \in \mathcal{B}$ because being $\mathcal{F}_\alpha$ locally finite then $C(p, \mathcal{F}_\alpha)$ is a finite intersection of members of $\mathcal{T}$ containing p. This proves that $\mathcal{B}(p)$ subconverges to the compact $C(p) \subset Y$. An appeal to Lemma 3 ensures us that

$$p \in \bigcap\{\overline{F}^{\beta Y} : F \in \mathcal{B}(p)\} = \bigcap\{F : F \in \mathcal{B}(p)\} = C(p) \subset Y.$$

Hence the family $\{\overline{F}^{\beta Y} : F \in \mathcal{F}\}$ satisfies property (ii) in Proposition 6 and since $|\mathcal{F}| \leq \sup\{|A|, \ell(Y)\}$, where $|A|$ is the cardinal of an arbitrary strong Σ-net of Y we obtain $Nag(Y) \leq \max\{\Sigma(Y), \ell(Y)\}$ and the proof is finished.

Proposition 26 (Proposition 4.1, [15]) *Every regular topological space Y has a strong Σ-net $\{\mathcal{F}_\alpha : \alpha \in A\}$ with $|A| \leq nw(Y)$. In particular, $\Sigma(Y) \leq nw(Y)$.*

Proof Let $\mathcal{N} = \{N_\alpha : \alpha \in A\}$ a network for Y with $|A| = nw(Y)$. For every $\alpha \in A$ we take $\mathcal{F}_\alpha = \{\overline{N_\alpha}, Y\}$. Then $\{\mathcal{F}_\alpha : \alpha \in A\}$ is a strong Σ-net.

Corollary 27 *Let Y be a completely regular topological space, then $Nag(Y) \leq nw(Y)$.*

Proof After [13, Theorem 3.8.12], we have $\ell(Y) \leq nw(Y)$. On the other hand, the previous proposition tell us that $\Sigma(Y) \leq nw(Y)$. Thus, the inequality $Nag(Y) \leq nw(Y)$ straightforwardly follows now from Theorem 25.

See [20] for a result related to the previous corollary.

Corollary 28 *Let Y be a completely regular topological space. Then, Y is Lindelöf (i.e. $\ell(Y) = \omega$) and Σ-space (i.e. $\Sigma(Y) = \omega$) if, and only if, $Nag(Y) = \ell\Sigma(Y) = \omega$.*

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References

1. Arhangel′skiĭ, A.V.: On some topological spaces that occur in functional analysis. Russ. Math. Surv. **31**(5), 14–30 (1976)
2. Arhangel′skiĭ, A.V.: Structure and classification of topological spaces and cardinal invariants. Russ. Math. Surv. **33**(6), 33–96 (1978)
3. Arhangel′skiĭ, A.V.: Topological function spaces. In: Mathematics and its Applications (Soviet Series), vol. 78. Kluwer, Dordrecht (1992). Translated from the Russian by R.A.M. Hoksbergen
4. Avilés, A.: The number of weakly compact sets which generate a Banach space. Isr. J. Math. **159**, 189–204 (2007)
5. Cascales, B., Oncina, L.: Compactoid filters and USCO maps. J. Math. Anal. Appl. **282**(2), 826–845 (2003)
6. Cascales, B., Orihuela, J.: On compactness in locally convex spaces. Math. Z. **195**(3), 365–381 (1987)
7. Cascales, B., Orihuela, J.: On pointwise and weak compactness in spaces of continuous functions. Bull. Soc. Math. Belg. Sér. B **40**(3), 331–352 (1988)
8. Cascales, B., Orihuela, J.: A sequential property of set-valued maps. J. Math. Anal. Appl. **156**(1), 86–100 (1991)
9. Cascales, B., Orihuela, J., Tkachuk, V.: Domination by second countable spaces and lindelöf σ-property. Topol. Appl. **158**(2), 204–214 (2011)
10. Christensen, J.P.R.: Topology and Borel structure. North-Holland, Amsterdam (1974). Descriptive topology and set theory with applications to functional analysis and measure theory. North-Holland Mathematics Studies, vol. 10. Notas de Matemática, No. 51
11. Deville, R., Godefroy, G., Zizler, V.: Smoothness and renormings in Banach spaces. In: Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 64. Longman Scientific & Technical, Harlow (1993)
12. Drewnowski, L., Labuda, I.: On minimal upper semicontinuous compact-valued maps. Rocky Mount. J. Math. **20**(3), 737–752 (1990)
13. Engelking, R.: General Topology. PWN—Polish Scientific Publishers, Warsaw (1977). Translated from the Polish by the author, Monografie Matematyczne, Tom 60. [Mathematical Monographs, vol. 60]
14. Floret, K.: Weakly compact sets. In: Lecture Notes in Mathematics, vol. 801. Springer, Berlin (1980). Lectures held at S.U.N.Y., Buffalo, in Spring 1978
15. Hödel, R.: On a theorem of Arhangel′skii concerning Lindelöf p-spaces. Can. J. Math. **27**, 459–468 (1975)
16. Hödel, R.E.: Cardinal functions I. In: Handbook of Set-Theoretic Topology, pp. 1–61. North-Holland, Amsterdam (1984)
17. Jayne, J.E., Rogers, C.A.: k-analytic sets. In: Analytic Sets. Lectures delivered at a Conference held at University College, University of London, London, July 16–29, 1978, pp. 1–181. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London (1980)
18. Kąkol, J., Kubiś, W., López-Pellicer, M.: Descriptive topology in selected topics of functional analysis. Developments in Mathematics, vol. 24. Springer, New York (2011)
19. Kelley, J.L.: General topology. Springer, New York (1975). Reprint of the 1955 edition [Van Nostrand, Toronto, Ont.], Graduate Texts in Mathematics, No. 27
20. Kortezov, I.: Fragmentability and cardinal invariants. Topol. Appl. **101**(2), 93–106 (2000)
21. Köthe, G.: Topological vector spaces I. Translated from the German by D.J.H. Garling. Die Grundlehren der mathematischen Wissenschaften, Band 159. Springer-Verlag New York Inc., New York (1969)
22. Lechicki, A., Levi, S.: Extensions of semicontinuous multifunctions. Forum Math. **2**(4), 341–360 (1990)
23. van Mill, J.: The infinite-dimensional topology of function spaces. North-Holland Mathematical Library, vol. 64. North-Holland, Amsterdam (2001)
24. Muñoz, M.: Indice de K-determinación de espacios topológicos y σ-fragmentabilidad de aplicaciones. Ph.D. thesis, Universidad de Murcia (2004)
25. Nagami, K.: Σ-Spaces. Fundam. Math. **65**, 169–192 (1969)
26. Orihuela, J.: Pointwise compactness in spaces of continuous functions. J. Lond. Math. Soc. (2) **36**(1), 143–152 (1987)
27. Pytkeev, E.G.: Tightness of spaces of continuous functions. Uspekhi Mat. Nauk. **37**(1(223)), 157–158 (1982)
28. Srivastava, S.M.: A course on Borel sets. Graduate Texts in Mathematics, vol. 180. Springer, New York (1998)
29. Talagrand, M.: Espaces de Banach faiblement $\mathcal{K}$-analytiques. Ann. Math. (2) **110**(3), 407–438 (1979)
30. Tkachuk, V.V.: Growths over discretes: some applications. Mosk. Univ. Math. Bull. **45**(4), 19–21 (1990)
31. Tkachuk, V.V.: A space $C_p(X)$ is dominated by irrationals if and only if it is K-analytic. Acta Math. Hungar. **107**(4), 253–265 (2005). doi:10.1007/s10474-005-0194-y