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# The Connection Between General Observers and Lanczos Potential

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We present some algorithms to find the explicit form of the Lanczos potential in an arbitrary geometry.

## 1. INTRODUCTION

In the early 1960s Cornelius Lanczos [1] made the remark that in any Riemannian geometry the Weyl conformal tensor, $W_{\alpha\beta\mu\nu}$ (i.e., the traceless part of the curvature tensor $R_{\alpha\beta\mu\nu}$] can be written as first derivatives of a third-order potential $L_{\alpha\beta\mu}$. All attempts to generalize this result to the curvature tensor have failed. Although research on the Weyl tensor became very important in gravitational theory, the same did not occur with Lanczos potential. There are two main reasons for this. The first one, of more general character, was just due to the suspicion (linked to the particular demonstration used by Lanczos) of the nonexistence of $L_{\alpha\beta\mu}$ in every Riemannian geometry. Lanczos used a variational principle to obtain Bianchi's identities and in this way, the potential $L_{\alpha\beta\mu}$ appeared as Lagrange multipliers. There remained some doubt on the generality of this procedure. Twenty years after the first Lanczos paper on this subject, Bampi and Caviglia [2] gave a completely new proof. However, both demonstrations were not able to provide an algorithm which could be used to obtain the form of $L_{\alpha\beta\mu}$ in a given geometry. This indeed is the second main reason which made Lanczos tensor be so seldom employed until

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recently. The purpose of the present paper is precisely to remedy this situation, searching for general methods to obtain Lanczos potential for an arbitrary Riemannian geometry.

## 2. NOTATION AND SOME USEFUL FORMULAS

We denote by a semicolon; the covariant derivative in the four-dimensional Riemannian space-time (ST):

$$\text{Antisymmetrization:} \qquad A_{[\mu\nu]} \equiv A_{\mu\nu} - A_{\nu\mu}$$

$$\text{Symmetrization:} \qquad A_{(\mu\nu)} \equiv A_{\mu\nu} + A_{\nu\mu}$$

The Weyl conformal tensor ($W_{\alpha\beta\mu\nu}$), the curvature tensor ($R_{\alpha\beta\mu\nu}$), and the contracted tensor ($R_{\mu\nu}$) are related by the formula

$$W_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - M_{\alpha\beta\mu\nu} + \tfrac{1}{6} R g_{\alpha\beta\mu\nu} \tag{1}$$

in which

$$g_{\alpha\beta\mu\nu} = g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu}$$

$$M_{\alpha\beta\mu\nu} = \tfrac{1}{2}(R_{\alpha\mu} g_{\beta\nu} + R_{\beta\nu} g_{\alpha\mu} - R_{\alpha\nu} g_{\beta\mu} - R_{\beta\mu} g_{\alpha\nu})$$

The dual operation, represented by an asterisk * is defined by

$$A_{\mu * \nu} = \tfrac{1}{2} \eta_{\mu\nu}{}^{\rho\sigma} A_{\rho\sigma}$$

for any antisymmetric tensor. $\eta_{\mu\nu\rho\sigma}$ is the Levi–Civita completely antisymmetric object.

We have

$$\overset{*}{g}_{\alpha\beta\mu\nu} \equiv g_{\alpha * \beta\mu\nu} = g_{\alpha\beta\mu * \nu} = \eta_{\alpha\beta\mu\nu}$$

$$\overset{*}{\eta}_{\alpha\beta\mu\nu} = -g_{\alpha\beta\mu\nu}$$

The symbol $g_{\alpha\beta\mu\nu}$ is a sort of metric for bivectors, in the sense that, for any bivector $A_{\mu\nu} = -A_{\nu\mu}$ we have

$$g_{\mu\nu\alpha\beta} A^{\alpha\beta} = 2A_{\mu\nu}$$

The electric ($E_{\mu\nu}$) and the magnetic ($H_{\mu\nu}$) parts of the Weyl tensor are defined for an arbitrary timelike vector by

$$E_{\mu\nu} = -W_{\mu\alpha\nu\beta} V^{\alpha} V^{\beta} \tag{2a}$$

$$H_{\mu\nu} = -\overset{*}{W}_{\mu\alpha\nu\beta} V^{\alpha} V^{\beta} \tag{2b}$$

Due to the fact that the Weyl tensor is traceless we can show that

$$\overset{*}{W}_{\alpha\beta\mu\nu} \equiv W_{\alpha^*\beta\mu\nu} = W_{\alpha\beta\mu^*\nu}$$

The 10 d.f. of $W_{\alpha\beta\mu\nu}$ are equally distributed among its electric and magnetic parts. Indeed, (2) implies that

$$\begin{aligned} E_{\mu\nu} &= E_{\nu\mu} \\ E_{\mu\nu} V^{\mu} &= 0 \\ E_{\mu}{}^{\mu} &= 0 \\ H_{\mu\nu} &= H_{\nu\mu} \\ H_{\mu\nu} V^{\mu} &= 0 \\ H_{\mu}{}^{\mu} &= 0 \end{aligned} \tag{3}$$

We define the projector tensor $h_{\mu\nu}$

$$h_{\mu\nu} = g_{\mu\nu} - V_{\mu} V_{\nu}$$

which projects any tensor in the three-dimensional rest space of the observer $V^{\mu}$, with the properties

$$\begin{aligned} h_{\mu}{}^{\lambda} h_{\lambda\nu} &= h_{\mu\nu} \\ h_{\mu\nu} &= h_{\nu\mu} \\ h_{\mu\nu} g^{\mu\nu} &= 3 \end{aligned}$$

The covariant derivative of $V^{\mu}$ is separated in its irreducible parts by the expression

$$V_{\mu;\nu} = \sigma_{\mu\nu} + (\theta/3) h_{\mu\nu} + \omega_{\mu\nu} + a_{\mu} V_{\nu} \tag{4}$$

in which the symmetric shear $\sigma_{\mu\nu}$ is given by

$$\sigma_{\mu\nu} = \tfrac{1}{2} h_{(\mu}{}^{\alpha} h_{\nu)}{}^{\beta} V_{\alpha;\beta} - \tfrac{1}{3}\theta h_{\mu\nu} \tag{5a}$$

the antisymmetric vorticity $\omega_{\mu\nu}$ by

$$\omega_{\mu\nu} = \tfrac{1}{2} h_{[\mu}{}^{\alpha} h_{\nu]}{}^{\beta} V_{\alpha;\beta} \tag{5b}$$

the expansion $\theta$

$$\theta = V^{\mu}{}_{;\mu} \tag{5c}$$

and the acceleration $a_\mu$

$$a_\mu = V_{\mu;\lambda} V^\lambda \tag{5d}$$

with the properties

$$\begin{aligned} \sigma_{\mu\nu} V^\mu &= 0 \\ \omega_{\mu\nu} V^\mu &= 0 \\ a_\mu V^\mu &= 0 \\ \sigma_{\mu\nu} g^{\mu\nu} &= 0 \end{aligned} \tag{6}$$

From $\omega_{\mu\nu}$ we can define the corresponding vector $\omega_\mu$

$$\omega^\tau = \tfrac{1}{2}\eta^{\alpha\beta\rho\tau} \omega_{\alpha\beta} V_\rho \tag{7a}$$

or, inversely

$$\omega_{\alpha\beta} = \eta_{\alpha\beta\mu\nu} \omega^\mu V^\nu \tag{7b}$$

In any Riemannian ST these quantities obey three constraints and three evolution equations.

## Equations of Constraint

$$\tfrac{2}{3}\theta_{,\mu} h^\mu{}_\lambda - (\sigma^\alpha{}_\beta + \omega^\alpha{}_\beta)_{;\alpha} h^\beta{}_\lambda - a^\alpha(\sigma_{\lambda\alpha} + \omega_{\lambda\alpha}) = R_{\mu\alpha} V^\mu h^\alpha{}_\lambda \tag{8a}$$

$$\omega^\alpha{}_{;\alpha} + 2\omega^\alpha a_\alpha = 0 \tag{8b}$$

$$\tfrac{1}{2}h^\varepsilon{}_{(\rho} h^\alpha{}_{\sigma)} \eta_\varepsilon{}^{\beta\gamma\nu} V_\nu (\omega_{\alpha\beta} + \sigma_{\alpha\beta})_{;\gamma} - a_{(\rho}\omega_{\sigma)} = -H_{\rho\sigma} \tag{8c}$$

## Dynamical Equations

$$\dot{\theta} + (\theta^2/3) + \sigma^2 - \omega^2 - a^\mu{}_{;\mu} = R_{\mu\nu} V^\mu V^\nu \tag{9a}$$

$$\begin{aligned} &h_\alpha{}^\mu h_\beta{}^\nu \dot{\sigma}_{\mu\nu} + \tfrac{1}{3}h_{\alpha\beta}[\tfrac{1}{2}\omega^2 - \sigma^2 + a^\lambda{}_{;\lambda}] + a_\alpha a_\beta - \tfrac{1}{2}h_\alpha{}^\mu h_\beta{}^\nu a_{(\mu;\nu)} \\ &\quad + \tfrac{2}{3}\theta\sigma_{\alpha\beta} + \sigma_{\alpha\mu}\sigma^\mu{}_\beta - \omega_\alpha\omega_\beta = R_{\alpha\varepsilon\beta\nu} V^\varepsilon V^\nu - \tfrac{1}{3}R_{\mu\nu} V^\mu V^\nu h_{\alpha\beta} \end{aligned} \tag{9b}$$

$$h_\alpha{}^\mu h_\beta{}^\nu \dot{\omega}_{\mu\nu} - \tfrac{1}{2}h_\alpha{}^\mu h_\beta{}^\nu a_{[\mu;\nu]} + \tfrac{2}{3}\theta\omega_{\alpha\beta} + \sigma_{\alpha\mu}\omega^\mu{}_\beta - \sigma_{\beta\mu}\omega^\mu{}_\alpha = 0 \tag{9c}$$

in which $\sigma^2 \equiv \sigma_{\mu\nu}\sigma^{\mu\nu}$ and $\omega^2 = \omega_{\alpha\beta}\omega^{\alpha\beta} = -2\omega_\mu\omega^\mu$.

An overdot means derivative projected in the $V^\mu$ direction, i.e., $\dot{\theta} = \theta_{,\mu} V^\mu$.

## Einstein's Equations

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = -kT_{\mu\nu} - \Lambda g_{\mu\nu} \tag{10}$$

In this paper we use $k = 1$.

The right-hand side is represented by a perfect fluid

$$T_{\mu\nu} = \rho V_\mu V_\nu - ph_{\mu\nu} \tag{11}$$

The generalization of the subsequent results for more general kinds of fluids is straightforward, although rather tedious.

There are two important consequences of this hypothesis in the above equations. Using (10), (11) and (9a), this latter equation takes the form

$$\dot{\theta} + (\theta^2/3) + \sigma^2 - \omega^2 - a^\mu{}_{;\mu} = -\tfrac{1}{2}(\rho + 3p) + \Lambda \tag{9a'}$$

and for the evolution of the shear

$$\begin{aligned} &h_\alpha{}^\mu h_\beta{}^\nu \dot{\sigma}_{\mu\nu} + \tfrac{1}{3}h_{\alpha\beta}(-\tfrac{1}{2}\omega^2 - \sigma^2 + a^\lambda{}_{;\lambda}) \\ &\quad + a_\alpha a_\beta - \tfrac{1}{2}h_\alpha{}^\mu h_\beta{}^\nu a_{(\mu;\nu)} + \tfrac{2}{3}\theta\sigma_{\alpha\beta} + \sigma_\alpha{}^\mu \sigma_{\mu\beta} - \omega_\alpha \omega_\beta = -E_{\alpha\beta} \end{aligned} \tag{9b'}$$

and for (8a)

$$\tfrac{2}{3}\theta_{,\mu}h^\mu{}_\lambda - (\sigma^\alpha{}_\beta + \omega^\alpha{}_\beta)_{;\alpha}h^\beta{}_\lambda - a^\alpha(\sigma_{\lambda\alpha} + \omega_{\lambda\alpha}) = 0 \tag{8a'}$$

## 3. LANCZOS POTENTIAL

**Definition.** A tensor $L_{\mu\nu\rho}$ which has the symmetries

$$L_{\alpha\beta\mu} + L_{\beta\alpha\mu} = 0 \tag{12a}$$

$$L_{\alpha\beta\mu} + L_{\beta\mu\alpha} + L_{\mu\alpha\beta} = 0 \tag{12b}$$

and from which the Weyl conformal tensor $W_{\alpha\beta\mu\nu}$ can be obtained by the formula

$$\begin{aligned} W_{\alpha\beta\mu\nu} &= L_{\alpha\beta[\mu;\nu]} + L_{\mu\nu[\alpha;\beta]} \\ &\quad + \tfrac{1}{2}[L_{(\alpha\nu)}g_{\beta\mu} + L_{(\beta\mu)}g_{\alpha\nu} - L_{(\alpha\mu)}g_{\beta\nu} - L_{(\beta\nu)}g_{\alpha\mu}] + \tfrac{2}{3}L^{\sigma\lambda}{}_{\sigma;\lambda}\, g_{\alpha\beta\mu\nu} \end{aligned} \tag{13}$$

is called a Lanczos potential. In this expression

$$L_{\alpha\mu} \equiv L_\alpha{}^\sigma{}_{\mu;\sigma} - L_\alpha{}^\sigma{}_{\sigma;\mu} \tag{14}$$

Note that although the Weyl tensor has only 10 d.f. a tensor $L_{\alpha\beta\mu}$ which obeys (12) has 20 independent components. This means that there are 10 d.f. Lanczos chooses to fix this arbitrariness by imposing extra conditions, e.g., $L_{\alpha\beta}{}^{\beta}=0$ and $L_{\alpha\beta}{}^{\mu}{}_{;\mu}=0$, which gives precisely 10 more equations to eliminate the freedom. The trace-free condition comes from the invariance of $W_{\alpha\beta\mu\nu}$ under the map

$$L_{\alpha\beta\mu} \to \tilde{L}_{\alpha\beta\mu} = L_{\alpha\beta\mu} + M_\alpha\, g_{\beta\mu} - M_\beta\, g_{\alpha\mu} \tag{15}$$

for an arbitrary vector $M_\alpha$. The second (divergence-free) condition comes from the observation that in (13) this divergence is completely absent. It is clear, however, that there is no compelling argument for imposing such a gauge, and it remains as arbitrary as any other.

From (13) it follows that there is no local relationship between $L_{\alpha\beta\mu}$ and the metric $g_{\mu\nu}$ (although such a relation can be exhibited in the quasi-Minkowskian ST in the approximation $g_{\mu\nu} \approx \eta_{\mu\nu} + \varepsilon\psi_{\mu\nu}$ for $\varepsilon^2 \ll \varepsilon$. In this case it was shown that $L_{\alpha\beta\mu} = \frac{1}{4}(\psi_{\alpha\mu,\beta} - \psi_{\beta\mu,\alpha} + \frac{1}{6}\psi_{,\alpha}\eta_{\mu\beta} - \frac{1}{6}\psi_{,\beta}\eta_{\mu\alpha})$, and this seems to turn the role of $L_{\alpha\beta\mu}$ somehow mysterious [3][4].[2] This situation can be overcome by exhibiting $L_{\alpha\mu\beta}$ for any geometry. We investigate this approach and prove some useful lemmas. Let us remind the reader that throughout this paper we call space-time (ST) any four-dimensional Riemannian geometry that satisfies Einstein's equations with a perfect fluid as its source.

**Lemma 1.** If in a given ST there is a field of observers $V^\mu$ that is shear-free and irrotational, then the magnetic part of the Weyl tensor vanishes for the observers ($H_{\mu\nu}=0$)

*Proof.* Trivial (use eq. 8c).

**Lemma 2.** If in a given ST there is a field of observers $V^\mu$ that is shear-free and irrotational, then the Lanczos potential is given by

$$L_{\alpha\beta\mu} = a_\alpha V_\beta V_\mu - a_\beta V_\alpha V_\mu \tag{16}$$

up to a gauge.

*Proof.* A direct manipulation of the above kinematical equations are used. Suppose (16) applies. Then we have

$$\begin{aligned} L_{\alpha\beta\mu;\nu} = {} & a_{\alpha;\nu} V_\beta V_\mu + a_\alpha[(\theta/3)h_{\beta\nu} + a_\beta V_\nu]V_\mu + a_\alpha V_\beta[(\theta/3)h_{\mu\nu} + a_\mu V_\nu] \\ & - a_\beta V_\alpha[(\theta/3)h_{\mu\nu} + a_\mu V_\nu] - a_\beta[(\theta/3)h_{\alpha\nu} + a_\alpha V_\nu]V_\mu \end{aligned} \tag{17}$$

[2] After this work was completed we received a preprint [4] in which Lanczos potentials for some geometries (Gödel, Minkowski, Schwarzschild, Kasner, Taub, and Bertotti) have independently been obtained in a different way.

Contracting with $V^\beta V^\nu$

$$L_{\alpha\beta\mu;\nu} V^\beta V^\nu = \dot{a}_\alpha V_\mu + a_\alpha a_\mu + a^2 V_\alpha V_\mu$$

Then, in an analogous way

$$L_{\alpha\beta\nu;\mu} V^\beta V^\nu = a_{\alpha;\mu} + (\theta/3) a_\mu V_\alpha + a^2 V_\alpha V_\mu$$
$$L_{\mu\nu\alpha;\beta} V^\beta V^\nu = \dot{a}_\mu V_\alpha + a_\mu a_\alpha + a^2 V_\mu V_\alpha$$
$$L_{\mu\nu\beta;\alpha} V^\beta V^\nu = a_{\mu;\alpha} + (\theta/3) a_\alpha V_\mu + a^2 V_\alpha V_\mu$$

Then

$$(L_{\alpha\beta[\mu;\nu]} + L_{\mu\nu[\alpha;\beta]})\, V^\beta V^\nu = \dot{a}_{(\alpha} V_{\mu)} + 2a_\alpha a_\mu - a_{(\alpha;\mu)} - (\theta/3)\, a_{(\mu} V_{\alpha)} \quad (18)$$

The contracted tensor is

$$L_{\alpha\mu} = L_\alpha{}^\sigma{}_{\mu;\sigma} - L_\alpha{}^\sigma{}_{\sigma;\mu}$$

In this case, from (16)

$$L_\alpha{}^\sigma{}_\sigma = a_\alpha$$

then

$$L_{\alpha\mu} = \dot{a}_\alpha V_\mu + \tfrac{2}{3}\theta a_\alpha V_\mu + a_\alpha a_\mu - a^\varepsilon{}_{;\varepsilon} V_\alpha V_\mu - a_{\alpha;\mu}$$

Then, we have

$$\tfrac{1}{2}L_{(\alpha\mu)}\, g_{\beta\nu} V^\beta V^\nu = \tfrac{1}{2}L_{(\alpha\mu)}$$
$$\tfrac{1}{2}L_{(\beta\nu)}\, g_{\alpha\mu} V^\beta V^\nu = -a^\varepsilon{}_{;\varepsilon}\, g_{\alpha\mu}$$
$$\tfrac{1}{2}L_{(\alpha\nu)}\, g_{\beta\mu} V^\beta V^\nu = (\theta/3) a_\alpha V_\mu - a^\varepsilon{}_{;\varepsilon} V_\alpha V_\mu$$
$$\tfrac{1}{2}L_{(\beta\mu)}\, g_{\alpha\nu} V^\beta V^\nu = (\theta/3)\, a_\mu V_\alpha - a^\varepsilon{}_{;\varepsilon} V_\alpha V_\mu$$

Then

$$\tfrac{1}{2}[L_{(\alpha\nu)}\, g_{\beta\mu} + L_{(\beta\mu)}\, g_{\alpha\nu} - L_{(\alpha\mu)}\, g_{\beta\nu} - L_{(\beta\nu)}\, g_{\alpha\mu}]\, V^\beta V^\nu$$
$$= -\tfrac{1}{2}\dot{a}_{(\alpha} V_{\mu)} + \tfrac{1}{6}\theta a_{(\alpha} V_{\mu)} - a_\alpha a_\mu + a^\lambda{}_{;\lambda} h_{\alpha\mu} + \tfrac{1}{2}a_{(\mu;\alpha)} \quad (19)$$

And

$$\tfrac{2}{3}L^{\alpha\lambda}{}_{\sigma;\lambda}\, g_{\alpha\beta\mu\nu} V^\beta V^\nu = -\tfrac{2}{3}a^\lambda{}_{;\lambda} h_{\alpha\mu} \quad (20)$$

Collecting all terms (18)–(20), we obtain

$$-E_{\mu\nu} = \tfrac{1}{2}\dot{a}_{(\mu} V_{\nu)} + a_\mu a_\nu - \tfrac{1}{2}a_{(\mu;\nu)} - \tfrac{1}{6}\theta a_{(\mu} V_{\alpha)} + \tfrac{1}{3}a^\varepsilon{}_{;\varepsilon} h_{\mu\nu}$$

Comparing this expression with (9b) we see that (16) yields the correct electric part of the Weyl tensor. It remains to show (by Lemma 1) that (16) takes one to the vanishing of the magnetic part. We have

$$(L_{\alpha\beta[\mu;\nu]} + L_{\mu\nu[\alpha;\beta]}) V_\lambda V^\nu \eta^{\sigma\lambda\alpha\beta} = -\tfrac{2}{3}\theta a_\alpha V_\lambda \eta^{\delta\lambda\alpha}{}_\mu$$

and

$$\tfrac{1}{2}(L_{(\nu\alpha)} g_{\beta\mu} + L_{(\beta\mu)} g_{\alpha\nu} - L_{(\alpha\nu)} g_{\beta\nu} - L_{(\beta\nu)} g_{\alpha\mu}) V_\lambda V^\nu \eta^{\sigma\lambda\alpha\beta} = +\tfrac{2}{3}\theta a_\alpha V_\lambda \eta^{\sigma\lambda\alpha}{}_\mu$$

and

$$L^{\delta\varepsilon}{}_{\delta;\varepsilon}\, g_{\alpha\beta\mu\nu} V_\lambda V^\nu \eta^{\sigma\lambda\alpha\beta} = 0$$

Adding those terms, we obtain $H_{\mu\nu} = 0$, which ends the proof.

**Example.** Lanczos potential for Schwarzschild geometry. Consider the metric

$$ds^2 = (1 - 2M/r)\, dt^2 - dr^2/(1 - 2M/r) - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{21}$$

Let us choose a frame defined by the observer $V^\mu$ which, in the system $(t, r, \theta, \phi)$, has components

$$V^\mu = \delta^\mu{}_0 (g_{00})^{-1/2} = \delta^\mu{}_0 \left(1 - \frac{2M}{r}\right)^{-1/2} \tag{22}$$

This observer is shear-free and irrotational and has a nonvanishing acceleration given by

$$a_\mu = \left(0,\ -M\Big/r^2\left(1 - \frac{2M}{r}\right), 0, 0\right) \tag{23}$$

A direct application of Lemma 2 implies that the Lanczos potential has the form $L_{\alpha\beta\mu} = a_{[\alpha} V_{\beta]} V_\mu$, which yields for the only nonvanishing term

$$L_{010} = M/r^2 \tag{24}$$

If we want to exhibit $L_{\alpha\beta\mu}$ in the Lanczos gauge, we have just to subtract the trace or to set

$$\tilde{L}_{\alpha\beta\mu} = (a_\alpha V_\beta V_\mu - a_\beta V_\alpha V_\mu) - \tfrac{1}{3}(a_\alpha g_{\beta\mu} - a_\beta g_{\alpha\mu}) \tag{25}$$

Remark that in this form we have incidently both Lanczos conditions satisfied

$$\tilde{L}_{\alpha\beta}{}^{\beta} = 0$$

$$\tilde{L}_{\alpha\beta}{}^{\mu}{}_{;\mu} = 0$$

In this gauge

$$\tilde{L}_{010}=\frac{2}{3}\frac{M}{r^2}$$

$$\tilde{L}_{122}=-\frac{1}{3}\frac{M}{1-(2M/r)}$$

$$\tilde{L}_{133}=-\frac{1}{3}\frac{M\sin^2\theta}{1-(2M/r)}$$

**Lemma 3.** If in a given ST there is a field of free observers (geodetic) $V^\mu$ which is irrotational and such that either $H_{\mu\nu}\neq 0$ and

$$\text{(i)}\quad \sigma_\mu{}^\varepsilon\sigma_{\varepsilon\nu}-\tfrac{1}{3}\sigma^2 h_{\mu\nu}-\tfrac{1}{3}\theta\sigma_{\mu\nu}=0 \tag{26}$$

or $H_{\mu\nu}=0$ and

$$\text{(ii)}\quad \dot{\sigma}_{\mu\nu}+\theta\sigma_{\mu\nu}=0 \tag{27}$$

then the Lanczos potential is given, respectively, by

$$L_{\alpha\beta\mu}=\sigma_{\mu\alpha}V_\beta-\sigma_{\mu\beta}V_\alpha \tag{28}$$

in case (i), and by

$$L_{\alpha\beta\mu}=\tfrac{1}{3}(\sigma_{\mu\alpha}V_\beta-\sigma_{\beta\mu}V_\alpha) \tag{29}$$

in case (ii), up to a gauge.

*Proof.* The procedure is the same as in the precedent Lemma. Suppose

$$L_{\alpha\beta\mu}=\sigma_{\mu\alpha}V_\beta-\sigma_{\mu\beta}V_\alpha$$

then

$$(L_{\alpha\beta[\mu;\nu]}+L_{\mu\nu[\alpha;\beta]})\,\eta_\sigma{}^{\lambda\alpha\beta}V_\lambda V^\nu=\eta_\sigma{}^{\lambda\alpha\beta}\sigma_{\mu\alpha;\beta}V_\lambda$$

all the other terms vanish identically when symmetrized in $\sigma$, $\mu$. Then we obtain

$$H_{\sigma\mu}=-\tfrac{1}{2}\eta_{(\sigma}{}^{\beta\lambda\nu}\sigma_{\mu)\beta;\lambda}V_\nu$$

which is the value of the magnetic part of the Weyl tensor in the case of absence of vorticity (cf., eq. 8c).

From (28) we obtain the electric part as given by

$$E_{\mu\nu}=-\dot{\sigma}_{\mu\nu}-3\sigma_\mu{}^\varepsilon\sigma_{\varepsilon\nu}+\sigma^2 h_{\mu\nu}$$

which will be compatible with expression (9b)′ of the evolution of shear only if (26) applies. This ends the proof. In a similar way we show that in the case of vanishing $H_{\mu\nu}$ the form of Lanczos tensor is given by (29) under condition (27).

**Example 1.** In order to exhibit more directly the interdependence of the Lanczos potential and an observer, we analyze an example to illustrate the present case in the same geometry as above (Schwarzschild) as viewed by a distinct observer. We choose here the path of an observer to be geodetic and irrotational. For instance, take

$$V_\mu = \left(1, \frac{(2M/r)^{1/2}}{1-(2M/r)}, 0, 0\right) \tag{30}$$

The shear is given by (the system of coordinates is the same as in eq. 21)

$$\begin{aligned}
\sigma_{00} &= -\frac{1}{r}\left(\frac{2M}{r}\right)^{3/2} \\
\sigma_{01} &= -\frac{2M}{r^2[1-(2M/r)} \\
\sigma_{11} &= -\left(\frac{2M}{r}\right)^{1/2} \frac{1}{r[1-(2M/r)]^2} \\
\sigma_{22} &= (Mr/2)^{1/2} \\
\sigma_{33} &= (Mr/2)^{1/2} \sin^2\theta
\end{aligned} \tag{31}$$

and, consequently, applying our Lemma 3, the non-null components of Lanczos potential are

$$\begin{aligned}
L_{010} &= \frac{2}{3}\frac{M}{r^2} \\
L_{011} &= \frac{1}{3}\left(\frac{2M}{r}\right)^{1/2} \frac{1}{r[1-(2M/r)]} \\
L_{022} &= -\frac{1}{6}(2Mr)^{1/2} \\
L_{033} &= -\frac{1}{6}(2Mr)^{1/2}\sin^2\theta \\
L_{122} &= -\frac{1}{3}\frac{M}{[1-(2M/r)]} \\
L_{133} &= -\frac{1}{3}\frac{M}{[1-(2M/r)]}\sin^2\theta
\end{aligned} \tag{32}$$

Compare this expression of Lanczos potential with the one obtained in (24) by choosing an accelerated observer (without shear and vorticity). As the geometry is the same and the system of coordinates is also the same, the components of the Weyl tensor must be the same. What is then the difference between the very simple expression (24) and the rather long one (32) for Lanczos potential? The answer is simple: It is due to the different gauge choices made in these cases.

**Example 2.** Kasner geometry. A rather simple case is Kasner geometry, where a geodetic irrotational observer is well-known

$$V^{\mu} = \delta^{\mu}{}_{0} \tag{33}$$

We write the metric in the standard Gaussian system of coordinates

$$ds^2 = dt^2 - t^{2p_1}\, dx^2 - t^{2p_2}\, dy^2 - t^{2p_3}\, dz^2 \tag{34}$$

with

$$p_1 + p_2 + p_3 = 1 \qquad \text{and} \qquad (p_1)^2 + (p_2)^2 + (p_3)^2 = 1$$

The nonvanishing components of the shear for (33) are

$$\begin{aligned} \sigma_{11} &= (1 - 3p_1/3)\, t^{2p_1 - 1} \\ \sigma_{22} &= (1 - 3p_2/3)\, {}^{2p_2 - 1} \\ \sigma_{33} &= (1 - 3p_3/3)\, t^{2p_3 - 1} \end{aligned} \tag{35}$$

In this case, a direct calculation shows that

$$H_{\mu\nu}[V] = 0 \tag{36a}$$

and

$$\dot{\sigma}_{\mu\nu} + \theta\sigma_{\mu\nu} = 0 \tag{36b}$$

We can then apply Lemma 3 for this observer to obtain

$$\begin{aligned} L_{011} &= \tfrac{1}{3}(p_1 - \tfrac{1}{3})\, t^{2p_1 - 1} \\ L_{022} &= \tfrac{1}{3}(p_2 - \tfrac{1}{3})\, t^{2p_2 - 1} \\ L_{033} &= \tfrac{1}{3}(p_3 - \tfrac{1}{3})\, t^{2p_3 - 1} \end{aligned} \tag{37}$$

Incidently, in this case we can easily see that this form is in Lanczos gauge ($L_{\alpha}{}^{\mu}{}_{\mu} = 0$ and $L_{\alpha\beta}{}^{\mu}{}_{;\mu} = 0$).

**Lemma 4.** If in a given ST there is a field of free observers (geodetic) $V^\mu$ which is nonexpanding and shear-free, then the magnetic part of the Weyl tensor vanishes for this observer ($H_{\mu\nu}=0$), if $\omega_{\mu;\nu}=0$.

*Proof.* Using (8c) and the hypothesis that

$$V_{\mu;\nu}=\omega_{\mu\nu} \tag{38}$$

we have

$$H_{\rho\sigma}=-\eta_\rho{}^{\beta\gamma\nu}V_\nu\omega_{\sigma\beta;\gamma}=2V^\nu\omega_{\sigma\nu^*;\rho} \tag{39}$$

But

$$\begin{aligned}
\omega_{\mu\alpha;\beta}\eta^{\sigma\varepsilon\alpha\beta}V_\varepsilon &= (\eta_{\mu\alpha}{}^{\rho\lambda}\omega_\rho V_\lambda)_{;\beta}\eta^{\sigma\varepsilon\alpha\beta}V_\varepsilon\\
&=\eta_{\mu\alpha\rho\lambda}\eta^{\sigma\varepsilon\alpha\beta}(\omega^\rho V^\lambda)_{;\beta}V_\varepsilon\\
&=\delta^{\sigma\varepsilon\beta}_{\mu\rho\lambda}(\omega^\rho V^\lambda)_{;\beta}V_\varepsilon\\
&=(\delta^\sigma_\mu\delta^\varepsilon_\rho\delta^\beta_\lambda-\delta^\sigma_\mu\delta^\varepsilon_\lambda\delta^\beta_\rho+\delta^\sigma{}_\lambda\delta^\varepsilon_\mu\delta^\beta_\rho-\delta^\sigma_\rho\delta^\varepsilon_\mu\delta^\beta_\lambda+\delta^\sigma{}_\rho\delta^\varepsilon_\lambda\delta^\beta_\mu\\
&\quad-\delta^\sigma_\lambda\delta^\varepsilon_\rho\delta^\beta_\mu)(\omega^\rho V^\lambda)_{;\beta}V_\varepsilon\\
&=\delta^\sigma_\mu[(\omega^\varepsilon V^\beta)_{;\beta}V_\varepsilon-(\omega^\beta V^\varepsilon)_{;\beta}V_\varepsilon]+(\omega^\beta V^\sigma)_{;\beta}V_\mu\\
&\quad-(\omega^\sigma V^\beta)_{;\beta}V_\mu+(\omega^\sigma V^\varepsilon)_{;\mu}V_\varepsilon-(\omega^\varepsilon V^\sigma)_{;\mu}V_\varepsilon
\end{aligned}$$

Using (8b), which implies that $\omega^\mu{}_{;\mu}=0$ and the identity $\omega_{\alpha\mu}\omega^\mu=0$, we arrive at

$$\omega_{\mu\alpha;\beta}\eta^{\sigma\varepsilon\alpha\beta}V_\varepsilon=0 \tag{40}$$

or, equivalently

$$H_{\mu\nu}=0$$

which ends the proof.

**Lemma 5.** If in a given ST there is a field of free observers (geodetic) $V^\mu$ which is shear-free and nonexpanding and such that

$$\omega_{\mu;\nu}=0 \tag{41}$$

then the Lanczos potential is given by

$$L_{\alpha\beta\mu}=\tfrac{2}{9}[\omega_{\alpha\beta}V_\mu+\tfrac{1}{2}\omega_{\alpha\mu}V_\beta-\tfrac{1}{2}\omega_{\beta\mu}V_\alpha] \tag{42}$$

*Proof.* Let us evaluate the electric part ($E_{\mu\nu}$) from (42). We have

$$L_{\alpha\beta\mu;\nu}V^\beta V^\nu=\tfrac{1}{9}\dot\omega_{\alpha\mu}=0$$

which vanishes by (9c)

$$L_{\alpha\beta\nu;\mu}V^{\beta}V^{\nu}=-\tfrac{1}{3}\omega_{\alpha\beta}\omega^{\beta}{}_{\mu}$$
$$L_{\mu\nu\alpha;\beta}V^{\beta}V^{\nu}=\tfrac{1}{9}\dot{\omega}_{\mu\alpha}=0$$
$$L_{\mu\nu\beta;\alpha}V^{\beta}V^{\nu}=-\tfrac{1}{3}\omega_{\mu\beta}\omega^{\beta}{}_{\alpha}$$

Then

$$(L_{\alpha\beta[\mu;\nu]}+L_{\mu\nu[\alpha;\beta]})\,V^{\beta}V^{\nu}=\tfrac{2}{3}\omega_{\alpha}{}^{\lambda}\omega_{\lambda\mu}$$

From (42)

$$L_{(\alpha\mu)}=\tfrac{1}{3}\omega_{(\alpha}{}^{\sigma}{}_{;\sigma}V_{\mu)}-\tfrac{2}{3}\omega_{\mu}{}^{\sigma}\omega_{\sigma\alpha}=-\tfrac{2}{3}\omega_{\mu}{}^{\sigma}\omega_{\sigma\alpha}$$

since from (8b) $\omega_{\alpha}{}^{\sigma}{}_{;\sigma}=0$.

Then

$$L_{(\alpha\mu)}g_{\beta\nu}V^{\beta}V^{\nu}=-\tfrac{2}{3}\omega_{\mu}{}^{\varepsilon}\omega_{\varepsilon\alpha}$$
$$L_{(\beta\nu)}g_{\alpha\mu}V^{\beta}V^{\nu}=-\tfrac{2}{3}\omega^{2}g_{\alpha\mu}$$
$$L_{(\alpha\nu)}g_{\beta\mu}V^{\beta}V^{\nu}=\tfrac{1}{3}\omega_{\alpha}{}^{\sigma}{}_{;\sigma}V_{\mu}-\tfrac{1}{3}\omega^{2}V_{\alpha}V_{\mu}=-\tfrac{1}{3}\omega^{2}V_{\alpha}V_{\mu}$$
$$L_{(\beta\mu)}g_{\alpha\nu}V^{\beta}V^{\nu}=-\tfrac{1}{3}\omega^{2}V_{\alpha}V_{\mu}$$

The trace $L_{\alpha\beta}{}^{\beta}=0$; then we obtain from Lanczos formula of $W_{\alpha\beta\mu\nu}$

$$-E_{\mu\alpha}=\omega_{\alpha}{}^{\varepsilon}\omega_{\varepsilon\mu}+\tfrac{1}{3}\omega^{2}h_{\mu\alpha}$$

But from (7a, b)

$$\omega_{\alpha}{}^{\varepsilon}\omega_{\varepsilon\mu}=-\tfrac{1}{2}\omega^{2}h_{\alpha\mu}+\omega_{\alpha}\omega_{\mu}$$

Thus we can write

$$E_{\mu\alpha}=\tfrac{1}{6}\omega^{2}h_{\alpha\mu}-\omega_{\alpha}\omega_{\mu}\tag{43}$$

which is precisely the expression for the electric part of the Weyl tensor as given by (9b)′ in the present case.

It remains to show (by the previous Lemma 4) that

$$H_{\mu\nu}=0$$

From (42) we have

$$L_{\alpha\beta\mu;\nu}V_{\lambda}V^{\nu}\eta_{\sigma}{}^{\lambda\alpha\beta}=0$$
$$L_{\alpha\beta\nu;\mu}V_{\lambda}V^{\nu}\eta_{\sigma}{}^{\lambda\alpha\beta}=\tfrac{4}{3}\omega_{\sigma*\lambda;\mu}V^{\lambda}$$
$$L_{\mu\nu\alpha;\beta}V_{\lambda}V^{\nu}\eta_{\sigma}{}^{\lambda\alpha\beta}=\tfrac{2}{9}\omega_{\mu\sigma*;\lambda}V^{\lambda}=0 \qquad \text{(use eq. 40)}$$

Thus

$$(L_{\alpha\beta[\mu;\nu]} + L_{\mu\nu[\alpha;\beta]})\, V_\lambda V^\nu \eta_\sigma{}^{\lambda\alpha\beta} = \tfrac{4}{9}\omega_{\sigma*\lambda;\mu} V^\lambda$$

The trace $L_{\alpha\mu}{}^{\mu} = 0$, and thus

$$L_{(\alpha\mu)} = -\tfrac{2}{3}\omega_\alpha{}^\varepsilon \omega_{\varepsilon\mu}$$

Then, putting all terms together

$$\tfrac{1}{2}[L_{(\alpha\nu)}\, g_{\beta\mu} + L_{(\beta\mu)}\, g_{\alpha\nu} - L_{(\alpha\mu)}\, g_{\beta\nu} - L_{(\beta\nu)}\, g_{\alpha\mu}]\, V_\lambda V^\nu \eta_\sigma{}^{\lambda\alpha\beta} = 0$$

Finally, the magnetic part of the Weyl tensor constructed via the Lanczos potential yields

$$H_{\mu\sigma} = -\tfrac{2}{9}\omega_{\sigma*\lambda;\mu} V^\lambda \tag{44}$$

which vanishes by (41). Indeed, we have

$$\begin{aligned}
\omega_{\sigma*\lambda;\mu} V^\lambda &= \tfrac{1}{2}\eta_{\sigma\lambda}{}^{\alpha\beta}\omega_{\alpha\beta;\mu} V_\lambda = \tfrac{1}{2}(\eta_{\sigma\lambda}{}^{\alpha\beta}\omega_{\alpha\beta})_{;\mu} V^\lambda \\
&= \tfrac{1}{2}(\eta_{\sigma\lambda}{}^{\alpha\beta}\eta_{\alpha\beta\varepsilon\tau}\omega^\varepsilon V^\tau)_{;\mu} V^\lambda \\
&= (\delta^\sigma_\tau \delta^\lambda_\varepsilon - \delta^\sigma_\varepsilon \delta^\lambda_\tau)(\omega^\varepsilon V^\tau)_{;\mu} V^\lambda \\
&= -\omega_{\sigma;\mu} - \omega_0 \omega_{\lambda\mu} V^\lambda + \omega_{\lambda;\mu} V_\sigma V^\lambda \\
&= -\omega_{\sigma;\mu}
\end{aligned}$$

This ends the proof of Lemma 5.

**Example.** Gödel geometry. Let us consider Gödel's metric in the system of coordinates $(t, x, y, z)$

$$ds^2 = dt^2 - dx^2 + 2e^{ax}\, dt\, dy + \tfrac{1}{2}e^{2ax}\, dy^2 - dz^2 \tag{45}$$

Choose the observers $V^\mu$ co-moving with the matter that is responsible for the curvature of the ST

$$V^\mu = \delta^\mu{}_0 \tag{46}$$

The only nonvanishing component of the vorticity $\omega_{\mu\nu}$ is

$$\omega_{12} = -\omega_{21} = -(a/2)\, e^{ax} \tag{47}$$

and then

$$\omega^\mu = (0, 0, 0, 2^{1/2} a) \tag{48}$$

We see that for this vector $\omega_{\mu;\nu}=0$, which is (41). Thus, we can obtain Lanczos potential, using Lemma 5, by the formula

$$L_{\alpha\beta\mu}=\tfrac{2}{9}[\omega_{\alpha\beta}V_{\mu}+\tfrac{1}{2}\omega_{\alpha\mu}V_{\beta}-\tfrac{1}{2}\omega_{\beta\mu}V_{\alpha}] \tag{49}$$

which gives, for the nonnull components

$$\begin{aligned} L_{012} &= (a/18)\, e^{ax} \\ L_{021} &= -L_{012} \\ L_{120} &= -2L_{012} \\ L_{122} &= -3L_{012} \end{aligned} \tag{50}$$

Incidently, we can see that (46) gives $L_{\alpha\beta\mu}$ in Lanczos' gauge ($L_{\alpha\beta}{}^{\beta}=0$ and $L_{\alpha\beta}{}^{\mu}{}_{;\mu}=0$).[3]

## REFERENCES

1. Lanczos, C. (1962). *Rev. Mod. Phys.*, **34**, 379.
2. Bampi, E., and Caviglia, G. (1983). *Gen. Rel. Grav.*, **15**, 375.
3. Novello, M., and Rodrigues, Ligia M.C.S. (1985). *Lett. N. Cim.*, **43**, 292.
4. Chapon, José L. Fernandez, Bonilla, José L. Lopez, Zuñiga, Gerardo A. Ovando, and Medina, Marco A. Rosales (1986). Preprint, Universidad Autonoma Metropolitana, Mexico.

[3] In the calculations of the Lanczos potential and the correspondent Weyl tensor, and using the expression given by the Lemmas for the examples, we used the computational resources of REDUCE.