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CONSTRUCTION OF THE PARTITION FUNCTION FOR LATTICE GAUGE THEORY

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We review a new method to study lattice simulations of a gauge theory.The method is based on a direct numerical construction of the partition function of the system by computing its spectral density function. This leads to a knowledge of the partition function for a large range of couplings simultaniously including couplings that are complex. This allows for a study of various properties of the system which includes locating phase transitions as well as the determination of their critical exponents using the finite size scaling properties of the relevant nearest zeros of the partition function in the complex coupling plane. We restrict our discussion here to SU(2) lattice gauge theory and give results for its finite temperature phase transition.

1. INTRODUCTION

A new method[1,2] for studying lattice gauge and spin systems, and which relies on direct computation of the corresponding partition function, has been recently introduced. We describe this method here briefly and review some of the interesting qualitative and quantitave results obtained using it so far. We shall concentrate here on its use for lattice gauge systems and point the reader to Ref. 1 for its many applications to spin systems as well. Amoung its results is the determination of the positions of phase transitions of the system as well as the pattern of zeros in the complex coupling plane that is associated with it. This allows one to clearly distinguish between phase transitions and crossover regions. In the case of SU(2) the existence of the nontrivial transition at infinite β is also clearly indicated for the first time. On the the other hand the finite size scaling of these zeros[2,3] leads to a direct determination of the relevant critical exponents in a way that is independent of fitting methods employed in similar cases so far.

2. DESCRIPTION OF THE METHOD FOR LATTICE GAUGE THEORY

The lattice gauge theory action is taken here as the standard Wilson action S_w given by

$$S_w = 1 - (1/2)tr(U_P) \qquad (1)$$

where U_p is the ordered product of the link matrices around a plaquette. The partition function $Z(\beta)$ is then given as the sum (integral)

$$Z(\beta) = \sum_{U_l} \exp[-\beta \sum_{p} S(U_p)]. \qquad (2)$$

It can also be written as a sum over all possible values of the total action S . Thus,

$$Z(\beta) = \sum_{S} P(S)\exp -\beta S. \qquad (3)$$

Since the group is compact then for a finite lattice E varies over a limited range of values with a bounded maximum $E_{max} = N_p s_{max}$ where N_p is the number of plaquettes on the lattice and s_{max} is the maximum value of the action per plaquette. $N_p = 6L^3T$ where L is the length of the lattice in the spatial directions and T its length in the 'temporal' direction.

Our method starts of by dividing the range of $(0, S_{max})$ of S to many intervals (of the order of the volume of the system) and then dividing each of these intervals into several bins (in our case three or four depending on the system).The adjacent intervals overlap on both sides with their neigbours by one bin. The method of determining the partition function then pro-

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ceeds as follows. We generate states of the system that restricts the total action to a given interval. We then count the number of times the action value lies in one or the other of the bins in that interval. For an appropriate updating procedure to be described shortly the relatve occupation numbers of the bins in an interval are unbiased estimators of the relative $P(S)$ values of eq(3). This updating and histogramming procedure is repeated over all intervals. The overlap in action value at the boundary bins of neighbouring intervals gives the relative normalization of the P's between these intervals. This generates the full $P(S)$ and hence the partition function for all couplings up to an over all normalization which is irrelevant for most useful purposes.

To initialize the lattice one starts from any arbitrary link configuration. This initial configuration is then further initialized by changing the gauge elements on the links to arbitrary elements of the gauge group and accepting those changes that keep the total action of the system in the range of action values of any specific interval. By choosing the size of the initial interval appropriately one allows the system to migrate to any starting interval of interest. One then proceeds from one interval to its neighbour in the order preferred by the system so as to have the configurations of the next interval as a natural extension of those in the preceeding one. In other words one allows the system to migrate in the direction of increasing $P(S)$.

After every trial update(or a fixed number of trial updates), the lattice action is recorded and a histogram of its distribution over the chosen number of bins in the interval of interest is built. This is done regardless of whether the updates were accepted or not. This is an important aspect of the method which if neglected would give otherwise wrong estimators of $P(S)$. For this takes account of transitions into the interval from the outside.

It is important to note here that for a large system, and if one is interested in only a particular range of inverse coupling β, one may need to construct $P(S)$ for only a part of the full range of S. For if one considers the expression for $Z(\beta)$ it is clear that one may rewrite it in the following form:

$$Z(\beta) = \sum_S \exp[-\beta S + LnP(S)] \tag{4}$$

If one then expands $LnP(S)$ around some point S_0 it is clear that the main contribution to the integral(Sum) comes from those regions in S where

$$\beta = \frac{d}{dS} LnP(S) \tag{5}$$

At such a point the integral is dominantly gaussian for large lattices and is exactly so only in the infinite volume limit. In fact Monte Carlo computations are simply performing an important sampling estimate of this intgral and hence are only probing a small range of S in $P(S)$. The relevant range of S over which the integral gets its significant contribution is then the only range over which one needs to know $P(S)$ for a particular value of β. For a range of β values one can determine this range empirically after one has constructed a first sample partition function and then concentrates on this range for further samples to be constructed. We show in fig.(1) a plot of $Ln(P(S))/V$ vs. S/V for $L = 6$ where V is the volume of the system $V = L^3T$. The sections relevant for the phase transition region surround the point where the slope of the curve is near 1.90 . This is the region from approximately $S/V = 2.0$ to $S/V = 4.0$. Sections of $Ln(P(S))/V$ in this range constructed for

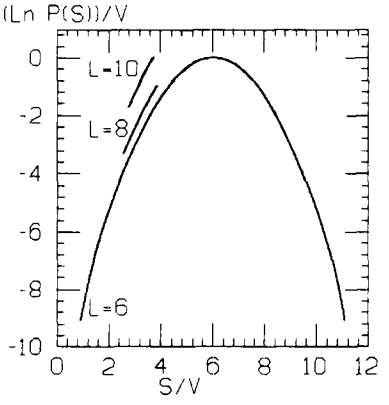

Figure 1. Partition function for L=6 and T=2. Sections relevant for the phase transition region for L=8 and L=10 are also shown.

$L = 8, 10$ are also shown. Note that all these curves are very smooth functions of their variable and hence can be determined very accurately. There are no fast variations to contend with inspite of this being the phase transition region. The fast behaviour is a consequence of folding in the exponential factor $\exp -\beta S$. This we do in our method seperately whereas in Monte Carlo simulations it is built in. Note that the sections to the right of $S/V = 6$ contribute mainly to the partition function for negative β . Furthermore, in all cases,the normalization of the curves for $Ln(P(S))/V$ are unimportant for any of our calculations.

Another quantity of interest in our study are the locations of the zeros of the partition function $Z(\beta)$ in the complex β plane or,of more interest as regards finite size scaling analysis, the variable $u = \exp -\beta$. We located these zeros using two methods. In terms of the variable u and in our binning approximation the partition function becomes effectively a polynomial of very high order. A program based on the Newton Raphson method for finding the complex zeros of such polynomials was used.A more direct method relies on finding in the complex β plane the locus of points where the real and imaginary parts of $Z(\beta)$ vanish (in our case change sign). The intersections of these curves determine the location of the zeroes in β.

3. RESULTS AND DISCUSSION

As examples we present here the results for SU(2) lattice gauge theory for a variety of lattice actions on very small lattices. In Figs.(2,3) the spectral density, plaquette energy and specific heat are shown for the adjoint and icosahedral actions on a 4^4 lattice showing clearly all expected phase transitions. Fig. (4) shows the distribution of zeros for these as well as the Wilson action. The freezing transition as well as the first order transitions for the first two choices are clearlty indicated by a well defined line of zeros in the $u = \exp -\beta$ complex planes. This is to be contrasted by the mere approach of a group of zeros to the real axis in the case of the cross over regions of both the Wilson action and the pure adjoint action (for u less than that for the line of first order transition). A more dramatic signal is the

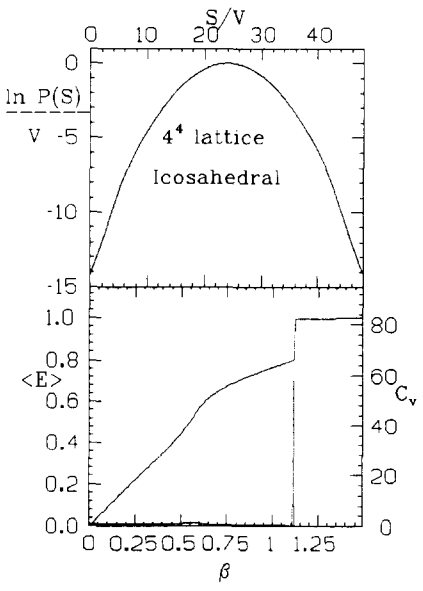

Figure 2. Spectral Density, Plaquette and Specific heat expectation values for Icosahedral actions on 4^4 Lattice.

cluster of zeros at the point $u = 0$ in these two cases, clearly pointing to the expected nontrivial phase transition at infinite β, and its marked absence in the case of the Icosahedral action where the freezing transition takes over at finite β. Such qualitative behaviour has not been seen before using Monte Carlo methods.

To demonstrate the use of the finite size scaling of the zeros of $Z(\beta)$ to determine critical exponents we present the results of a study[2] of the finite temperature phase transition for lattice SU(2) gauge theory. We construct the relevant sections of the spectral density on relatively larger lattices using $T = 2$ and $L = 6, 7, 8, 10$ and use the specific heat as a signal for the transition. The position of the transition zeros were measured and this is repeated for several samples of each system in order to get an estimate of the errors of the measurements as well. The region of the phse transition is seen to be controlled by a pair of zeros whose real parts are clearly resolved. This reflects itself in that the peak of the specific heat t is consequently resolved into a double peak.

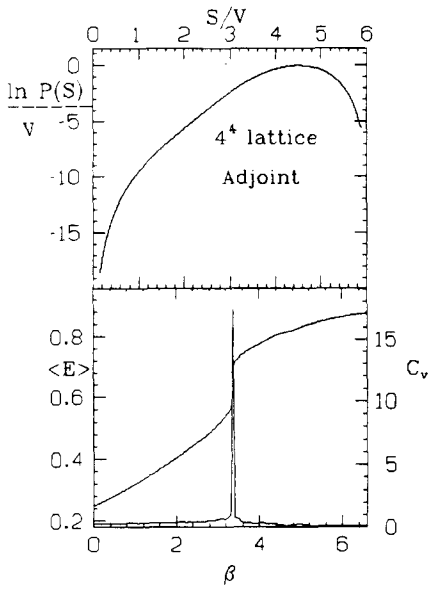

Figure 3. Spectral Density, Plaquette and Specific heat expectation values for Adjoint actions on 4^4 Lattice.

The basic relationship between these zeros and the critical exponent was pointed out in Ref. (3) and indicates that if $u_1(L)$ represents the nearest zero in the complex u plane and if u_c represents such point in the infinite volume limit we then have

$$|u_1(L) - u_c| = aL^{-1/\nu} + bL^{-c} \qquad (6)$$

The value of u_c is not available to us in this case and we assume it is mostly real and is very close to $Reu_1(L)$ for large L. One expects then, in the large L limit,that the imaginary[3] parts of the nearest zero to scale as above and furnish an estimate of ν. we use the data in this fashion and use the points with smallest value of Imu to determine ν by a straight line fit to the points ln(Im u) vs Ln(L). The results are shown in fig.(5) and the value of $\nu = 0.62(3)$ is then obtained. If one determines the average Imu from all points for a given L independent of their real parts one gets a smaller value of$\nu = 0.61(3)$. In any case it must be emphasized here that this fit is very sensitive to the errors in the measurement of Imu which is the source of error estimate given in the cases above. Clearly this is consistent with the fact that the system is in the same universality class as the three dimensional Ising system which is well known.

4. CONCLUSIONS

In conclusion we have demonstrated that the method of simulating lattice gauge systems by numerically constructing the partition function from the density of states works well even for larger size lattices. In the case at hand it provided, through finite size scaling of the zeroes of the partition function, a direct measurement of the critical exponent ν for the finite temperature transition of SU(2) lattice gauge theory. It also, by virtue of

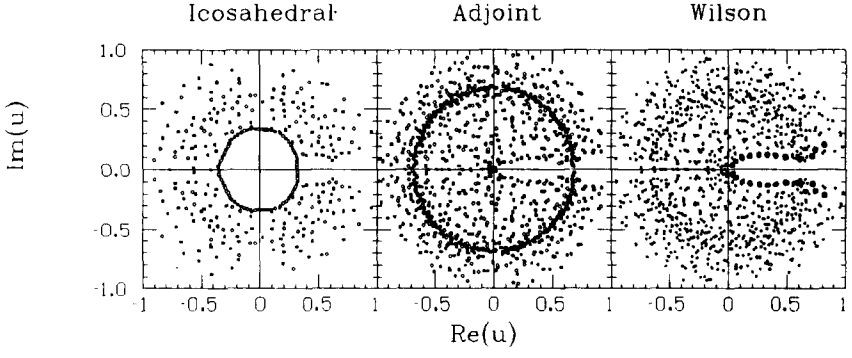

Figure 4. Distribution of zeros in the complex u plane for icosahedral, adjoint and and Wilson actions.

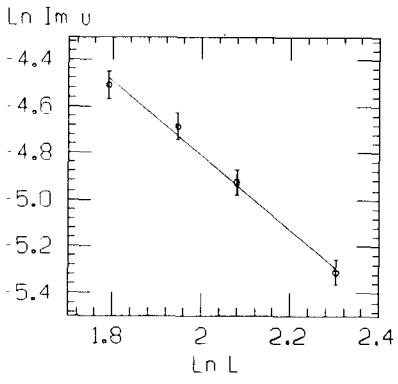

Figure 5. Straight line fit to our data of LnImu vs. LnL.

its detailed look at the density of states which is a relatively smooth function in the phase transition region, is able to provide more detailed information concerning the specific heat and other order parameters than ordinary Monte Carlo methods. This is Clearly so for by factoring out the exponential $\exp -\beta S$ from the partition function and measuring $P(S)$ directly we are dealing with a smoother and much more manageable quantity.

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