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POLYSPECTRAL ANALYSIS AND OPTIMIZATION OF THE FREQUENCY-DOMAIN DIFFERENTIAL PHASE-SHIFT KEYING

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Using the mathematical tool of high-order spectra (polyspectra), we consider the synthesis of optimal variants of high-order differential phase-shift keying in the frequency domain. The proposed approach allows all possible variants of the second-order differential phase-shift keying to be reduced to phase modulation of a special form of the fourth-order spectrum (trispectrum) — triphase modulation. The optimal triphase shift keying variants which have the best spectral efficiency and outperform the known variants of the second-order differential phase-shift keying in noise immunity are devised. It is shown that the noise immunity of proposed modulation is higher than that of the first-order differential phase-shift keying in the channels with simultaneous time and frequency offsets.

1. INTRODUCTION

At present, communication systems with orthogonal frequency division multiplexing, or OFDM systems, which use a set of mutually orthogonal sections of harmonic oscillations (subcarriers) as the transmitted signal, have become widespread in digital communication engineering in different frequency ranges. This is due to their merits, such as the absence of intersymbol interference in a multipath circuit, which is provided by the introduction of a time guard with high spectral efficiency preserved and the relative simplicity of the formation and reception of a signal due to the use of computationally efficient algorithms of fast Fourier transform. At the same time, the OFDM systems have an elevated sensitivity to phase distortions due to the instability of phase characteristics of the channel as well as time and frequency offsets [1, 2].

The invention of differential phase modulation (or differential phase-shift keying) and the development of its methods have been milestones in the solution of the "phase problem" [3–5]. It appears that the phase distortion effect in the data transmission channel can be removed or attenuated by using invariant differential phase-shift keying of a certain order and the corresponding algorithms of incoherent reception [5–8]. The OFDM systems can use both the conventional time-domain differential phase-shift keying of the kth order (TDPSK-k) and the frequency-domain differential phase-shift keying of the kth order (FDPSK-k) [9, 10]. In the first case, the information parameter is the kth-order phase difference of the same subcarrier in different symbol intervals and in the second case, the information parameter is the kth-order difference of different subcarriers inside the same symbol interval. The first-order differential phase-shift keying, which ensures the invariance to the total phase rotation of the subcarriers, and the second-order differential phase-shift keying, which is invariant to the linear time–frequency phase shift of the subcarriers of a signal, have become most widespread.

The increased noise immunity of reception of the kth-order differential phase-shift keying (DPSK-k) variants with diverse phase differences of the $(k-1)$th order relative to the classical DPSK-k with conjugate

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differences [4], which was mentioned in [6, 8, 9], leads to the idea of increasing the reception noise immunity due to optimization of the differential phase-shift keying. In [8, 9], the methods of differential phase-shift keying were analyzed on the basis of the mathematical tool of high-order instantaneous moments and the causality principle. However, since the reception of differential phase-shift keying signals is inherently block-oriented, the causality principle for them is a redundant constraint on the possible methods of choosing the phase differences.

In this paper, we propose that the problems of synthesis of optimal FDPSK variants of the higher orders are considered on the basis of the mathematical tool of the high-order spectra (polyspectra). Polyspectral analysis is widely used in physics and engineering as an adequate method for obtaining brand new information on the nonlinear properties of objects and media, in the problems of nonlinear spectroscopy, diagnostics, identification, and data transfer, which completes the spectral-correlation approach to the solution of similar problems (see monographs [11–13] and the references in [14]). Such an analysis is an adequate tool for a study of stationary (time offset-invariant) phase relations of the different-frequency components of complex information signals [15–18]. In the way various methods of TDPSK-k were devised in [8] by assigning a special form to the "instantaneous moments of the 2^kth order," different methods of FDPSK-k for $k > 1$ can be obtained from an analysis of the methods of assigning a special form to the spectra of order 2^k.

This paper aims at a polyspectral analysis and synthesis of optimal variants for the simplest higher-order frequency-domain differential phase-shift keying, namely, FDPSK-2.

In Sec. 2, we describe in a unifed way the use of the first- and second-order time- and frequency-domain differential phase-shift keying in OFDM systems and compare the spectral efficiencies and signal-to-noise ratios at the outputs of the corresponding incoherent demodulators as functions of time delays and Doppler frequency shifts in communication lines. The method of improving the noise immunity of FDPSK-2 signals, which is based on the development of new families of signals with reference subcarriers that are common for all DPSK-2 symbols, has been substantiated. In Sec. 3, the possible methods of choosing the second-order phase differences are studied by analyzing the adequate FDPSK-2 form of the fourth-order spectrum (trispectrum), and the class of methods with two reference subcarriers is singled out among them. Section 4 represents the results of a numerical study of the noise immunity of new forms of FDPSK-2 with two reference subcarriers in comparison with known forms of time- and frequency-domain differential phase-shift keying, demonstrating their merits in the channels with additive Gaussian white noise, indefinite phase, and time–frequency offsets. The main obtained results are summarized in Conclusions.

2. THE MODEL OF AN OFDM SYSTEM WITH DIFFERENTIAL PHASE-SHIFT KEYING

2.1. Modulation and formation of the transmitted signal

Consider an OFDM system with N subcarriers located in the frequency band F with the $\Delta f = F/N = 1/T$ step. In the transmitter, a message of volume $I = K_I N_I \log_2 M$ bits with the use of the M-position modulation is divided into K_I blocks with $\mathbf{m}[k] = (m_0[k], \ldots, m_{N_I-1}[k])^{\mathrm{T}}$, each having N_I elements, where the superscript T denotes transposition, $0 \leq k \leq K_I - 1$, and each element can take one of the M values: $m_n[k] = 0, \ldots, M-1$. Depending on the form of modulation for the transmission of K_I blocks, K OFDM symbols are used. The block transmitted in the kth OFDM symbol encodes the set N of complex quantities: $\mathbf{X}[k] = (X_0[k], \ldots, X_{N-1}[k])^{\mathrm{T}}$. The complex envelope of the kth symbol is formed via the inverse Fourier transform.

$$x^{[k]}(t) = \sum_{n=0}^{N-1} X_n[k] \exp(i\, 2\pi n t/T); \tag{1}$$

hence, $X_n[k] = A_n[k] \exp(i\theta_n[k])$ is the complex amplitude of the nth subcarrier when the kth signal is transmitted. For the phase forms of modulation, the amplitudes $A_n[k]$ are fixed and are usually identical for any n and k, while information is stored in the values of the phases $\theta_n[k]$ or their combinations.

We note that the OFDM system with time-domain differential phase-shift keying can be considered a multichannel system with frequency division, where N subcarriers form $N_I = N$ independent subchannels with differential phase-shift keying (DPSK subchannels in what follows), which transmit blocks of $L_I = K_I$ message elements. In a similar way, the frequency-domain differential phase-shift keying system is a multichannel system with time division, where K OFDM symbols form $K_I = K$ independent DPSK subchannels transmitting blocks of $L_I = N_I$ information elements.

In order to describe the time- and frequency-domain differential phase-shift keying in a unified way, we introduce the following designations: vector column $\mathbf{r} = (n,k)^{\mathrm{T}}$, whose elements n and k are equal to the number of the subcarrier and the number of the OFDM symbol, respectively, where $0 \le n \le N-1$ and $0 \le k \le K-1$; $\Delta\mathbf{r} = (0,1)^{\mathrm{T}}$ for the TDPSK and $\Delta\mathbf{r} = (1,0)^{\mathrm{T}}$ for the FDPSK. Let $s^{(j)}[\mathbf{r}] = S^{(j)}[\mathbf{r}]\exp(i\theta^{(j)}[\mathbf{r}]) = s_n^{(j)}[k]$ be a DPSK symbol put into correspondence with the information element $m[\mathbf{r}] = m_n[k]$ for the differential phase-shift keying of the jth order, where $\theta^{(j)}[\mathbf{r}] = (2\pi/M)m[\mathbf{r}]$ is the information phase and $S^{(j)}[\mathbf{r}]$ is an arbitrarily specified amplitude of the DPSK symbol. Such an abbreviated designation system will be used for other quantities, too. For example, $X[\mathbf{r}] = X_n[k]$, $A[\mathbf{r}] = A_n[k]$, and $\theta[\mathbf{r}] = \theta_n[k]$.

For DPSK-1, the information phases are the differences

$$\theta^{(1)}[\mathbf{r}] = \theta[\mathbf{r} + \Delta\mathbf{r}] - \theta[\mathbf{r}]. \tag{2}$$

For DPSK-2, the message encodes the differences of the quantities described by Eq. (2). There are two forms of DPSK-2. The direct generalization of DPSK-1 is the conventional DPSK-2 scheme, for which the information phases are the differences of the adjacent phase differences of the first order (AD DPSK-2)

$$\theta^{(2.1)}[\mathbf{r}] = \theta^{(1)}[[\mathbf{r} + \Delta\mathbf{r}] - \theta^{(1)}[\mathbf{r}] = \theta[\mathbf{r} + 2\,\Delta\mathbf{r}] - 2\theta[\mathbf{r} + \Delta\mathbf{r}] + \theta[\mathbf{r}]. \tag{3a}$$

An alternative way is to manipulate the second-order phase differences by using separated phase differences of the first order (SD DPSK-2):

$$\theta^{(2.2)}[\mathbf{r}] = \theta^{(1)}[[\mathbf{r} + 2\Delta\mathbf{r}] - \theta^{(1)}[\mathbf{r}] = \theta[\mathbf{r} + 3\,\Delta\mathbf{r}] - \theta[\mathbf{r} + 2\,\Delta\mathbf{r}] - \theta[\mathbf{r} + \Delta\mathbf{r}] + \theta[\mathbf{r}]. \tag{3b}$$

Compared with the previous one, this scheme of DPSK-2 requires an additional reference symbol or a subcarrier for the transmission initialization. However, it is known [8] that such a scheme ensures a higher signal-to-noise ratio at the demodulator output. In Eq. (3) and elsewhere below, designations of the form $\theta^{(2.l)}$, where $l = 1$ for AD-DPSK-2 and $l = 2$ for SD-DPSK-2 are used for differentiation between characteristics of different forms of DPSK-2.

The recurrent algorithms of the formation of complex amplitudes of the subcarriers for the signals with DPSK-1 and DPSK-2 have the form

$$X[\mathbf{r} + \Delta\mathbf{r}] = s^{(1)}[\mathbf{r}]/X^*[\mathbf{r}], \tag{4a}$$

$$X[\mathbf{r} + (l+1)\,\Delta\mathbf{r}] = s^{(2.l)}[\mathbf{r}]/(X^*[\mathbf{r} + l\,\Delta\mathbf{r}]X^*[\mathbf{r} + \Delta\mathbf{r}]X[\mathbf{r}]) \qquad (l = 1,2), \tag{4b}$$

where the asterisks denote complex conjugation.

A time guard is inserted for protection against the intersymbol interference between successive symbols. Its duration T_{G} should be longer than the maximum expected time of scattering in the channel. Usually, the time guard is used for transmission of the so-called cyclic prefix, which represents the last T_{G} seconds of the next symbol. In this case, the complex envelope of a low-frequency signal that transmits the whole message over the time $T_\Sigma = KT_{\mathrm{S}}$ with the clock interval $T_{\mathrm{S}} = T + T_{\mathrm{G}}$ can be represented in the form

$$x(t) = \sum_{k=0}^{K-1} x^{[k]}(t - T_{\mathrm{G}} - T_{\mathrm{S}}k)\Pi_{T_{\mathrm{S}}}(t - T_{\mathrm{S}}k) = \sum_{k=0}^{K-1}\sum_{n=0}^{N-1} X_n[k]\exp[i\,2\pi n(t - T_{\mathrm{G}} - T_{\mathrm{S}}k)/T]\Pi_{T_{\mathrm{S}}}(t - T_{\mathrm{S}}k), \tag{5}$$

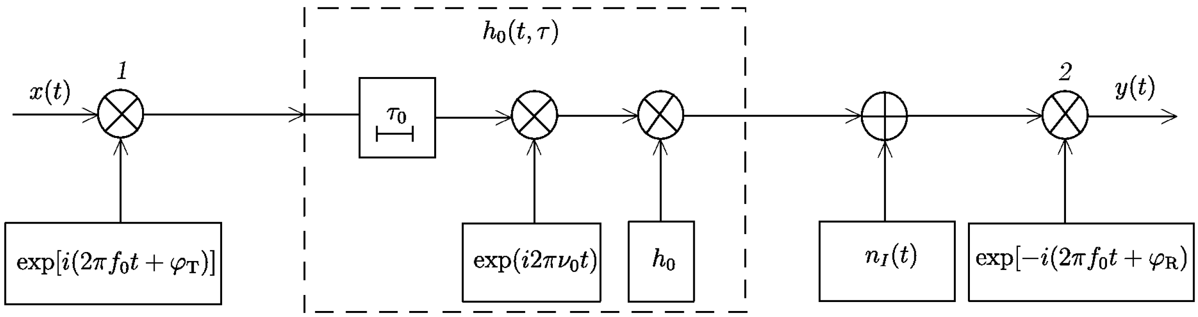

Fig. 1. Model of the communication line.

where $\Pi_T(t) = \{1, t \in [0,T]$, and $0, t \notin [0,T]\}$ denotes a rectangular pulse with unit amplitude and duration T.

Compare spectral efficiencies (4) of the modulation techniques, i.e., the specific rates $c \equiv I/(T_\Sigma F)$:

$$c^{(j)} = c^{(0)}/(1 + \alpha^{(j)}/L_I), \tag{6}$$

where $c^{(0)} = \log_2 M/(1 + T_{\mathrm{G}}/T)$ is the specific rate of the absolute M-position phase modulation, $L_I = K - \alpha^{(j)}$ for TDPSK and $L_I = N - \alpha^{(j)}$ for FDPSK, $\alpha^{(1)} = 1$, and $\alpha^{(2.l)} = l + 1$ $(l = 1, 2)$. It follows from Eq. (6) that TDPSK has the higher spectral efficiency for long sessions with a large number of transmitted OFDM symbols (for $L_I = K_I > N_I = N$), while FDPSK is more efficient for short sessions with a high degree of alignment (for $L_I = N_I > K_I = K$).

2.2. Model of the communication line

The idealized model of a single-path communication line with time delay and Doppler frequency shift is presented in Fig. 1. In mixer *1*, the low-frequency signal (5) on the transmitting side modulates a carrier oscillation with frequency f_0. The inverse operation is performed in mixer *2* on the receiving side. The phases φ_{T} and φ_{R} of the carrier-oscillation generators are treated as random quantities, so that their time fluctuations are neglected. Multiplicative distortions of a signal during its propagation are described by a linear channel with the nonstationary pulse characteristic $h_0(t,\tau) = h_0 \exp(i\,2\pi\nu_0 t)\,\delta(\tau - \tau_0)$, where $\delta(t)$ is a Dirac delta function, ν_0 and τ_0 are the Doppler frequency shift and the time delay, respectively, and h_0 is a complex transmission coefficient which models general fadings. At the mixer input, a signal in the receiver is distorted by additive Gaussian white noise $n_I(t)$, which generates additive Gaussian white noise $n(t) = n_I(t)\exp[-i\,(2\pi f_0 t + \varphi_R)]$ with $\langle n(t)\rangle = 0$ $\langle n(t)n(t+\tau)\rangle = 0$ and $\langle n(t)n^*(t+\tau)\rangle = N_0\delta(\tau)$ at the output.

2.3. Message recovery in the receiver

The transmitted signal

$$y(t) = \exp\bigl\{i\,(\varphi_{\mathrm{T}} - \varphi_{\mathrm{R}} - 2\pi f_0\tau_0 + 2\pi\nu_0 t)\bigr\}h_0 x(t - \tau_0) + n(t) \tag{7}$$

is gated, and the segments $y^{[k]}(t) = y(t + T_{\mathrm{G}} + kT_{\mathrm{S}})\Pi_T(t)$ corresponding to separate OFDM symbols with neglected prefix are singled out. On their basis, the sets $\mathbf{Y}[k] = (Y_0[k], \ldots, Y_{N-1}[k])^{\mathrm{T}}$ of complex amplitudes of the subcarriers are calculated:

$$\mathbf{Y}[k] = Y_n[k] = \frac{1}{T}\int_0^T \exp(-i\,2\pi nt/T)y^{[k]}(t)\,\mathrm{d}t. \tag{8}$$

In the case $T_{\mathrm{G}} > \tau_0$, where the time guard is sufficient for complete suppression of the intersymbol interference, complex amplitudes of the subcarriers have the form

$$Y_n[k] = h_0 \exp\left(i\xi_0 + ik\zeta_0^{(\mathrm{TD})}\right) \sum_{m=0}^{N-1} X_m[k] \exp\left(-im\zeta_0^{(FD)}\right) (-1)^{n-m} \operatorname{sinc}[\pi\,(n-m-\nu_0 T)] + N_n[k], \qquad (9)$$

where $\operatorname{sinc} x = \sin(x)/x$;

$$\xi_0 = \varphi_{\mathrm{T}} - \varphi_{\mathrm{R}} - 2\pi f_0 \tau_0 + \pi\nu_0\,(2T_{\mathrm{G}} + T), \qquad \zeta_0^{(\mathrm{TD})} = 2\pi\nu_0 T_{\mathrm{S}}, \qquad \zeta_0^{(\mathrm{FD})} = 2\pi\tau_0/T \qquad (10)$$

are the total phase rotation and the linear increments in time (TD) and frequency (FD), respectively, in the subcarrier phases. Additive noise $N_n[k] = (1/T)\int_0^T \exp(-i\,2\pi nt/T) n(t + T_{\mathrm{G}} + kT_{\mathrm{S}})\,dt$ represents a sequence of independent Gaussian random quantities with $\langle N_n[k]\rangle = 0$, $\langle N_n^2[k]\rangle = 0$, and $\langle |N_n[k]|^2\rangle = N_0/T$. If $\nu_0 T \ll 1$, then the maximum interference can be neglected and, instead of Eq. (9), one can write in abbreviated form

$$Y[\mathbf{r}] = X[\mathbf{r}]\alpha_0 \exp(i\xi_0 + i\mathbf{r}^{\mathrm{T}}\zeta_0) + N[\mathbf{r}], \qquad (11)$$

where $\alpha_0 = h_0 \operatorname{sinc}(\pi\nu_0 T)$ and $\zeta_0 = (-\zeta_0^{(\mathrm{FD})}, \zeta_0^{(\mathrm{TD})})$.

In the case of symbol-by-symbol detection using algorithms inverse of the synthesis rules (4), one calculates the estimates of transmitted DPSK symbols

$$\hat{s}^{(1)}[\mathbf{r}] = Y[\mathbf{r} + \Delta\mathbf{r}]Y^*[\mathbf{r}], \qquad (12a)$$

$$\hat{s}^{(2.l)}[\mathbf{r}] = Y[\mathbf{r} + (l+1)\,\Delta\mathbf{r}]Y^*[\mathbf{r} + l\,\Delta\mathbf{r}]Y^*[\mathbf{r} + \Delta\mathbf{r}]Y[\mathbf{r}] \qquad (l = 1, 2), \qquad (12b)$$

on the basis of which the decisions are taken:

$$\hat{m}[\mathbf{r}] = \arg \min_{0 \le m \le M-1} \left| \arg \hat{s}^{(j)}[\mathbf{r}] - \frac{2\pi m}{M} \right|. \qquad (13)$$

2.4. On the signal-to-noise ratio for demodulators of the phase difference signals

Compare the signal-to-noise ratios for statistics (12). Since the amplitudes $\hat{s}^{(j)}[\mathbf{r}]$ of the statistics do not affect solutions (13), the signal-to-noise ratio can be defined by the expression

$$\rho^{(j)}[\mathbf{r}] = \frac{|\langle \hat{s}^{(j)}[\mathbf{r}]\rangle|^2}{\left\langle \left|\hat{s}^{(j)}[\mathbf{r}] - \left|\hat{s}^{(j)}[\mathbf{r}]\right| \exp(i \arg s^{(j)}[\mathbf{r}])\right|^2 \right\rangle}. \qquad (14)$$

In the denominator, the deviation $\hat{s}^{(j)}[\mathbf{r}]$ of the statistics from the expected value for the phase-unbiased estimate serves as the noise power.

It can be shown that under conditions of validity of Eq. (11),

$$1/\rho^{(1)}[\mathbf{r}] = 4\sin^2(\Delta\mathbf{r}^{\mathrm{T}}\zeta_0/2) + (1 + d_0^{-1}[\mathbf{r}])\,(1 + d_1^{-1}[\mathbf{r}]) - 1, \qquad (15a)$$

$$1/\rho^{(2.1)}[\mathbf{r}] = (1 + d_0^{-1}[\mathbf{r}])\,(1 + 4d_1^{-1}[\mathbf{r}] + 2d_1^{-2}[\mathbf{r}])\,(1 + d_2^{-1}[\mathbf{r}]) - 1, \qquad (15b)$$

$$1/\rho^{(2.2)}[\mathbf{r}] = \prod_{k=0}^{3} \left(1 + d_k^{-1}[\mathbf{r}]\right) - 1, \qquad (15c)$$

where $d_l[\mathbf{r}] = |\alpha_0|^2 A^2[\mathbf{r} + l\,\Delta\mathbf{r}]T/N_0$ is the signal-to-noise ratio at the input in the $1/T$ band. If all the amplitudes of the transmitted signal are identical, i.e., $A[\mathbf{r}] \equiv A$, then $d_l[\mathbf{r}] = d^{(0)}/(1 + \alpha^{(j)}/L_l)$, where $d^{(0)} = c^{(0)}\,|\alpha_0|^2\,E_{\mathrm{b}}/N_0$ is the signal-to-noise ratio in the subchannels of symbol-by-symbol detection

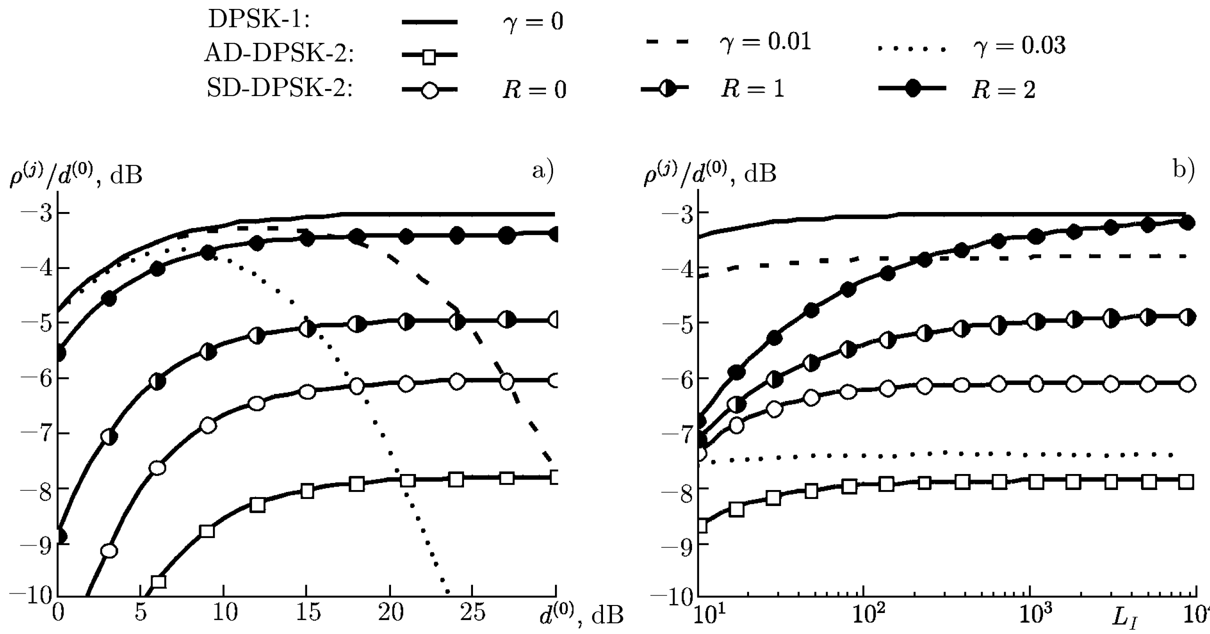

Fig. 2. Signal-to-noise ratio at the outputs of symbol-by-symbol demodulators for OFDM systems with differential phase-shift keying, which is calculated for the channels with the signal offset in frequency (ν_0) and time (τ_0) with respect to the signal-to-noise ratio ($d^{(0)}$) of a phase-modulated system with equivalent energy in the case of ideal compensation for phase distortions. The diagrams show the dependences of the signal-to-noise ratio on $d^{(0)}$ for the information block of length $L_I = 1024$ (*a*) in an independent DPSK subchannel and on L_I for $d^{(0)} = 20$ dB (*b*); $\gamma = \nu_0 T_S$ for TDPSK-1 and $\gamma = \tau_0/T$ for FDPSK-1. The curves $R = 1$ and $R = 2$ demonstrate the possibility of increasing the noise immunity of the systems with DPSK-2 in the case of using new forms of SD-DPSK-2 with one and two reference components, respectively, and optimal redistribution of the signal power.

for a coherent phase-modulated OFDM system that is equivalent in E_b/N_0 (relative energy expense for transmission of one bit of message) in the case of ideal compensation for phase distortions.

Equation (15a) demonstrates that due to the invariance of DPSK-1 to the total phase rotation and the absence of linear phase distortions in the corresponding DPSK subchannels, TDPSK-1 is insensitive to a time delay and FDPSK-1 is insensitive to a frequency shift. The minimum loss of the differential phase-shift keying systems with respect to systems with phase modulation and ideal compensation for phase distortions is ensured in asymptotics of sufficiently large input signal-to-noise ratios and long information blocks in the DPSK subchannels (see Fig. 2). For DPSK-1, in the absence of linear phase distortions in the DPSK subchannels, this loss amounts to 3 dB. However, as is demonstrated by Fig. 2, the presence of even weak linear phase distortions impairs dramatically the noise immunity of DPSK-1.

The additional invariance of DPSK-2 to linear phase distortions, as is evidenced by Eqs. (15b) and (15c) and Fig. 2, leads to the insensitivity of any DPSK-2 both to time and frequency offsets. However, this advantage of DPSK-2 is reached due to a significant decrease in the signal-to-noise ratio at the demodulator output. While for the conventional AD-DPSK-2 the minimum loss in phase modulation amounts to 7.8 dB, this value for SD-DPSK-2 decreases to 6 dB.

To improve the noise immunity of OFDM signals with FDPSK-2, we propose new forms of signals with reference subcarriers, which are common for all DPSK-2 symbols. Optimal redistribution of power of an OFDM signal between the subcarriers in favor of the reference ones is possible in such signals. Note that for TDPSK-2, changing the power of separate OFDM signals could be equivalent, but this is inexpedient for a given power of the transmitter.

The substantiation of our approach is presented in Fig. 2 by dependences of the output signal-to-noise ratios for hypothetic forms of SD-DPSK-2 with one and two reference components in the case of optimal redistribution of the signal power. The dependences were calculated by Eq. (15c) for $d_l[\mathbf{r}] = pd[\mathbf{r}]$ for R of the reference components and $d_l[\mathbf{r}] = d[\mathbf{r}] = d^{(0)}/[1 + (3 - R + Rp)/L_I]$ for other components, where $p = \sqrt{(L_I + 2 - R)/(4 - R)}$ is the value of the relative power of the reference components, which is asymptotically optimal for $d^{(0)} \to \infty$. It is seen that SD-DPSK-2 with two reference components is the best form of DPSK-2 among those presented in Fig. 2 and tends to DPSK-1 in noise immunity for fairly large input signal-to-noise ratios and long information blocks in the DPSK subchannels.

3. TRISPECTRUM SYNTHESIS OF FDPSK-2 SIGNALS

3.1. Generalized FDPSK-2 and its adequate trispectrum form

The class of generalized TDPSK-2, which was deduced in [8] on the basis of the mathematical tool of high-order instantaneous moments, was transformed in [9] by means of a transition from the time to frequency domain into the class of generalized FDPSK-2 with the information phases

$$\theta_m^{[2.(m_1,m_2)]} = \theta_m - \theta_{m-m_1} - \theta_{m-m_2} + \theta_{m-m_1-m_2}, \tag{16}$$

where (m_1, m_2) are the parameters determining a separate element of the FDPSK-2-$(m_1,\, m_2)$ class. Hereafter we consider a separate OFDM symbol and, therefore, the dependence on the symbol number is omitted. By virtue of the causality principle, the author of [9] limited this class by the condition $m > m_2 \geq m_1 \geq 1$, with allowance of which the domain of definition for the current index m in Eq. (16) can easily be found: $m_1 + m_2 \leq m \leq N - 1$.

Since in the frequency domain the causality principle is meaningless and the reception of FDPSK signals is inherently block-oriented, the FDPSK-2 class with the information phases

$$\theta^{(2)}_{n_1,n_2,n_3,n_4} = \theta_{n_1} - \theta_{n_3} - \theta_{n_4} + \theta_{n_2} \tag{17}$$

is more general. Here, the invariance to the total phase rotation of the subcarriers $\theta_{n_l} \to \theta_{n_l} + \xi_0$ is realized automatically, while the invariance to the linear phase shift $\theta_{n_l} \to \theta_{n_l} + n_l\zeta_0$ is provided by a special relation between the numbers (i.e., dimensionless frequencies) of the subcarriers:

$$n_1 + n_2 = n_3 + n_4. \tag{18}$$

Hence, information phase (17) and the corresponding FDPSK-2 symbol can be considered a function of three variables, e.g.,

$$s^{(2)}_{\mathbf{n}} = s^{(2)}_{n_1,n_2,n_3} = X_{n_1} X_{n_2} X^*_{n_3} X^*_{n_1+n_2-n_3}. \tag{19}$$

In Eq. (19) and elsewhere below the abbreviated notation $\mathbf{n} = (n_1, n_2, n_3)$ is used.

In principle, using additional notions is unnecessary when the problems of construction of OFDM signals with the optimal method of FDPSK-2 are addressed, and optimal sets of information symbols $\{s^{(2)}_{\mathbf{n}}\}$ and the rules of formation of the spectral components $\{X_n\}$ for them can be found by analyzing Eq. (17).

At the same time, it is easy to interpret Eqs. (17)–(19) by using the notions which are known from the mathematical tool of polyspectral analysis [17, 18]. Indeed, Eqs. (17) and (18) can be understood as conditions of polyspectral boundedness of the spectral components for a specific form of the fourth-order spectrum (i.e., trispectrum) of T-finite complex signals [17]

$$G_T(f_1, f_2, f_3) = (1/T)\, \langle X_T(f_1) X_T(f_2) X^*_T(f_3) X^*_T(f_1 + f_2 - f_3)\rangle. \tag{20}$$

The spectral components $X_T(f) = \int_0^T \exp(-i\,2\pi f t)\, x(t)\, \mathrm{d}t$ for the T-finite analytical signal are unambiguously determined by the coefficients $X_n = X_T(n/T)/T$ of Fourier series (1), for $N \to \infty$ in general. Obviously, for signals with a finite number of terms in expansion (1), the trispectrum given by Eq. (20) is unambiguously determined by the discrete samples

$$G_{\mathbf{n}} = T^3 G_T(n_1/T, n_2/T, n_3/T) = \langle X_{n_1} X_{n_2} X^*_{n_3} X^*_{n_1+n_2-n_3} \rangle, \tag{21}$$

$\mathbf{n} \in \Lambda_N$, where $\Lambda_N = \{\mathbf{n} \mid 0 \le n_k < N, 1 \le k \le 4, n_4 = n_1 + n_2 - n_3\}$. Note that for the determinate amplitudes $|X_n|$ and specified

$$\psi_{\mathbf{n}} = \arg G_{\mathbf{n}} = \theta_{n_1} + \theta_{n_2} - \theta_{n_3} - \theta_{n_1+n_2-n_3} \tag{22}$$

the parentheses of statistical averaging on the right-hand side of Eq. (21) can be omitted. Taking this into account and comparing Eqs. (17) and (19) with Eqs. (22) and (21), we come to the conclusion that the information combinations of phases (17) are phases (22), which are called triphases, and the generalized FDPSK-2 is essentially triphase modulation.

By virtue of the symmetry properties of trispectrum (21), namely,

$$\begin{aligned} G_T(f_1, f_2, f_3) &= G_T(f_2, f_1, f_3) \\ &= G_T(f_2, f_1, f_1 + f_2 - f_3), \\ G_T^*(f_1, f_2, f_3) &= G_T(f_3, f_1 + f_2 - f_3, f_2) \\ &= G_T(f_1 + f_2 - f_3, f_3, f_1), \end{aligned} \tag{23}$$

its definition domain for a signal in the band from 0 to $F_N = (N-1)/T$ is divided into eight symmetry domains (see Fig. 3), each representing a tetrahedron. Specifying a trispectrum in any of these domains is sufficient for its full description. For example, as the base domain, one can choose a tetrahedron $0 \le f_3 \le f_2 \le f_1 \le F_N - f_2 + f_3$ (whose edges are bold in Fig. 3). Note that on the face $f_2 = f_3$ of the base domain, the trispectrum takes real values and the triphases of the samples on this face cannot be information carriers. Hence, in what follows we consider only the trispectrum samples with coordinates from the set $\Omega_N = \{\mathbf{n} \mid 0 \le n_3 < n_2 \le n_1 < N - 1 - n_2 + n_3\}$ containing $M = [(N^2 - 1)(2N - 3)/24]$ elements (hereafter, $[x]$ is the integer part of the number x).

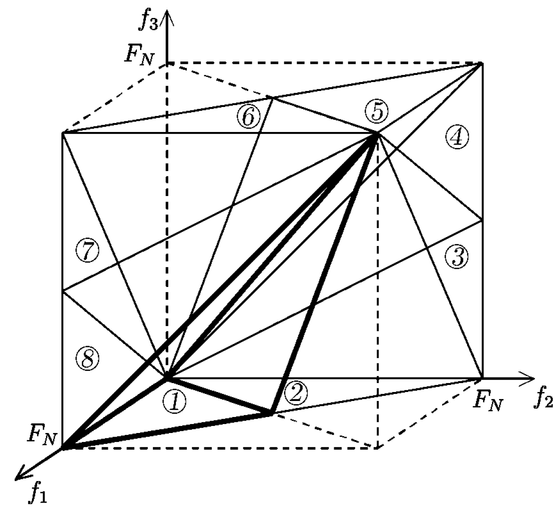

Fig. 3. Regions of symmetry of the trispectrum of an analytical signal in the band from 0 to F_N.

3.2. General principles and algorithms of synthesis of triphase-modulation signals

Introducing the vectors of the phases $\tilde{\boldsymbol{\theta}}_N = (\theta_0, \theta_1, \ldots, \theta_{N-1})^{\mathrm{T}}$ and triphases $\boldsymbol{\psi}_N$, we represent Eq. (22) in matrix form:

$$\tilde{\mathbf{A}}_N \tilde{\boldsymbol{\theta}}_N = \boldsymbol{\psi}_N, \tag{24}$$

where $\tilde{\mathbf{A}}_N = \{a_{mn}\} = (\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{N-1}) = (\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_{M-1})^{\mathrm{T}}$ is an $(M \times N)$ matrix and a_{mn} is its element at the intersection of the mth row ($\mathbf{b}_m^{\mathrm{T}}$) and the nth column ($\mathbf{a}_n$). Let us study the properties of this matrix.

Note that each row $\mathbf{b}_m^{\mathrm{T}} = \mathbf{b}_{\mathbf{n}}^{\mathrm{T}}$ corresponds to Eq. (22) for some trispectrum sample with the coordinates $\mathbf{n} \in \Omega_N$ and has the following properties:

$$\mathbf{b}_m^{\mathrm{T}} \mathbf{1}_N = 0, \qquad \mathbf{b}_m^{\mathrm{T}} \mathbf{f}_N = 0, \tag{25}$$

where $\mathbf{1}_N = (1, 1, \ldots, 1)^{\mathrm{T}}$ is a vector of length N with unit elements, $\mathbf{f}_N = (0, 1, \ldots, N-1)^{\mathrm{T}}$ is the vector column of dimensionless frequencies of the subcarriers. The first of Eqs. (25) ensures the invariance of the triphase to the total phase rotation, $\psi_{\mathbf{n}} = \mathbf{b}_{\mathbf{n}}^{\mathrm{T}}(\tilde{\boldsymbol{\theta}}_N + \xi_0 \mathbf{1}_N) = \mathbf{b}_{\mathbf{n}}^{\mathrm{T}}\tilde{\boldsymbol{\theta}}_N$, and the second equation ensures its invariance to a linear phase shift, $\psi_{\mathbf{n}} = \mathbf{b}_{\mathbf{n}}^{\mathrm{T}}(\tilde{\boldsymbol{\theta}}_N + \zeta_0 \mathbf{f}_N) = \mathbf{b}_{\mathbf{n}}^{\mathrm{T}}\tilde{\boldsymbol{\theta}}_N$. Using Eqs. (25), one can easily find the corresponding relations for the matrix as a whole and its columns:

$$\tilde{\mathbf{A}}_N \mathbf{1}_N = \sum_{n=0}^{N-1} \mathbf{a}_n = \mathbf{0}_M, \qquad \tilde{\mathbf{A}}_N \mathbf{f}_N = \sum_{n=0}^{N-1} n\mathbf{a}_n = \mathbf{0}_M, \tag{26}$$

where $\mathbf{0}_M = (0, 0, \ldots, 0)^{\mathrm{T}}$ is a zero vector of length M. Combining Eqs. (26), we have

$$\sum_{n=0}^{N-1} (n - c)\, \mathbf{a}_n = \mathbf{0}_M. \tag{27}$$

For the integer values $c \in \{0, 1, \ldots, N-1\}$, Eq. (27) expresses the linear relationship between $N-1$ columns $\{a_n \mid n \neq c\}$, and for the other values of c, the linear relationship between all N columns. Hence, the rank of the matrix $\tilde{\mathbf{A}}_N$ does not exceed $N-2$.

Consider the recurrent (with increasing band of a signal) method of ordering of the vector elements $\boldsymbol{\psi}_N$ and, therefore, rows of the matrix $\tilde{\mathbf{A}}_N$ by "cuts" composed of trispectrum samples with the same value of the frequency $n_4 = n_1 + n_2 - n_3$ of the fourth subcarrier that forms them. This method of ordering has the following recurrent form of record of the set of coordinates of trispectrum samples:

$$\Omega_{N+1} = \Omega_N \cup \left\{ \mathbf{n} \mid 0 \leq n_3 \leq N-2, n_3 + 1 \leq n_2 \leq \left[\frac{n_3 + N}{2} \right], n_1 = N + n_3 - n_2 \right\}. \tag{28}$$

Equation (28) shows that when another subcarrier at frequency N is added to a signal composed of N subcarriers, the samples with the same frequency $n_4 = N$ are added to the set of trispectrum samples. Hence, the recurrent relation

$$\tilde{\mathbf{A}}_{N+1} = \begin{pmatrix} \tilde{\mathbf{A}}_N & \mathbf{0}_M \\ \tilde{\mathbf{C}}_N & -\mathbf{1}_{[N^2/4]} \end{pmatrix}, \tag{29}$$

where $\tilde{\mathbf{C}}_N$ is an $[N^2/4] \times N$ matrix, is valid. Since $\tilde{\mathbf{A}}_3 = (-1\,2\,-1)$ and $\operatorname{rank} \tilde{\mathbf{A}}_3 = 1$, it can easily be shown on the basis of Eq. (29) that $\operatorname{rank} \tilde{\mathbf{A}}_{N+1} = \operatorname{rank} \tilde{\mathbf{A}}_N + 1$ for any $N \geq 3$ and thereby prove by induction the property

$$\operatorname{rank} \tilde{\mathbf{A}}_N = L_I = N - 2. \tag{30}$$

This means that among N triphases, only L_I triphases are linearly independent. A set of independent triphases form the basis Ψ_n^I, which in ordered form represents the L_I vector of the informative triphases, i.e., $\boldsymbol{\psi}_N^I$. The other triphases comprise the $(M - L_I)$ vector of the accompanying triphases, i.e., $\boldsymbol{\psi}_N^C$.

In order to determine the algorithms of synthesis of a triphase-modulation OFDM signal for the given informative triphases, it is necessary to express through them the phases of the subcarriers $\tilde{\boldsymbol{\theta}}_N$. Due to Eq. (30), when Eq. (24) is solved relative to the elements of the vector $\tilde{\boldsymbol{\theta}}_N$, the phases θ_r and θ_g should be specified for any two "initializing" subcarriers ($r < g$) and then the phases $\boldsymbol{\theta}_N = [\theta_0, \ldots, \theta_{r-1}, \theta_{r+1}, \ldots, \theta_{g-1}, \theta_{g+1}, \ldots, \theta_{N-1}]^{\mathrm{T}}$ of other subcarriers should be found. We now determine the matrix $\mathbf{A}_N$ by removing the columns $\mathbf{a}_r$ and $\mathbf{a}_g$, which correspond to the initializing subcarriers, from the matrix $\tilde{\mathbf{A}}_N$. By replacement of lines in the set of Eqs (24), we derive relationships for the informative and accompanying triphases,

$$\begin{pmatrix} \boldsymbol{\psi}_N^I \\ \boldsymbol{\psi}_N^C \end{pmatrix} = \begin{pmatrix} \mathbf{A}_N^I \\ \mathbf{A}_N^C \end{pmatrix} \theta_N + \begin{pmatrix} \mathbf{a}_r^I \\ \mathbf{a}_r^C \end{pmatrix} \theta_r + \begin{pmatrix} \mathbf{a}_g^I \\ \mathbf{a}_g^C \end{pmatrix} \theta_g, \tag{31}$$

from which for arbitrary values of the phases of the initializing subcarriers we find

$$\boldsymbol{\theta}_N = (\mathbf{A}_N^I)^{-1} (\boldsymbol{\psi}_N^I - \mathbf{a}_r^I \theta_r - \mathbf{a}_g^I \theta_g). \tag{32}$$

This expression describes the synthesis of triphase-modulation signals for any choice of the set of informative triphases and initializing subcarriers. For a given value of N, there is a large number of methods of choosing the basis from the entire set of triphases and, therefore, different variants of triphase modulation. Formula (32), being general, expresses the relation in matrix form and is inconvenient for practical use. In each particular case, replacing the rows in the set of Eqs. (31) and reducing the matrix $\mathbf{A}_N^I$ to a triangular form, from Eq. (32) we can obtain a recurrent algorithm for calculation of the subcarrier phases.

Consider as an example the class of triphase modulation variants with the recurrent method of the basis formation in the direction of increasing frequency, which corresponds to ordering (28) of elements of the vector $\boldsymbol{\psi}_N$. In this case, the subcarriers with numbers 0 and 1 will be initializing, and the recurrent relation

$$\mathbf{A}_{m+1}^I = \begin{pmatrix} \mathbf{A}_m^I & \mathbf{0}_{m-2} \\ \mathbf{c}_m^{\mathrm{T}} & -\mathbf{1} \end{pmatrix} \tag{33}$$

with the initial condition $\mathbf{A}_3^I = (-1)$ corresponding to the initial element $\psi_0^I = (\boldsymbol{\psi}_N^I)_0 = \psi_{1,1,0}$ will be valid for the informative-triphase matrix. For other elements of $\psi_{m-2}^I = (\boldsymbol{\psi}_N^I)_{m-2} = \psi_{n_1,n_2,n_1+n_2-m}$, where $3 \le m \le N-1$, the choice of one of the triphases of the cut $n_4 = m$ as an informative triphase, i.e., the choice of the row $\mathbf{c}_m^{\mathrm{T}}$ from the matrix $\tilde{\mathbf{C}}_m$ of size $[m^2/4] \times (m-2)$ without the columns with numbers 0 and 1, can be arbitrary. This means that in such a way one can construct $\prod_{m=3}^{N-1}[m^2/4]$ different variants of triphase modulation.

Using conversion (33) in Eq. (32), $(\mathbf{A}_{m+1}^I)^{-1} = \begin{pmatrix} (\mathbf{A}_m^I)^{-1} & \mathbf{0}_{m-2} \\ \mathbf{c}_m^{\mathrm{T}}\,(\mathbf{A}_m^I)^{-1} & -1 \end{pmatrix}$, we obtain the formula

$$\boldsymbol{\theta}_{m+1} = \begin{pmatrix} \boldsymbol{\theta}_m \\ \theta_m \end{pmatrix} = \begin{pmatrix} (\mathbf{A}_m^I)^{-1} & \mathbf{0}_{m-2} \\ \mathbf{c}_m^{\mathrm{T}}\,(\mathbf{A}_m^I)^{-1} & -1 \end{pmatrix} \left\{ \begin{pmatrix} \boldsymbol{\psi}_m^I \\ \psi_{m-2}^I \end{pmatrix} - \begin{pmatrix} \mathbf{a}_0^I \\ a_{m,0}^I \end{pmatrix} \theta_0 - \begin{pmatrix} \mathbf{a}_1^I \\ a_{m,1}^I \end{pmatrix} \theta_1 \right\},$$

from which we find a recurrent algorithm for calculation of the subcarrier phases

$$\theta_2 = 2\theta_1 - \theta_0 - \psi_0^I; \qquad \theta_m = \mathbf{c}_m^{\mathrm{T}}\boldsymbol{\theta}_m - \psi_{m-2}^I + a_{m,0}^I\theta_0 + a_{m,1}^I\theta_1, \qquad 3 \le m \le N-1. \tag{34}$$

To decrease the complexity of the synthesis and reception of a signal, it is expedient to choose informative triphases in a unified way from cut to cut by a certain law. In the simplest case where the relation between frequency triples of the subcarriers forming different trispectrum samples of the basis is linear, these samples will lie on a certain straight line in 3D frequency space. For the considered recurrent method of the basis formation, this line should pass through some sample of each cut $n_4 = m$. The cuts $n_4 = 2$ and $n_4 = 3$ contain one and two samples, respectively, through which two lines can be drawn. The basis samples on the first line ($n_2 = 1$ and $n_3 = 0$) correspond to one of the possible forms of optimal triphase modulation with common subcarriers (see the text below) and the second line $n_1 = n_2 = n_3 + 1$, to AD-TDPSK-2.

We note that AD-TDPSK-2, as well as FDPSK-2-$(1,1)$, is a unique element of the FDPSK-2 class from [9], which has the complete information capacity of triphase modulation. Indeed, FDPSK-2-(m_1, m_2), which is a separate element of this class, contains $N - m_1 - m_2$ information phases $\theta_n^{(2.(m_1,m_2))} = -\psi_{n-m_1,n-m_2,n-m_1-m_2}$, which form (with accuracy up to insignificant inversion) a set of triphases corresponding to the trispectrum samples on a section of the line ($n_1 - n_2 = m_2 - m_1$ and $n_2 - n_3 = m_1$). Since this section begins with the cut $n_4 = m_1 + m_2$, for the conversion of this form of FDPSK-2 into one of the variants of triphase modulation with the recurrent basis formation in the direction of frequency doubling, one sample from each cut $n_4 = n$ for $2 \le n < m_1 + m_2$ should be added as the lacking element of the basis. For example, to extend the set of information phases for SD-TDPSK-2 (i.e., FDPSK-2-$(1,2)$) to the basis, a sample with the triphase $\psi_{1,1,0}$ should be added to it.

It is clear that the considered class of triphase modulation variants is not unique. In particular, the class of triphase modulation variants with the recurrent method of basis formation in the direction of

TABLE 1. The set L_{ij}, its power l_{ij}, and the domain of definition D_{ij} over parameters (p,q) and sets $K_{ij}^{(m)} (m \notin \{i,j\})$.

1	$L_{32} = \{\mathbf{n} \mid n_3 = p, n_2 = p+q, n_1 \in K_{32}^{(1)}\}; \quad l_{32} = \max(0, N-p-2q);$ $D_{32} = \{(p,q) \mid 0 \le p \le N-3, 1 \le q \le [(N-p-1)/2]\};$ $K_{32}^{(1)} = \{n \mid p+q \le n \le N-1-q\}; \quad K_{32}^{(4)} = \{n \mid p+2q \le n \le N-1\}.$
2	$L_{31} = \{\mathbf{n} \mid n_3 = p, n_1 = p+q, n_2 \in K_{31}^{(2)}\}; \quad l_{31} = \min(q, N-p-q)-1;$ $D_{31} = \{(p,q) \mid 0 \le p \le N-4, 2 \le q \le N-2-p\};$ $K_{31}^{(2)} = \{n \mid p+1 \le n \le \min(p+q, N-q)-1\};$ $K_{31}^{(4)} = \{n \mid p+q+1 \le n \le \min(p+2q, N)-1\}.$
3	$L_{34} = \{\mathbf{n} \mid n_3 = p, n_2 \in K_{34}^{(2)}, n_1 = 2p+q-n_2\}; \quad l_{34} = [q/2];$ $D_{34} = \{(p,q) \mid 0 \le p \le N-3, 2 \le q \le N-1-p\};$ $K_{34}^{(2)} = \{n \mid p+1 \le n \le p+[q/2]\}; \quad K_{34}^{(1)} = \{n \mid p+[(q+1)/2] \le n \le p+q-1\}.$
4	$L_{21} = \{\mathbf{n} \mid n_2 = p, n_1 = p+q, n_3 \in K_{21}^{(3)}\}; \quad l_{21} = \min(p, N-1-p-q);$ $D_{21} = \{(p,q) \mid 1 \le p \le N-3, 1 \le q \le N-2-p\};$ $K_{21}^{(3)} = \{n \mid \max(2p+q+1-N, 0) \le n \le p-1\};$ $K_{21}^{(4)} = \{n \mid p+q+1 \le n \le \min(2p+q, N-1)\}.$
5	$L_{24} = \{\mathbf{n} \mid n_2 = p, n_1 \in K_{24}^{(1)}, n_3 = n_1 - q\}; \quad l_{24} = \min(q-1, p);$ $D_{24} = \{(p,q) \mid 1 \le p \le N-3, 2 \le q \le N-1-p\};$ $K_{24}^{(3)} = \{n \mid \max(p-q+1, 0) \le n \le p-1\}; \quad K_{24}^{(1)} = \{n \mid \max(p+1, q) \le n \le p+q-1\}.$
6	$L_{14} = \{\mathbf{n} \mid n_1 = p, n_2 \in K_{14}^{(2)}, n_3 = n_2 - q\}; \quad l_{14} = \max(0, p-q+1);$ $D_{14} = \{(p,q) \mid 1 \le p \le N-2, 1 \le q \le \min(p, N-1-p)\};$ $K_{14}^{(3)} = \{n \mid 0 \le n \le p-q\}; \quad K_{14}^{(2)} = \{n \mid q \le n \le p\}.$

frequency doubling is of absolutely equal value. This method corresponds to ordering of trispectrum samples by the cuts $n_3 = m$ with the same value of the minimum frequencies for quadruples of the subcarriers forming the trispectrum samples in the direction from $m = N-3$ to $m = 0$. It can easily be shown that due to the invariance of the boundedness of frequencies (18) to the replacement $n_i \to N-1-n_i$ $(1 \le i \le 4)$, each variant of triphase modulation of this class corresponds one-to-one to a certain variant from the above-considered class, and the transition from one variant to another is performed by replacement of the coordinates of the informative trispectrum samples: $(n_1, n_2, n_3) \to (N-1-n_2, N-1-n_1, N-1-n_1-n_2+n_3)$.

3.3. Synthesis of the signals with optimal triphase modulation

We now pass to the synthesis of optimal forms of triphase modulation with the reference subcarriers used during the formation of all elements of the basis Ψ_N^I. Note that since the fixing of triples of subcarriers specifies a unique trispectrum sample, no variants of triphase modulation with three reference subcarriers exist. Hence, as was shown in Sec. 2.4, the variants of triphase modulation with two reference subcarriers (RTPSK) are optimal with respect to the signal-to-noise ratio. We now examine and classify these variants.

Obviously, the fixing of a pair of reference-subcarrier frequencies $(p, p+q)$ extracts in Ω_N the subsets of coordinates of trispectrum samples in $L_{ij} = L_{ij}(p,q) = \Omega_N \cap (n_i = p, n_j = p+q)$, which lie on sections of lines in 3D space. Table 1 presents six non-overlapping sets L_{ij} with nonzero power $l_{ij} = |L_{ij}|$ for the parameters $(p,q) \in D_{ij}$. Note that during the construction of these sets, the "diagonal" element $(p+q, p+q, p) = (n_3 = p, n_2 = p+q) \cap (n_3 = p, n_1 = p+q)$ is related to the set L_{32}, and the element $(p, p, p-q) = (n_1 = p, n_4 = p+q) \cap (n_2 = p, n_4 = p+q)$, to the set L_{14}.

TABLE 2. Recurrent algorithm of the phase formation for RTPSK-0-(p, q), where $(p, q) \in D_B$.

Step 1. Specify the values of the triphases from the set $\Psi_B = \Psi_{32} \cup \Psi_{31} \cup \Psi_{34} \cup \Psi_{24} \cup \Psi_{14}$ and initial phases $\{\theta_p, \theta_{p+q}\}$.
Step 2. If $q = 2$, then calculate, using Eq. (35), the existing phases from the set $\{\theta_{p+1}, \theta_{p-1}, \theta_{p+3}\}$.
Step 3. a) If $p \le N - 1 - 2q$, then recurrently calculate, using Eq. (36a), the phases $\{\theta_n \mid p + 2q \le n \le N - 1\}$, beginning with $n = p + 2q$. b) If $p \ge q$, then recurrently calculate, using Eq. (36b), the phases $\{\theta_n \mid 0 \le n \le p - q\}$, beginning with $n = p - q$.

For each set L_{ij}, it is expedient to determine four sets $K_{ij}^{(m)} = \{n_m \mid \mathbf{n} \in L_{ij}, n_4 = n_1 + n_2 - n_3\}$. Combining $K_{ij} = \bigcup_{m=1}^{4} K_{ij}^{(m)}$ forms a set of frequencies of all the subcarriers which participate in the formation of trispectrum samples with the triphases from the set $\Psi_{ij} = \{\psi_{\mathbf{n}} \mid \mathbf{n} \in L_{ij}\}$. To simplify records and facilitate the understanding of the structure of the relationships with triphases, we introduce designations of form $\psi_{\mathbf{n}}^{(ij)}$, where the upper index indicates that $\psi_{\mathbf{n}} \in \Psi_{ij}$. Note that for any set Ψ_{ij}, each triphase $\psi_{\mathbf{n}}^{(ij)}$ corresponds one-to-one to the frequencies of the sets $K_{ij}^{(m)}$ and $K_{ij}^{(l)}$ presented in Table 1, where $m \notin \{i, j\}$, $l \notin \{i, j\}$, and $K_{ij}^{(m)} \backslash (K_{ij}^{(m)} \cap K_{ij}^{(l)}) \neq \emptyset$. Hence, the set Ψ_{ij} comprises at least one triphase formed with the participation of the subcarrier phase which does not participate in the formation of other triphases and, therefore, the elements of the set Ψ_{ij} are linearly independent.

From the relationships $K_{24}^{(3)} \cap K_{34} = \emptyset$, $K_{31}^{(4)} \cap (K_{34} \cup K_{24}) = \emptyset$, $K_{32}^{(4)} \cap (K_{31} \cup K_{34} \cup K_{24}) = \emptyset$, and $K_{14}^{(3)} \cap (K_{32} \cup K_{31} \cup K_{34} \cup K_{24}) = \emptyset$ it follows that for each set Ψ_{ij} combined into $\Psi_B = \Psi_{32} \cup \Psi_{31} \cup \Psi_{34} \cup \Psi_{24} \cup \Psi_{14}$, there is an own set of subcarriers, whose phases participate in the formation of triphases only from the given set. Hence, the elements of of the combination Ψ_B are linearly independent. Since $K_{21}^{(3)} \subset (K_{24}^{(3)} \cup K_{14}^{(3)})$ and $K_{21}^{(4)} \subset (K_{32}^{(4)} \cup K_{31}^{(4)})$, the set Ψ_{21} is linearly dependent on the set Ψ_B.

Thus, the maximum number of linearly independent triphases with two common subcarriers is $|\Psi_B| = L_I - [(q-1)/2]$ and, therefore, the basis of L_I informative triphases with two common reference subcarriers is possible only if the parameters (p, q) from the set $D_B = \{(p, q) \mid q \in \{1, 2\}, 0 \le p \le N - 1 - q\}$ are specified.

Combining all variants of RTPSK with the bases $\Psi_N^I = \Psi_B$ for $(p, q) \in D_B$ forms the basis subclass of RTPSK, which we will denote RTPSK-0. Each of its elements is unambiguously specified by the parameters (p, q) and can therefore be denoted RTPSK-0-(p, q). A common algorithm for the formation of the subcarrier phases for RTPSK-0 is formulated in Table 2. Such an algorithm is easily deduced on the basis of the following considerations. For RTPSK, the choice of the phases θ_p and θ_{p+q} of the reference subcarriers as the initializing phases is natural. For $q = 1$, the sets Ψ_{31}, Ψ_{34}, and Ψ_{24} are empty, and for $q = 2$ they comprise the triphases $\psi_{p+2,p+1,p}^{(31)}$, $\psi_{p+1,p+1,p}^{(34)}$, and $\psi_{p+1,p,p-1}^{(24)}$. Using the latter, one can calculate the phases

$$\theta_{p+1} = (\theta_p + \theta_{p+2} + \psi_{p+1,p+1,p}^{(34)})/2; \qquad \theta_{p-1} = \theta_{p+1} + \theta_p - \theta_{p+2} - \psi_{p+1,p,p-1}^{(24)} (p \ge 1);$$
$$\theta_{p+3} = \theta_{p+2} + \theta_{p+1} - \theta_p - \psi_{p+2,p+1,p}^{(31)} (p \le N - 4). \tag{35}$$

The remaining phases can be found on the basis of the formulas

$$\theta_n = \theta_{n-q} + \theta_{p+q} - \theta_p - \psi_{n-q,p+q,p}^{(32)} \quad (n \in K_{32}^{(4)}), \tag{36a}$$

$$\theta_n = \theta_p + \theta_{n+q} - \theta_{p+q} - \psi_{p,n+q,n}^{(14)} \quad (n \in K_{14}^{(3)}), \tag{36b}$$

which recurrently express the phases of the subcarriers from the own sets $K_{32}^{(4)}$ and $K_{14}^{(3)}$ via the triphases

$\psi^{(32)}_{n-q,p+q,p}$ and $\psi^{(14)}_{p,n+q,n}$, respectively, when the recursion is directed from the subcarriers neighboring the reference ones to the boundary subcarriers. Note that the steps 3a and 3b in Table 2 can be performed in any sequence.

The remaining variants of RTPSK for specified $(p,q) \in D_B$ can be synthesized from RTPSK-0-(p,q) by replacement of a part of the triphases from the set Ψ_B by the triphases from Ψ_{21}. The rules of such a replacement can be formulated on the basis of the relationships

$$\psi^{(21)}_{p+q,p,n} = \psi^{(24)}_{n+q,p,n} + \psi^{(31)}_{p+q,2p-n,p} - \psi^{(34)}_{2p-n,n+q,p}, \qquad n \in K^{(3)}_{21} \cap K^{(3)}_{24}; \tag{37a}$$

$$\psi^{(21)}_{p+q,p,n} = \sum_{m=0}^{c-1} (\psi^{(14)}_{p,n+mq+q,n+mq} + \psi^{(32)}_{2p-n-mq,p+q,p}) + (1-\delta_{n+qc,p})\times$$

$$\times(\psi^{(24)}_{n+cq+q,p,n+cq} + \psi^{(31)}_{p+q,2p-n-cq,p} - \psi^{(34)}_{2p-n-cq,n+cq+q,p}), \qquad n \in K^{(3)}_{21} \cap K^{(3)}_{14}; \tag{37b}$$

where $c = [(p-n)/q]$, and $\delta_{nm} = \{1, n = m; 0, n \neq m\}$ is the Kronecker symbol. These relationships can be obtained by expressing the phases θ_n and θ_{2p+q-n}, which enter the combination $\psi^{(21)}_{p+q,p,n}$, via the triphases from Ψ_B. It should be mentioned that Eq. (37b) can most easily be deduced by recurrently using the relationships

$$\psi^{(21)}_{p+q,p,n} = \psi^{(14)}_{p,n+q,n} + \psi^{(32)}_{2p-n,p+q,p} + \psi^{(21)}_{p+q,p,n+q}, \qquad n \in K^{(3)}_{21} \cap K^{(3)}_{14}. \tag{38}$$

Among a large number of RTPSK variants, whose basis contains triphases from the set Ψ_{21}, we single out the class of variants with the bases formed from the set Ψ_B by replacing entirely its one or two subsets Ψ_{ij} by the set Ψ_{21}. Denote this class RTPSK-1.

According to Eq. (37a) for $q = 2$ and $1 \leq p \leq N-4$, there are three variants of replacement of one of the triphases $\psi^{(34)}_{p+1,p+1,p}$, $\psi^{(24)}_{p+1,p,p-1}$, or $\psi^{(31)}_{p+2,p+1,p}$ by $\psi^{(21)}_{p+2,p,p-1}$, which correspond to the following formulas for calculation of the phases $\{\theta_{p+1}, \theta_{p-1}, \theta_{p+3}\}$:

$$\theta_{p+1} = (\theta_p + \theta_{p+2} + \psi^{(31)}_{p+2,p+1,p} + \psi^{(24)}_{p+1,p,p-1} - \psi^{(21)}_{p+2,p,p-1})/2,$$

$$\theta_{p+3} = \theta_{p+2} + \theta_{p+1} - \theta_p - \psi^{(31)}_{p+2,p+1,p}, \qquad \theta_{p-1} = \theta_{p+2} + \theta_p - \theta_{p+3} - \psi^{(21)}_{p+2,p,p-1}; \tag{39a}$$

$$\theta_{p+1} = (\theta_p + \theta_{p+2} + \psi^{(34)}_{p+1,p+1,p})/2; \qquad \theta_{p+3} = \theta_{p+2} + \theta_{p+1} - \theta_p - \psi^{(31)}_{p+2,p+1,p};$$

$$\theta_{p-1} = \theta_{p+2} + \theta_p - \theta_{p+3} - \psi^{(21)}_{p+2,p,p-1}; \tag{39b}$$

$$\theta_{p+1} = (\theta_p + \theta_{p+2} + \psi^{(34)}_{p+1,p+1,p})/2; \qquad \theta_{p-1} = \theta_{p+1} + \theta_p - \theta_{p+2} - \psi^{(24)}_{p+1,p,p-1};$$

$$\theta_{p+3} = \theta_{p+2} + \theta_p - \theta_{p-1} - \psi^{(21)}_{p+2,p,p-1}. \tag{39c}$$

From Eq. (38) it follows that the replacement $\Psi_{14} \to \{\psi^{(21)}_{p+q,p,n} \mid n \in K^{(3)}_{21} \cap K^{(3)}_{14}\}$ is possible under the condition $K^{(3)}_{21} \cap K^{(3)}_{14} = K^{(3)}_{14}$, which is satisfied if $1 \leq p \leq (N-1-q)/2$. Expressing $\psi^{(14)}_{p,n+q,n}$ via $\psi^{(21)}_{p+q,p,n}$ and the phases determining other triphases on the right-hand side of Eq. (38) with the help of Eq. (38) and substituting the result into Eq. (36b), we obtain the following rule of recurrent calculation of the phases of low-frequency subcarriers:

$$\theta_n = \theta_p + \theta_{p+q} - \theta_{2p+q-n} - \psi^{(21)}_{p+q,p,n} \qquad (n \in K^{(3)}_{14}). \tag{40}$$

Note that for calculations, the phases $\{\theta_{2p+q-n}\}$ of high-frequency subcarriers have already been determined on the basis of Eq. (36a). In a similar way, for the alternative replacement $\Psi_{32} \to\to \{\psi^{(21)}_{p+q,p,2p+q-n} \mid n \in$

TABLE 3. Recurrent algorithm of the phase formation for RTPSK-1-(p,q,m), where $m \in \{1,2,3\}$, $q \in \{1,2\}$, and $1 \le p \le N-2-q$.

Step 1. Specify the values for the triphases from the basis $\Psi_{21} \cup \Psi_L \cup \Psi_S$ and initial phases $\{\theta_p, \theta_{p+q}\}$, where $\Psi_L = \begin{cases} \Psi_{32}, & p \le (N-1-q)/2; \\ \Psi_{14}, & p > (N-1-q)/2; \end{cases}$ $\Psi_S = \begin{cases} \Psi_{24} \cup \Psi_{31}, & q = 2 \wedge m = 1; \\ \Psi_{31} \cup \Psi_{34}, & q = 2 \wedge m = 2; \\ \Psi_{34} \cup \Psi_{24}, & q = 2 \wedge m = 3; \\ \emptyset, & q = 1. \end{cases}$
Step 2. If $q = 2$, then calculate the phases from the set $\{\theta_{p+1}, \theta_{p-1}, \theta_{p+3}\}$ using Eq. (39a) for $m = 1$, Eq. (39b) for $m = 2$, and Eq. (39c) for $m = 3$.
Step 3. a) If $p \le (N-1-q)/2$, then recurrently calculate, using Eq. (36a), the phases $\{\theta_n \mid p+2q \le n \le N-1\}$, beginning with $n = p+2q$. Otherwise, recurrently calculate, using Eq. (36b), the phases $\{\theta_n \mid 0 \le n \le p-q\}$, beginning with $n = p-q$.
b) If $p \le (N-1-q)/2$, then recurrently calculate, using Eq. (40), the phases $\{\theta_n \mid 0 \le n \le p-q\}$, beginning with $n = p-q$. Otherwise, recurrently calculate, using Eq. (41), the phases $\{\theta_n \mid p+2q \le n \le N-1\}$, beginning with $n = p+2q$.

$K_{21}^{(4)} \cap K_{32}^{(4)}\}$ the condition $K_{21}^{(4)} \cap K_{32}^{(4)} = K_{32}^{(4)}$, which is valid if $(N-1-q)/2 \le p \le N-2-q$, should be fulfilled. In this case, using Eqs. (36a) and (38), one can find the relationship for recurrent calculation of the phases of high-frequency subcarriers

$$\theta_n = \theta_p + \theta_{n+q} - \theta_{2p+q-n} - \psi^{(21)}_{p+q,p,2p+q-n} \qquad (n \in K_{32}^{(4)}) \tag{41}$$

after the calculation of the phases $\{\theta_{2p+q-n}\}$ of low-frequency subcarriers by using Eq. (36b).

To simplify the parameterization of the elements of the RTPSK-1 class combining the above-considered methods of replacement of the sets Ψ_{ij} by the set Ψ_{21}, it is convenient to neglect, in the case $p = (N-1-q)/2$, the possibility of both replacements $\Psi_{14} \to \{\psi^{(21)}_{p+q,p,n} \mid n \in K_{14}^{(3)}\}$ and $\Psi_{32} \to \{\psi^{(21)}_{p+q,p,2p+q-n} \mid n \in K_{32}^{(4)}\}$ in favor of, e.g., the latter one. With allowance for this constraint, each element of this class can be specified by a triple of parameters (p,q,m) and denoted RTPSK-1-(p,q,m), where m is a parameter that is significant only for $q = 2$, since it is the number of one of the variants corresponding to Eqs. (39). Table 3 presents the rule of choice of the basis as a function of the parameters (p,q,m) and formulates a common algorithm for the formation of phases of the subcarriers for RTPSK-1. Note that, unlike RTPSK-0, step 3b in Table 3 should be performed after step 3a.

4. STUDY OF THE NOISE IMMUNITY

The study and comparative analysis of the noise immunity of incoherent reception of OFDM signals with DPSK-1, DPSK-2, and RTPSK were performed by computer simulation in the low-frequency single-path model of a channel with indefinite phase, time delay, Doppler frequency shift, and additive Gaussian white noise. The simulation was done at the sampling frequency of complex signals $F_s = 132/T$ and the relative length of the time guard $T_G/T = 0.1$. The number of information elements $L_I = 128$ transmitted by an OFDM symbol with different forms of the time-domain differential phase-shift keying was equal to the number of subcarriers for TDPSK-1, and the position of the information elements was $M = 4$. As SD-TDPSK-2, we examined the variant with full information capacity provided by an additional sample with the triphase $\psi_{1,1,0}$, and as RTPSK, the variant RTPSK-0-(0,1). During the simulation, the bit-error probability P_b was estimated as a function of relative energy expenses for the transmission of one bit of message E_b/N_0. The results are presented in Figs. 4 and 5.

The noise immunity curves for AD-TDPSK-2, SD-TDPSK-2, and RTPSK in Fig. 4 confirm that among these variants of triphase modulation, RTPSK has the best noise immunity. In a channel with

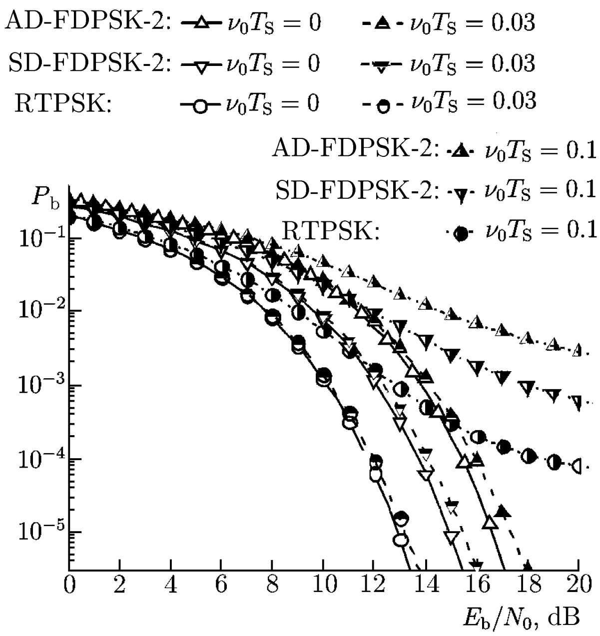

Fig. 4. Noise immunity of AD-TDPSK-2, SD-TDPSK-2, and RTPSK in the channels with additive Gaussian white noise and Doppler frequency shift ν_0, $T_G/T = 0.1$, and $\tau_0/T = 0$.

additive Gaussian white noise in the absence of the frequency shift, the energy gain of RTPSK amounts to no less than 2 dB with repsect to SD-TDPSK-2 and more than 3.5 dB with respect to AD-TDPSK-2. The advantage of RTPSK increases if the frequency shift is present.

The noise immunity diagrams for TDPSK-1, FDPSK-1, and RTPSK are compared in Fig. 5. Figure 5*a* demonstrates high sensitivity of FDPSK-1 to a time delay, and Fig. 5*b*, strong instability of TDPSK-1 to a frequency delay. The noise immunity of FDPSK-1 (TDPSK-1) is significantly impaired even in the case of small time (frequency) shifts. The expected invariance of TDPSK and RTPSK to time delays (see Figs. 5*a* and 5*c*) is realized due to the effective prevention of intersymbol interference when the length of the time guard exceeds the possible detays. Violation of the invariance of FDPSK and RTPSK due to the interchannel interference (see Figs. 5*b* and 5*d*) becomes significant for $\nu_0 T_S > 0.03$, and for the greater frequency offsets ($\nu_0 T_S = 0.1$) this leads to a dramatic decrease in the noise immunity and the appearance of an irreducible error probability for the greater values of E_b/N_0. Figures 5*c* and 5*d* demonstrate that even for fairly small simultaneous time and frequency offsets ($\tau_0 = 0.03T$ and $\nu_0 = 0.03/T_S$), RTPSK exceeds both TDPSK-1 and FDPSK-1 in noise immunity by more than 1 dB for $P_b \sim 10^{-4}$. While for a fixed offset ν_0 and a further increase in the time offset the advantage of RTPSK increases only with respect to FDPSK-1, it becomes greater with respect to both forms of DPSK-1 for a fixed delay τ_0 and an increase in the frequency shift.

5. CONCLUSIONS

It has been established that the triphase modulation forms the most general class of FDPSK-2, describing all possible methods of choosing the second-order phase differences of the subcarriers of an OFDM symbol. On the basis of the mathematical tool of trispectrum analysis, we explored the principles and obtained algorithms for the synthesis of triphase-modulation OFDM signals. It is shown that the subclass of triphase modulation variants with two reference subcarriers (RTPSK), which are used during the formation of all information phase differences of the second order in the channels with additive Gaussian white noise and with time and frequency scattering in the case of optimal redistribution of the signal energy in favor of the reference subcarriers ensures the best noise immunity among the different forms of triphase modulation. An asymptotically optimal value of the relative power of the reference components by the criterion of maximizing of the signal-to-noise ratio at the output of a symbol-by-symbol incoherent demodulator has been obtained. Different variants of RTPSK have been explored and classified, and recurrent algorithms for the synthesis of OFDM signals with RTPSK have been deduced. By means of simulation, it is demonstrated that for fairly large simultaneous time and frequency offsets, RTPSK exceeds both TDPSK-1 and FDPSK-1 in noise immunity, and this advantage increases with increasing offsets.

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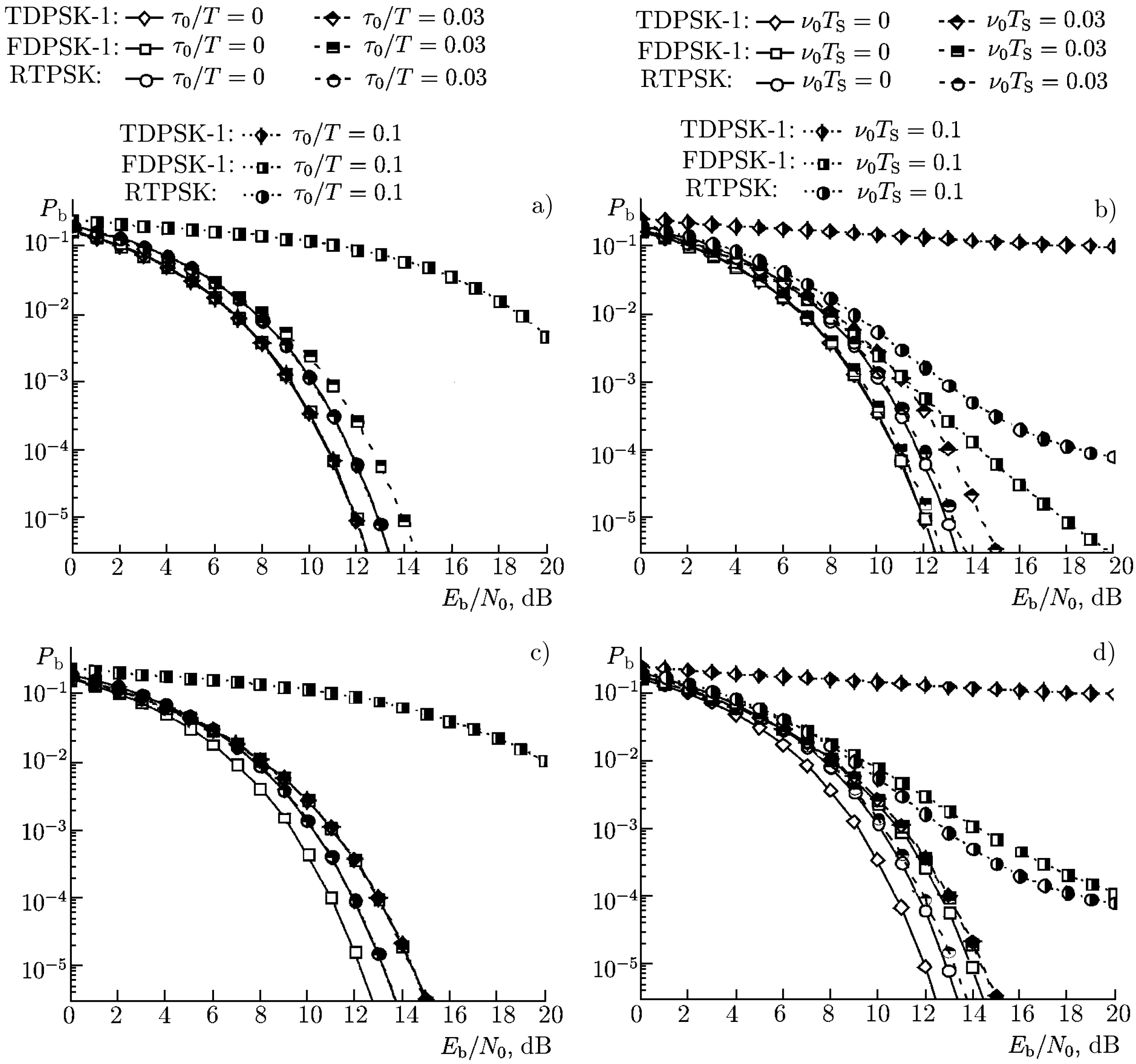

Fig. 5. Noise immunity of TDPSK-1, FDPSK-1, and RTPSK in the channels with additive Gaussian white noise and offsets in frequency ν_0 and time τ_0 for $T_G/T = 0.1$ under the following conditions: $\nu_0 T_S = 0$ (*a*), $T_G/T = 0$ (*b*), $\tau_0/T_S = 0.03$ (*c*), and $\tau_0/T = 0.03$ (*d*). Designations of the curves for fixed values of τ_0/T correspond to the panels (*a*) and (*c*), and for fixed values of $\nu_0 T_S$, to the panels (*b*) and (*d*).

REFERENCES

1. R. Van Nee and R. Prasad, *OFDM for Wireless Multimedia Communication*, Artech House Publishers, Boston, London (2000).

2. A. R. S., B. R. Saltzberg, and M. Ergen, *Multi-Carrier Digital Communications*, Springer Science, Boston (2004).

3. N. T. Petrovich, *Transmission of Discrete Information in Phase-Manipulation Channels* [in Russian], Sovetskoe Radio, Moscow (1965).

4. A. M. Zaezdny, Yu. B. Okunev, and L. M. Rakhovich, *Differential Phase-Shift Keying and Its Use for the Transmission of Discrete Information* [in Russian], Svyaz', Moscow (1967).

5. Yu. B. Okunev, *Digital Transmission of Information by Phase-Modulated Signals* [in Russian], Radio i Svyaz', Moscow (1991).

6. D. Divsalar and M. K. Simon, *IEEE Trans. Commun.*, **40**, No. 2, 278 (1992).

7. D. K. Van Alphen and W. C. Lindsey, *IEEE Trans. Commun.*, **42**, Nos. 2–4, 440 (1994).

8. F. Gini and G. B. Giannakis, *IEEE Trans. Signal Processing*, **46**, No. 11, 2967 (1998).

9. M. J. Dehghani, *EURASIP J. Wireless Commun. Networking*, **2006**, No. 2, 6 (2006).

10. M. Lott, in: *Proc. IEEE 49th Veh. Technol. Conf. Houston, 1999*, Vol. 2, p. 877.

11. C. L. Nikias, A. P. Petropulu, *Higher-Order Spectral Analysis: A Nonlinear Signal Processing Framework*, Prentice-Hall, New Jersey (1993).

12. A. K. Novikov, *Polyspectral Analysis* [in Russian], A. N. Krylov Central Research Institute, St. Petersburg (2002).

13. G. N. Bochkov and K. V. Gorokhov, "Polyspectral analysis and synthesis of signals," a manual for the advanced training program "New approaches to the problems of generation, processing, transmission, storage, and protection of information and their applications" [in Russian], Nizhny Novgorod (2007). (http://www.unn.ru/pages/e-library/aids/2007/26.pdf).

14. A. Swami, G. B. Giannakis, and G. Zhou, *Signal Processing*, **60**, 65 (1997).

15. G. N. Bochkov, K. V. Gorokhov, and A. A. Dubkov, *Radiophys. Quantum Electron.*, **48**, No. 2, 142 (2005).

16. G. N. Bochkov and K. V. Gorokhov, *Radiophys. Quantum Electron.*, **40**, No. 11, 935 (1997).

17. G. N. Bochkov, K. V. Gorokhov, and A. V. Kolobkov, *Vestnik of N. I. Lobachevsky State University*, No. 2, 62 (2009).

18. P. J. Schreier and L. L. Scharf, *Signal Processing*, **86**, 3321 (2006).