Acta Appl Math (2009) 108: 315–351
DOI 10.1007/s10440-008-9314-3

Vertex-Compressed Subalgebras of a Graph von Neumann Algebra

Ilwoo Cho

Received: 22 August 2008 / Accepted: 29 August 2008 / Published online: 10 October 2008

Abstract In Cho (Acta Appl. Math. 95:95–134, 2007 and Complex Anal. Oper. Theory 1:367–398, 2007), we introduced Graph von Neumann Algebras which are the (groupoid) crossed product algebras of von Neumann algebras and graph groupoids via graph-representations, which are groupoid actions. In Cho (Acta Appl. Math. 95:95–134, 2007), we showed that such crossed product algebras have the amalgamated reduced free probabilistic properties, where the reduction is totally depending on given directed graphs. Moreover, in Cho (Complex Anal. Oper. Theory 1:367–398, 2007), we characterize each amalgamated free blocks of graph von Neumann algebras: we showed that they are characterized by the well-known von Neumann algebras: Classical group crossed product algebras and (operator-valued) matricial algebras. This shows that we can provide a nicer way to investigate such groupoid crossed product algebras, since we only need to concentrate on studying graph groupoids and characterized algebras. How about the compressed subalgebras of them? i.e., how about the inner (cornered) structures of a graph von Neumann algebra? In this paper, we will provides the answer of this question. Consequently, we show that vertex-compressed subalgebras of a graph von Neumann algebra are characterized by other graph von Neumann algebras. This gives the full characterization of the vertex-compressed subalgebras of a graph von Neumann algebra, by other graph von Neumann algebras.

Keywords Graph groupoids · Graph von Neumann algebras · Amalgamated diagonal graph · W^*-probability spaces · Vertex-compressions · Vertex-compressed subalgebras

Mathematics Subject Classification (2000) 05C90 · 20E06 · 20L05 · 46L54 · 47L65

In [9], we introduced a graph von Neumann algebra $\mathbb{M}_G = M \times_\alpha \mathbb{G}$, over a fixed von Neumann algebra M, where $\mathbb{G}$ is a graph groupoid induced by a countable directed graph G and α is a graph-representation of G, which is nothing but a groupoid action of $\mathbb{G}$. We obtained the amalgamated reduced free probabilistic properties of $\mathbb{M}_G$, where the reduction is

I. Cho (✉)
Dep. of Math, Saint Ambrose University, 421 AH, 518 W. Locust St., Davenport, IA 52803, USA
e-mail: chowoo@sau.edu

totally depending on the given graph G (or the binary operation on $\mathbb{G}$). Such amalgamated free probabilistic data of a graph von Neumann algebra $\mathbb{M}_G$ makes us deal with $\mathbb{M}_G$ as a reduced free product of certain amalgamated free blocks. Since the free blocks are characterized in [10], we have all tools to study such crossed product algebras, in terms of Free Probability.

In this paper, we will consider the compressed W^*-subalgebras $p\mathbb{M}_G p$ of a graph von Neumann algebra $\mathbb{M}_G$, where p is a projection in $\mathbb{M}_G$. i.e., we will observe the inner cornered structures of $\mathbb{M}_G$. In particular, we are interested in the amalgamated reduced free probabilistic information of $p\mathbb{M}_G p$. In particular, we will consider the so-called vertex-compressions of a graph von Neumann algebra $\mathbb{M}_G = M \times_\alpha \mathbb{G}$. Let $V = \{v_1, \ldots, v_N\}$ be a subset of the vertex set of the given graph G and $P_V = \sum_{j=1}^{N} L_{v_j}$, the corresponding projection in $\mathbb{M}_G$. Then we can construct the V-compressed algebra $P_V \mathbb{M}_G P_V$. We provides the characterization of those vertex-compressed algebras of $\mathbb{M}_G$: the V-compressed algebra $P_V \mathbb{M}_G P_V$ is $*$-isomorphic to a graph von Neumann algebra $\mathbb{M}_F = M \times_{\alpha_F} \mathbb{F}$, where F is a directed graph with its graph groupoid $\mathbb{F}$ having its vertex set V.

1 Introduction

Recently, the countable directed graphs have been studied in Pure and Applied Mathematics, because not only that they are involved by a certain noncommutative structures but also that they visualize such structures. Furthermore, the visualization has nice matricial expressions, (sometimes, the operator-valued matricial expressions depending on) adjacency matrices, or incidence matrices or graph-matrices in the sense of [12]. In particular, the certain partial isometries in an operator algebra can be expressed and visualized by directed graphs (see [9, 10, 13] and [12]). Especially, starting with a countable directed graph, we assign partial isometries to the directed edges and projections (on the initial or final spaces of the partial isometries) to vertices. Moreover, we read off the structural properties of the graph from our operator algebraic tools. Since the adjoints of partial isometries are again partial isometries, our approach is both natural and useful for directed graphs. It leads to what we call graph groupoids. We study their representations as algebras of operators on a Hilbert space. A key tool in our operator algebras is the notion of (Groupoid) Crossed Product (see [9] and [10]). In particular, in [9], we showed that the representations of directed graphs can be formulated naturally in terms of such groupoid crossed products, extending the more familiar context of classical (group) crossed products. Also, we could show that each graph von Neumann algebra is $*$-isomorphic to an amalgamated reduced free product of certain amalgamated free blocks determined by the directed edges. In [10], we could characterize the amalgamated free blocks of a graph von Neumann algebra. This shows that we can provide a nicer way to investigate such groupoid crossed product algebras, since we only need to concentrate on studying graph groupoids and characterized algebras. Also, in [12], the author and Jorgensen use different representations of graph groupoids to characterize C^*-subalgebras generated by certain partial isometries in a fixed operator algebra $B(H)$.

Iterated function systems (abbreviated IFS, see [21]) are typically built from infinite graphs, as well as limits of systems of finite graphs. This serves in two ways as a link between analysis on discrete systems on one side and operator algebras on the other. Recall that IFSs generate fractal images arising in numerous applications, computer graphics, and theoretical computer science: For example, some IFS-fractals may be built as limits of iterated backwards trajectories of a dynamical system associated to a fixed endomorphism $T : X \to X$ in a metric space X. The generation of the fractals is via recursive procedures applied to branches of a choice of inverse mappings for T. This may often be done in a way that makes the inverse branches contractive: As attractors, we then get limit fractal-sets and fractal measures μ. So in this way the Hilbert space $L^2(\mu)$ arises as a limit of Hilbert space; starting with a graph and passing to the limit. On the discrete side, the graph G has vertices in the vertex set $V(G)$ of G and edges in the edge set $E(G)$ of G. The first approach is to model IFSs with infinite vertex set $V(G)$, and associated Hilbert spaces of functions on $V(G)$. In the second approach (e.g., [20]) one starts with an IFS, and then there is an associated graph G with the vertex set $V(G)$ set a singleton, but instead with edges made up of an infinite set $E(G)$ of self-loops. Thus there are two interesting and interdisciplinary links to operator algebras in symmetric Hilbert spaces, C^*-algebras and von Neumann algebras. It is via operators in these Hilbert spaces which are built on infinite discrete spaces.

A graph is a set of objects called vertices (or points or nodes) connected by links called edges (or lines). In a directed graph, the two directions are counted as being distinct directed edges (or arcs). A graph is depicted in a diagrammatic form as a set of dots (for vertices), jointed by curves (for edges). Similarly, a directed graph is depicted in a diagrammatic form as a set of dots jointed by arrowed curves, where the arrows point the direction of the directed edges.

Throughout this paper, let G be a countable directed graph and let $G^{\wedge} = G \cup G^{-1}$ be the shadowed graph, where G^{-1} is the shadow of G, which is defined to be the opposite directed graph of G. Then we have a free semigroupoid $\mathbb{F}^+(G^{\wedge})$ of the shadowed graph $G^{\wedge}$, as a set of all (non-reduced) words in $E(G^{\wedge})$ union with $V(G^{\wedge})$. By defining the reduction on $\mathbb{F}^+(G^{\wedge})$, we construct the graph groupoid $\mathbb{G}$. i.e., the graph groupoid $\mathbb{G}$ is a set of all reduced words in $E(G^{\wedge})$, where $E(G^{\wedge})$ is the edge set of $G^{\wedge}$. Indeed, this graph groupoid $\mathbb{G}$ is a categorial groupoid with its base $V(G^{\wedge})$ and its morphisms $FP_r(G^{\wedge})$, where $V(G^{\wedge})$ is the vertex set of $G^{\wedge}$ and $FP_r(G^{\wedge})$ is the reduced-finite-path set defined by

$$FP_r(G^{\wedge}) \overset{def}{=} \mathbb{G} \setminus (V(G^{\wedge}) \cup \{\emptyset\}),$$

where $\emptyset$ is the empty word of $\mathbb{G}$ (see [14]). For an arbitrary fixed von Neumann algebra M in the operator algebra $B(K)$ of all bounded operators on a Hilbert space K, we define the crossed product algebra $\mathbb{M}_G = M \times_\alpha \mathbb{G}$ of M and $\mathbb{G}$ via a graph-representation $\alpha : \mathbb{G} \to B(K \otimes H_G)$, where $B(K \otimes H_G)$ is the operator algebra consisting of all bounded operators on the Hilbert space $K \otimes H_G$, where H_G is the Hilbert space with its Hilbert basis $\{\xi_w : w \in FP_r(G^{\wedge})\}$. The graph-representation α determines bounded operators α_w on $K \otimes H_G$ satisfying that

$$\alpha_w(m) L_w L_w^* = L_w^* m L_w, \quad \text{for all } w \in FP_r(G^{\wedge}),$$

and

$$\alpha_v(m) = m, \quad \text{for all } v \in V(G^{\wedge}),$$

for all $m \in M$, where L_w is the multiplication operator on H_G, which is a partial isometry, with its symbol ξ_w and where L_w^* is the corresponding adjoint of L_w, such that $L_w^* = L_{w^{-1}}$, for all $w \in \mathbb{G}$. Note that L_w and L_w^*, for $w \in \mathbb{G}$, are regarded as $1 \otimes L_w$ and $1 \otimes L_w^*$ in $\mathbb{M}_G$. By definition, we always assume that an operator $\alpha_w(m)$ is contained in M, for $m \in M$ and $w \in \mathbb{G}$. The crossed product algebra $\mathbb{M}_G$ is said to be a graph von Neumann algebra induced by G over M.

1.1 Preliminaries and Basic Definitions

Free Probability has been developed since mid 1980s. In this paper, we will follow Speicher's combinatorial approach (see [4, 18] and [19]). Let $B \subset A$ be von Neumann algebras with $1_B = 1_A$ and assume that there is a conditional expectation $E_B : A \to B$ satisfying that (i) E_B is a $\mathbb{C}$-linear map, (ii) $E_B(b) = b$, for all $b \in B$, (iii) $E_B(b_1\ a\ b_2) = b_1 E_B(a) b_2$, for all $b_1, b_2 \in B$ and $a \in A$, (iv) E_B is continuous under the given topology and (v) $E_B(a^*) = E_B(a)^*$, for all $a \in A$. The algebraic pair (A, E_B) is said to be a B-valued W^*-probability space (over B). Every operator in (A, E_B) is called a B-valued random variable. Any arbitrary chosen B-valued random variables have their free distributional data called B-valued $*$-moments and B-valued $*$-cumulants. Suppose $a_1, \ldots, a_s$ are B-valued random variables in (A, E_B), where $s \in \mathbb{N}$. The $(i_1, \ldots, i_n)$-th joint B-valued $*$-moments of $a_1, \ldots, a_s$ are defined by

$$E_B\big(d_1 a_{i_1}^{r_{i_1}} d_2\, a_{i_2}^{r_{i_2}} \ldots d_n\ a_{i_n}^{r_{i_n}}\big),$$

and the $(j_1, \ldots j_k)$-th joint B-valued $*$-cumulants of $a_1, \ldots, a_s$ are defined by

$$k_k^B\big(d_1\ a_{j_1}^{r_{j_1}}, d_2 a_{j_2}^{r_{j_2}}, \ldots, d_n a_{j_k}^{r_{j_k}}\big) = \sum_{\pi \in NC(k)} E_{B:\pi}\big(d_1 a_{j_1}^{r_{j_1}}, \ldots, d_n a_{j_k}^{r_{j_k}}\big)\mu(\pi, 1_n),$$

for all $(i_1, \ldots, i_n) \in \{1, \ldots, s\}^n$ and for all $(j_1, \ldots, j_k) \in \{1, \ldots, s\}^k$, where $r_{i_1}, \ldots, r_{i_n}$, $r_{j_1}, \ldots, r_{j_k} \in \{1, *\}$, and $d_j \in B$ are arbitrary, and $NC(n)$ is the lattice of all noncrossing partitions with its minimal element $0_n = \{(1), (2), \ldots, (n)\}$ and its maximal element $1_n = \{(1, 2, \ldots, n)\}$, and where μ is the Moebius functional in the incidence algebra $\mathcal{I}$, satisfying that $\sum_{\pi \in NC(n)} \mu(\pi, 1_n) = 0$. Here $E_{B:\pi}(\ldots)$ is the partition-depending B-valued moment. For example, if $\pi = \{(1, 4), (2, 3), (5)\}$ in $NC(5)$, then

$$E_{B:\pi}(a_1, a_2, a_3, a_4, a_5) = E_B(a_1 E_B(a_2 a_3) a_4) E_B(a_5).$$

Recall that the set $NC(n)$ of all noncrossing partitions over $\{1, \ldots, n\}$ is a lattice under the following ordering:

$$\pi \leq \theta \quad \overset{def}{\Longleftrightarrow} \quad \forall V \in \pi, \exists B \in \theta \quad \text{such that} \quad V \subseteq B,$$

where "$\subseteq$" is the usual set-inclusion.

The B-valued freeness on (A, E_B) is characterized by the B-valued cumulants (see [18]). Let A_1 and A_2 be W^*-subalgebras of A having their common subalgebra B. We say that A_1 and A_2 are free over B in (A, E_B) if all mixed B-valued $*$-cumulants of A_1 and A_2 vanish. The subsets X_1 and X_2 of A are said to be free over B in (A, E_B) if the von Neumann algebras $vN(X_1, B)$ and $vN(X_2, B)$ are free over B in (A, E_B), where $vN(S_1, S_2)$ means the von Neumann algebra generated by sets S_1 and S_2. Similarly, we say that the B-valued random variables x and y are free over B in (A, E_B) if the subsets $\{x\}$ and $\{y\}$ are free over B. Let A_1 and A_2 be W^*-subalgebras of A containing their common W^*-subalgebra B. Assume that they are free over B in (A, E_B). Then the W^*-subalgebra $vN(A_1, A_2)$ generated by A_1 and A_2 is denoted by $A_1 *_B A_2$. Suppose A is given as above and let $A = vN(\bigcup_{i \in I} A_i)$, where $\{A_i \supseteq B : i \in I\}$ is a family of W^*-subalgebras of A (over B). If A_i's are free over B from each other, for $i \in I$, then A is denoted by $*_{B\, i \in I} A_i$, and it is called a B-valued free (or B-free) product algebra of A_i's.

Let G be a countable directed graph with its vertex set $V(G)$ and its edge set $E(G)$. Denote the set of all finite paths of G by $FP(G)$. Clearly, the edge set $E(G)$ is contained in $FP(G)$. Let w be a finite path in $FP(G)$. Then it is represented as a word in $E(G)$. If $e_1, \ldots, e_n$ are connected directed edges in $E(G)$, for $n \in \mathbb{N}$, then we have a finite path $w = e_1 \ldots e_n$ in $FP(G)$. If there exists a finite path $w = e_1 \ldots e_n$ in $FP(G)$, then we say that the directed edges $e_1, \ldots, e_n$ are admissible (or connected in the order $(e_1, \ldots, e_n)$) and then the length $|w|$ of w is defined to be n, which is the cardinality of the admissible edges constructing w. Also, we say that finite paths $w_1 = e_{11} \ldots e_{1k_1}$ and $w_2 = e_{21} \ldots e_{2k_2}$ are admissible (or connected in the order (w_1, w_2)), if $w_1 w_2 = e_{11} \ldots e_{1k_1} e_{21} \ldots e_{2k_2}$ is again in $FP(G)$, where $e_{11}, \ldots, e_{1k_1}, e_{21}, \ldots, e_{2k_2} \in E(G)$. Otherwise, we say that w_1 and w_2 are not admissible. By definition, $|w_1 w_2| = |w_1| + |w_2|$, whenever w_1 and w_2 are admissible in $FP(G)$, for $w_1, w_2 \in FP(G)$. Suppose that w is a finite path in $FP(G)$ with its initial vertex v_1 and its terminal vertex v_2. We will write $w = v_1 w$ or $w = w v_2$ or $w = v_1 w v_2$, for emphasizing the initial vertex of w respectively the terminal vertex of w respectively both the initial vertex and the terminal vertex of w. Suppose $w = v_1 w v_2$ in $FP(G)$ with $v_1, v_2 \in V(G)$. Then we say that [v_1 and w are admissible] and [w and v_2 are admissible]. Notice that even though finite paths w_1 and w_2 are admissible, w_2 and w_1 are not admissible, in general. For instance, if $e_1 = v_1 e_1 v_2$ is an edge with $v_1 \neq v_2$ in $V(G)$ and $e_2 = v_2 e_2 v_2$ is a loop-edge in $E(G)$, then there is a finite path $e_1 e_2$ in $FP(G)$, but there is no (nonempty) finite path $e_2 e_1$ in $FP(G)$.

The free semigroupoid $\mathbb{F}^+(G)$ of G is defined by a set

$$\mathbb{F}^+(G) = \{\emptyset\} \cup V(G) \cup FP(G),$$

with its binary operation $(\cdot)$ on $\mathbb{F}^+(G)$, defined by

$$(w_1, w_2) \longmapsto w_1 \cdot w_2 = \begin{cases} w_1 & \text{if } w_1 = w_2 \text{ in } V(G) \\ w_1 & \text{if } w_1 \in FP(G),\ w_2 \in V(G) \text{ and } w_1 = w_1 w_2 \\ w_2 & \text{if } w_1 \in V(G),\ w_2 \in FP(G) \text{ and } w_2 = w_1 w_2 \\ w_1 w_2 & \text{if } w_1,\ w_2 \text{ in } FP(G) \text{ and } w_1 w_2 \in FP(G) \\ \emptyset & \text{otherwise,} \end{cases}$$

where $\emptyset$ is the empty word in $V(G) \cup E(G)$. The binary operation on $\mathbb{F}^+(G)$ is called the admissibility. i.e., the pair $(\mathbb{F}^+(G), \cdot)$ is the free semigroupoid of G. For convenience, we just denote the free semigroupoid of G by $\mathbb{F}^+(G)$.

For the given countable directed graph G, we can define a new countable directed graph G^{-1} which is the opposite directed graph of G, with

$$V(G^{-1}) = V(G) \quad \text{and} \quad E(G^{-1}) = \{e^{-1} : e \in E(G)\},$$

where $e^{-1} \in E(G^{-1})$ is the opposite directed edge of $e \in E(G)$. i.e., if $e = v_1 e v_2$ in $E(G)$, with $v_1, v_2 \in V(G)$, then $e^{-1} = v_2 e^{-1} v_1$ in $E(G^{-1})$, with $v_2, v_1 \in V(G^{-1}) = V(G)$. The graph G^{-1} is called the shadow of G. Similar to the previous paragraph, we can construct the set $FP(G^{-1})$ of all finite paths of G^{-1} and the free semigroupoid $\mathbb{F}^+(G^{-1})$ of G^{-1}. The admissibility on $\mathbb{F}^+(G^{-1})$ is oppositely preserved by that on $\mathbb{F}^+(G)$. In other words, if $w \in \mathbb{F}^+(G)$, then there is the corresponding shadow $w^{-1} \in \mathbb{F}^+(G^{-1})$, and vice versa. It is trivial that $(G^{-1})^{-1} = G$.

A new countable directed graph $\hat{G}$ is called the shadowed graph of G if it is a directed graph with its vertex set

$$V(\hat{G}) = V(G) = V(G^{-1})$$

and its edge set

$$E(G^{\wedge}) = E(G) \cup E(G^{-1}).$$

The free semigroupoid $\mathbb{F}^{+}(G^{\wedge})$ of $G^{\wedge}$ satisfies that

$$\mathbb{F}^{+}(G^{\wedge}) \supset (\mathbb{F}^{+}(G) \cup \mathbb{F}^{+}(G^{-1})),$$

in general.

Definition 1.1 Let G be a countable directed graph and $G^{\wedge}$, the shadowed graph of G, and let $\mathbb{F}^{+}(G^{\wedge})$ be the free semigroupoid of $G^{\wedge}$. Define the reduction (RR) on $\mathbb{F}^{+}(G^{\wedge})$ by

$$\text{(RR)} \qquad ww^{-1} = v \quad \text{and} \quad w^{-1}w = v',$$

whenever $w = vwv'$ in $FP(G^{\wedge})$ with $v, v' \in V(G^{\wedge})$. The set $\mathbb{F}^{+}(G^{\wedge})$ with this reduction (RR) is denoted by $\mathbb{F}^{+}_{r}(G^{\wedge})$ and this set with inherited admissibility from $\mathbb{F}^{+}(G^{\wedge})$ is called a graph groupoid of G. Denote this graph groupoid $(\mathbb{F}^{+}_{r}(G^{\wedge}), \cdot)$ of G by $\mathbb{G}$.

Now, construct the graph Hilbert space H_G induced by the given graph G.

Definition 1.2 Let G be a given countable directed graph and let $\mathbb{G}$ be the corresponding graph groupoid. Define the Hilbert space H_G by

$$H_G \overset{def}{=} \bigoplus_{w \in FP_r(G^{\wedge})} (\mathbb{C}\xi_w)$$

with its Hilbert basis $\mathcal{B}_{H_G} = \{\xi_w : w \in FP_r(G^{\wedge})\}$ in H_G, where

$$FP_r(G^{\wedge}) \overset{def}{=} \mathbb{G} \setminus (V(G^{\wedge}) \cup \{\emptyset\}).$$

On H_G, we have the following multiplication rule:

$$\xi_{w_1}\xi_{w_2} = \begin{cases} \xi_{w_1 w_2} & \text{if } w_1 \text{ and } w_2 \text{ are admissible in } \mathbb{G} \\ \xi_{\emptyset} = 0_{H_G} & \text{otherwise,} \end{cases}$$

for all $w_1, w_2 \in \mathbb{G}$, by the admissibility on $\mathbb{G}$. Notice that the product $w_1 w_2$ of w_1 and w_2 means the reduced word gotten from the product $w_1 \cdot w_2$ in $\mathbb{G}$. For instance, if $w_1 = e_1 e_2$ and $w_2 = e_2^{-1}$, then the product $w_1 w_2$ means e_1 in $\mathbb{G}$.

The above multiplication rule on H_G guarantees the existence of multiplication operators on H_G.

Definition 1.3 The operator L_w is defined by the multiplication operator with its symbol ξ_w on H_G, for $w \in \mathbb{G}$. i.e.,

$$L_w \xi_{w'} \overset{def}{=} \begin{cases} \xi_w \xi_{w'} = \xi_{ww'} & \text{if } ww' \neq \emptyset \\ \xi_{\emptyset} = 0_{H_G} & \text{otherwise,} \end{cases}$$

for all $w, w' \in \mathbb{G}$. The adjoint L_w^{*} of L_w is defined by $L_w^{*} = L_{w^{-1}}$, for all $w \in \mathbb{G}$.

For $w \in FP_r(G^\wedge)$, we have $\xi_w \xi_{w^{-1}} = \xi_{v_1}$ and $\xi_{w^{-1}} \xi_w = \xi_{v_2}$, whenever $w = v_1 w v_2$ with $v_1, v_2 \in V(G^\wedge)$. Thus, for any vertex $v \in V(G^\wedge)$, we can define a multiplication operator L_v on H_G with its symbol $\xi_v \in H_G$. Therefore, for any $w \in \mathbb{G}$, we can construct a corresponding multiplication operator L_w on H_G.

Let L_{w_1} and L_{w_2} be the multiplication operators on H_G, where $w_1, w_2 \in \mathbb{G}$. Then

$$L_{w_1} L_{w_2} = L_{w_1 w_2} \quad \text{on } H_G.$$

Therefore, we can get that the multiplication operators L_v's are projections, for all $v \in V(G^\wedge)$. Indeed, we have that $L_v^* = L_{v^{-1}} = L_v = L_{v^2} = L_v^2$. And hence, the multiplication operators L_w's and L_w^*'s are partial isometries, for all $w \in FP_r(G^\wedge)$, since

$$L_w L_w^* = L_w L_{w^{-1}} = L_{ww^{-1}} \quad \text{and} \quad L_w^* L_w = L_{w^{-1}} L_w = L_{w^{-1}w},$$

since $ww^{-1}, w^{-1}w \in V(G^\wedge)$, by the reduction (RR).

Definition 1.4 Let $\mathbb{G}$ be a graph groupoid of a countable directed graph G and let M be a von Neumann algebra acting on a Hilbert space K (i.e., $M \subseteq B(K)$). Define a graph-representation (in short, a G-representation) $\alpha : \mathbb{G} \to B(K \otimes H_G)$ by a nonunital partial representation satisfying that

$$\alpha_w(m) L_w L_w^* = L_w^* m L_w = L_{w^{-1}} m L_w, \tag{1.1}$$

$$\alpha_v(m) = m \tag*{(1.1)$'$}$$

for all $m \in M$ and $w \in FP_r(G^\wedge)$ and $v \in V(G^\wedge)$, where L_w is a multiplication operator on H_G. We can regard L_w as elements $1_M \otimes L_w$ in $B(K \otimes H_G)$, for all $w \in \mathbb{G}$.

Notation For convenience, denote $\alpha_w(m)$ by m^w, for $w \in \mathbb{G}$ and $m \in M$.

Definition 1.5 Define the crossed product $\mathbb{M}_G = M \times_\alpha \mathbb{G}$ of M and $\mathbb{G}$ via a G-representation α by the von Neumann algebra generated by M and $\{L_w : w \in \mathbb{G}\}$ in $B(K \otimes H_G)$, satisfying (1.1) and (1.1)$'$. This von Neumann algebra $\mathbb{M}_G$ is called a graph von Neumann algebra generated by G over M.

1.2 Graph von Neumann Algebras

Let G be a countable directed graph with its graph groupoid $\mathbb{G}$, and let $\mathbb{M}_G = M \times_\alpha \mathbb{G}$ be a graph von Neumann algebra, where M is an arbitrary von Neumann algebra and α is a graph-representation of $\mathbb{G}$. Every element x in $\mathbb{M}_G$ has its expression,

$$x = \sum_{w \in \mathbb{G}} m_w L_w, \quad \text{for } m_w \in M.$$

Each graph von Neumann algebra $\mathbb{M}_G$ always has its W^*-subalgebra $\mathbb{D}_G$ defined by

$$\mathbb{D}_G \overset{def}{=} \bigoplus_{v \in V(G^\wedge)} (M \cdot L_v).$$

Notice that the construction of $\mathbb{D}_G$ is independent from the choice of graph-representations. So, we will call this W^*-subalgebra $\mathbb{D}_G$ of $\mathbb{M}_G$, the M-diagonal subalgebra of $\mathbb{M}_G$.

We can construct the canonical conditional expectation $E : \mathbb{M}_G \to \mathbb{D}_G$ by

$$E\left(\sum_{w \in \mathbb{G}} m_w L_w\right) \overset{def}{=} \sum_{v \in V(G^\wedge)} m_v L_v.$$

Then the pair $(\mathbb{M}_G,\ E)$ is called the M-diagonal graph W^*-probability space over $\mathbb{D}_G$, which is a well-defined $\mathbb{D}_G$-valued W^*-probability space.

In [9], we computed $\mathbb{D}_G$-valued moments and cumulants of $\mathbb{D}_G$-valued random variables $m_1 L_{w_1}, \ldots, m_n L_{w_n}$ in the M-diagonal graph W^*-probability space $(\mathbb{M}_G, E)$, where $m_1, \ldots, m_n \in M$:

$$E(m_1 L_{w_1} \ldots m_n L_{w_n}) = m_0 E(L_{w_1 \ldots w_n})$$

and

$$k_n(m_1 L_{w_1}, \ldots, m_n L_{w_n}) = (\mu_0 \cdot m_0) E(L_{w_1 \ldots w_n}),$$

where

$$m_0 \overset{def}{=} m_1 m_2^{w_1} m_3^{(w_1 w_2)^{-1}} \ldots m_n^{(w_1 \ldots w_{n-1})^{-1}} \in M$$

and

$$\mu_0 \overset{def}{=} \sum_{\pi \in NC(w_1, \ldots, w_n)} \mu(\pi, 1_n) \in \mathbb{C},$$

where m^w means $\alpha_w(m)$, for all $m \in M$ and $w \in \mathbb{G}$ and where

$$NC(w_1, \ldots, w_n) \overset{def}{=} \{\theta \in NC(n) : E_\theta(L_{w_1}, \ldots, L_{w_n}) = E(L_{w_1 \ldots w_n})\}.$$

Based on those computations, we showed that:

Lemma 1.1 (See [9]) *The subsets $M \cdot L_{w_1}$ and $M \cdot L_{w_2}$ are free over $\mathbb{D}_G$ in $(\mathbb{M}_G, E)$ if and only if w_1 and w_2 are diagram-distinct in the sense that* (i) $w_1 \neq w_2^{-1}$ *in $\mathbb{G}$, and* (ii) *the diagrams* (*or the graphical images*) *of w_1 and w_2 are distinct on G.*

And, by the previous lemma, we could get the following theorem.

Theorem 1.2 (See [9] and [10]) *Let $\mathbb{M}_G = M \times_\alpha \mathbb{G}$ be a graph von Neumann algebra. Then*

$$\mathbb{M}_G \overset{*\text{-isomorphic}}{=} \underset{e \in E(G)}{*^r_{\mathbb{D}_G}} \mathbb{M}_e,$$

where $\mathbb{M}_e = \mathbb{M}_{\{e, e^{-1}\}}$, where "$*^r_{\mathbb{D}_G}$" means the amalgamated reduced free product over $\mathbb{D}_G$.

This shows that our graph von Neumann algebra is $*$-isomorphic to an amalgamated reduced free product algebra. And this theorem will be used to consider a compressed subalgebra $p\mathbb{M}_G p$ of $\mathbb{M}_G$, where p is a projection in $\mathbb{D}_G$. i.e., we can observe the $(p\mathbb{D}_G p)$-(compressed-)free structures of $p\mathbb{M}_G p$. And then we may realize that the compressed algebra $p\mathbb{M}_G p$ has certain amalgamated free block structures. If there is a nice amalgamated free structures inside $p\mathbb{M}_G p$, then we can conclude that this compressed subalgebra is also $*$-isomorphic to amalgamated (reduced) free product algebra.

However, the above theorem does not explain how the reduction works on the $\mathbb{D}_G$-free product "$*^r_{\mathbb{D}_G}$". To see that, we need to consider the Banach space expression of the reduced $\mathbb{D}_G$-valued free product algebra $*^r_{\mathbb{D}_G\, e\in E(G)}\mathbb{M}_e$.

Let $A = *_{B\, i\in I} A_i$ be a B-valued free product algebra. Then A has the following Banach space expression,

$$B \oplus \left(\bigoplus_{n=1}^{\infty} \left(\bigoplus_{i_1\neq i_2, i_2\neq i_3, \ldots, i_{n-1}\neq i_n} (A^o_{i_1} \otimes_B \cdots \otimes_B A^o_{i_n}) \right) \right),$$

where $A^o_{i_j} \overset{def}{=} A_{i_j} \ominus B$, for all $j = 1, \ldots, n$.

Since our graph von Neumann algebra $\mathbb{M}_G$ is $*$-isomorphic to a $\mathbb{D}_G$-free product algebra of $\mathbb{M}_e$'s, it should have the similar Banach space expression. However, we can check that if e_1 and e_2 are totally disjoint edges in $E(G)$, in the sense that: neither $[e_1$ and $e_2^{\pm 1}]$ nor $[e_2$ and $e_1^{\pm 1}]$ are admissible, then the $\mathbb{D}_G$-tensor product Banach space $\mathbb{M}^o_{e_1} \otimes_{\mathbb{D}_G} \mathbb{M}^o_{e_2}$ is Banach-space isomorphic to the trivial space $\{0_{\mathbb{D}_G}\}$. Thus, we can get the following theorem.

This theorem explains how the reduction of the free product "$*^r_{\mathbb{D}_G}$" works.

Theorem 1.3 (See [9] and [10]) *The graph von Neumann algebra $\mathbb{M}_G$ is expressed as a Banach space,*

$$\mathbb{D}_G \oplus \left(\bigoplus_{w^*\in E(G)^*_r} \mathbb{M}^o_{w^*} \right),$$

where

$$E(G^\wedge)^*_r \overset{def}{=} E(G^\wedge) \cup \left(\bigcup_{k=2}^{\infty} \left\{ e_1 e_2 \ldots e_k \in FP_r(G^\wedge) \,\middle|\, \begin{array}{c} e_1 \neq e_2^{\pm 1}, \\ e_2 \neq e_3^{\pm 1}, \\ \ldots e_{k-1} \neq e_k^{\pm 1} \end{array} \right\} \right)$$

under the fact that $\mathbb{M}_e = \mathbb{M}_{e^{-1}}$, for all $e \in E(G)$, and

$$\mathbb{M}^o_{w^*} \overset{def}{=} \mathbb{M}^o_{e_1} \otimes_{\mathbb{D}_G} \cdots \otimes_{\mathbb{D}_G} \mathbb{M}^o_{e_n}, \quad \textit{with } \mathbb{M}^o_{e_k} \overset{def}{=} \mathbb{M}_{e_k} \ominus \mathbb{D}_G,$$

whenever $w^* = e_1 \ldots e_n$ in $E(G^\wedge)^*_r$.

The above theorem shows that the reduction is totally depending on the admissibility on $\mathbb{G}$.

It is easy to see that if a fixed von Neumann algebra M is $\mathbb{C}$, then graph von Neumann algebras $\mathbb{C} \times_\alpha \mathbb{G}$ are all $*$-isomorphic to $\overline{\mathbb{C}[\mathbb{G}]}^w$, for all graph-representations α, by the $\mathbb{C}$-linearity of α on $\mathbb{C}$.

If G_e is a directed graph with only one edge e, then the graph von Neumann algebra $\overline{\mathbb{C}[\mathbb{G}]}^w$ is $*$-isomorphic to either $L^\infty(\mathbb{T})$ or $M_2(\mathbb{C})$, where $\mathbb{T}$ is the unit circle in $\mathbb{C}$ and $M_2(\mathbb{C})$ is the matricial algebra generated by all (2×2)-matrices. More generally, a graph von Neumann algebra $\mathbb{M}_{G_e} = M \times_\alpha \mathbb{G}_e$ is $*$-isomorphic to either a classical crossed product algebra $M \times_\gamma \mathbb{Z}$ or a W^*-subalgebra $M_2^\alpha(M)$ of the M-valued matricial algebra $M_2(M) = M \otimes_{\mathbb{C}} M_2(\mathbb{C})$.

In [10], we proved that:

Theorem 1.4 (See [10]) *Let $\mathbb{M}_G = M \times_\alpha \mathbb{G}$ be a graph von Neumann algebra, with its $\mathbb{D}_G$-free blocks $\{\mathbb{M}_e : e \in E(G)\}$. If e is a loop edge, then $\mathbb{M}_e = vN(M \times_{\lambda^{(e)}} \mathbb{Z}, \mathbb{D}_G)$, where the group action $\lambda^{(e)}$ is identified with $\alpha\,|_{\mathbb{G}_e}$, on M, where $\mathbb{G}_e$ is the substructure of $\mathbb{G}$ consisting of all reduced words in $\{e, e^{-1}\}$. If e is a non-loop edge, then $\mathbb{M}_e = vN(M_2^{\alpha_e}(M), \mathbb{D}_G)$, where*

$$M_2^{\alpha_e}(M) \overset{def}{=} vN\left(M,\ \left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\} \,\middle|\, \begin{array}{c}\begin{pmatrix} \alpha_e(m) & 0_M \\ 0_M & 0_M \end{pmatrix} = \begin{pmatrix} 0_M & 0_M \\ 0_M & m \end{pmatrix}, \\ \forall m \in M \end{array}\right),$$

in $M_2(M) = M \otimes_{\mathbb{C}} M_2(\mathbb{C})$. In particular, if $M = \mathbb{C}$, then $\mathbb{M}_e$ is $*$-isomorphic to either $vN(L^\infty(\mathbb{T}), \mathbb{D}_G)$ or $vN(M_2(\mathbb{C}), \mathbb{D}_G)$.

We can see that the free group factor $L(F_N)$ generated by the free group F_N with N-generators is $*$-isomorphic to our graph von Neumann algebra $\mathbb{C} \times_\alpha \mathbb{G}_N = \overline{\mathbb{C}[\mathbb{G}_N]}^w$ (for any α), where $\mathbb{G}_N$ is the graph groupoid of the one-vertex-N-loop-edge graph G_N. We can check that $L(F_N)$ is $*$-isomorphic to $*_{j=1}^N L(\mathbb{Z})_j$, where $L(\mathbb{Z})_j = L(\mathbb{Z})$, for all $j = 1, \ldots, N$, and hence it is easily shown that $L(F_N)$ is $*$-isomorphic to $L(F_{k_1}) * L(F_{k_2})$, for all $k_1, k_2 \in \mathbb{N}$ such that $k_1 + k_2 = N$.

If C_N is the one-flow circulant graph with

$$V(C_N) = \{v_1, \ldots, v_N\}$$

and

$$E(C_N) = \{e_j = v_j e_j v_{j+1} : j = 1, \ldots, N, v_{N+1} \overset{def}{=} v_1\},$$

then

$$\mathbb{C}_{C_N} \overset{def}{=} \mathbb{C} \times_\alpha \Delta_N \underset{k=1}{\overset{N}{*_{D_{C_N}}}} \mathbb{C}_{e_j} = D_{C_N} \oplus \left(\bigoplus_{w \in FP(C_N) \cup FP(C_N^{-1})} \mathbb{C}_w^o \right),$$

where Δ_N is the graph groupoid of C_N and $\mathbb{C}_w^o = \mathbb{C}_{e_1}^o \otimes_{D_{C_N}} \cdots \otimes_{D_{C_N}} \mathbb{C}_{e_n}^o$, for all $w = e_1 \ldots e_n \in FP(G)$, where $\mathbb{C}_{e_k}^o = M_2(\mathbb{C}) \ominus D_2(\mathbb{C})$, where $D_2(\mathbb{C})$ is the matricial algebra generated by all (2×2)-diagonal matrices in $M_2(\mathbb{C})$.

1.3 $*$-Isomorphic Graph von Neumann Algebras

In this section, we will consider $*$-isomorphic graph von Neumann algebras. We want to characterize the graph von Neumann algebra up to combinatorial morphisms. However, finding such characterization is still open. In [10] and [11], we could show that:

Theorem 1.5 (See [10] and [11]) *If G_1 and G_2 are countable directed graphs having graph-isomorphic shadowed graphs $G_1^\wedge$ and $G_2^\wedge$, and if the graph-representations α_1 and α_2 of the corresponding graph von Neumann algebras $\mathbb{M}_{G_1} = M \times_{\alpha_1} \mathbb{G}_1$ and $\mathbb{M}_{G_2} = M \times_{\alpha_2} \mathbb{G}_2$ satisfy that*

$$(\alpha_2)_{\Phi(w)}(m) = (\alpha_1)_w(m), \quad \textit{for all } m \in M,$$

for all $w \in \mathbb{G}_1$, *then* $\mathbb{M}_{G_1}$ *and* $\mathbb{M}_{G_2}$ *are* $*$*-isomorphic*, *where* $\Phi : \mathbb{G}_1 \to \mathbb{G}_2$ *is a groupoid-isomorphism induced by the graph-isomorphism of* $G_1^{\wedge}$ *and* $G_2^{\wedge}$.

Later, the above theorem also provides the tool to classify the $*$-isomorphic compressed subalgebras of a fixed von Neumann algebra $\mathbb{M}_G$.

Recall that two directed graphs G_1 and G_2 are graph-isomorphic, if there exists a morphism

$$g : V(G_1) \cup E(G_1) \to V(G_2) \cup E(G_2)$$

such that (i) g is bijective from $V(G_1)$ onto $V(G_2)$, (ii) g is bijective from $E(G_1)$ onto $E(G_2)$, and (iii) if $e = v_1 e v_2$ in $E(G_1)$ with $v_1, v_2 \in V(G_1)$, then $g(e) = g(v_1)g(e)g(v_2)$ in $E(G_2)$. The morphism g is said to be a graph-isomorphism.

Let G_1 and G_2 be given as above and let $G_1^{\wedge}$ and $G_2^{\wedge}$ be their shadowed graphs. Suppose that $G_1^{\wedge}$ and $G_2^{\wedge}$ are graph-isomorphic, with a graph-isomorphism $g : G_1^{\wedge} \to G_2^{\wedge}$. Then we can easily construct a corresponding groupoid-isomorphism $\Phi : \mathbb{G}_1 \to \mathbb{G}_2$, as follows,

$$\Phi(w) \overset{def}{=} \begin{cases} g(w) & \text{if } w \in V(G_1^{\wedge}) \cup E(G_1^{\wedge}) \\ g(e_1)\dots g(e_n) & \text{if } w = e_1 \dots e_n \in FP_r(G^{\wedge}) \\ \emptyset & \text{if } w = \emptyset, \end{cases}$$

in $\mathbb{G}_2$, for all $w \in \mathbb{G}_1$. As in [11] and [12], the morphism Φ is indeed a groupoid-isomorphism.

Consider the graph von Neumann algebras

$$M_{G_k} \overset{def}{=} \overline{\mathbb{C}[\mathbb{G}_k]}^{w}, \quad \text{for } k = 1, 2,$$

where $\mathbb{G}_k$ are the graph groupoids of G_k, for $k = 1, 2$. Then we can get the following corollary.

Corollary 1.6 (See [10] and [11]) *Let* G_1 *and* G_2 *be countable directed graphs with their graph groupoids* $\mathbb{G}_1$ *and* $\mathbb{G}_2$, *If the shadowed graphs* $G_1^{\wedge}$ *and* $G_2^{\wedge}$ *are graph-isomorphic*, *then the graph von Neumann algebras* M_{G_1} *and* M_{G_2} *are* $*$*-isomorphic.*

Unfortunately, we still do not know the converse of the previous theorem and corollary hold true or not (see [11] and [19]).

1.4 Main Results

In this section, we introduce our main results of this paper. Let L_v be a projection in $\mathbb{M}_G$, for $v \in V(G^{\wedge})$. Then we can consider the compressed W^*-subalgebra $L_v \mathbb{M}_G L_v$. We can see that such compressed algebra consists of the elements

$$\sum_{w \in loop_v(G^{\wedge})} m_w L_w, \quad \text{where } m_w \in M$$

and where

$$loop_v(G^{\wedge}) \overset{def}{=} \{v\} \cup \{w \in \mathbb{G} : w = vwv\}.$$

Define a subset $Loop_v(G^{\wedge}) \setminus \{v\}$ of $loop_v(G^{\wedge})$, by a set satisfying

$$Loop_v(G^{\wedge}) \cup (Loop_v(G^{\wedge}))^{-1} = loop_v(G^{\wedge}) \setminus \{v\}$$

and

$$Loop_v(G^{\wedge}) \cap (Loop_v(G^{\wedge}))^{-1} = \varnothing.$$

Notice that the choice of $Loop_v(G^{\wedge})$ in $loop_v(G^{\wedge})$ is not unique. There are many possibility to choose $Loop_v(G^{\wedge})$. Assume that we fix one $Loop_v(G^{\wedge})$. Then we can define the subset $Loop_v^{\delta}(G^{\wedge})$ of $Loop_v(G^{\wedge})$ by

$$Loop_v^{\delta}(G^{\wedge}) = \{\delta(w) : w \in Loop_v(G^{\wedge})\},$$

where $\delta(w)$ means the diagram, which is the graphical image, of w, for all $w \in \mathbb{G}$. Then we can characterize the compressed subalgebra $L_v \mathbb{M}_G L_v$ of $\mathbb{M}_G$ by the following theorem.

Theorem 1.7 (see Sect. 2.1) *The compressed W^*-subalgebra $L_v\ \mathbb{M}_G\ L_v$ is $*$-isomorphic to either M or*

$$\underset{j=1}{\overset{k}{*_M}} (M \times_{\lambda_j} \mathbb{Z}) \overset{*\text{-isomorphic}}{=} M \times_{\lambda} F_k,$$

where k is the cardinality of "basic" loops connecting v to v. Here, a von Neumann algebra $M \times_{\lambda_j} \mathbb{Z}$ are classical crossed product algebras of M and the group $\mathbb{Z}$ via the group actions $\lambda_j : \mathbb{Z} \to Aut(M)$, for all $j = 1, \ldots, k$, induced by the G-representation α. And a von Neumann algebra $M \times_{\lambda} F_k$ is a classical crossed product algebra of M and the free group F_k with k-generators via the group action $\lambda : F_k \to Aut(M)$ induced by α, where $Aut(M)$ is the automorphism group of M. In particular,

$$k = |Loop_v^{\delta}(G^{\wedge})|,$$

where

$$Loop_v^{\delta}(G^{\wedge}) \overset{def}{=} \{\delta(w) \in FP_r(G^{\wedge}) : w \in Loop_v(G^{\wedge})\}.$$

Here, the element $\delta(w)$ of w in $FP_r(G^{\wedge})$ means the diagram (or the graphical image) of w in the sense of [9].

Remark 1.1 Notice that, even though the choice of $Loop_v(G^{\wedge})$ is not uniquely determined, the compressed algebra $L_v \mathbb{M}_G L_v$ is $*$-isomorphic to $*_M{}_{j=1}^{|Loop_v^{\delta}(G^{\wedge})|} (M \times_{\lambda_j} \mathbb{Z})$, by Sect. 1.3, whenever $Loop_v(G^{\wedge}) \neq \varnothing$.

As a corollary of the previous theorem, we can get that the compressed W^*-subalgebra $L_v M_G L_v$ of the graph von Neumann algebra $M_G = \overline{\mathbb{C}[\mathbb{G}]}^w$ is $*$-isomorphic to either $\mathbb{C}$ or $L(F_k)$, where k is the cardinality of loops connecting v to v.

Now, let $L_{v_1}, \ldots, L_{v_n} \in \mathbb{M}_G$ be projections, where $V = \{v_1, \ldots, v_n\} \subseteq V(G^{\wedge})$ and $n > 1$. Then we can construct a new projection P_V in $\mathbb{M}_G$,

$$P_V \overset{def}{=} \sum_{k=1}^{n} L_{v_k}.$$

Let $n = 2$ and assume we have projections L_{v_1} and L_{v_2}, for $\{v_1, v_2\} \subseteq V(G^{\wedge})$. Then we can create a projection $P_V = L_{v_1} + L_{v_2}$. Indeed, the operator $P \in \mathbb{M}_G$ satisfies that

$$\begin{aligned} P_V^2 &= (L_{v_1} + L_{v_2})^2 = L_{v_1}^2 + L_{v_2}^2 = P_V \\ &= L_{v_1} + L_{v_2} = (L_{v_1} + L_{v_2})^* = P_V^*. \end{aligned}$$

Then we can construct a compressed W^*-subalgebra $P_V \mathbb{M}_G P_V$ of a graph von Neumann algebra $\mathbb{M}_G = M \times_\alpha \mathbb{G}$. We can see that every element $y \in P_V \mathbb{M}_G P_V$ has its expression

$$y = y_{(D:V)} + y_{(O:V)}$$

with

$$y_{(D:V)} \sum_{j=1}^{n} \left(\sum_{w \in loop_{v_j}(G^{\wedge})} m_w L_w \right) \overset{denote}{=} \sum_{j=1}^{N} y_j$$

and

$$y_{(O:V)} = \sum_{i<j \in \{1,\ldots,n\}} \left(\sum_{w = v_i w v_j \in FP_r(G^{\wedge})} m_w L_w \right) \overset{denote}{=} \sum_{i<j \in \{1,\ldots,n\}} y_{ij}.$$

The summands $y_{(D:V)}$ and $y_{(O:V)}$ are called the diagonal compressed part and the off-diagonal compressed part of y, respectively. Moreover, we can see that the operators

$$\{y_j : j = 1, \ldots, n\} \quad \text{and} \quad \{y_{ij} : i < j \in \{1, \ldots, n\}\}$$

are free over $P_V \mathbb{D}_G P_V$ in the compressed W^*-probability space $(P_V \mathbb{M}_G P_V, E)$. This observation provides the motive of the following theorem.

Theorem 1.8 (See Sect. 2.3) *Let $\mathbb{M}_G = M \times_\alpha \mathbb{G}$ be a graph von Neumann algebra and P_V, the corresponding projection of a subset $V = \{v_1, \ldots, v_N\} \subseteq V(G^{\wedge})$, and let $\mathbb{M}_{G:V} = P_V \mathbb{M}_G P_V$ be the V-compressed algebra of $\mathbb{M}_G$. Then $\mathbb{M}_{G:V}$ is $*$-isomorphic to*

$$\underset{w \in \mathcal{I}_V}{\overset{r}{*}}_{\mathbb{D}_{G:V}} \mathbb{M}_w^{(V)},$$

with $\mathbb{M}_w^{(V)} \overset{def}{=} vN(M \times_\alpha \mathbb{G}_w, \mathbb{D}_{G:V})$, where $\mathbb{D}_{G:V} \overset{def}{=} \oplus_{j=1}^{N} (M \cdot L_{v_j})$ and $\mathbb{G}_w$ is the subset of $\mathbb{G}$ consisting of all reduced words in $\{w, w^{-1}\}$, for $w \in \mathcal{I}_V$, where

$$\mathcal{I}_V \overset{def}{=} \left(\left(\bigcup_{j=1}^{N} Loop_{v_j}^{\delta}(G^{\wedge}) \right) \cup \left(\bigcup_{i<j \in \{1,\ldots,N\}} (\mathbb{G}_\delta(v_i, v_j)) \right) \right),$$

where $\mathbb{G}(v_i, v_j) \overset{def}{=} \{w \in \mathbb{G} : w = v_i w v_j\}$ and

$$\mathbb{G}_\delta(v_i, v_j) \overset{def}{=} \left\{ w \in \mathbb{G}(v_i, v_j) : \delta(w) \notin \bigcup_{j=1}^{N} Loop_{v_j}^{\delta}(G^{\wedge}) \right\}.$$

By the previous theorem, we now derive our main result of this paper.

Theorem 1.9 (See Sect. 2.3) *Let* $V = \{v_1, \ldots, v_n\} \subseteq V(\widehat{G})$ *and let* $P_V \overset{def}{=} \sum_{j=1}^{n} L_{v_j} \in \mathbb{M}_G$ *be a* V*-depending projection. Then the compressed* W^**-algebra* $P_V \mathbb{M}_G P_V$ *of a graph von Neumann algebra* $\mathbb{M}_G = M \times_\alpha \mathbb{G}$ *is* $*$*-isomorphic to a graph von Neumann algebra* $\mathbb{M}_F = M \times_{\alpha_F} \mathbb{F}$*, where* $\mathbb{F}$ *is the graph groupoid of a certain graph* F *with its vertex set* $V(F) = V$ *and its edge set*

$$E(H) = \left(\left(\bigcup_{j=1}^{N} Loop_{v_j}^{\delta}(\widehat{G}) \right) \cup \left(\bigcup_{i<j \in \{1,\ldots,N\}} \mathbb{G}_\delta(v_i,\ v_j) \right) \right).$$

The above theorem shows that every vertex-compressed graph von Neumann algebra $P_V \mathbb{M}_G P_V$, for $V \subseteq V(\widehat{G})$, is an another graph von Neumann algebra $\mathbb{M}_F$ induced by a certain graph F. Moreover, this graph F has its graph groupoid $\mathbb{F}$ which is the substructure of $\mathbb{G}$ having the same admissibility.

2 Vertex-Compressions of a Graph von Neumann Algebra

Throughout this chapter, let G be a countable directed graph with its graph groupoid $\mathbb{G}$ and let M be a fixed arbitrary von Neumann algebra acting on a Hilbert space K. Also, let $\mathbb{M}_G = M \times_\alpha \mathbb{G}$ be a graph von Neumann algebra induced by G over M, via a G-representation α. We will consider certain types of projections Q of $\mathbb{M}_G$ and construct a compressed algebra $Q\mathbb{M}_G Q$. In Sect. 2.1, we observe the case where the projection Q is L_v, for $v \in V(\widehat{G})$. We give the full characterization of such compressed algebras, in terms of other graph von Neumann algebras. In Sect. 2.2, we study the case where $P = \sum_{j=1}^{N} L_{v_j}$, where $v_j \in V(\widehat{G})$, for $j = 1, \ldots, N$, especially, $N > 1$. Then the compressed algebra $P\mathbb{M}_G P$ is decomposed by the diagonal compressed part and the off-diagonal compressed part. Each part has its direct summands and those direct summands would be considered. In Sect. 2.3, we characterize the compressed W^*-subalgebras, considered in Sect. 2.2, in terms of certain graph von Neumann algebras.

2.1 Single Vertex-Compressions of $\mathbb{M}_G$

Throughout this section, let G be a countable directed graph with its graph groupoid $\mathbb{G}$. Let $\mathbb{M}_G = M \times_\alpha \mathbb{G}$ be a graph von Neumann algebra induced by G over M via a graph-representation α. Let v be a fixed vertex in $V(\widehat{G})$. Then the multiplication operator L_v in $\mathbb{M}_G$ is a projection. So, we can consider a compressed algebra $L_v \mathbb{M}_G L_v$ of $\mathbb{M}_G$.

Definition 2.1 A compressed W^*-subalgebra $L_v\ \mathbb{M}_G\ L_v$ of a graph von Neumann algebra $\mathbb{M}_G$ is said to be a single vertex-compressed algebra (in short, the v-compressed algebra), for $v \in V(\widehat{G})$.

Recall that every element x in a graph von Neumann algebra $\mathbb{M}_G$ has its expression,

$$x = \sum_{w \in \mathbb{G}} m_w L_w \quad \text{with } m_w \in M.$$

Thus the compressed element $L_v x L_v$ in the v-compressed algebra $L_v \mathbb{M}_G L_v$ of $\mathbb{M}_G$ has its expression,

$$L_v x L_v = L_v \left(\sum_{w \in \mathbb{G}} m_w L_w \right) L_v = \sum_{w \in \mathbb{G}} L_v m_w L_w L_v$$
$$= \sum_{w \in \mathbb{G}} m_w^{v^{-1}} L_v L_w L_v$$

where $m_w^{v^{-1}} = m_w^v = \alpha_v(m_w)$, for all $w \in \mathbb{G}$

$$= \sum_{w \in \mathbb{G}} m_w L_v L_w L_v$$

since $\alpha_v(m) = m$, for all $m \in M$ and $v \in V(G^\wedge)$, by definition

$$= \sum_{w \in \mathbb{G},\, w = vwv} m_w L_w.$$

Define a subset $loop_v(G^\wedge)$ of the graph groupoid $\mathbb{G}$ by

$$loop_v(G^\wedge) \overset{def}{=} \{v\} \cup \{w \in FP_r(G^\wedge) : w = vwv\}.$$

Notice that the subset $loop_v(G^\wedge)$ of $\mathbb{G}$ is not only a subset but also a substructure of $\mathbb{G}$, with the inherited admissibility on $\mathbb{G}$. We checked that every graph groupoid is a (categorial) groupoid in [11] and [12]. So, the subset $loop_v(G^\wedge)$, with the inherited admissibility on $\mathbb{G}$, is an algebraic substructure of $\mathbb{G}$. i.e., a substructure $loop_v(G^\wedge)$ is a subgroupoid of $\mathbb{G}$, in the sense of [11] and [12].

Recall that, in [9–11] and [12], we defined the so-called the diagram map $\delta : \mathbb{G} \to \mathbb{G}$ by $w \mapsto \delta(w)$, where $\delta(w)$ is the diagram (which is the graphical image) of w (on the graph G, as a geometric object in $\mathbb{R}^2$). For instance, if e is a loop edge, then $\delta(e) = e = \delta(e^n)$, for all $n \in \mathbb{N}$. Suppose $w \in \mathbb{G}$ and $\delta(w) = w$. Then we will say that w is a basic element in $\mathbb{G}$. For example, all vertices and edges are basic.

Define the subset $loop_v^\delta(G^\wedge)$ of $loop_v(G^\wedge)$, for all $v \in V(G^\wedge)$, by

$$loop_v^\delta(G^\wedge) \overset{def}{=} \{\delta(w) \in \mathbb{G} : w \in loop_v(G^\wedge)\}.$$

i.e., the subset $loop_v^\delta(G^\wedge)$ of $loop_v(G^\wedge)$ is the collection of all basic elements in $loop_v(G^\wedge)$. By the help of new definitions, we can re-write the compressed element $L_v x L_v$ by

$$L_v x L_v = \sum_{w \in loop_v(G^\wedge)} m_w L_w, \quad \text{whenever } x = \sum_{w \in \mathbb{G}} m_w L_w \text{ in } \mathbb{M}_G.$$

Definition 2.2 Define a subset $Loop_v^\delta(G^\wedge)$ of $loop_v^\delta(G^\wedge) \setminus \{v\}$, by the subset of $loop_v^\delta(G^\wedge)$ satisfying that

$$Loop_v^\delta(G^\wedge) \cup Loop_v^\delta(G^\wedge) = loop_v^\delta(G^\wedge) \setminus \{v\}$$

and

$$Loop_v^\delta(G^\wedge) \cap (Loop_v^\delta(G^\wedge))^{-1} = \varnothing,$$

where

$$(Loop_v^{\delta}(G^{\wedge}))^{-1} = \{w^{-1} \in loop_v^{\delta}(G^{\wedge}) \setminus \{v\} : w \in Loop_v^{\delta}(G^{\wedge})\}.$$

Remark 2.1 Notice that the choice of the subset $Loop_v^{\delta}(G^{\wedge})$ of $loop_v^{\delta}(G^{\wedge}) \setminus \{v\}$ is not uniquely determined. In other words, we can take many different $Loop_v^{\delta}(G^{\wedge})$'s satisfying the above partitioning property of $loop_v^{\delta}(G^{\wedge}) \setminus \{v\}$. Suppose there are two distinct subset $\mathcal{L}_1$ and $\mathcal{L}_2$ satisfying the $Loop_v^{\delta}(G^{\wedge})$'s partitioning property. Then they generate same (sub)groupoid $loop_v(G^{\wedge})$ of $\mathbb{G}$. So, even though there can be many different choice of $Loop_v^{\delta}(G^{\wedge})$, under the admissibility on $\mathbb{G}$, we can generate same groupoid $loop_v(G^{\wedge})$. Without loss of generality, we can assume that $Loop_v^{\delta}(G^{\wedge})$ is chosen arbitrary and fixed in $loop_v^{\delta}(G^{\wedge}) \setminus \{v\}$.

The following theorem provides the full characterization of single vertex-fixed compressed algebras of a graph von Neumann algebra $\mathbb{M}_G$.

Theorem 2.1 *Let $\mathbb{M}_G = M \times_{\alpha} \mathbb{G}$ be a graph von Neumann algebra and let $L_v \mathbb{M}_G L_v$ be the v-compressed algebra of $\mathbb{M}_G$, for a fixed vertex $v \in V(G^{\wedge})$. Then this von Neumann algebra $L_v \mathbb{M}_G L_v$ is $*$-isomorphic to either a M-valued free product algebra*

$$\underset{w \in Loop_v^{\delta}(G^{\wedge})}{*_M} (M \times_{\lambda^{(w)}} \mathbb{Z})$$

of classical crossed product algebras $M \times_{\lambda^{(w)}} \mathbb{Z}$'s, or the fixed von Neumann algebra M. If $L_v \mathbb{M}_G L_v$ is $*$-isomorphic to the above M-valued free product algebra, then the group actions $\lambda^{(w)}$ of $\mathbb{Z}$, acting on $Aut(M)$, satisfies that

$$\lambda^{(w)} = \alpha|_{\mathbb{G}_w} \quad on\ M,$$

where $\mathbb{G}_w$ is the subset of $\mathbb{G}$ consisting of all reduced words in $\{w, w^{-1}\}$, for all $w \in Loop_v^{\delta}(G^{\wedge})$.

Proof Since a graph von Neumann algebra $\mathbb{M}_G$ is $*$-isomorphic to the amalgamated reduced free product algebra $*^{r}_{\mathbb{D}_G\, e \in E(G)} \mathbb{M}_e$, over the M-diagonal subalgebra $\mathbb{D}_G$, we can get that

$$\begin{aligned} L_v \mathbb{M}_G L_v &= L_v \left(\underset{e \in E(G)}{*^{r}_{\mathbb{D}_G}} \mathbb{M}_e \right) L_v \quad (*\text{-isomorphic}) \\ &= L_v \left(\mathbb{D}_G \oplus \left(\bigoplus_{w^* \in E(G^{\wedge})^*_r} \mathbb{M}^{o}_{w^*} \right) \right) L_v \quad (\text{Banach-Space isomorphic}) \end{aligned}$$

by Sect. 1.2

$$\begin{aligned} &= (L_v \mathbb{D}_G L_v) \oplus \left(\bigoplus_{w^* \in E(G^{\wedge})^*_r} L_v \mathbb{M}^{o}_{w^*} L_v \right) \\ &= (M \cdot L_v) \oplus \left(\bigoplus_{w^* = e_1 \ldots e_n \in E(G^{\wedge})^*_r} L_v (\mathbb{M}^{o}_{e_1} \otimes_B \cdots \otimes_B \mathbb{M}^{o}_{e_n}) L_v \right) \\ &= (M \cdot L_v) \oplus \left(\bigoplus_{e_1 \ldots e_n \in E(G^*)^*_r \cap loop_v(G^{\wedge})} (\mathbb{M}^{o}_{e_1} \otimes_B \cdots \otimes_B \mathbb{M}^{o}_{e_n}) \right). \end{aligned} \tag{2.1}$$

So, the Banach space expression (2.1) of $L_v \mathbb{M}_G L_v$ is Banach-space isomorphic to the $(M \cdot L_v)$-valued (reduced) free product algebra,

$$\underset{w \in Loop_v^\delta(G^\wedge)}{*^r_{M \cdot L_v}} (\mathbb{M}_w^{(v)}), \tag{2.2}$$

with

$$\mathbb{M}_w^{(v)} \overset{def}{=} vN(M \times_\alpha \mathbb{G}_w, M \cdot L_v),$$

for all $w \in Loop_v^\delta(G^\wedge)$. i.e., if $w \in loop_v^\delta(G^\wedge) \setminus \{v\}$, then $w^{-1} \in loop_v^\delta(G^\wedge) \setminus \{v\}$ and $\mathbb{M}_w^{(v)}$ is identically same as $\mathbb{M}_{w^{-1}}^{(v)}$, and vice versa. Thus we need to choose one of $\{w, w^{-1}\}$ to create a amalgamated reduced free product. Remark that $\mathbb{M}_w^{(v)} = M \times_\alpha \mathbb{G}_w$ is the W^*-subalgebra of $\mathbb{M}_G$, where $\mathbb{G}_w$ is a subset of $\mathbb{G}$ consisting of all reduced words in $\{w, w^{-1}\}$. (In fact, the subset $\mathbb{G}_w$ is a sub-groupoid of $\mathbb{G}$.) Notice that if $\delta(w) = w_0$ in $\mathbb{G}$, then the von Neumann algebra $M \times_\alpha \mathbb{G}_w$ is a W^*-subalgebra of $M \times \mathbb{G}_{w_0}$. So, a von Neumann algebra $M \times_\alpha \mathbb{G}_{w_0}$ contains all $M \times_\alpha \mathbb{G}_w$'s, as its W^*-subalgebras, whenever $\delta(w) = w_0$.

Notice also that the choice of the set $Loop_v^\delta(G^\wedge)$ is not unique. However, we know that if two directed graphs G_1 and G_2 have the graph-isomorphic shadowed graphs $G_1^\wedge$ and $G_2^\wedge$, via a graph-isomorphism g, then the corresponding graph von Neumann algebras $\mathbb{M}_{G_1} = M \times_{\alpha_1} \mathbb{G}_1$ and $\mathbb{M}_{G_2} = M \times_{\alpha_2} \mathbb{G}_2$ are $*$-isomorphic, whenever

$$(\alpha_2)_{\Phi(w)}(m)(\alpha_1)_w(m) \in M, \quad \text{for all } m \in M, \quad \text{and} \quad w \in \mathbb{G}_1,$$

where $\mathbb{G}_k$ are the graph groupoids of G_k, and $\Phi : \mathbb{G}_1 \to \mathbb{G}_2$ is the groupoid-isomorphism induced g (see Sect. 1.3). So, the W^*-subalgebra (2.2) of $\mathbb{M}_G$ is $*$-isomorphic from each other, for the different choices of $Loop_v^\delta(G^\wedge)$'s.

Clearly, the von Neumann algebra $M \cdot L_v$ is $*$-isomorphic to M. Also, the von Neumann algebra $M \times_\alpha \mathbb{G}_w$ is $*$-isomorphic to the classical crossed product algebra $M \times_{\lambda^{(w)}} \mathbb{Z}$ of M and the group $\mathbb{Z}$, via a group action $\lambda^{(w)} : \mathbb{Z} \to Aut(M)$ defined by

$$\lambda^{(w)} = \alpha|_{\mathbb{G}_w} \quad \text{on } M, \tag{2.3}$$

by [10], where $Aut(M)$ is the automorphism group of M. i.e., the M-free blocks $\mathbb{M}_w^{(v)}$ of $L_v \mathbb{M}_G L_v$ are $*$-isomorphic to the classical crossed product algebras $M \times_{\lambda^{(w)}} \mathbb{Z}$, for all $w \in Loop_v^\delta(G^\wedge)$. Therefore, (2.2) can be re-written by

$$\underset{w \in Loop_v^\delta(G^\wedge)}{*^r_M} (M \times_{\lambda^{(w)}} \mathbb{Z}). \tag{2.4}$$

Let's assume that $Loop_v^\delta(G^\wedge)$ is empty in $\mathbb{G}$ (equivalently, assume that there is no loop finite paths having their initial and terminal vertices v). Then (2.4) is $*$-isomorphic to the fixed von Neumann algebra M of $\mathbb{M}_G$.

Now, observe that the M-valued reduced free product "$*^r_M$" in (2.4) is in fact the usual M-free product. By definition, if the set $Loop_v^\delta(G^\wedge)$ is nonempty and if $w_1 \neq w_2$ are any arbitrary chosen two elements in it, then both $w_1 w_2$ and $w_2 w_1$ are well-determined loop finite path having their initial and terminal vertices v. In other words, all distinct elements $Loop_v^\delta(G^\wedge)$ are admissible via v, equivalently, the nonempty set $loop_v^\delta(G^\wedge)$ does not contain the empty word $\emptyset$. Therefore, we can construct (2.4) simply by the M-valued non-reduced free product algebra,

$$\underset{w \in Loop_v^\delta(G^\wedge)}{*_M} (M \times_{\lambda^{(w)}} \mathbb{Z}).$$

□

The above theorem provides the characterization of the v-compressed algebras $L_v \mathbb{M}_G L_v$ of $\mathbb{M}_G$, for all $v \in V(G^\wedge)$. If v has at least one incident loop finite paths, then the v-compressed algebra is characterized by the group crossed product algebras, and if not, then it is identified with M. As corollary, we can get the following result. We will omit the proof, because the generalized fact will be introduced and proven in Sect. 2.3.

Corollary 2.2 *Let $\mathbb{M}_G$ be a graph von Neumann algebra and $L_v \mathbb{M}_G L_v$, the v-compressed algebra, for $v \in V(G^\wedge)$. Assume that*

$$|Loop_v^\delta(G^\wedge)| = N,$$

for $N \in \mathbb{N} \cup \{0\}$. Then the v-compressed algebra $L_v \mathbb{M}_G L_v$ is $*$-isomorphic to a graph von Neumann algebra $\mathbb{M}_{G_N} = M \times_{\alpha_N} \mathbb{G}_N$, where $\mathbb{G}_N$ is the graph groupoid of the one-vertex-N-loop-edge graph G_N (which is a group, group-isomorphic to the free group F_N) and α_N satisfies that

$$(\alpha_N)_w(m) = \alpha_w(m), \quad \textit{for all } m \in M,$$

where $w \in Loop_v^\delta(G^\wedge)$.

Recall that if a fixed von Neumann algebra M is identical to $\mathbb{C}$, then every graph von Neumann algebra $\mathbb{C} \times_\alpha \mathbb{G}$ is $*$-isomorphic to $\overline{\mathbb{C}[\mathbb{G}]}^w$, for all graph-representations α. According to the previous theorem, we can get the following corollary.

Corollary 2.3 *Let M_G be the graph von Neumann algebra $\overline{\mathbb{C}[\mathbb{G}]}^w$ and let $v \in V(G^\wedge)$ be a fixed vertex. Then the v-compressed algebra $L_v M_G L_v$ is $*$-isomorphic to either $L(F_N)$ or $\mathbb{C}$, for some $N \in \mathbb{N}$, where $L(F_N)$ is the free group factor which is a group von Neumann algebra generated by the free group F_N with N-generators.*

Proof By the previous theorem, we can have that the v-compressed algebra $L_v\ M_G L_v$ is $*$-isomorphic to

$$\underset{w \in Loop_v^\delta(G^\wedge)}{*_{\mathbb{C}}} (\mathbb{C} \times_{\lambda(w)} \mathbb{Z}) = \underset{w \in Loop_v^\delta(G^\wedge)}{*_{\mathbb{C}}} (\overline{\mathbb{C}[\mathbb{Z}]}^w)_w$$

where $(\overline{\mathbb{C}[\mathbb{Z}]}^w)_w = \overline{\mathbb{C}[\mathbb{Z}]}^w$, for all $w \in Loop_v^\delta(G^\wedge)$

$$= \underbrace{L(\mathbb{Z}) *_{\mathbb{C}} \ \ldots\ldots\ldots\ldots \ *_{\mathbb{C}}\ L(\mathbb{Z})}_{Loop_v^\delta(G^\wedge)\text{-times}}, \tag{2.5}$$

since $L(\mathbb{Z}) = \overline{\mathbb{C}[\mathbb{Z}]}^w$. Say $N = Loop_v^\delta(G^\wedge) \in \mathbb{N} \cup \{\infty\}$. In general, if Γ_i's are groups, for $i \in I$, then the free product algebra $*_{\mathbb{C}i \in I} L(\Gamma_i)$ of group von Neumann algebras $L(\Gamma_i)$'s is $*$-isomorphic to the group von Neumann algebra $L(*_{i \in I} \Gamma_i)$, where $*_{i \in I} \Gamma_i$ is the group free product of Γ_i's, by Voiculescu. Therefore, the formula (2.5) is $*$-isomorphic to

$$L\left(\underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{N\text{-times}}\right) = L(F_N).$$

Clearly, if $N = 0$ (equivalently, if $Loop_v^\delta(G^\wedge)$ is empty), then $L_v M_G L_v$ is $*$-isomorphic to $\mathbb{C}$. $\square$

The above theorem shows that if a graph groupoid $\mathbb{G}$ has at least one vertex v having k-incident loop finite basic element(s), where $k \neq 0$, then a graph von Neumann algebra $\mathbb{M}_G = M \times_\alpha \mathbb{G}$ contains the free group factor $L(F_k)$, as its W^*-subalgebra.

Example 2.1 Let G be a one-flow circulant graph with its vertex set $V(G) = \{v_1, v_2, v_3\}$ and its edge set $\{e_{12}, e_{23}, e_{31}\}$, where e_{ij} means the edge connecting v_i to v_j. For the fixed von Neumann algebra M, we can construct a graph von Neumann algebra $\mathbb{M}_G = M \times_\alpha \mathbb{G}$. Then the v_1-compressed algebra $L_{v_1}\mathbb{M}_G L_{v_1}$ is $*$-isomorphic to $M \times_\alpha \mathbb{H}_1$, which is $*$-isomorphic to a classical crossed product algebra $M \times_{\lambda_1} \mathbb{Z}$, where $\lambda_1 = \alpha|_{\mathbb{H}_1}$, on M, where $\mathbb{H}_1$ is the collection of all reduced words in $\{e_{12}e_{23}e_{31}, e_{31}^{-1}e_{23}^{-1}e_{12}^{-1}\}$. Notice that the subset $\mathbb{H}_1$ can be regarded as the graph groupoid of a graph H with $V(H) = \{v_1\}$ and $E(H) = \{w = e_{12}e_{23}e_{31}\}$. It is easy to check that

$$loop_{v_1}(G^\wedge) = \{w^n : n \in \mathbb{Z}\} \cup \{v_1\}$$

and

$$loop^\delta_{v_1}(G^\wedge) = \{w, w^{-1}\} \cup \{v_1\},$$

where $w = e_{12}e_{23}e_{31}$. Thus we have that

$$Loop^\delta_{v_1}(G^\wedge) \quad \text{is either} \quad \{w\} \quad \text{or} \quad \{w^{-1}\}.$$

Similarly, we can have that the v_2-compressed algebra $L_{v_2}\mathbb{M}_G L_{v_2}$ is $*$-isomorphic to $M \times_\alpha \mathbb{H}_2$ which is $*$-isomorphic to $M \times_{\lambda_2} \mathbb{Z}$ (with $\lambda_2 = \alpha \mid_{\mathbb{H}_2}$ on M, where $\mathbb{H}_2$ is the collection of all reduced words in $\{e_{23}e_{31}e_{12}, e_{12}^{-1}e_{31}^{-1}e_{23}^{-1}\}$) and $L_{v_3}\mathbb{M}_G L_{v_3}$ is $*$-isomorphic to $M \times_\alpha \mathbb{H}_3$ which is $*$-isomorphic to $M \times_{\lambda_3} \mathbb{Z}$ (with $\lambda_3 = \alpha \mid_{\mathbb{H}_3}$ on M, where $\mathbb{H}_3$ is the collection of all reduced words in $\{e_{31}e_{12}e_{23}, e_{23}^{-1}e_{12}^{-1}e_{31}^{-1}\}$). Therefore, the v_k-compressed algebras $L_{v_k}\mathbb{M}_G L_{v_k}$'s are $*$-isomorphic from each other, by [10] and [11].

Suppose $M = \mathbb{C}$. Then all vertex-compressed algebras $L_{v_k}\mathbb{M}_G L_{v_k}$ are $*$-isomorphic to the group von Neumann algebra $L(\mathbb{Z})$ which is $*$-isomorphic to $L^\infty(\mathbb{T})$, for all $k = 1, 2, 3$, where $\mathbb{T}$ is the unit circle of $\mathbb{C}$.

Example 2.2 Suppose G is a directed graph with its isolated vertex v_0. Recall that v_0 is an isolated vertex if there is no incident edges of v_0. Then the v_0-compressed algebra $L_{v_0}\mathbb{M}_G L_{v_0}$ is $*$-isomorphic to M, whenever $\mathbb{M}_G = M \times_\alpha \mathbb{G}$, since $loop_{v_0}(G^\wedge) \setminus \{v_0\}$ is empty in $\mathbb{G}$.

2.2 Vertex-Compressions

In the previous section, we characterize "single" vertex-compressed algebras of a given graph von Neumann algebra. In this section, we consider the general case. As before, we will let M be a fixed von Neumann algebra acting on a Hilbert space K and G, a countable directed graph with its graph groupoid $\mathbb{G}$, and let $\mathbb{M}_G = M \times_\alpha \mathbb{G}$ be a graph von Neumann algebra induced by G over M, via a graph-representation α.

Let $V = \{v_1, \ldots, v_N\}$ be a subset of the vertex set $V(G^\wedge)$, for $N \in \mathbb{N}$. Then this set has its corresponding family $\{L_{v_1}, \ldots, L_{v_N}\}$ of projections in a graph von Neumann algebra $\mathbb{M}_G$. Then we can define a projection P_V by adding those projections. i.e.,

$$P_V \overset{def}{=} \sum_{j=1}^{N} L_{v_j} \in \mathbb{M}_G.$$

Since L_{v_j}'s are orthogonal from each other, the operator P_V is again a projection in $\mathbb{M}_G$. Indeed,

$$
\begin{aligned}
P_V^2 &= \left(\sum_{j=1}^{N} L_{v_j}\right)^2 = \sum_{j=1}^{N} L_{v_j}^2 = \sum_{j=1}^{N} L_{v_j} \\
&= P_V = \sum_{j=1}^{N} L_{v_j} \\
&= \sum_{j=1}^{N} L_{v_j^{-1}} = \sum_{j=1}^{N} L_{v_j}^* = \left(\sum_{j=1}^{N} L_{v_j}\right)^* = P_V^*.
\end{aligned}
$$

Thus, with respect to this projection $P_V \in \mathbb{M}_G$, we can determine the corner of our graph von Neumann algebra $\mathbb{M}_G$, which is a compressed W^*-subalgebra $P_V \mathbb{M}_G P_V$.

Definition 2.3 Let $P_V = \sum_{j=1}^{N} L_{v_j}$ be a projection in a graph von Neumann algebra $\mathbb{M}_G = M \times_\alpha \mathbb{G}$, induced by the subset $V = \{v_1, \ldots, v_N\}$ of $V(G^\wedge)$, where $N \in \mathbb{N}$. Then the compressed W^*-algebra $P_V \mathbb{M}_G P_V$ of $\mathbb{M}_G$ is called a vertex-compressed algebra of $\mathbb{M}_G$ induced by V (in short, the V-compressed algebra of $\mathbb{M}_G$). The corresponding compressed W^*-probability space $(P_V \mathbb{M}_G P_V, E)$ over $P_V \mathbb{D}_G P_V$ is called the V-compressed graph W^*-probability space. Here, the conditional expectation E is in fact the restriction $E \mid_{P_V \mathbb{M}_G P_V}$ of the canonical conditional expectation E of $\mathbb{M}_G$ over $\mathbb{D}_G$.

Consider a compressed element $P_V x P_V$ in the V-compressed algebra $P_V \mathbb{M}_G P_V$, where $V = \{v_1, \ldots, v_N\} \subseteq V(G^\wedge)$:

$$
\begin{aligned}
P_V x P_V &= \left(\sum_{j=1}^{N} L_{v_j}\right)\left(\sum_{w \in \mathbb{G}} m_w L_w\right)\left(\sum_{j=1}^{N} L_{v_j}\right) \\
&= \sum_{(i,j) \in \{1,\ldots,N\}^2} \left(\sum_{w \in \mathbb{G}} L_{v_i} m_w L_w L_{v_j}\right) \\
&= \sum_{(i,j) \in \{1,\ldots,N\}^2} \left(\sum_{w \in \mathbb{G}} m_w^{v_i} L_{v_i} L_w L_{v_j}\right)
\end{aligned}
$$

where $m_w^{v_i} = m_w^{v_i^{-1}} = \alpha_{v_i}(m_w) = m_w$, for all $w \in \mathbb{G}$

$$
\begin{aligned}
&= \sum_{(i,j) \in \{1,\ldots,N\}^2} \left(\sum_{w \in \mathbb{G}} m_w L_{v_i w v_j}\right) \\
&= \sum_{(i,j) \in \{1,\ldots,N\}^2} \left(\sum_{w \in \mathbb{G},\, w = v_i w v_j} m_w L_w\right) \\
&= \sum_{w \in (\bigcup_{(i,j) \in \{1,\ldots,N\}^2} \mathbb{G}(v_i, v_j))} m_w L_w, \qquad (2.6)
\end{aligned}
$$

where

$$\mathbb{G}(v_i, v_j) \overset{def}{=} \begin{cases} \{w \in FP_r(G^{\wedge}) : w = v_i w v_j\} & \text{if } i \neq j \\ loop_{v_i}(G^{\wedge}) & \text{if } i = j, \end{cases} \tag{2.7}$$

for all $(i, j) \in \{1, \ldots, N\}^2$.

Recall that two elements $w_1, w_2 \in \mathbb{G}$ are diagram-distinct if (i) $w_1 \neq w_2^{-1}$ and (ii) the diagrams $\delta(w_1)$ and $\delta(w_2)$ are distinct. Similarly, arbitrary two subsets X_1 and X_2 of $\mathbb{G}$ are diagram-distinct, if the elements w_1 and w_2 are diagram-distinct, for all pairs $(w_1, w_2) \in X_1 \times X_2$. In [9], we showed that the subsets $M \cdot L_{w_1}$ and $M \cdot L_{w_2}$ are free over $\mathbb{D}_G$ in $(\mathbb{M}_G, E)$ if and only if w_1 and w_2 are diagram-distinct (also, see Sect. 1.1). More generally, two subsets $\bigcup_{w_1 \in X_1}(M \cdot L_{w_1})$ and $\bigcup_{w_2 \in X_2}(M \cdot L_{w_2})$ are free over $\mathbb{D}_G$ in $(\mathbb{M}_G, E)$ if and only if X_1 and X_2 are diagram-distinct.

Lemma 2.4 *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be a W^*-subalgebra of a graph von Neumann algebra $\mathbb{M}_G$, and assume that they contain the M-diagonal subalgebra $\mathbb{D}_G$, as their common W^*-subalgebra. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are free over $\mathbb{D}_G$ in $(\mathbb{M}_G, E)$, then the corresponding V-compressed algebras $P_V \mathcal{M}_1 P_V$ and $P_V \mathcal{M}_2 P_V$ are free over $P_V \mathbb{D}_G P_V$ in the compressed W^*-probability space $(P_V \mathbb{M}_G P_V, E)$, too.*

Proof Recall that a graph von Neumann algebra $\mathbb{M}_G$ is $*$-isomorphic to the $\mathbb{D}_G$-valued reduced free product algebra $*^r_{\mathbb{D}_G\, e \in E(G)} \mathbb{M}_e$ of the $\mathbb{D}_G$-free blocks $\mathbb{M}_e$'s, for all $e \in E(G)$. Suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are free over $\mathbb{D}_G$ in $(\mathbb{M}_G, E)$. Then, without loss of generality, we can assume that

$$\mathcal{M}_k \overset{*\text{-isomorphic}}{=} \underset{e \in X_k}{*^r_{\mathbb{D}_G}} \mathbb{M}_e, \quad \text{for all } k = 1, 2,$$

where X_1 and X_2 are the subsets of $E(G)$ which are diagram-distinct in $\mathbb{G}$. Then we can get that

$$P_V \mathcal{M}_k P_V \overset{*\text{-isomorphic}}{=} \underset{e \in Y_k}{*^r_{\mathbb{D}_G}} (P_V \mathbb{M}_e P_V),$$

for $k = 1, 2$, where

$$Y_k \overset{def}{=} \left\{ e \in X_k \,\middle|\, \begin{array}{c} e = v_i e v_j, \ v_i, v_j \in V, \\ \text{for all } (i, \ j) \in \{1, \ldots, N\}^2 \end{array} \right\}.$$

Therefore,

$$P_V \mathcal{M}_k P_V \overset{*\text{-isomorphic}}{=} \underset{e \in Y_k}{*^r_{P_V \mathbb{D}_G P_V}} \mathbb{M}_e, \quad \text{for } k = 1, 2,$$

and hence $P_V \mathcal{M}_1 P_V$ and $P_V \mathcal{M}_2 P_V$ are free over $P_V \mathbb{D}_G P_V$ in $(P_V \mathbb{M}_G P_V, E)$. □

By the previous lemma, we can get the following proposition.

Proposition 2.5 *Let $x = \sum_{w \in \mathbb{G}} m_w L_w$ be an element in a graph von Neumann algebra $\mathbb{M}_G = M \times_\alpha \mathbb{G}$ and let $P_V \sum_{j=1}^{N} L_{v_j}$ be the projection induced by $V = \{v_1, \ldots, v_N\} \subseteq$*

$V(G^{\wedge})$ in $\mathbb{M}_G$. Then the compressed element $P_V x P_V$ of x is additively decomposed by the diagonal compressed part

$$x_{(D:V)} = \sum_{j=1}^{N} \left(\sum_{w \in loop_{v_j}(G^{\wedge})} m_w L_w \right),$$

and the off-diagonal compressed part

$$x_{(O:V)} = \sum_{i \neq j \in \{1,\dots,N\}} \left(\sum_{w \in \mathbb{G}(v_i, v_j)} m_w L_w \right).$$

In particular, the diagonal part $x_{(D:V)}$ and the off-diagonal part $x_{(O:V)}$ are free over $P_V \mathbb{D}_G P_V$ in the V-compressed graph W^*-probability space $(P_V \mathbb{M}_G P_V, E)$.

Proof By (2.6), the compressed element $P_V x P_V$ is expressed in the V-compressed algebra $P_V \mathbb{M}_G P_V$, by

$$P_V x P_V = \sum_{(i,j) \in \{1,\dots,N\}^2} \left(\sum_{w \in \mathbb{G}(v_i, v_j)} m_w L_w \right),$$

where $\mathbb{G}(v_i, v_j)$'s are defined in (2.7). This compressed element can be rewritten by

$$\begin{aligned} P_V x P_V &= \sum_{j=1}^{N} \left(\sum_{w \in \mathbb{G}(v_j, v_j)} m_w L_w \right) + \sum_{i \neq j \in \{1,\dots,N\}} \left(\sum_{w \in \mathbb{G}(v_i, v_j)} m_w L_w \right) \\ &= \sum_{j=1}^{N} \left(\sum_{w \in loop_{v_j}(G^{\wedge})} m_w L_w \right) + \sum_{i \neq j \in \{1,\dots,N\}} \left(\sum_{w \in \mathbb{G}(v_i, v_j)} m_w L_w \right) \\ &= x_{(D:V)} + x_{(O:V)}. \end{aligned}$$

Therefore, the compressed element $P_V x P_V = x_{(D:V)} + x_{(O:V)}$, where $x_{(D:V)}$ is the diagonal compressed part and $x_{(O:V)}$ is the off-diagonal compressed part of $P_V x P_V$.

Notice that the two subsets

$$D_V = \bigcup_{i=1}^{N} (loop_{v_i}(G^{\wedge})) \quad \text{and} \quad O_V = \bigcup_{i \neq j \in \{1,\dots,N\}} (\mathbb{G}(v_i, v_j))$$

are diagram-distinct in $\mathbb{G}$, since each element in D_V is a loop finite path and each element in O_V is a non-loop finite path (and hence they are diagram-distinct). Therefore, the parts $x_{(D:V)}$ and $x_{(O:V)}$ of $P_V x P_V$ are free over $\mathbb{D}_G$ in the M-diagonal graph W^*-probability space $(\mathbb{M}_G, E)$. By the previous lemma, the diagonal part $x_{(D:V)}$ and the off-diagonal part $x_{(O:V)}$ are free over $P_V \mathbb{D}_G P_V$ in the V-compressed graph W^*-probability space $(P_V \mathbb{M}_G P_V, E)$, too. □

Let $x = \sum_{w \in \mathbb{G}} m_w L_w$ in a graph von Neumann algebra $\mathbb{M}_G$. Define its support $Supp(x)$ by a subset

$$Supp(x) = \{w \in \mathbb{G} : m_w \neq 0_M\}$$

of $\mathbb{G}$. Clearly, if $x_k = \sum_{w_k \in Supp(x_k)} m_{w_k} L_{w_k}$ are operators in $\mathbb{M}_G$, for $k = 1, 2$, and if $Supp(x_1)$ and $Supp(x_2)$ are diagram-distinct, then these operators x_1 and x_2 are free over $\mathbb{D}_G$ in $(\mathbb{M}_G, E)$.

By the previous proposition, we can partition the support $Supp(P_V x P_V)$ of the V-compressed element $P_V x P_V$. By the existence of diagonal compressed parts and off-diagonal compressed parts, we can decompose the support $Supp(P_V x P_V)$ by

$$Supp(P_V x P_V) = \left(\bigcup_{j=1}^{N} (loop_{v_j}^{(x)}(G^{\wedge}))\right) \cup \left(\bigcup_{i \neq j \in \{1,\dots,N\}} (\mathbb{G}^{(x)}(v_i, v_j))\right),$$

where

$$loop_{v_j}^{(x)}(G^{\wedge}) \overset{def}{=} loop_{v_j}(G^{\wedge}) \cap Supp(x)$$

and

$$\mathbb{G}^{(x)}(v_i, v_j) \overset{def}{=} \mathbb{G}(v_i, v_j) \cap Supp(x).$$

We will say that the set

$$\bigcup_{j=1}^{N} (loop_{v_j}^{(x)}(G^{\wedge})) \overset{denote}{=} Supp_D(P_V x P_V)$$

is the diagonal compressed part of $Supp(P_V x P_V)$, and we will also say that

$$\bigcup_{i \neq j \in \{1,\dots,N\}} (\mathbb{G}^{(x)}(v_i,\ v_j)) \overset{denote}{=} Supp_O(P_V x P_V)$$

is the off-diagonal compressed part of $Supp(P_V x P_V)$.

We want to decompose $Supp_D(P_V x P_V)$ and $Supp_O(P_V x P_V)$ by the diagram-distinct subsets of $\mathbb{G}$. By the very definitions, we have that

$$Supp(x_{(D:V)}) = Supp_D(P_V x P_V)$$

and

$$Supp(x_{(O:V)}) = Supp_O(P_V x P_V),$$

where $x_{(D:V)}$ and $x_{(O:V)}$ are the diagonal compressed part and the off-diagonal compressed part of $P_V x P_V$, respectively, given as in the previous proposition. Thus we can conclude that every compressed element $P_V x P_V$ in the V-compressed algebra $P_V \mathbb{M}_G P_V$ satisfies that

$$\begin{aligned} Supp(P_V x P_V) &= Supp(x_{(D:V)} + x_{(O:V)}) \\ &= Supp(x_{(D:V)}) \cup Supp(x_{(O:V)}) \\ &= Supp_D(P_V x P_V) \cup Supp_O(P_V x P_V). \end{aligned}$$

By the previous observation, we can get the following proposition.

Proposition 2.6 *Let $x = \sum_{w \in \mathbb{G}} m_w L_w$ be an operator in a graph von Neumann algebra $\mathbb{M}_G = M \times_\alpha \mathbb{G}$ and let $P_V = \sum_{j=1}^{N} L_{v_j}$ be the corresponding projection of $V = \{v_1, \ldots, v_N\} \subseteq V(G^\wedge)$, for $N \in \mathbb{N}$. Then each compressed element $P_V x P_V$ in the V-compressed algebra $P_V \mathbb{M}_G P_V$ can be expressed by*

$$P_V x P_V = \sum_{j=1}^{N} x_j + \sum_{i<j \in \{1,\ldots,N\}} x_{ij}$$

with

$$x_j = \sum_{w \in loop_{v_j}^{(x)}(G^\wedge)} m_w L_w, \quad \forall j = 1, \ldots, N$$

and

$$x_{ij} = \sum_{w \in \mathbb{G}^{(x)}(v_i, v_j) \cup \mathbb{G}^{(x)}(v_j, v_i)} m_w L_w, \quad \forall i < j \in \{1, \ldots, N\}.$$

Furthermore, the families $\{x_j : j = 1, \ldots, N\}$ and $\{x_{ij} : i < j \in \{1, \ldots, N\}\}$ are $(P_V \mathbb{D}_G P_V)$-free families in the V-compressed graph W^*-probability space $(P_V\ \mathbb{M}_G P_V, E)$.

Proof By the previous proposition, the compressed operator $P_V x P_V$ in the V-compressed algebra $P_V \mathbb{M}_G P_V$ is decomposed by its diagonal compressed part $x_{(D:V)}$ and its off-diagonal compressed part $x_{(O:V)}$. Also, the support $Supp(P_V\ x\ P_V)$ is decomposed by $Supp(x_{(D:V)})$ and $Supp(x_{(O:V)})$. As we observed before, the set $Supp(x_{(D:V)})$ is identified with the diagonal compressed part $Supp_D(P_V x P_V)$ of the support $Supp(P_V\ x P_V)$ and the set $Supp(x_{(O:D)})$ is identified with the off-diagonal compressed part $Supp_O(P_V x P_V)$ of $Supp(P_V x P_V)$. Since $Supp_D(P_V x P_V)$ is again partitioned by

$$\{loop_{v_j}^{(x)}(G^\wedge) : j = 1, \ldots, N\}$$

and since $Supp_O(P_V x P_V)$ is partitioned by

$$\{\mathbb{G}^{(x)}(v_i, v_j) \cup \mathbb{G}^{(x)}(v_j, v_i) : i < j \text{ in } \{1, \ldots, N\}\},$$

we can have that

$$\begin{aligned}
x &= x_{(D:V)} + x_{(O:V)} \\
&= \sum_{j=1}^{N} \left(\sum_{w \in loop_{v_j}^{(x)}(G^\wedge)} m_w L_w \right) + \sum_{i<j \in \{1,\ldots,N\}} \left(\sum_{w \in \mathbb{G}^{(x)}(v_i,v_j) \cup \mathbb{G}^{(x)}(v_j,v_i)} m_w L_w \right), \\
&= \sum_{j=1}^{N} x_j + \sum_{i<j \in \{1,\ldots,N\}} x_{ij}.
\end{aligned}$$

Consider the subsets $loop_{v_k}^{(x)}(G^\wedge)$ and $\mathbb{G}^{(x)}(v_i, v_j) \cup \mathbb{G}^{(x)}(v_j, v_i)$ of $\mathbb{G}$, for $k, i < j \in \{1, \ldots, N\}$. Then they are diagram-distinct from each other. So, the elements x_k and x_{ij} are free from each other, over $\mathbb{D}_G$ in $(\mathbb{M}_G, E)$. And hence they are free from each other, over $P_V \mathbb{D}_G P_V$ in $(P_V \mathbb{M}_G P_V, E)$. □

The above proposition provides the motivation of Sect. 2.3. Now, we will characterize each vertex-compressed algebra $P_V \mathbb{M}_G P_V$ of a graph von Neumann algebra $\mathbb{M}_G = M \times_\alpha \mathbb{G}$, in terms of a certain graph von Neumann algebra $\mathbb{M}_H = M \times_{\alpha_H} \mathbb{H}$, where $\mathbb{H}$ is the graph groupoid of a suitable graph H, induced by G and $\alpha_H = \alpha \mid_{\mathbb{H}}$.

2.3 Vertex-Compressed Graph von Neumann Algebras

Throughout this section, let G be a countable directed graph with its graph groupoid $\mathbb{G}$ and let $\mathbb{M}_G = M \times_\alpha \mathbb{G}$ be a graph von Neumann algebra induced by G over M, via a graph-representation α. We will consider a compressed algebra $P_V \mathbb{M}_G P_V$ of a graph von Neumann algebra $\mathbb{M}_G = M \times_\alpha \mathbb{G}$, where a projection P_V is induced by a subset V of $V(G^\wedge)$. We will fix a subset $V = \{v_1, \ldots, v_N\}$ of $V(G^\wedge)$, where $N \in \mathbb{N}$, and its corresponding projection $P_V = \sum_{j=1}^N L_{v_j}$.

In Sect. 2.2, we observed the expression of compressed operators in the V-compressed algebra $P_V \mathbb{M}_G P_V$. In particular, we showed that each operator $P_V x P_V$ is additively decomposed by its diagonal compressed part

$$x_{(D:V)} = \sum_{w \in Supp_D(P_V x P_V)} m_w L_w$$

and its off-diagonal compressed part

$$x_{(O:V)} = \sum_{w \in Supp_O(P_V x P_V)} m_w L_w,$$

where the support $Supp(P_V x P_V)$ of $P_V x P_V$ satisfies

$$Supp(P_V x P_V) = Supp_D(P_V x P_V) \cup Supp_O(P_V x P_V)$$

with

$$Supp_D(P_V x P_V) = \bigcup_{j=1}^{N} (loop_{v_j}^{(x)}(G^\wedge))$$

and

$$Supp_O(P_V x P_V) = \bigcup_{i<j \in \{1,\ldots,N\}} (\mathbb{G}^{(x)}(v_i, v_j) \cup \mathbb{G}^{(x)}(v_j, v_i)).$$

By the diagram-distinctness of the subsets

$$\{loop_{v_j}^{(x)}(G^\wedge) : j = 1, \ldots, N\}$$

and

$$\{\mathbb{G}^{(x)}(v_i, v_j) \cup \mathbb{G}^{(x)}(v_j, v_i) : i < j \in \{1, \ldots, N\}\}$$

of $\mathbb{G}$, we can determine the $(P_V \mathbb{D}_G P_V)$-free summands

$$\{x_k : j = 1, \ldots, N\} \quad \text{and} \quad \{x_{ij} : i < j \in \{1, \ldots, N\}\}$$

of $P_V x P_V$ in $(P_V \mathbb{M}_G P_V, E)$, where

$$x_k \overset{def}{=} \sum_{w \in loop_{v_k}^{(x)}(G^\wedge)} m_w L_w$$

and

$$x_{ij} \overset{def}{=} \sum_{w \in \mathbb{G}^{(x)}(v_i, v_j) \cup \mathbb{G}^{(x)}(v_j, v_i)} m_w L_w.$$

This shows us that the V-compressed algebra $P_V \mathbb{M}_G P_V$ is a $(P_V \mathbb{D}_G P_V)$-valued reduced free product of its $(P_V \mathbb{D}_G P_V)$-free blocks

$$\{\mathbb{M}_w^{(V)} \mid w = v_i w v_j, (i, j) \in \{1, \ldots, N\}^2\},$$

where $\mathbb{M}_w^{(V)}$ are certain W^*-subalgebras of $P_V \mathbb{M}_G P_V$ containing their common W^*-subalgebra $P_V \mathbb{D}_G P_V$, where the reduction is still depending on the admissibility on $\mathbb{G}$.

Notation Throughout this section, we will use the following notations, for convenience:

$$P_V \mathbb{M}_G P_V \overset{denote}{=} \mathbb{M}_{G:V}, \qquad P_V \mathbb{D}_G P_V \overset{denote}{=} \mathbb{D}_{G:V},$$

and

$$P_V x P_V \overset{denote}{=} x_V, \quad \text{for all } x \in \mathbb{M}_G.$$

It is clear that the W^*-subalgebra $\mathbb{D}_{G:V}$ of the M-diagonal subalgebra $\mathbb{D}_G$ is nothing but $\oplus_{j=1}^N (M \cdot L_{v_j})$. The following theorem is the main result of this paper. It provides not only the generalization of the main result of Sect. 2.1, but also the characterization of arbitrary vertex-compressed algebras of a graph von Neumann algebra.

Theorem 2.7 *Let $\mathbb{M}_G = M \times_\alpha \mathbb{G}$ be a graph von Neumann algebra and P_V, the corresponding projection of $V = \{v_1, \ldots, v_N\} \subseteq V(G^\wedge)$, and let $\mathbb{M}_{G:V}$ be the corresponding V-compressed algebra of $\mathbb{M}_G$. Then $\mathbb{M}_{G:V}$ is $*$-isomorphic to*

$$\underset{w \in \mathcal{I}_V}{*^r_{\mathbb{D}_{G:V}}} \mathbb{M}_w^{(V)}, \tag{2.8}$$

with

$$\mathbb{M}_w^{(V)} \overset{def}{=} vN(M \times_\alpha \mathbb{G}_w, \mathbb{D}_{G:V}),$$

where $\mathbb{G}_w$ is the subset of $\mathbb{G}$, consisting of all reduced words in $\{w, w^{-1}\}$, for all $w \in \mathcal{I}_V$, and where

$$\mathcal{I}_V \overset{def}{=} \left(\bigcup_{j=1}^N Loop_{v_j}^\delta (G^\wedge) \right) \cup \left(\bigcup_{i<j \in \{1,\ldots,N\}} (\mathbb{G}_\delta(v_i, v_j)) \right).$$

Here, the subsets $Loop_{v_j}^\delta(G^\wedge)$'s of $\mathcal{I}_V$ are defined in Sect. 2.1, for $j = 1, \ldots, N$, and the subsets $\mathbb{G}_\delta(v_i, v_j)$'s of $\mathcal{I}_V$ are defined by

$$\mathbb{G}_\delta(v_i, v_j) \overset{def}{=} \left\{ w \in \mathbb{G}(v_i, v_j) : \delta(w) \notin \bigcup_{j=1}^N (Loop_{v_j}^\delta (G^\wedge)) \right\},$$

for all $i < j \in \{1, \ldots, N\}$.

Proof By the construction of vertex-compressions, we can conclude that the V-compressed algebra $\mathbb{M}_{G:V}$ is a W^*-subalgebra of the von Neumann algebra

$$M_0 \overset{def}{=} vN(S_1,\ S_2,\ \mathbb{D}_{G:V})$$

generated by

$$S_1 \overset{def}{=} \bigcup_{j=1}^{N} (\mathbb{M}_{G:\{v_j\}})$$

and

$$S_2 \overset{def}{=} \bigcup_{i<j\in\{1,\ldots,N\}} (L_{v_i}\mathbb{M}_G L_{v_j} \bigcup L_{v_j}\mathbb{M}_G L_{v_i}).$$

Here, $\mathbb{M}_{G:\{v_j\}}$'s are the well-defined v_j-compressed algebras $L_{v_j}\mathbb{M}_G L_{v_j}$, observed in Sect. 2.1, and $L_{v_i}\mathbb{M}_G L_{v_j}$'s are "subsets" (which are not even an algebra) of $\mathbb{M}_G$. By the diagram-distinctness of

$$\bigcup_{j=1}^{N} (loop_{v_j}(G^{\wedge}) \setminus \{v_j\})$$

and

$$\bigcup_{i<j\in\{1,\ldots,N\}} (\mathbb{G}^{\pm 1}(v_i, v_j)),$$

where

$$\mathbb{G}^{\pm 1}(v_i, v_j) \overset{denote}{=} \mathbb{G}(v_i, v_j) \cup \mathbb{G}(v_j, v_i), \quad \text{for } i < j,$$

the subsets S_1 and S_2 are free over $\mathbb{D}_G$ in $(\mathbb{M}_G, E)$, and hence they are free over $\mathbb{D}_{G:V}$ in the $\mathbb{D}_{G:V}$-valued W^*-probability space (M_0, E_0), where $E_0 = E|_{M_0}$ (see Sect. 2.2). Define the families $\mathcal{S}_1$ and $\mathcal{S}_2$ of W^*-subalgebras of $\mathbb{M}_G$ by

$$\mathcal{S}_1 \overset{def}{=} \{\mathbb{M}^{(V)}_{G:\{v_j\}} \overset{def}{=} vN(\mathbb{M}_{G:\{v_j\}}, \mathbb{D}_{G:V}) : j = 1, \ldots, N\}$$

and

$$\mathcal{S}_2 \overset{def}{=} \left\{ \mathbb{M}^{(V)}_{G:i<j} \overset{def}{=} vN(L_{v_i}\mathbb{M}_G L_{v_j} \cup L_{v_j}\mathbb{M}_G L_{v_i}, \mathbb{D}_{G:V}) \,\middle|\, \begin{array}{c} i < j \text{ in} \\ \{1, \ldots, N\} \end{array} \right\}.$$

Then the families $\mathcal{S}_1$ and $\mathcal{S}_2$ are free over $\mathbb{D}_{G:V}$ in $\mathbb{M}_{G:V}$. And hence, M_0 is identified with $vN(\mathcal{S}_1, \mathcal{S}_2)$, which is $*$-isomorphic to

$$M_0 = vN(\mathcal{S}_1, \mathcal{S}_2) \overset{*\text{-isomorphic}}{=} vN(\mathcal{S}_1) *^r_{\mathbb{D}_{G:V}} vN(\mathcal{S}_2).$$

Since $\{L_{v_j} : j = 1, \ldots, N\}$ is a family of mutually orthogonal projections, we have that

$$vN(\mathcal{S}_1) \overset{*\text{-isomorphic}}{=} \bigoplus_{j=1}^{N} \mathbb{M}^{(V)}_{G:\{v_j\}} \quad \text{in } M_0. \tag{2.9}$$

Furthermore, by Sect. 2.1, each direct summand $\mathbb{M}_{G:\{v_j\}}^{(V)}$ is a von Neumann algebra generated by the v_j-compressed algebra $\mathbb{M}_{G:\{v_j\}}$ (which are $*$-isomorphic to either $M = M \cdot L_{v_j}$ or $M \times_{\lambda_j} F_{n_j}$, for some $n_j \in \mathbb{N}$) and $\mathbb{D}_{G:V}$. In particular, if the direct summand $\mathbb{M}_{G:\{v_j\}}^{(V)}$ is $*$-isomorphic to $vN(M \cdot L_{v_j}, \mathbb{D}_{G:V})$, then it is nothing but $\mathbb{D}_{G:V}$.

Now, let's concentrate on observing $\mathcal{S}_2$. Take $\mathbb{M}_{G:i_1<j_1}^{(V)}$ and $\mathbb{M}_{G:i_2<j_2}^{(V)}$, where $(i_1, j_1) \neq (i_2, j_2)$ in $\{1, \ldots, N\}^2$. Then, for all pairs (x_1, x_2) in $\mathbb{M}_{G:i_1<j_1}^{(V)} \times \mathbb{M}_{G:i_2<j_2}^{(V)}$, the support $Supp(x_1)$ and $Supp(x_2)$ are diagram-distinct, because there exist $x'_k = \sum_{w_k \in \mathbb{G}} m_{w_k} L_{w_k} \in \mathbb{M}_G$, for $k = 1, 2$, such that

$$\begin{aligned}
x_k &= L_{v_{i_k}} x'_k L_{v_{j_k}} \\
&= L_{v_{i_k}} \Big(\sum_{w_k \in \mathbb{G}} m_{w_k} L_{w_k} \Big) L_{v_{j_k}} + L_{v_{j_k}} \Big(\sum_{w_k \in \mathbb{G}} m_{w_k} L_{w_k} \Big) L_{v_{i_k}} \\
&= \sum_{w_k \in \mathbb{G}} L_{v_{i_k}} m_{w_k} L_{w_k} L_{v_{j_k}} + \sum_{w_k \in \mathbb{G}} L_{v_{j_k}} m_{w_k} L_{w_k} L_{v_{i_k}} \\
&= \sum_{w_k \in \mathbb{G}} m_{w_k}^{v_{i_k}} L_{v_{i_k}} L_{w_k} L_{v_{j_k}} + \sum_{w_k \in \mathbb{G}} m_{w_k}^{v_{j_k}} L_{v_{j_k}} L_{w_k} L_{v_{i_k}}
\end{aligned}$$

where $m_w^v = \alpha_v(m_w) = m_w$, for all $m \in M$ and $v \in V(G^\wedge)$

$$= \sum_{w = v_{i_k} w v_{j_k} \text{ or } w = v_{j_k} w v_{i_k} \in \mathbb{G}} m_w L_w,$$

for all $k = 1, 2$. Therefore,

$$Supp(x_k) \subseteq \{w \in \mathbb{G} : w = v_{i_k} w v_{j_k} \text{ or } w = v_{j_k} w v_{i_k}\} \cup \{v_i, v_j\},$$

for $k = 1, 2$. i.e., it is contained in $\mathbb{G}^{\pm 1}(v_i, v_j)$. Thus the subsets

$$Supp(x_1) \setminus \{v_{i_1}, v_{j_1}\} \quad \text{and} \quad Supp(x_2) \setminus \{v_{i_2}, v_{j_2}\}$$

of $\mathbb{G}$ are diagram-distinct, and hence x_1 and x_2 are free over $\mathbb{D}_G$ in $(\mathbb{M}_G, E)$. Therefore, the W^*-subalgebras $\mathbb{M}_{G:i_1<j_1}^{(V)}$ and $\mathbb{M}_{G:i_2<j_2}^{(V)}$ are free over $\mathbb{D}_{G:V}$ in (M_0, E_0). This shows that:

$$vN(\mathcal{S}_2) \overset{W^*\text{-subalgebra}}{\subseteq} \Bigg(\underset{(i,j) \in \{1,\ldots,N\}^2,\ i<j}{*^r_{\mathbb{G}_{G:V}}} \mathbb{M}_{G:i<j}^{(V)} \Bigg). \tag{2.10}$$

Observe now the $\mathbb{D}_{G:V}$-valued W^*-probability space $(vN(\mathcal{S}_2), E_0)$, where $E_0 = E_0|_{vN(\mathcal{S}_2)}$. Then we can easily check that the W^*-subalgebras $\{\mathbb{M}_{G:i,j}^{(V)} : i < j\}$ are free over $\mathbb{D}_{G:V}$ from each other in $(vN(\mathcal{S}_2), E_0)$, by the diagram-distinctness. And, by definition, they generates $vN(\mathcal{S}_2)$. So, we can have that

$$\underset{(i,j) \in \{1,\ldots,N\}^2, i<j}{*^r_{\mathbb{D}_{G:V}}} \mathbb{M}_{G:i<j}^{(V)} \overset{W^*\text{-subalgebra}}{\subseteq} vN(\mathcal{S}_2). \tag{2.11}$$

By (2.10) and (2.11), the von Neumann algebra $vN(\mathcal{S}_2)$ is $*$-isomorphic to the $\mathbb{D}_{G:V}$-valued reduced free product algebra $*^r_{\mathbb{D}_{G:V}\, i<j} \mathbb{M}_{G:i<j}^{(V)}$.

Since

$$M_0 = vN(\mathcal{S}_1, \mathcal{S}_2) \overset{*\text{-isomorphic}}{=} vN(\mathcal{S}_1) *^r_{\mathbb{D}_{G:V}} vN(\mathcal{S}_2)$$

we can conclude that the W^*-algebra M_0 of $\mathbb{M}_G$ is $*$-isomorphic to

$$\begin{aligned} M_0 &\overset{*\text{-isomorphic}}{=} \left(\bigoplus_{j=1}^{N} \mathbb{M}^{(V)}_{G:\{v_j\}}\right) *^r_{\mathbb{D}_{G:V}} \left(\underset{(i,j)\in\{1,\ldots,N\}^2,\ i<j}{*^r_{\mathbb{D}_{G:V}}} \mathbb{M}^{(V)}_{G:i<j}\right) \\ &\overset{*\text{-isomorphic}}{=} \left(\underset{j=1}{\overset{N}{*^r_{\mathbb{D}_{G:V}}}} \mathbb{M}^{(V)}_{G:\{v_j\}}\right) *^r_{\mathbb{D}_{G:V}} \left(\underset{(i,j)\in\{1,\ldots,N\}^2,\ i<j}{*^r_{\mathbb{D}_{G:V}}} \mathbb{M}^{(V)}_{G:i<j}\right) \\ &= \underset{w\in(\bigcup_{j=1}^{N}(Loop^{\delta}_{v_j}(G^{\wedge})))\cup(\bigcup_{i<j}\mathbb{G}(v_i,v_j))}{*^r_{\mathbb{D}_{G:V}}} \mathbb{M}^{(V)}_{w}, \end{aligned} \tag{2.12}$$

where

$$\mathbb{M}^{(V)}_{w} = \mathbb{M}^{(V)}_{G:\{v_j\}} \quad \text{if } w \in \bigcup_{j=1}^{N} Loop^{\delta}_{v_j}(G^{\wedge}),$$

and

$$\mathbb{M}^{(V)}_{w} = \mathbb{M}^{(V)}_{G:i<j} \quad \text{if } w \in \bigcup_{i<j\in\{1,\ldots,N\}} \mathbb{G}(v_i, v_j).$$

Under the reduction, if $w = v_i w v_j \in \mathbb{G}(v_i, v_j)$ satisfies that its diagram $\delta(w) \in Loop^{\delta}_{v_j}(G^{\wedge}) \cup Loop^{\delta}_{v_j}(G^{\wedge})$, then there exists $w_{ij} \in \mathbb{G}(v_i, v_j)$ and $k_w \in \mathbb{Z} \setminus \{0\}$ such that (i) the diagram $\delta(w_{ij})$ is not a diagram of a loop finite path, and (ii) $w = w_{ij}\delta(w)^{k_w}$ or $w = \delta(w)^{k_w} w_{ij}$. (See the example at the end of this section.) So, the operator L_w can be regarded as $L_{w_{ij}\delta(w)^k} = L_{w_{ij}} L_{\delta(w)^k}$ or $L_{\delta(w)^k w_{ij}} = L_{\delta(w)^k} L_{w_{ij}}$. So, the operator L_w is contained in $\mathbb{M}^{(V)}_{\delta(w)} *^r_{\mathbb{D}_{G:V}} \mathbb{M}^{(V)}_{w_{ij}}$. i.e.,

$$\mathbb{M}^{(V)}_{w} \subseteq \mathbb{M}^{(V)}_{\delta(w)} *^r_{\mathbb{D}_{G:V}} \mathbb{M}^{(V)}_{w_{ij}}.$$

Therefore, (2.12) can be re-written by

$$\underset{w\in(\bigcup_{j=1}^{N}(Loop^{\delta}_{v_j}(G^{\wedge})))\cup(\bigcup_{i<j}\mathbb{G}_{\delta}(v_i,v_j))}{*^r_{\mathbb{D}_{G:V}}} \mathbb{M}^{(V)}_{w}, \tag{2.12$'$}$$

where

$$\mathbb{G}_{\delta}(v_i, v_j) \overset{def}{=} \left\{ w \in \mathbb{G}(v_i, v_j) : \delta(w) \notin \bigcup_{j=1}^{N} (Loop^{\delta}_{v_j}(G^{\wedge})) \right\}$$

Denote the index set of (2.12)$'$ simply by $\mathcal{I}_V$.

Now, we need to show that the W^*-subalgebra M_0 is $*$-isomorphic to our V-compressed algebra $\mathbb{M}_{G:V}$. Notice that $\mathbb{M}_{G:V}$ is W^*-subalgebra of M_0, since M_0 is $*$-isomorphic to the $\mathbb{D}_{G:V}$-valued reduced free product algebra $*^r_{\mathbb{D}_{G:V}\, M\in\mathcal{S}_1\cup\mathcal{S}_2} M$.

By (2.12)′, the W^*-subalgebra M_0 has its Banach space expression,

$$\mathbb{D}_{G:V} \oplus \left(\bigoplus_{w^* \in E(G^\wedge)^*_r \cap ((\bigcup_{j=1}^N loop_v(G^\wedge)) \cup \mathcal{W}_r(\bigcup_{i<j\in\{1,\ldots,N\}} \mathbb{G}^{\pm 1}(v_i, v_j)))} \mathbb{M}_{w^*}^{(V)\,o} \right), \tag{2.13}$$

with

$$\mathbb{M}_{w^*}^{(V)\,o} = \mathbb{M}_{e_1}^o \otimes_{\mathbb{D}_{G:V}} \cdots \otimes_{\mathbb{D}_{G:V}} \mathbb{M}_{e_n}^o,$$

whenever $w^* e_1 \ldots e_n$, where $\mathbb{M}_e^o = \mathbb{M}_e \ominus \mathbb{D}_G$, for all $e \in E(G)$. Here, $\mathbb{M}_e$'s are the $\mathbb{D}_G$-free blocks of the graph von Neumann algebra $\mathbb{M}_G$, for all $e \in E(G)$, and the set

$$\mathcal{W}_r\left(\bigcup_{i<j\in\{1,\ldots,N\}} \mathbb{G}^{\pm 1}(v_i, v_j) \right)$$

means the all reduced words in $\bigcup_{i<j\in\{1,\ldots,N\}} \mathbb{G}^{\pm 1}(v_i, v_j)$. But notice that

$$\mathcal{W}_r\left(\bigcup_{i<j\in\{1,\ldots,N\}} \mathbb{G}^{\pm 1}(v_i, v_j) \right) = \{\emptyset\} \cup \left(\bigcup_{j=1}^N loop_v(G^\wedge) \right) \cup \left(\bigcup_{i<j\in\{1,\ldots,N\}} \mathbb{G}^{\pm 1}(v_i, v_j) \right),$$

inside $\mathbb{G}$, and hence this set without the empty word $\emptyset$ is contained in the index set of (2.13). (For instance, if $w \in \mathbb{G}^{\pm 1}(v_i, v_j)$ and $w' \in \mathbb{G}^{\pm 1}(v_j, v_k)$, then $w\ w' \in \mathcal{W}_r(\bigcup_{i<j} \mathbb{G}^{\pm 1}(v_i, v_j))$ is either the empty word $\emptyset$ or a reduced word in $\mathbb{G}^{\pm 1}(v_i, v_k) \cup \mathbb{G}^{\pm 1}(v_j, v_k)$.) Therefore, this amalgamated reduced free product algebra M_0 is a W^*-subalgebra of $\mathbb{M}_{G:V}$. This shows that our V-compressed algebra $\mathbb{M}_{G:V}$ is $*$-isomorphic to M_0, which is $*$-isomorphic to the $\mathbb{D}_{G:V}$-valued reduced free product algebra $\ast^r_{\mathbb{D}_{G:V}\, w\in\mathcal{I}_V} \mathbb{M}_w^{(V)}$. □

By the previous theorem, we can conclude that a vertex-compressed algebra $\mathbb{M}_{G:V}$ of a graph von Neumann algebra $\mathbb{M}_G = M \times_\alpha \mathbb{G}$ is again an amalgamated reduced free product algebra $\ast^r_{\mathbb{D}_{G:V}\, w\in\mathcal{I}_V} \mathbb{M}_w^{(V)}$, where the reduction is still depending on the admissibility on the graph groupoid $\mathbb{G}$ of the given graph G. The reduction-preserving property gives us a motive that such amalgamated reduced free product algebra which is $*$-isomorphic to our vertex-compressed algebra $\mathbb{M}_{G:V}$ is another graph von Neumann algebra $M \times_{\alpha_F} \mathbb{F}$, where the graph groupoid $\mathbb{F}$ of a certain graph F is induced by the graph groupoid $\mathbb{G}$ (or it is a sub-groupoid of $\mathbb{G}$). The next theorem shows that indeed it is true. This means that all vertex-compressed algebras of a graph von Neumann algebra $\mathbb{M}_G$ are other graph von Neumann algebras induced by graphs which are induced by G.

Theorem 2.8 *Let $\mathbb{M}_G = M \times_\alpha \mathbb{G}$ be a graph von Neumann algebra and let $V = \{v_1, \ldots, v_N\} \subseteq V(G)$ and $P_V = \sum_{j=1}^N L_{v_j}$. Let $\mathbb{M}_{G:V}$ be the V-compressed algebra of $\mathbb{M}_G$. Then there exists a graph F having its graph groupoid $\mathbb{F}$ such that* (i) *$\mathbb{F}$ is an algebraic substructure* (*or a sub-groupoid*) *of $\mathbb{G}$ having the same admissibility, and* (ii) *this W^*-subalgebra $\mathbb{M}_{G:V}$ of $\mathbb{M}_G$ is $*$-isomorphic to a graph von Neumann algebra $M \times_{\alpha_F} \mathbb{F}$, where*

$\alpha_F = \alpha \mid_{\mathbb{F}}$ *on* M. *In particular, the graph* F *has its vertex set* $V(F) = V$ *and its edge set*

$$E(F) = \left(\bigcup_{j=1}^{N} Loop_{v_j}^{\delta}(G^{\wedge})\right) \cup \left(\bigcup_{i<j\in\{1,\ldots,N\}} \mathbb{G}_{\delta}(v_i,\ v_j)\right).$$

Proof We will use the same notations used in the proof of the previous theorem. Construct a countable directed graph F by a graph with its vertex set $V(F) = V$ and its edge set

$$E(F) = \left(\bigcup_{j=1}^{N} Loop_{v_j}^{\delta}(G^{\wedge})\right) \cup \left(\bigcup_{i<j\in\{1,\ldots,N\}} \mathbb{G}_{\delta}(v_i,\ v_j)\right).$$

Then the graph groupoid $\mathbb{F}$ of F satisfies that

$$\begin{aligned}\mathbb{F} &= \left(\bigcup_{j=1}^{N} (loop_{v_j}(G^{\wedge}))\right) \cup \mathcal{W}_r\left(\bigcup_{i<j\in\{1,\ldots,N\}} \mathbb{G}^{\pm 1}(v_i, v_j)\right) \\ &= \left(\bigcup_{j=1}^{N} (loop_{v_j}(G^{\wedge}))\right) \cup \left(\bigcup_{i<j\in\{1,\ldots,N\}} \mathbb{G}^{\pm 1}(v_i, v_j)\right),\end{aligned} \tag{2.14}$$

as a substructure having the same admissibility of the graph groupoid $\mathbb{G}$ of G.

Now, let $\alpha_F : \mathbb{F} \to B(K \otimes H_F)$ be a F-representation defined by $\alpha_F = \alpha \mid_{\mathbb{F}}$ on M, where H_F is the graph Hilbert space induced by the graph F. Then we can construct a corresponding graph von Neumann algebra $\mathbb{M}_F M \times_{\alpha_F} \mathbb{F}$. Then it has its M-diagonal subalgebra $\mathbb{D}_F \overset{def}{=} \bigoplus_{v\in V(F)}(M \cdot L_v) = \mathbb{D}_{G:V}$, and we can construct the M-diagonal graph W^*-probability space $(\mathbb{M}_F, E_F)$ over $\mathbb{D}_F$. By [9] and [10], we can have that this graph von Neumann algebra $\mathbb{M}_F$ is $*$-isomorphic to the $\mathbb{D}_F$-valued reduced free product algebra

$$\underset{e\in E(F)}{*^{r}_{\mathbb{D}_F}} \mathbb{M}_e^{(F)} \tag{2.15}$$

with

$$\mathbb{M}_e^{(F)} \overset{def}{=} vN(M \times_{\alpha_F} \mathbb{F}_e, \mathbb{D}_F),$$

where $\{\mathbb{M}_e^{(F)} : e \in E(F)\}$ are the $\mathbb{D}_F$-free blocks of $\mathbb{M}_F$, and where $\mathbb{F}_e$ are the subset of $\mathbb{F}$ consisting of all reduced words in $\{e, e^{-1}\}$, for all $e \in E(F)$. Notice that $\mathbb{D}_F = \mathbb{D}_{G:V}$ and $\mathbb{M}_e^{(F)} = \mathbb{M}_e^{(V)}$, by regarding e as an element in $\mathbb{G}$, where "$=$" means "being identical to". Especially, the edge set $E(F) = \mathcal{I}_V$, where $\mathcal{I}_V$ is defined in (2.12)$'$. Therefore, the $\mathbb{D}_F$-valued reduced free product algebra in (2.15) is $*$-isomorphic to the $\mathbb{D}_{G:V}$-valued reduced free product algebra $*^{r}_{\mathbb{D}_{G:V}\, w\in\mathcal{I}_V} \mathbb{M}_w^{(V)}$ in (2.12)$'$. This shows that the graph von Neumann algebra $\mathbb{M}_F$ is $*$-isomorphic to the V-compressed algebra $\mathbb{M}_{G:V}$. □

The above theorem shows that every vertex-compressed algebra $\mathbb{M}_{G:V}$ of a graph von Neumann algebra $\mathbb{M}_G$ is $*$-isomorphic to another graph von Neumann algebra $\mathbb{M}_F$, where the corresponding graph F is determined by the given graph G. Therefore, the above theorem provides the full (amalgamated reduced free probabilistic) characterization of all vertex-compressed algebras of a graph von Neumann algebras, in terms of [9] and [10]. Also, by the previous two theorems, we can get the following corollary.

Corollary 2.9 *Let $\mathbb{M}_G = M \times_\alpha \mathbb{G}$ be a graph von Neumann algebra and let $V = \{v_1, \ldots, v_N\} \subseteq V(G)$ and $P_V = \sum_{j=1}^{N} L_{v_j}$. If $\mathbb{M}_{G:V}$ is the V-compressed algebra of $\mathbb{M}_G$, then it has its Banach space expression,*

$$\mathbb{D}_{G:V} \oplus \left(\bigoplus_{w \in \mathcal{J}_V} \mathbb{M}_w^{(V)} \right) \tag{2.16}$$

with

$$\mathbb{M}_w^{(V)} \overset{def}{=} vN(M \times_\alpha \mathbb{G}_w, \mathbb{D}_{G:V})$$

where $\mathbb{G}_w$ are the subset of $\mathbb{G}$ consisting of all reduced words in $\{w, w^{-1}\}$, for all $w \in \mathcal{J}_V$. Here, the index set $\mathcal{J}_V$ is determined by

$$\mathcal{J}_V \overset{def}{=} E(G^{\wedge})_r^* \cap \left(\left(\bigcup_{j=1}^{N} (loop_{v_j}(G^{\wedge}) \setminus \{v_j\}) \right) \cup \left(\bigcup_{i<j \in \{1,\ldots,N\}} \mathbb{G}^{\pm 1}(v_i, v_j) \right) \right).$$

Example 2.3 Let G be the one-flow circulant graph with $V(G) = \{v_1, v_2, v_3\}$ and $E(G) = \{e_{12}, e_{23}, e_{31}\}$, where e_{ij} means an edge connecting v_i to v_j. Let $V = \{v_1, v_2\}$ and let $P_V = L_{v_1} + L_{v_2}$ be the corresponding projection in a graph von Neumann algebra $\mathbb{M}_G = M \times_\alpha \mathbb{G}$, where M is an arbitrary fixed von Neumann algebra. Then the V-compressed algebra $\mathbb{M}_{G:V}$ is $*$-isomorphic to

$$\mathbb{M}_{v_1}^{(V)} *^r_{\mathbb{D}_{G:V}} \mathbb{M}_{v_2}^{(V)} *^r_{\mathbb{D}_{G:V}} (\mathbb{M}_{e_{12}}^{(V)}),$$

where

$$\mathbb{M}_{v_1}^{(V)} = vN(M \times_{\lambda_1} \mathbb{Z}, \mathbb{D}_{G:V}) = vN(M \times_{\lambda_1} \mathbb{Z}, M^{\oplus 2}),$$
$$\mathbb{M}_{v_2}^{(V)} = vN(M \times_{\lambda_2} \mathbb{Z}, \mathbb{D}_{G:V}) = vN(M \times_{\lambda_2} \mathbb{Z}, M^{\oplus 2}),$$

and

$$\mathbb{M}_{e_{12}}^{(V)} = vN(M \times_\alpha \mathbb{G}_{e_{12}}, \mathbb{D}_{G:V}) = vN(M_2^{\alpha_{e_{12}}}(M), M^{\oplus 2}),$$

by Sect. 2.1, and by the fact that $M \times_\alpha \mathbb{G}_{e_{12}}$ is $*$-isomorphic to

$$M_2^{\alpha_{e_{12}}}(M) \subseteq M_2(M) = M \otimes_{\mathbb{C}} M_2(\mathbb{C}),$$

where $\mathbb{D}_{G:V}(M \cdot L_{v_1}) \oplus (M \cdot L_{v_2})$, which is $*$-isomorphic to $M^{\oplus 2}$. Construct a directed graph F with $V(F) = \{v_1, v_2\}$ and

$$E(F) = (Loop_{v_1}^{\delta}(G^{\wedge})) \cup (Loop_{v_2}^{\delta}(G^{\wedge})) \cup \mathbb{G}_\delta(v_1, v_2)$$

where

$$Loop_{v_1}^{\delta}(G^{\wedge}) = \{w_1 e_{12} e_{23} e_{31}\},$$
$$Loop_{v_2}^{\delta}(G^{\wedge}) = \{w_2 = e_{23} e_{31} e_{12}\},$$
$$\mathbb{G}(v_1, v_2) = \left\{ w_1^n e_{12} : n \in \mathbb{N} \cup \{0\}, w_1^0 \overset{def}{=} v_1 \right\}$$

and

$$\mathbb{G}_{\delta}(v_1, v_2) = \{e_{12}\}.$$

So, the V-compressed algebra $\mathbb{M}_{G:V}$ has its Banach space expression,

$$M^{\oplus 2} \oplus \left(\bigoplus_{w \in E(G^{\wedge})_r^* \cap ((loop_{v_1}(G^{\wedge}) \setminus \{v_1\}) \cup (loop_{v_2}(G^{\wedge}) \setminus \{v_2\}) \cup \mathbb{G}^{\pm 1}(v_1, v_2))} \mathbb{M}_w^{(V)\,o} \right),$$

with

$$\mathbb{M}_w^{(V)\,o} \overset{def}{=} \mathbb{M}_{e_1}^{(V_1)\,o} \otimes_{M^{\oplus 2}} \cdots \otimes_{M^{\oplus 2}} \mathbb{M}_{e_n}^{(V)\,o},$$

whenever $w = e_1 \dots e_n \in E(G^{\wedge})_r^*$ in

$$(loop_{v_1}(G^{\wedge}) \setminus \{v_1\}) \cup (loop_{v_2}(G^{\wedge}) \setminus \{v_2\}) \cup \mathbb{G}^{\pm 1}(v_1, v_2),$$

where $\mathbb{M}_{e_k}^{(V)\,o} \overset{def}{=} \mathbb{M}_{e_k}^{(V)} \ominus M^{\oplus 2}$, for $e_k \in E(F)$, for $k = 1, \dots, n$, $n \in \mathbb{N}$.

Assume now that $M = \mathbb{C}$. Then the V-compressed algebra $\mathbb{M}_{G:V}$ is $*$-isomorphic to

$$M_1 \ *_{\mathbb{C}^{\oplus 2}} M_2 *_{\mathbb{C}^{\oplus 2}} M_3,$$

where

$$M_1 \overset{*\text{-isomorphic}}{=} vN(L(\mathbb{Z}), \mathbb{C}^{\oplus 2}) \overset{*\text{-isomorphic}}{=} M_2$$

and

$$M_3 \overset{*\text{-isomorphic}}{=} vN(M_2(\mathbb{C}), \mathbb{C}^{\oplus 2}) = M_2(\mathbb{C}),$$

where $L(\mathbb{Z})$ is the group von Neumann algebra generated by $\mathbb{Z}$, which is $*$-isomorphic to $L^{\infty}(\mathbb{T})$.

3 More Examples

In this chapter, we will consider more examples about vertex-compressed algebras of a graph von Neumann algebra. Let G be a countable directed graph with its graph groupoid $\mathbb{G}$, and let $\mathbb{M}_G = M \times_{\alpha} \mathbb{G}$ be a graph von Neumann algebra induced by G over an arbitrary von Neumann algebra M.

Let $V = \{v_1, \dots, v_N\} \subseteq V(G)$ be an arbitrary fixed subset and $P_V = \sum_{j=1}^{N} L_{v_j}$, the corresponding projection of V in $\mathbb{M}_G$. Recall that every vertex-compressed algebra $\mathbb{M}_{G:V} = P_V \mathbb{M}_G P_V$ of $\mathbb{M}_G$ is $*$-isomorphic to a new graph von Neumann algebra $\mathbb{M}_F = M \times_{\alpha_F} \mathbb{F}$, where

(i) $\mathbb{F}$ is a graph groupoid of a certain graph F, which is a substructure of $\mathbb{G}$, under the same admissibility, and
(ii) $\alpha_F = \alpha \mid_{\mathbb{F}}$, on M.

In particular, the graph F is a directed graph with its vertex set $V(F) = V$ and its edge set

$$E(F) = \left(\bigcup_{j=1}^{N} Loop_{v_j}^{\delta}(G^{\wedge}) \right) \cup \left(\bigcup_{i<j \in \{1,\ldots,N\}} \mathbb{G}_{\delta}(v_i,\ v_j) \right).$$

Notice that the graph groupoid $\mathbb{F}$ of F satisfies

$$\mathbb{F} = \left(\bigcup_{j=1}^{N} loop_{v_j}(G^{\wedge}) \right) \cup \left(\bigcup_{i<j \in \{1,\ldots,N\}} \mathbb{G}^{\pm 1}(v_i, v_j) \right),$$

set-theoretically, where

$$loop_v(G^{\wedge}) = \{v\} \cup \{w \in FP_r(G^{\wedge}) : w = vwv\}$$

for all $v \in V(G)$, and

$$\mathbb{G}(v_i, v_j) = \{w \in FP_r(G^{\wedge}) : w = v_i w v_j\},$$
$$\mathbb{G}^{\pm}(v_i, v_j) = \mathbb{G}(v_i, v_j) \cup \mathbb{G}(v_j, v_i),$$

for all $i < j \in \{1, \ldots, N\}$, and

$$\mathbb{G}_{\delta}(v_i, v_j) = \left\{ w \in \mathbb{G}(v_i, v_j) : \delta(w) \notin \bigcup_{j=1}^{N} Loop_{v_j}^{\delta}(G^{\wedge}) \right\},$$

where $\delta(w)$ means the diagram of w on the shadowed graph $G^{\wedge}$, for all $w \in \mathbb{G}$.

In the following examples, we will use the above notations.

Example 3.1 Suppose that there exist $n_1, \ldots, n_k \in \{1, \ldots, N\}$, for $k \in \mathbb{N}$, such that $loop_{v_{n_j}}(G^{\wedge})$ is nonempty, for $j = 1, \ldots, k$. Then we can conclude that the V-compressed algebra $\mathbb{M}_{G:V}$ contains its W^*-subalgebra, $*$-isomorphic to

$$\mathbb{M}_{G:V}^{loop} \overset{def}{=} \underset{w \in \bigcup_{j=1}^{k} loop_{v_{n_j}}^{\delta}(G^{\wedge})}{\overset{r}{*}_{\mathbb{D}_{G:V}}} \mathbb{M}_w^{(V)}.$$

By the mutual orthogonality of $\{M \cdot L_{v_{n_j}} : j = 1, \ldots, k\}$, the W^*-subalgebra $\mathbb{M}_{G:V}^{loop}$ is $*$-isomorphic to $\bigoplus_{j=1}^{k} \mathbb{M}_{G:\{v_{n_j}\}}^{(V)}$ inside $\mathbb{M}_G$, where the v_{n_j}-compressed algebras $\mathbb{M}_{G:\{v_{n_j}\}}^{(V)}$ are $*$-isomorphic to the classical crossed product algebras

$$vN(M \times_{\lambda_{n_j}} F_{N_j}, \mathbb{D}_{G:V}), \quad \text{for } j = 1, \ldots, k,$$

where F_{N_j} are the free group with N_j-generators and the group actions $\lambda_{n_j} = \alpha \mid_{\mathbb{G}_{N_j}}$ on M, where

$$N_j = \left| Loop_{v_{n_j}}^{\delta}(G^{\wedge}) \right|, \quad \text{for all } j = 1, \ldots, k,$$

by Sect. 2.1. Thus, the W^*-subalgebra $\mathbb{M}_{G:V}^{loop}$ is $*$-isomorphic to

$$\bigoplus_{j=1}^{k} \big(vN(M \times_{\lambda_{n_j}} F_{N_j}, \mathbb{D}_{G:V}) \big),$$

where $\mathbb{D}_{G:V} = \bigoplus_{j=1}^{N} (M \cdot L_{v_j})$. Notice that each crossed product algebras $M \times_{\lambda_{n_j}} F_{N_j}$ are $*$-isomorphic to the M-valued free product algebra $*_{M}{}_{j=1}^{N_j} (M \times_{\lambda_{n_j}} \mathbb{Z})$. In particular, if $M = \mathbb{C}$, then the V-compressed algebra $\mathbb{M}_{G:V}$ contains its W^*-subalgebra $\bigoplus_{j=1}^{k} vN(L(F_{N_j}), \mathbb{D}_{G:V})$, where $L(F_{N_j})$'s are the free group factors, for $j = 1, \ldots, k$.

Example 3.2 Let G be the 2-regular, 2-story growing tree with its base point (or its root) v_1. i.e.,

$$G = \begin{tikzcd}[column sep=small, row sep=small]
 & & & v_{111} \\
 & & v_{11} \arrow[ur] \arrow[r] & v_{112} \\
v_1 \arrow[urr] \arrow[drr] & & & \\
 & & v_{12} \arrow[r] \arrow[dr] & v_{121} \\
 & & & v_{122}
\end{tikzcd}.$$

For convenience, we will denote the edge e connecting a vertex v to another vertex v' by $[v, v']$. Fix the vertex v_{11} and the corresponding projection $L_{v_{11}}$ in a graph von Neumann algebra $\mathbb{M}_G = M \times_\alpha \mathbb{G}$. The v_{11}-compressed algebra $\mathbb{M}_{G:\{v_{11}\}} = L_{v_{11}} \mathbb{M}_G L_{v_{11}}$ is $*$-isomorphic to $M \cdot L_{v_{11}}$ which is $*$-isomorphic to M, since $loop_{v_{11}}(G^\wedge) = \{v_{11}\}$ and hence $Loop_{v_{11}}^{\delta}(G^\wedge)$ is empty.

Now, fix a subset $\{v_1, v_{11}, v_{12}\}$ in $V(G)$, and the corresponding projection $P_V = L_{v_1} + L_{v_{11}} + L_{v_{12}}$ in $\mathbb{M}_G$. Then the V-compressed algebra $\mathbb{M}_{G:V} = P_V \mathbb{M}_G P_V$ is $*$-isomorphic to

$$\mathbb{M}_{[v_1, v_{11}]}^{(V)} *_{\mathbb{D}_{G:V}}^{r} \mathbb{M}_{[v_1, v_{12}]}^{(V)}.$$

since

$$loop_{v_1}^{\delta}(G^\wedge) = \{v_1\} \quad \text{and} \quad loop_{v_{1i}}^{\delta}(G^\wedge) = \{v_{1i}\}, \quad \text{for } i = 1, 2,$$

and

$$\mathbb{G}_\delta(v_1, v_{11}) = \{[v_1, v_{11}]\} \quad \text{and} \quad \mathbb{G}_\delta(v_1, v_{12}) = \{[v_1, v_{12}]\}.$$

Example 3.3 Let G be the 2-regular, ∞-story growing tree with its base point v_1. i.e.,

$$
G = \begin{array}{lllllll}
 & & & & & & \cdots \\
 & & & & & \nearrow & \\
 & & & & v_{111} & \rightarrow & \cdots \\
 & & & \nearrow & & & \\
 & & v_{11} & \rightarrow & v_{112} & \underset{\searrow}{\rightarrow} & \cdots \\
 & \nearrow & & & & & \cdots \\
v_1 & & & & & & \\
 & \searrow & & & & & \cdots \\
 & & v_{12} & \rightarrow & v_{121} & \underset{\rightarrow}{\nearrow} & \cdots \\
 & & & \searrow & & & \\
 & & & & v_{122} & \rightarrow & \cdots \\
 & & & & & \searrow & \\
 & & & & & & \cdots
\end{array}.
$$

Fix a subset $V = \{v_{11}, v_{111}, v_{112}, v_{1111}, v_{1112}, v_{1121}, v_{1122}, \ldots\}$ in $V(G)$, and the corresponding projection P_V. We can construct a graph F as a 2-regular, ∞-story growing tree with its base point v_{11}.

$$
F = \begin{array}{lllllll}
 & & & & & & \cdots \\
 & & & & & \nearrow & \\
 & & & & v_{1111} & \rightarrow & \cdots \\
 & & & \nearrow & & & \\
 & & v_{111} & \rightarrow & v_{1112} & \underset{\searrow}{\rightarrow} & \cdots \\
 & \nearrow & & & & & \cdots \\
v_{11} & & & & & & \\
 & \searrow & & & & & \cdots \\
 & & v_{112} & \rightarrow & v_{1121} & \underset{\rightarrow}{\nearrow} & \cdots \\
 & & & \searrow & & & \\
 & & & & v_{1122} & \rightarrow & \cdots \\
 & & & & & \searrow & \\
 & & & & & & \cdots
\end{array}.
$$

Then the V-compressed algebra $\mathbb{M}_{G:V}$ is $*$-isomorphic to the graph von Neumann algebra $\mathbb{M}_F = M \times_\alpha \mathbb{F}$, where $\mathbb{F}$ is the graph groupoid of F. Clearly, the graph von Neumann algebras $\mathbb{M}_G$ and $\mathbb{M}_F$ are $*$-isomorphic, since G and F are graph-isomorphic (and hence their shadowed graphs are graph-isomorphic). Therefore, the V-compressed algebra $\mathbb{M}_{G:V}$ is $*$-isomorphic to the given graph von Neumann algebra $\mathbb{M}_G$.

This example shows that a vertex-compressed algebra of a graph von Neumann algebra can be $*$-isomorphic to the given graph von Neumann algebra.

Acknowledgements Special thanks to Prof. Palle Jorgensen.

References

1. Myasnikov, A.G., Shapilrain, V. (eds.): Group Theory, Statistics and Cryptography. Contemporary Math., vol. 360. AMS, Providence (2003)

2. Bollobás, B.: Modern Graph Theory. Graduate Tests in Math., vol. 184. Springer, New York (1998). ISBN:0-387-98488-7
3. Solel, B.: You can see the arrows in a Quiver Operator Algebras. Preprint (2000)
4. Voiculescu, D., Dykemma, K., Nica, A.: Free Random Variables. CRM Monograph Series, vol. 1 (1992)
5. Kribs, D.W., Jury, M.T.: Ideal structure in free semigroupoid algebras from directed graphs. Preprint
6. Kribs, D.W., Power, S.C.: Free semigroupoid algebras. Preprint
7. Cho, I.: Direct producted W^*-probability spaces and corresponding amalgamated stochastic integration. Bull. KMS **44**(1), 131–150 (2007)
8. Cho, I.: Measures on graphs and certain groupoid measures, Complex Anal. Oper. Theory (2006, to appear)
9. Cho, I.: Graph von Neumann algebras. Acta Appl. Math. **95**, 95–134 (2007)
10. Cho, I.: Characterization of amalgamated free blocks of a graph von Neumann algebra. Complex Anal. Oper. Theory **1**, 367–398 (2007)
11. Cho, I.: Towers of graph von Neumann algebras. Preprint (2007)
12. Cho, I., Jorgensen, P.: C^*-algebras generated by partial isometries. JAMC (2007, to appear)
13. Raeburn, I.: Graph Algebras. CBMS Regional Conference Series in Math., vol. 103. AMS, Providence (2005). ISBN:0-8218-3660-9
14. Stallings, J.: Centerless groups-an algebraic formulation of Gottlieb's theorem. Topology **4**, 129–134 (1965)
15. Siestel, R.: Graph Theory. Graduate Texts in Math., vol. 173, 3rd edn. Springer, Berlin (2005). ISBN:978-3-540-26182-7
16. Exel, R.: A new look at the crossed-product of a C^*-algebra by a semigroup of endomorphisms. Preprint (2005)
17. Gliman, R., Shpilrain, V., Myasnikov, A.G. (eds.): Computational and Statistical Group Theory. Contemporary Math., vol. 298. AMS, Providence (2001)
18. Speicher, R. (eds.): Combinatorial Theory of the Free Product with Amalgamation and Operator-Valued Free Probability Theory. AMS Mem, vol. 132, Nr. 627 (1998)
19. Radulescu, F.: Random matrices, amalgamated free products and subfactors of the von Neumann algebra of a free group, of noninteger index. Invent. Math. **115**, 347–389 (1994)
20. Kessebohmer, M., Urbanski, M.: Higher-dimensional multifractal value sets for conformal infinite graph directed Markov Systems. Nonlinearity **20**(8), 1969–1985 (2007)
21. Hutchinson, J.E.: Fractals and self-similarity. Indiana Univ. Math. J. **30**(5), 713–747 (1981)