Journal of Applied Mathematics and Mechanics 78 (2014) 518–523

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

Numerical modelling of the plane problem of the stress state of a tube immersed in a liquid☆

A.O. Kazakova, A.G. Terent'ev

Cheboksary, Russia

ARTICLE INFO

Article history:
Received 10 September 2014

ABSTRACT

A numerical method for solving the plane problem of determining the stress state of a tube of arbitrary section immersed in a homogeneous incompressible liquid is proposed. The change from the boundary conditions for this problem to the boundary conditions for a biharmonic stress function is carried out, which enables the algorithm for solving boundary value problems in the case of a polyharmonic function developed earlier to be used to solve the problem under consideration. It is shown that the boundary conditions for doubly-connected domains contain three unknown constants. The conditions for finding these constants in a form that is convenient for the implementation of a numerical algorithm are obtained. Tubes with sections in the form of concentric, eccentric and elliptic rings are considered as examples.

Methods for calculating cylindrical casings under a variable external pressure are of interest in connection with calculations of the strength of submerged objects of large displacement.[1,2] If a long cylindrical body is placed horizontally in a liquid, the pressure is constant along each of its generating lines and the problem reduces to a plane problem of the theory of elasticity, that is, to the solution of a biharmonic equation. The most complete study of biharmonic functions as a special case of polyharmonic functions is given by Vekua.[3] A number of new problems were subsequently considered using analytical methods.[4,5] However, it should be noted that analytical methods are not always available for such calculations, and results are therefore mainly obtained for certain domains of special form.

A numerical algorithm for solving boundary value problems in the case of a polyharmonic equation developed by the authors[6] is used below to solve the problem of determining the plane stress state of a tube of arbitrary cross section immersed in a ponderable liquid. In this case, the boundary conditions obtained for the corresponding biharmonic equation contain three unknown constants and, in order to find these, conditions for the uniqueness of the displacements are obtained in a form convenient for the use of the numerical algorithm. The proposed method is simple to implement numerically, it possesses a high degree of accuracy and enables a plane problem in the theory of elasticity in an arbitrary doubly-connected domain with arbitrary boundary conditions to be solved.

1. Statement of the problem

Suppose a homogeneous elastic body, that is a hollow cylinder of arbitrary cross section, is immersed in a quiescent homogeneous ponderable liquid of density ρ such that, along each of its generating lines, the hydrostatic pressure acting on the outer surface of the body is constant and the internal pressure P_1 is constant over the whole of the inner surface of the body. Suppose h and H are the minimum and maximum values of the height at which the points of the body immersed in the liquid are located. The hydrostatic pressure acts on the outer surface of the body along a normal and varies linearly: $P(y) = \rho g y$, $y \in [h, H]$. The section of a cylinder with a plane perpendicular to its lateral surface is a plane doubly-connected domain. Suppose this domain is arranged in a Cartesian system of coordinates such that the line $y = 0$ coincides with the free boundary (Fig. 1).

☆ *Prikl. Mat. Mekh.*, Vol. 78, No. 5, pp. 721–727, 2014.
E-mail addresses: agterent@Rambler.Ru, kazakova_anastasia@bk.ru (A.O. Kazakova).

http://dx.doi.org/10.1016/j.jappmathmech.2015.03.011
0021-8928/

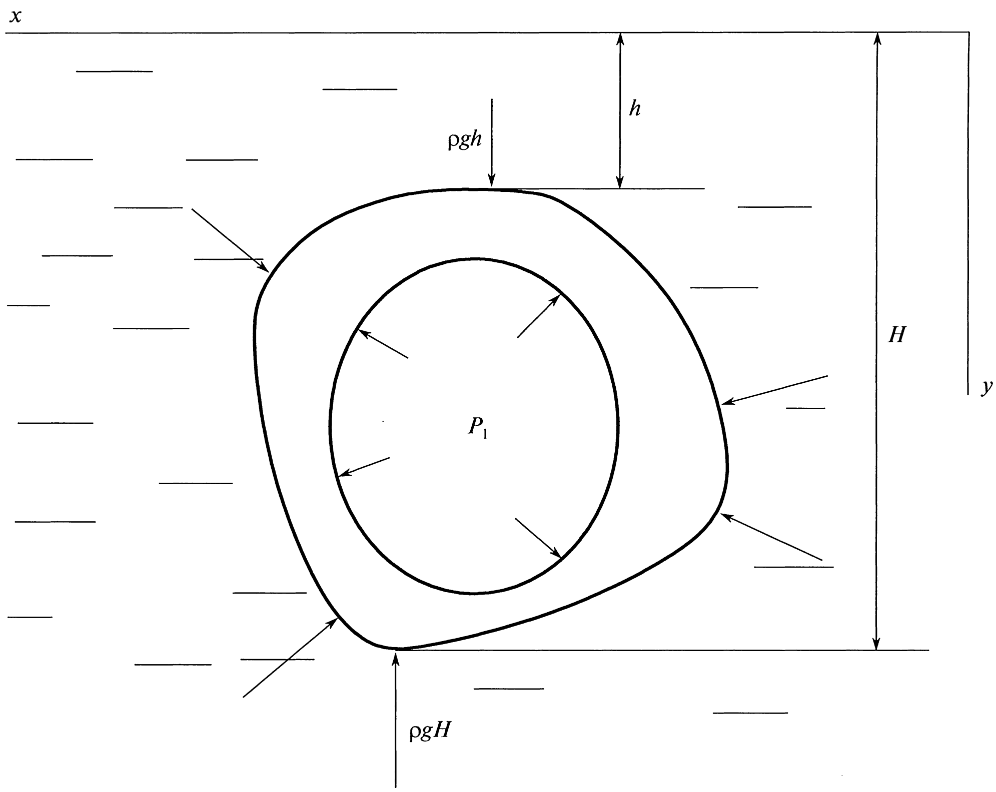

Fig. 1.

The problem of determining the plane stress state of the body considered can then be formulated as follows: *it is required to determine the components of the stress tensor at each point of an arbitrary doubly-connected plane domain D, the boundary* of which $\partial D = (\partial D)_1 \cup (\partial D)_2$ *is specified by the functions of a curved coordinate:*

$$x = x_k(s), \quad y = y_k(s), \quad s \in (\partial D)_k, \quad k = 1, 2$$

The external stress vector on the boundary

$$\mathbf{p} = [p_x(s), p_y(s)] = [P(s)n_x, P(s)n_y]; \quad P(s) = \begin{cases} P_1, & s \in (\partial D)_1 \\ \rho g y_2(s), & s \in (\partial D)_2 \end{cases}$$

is specified, where $\mathbf{n} = (n_x, n_y)$ *is the unit vector of the outward normal to the boundary of the domain D.*

The external stress is expressed in terms of the components of the stress tensor in the form of the equalities

$$\sigma_{xx}n_x + \sigma_{xy}n_y = p_x, \quad \sigma_{xy}n_x + \sigma_{yy}n_y = p_y \tag{1.1}$$

that serve as boundary conditions for finding the stresses.

2. Change to the boundary conditions for a biharmonic stress function

It is well-known that, in a plane problem of the theory of elasticity, all three stresses within a domain *D* can be expressed in terms of a single biharmonic function φ (the stress function)

$$\sigma_{xx} = \frac{\partial^2 \varphi}{\partial y^2}, \quad \sigma_{xy} = -\frac{\partial^2 \varphi}{\partial x \partial y}, \quad \sigma_{yy} = \frac{\partial^2 \varphi}{\partial x^2} \tag{2.1}$$

Boundary conditions (1.1) then become

$$\begin{aligned} \frac{\partial^2 \varphi}{\partial y^2} n_x - \frac{\partial^2 \varphi}{\partial x \partial y} n_y &= \frac{\partial}{\partial s}\left(\frac{\partial \varphi}{\partial y}\right) = p_x(s) \\ \frac{\partial^2 \varphi}{\partial x^2} n_y - \frac{\partial^2 \varphi}{\partial x \partial y} n_x &= -\frac{\partial}{\partial s}\left(\frac{\partial \varphi}{\partial x}\right) = p_y(s) \end{aligned} \tag{2.2}$$

whence it is possible to obtain the normal and tangential derivatives on the boundaries

$$\left.\frac{\partial\varphi}{\partial n}\right|_{(\partial D)_1} = (P_1 x_1(s) + C_1^{(1)})\frac{dy_1}{ds} - (P_1 y_1(s) + C_2^{(1)})\frac{dx_1}{ds}$$
$$\left.\frac{\partial\varphi}{\partial n}\right|_{(\partial D)_2} = (-F_y(s) + C_1^{(2)})\frac{dy_2}{ds} - (F_x(s) + C_2^{(2)})\frac{dx_2}{ds} \tag{2.3}$$

$$\left.\frac{\partial\varphi}{\partial s}\right|_{(\partial D)_1} = (P_1 x_1(s) + C_1^{(1)})\frac{dx_1}{ds} + (P_1 y_1(s) + C_2^{(1)})\frac{dy_1}{ds}$$
$$\left.\frac{\partial\varphi}{\partial s}\right|_{(\partial D)_2} = (-F_y(s) + C_1^{(2)})\frac{dx_2}{ds} + (F_x(s) + C_2^{(2)})\frac{dy_2}{ds} \tag{2.4}$$

where $C_k^{(1)}, C_k^{(2)}(k=1,2)$ are integration constants and $F_x(s)$ and $F_y(s)$ are the projections onto the corresponding axes of the resultant of all the forces on the segment from 0 to s of the external contour $(\partial D)_2$.

The stress function on the boundaries

$$s \in (\partial D)_1: \quad \varphi(s) = P_1\left(\frac{x_1^2(s)}{2} + \frac{y_1^2(s)}{2}\right) + C_1^{(1)}x_1(s) + C_2^{(1)}y_1(s) + C_3^{(1)}$$

$$s \in (\partial D)_2: \quad \varphi(s) = F_x(s)y_2(s) - F_y(s)x_2(s) - \frac{\rho g}{3}y_2^3(s)$$

$$-\rho g\int_0^s y_2(s)x_2(s)\,dx_2(s) + C_1^{(2)}x_2(s) + C_2^{(2)}y_2(s) + C_3^{(2)} \tag{2.5}$$

is determined from equalities (2.4).

It is seen from this that the function φ on the boundaries is determined, apart from a linear term that is unimportant for the stress distribution. However, in a doubly-connected domain, the constants can only be arbitrarily chosen for one contour and, on the other contour, they have to be found from the conditions for the uniqueness of the displacements (see Ref. 7).

Hence, the problem in the theory of elasticity considered is equivalent to the fundamental boundary value problem for a biharmonic function with specified boundary conditions (2.3) and (2.5) defined apart from three constants.

3. Determination of the unknown constants occurring in the boundary conditions

In this problem it is convenient to put the constants on the outer boundary equal to zero. The boundary conditions then contain three unknown constants (on the internal boundary) that can be found from the conditions for the displacements to be unique. The displacement of any element of the domain D is comprised of a translational displacement $\mathbf{u} = (u_x, u_y)$ and a rotation about the z axis. The equalities [5]

$$u_x = \frac{1}{2G}\left(-\frac{\partial\varphi}{\partial x} + \frac{4}{1+\nu}p\right), \quad u_y = \frac{1}{2G}\left(-\frac{\partial\varphi}{\partial y} + \frac{4}{1+\nu}q\right)$$

hold for the displacements u_x and u_y, where G is the shear modulus, ν is Poisson's ratio, and p and q are adjoint harmonic functions satisfying the conditions

$$\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y} = \frac{\Delta\varphi}{4}, \quad \frac{\partial p}{\partial y} = -\frac{\partial q}{\partial x}$$

In view of the single-valuedness of the functions u_x, the integral of its total differential along a closed contour must be equal to zero and therefore

$$\oint_{(\partial D)_1} d\left(-\frac{\partial\varphi}{\partial x} + \frac{4}{1+\nu}p\right) = -\oint_{(\partial D)_1}\left(\frac{\partial^2\varphi}{\partial x^2}dx + \frac{\partial^2\varphi}{\partial x\partial y}dy\right)$$
$$+\frac{4}{1+\nu}\oint_{(\partial D)_1}\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy\right) = \oint_{(\partial D)_1} p_y(s)ds - \frac{4}{1+\nu}\oint_{(\partial D)_1}\frac{\partial q}{\partial n}ds = 0$$

Consequently,

$$\oint_{(\partial D)_1}\frac{\partial q}{\partial n}ds = \frac{1+\nu}{4}\oint_{(\partial D)_1} p_y(s)ds = 0 \tag{3.1}$$

and, similarly,

$$\oint_{(\partial D)_1} \frac{\partial p}{\partial n} ds = -\frac{1+\nu}{4} \oint_{(\partial D)_1} p_x(s) ds = 0 \tag{3.2}$$

The angle of rotation about the z axis is equal to [5]

$$\omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) = \frac{1}{G(1+\nu)}\left(Q(x,y) - \frac{\partial p}{\partial y}\right) = \frac{2Q(x,y)}{G(1+\nu)}; \quad Q(x,y) = \frac{\partial q}{\partial x}$$

whence it follows that the function Q must be single-valued. Since $\Delta\varphi$ and Q are adjoint harmonic functions then

$$Q(s) = \int_0^s dQ = \int_0^s \left(\frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial y} dy\right) = \int_0^s \left(\frac{\partial \Delta\varphi}{\partial x} dy - \frac{\partial \Delta\varphi}{\partial y} dx\right) = \int_0^s \frac{\partial \Delta\varphi}{\partial n} ds$$

The third condition for the displacements to be unique

$$\oint_{(\partial D)_1} \frac{\partial \Delta\varphi}{\partial n} ds = 0 \tag{3.3}$$

follows from this.

Three equations (3.1) - (3.3) have therefore been obtained for determining the three unknown constants on the internal contour $(\partial D)_1$ of the domain D.

4. The use of the boundary element method

Integral relations for polyharmonic functions have been obtained from Green's integral formula.[6] In particular, for a biharmonic function they have the form

$$\begin{aligned} \varepsilon\hat{u}(P) &= \int_{\partial D} (\hat{\upsilon} G_0 - \hat{u} H_0) ds \\ \varepsilon u(P) &= \int_{\partial D} (\upsilon G_0 - u H_0 + \hat{\upsilon} G_1 - \hat{u} H_1) ds \end{aligned} \tag{4.1}$$

where

$$\hat{u} = \Delta u, \quad \upsilon = \frac{\partial u}{\partial n}, \quad \hat{\upsilon} = \frac{\partial \hat{u}}{\partial n}$$

$$G_0 = \frac{1}{2\pi} \ln\frac{1}{r}, \quad G_1 = \frac{r^2}{8\pi}\left(1 + \ln\frac{1}{r}\right), \quad H_0 = \frac{\partial G_0}{\partial n}, \quad H_1 = \frac{\partial G_1}{\partial n}$$

u is a biharmonic function, r is the distance between the point P and the variable point of integration and the factor $\varepsilon = 1/2$ for the point P on the smooth boundary and $\varepsilon = 1$ for an internal point.

Using the boundary element method, Eqs (4.1) can be represented in the form of the two matrix equations

$$\begin{aligned} &(\varepsilon\mathbf{E} + \mathbf{A})\hat{\mathbf{U}} - \mathbf{B}\hat{\mathbf{V}} = 0 \\ &(\varepsilon\mathbf{E} + \mathbf{A})\mathbf{U} - \mathbf{B}\mathbf{V} + \hat{\mathbf{A}}\hat{\mathbf{U}} - \hat{\mathbf{B}}\hat{\mathbf{V}} = 0 \end{aligned} \tag{4.2}$$

where $\mathbf{E}$ is the unit matrix, $\mathbf{U}$, $\mathbf{V}$, $\hat{\mathbf{U}}$, $\hat{\mathbf{V}}$ are column vectors, the components of which are the values of the functions at the control points

$$U_j = u(P_j), \quad V_j = \upsilon(P_j), \quad \hat{U}_j = \hat{u}(P_j), \quad \hat{V}_j = \hat{\upsilon}(P_j), \quad j = 1, 2, \ldots, N$$

and $\mathbf{A}$, $\mathbf{B}$, $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$ are matrices, the elements of which are calculated by integrating the corresponding functions over the rectilinear boundary elements Γ_j:

$$A_{i,j} = \int_{\Gamma_j} H_0 ds, \quad B_{i,j} = \int_{\Gamma_j} G_0 ds, \quad \hat{A}_{i,j} = \int_{\Gamma_j} H_1 ds, \quad \hat{B}_{i,j} = \int_{\Gamma_j} G_1 ds; \quad i, j = 1, 2, \ldots, N$$

System (4.2) is a system of $2N$ linear algebraic equations in the $2N$ unknown components of the vectors $\hat{\mathbf{U}}$ and $\hat{\mathbf{Y}}$. The elements of the vectors $\mathbf{U}$ and $\mathbf{V}$ are known from the formulation of the problem. In the case of a doubly-connected domain, conditions (3.1)–(3.3) have to be added in order to find the unknown constants. To do this, it is advisable to reduce them to equations that only contain the unknown values of the functions $\hat{u}$ and $\hat{\upsilon}$.

It follows from equality (3.1) that (henceforth summation is carried out from $j = 1$ to $j = N$)

$$\sum \hat{U}_j^{(1)} \int_{\Gamma_j^{(1)}} y'_s ds + \sum Q_j^{(1)} \int_{\Gamma_j^{(1)}} x'_s ds = 0 \tag{4.3}$$

where $\mathbf{Q}^{(1)}$ is the column vector of the values of the function $Q = \partial q/\partial x$ at the control points of the contour $(\partial D)_1$ and, after some transformations, Eq. (4.3) therefore reduces to the equation

$$\sum (h_j^{(l)}(x_1^{(l)} - x_j^{(l)})\hat{V}_j^{(l)} + (y_{j+1}^{(l)} - y_j^{(l)})\hat{U}_j^{(l)}) = 0 \tag{4.4}$$

Similarly, condition (3.2) is transformed to the form

$$\sum (h_j^{(l)}(y_1^{(l)} - y_j^{(l)})\hat{V}_j^{(l)} + (x_{j+1}^{(l)} - x_j^{(l)})\hat{U}_j^{(l)}) = 0 \tag{4.5}$$

and condition (3.3) reduces to the linear equation

$$\sum \hat{V}_j^{(l)} \int\limits_{\Gamma_j^{(l)}} ds = 0 \Leftrightarrow \sum h_j^{(l)}\hat{V}_j^{(l)} = 0 \tag{4.6}$$

Three linear equations (4.4) - (4.6) are therefore obtained for the inner contour of the doubly-connected domain D that, together with system (4.2), form the complete system of equations for finding the values of the functions $\hat{u}$, $\hat{v}$ and the three unknown constants on the inner contour of the boundary of the doubly-connected domain D.

5. Determination of the stresses

Solution of the system of linear equations obtained gives the values of the auxiliary functions $\hat{u}$ and $\hat{v}$ at the control points on the boundary of the domain. The value of the stress function at the internal point $P\,(\varepsilon = 1)$ is determined from the second equation of (4.1), on the basis of which it can be assumed that

$$\varphi(P) = \sum \left[V_j \int\limits_{\Gamma_j} G_0 ds - U_j \int\limits_{\Gamma_j} H_0 ds + \hat{V}_j \int\limits_{\Gamma_j} G_1 ds - \hat{U}_j \int\limits_{\Gamma_j} H_1 ds \right] \tag{5.1}$$

However, for the complete solution of the problem it is also necessary to calculate the stresses using formulae (2.1), that is, the second partial derivatives of the stress function have to be calculated. It is seen from their definition that the functions G_0, G_1, H_0 and H_1 in this case depend on the coordinates (x,y) of the internal point P and on the coordinates of the point of integration. The integrals in equality

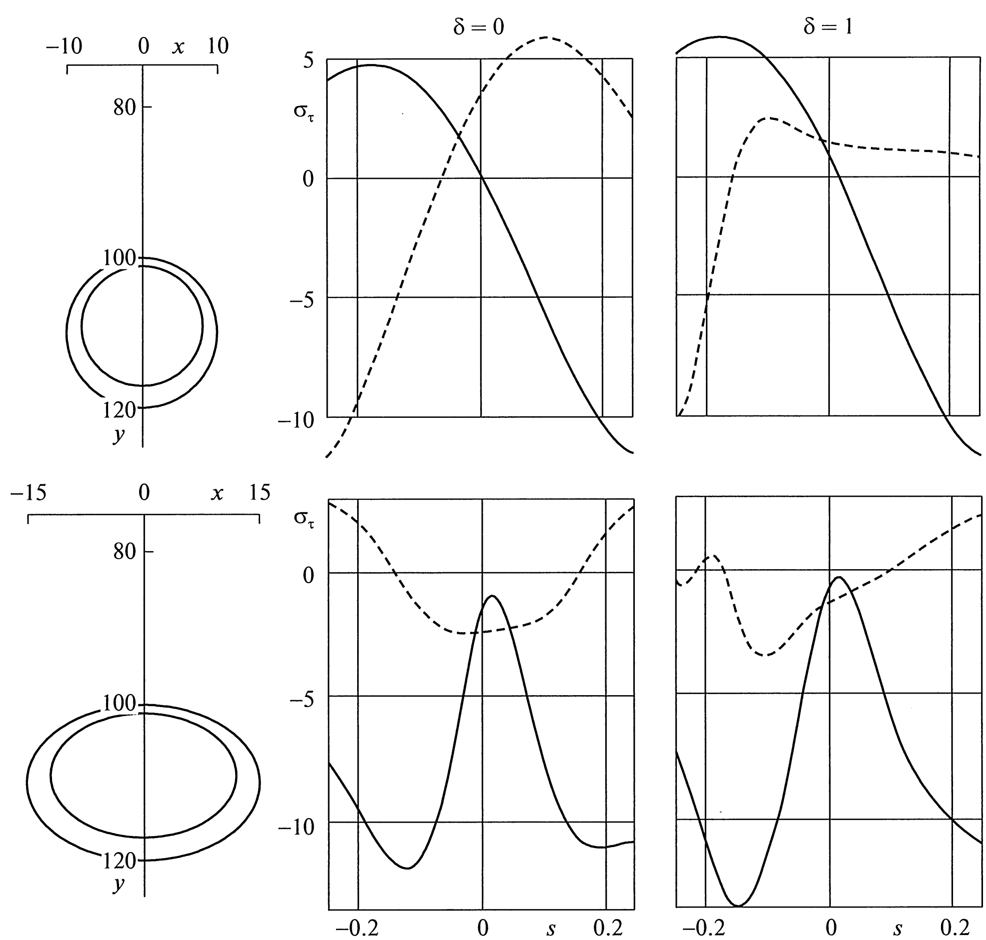

Fig. 2.

(5.1) can therefore be calculated, and the right-hand side of this equality reduces to an expression that only depends on x and y and does not contain integrals. The derivatives of the function φ are then easily calculated, and this means that both the stress function φ as well as the stresses σ_{xx}, σ_{xy}, σ_{yy} themselves can be determined at any point of the domain D.

6. Some special cases

Problems on determining the plane-stress state of a tube with sections in the form of concentric, eccentric and elliptic rings immersed in water at a depth $h = 100$ m ($\rho = 1000$ kg/m^3, $g = 9.8$ N/m) were considered as examples. The calculations were carried out for $N = 60$. The internal pressure P_1 was assumed to be equal to the atmospheric pressure. Using the algorithm described above, after changing from the boundary conditions of the problem to the boundary conditions for the stress function, using the boundary element method and determining the unknown constants on the internal boundary, the value of the stress function φ as well as the components of the stress tensor can be calculated at any point of the domain D.

The sections considered and graphs of the relation between the stress $\sigma_\tau = \partial^2\varphi/\partial n^2$ on the tube boundary and the normalized curved coordinate s are shown in Fig. 2. The results for sections in the form of concentric and eccentric rings with radii of 10m and 8m are presented in the upper part of Fig. 2 and those for elliptic rings with external contour semi-axes of 15m and 10m and internal contour semi-axes of 12m and 8m are presented in the lower part; δ is the distance between the centres of the internal and external boundaries. Plots of the stress σ_τ on the external contour of the section (MPa) are denoted by solid lines and, on the internal contour (kPa) by dashed lines. Graphs are only presented for $s \in [-1/4, 1/4]$, that corresponds to values of the polar angle $\theta \in [-\pi/2, \pi/2]$. The values for the remaining part of the boundary are easily determined by virtue of the symmetry of the domains considered.

References

1. Terent'ev AG, Terent'ev AA. The Global Problem of Mankind and Ways of Solving it. in: *Innovation in the Educational Process*. Cheboksary: MGOU 2011:9: 3–9.
2. Terentiev AG. *Deep water technology: problems and solutions*. Saint-Petersburg: World Maritime Technology Conf.; 2012. p. 1–7.
3. Vekua IN. *New Methods of Solving Elliptic Equations*. Moscow: Gostekhizd; 1948.
4. Lavrent'ev MA, Shabat BV. *Methods in the Theory of Functions of a Complex Variable*. Moscow: Nauka; 1965.
5. Timoshenko SP, Goodier JN. *Theory of Elasticity*. N.Y: McGraw-Hill; 1970.
6. Kazakova AO, Terent'ev AG. Numerical solution of boundary value problems for a polyharmonic equation. *Zh Vychisl Mat i Mat Fiz* 2012;**52**(11):2050–9.
7. Amenadze YuA. *The Theory of Elasticity*. Moscow: Vyssh. Shk; 1976.