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CHROMATIC SUMS OF SINGULAR MAPS ON SOME SURFACES*

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ABSTRACT. A map is singular if each edge is on the same face on a surface (i.e., those have only one face on a surface). Because any map with loop is not colorable, all maps here are assumed to be loopless. In this paper povides the explicit expression of chromatic sum functions for rooted singular maps on the projective plane, the torus and the Klein bottle. From the explicit expression of chromatic sum functions of such maps, the explicit expression of enumerating functions of such maps are also derived.

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1. Introduction

On chromatic sums, the first paper which was published in 1973 by W. T. Tutte [9] is for rooted planar triangulations. Since 1973, Tutte has published a series of papers [9-13] on chromatic sums for rooted planar triangulations to tackle the coloring average problem. And then, Liu [2-5] has published a series of papers on chromatic sums for rooted outerplanar maps and rooted cubic planar maps. All results of chromatic sums having been published are on the plane. In this paper we study the chromatic sums for singular maps on the projective plane, the torus and the Klein bottle. It is well-known that any kind of non planar map is very difficult to count in an exact way, and chromatic sums of non planar maps is more difficult to be determined.

A *surface* is a compact 2-manifold. An(A) *orientable*(*non-orientable*) surface of genus g is homeomorphic to the sphere with g *handles* (*crosscaps*) and is denoted by $S_g(N_g)$, S_1 denotes the torus, N_1 denotes the projective plane, N_2

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denotes the Klein bottle. A circuit C on a surface $\sum$ is called *essential* if $\sum -C$ has no connected region homeomorphic to a disc, otherwise it is planar. Because any map with loop is not colorable, all maps here are always assumed to be loopless. Embedding theory says that any up-embeddability of a graph on a nonorientable surface must have exactly one face and an orientable surface must have exactly one or two faces [6]. Hence, singular maps play a key role in the embeddings of the graphs. Singular maps on the sphere are plane tree. Ren [8] has studied the enumeration of singular maps on some surfaces.

Let $\mathcal{S}$, $\mathcal{P}$, $\mathcal{T}$ and $\mathcal{K}$ be, respectively, the sets of all rooted singular maps on the sphere, the projective plane, the torus and the Klein bottle. Their chromatic sum functions are, respectively,

$$\begin{aligned} g &= g(x,y;\lambda) = \sum_{M\in\mathcal{S}} P(M;\lambda)x^{m(M)}y^{n(M)};\\ f &= f(x,y;\lambda) = \sum_{M\in\mathcal{P}} P(M;\lambda)x^{m(M)}y^{n(M)};\\ F &= F(x,y;\lambda) = \sum_{M\in\mathcal{T}} P(M;\lambda)x^{m(M)}y^{n(M)};\\ h &= h(x,y;\lambda) = \sum_{M\in\mathcal{K}} P(M;\lambda)x^{m(M)}y^{n(M)}. \end{aligned}$$

Where $m(M)$, $n(M)$, $P(M;\lambda)$ be, respectively, the number of nonrooted vertices of M, the number of edges of M and the chromatic polynomial of M.

Now we introduce two well known formulae on chromatic polynomials of maps for further use. The first one is

$$P(M;\lambda) = P(M-R;\lambda) - P(M\bullet R;\lambda), \tag{1}$$

for any map M, where R is an edge of M, $M-R$ and $M\bullet R$ stand for the resultant maps of deleting and contracting R from M respectively. The second one is

$$P(M_1\cup M_2;\lambda) = \frac{1}{\lambda(\lambda-1)\cdots(\lambda-i+1)}P(M_1;\lambda)P(M_2;\lambda), \tag{2}$$

provided that $M_1\cap M_2 = K_i$, the complete graph of order i, and $i\geq 1$.

Theorem 1. *The chromatic sum function of rooted singular maps on the sphere satisfies the following equation*

$$\Big(1-xy\Big(1-\frac{1}{\lambda}\Big)g\Big)g = \lambda$$

and furthermore, for $s \geq 1$

$$g = \sum_{n\geq 0} \frac{\lambda(\lambda-1)^n(2n)!}{n!(n+1)!} x^n y^n;$$

$$g^s = \sum_{n\geq s} \frac{s(2n-s-1)!}{n!(n-s)!} \lambda^s(\lambda-1)^{n-s} x^{n-s} y^{n-s}.$$

Lemma 1. *Let M be a map of k-cycle (with k vertex) . Then, the chromatic polynomial of M is*

$$P(M;\lambda) = (\lambda-1)^k + (-1)^k(\lambda-1).$$

Proof. It can be recursively obtained by (1). □

Given two maps M_1 and M_2 with roots $r_1 = r(M_1)$ and $r_2 = r(M_2)$, respectively, we define $M = M_1 \odot M_2$ to be the map obtained by identifying the root-vertex , the root-edge of M is same as those of M_1, but the root-face of M is the composition of $f_r(M_1)$ and $f_r(M_2)$, where $f_r(M_i)$ is the root-face of $M_i (i = 1, 2)$. The operation for getting M from M_1 and M_2 is called a $1v$*–production*. Further, for two sets of maps $\mathcal{M}_1$ and $\mathcal{M}_2$, the set of maps

$$\mathcal{M}_1 \odot \mathcal{M}_2 = \{M_1 \odot M_2 \,|M_i \in \mathcal{M}_i, i = 1, 2\}$$

is said to be the $1v$*–production* of $\mathcal{M}_1$ and $\mathcal{M}_2$.

2. Chromatic sum for maps on N_1

We define that ϑ*-map* on the projective plane is a nonseparable singular map on the projective plane with one vertex of valency 2.

Fact 1. There is only one kind of rooted ϑ*-map* on the projective plane.

Let $\mathcal{P}_{(ns)}$ denote the set of maps, which is obtained by replacing the edge of ϑ-map on the projective plane with a path. We place a singular map on the sphere at each corner around every vertex of $\mathcal{P}_{(ns)}$. According to this, we can obtain all maps of $\mathcal{P}$.

We divide the set $\mathcal{P}$ into two parts as

$$\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2,$$

where

$$\mathcal{P}_1 = \{M | M \in \mathcal{P}, \quad e_r(M) \text{ is not on an essential circuit}\};$$
$$\mathcal{P}_2 = \{M | M \in \mathcal{P}, \quad e_r(M) \text{ is on an essential circuit}\}.$$

Lemma 2. *$\mathcal{P}_1 = \mathcal{P}(1) \odot \mathcal{S} + \mathcal{S}(1) \odot \mathcal{P}$, where $\mathcal{P}(1)$ and $\mathcal{S}(1)$ be respectively the set of maps in $\mathcal{P}$ and $\mathcal{S}$ with root-valency 1.*

Proof. For any map in $\mathcal{P}_1$, the root-edge $e_r(M)$ is an isthmus. The two submaps determined by $e_r(M)$ are respectively in $\mathcal{P}(1)$ and $\mathcal{S}$, or in $\mathcal{S}(1)$ and $\mathcal{P}$. Hence ,$\mathcal{P}_1$ is a subset of $\mathcal{P}(1) \odot \mathcal{S} + \mathcal{S}(1) \odot \mathcal{P}$.

Conversely, any map M in $\mathcal{P}(1)\odot\mathcal{S}+\mathcal{S}(1)\odot\mathcal{P}$ must has its root-edge separable by the definition of the $1v-$*production* of two maps. Hence, $\mathcal{P}(1)\odot\mathcal{S}+\mathcal{S}(1)\odot\mathcal{P} \subseteq \mathcal{P}_1$. □

According to Lemma 2 and (2), the chromatic sum function of $\mathcal{P}_1$ is

$$f_1 = \frac{2xy(\lambda-1)fg}{\lambda}. \tag{3}$$

Applying Lemma 1 and (2), the chromatic sum function of $\mathcal{P}_2$ is

$$f_2 = \sum_{k\geq 2}[(\lambda-1)^k + (-1)^k(\lambda-1)]x^{k-1}y^k\left(\frac{g}{\lambda}\right)^{2k}, \tag{4}$$

where g is defined in Theorem 1.

Applying (3), (4) and Theorem 1, we have

Theorem 2. *The chromatic sum function of rooted singular maps on the projective plane is*

$$\begin{aligned} f \;=\; & \sum_{k\geq 2, n\geq 2k} \frac{2k(2n-2k-1)!}{n!(n-2k)!}\left[(\lambda-1)^{n-k} + (-1)^k(\lambda-1)^{n-2k+1}\right]x^{n-k-1}y^{n-k} \\ & + \sum_{\substack{k\geq 2, n\geq 2k \\ l\geq s\geq 1}} \frac{2^{s+1}ks(2l-s-1)!(2n-2k-1)!}{n!l!(n-2k)!(l-s)!} \\ & \times\left[\ (\lambda-1)^{n+l-k} + (-1)^k(\lambda-1)^{n+l-2k+1}\ \right]x^{n+l-k-1}y^{n+l-k}. \end{aligned}$$

3. Chromatic sum for maps on S_1

We define that *ϑ-map* on the torus is a nonseparable singular map on the torus without vertices of valency 2.

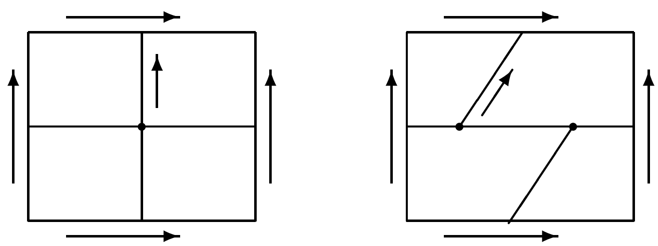
Fact 2. There are only two kinds of rooted *ϑ-maps* on the torus(shown in Fig. 1). Based on the two *ϑ-maps* in Fig. 1, we can obtain rooted singular maps on the torus (root is on an essential circuit).

We divide the set $\mathcal{T}$ into two parts as

$$\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2,$$

where

$$\mathcal{T}_1 = \{M | M \in \mathcal{T}, \quad e_r(M) \text{ is not on an essential circuit}\};$$
$$\mathcal{T}_2 = \{M | M \in \mathcal{T}, \quad e_r(M) \text{ is on an essential circuit}\}.$$



θ_1 θ_2

Fig. 1

As we have reasoned in Lemma 2, we obtain the following:

Lemma 3. $\mathcal{T}_1 = \mathcal{T}(1) \odot \mathcal{S} + \mathcal{S}(1) \odot \mathcal{T}$, *where* $\mathcal{T}(1)$ *and* $\mathcal{S}(1)$ *be respectively the set of maps in* $\mathcal{T}$ *and* $\mathcal{S}$ *with root-valency* 1.

According to Lemma 3 and (2), the chromatic sum function of $\mathcal{T}_1$ is

$$F_1 = \frac{2xy(\lambda - 1)Fg}{\lambda}. \tag{5}$$

We divide the set $\mathcal{T}_2$ into two parts as

$$\mathcal{T}_2 = \mathcal{T}_2^1 + \mathcal{T}_2^2,$$

where

$$\mathcal{T}_2^1 = \{M | M \in \mathcal{T}_2, \quad M \text{ is obtained by } \theta_1\};$$
$$\mathcal{T}_2^2 = \{M | M \in \mathcal{T}_2, \quad M \text{ is obtained by } \theta_2\}.$$

Let $\mathcal{T}_{2(ns)}^i (i = 1, 2)$ denote the set of maps obtained by replacing each edge of θ_i(Fig. 1) with a path. We place a singular map on the sphere at each corner around every vertex of $\mathcal{T}_{2(ns)}^i$. According to this, we can obtain all maps of $\mathcal{T}_2^i$.

Now let us begin to calculate F_{21}, the chromatic sum function of $\mathcal{T}_2^1$. Notice that the maps in $\mathcal{T}_{2(ns)}^1$ are either with their root-valency 4 or 2, applying Lemma

1 and (2), the contribution of $\mathcal{T}_2^1$ is

$$F_{21} = \sum_{k_1\geq 2, k_2\geq 2} \frac{k_1}{\lambda}\left[(\lambda-1)^{k_1}+(-1)^{k_1}(\lambda-1)\right]\left[(\lambda-1)^{k_2}+(-1)^{k_2}(\lambda-1)\right] \times\left(\frac{g}{\lambda}\right)^{2(k_1+k_2)} x^{k_1+k_2-2} y^{k_1+k_2}. \quad (6)$$

Let $A(k_1,k_2,k_3) = [(\lambda-1)^{k_1+1}+(-1)^{k_1+1}(\lambda-1)][(\lambda-1)^{k_2+1}+(-1)^{k_2+1}(\lambda-1)][(\lambda-1)^{k_3+1}+(-1)^{k_3+1}(\lambda-1)]$.

Theorem 3. *Let $\mathcal{M}_1$ be a set of maps (shown in Fig.2, M_1), which is the composition of two cycles, and the two cycles have only one common path. Then, the chromatic polynomial of $\mathcal{M}_1$ is*

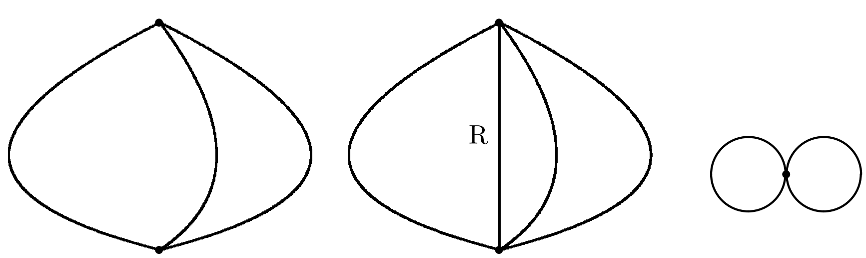
$$P(\mathcal{M}_1;\lambda) = \sum_{\substack{k_1\geq 0, k_2\geq 0\\ k_3\geq 0}} \frac{A(k_1+1,k_2+1,k_3+1)}{\lambda^2(\lambda-1)^2} + \sum_{\substack{k_1\geq 1, k_2\geq 1\\ k_3\geq 1}} \frac{A(k_1,k_2,k_3)}{\lambda^2}.$$

Proof. $\forall M_1 \in \mathcal{M}_1$, we add an edge R on M_1, and obtain a map M_2 (shown in Fig. 2, M_2). We contract R from M_2 and obtain a map M_3 (shown in Fig. 2, M_3).

Applying (1) and (2), we have

$$P(\mathcal{M}_1;\lambda) = \sum_{\substack{k_1\geq 0, k_2\geq 0\\ k_3\geq 0}} \frac{A(k_1+1,k_2+1,k_3+1)}{\lambda^2(\lambda-1)^2} + \sum_{\substack{k_1\geq 1, k_2\geq 1\\ k_3\geq 1}} \frac{A(k_1,k_2,k_3)}{\lambda^2}.$$

□



M_1 M_2 M_3

Fig. 2

Now let us begin to calculate F_{22}, the chromatic sum function of $\mathcal{T}_2^2$. Notice that the maps in $\mathcal{T}_{2(ns)}^2$ are either with their root-valency 3 or 2, applying Theorem 3 and (2), the contribution of $\mathcal{T}_2^2$ is

$$\begin{aligned} F_{22} &= \sum_{\substack{k_1\geq 1,k_2\geq 1\\ k_3\geq 1}} \frac{k_1 A(k_1,k_2,k_3)}{\lambda^2(\lambda-1)^2} x^{k_1+k_2+k_3-2} y^{k_1+k_2+k_3} \left(\frac{g}{\lambda}\right)^{2(k_1+k_2+k_3)} \\ &\quad + \sum_{\substack{k_1\geq 2,k_2\geq 2\\ k_3\geq 2}} \frac{k_1 A(k_1-1,k_2-1,k_3-1)}{\lambda^2} x^{k_1+k_2+k_3-2} y^{k_1+k_2+k_3} \\ &\quad \times \left(\frac{g}{\lambda}\right)^{2(k_1+k_2+k_3)}. \end{aligned} \tag{7}$$

Applying (5)-(7) and Theorem 1, we have

Theorem 4. *The chromatic sums function of rooted singular maps on the torus is*

$$F = \sum_{l\geq s\geq 1} \frac{2^s s(2l-s-1)!}{l!(l-s)!}(\lambda-1)^l x^l y^l F_2 + F_2,$$

where

$$\begin{aligned} F_2 &= F_{21} + F_{22} \\ &= \sum_{\substack{k_1\geq 2,k_2\geq 2\\ n\geq 2(k_1+k_2)}} \frac{2k_1(k_1+k_2)(2n-2k_1-2k_2-1)!}{\lambda n!(n-2k_1-2k_2)!}(\lambda-1)^{n-2k_1-2k_2} \\ &\quad \times \left[(\lambda-1)^{k_1} + (-1)^{k_1}(\lambda-1)\right]\left[(\lambda-1)^{k_2} + (-1)^{k_2}(\lambda-1)\right] x^{n-k_1-k_2-2} \\ &\quad \times y^{n-k_1-k_2} + \sum_{\substack{k_1\geq 1,k_2\geq 1,k_3\geq 1\\ n\geq 2(k_1+k_2+k_3)}} \frac{2k_1(k_1+k_2+k_3)(2n-2k_1-2k_2-2k_3-1)!}{n!(n-2k_1-2k_2-2k_3)!\lambda^2} \\ &\quad \times A(k_1,k_2,k_3)(\lambda-1)^{n-2k_1-2k_2-2k_3-2} x^{n-k_1-k_2-k_3-2} y^{n-k_1-k_2-k_3} \\ &\quad + \sum_{\substack{k_1\geq 2,k_2\geq 2,k_3\geq 2\\ n\geq 2(k_1+k_2+k_3)}} \frac{2k_1(k_1+k_2+k_3)(2n-2k_1-2k_2-2k_3-1)!}{n!(n-2k_1-2k_2-2k_3)!\lambda^2} \\ &\quad \times A(k_1-1,k_2-1,k_3-1)(\lambda-1)^{n-2k_1-2k_2-2k_3} x^{n-k_1-k_2-k_3-2} \\ &\quad \times y^{n-k_1-k_2-k_3}. \end{aligned}$$

4. Chromatic sum for maps on N_2

A *ϑ-map* on the Klein bottle is a nonseparable singular map on the Klein bottle without vertices of valency 2. It is easy to see that

Fact 3. There are only ten kinds of rooted ϑ-maps on the Klein bottle (shown in Fig. 3).

Based on the ten rooted ϑ-maps in Fig. 3, we can obtain all rooted singular maps on the Klein bottle.

We divide the set $\mathcal{K}$ into two parts as

$$\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2,$$

where

$$\begin{aligned}\mathcal{K}_1 &= \{M|M \in \mathcal{K},\ e_r(M) \text{ is not on an essential circuit}\};\\ \mathcal{K}_2 &= \{M|M \in \mathcal{K},\ e_r(M) \text{ is on an essential circuit}\}.\end{aligned}$$

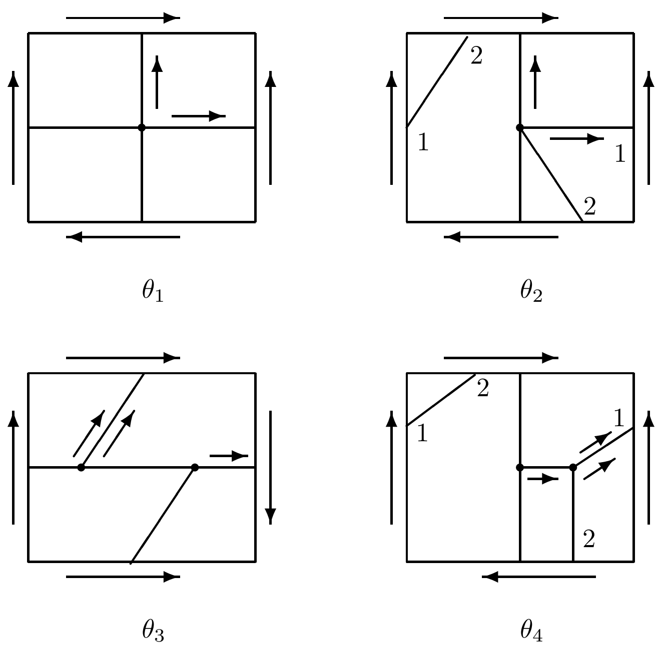


Fig. 3

Lemma 4. $\mathcal{K}_1 = \mathcal{K}(1) \odot \mathcal{S} + \mathcal{S}(1) \odot \mathcal{K}$, *where* $\mathcal{K}(1)$ *and* $\mathcal{S}(1)$ *be, respectively, the set of maps in* $\mathcal{K}$ *and* $\mathcal{S}$ *with root-valency* 1.

Proof. For any map in $\mathcal{K}_1$, the root-edge $e_r(M)$ is an isthmus. The two submaps determined by $e_r(M)$ are, respectively, in $\mathcal{K}(1)$ and $\mathcal{S}$, or in $\mathcal{S}(1)$ and $\mathcal{K}$. Hence, $\mathcal{K}_1$ is a subset of $\mathcal{K}(1) \odot \mathcal{S} + \mathcal{S}(1) \odot \mathcal{K}$.

Conversely, any map M in $\mathcal{K}(1)\odot\mathcal{S}+\mathcal{S}(1)\odot\mathcal{K}$ must has its root-edge separable by the definition of the $1v-production$ of two maps. Hence, $\mathcal{K}(1)\odot\mathcal{S}+\mathcal{S}(1)\odot\mathcal{K} \subseteq \mathcal{K}_1$. $\square$

According Lemma 4 and (2), the chromatic sum function of $\mathcal{K}_1$ is

$$h_1 = \frac{2xy(\lambda-1)hg}{\lambda}. \tag{8}$$

We divide the set $\mathcal{K}_2$ into four parts as

$$\mathcal{K}_2 = \mathcal{K}_2^1 + \mathcal{K}_2^2 + \mathcal{K}_2^3 + \mathcal{K}_2^4,$$

where

$$\begin{aligned}
\mathcal{K}_2^1 &= \{M|M \in \mathcal{K}_2, \quad M \quad \text{is obtained by} \quad \theta_1\};\\
\mathcal{K}_2^2 &= \{M|M \in \mathcal{K}_2, \quad M \quad \text{is obtained by} \quad \theta_2\};\\
\mathcal{K}_2^3 &= \{M|M \in \mathcal{K}_2, \quad M \quad \text{is obtained by} \quad \theta_3\};\\
\mathcal{K}_2^4 &= \{M|M \in \mathcal{K}_2, \quad M \quad \text{is obtained by} \quad \theta_4\}.
\end{aligned}$$

Let $\mathcal{K}^i_{2(ns)} (i = 1, 2, 3, 4)$ denote the set of maps obtained by replacing each edge of θ_i(in $Fig.$3) with a path. We place a singular map on the sphere at each corner around every vertex of $\mathcal{K}^i_{2(ns)}$. According to this, we can obtain all maps of $\mathcal{K}^i_2$.

Now let us begin to calculate h_{21}, the chromatic sum function of $\mathcal{K}^1_2$. Notice that the maps in $\mathcal{K}^1_{2(ns)}$ are either with their root-valency 4 or 2, applying Lemma 1 and (2), the contribution of $\mathcal{K}^1_2$ to $\mathcal{K}$ is

$$\begin{aligned}
h_{21} &= \sum_{k_1\geq 2, k_2\geq 2} \frac{(k_1+k_2)}{\lambda}[(\lambda-1)^{k_1} + (-1)^{k_1}(\lambda-1)][(\lambda-1)^{k_2} \\
&\quad +(-1)^{k_2}(\lambda-1)]\left(\frac{g}{\lambda}\right)^{2(k_1+k_2)} x^{k_1+k_2-2}y^{k_1+k_2}, \qquad (9)\\
h_{22} &= h_{21}. \qquad (10)
\end{aligned}$$

Applying Theorem 3 and (2), we obtain

$$\begin{aligned}
h_{23} &= \sum_{\substack{k_1\geq 1, k_2\geq 1\\ k_3\geq 1}} \frac{(2k_1+k_2)A(k_1,k_2,k_3)}{\lambda^2(\lambda-1)^2} x^{k_1+k_2+k_3-2}y^{k_1+k_2+k_3}\left(\frac{g}{\lambda}\right)^{2(k_1+k_2+k_3)} \\
&\quad + \sum_{\substack{k_1\geq 2, k_2\geq 2\\ k_3\geq 2}} \frac{(2k_1+k_2)A(k_1,k_2,k_3)}{\lambda^2} x^{k_1+k_2+k_3-2}y^{k_1+k_2+k_3} \\
&\quad \times\left(\frac{g}{\lambda}\right)^{2(k_1+k_2+k_3)}. \qquad (11)
\end{aligned}$$

Lemma 5. *Let $\mathcal{M}$ be a set of maps, which are the composition of two cycles and one path. The two cycles are connected by the path(every cycle has more than two vertices). Then, the chromatic polynomial of $\mathcal{M}$ is*

$$\begin{aligned} P(\mathcal{M};\lambda) &= \sum_{\substack{k_1\geq 1, k_2\geq 0\\ k_3\geq 1}} \frac{[(\lambda-1)^{k_1+1}+(-1)^{k_1+1}(\lambda-1)]}{\lambda} \\ &\quad \times \frac{[(\lambda-1)^{k_3+1}+(-1)^{k_3+1}(\lambda-1)](\lambda-1)^{k_2+1}}{\lambda}. \end{aligned}$$

Proof. It can be obtained by Lemma 1 and (2). □

Now let us begin to calculate h_{24}, the chromatic sum function of $\mathcal{K}_2^4$. Applying Lemma 5 and (2), we can obtain

$$\begin{aligned} h_{24} &= \sum_{\substack{k_1\geq 2, k_2\geq 1\\ k_3\geq 2}} \frac{(2k_1+k_2)[(\lambda-1)^{k_1}+(-1)^{k_1}(\lambda-1)][(\lambda-1)^{k_3}}{\lambda} \\ &\quad + \frac{(-1)^{k_3}(\lambda-1)](\lambda-1)^{k_2}}{\lambda} x^{k_1+k_2+k_3-2} y^{k_1+k_2+k_3} \left(\frac{g}{\lambda}\right)^{2(k_1+k_2+k_3)}. \end{aligned} \tag{12}$$

Applying (8)-(12), we have

Theorem 5. *The chromatic sum function of rooted singular maps on the Klein bottle is*

$$h = \sum_{l\geq s\geq 1} \frac{2^s s(2l-s-1)!}{l!(l-s)!}(\lambda-1)^l x^l y^l h_2 + h_2,$$

where

$$\begin{aligned} h_2 &= h_{21}+h_{22}+h_{23}+h_{24} \\ &= \sum_{\substack{k_1\geq 2, k_2\geq 2\\ n\geq 2(k_1+k_2)}} \frac{4(k_1+k_2)^2(2n-2k_1-2k_2-1)!}{\lambda n!(n-2k_1-2k_2)!} \\ &\quad \times(\lambda-1)^{n-2k_1-2k_2}[(\lambda-1)^{k_1}+(-1)^{k_1}(\lambda-1)][(\lambda-1)^{k_2} \\ &\quad +(-1)^{k_2}(\lambda-1)]x^{n-k_1-k_2-2}y^{n-k_1-k_2} \\ &\quad + \sum_{\substack{k_1\geq 1, k_2\geq 1, k_3\geq 1\\ n\geq 2(k_1+k_2+k_3)}} \frac{2(2k_1+k_2)(k_1+k_2+k_3)(2n-2k_1-2k_2-2k_3-1)!}{n!(n-2k_1-2k_2-2k_3)!\lambda^2} \end{aligned}$$

$$\times A(k_1,k_2,k_3)(\lambda-1)^{n-2k_1-2k_2-2k_3-2}x^{n-k_1-k_2-k_3-2}y^{n-k_1-k_2-k_3}$$
$$+\sum_{\substack{k_1\geq 2,k_2\geq 2,k_3\geq 2\\ n\geq 2(k_1+k_2+k_3)}}\frac{2(2k_1+k_2)(k_1+k_2+k_3)(2n-2k_1-2k_2-2k_3-1)!}{n!(n-2k_1-2k_2-2k_3)!\lambda^2}$$
$$\times A(k_1-1,k_2-1,k_3-1)(\lambda-1)^{n-2k_1-2k_2-2k_3}x^{n-k_1-k_2-k_3-2}$$
$$\times y^{n-k_1-k_2-k_3}$$
$$+\sum_{\substack{k_1\geq 2,k_2\geq 1,k_3\geq 2\\ n\geq 2(k_1+k_2+k_3)}}\frac{2(2k_1+k_2)(k_1+k_2+k_3)(2n-2k_1-2k_2-2k_3-1)!}{n!(n-2k_1-2k_2-2k_3)!\lambda}$$
$$\times[(\lambda-1)^{k_1}+(-1)^{k_1}(\lambda-1)][(\lambda-1)^{k_3}+(-1)^{k_3}(\lambda-1)]$$
$$\times(\lambda-1)^{n-2k_1-k_2-2k_3}x^{n-k_1-k_2-k_3-2}y^{n-k_1-k_2-k_3}.$$

5. The case: $\lambda=\infty$

In this section, we use one polynomial $R(M;\gamma)$ stand for the chromatic polynomial $P(M;\lambda)$
$$P(M;\lambda)=\lambda^{m(M)+1}R(M;\gamma),$$
where $\gamma=\lambda^{-1}$, $R(M;0)=1$.

We construct one function g^* correspond to the chromatic sum function g.
$$g^*=g^*(x,y;\gamma)=\sum_{M\in\mathcal{S}}R(M;\gamma)x^{m(M)}y^{n(M)}. \tag{13}$$

Let $X=\lambda^{-1}x$. Then, we obtain
$$g^*(x,y;\gamma)=\lambda^{-1}g(X,y;\lambda). \tag{14}$$

Applying (14) and Theorem 1, we obtain that

Corollary 1. (Tutte [14]) *The enumerating function $g^*(x,y;0)$ of rooted trees has the following explicit expression:*
$$g^*(x,y;0)=\sum_{n\geq 0}\frac{(2n)!}{n!(n+1)!}x^ny^n.$$

We construct one function f^* correspond to the chromatic sum function f.
$$f^*=f^*(x,y;\gamma)=\sum_{M\in\mathcal{P}}R(M;\gamma)x^{m(M)}y^{n(M)}.$$

Let $X=\lambda^{-1}x$. Then, we obtain

$$f^*(x,y;\gamma)=\lambda^{-1}f(X,y;\lambda). \tag{15}$$

Applying (15) and Theorem 2, we obtain that

Corollary 2. *The enumerating function $f^*(x,y;0)$ of singular maps on the projective plane has the following explicit expression:*

$$\begin{aligned} f^*(x,y;0) &= \sum_{k\geq 2, n\geq 2k} \frac{2k(2n-2k-1)!}{n!(n-2k)!}x^{n-k-1}y^{n-k} \\ &+ \sum_{\substack{k\geq 2, n\geq 2k \\ l\geq s\geq 1}} \frac{2^{s+1}ks(2l-s-1)!(2n-2k-1)!}{n!l!(n-2k)!(l-s)!}x^{n+l-k-1}y^{n+l-k}. \end{aligned}$$

We construct one function F^* correspond to the chromatic sum function F.

$$F^*=F^*(x,y;\gamma)=\sum_{M\in\mathcal{T}} R(M;\gamma)x^{m(M)}y^{n(M)}.$$

Similarly, we obtain that

Corollary 3. *The enumerating function $F^*(x,y;0)$ of singular maps on the torus has the following explicit expression:*

$$F^*(x,y;0)=\sum_{l\geq s\geq 1}\frac{2^s s(2l-s-1)!}{l!(l-s)!}x^l y^l F_2^* + F_2^*,$$

where

$$\begin{aligned} F_2^* &= \sum_{\substack{k_1\geq 2, k_2\geq 2 \\ n\geq 2(k_1+k_2)}} \frac{2k_1(k_1+k_2)(2n-2k_1-2k_2-1)!}{n!(n-2k_1-2k_2)!} \\ &\quad \times x^{n-k_1-k_2-2}y^{n-k_1-k_2} \\ &+ \sum_{\substack{k_1\geq 1, k_2\geq 1, k_3\geq 1 \\ n\geq 2(k_1+k_2+k_3)}} \frac{2k_1(k_1+k_2+k_3)(2n-2k_1-2k_2-2k_3-1)!}{n!(n-2k_1-2k_2-2k_3)!} \\ &\quad \times x^{n-k_1-k_2-k_3-2}y^{n-k_1-k_2-k_3}. \end{aligned}$$

We construct one function h^* correspond to the chromatic sum function h.

$$h^*=h^*(x,y;\gamma)=\sum_{M\in\mathcal{K}} R(M;\gamma)x^{m(M)}y^{n(M)}.$$

Similarly, we obtain that

Corollary 4. *The enumerating function $h^*(x,y;0)$ of singular maps on the Klein bottle has the following explicit expression:*

$$h^*(x,y;0) = \sum_{l \geq s \geq 1} \frac{2^s s(2l-s-1)!}{l!(l-s)!} x^l y^l h_2^* + h_2^*,$$

where

$$\begin{aligned} h_2^* \;=\; & \sum_{\substack{k_1 \geq 2, k_2 \geq 2 \\ n \geq 2(k_1+k_2)}} \frac{4(k_1+k_2)^2(2n-2k_1-2k_2-1)!}{n!(n-2k_1-2k_2)!} \\ & \times x^{n-k_1-k_2-2} y^{n-k_1-k_2} \\ & + \sum_{\substack{k_1 \geq 1, k_2 \geq 1, k_3 \geq 1 \\ n \geq 2(k_1+k_2+k_3)}} \frac{2(2k_1+k_2)(k_1+k_2+k_3)(2n-2k_1-2k_2-2k_3-1)!}{n!(n-2k_1-2k_2-2k_3)!} \\ & \times x^{n-k_1-k_2-k_3-2} y^{n-k_1-k_2-k_3} \\ & + \sum_{\substack{k_1 \geq 2, k_2 \geq 1, k_3 \geq 2 \\ n \geq 2(k_1+k_2+k_3)}} \frac{2(2k_1+k_2)(k_1+k_2+k_3)(2n-2k_1-2k_2-2k_3-1)!}{n!(n-2k_1-2k_2-2k_3)!} \\ & \times x^{n-k_1-k_2-k_3-2} y^{n-k_1-k_2-k_3}. \end{aligned}$$

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