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Higgs Bundles and the Fourier-Mukai Transform

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Abstract. The Fourier-Mukai transform is extended to the context of Higgs bundles under certain conditions. Some results valid for sheaves on abelian varieties and K3 surfaces are extended to the situation of Higgs bundles. An application to the relative setting is given.

1. Introduction.

The Fourier-Mukai transform was originally defined by Mukai [9] as a functor between the derived categories of $\mathcal{O}$-modules on an abelian variety and on the dual variety; this functor was proved to be an equivalence of triangulated categories. Subsequently, the notion of Fourier-Mukai transform was extended to $K3$ surfaces in [2]. More generally, it is possible to define Fourier-Mukai functors, which may fail to be invertible, from the derived category of $\mathcal{O}$-modules on a projective variety X to the derived category of $\mathcal{O}$-modules on a projective variety Y, whenever a sheaf $\mathscr{P}$ is defined on the product variety $X \times Y$ (see [4] for a discussion on the invertibility).

Fourier-Mukai functors, if invertible, preserve the Ext groups. Consequently, they induce morphisms between moduli spaces of simple sheaves on X and Y, and these morphisms turn out to be useful tools in the study of the geometrical structure of the moduli space.

Let us consider two projective manifolds X and Y and a vector bundle over their Cartesian product $X \times Y$, together with a morphism

$$\Phi : p_1^*\Omega_X^1 \to p_2^*\Omega_Y^1$$

between the pull-backs of the sheaves of 1-forms, where p_1 (respectively, p_2) is the canonical projections of $X \times Y$ to X (respectively, Y). The purpose of this note is to show how to construct an analog of the Fourier-Mukai functor between Higgs bundles over X and Y (for a definition of Higgs bundles see Section 2). This construction appears to be quite natural in several respects, and it enables us to extend some results valid for sheaves on abelian varieties and K3 surfaces to the situation of Higgs bundles (Sections 2 and 3). In the final Section, an application to the relative setting is given.

2. Fourier-Mukai Transformation of a Higgs Bundle.

Let X be a compact connected complex manifold. A *Higgs sheaf* over X is a pair of the form (E, ϕ), where E is a coherent analytic sheaf over X and ϕ is a $\mathcal{O}_X$-linear homomorphism

$$\phi \, : \, E \to E \otimes \Omega^1_X$$

satisfying the condition that the composition

$$E \xrightarrow{\phi} E \otimes \Omega^1_X \xrightarrow{\bar{\phi}} E \otimes \Omega^1_X \tag{2.1}$$

vanishes identically, where $\bar{\phi}$ sends any vector $e \otimes \omega$ in a fiber of $E \otimes \Omega^1_X$ to the vector $\phi(e) \wedge \omega$. If E is locally free, i.e., it is a holomorphic vector bundle, then the pair (E, ϕ) is called a *Higgs bundle*. The homomorphism ϕ is called the *Higgs field*.

Take a compact connected complex manifold Y. The projection of $X \times Y$ onto X (respectively, Y) will be denoted by p_1 (respectively, p_2).

Let

$$T \, : \, p_1^* \Omega^1_X \to p_2^* \Omega^1_Y \tag{2.2}$$

be a homomorphism of vector bundles.

Fix a holomorphic line bundle $\mathscr{P}$ over $X \times Y$.

Take a Higgs sheaf (E, ϕ) over X. Now consider the i-th direct image

$$\mathscr{F}^i(E) := R^i p_{2*}(p_1^* E \otimes \mathscr{P}), \tag{2.3}$$

which is a coherent analytic sheaf over Y.

Let $\overline{T} \, : \, p_1^* E \otimes p_1^* \Omega^1_X \to p_1^* E \otimes p_2^* \Omega^1_Y$ be the homomorphism, which sends $e \otimes \omega$ to $e \otimes T(\omega)$. Therefore, we have

$$\begin{aligned} \bar{\phi}_T := (\overline{T} \circ p_1^* \phi) \otimes Id_{\mathscr{P}} \, : \, p_1^* E \otimes \mathscr{P} &\to p_1^* E \otimes p_2^* \Omega^1_Y \otimes \mathscr{P} \\ &= p_1^* E \otimes \mathscr{P} \otimes p_2^* \Omega^1_Y, \end{aligned} \tag{2.4}$$

which sends any $e \otimes v$ to $\overline{T} \circ p_1^* \phi(e) \otimes v$.

Taking the direct image using using p_2, the homomorphism ϕ_T induces a homomorphism

$$\phi_T := R^i\bar{\phi}_T \, : \, \mathscr{F}^i(E) \to R^i p_{2*}(p_1^* E \otimes p_2^* \Omega_Y^1) = \mathscr{F}^i(E) \otimes \Omega_Y^1, \tag{2.5}$$

where $\mathscr{F}^i(E)$ is defined in (2.3); the last equality follows from the projection formula.

Theorem 2.6. *The homomorphism*

$$\phi_T \, : \, \mathscr{F}^i(E) \to \mathscr{F}^i(E) \otimes \Omega_Y^1,$$

constructed in (2.5)*, is a Higgs field on* $\mathscr{F}^i(E)$.

Proof. Let

$$\hat{\phi}_T \, : \, p_1^* E \otimes p_2^* \Omega_Y^1 \to p_1^* E \otimes p_2^* \Omega_Y^2$$

be the homomorphism which sends any $e \otimes \omega$ to $\bar{\phi}_T(e) \wedge \omega$, where $\bar{\phi}_T$ is defined in (2.4).

The composition $\hat{\phi}_T \circ \bar{\phi}_T$ evidently coincides with $\bigwedge^2 T \circ p_1^*(\bar{\phi} \circ \phi)$, where $\bar{\phi}$ is defined in (2.1), and $\bigwedge^2 T \, : \, p_1^* \Omega_X^2 \to p_2^* \Omega_Y^2$ is the homomorphism induced by T, i.e., $\bigwedge^2 T(\omega \wedge v) = T(\omega) \wedge T(v)$. Consequently, from the assumption $\bar{\phi} \circ \phi = 0$ we conclude that $\hat{\phi}_T \circ \bar{\phi}_T = 0$.

Let

$$\phi_T' \, : \, \mathscr{F}^i(E) \otimes \Omega_Y^1 \to \mathscr{F}^i(E) \otimes \Omega_Y^2$$

be the homomorphism that sends any $e \otimes \omega$ to $\phi_T(e) \wedge \omega$.

Since $R^i p_{2*}(\phi_T' \circ \phi_T) = \hat{\phi}_T \circ \bar{\phi}_T = 0$, we conclude that ϕ_T is a Higgs field over $\mathscr{F}^i(E)$. This completes the proof of the theorem. □

Therefore, starting with a Higgs sheaf (E, ϕ) over X, using the homomorphism T we have a Higgs sheaf $\left(\mathscr{F}^i(E), \phi_T\right)$ over Y.

3. Higgs Bundles over an Abelian Surface.

Let X be an abelian variety over $\mathbb{C}$ of dimension d. The identity element in the group X will be denoted e.

Let $\hat{X}$ be the dual abelian variety, which is the connected component of the identity element in the Picard group $\mathrm{Pic}(X)$. The group X acts freely transitively on $\hat{X}$. The action of $g \in X$ sends a line bundle ξ over X to $(T_g^{-1})^*\xi$, where T_g is the translation of X given by g, i.e., $T(x) = g + x$. Therefore, for any $x \in X$ and any $y \in \hat{X}$, the tangent spaces $T_x X$ and $T_y \hat{X}$ are identified in a canonical fashion, and these tangent spaces are canonically identified with $H^1(X, \mathcal{O}_X)$.

The identity element of the group $\hat{X}$ will be denoted by $\hat{e}$.

Set Y in Section 2 to be $\hat{X}$. The natural identification of the cotangent space $T_x^* X$ with $T_y^* \hat{X}$ (both of them are naturally isomorphic to $H^1(X, \mathcal{O}_X)^*$) induces a homomorphism T as in (2.2).

The line bundle $\mathscr{P}$ over the product $X \times \hat{X}$ will be the Poincaré bundle, normalized in such a way that its restriction to $X \times \{\hat{e}\}$ is trivial.

Fix once and for all a basis $\{v_1, v_2, \ldots, v_d\}$ of the vector space $H^1(X, \mathcal{O}_X)^*$.

Using the above basis of $H^1(X, \mathcal{O}_X)^*$, a Higgs field on any coherent analytic sheaf E over X is now simply a data $\{\phi_1, \phi_2, \ldots, \phi_d\}$, where $\phi_i \in H^0\big(X, \mathrm{End}(E)\big)$, and $\phi_i \phi_j = \phi_j \phi_i$ for all $i, j \in [1, d]$. Similarly, a Higgs field on a sheaf E over $\hat{X}$ is a data $\{\psi_1, \ldots, \psi_d\}$, where $\psi_i \in H^0\big(\hat{X}, \mathrm{End}(E)\big)$, and $\psi_i \psi_j = \psi_j \psi_i$ for all $i, j \in [1, d]$.

Take a Higgs sheaf $(E, \{\phi_1, \ldots, \phi_d\})$ over X. Let $\big(\mathscr{F}^i(E), \{\psi_1, \ldots, \psi_d\}\big)$ be the corresponding Higgs sheaf over $\hat{X}$ constructed in Section 2. The endomorphism ϕ_j of E naturally induces an endomorphism of $\mathscr{F}^i(E)$. Indeed, the direct image of the endomorphism $p_1^* \phi_j \otimes Id_{\mathscr{P}}$ of $p_1^* E \otimes \mathscr{P}$ is an endomorphism of $\mathscr{F}^i(E)$.

Proposition 3.1. *For any $j \in [1, d]$, the endomorphism $\psi_j \in H^0\big(X, End\big(\mathscr{F}^i(E)\big)\big)$ coincides with the one induced by the endomorphism ϕ_j of E.*

Proof. Since the identifications of $T_e^* X$ and $T_{\hat{e}}^* \hat{X}$ with $H^1(X, \mathcal{O}_X)^*$ commute with the identification of $T_e^* X$ with $T_{\hat{e}}^*$ that has been used in the construction of the homomorphism T, the proposition follows from the construction of the Higgs field ϕ_T done in (2.5). □

The content of the above proposition can also be described without choosing a basis of $H^1(X, \mathcal{O}_X)^*$. A Higgs field on a sheaf E over X (respectively, $\hat{X}$) is an element of $H^1(X, \mathcal{O}_X)^* \otimes H^0\big(X, \mathrm{End}(E)\big)$ (respectively, $H^1(X, \mathcal{O}_X)^* \otimes H^0\big(\hat{X}, \mathrm{End}(E)\big)$). We already observed that an endomorphism of E gives an endomorphism of $\mathscr{F}^i(E)$. In other words, we have a homomorphism

$$h : \mathrm{End}(E) \to \mathrm{End}\big(\mathscr{F}^i(E)\big)$$

Proposition 3.1 says that for a Higgs sheaf (E, ϕ) on X, the Higgs field ϕ_T on $\mathscr{F}^i(E)$, constructed in (2.5), coincides with $(Id \otimes h)(\phi)$, where Id is the identity automorphism of $H^1(X, \mathcal{O}_X)^*$.

Using Proposition 3.1, various properties of the Fourier-Mukai transform of sheaves over abelian varieties (cf. [7]) easily extend to the context of Higgs bundles. For example, any stable zero-degree μ-stable Higgs bundle (E, ϕ) of rank more than one on an abelian surface X has $\mathscr{F}^0(E) = \mathscr{F}^2(E) = 0$, while $\mathscr{F}^1(E)$ is locally free, and $\big(\mathscr{F}^1(E), \phi_T\big)$ is a degree zero μ-stable Higgs bundle on $\hat{X}$ (cf. [5]). The polarization on $\hat{X}$ is obtained from the polarization on X via the Fourier-Mukai transform.

4. More Examples of Fourier-Mukai Transform of Higgs Bundles.

Let Y be a connected complex projective manifold of dimension d. Fix a polarization L over Y. A vector bundle E over Y is called *μ-stable* if for any subsheaf $F \subset E$, with E/F nonzero and torsionfree, the inequality

$$\mu(F) := \frac{\int_Y c_1(F) \cup c_1(L)^{d-1}}{\mathrm{rank}(F)} < \frac{\int_Y c_1(F) \cup c_1(L)^{d-1}}{\mathrm{rank}(F)} =: \mu(E)$$

is valid. A *μ-polystable* vector bundle is a direct sum of stable vector bundles with same μ.

Fix a class $c \in NS(Y)$ in the Neron-Severi group of Y. Let $\mathscr{M}$ denote the moduli space of μ-polystable vector bundles E of rank r over Y, with

$$c_1(E) = c \quad \text{and} \quad c_2(E) = \frac{(r-1)c^2}{2r}. \tag{4.1}$$

The above condition on $c_2(E)$ is equivalent to the condition that $c_2(\mathrm{End}(E)) = 0$. We note that the conditions on the Chern classes of E ensure that for the polystable vector bundle $\mathrm{End}(E)$, we have

$$c_1(\mathrm{End}(E)) = 0 = c_2(\mathrm{End}(E)).$$

Therefore, $\mathrm{End}(E)$ admits a unitary flat connection.

The moduli space $\mathscr{M}$ is a complete projective variety. Note that any torsionfree coherent polystable sheaf E with $c_2(\mathrm{End}(E)) = 0$ must be locally free. We assume that the pair Y and c is such that there is an universal vector bundle over $Y \times \mathscr{M}$. This means that there is a vector bundle $\mathscr{P}$ of rank r over $Y \times \mathscr{M}$, such that for any $t \in \mathscr{M}$, the restriction of the pair $\mathscr{P}$ to $Y \times t$ is a polystable vector bundle represented by the point t of the moduli space.

If $\dim Y = 1$ and the integer $c \in NS(Y) = \mathbb{Z}$ is coprime to r, then there is a universal vector bundle. Take a vector bundle E over Y represented by a point point $t \in \mathscr{M}$. The Zariski tangent space $T_t\mathscr{M}$ is $H^1(Y, \mathrm{End}(E))$. Taking the dual we have the isomorphism

$$f_E : T_t^*\mathscr{M} \to H^1(Y, \mathrm{End}(E))^*.$$

Now $H^1(Y, \mathrm{End}(E))^* = H^0(Y, \mathrm{End}(E) \otimes \Omega_Y^1)$ [10, Lemma 2.5], and the pairing

$$(\omega, \alpha) \mapsto \int_X \mathrm{trace}(\omega, \alpha) c_1(L)^{d-1},$$

where $\omega \in H^0(Y, \mathrm{End}(E) \otimes \Omega_Y^1)$ and $\alpha \in H^1(Y, \mathrm{End}(E))$, defines the duality.

For any $y \in Y$, let

$$e_y : H^0\big(Y, \mathrm{End}(E) \otimes \Omega^1_Y\big) \to T^*_y Y$$

be the homomorphism that sends any ω to $\mathrm{trace}(\omega)(y)$.

The pointwise construction of the homomorphism $e_y \circ f_E : T^*_t \mathscr{M} \to T^*_y Y$ gives a homomorphism

$$T : p_1^* \Omega^1_{\mathscr{M}} \to p_2^* \Omega^1_Y, \tag{4.2}$$

where p_1 (respectively, p_2) is the projection of $\mathscr{M} \times Y$ onto $\mathscr{M}$ (respectively, Y).

Therefore, using the above homomorphism T, and the universal vector bundle $\mathscr{P}$, there is a natural Fourier-Mukai transform from the Higgs sheaves over $\mathscr{M}$ to Higgs sheaves over Y.

Another example can be constructed in the following way. Let X be an algebraic K3 surface polarized by an ample divisor H. We assume that $H^2 = 2$—i.e., that X is a double cover of $\mathbb{P}^2$—and that there exists a class $c \in NS(Y)$ such that $c \cdot H = 0$ and $c^2 = -12$. These assumptions are satisfied by a family of K3 surfaces having codimension 1 in the moduli space of all polarized K3 surfaces. If the divisor $c + 2h$ is not effective, the moduli space $\mathscr{M}$ of μ-stable vector bundles on X having Chern character $(2, c, -5)$ is a K3 surfaces isomorphic to X [9, 2]. Indeed, for every $E \in \mathscr{M}$ there is exactly one exact sequence

$$0 \to \mathscr{O} \to E \otimes \mathscr{O}(H) \to \mathscr{I}_x \otimes \mathscr{O}(c + 2H) \to 0,$$

where $\mathscr{I}_x$ is the ideal sheaf of the point $x \in X$. We denote by $\Phi : X \to \mathscr{M}$ the corresponding isomorphism. On the product $X \times \mathscr{M}$ there is a universal rank-two bundle $\mathscr{P}$, which can be normalized in such a way that $R^1 p_{2*}(\mathscr{P}) = \mathscr{O}_{\mathscr{M}}$. In terms of the isomorphism induced by Φ between $p_1^* \Omega^1_{\mathscr{M}}$ and $p_2^* \Omega^1_X$, we can therefore define a Fourier-Mukai transform from Higgs sheaves over $\mathscr{M}$ to the Higgs sheaves over X.

5. Relative Fourier-Mukai Transform.

Let X be a smooth projective variety and $p : X \to B$ an elliptic fibration over a smooth projective variety B, whose fibers are geometrically integral Gorenstein curves of arithmetic genus 1. We assume that p has a section s which does not intersect the singular locus of p. The image $H := s(B)$ is a relative polarization. We shall denote by X_t the fibre of p over the point $t \in B$. The relative dualizing sheaf of the fibration is $\omega_{X/B} = p^* \omega^{-1}$, where $\omega = R^1 p_* \mathscr{O}_X$.

Let $\hat{X}$ the compactified relative Jacobian of X (cf. [1]). So $\hat{X}$ is an elliptic fibration over B, $\hat{p} : \hat{X} \to B$, and there is a natural isomorphism of B-schemes $\varsigma : X \to \hat{X}$, given by $x \mapsto \mathfrak{m}_x^* \otimes \mathscr{O}_{X_t}\big(-s(t)\big)$, where $\mathfrak{m}_x$ is the ideal sheaf of the point $x \in X_t$. We denote by $\hat{s}$ the section of $\hat{p} : \hat{X} \to B$.

On the fiber product $X \times_B \hat{X}$ we can define a Poincaré sheaf $\mathscr{P}$, which is reflexive and flat both over X and $\hat{X}$. We normalize $\mathscr{P}$ in such a way that

$$\mathscr{P}_{|H \times \hat{X}} \simeq \mathscr{O}_{\hat{X}}.$$

Let us denote by $\pi_1 : X \times_B \hat{X} \to X$ and by $\pi_2 : X \times_B \hat{X} \to \hat{X}$ the two natural projections. If E is a coherent sheaf on X flat over B, the i-th direct image

$$\mathscr{F}^i(E) := R^i \pi_{2*}(\pi_1^* E \otimes \mathscr{P})$$

is a coherent sheaf over $\hat{X}$ which is flat over B. Notice that, if $\mathscr{F}^1(E) = 0$ and $\mathscr{F}^0(E) \neq 0$, then the relative degree $d(E)$ of E is necessarily greater than zero: $d(E) > 0$ (cf. [3]).

The above isomorphism $\varsigma : X \to \hat{X}$ induces an isomorphism

$$T : \pi_1^* \Omega_X^1 \to \pi_2^* \Omega_{\hat{X}}^1.$$

Via this isomorphism T it is then possible to define a relative Fourier-Mukai transform from Higgs sheaves over the elliptic fibration X to Higgs sheaves over the dual elliptic fibration $\hat{X}$.

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