Math. Z. 191, 165–199 (1986)

Mathematische
Zeitschrift

On a Global Existence Theorem of Small Amplitude Solutions for Nonlinear Wave Equations in an Exterior Domain

Yoshihiro Shibata[1] and Yoshio Tsutsumi[2]

[1] Department of Mathematics, University of Tsukuba, Ibaraki 305, Japan
[2] Faculty of Integrated Arts and Sciences, Hiroshima University, Hiroshima 730, Japan

1. Introduction

Let Ω be an unbounded domain in an n-dimensional Euclidean space $\mathbb{R}^n$ with compact and C^∞ boundary $\partial\Omega$. Let us denote time variable by t or x_0 and space variables by $x=(x_1, \dots, x_n)$, respectively. For differentiation, we use the symbols $\partial_t=\partial_0=\partial/\partial t$ and $\partial_x=(\partial_1, \dots, \partial_n)$ with $\partial_j=\partial/\partial x_j$, $j=1, \dots, n$. In this paper we shall consider the mixed problem:

$$\begin{aligned} &\Phi(u)=\square u+F(t, x, \tilde{D}^1 D^1 u)=f(t, x) && \text{in } [0, \infty)\times\Omega, \\ &u=0 && \text{on } [0, \infty)\times\partial\Omega, \\ &u(0, x)=u_0(x), \quad (\partial_t u)(0, x)=u_1(x) && \text{in } \Omega, \end{aligned} \tag{1.1}$$

where $\square=\partial_t^2-\Delta=\partial_t^2-\sum_{j=1}^n \partial_j^2$, F is a real-valued nonlinear function and $\tilde{D}^1 D^1 u=(\partial_t^j \partial_x^\alpha u;\ 1\leqq j+|\alpha|\leqq 2)$.

It is important to consider the mixed problem (1.1) in an exterior domain Ω in order to study the scattering of a reflecting object for the nonlinear wave equation. In this paper we shall prove that there exists one and only one global classical solution of the Eq. (1.1) if data u_0, u_1 and f are small and smooth in some sense. And also, this is an extension of results due to Shatah [15] for the Cauchy problem to the mixed problem (1.1) (see also, Klainerman [5] and Klainerman and Ponce [6]).

The proof is divided into two main steps. The first step is to get an existence theorem of time local solutions. As the nonlinear term is fully nonlinear, if we use the usual contraction mapping principle we encounter the so-called derivative loss. One of the methods to overcome this difficulty is to use the "Nash-Moser technique". In fact, using this technique, we have already proved the global unique existence theorem of classical solutions for the Eq. (1.1) in Shibata-Tsutsumi [18] (see also, [13, 16] and [17]). But, for the regularity of solutions, the results based on the Nash-Moser technique seem to be somewhat rough. For the Cauchy problem it is well-known that we can reduce a fully nonlinear equation to a quasilinear system, which is a method due to Dionne [2]. So, using this method, we can overcome the derivative loss (see also, Klainerman and Ponce [6] and Shatah [15]). But, for the mixed problem we can not use Dionne's method because of the boundary condition. From this point of view, we need a new method. In Shibata and Tsutsumi [19], the corresponding method was developed. The idea is the following. Differentiating $\Phi(u)$ with respect to t and putting $\partial_t u = v$, we get a system of two equations for u and v from the equations $\Phi(u)=f$ and $\partial_t[\Phi(u)] = \partial_t f$. We can regard the first equation as a fully nonlinear elliptic equation with respect to u and the second as a quasilinear hyperbolic equation with respect to v. Thus, we get an existence theorem of time local solution by the usual contraction mapping principle. In particular, if data are sufficiently smooth, time local solution has the same regularity as that of data (see Sect. 3 below).

The second step is to show "a priori estimate" of local solutions. For this purpose the uniform decay estimate plays an important role. For the Cauchy problem we can derive this estimate by using the concrete representation formula of solutions of the equation: $\square u = 0$ in $\mathbb{R}^n$ (see [5, 15] and [23]). But, we can not expect to get the concrete representation formula of solutions for the mixed problem. Thus, we have to develop a new method to get the uniform decay estimate. This point will be discussed in Sect. 4. In our method, we have to assume that the domain Ω is non-trapping. The notion of Ω being "non-trapping" will be defined precisely in Sect. 2.

2. Notations and Main Results

First, we introduce the notations used in what follows, and then we state assumptions and main results exactly. For any integer $N \geqq 0$, we write

$$D_x^N u = (\partial_x^\alpha u;\, |\alpha| = N), \qquad \bar{D}_x^N u = (\partial_x^\alpha u;\, |\alpha| \leqq N),$$
$$D^N u = (\partial_t^j \partial_x^\alpha u;\, j + |\alpha| = N), \qquad \bar{D}^N u = (\partial_t^j \partial_x^\alpha u;\, j + |\alpha| \leqq N).$$

Let $\mathcal{O}$ be an arbitrary open set in $\mathbb{R}^n$. For any p with $1 \leqq p \leqq \infty$, we denote the standard L^p space defined on $\mathcal{O}$ and its norm by $L^p(\mathcal{O})$ and $\|\cdot\|_{L^p(\mathcal{O})}$, respectively. For a vector valued function $h = (h_1, \ldots, h_s)$ put

$$|h| = \sum_{j=1}^{s} |h_j|, \qquad \|h\|_{L^p(\mathcal{O})} = \sum_{j=1}^{s} \|h_j\|_{L^p(\mathcal{O})}.$$

Especially, we write

$$\|f\|_{p,N} = \|\bar{D}_x^N f\|_{L^p(\Omega)}, \qquad \|f\|_p = \|f\|_{L^p(\Omega)}, \qquad \|f\|'_{p,N} = \|\bar{D}_x^N f\|_{L^p(\mathbb{R}^n)},$$
$$\|f\|'_p = \|f\|_{L^p(\mathbb{R}^n)}, \qquad \|h\|_{p,N} = \sum_{j=1}^{s} \|h_j\|_{p,N}, \qquad \|h\|_p = \sum_{j=1}^{s} \|h_j\|_p,$$
$$\|h\|'_{p,N} = \sum_{j=1}^{s} \|h_j\|'_{p,N}, \qquad \|h\|'_p = \sum_{j=1}^{s} \|h_j\|'_p.$$

We set $H_p^N(\mathcal{O})=\{f\in L^p(\mathcal{O});\ \|\bar{D}_x^N f\|_{L^p(\mathcal{O})}<\infty\}$. Note that $H_p^0(\mathcal{O})=L^p(\mathcal{O})$. Let r_0 be a fixed positive constant such that $\partial\Omega$ is contained in $\{x\in\mathbb{R}^n;\ |x|\leqq r_0\}$. For any $r\geqq r_0$ we denote the subset $\{x\in\Omega;\ |x|<r\}$ by Ω_r. For any $r\geqq r_0$ we put $L_r^2(\Omega)=\{u\in L^2(\Omega);\ \operatorname{supp} u\subset\bar{\Omega}_r\}$. By $H_{\mathcal{D}}(\Omega)$ we denote the completion of $C_0^\infty(\Omega)$ in the Dirichlet norm: $\|D_x^1 u\|_2$. By $\mathscr{B}^N(\bar{\mathcal{O}})$ we denote the set of all $C^N(\bar{\mathcal{O}})$ functions having all derivatives of order $\leqq N$ bounded in $\bar{\mathcal{O}}$. For an interval $I\subset\mathbb{R}^1$ and a Banach space X we denote the set of all m-times continuously differentiable X-valued functions on I by $C^m(I;X)$. For any numbers t, T, T' such that $t>0$ and $0\leqq T<T'$, non-negative integer N, extended real number p with $1\leqq p\leqq\infty$ and real number k we put

$$\mathscr{E}_p^N([T,T']\times\Omega)=\bigcap_{j=0}^{N} C^j([T,T'];H_p^{N-j}(\Omega)),$$

$$|u|_{p,k,N,t}=\sup_{0\leqq s\leqq t}(1+s)^k\|\bar{D}^N u(s,\cdot)\|_p,\qquad |||u|||_{p,N,T,T'}=\sup_{T\leqq s\leqq T'}\|\bar{D}^N u(s,\cdot)\|_p.$$

For positive integers s, i, vectors $u=(u_1,\dots,u_s)$, $v^j=(v_i^j,\dots,v_s^j)$ $(1\leqq j\leqq i)$ and a scalar function $H(t,x,\omega)$ $(\omega\in\mathbb{R}^s)$ we put

$$(d_\omega^i H)(t,x,u)(v^1,\dots,v^i)=\left(\frac{\partial^i H}{\partial\theta_1\dots\partial\theta_i}\right)\left(t,x,u+\sum_{j=1}^{i}\theta_j v^j\right)\Bigg|_{\theta_1=\dots=\theta_i=0}.$$

We put

$$M(n)=2n+3,\quad \sigma(n)=\begin{cases}1, & n\geqq 6\\ 2, & 3\leqq n\leqq 5,\end{cases}\quad p(n)=\begin{cases}4, & n\geqq 6,\\ 6, & 3\leqq n\leqq 5,\end{cases}$$

$$q(n)=\begin{cases}4/3, & n\geqq 6,\\ 6/5, & 3\leqq n\leqq 5,\end{cases}\qquad d(n)=\begin{cases}(n-1)/4, & n\geqq 6,\\ 4/3, & n=5,\\ 1, & n=4,\\ 4/7, & n=3.\end{cases}$$

$S(t,x;\mathscr{d})=S(\mathscr{d})$ and $\mathscr{d}=(u_0,u_1,f)$ denote a solution and data of the mixed problem:

$$\Box u=f\ \text{ in } [0,\infty)\times\Omega,\qquad u=0\ \text{ on } [0,\infty)\times\partial\Omega,$$
$$u(0,x)=u_0(x)\ \text{ and }\ (\partial_t u)(0,x)=u_1(x)\ \text{ in } \Omega,$$

respectively. $S_0(t,x;\mathscr{d}_0)=S_0(\mathscr{d}_0)$ and $\mathscr{d}_0=(v_0,v_1,g)$ denote a solution and data of the Cauchy problem:

$$\Box v=g\ \text{ in } [0,\infty)\times\mathbb{R}^n,\qquad v(0,x)=v_0(x)\ \text{ and }\ (\partial_t v)(0,x)=v_1(x)\ \text{ in } \mathbb{R}^n,$$

respectively. In the course of calculations below various constants will be simply denoted by C. Especially $C(\dots)$ denotes a constant depending essentially only on the quantities appearing in parentheses.

Now, we make the following assumptions.

Assumption 2.1 (1) *The spatial dimension* $n\geqq 3$.

(2) *The nonlinear function F is real-valued and belongs to*

$$\mathscr{B}^\infty([0,\infty)\times\bar{\Omega}\times\{\lambda\in\mathbb{R}^{(n+1)(n+2)};\ |\lambda|\leqq 1\}).$$

$$(3) \qquad F(t,x,\lambda)=\begin{cases} O(|\lambda|^2) & \textit{near } \lambda=0 \textit{ if } n\geqq 6,\\ O(|\lambda|^3) & \textit{near } \lambda=0 \textit{ if } 3\leqq n\leqq 5.\end{cases}$$

(4) *The exterior domain Ω is "non-trapping" in the following sense: Let $G(t,x,y)=S(t,x,\mathscr{A}(y))$, $\mathscr{A}(y)=(0,\delta(\cdot-y),0)$ be the Green function, where δ is the Dirac delta function and y is an arbitrary point in Ω. Let a and b be arbitrary positive constants with $r_0\leqq a\leqq b$. For any $v\in L^2_a(\Omega)$ we put $(\mathscr{G}v)(t,x)=\int_\Omega G(t,x,y)v(y)\,dy$. Then, there exists a $T_0>0$ depending only on n, a, b and Ω such that $(\mathscr{G}v)(t,x)\in C^\infty([T_0,\infty)\times\bar{\Omega}_b)$ for any $v\in L^2_a(\Omega)$.*

Remark 2.2. (1) It is well-known that if the complement of Ω is convex then Assumption 2.1(4) is satisfied (see [10] and [24]).

(2) In order to get an existence theorem of global smooth solutions, in general we can not relax Assumption 2.1(3) if $n=3$, which is due to John [4].

(3) The above definition of Ω being "non-trapping" is essentially due to Vainberg [22] (see also [21, Remark 1.1]).

Now, we state the main results and examples.

Main Theorem. *Assume that Assumption* 2.1 *holds. Let N be an integer $\geqq 4n+8$. Then, for any $\varepsilon_1>0$, there exists a positive number ε_2 depending essentially on ε_1, n, N, F and Ω such that if $u_0\in H^N_2(\Omega)\cap H^{N-1}_{q(n)}(\Omega)$, $u_1\in H^{N-1}_2(\Omega)\cap H^{N-2}_{q(n)}(\Omega)$ and $f\in\mathscr{E}^{N-1}_2([0,\infty)\times\Omega)\cap\mathscr{E}^{N-2}_{q(n)}([0,\infty)\times\Omega)$ satisfy the compatibility condition of order $N-1$ and the inequality:*

$$\begin{aligned}&\|u_0\|_{2,N}+\|u_1\|_{2,N-1}+|f|_{2,d(n)\sigma(n),N-1,\infty}\\ &\quad+\|u_0\|_{q(n),N-1}+\|u_1\|_{q(n),N-2}+|f|_{q(n),d(n)\sigma(n),N-2,\infty}\leqq\varepsilon_2,\end{aligned}$$

then there exists one and only one solution $u\in\mathscr{E}^N_2([0,\infty)\times\Omega)\cap\mathscr{E}^{N-M(n)-1}_{p(n)}([0,\infty)\times\Omega)$ of the mixed problem (1.1) *having the properties:*

$$|D^1u|_{2,0,N-1,\infty}+|D^1u|_{p(n),d(n),N-M(n)-1,\infty}<\varepsilon_1, \qquad |\bar{D}^1D^1u|_{\infty,0,0,\infty}\leqq 1.$$

Here the compatibility condition will be defined in Definition 3.7 *of Sect.* 3 *below.*

In addition, let $u_j\in H^\infty_2(\Omega)$, $j=0,1$ and $f\in\mathscr{E}^\infty_2([0,\infty)\times\Omega)$ and let u_0, u_1 and f satisfy the compatibility condition of infinite order. Then, the above solution is in $\mathscr{E}^\infty_2([0,\infty)\times\Omega)$.

Example 2.3. $\Phi(u)=\partial_t^2u-\sum_{j=1}^n\partial_j(\partial_ju(1+|D^1_xu|^2)^{-1/2})$,

$$\Phi(u)=\square u+\sum_{i,j=0}^n(\partial_i\partial_ju)^a, \qquad a=\begin{cases}2, & n\geqq 6,\\ 3, & 3\leqq n\leqq 5.\end{cases}$$

3. Local Existence Theorem

In this section, we shall state a time local existence theorem of solutions of the mixed problem:

$$\begin{aligned} &\Phi(v)=\Box v+F(t, x, \bar{D}^1 D^1 v)=g(t, x) && \text{in } [T, T']\times\Omega, \\ &v=0 && \text{on } [T, T']\times\partial\Omega, \qquad (3.1)\\ &v(T, x)=v_0(x), \quad (\partial_t v)(T, x)=v_1(x) && \text{in } \Omega, \end{aligned}$$

where $0\leqq T<T'$. Below, we shall quote a result in Shibata-Tsutsumi [19], where more general operator is treated. Since $F(t, x, 0)=0$, $(d_\lambda F)(t, x, 0)=0$, if we write

$$\begin{aligned} (d_\lambda F)(t, x, \bar{D}^1 D^1 w)\bar{D}^1 D^1 v = \sum_{j=0}^{n} F_2^j(t, x, \bar{D}^1 D^1 w)\,\partial_t\partial_j v \\ -\sum_{i,j=1}^{n} F_2^{ij}(t, x, \bar{D}^1 D^1 w)\,\partial_i\partial_j v+\sum_{j=0}^{n} F_1^j(t, x, \bar{D}^1 D^1 w)\,\partial_j v, \end{aligned} \tag{3.2}$$

there exists a $\lambda_0\in(0, 1/3]$ depending only on F such that

$$1/2\leqq 1+F_2^0(t, x, \lambda)\leqq 3/2, \tag{3.3}$$

$$\sum_{i,j=1}^{n}(\delta_{ij}+F_2^{ij}(t, x, \lambda))\,\xi_i\xi_j\geqq\tfrac{1}{2}|\xi|^2, \quad {}^\forall\xi=(\xi_1, \dots, \xi_n)\in\mathbb{R}^n, \tag{3.4}$$

for all $(t, x)\in[0, \infty)\times\bar{\Omega}$ and λ with $|\lambda|\leqq 3\lambda_0$, where δ_{ij} denotes Kronecker's delta sign, that is, $\delta_{ij}=1$ if $i=j$ and $=0$ if $i\neq j$. To state the local existence theorem according to [19], we have to define a certain class of data and compatibility condition.

Definition 3.1. We shall say that a triple of functions $(v_0(x), v_1(x), g(t, x))$ with $v_0(x)\in\mathscr{B}^2(\bar{\Omega})$, $v_1(x)\in\mathscr{B}^1(\bar{\Omega})$ and $g(T, x)\in\mathscr{B}^0(\bar{\Omega})$ belongs to class $\mathscr{D}_T$ with v_2 if there exists a function $v_2=v_2(T, x)\in\mathscr{B}^0(\bar{\Omega})$ (T being regarded as a parameter) such that v_2 satisfies the condition:

$$\|\bar{D}_x^1 D_x^1 v_0\|_\infty+\|\bar{D}_x^1 v_1\|_\infty+\|v_2\|_\infty\leqq\lambda_0,$$

and the equation:

$$v_2(T, x)-\Delta v_0(x)+F(T, x, \bar{D}_x^1 D_x^1 v_0(x), \bar{D}_x^1 v_1(x), v_2(T, x))=g(T, x), \qquad x\in\Omega.$$

Remark 3.2. Since the non-linear term F contains only derivatives of v, we need not impose any conditions on $\|v_0\|_\infty$ in Definition 3.1.

In order to state the compatibility condition, let us introduce some notations. For a smooth function $w(t, x)$ and integer $j\geqq 2$, we put $w_j=(\partial_t^j w)(t, x)$. Let us define functions $G^k_{\alpha^k, \beta^k, \gamma^k}(t, x, \lambda)$, $|\lambda|\leqq 3\lambda_0$, as follows:

$$\begin{aligned} \partial_t^{j-2}[F(t, x, \bar{D}^1 D^1 w)] &= F_2^0(t, x, \bar{D}_x^1 D_x^1 w_0, \bar{D}_x^1 w_1, w_2)\, w_j \\ &+\sum_{k=1}^{j-2}\sum_{\alpha^k, \beta^k, \gamma^k} G^k_{\alpha^k, \beta^k, \gamma^k}(t, x, \bar{D}_x^1 D_x^1 w_0, \bar{D}_x^1 w_1, w_2)(\bar{D}_x^1 D_x^1 w_1)^{\alpha_1^k}\dots(\bar{D}_x^1 D_x^1 w_k)^{\alpha_k^k} \\ &\times(\bar{D}_x^1 w_2)^{\beta_1^k}\dots(\bar{D}_x^1 w_{k+1})^{\beta_k^k}(w_3)^{\gamma_1^k}\dots(w_{k+2})^{\gamma_k^k} \\ &+(\partial_t^{j-2}F)(t, x, \bar{D}_x^1 D_x^1 w_0, \bar{D}_x^1 w_1, w_2), \qquad j\geqq 3, \end{aligned}$$

where $\alpha^k=(\alpha_1^k, \dots, \alpha_k^k)$, $\beta^k=(\beta_1^k, \dots, \beta_k^k)$, $\gamma^k=(\gamma_1^k, \dots, \gamma_k^k)$, α_j^k and β_j^k, $j=1, \dots, k$, are multi-indices such that

$$\sum_{m=1}^{k} m(|\alpha_m^k|+|\beta_m^k|+\gamma_m^k)=k \quad \text{and} \quad \gamma_{j-2}^{j-2}=0.$$

Now, for $(v_0, v_1, g)\in\mathscr{D}_T$ with v_2 let us define $v_j=V_j(T, v_0, v_1, v_2, g; x)$, $j\geqq 3$, successively as follows:

$$\begin{aligned} v_j=V_j(T, v_0, v_1, v_2, g; x)= &-\{1+F_2^0(t, x, \bar{D}_x^1 D_x^1 v_0, \bar{D}_x^1 v_1, v_2)\}^{-1} \\ &\times\Bigg[\Delta v_{j-2}+\sum_{k=1}^{j-2}\sum_{\alpha^k, \beta^k, \gamma^k} G_{\alpha^k, \beta^k, \gamma^k}^k(T, x, \bar{D}_x^1 D_x^1 v_0, \bar{D}_x^1 v_1, v_2) \\ &\times(\bar{D}_x^1 D_x^1 v_1)^{\alpha_1^k}\dots(\bar{D}_x^1 D_x^1 v_k)^{\alpha_k^k}(\bar{D}_x^1 v_2)^{\beta_1^k}\dots(\bar{D}_x^1 v_{k+1})^{\beta_k^k}(v_3)^{\gamma_1^k}\dots(v_{k+2})^{\gamma_k^k} \\ &+(\partial_t^{j-2}F)(T, x, \bar{D}_x^1 D_x^1 v_0, \bar{D}_x^1 v_1, v_2)-(\partial_t^{j-2}g)(T, x)\Bigg], \quad j\geqq 3, \end{aligned} \tag{3.5}$$

where v_2 is the same as in Definition 3.1. Our compatibility condition for local solvability is defined as follows.

Definition 3.3. Let a triple (v_0, v_1, g) belong to class $\mathscr{D}_T$ with v_2. We shall say that v_0, v_1 and g satisfy the compatibility condition of order $N-1$ at time T if v_0, v_1 and g satisfy the conditions:

$$v_j\in H_{\Gamma}(\Omega)\cap L^2(\Omega), \quad j=0, 1, \dots, N-1, \ v_N\in L^2(\Omega),$$

where $v_j=V_j(T, v_0, v_1, v_2, g; x)$, $3\leqq j\leqq N$.

Now, let us summarize properties about v_j, $j\geqq 2$.

Lemma 3.4. *Let λ_0 be the constant appearing in* (3.3) *and* (3.4). *Let $v_j\in\mathscr{B}^{2-j}(\bar{\Omega})$, $j=0, 1, 2$ and $g\in\mathscr{B}^0([T, T']\times\bar{\Omega})$ be functions satisfying the equation:*

$$v_2(x)-\Delta v_0(x)+F(T, x, \bar{D}_x^1 D_x^1 v_0(x), \bar{D}_x^1 v_1(x), v_2(x))=g(T, x) \quad in \ \Omega, \tag{3.6}$$

and the condition:

$$\|\bar{D}_x^1 D_x^1 v_0\|_\infty+\|\bar{D}_x^1 v_1\|_\infty+\|v_2\|_\infty\leqq 3\lambda_0. \tag{3.7}$$

Let $v_j=V_j(T, v_0, v_1, v_2, g; x)$, $j\geqq 3$ (*cf.* (3.5)). *Let p be an extended real number >1 and N an integer $\geqq 2$. Then, the following assertions are valid.*

(I) *Let $w\in\mathscr{B}^0(\bar{\Omega})$ be a function satisfying the equation:*

$$w(x)-\Delta v_0(x)+F(T, x, \bar{D}_x^1 D_x^1 v_0(x), \bar{D}_x^1 v_1(x), w(x))=g(T, x) \quad in \ \Omega, \tag{3.8}$$

and the condition:

$$\|\bar{D}_x^1 D_x^1 v_0\|_\infty+\|\bar{D}_x^1 v_1\|_\infty+\|w\|_\infty\leqq 3\lambda_0. \tag{3.9}$$

Then, $w=v_2$.

(II) *Assume that $v_j\in H_p^{N-j}(\Omega)$, $j=0, 1, 2$ and $g(T, \cdot)\in H_p^{N-2}(\Omega)$. Then, the following estimate holds:*

$$\|v_2\|_{p, N-2}\leqq C(N, p, F)[\|v_0\|_{p, N}+\|v_1\|_{p, N-1}+\|g(T, \cdot)\|_{p, N-2}].$$

(III) *Assume that all norms of* v_j, $j=0, 1, 2$, *and* g *appearing below are finite. Then, the following estimates hold:*

(i) $\|v_j\|_{p, N-j}$

$$\leqq C(n, p, F)\left(1+\sum_{k=0}^{2}\|v_k\|_{p,[n/p]+3-k}+\|(\bar{D}^{[n/p]+1}g)(T,\ \cdot)\|_p\right)^{M(j)}$$

$$\times\left(\sum_{k=0}^{2}\|v_k\|_{p,[n/p]+3-k}+\|(\bar{D}^{[n/p]+1}g)(T,\ \cdot)\|_p\right),$$

$$3\leqq j\leqq N\leqq[n/p]+2,$$

(ii) $\|v_j\|_{p, N-j}$

$$\leqq C(N, p, F)\left(1+\sum_{k=0}^{2}\|v_k\|_{p,N-k}+\|(\bar{D}^{N-2}g)(T,\ \cdot)\|_p\right)^{M(j)}$$

$$\times\left(\sum_{k=0}^{2}\|v_k\|_{p,N-k}+\|(\bar{D}^{N-2}g)(T,\ \cdot)\|_p\right),$$

$$3\leqq j\leqq[n/p]+2<N \text{ or } [n/p]+3\leqq j\leqq N,$$

where $M(j)$ *is some positive integer depending only on* j.

(IV) *Assume that* $N\geqq[n/2]+3$. *Let* $v\in\mathscr{E}_2^N([T, T']\times\Omega)$ *be a solution of* (3.1) *for data* (v_0, v_1, g) *satisfying the condition:*

$$|||\bar{D}^1 D^1 v|||_{\infty, 0, T, T'}\leqq 2\lambda_0. \tag{3.10}$$

Then, we have that $(\partial_t^j v)(T, x)=v_j(x)$, $2\leqq j\leqq N$.

Proof. Using notations defined in (3.2) and applying Taylor expansion to (3.6) and (3.8), we can write

$$\left\{1+\int_0^1 F_2^0(T, x, \bar{D}_x^1 D_x^1 v_0, \bar{D}_x^1 v_1, \theta w+(1-\theta)v_2)\,d\theta\right\}(w-v_2)=0.$$

Since it follows from (3.7) and (3.9) that $\|\bar{D}_x^1 D_x^1 v_0\|_\infty+\|\bar{D}_x^1 v_1\|_\infty+\|\theta w+(1-\theta)v_2\|_\infty\leqq 3\lambda_0$, $0\leqq\theta\leqq 1$, the assertion (I) follows from (3.3) immediately. Other assertions will be proved in Sect. 5. Q.E.D.

From a theorem due to Shibata-Tsutsumi [19] we have

Theorem 3.5 *(local existence theorem).* (I) *Let* F *be a function satisfying Assumption* 2.1(2) *and* (3), T, T' *real numbers with* $0\leqq T<T'$, λ_0 *the constant appearing in* (3.3) *and* (3.4), *and* $L_0=[n/2]+8$. *For any positive number* B *and integer* $L\geqq L_0$ *we define the set* $\mathbb{D}(L, B, T)$ *of data for the problem* (3.1) *as follows:*

$$\begin{aligned}\mathbb{D}(L, B, T)=\{&(v_0, v_1, g);\ v_0\in H_2^L(\Omega),\ v_1\in H_2^{L-1}(\Omega),\\ &g\in C^{L-1}([T, T'];\ L^2(\Omega))\cap\mathscr{E}_2^{L-2}([T, T']\times\Omega),\end{aligned}$$

$$\|v_0\|_{2, L_0}+\|v_1\|_{2, L_0-1}+|||g|||_{2, L_0-2, T, T'}+|||\partial_t^{L_0-1}g|||_{2, 0, T, T'}\leqq B,$$

there exists a function $v_2\in\mathscr{B}^0(\bar{\Omega})$ *such that a triple* (v_0, v_1, g) *belongs to* $\mathscr{D}_T$ *with* v_2 *and* v_0, v_1 *and* g *satisfy the compatibility condition of order* $L-1$ *at time* $T\}$.

Then, there exists a $T'' \in (0, T'-T]$ *depending only on* n, Ω, F, B *and* λ_0 *such that for any data* $(v_0, v_1, g) \in \mathbb{D}(L, B, T)$ *the Eq.* (3.1) *has a unique time local solution* $v \in \mathscr{E}_2^L([T, T+T''] \times \Omega)$ *with* $|||\bar{D}^1 D^1 v|||_{\infty, 0, T, T+T''} \leqq 2\lambda_0$.

(II) *In addition to all assumptions of* (I), *let* $v_0 \in H_2^\infty(\Omega)$, $v_1 \in H_2^\infty(\Omega)$ *and* $g \in \mathscr{E}_2^\infty([T, T+T''] \times \Omega)$ *and* v_0, v_1, g *satisfy the compatibility condition of infinite order at time* T. *Then, the above solution is in* $\mathscr{E}_2^\infty([T, T+T''] \times \Omega)$. *In particular,* $v \in C^\infty([T, T+T''] \times \bar{\Omega})$.

Now, let us give a lemma which will be used in latter sections.

Lemma 3.6. *Let* N *be an integer* $\geqq [n/2]+3$ *and* $v \in \mathscr{E}_2^N([T, T'] \times \Omega)$ *be a solution of* (3.1) *for data* v_0, v_1 *and* g.

(I) *If* v *satisfies the condition:*

$$|||\bar{D}^1 D^1 v|||_{\infty, 0, T, T'} \leqq \lambda_0, \tag{3.11}$$

then the following assertions are valid.

(i) *A triple* $(v(t,x), (\partial_t v)(t,x), g(t,x))$ *belongs to* $\mathscr{D}_t$ *with* $(\partial_t^2 v)(t,x)$ *for any* $t \in [T, T']$.

(ii) *The functions* $v(t,x)$, $(\partial_t v)(t,x)$ *and* $g(t,x)$ *satisfy the compatibility condition of order* $N-1$ *at time* t *for any* $t \in [T, T']$.

(II) *If* v *satisfies the condition:*

$$|||\bar{D}^1 D^1 v|||_{\infty, 0, T, T'} \leqq 2\lambda_0, \tag{3.12}$$

and the data v_0, v_1 *and* g *satisfy the condition:*

$$\|v_0\|_{2,N} + \|v_1\|_{2,N-1} + \|(\bar{D}^{N-2} g)(T, \cdot)\|_2 \leqq 1, \tag{3.13}$$

then the following estimates hold:

$$\|(\bar{D}^N v)(T, \cdot)\|_2 \leqq C(N, F)[\|v_0\|_{2,N} + \|v_1\|_{2,N-1} + \|(\bar{D}^{N-2} g)(T, \cdot)\|_2].$$

Proof. (I) (i) By (3.11) we have that

$$\|\bar{D}_x^1 D_x^1 v(t, \cdot)\|_\infty + \|\bar{D}_x^1 (\partial_t v)(t, \cdot)\|_\infty + \|(\partial_t^2 v)(t, \cdot)\|_\infty \leqq \lambda_0 \tag{3.14}$$

for any $t \in [T, T']$. Thus, by (3.14) and the fact that v is a solution of (3.1) we see easily that a triple $(v(t,x), (\partial_t v)(t,x), g(t,x))$ belongs to $\mathscr{D}_t$ with $(\partial_t^2 v)(t,x)$ for any $t \in [T, T']$.

(ii) Since $N \geqq [n/2]+3$, $v \in \mathscr{E}_2^N([T, T'] \times \Omega)$ is a solution of (3.1) and (3.12) is valid, by Lemma 3.4(IV) we have that

$$(\partial_t^j v)(t,x) = v_j(t,x), \quad 0 \leqq j \leqq N, \tag{3.15}$$

where $v_j(t,x) = (\partial_t^j v)(t,x)$, $j = 0, 1, 2$ and $v_j(t,x) = V_j(t, v(t, \cdot), (\partial_t v)(t, \cdot), (\partial_t^2 v)(t, \cdot), g; x)$, $3 \leqq j \leqq N$ (cf. (3.5)). Since $v \in \mathscr{E}_2^N([T, T'] \times \Omega)$, we have that

$$(\partial_t^j v)(t, \cdot) \in H_2^{N-j}(\Omega), \quad 0 \leqq j \leqq N. \tag{3.16}$$

Since $v(t, x)=0$ on $\partial\Omega$, by (3.16) and the trace theorem we have that

$$(\partial_t^j v)(t, x)=0 \quad \text{on } \partial\Omega, \qquad 0 \leqq j \leqq N-1. \tag{3.17}$$

Combining (3.15), (3.16) and (3.17), we obtain the assertion (ii).

(II) Since v is a solution of (3.1) and (3.12) is valid, we have that $(\partial_t^j v)(T, x)$, $j=0, 1, 2$, and $g(T, x)$ satisfy the equation:

$$(\partial_t^2 v)(T, x) - \Delta v_0(x) + F(T, x, \bar{D}_x^1 D_x^1 v_0(x), \bar{D}_x^1 v_1(x), (\partial_t^2 v)(T, x)) = g(T, x) \quad \text{in } \Omega,$$

and the condition: $\|\bar{D}_x^1 D_x^1 v_0\|_\infty + \|\bar{D}_x^1 v_1\|_\infty + \|(\partial_t^2 v)(T, \cdot)\|_\infty \leqq 2\lambda_0$. Here we have used the facts that $v(T, x)=v_0(x)$ and $(\partial_t v)(T, x)=v_1(x)$. Since $v \in \mathscr{E}_2^N([T, T'] \times \Omega)$, we have that $v_0 \in H_2^N(\Omega)$, $v_1 \in H_2^{N-1}(\Omega)$, $(\partial_t^2 v)(T, x) \in H_2^{N-2}(\Omega)$ and $g(T, \cdot) \in H_2^{N-2}(\Omega)$. Thus, by Lemma 3.4(II) we have

$$\|(\partial_t^2 v)(T, \cdot)\|_{2, N-2} \leqq C(N, F)[\|v_0\|_{2, N} + \|v_1\|_{2, N-1} + \|g(T, \cdot)\|_{2, N-2}]. \tag{3.18}$$

Since $N \geqq [n/2]+3$, using (3.15), by Lemma 3.4(III)(ii) we have that

$$\|(\partial_t^2 v)(T, \cdot)\|_{2, N-2} \leqq C(N, F)[\|v_0\|_{2, N} + \|v_1\|_{2, N-1} + \|g(T, \cdot)\|_{2, N-2}]. \tag{3.18}$$

Since $N \geqq [n/2]+3$, using (3.15), by Lemma 3.4(III)(ii) we have that

$$\|(\partial_t^j v)(T, \cdot)\|_{2, N-j} \leqq C(N, F)[\|v_0\|_{2, N} + \|v_1\|_{2, N-1} + \|(\bar{D}^{N-2} g)(T, \cdot)\|_2), \quad 3 \leqq j \leqq N. \tag{3.19}$$

Here we have used the assumption (3.13). Combining (3.18) and (3.19), we obtain the assertion (II). This completes the proof.

Finally, we shall discuss about the compatibility condition for the original problem (1.1). First, let us consider the non-linear equation:

$$\mu_{00} - \sum_{j=1}^n \mu_{jj} + F(0, x, \mu', \mu_{00}) = g$$

for $x \in \bar{\Omega}$ and $(\mu', \mu_{00}, g) \in \mathbb{R}^{n^2+2n+3}$. Here, $\mu' = (\mu_j; j=1, \dots, n, \mu_{ij}; \ i, j=1, \dots, n, \mu_0, \mu_{0i}; \ i=1, \dots, n)$ and μ_i, $i=0, 1, \dots, n$, and μ_{ij}, $i, j=0, 1, \dots, n$, are variables corresponding to $\partial_i u$, $i=0, 1, \dots, n$, and $\partial_i \partial_j u$, $i, j=0, 1, \dots, n$, respectively. Note that $\lambda=(\mu', \mu_{00})$. Since $F(t, x, 0)=0$ and (3.3) holds, it follows from the implicit function theorem that there exists a $\delta_0 \in (0, 3\lambda_0]$ and a function $v(x, \mu', g)$ in $\mathscr{B}^\infty(\bar{\Omega} \times \{(\mu', g); |\mu'|+|g| \leqq \delta_0\})$ such that

$$v(x, \mu', g) - \sum_{j=1}^n \mu_{jj} + F(0, x, \mu', v(x, \mu', g)) = g, \tag{3.20}$$

$$v(x, 0, 0) = 0, \tag{3.21}$$

$$\sum_{|\alpha|+|\beta|+k \leqq L} |\partial_x^\alpha \partial_{\mu'}^\beta \partial_g^k v(x, \mu', g)| \leqq C(L, F), \qquad {}^\forall L \geqq 0, \tag{3.22}$$

for any $x\in\bar{\Omega}$ and (μ', g) with $|\mu'|+|g|\leqq\delta_0$, where δ_0 is depending only on n, Ω and F. Thus, if $u_j(x)\in\mathscr{B}^{2-j}(\bar{\Omega})$, $j=0, 1$, and $f(t, x)\in\mathscr{B}^0([0, \infty)\times\bar{\Omega})$ satisfy the condition:

$$\|\bar{D}_x^1 D_x^1 u_0\|_\infty+\|\bar{D}_x^1 u_1\|_\infty+\|f(0, \cdot)\|_\infty\leqq\delta_0, \tag{3.23}$$

putting

$$u_2(x)=v(x, \bar{D}_x^1 D_x^1 u_0(x), \bar{D}_x^1 u_1(x), f(0, x)), \tag{3.24}$$

we have by (3.20) and (3.24) that

$$u_2(x)-\Delta u_0(x)+F(0, x, \bar{D}_x^1 D_x^1 u_0(x), \bar{D}_x^1 u_1(x), u_2(x))=f(0, x) \quad \text{in } \Omega. \tag{3.25}$$

Since (3.21) and (3.22) hold, by Taylor expansion we have that there exists a $\delta_1\in(0, \delta_0]$ depending only on F such that if

$$\|\bar{D}_x^1 D_x^1 u_0\|_\infty+\|\bar{D}_x^1 u_1\|_\infty+\|f(0, \cdot)\|_\infty\leqq\delta_1, \tag{3.26}$$

then

$$\|\bar{D}_x^1 D_x^1 u_0\|_\infty+\|\bar{D}_x^1 u_1\|_\infty+\|u_2\|_\infty\leqq\delta_0. \tag{3.27}$$

In particular, by (3.25) and (3.27) we see that if $u_j\in\mathscr{B}^{2-j}(\bar{\Omega})$, $j=0, 1$, and $f\in\mathscr{B}^0([0, \infty)\times\bar{\Omega})$ satisfy the condition (3.26), then a triple (u_0, u_1, f) belongs to $\mathscr{D}_0$ with u_2. Let $u_j=V_j(0, u_0, u_1, u_2, f; x)$, $j\geqq 3$, and N be an integer $\geqq[n/2]+3$. If $u_j\in H_2^{N-j}(\Omega)$, $j=0, 1$, and $f\in\mathscr{E}_2^{N-2}([0, \infty)\times\Omega)$, noting (3.21) and applying Moser's lemma (cf. Lemma 5.5 below) to (3.24), we have that $u_2\in H_2^{N-2}(\Omega)$. By Lemma 3.4(III) we have that $u_j\in H_2^{N-j}(\Omega)$, $3\leqq j\leqq N$. From this point of view, we define our compatibility condition for the original problem as follows.

Definition 3.7. Let N be an integer $\geqq[n/2]+3$. Let δ_1 be the constant appearing in (3.26). We shall say that the data $u_0\in H_2^N(\Omega)$, $u_1\in H_2^{N-1}(\Omega)$ and $f\in\mathscr{E}_2^{N-2}([0, \infty)\times\Omega)$ satisfy the compatibility condition of order $N-1$ if u_0, u_1 and f satisfy the conditions:

(i) $\|\bar{D}_x^1 D_x^1 u_0\|_\infty+\|\bar{D}_x^1 u_1\|_\infty+\|f(0, \cdot)\|_\infty\leqq\delta_1$,

(ii) $u_j(x)\in H_\gamma(\Omega)\cap L^2(\Omega)$, $j=0, 1, \ldots, N-1$, and $u_N\in L^2(\Omega)$.

Here, u_j, $2\leqq j\leqq N-1$, be the functions stated above.

4. L^p-L^q Estimates

In this section, we shall derive L^p-L^q estimates of solutions of the mixed problem for the d'Alembertian equation. These play the most important role to get our "a priori estimate" of local solutions. The following is a main result in this section.

Theorem 4.1. *Assume that $n\geqq 3$ and Ω is non-trapping. Let L be an integer $\geqq 1$ and u_0, u_1 and g functions satisfying the conditions:*

(i) $u_0\in H_2^{L+n+1}(\Omega)\cap H_{q(n)}^{L+M(n)-1}(\Omega)$, $u_1\in H_2^{L+n}(\Omega)\cap H_{q(n)}^{L+M(n)-2}(\Omega)$, $g\in\mathscr{E}_2^{L+n}([0, \infty)\times\Omega)\cap\mathscr{E}_{q(n)}^{L+n-2}([0, \infty)\times\Omega)$.

(ii) *u_0, u_1 and g satisfy the compatibility condition of order $L+n$ for the d'Alembertian equation in Ω (see Remark 4.2 below). Then, there exists a con-*

stant $C=C(L, n, \Omega)$ *such that*

$$|D^1 S(d)|_{p(n), d(n), L-1, t} \leqq C[\|u_0\|_{q(n), L+M(n)-1} + \|u_1\|_{q(n), L+M(n)-2} + |g|_{q(n), d(n)\sigma(n), L+M(n)-2, t}], \quad t \geqq 0, \ d=(u_0, u_1, g).$$

Remark 4.2. For data $u_0 \in H_2^N(\Omega)$, $u_1 \in H_2^{N-1}(\Omega)$, $g \in \mathscr{E}_2^{N-1}([0, \infty) \times \Omega)$, N being an integer $\geqq 2$, we define $u_j = U_j(u_0, u_1, g; x)$, $j \geqq 2$, successively as follows:

$$u_j = U_j(u_0, u_1, g; x) = \Delta u_{j-2} + (\partial_t^{j-2} g)(0, x). \tag{4.1}$$

If $u_j \in H_\gamma(\Omega) \cap L^2(\Omega)$, $j=0, 1, \ldots, N-1$, and $u_N \in L^2(\Omega)$, then we say usually that u_0, u_1 and g satisfy the compatibility condition of order $N-1$ for the d'Alembertian equation in Ω. It is well-known that there exists a unique solution $S(t, x; d) \in \mathscr{E}_2^N([0, \infty) \times \Omega)$, $d=(u_0, u_1, g)$ (see Mizohata [12] and Ikawa [3]).

In order to prove Theorem 4.1, we need the following two lemmas.

Lemma 4.3 *(local energy decay). Let γ, a and b be any real numbers with $0<\gamma \leqq n-1$ and $a, b \geqq r_0$. Let M be an integer $\geqq 2$. Let u_0, u_1 and g be functions satisfying the conditions:*

(i) $u_0 \in H_2^M(\Omega)$, $u_1 \in H_2^{M-1}(\Omega)$, $g \in \mathscr{E}_2^{M-1}([0, \infty) \times \Omega)$,

(ii) *u_0, u_1 and g satisfy the compatibility condition of order $M-1$ for the d'Alembertian equation in Ω,*

(iii) $\operatorname{supp} u_i \subset \Omega_a$, $i=0, 1$, $\operatorname{supp} g \subset [0, \infty) \times \Omega_a$.

Then, there exists a constant $C=C(M, a, b, n, \Omega)$ such that

$$\|\bar{D}^M S(t, \cdot; d)\|_{L^2(\Omega_b)} \leqq C(1+t)^{-\gamma}[\|u_0\|_{2,M} + \|u_1\|_{2,M-1} + |g|_{2,\gamma,M-1,t}], \quad {}^\forall t>0, \ d=(u_0, u_1, g).$$

Lemma 4.3 will be proved in Appendix I.

Lemma 4.4 *(L^p-L^q estimate in $\mathbb{R}^n$). Let p be an extended real number with $2 \leqq p \leqq \infty$ and $q=p/(p-1)$. Then for any integer $M \geqq 0$ there exists a constant $C=C(p, n, M)$ such that if Cauchy data v_0 and v_1 are in C^∞ and their norms appearing below are finite, then*

$$\|\bar{D}^M S_0(t, \cdot; d_0)\|'_p \leqq C(1+t)^{-\frac{n-1}{2}\left(1-\frac{2}{p}\right)}[\|v_0\|'_{q, M+n+1} + \|v_1\|'_{q, M+n}], \quad t>0, \ p>\frac{2(n+1)}{n-1}, \ d_0=(v_0, v_1, 0),$$

$$\|\bar{D}^M D^1 S_0(t, \cdot; d_0)\|'_p \leqq C(1+t)^{-\frac{n-1}{2}\left(1-\frac{2}{p}\right)}[\|v_0\|'_{q, M+n+2} + \|v_1\|'_{q, M+n+1}], \quad t>0, \ 2 \leqq p \leqq \infty, \ d_0=(v_0, v_1, 0).$$

Remark 4.5. (i) We can show Lemma 4.5 using a concrete representation formula of solutions of the d'Alembertian equation: $\Box u=0$ in $\mathbb{R}^n$ and interpolation theorem (see, von Wahl [23], Klainerman [5] and Shatah [15]).

(ii) By the limiting process we can remove the assumption that v_0 and v_1 are in C^∞.

Now, we shall prove Theorem 4.1. For this we begin with

Lemma 4.6. *Assume that Ω is non-trapping and $n \geqq 3$. Let M be an integer $\geqq 1$, p an extended real number with $2 \leqq p \leqq \infty$ and $q = p/(p-1)$. Let α be the same constant as in (4.6) below. Let u_0 and u_1 be functions satisfying the conditions:*

(i) $u_0 \in H_q^{M+2n+2}(\Omega)$, $u_1 \in H_q^{M+2n+1}(\Omega)$,

(ii) $\operatorname{supp} u_i \subset \mathbb{R}^n - \Omega_{r_0+2}$, $i = 0, 1$.

Then, there exists a constant $C = C(p, n, \alpha, M, \Omega) > 0$ such that

$$\|\bar{D}^{M-1} D^1 S(t, \cdot; \mathscr{d})\|_p$$
$$\leqq C(1+t)^{-\alpha\left(1-\frac{2}{p}\right)}[\|u_0\|_{q, M+2n+2} + \|u_1\|_{q, M+2n+1}], \qquad \mathscr{d} = (u_0, u_1, 0).$$

Proof. Choose $\psi(x) \in C^\infty(\mathbb{R}^n)$ so that $\psi(x) = 1$ if $x \in \mathbb{R}^n - \Omega_{r_0+2}$ and $= 0$ if $x \in \Omega_{r_0+1}$ and define the linear operator $T(v_0, v_1)$ by

$$T(v_0, v_1) = T(v_0, v_1)(t, x) = S(t, x; \psi \mathscr{d}_0)$$

where $\psi \mathscr{d}_0 = (\psi v_0, \psi v_1, 0)$. Below, for notational convenience the same letter C is used to denote constants depending on Ω, ψ, n, p, α and M. First, we shall prove that the estimate:

$$\|\bar{D}^M T(v_0, v_1)(t, \cdot)\|_\infty \leqq C(1+t)^{-\alpha}[\|v_0\|'_{1, M+2n+2} + \|v_1\|'_{1, M+2n+1}], \tag{4.2}$$

holds for any $v_j \in C_0^\infty(\mathbb{R})$, $j = 0, 1$. By Lemma 4.4,

$$\|\bar{D}^N S_0(t, \cdot; \psi \mathscr{d}_0)\|'_\infty \leqq C(N)(1+t)^{-\frac{n-1}{2}}[\|v_0\|'_{1, N+n+1} + \|v_1\|'_{1, N+n}] \tag{4.3}$$

for any integer $N \geqq 0$. Choose $\rho(x) \in C_0^\infty(\mathbb{R}^n)$ so that $\rho(x) = 1$ if $x \in \mathbb{R}^n - \Omega_{r_0+1}$ and $= 0$ if $x \in \Omega_{r_0}$. By the uniqueness of solutions we can write

$$T(v_0, v_1)(t, x) = \rho(x)\, v(t, x) + S(t, x; \mathscr{d}_1)$$

where $v(t, x) = S_0(t, x; \psi \mathscr{d}_0)$ and

$$\mathscr{d}_1 = \left(0, 0, 2 \sum_{j=1}^n \partial_j \rho(x) \cdot \partial_j v(t, x) + \Delta \rho(x) \cdot v(t, x)\right).$$

It remains only to evaluate $S(t, x; \mathscr{d}_1)$ to get (4.2). First, let us evaluate $S(t, x; \mathscr{d}_1)$ for $x \in \Omega_{r_0+2}$. Note that the elements of $\mathscr{d}_1$ are in $C_0^\infty(\Omega)$ for all t. Namely, they satisfy the compatibility condition of infinite order for the d'Alembertian equation in Ω. And also, note that their supports are contained in Ω_{r_0+1} for all t. We can apply Lemma 4.3 to evaluate the local energy decay of $S(t, x; \mathscr{d}_1)$. Then, by Lemma 4.3 and (4.3) we have

$$\|\bar{D}^{M+n+1} S(t, \cdot; \mathscr{d}_1)\|_{L^2(\Omega_{r_0+2})}$$
$$\leqq C(1+t)^{-\frac{n-1}{2}} \sup_{0 \leqq s \leqq t} (1+s)^{(n-1)/2} \|\bar{D}^{M+n+1} S_0(t, \cdot; \mathscr{d}_0)\|_{L^2(\Omega_{r_0+1})}$$
$$\leqq C(1+t)^{-\frac{n-1}{2}}[\|v_0\|'_{1, M+2n+2} + \|v_1\|'_{1, M+2n+1}]. \tag{4.4}$$

Next, let us evaluate $S(t, x; \mathscr{d}_1)$ for $x \notin \Omega_{r_0+2}$. Choose $\mu \in C^\infty(\mathbb{R}^n)$ so that $\mu(x)=1$ when $|x| \geqq r_0+2$ and $=0$ when $|x| \leqq r_0+1$. By the definition of $\mu(x)$ and the uniqueness of solutions for Cauchy problem we have that $\mu(x) S(t, x; \mathscr{d}_1) = S_0(t, x; \mathscr{d}'_0)$ where $\mathscr{d}'_0=(0, 0, h)$, $h=h(t, x)= -2\sum_{j=1}^{n} \partial_j \mu(x) \cdot \partial_j w(t, x) - \Delta\mu(x) \cdot w(t, x)$ and $w(t, x)=S(t, x; \mathscr{d}_1)$ when $x \in \Omega$ and $=0$ when $x \notin \Omega$. By Duhamel's principle we can write

$$\mu(x) S(t, x; \mathscr{d}_1) = \int_0^t S_0(t-s, x; \mathscr{d}''_0(x))\, ds,$$

where $\mathscr{d}''_0(s)=(0, h(s, \cdot), 0)$. Since the supports of $\partial_j \mu$ and $\Delta\mu$ are contained in Ω_{r_0+2}, it follows from Hölder's inequality, Lemma 4.4, Sobolev's inequality and (4.5) that

$$\begin{aligned}
&\|\bar{D}^M(\mu(\cdot) S(t, \cdot; \mathscr{d}_1)\|_\infty \\
&\leqq C\left\{(1+t)^{-\frac{n-1}{2}} + \int_0^t (1+(t-s))^{-\frac{n-1}{2}}(1+s)^{-\frac{n-1}{2}} ds\right\} \\
&\quad \cdot [\|v_0\|'_{1, M+2n+2} + \|v_1\|'_{1, M+2n+1}] \\
&\leqq C(\alpha)(1+t)^{-\alpha}[\|v_0\|'_{1, M+2n+2} + \|v_1\|'_{1, M+2n+1}].
\end{aligned} \tag{4.5}$$

Here we have put

$$\alpha = \frac{n-1}{2} \quad \text{when } n \geqq 4 \quad \text{and} \quad =1-\varepsilon \text{ when } n=3 \tag{4.6}$$

for any sufficiently small positive number ε. Combining (4.3), (4.4) and (4.5) and using Sobolev's inequality we have (4.2). By (4.2) we can extend $T(v_0, v_1)$ to a bounded linear operator $\bar{T}: (v_0, v_1) \to \bar{T}(v_0, v_1)(t, \cdot)$ from $H_1^{M+2n+2}(\mathbb{R}^n) \times H_1^{M+2n+1}(\mathbb{R}^n)$ into $L^\infty(\Omega)$ where t is regarded as a parameter.

On the other hand, for any integer $L \geqq 2$ we have that $T(v_0, v_1)(t, x) \in \mathscr{E}_2^L([0, \infty) \times \Omega)$, if $v_j \in H_2^{L-j}(\mathbb{R}^n)$, $j=0, 1$. By the usual energy method (see the proof of Lemma 6.3 in Appendix II, below) we have

$$\|\bar{D}^{L-1} D^1 T(v_0, v_1)(t, \cdot)\|'_2 \leqq C[\|v_0\|'_{2, L} + \|v_1\|'_{2, L-1}]. \tag{4.7}$$

By Sobolev's imbedding theorem we see that $H_1^{K+[n/2]+1}(\mathbb{R}^n)$ is continuously imbedded into $H_2^K(\mathbb{R}^n)$ for $K \geqq 0$. By (4.7) and Sobolev's inequality

$$\begin{aligned}
\|\bar{D}^M T(v_0, v_1)(t, \cdot)\|'_\infty &\leqq (1+t)[\|v_0\|'_{2, M+[n/2]+1} + \|v_1\|'_{2, M+[n/2]}] \\
&\leqq (1+t)[\|v_0\|'_{1, M+2n+2} + \|v_1\|'_{1, M+2n+1}],
\end{aligned} \tag{4.8}$$

if $v_j \in H_1^{M+2n+2-j}(\mathbb{R}^n)$, $j=0, 1$. Combining (4.2) and (4.8), we see that $\bar{T}(v_0, v_1) = T(v_0, v_1)$ and (4.2) holds if $v_j \in H_1^{M+2n+2-j}(\mathbb{R}^n)$, $j=0, 1$. Therefore, interpolating (4.2) and (4.7) with $L=M$, we have

$$\begin{aligned}
&\|\bar{D}^{M-1} D^1 T(v_0, v_1)(t, \cdot)\|_p \\
&\leqq C(1+t)^{-\alpha\left(1-\frac{2}{p}\right)}[\|v_0\|'_{q, M+2n+2} + \|v_1\|'_{q, M+2n+1}]
\end{aligned} \tag{4.9}$$

for any $v_j \in H_q^{M+2n+2-j}(\mathbb{R}^n)$, $j=0,1$. Let u_j, $j=0,1$, be the functions satisfying assumptions (i) and (ii). Put $\bar{u}_j(x)=u_j(x)$ if $x\in\Omega$ and $=0$ if $x\notin\Omega$. Since $\bar{u}_j\in H_q^{M+2n+2-j}(\mathbb{R}^n)$, $j=0,1$ and since

$$T(\bar{u}_0, \bar{u}_1)(t, x)=S(t, x; \mathscr{d}), \qquad \mathscr{d}=(u_0, u_1, 0),$$

we have the lemma by (4.9).

Lemma 4.7. *Assume that Ω is non-trapping and $n\geqq 3$. Let M be an integer $\geqq 1$, p an extended real number with $2\leqq p\leqq\infty$ and $q=p/(p-1)$. Let α and β be the same constants as in* (4.6) *and* (4.11) *below, respectively. Let g be a function satisfying the conditions:*

(i) $g\in\mathscr{E}_q^{M+2n+1}([0,\infty)\times\Omega)$,

(ii) $\operatorname{supp} g\subset[0,\infty)\times(\mathbb{R}^n-\Omega_{r_0+2})$.

Then, there exists a constant $C=C(p, n, \alpha, \beta, M, \Omega)>0$ such that

$$\|\bar{D}^{M-1} D^1 S(t, \cdot; \mathscr{d})\|_p\leqq C(1+t)^{-\alpha\left(1-\frac{2}{p}\right)}|g|_{q,\beta,M+2n+1,t}, \qquad {}^\forall t\geqq 0,\ \mathscr{d}=(0,0,g).$$

Proof. By (ii) we can use Duhamel's principle for the mixed problem. Thus, we have

$$S(t, x; \mathscr{d})=\int_0^t S(t-s, x; \mathscr{d}'(s))\,ds, \qquad \mathscr{d}'(s)=(0, g(s, \cdot), 0). \tag{4.10}$$

Note that

$$\int_0^t (1+(t-s))^{-\alpha\left(1-\frac{2}{p}\right)}(1+s)^{-\beta}\,ds\leqq C(1+t)^{-\alpha\left(1-\frac{2}{p}\right)}.$$

Here, we have put

$$\beta=\alpha\left(1-\frac{2}{p}\right) \text{ when } \alpha\left(1-\frac{2}{p}\right)>1 \quad\text{and}\quad 1+\kappa \text{ when } \alpha\left(1-\frac{2}{p}\right)\leqq 1 \tag{4.11}$$

for any sufficiently small positive number κ. Applying Lemma 4.6 to (4.10), we have the lemma.

Combining Lemmas 4.6 and 4.7, we have

Corollary 4.8. *Assume that Ω is non-trapping and $n\geqq 3$. Let L be an integer $\geqq 1$. Let u_0, u_1 and g be functions satisfying the conditions:*

(i) $u_0\in H_{q(n)}^{L+M(n)-1}(\Omega)$, $u_1\in H_{q(n)}^{L+M(n)-2}(\Omega)$, $g\in\mathscr{E}_{q(n)}^{L+M(n)-2}([0,\infty)\times\Omega)$,

(ii) $\operatorname{supp} u_i\subset\mathbb{R}^n-\Omega_{r_0+2}$, $\operatorname{supp} g\subset[0,\infty)\times(\mathbb{R}^n-\Omega_{r_0+2})$.

Then, there exists a constant $C=C(L, n, \Omega)>0$ such that

$$\begin{aligned}|D^1 S(\mathscr{d})|_{p(n), L-1, d(n), t} &\leqq C[\|u_0\|_{q(n), L+M(n)-1}+\|u_1\|_{q(n), L+M(n)-2}\\ &\quad+|g|_{q(n), d(n)\sigma(n), L+M(n)-2, t}], \qquad t\geqq 0,\ \mathscr{d}=(u_0, u_1, g).\end{aligned}$$

Proof. Noting that $M(n)=2n+3$ and putting $\alpha\left(1-\dfrac{2}{p(n)}\right)=d(n)$, $\beta=d(n)\,\sigma(n)$, by Lemmas 4.6 and 4.7 we have the corollary.

Lemma 4.9. *Assume that Ω is non-trapping and $n \geqq 3$. Let L be an integer $\geqq 1$ and u_0, u_1 and g functions satisfying the conditions:*

(i) $u_0 \in H_2^{L+n+1}(\Omega)$, $u_1 \in H_2^{L+n}(\Omega)$, $g \in \mathscr{E}_2^{L+n}([0, \infty) \times \Omega)$,

(ii) $\operatorname{supp} u_i \subset \Omega_{r_0+3}$, $i=0, 1$, $\operatorname{supp} g \subset [0, \infty) \times \Omega_{r_0+3}$,

(iii) *u_0, u_1 and g satisfy the compatibility condition of order $L+n$ for the d'Alembertian equation in Ω.*

Then, there exists a constant $C = C(n, L, \Omega) > 0$ such that

$$|D^1 S(\mathscr{d})|_{p(n), d(n), L-1, t} \leqq C[\|u_0\|_{2, L+n+1} + \|u_1\|_{2, L+n} + |g|_{2, d(n)\sigma(n), L+n, t}], \quad t \geqq 0, \ \mathscr{d} = (u_0, u_1, g).$$

Proof. By (i) and (iii), we have that $S(t, x; \mathscr{d}) \in \mathscr{E}_2^{L+n+1}([0, \infty) \times \Omega)$. By (ii) we can use Lemma 4.3. Thus, we have

$$\|\bar{D}^{L+n+1} S(t, \cdot; \mathscr{d})\|_{L^2(\Omega_{r_0+5})} \leqq C(1+t)^{-d(n)\sigma(n)} \cdot [\|u_0\|_{2, L+n+1} + \|u_1\|_{2, L+n} + |g|_{2, d(n)\sigma(n), L+n, t}], \quad t \geqq 0. \tag{4.12}$$

Choose $\rho \in C^\infty(\mathbb{R}^n)$ so that $\rho(x) = 1$ when $|x| \geqq r_0 + 4$ and $=0$ when $|x| \leqq r_0 + 3$. By the definition of $\rho(x)$ and the uniqueness of solutions for Cauchy problem, we have that $\rho(x) S(t, x; \mathscr{d}) = S_0(t, x; \mathscr{d}_0)$, where $\mathscr{d}_0 = (0, 0, h)$, $h = h(t, x) = -2 \sum_{j=1}^{n} \partial_j \rho(x) \cdot \partial_j w(t, x) - \Delta \rho(x) \cdot w(t, x)$ and $w(t, x) = S(t, x; \mathscr{d})$ when $x \in \Omega$ and $=0$ when $x \notin \Omega$. By Duhamel's principle we have

$$\rho(x) S(t, x; \mathscr{d}) = \int_0^t S_0(t-s, x; \mathscr{d}_0'(s))\, ds, \qquad \mathscr{d}_0'(s) = (0, h(s, \cdot), 0). \tag{4.13}$$

Noting that the supports of $\partial_j \rho$ and $\Delta \rho$ are contained in Ω_{r_0+4} and applying Lemma 4.4 to (4.13), we have by (4.12)

$$\begin{aligned} &\|\bar{D}^{L-1} D^1 [\rho(\cdot) S(t, \cdot; \mathscr{d})]\|_{p(n)} \\ &\leqq C(1+t)^{-d(n)} \sup_{0 \leqq s \leqq t} (1+s)^{d(n)\sigma(n)} \|\bar{D}^{L+n+1} S(s, \cdot; \mathscr{d})\|_{L^2(\Omega_{r_0+4})} \\ &\leqq C(1+t)^{-d(n)} [\|u_0\|_{2, L+n+1} + \|u_1\|_{2, L+n} + |g|_{2, d(n)\sigma(n), L+n, t}]. \end{aligned} \tag{4.14}$$

Combining (4.12) and (4.14) and using Sobolev's inequality we obtain the lemma.

Proof of Theorem 4.1. Choose $\rho \in C_0^\infty(\mathbb{R}^n)$ so that $0 \leqq \rho \leqq 1$, $\rho(x) = 1$ when $x \in \Omega_{r_0+2}$ and $=0$ when $x \notin \Omega_{r_0+3}$. Put $\mathscr{d}' = ((1-\rho) u_0, (1-\rho) u_1, (1-\rho) g)$ and $\mathscr{d}'' = (\rho u_0, \rho u_1, \rho g)$. By the uniqueness of solutions we have that $S(t, x; \mathscr{d}) = S(t, x; \mathscr{d}') + S(t, x; \mathscr{d}'')$. Apply Corollary 4.8 and Lemma 4.9 to $S(t, x; \mathscr{d}')$ and $S(t, x; \mathscr{d}'')$, respectively. Then, using Sobolev's inequality and noting that $M(n) = 2n+3$, we have

$$\begin{aligned} &|D^1 S(\mathscr{d}')|_{p(n), d(n), L-1, t} \\ &\leqq C[\|u_0\|_{q(n), L+M(n)-1} + \|u_1\|_{q(n), L+M(n)-2} \\ &\quad + |g|_{q(n), d(n)\sigma(n), L+M(n)-2, t}], \end{aligned} \tag{4.15}$$

$$
\begin{aligned}
|D^1 S(\mathscr{A}'')|_{p(n),\, d(n),\, L-1,\, t} &\leqq C[\|u_0\|_{2,\, L+n+1} + \|u_1\|_{2,\, L+n} + |g|_{2,\, d(n)\sigma(n),\, L+n,\, t}] \\
&\leqq C[\|u_0\|_{q(n),\, L+M(n)-1} + \|u_1\|_{q(n),\, L+M(n)-2} \\
&\quad + |g|_{q(n),\, d(n)\sigma(n),\, L+M(n)-2,\, t}]. \qquad (4.16)
\end{aligned}
$$

Combining (4.15) and (4.16), we have the theorem.

5. The Estimates of Nonlinear Terms

In this section, we shall evaluate non-linear terms which will be useful in obtaining a priori estimates in Sect. 6 below. And also, we shall prove Lemma 3.4. The main result in this section is Theorem 5.3, whose proof is based on Lemma 5.1. Since the proof of Lemma 5.1 is analogous to that in Mizohata [11, Sect. 7] which is based on Sobolev's inequality, we may omit the proof.

Lemma 5.1. *Let $u_j(t, x)$, $j=1, \dots, k$, be functions whose norms appearing below are all finite.*

1° *Let p be an extended real number with $1<p\leqq\infty$. Let k and N be integers such that $k\geqq 2$ and $N\geqq[n/p]+1$ and let v_j, $j=1, \dots, k$, be non-negative integers such that $0\leqq v_j\leqq v_k$, $j=1, \dots, k-1$ and $v_1+\dots+v_k\leqq N$. Then, there exists a constant $C=C(p, N, \Omega)$ such that*

$$\left\| \prod_{j=1}^{k} |D^{v_j} u_j(t, \cdot)| \right\|_p \leqq C \prod_{j=1}^{k} \|\bar{D}^N u_j(t, \cdot)\|_p.$$

2° *Let p be an extended real number with $2\leqq p\leqq\infty$. Let k, M and N be integers such that $1\leqq k\leqq\frac{p}{2}+1$, $M\geqq 0$ and $N-M-\frac{n}{p}>0$, and let v_j, $j=1, \dots, k$, be integers such that $0\leqq v_j<N-M$ $(j=1, \dots, k-1)$ and $v_1+\dots+v_k\leqq N$. Then, there exists a constant $C=C(p, N, \Omega)$ such that*

$$\left\| \prod_{j=1}^{k} |D^{v_j} u_j(t, \cdot)| \right\|_2 \leqq C \prod_{j=1}^{k-1} \|\bar{D}^{N-M} u_j(t, \cdot)\|_p \cdot \|\bar{D}^N u_k(t, \cdot)\|_2.$$

3° *Let p be an extended real number with $2\leqq p\leqq\infty$ and $q=p/(p-1)$. Let k, N and M be integers such that $p\geqq k\geqq p/2$, $M\geqq 0$ and $N-M-\frac{n}{p}>0$, and v_j, $j=1, \dots, k$, be integers such that $0\leqq v_j<N-M$ $(j=1, \dots, k-1)$ and $v_1+\dots+v_k\leqq N$. Then, there exists a constant $C=C(p, N, \Omega)$ such that*

$$\left\| \prod_{j=1}^{k} |D^{v_j} u_j(t, \cdot)| \right\|_q \leqq C \prod_{j=1}^{k-1} \|\bar{D}^{N-M} u_j(t, \cdot)\|_p \cdot \|\bar{D}^N u_k(t, \cdot)\|_2.$$

The following lemma is concerned with estimates of composed functions and proved by applying Lemma 5.1-1° to the representation formula of derivatives of composed functions (see, e.g., [19, Sect. 5]).

Lemma 5.2. *Let $G(t, x, \mu)$ $(\mu=(\mu_1, \dots, \mu_s))$ be a function defined on $[0, \infty)\times\bar{\Omega}\times\{\mu\in\mathbb{R}^s;\ |\mu|\leqq 1\}$ and in $\mathscr{B}^\infty$. Assume that for some integer $k\geqq 1$ $G(t, x, \mu)=O(|\mu|^k)$ near $\mu=0$. Let p be an extended real number with $1\leqq p\leqq\infty$, N an integer $\geqq[n/p]+1$ and $u(t, x)=(u_1(t, x), \dots, u_s(t, x))$ a vector valued function such that its norms appearing below are all finite and $\|u(t, \cdot)\|_\infty\leqq 1$. Then, for any integer L with $0\leqq L\leqq N$ there exists a constant $C=C(N, \Omega)>0$ such that*

$$\|D^L G(t, \cdot, u(t, \cdot))\|_p\leqq C\left(1+\sum_{j=1}^{s}\|\bar{D}^N u_j(t, \cdot)\|_p\right)^L\cdot\left(\sum_{j=1}^{s}\|\bar{D}^N u_j(t, \cdot)\|_p\right)^k.$$

By Lemmas 5.1 and 5.2 and Leibnitz's rule we have

Theorem 5.3. *Let F be a function satisfying conditions (2) and (3) in Assumption 2.1. Let $u(t, x)$ be a function such that its norms appearing below are all finite and $\|\bar{D}^1 D^1 u(t, \cdot)\|_\infty\leqq 1$. Let M and N be integers such that $M\geqq 2$ and $N\geqq\max(M+2+[n/p(n)],\ 2M,\ [n/2]+1)+2$. Then, for any integer L with $0\leqq L\leqq N-2$*

1° $\|D^L F(t, \cdot, \bar{D}^1 D^1 u(t, \cdot)\|_2$
$\leqq C(N)(1+\|\bar{D}^{N-1} D^1 u(t, \cdot)\|_2)^L\cdot\|\bar{D}^{N-1} D^1 u(t, \cdot)\|_2^{\sigma(n)+1}, \qquad t>0,$

2° $\|D^L F(t, \cdot, \bar{D}^1 D^1 u(t, \cdot)\|_2+\|D^L F(t, \cdot, \bar{D}^1 D^1 u(t, \cdot)\|_{q(n)}$
$\leqq C(N)(1+\|\bar{D}^{N-1} D^1 u(t, \cdot)\|_2+\|\bar{D}^{N-M-1} D^1 u(t, \cdot)\|_{p(n)})^{N-2}$
$\cdot\|\bar{D}^{N-M-1} D^1 u(t, \cdot)\|_{p(n)}^{\sigma(n)}\cdot\|\bar{D}^{N-1} D^1 u(t, \cdot)\|_2, \qquad t>0,$

3° $\|\bar{D}^1[\partial_t^{N-2}\{F(t, \cdot, \bar{D}^1 D^1 u(t, \cdot))\}$
$-(d_\lambda F)(t, \cdot, \bar{D}^1 D^1 u(t, \cdot))\bar{D}^1 D^1\partial_t^{N-2} u(t, \cdot)]\|_2$
$\leqq C(N)(1+\|\bar{D}^{N-1} D^1 u(t, \cdot)\|_2+\|\bar{D}^{N-M-1} D^1 u(t, \cdot)\|_{p(n)})^{N-2}$
$\cdot\|\bar{D}^{N-M-1} D^1 u(t, \cdot)\|_{p(n)}^{\sigma(n)}\cdot\|\bar{D}^{N-1} D^1 u(t, \cdot)\|_2, \qquad t>0.$

Now, we shall prove Lemma 3.4. For this, we need two lemmas. The first lemma (Lemma 5.4) can be proved by using Sobolev's inequality in the analogous way to Mizohata [11, Sect. 7]. So, we may omit its proof. The second lemma (Lemma 5.5) is well-known as Moser's lemma (see, e.g., Klainerman [5, Lemma 5.11]).

Lemma 5.4. *Let p be an extended real number with $1<p\leqq\infty$. Let $r_1, r_2, \dots, r_m$ (m being an integer $\geqq 2$), K, L and M be integers such that $L\geqq 1$, $\sum_{j=1}^{m} r_j=L$, $0\leqq r_1\leqq r_2\leqq\dots\leqq r_m\leqq L$, $M\geqq[n/p]+1$ and $K+L\leqq M$. Then, for any $u_j\in H_p^{M-r_j}(\Omega)$, $j=1, \dots, m$, we have*

$$\left\|\prod_{j=1}^{m} u_j\right\|_{p, K}\leqq C\prod_{j=1}^{m}\|u_j\|_{p, M-r_j}.$$

Lemma 5.5. *Let $G(x, y)$ be a function in $\mathscr{B}^\infty(\bar{\Omega}\times\{y=(y_1, \dots, y_m)\in\mathbb{R}^m;\ |y|\leqq 1\})$ with $G(x, 0)=0$. Let M be a non-negative integer and p an extended real number with $1\leqq p\leqq\infty$. Then, for any $u(x)=(u_1(x), \dots, u_m(x))$ with $\|u\|_\infty\leqq 1$ we have*

$$\|G(\cdot, u(\cdot))\|_{p, M}\leqq C(p, M, \Omega)\|u\|_{p, M}.$$

Proof of Lemma 3.4. (II) Since $F(t, x, 0)=0$, using the notations defined in (3.2) and applying Taylor expansion to (3.6), we can write

$$v_2(x)=\left(1+\int_0^1 F_2^0(T, x, \bar{D}_x^1 D_x^1 v_0(x), \bar{D}_x^1 v_1(x), \theta v_2(x))\, d\theta\right)^{-1}(\Delta v_0(x)+g(T, x)).$$

Since it follows from (3.7) that $\|\bar{D}_x^1 D_x^1 v_0\|_\infty+\|\bar{D}_x^1 v_1\|_\infty+\|\theta v_2\|_\infty \leqq 3\lambda_0$, $0\leqq\theta\leqq 1$, by (3.3) we have

$$\|v_2\|_p \leqq 2(\|\Delta v_0\|_p+\|g(T, \cdot)\|_p) \leqq 2(\|v_0\|_{p,2}+\|v_1\|_{p,1}+\|g(T, \cdot)\|_p). \tag{5.1}$$

Differentiating the identity (3.6), we can write symbolically

$$\begin{aligned} D_x^1 v_2 = {} & H_1^1(T, x, V) D_x^1 \bar{D}_x^1 D_x^1 v_0 + H_2^1(T, x, V) D_x^1 \bar{D}_x^1 v_1 \\ & + H_3^1(T, x, V) D_x^1 g(T, x) + H_0^1(T, x, V) \end{aligned} \tag{5.2}$$

where $V=(\bar{D}_x^1 D_x^1 v_0, \bar{D}_x^1 v_1, v_2)$, $H_j^1(T, x, V)$, $j=1,2,3$, are vector valued functions whose components are all in $\mathscr{B}^\infty(\bar{\Omega}\times\{V; |V|\leqq 3\lambda_0\})$ and $H_0^1(T, x, V)$ is a function in $\mathscr{B}^\infty(\bar{\Omega}\times\{V; |V|\leqq 3\lambda_0\})$ such that $H_0^1(T, x, 0)=0$. Using (5.2), by inductive arguments we see easily that we can write symbolically for any integer $L\geqq 1$

$$\begin{aligned} D_x^L v_2 = {} & \sum_{K=0}^{L} \sum_{\alpha^K, \beta^K, \gamma^K} H^L_{\alpha^K, \beta^K, \gamma^K}(T, x, V) \\ & \cdot \prod_{j=1}^{K} (D_x^j \bar{D}_x^1 D_x^1 v_0)^{\alpha_j^K} (D_x^j \bar{D}_x^1 v_1)^{\beta_j^K} (D_x^j g(T, x))^{\gamma_j^K} + H_0^L(T, x, V), \end{aligned} \tag{5.3}$$

where $\alpha^K=(\alpha_1^K, \dots, \alpha_K^K)$, $\beta^K=(\beta_1^K, \dots, \beta_K^K)$, $\gamma^K=(\gamma_1^K, \dots, \gamma_K^K)$, α_j^K, β_j^K, $j=1, \dots, K$, are multi-indices such that $\sum_{j=1}^{K} j(|\alpha_j^K|+|\beta_j^K|+\gamma_j^K)=K$, $H^L_{\alpha^K, \beta^K, \gamma^K}(T, x, V)$ are all vector valued functions whose components are all in $\mathscr{B}^\infty(\bar{\Omega}\times\{V; |V|\leqq 3\lambda_0\})$ and $H_0^L(T, x, V)$ is a function in $\mathscr{B}^\infty(\bar{\Omega}\times\{V; |V|\leqq 3\lambda_0\})$ such that $H_0^L(T, x, 0)=0$. Note the Gagliardo and Nirenberg inequality:

$$\|D_x^i w\|_{rp/i} \leqq C\|w\|_\infty^{1-(i/r)} \|w\|_{p,r}^{i/r}, \quad 0<i<r, \ 1\leqq p\leqq\infty. \tag{5.4}$$

Thus, in the same way as in the proof of Lemma 5.1 in Klainerman [5], applying (5.4) to (5.3) and using (3.7), we have the assertion (II).

(III) Note that by Taylor expansion we can write

$$(1+F_2^0(T, x, V))^{-1} = 1-\int_0^1 (1+F_2^0(T, x, \theta V))^{-2}\, d\theta \cdot F_2^0(T, x, V),$$

$$G^k_{\alpha^k, \beta^k, \gamma^k}(T, x, V) = G^k_{\alpha^k, \beta^k, \gamma^k}(T, x, 0) + \int_0^1 (d_\lambda G^k_{\alpha^k, \beta^k, \gamma^k})(T, x, \theta V)\, d\theta \cdot V$$

(cf. (3.5)), where $V=(\bar{D}_x^1 D_x^1 v_0, \bar{D}_x^1 v_1, v_2)$. Noting (3.3), $\|V\|_\infty \leqq 3\lambda_0$ (by (3.7)) and $F_2^0(T, x, 0)=0$, applying Lemma 5.4 with $K=S-j$, $M=S-2$ and $L=k$ $(1\leqq k\leqq j$

$-2)$ to (3.5) and using Lemma 5.5, we have

$$\|v_j\|_{p,S-j} \leqq C(p, N, F)\left(1+\sum_{k=0}^{j-1} \|v_k\|_{p,S-k}\right)^{j-2} \left(\sum_{k=0}^{j-1} \|v_k\|_{p,S-k}\right) + \left(1+\sum_{k=0}^{2} \|v_k\|_{p,S-k}\right) \cdot \|(\bar{D}^{S-2} g)(T, \cdot)\|_p, \quad j \geqq 3, \tag{5.5}$$

provided that (3.7) is valid and that S is an integer $\geqq [n/p]+3$. Using (5.5), we can see that the Assertion (III) is valid by inductive arguments.

(IV) Since $C^\infty([T, T'] \times \bar{\Omega}) \cap \mathscr{E}_2^N([T, T'] \times \Omega)$ is dense in $\mathscr{E}_2^N([T, T'] \times \Omega)$, since $N \geqq [n/2]+3$ and since (3.10) holds, there exists a sequence $\{v^m\}$ in $C^\infty([T, T'] \times \bar{\Omega}) \cap \mathscr{E}_2^N([T, T'] \times \Omega)$ such that

$$|||v^m - v|||_{2, N, T, T'} \to 0 \quad \text{as } m \to \infty, \tag{5.6}$$

$$|||\bar{D}^1 D^1 v^m|||_{\infty, 0, T, T'} \leqq 3\lambda_0 \quad \text{for all } m. \tag{5.7}$$

Put $g^m = \Phi(v^m)$. By the definition of $G^k_{\alpha^k, \beta^k, \gamma^k}$ we see easily that

$$\begin{aligned} v_j^m(T, x) = &-\{1+F_2^0(T, x, V^m)\}^{-1} \\ &\cdot \left[\Delta v_{j-2}^m + \sum_{k=1}^{j-2} \sum_{\alpha^k, \beta^k, \gamma^k} G^k_{\alpha^k, \beta^k, \gamma^k}(T, x, V^m)\right. \\ &\cdot \prod_{i=1}^{k} (\bar{D}_x^1 D_x^1 v_i^m)^{\alpha_i^k} (\bar{D}_x^1 v_{i+1}^m)^{\beta_i^k} (v_{i+2}^m)^{\gamma_i^k} \\ &\left. + (\partial_t^{j-2} F)(T, x, V^m) - (\partial_t^{j-2} g^m)(T, x)\right], \quad j \geqq 3, \end{aligned} \tag{5.8}$$

where $v_k^m = (\partial_t^k v^m)(T, x)$ and $V^m = (\bar{D}_x^1 D_x^1 v_0^m, \bar{D}_x^1 v_1^m, v_2^m)$. By Taylor expansion we can write

$$\begin{aligned} g^m - g = \Phi(v^m) - \Phi(v) = \square (v^m - v) \\ + \int_0^1 (d_\lambda F)(T, x, \theta \bar{D}^1 D^1 v^m + (1-\theta) \bar{D}^1 D^1 v)\, d\theta \cdot \bar{D}^1 D^1 (v^m - v). \end{aligned} \tag{5.9}$$

By (5.7) we have

$$|||\theta \bar{D}^1 D^1 v^m + (1-\theta) \bar{D}^1 D^1 v|||_{\infty, 0, T, T'} \leqq 3\lambda_0, \quad {}^\forall m, \ 0 \leqq {}^\forall \theta \leqq 1. \tag{5.10}$$

Applying Leibniz's rule to (5.9), we have

$$\begin{aligned} |\bar{D}^{N-2}(g^m - g)| \leqq &|\bar{D}^{N-2} \square (v^m - v)| \\ &+ C(N) \sum_{M=0}^{N-2} \left| \bar{D}^{N-2-M} \left[\int_0^1 (d_\lambda F)(T, x, \theta \bar{D}^1 D^1 v^m + (1-\theta) \bar{D}^1 D^1 v)\, d\theta \right] \right| \\ &\cdot |\bar{D}^M \bar{D}^1 D^1 (v^m - v)|. \end{aligned} \tag{5.11}$$

Noting that $(d_\lambda F)(T, x, 0) = 0$, (5.10) and the fact that $N-2 \geqq [n/2]+1$ and applying Lemmas 5.1 and 5.2 to (5.11), we have

$$|||g^m - g|||_{2, N-2, T, T'} \leqq C(N, F)(1 + |||v^m - v|||_{2, N, T, T'})^{N-2} |||v^m - v|||_{2, N, T, T'}. \tag{5.12}$$

Combining (5.6) and (5.12), we have that

$$|||g^m - g|||_{2, N-2, T, T'} \to 0 \quad \text{as } m \to \infty. \tag{5.13}$$

By (5.6) we have

$$\|v_k^m(T, \cdot) - (\partial_t^k v)(T, \cdot)\|_{2, N-k} \to 0 \quad \text{as } m \to \infty. \tag{5.14}$$

Passing to the limit in (5.8), by Lemma 5.4, (5.13) and (5.14) we see that

$$\begin{aligned}(\partial_t^j v)(T, x) = &-\{1 + F_2^0(T, x, V)\}^{-1} \\ &\cdot \Bigg[\Delta(\partial_t^{j-2} v)(T, x) + \sum_{k=1}^{j-2} \sum_{\alpha^k, \beta^k, \gamma^k} G^k_{\alpha^k, \beta^k, \gamma^k}(T, x, V) \\ &\cdot \prod_{i=1}^{k} (\bar{D}_x^1 D_x^1 \partial_t^i v(T, x))^{\alpha_i^k} (\bar{D}_x^1 \partial_t^{i+1} v(T, x))^{\beta_i^k} (\partial_t^{i+2}(T, x))^{\gamma_i^k} \\ &+ (\partial_j^{j-2} F)(T, x, V) - (\partial_t^{j-2} g)(T, x) \Bigg], \quad 3 \leqq j \leqq N, \end{aligned} \tag{5.15}$$

where $V = (v(T, x), (\partial_t v)(T, x), (\partial_t^2 v)(T, x))$. Since v is a solution of (3.1), we have

$$v(T, x) = v_0(x), \quad (\partial_t v)(T, x) = v_1(x). \tag{5.16}$$

And also, by (5.16) and (3.10) we have

$$\begin{aligned} &(\partial_t^2 v)(T, x) - \Delta v_0(x) \\ &\quad + F(T, x, \bar{D}_x^1 D_x^1 v_0(x), \bar{D}_x^1 v_1(x), (\partial_t^2 v)(T, x)) = g(T, x) \quad \text{in } \Omega, \end{aligned} \tag{5.17}$$

$$\|\bar{D}_x^1 D_x^1 v_0\|_\infty + \|\bar{D}_x^1 v_1\|_\infty + \|(\partial_t^2 v)(T, \cdot)\|_\infty \leqq |||\bar{D}^1 D^1 v|||_{\infty, 0, T, T'} \leqq 2\lambda_0. \tag{5.18}$$

By (5.17), (5.18) and Lemma 3.4(I) which is already proved we have

$$v_2(x) = (\partial_t^2 v)(T, x). \tag{5.19}$$

Noting the definition of v_j, $j \geqq 3$, (*cf.* (3.5)) and combining (5.15), (5.16) and (5.19), by inductive arguments we have

$$(\partial_t^j v)(T, x) = v_j(x), \quad 3 \leqq j \leqq N,$$

which completes the proof.

6. A Priori Estimate

In this section, we shall derive a priori estimate of time local solutions. The following is a main result in this section.

Theorem 6.1. *Assume that Assumption* 2.1 *holds. Let T be a positive number, N an integer $\geqq 2M(n) + 2 = 4n + 8$, λ_0 a positive constant appearing in* (3.3) *and* (3.4). *Let $u \in \mathscr{E}_2^N([0, T] \times \Omega)$ be a solution of the equation:*

$$\begin{aligned}
&\Phi(u)=\Box u+F(t,x,\tilde{D}^1 D^1 u)=f(t,x) && \text{in } [0,T]\times\Omega,\\
&u=0 && \text{on } [0,T]\times\partial\Omega, \qquad (6.1)\\
&u(0,x)=u_0(x),\quad (\partial_t u)(0,x)=u_1(x) && \text{in } \Omega.
\end{aligned}$$

Suppose that

$$|||\tilde{D}^1 D^1 u|||_{\infty,0,0,T}\leqq 2\lambda_0. \tag{6.2}$$

Put

$$X(t)=|D^1 u|_{2,0,N-1,t}+|D^1 u|_{p(n),d(n),N-M(n)-1,t},$$

$$\begin{aligned}\varepsilon=&\|u_0\|_{2,N}+\|u_1\|_{2,N-1}+|f|_{2,d(n)\sigma(n),N-1,\infty}\\
&+\|u_0\|_{q(n),N-1}+\|u_1\|_{q(n),N-2}+|f|_{q(n),d(n)\sigma(n),N-2,\infty}.\end{aligned}$$

Then, for any $\varepsilon_1>0$ there exists an $\varepsilon_2>0$ independent of T and depending only on ε_1, N, n, Ω, λ_0 and F such that if

$$\varepsilon\leqq\varepsilon_2, \tag{6.3}$$

then

$$X(t)\leqq\varepsilon_1. \tag{6.4}$$

To prove the theorem, we need the following three lemmas.

Lemma 6.2 (*see, e.g.* [7]). *Let L be an integer $\geqq 2$ and $v\in H_2^L(\Omega)\cap H_{\mathscr{V}}(\Omega)$. Then,*

$$\|D_x^L v\|_2\leqq C(L,\Omega)[\|\Delta v\|_{2,L-2}+\|D_x^1 v\|_2].$$

Lemma 6.3. *Let L be an integer $\geqq 2$ and $u_j\in H_2^{L-j}(\Omega)$, $j=0,1$, $g\in\mathscr{E}_2^{L-1}([0,T]\times\Omega)$. Assume that u_0, u_1 and g satisfy the compatibility condition of order $L-1$ for the d'Alembertian equation in Ω (cf. Remark 4.2 in Sect. 4). Then,*

$$\begin{aligned}\|\tilde{D}^{L-1}D^1 S(t,\cdot;\mathscr{d})\|_2\leqq& C(L,n,\Omega)[\|u_0\|_{2,L}+\|u_1\|_{2,L-1}\\
&+\|(\tilde{D}^{L-2}g)(0,\cdot)\|_2+|g|_{2,d(n)\sigma(n),L-1,t}],\qquad \mathscr{d}=(u_0,u_1,g).\end{aligned}$$

Lemma 6.4. *Let T be a positive constant and F a function satisfying conditions* (2) *and* (3) *in Assumption* 2.1 *and λ_0 a constant appearing in* (3.3) *and* (3.4). *Let $v_0\in H_2^2(\Omega)\cap H_{\mathscr{V}}(\Omega)$, $v_1\in H_{\mathscr{V}}(\Omega)\cap L^2(\Omega)$ and $g\in C^1([0,T];L^2(\Omega))$. Then, there exists a $\delta_2>0$ depending only on n, λ_0, F and Ω such that if $v\in\mathscr{B}^4([0,T]\times\bar{\Omega})$ with $|||\tilde{D}^1 D^1 v|||_{\infty,0,0,T}\leqq\min(3\lambda_0,\delta_2)$ and $w\in\mathscr{E}_2^2([0,T]\times\Omega)$ is a solution of the mixed problem:*

$$\begin{aligned}
&\Box w+(d_\lambda F)(t,x,\tilde{D}^1 D^1 v(t,x))\tilde{D}^1 D^1 w=g(t,x) && \text{in } [0,T]\times\Omega,\\
&w=0 && \text{on } [0,T]\times\partial\Omega,\\
&w(0,x)=v_0(x),\quad (\partial_t w)(0,x)=v_1(x) && \text{in } \Omega,
\end{aligned}$$

then w satisfies the estimate:

$$\begin{aligned}\|\tilde{D}^1 D^1 w(t,\cdot)\|_2\leqq& C_1(1+|D^1 v|_{\infty,d(n),2,t}^{\sigma(n)})(\exp C_2|D^1 v|_{\infty,d(n),2,t}^{\sigma(n)})\\
&\cdot[\|v_0\|_{2,2}+\|v_1\|_{2,1}+\|g(0,\cdot)\|_2+|g|_{2,d(n)\sigma(n),1,t}]\end{aligned}$$

where C_1 and C_2 are some positive constants depending only on n, Ω, λ_0 and F and are bounded as $\delta_2\to 0$.

Lemmas 6.3 and 6.4 will be proved in Appendix II. To prove Theorem 6.1, first we shall show the following lemma by using Lemmas 6.2, 6.3 and 6.4.

Lemma 6.5. *Let T' be a positive number $\leqq T$ and δ_2 the same as in Lemma 6.4. In addition to all assumptions of Theorem 6.1, assume that*

$$|||\bar{D}^1 D^1 u|||_{\infty, 0, 0, T'} \leqq \delta_2, \tag{6.5}$$

$$E = \|u_0\|_{2,N} + \|u_1\|_{2,N-1} + \|\bar{D}^{N-2} f(0, \cdot)\|_2 \leqq 1. \tag{6.6}$$

Then, there exists a $C_3 > 0$ depending only on n, Ω and F such that the inequality:

$$X(t) \leqq C_3(1+X(t))^{\sigma(n)}(\exp C_3 X(t)^{\sigma(n)})[\varepsilon + (1+X(t))^{N-2} X(t)^{\sigma(n)+1}], \tag{6.7}$$

holds for all $t \in [0, T']$. Here u, $X(t)$ and ε are the same as in Theorem 6.1.

Proof. First, let us note that Theorem 4.1 and Lemma 6.3 can be used without checking the compatibility condition. In fact, since $N \geqq [n/2]+3$, $u \in \mathscr{E}_2^N([0, T] \times \Omega)$ is a solution of (6.1) and satisfies (6.2), by Lemma 3.4(IV) we have

$$(\partial_t^j u)(0, x) = v_j(x), \quad 3 \leqq j \leqq N,$$

where $v_j(x) = V_j(0, u(0, \cdot), (\partial_t u)(0, \cdot), (\partial_t^2 u)(0, \cdot), f; x)$, $3 \leqq j \leqq N$ (cf. (3.5)).

On the other hand, put $g(t, x) = f(t, x) - F(t, x, \bar{D}^1 D^1 u(t, x))$, $u_0(x) = u(0, x)$, $u_1(x) = (\partial_t u)(0, x)$ and $u_j(x) = U_j(u_0, u_1, g; x)$, $j \geqq 2$, (cf. (4.1)). In particular we see that

$$\begin{aligned} u_2(x) &= \Delta u_0(x) + g(0, x) \quad \text{(by (4.1))} \\ &= \Delta u(0, x) + f(0, x) - F(0, x, \bar{D}^1 D^1 u|_{t=0}) \\ &= (\partial_t^2 u)(0, x). \end{aligned}$$

Further, using (3.5) and (4.1), by inductive arguments we have

$$u_j(x) = v_j(x) = (\partial_t^j u)(0, x), \quad 3 \leqq j \leqq N.$$

Since $u \in \mathscr{E}_2^N([0, T] \times \Omega)$ and $u = 0$ on $\partial\Omega$, we have

$$u_j \in H_V(\Omega) \cap L^2(\Omega), \quad j = 0, 1, \ldots, N-1, \ u_N \in L^2(\Omega).$$

This implies that the data $u_0(x) = u(0, x)$, $u_1(x) = (\partial_t u)(0, x)$ and $g(t, x) = f(t, x) - F(t, x, \bar{D}^1 D^1 u(t, x))$ satisfy the compatibility condition of order $N-1$ for the d'Alembertian equation in Ω.

Now, we shall prove the lemma. In the course of the proof, for notational convenience the same letter C is used to denote constants on n, Ω, N and F. Since $u \in \mathscr{E}_2^N([0, T] \times \Omega)$ is a solution of (6.1) and satisfies (6.2) and since $N \geqq [n/2]+3$ and (6.6) holds, by Lemma 3.6(II) we have

$$\|(\bar{D}^{N-1} D^1 u)(0, \cdot)\|_2 \leqq CE. \tag{6.8}$$

Applying Lemma 6.3 with $L = N-1$ and $g = f - F$ to (6.1), we have

$$\begin{aligned} \|\bar{D}^{N-2} D^1 u(t, \cdot)\|_2 \leqq C[E &+ |f|_{2, d(n)\sigma(n), N-2, t} + \|\bar{D}^{N-2} F|_{t=0}\|_2 \\ &+ |F|_{2, d(n)\sigma(n), N-2, t}], \quad t \in [0, T']. \end{aligned} \tag{6.9}$$

By Lemma 5.2, Theorem 5.3 and (6.8), we have

$$\|\bar{D}^{N-2}F|_{t=0}\|_2 \leqq C(1+\|\bar{D}^{N-1}D^1 u(0,\ \cdot)\|_2)^{N-2}\|\bar{D}^N u(0,\ \cdot)\|_2^{\sigma(n)+1}$$
$$\leqq C(1+E)^{N-2}E^{\sigma(n)+1} \leqq CE, \tag{6.10}$$

$$|F|_{2,\,d(n)\sigma(n),\,N-2,\,t} \leqq C(1+\|\bar{D}^{N-1}D^1 u(t,\ \cdot)\|_2 + \|\bar{D}^{N-M(n)-1}D^1 u(t,\ \cdot)\|_{p(n)})^{N-2}$$
$$\cdot(|D^1 u|_{p(n),\,d(n),\,N-M(n)-1,\,t})^{\sigma(n)}|D^1 u|_{2,\,0,\,N-1,\,t}$$
$$\leqq C(1+X(t))^{N-2}X(t)^{\sigma(n)+1} \qquad t\in[0,\ T'], \tag{6.11}$$

where we have used (6.8) in (6.10). Combining (6.9)–(6.11), we have

$$|D^1 u|_{2,\,0,\,N-2,\,t} \leqq C[E+|f|_{2,\,d(n)\sigma(n),\,N-2,\,t}+X(t)^{\sigma(n)+1}(1+X(t))^{N-2}]. \tag{6.12}$$

Next, differentiating (6.1) $N-2$ times with respect to t, we have

$$\Box(\partial_t^{N-2}u)+(d_\lambda F)(t,\ x,\ \bar{D}^1 D^1 u(t,\ x))\,\bar{D}^1 D^1(\partial_t^{N-2}u)=\partial_t^{N-2}f-h \tag{6.13}$$

where $h=\partial_t^{N-2}[F(t,\ x,\ \bar{D}^1 D^1 u(t,\ x))]-(d_\lambda F)(t,\ x,\ \bar{D}^1 D^1 u(t,\ x))\,\bar{D}^1 D^1 \partial_t^{N-2}u(t,\ x)$. By Theorem 5.3,

$$|\bar{D}^1 h|_{2,\,d(n)\sigma(n),\,1,\,t} \leqq C(1+X(t))^{N-2}X(t)^{\sigma(n)+1}, \qquad t\in[0,\ T']. \tag{6.14}$$

Since $N \geqq [n/2]+5$, by Sobolev's imbedding theorem we have that $u\in\mathscr{B}^4([0,\ T']\times\bar{\Omega})$. And also, (6.5) holds. So, we can use Lemma 6.4 with $v=u$, $w=\partial_t^{N-2}u$, $g=\partial_t^{N-2}f-h$, $v_0=\partial_t^{N-2}u|_{t=0}$ and $v_1=\partial_t^{N-1}u|_{t=0}$. Applying Lemma 6.4 to (6.13), we have

$$\sum_{j=0}^{2}\|D_x^j\partial_t^{N-j}u(t,\ \cdot)\|_2 \leqq C(1+X(t)^{\sigma(n)})(\exp CX(t)^{\sigma(n)})$$
$$\cdot[E+|f|_{2,\,d(n)\sigma(n),\,N-1,\,t}+(1+X(t))^{N-2}X(t)^{\sigma(n)+1}], \qquad t\in[0,\ T']. \tag{6.15}$$

Further, differentiating the equation:

$$\Delta u=\partial_t^2 u+F(t,\ x,\ \bar{D}^1 D^1 u(t,\ x))-f(t,\ x)$$

L-times $(0\leqq L\leqq N-3)$ with respect to t and applying Lemma 6.2, we have

$$\sum_{L=0}^{N-3}\|D_x^{N-L}\partial_t^L u(t,\ \cdot)\|_2$$
$$\leqq C[\|\bar{D}^{N-2}F\|_2+\|\bar{D}^{N-2}f\|_2+\sum_{j=0}^{2}\|D_x^j\partial_t^{N-j}u(t,\ \cdot)\|_2$$
$$+\|\bar{D}^{N-2}D^1 u(t,\ \cdot)\|_2], \qquad t\in[0,\ T']. \tag{6.16}$$

Combining (6.11), (6.12), (6.15) and (6.16), we have

$$|D^1 u|_{2,\,0,\,N-1,\,t} \leqq C(1+X(t)^{\sigma(n)})(\exp CX(t)^{\sigma(n)})$$
$$\cdot[E+|f|_{2,\,d(n)\sigma(n),\,N-2,\,t}+(1+X(t))^{N-2}X(t)^{\sigma(n)+1}]. \tag{6.17}$$

Now, we are going to evaluate the term: $|D^1 u|_{p(n), d(n), N-M(n)-1, t}$. Since $u(t, x)=S(t, x; \mathscr{d})$, $\mathscr{d}=(u_0, u_1, f-F)$, applying Theorem 4.1, we have

$$|D^1 u|_{p(n), M-M(n)-1, d(n), t} \leqq C[\|u_0\|_{q(n), N-1} + \|u_1\|_{q(n), N-2} + |f|_{q(n), d(n)\sigma(n), N-2, t} + |F|_{q(n), d(n)\sigma(n), N-2, t}], \quad t \in [0, T']. \tag{6.18}$$

By Theorem 5.3,

$$|F|_{q(n), d(n)\sigma(n), N-2, t} \leqq C(1+X(t))^{N-2} X(t)^{\sigma(n)+1}, \quad t \in [0, T']. \tag{6.19}$$

Combining (6.18) and (6.19), we have

$$|D^1 u|_{p(n), N-M(n)-1, d(n), t} \leqq C[\|u_0\|_{q(n), N-1} + \|u_1\|_{q(n), N-2} + |f|_{q(n), d(n)\sigma(n), N-2, t} + (1+X(t))^{N-2} X(t)^{\sigma(n)+1}], \quad t \in [0, T']. \tag{6.20}$$

Combining (6.17) and (6.20), we have (6.7), which completes the proof.

Now, using (6.7), we shall prove Theorem 6.1. Our arguments here are analogous to those in Chadam [1], Segal [14] and Matsumura-Nishida [9]. First, note that $X(t)$ is continuous in $[0, T]$, which plays important role below. Choose $\varepsilon_2 > 0$ so that

$$\varepsilon_2 \leqq 1. \tag*{[A.1]}$$

Since $E \leqq \varepsilon \leqq \varepsilon_2 \leqq 1$, (6.6) holds. By Sobolev's inequality, (6.6) and (6.8),

$$X(0) \leqq C_4 \varepsilon_2 \tag{6.21}$$

where C_4 is some constant depending only on n and Ω. By continuity of $X(t)$ we can choose $\kappa \in (0, T]$ so small that

$$X(t) \leqq (1+C_4)\varepsilon_2, \quad t \in [0, \kappa]. \tag{6.22}$$

By Sobolev's inequality, we have

$$\||\bar{D}^1 D^1 u\||_{\infty, 0, 0, t} \leqq C_5 X(t), \quad t \in [0, T] \tag{6.23}$$

where C_5 is some constant depending only on n and Ω. Choose ε_2 so that

$$(1+C_4)\varepsilon_2 \leqq 1, \quad (1+C_4) C_5 \varepsilon_2 \leqq \delta_2. \tag*{[A.2]}$$

By [A.2], (6.22) and (6.23), we have that $\||\bar{D}^1 D^1 u\||_{\infty, 0, 0, \kappa} \leqq \delta_2$ and that

$$X(t) \leqq 1, \quad t \in [0, \kappa]. \tag{6.24}$$

By Lemma 6.5 with $T' = \kappa$, (6.3) and (6.24) we have

$$X(t) \leqq C_6[\varepsilon_2 + X(t)^2], \quad t \in [0, \kappa], \tag{6.25}$$

where $C_6 = 2^{N-2+\sigma(n)} C_3 \exp C_3$. Without loss of generality, we may assume that $C_6 \geqq 1$. We need the following elementary result, which can be proved by elementary calculation.

Lemma 6.6. *Let C_6 be the same as in (6.25). Let A and B be positive nembers. Assume that*

$$A \leqq (2\sqrt{2}\, C_6)^{-2}, \quad B < (\sqrt{2}+1)(2\sqrt{2}\, C_6)^{-1}, \quad B \leqq C_6(A+B^2).$$

Then, we have

$$B \leqq (\sqrt{2}\, C_6)\, A.$$

In order to use Lemma 6.6, choose ε_2 so that

$$(1+C_4)\,\varepsilon_2 < (\sqrt{2}+1)(2\sqrt{2}\, C_6)^{-1}, \qquad [A.3]$$

$$\varepsilon_2 \leqq (2\sqrt{2}\, C_6)^{-2}. \qquad [A.4]$$

It follows from (6.22) and [A.3] that

$$X(t) < (\sqrt{2}+1)(2\sqrt{2}\, C_6)^{-1}, \quad t \in [0, \kappa]. \tag{6.26}$$

In view of (6.25), [A.4] and (6.26), applying Lemma 6.6 with $A = \varepsilon_2$ and $B = X(t)$, we have

$$X(t) \leqq (\sqrt{2}\, C_6)\, \varepsilon_2, \quad t \in [0, \kappa]. \tag{6.27}$$

Now, let us define the subset I of the open interval $(0, T)$ as follows:

$$I = \{\tau \in (0, T);\ X(t) \leqq (\sqrt{2}\, C_6)\, \varepsilon_2,\ t \in [0, \tau]\}. \tag{6.28}$$

We have already proved that $I \supset (0, \kappa]$, which implies that I is non-empty. By continuity of $X(t)$ we see easily that I is closed. Since $(0, T)$ is connected, if we prove that I is open, then $I = (0, T)$. So, we shall show that I is open below. Let τ be any point in I. By continuity of $X(t)$ we can choose τ' so that $0 < \tau < \tau' < T$ and

$$X(t) \leqq (2\sqrt{2}\, C_6)\, \varepsilon_2, \quad t \in [0, \tau']. \tag{6.29}$$

Choose ε_2 so that

$$(2\sqrt{2}\, C_5 \cdot C_6)\, \varepsilon_2 \leqq \delta_2. \qquad [A.5]$$

Since it follows from (6.23), (6.29) and [A.5] that $|||\tilde{D} D^1 u|||_{\infty, 0, 0, \tau'} \leqq \delta_2$ and since $E \leqq 1$, applying Lemma 6.5 with $T' = \tau'$ we have

$$\begin{aligned} X(t) \leqq\ & C_3(1 + X(t)^{\sigma(n)})(\exp C_3 X(t)^{\sigma(n)}) \\ & \cdot [\varepsilon_2 + (1 + X(t))^{N-2} X(t)^{\sigma(n)+1}] \end{aligned} \tag{6.30}$$

for $t \in [0, \tau']$. Since $C_6 \geqq 1$, by (6.29) and [A.4] we have

$$X(t) \leqq (2\sqrt{2}\, C_6)^{-1} \leqq 1, \quad t \in [0, \tau']. \tag{6.31}$$

Combining (6.30) and (6.31) and noting the definition of C_6 (cf. (6.25)), we have

$$X(t) \leqq C_6[\varepsilon_2 + X(t)^2], \quad t \in [0, \tau']. \tag{6.32}$$

By (6.31) we have

$$X(t) \leqq (2\sqrt{2}\, C_6)^{-1} < (\sqrt{2}+1)(2\sqrt{2}\, C_6)^{-1}, \qquad t \in [0, \tau']. \tag{6.33}$$

In view of [A.4], (6.32) and (6.33), applying Lemma 6.6 we have

$$X(t) \leqq (\sqrt{2}\, C_6)\, \varepsilon_2, \qquad t \in [0, \tau'].$$

Thus, we have that $\tau \in (0, \tau') \subset I$. This implies that I is open, and then $I = (0, T)$. From this and the continuity of $X(t)$ in $[0, T]$ we obtain that

$$X(t) \leqq (\sqrt{2}\, C_6)\, \varepsilon_2, \qquad t \in [0, T].$$

Finally, choose ε_2 so that

$$(\sqrt{2}\, C_6)\, \varepsilon_2 \leqq \varepsilon_1. \tag{A.6}$$

Since C_4, C_5 and C_6 depend only on n, N, Ω and F, we can choose ε_2 so small that ε_2 satisfies the conditions [A.1]–[A.6] and depends only on n, N, Ω, F and ε_1. This completes the proof of Theorem 6.1.

7. Proof of Main Theorem

Let N be an integer $\geqq 2M(n)+2 = 4n+8$. Let $u_0 \in H_2^N(\Omega) \cap H_{q(n)}^{N-1}(\Omega)$, $u_1 \in H_2^{N-1}(\Omega) \cap H_{q(n)}^{N-2}(\Omega)$ and $f \in \mathscr{E}_2^{N-1}([0, \infty) \times \Omega) \cap \mathscr{E}_{q(n)}^{N-2}([0, \infty) \times \Omega)$ be the data for the problem (1.1) which satisfy the compatibility condition of order $N-1$ which is defined in Definition 3.7. Let ε and $X(t)$ be the same as in Theorem 6.1, below. We shall prove that there exists a solution $u \in \mathscr{E}_2^N([0, \infty) \times \Omega)$ of the equation (1.1) if ε is small enough. For simplicity, we shall say that u is a solution in $[0, T_1]$ if $u \in \mathscr{E}_2^N([0, T_1] \times \Omega)$ and u satisfies the equation:

$$\begin{gathered} \square u + F(t, x, \tilde{D}^1 D^1 u) = f(t, x) \quad \text{in } [0, T_1] \times \Omega, \qquad u = 0 \quad \text{on } [0, T_1] \times \partial\Omega, \\ u(0, x) = u_0(x), \qquad (\partial_t u)(0, x) = u_1(x) \quad \text{in } \Omega, \end{gathered} \tag{7.1}$$

and the condition: $|||\tilde{D}^1 D^1 u|||_{\infty, 0, 0, T_1} \leqq 2\lambda_0$. By Theorem 3.5 we know that a solution in $[0, T'']$ exists for some $T'' > 0$ small enough. Let $\bar{T}$ be the supremum of the number $T > 0$ so that a solution in $[0, T]$ exists. Suppose that $\bar{T} < \infty$. Let ε_1 be any positive number satisfying the assumption

$$C_5\, \varepsilon_1 \leqq \lambda_0 \tag{A.7}$$

where λ_0 is the constant appearing in (3.3) and (3.4) and C_5 is the same as in (6.23). By Theorem 6.1 we see that for any $T \in (0, \bar{T})$ a solution u in $[0, T]$ has the estimate:

$$X(t) \leqq \varepsilon_1, \qquad t \in [0, T], \qquad \text{if } \varepsilon \leqq \varepsilon_2, \tag{7.2}$$

where ε_2 is a positive constant depending only on n, N, Ω, F and ε_1 and independent of T. Put $L_0 = [n/2] + 8$ (cf. Theorem 3.5). By (7.2) we have

$$\begin{aligned}&\|u(t,\ \cdot)\|_{2,L_0}+\|(\partial_t u)(t,\ \cdot)\|_{2,L_0-1}+|||f|||_{2,L_0-1,0,t}\\&\leqq\varepsilon_2+(1+\bar{T})\,\varepsilon_1,\qquad t\in[0,T],\end{aligned}\tag{7.3}$$

for any $T\in(0,\bar{T})$ if $\varepsilon\leqq\varepsilon_2$. Here we have used the fact that

$$\|u(t,\ \cdot)\|_2\leqq\|u_0\|_2+\int_0^t\|(\partial_t u)(s,\ \cdot)\|_2\,ds.$$

Combining [A.7] and (6.23) we have

$$|||\bar{D}^1 D^1 u|||_{\infty,0,0,T}\leqq\lambda_0.\tag{7.4}$$

Thus, by Lemma 3.6(I) we see that a triple $(u(T,x),(\partial_t u)(T,x),f(t,x))$ belongs to to $\mathscr{D}_T$ with $(\partial_t^2 u)(T,x)$ and that $u(T,x)$, $(\partial_t u)(T,x)$ and $f(t,x)$ satisfy the compatibility condition of order $N-1$ at time T. From this and (7.3) it follows that

$$(u(T,x),(\partial_t u)(T,x),f(t,x))\in\mathbb{D}(N,\varepsilon_2+(1+\bar{T})\,\varepsilon_1,T)\tag{7.5}$$

for any $T\in(0,\bar{T})$. By Theorem 3.5 we know that there exists a $T''>0$ depending on n, Ω, F, ε_1, ε_2 and $\bar{T}$ such that T'' is independent of $T\in(0,\bar{T})$ and there exists a function $v\in\mathscr{E}_2^N([T,T+T'']\times\Omega)$ satisfying the condition:

$$\begin{aligned}&\square v+F(t,x,\bar{D}^1 D^1 v)=f &&\text{in}\ \ [T,T+T'']\times\Omega,\\&v=0 &&\text{on}\ \ [T,T+T'']\times\partial\Omega,\\&v(T,x)=u(T,x),\quad(\partial_t v)(T,x)=(\partial_t u)(T,x) &&\text{in}\ \ \Omega,\end{aligned}\tag{7.6}$$

and the condition: $|||\bar{D}^1 D^1 v|||_{\infty,0,T,T+T''}\leqq 2\lambda_0$. Since u is a solution in $[0,T]$ and v is a solution of (7.6), and since u and v satisfy the inequalities:

$$\begin{aligned}&\|\bar{D}_x^1 D_x^1 u(T,\ \cdot)\|_\infty+\|\bar{D}_x^1(\partial_t u)(T,\ \cdot)\|_\infty+\|(\partial_t^2 u)(T,\ \cdot)\|_\infty\leqq\lambda_0,\\&\|\bar{D}_x^1 D_x^1 u(T,\ \cdot)\|_\infty+\|\bar{D}_x^1(\partial_t u)(T,\ \cdot)\|_\infty+\|(\partial_t^2 v)(T,\ \cdot)\|_\infty\\&\quad=\|\bar{D}_x^1 D_x^1 v(T,\ \cdot)\|_\infty+\|\bar{D}_x^1(\partial_t v)(T,\ \cdot)\|_\infty+\|(\partial_t^2 v)(T,\ \cdot)\|_\infty\\&\quad\leqq|||\bar{D}^1 D^1 v|||_{\infty,0,T,T+T''}\leqq 2\lambda_0,\end{aligned}$$

by Lemma 3.4(I) we have

$$(\partial_t^2 v)(T,x)=(\partial_t^2 u)(T,x).\tag{7.7}$$

Since $N\geqq[n/2]+3$, by Lemma 3.4(IV) and (7.7) we have that

$$(\partial_t^j u)(T,x)=(\partial_t^j v)(T,x)=v_j(x),\qquad 3\leqq j\leqq N,\tag{7.8}$$

where $v_j=V_j(T,u(T,\ \cdot),(\partial_t u)(T,\ \cdot),(\partial_t^2 u)(T,\ \cdot),f;x)$, $3\leqq j\leqq N$ (cf. (3.5)). Thus, if we put

$$u'(t,x)=u(t,x)\quad\text{when}\ \ 0\leqq t\leqq T\quad\text{and}\quad=v(t,x)\quad\text{when}\ \ T\leqq t\leqq T+T'',$$

by (7.7) and (7.8) we see that $u'(t,x)\in\mathscr{E}_2^N([0,T+T'']\times\Omega)$ and that $u'(t,x)$ satisfies the equation (7.1) with $T_1=T+T''$ and the condition:

$|||\bar{D}^1 D^1 u'|||_{\infty, 0, 0, T+T''} \leqq 2\lambda_0$. Namely, u' is a solution in $[0, T+T'']$. Since T'' is independent of T we can choose T so that $0<T=\bar{T}-(T''/2)<\bar{T}$, which contradicts the maximality of $\bar{T}$. Thus, we have that $\bar{T}=\infty$ if $\varepsilon \leqq \varepsilon_2$. This implies the existence of time global solutions. Other assertions of Main Theorem follow immediately from Theorems 3.5 and 6.1.

Concluding Remark. Our proof is based on the ellipticity of the Laplacian and the uniform decay of solutions of the mixed problem for the d'Alembertian. Therefore, the argument developed in the present paper can be applied to the nonlinear Klein-Gordon equation, the nonlinear wave equation with first order dissipation and the nonlinear Schrödinger equation (see, e.g., Shibata [16, 17] and Tsutsumi [20]).

Appendix I. Proof of Lemma 4.3

Let us define three Hilbert spaces $\mathscr{H}$, $\mathscr{H}_b$ and $\mathscr{H}_0^a$, $a, b \geqq r_0$, as follows:

$$\mathscr{H}=\{(u, v)';\ u \in H_{\nabla}(\Omega),\ v \in L^2(\Omega)\},$$
$$\mathscr{H}_b=\{(u, v)';\ u \in H_2^1(\Omega_b),\ v \in L^2(\Omega_b),\ u=0 \text{ on } \partial\Omega\},$$
$$\mathscr{H}_0^a=\{(u, v)' \in \mathscr{H};\ \operatorname{supp} u,\ \operatorname{supp} v \subset \Omega_a\},$$

where $(u, v)'$ denotes the transpose vector of $\begin{pmatrix} u \\ v \end{pmatrix}$. $\mathscr{H}$, $\mathscr{H}_0^a$ and $\mathscr{H}_b$ are equipped with norms:

$$|||(u, v)'|||_{\mathscr{H}}^2 = |||(u, v)'|||_{\mathscr{H}_0^a}^2 = \|D_x^1 u\|_2^2 + \|v\|_2^2,$$
$$|||(u, v)|||_{\mathscr{H}_b}^2 = \|\bar{D}_x^1 u\|_{L^2(\Omega_b)}^2 + \|v\|_{L^2(\Omega_b)}^2,$$

respectively. Let us denote the norm of operators from $\mathscr{H}_0^a$ to $\mathscr{H}_b$ by $\|\cdot\|_{a \to b}$. Put $A=\begin{pmatrix} 0, & 1 \\ \Delta, & 0 \end{pmatrix}$. A is a skew adjoint operator with domain $\mathscr{D}(A)=\{(u, v)' \in \mathscr{H};\ v \in H_{\nabla}(\Omega),\ \Delta u \in L^2(\Omega)\}$. By Lemma 6.2 we see that

$$\mathscr{D}(A)=\{(u, v)' \in \mathscr{H};\ D_x^2 u \in L^2(\Omega),\ v \in H_{\nabla}(\Omega)\}.$$

By Stone's theorem we know that A generates the one parameter unitary group $\{U(t);\ t \in \mathbb{R}^1\}$.

Lemma Ap. 1 *(local energy decay of the first energy). Assume that Ω is non-trapping and $n \geqq 3$. Then, for any $a, b \geqq r_0$ we have*

$$\|U(t)\|_{a \to b} \leqq C(a, b, n, \Omega)\, \Gamma(t)$$

where $\Gamma(t)=\mathrm{e}^{-c|t|}$ if n is odd and $=(1+|t|)^{-n+1}$ if n is even. Here c is some positive number depending only on n, a, b and Ω.

Remark Ap. 2. The lemma is proved in Shibata-Tsutsumi [18, Sect. 2.1]. If $\mathbb{R}^n - \Omega$ is star-shaped and n is odd, the result is well-known (see, e.g., [8]). And also, if $\mathbb{R}^n - \Omega$ is convex, then Ω is non-trapping in our sense (see [24] and

[10]). Melrose [10] proves that

$$\Gamma(t)=e^{-c|t|} \text{ if } n \text{ is odd and } =(1+|t|)^{-n/2} \text{ if } n \text{ is even}$$

under the similar assumption. But, our result in [18] is sharper than that in [10] when n is even.

The following lemma is well-known.

Lemma Ap. 3 (see, e.g., [12]). *Let* $(u_0, u_1)' \in \mathscr{D}(A)$ *and* $g(t, x) \in C^1([0, \infty); L^2(\Omega))$. *Put* $\mathscr{A}=(u_0, u_1, g)$. *Then*

$$(S(t, \cdot; \mathscr{A}), \partial_t S(t, \cdot; \mathscr{A}))' = U(t)(u_0, u_1)' + \int_0^t U(t-s)(0, g(s, \cdot))' \, ds.$$

Combining Lemmas Ap. 1 and Ap. 3, we have

Lemma Ap. 4. *Assume that Ω is non-trapping and $n \geqq 3$. Let a, b and γ be any positive numbers such that $0<\gamma \leqq n-1$ and $a, b \geqq r_0$. Let u_0, u_1 and g be functions satisfying the conditions:*

(i) $u_0 \in H_2^2(\Omega) \cap H_\Gamma(\Omega)$, $u_1 \in H_\Gamma(\Omega) \cap L^2(\Omega)$, $g \in C^1([0, \infty); L^2(\Omega))$,

(ii) $\operatorname{supp} u_i \subset \Omega_a$, $i=0, 1$, $\operatorname{supp} g \subset [0, \infty) \times \Omega_a$.

Then, we have

$$\begin{aligned} \|\bar{D}^1 S(t, \cdot; \mathscr{A})\|_{L^2(\Omega_b)} \leqq\ & C(a, b, n, \gamma, \Omega)[\Gamma(t)\{\|u_0\|_{2,1} + \|u_1\|_2\} \\ & +(1+t)^{-\gamma} |g|_{2, \gamma, 0, t}], \quad {}^\forall t \geqq 0, \ \mathscr{A}=(u_0, u_1, g). \end{aligned} \tag{Ap. 1.1}$$

Here, $\Gamma(t)$ is the same as in Lemma Ap. 1.

Remark Ap. 5. The condition (i) means that data u_0, u_1 and g satisfy the compatibility condition of order 1 for the d'Alembertian equation in Ω.

Now, under the assumptions of Lemma 4.3 let us evaluate the higher order derivatives. Put $u=S(t, x; \mathscr{A})$, $\mathscr{A}=(u_0, u_1, g)$. By the Assumptions (i), (ii) and (iii) we know that

$$u \in \mathscr{E}_2^M([0, \infty) \times \Omega), \qquad g \in \mathscr{E}_2^{M-1}([0, \infty) \times \Omega), \tag{Ap. 1.2}$$

$$u=0 \quad \text{on} \quad \partial\Omega, \tag{Ap. 1.3}$$

$$\operatorname{supp} u(0, x), \quad \operatorname{supp}(\partial_t u)(0, x) \subset \Omega_a, \quad \operatorname{supp} g \subset [0, \infty) \times \Omega_a. \tag{Ap. 1.4}$$

First, we shall evaluate $\|\bar{D}^1 \partial_t^K u(t, \cdot)\|_{L^2(\Omega_b)}$, $b \geqq r_0$, $0 \leqq K \leqq M-1$. Choose $\rho(t) \in C_0^\infty(\mathbb{R}^1)$ so that

$$0 \leqq \rho \leqq 1, \quad \operatorname{supp} \rho \subset [-2, -1], \quad \int_{-\infty}^{\infty} \rho(s) \, ds = 1. \tag{Ap. 1.5}$$

For any $\varepsilon \in (0, 1)$, let us put

$$u_\varepsilon(t, x) = \int_{-\infty}^{\infty} u(t-s, x) \, \rho_\varepsilon(s) \, ds,$$

$$g_\varepsilon(t, x) = \int_{-\infty}^{\infty} g(t-s, x) \, \rho_\varepsilon(s) \, ds, \qquad \rho_\varepsilon(t) = \varepsilon^{-1} \rho(\varepsilon^{-1} t).$$

Since the solution u has the propagation speed 1, by (Ap. 1.4) and (Ap. 1.5) we have

$$\operatorname{supp}(\partial_t^K u_\varepsilon)(0, x) \subset \{x \in \mathbb{R}^n; |x| \leqq a+2\}, \quad \varepsilon \in (0, 1), \ {}^\forall K \geqq 0. \tag{Ap. 1.6}$$

By (Ap. 1.2), (Ap. 1.3) and (Ap. 1.4) we have

$$\begin{aligned} &u_\varepsilon \in C^\infty([0, \infty); H_2^M(\Omega) \cap H_\gamma(\Omega)), \\ &g_\varepsilon \in C^\infty([0, \infty); H_2^{M-1}(\Omega)), \quad \operatorname{supp} g_\varepsilon \subset [0, \infty) \times \Omega_a. \end{aligned} \tag{Ap. 1.7}$$

By (Ap. 1.6) and (Ap. 1.7) we have that the functions $(\partial_t^K u_\varepsilon)(0, x)$, $(\partial_t^{K+1} u_\varepsilon)(0, x)$ and $(\partial_t^K g_\varepsilon)(t, x)$ satisfy the conditions (i) and (ii) of Lemma Ap. 4. Since $\partial_t^K u = S(t, x; \mathscr{d}_\varepsilon^K)$ where $\mathscr{d}_\varepsilon^K = ((\partial_t^K u_\varepsilon)(0, x), (\partial_t^{K+1} u_\varepsilon)(0, x), (\partial_t^K g_\varepsilon)(t, x))$, by Lemma Ap. 4 we have

$$\begin{aligned} \|\bar{D}^1 \partial_t^K u_\varepsilon(t, \cdot)\|_{L^2(\Omega_{2b})} \leqq C(a, b, n, \gamma, \Omega)[\Gamma(t)\{\|(\partial_t^K u_\varepsilon)(0, \cdot)\|_{2,1} \\ + \|(\partial_t^{K+1} u_\varepsilon)(0, \cdot)\|_2\} + (1+t)^{-\gamma} |\partial_t^K g_\varepsilon|_{2, \gamma, 0, t}]. \end{aligned} \tag{Ap. 1.8}$$

Thus, passing to the limit in (Ap. 1.8), by (Ap. 1.2) and (Ap. 1.5) we have

$$\begin{aligned} &\|\bar{D}^1 \partial_t^K u(t, \cdot)\|_{L^2(\Omega_{2b})} \\ &\quad \leqq C(a, b, n, \gamma, \Omega)[\Gamma(t)\{\|(\partial_t^K u)(0, \cdot)\|_{2,1} + \|(\partial_t^{K+1} u)(0, \cdot)\|_2\} \\ &\qquad + (1+t)^{-\gamma} |\partial_t^K g|_{2, \gamma, 0, t}], \quad {}^\forall t \geqq 0, \ 0 \leqq {}^\forall K \leqq M-1. \end{aligned} \tag{Ap. 1.9}$$

Since $u \in \mathscr{E}_2^M([0, \infty) \times \Omega)$ and u satisfies the equation: $\partial_t^2 u = \Delta u + g$, we have

$$(\partial_t^{j+2} u)(0, x) = (\partial_t^j u)(0, x) + (\partial_t^j g)(0, x), \quad 0 \leqq j \leqq M-2. \tag{Ap. 1.10}$$

Using (Ap. 1.10) and the facts that $u(0, x) = u_0(x)$ and $(\partial_t u)(0, x) = u_1(x)$, by inductive arguments we have

$$\begin{aligned} &\|(\partial_t^j u)(0, \cdot)\|_{2, L} \\ &\quad \leqq C(j, L)[\|u_0\|_{2, L+j} + \|u_1\|_{2, L+j-1} + \|(\bar{D}^{L+j-2} g)(0, \cdot)\|_2], \end{aligned} \tag{Ap. 1.11}$$

for any integers L and j such that $2 \leqq j \leqq M$ and $0 \leqq L \leqq M-j$. Combining (Ap. 1.9) and (Ap. 1.11) we have

$$\begin{aligned} &\|\bar{D}^1 \partial_t^K u(t, \cdot)\|_{L^2(\Omega_{2b})} \\ &\quad \leqq C(a, b, n, \gamma, \Omega)[\Gamma(t)\{\|u_0\|_{2, K+1} + \|u_1\|_{2, K}\} \\ &\qquad + (1+t)^{-\gamma} |g|_{2, \gamma, M-1, t}], \quad {}^\forall t \geqq 0, \ 0 \leqq {}^\forall K \leqq M-1. \end{aligned} \tag{Ap. 1.12}$$

Here, we have used the fact that $\|(\bar{D}^K g)(0, \cdot)\|_2 \leqq |g|_{2, \gamma, M-1, t}$, ${}^\forall t \geqq 0$, $0 \leqq {}^\forall K \leqq M-1$.

Finally, let us evaluate the higher order derivatives of u with respect to x. For this, we need the following well-known result.

Lemma Ap. 6. *Let r and r' be any positive numbers with $r_0 \leqq r < r'$. Let L be any integer $\geqq 2$. Then, there exists a constant $C = C(r, r', L, \Omega) > 0$ such that for any $v \in H_2^L(\Omega) \cap H_\gamma(\Omega)$ the following estimate holds:*

$$\|\bar{D}_x^L v\|_{L^2(\Omega_r)} \leqq C[\|\bar{D}_x^{L-2} \Delta v\|_{L^2(\Omega_{r'})} + \|\bar{D}_x^1 v\|_{L^2(\Omega_{r'})}].$$

Let K and L be integers such that $0 \leqq K \leqq L \leqq M-2$. Since $\partial_t^K u$ can be regarded as a solution of the equation:

$$\Delta(\partial_t^K u) = \partial_t^{K+2} u - \partial_t^K g \quad \text{in } \Omega$$

and since $\partial_t^K u \in H_\nabla(\Omega) \cap H_2^{M-K}(\Omega)$, by Lemma Ap. 6 we have

$$\begin{aligned} &\|\bar{D}_x^{M-L} \partial_t^K u(t, \cdot)\|_{L^2(\Omega_b)} \\ &\quad \leqq C(n, b, \varepsilon, \Omega)[\|\bar{D}_x^{M-L-2} \partial_t^{K+2} u(t, \cdot)\|_{L^2(\Omega_{b+2\varepsilon})} + \|\bar{D}_x^1 \partial_t^K u(t, \cdot)\|_{L^2(\Omega_{b+2\varepsilon})} \\ &\quad + (1+t)^{-\gamma} |g|_{2,\gamma,M-2,t}], \qquad t \geqq 0,\ 0 \leqq {}^{\forall} L \leqq M-2, \end{aligned}$$

where ε is any small positive constant. Repeating the argument, we have

$$\begin{aligned} &\|\bar{D}_x^{M-L} \partial_t^K u(t, \cdot)\|_{L^2(\Omega_b)} \\ &\quad \leqq C(n, b, M, \varepsilon, \Omega)\left[\sum_{j=0}^{M-1} \|(\bar{D}^1 \partial_t^j u)(t, \cdot)\|_{L^2(\Omega_{b+(M-L)\varepsilon})} \right. \\ &\quad \left. + (1+t)^{-\gamma} |g|_{2,\gamma,M-2,t}\right]. \end{aligned} \tag{Ap. 1.13}$$

Taking $(M-L)\varepsilon = b$ in the right-hand side of (Ap. 1.13) and substituting (Ap. 1.12), we have Lemma 4.3.

Appendix II. Proofs of Lemmas 6.3 and 6.4

Proof of Lemma 6.3. Put $u = S(t, x; \mathscr{A})$. As is well-known, by using the mollifier with respect to t we may assume that $u, \partial_t u \in \mathscr{E}_2^L([0, \infty) \times \Omega)$ (see the proof of Lemma 4.3 in Appendix I). In view of Remark 4.2(i), by inductive arguments we have

$$\begin{aligned} \|(\partial_t^j u)(0, \cdot)\|_{2,K} \leqq C(K, j)[\|u_0\|_{2,K+j} + \|u_1\|_{2,K+j-1} \\ + \|(\bar{D}^{K+j-2} g)(0, \cdot)\|_2], \end{aligned} \tag{Ap. 2.1}$$

for any integers $j \geqq 2$ and $K \geqq 0$ with $j + K \leqq L$. Note that u satisfies the equation:

$$\begin{aligned} \Box u = g \text{ in } [0, \infty) \times \Omega, \quad u = 0 \text{ on } [0, \infty) \times \partial\Omega, \\ u(0, x) = u_0(x), \quad (\partial_t u)(0, x) = u_1(x) \text{ in } \Omega. \end{aligned} \tag{Ap. 2.2}$$

Let M be an integer with $0 \leqq M \leqq L-1$. Differentiating (Ap. 2.2) M-times with respect to t, multiplying the resulting equation by $\partial_t^{M+1} u$ and integrating over $[0, t] \times \Omega$, by (Ap. 2.1) we have

$$\begin{aligned} &\tfrac{1}{2}\|D^1 \partial_t^M u(t, \cdot)\|_2^2 \\ &\quad \leqq \tfrac{1}{2}\|D^1 \partial_t^M u(0, \cdot)\|_2^2 + \int_0^t \|(\partial_t^M g)(s, \cdot)\|_2 \cdot \|(\partial_t^{M+1} u)(s, \cdot)\|_2\, ds \\ &\quad \leqq C[\|u_0\|_{2,M+1} + \|u_1\|_{2,M} + \mu(M)\|\bar{D}^{M-1} g(0, \cdot)\|_2 + |g|_{2,d(n)\sigma(n),M,\infty}]^2 \\ &\quad + \tfrac{1}{2}\int_0^t (1+s)^{-d(n)\sigma(n)} \|(\partial_t^{M+1} u)(s, \cdot)\|_2^2\, ds \end{aligned} \tag{Ap. 2.3}$$

where $\mu(M)$ denotes the constant such that $\mu(M)=1$ if $M \geqq 1$ and $=0$ if $M=0$. Here we have used the fact that $d(n)\sigma(n)>1$. Applying Gronwall's inequality to (Ap. 2.3) we have

$$\|D^1 \partial_t^M u(t, \cdot)\|_2 \leqq C[\|u_0\|_{2,M+1}+\|u_1\|_{2,M} + \mu(M)\|\bar{D}^{M-1} g(0, \cdot)\|_2+|g|_{2,d(n)\sigma(n),M,\infty}]. \qquad \text{(Ap. 2.4)}$$

On the other hand, applying Lemma 6.2 to the equation:

$$\Delta(\partial_t^M u) = -\partial_t^M g + \partial_t^{M+2} u \text{ in } \Omega, \qquad \partial_t^M u = \text{ on } \partial\Omega,$$

t being regarded as a parameter, we have

$$\|D_x^{K-M} \partial_t^M u(t, \cdot)\|_2 \leqq C[\|(\bar{D}^{K-2} g)(t, \cdot)\|_2 + \|(D_x^{K-M-2}\partial_t^{M+2} u)(t, \cdot)\|_2 + \|(\bar{D}^{K-2} D^1 u)(t, \cdot)\|_2] \qquad \text{(Ap. 2.5)}$$

for any integers $K \geqq 2$ and $M \geqq 0$ with $M+2 \leqq K \leqq L$. Combining (Ap. 2.4) and (Ap. 2.5) by inductive arguments we have Lemma 6.3.

Proof of Lemma 6.4. Using the notations defined in (3.2) and putting

$$\begin{aligned} F_2^j(t, x, \bar{D}^1 D^1 v) &= a_j(t, x), \quad j=0, 1, \dots, n, \\ F_2^{ij}(t, x, \bar{D}^1 D^1 v) &= a_{ij}(t, x), \quad i, j=1, \dots, n, \\ F_1^j(t, x, \bar{D}^1 D^1 v) &= b_j(t, x), \quad j=0, 1, \dots, n, \end{aligned}$$

we can write

$$\begin{aligned} &(1+a_0(t,x))\partial_t^2 w + \sum_{j=1}^n a_j(t,x)\partial_j\partial_t w \\ &\quad - \sum_{i,j=1}^n (\delta_{ij}+a_{ij}(t,x))\partial_i\partial_j w \\ &\quad + \sum_{j=0}^n b_j(t,x)\partial_j w = g(t,x) && \text{in } [0,\infty)\times\Omega, \qquad \text{(Ap. 2.6)} \\ &w=0 && \text{on } [0,\infty)\times\partial\Omega, \\ &w(0,x)=v_0(x), \quad (\partial_t w)(0,x)=v_1(x) && \text{in } \Omega. \end{aligned}$$

Since $|||\bar{D}^1 D^1 v|||_{\infty,0,0,T} \leqq 3\lambda_0$, by (3.3) and (3.4) we have

$$1/2 \leqq 1+a_0(t,x) \leqq 3/2, \qquad \sum_{i,j=1}^n (\delta_{ij}+a_{ij}(t,x))\xi_i\xi_j \geqq \tfrac{1}{2}|\xi|^2 \qquad \text{(Ap. 2.7)}$$

for any $(t, x) \in [0, T]\times\bar{\Omega}$ and $\xi=(\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. Since $v \in \mathscr{B}^4$, we have that a_j, a_{ij}, $b_j \in \mathscr{B}^2$. Thus, as is well-known, by using the mollifier with respect to t, we may assume that $w, \partial_t w \in \mathscr{E}_2^2([0, T]\times\Omega)$ (see Ikawa [3] and also the proof of Lemma 4.3 in Appendix I). By Assumption 2.1(3), $F_2^j(t, x, \lambda)=F_2^{ij}(t, x, \lambda) = F_1^j(t, x, \lambda) = O(|\lambda|^{\sigma(n)})$. From this and the fact that $|||\bar{D}^1 D^1 v|||_{\infty,0,0,T} \leqq 3\lambda_0$ we have

$$\sum_{i,j=1}^n |||a_{ij}|||_{\infty,0,0,T} + \sum_{j=0}^n |||a_j|||_{\infty,0,0,T} + \sum_{j=0}^n |||b_j|||_{\infty,0,0,T} \leqq C, \qquad \text{(Ap. 2.8)}$$

$$\sum_{i,j=1}^{n} \|\bar{D}^1 a_{ij}(t,\cdot)\|_\infty + \sum_{j=0}^{n} \|\bar{D}^1 a_j(t,\cdot)\|_\infty$$
$$+ \sum_{j=0}^{n} \|\bar{D}^1 b_j(t,\cdot)\|_\infty \leqq C(\|\bar{D}^2 D^1 v(t,\cdot)\|_\infty)^{\sigma(n)}. \qquad \text{(Ap. 2.9)}$$

Here and hereafter, for notational convenience the same letter C is used to denote constants depending essentially on n, Ω, λ_0 and F. For simplicity, put

$$a(t) = (\|\bar{D}^2 D^1 v(t,\cdot)\|_\infty)^{\sigma(n)}.$$

Multiplying (Ap. 2.6) by $\partial_t w$ and integrating over $[0,t]\times\Omega$, by (Ap. 2.7), (Ap. 2.8) and (Ap. 2.9) we have

$$\|D^1 w(t,\ \cdot)\|_2^2$$
$$\leqq C[\|v_0\|_{2,1} + \|v_1\|_2] + \tfrac{1}{2}\int_0^t (1+s)^{d(n)\sigma(n)} \|g(s,\ \cdot)\|_2^2\, ds$$
$$+ \int_0^t [Ca(s) + \tfrac{1}{2}(1+s)^{-d(n)\sigma(n)}]\, \|D^1 w(s,\ \cdot)\|_2^2\, ds. \qquad \text{(Ap. 2.10)}$$

Applying Lemma 6.2 to the equation:

$$\Delta w = (1+a_0)\,\partial_t^2 w + \sum_{j=1}^{n} a_j \partial_j \partial_t w + \sum_{j=0}^{n} b_j \partial_j w - g - \sum_{i,j=1}^{n} a_{ij}\partial_i\partial_j w \quad \text{in } \Omega,$$
$$w = 0 \quad \text{on } \partial\Omega,$$

t being regarded as a parameter, we have

$$\|D_x^2 w(t,\ \cdot)\|_2 \leqq C_7\left(\sum_{i,j=1}^{n} \|a_{ij}(t,\ \cdot)\|_\infty\right) \|D_x^2 w(t,\ \cdot)\|_2$$
$$+ C[\|g(t,\ \cdot)\|_2 + \|D^1 \partial_t w(t,\ \cdot)\|_2 + \|D^1 w(t,\ \cdot)\|_2] \qquad \text{(Ap. 2.11)}$$

where C_7 is some constant depending only on n and Ω. Choose $\delta_2' > 0$ so small that

$$\delta_2' C_7 = \tfrac{1}{2}. \qquad \text{(Ap. 2.12)}$$

Combining (Ap. 2.11) and (Ap. 2.12), we have

$$\|D_x^2 w(t,\ \cdot)\|_2 \leqq C[\|g(t,\ \cdot)\|_2 + \|D^1 \partial_t w(t,\ \cdot)\|_2 + \|D^1 w(t,\ \cdot)\|_2] \qquad \text{(Ap. 2.13)}$$

provided that $\sum_{i,j=1}^{n} \|a_{ij}(t,\ \cdot)\|_\infty \leqq \delta_2'$, $t\in[0,T]$. Since

$$\sum_{i,j=1}^{n} \|a_{ij}(t,\ \cdot)\|_\infty \leqq C(|\!|\!|\bar{D}^1 D^1 v|\!|\!|_{\infty,0,0,T})^{\sigma(n)}, \qquad t\in[0,T],$$

we have that there exists a $\delta_2 > 0$ depending only on n, λ_0, F and Ω such that if $|\!|\!|\bar{D}^1 D^1 v|\!|\!|_{\infty,0,0,T} \leqq \delta_2$ then (Ap. 2.13) is valid. Differentiating (Ap. 2.6) with respect to t, multiplying the resulting equation by $\partial_t^2 w$ and integrating over $[0, T] \times \Omega$, by (Ap. 2.7), (Ap. 2.8) and (Ap. 2.9) we have

$$\begin{aligned}\|D^1 \partial_t w(t, \cdot)\|_2^2 \leqq & C[\|v_0\|_{2,2}^2 + \|v_1\|_{2,1}^2 + \|g(0, \cdot)\|_2^2] \\ & + \tfrac{1}{2}\int_0^t (1+s)^{d(n)\sigma(n)} \|(\partial_t g)(s, \cdot)\|_2^2 \, ds \\ & + \int_0^t [Ca(s) + \tfrac{1}{2}(1+s)^{-d(n)\sigma(n)}] \|\bar{D}^1 D^1 w(s, \cdot)\|_2^2 \, ds. \end{aligned} \tag{Ap. 2.14}$$

Here we have used the fact that

$$\|(\partial_t^2 w)(0, \cdot)\|_2 \leqq C[\|v_0\|_{2,2} + \|v_1\|_{2,1} + \|g(0, \cdot)\|_2].$$

Combining (Ap. 2.10), (Ap. 2.13) and (Ap. 2.14), we have

$$\begin{aligned}\|\bar{D}^1 D^1 w(t, \cdot)\|_2^2 \leqq & C\Big[\|v_0\|_{2,2}^2 + \|v_1\|_{2,1}^2 + \|g(0, \cdot)\|_2^2 \\ & + \int_0^t (1+s)^{d(n)\sigma(n)} \sum_{k=0}^{1} \|(\partial_t^k g)(s, \cdot)\|_2^2 \, ds\Big] \\ & + \int_0^t C[a(s) + (1+s)^{-d(n)\sigma(n)}] \|\bar{D}^1 D^1 w(s, \cdot)\|_2^2 \, ds. \end{aligned} \tag{Ap. 2.15}$$

Since $a(s) \leqq (1+s)^{-d(n)\sigma(n)}(|D^1 v|_{\infty, d(n), 2, t})^{\sigma(n)}$, $0 \leqq {}^\forall s \leqq t$, noting that $d(n)\sigma(n) > 1$ and applying Gronwall's inequality to (Ap. 2.15), we have Lemma 6.4.

References

1. Chadam, J.: Asymptotic for $\Box u = m^2 u + G(t, x, u, u_x, u_t)$. I. Ann. Scuola Norm. Sup., Pisa **26**, 33–65 (1972)
2. Dionne, P.: Sur les problème de Cauchy hyperboliques bién poses. J. Analyse Math. **10**, 1–90 (1962)
3. Ikawa, M.: Mixed problems for hyperbolic equations of second order. J. Math. Soc. Japan **20**, 580–608 (1968)
4. John, F.: Blow-up for quasi-linear wave equations in three space dimension. Commun. Pure Appl. Math. **34**, 29–51 (1981)
5. Klainerman, S.: Global existence for nonlinear wave equations. Ibid. **33**, 43–101 (1980)
6. Klainermann, S., Ponce, G.: Global small amplitude solutions to nonlinear evolution equations. Ibid. **36**, 133–141 (1983)
7. Lax, P.D., Phillips, R.S.: Scattering Theory. New York & London: Academic Press 1967
8. Lax, P.D., Morawetz, C.S., Phillips, R.S.: Exponential decay of solutions of the wave equations in the exterior of a star-shaped obstacle. Ibid. **16**, 477–486 (1963)
9. Matsumura, A., Nishida, T.: The initial value problem for the equations of motion of viscous and heat-conductive gases. J. Math. Kyoto Univ. **20**, 67–104 (1980)
10. Melrose, R.B.: Singularities and energy decay in acoustical scattering. Duke Math. J. **46**, 43–59 (1979)
11. Mizohata, S.: The theory of partial differential equations. London: Cambridge Univ., Press 1973
12. Mizohata, S.: Quelque problemès au bord, du type mixte, pour des équations hyperboliques, Séminair sur les équations aux derivées partielles. Collège de France, 23–60 (1966/67)

13. Rabinowitz, P.H.: Periodic solutions of nonlinear hyperbolic partial differential equations II. Commun. Pure Appl. Math. **22**, 15–39 (1969)
14. Segal, I.: Non-linear semi-groups. Ann. Math. **78**, 339–364 (1963)
15. Shatah, J.: Global existence of small solutions to nonlinear evolution equations. J. Diff. Eqs. **46**, 409–425 (1982)
16. Shibata, Y.: On the global existence theorem of classical solutions of mixed problem for some second order non-linear hyperbolic operators with dissipative term in the interior domain. Funk. Ekva. **25**, 303–345 (1982)
17. Shibata, Y.: On the global existence theorem of classical solutions of second order fully nonlinear hyperbolic equations with first order dissipation in the exterior domain. Tsukuba J. Math. **7**, 1–68 (1983)
18. Shibata, Y., Tsutsumi, Y.: Global existence theorem of nonlinear wave equation in the exterior domain. Lecture Notes in Num. Appl. Anal. **6**, 155–196 (1983), Kinokuniya/North-Holland
19. Shibata, Y., Tsutsumi, Y.: Local existence of solution for the initial-boundary value problem of fully nonlinear wave equation. To appear
20. Tsutsumi, Y.: Global solutions of the nonlinear Schrödinger equation in exterior domains. Commun. P.D.E. **8**, 1337–1374 (1983)
21. Tsutsumi, Y.: Local energy decay of solutions to the free Schrödinger equation in exterior domain. J. Fac. Sci. Univ. Tokyo Sec. IA, **31**, 97–108 (1984)
22. Vainberg, B.R.: On the short wave asymptotic behaviour of solutions of stationary problems and the asymptotic behaviour as $t \to \infty$ of solutions of nonstationary problems. Russian Math. Survey **30** (2), 1–58 (1975)
23. von Wahl, W.: L^p-decay rates for homogeneous wave equations. Math. Z. **120**, 93–106 (1971)
24. Yamamoto, K.: Characterization of convex obstacle by singularities of the scattering kernel. To appear

Received February 4, 1985