Some Exact Solutions of Unsteady Boundary Layer Equations - I

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Summary – Unsteady boundary layer flows generated in a homogeneous, non-rotating viscous fluid are considered. The method of Laplace transform is used to obtain exact solutions of the unsteady boundary layer equations in a more general situation. The structures of the unsteady velocity field and the associated boundary layers are determined. Several particular solutions are recovered as special cases of the present general theory. The physical implications of the mathematical results are investigated.

1. Introduction

PRANDTL first introduced the concept of boundary layers and initiated the study of boundary layer theory which subsequently received attention of several authors. PRANDTL and his student BLASIUS developed the basic mathematical theory of boundary layer flows. Following their pioneer work, considerable attention was given to the development, structure and stability of laminar steady and unsteady boundary layers including various experimental work.

The boundary layer equations with appropriate boundary conditions are often solved in various situations subject to the assumptions of steady motion or the separableness of space variables from time. Such methods of treatment greatly simplify the governing equations and subsequently a solution of interest can be exactly or approximately obtained by analytical or numerical methods. Nevertheless, the resulting solution has some interest and application in predicting the structures of the unsteady boundary layers.

In spite of some interest in the boundary layer flows based on the above assumptions, the formulation and method of solution of the problem appear to be physically unrealistic and mathematically unjustified for various reasons. First, there seems to be no prior justification of separation of variables other than mathematical convenience and simplification of the problem. In some situations of interest, this treatment is not satisfactory and leads to erroneous conclusions. Second, although the steady boundary layer flow has some interest and application, it cannot describe the development of the flow for various time scales, but only in the limiting situation.

Due to the above considerations, it would be highly desirable to give some attention to the unsteady boundary layer flows in various geometric configurations

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so that the above stated difficulties can be resolved. Some study in this direction has already been undertaken for some special situations. The object of the present paper is to investigate the unsteady boundary layer phenomenon in a more general situation by a simple straight-forward and elegant mathematical treatment. The steady and the unsteady velocity fields together with a description of the associated boundary layers are obtained. The results of the earlier investigators are recovered as particular cases of the present general theory. In addition to the generalizations of earlier problems, some new results of interest are discussed and their physical implications interpreted. The method of Laplace transform combined with the theory of residues is used to investigate the unsteady boundary layer flows.

2. *Formulation of the problem*

Unsteady boundary layer flows are here considered (i) in a semi-infinite expanse of a homogeneous non-rotating viscous fluid bounded by an infinite horizontal plate at $z=0$ and (ii) in a homogeneous viscous fluid between two parallel horizontal rigid plates, one at $z=0$ and the other at $z=D$.

The motion is generated in the fluid from rest by moving the plate(s) impulsively in its (their) own plane(s) with a prescribed velocity as

$$\text{(i)}\quad U(t)=U\,H(t)\,f(t)\quad \text{on}\quad z=0, \tag{2.1}$$

and

$$\text{(ii)}\quad U(t)=U^*\,g(t)\,H(t)\quad \text{on}\quad z=D, \tag{2.2}$$

where U and U^* are constants, $f(t)$ and $g(t)$ are arbitrary functions of time t, and $H(t)$ is the Heaviside unit function of t.

In the two dimensional Cartesian coordinates (x, z), the only non-zero component of velocity is that parallel to the plate(s). Both the velocity and the pressure field will be functions of z and t. The governing equation of motion [1][2]) is

$$\frac{\partial u}{\partial t}=v\,\frac{\partial^2 u}{\partial z^2},\qquad t>0 \tag{2.3}$$

where $0\leqq z\leqq\infty$ or $0\leqq z\leqq D$ according to configuration (i) or (ii), and v is the kinematic viscosity of the fluid.

Assuming no slippage occurs between the plate and the fluid, the appropriate boundary conditions for configuration (i) are

$$u(z,t)=U\,f(t)\quad \text{on}\quad z=0,\quad \text{for all}\quad t>0, \tag{2.4}$$

and

$$u(z,t)\to 0\quad \text{as}\quad z\to\infty,\quad \text{for all}\quad t>0. \tag{2.5}$$

[2]) Numbers in brackets refer to References, page 174.

The initial condition is

$$u(z, t) = 0, \quad \text{at} \quad t = 0 \quad \text{for} \quad z > 0. \tag{2.6}$$

The boundary conditions for configuration (ii) are

$$u(z, t) = U f(t), \quad \text{on} \quad z = 0 \quad \text{for all} \quad t > 0, \tag{2.7}$$

and

$$u(z, t) = U^* g(t) \quad \text{on} \quad z = D \quad \text{for all} \quad t > 0, \tag{2.8}$$

The initial condition for configuration (ii) is

$$u(z, t) = 0 \quad \text{at} \quad t = 0, \quad \text{for all} \quad z \quad \text{in} \quad 0 \leqq z \leqq D. \tag{2.9}$$

3. *The solution of the problem*

It is convenient to introduce the Laplace transform [2] defined by the integral

$$\bar{u}(z, s) = \int_0^\infty e^{-st} u(z, t)\, dt. \tag{3.1}$$

The inverse Laplace transformation [2] is given by

$$u(z, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \bar{u}(z, s)\, ds, \qquad c > 0. \tag{3.2}$$

In view of the transformation and the initial condition, the solution for configuration (i) has the form

$$u(z, t) = \frac{U}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{f}(s) \exp(s t - \lambda z)\, ds, \ \lambda = \sqrt{\frac{s}{\nu}} \tag{3.3}$$

which is, by the convolution theorem of the Laplace transform,

$$u(z, t) = \frac{U}{2\sqrt{\pi \nu}} \int_0^t f(t - \tau)\, \tau^{-3/2} \exp\left(-\frac{z^2}{4 \nu \tau}\right) d\tau. \tag{3.4}$$

This is the most general solution for configuration (i) with an arbitrary functional form of time, $f(t)$.

Similarly, for configuration (ii), the solution is given by

$$\left.\begin{aligned} u(z,t) &= \frac{U}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{f}(s)\,\frac{\sinh\lambda(D-z)}{\sinh\lambda D}\,e^{st}\,ds \\ &+ \frac{U^*}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{g}(s)\,\frac{\sinh\lambda z}{\sinh\lambda D}\,e^{st}\,ds. \end{aligned}\right\} \tag{3.5}$$

In order to obtain an explicit solution for $u(z, t)$ and to determine the principal features of boundary layer flows in both the configurations, it is necessary to consider some special cases of interest. We consider the following particular cases:
(a) $f(t)=1$,
(b) $f(t)=t$,
(c) $f(t)=e^{\alpha t}$, $(\alpha>0)$,
(d) $f(t)=t^n$, $(n\geqq 0)$ and
(e) $f(t)=\delta(t)$, the Dirac delta function;
so that the Laplace transforms are $1/s$, $1/s^2$, $1/(s-\alpha)$, $\Gamma(n+1)/s^{n+1}$ and 1 respectively, where $\Gamma(x)$ is the Gamma function.

For case (a) the inversion integral (3.3) can be evaluated exactly and the solution has the form

$$u(z, t) = U\,\mathrm{erfc}(\zeta), \tag{3.6}$$

where $\zeta \equiv z/2\sqrt{\nu t}$ is the familiar similarity variable of the viscous boundary layer theory and $\mathrm{erf}(x)$, $\mathrm{erc}(x) = 1 - \mathrm{erf}(x)$ are the standard error and complementary error functions [3] defined by

$$\mathrm{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}}\int_0^x e^{-\sigma^2}\,d\sigma. \tag{3.7}$$

Solution (3.6) represents the exact solution first discovered by RAYLEIGH [4] which is generally referred to as the Rayleigh solution.

The velocity distribution related to case (b) is given by

$$u(z, t) = \frac{U}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{s^2}\exp(s\,t - \lambda\,z)\,ds, \tag{3.8}$$

which, by CAMPBELL and FOSTER [3], is

$$u(z, t) = U\,t[(1 + 2\,\zeta^2)\,\mathrm{erfc}(\zeta) - \frac{2\,\zeta}{\sqrt{\pi}}\exp(-\,\zeta^2)], \tag{3.9}$$

This solution was originally due to BLASIUS [5] who obtained it by an independent procedure.

For case (c), the velocity field is

$$u(z, t) = \frac{U}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{s-\alpha} \exp(s\,t - \lambda\,z)\,ds, \tag{3.10}$$

which, again by CAMPBELL and FOSTER [3], is

$$\left.\begin{aligned} u(z, t) = \tfrac{1}{2}\,U\,e^{\alpha t}\Big[&\exp\Big(-z\sqrt{\frac{\alpha}{\nu}}\Big)\operatorname{erfc}(\zeta - \sqrt{\alpha\,t}) \\ &+ \exp\Big(z\sqrt{\frac{\alpha}{\nu}}\Big)\operatorname{erfc}(\zeta + \sqrt{\alpha\,t})\Big]. \end{aligned}\right\} \tag{3.11}$$

This solution is believed to be new. In the limit $t \to \infty$, $\operatorname{erfc}(\zeta - \sqrt{\alpha\,t}) \to 2$ and $\operatorname{erfc}(\zeta + \sqrt{\alpha\,t}) \to 0$ so that the solution tends to the steady state solution $u_{st}(z, t)$ where

$$u_{st}(z, t) \sim f(t)\exp\Big(-z\sqrt{\frac{\alpha}{\nu}}\Big). \tag{3.12}$$

This result was reported by WATSON [6] by a different method.

The solution for case (d) is

$$u(z, t) = U \int_0^t (t-\tau)^n \frac{z}{2\sqrt{\pi\,\nu\,\tau^3}} \exp\Big(-\frac{z^2}{4\,\nu\,\tau}\Big)\,d\tau. \tag{3.13}$$

Making the substitution $\alpha = z/2\sqrt{\nu\,\tau}$, and $\zeta = z/2\sqrt{\nu\,t}$, we obtain

$$u(z, t) = \frac{2\,U}{\sqrt{\pi\,\nu}} \int_\zeta^\infty \Big(\frac{z^2}{4\,\nu\,\zeta^2} - \frac{z^2}{4\,\nu\,\alpha^2}\Big)^n e^{-\alpha^2}\,d\alpha. \tag{3.14}$$

An integral solution similar to this is also obtained by WATSON [6] by an independent treatment. He has further presented numerical results for u/U with a set of values for the parameter ζ and n. These results infer a boundary layer nature. In view of the similarities, any further discussion here would be redundant.

The velocity distribution for case (e) is given by

$$u(z, t) = \frac{U\,z}{2\sqrt{\pi\,\nu\,t^3}} \exp(-\zeta^2). \tag{3.15}$$

This corresponds to the transient motion only and decays to zero as $t \to \infty$, which are really expected.

In order to examine the structure of the boundary layers for configuration (ii), we take the following special cases:

(a′) $f(t)=1$ and $g(t)\equiv 0$ with $D\to\infty$,
(b′) $f(t)=1$ and $g(t)=1$,
(c′) $f(t)=e^{\alpha t}$ and $g(t)=e^{\beta t}$, α, β are real constants; and
(d′) $f(t)=e^{\alpha t}$ and $g(t)\equiv 0$ with $D\to\infty$.
The corresponding Laplace transforms of these functions can be found in [2].

The solution for case (a′) is given by

$$u(z, t) = U \operatorname{erfc}(\zeta). \tag{3.16}$$

This is a recovery of the Rayleigh solution as configuration (ii) conforms to configuration (i).

The solution for case (b′) can be obtained by using residue theory and has the form

$$\left.\begin{aligned} u(z, t) &= U\left(1-\frac{z}{D}\right)+\frac{U^* z}{D} \\ &+ U \sum_{n=1}^{\infty} \frac{(-1)^n 2\pi n \nu \sin\left[n\pi\left(1-\frac{z}{D}\right)\right] \exp\left[(-t(n^2\pi^2\nu/D^2)\right]}{n^2\pi^2\nu} \\ &+ U^* \sum_{n=1}^{\infty} \frac{(-1)^n 2\pi n \nu \sin\left(\frac{n\pi z}{D}\right) \exp\left[-t(n^2\pi^2\pi\nu/D^2)\right]}{n^2\pi^2\nu}. \end{aligned}\right\} \tag{3.17}$$

This result consists of the steady state and the transient solutions. The first two terms represent the former and the two infinite series correspond to the latter. Proceeding to the limit $t\to\infty$, the latter component decays very rapidly and the steady state is reached in the limit. The steady solution represents the well-known Couette flow between two plates.

The velocity distribution related to case (c′) can be evaluated by the calculus of residues and has the representation

$$\left.\begin{aligned} u(z, t) &= \frac{U \sinh\left[\sqrt{\frac{\alpha}{\nu}}(D-z)\right]\exp(\alpha t)}{\sinh\left(\sqrt{\frac{\alpha}{\nu}} z\right)} + \frac{U^* \sinh\left[\sqrt{\frac{\beta}{\nu}} z\right]\exp(\beta t)}{\sinh\left(\sqrt{\frac{\beta}{\nu}} z\right)} \\ &+ U \sum_{n=1}^{\infty} \frac{(-1)^n 2\pi n \nu \sin\left[n\pi\left(1-\frac{z}{D}\right)\right] \exp\left[-t(n^2\pi^2\nu/D^2)\right]}{n^2\pi^2\nu+\alpha D^2} \\ &+ U^* \sum_{n=1}^{\infty} \frac{(-1)^n 2\pi n \nu \sin\left(\frac{n\pi z}{D}\right) \exp\left[-t(n^2\pi^2\nu/D^2)\right]}{n^2\pi^2\nu+\beta D^2}. \end{aligned}\right\} \tag{3.18}$$

With exceptions of the steady terms, this solution is very much similar to that of (3.17). In the limit $t \to \infty$, the last two infinite series representing the transient solution fade away exponentially and the ultimate steady state is established.

Making reference to CAMPBELL and FOSTER [3], the solution for case (d′) has the form

$$\left.\begin{aligned} u(z, t) = \frac{U\, e^{\alpha t}}{2}\Bigg[&\exp\left(z\sqrt{\frac{\alpha}{\nu}}\right)\operatorname{erfc}\left(\zeta + \sqrt{\alpha\, t}\right) \\ &+ \exp\left(-z\sqrt{\frac{\alpha}{\nu}}\right)\operatorname{erfc}\left(\zeta - \sqrt{\alpha\, t}\right)\Bigg], \end{aligned}\right\} \tag{3.19}$$

which is, in the limit $t \to \infty$,

$$u(z, t) \sim U\, e^{\alpha t} \exp\left(-z\sqrt{\frac{\alpha}{\nu}}\right). \tag{3.20}$$

Each of the solutions (3.17)–(3.19) is believed to be new and describes the general features of the unsteady boundary layer flows.

4. *Discussion and conclusions*

The above analysis reveals that the exact general solution for configuration (i) includes several interesting special cases. The most important physical quantity related to the problems is the skin friction, defined by

$$\tau_0 = \mu\left(\frac{\partial u}{\partial z}\right)_{z=0}, \tag{4.1}$$

where μ is the dynamic viscosity. This can readily be calculated from the known velocity field $u(z, t)$. Some properties of the skin friction can easily be discussed.

It may be of interest to point out the general nature of the solution for configuration (ii), which consists of steady and transient components. In the limit $t \to \infty$, the latter decay exponentially and the ultimate steady state is established. The former indicates the formation of boundary layers adjacent to both the disks. It follows from (3.18) that the thickness of these layers is of the order $\sqrt{\nu/\alpha}$ on $z=0$ and $\sqrt{\nu/\beta}$ on $z=D$.

As in the case of configuration (i), the skin friction on both disks can be determined from the results

$$\tau_0 = \mu\left(\frac{\partial u}{\partial z}\right)_{z=0} \quad \text{and} \quad \tau_D = \mu\left(\frac{\partial u}{\partial z}\right)_{z=D}. \tag{4.2a, b}$$

The implication of these results can then be described without difficulty.

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REFERENCES

[1] G. K. BATCHELOR, *An Introduction to Fluid Dynamics* (Cambridge University Press, 1967).
[2] C. J. TRANTER, *Integral Transforms in Mathematical Physics* (Methuen's Monographs, 1966).
[3] G. K. CAMPBELL and R. M. FOSTER, *Fourier Integrals for Practical Applications* (D. van Nostrand, 1948).
[4] LORD RAYLEIGH, *On the motion of solid bodies through viscous fluid*, Phil. Mag. *21* (1911), 697–711.
[5] H. BLASIUS, *Grenzschichten in Flüssigkeiten mit kleiner Reibung*, Z. Math. Phys. *56* (1908), 1–37. English transl. in NACA TM 1256.
[6] E. J. WATSON, *Boundary layer growth*. Proc. Roy. Soc. Lond. *A231* (1955), 104–114.

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