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A general nonlocal beam theory: Its application to nanobeam bending, buckling and vibration

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ABSTRACT

In the present study, a generalized nonlocal beam theory is proposed to study bending, buckling and free vibration of nanobeams. Nonlocal constitutive equations of Eringen are used in the formulations. After deriving governing equations, different beam theories including those of Euler–Bernoulli, Timoshenko, Reddy, Levinson and Aydogdu [Compos. Struct., 89 (2009) 94] are used as a special case in the present compact formulation without repeating derivation of governing equations each time. Effect of nonlocality and length of beams are investigated in detail for each considered problem. Present solutions can be used for the static and dynamic analyses of single-walled carbon nanotubes.

1. Introduction

Due to small length scale in micro- and nano-applications of beam, plate and shell-type structures nonlocal elasticity has been used in recent years. Basic difference between classical elasticity and nonlocal elasticity is definition of stress: stress at a point is function of strain at that point in local elasticity, whereas in local elasticity stress at a point is function of strains at all points in the continuum. In nonlocal elasticity, forces between atoms and internal length scale are considered in construction of constitutive equations [1–4].

Nonlocal elasticity has been used to study wave propagation in composites, elastic waves, dislocation mechanics, dynamic and static analysis of carbon nanotubes and nanorods [1–9]. Recently, molecular dynamic simulations and nonlocal continuum models are compared for wave propagation in single- and double-walled carbon nanotubes [10] and elastic buckling of single-layered graphene sheet [11]. Good agreement is observed between molecular dynamic simulations and nonlocal continuum modeling.

Recently, Reddy [12] used different beam theories including those of Euler–Bernoulli, Timoshenko, Levinson [13] and Reddy [14] to analyze bending, buckling and vibration of nonlocal beams. In his study, different displacement functions are chosen in the first step and then all steps are repeated when deriving beam equilibrium and equations of motion. Also in his study length-scale effect cannot be observed due to constant beam length L, and free vibration frequencies are compared only for the fundamental frequency.

In the present study, a compact generalized beam theory [15–19] is used to analyze bending, buckling and vibration of nanoscale beams using local and nonlocal elasticity. In the formulation of beam theories transverse shear deformation is modeled by help of a general function. After general formulation, each beam theory is found as a special case. Also a new shear deformation theory [20] proposed by Aydogdu in a previous study is used in the formulations. Free vibration results are given for the first three modes to see the effect of nonlocality in the higher modes. Also L is changed in the analyses to see the length-scale effect in the investigated problems. Present formulation also can be used for nonlocal analyses of nanocomposite beams.

2. Beam theories

In this section, derivation of the governing equations for the nanobeams is explained. Governing equations are derived for nanocomposite beams and numerical results are given for an

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isotropic nanobeam. Consider a straight uniform composite beam having length L with thickness h. The beam is assumed to be constructed of arbitrary number, N, of linearly elastic transversely isotropic layers. Therefore, the state of stress in each layer is given by [21]

$$\sigma_x^{(k)} = Q_{11}^{(k)}\varepsilon_x, \quad \tau_{xz}^{(k)} = Q_{55}^{(k)}\gamma_{xz}, \tag{1}$$

where ${Q_{ij}}^{(k)}$ are well-known reduced stiffnesses [21] and k is the number of layers. Assuming that the deformations of the beam take place in the x–z plane and upon denoting the displacement components along the x-, y- and z-directions by U, V and W, respectively, the following displacement field can be written

$$\begin{aligned} &U(x,z;t) = u(x;t) - zw_{,x} + f(z)u_1(x;t),\\ &V(x,z;t) = 0,\\ &W(x,z;t) = w(x;t). \end{aligned} \tag{2}$$

The displacement model (2) yields the following kinematic relations:

$$\begin{aligned} &\varepsilon_x = u_{,x} - zw_{,xx} + f(z)u_{1,x},\\ &\gamma_{xz} = f' u_1, \end{aligned} \tag{3}$$

where a prime denotes the derivative with respect to z and "$_{,x}$" represents partial derivative with respect to x.

Although different shape functions are applicable, only the ones which convert the present theory to the corresponding Euler–Bernoulli beam theory (EBT), parabolic shear deformation beam theory (RBT) of Reddy [14], first order shear deformation beam theory (TBT) of Timoshenko [12] and general exponential shear deformation beam theory (ABT) of Aydogdu [20] are employed in the present study. This is achieved by choosing the shape functions as follows:

$$\begin{aligned} EBT &: f(z) = 0,\\ TBT &: f(z) = z,\\ RBT &: f(z) = z(1 - 4z^2/3h^2),\\ ABT &: f(z) = (z)(3)^{-2(z/h)^2/\ln 3}. \end{aligned} \tag{4}$$

The principle of virtual displacement

$$\begin{aligned} 0 = \int\int\Bigg[&\rho_0\left(\frac{\partial u}{\partial t}\frac{\partial \delta u}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial \delta w}{\partial t}\right) + \rho_{01}\left(\frac{\partial u}{\partial t}\frac{\partial \delta u_1}{\partial t} + \frac{\partial u_1}{\partial t}\frac{\partial \delta u}{\partial t}\right)\\ &+ \rho_2\left(\frac{\partial^3 w}{\partial x\partial t^2} + \frac{\partial^3 \delta w}{\partial x\partial t^2}\right) - \rho_{11}\left(\frac{\partial u_1}{\partial t}\frac{\partial^2 \delta w}{\partial x\partial t} + \frac{\partial \delta u_1}{\partial t}\frac{\partial^2 w}{\partial x\partial t}\right)\\ &+ \rho_{02}\frac{\partial u_1}{\partial t}\frac{\partial \delta u_1}{\partial t} - N_x\delta\varepsilon_x^0 - M_x^c\delta k_x^c - M_x^a\delta k_x^a\\ &- Q_x\delta u_1 - f\delta u + q\delta w + N_x^e\frac{\partial w}{\partial x}\frac{\partial \delta}{\partial x}\Bigg]dt\,dx, \end{aligned} \tag{5}$$

where the force and the moment resultants are defined in the following form:

$$\begin{aligned} &(N_x^c) = \int_{-h/2}^{h/2}(\sigma_x)dz, \quad (M_x^c) = \int_{-h/2}^{h/2}\sigma_x z\,dz,\\ &(M_x^{sd}) = \int_{-h/2}^{h/2}(\sigma_x)f(z)dz, \quad (Q_x^{sd}) = \int_{-h/2}^{h/2}(\tau_{xz})f'(z)dz. \end{aligned} \tag{6}$$

$$\begin{aligned} &\rho_i = \textstyle\int_{-h/2}^{h/2}\rho z^i dz, \qquad (i = 0,1,2),\\ &\rho_{jm} = \textstyle\int_{-h/2}^{h/2}\rho z^j f_j^m dz, \quad (j = 0,1;\ \ m = 1,2) \end{aligned}$$

The resultants denoted with a superscript 'c' are the conventional ones of the classical beam theories, whereas the remaining ones with superscript 'sd' are additional quantities incorporating the shear deformation effects. By substituting the stress–strain relations into the definitions of the force and the moment resultants of the present theory, the following constitutive equations are obtained [15–20]:

$$\begin{bmatrix} N_x^c\\ M_x^c\\ M_x^{sd} \end{bmatrix} = \begin{bmatrix} A_{11} & B_{11} & E_{11}\\ & D_{11} & F_{11}\\ Sim & & H_{11} \end{bmatrix}\begin{bmatrix} u_{,x}\\ -w_{,xx}\\ u_{1,x} \end{bmatrix}\left[Q_x^{sd}\right] = [A_{55}][u_1]. \tag{7}$$

The extensional, coupling, bending and transverse shear rigidities are defined as follows:

$$\begin{aligned} &A_{i11} = \int_{-h/2}^{h/2} Q_{11}^{(k)}dz, \quad A_{55} = \int_{-h/2}^{h/2} Q_{55}^{(k)}(f')^2 dz, \quad B_{11} = \int_{-h/2}^{h/2} Q_{ij}^{(k)}z\,dz,\\ &E_{11} = \int_{-h/2}^{h/2} Q_{ij}^{(k)}f(z)dz, \quad D_{11} = \int_{-h/2}^{h/2} Q_{ij}^{(k)}z^2 dz, \quad F_{11} = \int_{-h/2}^{h/2} Q_{ij}^{(k)}f(z)z\,dz,\\ &H_{11} = \int_{-h/2}^{h/2} Q_{ij}^{(k)}(f)^2 dz \quad ()' = d()/dz. \end{aligned} \tag{8}$$

Upon employing principle of virtual displacement, the three variationally consistent governing equations of the beam are obtained as

$$\begin{aligned} N_{x,x}^c &= (\rho_0 u + \rho_{01}u_1 - \rho_1 w_{,x})_{,tt},\\ M_{x,xx}^c &= (\rho_1 u_{,x} + \rho_{11}u_{1,x} + \rho_0 w - \rho_2 w_{,xx})_{,tt} + N_x^e w_{,xx} + q,\\ M_{x,x}^{sd} - Q_x^{sd} &= (\rho_{01}u + \rho_{02}u_1 - \rho_{11}w_{,x})_{,tt} \end{aligned} \tag{9}$$

where $_{,tt}$ denotes time derivatives.

Moreover, the following sets of boundary conditions at the edges of the beam are obtained by the application of the virtual displacement principle

$$\begin{aligned} &\text{at } x = 0,\ L\\ &\text{either } u \text{ or } N_x^c \text{ prescribed},\\ &\text{either } w \text{ or } M_{x,x}^c \text{ prescribed},\\ &\text{either } w_x \text{ or } M_x^c \text{ prescribed},\\ &\text{either } u_1 \text{ or } M_x^{sd} \text{ prescribed}. \end{aligned} \tag{10}$$

3. Nonlocal beam theories

Response of materials at the nanoscale is different from those of their bulk counterparts. Nonlocal elasticity is first considered by Eringen [1–4]. He assumed that the stress at a reference point is a functional of the strain field at every point of the continuum. Nonlocal stress tensor t at point x' is defined by

$$\sigma = \int_V K(|x - x'|,\ \tau)S(x)dx, \tag{11}$$

where $S(x')$ is the classical, macroscopic stress tensor at point x', $K(|x–x'|, \tau)$ is the kernel function and τ is a material constant that depends on internal and external characteristic length (such as the lattice spacing and wave length).

Nonlocal constitutive relations for present nanobeams can be written as

$$\begin{aligned} &\sigma_{xx} - \mu\frac{\partial^2\sigma_{xx}}{\partial x^2} = Q_{11}\varepsilon_{xx},\\ &\sigma_{xz} - \mu\frac{\partial^2\sigma_{xz}}{\partial x^2} = Q_{55}\varepsilon_{xz}, \end{aligned} \tag{12}$$

where $\mu = (e_0 a)^2$ is nonlocal parameter, a an internal characteristic length and e_0 a constant. Choice of $e_0 a$ (in dimension of length) is crucial to ensure the validity of nonlocal models. This parameter was determined by matching the dispersion curves based on the atomic models [1–4]. For a specific material, the corresponding nonlocal parameter can be estimated by fitting the results of atomic lattice dynamic and experiment. A conservative estimate of the scale coefficient $e_0 a < 2.0$ nm for a single-walled carbon nanotube is proposed [22]. In this study, $0 \le \mu \le 4$ is chosen to investigate nonlocality effects. Using Eqs. (7) and (8) following force-strain and moment strain relations are obtained:

$$N_x^c - \mu \frac{\partial^2 N_x^c}{\partial x^2} = A_{11}\varepsilon_{xx} + B_{11}k_x^c + E_{11}k_x^{sd},$$
$$M_x^c - \mu \frac{\partial^2 M_x^c}{\partial x^2} = B_{11}\varepsilon_{xx} + D_{11}k_x^c + F_{11}k_x^{sd},$$
$$M_x^{sd} - \mu \frac{\partial^2 M_x^{sd}}{\partial x^2} = E_{11}\varepsilon_{xx} + D_{11}k_x^c + H_{11}k_x^{sd}. \quad (13)$$

Using Eqs. (7), (9), (12) and (13) following governing equations can be found in terms of displacements:

$$A_{11}u_{,xx} - B_{11}w_{,xxx} + E_{11}u_{1,xx} = \left(1 - \mu\frac{d^2}{dx^2}\right)(\rho_0 u + \rho_{01}u_1 - \rho_1 w_{,x})_{,tt},$$
$$B_{11}u_{,xxx} - D_{11}w_{,xxxx} + F_{11}u_{1,xxx} = \left(1 - \mu\frac{d^2}{dx^2}\right)[(\rho_1 u_{,x} + \rho_{11}u_{1,x} + \rho_0 w - \rho_2 w_{,xx})_{,tt} + N_x^e w_{,xx} + q],$$
$$E_{11}u_{,xx} - F_{11}w_{,xxx} + H_{11}u_{1,xx} - A_{55}u_1 = \left(1 - \mu\frac{d^2}{dx^2}\right)(\rho_{01}u + \rho_{02}u_1 - \rho_{11}w_{,x})_{,tt} \quad (14)$$

4. Bending, buckling and vibration of simply supported nanobeams

In this study, analytical solutions are given for simply supported isotropic nanobeams for bending, buckling and free vibration.

The boundary conditions of simply supported beams are

$$w = M_x^c = M_x^{sd} = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad L. \quad (15)$$

The following displacement field satisfies boundary conditions and governing equations.

$$w(x,t) = \sum_{m=1}^{\infty} W_m \sin\frac{m\pi x}{L}\sin\omega t,$$
$$Lu_1(x,t) = \sum_{m=1}^{\infty} U_{1m}\cos\frac{m\pi x}{L}\sin\omega t. \quad (16)$$

For bending problem $N_x{}^e$ and all time derivatives set to zero the transverse load acting on nanobeam can be written as

$$q(x) = \sum_{m=1}^{\infty} Q_m \sin\frac{m\pi x}{L}, \quad Q_m = \frac{2}{L}\int_0^L q(x)\sin\frac{m\pi x}{L}\,dx. \quad (17)$$

The Fourier coefficients Q_m associated with uniform and point loads are given

$$q(x) = q_0, \quad Q_m = \frac{4q_0}{m\pi}, \quad n = 1,3,5\ldots \quad \text{for uniform load,} \quad (18)$$

$$q(x) = Q_0\delta(x - x_0), \quad Q_m = \frac{2Q_0}{L}\sin\frac{\pi x_0}{L}, \quad n = 1,2,3\ldots \quad \text{for point load,} \quad (19)$$

where q_0 is the density of uniform load, δ Dirac Delta function, Q_0 magnitude of point load and x_0 $(0 \le x_0 \le L)$ application position of point load.

For buckling, we set q and all time derivatives to zero, and for the free vibration we set $N_x{}^e$ and q to zero.

4.1. Bending

For bending, we set $N_x{}^e$ and all time derivatives to zero. Inserting Eqs. (16)–(19) in to Eq. (14) following static deflections can be obtained.

Uniform load:

$$w = \sum_{m=1}^{\infty} \frac{4[1 + (\mu[(2m-1)\pi]^2/L^2)]q_0L^4}{d_{11}[(2m-1)\pi]^5 EI} \quad \text{EBT,} \quad (20)$$

$$w = \sum_{m=1}^{\infty} \frac{4[1 + (\mu[(2m-1)\pi]^2/L^2)]q_0L^4}{[-d_{11}[(2m-1)\pi]^5 + f_{11}^2([(2m-1)\pi]^7/h_{11}[(2m-1)\pi]^2 + \kappa a_{55}L^2)]EI} \quad \text{RBT, ABT and TBT,} \quad (21)$$

where $\kappa = \frac{5}{6}$ for TBT and $\kappa = 1$ for RBT and ABT.

$$w = \sum_{m=1}^{\infty} \frac{4[1 + (\mu[(2m-1)\pi]^2/L^2)][h_{11}[(2m-1)\pi]^2 + \kappa a_{55}L^2]q_0L^4}{[-d_{11}[(2m-1)\pi]^5 a_{55}L^2]EI} \quad \text{LBT.} \quad (22)$$

Point load:

$$w = \sum_{m=1}^{\infty} \frac{2[1 + (\mu[m\pi]^2/L^2)]Q_0L^3}{d_{11}[m\pi]^4 EI} \quad \text{EBT,} \quad (23)$$

$$w = \sum_{m=1}^{\infty} \frac{2[1 + (\mu[m\pi]^2/L^2)]Q_0L^3}{[-d_{11}[m\pi]^4 + f_{11}^2([m\pi]^6/h_{11}[m\pi]^2 + \kappa a_{55}L^2)]EI} \quad \text{TBT, RBT and ABT,} \quad (24)$$

$$w = \sum_{m=1}^{\infty} \frac{2[1 + (\mu[m\pi]^2/L^2)][h_{11}[m\pi]^2 + \kappa a_{55}L^2]Q_0L^3}{[-d_{11}[m\pi]^4 a_{55}L^2]EI} \quad \text{LBT,} \quad (25)$$

where $(d_{11}, f_{11}, h_{11}, a_{55}) = ((D_{11}, F_{11}, H_{11}, A_{55})/Do)$, $Do = D_{11}$.

4.2. Buckling and vibration

For buckling, q and all time derivatives are set to zero, and for vibration we set $N_x{}^e$ and q to zero in Eq. (14). Substituting Eq. (16) in Eq. (14) gives following eigen-value problem:

$$\{[K] - \lambda^2[M]\}\{\Delta\} = 0, \quad (26)$$

where K is the stiffness, M the inertia and load matrices for vibration and buckling, respectively, and Δ the column vector of unknown coefficients of Eq. (16). The eigen values (λ) for which the determinant of coefficient matrix of Eq. (26) is zero, leads to the free vibration frequencies and critical buckling load parameter in the case of buckling problem (it should be noted that in the case of buckling problem λ^2 is critical buckling load parameter).

5. Numerical results

In this section, numerical results are given for analytical solutions given in Section 2. Due to nondimensionalization, only the following material and geometrical properties are required in computations.

$\nu = 0.3$ and $h = 1$ nm.

Nondimensional terms are chosen in the following form:

$\bar{w} = 100w\frac{EI}{Q_0L^3}$ for point load,

$\bar{w} = 100w\frac{EI}{q_0L^4}$ for uniform load,

$\bar{\omega} = \omega L^2\sqrt{\frac{\rho}{EI}}$ frequency parameter,

$\bar{N} = N_x^e\frac{L^2}{EI}$ critical buckling load parameter.

The numerical results for bending under point load at the center and uniform load are given in Tables 1 and 2, respectively. Results

Table 1
Comparison of dimensionless maximum center deflection under uniform load for simply supported beams.

L/h	μ	EBT	TBT	RBT	LBT	ABT
10	0	1.3130	1.3483	1.3483	1.3487	1.3480
	1	1.4487	1.4949	1.4927	1.4954	1.4921
	2	1.5844	1.6414	1.6371	1.6421	1.6362
	3	1.7201	1.7880	1.7815	1.7888	1.7802
	4	1.8558	1.9345	1.9260	1.9355	1.9243
20	0	1.3130	1.3218	1.3218	1.3219	1.3218
	1	1.3469	1.3564	1.3566	1.3566	1.3563
	2	1.3808	1.3910	1.3909	1.3912	1.3908
	3	1.4148	1.4256	1.4254	1.4258	1.4253
	4	1.4487	1.4602	1.4599	1.4604	1.4598
50	0	1.3130	1.3144	1.3144	1.3144	1.3144
	1	1.3184	1.3199	1.3199	1.3199	1.3199
	2	1.3239	1.3253	1.3253	1.3253	1.3253
	3	1.3313	1.3307	1.3307	1.3308	1.3308
	4	1.3347	1.3362	1.3362	1.3362	1.3362

Table 2
Comparison of dimensionless maximum center deflection under point load at the center for simply supported beams.

L/h	μ	EBT	TBT	RBT	LBT	ABT
10	0	2.2222	2.3084	2.3074	2.3094	2.3066
	1	2.5535	3.1597	2.9495	3.1669	2.9099
	2	2.8848	4.0110	3.5915	4.0244	3.5133
	3	3.2162	4.8623	4.2336	4.8819	4.1167
	4	3.5475	5.7136	4.8757	5.7394	4.7200
20	0	2.2222	2.2438	2.2437	2.2440	2.2435
	1	2.3050	2.3591	2.3531	2.3597	2.3510
	2	2.3879	2.4744	2.4626	2.4754	2.4585
	3	2.4707	2.5897	2.5721	2.5912	2.5660
	4	2.5535	2.7051	2.6815	2.7069	2.6735
50	0	2.2222	2.2257	2.2257	2.2257	2.2256
	1	2.2355	2.2398	2.2397	2.2398	2.2396
	2	2.2487	2.2538	2.2538	2.2539	2.2537
	3	2.2620	2.2680	2.2678	2.2680	2.2677
	4	2.2752	2.2820	2.2819	2.2820	2.2818

are obtained using 100 terms in the series Eqs. (20)–(25). According to these tables, length scale is more obvious for lower L (see results for $L = 10$) and it decreases with increasing L. This result can be seen from Eqs. (20)–(25). Increasing nonlocal parameter μ decreases stiffness of the beams and bending results increase. Nonlocality effects are more pronounced for point load. (see Table 2, $L = 10$). EBT predicts lower results for lower L and L/h ratios. With increasing L results are converging to a certain value. Small differences are observed between different theories. Present results are compared with that of Reddy [12]. Good agreement is observed for uniform load but there are some deviations for point loads. This may be due to misprint in the Reddy's study. It should be noted that Levinson beam theory is used only for bending problem in the present study.

The nondimensional critical buckling loads are presented in Table 3. According to this table buckling loads decrease with increasing nonlocal parameter μ. Critical buckling load parameters are insensitive to used theory. Similar to bending results, for lower L, nonlocality is important and this effect is lost for higher L. Present results are compared with results of Reddy [12] and good agreement is observed.

Table 3
Dimensionless critical buckling load parameter for simply supported nanobeams.

L/h	μ	EBT	TBT	RBT	ABT
10	0	9.8696	9.6227	9.6228	9.6242
	1	8.9830	8.7583	8.7583	8.7597
	2	8.2426	8.0364	8.0364	8.0377
	3	7.6149	7.4244	7.4245	7.4256
	4	7.0761	6.8990	6.8991	6.9001
20	0	9.8696	9.8067	9.8067	9.8070
	1	9.6319	9.5705	9.5706	9.5709
	2	9.4055	9.3455	9.3455	9.3459
	3	9.1894	9.1308	9.1308	9.1312
	4	8.9830	8.9258	8.9258	8.9261
50	0	9.8696	9.8595	9.8595	9.8595
	1	9.8308	9.8207	9.8207	9.8208
	2	9.7923	9.7822	9.7822	9.7823
	3	9.7541	9.7440	9.7441	9.7441
	4	9.7161	9.7062	9.7062	9.7063

Table 4
Dimensionless frequency parameters for simply supported nanobeams (fundamental frequency).

L/h	μ	EBT	TBT		RBT		ABT	
		11	11	12	11	12	11	12
10	0	9.8696	9.7443	62.8169	9.7425	50.2695	9.7426	47.1004
	1	9.4124	9.2931	62.5620	9.2916	50.0646	9.2917	46.9082
	2	9.0133	8.8994	62.3105	8.8980	49.8626	8.8982	46.7187
	3	8.6611	8.5517	62.0621	8.5505	49.6633	8.5507	46.5318
	4	8.3472	8.2419	61.8169	8.2408	49.4665	8.2410	46.3472
20	0	9.8696	9.8381	175.9736	9.8380	140.7847	9.8382	131.9011
	1	9.7498	9.7187	175.8843	9.7186	140.7133	9.7188	131.8342
	2	9.6343	9.6036	175.7952	9.6035	140.6420	9.6036	131.7674
	3	9.5228	9.4924	175.7063	9.4923	140.5708	9.4925	131.7007
	4	9.4150	9.3850	175.6175	9.3849	140.4997	9.3851	131.6341
50	0	9.8696	9.8645	693.7310	9.8645	554.9881	9.8645	519.9673
	1	9.8501	9.8451	693.7085	9.8451	554.9701	9.8451	519.9504
	2	9.8308	9.8258	693.6859	9.8258	554.9521	9.8258	519.9335
	3	9.8117	9.8066	693.6635	9.8066	554.9341	9.8066	519.9166
	4	9.7925	9.7875	693.6409	9.7875	554.9161	9.7875	519.8998

Table 5
Dimensionless frequency parameters for simply supported nanobeams (second frequency).

L/h	μ	EBT	TBT		RBT		ABT	
		11	11	12	11	12	11	12
10	0	39.4784	36.8406	66.4670	36.0049	54.4382	35.6000	51.5980
	1	33.2731	31.2366	62.1584	30.7677	50.5122	30.5544	47.6689
	2	29.2992	27.5870	58.6955	27.2651	47.5364	27.1251	44.7794
	3	26.4764	24.9727	55.7890	24.7271	45.0984	24.6238	42.4419
	4	24.3384	22.9826	53.2909	22.7831	43.0285	22.7013	40.4701
20	0	39.4784	38.9645	177.7280	38.9495	142.2578	38.9467	133.3007
	1	37.6496	37.1614	176.2940	37.1483	141.1053	37.1461	132.2190
	2	36.0535	35.5875	174.8954	35.5759	139.9820	35.5741	131.1651
	3	34.6445	34.1979	173.5304	34.1876	138.8863	34.1862	130.1373
	4	33.3888	32.9594	172.1977	32.9501	137.8170	32.9489	129.1343
50	0	39.4784	39.3976	694.7988	39.3974	555.8544	39.3978	520.7795
	1	39.1699	39.0897	694.4381	39.0896	555.5658	39.0900	520.5091
	2	38.8686	38.7890	694.0779	38.7888	555.2776	38.7893	520.2391
	3	38.5741	38.4951	693.7183	38.4949	554.9899	38.4954	519.9695
	4	38.2862	38.2078	693.3592	38.2076	554.7026	38.2081	519.7003

Table 6
Dimensionless frequency parameters for simply supported nanobeams (third frequency).

L/h	μ	EBT	TBT		RBT		ABT	
		11	11	12	11	12	11	12
10	0	88.8264	57.4499	95.9180	47.6662	92.6011	44.9838	91.9910
	1	63.5445	43.6798	70.1020	36.6399	66.9404	34.6447	66.3713
	2	52.0886	36.5965	58.0353	30.8545	55.1370	29.2008	54.6185
	3	45.1995	32.1140	50.6460	27.1541	47.9770	25.7122	47.5010
	4	40.4759	28.9541	45.5199	24.5282	43.0402	23.2337	42.5985
20	0	88.8264	85.7483	181.7154	85.1605	146.4239	84.8989	137.6299
	1	80.2169	77.5291	174.4723	77.0985	140.4033	76.9158	131.8781
	2	73.7066	71.2922	168.0723	70.9527	135.1452	70.8135	126.8872
	3	68.5611	66.3515	162.3490	66.0707	130.4735	65.9587	122.4682
	4	64.3618	62.3129	157.1847	62.0728	126.2749	61.9791	118.5050
	5	60.8503	58.9316	152.4906	58.7212	122.4689	58.6407	114.9173
50	0	88.8264	88.4147	696.6058	88.4109	557.3386	88.4119	522.1777
	1	87.2848	86.8804	694.7794	86.8768	555.8766	86.8778	520.8076
	2	85.8207	85.4233	692.9673	85.4199	554.4262	85.4209	519.4484
	3	84.4280	84.0372	691.1695	84.0339	552.9871	84.0349	518.0999
	4	83.1009	82.7164	689.3856	82.7132	551.5593	82.7143	516.7620
	5	81.8345	81.4560	687.6156	81.4530	550.1425	81.4540	515.4344

First three flexural nondimensional frequencies are presented in Tables 4–6 for different nonlocal parameter μ, L, L/h ratios and for different theories. In Table 4, results are given for different theories. For given m, two frequencies sets are obtained for shear deformation theories, whereas only one set is found for E–B theory. According to these tables, fundamental frequency is insensitive to used theory for $L = 10$ and results are approaching for increasing L. Difference between theories increases with increasing mode number (Tables 5 and 6). This is due to small wavelength effect for higher modes.

For second set important differences are observed between TBT and RBT, ABT. This may be due to second spectrum.

6. Conclusion

A generalized nonlocal beam theory is used to study bending, buckling and free vibration of nanobeams. Nonlocal constitutive equations of Eringen are used in the formulations. Different beam theories including those of Euler–Bernoulli, Timoshenko, Reddy, Levinson and Aydogdu are used as a special case in the present formulation. Effect of nonlocality and length are investigated in detail for each considered problem. Present formulation can be extended to other classical boundary conditions.

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