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Remark on linear forms

By

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Introduction. Let $A = \{a_1, \ldots, a_k\} \subset \mathbb{N}$, $m \in \mathbb{N}$ such that

$$\gcd(m a_1, a_2 - a_1, a_3 - a_2, \ldots, a_k - a_{k-1}) = 1.$$

For $j \in \{1, \ldots, m\}$ a given number $n \in \mathbb{N}_0$ is called *j-omitted* if it has *no* representation

$$(1) \qquad n = \sum_{i=1}^{k} x_i a_i, \quad x_i \in \mathbb{N}_0$$

such that

$$(2) \qquad \sum_{i=1}^{k} x_i \equiv j \pmod{m}.$$

A number $n \in \mathbb{N}_0$ is called *omitted* if it is j-omitted for some $j \in \{1, \ldots, m\}$. By $\omega(m; a_1, \ldots, a_k) = \omega(m, A) = \omega$ we denote the number of omitted numbers $n \in \mathbb{N}_0$. If there is an omitted number, then we have a *greatest* omitted number which is denoted by $g(m; a_1, \ldots, a_k) = g(m, A) = g$. Otherwise we write $g(m, A) = -1$; in this case we must have $1 \in A$ and $m = 1$. The problem of determining $\omega(m, A)$ and $g(m, A)$ was introduced by Skupień [6]. For $m = 1$, it is well-known that the Frobenius number $g(1, A) = g(A)$ really exists. The same is proved by Skupień for arbitrary $m \in \mathbb{N}$.

In this note we give formulas for $\omega(m, A)$ and $g(m, A)$ for arithmetic progressions $A: a, a + d, \ldots, a + \kappa d$, where $\gcd(am, d) = 1$. Furthermore we show that in this case there is just one $j (1 \leqq j \leqq m)$ such that there is no representation (1) with (2) for $n = g(m, A)$. This j is also given explicitly.

Theorem 1. *Let* $a, d, m \in \mathbb{N}$ *with* $\gcd(am, d) = 1$. *Then*

$$g(m; a, a + d, \ldots, a + \kappa d) = \left\lfloor \frac{ma - 2}{\kappa} \right\rfloor a + (ma - 1) d,$$

where $\lfloor x \rfloor$ *denotes the greatest integer* $\leqq x$.

Proof. We set $a_i := a + id$ and $b_i := a_i - a_{i-1}$ for $i = 0, 1, \ldots, \kappa$ $(a_{-1} := 0)$. A representation

$$n = \sum_{i=0}^{\kappa} x_i a_i, \quad x_i \in \mathbb{N}_0$$

such that

$$\sum_{i=0}^{\kappa} x_i = j \quad \text{for some } j \in \mathbb{N}_0$$

is equivalent to the representation

$$n = \sum_{i=0}^{\kappa} y_i b_i, \quad y_i \in \mathbb{N}_0$$

with

$$j = y_0 \geqq y_1 \geqq \dots \geqq y_\kappa \geqq 0$$

simply by the relation

$$y_i = \sum_{j=i}^{\kappa} x_j \quad \text{for } i = 0, 1, \dots, \kappa.$$

In our case the sequence $(b_0, b_1, \dots, b_\kappa)$ has the form $(a, \underbrace{d, \dots, d}_{\kappa})$.

Now, each $n \in \mathbb{Z}$ has a unique representation

$$n = \alpha a + \delta d, \quad 0 \leqq \delta \leqq a - 1 \tag{3}$$

and therefore

$$n = (\alpha - id) a + (\delta + ia) d, \quad i = 0, 1, \dots.$$

This can be rewritten as

$$n = y_0^{(i)} a + y_1^{(i)} d + \dots + y_\kappa^{(i)} d \tag{4}$$

with $\alpha - id = y_0^{(i)} \geqq y_1^{(i)} \geqq \dots \geqq y_\kappa^{(i)} \geqq 0$ as long as

$$\alpha - id \geqq \left\lceil \frac{\delta + ia}{\kappa} \right\rceil. \tag{5}$$

To see this we write for each i:

$$\delta + ia = q\kappa + \kappa_1, \quad 0 \leqq \kappa_1 < \kappa$$
$$\kappa_2 := \kappa - \kappa_1.$$

Then

$$\begin{aligned} \delta + ia &= \left\lceil \frac{\delta + ia}{\kappa} \right\rceil \kappa_1 + \left\lfloor \frac{\delta + ia}{\kappa} \right\rfloor \kappa_2 \\ &= \underbrace{\left\lceil \frac{\delta + ia}{\kappa} \right\rceil 1 + \dots + \left\lceil \frac{\delta + ia}{\kappa} \right\rceil 1}_{\kappa_1} + \underbrace{\left\lfloor \frac{\delta + ia}{\kappa} \right\rfloor 1 + \dots + \left\lfloor \frac{\delta + ia}{\kappa} \right\rfloor 1}_{\kappa_2}. \end{aligned}$$

This is a special representation of the type

$$\delta + ia = y_1 1 + \dots + y_\kappa 1, \quad y_1 \geqq \dots \geqq y_\kappa \geqq 0$$

with y_1 being minimal among all representations of $\delta + ia$ by $(1, \dots, 1)$. The integers α, $\alpha - d, \dots, \alpha - (m-1)d$ constitute a complete residue system mod m. We ask for the greatest α, such that a $\delta\,(0 \leqq \delta \leqq a-1)$ exists with

$$\alpha - id \geqq \left\lceil \frac{\delta + ia}{\kappa} \right\rceil \quad \text{for all } i = 0, 1, \dots, m-2$$

and

$$\alpha - (m-1)d < \left\lceil \frac{\delta + (m-1)a}{\kappa} \right\rceil. \tag{6}$$

In order to maximize α in (6) we have to maximize δ giving

$$\delta = a - 1.$$

This gives for the maximal α in (6)

$$\alpha = \left\lceil \frac{a - 1 + (m-1)a}{\kappa} \right\rceil + (m-1)d - 1. \tag{7}$$

Thus, the greatest n such that $y_0^{(i)}$ in (4) does not run through a complete residue system mod m under the condition (5) is given by

$$\begin{aligned} n &= \left(\left\lceil \frac{a - 1 + (m-1)a}{\kappa} \right\rceil + (m-1)d - 1 \right) a + (a-1)d \\ &= \left(\left\lceil \frac{ma - 1}{\kappa} \right\rceil - 1 \right) a + (ma - 1)d \\ &= \left\lfloor \frac{ma - 2}{\kappa} \right\rfloor a + (ma - 1)d. \quad \square \end{aligned}$$

Corollary 1. *The only j for which there is no representation*

$$g(m; a, a+d, \dots, a+\kappa d) = \sum_{i=0}^{\kappa} x_i^{(j)}(a + id)$$

with $x_i^{(j)} \in \mathbb{N}_0$ and $\sum_{i=0}^{\kappa} x_i^{(j)} \equiv j \pmod{m}$ is given by

$$j = \left\lfloor \frac{ma - 2}{\kappa} \right\rfloor.$$

P r o o f. We simply have to show that (5) holds for $i = 0, 1, \dots, m-2$, when α is given by (7). Obviously we have

$$\left\lceil \frac{ma - 1}{\kappa} \right\rceil + d - 1 \geqq \left\lceil \frac{ma - a - 1}{\kappa} \right\rceil$$

or

$$\alpha - (m-2)d \geqq \left\lceil \frac{a - 1 + (m-2)a}{\kappa} \right\rceil. \quad \square$$

Corollary 2. *We have*

$$\omega(m; a, a+d, \dots, a+\kappa d) = \sum_{\delta=0}^{a-1}\left(\left\lceil\frac{\delta+(m-1)a}{\kappa}\right\rceil + \left\lfloor\frac{\delta d}{a}\right\rfloor\right) + (m-1)ad. \tag{8}$$

Proof. We go back to (3) and count all $n \geqq 0$ for which (6) holds. For a given δ $(0 \leqq \delta \leqq a-1)$ we have

$$\omega_\delta := \left\lceil\frac{\delta+(m-1)a}{\kappa}\right\rceil + (m-1)d - 1 + \left\lfloor\frac{\delta d}{a}\right\rfloor + 1$$

omitted numbers $n = \alpha a + \delta d$ $\left(\text{since } n \geqq 0 \text{ is equivalent to } \alpha \geqq -\frac{\delta d}{a} \text{ or } \alpha \geqq -\left\lfloor\frac{\delta d}{a}\right\rfloor\right)$. □

In order to simplify the right hand expression in (8) we need two lemmas. The first one is given by Grant [3], the second one we state without proof.

Lemma 1 (Grant [3]). *If* $a, d \in \mathbb{N}$ *with* $\gcd(a, d) = 1$ *then*

$$\sum_{\delta=0}^{a-1}\left\lfloor\frac{\delta d}{a}\right\rfloor = \frac{1}{2}(a-1)(d-1).$$

Lemma 2. *Let* $a, c \in \mathbb{N}$, $b \in \mathbb{Z}$. *Write*

$$b = q_1 c - r_1, \quad 0 \leqq r_1 < c,$$
$$a - 1 - r_1 = q_2 c + r_2, \quad 0 \leqq r_2 < c,$$

then

$$\sum_{\delta=0}^{a-1}\left\lceil\frac{\delta+b}{c}\right\rceil = q_1 a + \frac{c}{2}q_2(q_2+1) + r_2(q_2+1).$$

Together with Corollary 2 we obtain

Theorem 2. *Let* $a, d, m \in \mathbb{N}$ *with* $\gcd(am, d) = 1$. *Write*

$$(m-1)a = q_1\kappa - r_1, \quad 0 \leqq r_1 < \kappa$$
$$a - 1 - r_1 = q_2\kappa + r_2, \quad 0 \leqq r_2 < \kappa,$$

then

$$\begin{aligned}\omega(m; a, a+d, \dots, a+\kappa d) &= \\ &= \frac{1}{2}(a-1)(d-1) + (m-1)ad + q_1 a + \frac{1}{2}(a-1-r_1+r_2)(q_2+1) \\ &= \frac{1}{2}\Big((a-1)(q_2+d) + (r_2-r_1)(q_2+1)\Big) + \Big((m-1)d + q_1\Big)a.\end{aligned}$$

Remark. With $\kappa = 1$ and $a + d = b$, Theorems 1 and 2 cover the general case of two basis elements:

$$g(m; a, b) = ma - a - b$$
$$\omega(m; a, b) = \frac{1}{2}(a-1)(b-1) + (m-1)ab.$$

These formulas were also proved by Skupień [6] (his Theorem 3.1). Furthermore the first expressions for $\omega(m; a, a+d, \ldots, a+\kappa d)$ in Theorem 2 is for $m = 1$ that of Grant [3], while the second one gives for $m = 1$ that of Selmer [5]. The method used here is essentially the one of Djawadi and Hofmeister [2]. For $m = 1$ Theorem 1 gives the well-known formula of Roberts [4], see also Bateman [1] whose proof is fundamental to our's.

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