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Invariant Imbedding Theory of Neutron Transport: Correlation Functions*

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I. INTRODUCTION

In a series of papers [1-4], invariant imbedding techniques were applied to investigate reflection and transmission phenomena for a variety of physical situations. In [2] the energy-dependent and time-dependent cases of the neutron transport problem were investigated, and the important idea of moment density functions was introduced to calculate the average reflected and transmitted flux. It is the purpose of this paper to introduce the correlation functions in t-space and also in energy space, and to write out functional equations for these functions directly, using the standard arguments of imbedding theory. This extension is important in analyzing the fluctuation problems of the time-dependent or energy-dependent cases and in the applications of this theory.

II. DEFINITIONS OF THE DENSITY FUNCTIONS

We shall use the physical model and nomenclature given in [2], where the function $u_1(x, t)$, which is the average number of reflected neutrons at time t from the rod of length x, was defined. This function is actually the product density of degree one in the infinite t-space [5]. In terms of a single particle, it is the probability that a particle will be reflected in the time interval from t to $t + dt$. Similarly the correlation function $u_2(x, t_1, t_2)\, dt_1\, dt_2$, which is the probability that a particle will be reflected from the rod between times t_1 and $t_1 + dt_1$ and between times t_2 and $t_2 + dt_2$, is the product density of degree two. It has been shown in [5] that

$$\begin{aligned}\mathscr{E}(n) &= \text{average number of particles reflected over all time}\\ &= E(x) = \int_0^\infty u_1(x, t)\, dt, \qquad (2.1)\end{aligned}$$

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and the mean square number of particles reflected over all time is

$$\mathscr{E}(n^2) = \int_0^\infty u_1(x, t)\, dt + \int_0^\infty \int_0^\infty u_2(x, t_1, t_2)\, dt_1, dt_2 . \tag{2.2}$$

III. Functional Equations

We know that $u_1(x, t)$ satisfies the following equation:

$$u_1 \left(x + \Delta, t + \frac{2\Delta}{c}\right)$$

$$= (1 - \lambda^{-1}\Delta) \left[u_1(x, t) + \lambda^{-1}\Delta \int_0^t u_1(x, s)\, u_1(x, t - s)\, ds\right]$$

$$+ \lambda^{-1}\Delta \left[\delta(t) + u_1(x, t) + \lambda^{-t}\Delta \int ...\right] , \tag{3.1}$$

where c is the speed of the particles. In the limit, (3.1) becomes

$$\frac{\partial u_1}{\partial x} + \frac{2}{c}\frac{\partial u_1}{\partial t} = \lambda^{-1} \int_0^t u_1(x, s)\, u_1(x, t - s)\, ds + \lambda^{-1}\delta(t).$$

Defining the Laplace transform of $u_1(x, t)$ as

$$\int_0^\infty u_1(x, t)\, dte^{-pt} = G_1(p, x)$$

and using the initial condition

$$u_1(x, 0) = 0, \tag{3.2}$$

we obtain

$$\frac{\partial G_1}{\partial x} + \frac{2p}{c} G_1(p, x) = \lambda^{-1}\, G_1^2(p, x) + \lambda^{-1}. \tag{3.3}$$

When we put $p = 0$, then $G_1(p, x)\,|_{p=0} = E(x)$ and satisfies the usual Riccati equation

$$\frac{\partial^2 E}{\partial x} = \lambda^{-1}[E^2 + 1]. \tag{3.4}$$

Let us now write out the equation satisfied by the correlation function, correct up to terms involving Δ^2:

$$
\begin{aligned}
u_2\left(x+\Delta; t_1+\frac{2\Delta}{c}, t_2+\frac{2\Delta}{c}\right) = (1-\lambda^{-1}\Delta)\Big[&u_2(x; t_1, t_2) \\
&+ \lambda^{-1}\Delta \int_0^{t_1} u_2(x; s, t_2)\, u_1(x, t_1 - s)\, ds \\
&+ \lambda^{-1}\Delta \int_0^{t_2} u_2(x, t_1, s)\, u_1(x, t_2 - s)\, ds \\
&+ \lambda^{-1}\Delta \int_0^{\min(t_1,t_2)} u_1(x, s)\, u_2(x, t_1 - s, t_2 - s)\, ds \\
&+ u_1(x, t_1)\, \lambda^{-1}\Delta u_1(x, t_2 - t_1) + u_1(x, t_2)\, \lambda^{-1}\Delta u_1(x, t_1 - t_2)\Big] \\
&+ \lambda^{-1}\Delta[\delta(t_1)\, u_1(x, t_2) + \delta(t_2)\, u_1(x, t_1) + o(\Delta)].
\end{aligned} \tag{3.5}
$$

Here we use the fact that $u_1(x, t) = 0$ if $t < 0$, and $u_2(x, t_1, t_2) = 0$ if either t_1 or $t_2 < 0$. In (3.5) the fourth term on the right-hand side has the limit t_1 or t_2 for the integral according as t_1 or t_2 is the smaller of the two. In the fourth and fifth terms, $u_1(x, t_1 - t_2)$ will become zero if $(t_1 - t_2)$ is negative.

Now let us define the double Laplace transform

$$\int_0^\infty \int_0^\infty u_2(x, t_1, t_2)\, e^{-p_1 t_1}\, e^{-p_2 t_2}\, dt_1\, dt_2 = G_2(p_1, p_2, x) \tag{3.6}$$

and rewrite Eq. (3.5) as

$$
\begin{aligned}
\frac{dG_2(p_1, p_2, x)}{dx} + \frac{2p_1 \partial G_2}{c} + \frac{2p_2 \partial G_2}{c} \\
= \lambda^{-1} G_2(p_1, p_2, x)\, G_1(p_1, x) + \lambda^{-1} G_2(p_1, p_2, x)\, G_1(p_2, x) \\
+ \lambda^{-1} G_1(p_1 + p_2, x)\, G_2(x, p_1, p_2) \\
+ \lambda^{-1} G_1(p_1 + p_2, x)\, G_1(p_2, x) \\
+ \lambda^{-1} G_1(p_1 + p_2, x)\, G_1(p_1, x) \\
+ \lambda^{-1} G_1(p_2, x) + \lambda^{-1} G_1(p_1, x).
\end{aligned} \tag{3.7}
$$

The fourth term of (3.5) has the same Laplace transform whether the upper limit of the integral is either t_1 or t_2. Since we are interested in the quantity

$$\int_0^\infty \int_0^\infty u_2(x, t_1, t_2) = S(x)^1 = G(0, 0, x),$$

[1] Referred to in [2].

we put $p_1 = p_2 = 0$ in Eq. (3.7) and obtain

$$\frac{d}{dt} S(x) = \lambda^{-1}\{3S(x) E(x) + 2E^2(x) + 2E(x)\}. \tag{3.8}$$

Notice that this gives a proof of the equation (2.2).

Solving for $S(x)$ will lead to the mean square number of reflected neutrons. Solving for $G_1(p_1, x)$ and $G_2(p_1, p_2, x)$ from (3.2) and (3.7) will lead to the determination of the correlation function $u_2(x, t_1, t_2)$ also.

Similar correlation functions for the transmitted flux $v_2(x, t_1, t_2)$ can also be defined and analogous equations written down.

IV. Energy Dependence

Let us now consider the energy-dependent case, which has been analyzed in [2]. But so as to take into account the correlation functions, we shall enlarge the meaning of some of the functions.

We define the following functions:

$u_1(x; w; r)\, dr$ = expected number of neutrons reflected over all time from the rod of length x, with energies in the ranges from r to $r + dr$, and with w as the energy of the initial impinging neutron;

$Q(E; s, t)\, ds\, dt$ = the probability of energy change from E to between s and $s + ds$ for the particle moving in the same direction as the original particle, and to between t and $t + dt$ for the particle moving in the opposite direction in the rod due to fissioning;

$\Delta/\lambda(E)$ = the probability of fissioning in a segment of length Δ;

$u_2(x; w; r_1; r_2)\, dr_1\, dr_2$ = correlation function for the reflected flux of neutrons emerging with energies in the ranges $(r_1, r_1 + dr_1)$ and $(r_2, r_2 + dr_2)$ due to original neutron of energy w.

Of course u_1 and u_2 are moment density functions [5] in the continuous energy space, and the functional equations for u_1 and u_2 are as follows, using familiar reasoning:

$$\begin{aligned} u_1(x; w; r) &= \left(1 - \frac{\Delta}{\lambda(w)}\right) u_1(x - \Delta, w; r) \left(1 - \frac{\Delta}{\lambda(r)}\right) \\ &+ \left(1 - \frac{\Delta}{\lambda(w)}\right) \Delta \int_0^\infty \int_0^\infty u_1(x - \Delta, w; s) \cdot Q(s; r, t)\, dt \frac{ds}{\lambda(s)} \\ &+ \left(1 - \frac{\Delta}{\lambda(w)}\right) \Delta \iint \int_0^\infty u_1(x - \Delta, w; s) \end{aligned} \tag{4.1}$$

$$\cdot Q(s; l; t)\, u_1(x - \Delta; t, r) \frac{ds}{\lambda(s)}\, dl\, dt$$

$$+ \frac{\Delta}{\lambda(w)} \left[\int_0^\infty Q(w; s; r)\, ds + \iint Q(w; s; z) \cdot u_1(x; z; r)\, ds\, dz \right].$$

Passing to the limit as $\Delta \to 0$, we can obtain the differential equations satisfied by u. If necessary we can even designate

$$\int_0^\infty Q(E, s; t)\, ds = q_1(E; t)$$

and

$$\int_0^\infty Q(E, s; t)\, dt = q_2(E; t).$$

Now we can write the functional equation for $u_2(x; w; r_1, r_2)$ as follows:

$$\begin{aligned}
u_2(x; w; r_1, r_2) &= \left(1 - \frac{\Delta}{\lambda(w)}\right) u_2(x - \Delta, w; r_1, r_2) \left[1 - \frac{\Delta}{\lambda(r_1)} - \frac{\Delta}{\lambda(r_2)}\right] \\
&+ \left(1 - \frac{\Delta}{\lambda(w)}\right) \Delta \left[\int_0^\infty \int_0^\infty u_2(x - \Delta, w; s, r_2)\, Q(s; r_1; l)\, dl \frac{ds}{\lambda(s)} \right. \\
&\qquad \left. + \int_0^\infty \int_0^\infty u_2(x - \Delta; w; r_1, s)\, Q(s; r_2; l)\, dl \frac{ds}{\lambda(s)} \right] \\
&+ \left(1 - \frac{\Delta}{\lambda(w)}\right) \Delta \left[\int_0^\infty \int_0^\infty \int_0^\infty u_2(x - \Delta, w; s, r_2)\, Q(s; l, t) \right. \\
&\qquad \cdot\ u_1(x - \Delta; t; r_1)\, dl\, dt \frac{ds}{\lambda(s)} \qquad (4.2) \\
&\qquad \left. + \int_0^\infty \int_0^\infty \int_0^\infty u_2(x - \Delta, w; r_1, s)\, Q(s; l, t) \cdot u_1(x - \Delta; t, r_2)\, dl\, dt \frac{ds}{\lambda(s)} \right] \\
&+ \left(1 - \frac{\Delta}{\lambda(w)}\right) \Delta \left[\int_0^\infty u_1(x - \Delta; w, r_1)\, Q(r_1; r_1, s)\, u_1(x - \Delta, s; r_2) \frac{ds}{\lambda(r_1)} \right. \\
&\qquad \left. + \int_0^\infty u_1(x - \Delta; w, r_2)\, Q(r_2; r_2, s)\, u_1(x - \Delta, s; r_1) \frac{ds}{\lambda(r_2)} \right] \\
&+ \int u_1(x - \Delta; w, r_1)\, Q(r_1; r_2; \Delta)\, u_1(x - \Delta, s; r_2) \frac{ds}{\lambda(r_1)} \\
&+ \int u_1(x - \Delta, w; r_2)\, Q(r_2; r_1; s)\, u_1(s, x, r_2) \frac{ds}{\lambda(s)}
\end{aligned}$$

$$+ \frac{\Delta}{\lambda(w)} \left[\int_0^\infty Q(w; z, r_1)\, u_1(x; z; r_2)\, dz + \int_0^\infty Q(w; z; r_2)\, u_1(x; z, r_1)\, dz \right.$$
$$\left. + \int_0^\infty \int_0^\infty Q(w, l, z)\, u_2(x; l; r_1, r_2)\, dl\, dz \right].$$

Passing to the limit as $\Delta \to 0$, we can write the differential equation for u_2 as

$$\begin{aligned}
\frac{\partial u_2}{\partial t} = & \left(-\frac{1}{\lambda(w)} - \frac{1}{\lambda(r_1)} - \frac{1}{\lambda(r_2)} \right) u_2 \\
& + \iint u_2(x, w; s, r_2)\, Q(s; r, l) \frac{dl\, ds}{\lambda(s)} \\
& + \iint u_2(x, w; r_1, s)\, Q(s; r_2, l) \frac{dl\, ds}{\lambda(s)} \\
& + \iiint u_2(x; w, s, r_2)\, Q(s; l, t)\, u_1(x, t; r_1) \frac{dl\, dt\, ds}{\lambda(s)} \\
& + \iiint u_2(x; w; r_1, s)\, Q(s; l, t)\, u_1(x, t; r_2) \frac{dl\, dt\, ds}{\lambda(s)} \\
& + \int_0^\infty u_1(x; w; r_1)\, Q(r_1; r_1, s)\, u_1(x, s; r_2) \frac{ds}{\lambda(r_1)} \\
& + \int_0^\infty u_1(x; w; r_2)\, Q(r_2, r_2, s)\, u_1(x, s; r_1) \frac{ds}{\lambda(r_2)} \\
& + \int_0^\infty u_1(x; w; r_1)\, Q(r_1; r_2, s)\, u_1(x, s; r_1) \frac{ds}{\lambda(r_1)} \\
& + \int_0^\infty u_1(x, w, r_2)\, Q(r_2, r_1, s)\, u_1(x, s; r_2) \frac{ds}{\lambda(r_2)} \\
& + \frac{1}{\lambda(w)} \left[\int_0^\infty Q(w; z, r_1)\, u_1(x; z; r_2)\, dz + \int Q(w; z; r_2)\, u_1(x; z; r_1)\, dz \right. \\
& \qquad \left. + \iint Q(w, l, z)\, u_2(x; l; r_1, r_2)\, dl\, dz \right].
\end{aligned} \tag{4.3}$$

Similar equations can be written down for the transmitted fluxes also. Equations (4.3) and (4.1) have to be solved simultaneously for u_1 and u_2. Then, knowing these quantities, we can obtain more interesting information regarding the total amount of energy flux that comes out of the rod over all time by starting the process with an initial neutron of energy w. That is, if we are interested in the *average amount* of total energy of all neutrons reflected from the rod, we calculate

$$\mathscr{E}[(E)] = \int_0^\infty u_1(x; w; r_1)\, r_1 dr_1\,, \tag{4.4}$$

where $\mathscr{E}$ stands for expectation value. Similarly the average fluctuation, or the average mean square, of the total reflected energy of all the neutrons over all time is given by

$$\mathscr{E}[(E^2)] = \int_0^\infty u_1(x; w; r_1)\, r_1^2 dr_1 + \int_0^\infty \int_0^\infty u_2(x; w; r_1, r_2)\, r_1 r_2 dr_1 dr_2\,. \tag{4.5}$$

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