Finite Elements in Analysis and Design 46 (2010) 257–264

Contents lists available at ScienceDirect

Finite Elements in Analysis and Design

journal homepage: www.elsevier.de/locate/finel

Solving inverse couple-stress problems via an element-free Galerkin (EFG) method and Gauss–Newton algorithm

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ARTICLE INFO

Article history:
Received 10 October 2008
Received in revised form
1 July 2009
Accepted 25 September 2009
Available online 18 November 2009

Keywords:
Inverse problems
Couple stress
Element-free Galerkin method
Gauss–Newton algorithm

ABSTRACT

This paper focuses on the identification of constitutive parameters in the couple stress problem. The direct problem is modeled by element-free Galerkin method (EFGM), thus the inconvenience that may be caused by C^1 continuity requirement in the implementation of FEM can be avoided, and the sensitivity analysis that is required for the solution process of the inverse problem can be carried out conveniently. The inverse problem is solved via the Gauss–Newton technique. The proposed method is verified in the cases of slight and strong regional inhomogeneity. The effects of initial guesses, noisy data and location of the measured points on the solutions are investigated, and satisfactory results are achieved.

1. Introduction

The existence of couple stress was originally postulated by Voigt in 1887. In 1909, the brothers E. and F. Cosserat first set up a framework of couple stress theory which has been further developed since then [1–4]. Couple stress theory is an extended continuum theory that includes the effects of a couple per unit area on a material volume [5]. Accordingly, a group of variables including moments, curvatures, and characteristic length are introduced within a continuum framework [3].

One important application of couple stress theory was to describe the materials with microstructures, such as the materials with granular [6], fibrous [7] and lattice structures [8]. In the addition, in the some cases where the size effects have to be taken into account [9], the theory was employed to explain the variation of hardening behavior [10], and local singularities [11].

The study of this paper is motivated by a question that if a continuum couple stress model is adopted, how to determine relevant constitutive parameters, including the so-called characteristic length ℓ?

One of the solutions is to treat this issue as an inverse problem with unknown constitutive parameters. This inverse problem can be investigated under the framework of inverse problems in elasticity for which a comprehensive review was given by Bonnet [12]. If the sufficient 'measurement' message, such as the displacements, strains etc. is provided, all the unknowns are able to be determined analytically or numerically. In comparison with the previous work based on the classical elasticity, the parameters identification of the inverse couple stress problem includes both constitutive parameters appearing in the classical model and those additional items describing the constitutive relationship of couple stress. To the best of the authors' knowledge, it seems there are no reports directly relevant to this matter.

Since the displacement that is usually reliable and is easy to measure [13], it is employed as 'measurement' message in this paper. We propose a numerical model that consists of two parts, one is concerned with the direct problem formulated by element-free Galerkin method [14], and the implementation of sensitivity analysis; the another is for the description of inverses problem that is treated as an optimization problem solved by the Gauss–Newton technique, the major issues concerned in this part include the combined identification, regional inhomogeneity, and computing accuracy with the consideration of noisy 'measurement' message and location of measured points.

2. Governing equations for direct couple stress problems

For plane couple stress problems in the absence of body forces and couples, the equilibrium equations are given by [15]

$$\begin{aligned} &\frac{1}{2}(\sigma_{ij}+\sigma_{ji})_{,j}+\frac{1}{2}(\sigma_{ij}-\sigma_{ji})_{,j}=0 \\ &\mu_{i,i}+\alpha_{ij3}\sigma_{ij}=0 \end{aligned} \qquad \text{in} \quad \Omega \qquad (1)$$

where σ_{ij} stands for the Cauchy component of the stress tensor, μ_i denotes the component of moments, α_{ij3} is the permutation symbol, subscript i and j range form 1 to 2.

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doi:10.1016/j.finel.2009.09.009

The relationship of displacement and strain is described by [15]

$$\{\varepsilon\}=[L]\{u\} \tag{2}$$

where $\{u\}=\{u_1,u_2\}^T$ represents the vector of displacement, $\{\varepsilon\}=\{\varepsilon_{11},\varepsilon_{22},\varepsilon_{12},\kappa_1,\kappa_2\}^T$ represents strain vector, κ_i designates curvature corresponding to μ_i and is specified by

$$\kappa_i=\theta_{,i}, \quad i=1,2. \tag{3}$$

where θ is a microrotation defined by

$$\theta=\frac{1}{2}\alpha_{ij3}u_{j,i} \tag{4}$$

$$[L]^T=\begin{bmatrix} \dfrac{\partial}{\partial x_1} & 0 & \dfrac{\partial}{\partial x_2} & -\dfrac{\partial^2}{2\partial x_1\partial x_2} & -\dfrac{\partial^2}{2\partial x_2^2} \\ 0 & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_1} & \dfrac{\partial^2}{2\partial x_1^2} & \dfrac{\partial^2}{2\partial x_1\partial x_2} \end{bmatrix} \tag{5}$$

The boundary conditions are specified by [15]

$$\begin{matrix} u_i=\tilde{u}_i \\ \theta=\widetilde{\theta} \end{matrix} \quad x\in\Gamma_u \tag{6}$$

$$\begin{cases} \sigma_{ij}n_j=T_i^0 \\ \mu_j n_j=q_i^0 \end{cases} \quad x\in\Gamma_\sigma \tag{7}$$

where $\{\tilde{u}\}$ and $\widetilde{\theta}$ are the prescribed values of $\{u\}$ and θ on Γ_u, T_i^0 and q_i^0 are the prescribed vectors of traction and moment on Γ_σ, n_j denotes the unit outside normal on the boundary, $\Gamma_u+\Gamma_\sigma=\Gamma$ designates the whole boundary of Ω, x represents a vector of coordinates. Subscripts u and σ refer to displacement and stress, respectively.

The constitutive relationship is described by

$$[D]=\begin{bmatrix} \dfrac{E}{1-v^2} & \dfrac{Ev}{1-v^2} & 0 & 0 & 0 \\ \dfrac{Ev}{1-v^2} & \dfrac{E}{1-v^2} & 0 & 0 & 0 \\ 0 & 0 & \dfrac{E}{2(1+v)} & 0 & 0 \\ 0 & 0 & 0 & 4\beta & 0 \\ 0 & 0 & 0 & 0 & 4\beta \end{bmatrix} \quad \text{for the plane stress problem [15]} \tag{8}$$

where $\beta=\ell^2G=\ell^2(E/2(1+v))$ is called the curvature modulus, E, v and ℓ are Young's modulus, Poisson's ratio and character length, respectively.

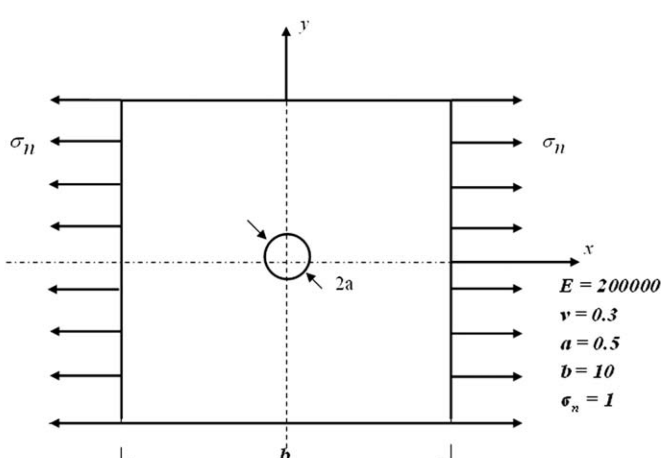

Fig. 1. A circular hole in a uniform tension field.

$[D]$ can be decomposed into

$$[D]=b_1[H_1]+b_1[H_2]+b_3[H_3] \tag{9}$$

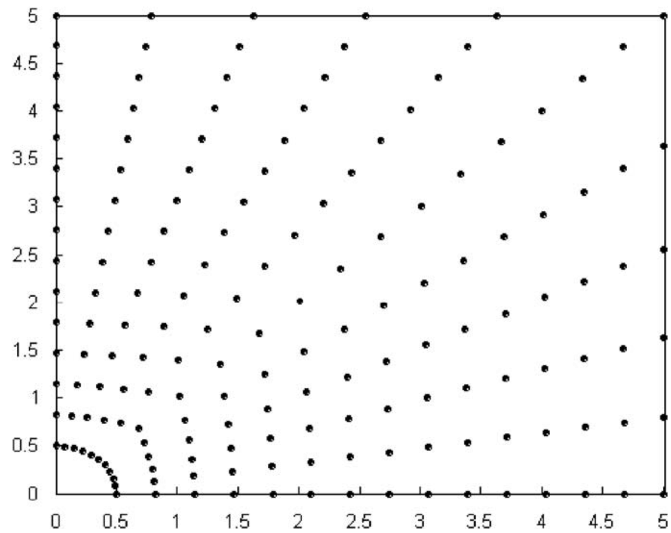

Fig. 2. Nodal arrangement (99 nodes).

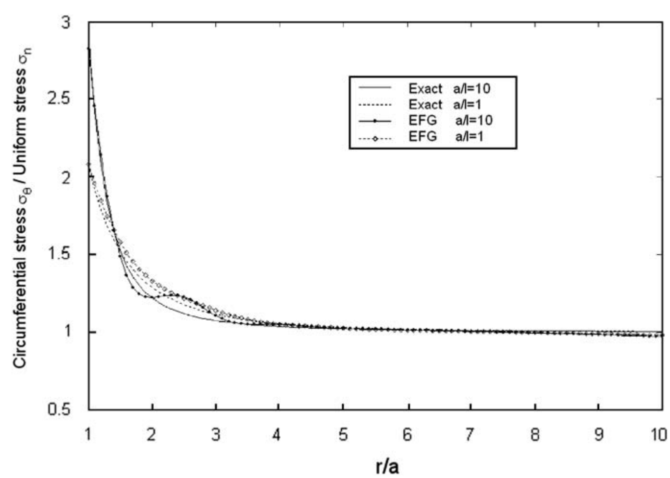

Fig. 3. The comparison of σ_θ/σ_n at θ=90°.

Table 1
The comparison of σ_θ/σ_n at θ=90°.

σ_θ/σ_n at θ=90°						
r/a	$\ell=0.5$			$\ell=0.05$		
	Exact	EFG	% error	Exact	EFG	% error
1.00	2.0666	2.0778	0.54	2.9130	2.8209	3.16
1.10	1.9129	1.9553	2.22	2.4220	2.4552	1.37
1.20	1.7853	1.8437	3.27	2.0737	2.1372	3.06
1.30	1.6796	1.7434	3.80	1.8279	1.8696	2.28
1.40	1.5916	1.6545	3.95	1.6522	1.6535	0.078
1.50	1.5181	1.5769	3.87	1.5241	1.4876	2.39
1.60	1.4562	1.5101	3.70	1.4287	1.3683	4.23
1.70	1.4038	1.4533	3.53	1.3563	1.2899	4.89
1.80	1.3591	1.4055	3.42	1.3001	1.2452	4.22
1.90	1.3207	1.3654	3.39	1.2559	1.2256	2.41
2.00	1.2876	1.3318	3.43	1.2206	1.2226	0.16

Table 2
The comparison of stress concentration factor.

a/ℓ	1	2	3	4	5	6	7	8	9	10
Exact	2.0666	2.3356	2.5292	2.6580	2.7439	2.8025	2.8438	2.8737	2.8960	2.9130
EFG	2.0778	2.4012	2.5803	2.6783	2.7340	2.7676	2.7890	2.8034	2.8135	2.8209
% error	0.54	2.80	2.02	0.76	0.36	1.24	1.92	2.45	2.85	3.16

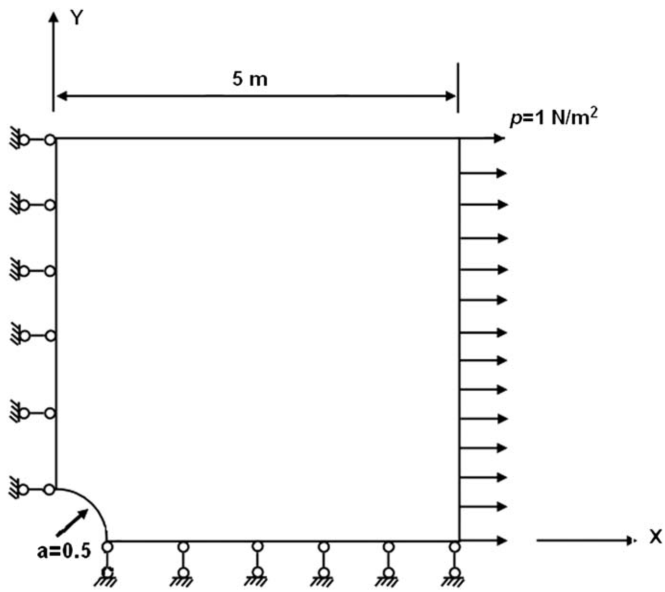

Fig. 4. A domain subjected to a uniform tension.

where $b_1 = E\nu/1-\nu^2$, $b_2 = E/1+\nu$, $b_3 = 4\beta$,

$$[H_1]=\begin{bmatrix}1&1&0&0&0\\1&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix},\quad [H_2]=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&\frac{1}{2}&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix},$$

$$[H_3]=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix} \tag{10}$$

$\{X\} = \{E, \nu, l\}$ can be obtained via $\{b\} = \{b_1, b_2, b_3\}^T$ uniquely.

3. Implementation of EFG method for couple stress problems

By utilizing a weighting residual technique, Eq. (1) with boundary condition Eqs. (6)–(7) can be written as [16]

$$\int_\Omega \left[\delta u_i \frac{1}{2}(\sigma_{ij}+\sigma_{ji})_{,j} + \delta u_i \frac{1}{2}(\sigma_{ij}-\sigma_{ji})_{,j} + \delta\theta(\mu_{i,i}+\alpha_{ij3}\sigma_{ij})\right] d\Omega$$
$$-\int_{\Gamma_\sigma}[\delta u_i(T_i - T_i^0) + \delta\theta(M - M^0)]\,d\Gamma + \int_{\Gamma_u}[\delta\lambda_i(u_i - \tilde{u}_i)$$
$$+\delta\xi(\theta - \tilde{\theta})]\,d\Gamma + \int_{\Gamma_u}(\delta u_i \cdot \lambda_i + \delta\theta \cdot \xi)\,d\Gamma = 0 \tag{11}$$

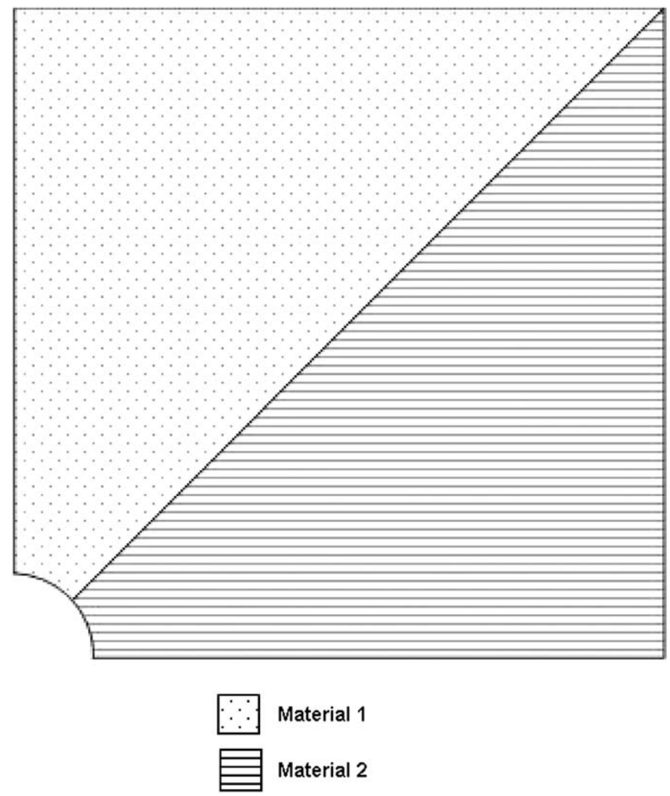

Fig. 5. Regional inhomogeneity of material.

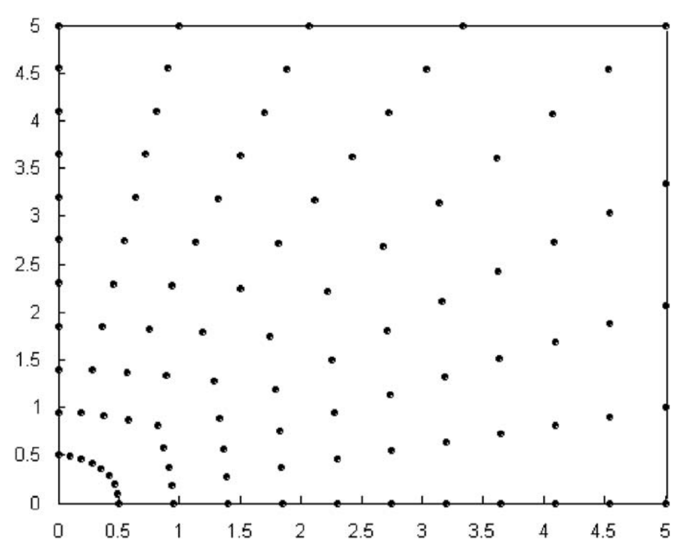

Fig. 6. Nodal arrangement.

The application of integral by parts for the above equation leads to

$$\int_{\Omega}\left[\delta u_{i,j}\frac{1}{2}(\sigma_{ij}+\sigma_{ji})+\delta\theta_i\cdot\mu_i+(\alpha_{ij3}\sigma_{ij})(\delta\theta-\frac{1}{2}(\alpha_{ij3}\cdot\delta u_{j,i}))\right]d\Omega$$
$$+\int_{\Gamma_\sigma}(\delta u_i T_i^0+\delta\theta\cdot M^0)\,d\Gamma+\int_{\Gamma_u}[\delta\lambda_i(u_i-\tilde{u}_i)+\delta\xi(\theta-\tilde{\theta})]\,d\Gamma$$
$$+\int_{\Gamma_u}(\delta u_i\cdot\lambda_i+\delta\theta\cdot\xi)\,d\Gamma=0 \quad (12)$$

where λ_i (i=1,2) and ξ represent Lagrange multipliers.

u can be approximated by [14]

$$\{u\}=[\Phi]\{\overline{u}\} \quad (13)$$

$$\{\theta\}=[L_\phi][\Phi]\{\overline{u}\} \quad (14)$$

where $\{\overline{u}\}$ is nodal parameter vectors of $\{u\}$,

$$[L_\phi]=\left[-\frac{1}{2}\frac{\partial}{\partial x_2}\quad \frac{1}{2}\frac{\partial}{\partial x_1}\right] \quad (15)$$

Φ is constructed via MLS technique [14], $(1,x_1,x_2,x_1x_2,x_1^2.x_2^2)^T$ is adopted as basis functions, and Gaussian weight function is chosen as weight function [14].

Substituting Eq. (13) and (14) into (12) yields

$$\begin{bmatrix}K & G^T\\ G & 0\end{bmatrix}\begin{Bmatrix}\overline{u}\\ \underline{\lambda}\end{Bmatrix}=\begin{Bmatrix}f\\ q\end{Bmatrix} \quad (16)$$

where

$$[K]=\int_{\Omega}[B]^T[D][B]\,d\Omega \quad (17)$$

Table 3
The effects of initial guesses on the results.

Constitutive parameters	1			2			3			Actual values
	Initial guesses	Final values	Number of iterations	Initial guesses	Final values	Number of iterations	Initial guesses	Final values	Number of iterations	
E_1	5.00×10^5	2.00×10^5	17	1.00×10^4	1.99×10^5	20	6.00×10^6	1.99×10^5	19	2.00×10^5
ν_1	4.00×10^{-1}	3.00×10^{-1}		2.50×10^{-1}	2.99×10^{-1}		3.20×10^{-1}	2.99×10^{-1}		3.00×10^{-1}
l_1	2.00×10^{-2}	9.99×10^{-4}		5.00×10^{-4}	1.00×10^{-3}		1.00×10^{-2}	1.00×10^{-3}		1.00×10^{-3}
E_2	5.00×10^5	9.99×10^5		5.00×10^6	1.00×10^6		2.50×10^4	1.00×10^6		1.00×10^6
ν_2	4.00×10^{-1}	3.50×10^{-1}		$4.00\times10-^{1}$	3.49×10^{-1}		2.00×10^{-1}	3.49×10^{-1}		3.50×10^{-1}
l_2	2.00×10^{-2}	3.00×10^{-3}		5.00×10^{-3}	2.99×10^{-3}		2.00×10^{-4}	2.99×10^{-3}		3.00×10^{-3}

Table 4
Identification with slight and strong regional inhomogeneity.

Constitutive parameters	1			2			3		
	Initial guesses	Final values	Actual values	Initial guesses	Final values	Actual values	Initial guesses	Final values	Actual values
E_1	6.0000×10^6	1.9999×10^5	$\mathbf{2.0000\times10^5}$	6.0000×10^6	1.9999×10^5	2.0000×10^5	6.0000×10^6	2.0100×10^5	$\mathbf{2.0100\times10^5}$
ν_1	4.0000×10^{-1}	3.0000×10^{-1}	3.0000×10^{-1}	4.0000×10^{-1}	2.9999×10^{-1}	3.0000×10^{-1}	4.0000×10^{-1}	3.0000×10^{-1}	3.0000×10^{-1}
l_1	5.0000×10^{-3}	1.0000×10^{-3}	1.0000×10^{-3}	5.0000×10^{-3}	2.0000×10^{-3}	$\mathbf{2.0000\times10^{-3}}$	5.0000×10^{-3}	5.9999×10^{-2}	$\mathbf{6.0000\times10^{-2}}$
E_2	2.5000×10^4	3.9999×10^7	$\mathbf{4.0000\times10^7}$	2.5000×10^4	5.0000×10^5	5.0000×10^5	2.5000×10^4	1.9999×10^5	$\mathbf{2.0000\times10^5}$
ν_2	2.0000×10^{-1}	3.4999×10^{-1}	3.5000×10^{-1}	2.0000×10^{-1}	3.4999×10^{-1}	3.5000×10^{-1}	2.0000×10^{-1}	3.5000×10^{-1}	3.5000×10^{-1}
l_2	5.0000×10^{-3}	2.9999×10^{-3}	3.0000×10^{-3}	5.0000×10^{-3}	2.0099×10^{-3}	$\mathbf{2.0100\times10^{-3}}$	5.0000×10^{-3}	3.0000×10^{-4}	$\mathbf{3.0000\times10^{-4}}$

Table 5
Combined identification of material parameters and load.

	Initial guesses	Final values	Actual values	Number of iteration		Initial guesses	Final values	Actual values	Number of iteration
p	2.00×10^{-1}	2.00×10^1	2.00×10^1	18	p	1.00×10^0	7.99×10^0	8.00×10^0	15
ν_1	2.00×10^{-1}	4.00×10^{-1}	4.00×10^{-1}		ν_2	2.00×10^{-1}	3.49×10^{-1}	3.50×10^{-1}	
l_2	1.00×10^{-2}	6.00×10^{-3}	6.00×10^{-3}		l_2	2.00×10^{-1}	4.99×10^{-2}	5.00×10^{-2}	

Table 6
The effect of noisy data on results.

Constitutive parameters	Initial guesses	Confidence intervals		Actual values
		δ=0.03	δ=0.05	
E_1	6.00×10^6	$200\,465\pm1936$	$200\,978\pm3267$	2.00×10^5
ν_1	4.00×10^{-1}	$2.99\times10^{-1}\pm7.87\times10^{-12}$	$2.99\times10^{-1}\pm1.28\times10^{-11}$	3.00×10^{-1}
l_1	5.00×10^{-3}	$3.00\times10^{-3}\pm6.47\times10^{-9}$	$3.00\times10^{-3}\pm1.08\times10^{-8}$	3.00×10^{-3}
E_2	2.50×10^4	$501\,163\pm4841$	$502\,445\pm8168$	5.00×10^5
ν_2	2.00×10^{-1}	$3.49\times10^{-1}\pm1.07\times10^{-11}$	$3.49\times10^{-1}\pm1.83\times10^{-11}$	3.50×10^{-1}
l_2	5.00×10^{-3}	$7.99\times10^{-3}\pm1.85\times10^{-9}$	$7.99\times10^{-3}\pm3.08\times10^{-9}$	8.00×10^{-3}

$$[G] = -\int_{\Gamma_u} [N]^T [B_1][\Phi]\, d\Gamma \quad (18)$$

$$[f] = \int_{\Gamma_\sigma} ([B_1][\Phi])^T \{F\}\, d\Gamma \quad (19)$$

$$[q] = -\int_{\Gamma_\sigma} [N]^T \{\hat{u}\}\, d\Gamma \quad (20)$$

$$[B] = [B_1,\ B_2,\ \ldots,\ B_N], [B]_I^T = \begin{bmatrix} \Phi_{I,x_1} & 0 & \Phi_{I,x_2} & -\frac{1}{2}\Phi_{I,x_1x_2} & -\frac{1}{2}\Phi_{I,x_2x_2} \\ 0 & \Phi_{I,x_2} & \Phi_{I,x_1} & \frac{1}{2}\Phi_{I,x_1x_1} & \frac{1}{2}\Phi_{I,x_1x_2} \end{bmatrix} \quad (21)$$

$$[B_1] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\dfrac{\partial}{2\partial x_2} & -\dfrac{\partial}{2\partial x_1} \end{bmatrix} \quad (22)$$

$$\{F\} = \{\tilde{T}_1,\ \tilde{T}_2,\ \tilde{M}\}^T \quad (23)$$

$$\{\hat{u}\} = \{\tilde{u}_1,\ \tilde{u}_2,\ \tilde{\theta}\}^T \quad (24)$$

$$[N] = [[N]_1\ \ [N]_2\ \ \ldots\ \ [N]_{Nu}], [N]_I = N_I[I]_3 \quad (25)$$

$$\{\underline{\lambda}\} = \{\lambda_1,\ \lambda_2,\ \xi\}^T \quad (26)$$

$$K = \int\int_\Omega [B]^T (b_1[H_1] + b_1[H_2] + b_3[H_3])[B]\, d\Omega \quad (27)$$

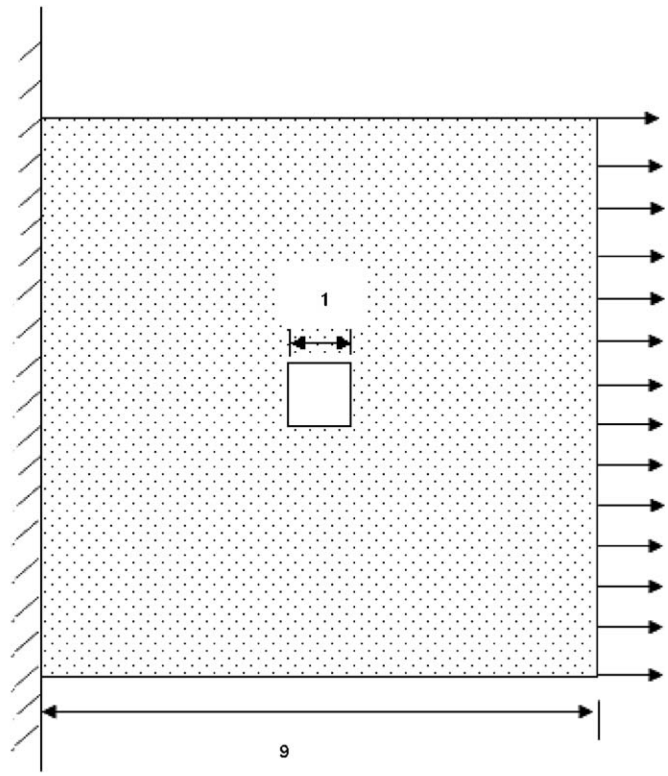

Fig. 7. A plate with a rectangle hole subjected to a uniform tension.

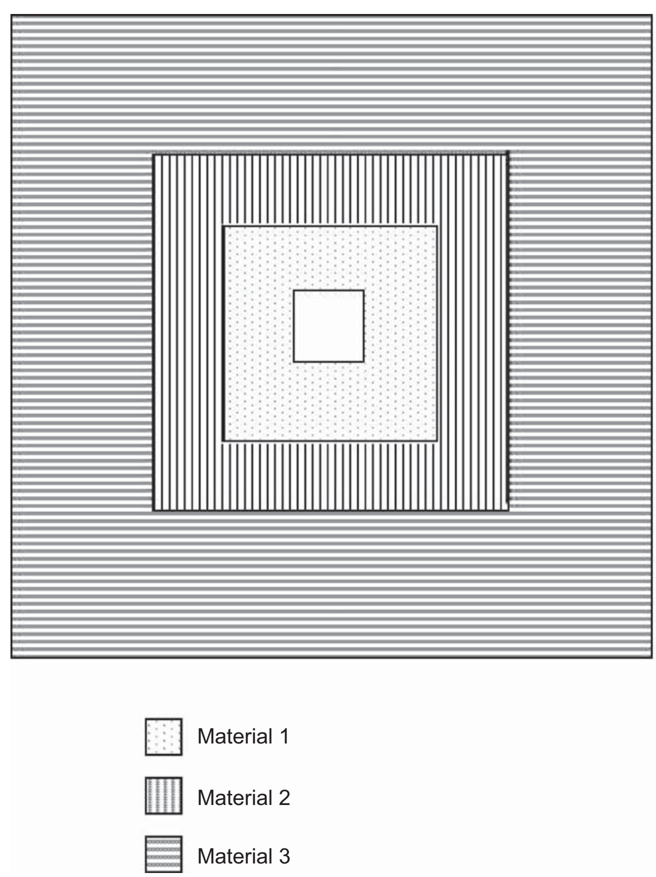

Fig. 8. Regional inhomogeneity of material.

4. Numerical examples of direct problems

A benchmark test is presented to verify the accuracy of the proposed EFG model.

For the simplicity, all the parameters and variables are assumed dimensionless.

Consider a circular hole in a uniform tension field as shown in Fig. 1 where Young's modulus and Poisson's ratio are 200 000 and 0.3, respectively, the radius of the hole a=0.5. The node arrangement is exhibited in Fig. 2.

Fig. 3 and Table 1 describe stress σ_θ at $\theta = 90^\circ$ in the cases $\ell = 0.5$ and 0.05 respectively. The maximum error is 4.89% in comparison with the analytical solution [2].

Table 2 shows the variation of concentration factor with a/ℓ, the maximum error is 3.16% in comparison with the analytical solution [2].

5. Inverse couple stress problems

The aim of this part is to determine unknown constitutive coefficients $\{b\} = \{b_1, b_2, b_3\}^T$ by utilizing the displacement that is partially obtained (either by experiment or by numerical simulation).

$\{b\}$ can be evaluated by minimizing a functional defined by

$$\Pi = \frac{1}{2}([J]\{u\} - \{u^*\})^T([J]\{u\} - \{u^*\}) = \frac{1}{2}[R]^T[R] \quad (28)$$

with constraints

$$\{u\} = [\Phi]\{\overline{u}\}$$

$$\begin{bmatrix} K & G^T \\ G & 0 \end{bmatrix} \begin{Bmatrix} \overline{u} \\ \underline{\lambda} \end{Bmatrix} = \begin{Bmatrix} f \\ q \end{Bmatrix} \quad (29, 30)$$

where $\{u^*\}$ stands for a vector of measured or simulated displacement, and $[J]$ is a transformation matrix mapping the relationship of location between $\{u\}$ and $\{u^*\}$.

Gauss–Newton algorithm is employed to solve Eqs. (28)–(30) with an iterative process [17]

$$\{b^{n+1}\} = \{b^n\} + \{\Delta b\} \tag{31}$$

$$\{\Delta b\} = -\left(\left(\frac{\partial\{u\}}{\partial\{b^n\}}\bigg|\right)^T \left(\frac{\partial\{u\}}{\partial\{b^n\}}\bigg|\right)\right)^{-1} \left(\frac{\partial\{u\}}{\partial\{b^n\}}\bigg|\right)^T (\{u\}_{(\{b^n\})} - \{u^*\}) \tag{32}$$

The major steps of computing include:

Step 1. Set $n=0$, initial guess $\{b^0\}$ and error bound ε.
Step 2. Calculate $\{u\}_{(\{b^n\})}$ via Eq. (15).
Step 3. Calculate $\partial u/\partial b_j$ via Eq. (33) and solve $\{\Delta b\}$ via Eq. (31).
Step 4. $\{b^{n+1}\} = \{b^n\} + \{\Delta b\}$ and $n = n+1$.
Step 5. if $\|\{\Delta b\}\| \le \varepsilon$, stop iteration; else, goto step 2.

The sensitivity of $\{u\}$ with respect to $\{b\}$ can be obtained by

$$\frac{\partial u}{\partial b_j} = [\Phi]\frac{\partial \overline{u}}{\partial b_j} \tag{33}$$

Table 7
The effects of the number of measured points on the results.

Constitutive parameters	Initial guesses	100 measured points		41 measured points		21 measured points		Actual values
		Final values	Number of iterations	Final values	Number of iterations	Final values	Number of iterations	
E_1	1.00×10^5	6.99×10^5	8	6.99×10^5	9	6.99×10^5	7	7.00×10^5
ν_1	3.00×10^{-1}	1.99×10^{-1}		2.00×10^{-1}		2.00×10^{-1}		2.00×10^{-1}
l_1	3.00×10^{-3}	9.99×10^{-3}		1.00×10^{-2}		1.00×10^{-2}		1.00×10^{-2}
E_2	1.00×10^5	5.00×10^5		5.00×10^5		5.00×10^5		5.00×10^5
ν_2	3.00×10^{-1}	3.50×10^{-1}		3.50×10^{-1}		3.49×10^{-1}		3.50×10^{-1}
l_2	3.00×10^{-3}	5.00×10^{-3}		4.99×10^{-3}		4.99×10^{-3}		5.00×10^{-3}
E_3	1.00×10^5	2.49×10^5		2.49×10^5		2.49×10^5		2.50×10^5
ν_3	3.00×10^{-1}	3.99×10^{-1}		3.99×10^{-1}		3.99×10^{-1}		4.00×10^{-1}
l_3	3.00×10^{-3}	2.00×10^{-3}		2.00×10^{-3}		1.99×10^{-3}		2.00×10^{-3}

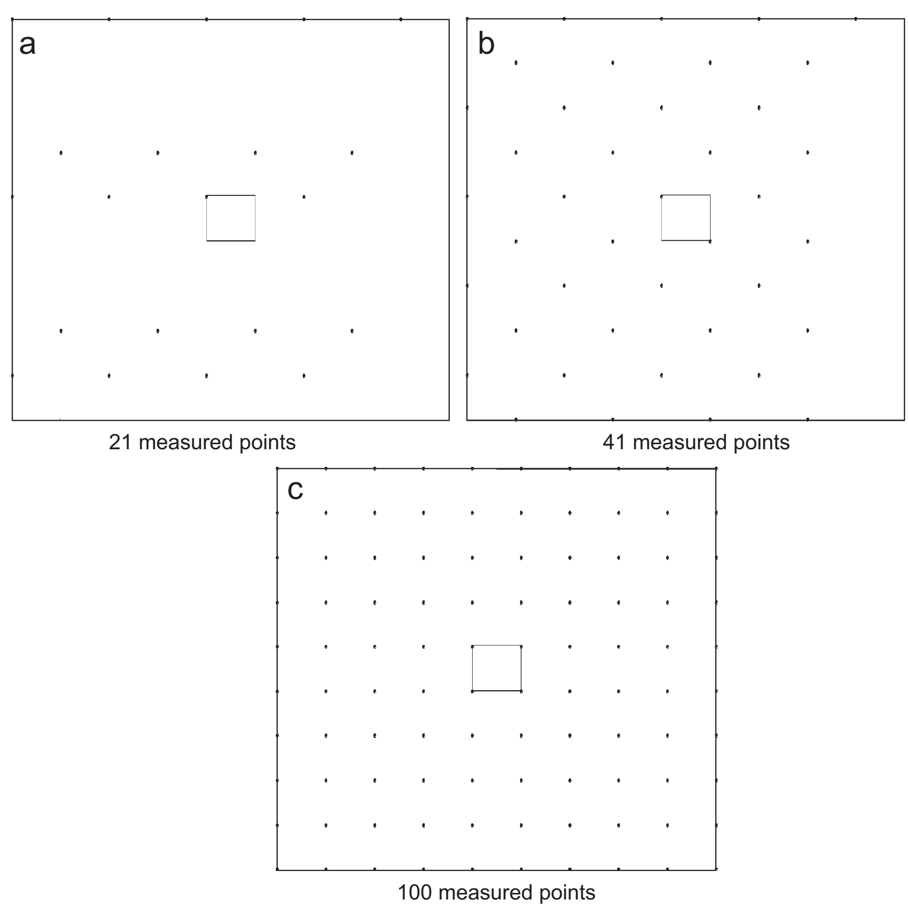

Fig. 9. The locations of measured points.

Table 8
The effect of noisy data on results.

Constitutive parameters	Initial guesses	Confidence intervals		Actual values
		δ=0.03	δ=0.05	
E_1	1.00×10^5	$697\,303\pm6215$	$696\,100\pm10374$	7.00×10^5
ν_1	3.00×10^{-1}	$2.00\times10^{-1}\pm1.13\times10^{-14}$	$2.00\times10^{-1}\pm1.09\times10^{-14}$	2.00×10^{-1}
l_1	3.00×10^{-3}	$1.00\times10^{-2}\pm1.42\times10^{-13}$	$1.00\times10^{-2}\pm1.28\times10^{-13}$	1.00×10^{-2}
E_2	1.00×10^5	$498\,073\pm4439$	$49\,721\pm7410$	5.00×10^5
ν_2	3.00×10^{-1}	$3.49\times10^{-1}\pm5.04\times10^{-15}$	$3.50\times10^{-1}\pm4.88\times10^{-15}$	3.50×10^{-1}
l_2	3.00×10^{-3}	$4.99\times10^{-3}\pm8.08\times10^{-13}$	$4.99\times10^{-3}\pm6.76\times10^{-13}$	5.00×10^{-3}
E_3	1.00×10^5	$249\,037\pm2219$	$248\,607\pm3705$	2.50×10^5
ν_3	3.00×10^{-1}	$3.99\times10^{-3}\pm3.27\times10^{-15}$	$3.99\times10^{-3}\pm2.85\times10^{-15}$	4.00×10^{-1}
l_3	3.00×10^{-3}	$2.49\times10^{-3}\pm1.48\times10^{-12}$	$2.49\times10^{-3}\pm1.38\times10^{-12}$	2.50×10^{-3}

$$\begin{Bmatrix} \dfrac{\partial \underline{u}}{\partial b_j} \\ \underline{\lambda} \end{Bmatrix} = -\begin{bmatrix} K & Gu \\ Gu^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{\partial K}{\partial b_j} & Gu \\ Gu^T & 0 \end{bmatrix} \begin{Bmatrix} \underline{u} \\ \underline{\lambda} \end{Bmatrix} \tag{34}$$

where

$$\frac{\partial K}{\partial b_j} = \int\int_\Omega [B]^T[H_j][B]\,d\Omega \tag{35}$$

The effect of noisy data is taken into account in the form [18]

$$\{u^*\} = (1+\xi\cdot\delta)\{u^e\} \tag{36}$$

where $\{u^e\}$ represents the vector of known displacement without noise, and is provided by solving Eq. (15) with actual constitutive parameters in this paper. ξ is a random variable which follows a normal distribution with zero mean and unit standard deviation, and δ denotes a deviation.

For each fixed value of δ, 40 groups of results are obtained with 40 ξ produced randomly.

The confidence interval is evaluated by [18]

$$\overline{x} \pm \frac{t_{(\beta/2,N-1)} * s}{\sqrt{N}} \tag{37}$$

where $\overline{x}$ represents the mean value of the identified parameters, s is a standard deviation of the identified parameters, t denotes a t distribution with the degree of freedom $(N-1)$, N is the capability of samples, and the confidence level is $1-\beta$.

6. Numerical examples of inverse problems

Example 1. Consider a domain subjected to a uniform tension p as shown in Fig. 4 where the distribution of constitutive parameters is inhomogeneous regionally, as illustrated in Fig. 5. Fig. 6 gives a description of nodes arrangement. Totally six different constitutive parameters need to be identified, and all the nodes are employed as the measured points in this example.

Table 3 provides the results with different initial guesses which seem no impact on the final solutions. Table 4 exhibits the results when the constitutive parameters in one region differs from another slightly (0.05%) and strongly (200 times). Table 5 shows a combined identification of constitutive parameter and load p. Table 6 presents the results with noisy data at 95% confidence level. The injection of noisy data with $\delta = 0.03$ and 0.05 results in a 11.55% maximum relative error of the additional information.

Example 2. This example refers to a plate with a rectangle hole as shown in Fig. 7 where the distribution of constitutive parameters is inhomogeneous regionally, as illustrated in Fig. 8. The plate is subjected to a uniform tension q=1. There are nine constitutive parameters to be identified.

Table 7 exhibits the results with different locations of the measured points as illustrated in Fig. 9, Table 8 shows the effect of noisy data on results at 95% confidence level, the injection of noisy data with $\delta = 0.03$ and 0.05 results in a 10.60% maximum relative error of the additional information.

Numerical verification indicates that

1. Initial guess seems no impact on the solution.
2. The solutions are available for both slight and strong regional inhomogeneity.
3. The solution is available within a certain range of noisy data, and the Young's modulus seems more sensitive to the measurement noisy than the Poisson's ratio and character length.
4. The decrease of number of measured points from 100 to 21 seems no impact on the solution.

7. Conclusion

This paper aims at developing a numerical model to identify unknown constitutive parameters in the couple stress problem. By modeling the direct problem via EFGM, the inconvenience that may be caused by C^1 continuity requirement in the implementation of FEM can be avoided, and the sensitivity analysis can be realized easily. In comparison with the inverse problem of classical elasticity, the difference of the problem concerned in this paper mainly comes from the increase of unknowns with the appearance of constitutive parameters describing couple stress. Numerical example indicates that such a difference seems not to cause a distinct impact on the identification process conducted using a conventional Gauss–Newton technique. The results obtained encourage authors to make further effort to improve the proposed model for its real application in practical engineering.

Acknowledgments

The research leading to this paper is funded by NSF (10421002), NSF (10472019), NSF (10772035), NSF (10721062), NKBRSF [2005CB321704,2010CB832703], and the fund of disciplines leaders of young and middle age faculty in colleges of Liaoning Province.

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