International Journal of Control, Automation, and Systems (2015) 13(5):1140-1149
DOI 10.1007/s12555-013-0403-3

ISSN:1598-6446 eISSN:2005-4092
http://www.springer.com/12555

Further Results on Robust Delay-range-dependent Stability Criteria for Uncertain Neural Networks with Interval Time-varying Delay

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Abstract: The problem of robust delay-range-dependent stability analysis of neural networks (NNs) with interval time-varying delay in a given range is investigated in this paper. The relationship between time-varying delay and its lower and upper bounds is taken into account when estimating the upper bound of the Lyapunov functional derivative. By defining a more general type of Lyapunov functionals, some new less conservative delay-dependent stability criteria are established in terms of linear matrix inequalities (LMIs), which can be computed and optimized easily. Meanwhile, the computational complexity of newly obtained stability conditions is reduced because fewer variables are involved. Finally, through four well-known numerical examples used in other literature, it will be shown that proposed stability criteria achieve the improvements over existing ones, as well as the effectiveness of the proposed idea.

Keywords: Interval time-varying delay, linear matrix inequalities (LMIs), maximum allowable delay bound (MADB), neural networks (NNs).

1. INTRODUCTION

Neural networks (NNs) have been extensively studied over the past few decades, finding application in a variety of areas: e.g., pattern recognition, associative memory, and combinatorial optimization. Such applications depend heavily on the dynamic behavior of the network. In recent years, considerable effort has been devoted to analyzing stability of neural networks with no time delay [2]. In reality, dynamics of a neural network often involves time delays due to finite switching speed of amplifiers in electronic neural networks, or to the finite signal propagation time in biological networks. Stability criterion is formulated in terms of linear matrix inequalities (LMIs) which can be solved numerically using effective interior-point algorithm [1]. Therefore, many researchers have focused on the study of stability analysis of delayed neural networks during the last decades [3-22].

Recently, a free weighting-matrix approach was proposed [4-7] in which free weighting matrices are employed to express the relationship between the terms of the Leibniz-Newton formula; and all the negative terms in the derivative of the Lyapunov functional are retained. This approach avoids the restriction on the derivative of a time-varying delay. In order to avoid conservatism involve by model transformation and bounding techniques for cross terms, free-weighting matrices method was used to derive stability criteria for neural networks with time-varying delay [5]. Results in [5] were further improved in [6] by considering some useful terms that were ignored in previous results when estimating the upper bound on the derivative of the Lyapunov functional. Using Jensen's inequality, some simplified stability criteria were proposed [13]. These criteria were equivalent to those in [6] but with less decision variables. By constructing an augmented Lyapunov functional, improved stability conditions have been established in [7,13]. However, the above results are still conservative to some extent and there exists room for further improvement. In practice, lower bound of delay is not always 0; delay considered in this paper is assumed to belong to a given interval.

On the other hand, range of time-varying delay for NNs considered in [4-6,9] is from 0 to upper bound. In practice, time-varying interval delay is often encountered, that is, the range of delay varies in an interval for which the lower bound is not restricted to 0. In this case, stability criteria for NNs with time-varying delay in [4-6,9,15] are conservative because they do not take into account the information of the lower bound of delay. To the best of the authors' knowledge, few stability results have been reported in literature for NNs with time-varying interval delay. Also, because of unavoidable factors, such as modeling error, external perturbation and parameter fluctuation, the neural network model certainly involve uncertainties like perturbation and component variation, which alter stability of neural networks. And in recent years, stability analysis issues for neural networks in the presence of parameter uncertainties perturbations have stirred some initial research attention [10,15-18].

This paper investigates the problem of robust delay-range-dependent stability for delayed NNs. To ensure

Manuscript received August 26, 2013; revised August 26, 2014; accepted December 14, 2014. Recommended by Associate Editor Shengyuan Xu under the direction of Editor Yoshito Ohta.

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larger delay bounds, a new type of Lyapunov functional is proposed and some new delay-range-dependent stability criteria derived in terms of linear matrix inequalities (LMIs). Newly obtained results prove less conservative or computationally complex than existing corresponding ones; these stability criteria are also more applicable. Finally, numerical examples show efficacy of the main results.

The rest of the paper will be organized as follows: In Section 2, stability description and preliminaries are given. In Section 3, the robust stability result is derived. In Section 4, four simulation examples are given to illustrate the effectiveness of the proposed approach. Finally, the conclusion is drawn in Section 5.

2. STABILITY DESCRIPTION AND PRELIMINARIES

Consider the following delayed neural networks with parameter uncertainties:

$$\begin{aligned}\dot{u}(t) &= -(C+\Delta C(t))u(t)+(A+\Delta A(t))f(u(t)) \\ &\quad +(B+\Delta B(t))f(u(t-h(t)))+J,\end{aligned} \tag{1}$$

where $u(t)=[u_1(t),\cdots,u_n(t)]^{\mathrm{T}}\in R^n$ is state vector with n neurons; $f(u(t))=[f_1(u_1(t)),\cdots,f_n(u_n(t))]^{\mathrm{T}}\in R^n$ is called an activation function indicating how the j-th neuron responses to its input; $C=diag(c_1,...,c_n)$ is a diagonal matrix with each $c_i>0$ controlling the rate with which the i-th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs; $A=(a_{ij})_{n\times n}$ and $B=(b_{ij})_{n\times n}$ are feedback and delayed feedback matrix, respectively; $J=[J_1,\cdots,J_n]^{\mathrm{T}}\in R^n$ is a constant input vector, $\Delta A(t)$, $\Delta B(t)$, and $\Delta C(t)$ are unknown matrices that represent the time-varying parameter uncertainties and $h(t)$ is time-varying delay. We consider two different cases for time varying delays:

Case 1: $h(t)$ is a differentiable function satisfying

$$0\le h_1\le h(t)\le h_2,\quad \dot{h}(t)\le h_d,\quad \forall t\ge 0, \tag{2}$$

Case 2: $h(t)$ is a differentiable function satisfying

$$0\le h_1\le h(t)\le h_2, \tag{3}$$

where h_1 and h_2 are lower and upper delay bounds, respectively; h_1, h_2 and h_d are constants. Here h_1, lower bound of delay may not equal 0, and when $h_d=0$ we have $h_1=h_2$. Both Cases 1 and 2 have considered upper and non-zero lower delay bounds of the interval time-varying delay. Case 1 is a special case of Case 2. If the time-varying delay is differentiable and $h_d<1$, one can obtain a less conservative result using Case 1 than that using Case 2.

In this paper, the neuron activation functions are assumed to be bounded and satisfy the following assumption.

Assumption 1: It is assumed that each of the activation functions $f_j(j=1,2,...,n)$ possess the following condition

$$\gamma_i\le\frac{f_i(\varsigma_1)-f_i(\varsigma_2)}{\varsigma_1-\varsigma_2}\le\sigma_i,\ \varsigma_1\ne\varsigma_2\in R,\ i=1,2,...,n, \tag{4}$$

where γ_i and σ_i are known constant scalars.

Remark 1: If the neuron activation functions satisfy Assumption 1, then they satisfy

$$|f_i(\varsigma_1)-f_i(\varsigma_1)|\le max\{|\gamma_i|,|\sigma_i|\}|\varsigma_1-\varsigma_2|=\rho_i|\varsigma_1-\varsigma_2|,\quad i=1,2...,n. \tag{5}$$

It is noted that assumption condition (5) has been investigated in many research papers [11,12]. However, we shall point out that this assumption is too strong and may lead to conservative conditions for delay-dependent stability analysis of delayed neural networks. For example, if $\gamma_i<k_i<0$, then the delay-dependent stability result obtained by using (4) is generally less conservative than the one obtained by using (5). This will be shown via numerical examples in Section 4 in this paper.

Next, the equilibrium point $u^*=[u_1^*,\cdots,u_n^*]^{\mathrm{T}}$ of system (1) is shifted to the origin via transformation $x(t)=u(t)-u^*$, then system (1) can be equivalently written as

$$\begin{aligned}\dot{x}(t) &= -(C+\Delta C(t))x(t)+(A+\Delta A(t))g(x(t)) \\ &\quad +(B+\Delta B(t))g(x(t-h(t))),\end{aligned} \tag{6}$$

where

$$\begin{aligned}&x(\cdot)=[x_1(\cdot),\cdots,x_n(\cdot)]^{\mathrm{T}},\\ &g(x(\cdot))=[g_1(x_1(\cdot)),\cdots,g_n(x_n(\cdot))]^{\mathrm{T}},\\ &g_i(x_i(\cdot))=f_i(x_i(\cdot)+u_i^*)-f_i(u_i^*),\qquad i=1,2,...,n.\end{aligned}$$

Matrices $\Delta C(t)$, $\Delta A(t)$, and $\Delta B(t)$ are uncertainties of the system and have the form

$$[\Delta C(t)\quad \Delta A(t)\quad \Delta B(t)]=DF(t)[E_c\quad E_a\quad E_b], \tag{7}$$

where E_c, E_a, and E_b are known constant real matrices with appropriate dimensions, $F(t)$ an unknown matrix function with Lebesgue-measurable elements bounded by

$$F^T(t)F(t)\le I,\qquad \forall t, \tag{8}$$

where I is an appropriately dimensioned identity matrix.

Now we state the following lemmas, more useful in the sequel.

Lemma 1 [15]**:** For any positive semi-definite matrices

$$X=\begin{bmatrix}X_{11} & X_{12} & X_{13}\\ X_{12}^T & X_{22} & X_{23}\\ X_{13}^T & X_{23}^T & X_{33}\end{bmatrix}\ge 0, \tag{9a}$$

the following integral inequality holds

$$-\int_{t-h(t)}^{t}\dot{x}^T(s)X_{33}\dot{x}(s)ds$$
$$\le \int_{t-h(t)}^{t}\begin{bmatrix} x^T(t) & x^T(t-h(t)) & \dot{x}^T(s)\end{bmatrix} \tag{9b}$$
$$\times\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & 0\end{bmatrix}\begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(s)\end{bmatrix}ds.$$

Secondly, we introduce the Schur complement, essential to proofs of our results.

Lemma 2 [1]**:** The following matrix inequality

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x)\end{bmatrix} < 0, \tag{10a}$$

where $Q(x)=Q^T(x)$, $R(x)=R^T(x)$ and $S(x)$ depend on affine on x, is equivalent to

$$R(x) < 0, \tag{10b}$$
$$Q(x) < 0, \tag{10c}$$

and

$$Q(x) - S(x)R^{-1}(x)S^T(x) < 0. \tag{10d}$$

Lemma 3 [1]**:** Given symmetric matrices Ω and D, E, of appropriate dimensions,

$$\Omega + DF(t)E + E^TF^T(t)D^T < 0, \tag{11a}$$

for all $F(t)$ satisfying $F^T(t)F(t) \le I$, if and only if there exists some $\varepsilon > 0$ such that

$$\Omega + \varepsilon DD^T + \varepsilon^{-1}E^TE < 0. \tag{11b}$$

This paper finds new stability criteria less conservative than existing results. A newly constructed Lyapunov functional contains information on the lower bound of delay h_1 and upper bound h_2. Theorem 1 presents a delay-range-dependent result in terms of LMIs, expressing relationships between terms of a Leibniz-Newton formula.

Theorem 1: For given scalars $\sum_2 = diag\{\sigma_1, \sigma_2, ..., \sigma_n\}$ and $\sum_1 = diag\{\gamma_1, \gamma_2, ..., \gamma_n\}$, h_1, h_2, h_d, nominal system (6) $(\Delta C(t) = \Delta A(t) = \Delta B(t) = 0)$ subject to (2) is asymptotically stable if there exist positive definite matrices $P = P^T > 0$, $Q_1 = Q_1^T > 0$, $Q_2 = Q_2^T > 0$, $Q_3 = Q_3^T > 0$, $R_1 = R_1^T > 0$, $R_2 = R_2^T > 0$ diagonal matrices $T_1 \ge 0$, $T_2 \ge 0$, $\Lambda = diag\{\lambda_1, \lambda_2, ..., \lambda_n\}$, $K = diag\{k_1, k_2, ..., k_n\}$,

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33}\end{bmatrix} \ge 0,$$
$$Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{12}^T & Y_{22} & Y_{23} \\ Y_{13}^T & Y_{23}^T & Y_{33}\end{bmatrix} \ge 0$$

such that the following LMIs hold:

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & 0 & \Xi_{15} & 0 & \Xi_{17} \\ \Xi_{12}^T & \Xi_{22} & \Xi_{23} & 0 & 0 & 0 & \Xi_{27} \\ \Xi_{13}^T & \Xi_{23}^T & \Xi_{33} & \Xi_{34} & 0 & 0 & \Xi_{37} \\ 0 & 0 & \Xi_{34}^T & \Xi_{44} & \Xi_{45} & \Xi_{46} & 0 \\ \Xi_{15}^T & 0 & 0 & \Xi_{45}^T & \Xi_{55} & 0 & 0 \\ 0 & 0 & 0 & \Xi_{46}^T & 0 & \Xi_{66} & 0 \\ \Xi_{17}^T & \Xi_{27}^T & \Xi_{37}^T & 0 & 0 & 0 & \Xi_{77}\end{bmatrix} < 0, \tag{12a}$$

and

$$R_1 - X_{33} \ge 0, \quad R_2 - Y_{33} \ge 0, \tag{12b}$$

where

$$\Xi_{11} = -C^TP - PC + \Sigma_1\Lambda C + C^T\Lambda\Sigma_1^T - \Sigma_2 KC$$
$$- C^TK^T\Sigma_2^T + Q_1 + Q_2 + Q_3 + h_1X_{11} + X_{13} + X_{13}^T,$$
$$\Xi_{12} = PA - \Sigma_1\Lambda A - C^T\Lambda + \Sigma_2KA + C^TK^T + T_1(\Sigma_1 + \Sigma_2),$$
$$\Xi_{13} = PB - \Sigma_1\Lambda B + \Sigma_2KB, \quad \Xi_{15} = h_1X_{12} - X_{13} + X_{23}^T,$$
$$\Xi_{17} = -C^T(h_1R_1 + h_{21}R_2),$$
$$\Xi_{22} = \Lambda K - KA - A^TK^T - 2T_1,$$
$$\Xi_{23} = \Lambda B + A^T\Lambda - KB, \quad \Xi_{27} = A^T(h_1R_1 + h_{21}R_2),$$
$$\Xi_{33} = -2T_2, \quad \Xi_{34} = (\Sigma_1 + \Sigma_2)T_2,$$
$$\Xi_{37} = B^T(h_1R_1 + h_{21}R_2),$$
$$\Xi_{44} = -(1-h_d)Q_3 + h_{21}Y_{22} - Y_{23} - Y_{23}^T$$
$$+ h_{21}Y_{11} + Y_{13} + Y_{13}^T - 2\Sigma_2T_2\Sigma_1,$$
$$\Xi_{45} = h_{21}Y_{12}^T - Y_{13}^T + Y_{23}, \quad \Xi_{46} = h_{21}Y_{12} - Y_{13} + Y_{23}^T,$$
$$\Xi_{55} = -Q_2 + h_{21}X_{22} - X_{23} - X_{23}^T,$$
$$\Xi_{66} = -Q_1 + h_{21}Y_{22} - Y_{23} - Y_{23}^T,$$
$$\Xi_{77} = -(h_1R_1 + h_{21}R_2).$$

Proof: Construct a Lyapunov-Krasovskii functional candidate as

$$\begin{aligned} V(x_t) &= x^T(t)Px(t) + 2\sum_{i=1}^{n}\{\lambda_i\int_0^{x_i(t)}(g_i(s) - r_i(s))ds \\ &+ \int_0^{x_i(t)}k_i(\sigma_i(s) - g_i(s))ds\} + \int_{t-h_1}^{t}x^T(s)Q_1x(s)ds \\ &+ \int_{t-h2}^{t}x^T(s)Q_2x(s)ds + \int_{t-h(t)}^{t}x^T(s)Q_3x(s)ds \\ &+ \int_{-h_1}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)R_1\dot{x}(s)dsd\theta \\ &+ \int_{-h_2}^{-h_1}\int_{t+\theta}^{t}\dot{x}^T(s)R_2\dot{x}(s)dsd\theta. \end{aligned} \tag{13}$$

Calculating the derivative of (13) with respect to $t > 0$ along the trajectories of nominal system (6) leads to

$$\dot{V}(x_t) = x^T(t)(-C^TP - PC)x(t)$$
$$+ 2x^T(t)P[Ag(x(t)) + Bg(x(t-h(t)))]$$

$$
\begin{aligned}
&+2[g^T(x(t))-x^T(t)\Sigma_1]\Lambda\dot{x}(t)\\
&+2[x^T(t)\Sigma_2-g^T(x(t))]K\dot{x}(t)\\
&+x^T(t)(Q_1+Q_2+Q_3)x(t)\\
&-x^T(t-h_1)Q_1x(t-h_1)-x^T(t-h_2)Q_2x(t-h_2)\\
&-x^T(t-h(t))(1-\dot{h}(t))Q_3x(t-h(t))+\dot{x}^T(t)h_1R_1\dot{x}(t)\\
&+\dot{x}^T(t)h_{21}R_2\dot{x}(t)-\int_{t-h_1}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds\\
&-\int_{t-h_2}^{t-h_1}\dot{x}^T(s)R_2\dot{x}(s)ds\\
\le{}& x^T(t)(-C^TP-PC)x(t)\\
&+2x^T(t)P[Ag(x(t))+Bg(x(t-h(t)))]\\
&+2[g^T(x(t))-x^T(t)\textstyle\sum_1]\Lambda[-Cx(t)+Ag(x(t))\\
&+Bg(x(t-h(t)))]\\
&+2[x^T(t)\Sigma_2-g^T(x(t))]K[-Cx(t)+Ag(x(t))\\
&+Bg(x(t-h(t)))]\\
&+x^T(t)(Q_1+Q_2+Q_3)x(t)-x^T(t-h_1)Q_1x(t-h_1)\\
&-x^T(t-h_2)Q_2x(t-h_2)\\
&-x^T(t-h(t))(1-h_d)Q_3x(t-h(t))\\
&+\dot{x}^T(t)[h_1R_1+h_{21}R_2]\dot{x}(t)-\int_{t-h_1}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds\\
&-\int_{t-h_2}^{t-h_1}\dot{x}^T(s)R_2\dot{x}(s)ds. \qquad (14)
\end{aligned}
$$

Alternatively, the following equations are true:

$$
\begin{aligned}
&-\int_{t-h_1}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds-\int_{t-h_2}^{t-h_1}\dot{x}^T(s)R_2\dot{x}(s)ds\\
&=-\int_{t-\delta}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds-\int_{t-h(t)}^{t-\delta}\dot{x}^T(s)R_2\dot{x}(s)ds\\
&\quad-\int_{t-h_2}^{t-h(t)}\dot{x}^T(s)R_2\dot{x}(s)ds\\
&=-\int_{t-h_1}^{t}\dot{x}^T(s)(R_1-X_{33})\dot{x}(s)ds\\
&\quad-\int_{t-h_2}^{t-h_1}\dot{x}^T(s)(R_2-Y_{33})\dot{x}(s)ds\\
&\quad-\int_{t-h_1}^{t}\dot{x}^T(s)X_{33}\dot{x}(s)ds-\int_{t-h_2}^{t-h_1}\dot{x}^T(s)Y_{33}\dot{x}(s)ds\\
&=-\int_{t-h_1}^{t}\dot{x}^T(s)(R_1-X_{33})\dot{x}(s)ds\\
&\quad-\int_{t-h_2}^{t-h_1}\dot{x}^T(s)(R_2-Y_{33})\dot{x}(s)ds\\
&\quad-\int_{t-h_1}^{t}\dot{x}^T(s)X_{33}\dot{x}(s)ds-\int_{t-h_2}^{t-h(t)}\dot{x}^T(s)Y_{33}\dot{x}(s)ds\\
&\quad-\int_{t-h(t)}^{t-h_1}\dot{x}^T(s)Y_{33}\dot{x}(s)ds. \qquad (15)
\end{aligned}
$$

By utilizing Lemma 1 and the Leibniz-Newton formula, we have

$$
\begin{aligned}
&-\int_{t-h_1}^{t}\dot{x}^T(s)X_{33}\dot{x}(s)ds\\
&\le x^T(t)[h_1X_{11}+X_{13}+X_{13}^T]x(t)\\
&\quad+x^T(t)[h_1X_{12}-X_{13}+X_{23}^T]x(t-h_1)\\
&\quad+x^T(t-h_1)[h_1X_{12}^T-X_{13}^T+X_{23}]x(t) \qquad (16)\\
&\quad+x^T(t-h_1)[h_1X_{22}-X_{23}-X_{23}^T]x(t-h_1).
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
&-\int_{t-h(t)}^{t-h_1}\dot{x}^T(s)Y_{33}\dot{x}(s)ds\\
&\le x^T(t-h_1)[h_{21}Y_{11}+Y_{13}^T+Y_{13}]x(t-h_1)\\
&\quad+x^T(t-h_1)[h_{21}Y_{12}-Y_{13}+Y_{23}^T]x(t-h(t)) \qquad (17)\\
&\quad+x^T(t-h(t))[h_{21}Y_{12}^T-Y_{13}^T+Y_{23}]x(t-h_1)\\
&\quad+x^T(t-h(t))[h_{21}Y_{22}-Y_{23}-Y_{23}^T]x(t-h(t)).
\end{aligned}
$$

and

$$
\begin{aligned}
&-\int_{t-h_2}^{t-h(t)}\dot{x}^T(s)Y_{33}\dot{x}(s)ds\\
&\le x^T(t-h(t))[h_{21}Y_{11}+Y_{13}+Y_{13}^T]x(t-h(t))\\
&\quad+x^T(t-h(t))[h_{21}Y_{12}-Y_{13}+Y_{23}^T]x(t-h_2) \qquad (18)\\
&\quad+x^T(t-h_2)[h_{21}Y_{12}^T-Y_{13}^T+Y_{23}]x(t-h(t))\\
&\quad+x^T(t-h_2)[h_{21}Y_{22}-Y_{23}-Y_{23}^T]x(t-h_2).
\end{aligned}
$$

With the operator for the term $\dot{x}^T(t)[h_1R_1+h_{21}R_2]\dot{x}(t)$ as follows:

$$
\begin{aligned}
&\dot{x}^T(t)[h_1R_1+h_{21}R_2]\dot{x}(t)\\
&=[-Cx(t)+Ag(x(t))+Bg(x(t-h(t)))]^T\\
&\quad\times[h_1R_1+h_{21}R_2]\\
&\quad\times[-Cx(t)+Ag(x(t))+Bg(x(t-h(t)))]\\
&=x^T(t)C^T[h_1R_1+h_{21}R_2]Cx(t)-x^T(t)C^T\\
&\quad\times[h_1R_1+h_{21}R_2]Ag(x(t))\\
&\quad-x^T(t)C^T[h_1R_1+h_{21}R_2]Bg(x(t-h(t))) \qquad (19)\\
&\quad-g^T(x(t))A^T[h_1R_1+h_{21}R_2]Cx(t)\\
&\quad+g^T(x(t))A^T[h_1R_1+h_{21}R_2]Ag(x(t))\\
&\quad+g^T(x(t))A^T[h_1R_1+h_{21}R_2]Bg(x(t-h(t)))\\
&\quad-g^T(x(t-h(t)))A^T[h_1R_1+h_{21}R_2]Cx(t)\\
&\quad+g^T(x(t-h(t)))A^T[h_1R_1+h_{21}R_2]Ag(x(t))\\
&\quad+g^T(x(t-h(t)))A^T[h_1R_1+h_{21}R_2]Bg(x(t-h(t))).
\end{aligned}
$$

By (5), it can be verified that

$$
\begin{aligned}
&-2g^T(x(t))T_1g(x(t))+2x^T(t)T_1(\Sigma_1+\Sigma_2)g(x(t))\\
&\qquad-2x^T(t)\Sigma_2T_1\Sigma_1x(t)\ge 0. \qquad (20)
\end{aligned}
$$

Similarly, there holds

$$
\begin{aligned}
&-2g^T(x(t-h(t)))T_2g(x(t-h(t)))\\
&\qquad+2x^T(t-h(t))T_2(\Sigma_1+\Sigma_2)g(x(t-h(t))) \qquad (21)\\
&\qquad-2x^T(t-h(t))\Sigma_2T_2\Sigma_1x(t-h(t))\ge 0.
\end{aligned}
$$

Substituting (15)-(21) into (14), we obtain

$$\dot{V}(x_t) \leq \xi^T(t)\Omega\xi(t) - \int_{t-h_1}^{t} \dot{x}^T(s)(R_1 - X_{33})\dot{x}(s)ds - \int_{t-h_2}^{t-h_1} \dot{x}^T(s)(R_2 - Y_{33})\dot{x}(s)ds, \tag{22}$$

where

$$\xi^T(t) = [x^T(t) \quad g^T(x(t)) \quad g^T(x(t-h(t))) \quad x^T(t-h(t)) \quad x^T(t-h_2) \quad x^T(t-h_1)]$$

and

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & 0 & \Omega_{15} & 0 \\ \Omega_{12}^T & \Omega_{22} & \Omega_{23} & 0 & 0 & 0 \\ \Omega_{13}^T & \Omega_{23}^T & \Omega_{33} & \Omega_{34} & 0 & 0 \\ 0 & 0 & \Omega_{34}^T & \Omega_{44} & \Omega_{45} & \Omega_{46} \\ \Omega_{15}^T & 0 & 0 & \Omega_{45}^T & \Omega_{55} & 0 \\ 0 & 0 & 0 & \Omega_{46}^T & 0 & \Omega_{66} \end{bmatrix}$$

with

$$\begin{aligned}
\Omega_{11} &= -C^T P - PC + \Sigma_1 \Lambda C + C^T \Lambda \Sigma_1^T \\
&\quad - \Sigma_2 KC - C^T K^T \Sigma_2^T + Q_1 + Q_2 + Q_3 \\
&\quad + h_1 X_{11} + X_{13} + X_{13}^T + C^T(h_1 R_1 + h_{21} R_2)C, \\
\Omega_{12} &= PA - \Sigma_1 \Lambda A - C^T \Lambda + \Sigma_2 KA + C^T K^T \\
&\quad + T_1(\Sigma_1 + \Sigma_2) - C^T(h_1 R_1 + h_{21} R_2)\mathrm{A}, \\
\Omega_{13} &= PB - \Sigma_1 \Lambda B + \Sigma_2 KB - C^T(h_1 R_1 + h_{21} R_2)B, \\
\Omega_{15} &= h_1 X_{12} - X_{13} + X_{23}^T, \\
\Omega_{22} &= \Lambda K - KA - A^T K^T - 2T_1 + A^T(h_1 R_1 + h_{21} R_2)A, \\
\Omega_{23} &= \Lambda B + A^T \Lambda - KB + A^T(h_1 R_1 + h_{21} R_2)B, \\
\Omega_{33} &= -2T_2 + B^T(h_1 R_1 + h_{21} R_2)B, \\
\Omega_{34} &= (\Sigma_1 + \Sigma_2)T_2, \quad \Omega_{37} = B^T(h_1 R_1 + h_{21} R_2), \\
\Omega_{44} &= -(1-h_d)Q_3 + h_{21} Y_{22} - Y_{23} - Y_{23}^T + h_{21} Y_{11} \\
&\quad + Y_{13} + Y_{13}^T - 2\Sigma_2 T_2 \Sigma_1, \\
\Omega_{45} &= h_{21} Y_{12}^T - Y_{13}^T + Y_{23}, \quad \Omega_{46} = h_{21} Y_{12} - Y_{13} + Y_{23}^T, \\
\Omega_{55} &= -Q_2 + h_{21} X_{22} - X_{23} - X_{23}^T, \\
\Omega_{66} &= -Q_1 + h_{21} Y_{22} - Y_{23} - Y_{23}^T.
\end{aligned}$$

Finally, using Schur complements, with some effort we show that (22) guarantees of $\dot{V}(x_t) < 0$. It is clear that if $\Omega < 0$, $R_1 - X_{33} \geq 0$, $R_2 - Y_{33} \geq 0$, then, $\dot{V}(x_t) < 0$ for any $\xi(t) \neq 0$. So the neural networks with interval time-varying delays nominal system (6) is asymptotically stable if linear matrix inequalities (12) are true. This completes the proof.

With time-varying delay $h(t)$ not differentiable or information on time derivative of delay is unknown, by eliminating Q_3 in Theorem 1, one can easily get the following corollary for Case 2.

Corollary 1: For given scalars $\Sigma_2 = diag\{\sigma_1, \sigma_2, ..., \sigma_n\}$, $\Sigma_1 = diag\{\gamma_1, \gamma_2, ..., \gamma_n\}$, h_1, h_2, nominal system (6) $(\Delta C(t) = \Delta A(t) = \Delta B(t) = 0)$ subject to (3) is asymptotically stable if there exist $P = P^T > 0$, $Q_1 = Q_1^T > 0$, $Q_2 = Q_2^T > 0$, $R_1 = R_1^T > 0$, $R_2 = R_2^T > 0$ diagonal matrices $T_1 \geq 0$, $T_2 \geq 0$, $\Lambda = diag\{\lambda_1, \lambda_2, ..., \lambda_n\}$, $K = diag\{k_1, k_2, ..., k_n\}$,

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix} \geq 0, \; Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{12}^T & Y_{22} & Y_{23} \\ Y_{13}^T & Y_{23}^T & Y_{33} \end{bmatrix} \geq 0$$

such that the following LMIs hold:

$$\bar{\Xi} = \begin{bmatrix} \bar{\Xi}_{11} & \Xi_{12} & \Xi_{13} & 0 & \Xi_{15} & 0 & \Xi_{17} \\ \Xi_{12}^T & \Xi_{22} & \Xi_{23} & 0 & 0 & 0 & \Xi_{27} \\ \Xi_{13}^T & \Xi_{23}^T & \Xi_{33} & \Xi_{34} & 0 & 0 & \Xi_{37} \\ 0 & 0 & \Xi_{34}^T & \bar{\Xi}_{44} & \Xi_{45} & \Xi_{46} & 0 \\ \Xi_{15}^T & 0 & 0 & \Xi_{45}^T & \Xi_{55} & 0 & 0 \\ 0 & 0 & 0 & \Xi_{46}^T & 0 & \Xi_{66} & 0 \\ \Xi_{17}^T & \Xi_{27}^T & \Xi_{37}^T & 0 & 0 & 0 & \Xi_{77} \end{bmatrix} < 0, \tag{23a}$$

and

$$R_1 - X_{33} \geq 0, \quad R_2 - Y_{33} \geq 0, \tag{23b}$$

where $\Xi_{ij}\,(i, j = 1, 2, ..., 7; i < j \leq 7)$ are defined in (12) and

$$\begin{aligned}
\bar{\Xi}_{11} &= -C^T P - PC + \Sigma_1 \Lambda C + C^T \Lambda \Sigma_1^T - \Sigma_2 KC \\
&\quad - C^T K^T \Sigma_2^T + Q_1 + Q_2 + h_1 X_{11} + X_{13} + X_{13}^T, \\
\bar{\Xi}_{44} &= h_{21} Y_{22} - Y_{23} - Y_{23}^T + h_{21} Y_{11} + Y_{13} + Y_{13}^T - 2\Sigma_2 T_2 \Sigma_1.
\end{aligned}$$

Proof: If the matrix $Q_3 = 0$ is selected in (13). This proof can be completed in a similar formulation to Theorem 1.

When $h_1 = 0$, Theorem 1 reduces to the following Corollary 2.

Corollary 2. For given scalars $\Sigma_2 = diag\{\sigma_1, \sigma_2, ..., \sigma_n\}$, $\Sigma_1 = diag\{\gamma_1, \gamma_2, ..., \gamma_n\}$, h, h_d, nominal system (6) subject to (2) is asymptotically stable if there exist $P = P^T > 0$, $Q = Q^T > 0$, $R = R^T > 0$, diagonal matrices $T_1 \geq 0$, $T_2 \geq 0$, $\Lambda = diag\{\lambda_1, \lambda_2, ..., \lambda_n\}$, $K = diag\{k_1, k_2, ..., k_n\}$,

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix} \geq 0,$$

such that the following LMIs hold for

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} & \Psi_{15} \\ \Psi_{12}^T & \Psi_{22} & \Psi_{23} & 0 & \Psi_{25} \\ \Psi_{13}^T & \Psi_{23}^T & \Psi_{33} & \Psi_{34} & \Psi_{35} \\ \Psi_{14}^T & 0 & \Psi_{34}^T & \Psi_{44} & 0 \\ \Psi_{15}^T & \Psi_{25}^T & \Psi_{35}^T & 0 & \Psi_{55} \end{bmatrix} < 0 \tag{24a}$$

and

$$R - X_{33} \geq 0, \tag{24b}$$

where

$$\Psi_{11} = -C^T P - PC + \Sigma_1 \Lambda C + C^T \Lambda \Sigma_1^T - \Sigma_2 KC - C^T K^T \Sigma_2^T + Q + hX_{11} + X_{13} + X_{13}^T,$$

$$\Psi_{12} = PA - \Sigma_1 \Lambda A - C^T \Lambda + \Sigma_2 KA + C^T K^T + T_1(\Sigma_1 + \Sigma_2),$$

$$\Psi_{13} = PB - \Sigma_1 \Lambda B + \Sigma_2 KB, \quad \Psi_{14} = h_1 X_{12} - X_{13} + X_{23}^T,$$

$$\Psi_{15} = -hC^T \mathrm{R}, \quad \Psi_{22} = \Lambda K - KA - A^T K^T - 2T_1,$$

$$\Psi_{23} = \Lambda B + A^T \Lambda - KB, \quad \Psi_{25} = hA^T R,$$

$$\Psi_{33} = -2T_2, \quad \Psi_{34} = (\Sigma_1 + \Sigma_2)T_2, \quad \Psi_{35} = hB^T R,$$

$$\Xi_{44} = -(1-h_d)Q + hX_{22} - X_{23} - X_{23}^T - 2\Sigma_2 T_2 \Sigma_1,$$

$$\Psi_{55} = -hR.$$

Proof: Choose the following new Lyapunov-Krasovskii functional candidate to be

$$V(t) = V_1(t) + V_2(t) + V_3(t), \tag{25}$$

where

$$V_1(t)) = x^T(t)Px(t) + 2\sum_{i=1}^{n} \{\lambda_i \int_0^{x_i(t)} (g_i(s) - r_i(s))ds + \int_0^{x_i(t)} k_i(\sigma_i(s) - g_i(s))ds\},$$

$$V_2(t) = \int_{t-h(t)}^{t} x^T(s)Qx(s)ds,$$

$$V_3(t) = \int_{-h}^{0} \int_{t+\theta}^{t} \dot{x}^T(s)R\dot{x}(s)dsd\theta.$$

Then, taking the time derivative of $V(t)$ with respect to t along the nominal system (6) yield

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t). \tag{26}$$

Then proof follows a linear similar to proof of Theorem 1 and thus is omitted here.

3. ROBUST STABILITY

Based on Theorem 1, we have the following result for neural networks with interval time-varying delays system (6) with parameter uncertainties:

Theorem 2: For given scalars $\Sigma_2 = diag\{\sigma_1, \sigma_2, ..., \sigma_n\}$, $\Sigma_1 = diag\{\gamma_1, \gamma_2, ..., \gamma_n\}$, h_1, h_2, h_d, system (6) subject to (2) is asymptotically stable if there exist $P = P^T > 0$, $Q_1 = Q_1^T > 0$, $Q_2 = Q_2^T > 0$, $Q_3 = Q_3^T > 0$, $R_1 = R_1^T > 0$, $R_2 = R_2^T > 0$, $\varepsilon > 0$, diagonal matrices $T_1 \geq 0$, $T_2 \geq 0$, $\Lambda = diag\{\lambda_1, \lambda_2, ..., \lambda_n\}$, $K = diag\{k_1, k_2, ..., k_n\}$, and positive semi-definite matrices

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix} \geq 0, \; Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{12}^T & Y_{22} & Y_{23} \\ Y_{13}^T & Y_{23}^T & Y_{33} \end{bmatrix} \geq 0$$

such that the following LMIs hold:

$$\bar{\bar{\Xi}} = \begin{bmatrix} \bar{\bar{\Xi}}_{11} & \bar{\bar{\Xi}}_{12} & \bar{\bar{\Xi}}_{13} & 0 & \Xi_{15} & 0 & \Xi_{17} & \Xi_{18} \\ \bar{\bar{\Xi}}_{12}^T & \bar{\bar{\Xi}}_{22} & \bar{\bar{\Xi}}_{23} & 0 & 0 & 0 & \Xi_{27} & 0 \\ \bar{\bar{\Xi}}_{13}^T & \bar{\bar{\Xi}}_{23}^T & \bar{\bar{\Xi}}_{33} & \Xi_{34} & 0 & 0 & \Xi_{37} & 0 \\ 0 & 0 & \Xi_{34}^T & \Xi_{44} & \Xi_{45} & \Xi_{46} & 0 & 0 \\ \Xi_{15}^T & 0 & 0 & \Xi_{45}^T & \Xi_{55} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Xi_{46}^T & 0 & \Xi_{66} & 0 & 0 \\ \Xi_{17}^T & \Xi_{17}^T & \Xi_{37}^T & 0 & 0 & 0 & \Xi_{77} & \Xi_{78} \\ \Xi_{18}^T & 0 & 0 & 0 & 0 & 0 & \Xi_{78}^T & \Xi_{88} \end{bmatrix} < 0, \tag{27a}$$

and

$$R_1 - X_{33} \geq 0, \quad R_2 - Y_{33} \geq 0, \tag{27b}$$

where $\Xi_{ij} (i, j = 1, 2, ..., 7, i < j \leq 7)$ are defined in (12) and

$$\bar{\bar{\Xi}}_{11} = \Xi_{11} + \varepsilon E_c^T E_c, \quad \bar{\bar{\Xi}}_{12} = \Xi_{12} - \varepsilon E_c^T E_a,$$

$$\bar{\bar{\Xi}}_{13} = \Xi_{13} - \varepsilon E_c^T E_b, \quad \bar{\bar{\Xi}}_{22} = \Xi_{22} + \varepsilon E_a^T E_a,$$

$$\bar{\bar{\Xi}}_{23} = \Xi_{23} + \varepsilon E_a^T E_b, \quad \bar{\bar{\Xi}}_{33} = \Xi_{33} + \varepsilon E_b^T E_b,$$

$$\Xi_{18} = PD, \quad \Xi_{78} = h_1 R_1 D + h_{21} R_2 D, \quad \Xi_{88} = -\varepsilon I.$$

Proof: Replacing A, B, and C in (12) with $A + DF(t)E_a$, $B + DF(t)E_b$, and $C + DF(t)E_c$, respectively, we apply Lemma 3 for system (6) is equivalent to the following condition:

$$\Xi + \Gamma_d F(t) \Gamma_e + \Gamma_e^T F(t) \Gamma_d < 0, \tag{28}$$

where

$$\Gamma_d = [PD \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad h_1 R_1 D + h_{21} R_2 D]$$

and $\Gamma_e = [E_c \quad E_a \quad E_b \quad 0 \quad 0 \quad 0 \quad 0]$.

According to Lemma 3, (28) is true if there exist a scalar $\varepsilon > 0$ such that the following inequality holds

$$\Xi + \varepsilon^{-1} \Gamma_d^T \Gamma_d + \varepsilon \Gamma_e^T \Gamma_e < 0. \tag{29}$$

Schur complement shows (28) equivalent to (28a), completing the proof.

With time-varying delay $h(t)$ not differentiable or information on time derivative of delay is unknown, by eliminating Q_3 in Theorem 2, one easily gets the following corollary for Case 2.

Corollary 3: For given scalars $\Sigma_2 = diag\{\sigma_1, \sigma_2, ..., \sigma_n\}$, $\Sigma_1 = diag\{\gamma_1, \gamma_2, ..., \gamma_n\}$, h_1, h_2, h_d, system (6) subject to (3) is asymptotically stable if there exist $P = P^T > 0$, $Q_1 = Q_1^T > 0$, $Q_2 = Q_2^T > 0$, $R_1 = R_1^T > 0$, $R_2 = R_2^T > 0$, $\varepsilon > 0$, diagonal matrices $T_1 \geq 0$, $T_2 \geq 0$, $\Lambda = diag\{\lambda_1, \lambda_2, ..., \lambda_n\}$, $K = diag\{k_1, k_2, ..., k_n\}$,

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix} \geq 0, \; Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{12}^T & Y_{22} & Y_{23} \\ Y_{13}^T & Y_{23}^T & Y_{33} \end{bmatrix} \geq 0$$

such that the following LMIs hold:

$$\hat{\Xi} = \begin{bmatrix} \hat{\Xi}_{11} & \bar{\bar{\Xi}}_{12} & \bar{\bar{\Xi}}_{13} & 0 & \Xi_{15} & 0 & \Xi_{17} & \Xi_{18} \\ \bar{\bar{\Xi}}_{12}^T & \bar{\bar{\Xi}}_{22} & \bar{\bar{\Xi}}_{23} & 0 & 0 & 0 & \Xi_{27} & 0 \\ \bar{\bar{\Xi}}_{13}^T & \bar{\bar{\Xi}}_{23}^T & \bar{\bar{\Xi}}_{33} & \Xi_{34} & 0 & 0 & \Xi_{37} & 0 \\ 0 & 0 & \Xi_{34}^T & \bar{\bar{\Xi}}_{44} & \Xi_{45} & \Xi_{46} & 0 & 0 \\ \Xi_{15}^T & 0 & 0 & \Xi_{45}^T & \Xi_{55} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Xi_{46}^T & 0 & \Xi_{66} & 0 & 0 \\ \Xi_{17}^T & \Xi_{17}^T & \Xi_{37}^T & 0 & 0 & 0 & \Xi_{77} & \Xi_{78} \\ \Xi_{18}^T & 0 & 0 & 0 & 0 & 0 & \Xi_{78}^T & \Xi_{88} \end{bmatrix} < 0, \quad (30a)$$

and

$$R_2 - Y_{33} \geq 0, \quad R_1 - X_{33} \geq 0, \quad (30b)$$

where $\Xi_{ij}\,(i,j=1,2,\ldots,8, i<j\leq 8)$ are defined in (12), (23),(27) and $\hat{\Xi}_{11} = \bar{\bar{\Xi}}_{11} + \varepsilon E_c^T E_c$,

$$\bar{\bar{\Xi}}_{44} = h_{21}Y_{22} - Y_{23} - Y_{23}^T + h_{21}Y_{11} + Y_{13} + Y_{13}^T - 2\Sigma_2 T_2 \Sigma_1.$$

Proof: If the matrix $Q_3 = 0$ is selected in (13). This proof can be completed in a similar formulation to Theorem 2.

Remark 1: It is interesting to note that h_1, h_2 appears linearly in (12) and (27). Thus a generalized eigenvalue problem (GEVP) as defined in Boyd, *et al.* [1] can be formulated to solve minimum acceptable $1/h_1$ (or $1/h_2$) and thus maximum h_1 (or h_2) to maintain robust stability as judged by these conditions. Our optimization problem becomes a standard generalized eigenvalue problem, then which can be solved using GEVP technique. From this discussion, we have the following Remark 2.

Remark 2: Theorem 1 provides robust delay-dependent asymptotic stability criteria for neural network systems with interval time-varying delay (6) in terms of solvability of LMIs [1]. Based on them, we can obtain maximum allowable delay bound (MADB) $h_2 (0 \leq h_1 \leq h(t) \leq h_2)$ such that (6) is stable by solving a convex optimization problem

$$\begin{cases} \text{Maximize} & h_2 \\ \text{Subject to} & \Xi < 0 \text{ (or } \bar{\bar{\Xi}} < 0) \\ & \text{and } R_1 - X_{33} \geq 0 \text{ and } R_2 - Y_{33} \geq 0. \end{cases} \quad (31)$$

Inequality (31) is a convex optimization problem, obtained efficiently by using the MATLAB LMI Toolbox.

4. ILLUSTRATIVE EXAMPLES

To illustrate the usefulness of our results, this section will provide four numerical examples. It will be shown that proposed results prove less conservative than recent ones [3-11,13-15,17-22]. It is worth pointing out that our criteria proved more efficient for computation.

Example 1: Consider a delayed recurrent neural network with parameters as follows:

$$\dot{x}(t) = -Cx(t) + Ag(x(t)) + Bg(x(t-h(t))), \quad (32)$$

where

$$C = \begin{bmatrix} 1.2769 & 0 & 0 & 0 \\ 0 & 0.6321 & 0 & 0 \\ 0 & 0 & 0.9230 & 0 \\ 0 & 0 & 0 & 0.4480 \end{bmatrix},$$

$$A = \begin{bmatrix} -0.0370 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{bmatrix},$$

$\Sigma_2 = \mathrm{diag}(0.1137, 0.1279, 0.7994, 0.2368)$, and
$\Sigma_1 = \mathrm{diag}(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$.

Solution: For $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 0$ in the case of $h_1 = 0$, this example was also studied in [5-7,11,13,15,19-22]; corresponding results appear in Table 1. For comparison, the obtained MADB by Corollary 2 are listed. Table 1 also provides comparison of the number of variables involved in Theorem 1, and in some existing results. Apparently, Theorem 1 can provide larger MADB than [5-7,11,13,15,19-22]. It is assumed that $\dot{h}(t) \leq h_d$. The corresponding upper bounds on h_2 for various h_d and h_1 calculated by Theorem 1 are listed in Table 2 compared with those in [3,7,10,17]. Table 2 also lists results for unknown h_d, where "−" means that the result is not applicable to the corresponding case, and "unknown h_d" means that h_d can be arbitrary values, even h_d is very large, or $h(t)$ is not differentiable. Results obtained by Theorem 1 in this paper are visibly less conservative than those in [3,7,10,17]. For this system, when $h_1 = 3$, MADB obtained by methods of [8,11,14] for various conditions of h_d are listed in Table 3, where one can see that our results improve the existing results both theoretically and numerically.

Table 1. Comparisons MADB $h_2 (h_1 = 0)$ for Example 1 (N_V Numbers of variables).

Method	$h_d = 0.1$	$h_d = 0.5$	$h_d = 0.9$	≥ 1	N_V
[5]	3.2775	2.1502	1.3164	1.2598	14
[6]	3.2793	2.2245	1.5847	1.5444	10
[7]	3.3039	2.5376	2.0853	2.0389	15
[13]	3.2819	2.2261	1.6035	1.5593	14
[21]	3.4183	2.5943	2.1306	2.0770	13
[22]	3.4058	2.4183	1.9007	1.8571	10
[19]	3.3574	2.5915	2.1306	2.0779	19
[20]	3.3981	2.6711	2.1783	-	22
[11]	4.0071	3.3960	3.3033	3.2827	21
[15]	7.6568	6.5690	3.0366	3.1626	13
This paper	8.8759	6.9994	3.8712	3.7811	8

Table 2. Upper bounds on h_2 for various h_1 and h_d.

h_1	Methods	h_d=0.5	h_d=0.9	unknown h_d
1	[7]	2.5802	2.2736	2.2393
	[17]	1.8832	1.7657	1.7651
	[10]	2.2958	1.9512	1.9224
	[3]	2.6869	2.3924	2.3540
	This paper	2.9062	2.4110	2.4104
2	[7]	2.7500	2.6468	2.6299
	[17]	2.4340	2.4003	2.4001
	[10]	2.5778	2.4849	2.4712
	[3]	2.8475	2.7375	2.7190
	This paper	3.9633	3.1990	2.9964
3	[7]	3.1733	3.1155	3.1042
	[17]	3.0956	3.0682	3.0671
	[10]	3.1321	3.0872	3.0786
	[3]	3.2429	3.1827	3.1711
	This paper	3.9587	3.9539	3.9537

Table 3. Delay bounds h_2 with $h_1=3$ and different h_d (Example 1)

h_d	0.1	0.5	0.9	$h_d\geq 1$
[8]	3.71	3.36	3.29	3.28
[14]	3.78	3.45	3.39	3.38
[11]	4.0130	3.4470	3.3403	3.3196
This paper	4.8805	3.9587	3.9539	3.9537

For $\gamma_1=0.1$, $\gamma_2=\gamma_3=0$, $\gamma_4=-0.2$, $h_1=0$, the MADB h that guarantees delayed NNs asymptotically stable is calculated as 4.3007. Using Theorem 1 in [12], we calculate allowable delay bound $h=3.3668$, is smaller than the result obtained by our methods. Therefore, our method is less conservative to some degree than that in [12]. By the preceding comparison, we can say that results in this brief are much better than those in [3,5-8,10-15,19-22] for system (32).

Example 2: Consider a delayed recurrent neural network with parameters as follows:

$$\dot{x}(t)=-Cx(t)+Ag(x(t))+Bg(x(t-h(t))), \tag{33}$$

where

$$C=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad A=\begin{bmatrix}-1&0.5\\0.5&-1.5\end{bmatrix},\quad B=\begin{bmatrix}-2&0.5\\0.5&-2\end{bmatrix}.$$

Neuron activation functions are assumed to satisfy Assumption 1 with $\Sigma_1=\mathrm{diag}\{0,0\}$, $\Sigma_2=diag\{0.4,0.8\}$.

Solution: Objective is to compute upper bound of h_2 for various h_1 and h_d. Table 4 lists compared results. It can be seen that the method proposed in this paper yields less conservative results than those in literature [3,7,10, 17].

Example 3: Consider a delayed recurrent neural network with parameters as follows:

$$\dot{x}(t)=-Cx(t)+Ag(x(t))+Bg(x(t-h(t))), \tag{34}$$

where

Table 4. Upper bounds on h_2 for various h_1 and h_d.

h_1	Method	h_d=0.8	h_d=0.9	unknown h_d
0.5	[7]	1.3566	1.1689	1.0263
	[17]	0.8262	0.8215	0.8183
	[10]	1.1217	0.9984	0.9037
	[3]	1.3600	1.1786	1.0437
	This paper	1.4508	1.4042	1.0862
0.75	[7]	1.3856	1.2110	1.0803
	[17]	0.9669	0.9625	0.9592
	[10]	1.2213 2	1.1021	1.0102
	[3]	1.3990	1.2241	1.0972
	This paper	1.4891	1.4789	1.1838
1	[7]	1.4578	1.2887	1.1641
	[17]	1.1152	1.1108	1.1075
	[10]	1.3432	1.2238	1.1318
	[3]	1.4692	1.2948	1.1774
	This paper	1.6892	1.6880	1.4000

Table 5. Upper bounds on h_2 for various h_1 and h_d.

h_1	Methods	h_d=0.8	h_d=0.9	unknown h_d
0	[4,7,9]	1.2281	0.8636	0.8298
	[6]	1.6831	1.1493	1.0880
	[7]	2.3534	1.6050	1.5103
	[15]	4.1626	3.9766	3.1690
	This paper	4.5433	4.3465	3.1795
1	[7]	3.2575	2.4769	2.3606
	[11]	4.8668	3.8047	3.6001
	This paper	5.7083	3.9192	3.6002
2	[7]	4.2552	3.4769	3.3606
	This paper	8.8559	4.0114	3.7611
100	[7]	102.2552	101.4769	101.3606
	[11]	103.8101	102.6887	102.2975
	This paper	106.0860	104.9996	102.3211

$$C=\begin{bmatrix}2&0\\0&2\end{bmatrix},\quad A=\begin{bmatrix}1&1\\-1&-1\end{bmatrix},\quad B=\begin{bmatrix}0.88&1\\1&1\end{bmatrix}.$$

Neuron activation functions are assumed to satisfy Assumption 1 with $\Sigma_1=\mathrm{diag}\{0,0\}$, $\Sigma_2=diag\{0.4,0.8\}$.

Solution: Table 5 lists corresponding upper bounds of h_2 for various h_1 and h_d derived Theorem 1 and methods in [4-7,9,15]. On the other hand, the previous results cannot handle the case for $h_2\geq 1$ with $h_d\geq 0.8$. However, it is seen that the calculated upper bounds of h_2 increases with h_1 even for $h_1=100$. Therefore, we can say that results in this paper are highly effective and less conservative than those in [7,11].

Example 4: Consider the following neural networks with interval time-varying delays and parameter uncertainties:

$$C=\begin{bmatrix}2&0\\0&2\end{bmatrix},\quad A=\begin{bmatrix}0.7&0.8\\-0.5&0.3\end{bmatrix},\quad B=\begin{bmatrix}0.2&0.2\\-0.6&0.1\end{bmatrix},$$

$$D=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad E_c=\begin{bmatrix}-0.2&0.1\\0.1&-0.2\end{bmatrix},\quad E_a=\begin{bmatrix}0.1&0.6\\0.6&-0.3\end{bmatrix},$$

$$E_b=\begin{bmatrix}-0.1&0.4\\0.4&-0.1\end{bmatrix},\quad h_d \text{ is unknown.}$$

Table 6. Allowable upper bound of h_2 with given h_1.

Methods	$h_1 = 0$	$h_1 = 0.1$	$h_1 = 0.5$	$h_1 = 0.8$
Qiu *et al*. [17]	2.082	2.182	2.582	2.882
Wu *et al*. [18]	3.883	3.905	4.139	4.333
This paper	3.8913	4.9186	4.2172	4.5112

Neuron activation functions are assumed to satisfy Assumption 1 with $\Sigma_1 = \text{diag}\{0,0\}$, $\Sigma_2 = diag\{0.5, 0.5\}$.

Solution: Under different levels of upper bounds of time delay, Table 6 lists results of the maximum allowable delay bounds. It is seen from Table 6 that results obtained by our method are less conservative than those obtained from [17,18]. We can say that results in this paper are highly effective and less conservative than those in [17,18].

5. CONCLUSION

This paper assesses robust delay-range-dependent stability problem of NNs with interval time-varying delay. By defining appropriate Lyapunov functional, new delay-dependent stability criteria are derived in terms of LMIs. Newly obtained results are less conservative, less computationally complex, and more applicable than existing ones. Numerical examples are given to illustrate effectiveness of presented criteria and their improvement over existing results.

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