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Quantum Fields on dS and AdS

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Abstract. We give a short account of general approach to de Sitter and anti–de Sitter Quantum Field Theories that is based on the assumption of analyticity properties of the n-point correlation functions. We briefly discuss the status of the AdS/CFT and dS/CFT correspondences in this context.

1 Introduction

Einstein's cosmological equations in vacuo are easily solved. When the cosmological constant Λ is positive one gets the de Sitter (dS) universe while for negative Λ the solution is the so-called anti-de Sitter (AdS) universe. The two-sided limiting case $\Lambda = 0$ is the Minkowski spacetime. The Minkowski, de Sitter and anti-de Sitter universes belong therefore to a single one-parameter family of general relativistic spacetimes, the parameter being the cosmological constant, which, in turn, is proportional to the (constant) scalar curvature. While there are similarities between these spacetimes, the most important being their maximal symmetry, they are however very different either geometrically or in their physical meaning. Let us start by discussing the geometry. The simplest way to visualize the de Sitter manifold is to think of it as a quadric embedded in a $(d+1)$-dimensional Minkowski spacetime M_{d+1} (here indices are Lorentz tensor indices in the ambient space):

$$dS_d = \{{x^0}^2 - {x^1}^2 - \ldots - {x^d}^2 = -R^2\}. \tag{1}$$

The anti-de Sitter manifold can also be visualized as a quadric:

$$AdS_d = \{{x^0}^2 - {x^1}^2 - \ldots - {x^{d-1}}^2 + {x^d}^2 = R^2\} \tag{2}$$

embedded in a linear space E_{d+1} endowed with the metric $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, \ldots, -1, +1)$. The metric is induced on both dS_d and AdS_d by their respective ambient spaces:

$$ds^2 = \left[(dx^0)^2 - (dx^1)^2 - \ldots - (dx^d)^2\right]\Big|_{dS} \qquad \text{de Sitter}, \tag{3}$$

$$ds^2 = \left[(dx^0)^2 - (dx^1)^2 - \ldots - (dx^{d-1})^2 + (dx^d)^2\right]\Big|_{AdS} \quad \text{anti-de Sitter}\,. \tag{4}$$

These expressions provide solutions of the Einstein's equations with suitable values of the cosmological constant. The physical picture described by the de Sitter spacetime with the coordinate system plotted at the left side of Fig. 1 is that of a closed universe exponentially contracting and then re-expanding. The other coordinate

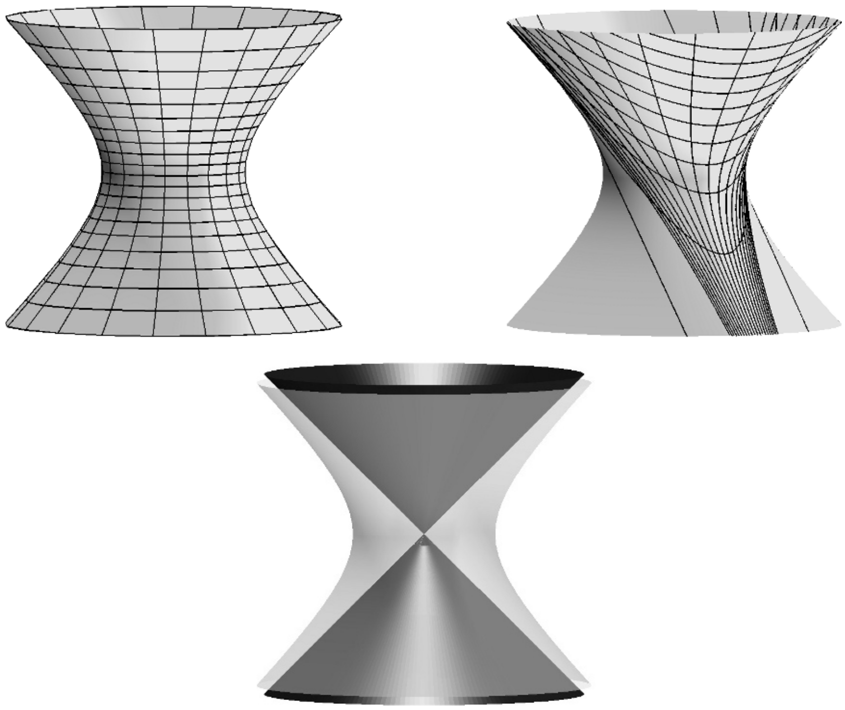

Figure 1: This figure represents the two-dimensional dS universe as an hyperboloid embedded in a three-dimensional space, with closed and flat coordinates. The asymptotic cone, that is the light cone of the origin in the ambient space, plays a major role in our constructions. Turning the figure by 90° the same figure represents also the AdS manifold with covering and Poincarè coordinates. The asymptotic cone in the ambient space can be regarded as a representation of the boundary at spacelike infinity of the AdS manifold and carries a natural action of the conformal group that is the basis for our treatment of the AdS-CFT correspondence.

systems describes one half of dS_d as an exponentially expanding flat universes. This interpretation lies at the basis of the central role that the de Sitter geometry plays in the inflationary paradigm, where the universe is thought to have undergone a phase of exponential expansion in the very early epochs of its life (see e.g. [1]). Recent observations [2] have also pointed towards an accelerated expansion of the universe and the existence of a positive cosmological constant (or of some form of "dark energy") now. The de Sitter physics may play again an important role. The anti-de Sitter universe has a completely different physical meaning. AdS_d is an example of a non-globally hyperbolic spacetime in that it admits closed timelike curves and has a boundary at spacelike infinity. The existence of that boundary is source of difficulties in quantizing fields on the AdS manifolds; on the other side it offers the geometrical basis for the famous AdS/CFT (Conformal Field Theory) correspondence [3].

2 Quantum Field Theory

Quantum fields are systems with infinitely many degrees of freedom; Von Neumann's theorem does not apply and there exist inequivalent “quantizations” (i.e. representations of the field algebra) of the same field theory. To illustrate this point let us briefly reconsider canonical quantization. The following remarks apply to general curved spacetimes but also in particular to the Minkowski manifold. Given a field equation, say the Klein-Gordon equation, one introduces a scalar product in the space of its solutions, and looks for a complete set of mode solutions $u_i(x)$. The field ϕ is then given the mode expansion $\phi(x) = \sum_i [a_i u_i(x) + a_i^\dagger \bar{u}_i(x)]$ and canonical quantization is achieved by assuming the canonical commutation relations and by choosing the corresponding vacuum. The procedure is inherently ambiguous: the previous mode expansion is based on an arbitrary choice of local coordinates. Different choices may produce inequivalent quantizations. Moreover, Bogolubov transformations also provide different vacua and again inequivalent quantizations. One needs therefore criteria to select, among the infinitely many inequivalent realizations of the (same) field algebra, those that are physically meaningful. This choice has striking consequences even in the case of free fields: the Hawking thermal radiation [4, 5], the Unruh effect [6] and the Gibbons and Hawking temperature [7] are the most famous examples.

Let us take now the viewpoint of the general theory of quantized fields. A theory is characterized by specifying the n-point ”vacuum” expectation values of the field (the word expresses here a mathematical concept, the GNS vacuum): $\mathcal{W}_n(x_1, \ldots, x_n) = \langle \Omega, \phi(x_1) \ldots \phi(x_n)\Omega\rangle$. In the Minkowskian case the general requirements of quantum mechanics and of special relativity are translated into mathematical properties of the n-point functions as follows (see [8] for a classic treatment of this subject):

1. Covariance: for any Poincaré transformation g

$$\mathcal{W}_n(gx_1, \ldots gx_n) = \mathcal{W}_n(x_1, \ldots x_n). \tag{5}$$

2. Local Commutativity: for x_j, x_{j+1} space-like separated

$$\mathcal{W}_n(x_1, \ldots x_j, x_{j+1}, \ldots, x_n) = \mathcal{W}_n(x_1, \ldots x_{j+1}, x_j, \ldots, x_n). \tag{6}$$

3. Positive Definiteness: this property warrants a quantum theoretical interpretation.

These properties are not sufficient to select among the vacua; an insight into the dynamical properties of the theory is needed. In Wightman's QFT one demands the existence of a complete set of states having positive or zero energy. This can

be reformulated as a property of the n-point functions by using their Fourier representation. By translation invariance $\mathcal{W}_n(x_1,\ldots,x_n) = W_n(\xi_1,\ldots,\xi_{n-1})$ where $\xi_i = x_i - x_{i+1}$, and the spectral condition becomes:
4. Spectral Property:

$$\mathrm{supp}\ \widetilde{W}_n(q_1,q_2,\ldots,q_{n-1}) \subset \overline{V^+}\times\ldots\times\overline{V^+} \tag{7}$$

where $\overline{V^+}$ is the closed forward light-cone (in momentum space).

At this point the crucial step is to introduce the complexification $M_d^{(c)}$ of the Minkowski spacetime; the spectral property then implies that the n-point function $\mathcal{W}_n(x_1,\ldots,x_n)$ is the distributional boundary value of a function $\mathrm{W}_n(z_1,\ldots,z_n)$ that is analytic in the tube

$$T_n^- = \{(z_1,\ldots,z_n)\in M^{(c)\times n} : \mathrm{Im}(z_{j+1}-z_j)\in\overline{V^+}\}. \tag{8}$$

(see [8]). To put our results in perspective it may be useful to review the role of the spectral condition within the usual Klein-Gordon QFT. A free field theory is characterized by the knowledge of the two-point function; in our case it must also solve the KG equation w.r.t. both variables:

$$(\Box_x + m^2)\mathcal{W}(x,y) = 0, \quad (\Box_y + m^2)\mathcal{W}(x,y) = 0. \tag{9}$$

By Fourier transforming the previous equations w.r.t. the difference variable $\xi = x-y$ one gets the KG equation in momentum space: $(p^2+m^2)\widetilde{W}(p) = 0$. The general solution can be written $\widetilde{W}(p) = a\,\theta(p^0)\delta(p^2-m^2) + b\,\theta(-p^0)\delta(p^2-m^2)$, and the spectral condition (7) sets $b=0$. The inverse Fourier transform finally gives the two-point function of the field:

$$\mathcal{W}(x,y) = W(x-y) = \frac{1}{2(2\pi)^{\frac{d}{2}}}\int e^{-ip\cdot(x-y)}\theta(p^0)\delta(p^2-m^2)\,dp. \tag{10}$$

The structure of the two-point function is that of a superposition of the plane waves $\exp(-ip\cdot x)$ and their complex conjugate $\exp(ip\cdot y)$, taken respectively at the points x and y; the momentum variable p is integrated w.r.t. the Lorentz invariant measure $d\mu = \theta(p^0)\delta(p^2-m^2)dp$ satisfying the spectral condition [8]. As a consequence of the spectral condition (or by direct inspection of the integral at the r.h.s. of Eq. (10)) the distribution $\mathcal{W}(x_1,x_2)$ is the boundary value of a function $\mathrm{W}(z_1,z_2)$ analytic in the backward tube T_2^-. A further application of the spectral property shows that $\mathrm{W}(z_1,z_2)$ is maximally analytic, i.e. it can be analytically continued to the cut-domain $\Delta = M_d^{(c)}\times M_d^{(c)}\setminus\{(z_1,z_2)\in M_d^{(c)}\times M_d^{(c)} : (z_1-z_2)^2 = \alpha\geq 0\}$.

When the background is curved, it is generally impossible to characterize the physically relevant vacuum states as the fundamental states of the energy operator in the previous sense and the analogue of the spectral condition is lacking. This happens in de Sitter case. For the anti-de Sitter case the situation is better in the sense that a global energy operator can be identified. On the other hand the situation is complicated by the lack of global hyperbolicity.

We describe in the following a general framework for the study of dS and AdS QFT which mimics as closely as possible the Minkowskian case [9, 10, 11, 12]. The requirements of dS or AdS covariance, local commutativity and positive-definiteness are easily adapted from the Minkowski case, but, as in the Minkowski case, these properties are not sufficient to characterize a QFT: we need a spectral condition. We have found that a general spectral condition for dS and AdS QFT's can be conveniently formulated in terms of the properties of the analytic continuation of the correlation functions in the complexified dS and AdS manifolds. Let us see how it works.

3 de Sitter

The complex de Sitter spacetime is identified with the complex hyperboloid

$$dS_d^{(c)} = \{z = x + iy \in M_{d+1}^{(c)} : {z^0}^2 - {z^1}^2 - \ldots - {z^d}^2 = -R^2\}, \tag{11}$$

equivalently characterized as the set $dS_d^{(c)} = \{(x, y) \in M_{d+1} \times M_{d+1} : x^2 - y^2 = -R^2,\ x \cdot y = 0\}$. The relevant one-point tubes are obtainable by intersecting the complex dS manifold with the future and past tubes of the ambient complex Minkowski space (see Fig. 2):

$$\mathcal{T}_1^+ = \mathrm{T}_1^+ \cap dS_d^{(c)}, \qquad \mathcal{T}_1^- = \mathrm{T}_1^- \cap dS_d^{(c)}, \tag{12}$$

where $\mathrm{T}_1^\pm = M_{d+1} + iV^\pm$ and $V^\pm$ are the future and past cone of the origin in the complex ambient space (see Fig. 1). $dS_d \bigcup \mathcal{T}_1^+ \bigcup \mathcal{T}_1^-$ contains the "euclidean sphere" $S_d = \{z \in dS_d^{(c)},\ z = (iy^0, x^1, \ldots x^d)\}$.

Let us consider the de Sitter Klein-Gordon equation $(\Box_{dS} + m^2)\phi = 0$, where m is a mass parameter. For any $\xi \in C^+ = \partial V^+$ (in the ambient space, see Fig. 1)) one can introduce plane-wave solutions:

$$\psi^\xi(z, s) = \left(\frac{z \cdot \xi}{R}\right)^\lambda, \quad \lambda \in \mathbf{C}. \tag{13}$$

These waves are indeed holomorphic for z in $\mathcal{T}_1^+$ or in $\mathcal{T}_1^-$. Physical values of the parameter s, corresponding to $m^2 > 0$, are given by $\lambda = -\frac{d-1}{2} + i\nu R$, with either $\nu \in \mathbf{R}$ or $\nu \in i\mathbf{R}$ with $|\nu| \leq \frac{d-1}{2R}$. We are now able to construct the two-point function of a dS KG field as a superposition of holomorphic plane waves, precisely as in Eq. (10):

$$\mathrm{W}_\nu(z_1, z_2) = c_{d,\nu} \int_\gamma (z \cdot \xi_\gamma)^{-\frac{d-1}{2} + i\nu} (\xi_\gamma \cdot z_2)^{-\frac{d-1}{2} - i\nu} \, d\mu_\gamma(\xi_\gamma), \tag{14}$$

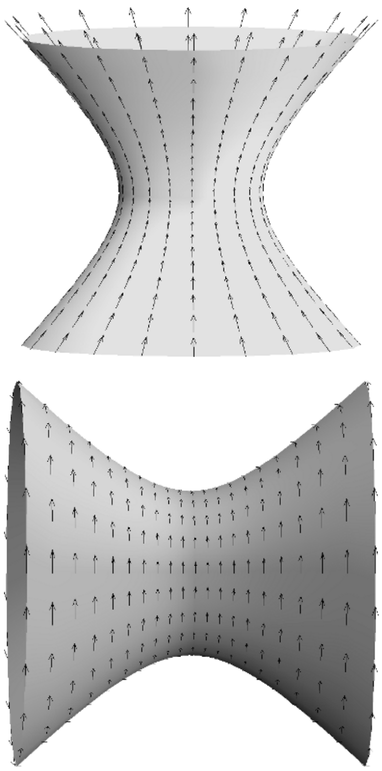

Figure 2: Sections of the future tubes of the complex dS and AdS manifolds. In the dS case the real part x of a point $x+iy$ of the tube $\mathcal{T}_1^+$ belongs to a de Sitter manifold of smaller radius $\{x^2 = -(R^2 - a^2)\}$, depending on on the length a of the imaginary part y: $y^2 = a^2 \leq R^2$. The imaginary part y is represented as a future directed tangent vector. The AdS future tube is characterized by its chirality. Now the real part of a point $x+iy$ of the AdS future tube $\mathcal{Z}_1^+$ belongs to a quadric $\{x^2 = R^2 + a^2\}$ whose radius is larger than R with no restriction on the length $y^2 = a^2$ of the imaginary part y.

where $z_1 \in \mathcal{T}_1^-$, $z_2 \in \mathcal{T}_1^+$. The integration is performed along any basis submanifold γ of the future cone C^+ w.r.t. a measure $d\mu_\gamma$ induced by the invariant measure on the cone; actually the integrand in (14) is the restriction to γ of a closed differential form. W_ν is a solution in both variables of the (complex) de Sitter Klein-Gordon equation which is analytic in the domain $\mathcal{T}_{12} = \{(z_1, z_2) \in dS_d^{(c)} \times dS_d^{(c)} : z_1 \in \mathcal{T}_1^-, z_2 \in \mathcal{T}_1^+\}$. It is possible to show that it is actually a function of the single de Sitter invariant variable $(z_1 - z_2)^2 = -2R^2 - 2z_1 \cdot z_2$ and that, as in the Minkowski case, it enjoys the maximal analyticity property: it can be analytically continued in the "cut-domain" $\Delta = dS_d^{(c)} \times dS_d^{(c)} \setminus \{(z_1, z_2) \in dS_d^{(c)} \times dS_d^{(c)} : (z_1 - z_2)^2 = \alpha \geq 0\}$.

What is the physical basis for these analyticity properties in the de Sitter case? It turns out that they can be understood in terms of a thermal spectral condition [7, 10]. For free fields this is an equivalence and can be proven easily by using the maximal analyticity property of the two point function. Consider indeed the geodesical observer through the base point $x_0 = (0, \dots, 0, R)$ contained in the (x^0, x^d)-plane. This choice singles out the region $\mathcal{U} = \{x \in dS_d : \quad x^d > |x^0|\}$ together with its event horizons $x^0 = \pm x^d, \ \ x^d > 0$. The region $\mathcal{U}$ is foliated by hyperbolic trajectories according to the following parametrization: $x(\tau, \vec{r}) = (x^0 = \sqrt{R^2 - r^2} \sinh \frac{\tau}{R}, \quad x^i = \vec{r},\ x^d = \sqrt{R^2 - r^2} \cosh \frac{\tau}{R})$; τ is the proper time of the observer. These curves are the orbits of the one-parameter group T of isometries of $\mathcal{U}$: $x^t \equiv T(t)[x(\tau, \vec{r})] = x(t + \tau, \vec{r}),\ t \in \mathbf{R}$. The complexified orbits of T have $(2i\pi R)$-periodicity in t and all their non-real points in the tubes $\mathcal{T}^+$ and $\mathcal{T}^-$.

If x_1 and x_2 are arbitrary events in $\mathcal{U}$, the maximal analyticity of $\mathrm{W}_\nu(z_1, z_2)$ implies that $\mathrm{W}_\nu(x_1, x_2^t)$ is a $2i\pi R$-periodic function of t analytic in the strip $\{t \in \mathbf{C},\ 0 < \mathrm{Im}\, t < 2i\pi R\}$ and satisfies the following K.M.S. relation at temperature T $= 1/2\pi R$:

$$\mathcal{W}_\nu(x_2^t, , x_1) = \lim_{\epsilon \to 0^+} \mathrm{W}_\nu(x_1, x_2^{t + 2i\pi R - i\epsilon}), \quad t \in \mathbf{R}, \tag{15}$$

where $\langle \Omega, \phi(x_1)\phi(x_2^t)\Omega\rangle = \mathcal{W}_\nu(x_1, x_2^t)$ and $\langle \Omega, \phi(x_2^t)\phi(x_1)\Omega\rangle = \mathcal{W}_\nu(x_2^t, x_1)$, The "energy operator" $\mathcal{E}$ associated with the observer is obtained by the spectral decomposition of the unitary representation of the time translation group T in the Hilbert space $\mathcal{H}$ of the theory, namely $U^t = \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}\omega t} dE(\omega)$, which yields $\mathcal{E} = \int_{-\infty}^{\infty} \omega dE(\omega)$. The previous K.M.S. condition is then equivalent to the fact that energy measurements performed by an observer at rest at the origin on states localized in $\mathcal{U}$ are exponentially damped by a factor $\exp(-2\pi R\omega)$ in the range of negative energies. In the limit of flat spacetime $R \to \infty$ this factor will kill all negative energies, so that one recovers the usual spectral condition of positivity of the energy.

In the general interacting case one cannot expect maximal analyticity properties to hold for n-point functions ($n > 2$). Let us consider the case of a theory that can be characterized by a sequence of Wightman distributions $\mathcal{W}_n(x_1, \dots, x_n) = \langle \Omega, \phi(x_1) \dots \phi(x_n)\Omega\rangle$ on $dS_d^{\times n}$. The first three properties the Wightman functions should satisfy are literal transcriptions to the dS spacetime of the corresponding Minkowskian properties: 1. de Sitter Covariance, 2. Local Commutativity and 3. Positive Definiteness. As regards the spectral condition we cannot directly formulate it in terms of an energy operator but we can propose analyticity properties similar to those equivalent the usual spectral condition in the Minkowskian case. Let us introduce the open set

$$\begin{aligned} &\mathcal{T}_{n-1}^- = dS_d^{(c)\,\times n} \cap T_{n-1} = \\ &\{z = (z_1, \dots, z_n);\ z_j = x_j + iy_j \in dS_d^{(c)},\ y_{j+1} - y_j \in V^+,\ 1 \le j \le n-1\}. \end{aligned} \tag{16}$$

Since $\mathcal{T}_{n-1}^-$ is a domain of $dS_d^{(c)\,\times n}$ which is moreover a tuboid (see [10] for a precise definition) above $dS_d^{(c)\,\times n}$, we can state the following axiom:

4. (Weak Spectral Condition). For each n the distribution $\mathcal{W}_n(x_1,\ldots,x_n)$ is the boundary value of an analytic function $\mathrm{W}_n(z_1,\ldots,z_n)$, defined in the tuboid $\mathcal{T}^-_{n-1}$ of the complex manifold $dS_d^{(c)\,\times n}$.

There are obviously other possible choices. But it is also true that our choice allows a (not so easy as in the free field case) proof of the thermal interpretation of the theory. Another feature that this approach assures the possibility of the transition to the euclidean sphere. In the case of the two-point functions these axioms are equivalent to the maximal analyticity property (see [11] for more details).

4 Anti-de Sitter

The complex anti-de Sitter spacetime is identified with the complex quadric

$$AdS_d^{(c)} = \{z = x + iy \in E_{d+1}^{(c)} : {z^0}^2 - {z^1}^2 - \ldots + {z^d}^2 = R^2\}, \tag{17}$$

equivalently characterized as the set $AdS_d^{(c)} = \{(x,y) \in E_{d+1} \times E_{d+1} : x^2 - y^2 = R^2,\ x\cdot y = 0\}$. The Euclidean submanifold of $AdS_d^{(c)}$ is defined by putting $z^0 = iy^0,\ x^1,\ldots,x^d$ real, $x^d > 0$. By changing z^μ to iz^μ for $0 < \mu < d$, $AdS_d^{(c)}$ becomes a complex sphere in $\mathbf{C}^{d+1}$, which has the same homotopy type as the real unit sphere. In particular $AdS_d^{(c)}$ is simply connected for $d \geq 2$. However the real manifold AdS_d admits a nontrivial covering space $\widetilde{AdS}_d$ whose "physical" role is to suppress the time-loops of pure AdS. This is visualized in the easiest way by using "covering parametrization" $x = x(r,\tau,\mathrm{e})$ (see Fig. 1) :

$$\left\{\begin{array}{lcll} x^0 & = & \sqrt{r^2+R^2}\sin\tau & \\ x^i & = & r\mathrm{e}^i & i = 1,\ldots,d-1 \\ x^d & = & \sqrt{r^2+R^2}\cos\tau & \end{array}\right. \tag{18}$$

with $\mathrm{e}^2 \equiv {\mathrm{e}^1}^2 + \ldots + {\mathrm{e}^{d-1}}^2 = 1$ and $r \geq 0$. For each fixed value of r, the corresponding "slice" $C_r = AdS_d \cap \{{x^0}^2 + {x^d}^2 = r^2 + R^2\}$ of AdS_d is a Lorentz manifold $\mathbb{S}_1 \times \mathbb{S}_{d-2}$. The relevant coverings $\widetilde{AdS}_d$ and $\widetilde{C}_r$ are obtained by unfolding the $2\pi-$periodic coordinate τ. The notion of space-like separation can now be extended and both these coverings admit a global causal ordering while in the pure AdS case this is possible only locally.

To introduce the relevant one-point tubes [12, 13, 14] let us consider the infinitesimal generator of the (covering of) the subgroup of rotations in the $(0,d)$-plane M_{0d}; such transformations can be interpreted as time translation relative to the covering coordinate system represented in Fig. (1). The set $\mathcal{C}_1 = \{\Lambda\, M_{0d}\, \Lambda^{-1} : \Lambda \in SO_0(2,d-1)\}$ is the subset of all the elements of the AdS Lie algebra $\mathcal{G}$ that are conjugate to M_{0d}; $\mathcal{C}_+$ denote the cone generated in $\mathcal{G}$ by $\mathcal{C}_1$, i.e. $\mathcal{C}_+ = \bigcup_{\rho>0} \rho\, \mathcal{C}_1$. The future tuboid $\mathcal{Z}_1^+$ and the past tuboid $\mathcal{Z}_1^-$ of $AdS_d^{(c)}$ are then introduced as follows:

$$\mathcal{Z}_1^+ = \{\exp(\tau M)\, x \ :\ M \in \mathcal{C}_1,\ \ x \in AdS_d,\ \ \tau \in \mathbf{C}_+\} = \left(\mathcal{Z}_1^-\right)^*. \tag{19}$$

where $\mathbf{C}_+ = \{\zeta \in \mathbf{C} \ : \ \mathrm{Im}\zeta > 0\} = -\mathbf{C}_-$. The tuboids $\mathcal{Z}_1^\pm$ (and their universal coverings $\widetilde{\mathcal{Z}}_1^\pm$) are domains of holomorphy. $\mathcal{Z}_1^\pm$ are equivalently charaterized by their chirality (see Fig. 2):

$$\mathcal{Z}_1^\pm = \{z = x + iy \in E_{d+1}^{(c)} \ : \ x^2 - y^2 = 1, \ \ (x \cdot y) = 0, \ y^2 > 0, \ \ \epsilon(z) = \pm\}, \tag{20}$$

where for $z = x + iy \in E_{d+1}^{(c)}$, we have defined $\epsilon(z) = y^0 x^d - x^0 y^d$. The form (20) shows that $\mathcal{Z}_1^\pm$ are open subsets of $AdS_d^{(c)}$ with empty intersection with each other, while their definition shows that they are connected. As it happens in the Minkowski and dS cases, the "Euclidean" AdS spacetime $AdS_d^{(\mathcal{E})}$ is contained in $AdS_d \cup \mathcal{Z}_{1+} \cup \mathcal{Z}_{1-}$.

An anti-de Sitterian QFT theory is characterized (as before) by a set of AdS invariant n-point functions $\{\mathcal{W}_n(x_1, \ldots, x_n)\}$ that satisfy the local commutativity in the (global) sense of the covering $\widetilde{AdS}_d$. The positive definiteness property allows the reconstruction of a Hilbert space $\mathcal{H}$ carrying a continuous unitary representation $g \mapsto U(g)$ of (the covering of) the AdS group $G_0 = SO_0(2, d-1)$. To every element M of the Lie algebra $\mathcal{G}$ one can associate the one-parameter subgroup $t \mapsto \exp tM$ of and a self-adjoint operator $\widehat{M}$ acting in $\mathcal{H}$ such that $\exp it\widehat{M} = U(\exp tM)$ for all $t \in \mathbf{R}$. In the AdS case the a spectral condition can be formulated as follows:

Spectral Condition: for every $M \in \mathcal{C}_+$ the operator $\widehat{M}$ has its spectrum contained in $\mathbf{R}_+$.

Let us consider now in particular a free field characterized by a two-point function $\mathcal{W}(x_1, x_2)$. The above spectrum condition gives rise to the following normal analyticity condition: the two-point function $\mathcal{W}(x_1, x_2)$ is the boundary value of a function $\mathrm{W}(z_1, z_2)$ which is holomorphic in the domain $\tilde{\mathcal{Z}}_1^- \times \tilde{\mathcal{Z}}_1^+$ of $\widetilde{AdS}_d^{(c)} \times \widetilde{AdS}^{(c)}$. AdS invariance and the normal analyticity condition together imply again the maximal analyticity property: $\mathrm{W}(z_1, z_2)$ can be continued in the covering $\widetilde{\Delta}$ of the cut-domain $\Delta = AdS_d^{(c)} \times AdS_d^{(c)} \setminus \{(z_1, z_2) \in AdS_d^{(c)} \times AdS_d^{(c)} : \zeta = (z_1 \cdot z_2)/R^2 \in [-1, 1]\}$. For special theories which are periodic in time W is in fact analytic in Δ itself. The simplest example of the previous structure is again given by the AdS Klein-Gordon fields. The corresponding two-point functions are expressed in terms of generalized Legendre functions [15]

$$Q_\lambda^{(d+1)}(\zeta) = \frac{\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)} \int_1^\infty (\zeta + it\sqrt{\zeta^2 - 1})^{-\lambda - d + 1} (t^2 - 1)^{\frac{d-3}{2}} dt \tag{21}$$

by the following formula:

$$W_{\lambda + \frac{d-1}{2}}(z_1, z_2) = w_{\lambda + \frac{d-1}{2}}(\zeta) = \frac{e^{-i\pi d}}{\pi^{\frac{d-1}{2}}} \Gamma\left(\frac{d+1}{2}\right) h_{d+1}(\lambda) Q_\lambda^{(d+1)}(\zeta), \quad \zeta = (z_1, z_2), \tag{22}$$

where the parameter λ is related to the mass by the formula $m^2 = \lambda(\lambda + d - 1) = \nu^2 - \frac{(d-1)^2}{4}$. The normalization can be obtained by imposing the local Hadamard condition, that gives

$$h_{d+1}(\lambda) = \frac{(2\lambda + d - 1)}{\Gamma(d)} \frac{\Gamma(\lambda + d - 1)}{\Gamma(\lambda + 1)}. \tag{23}$$

Theories with $\nu > -1$ are acceptable in the sense that they satisfy all the axioms including the positive-definiteness property. The previous two-point functions are holomorphic in Δ or $\widetilde{\Delta}$ according to whether λ is an integer or not. It is worthwhile to point out that also in the anti-de Sitter case we are able to provide a spectral representation of the two-point functions in terms of AdS plane waves, that is closely similar to that produced in Eq. (14). However, the construction of such representation is more tricky in the AdS case. We limit ourself to writing the formula without comments in the two-dimensional integer (i.e. periodical) case:

$$W(z_1, z_2) = c_\ell \int_{\gamma(z_1,z_2)} [z_1 \cdot \xi]^\ell \; [\xi \cdot z_2]^{-\ell-1} \mathrm{d}\mu_{\gamma^{(c)}}(\xi) \tag{24}$$

where $z_1 \in \mathcal{Z}_1^-$ and $z_2 \in \mathcal{Z}_1^+$ and the integration cycle $\gamma(z_1, z_2)$ that belongs to a certain class of cycles taken in the complexification of the asymptotic cone C.

To study interacting theory we do need to generalize the previous results and introduce certain n-point future tuboids $\mathcal{T}_n^+$ and past tuboids $\mathcal{T}_n^-$. The n-point functions are then assumed to be the the boundary value, in the sense of tempered distributions, of functions holomorphic and of tempered growth in such tubes. The construction is however rather elaborate an we refer the reader to [12] for the relevant details.

5 Correspondences

The AdS-CFT correspondence [3] is one of the most important advances in the recent panorama of theoretical high energy physics. Our framework allows for a very simple understanding of how, given an AdS QFT satisfying the general properties briefly discussed in the previous section, one can extract a Conformal Field Theory living on the boundary.

The heuristic idea is as follows: given an AdS QFT let us consider its restriction to the Lorentzian manifold $\widetilde{C}_r$. Alternatively one may consider its restriction to a Poincaré slice $\Pi_v = \{x \in AdS_d : x^{d-1} + x^d = \exp(v)\}$ (see Fig. 1). The problem is that, generally speaking, distributions cannot be restricted to lower dimensional manifolds. It is the AdS spectral condition that in our framework guarantees that the above restrictions are well-defined [12, 13]. In particular one can show that the restriction of an AdS QFT to the branes Π_v defines a one parameter family (the parameter being v) of Poincaré invariant QFT's satisfying all the Wightman axioms. If it is possible to perform the limit $v \to \infty$, this automatically gives rise to an acceptable Poincaré invariant Wightman's QFT on the boundary, identified here with a copy of a Minkowski spacetime. It remains to show the conformal invariance of the so-obtained QFT.

Similarly, it is meaningful to consider the restrictions of the distributions $\mathcal{W}_n$ to the submanifolds $\widetilde{C}_r$. These theories are obviously invariant under the product of the time translation group w.r.t. the τ parameter times the orthogonal group $SO(d-1)$ of space transformations acting on the sphere $\mathbb{S}_{d-2}$. Moreover, the local commutativity property of the original theory on $\widetilde{AdS}_d$ implies that the restricted theories also satisfy local commutativity in the sense of the spacetime manifolds $\widetilde{C}_r$. Finally, the spectrum of the generator of time-translations is positive.

The limit $r \to \infty$, if it is possible, generates a QFT on the boundary, identified here with the covering of the compactification of the Minkowski spacetime (this manifold is the right place where to study conformal invariance; on the contrary the conformal group acts on the Minkowski spacetime only infinitesimally). The invariance group of the limiting QFT is by construction $\mathbb{R} \times SO(d-1)$ and it remains the task of showing its conformal invariance.

To actually perform the previous limit some control on the power-decrease at infinity of the Wightman functions is needed; with the help of the coordinates (18) we assume the following *dimensional boundary condition at infinity:* a scalar QFT on $\widetilde{AdS}_d$ is said to be of asymptotic dimension Δ if the following limits exist in the sense of distributions:

$$\lim_{r\to+\infty} (r)^{n\Delta}\mathcal{W}_n(x_1(r,\tau_1,\mathrm{e}_1),\ldots,x_n(r,\tau_n,\mathrm{e}_n)) = \mathcal{W}_n^\infty((\tau_1,\mathrm{e}_1),\ldots,(\tau_n,\mathrm{e}_n)) \tag{25}$$

In a basic work by Lüscher and Mack [16] the concept of global conformal invariance in Minkowskian QFT has been associated with the general framework of QFT on the covering of a quadratic cone with signature $(+,+,-,\cdots,-)$ in one dimension more, which is precisely the asymptotic cone of the AdS quadric $\mathcal{C}_{2,d-1} = \{\eta = (\eta^0,\ldots,\eta^{(d)});\ {\eta^0}^2 - {\eta^1}^2 - \cdots - {\eta^{d-1}}^2 + {\eta^d}^2 = 0\}$ (see Fig. 1) . The covering coordinates $\eta = \eta(r,\tau,\mathrm{e})$ of the asymptotic cone are now

$$\left\{\begin{array}{lcll} \eta^0 & = & r\sin\tau & \\ \eta^i & = & r\mathrm{e}^i & i = 1,\ldots,d-1 \\ \eta^d & = & r\cos\tau & \end{array}\right. \tag{26}$$

with ${\mathrm{e}^1}^2 + \ldots + {\mathrm{e}^d}^2 = 1$ and $r \geq 0$. The parametrization (26) allows one to introduce the covering $\widetilde{\mathcal{C}}_{2,d-1}$ and of $\mathcal{C}_{2,d}$ by again unfolding the $2\pi-$periodic coordinate τ. Let us now consider a general QFT on $\widetilde{AdS}_d$ whose Wightman functions $\mathcal{W}_n$ satisfy AdS invariance together with the other properties described in the previous section and the dimensional boundary condition. In view of Eq. (25), by using the covering coordinates of the cone we can construct the following set of $n-$point distributions $\underline{\mathcal{W}}_n(\eta_1,\ldots,\eta_n)$ on $\widetilde{\mathcal{C}}_{2,d-1}$:

$$\underline{\mathcal{W}}_n(\eta_1,\ldots,\eta_n) = (r_1\cdots r_n)^{-\Delta}\mathcal{W}_n^\infty((\tau_1,\mathrm{e}_1),\ldots,(\tau_n,\mathrm{e}_n)). \tag{27}$$

The set of distributions $\underline{\mathcal{W}}_n$ satisfy the required positive-definiteness condition for defining a QFT on $\widetilde{\mathcal{C}}_{2,d-1}$ because the positivity of the QFT's on the spacetimes $\widetilde{C}_r$ is preserved in the limit. It follows from the reconstruction procedure [8] that the set of distributions $\underline{\mathcal{W}}_n$ define a quantum field $O(\eta)$ on $\widetilde{\mathcal{C}}_{2,d-1}$. $O(\eta)$ enjoys the local commutativity and the spectral condition as those of the Lüscher-Mack field theories [16]. The important result on which our AdS/CFT correspondence is based is that invariance of the AdS $n-$point functions, together with the other assumption we made, implies the conformal invariance of the field $O(\eta)$; more precisely, the Wightman functions $\underline{\mathcal{W}}_n$ of this field are invariant under the action on $\widetilde{\mathcal{C}}_{2,d-1}$ of the conformal group [16]:

$$\underline{\mathcal{W}}_n(g\eta_1, \ldots, g\eta_n) = \underline{\mathcal{W}}_n(\eta_1, \ldots, \eta_n) \tag{28}$$

for all conformal transformations g. A part of this invariance is trivial in view of how the limiting procedure is constructed: it is the invariance under the rotations in the $(0, d)-$plane (i.e. the translations in the time variables τ) and the invariance under the spatial orthogonal group of the subspace of variables $(\eta^1, \ldots, \eta^d)$ (acting on the sphere $\mathbb{S}_{d-1}$). For the proof of the nontrivial part we refer the reader to [13]. We have therefore a general AdS/CFT correspondence for QFT's:

$$\phi(x) \to \mathcal{O}(\eta) \tag{29}$$

between a scalar quantum field $\phi(x)$ on $\widetilde{AdS}_{d+1}$ satisfying our axioms and a conformally invariant local field $\mathcal{O}(\eta)$ on $\widetilde{\mathcal{C}}_{2,d-1}$ enjoying the Lüscher-Mack axioms; the degree of homogeneity (dimension) Δ of $\mathcal{O}(\eta)$ is equal to the asymptotic dimension of the AdS field $\Phi(X)$. The relationship of this approach with the standard way of understanding the AdS/CFT correspondence [17] has been elucidated in [18].

It has been also proposed a duality between a quantum theory on de Sitter space and a euclidean theory on its boundary [19] which should encode the de Sitterian quantum gravity degrees of freedom [19]. We have shown [20] that one can associate with a general (scalar) de Sitter quantum field theory satisfying a dimensional boundary condition a conformal Euclidean field theory on the boundary, again identified with a copy of the cone asymptotic to the de Sitter manifold in the embedding spacetime (see Fig. 1). However, the field theory that one gets this way does not in general satisfy reflection positivity, which is required to admit a physical interpretation. Therefore, the proposed construction can have in general a technical interest but no obvious holographic physical interpretation seems to be available.

6 Conclusions

We have shown that there exists a common way to undertake the study of Minkowski, de Sitter and anti-de Sitter QFT's that is based on the complexifications of those manifolds and the study of complex domains associated with the conditions on the spectra of the possible energy operators. In this context we are able to characterize general structural properties of the theories but the method is also useful to perform concrete calculations. The next step would be to test these ideas in perturbation theory, with important possible consequences for high energy physics and cosmology.

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