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# Eisenbud-Harris special non-hyperelliptic fibrations of genus 4

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**Abstract** We define certain index for the degenerate fibers in non-hyperelliptic fibrations of genus 4 but not Eisenbud-Harris general. Furthermore, we investigate the structure of the surfaces which have the pencil as above, and whose sum of the indices is equal to 0 or near 0.

**Keywords** Non-hyperelliptic curves · Index · Multiplicative map

**Mathematics Subject Classification (2000)** 14J29

## Introduction

Let $S$ be a smooth projective surface defined over the complex number field $\mathbb{C}$. Assume that $S$ has a pencil $f : S \to B$, where $B$ is a smooth projective curve, and that a general fiber of $f$ is a non-hyperelliptic curve of genus 4 which is not Eisenbud-Harris general. (See [2] for the definition of Eisenbud-Harris general.) In this case, a general fiber of $f$ is isomorphic to a curve $C$ obtained as the complete intersection of the quadric hypersurface $Q$ of rank 3 and the cubic hypersurface in $\mathbb{P}^3$. It is well-known that $Q$ is isomorphic to the cone over the smooth conic and that the minimal resolution $\tilde{Q}$ of $Q$ is isomorphic to the Hirzebruch surface $\Sigma_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$. Furthermore, $C$ does not go through the vertex of $Q$ and hence, we may consider as $C \subset \Sigma_2$. If $\Delta_0$ is the tautological divisor of $\Sigma_2$ (i.e. $\mu_* \mathcal{O}_{\Sigma_2}(\Delta_0) \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$, where $\mu : \Sigma_2 \to \mathbb{P}^1$ is the ruling), and if we consider $C$ as the divisor of $\Sigma_2$, then $C \sim 3\Delta_0$ holds.

We call the fibration $f : S \to B$ as above *Eisenbud-Harris special non-hyperelliptic fibration* (or E-H special non-hyperelliptic fibration, for short) of genus 4.

If $\omega_{S/B}$ is the relative dualizing sheaf, the direct image $E := f_* \omega_{S/B}$ is the locally free sheaf of rank 4 over $B$. Put $\Delta(f) := \deg E$, and let $K_{S/B}$ be the relative canonical divisor.

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For Eisenbud-Harris *general* fibration of genus 4, the following theorem is known:

**Theorem 1** (cf. [14]) *Let $f : S \to B$ be a non-hyperelliptic fibration of genus 4. Assume that a general fiber is Eisenbud-Harris general. Then for any $p \in B$, Horikawa index $\mathrm{Ind}(f^{-1}p)$ is defined, and the following equality holds:*

$$K_{S/B}^2 = \frac{7}{2}\Delta(f) + \sum_{p \in B} \mathrm{Ind}(f^{-1}p).$$

In Theorem 1, Horikawa index $\mathrm{Ind}(f^{-1}p)$ is given by the local data near $p$ and is non-negative. Furthermore, $\mathrm{Ind}(f^{-1}p) = 0$ holds except finite number of points in $B$.

Let's return to the case of E-H special non-hyperelliptic fibrations. In this paper, we are interested in the case where the relative canonical image of the surface is isomorphic to the relative canonical model. In particular, we assume the following:

**Assumption** The multiplicative map $\mathrm{Sym}^2 E \to f_*\omega_{S/B}^{\otimes 2}$ is surjective.

In Sect. 1, we define some kind of index for any fiber of $f$ which is similar to Horikawa index. In this paper, we don't call it the Horikawa index because we consider our problem under the above condition.

In Sect. 2, we classify our surfaces under further condition. In this case, we assume that the base curve $B$ is $\mathbb{P}^1$. We determine which vector bundle of rank 4 over $B$ is occur as the direct image of the relative dualizing sheaf.

In Sect. 3, we consider the problem of the geography. We give a lot of examples which have a lot of degenerate fibers in the pencil.

## 1 The index for the degenerate fiber

Let $f : S \to B$ be the surjective morphism of a smooth projective surface $S$ onto a smooth projective curve $B$ satisfying the following:

(1) Every fiber of $f$ is connected.
(2) $f$ is E-H special non-hyperelliptic fibration of genus 4.
(3) $f$ is relatively minimal.
(4) $f$ is not locally trivial.

Form (2), $E := f_*\omega_{S/B}$ is a locally free sheaf of ranf 4 over $B$. Let $\pi : W := \mathbb{P}(E) \to B$ be the $\mathbb{P}^3$-bundle defined by $E$, and $T$ the tautological divisor of $W$ (i.e. $\pi_*\mathcal{O}_W(T) \cong E$ holds.) Furthermore, let $\psi : S \cdots \to W$ be the rational map defined by the natural sheaf homomorphism $f^*E \to \omega_{S/B}$. By our assumption, $\psi$ gives the birational map of $S$ onto the image.

For every fiber $F$ of $f$, the multiplicative map

$$\mathrm{Sym}^2 H^0(\omega_F) \to H^0(\omega_F^{\otimes 2})$$

is defined, and hence, we obtain the multiplicative homomorphism

$$\varphi : \mathrm{Sym}^2 E \to f_*\omega_{S/B}^{\otimes 2}.$$

From Max-Noether's theorem and our assumption, $\varphi$ is generically surjective.

In this section, we assume that $\varphi$ is surjective.

**Lemma 1** *Let the notations and the conditions be as above. Then $\psi$ is a morphism and the image $S' := \psi(S)$ is isomorphic to its relative canonical model. In particular, $S'$ has at most rational double points as the singularities.*

*Proof* (due to Kazuhiro Konno.) Let $n$ be a sufficiently large integer and $\delta$ a divisor on $B$ with $\deg \delta = 2n$. Then we may assume $\dim H^0(\mathcal{O}_S(K_S + f^*\delta)) \geq 4$ and $(K_S + f^*\delta)^2 > 0$. Hence, for any $p \in S$, we have $|K_S + f^*\delta - 2p| \neq \emptyset$. Furthermore, it is proved by the same argument as [3, Chap. VII, Proposition 6.2] that a divisor $D \in |K_S + f^*\delta|$ is 2-connected. By the same argument as [4, Lemma 3.2], we obtain that $|2K_S + f^*\delta|$ is base point free. Let $F$ be a fiber of $f$. Since $|2K_S + f^*\delta|$ is base point free, so is $|2K_F|$ by the adjunction formula. Since we assume $\varphi$ is surjective, $|K_F|$ is also base point free. Hence, if we put $i = 0$ for [13, Corollary 1.2.3 (1)], we obtain that the map

$$H^0(K_F) \otimes H^0((j-1)K_F) \to H^0(jK_F)$$

is surjective for any $j > 0$. Hence, the canonical ring of $F$ is generated in degree 1, which leads us to the fact that $S'$ is isomorphic to its relative canonical model by the same argument as [10, Thoerem 1.1]. $\square$

Furthermore, there is no hyperelliptic curve in the pencil $f : S \to B$, and the relative canonical image $S'$ is the local complete intersection of the relative quadric hypersurface and the relative cubic hypersurface in $W$.

From now on, we define the index for a fiber of $f$. We prove three equations below.

1.1 Let $L$ be the kernel of $\varphi : \mathrm{Sym}^2 E \to f_*\omega_{S/B}^{\otimes 2}$. We have $\chi(f_*\omega_{S/B}^{\otimes 2}) = \chi(\omega_{S/B}^{\otimes 2})$ by Leray's spectral sequence. By applying Riemann-Roch theorem for both $\chi(f_*\omega_{S/B}^{\otimes 2})$ and $\chi(\omega_{S/B}^{\otimes 2})$, we obtain

$$\chi(f_*\omega_{S/B}^{\otimes 2}) = \deg\left(f_*\omega_{S/B}^{\otimes 2}\right) - 9(b-1),$$
$$\chi(\omega_{S/B}^{\otimes 2}) = K_S^2 + \chi(\mathcal{O}_S) - 36(b-1),$$

where $b$ is the genus of $B$. Since $K_S^2 = K_{S/B}^2 + 24(b-1)$ and $\chi(\mathcal{O}_S) = \Delta(f) + 3(b-1)$ (cf. e.g., [16, p. 6]), we obtain

$$\deg\left(f_*\omega_{S/B}^{\otimes 2}\right) = \Delta(f) + K_{S/B}^2.$$

Since $\deg \mathrm{Sym}^2 E = 5\Delta(f)$, we have

$$K_{S/B}^2 = 4\Delta(f) - \deg L. \tag{1}$$

1.2 By Konno's result (cf. [12], [14]), there exists a relative quadric hypersurface $Q \subset W$ containing $S'$. By the same argument as the proof of [14, Theorem 1], we obtain $Q \sim 2T - \pi^*\delta$, where $\delta$ is some divisor of $B$ with $\mathcal{O}_B(\delta) \cong L$. Since $f$ is E-H special non-hyperelliptic fibration of genus 4, a general fiber of $Q \to B$ is a cone over the smooth conic in $\mathbb{P}^3$. Hence, we obtain the relative vertex $B_0 \subset Q$. Clearly, we have $B_0 \cong B$. If we put $M := \mathcal{O}_W(T) \otimes_{\mathcal{O}_W} \mathcal{O}_{B_0}$, we obtain the surjective homomorphism $E \to M$. Let $E_0$ be the kernel of the homomorphism. Then we have the exact sequence

$$0 \to E_0 \to E \to M \to 0.$$

Let $\rho : \tilde{W} \to W$ be the blow-up along $B_0$. Then we obtain the following commutative diagram:

$$\begin{array}{ccc} \tilde{W} & \xrightarrow{\rho} & W \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{P}(E_0) & \xrightarrow[\mu]{} & B, \end{array}$$

where $\tilde{\pi} : \tilde{W} \to \mathbb{P}(E_0)$ is the $\mathbb{P}^1$-bundle and $\mu : \mathbb{P}(E_0) \to B$ is the $\mathbb{P}^2$-bundle. Let $\mathbb{E} := \rho^{-1}(B_0)$ be the exceptional divisor and $T_{E_0}$ the tautological divisor of $\mathbb{P}(E_0)$ (i.e. $\mu_* \mathcal{O}_{\mathbb{P}(E_0)}(T_{E_0}) \cong E_0$). Then we have $\rho^* T \sim \mathbb{E} + \tilde{\pi}^* T_{E_0}$. If $\tilde{Q}$ is the proper transform of $Q$ by $\rho$, then $\tilde{Q} \sim \tilde{\pi}^*(2T_{E_0} - \mu^*\delta)$ holds. Hence, there exists a relative quadric hypersurface $Q_0 \in |2T_{E_0} - \mu^*\delta|$ such that $\tilde{Q} = \tilde{\pi}^* Q_0$.

By our assumption, a general fiber of $Q_0 \to B$ is of rank 3. Let $q \in H^0(\mathcal{O}_{\mathbb{P}(E_0)}(2T_{E_0} - \mu^*\delta))$ be the global section defining $Q_0$. Since $q$ can be considered as the element of $H^0(B,\ (\mathrm{Sym}^2 E_0) \otimes L^{-1})$, we can consider that $q$ defines the homomorphism

$$q : E_0^\vee \to E_0 \otimes L^{-1}.$$

Consider the determinant map

$$\det(q) : \det E_0^\vee \to (\det E_0) \otimes L^{-3}.$$

By our assumption, we have $\det(q) \neq 0$. If we consider as

$$\det(q) \in H^0(B,\ (\det E_0)^{\otimes 2} \otimes L^{-3}),$$

then the divisor determined by $\det(q)$ is the discriminant locus of $Q_0$. If $\mathrm{Discr}(Q_0)$ is the discriminant locus of $Q_0$, then we have

$$\deg(\mathrm{Discr}(Q_0)) = 2 \deg E_0 - 3 \deg L. \tag{2}$$

1.3 If $S'$ intersects with $B_0$, then the intersection points are the singularities of $S'$. By our assumption, they are the rational double points. Hence, if we put $S'' = \rho^{-1}(S')$, then the map $S \to S'$ is factored as $S \to S'' \to S'$. Consider the commutative diagram

$$\begin{array}{ccc} \tilde{Q} & \xrightarrow{\rho_{\tilde{Q}}} & Q \\ {\scriptstyle\tilde{\pi}_{\tilde{Q}}}\downarrow & & \downarrow{\scriptstyle\pi_Q} \\ Q_0 & \xrightarrow[\mu_{Q_0}]{} & B, \end{array}$$

which is obtained as the restriction of the previous diagram. Let the notations of the morphism be as in the above diagram. Furthermore, put $\mathbb{E}_{\tilde{Q}} := \mathbb{E}_{|\tilde{Q}}$ ($\cong \mathbb{Q}_0$) and $T_{\tilde{Q}} := (\rho^* T)_{|\tilde{Q}}$. We consider the intersection of $S''$ with $\mathbb{E}_{\tilde{Q}}$ in $\tilde{Q}$.

Since $\mathcal{O}_{\tilde{W}}(\rho^* T) \otimes_{\mathcal{O}_{\tilde{W}}} \mathcal{O}_{\mathbb{E}} \cong (\tilde{\pi}^* \mu^* M)_{|\mathbb{E}}$, we have the following exact sequence:

$$0 \to \mathcal{O}_{\tilde{W}}(\tilde{\pi}^* T_{E_0}) \to \mathcal{O}_{\tilde{W}}(\rho^* T) \to (\tilde{\pi}^* \mu^* M)_{|\mathbb{E}} \to 0.$$

By taking the direct image by $\tilde{\pi}$ we have

$$0 \to \mathcal{O}_{\mathbb{P}(E_0)}(T_{E_0}) \to \mathcal{E} \to \mu^* M \to 0,$$

where $\mathcal{E} = \tilde{\pi}_* \mathcal{O}_{\tilde{W}}(\rho^* T)$. Hence, we have that $\det \mathcal{E} \cong \mathcal{O}_{\mathbb{P}(E_0)}(T_{E_0}) \otimes \mu^* M$. Let $m$ and $\delta_0$ be some divisors over $B$ with $\mathcal{O}_B(m) \cong M$ and $\mathcal{O}_B(\delta_0) \cong \det E_0$, respectively. Since $\tilde{\pi} : \tilde{W} \to \mathbb{P}(E_0)$ is a $\mathbb{P}^1$-bundle, we have $K_{\tilde{W}/B} \sim -2\rho^* T + \tilde{\pi}^*(\det \mathcal{E} + K_{\mathbb{P}(E_0)/B})$. Since $K_{\mathbb{P}(E_0)/B} \sim -3T_{E_0} + \mu^* \delta_0$, we obtain

$$\begin{aligned} K_{\tilde{W}/B} &\sim -2\rho^* T + \tilde{\pi}^*(T_{E_0} + \mu^* m - 3T_{E_0} + \mu^* \delta_0) \\ &\sim -2\rho^* T + \tilde{\pi}^*(-2T_{E_0} + \mu^*(m + \delta_0)). \end{aligned}$$

Hence, by the adjunction formula, we have

$$K_{\tilde{Q}/B} \sim -2T_{\tilde{Q}} + \tilde{\pi}^*_{\tilde{Q}} \mu^*(m + \delta_0 - \delta).$$

If we use the adjunction formula again, we obtain

$$K_{S''/B} \sim \left(S'' - 2T_{\tilde{Q}} + \tilde{\pi}^*_{\tilde{Q}} \mu^*(m + \delta_0 - \delta)\right)_{|S''}.$$

On the other hand, we have $K_{S''/B} \sim (T_{\tilde{Q}})_{|S''}$. Since $S'' \sim 3T_{\tilde{Q}} + \tilde{\pi}^*_{\tilde{Q}} D$ for some divisor $D$ over $\mathbb{P}(E_0)$ in $\tilde{Q}$, we have

$$S'' \sim 3T_{\tilde{Q}} + \tilde{\pi}^*_{\tilde{Q}} \mu^*_{Q_0}(\delta - \delta_0 - m)$$

in $\tilde{Q}$. If we let $H$ be the Cartier divisor over $\tilde{Q}$ with $H \sim T_{\tilde{Q}} - \tilde{\pi}^*_{\tilde{Q}} \mu^*_{Q_0} m$, then $H_{|\mathbb{E}} \sim 0$ holds and we have

$$S'' \sim 3H + \tilde{\pi}^*_{\tilde{Q}} \mu^*_{Q_0}(2m + \delta - \delta_0).$$

Hence, the intersection of $S''$ and $\mathbb{E}_{\tilde{Q}}$ consists of $\deg(2m + \delta - \delta_0)$ fibers of the ruling $\mathbb{E}_{\tilde{Q}} (\cong Q_0) \to B$. Namely, if we consider as $\mathbb{E}_{\tilde{Q}} = Q_0$, there exists a divisor $\tilde{\delta} \sim 2m + \delta - \delta_0$ such that $S''_{|\mathbb{E}_{\tilde{Q}}} = \mu^*_{Q_0} \tilde{\delta}$ holds. Since $\Delta(f) = \deg M + \deg E_0$, we have

$$\deg \tilde{\delta} = 2\Delta(f) + \deg L - 3 \deg E_0. \tag{3}$$

1.4 By eliminating $\deg L$ and $\deg E_0$ from (1), (2) and (3), we obtain

$$K^2_{S/B} = \frac{24}{7}\Delta(f) + \frac{2}{7} \deg \tilde{\delta} + \frac{3}{7} \deg(\mathrm{Discr}(Q_0)).$$

For any $p \in B$, we can define the index $\mathrm{ind}(f^{-1}p)$ as

$$\mathrm{ind}(f^{-1}p) := \frac{2}{7} \mathrm{mult}_p(\tilde{\delta}) + \frac{3}{7} \mathrm{mult}_p(\mathrm{Discr}(Q_0)),$$

and we obtain the following theorem:

**Theorem 2** *Let the notations and the conditions be as above. Then for any $p \in B$, the index $\mathrm{ind}(f^{-1}p)$ is defined and the following equality holds:*

$$K^2_{S/B} = \frac{24}{7}\Delta(f) + \sum_{p \in B} \mathrm{ind}(f^{-1}p).$$

## 2 Surfaces with small sum of indices

In this section, we consider the surfaces with E-H special non-hyperelliptic fibration of genus 4 whose sum of the indices in Theorem 2 is less than 1.

Even if we eliminate the assumption that the multiplicative map is surjective, we obtain the following:

**Lemma 2** *Let $f: S \to B$ be the E-H special non-hyperelliptic fibration of genus* 4. *Assume that the sum of the indices in* Theorem 2 *is less than* 1. *Then the multiplicative map* $\mathrm{Sym}^2 E \to f_*\omega_{S/B}^{\otimes 2}$ *is surjective.*

In order to prove Lemma 2, it is sufficient to prove that $\deg L = 4\Delta(f) - K_{S/B}^2$ holds. This can be proved by applying the proof of [12, Theorem 4.1], and by automatical calculations. So we leave the details to readers.

In the rest of this section, we consider the case where the base curve $B$ is isomorphic to $\mathbb{P}^1$. We put $r := \sum_{p \in B} \mathrm{ind}(f^{-1}p)$.

### 2.1 The case where $r = 0$

In the notations of the proof of Theorem 2, we have $E_0^\vee \cong E_0 \otimes L^{-1}$ and $M^{\otimes 2} \otimes L \cong \det E_0$. Hence, $\det E_0 \cong M^{\otimes 6}$ and $L \cong M^{\otimes 4}$ hold. Since $B = \mathbb{P}^1$, there exist a non-negative integer $k$ and a positive integer $d$ such that

$$E_0 \cong \mathcal{O}_{\mathbb{P}^1}(2d+k) \oplus \mathcal{O}_{\mathbb{P}^1}(2d) \oplus \mathcal{O}_{\mathbb{P}^1}(2d-k),$$

$E \cong E_0 \oplus \mathcal{O}_{\mathbb{P}^1}(d)$ and $Q_0 \sim 2T_{E_0} - 4d\mathcal{F}$ hold, where $\mathcal{F}$ is the fiber of $\mu : \mathbb{P}(E_0) \to \mathbb{P}^1$.

The non-negativity of $k$ and the positivity of $d$ follow from [5, Theorem 2.7] and [12, Lemma 2.5].

**Lemma 3** *Let the notations be as above. Then we have $Q_0 \cong \Sigma_k$, where $\Sigma_k = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(k))$ is the Hirzebruch surface.*

*Proof* When $k = 0$, $\mathbb{P}(E_0) \cong \mathbb{P}^1 \times \mathbb{P}^2$ holds and $Q_0$ is the inverse image of a smooth conic by the natural projection $\mathbb{P}(E_0) \to \mathbb{P}^2$, which implies that $Q_0$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \cong \Sigma_0$.

Assume $k > 0$. Let $Z_0, Z_1$ and $Z_2$ be global sections of $H^0(\mathcal{O}_{\mathbb{P}(E_0)}(T_{E_0} - (2d+k)\mathcal{F}))$, $H^0(\mathcal{O}_{\mathbb{P}(E_0)}(T_{E_0} - 2d\mathcal{F}))$ and $H^0(\mathcal{O}_{\mathbb{P}(E_0)}(T_{E_0} - (2d-k)\mathcal{F}))$, respectively, which define the homogeneous coordinates of each fiber of $\mu$. If $\Phi \in H^0(\mathcal{O}_{\mathbb{P}(E_0)}(2T_{E_0} - 4d\mathcal{F}))$ is the global section defining $Q_0$, then it is written as

$$\Phi = \phi_{00} Z_0^2 + c_{11} Z_1^2 + \phi_{01} Z_0 Z_1 + c_{02} Z_0 Z_2,$$

where $\phi_{00}$ and $\phi_{01}$ are the homogeneous forms of degree $2k$ and $k$, respectively, and $c_{11}$ and $c_{02}$ are the constants. Hence, we have that the base locus $B_1$ of $|2T_{E_0} - 4d\mathcal{F}|$ is the section of $\mu$ defined by $Z_0 = Z_1 = 0$. Note that $B_1$ is written as $B_1 = \mathbb{P}(E_0/E_1)$, where $E_1 := \mathcal{O}_{\mathbb{P}^1}(2d+k) \oplus \mathcal{O}_{\mathbb{P}^1}(2d)$. If $\rho_0 : X \to \mathbb{P}(E_0)$ is the blow-up along $B_1$, then we obtain the following commutative diagram:

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}(E_0) \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathbb{P}(E_1) & \longrightarrow & B. \end{array}$$

If $\xi$ is as in the above diagram, and if $\tilde{Q}_0$ is the proper transform of $Q_0$ by $\rho_0$, then $\tilde{Q}_0 \cong Q_0$ and $\xi_{|\tilde{Q}_0} : \tilde{Q}_0 \to \mathbb{P}(E_1)\ (\cong \Sigma_k)$ gives the isomorphism. □

**Lemma 4** *Let the notations be as in the proof of* Theorem 2*, and let the conditions be as in this section. Furthermore, let $\Delta_0$ be the tautological divisor of $Q_0(\cong \Sigma_k)$ (i.e. $\mu_{Q_0*}\mathcal{O}_{Q_0}(\Delta_0) \cong \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}$) and $\Gamma$ the fiber of $\mu_{Q_0}$. Then $\tilde{Q} \cong \mathbb{P}(\mathcal{O}_{\Sigma_k}(2\Delta_0 + (d-k)\Gamma) \oplus \mathcal{O}_{\Sigma_k})$.*

*Proof* From [15, Lemma 1], we obtain

$$\tilde{W} \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}(E_0)}(T_{E_0}) \oplus \mathcal{O}_{\mathbb{P}(E_0)}(d\mathcal{F})).$$

Hence, it is sufficient to prove

$$\mathcal{O}_{\mathbb{P}(E_0)}(T_{E_0}) \otimes_{\mathcal{O}_{\mathbb{P}(E_0)}} \mathcal{O}_{Q_0} \cong \mathcal{O}_{\Sigma_k}(2\Delta_0 + (2d-k)\Gamma).$$

If we use the same notations as in the proof of Lemma 3, it is equivalent to

$$\mathcal{O}_X(\rho_0^* T_{E_0}) \otimes_{\mathcal{O}_X} \mathcal{O}_{\tilde{Q}_0} \cong \mathcal{O}_{\Sigma_k}(2\Delta_0 + (2d-k)\Gamma),$$

Put $\mathbb{E}_0 := \rho_0^{-1}(B_1)$. Since $\rho_0^* T_{E_0} \sim \mathbb{E}_0 + \xi^*(\Delta_0 + 2d\Gamma)$, we have only to prove that $\mathcal{O}_{\tilde{Q}_0}(\mathbb{E}_0) \cong \mathcal{O}_{\Sigma_k}(\Delta_\infty)$, where $\Delta_\infty$ satisfies $\Delta_\infty \sim \Delta_0 - k\Gamma$.

Let $\tilde{Z}_0 \in H^0(\mathcal{O}_X(\rho_0^* T_{E_0} - \xi^*((2d-k)\Gamma)))$ and $\tilde{Z}_\infty \in H^0(\mathcal{O}_X(\rho_0^* T_{E_0} - \xi^*(\Delta_0 + 2d\Gamma)))$ be the global sections defining the homogeneous coordinates of each fiber of $\xi$. Note

$$\tilde{Q}_0 \sim \rho_0^* Q_0 - \mathbb{E}_0 \sim \rho_0^*(2T_{E_0} - 4d\Gamma) - \mathbb{E}_0 \sim \rho_0^* T_{E_0} + \xi^*(\Delta_0 - 2d\Gamma)$$

and

$$\begin{aligned} H^0(\mathcal{O}_X(\tilde{Q}_0)) &\cong H^0(\xi_* \mathcal{O}_X(\rho_0^* T_{E_0})) \otimes \mathcal{O}_{\Sigma_k}(\Delta_0 - 2d\Gamma)) \\ &\cong H^0(\mathcal{O}_{\Sigma_k}(\Delta_\infty)) \oplus H^0(\mathcal{O}_{\Sigma_k}(2\Delta_0)). \end{aligned}$$

If $\tilde{\Phi}$ is the global section defining $\tilde{Q}_0$, then $\tilde{\Phi}$ can be written as

$$\tilde{\Phi} = \tilde{\phi}_0 \tilde{Z}_0 + \tilde{\phi}_\infty \tilde{Z}_\infty,$$

where $\tilde{\phi}_0 \in H^0(\mathcal{O}_{\Sigma_k}(\Delta_\infty))$ and $\tilde{\phi}_\infty \in H^0(\mathcal{O}_{\Sigma_k}(2\Delta_0))$. Since the equation $\tilde{Z}_\infty = 0$ defines $\mathbb{E}_0$, the intersection of $\tilde{Q}_0$ and $\mathbb{E}_0$ coincides with the intersection of $\mathbb{E}_0$ and the subvariety defined by the equation $\tilde{\phi}_0 = 0$. Namely, we obtain the desired isomorphism. □

Let $S'$ be the image of the morphism $\psi : S \to W$ as in Sect. 1. We may consider as $\psi : S \to \tilde{Q}$ and $S' \subset \tilde{Q}$. Since

$$\tilde{Q} \cong \mathbb{P}(\mathcal{O}_{\Sigma_k}(2\Delta_0 + (d-k)\Gamma) \oplus \mathcal{O}_{\Sigma_k})$$

by Lemma 4, we have

$$\mathrm{Pic}(\tilde{Q}) \cong \mathbb{Z}H \oplus \mathbb{Z}\tilde{\pi}_{\tilde{Q}}^* \Delta_0 \oplus \mathbb{Z}\tilde{F},$$

where $\tilde{F}$ is the fiber of $\tilde{Q} \to B$, and $H$ is the divisor as in the proof of Theorem 2 (i.e. $H \sim T_{\tilde{Q}} - d\tilde{F}$). Since $S'$ is disjoint with $\mathbb{E}_Q$, we have $S' \sim 3H$.

**Lemma 5** *Let the notations and the conditions be as above. Then a general member of $|3H|$ is irreducible and has at most rational double points as the singularities if and only if $3d \geq 2k$.*

*Proof* Let $X_0 \in H^0(\mathcal{O}_{\tilde{Q}}(H))$ and $X_\infty \in H^0(\mathcal{O}_{\tilde{Q}}(\mathbb{E}_Q))$ be global sections defining the homogeneous coordinates of each fiber of $\tilde{\pi}_{\tilde{Q}} : \tilde{Q} \to \mathbb{P}(E_0)$. Then any global section $\Psi \in H^0(\mathcal{O}_{\tilde{Q}}(3H))$ can be written as

$$\Psi = c_0 X_0^3 + \psi_1 X_0^2 X_\infty + \psi_2 X_0 X_\infty^2 + \psi_3 X_\infty^3,$$

where $c_0 \in \mathbb{C}$ and $\psi_i \in H^0(\mathcal{O}_{\Sigma_k}(2i\Delta_0 + (di - ki)\Gamma))$ $(i = 1, 2, 3)$. Let $\Delta_\infty$ be the divisor with $\Delta_\infty \sim \Delta_0 - k\Gamma$. If $3d < 2k$, then $|6\Delta_0 + (3d - 3k)\Gamma| = 2\Delta_\infty + |4\Delta_0 + (3d - k)\Gamma|$ and $|4\Delta_0 + (2d - 2k)\Gamma| = \Delta_\infty + |3\Delta_0 + (2d - k)\Gamma|$. Hence, the divisor defined by $\Psi$ is singular along $\tilde{\pi}_{\tilde{Q}}^{-1}(\Delta_\infty) \cap H$.

If $2k \le 3d < 3k$, then $|6\Delta_0 + (3d - 3k)\Gamma| = \Delta_\infty + |5\Delta_0 + (3d - 2k)\Gamma|$, and hence, $\mathrm{Bs}\,|3H| = \tilde{\pi}_{\tilde{Q}}^{-1}(\Delta_\infty) \cap H$. If $\Psi$ is a general member, then the divisor defined by $\Psi$ has at most rational double points on the base locus.

If $d \ge k$, then $\mathrm{Bs}\,|3H| = \emptyset$. and a general member of $|3H|$ is nonsingular by Bertini's theorem. □

**Lemma 6** *Let the notations and the conditions be as in* Lemma 5. *If $|3H|$ has a member $S'$ which has at most rational double points as the singularities, then the minimal resolution $S$ of $S'$ satisfies the following:*

*(1)* $K_S^2 = 24d - 24$, $p_g(S) = 7d - 4$ *and* $q(S) = 0$.
*(2)* *If $f : S \to \mathbb{P}^1$ is the pencil obtained as the pull back of the linear system $|\Gamma|$ on $Q_0$ by $S \to Q_0$, then $f$ is E-H special non-hyperelliptic fibration of genus* 4.
*(3)* $f_*\omega_{S/\mathbb{P}^1} \cong E$ *holds.*

*Proof* Since $K_{\tilde{Q}} \sim -2H + \tilde{\pi}_{\tilde{Q}}^*((d-2)\Gamma)$, we have $K_{S'} \sim (H + \tilde{\pi}_{\tilde{Q}}^*((d-2)\Gamma))_{|S'}$, which leads us to

$$K_S^2 = K_{S'}^2 = 3H\left(H + \tilde{\pi}_{\tilde{Q}}^*((d-2)\Gamma)\right)^2 = 24d - 24.$$

Consider the exact sequence

$$0 \to \mathcal{O}_{\tilde{Q}}(K_{\tilde{Q}}) \to \mathcal{O}_{\tilde{Q}}\left(H + \tilde{\pi}_{\tilde{Q}}^*((d-2)\Gamma)\right) \to \mathcal{O}_{S'}(K_{S'}) \to 0.$$

We obtain

$$p_g(S) = \dim H^0\left(\mathcal{O}_{\tilde{Q}}(H + \tilde{\pi}_{\tilde{Q}}^*((d-2)\Gamma))\right) = 7d - 4,$$
$$q(S) = \dim H^1\left(\mathcal{O}_{\tilde{Q}}(H + \tilde{\pi}_{\tilde{Q}}^*((d-2)\Gamma))\right) = 0.$$

(2) is trivial. Since $\tilde{\pi}_{\tilde{Q}*}\mathcal{O}_{\tilde{Q}}(K_{\tilde{Q}}) = 0$ and $R^1\tilde{\pi}_{\tilde{Q}*}\mathcal{O}_{\tilde{Q}}(K_{\tilde{Q}}) = 0$, we obtain

$$f_*\omega_{S/\mathbb{P}^1} \cong \mu_{Q_0*}\tilde{\pi}_{\tilde{Q}*}\mathcal{O}_{\tilde{Q}}\left(H + \tilde{\pi}_{\tilde{Q}}^*(d\Gamma)\right) \cong E. \tag{4}$$

□

By summarizing the above argument, we obtain the following:

**Theorem 3** *Let $S$ be a smooth projective surface which has E-H special non-hyperelliptic fibration $f : S \to \mathbb{P}^1$ of genus* 4. *Assume $K_{S/\mathbb{P}^1}^2 = (24/7)\Delta(f)$. Then there exist a non-negative integer $k$ and a positive integer $d$ such that $3d \ge 2k$ and*

$$f_*\omega_{S/\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2d + k) \oplus \mathcal{O}_{\mathbb{P}^1}(2d) \oplus \mathcal{O}_{\mathbb{P}^1}(2d - k) \oplus \mathcal{O}_{\mathbb{P}^1}(d)$$

*hold. Furthermore, $S$ is the minimal resolution of a general member $S' \in |3H|$, where $H$ is the tautological divisor of* $\mathbb{P}(\mathcal{O}_{\Sigma_k}(2\Delta_0 + (d-k)\Gamma) \oplus \mathcal{O}_{\Sigma_k})$.

*Remark 1* (1) Consider the case where $d = 2$. Let $S$ be the obtained surface. In this case $k$ is one of 0, 1, 2 and 3. Furthermore, we have that $K_S^2 = 24$ and $p_g(S) = 10$ hold and that $S$ is canonical. Hence, if $S$ is minimal, then it is one of the surfaces which are classified in [9] and [11].

When $k = 0$ or $k = 2$, $S$ is one of type (3a) in [11], or of type (Ic) in [9]. When $k = 1$, $S$ is of type (3b) in [11]. When $k = 3$, $S$ has a $(-1)$-curve which intersects with every fiber of $f$ at a point transversally, and the minimal model $S_0$ of $S$ satisfies $K_{S_0}^2 = 3p_g(S_0) - 5$.

(2) Consider the case where $d = 1$. Then $k = 0$ or $k = 1$. $S$ satisfies $K_S^2 = 0$ and $p_g(S) = 3$.

When $k = 0$, $S$ is an elliptic surface. In particular, $\kappa(S) = 1$, where $\kappa(S)$ is the Kodaira dimension of $S$. When $k = 1$, $S$ has three $(-1)$-curves which intersect with every fiber at a point transversally, and the minimal model $S_0$ satisfies $K_{S_0}^2 = 3$. Namely, $S_0$ can be seen in [8].

### 2.2 The case where $r = 2/7$

We have $E_0^\vee \cong E_0 \otimes L^{-1}$. In this case, $\deg \tilde{\delta} = 1$ holds. Then we have $M^{\otimes 2} \otimes L \cong (\det E_0) \otimes \mathcal{O}_{\mathbb{P}^1}(1)$. Hence, we obtain $\det E_0 \cong M^{\otimes 6} \otimes \mathcal{O}_{\mathbb{P}^1}(-3)$ and $L \cong M^{\otimes 4} \otimes \mathcal{O}_{\mathbb{P}^1}(-2)$.

By the same argument as in the previous section, we obtain the following:

**Theorem 4** *Let $S$ be a smooth projective surface which has E-H special non-hyperelliptic fibration $f : S \to \mathbb{P}^1$ of genus 4. Assume $K_{S/\mathbb{P}^1}^2 = (24/7)\Delta(f) + (2/7)$. Then there exist a positive integer $d$ and a non-negative integer $k$ such that $3d \geq 2k + 2$ and*

$$f_*\omega_{S/\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2d+k-1) \oplus \mathcal{O}_{\mathbb{P}^1}(2d-1) \oplus \mathcal{O}_{\mathbb{P}^1}(2d-k-1) \oplus \mathcal{O}_{\mathbb{P}^1}(d)$$

*hold. Furthermore, $S$ is the minimal resolution of a general member $S' \in |3H + \tilde{F}|$, where $H$ is the tautological divisor of $\mathbb{P}(\mathcal{O}_{\Sigma_k}(2\Delta_0 + (d-k-1)\Gamma) \oplus \mathcal{O}_{\Sigma_k})$, and $\tilde{F}$ the fiber of $\tilde{Q} \to \mathbb{P}^1$.*

*Remark 2* (1) Consider the case where $d = 1$. We have $k = 0$, which leads us to $K_S^2 = -10$ and $p_g(S) = 0$. In this case, $S$ is obtained by blowing-up at eighteen points of $\Sigma_2$. These points are the intersection of two curves in $|3\Delta_0|$.
(2) Consider the case where $d = 2$. We have that $k$ is one of 0, 1 and 2, and that $K_S^2 = 14$ and $p_g(S) = 7$ hold.
When $k = 0$, we have $\tilde{Q} \cong \mathbb{P}(\mathcal{O}_{\Sigma_0}(2\Delta_0 + \Gamma) \oplus \mathcal{O}_{\Sigma_0})$. Let $\gamma : \Sigma_0 \to \mathbb{P}^1$ be the natural projection which is different from $\mu_{Q_0} : \Sigma_0 \to \mathbb{P}^1$. Then $\gamma \circ (\tilde{\pi}_{\tilde{Q}}|_{S'}) \circ \psi : S \to \mathbb{P}^1$ is the non-hyperelliptic fibration of genus 3. In particular, $S$ is the Castelnuovo surface of type (0, 2, 2) in [1].

When $k = 1$, we have $\tilde{Q} \cong \mathbb{P}(\mathcal{O}_{\Sigma_1}(2\Delta_0) \oplus \mathcal{O}_{\Sigma_1})$. There is a birational morphism $\tilde{Q} \to \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2})$ which maps $\tilde{\pi}_{\tilde{Q}}^{-1}(\Delta_\infty)$ $(\cong \mathbb{P}^1 \times \mathbb{P}^1)$ onto a fiber $l$ of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}) \to \mathbb{P}^2$. Let $T'$ be the tautological divisor of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2})$ and $L$ the pull back of a line by $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}) \to \mathbb{P}^2$. Then the relative canonical image $S'$ is contained in $|3T + L|$, and contains $l$. In particular, $S$ is the special case of the Castelnuovo surface of the case (3) in [1, Theorem 1.5].

When $k = 2$, then $S$ contains a $(-1)$-curve, and $\tilde{Q} \cong \mathbb{P}(\mathcal{O}_{\Sigma_2}(2\Delta_0 - \Gamma) \oplus \mathcal{O}_{\Sigma_2})$. The minimal model $S_0$ satisfies $K_{S_0}^2 = 3p_g(S_0) - 6$. It is proved that $S_0$ is the surface of type I-0: (s.1) in [11].

### 2.3 The case where $r = 3/7$

We have $(\det E_0^\vee) \otimes \mathcal{O}_{\mathbb{P}^1}(1) \cong (\det E_0) \otimes (L^{-1})^{\otimes 3}$. Furthermore, $M^{\otimes 2} \otimes L \cong \det E_0$ holds. Hence, we obtain $\det E_0 \cong M^{\otimes 6} \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$ and $L \cong M^{\otimes 4} \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$. There exist a non-negative integer $k$ and a positive integer $d$ such that

$$E_0 \cong \mathcal{O}_{\mathbb{P}^1}(2d + k - 1) \oplus \mathcal{O}_{\mathbb{P}^1}(2d) \oplus \mathcal{O}_{\mathbb{P}^1}(2d - k),$$

$E \cong E_0 \oplus \mathcal{O}_{\mathbb{P}^1}(d)$ and $Q_0 \sim 2T_{E_0} - (4d - 1)\mathcal{F}$ hold.

Let $Z_0 \in H^0(\mathcal{O}_{\mathbb{P}(E_0)}(T_{E_0} - (2d + k - 1)\mathcal{F}))$, $Z_1 \in H^0(\mathcal{O}_{\mathbb{P}(E_0)}(T_{E_0} - 2d\mathcal{F}))$ and $Z_2 \in H^0(\mathcal{O}_{\mathbb{P}(E_0)}(T_{E_0} - (2d - k)\mathcal{F}))$ be global sections defining the homogeneous coordinates of each fiber of $\mu : \mathbb{P}(E_0) \to \mathbb{P}^1$. Then by the same argument as in Lemma 3, the base locus $B_1 := \mathrm{Bs}\,|2T_{E_0} - (4d - 1)\mathcal{F}|$ is defined by the equation $Z_0 = Z_1 = 0$. Let $\rho_0 : X \to \mathbb{P}(E_0)$ be the blow-up along $B_1$, and $\tilde{Q}_0$ the proper transform of $Q_0$ by $\rho_0$. We have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\rho_0} & \mathbb{P}(E_0) \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \mu} \\ \Sigma_{k-1} & \xrightarrow[\zeta]{} & \mathbb{P}^1. \end{array}$$

Let $\xi$ and $\zeta$ be as in the above diagram.

**Lemma 7** *Let the notaions be as above. Then $\xi_{|\tilde{Q}_0} : \tilde{Q}_0 \to \Sigma_{k-1}$ is the blow-up at a point $q \in \Delta_\infty$, where $\Delta_\infty$ is the section of $\zeta$ with $\zeta_* \mathcal{O}_{\Sigma_{k-1}}(\Delta_\infty) \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1 - k)$.*

*Proof* Let $\mathbb{E}_{Q_0} := \rho_0^{-1}(B_1)$ be the exceptional divisor. Then $\rho_0^* T_{E_0} \sim \mathbb{E}_{Q_0} + \xi^*(\Delta_0 + 2d\Gamma)$ holds, where $\Delta_0$ is the tautological divisor of $\Sigma_{k-1}$ and $\Gamma$ the fiber of $\zeta$. Namely, $\Delta_0 \sim \Delta_\infty + (k - 1)\Gamma$. Let $\tilde{Z}_0 \in H^0(\mathcal{O}_X(\mathbb{E}_{Q_0}))$ be the global section defining $\mathbb{E}_{Q_0}$, and $\tilde{Z}_\infty \in H^0(\mathcal{O}_X(\rho_0^* T_{E_0} - (2d - k)\xi^*\Gamma))$ the global section such that they define the homogeneous coordinates of each fiber of $\xi$. Then the global section $\tilde{\Phi}$ defining $\tilde{Q}_0$ is written as

$$\tilde{\Phi} = \tilde{\phi}_0 \tilde{Z}_0 + \tilde{\phi}_\infty \tilde{Z}_\infty,$$

where $\tilde{\phi}_0 \in H^0(\mathcal{O}_{\Sigma_{k-1}}(2\Delta_0 + \Gamma))$ and $\tilde{\phi}_\infty \in H^0(\mathcal{O}_{\Sigma_{k-1}}(\Delta_\infty))$. Let $q \in \Sigma_{k-1}$ be the intersection point of $\Delta_\infty$ and the divisor defined by $\tilde{\phi}_0$, and put $p := \zeta(q)$. Then the fiber $F'$ over $p$ by $\tilde{Q}_0 \to B$ consists with two rational curves one of which is the fiber $\xi^{-1}(q)$. Any other fiber is isomorphic to $\mathbb{P}^1$. Since $F'$ is the degenerate fiber of $\tilde{Q}_0$, $p$ is the discriminant locus of $Q_0$. Furthermore, $\xi^{-1}(q)$ is contracted to a point $q$, and we are done. □

**Lemma 8** *Put $\nu : \tilde{Q}_0 \to \Sigma_{k-1}$ instead of $\xi_{|\tilde{Q}_0}$ in* Lemma 7*, and let $\tilde{\mathbb{E}} := \nu^{-1}(q)$ be the exceptional curve. Then we have*

$$\tilde{Q} \cong \mathbb{P}(\mathcal{O}_{\tilde{Q}_0}(\nu^*(2\Delta_0 + (d - k + 1)\Gamma) - \tilde{\mathbb{E}}) \oplus \mathcal{O}_{\tilde{Q}_0}).$$

*Proof* Since $\tilde{W} \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}(E_0)}(T_{E_0}) \oplus \mathcal{O}_{\mathbb{P}(E_0)}(d\tilde{F}))$, it is sufficient to prove $\mathcal{O}_{\tilde{Q}_0}(T_{E_0}) \cong \mathcal{O}_{\tilde{Q}_0}(\nu^*(2\Delta_0 + (2d - k + 1)\Gamma) - \tilde{\mathbb{E}})$.

By the same argument as the proof of Lemma 4, it is sufficient to prove $\mathcal{O}_{\tilde{Q}_0}(\mathbb{E}_{Q_0}) \cong \mathcal{O}_{\tilde{Q}_0}(\nu^*\Delta_\infty - \tilde{\mathbb{E}})$. This isomorphism is obtained by the equation of $\tilde{Q}_0$ in $\mathbb{P}(E_0)$ which can be seen in the proof of Lemma 7. □

Let $H$ be the tautological divisor of $\tilde{Q}$, namely,

$$\tilde{\pi}_{\tilde{Q}*}\mathcal{O}_{\tilde{Q}}(H) \cong \mathcal{O}_{Q_0}(\nu^*(2\Delta_0 + (d-k+1)\Gamma) - \tilde{\mathbb{E}}) \oplus \mathcal{O}_{Q_0}$$

holds. If $S' \subset \tilde{Q}$ is the image of $S$ by $\psi : S \to \tilde{Q}$, then $S' \sim 3H$.

**Lemma 9** *Let the notations be as above. Then a general member of* $|3H|$ *is irreducible and has at most rational double points as the singularities if and only if* $3d \geq 2k$.

*Proof* Let $X_0 \in H^0(\mathcal{O}_{\tilde{Q}}(H))$ and $X_\infty \in H^0(\mathcal{O}_{\tilde{Q}}(\mathbb{E}_Q))$ be global sections defining the homogeneous coordinates of each fiber of $\tilde{\pi}_{\tilde{Q}}$. Then any global section $\Psi \in H^0(\mathcal{O}_{\tilde{Q}}(3H))$ is written as

$$\Psi = c_0X_0^3 + \psi_1X_0^2X_\infty + \psi_2X_0X_\infty^2 + \psi_3X_\infty^3,$$

where $c_0$ is the constant and $\psi_i \in H^0(\mathcal{O}_{Q_0}(\nu^*(3i\Delta_0 + i(d-k+1)\Gamma) - i\mathbb{E}))$ $(i = 1, 2, 3)$. Let $\tilde{\Delta}_\infty$ be the proper transform of $\Delta_\infty$ by $\rho_0$.

If $3d < 2k$, then $|\nu^*(6\Delta_0+(3d-3k+3)\Gamma)-3\tilde{\mathbb{E}}| = 2\tilde{\Delta}_\infty+|\nu^*(4\Delta_0+(3d-k+1)\Gamma)-\tilde{\mathbb{E}}|$ and $|\nu^*(4\Delta_0+(2d-2k+2)\Gamma)-2\tilde{\mathbb{E}}| = \tilde{\Delta}_\infty+|\nu^*(3\Delta_0+(2d-k+1)\Gamma)-\tilde{\mathbb{E}}|$, and hence, any member of $|3H|$ is singular along $\tilde{\pi}_{\tilde{Q}}^{-1}(\tilde{\Delta}_\infty) \cap H$.

Assume $3d \geq 2k$. If $d < k$, then we have $\operatorname{Bs}|3H| = \tilde{\pi}_{\tilde{Q}}^{-1}(\tilde{\Delta}_\infty) \cap H$, and a general member of $|H|$ has rational double points on the base locus. If $d \geq k$, then $\operatorname{Bs}|3H| = \emptyset$ and a general member of $|3H|$ is irreducible and nonsingular. □

**Lemma 10** *Let the notations and the conditions be as in* Lemma 9. *If* $|3H|$ *has a member* $S'$ *which has at most rational double points as the singularities, the minimal resolution* $S$ *of* $S'$ *satisfies the following:*

*(1)* $K_S^2 = 24d - 27$, $p_g(S) = 7d - 5$ *and* $q(S) = 0$.
*(2) If* $f : S \to \mathbb{P}^1$ *is the pencil defined by* $\tilde{Q} \to \mathbb{P}^1$, *then* $f$ *is E-H special non-hyperelliptic fibration of genus* 4.
*(3)* $f_*\omega_{S/\mathbb{P}^1} \cong E$.

*Proof* (1) and (2) are obtained by the same argument as the proof of Lemma 6. Furthermore, by the same argument of Lemma 6, we have

$$\begin{aligned} f_*\omega_{S/\mathbb{P}^1} &\cong \mu_{Q_0*}\tilde{\pi}_{\tilde{Q}*}\mathcal{O}_{\tilde{Q}}(H + \nu^*(d\Gamma)) \\ &\cong \mu_{Q_0*}(\mathcal{O}_{Q_0}(\nu^*(2\Delta_0 + (2d-k)\Gamma) + \tilde{\Gamma}) \oplus \mathcal{O}_{Q_0}(\nu^*(d\Gamma))). \end{aligned}$$

Remember $Q_0 \subset \mathbb{P}(E_0)$ and that

$$T_{E_0}|_{Q_0} \sim \nu^*(2\Delta_0 + (2d-k+1)\Gamma) - \tilde{\mathbb{E}}$$

holds. (See Lemma 8) Consider the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}(E_0)}(T_{E_0} - Q_0) \to \mathcal{O}_{\mathbb{P}(E_0)}(T_{E_0}) \to \mathcal{O}_{Q_0}(\nu^*(2\Delta_0 + (2d-k)\Gamma) + \tilde{\Gamma}) \to 0.$$

Since $\mu_*\mathcal{O}_{\mathbb{P}(E_0)}(T_{E_0} - Q_0) = 0$ and $R^1\mu_*\mathcal{O}_{\mathbb{P}(E_0)}(T_{E_0} - Q_0) = 0$, we obtain

$$\mu_*\mathcal{O}_{\mathbb{P}(E_0)}(T_{E_0}) \cong \mu_{Q_0*}\mathcal{O}_{Q_0}(\nu^*(2\Delta_0 + (2d-k)\Gamma) + \tilde{\Gamma}),$$

and hence,

$$f_*\omega_{S/\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2d+k-1) \oplus \mathcal{O}_{\mathbb{P}^1}(2d) \oplus \mathcal{O}_{\mathbb{P}^1}(2d-k) \oplus \mathcal{O}_{\mathbb{P}^1}(d),$$

holds. □

Summarizing the above argument, we obtain the following:

**Theorem 5** *Let $S$ be a smooth projective surface which has E-H special non-hyperelliptic fibration of genus 4. Assume $K^2_{S/\mathbb{P}^1} = (24/7)\Delta(f) + (3/7)$. Then there exist positive integers $d$ and $k$ such that $3d \geq 2k$ and*

$$f_*\omega_{S/\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2d+k-1) \oplus \mathcal{O}_{\mathbb{P}^1}(2d) \oplus \mathcal{O}_{\mathbb{P}^1}(2d-k) \oplus \mathcal{O}_{\mathbb{P}^1}(d)$$

*hold. There exists a fiber $F_0$ of $f$ which is obtained as the complete intersection of a quadric hypersuface of rank 2 and a cubic hypersurface. Put $p_0 := f(F_0)$. Let $\Sigma_{k-1}$, $\Delta_0$ and $\Delta_\infty$ be as before, and let $\Gamma_0$ be a fiber of $\Sigma_{k-1} \to \mathbb{P}^1$ over $p_0$, and $q_0$ the intersection point of $\Delta_\infty$ and $\Gamma_0$. Furthermore, let $\nu : Q_0 \to \Sigma_{k-1}$ be the blow-up at $q_0$, and put $\tilde{\mathbb{E}} := \nu^{-1}(q_0)$. Then $S$ is obtained as a minimal resolution of a general member $S' \in |3H|$, where $H$ is the tautological divisor of $\mathbb{P}(\mathcal{O}_{Q_0}(\nu^*(2\Delta_0 + (d-k+1)\Gamma) - \tilde{\mathbb{E}}) \oplus \mathcal{O}_{Q_0})$.*

*Remark 3* Consider the case where $d = 2$. We have that $k$ is one of $1, 2$ and $3$, and that $K_S^2 = 21$ and $p_g(S) = 9$ hold. When $k = 1,\ 2$, $S$ is of type (3b) in [11]. When $k = 3$, $S$ has a $(-1)$-curve, and the minimal modle $S_0$ satisfies $K^2_{S_0} = 3p_g(S_0) - 5$.

### 2.4 The case where $r = 4/7$

We have $E_0^\vee \cong E_0 \otimes L^{-1}$ and $\deg \tilde{\delta} = 2$. Hence, $M^{\otimes 2} \otimes L \cong (\det E_0) \otimes \mathcal{O}_{\mathbb{P}^1}(2)$ holds, and hence, we obtain $L \cong M^{\otimes 2} \otimes \mathcal{O}_{\mathbb{P}^1}(-4)$ and $\det E_0 \cong M^{\otimes 6} \otimes \mathcal{O}_{\mathbb{P}^1}(-6)$.

By the same argument as Sect. 2.1, we obtain the following:

**Theorem 6** *Let $S$ be a smooth projective surface which has E-H special non-hyperelliptic fibration $f : S \to \mathbb{P}^1$ of genus 4. Assume $K^2_{S/\mathbb{P}^1} = (24/7)\Delta(f) + (4/7)$. Then there exist integers $d > 1$ and $k \geq 0$ such that $3d \geq 2k + 4$ and*

$$f_*\omega_{S/\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2d+k-2) \oplus \mathcal{O}_{\mathbb{P}^1}(2d-2) \oplus \mathcal{O}_{\mathbb{P}^1}(2d-k-2) \oplus \mathcal{O}_{\mathbb{P}^1}(d)$$

*hold. Furthermore, $S$ is the minimal resolution of a general member $S' \in |3H + 2\tilde{F}|$, where $H$ is the tautological divisor of $\mathbb{P}(\mathcal{O}_{\Sigma_k}(2\Delta_0 + (d-k-2)\Gamma) \oplus \mathcal{O}_{\Sigma_k})$.*

*Remark 4* Consider the case $d = 2$. In this case, $K_S^2 = 4$ and $p_g(S) = 4$ hold. We have $k = 0$ or $1$.

(I) When $k = 0$, $S$ has the double cover structure $h : S \to \Sigma_2$ whose branch locus $\mathcal{B}$ is linearly equivalent to $6\Delta_0$. In particular, $S$ is of type (2) for $p_g(S) = 4$ in [7].

If $S$ is generic among the surfaces we consider, then $S$ has two E-H special non-hyperelliptic fibrations of genus 4. One of them is mapped onto another one by the involution induced from the double cover.

On the other hand, if $S$ is special, then $S$ has only one E-H special non-hyperelliptic fibration of genus 4, which is mapped onto itself by the involution.

(II) When $k = 1$, $S$ has a $(-1)$-curve which is mapped onto $\mathbb{P}^1$ isomorphically by $f$. If $S_0$ is the minimal model of $S$, then $S_0$ is birational to a quintic surface which has a rational double point of type $A_3$. Namely, $S_0$ is the special surface among the surfaces in [6].

In order to avoid the confusion about the notations, we mention these cases in Sect. 4.

### 2.5 The case where $r = 5/7$

We have $E_0^\vee \otimes \mathcal{O}_{\mathbb{P}^1}(1) \cong E_0 \otimes L^{-1}$, and $M^{\otimes 2} \otimes L \cong (\det E_0) \otimes \mathcal{O}_{\mathbb{P}^1}(1)$. Hence, we obtain $L \cong M^{\otimes 4} \otimes \mathcal{O}_{\mathbb{P}^1}(-5)$ and $\det E_0 \cong M^{\otimes 6} \otimes \mathcal{O}_{\mathbb{P}^1}(-6)$.

By the same argument as in Sects. 2.1 and 2.3, we obtain the following:

**Theorem 7** *Let $S$ be a smooth projective surface with E-H special non-hyperelliptic fibration $f : S \to \mathbb{P}^1$ of genus 4. Assume $K^2_{S/\mathbb{P}^1} = (24/7)\Delta(f) + (5/7)$. Then there exist positive integers $d > 1$ and $k$ such that $3d \geq 2k + 3$ and*

$$f_*\omega_{S/\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2d + k - 2) \oplus \mathcal{O}_{\mathbb{P}^1}(2d - 1) \oplus \mathcal{O}_{\mathbb{P}^1}(2d - k - 1) \oplus \mathcal{O}_{\mathbb{P}^1}(d).$$

*Furthermore, $S$ is the minimal resolution of a general member $S' \in |3H + \tilde{F}|$, where $H$ is the tautological divisor of $\mathbb{P}(\mathcal{O}_{Q_0}(\nu^*(2\Delta_0 + (d - k)\Gamma) - \tilde{\mathbb{E}}) \oplus \mathcal{O}_{Q_0})$ in the same notations as in* Theorem 5.

*Remark 5* Consider the case where $d = 2$. We have $k = 1$, and $K_S^2 = 11$, $p_g(S) = 6$. In this case, $Q_0$ is obtained by blowing-up at a point of $\Sigma_0$ ($\cong \mathbb{P}^1 \times \mathbb{P}^1$). Let $p_i : \Sigma_0 \to \mathbb{P}^1$ be the natural projection for $i = 1, 2$. We may assume the pull back of the linear pencil $p_1 : \Sigma_0 \to \mathbb{P}^1$ to $S$ coincides with $f : S \to \mathbb{P}^1$. If $g : S \to \mathbb{P}^1$ is the linear pencil induced from $p_2$, then $g$ is the non-hyperelliptic fibration of genus 3. By the similar argument to the proof of Lemma 7, we obtain

$$p_{2*}\nu_*\mathcal{O}_{Q_0}(\nu^*(\Delta_0 + \Gamma) + \tilde{\Delta}_0) \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1),$$

where $\tilde{\Delta}_0$ is the proper transform of $\Delta_0$ by $\nu$. By considering the isomorphism, we obtain that the surface $S$ in this case is the Castelnuovo surface of type $(0, 1, 2)$ in [1].

### 2.6 The case where $r = 6/7$

There are two cases. First one is $\deg \tilde{\delta} = 3$ and $\deg \mathrm{Discr}(Q_0) = 0$. If $S$ is this case, we call $S$ is of type (I). Second one is $\deg \tilde{\delta} = 0$ and $\deg \mathrm{Discr}(Q_0) = 2$. If $S$ is this case, we call $S$ is of type (II).

#### *2.6.1 Type (I)*

By the same argument as in Sect. 2.1, we obtain the following:

**Theorem 8** *Let $S$ be a surface of type (I) as above. Then the non-negative integers $d$ and $k$ satisfy $3d \geq 2k + 6$. Furthermore, $S$ is the minimal resolution of a general menber of $|3H + 2\tilde{F}|$, where $H$ is the tautological divisor of $\mathbb{P}(\mathcal{O}_{\Sigma_k}(2\Delta_0 + (d - k - 2)\Gamma) \oplus \mathcal{O}_{\Sigma_k})$.*

*Remark 6* Consider the case $d = 2$. We have $k = 0$ and in this case $S$ has the Eisenbud-Harris *general* fibration of genus 4. This case can be seen in [11, type I-1:4.5].

#### *2.6.2 Type (II)*

Let $S$ be of a surface of type (II). Then we have the following two cases:

(1) $f_*\omega_{S/\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2d + k - 1) \oplus \mathcal{O}_{\mathbb{P}^1}(2d) \oplus \mathcal{O}_{\mathbb{P}^1}(2d - k - 1) \oplus \mathcal{O}_{\mathbb{P}^1}(d)$,
(2) $f_*\omega_{S/\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2d + k) \oplus \mathcal{O}_{\mathbb{P}^1}(2d - 1) \oplus \mathcal{O}_{\mathbb{P}^1}(2d - k - 1) \oplus \mathcal{O}_{\mathbb{P}^1}(d)$.

If $S$ satisfies (1), then we call $S$ is of type (IIa). If $S$ satisfies (2), then we call $S$ is of type (IIb).

Let $S$ be of type (IIa). If $S$ is generic, then $Q_0$ is obtained by blowing-up two different points $q_1$ and $q_2$ in the section $\Delta_\infty$ of $\mu : \Sigma_{k-1} \to \mathbb{P}^1$. Let $\nu : Q_0 \to \Sigma_{k-1}$ be the blow-up at $q_1$ and $q_2$, and put $\mathbb{E}_i := \nu^{-1}(q_i)$ $(i = 1, 2)$.

By the same argument as in Sect. 2.3, we obtain the following:

**Theorem 9** *Let $S$ be of type (IIa). Then the non-negative integers $d$ and $k$ satisfy $3d \geq 2k+2$. Furthermore, $S$ is the minimal resolution of a general member of $|3H|$, where $H$ is the tautological divisor of $\mathbb{P}(\mathcal{O}_{Q_0}(\nu^*(2\Delta_0 + (d-k+1)\Gamma) - \mathbb{E}_1 - \mathbb{E}_2) \oplus \mathcal{O}_{Q_0})$.*

Let $S$ be of type (IIb). Then $Q_0 \to \mathbb{P}^1$ has one degenerate fiber. Let $\mathcal{F}_0$ be the degenerate fiber of $Q_0$. Then $\mathcal{F}_0$ is written as $\mathcal{F}_0 = C_1 + C_2$, where $C_i$ is a line in the fiber $\tilde{F}_0 (\cong \mathbb{P}^2)$ of $\mu : \mathbb{P}(E_0) \to \mathbb{P}^1$. We have that $C_1 C_2 = 1$ in $\tilde{F}_0$ holds, and that $Q_0$ has a rational double point of type $A_1$ at the intersection point of $C_1$ and $C_2$.

Let $\tilde{q}$ be the singularity of $Q_0$. If we put $B_1 := \mathbb{P}(E_0/(\mathcal{O}_{\mathbb{P}^1}(2d+k) \oplus \mathcal{O}_{\mathbb{P}^1}(2d-1)))$ $(\subset \mathbb{P}(E_0))$, we have $\mathrm{Bs}\,|2T_{E_0} - (4d-2)\tilde{F}| = B_1$. Let $\rho_0 : X \to \mathbb{P}(E_0)$ be the blow-up along $B_1$. Then we obtain the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\rho_0} & \mathbb{P}(E_0) \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \mu} \\ \Sigma_{k+1} & \xrightarrow[\zeta]{} & \mathbb{P}^1. \end{array}$$

Let $\xi$ and $\zeta$ be as in the above diagram. The proper transform $\tilde{Q}_0$ of $Q_0$ is smooth. Put $\nu_1 : \tilde{Q}_0 \to Q_0$ and $\nu := \xi_{|\tilde{Q}_0}$. Then $\nu_1^{-1}(\mathcal{F}_0)$ is written as

$$\nu_1^{-1}(\tilde{F}_0) = \tilde{C}_1 + \tilde{C}_2 + \tilde{C},$$

where $\tilde{C}_i$ is the proper transform of $C_i$ by $\nu_1$ $(i = 1, 2)$, and $\tilde{C}$ is the inverse image of the singularity of $Q_0$ by $\nu_1$. Note that $\tilde{C}_i$ is contracted by $\nu$ to a point which is not contained in $\Delta_\infty$ $(\subset \Sigma_{k+1})$, and $\tilde{C}$ is mapped isomorphically onto a fiber of $\zeta$ by $\xi_{|\tilde{Q}_0}$.

By the same argument as in Sect. 2.3, we obtain the following:

**Theorem 10** *Let $S$ be of type (IIb) and let the notations be as above. Then two positive integers $d$ and $k$ satisfy $3d \geq 2k+2$. Furthermore, $S$ is the minimal resolution of a general member of $|3H|$, where $H$ is the tautological divisor of $\mathbb{P}(\mathcal{O}_{\tilde{Q}_0}(\nu^*(2\Delta_0 + (d-k+1)\Gamma) - \tilde{C}_1 - \tilde{C}_2) \oplus \mathcal{O}_{\tilde{Q}_0})$.*

*Remark 7* (1) Assume $d = 2$, Then we have $k = 1, 2$ when $S$ is of type (IIa), and that $k = 0, 1$ when $S$ is of type (IIb). Furthermore, we have $K_S^2 = 18$ and $p_g(S) = 8$. In either case, $S$ is the surface of type (3b) in [11].
(2) Assume $d = 1$. Then we have $k = 0$ in both cases of type (IIa) and (IIb). In this case, the surface $S$ is an elliptic surface with six $(-1)$-curves. Each $(-1)$-curve intersects with each fiber of $f$ at one point transversally. Furthermore, we have $p_g(S) = 1$.

## 3 Further examples

*Example 1* Let $\mu : \Sigma_k \to \mathbb{P}^1$ be the Hirzeburch surface, and $\Delta_0$, $\Delta_\infty$ and $\Gamma$ as before. Furthermore, let $\Gamma_1, \ldots, \Gamma_l$ be fibers of $\mu$, and $p_i$ the intersection point of $\Delta_\infty$ and $\Gamma_i$

($i = 1, \dots, l$). Consider the blow-up $\rho : Q_0 \to \Sigma_k$ at $\{p_i\}_{1 \le i \le l}$. Put $\mathbb{E}_i := \rho^{-1}(p_i)$, and let $\tilde{\Gamma}_i$ and $\tilde{\Delta}_\infty$ be the proper transforms of $\Gamma_i$ and $\Delta_\infty$ by $\rho$, respectively. Put $D := \rho^*(2\Delta_0 + a\Gamma) - \sum_{i=1}^{l} \mathbb{E}_i$ for an integer $a$. Let $H_0$ be the tautological divisor of the $\mathbb{P}^1$-bundle $\tilde{\pi} : \tilde{Q} := \mathbb{P}(\mathcal{O}_{Q_0}(D) \oplus \mathcal{O}_{Q_0}) \to Q_0$, $H_\infty$ the divisor with $H_0 \sim H_\infty + \tilde{\pi}^* D$ and $\tilde{F}$ the fiber of $\tilde{Q} \to \mathbb{P}^1$.

Consider the complete linear system $|3H_0 + n\tilde{F}|$. Let $Z_0$ and $Z_\infty$ be global sections for $H_0$ and $H_\infty$, respectively, which define the homogeneous coordinates of each fiber of $\tilde{\pi}$. If $\Psi$ is the global section for $3H_0 + n\tilde{F}$, then $\Psi$ is written as

$$\Psi = \psi_0 Z_0^3 + \psi_1 Z_0^2 Z_\infty + \psi_2 Z_0 Z_\infty^2 + \psi_3 Z_\infty^3,$$

where $\psi_i$ is the global section for $\rho^*(2i\Delta_0 + (ia + n)\Gamma) - i\sum_{i=1}^{l} \mathbb{E}_i$ ($i = 0, 1, 2, 3$). Put $\Lambda := |\rho^*(6\Delta_0 + (3a + n)\Gamma) - 3\sum_{i=1}^{l} \mathbb{E}_i|$. Note that $\tilde{\Delta}_\infty$ and $\cup_{i=1}^{l} \tilde{\Gamma}_i$ are disjoint, and $\rho^*(6\Delta_0 + (3a + n)\Gamma) - 3\sum_{i=1}^{l} \mathbb{E}_i$ is linearly equivalent to $\rho^*(6\Delta_0 + (3a + n - 3l)\Gamma) + \sum_{i=1}^{l} \tilde{\Gamma}_i$ and $\rho^*(3\Delta_0 + (3a + n + 3k)\Gamma) + 3\tilde{\Delta}_\infty$. Hence, if $3a + n \ge 3l$, then we have $\operatorname{Bs}|3H_0 + n\tilde{F}| = \emptyset$, and hence, there exists an irreducible and nonsingular member $S \in |3H_0 + n\tilde{F}|$. We say that the surface $S$ is of type $(n, l)$. By using the adjunction formula and easy calculations, we obtain

$$K^2_{S/\mathbb{P}^1} = 24(k + a) - 3l + 14n.$$

By considering the exact sequence

$$0 \to \mathcal{O}_{\tilde{Q}}(K_{\tilde{Q}}) \to \mathcal{O}_{\tilde{Q}}(S + K_{\tilde{Q}}) \to \mathcal{O}_S(K_S) \to 0,$$

we obtain

$$\Delta(f) = 7(k + a) - l + 4n.$$

Since every fiber with positive index in the above examples is the same type as one of the fibers in the previous section, we have that the multiplicative map for every surface in the above example is surjective. Furthermore, it is proved that E-H special fibration for every surface in the above example is relatively minimal but not locally trivial.

We obtain the following theorem:

**Theorem 11** *Let* $(x,\ y)$ *be an arbitrary pair of the positive integers with the following properties:*

*(i)* $(24/7)x \le y \le (18/5)x$.
*(ii)* $y \ne (24/7)x + (1/7)$.
*(iii) Assume* $7y - 24x$ *is odd. Put* $n := (7y - 24x - 3)/7$.

*When* $n \equiv 0 \pmod 3$*, then* $x \ge (5/3)n + 7$.
*When* $n \equiv 1 \pmod 3$*, then* $x \ge (5n + 28)/3$.
*When* $n \equiv 2 \pmod 3$*, then* $x \ge (5n + 35)/3$.
*Then there exists a smooth projective surface S with the following properties:*

*(1) S has E-H special non-hyperelliptic fibration* $f : S \to \mathbb{P}^1$ *of genus* 4.
*(2)* $f$ *is relatively minimal but not locally trivial.*
*(3)* $\Delta(f) = x$ *and* $K^2_{S/\mathbb{P}^1} = y$ *hold.*
*(4) The multiplicative map* $\operatorname{Sym}^2 f_*\omega_{S/\mathbb{P}^1} \to f_*\omega_{S/\mathbb{P}^1}^{\otimes 2}$ *is surjective.*

*Proof* First, consider the case of type $(n, 0)$. Namely, consider the surface with $K^2_{S/\mathbb{P}^1} = (24/7)\Delta(f) + (2n)/7$ for some $n \in \mathbb{N}$. If $S$ is one of the surfaces, then we have $K^2_{S/\mathbb{P}^1} = 24(k+a)+14n$ and $\Delta(f) = 7(k+a)+4n$. Since $3a+n \geq 0$, we obtain $K^2_{S/\mathbb{P}^1} \geq 24k+6n$ and $\Delta(f) \geq 7k+(5/3)n$. Consider the case where $K^2_{S/\mathbb{P}^1} = 24k+6n$ and $\Delta(f) = 7k+(5/3)n$. If $k$ moves in $\mathbb{Z}_{\geq 0}$, then we obtain the surface $S$ with $(\Delta(f),\ K^2_{S/\mathbb{P}^1}) = (x,\ y)$ for any lattice point $(x,\ y)$ with $y = (24/7)x + (2n)/7 \leq (18/5)x$.

Next, consider the case of type $(n,\ 1)$. Namely, consider the surface with $K^2_{S/\mathbb{P}^1} = (24/7)\Delta(f) + (2n+3)/7$ for some $n \in \mathbb{Z}_{\geq 0}$. If $S$ is one of the surfaces, then we have $K^2_{S/\mathbb{P}^1} = 24(k+a) - 3 + 14n$ and $\Delta(f) = 7(k+a) - 1 + 4n$. Since $3a+n \geq 3$, we obtain $K^2_{S/\mathbb{P}^1} \geq 24k + 6n + 21$ and $\Delta(f) \geq 7k + (5/3)n + 7$. Consider the case where $K^2_{S/\mathbb{P}^1} = 24k + 6n + 21$ and $\Delta(f) = 7k + (5/3)n + 7$. Then the statement of the theorem is obtained by easy calculations. □

*Example 2* There exist another examples with $K^2_{S/\mathbb{P}^1} = (18/5)\Delta(f)$.

Let the notations be as above. Furthermore, let $\mathcal{D} \in |2\Delta_0|$ be the irreducible, nonsingular member and $\Gamma_i$ $(i = 1, \ldots, l)$ the fiber of $\mu$. We may assume $\mathcal{D}$ and $\Gamma_i$ intersect at two points $p_{ij}$ $(i = 1, \ldots, l,\ j = 1,\ 2)$ transversally. Let $\nu : Q_0 \to \Sigma_k$ be the blow-up at $\{p_{ij}\}_{1\leq i\leq l,\ j=1,2}$ and $\mathbb{E}_{ij} := \nu^{-1}(p_{ij})$ the exceptional curve. Put $L := \nu^*(2\Delta_0 + a\Gamma) - \sum_{i=1}^{l}\sum_{j=1}^{2}\mathbb{E}_{ij}$, and let $\tilde{\pi} : \tilde{Q} := \mathbb{P}(\mathcal{O}_{Q_0}(L) \oplus \mathcal{O}_{Q_0}) \to Q_0$ be the $\mathbb{P}^1$-bundle, $H_0$ the tautological divisor of $\tilde{Q}$. By the same argument as above, we obtain that $\mathrm{Bs}\,|3H_0| = \emptyset$ when $a \geq l$ and that $K^2_{S/\mathbb{P}^1} = 24(k+a) - 6l$ and $\Delta(f) = 7(k+a) - 2l$ hold. If we put $k = 0$ and $l = a$, then we obtain the surface $S$ with $K^2_{S/\mathbb{P}^1} = (18/5)\Delta(f)$.

In this case, every fiber with positive index is the same type as the fiber which can be seen in Sect. 2.6 (IIb). Hence, the multiplicative map is surjective.

## 4 Appendix

In this section, we consider the surfaces which appear in Remark 4 in detail.

### 4.1 The case (I)

In this subsection, we argue the case that $S$ is in Remark 4 (I). Let the notations be as in the remark. We have the morphism

$$\psi : S \to \tilde{Q} := \mathbb{P}(\mathcal{O}_{\Sigma_0}(2\Delta_0) \oplus \mathcal{O}_{\Sigma_0}),$$

which maps $S$ onto the image birationally. Note that

$$\tilde{Q} \cong \Sigma_0 \times_{\mathbb{P}^1} \Sigma_2 \cong \Sigma_2 \times \mathbb{P}^1$$

holds. If $\xi : \tilde{Q} \to \mathbb{P}^1$ is the morphism defining $f : S \to \mathbb{P}^1$, and if $\tilde{F}$ is a fiber of $\xi$, then the image $S' = \psi(S)$ satisfies $S' \sim 3H + 2\tilde{F}$, where $H$ is the tautological divisor of the $\mathbb{P}^1$-bundle $Q \to \Sigma_0$.

Note that $\Phi_{|K_S|} = \Phi_{|H|} \circ \psi$ holds and that $\Phi_{|H|}$ coincides with the projection $\tilde{Q} \to \Sigma_2$. Hence, $\Phi_{|K_S|}$ gives the double cover $h : S \to \Sigma_2$, whose branch locus is linearly equivalent to $6\Delta_0$, where $\Delta_0$ is the tautological divisor of $\Sigma_2$. We can consider $\tilde{F}$ as the tautological divisor of the $\mathbb{P}^1$-bundle $\zeta : \tilde{Q} \to \Sigma_2$. Let $X_0,\ X_\infty \in H^0(\mathcal{O}_{\tilde{Q}}(\tilde{F}))$ be different two global sections. Then they define the homogeneous coordinates of each fiber of $\zeta$.

Since $\zeta_*\mathcal{O}_{\tilde{Q}}(\tilde{F}) \cong \mathcal{O}_{\Sigma_2}^{\oplus 2}$, the global section $\Psi$ defining $S'$ can be written as

$$\Psi = \psi_0 X_0^2 + \psi_1 X_0 X_\infty + \psi_2 X_\infty^2,$$

where $\psi_i \in H^0(\mathcal{O}_{\Sigma_2}(3\Delta_0))$ $(i = 0,\ 1,\ 2)$. The branch locus $\mathcal{B}$ of $h$ is defined by the discriminant $\psi_1^2 - 4\psi_0\psi_2$. Let $\tilde{F}_{(\alpha:\beta)}$ be a fiber of $\xi$ corresponding to $(\alpha : \beta) \in \mathbb{P}^1$. Then $\tilde{F}_{(\alpha:\beta)}$ is defined by

$$\alpha^2\psi_0 + \alpha\beta\psi_1 + \beta^2\psi_2 \in H^0(\mathcal{O}_{\Sigma_2}(3\Delta_0)).$$

By an easy calculation, we obtain that $\mathcal{B}$ and $\tilde{F}_{(\alpha:\beta)}$ intersect each other with intersection multiplicity two at each intersection point of them.

Consider the case of $\psi_1 \neq 0$. We may assume neither $\psi_0$ nor $\psi_2$ can be written as $c\psi_1$ with a constant $c$. If $S$ is sufficiently general, we may assume $\mathcal{B}$ is not the sum of two divisors both of which are linearly equivalent to $3\Delta_0$. If $(\alpha : \beta)$ varies, then the intersection points also vary. By Horikawa's result (cf. [7]), $\mathcal{B}$ has at most simple singularities. Hence, a general fiber $\tilde{F}_{(\alpha:\beta)}$ contacts with $\mathcal{B}$ at eighteen points with intersection multiplicity two. In particular, we obtain the following:

**Theorem 12** *Let the notations and conditions be as above. Assume $\psi_1 \neq 0$. Then $S$ has another E-H special non-hyperelliptic fibration of genus 4 other than $f$.*

*Proof* The pull back of the image of $\tilde{F}_{(\alpha:\beta)}$ by $h$ can be written as $\tilde{F}_{(\alpha:\beta)} + \tilde{F}'_{(\alpha:\beta)}$, where $\tilde{F}'_{(\alpha:\beta)}$ is also an E-H special non-hyperelliptic curve of genus 4. Note that $\tilde{F}_{(\alpha:\beta)}$ and $\tilde{F}'_{(\alpha:\beta)}$ intersect each other at eightteen points transversally. Hence, we obtain the new E-H special non-hyperelliptic pencil $\{\tilde{F}'_{(\alpha:\beta)}\}$ of genus 4. □

Consider the case $\psi_1 = 0$. In this case, $\mathcal{B}$ consists of two smooth curves $C_1$ and $C_2$ both of which are linearly equivalent to $3\Delta_0$. Furthermore, $C_1$ and $C_2$ intersect at eighteen points. Let $\nu : \Sigma' \to \Sigma_2$ be the blow-up at these eighteen points, and $\tilde{C}_i$ the proper transform of $C_i$ $(i = 1,\ 2)$ by $\nu$. $S$ is isomorphic to the double cover over $\Sigma'$ branched along $\tilde{C}_1 + \tilde{C}_2$, and the fibration $f : S \to \mathbb{P}^1$ is obtained as the pull back of $|\tilde{C}_i|$. In this case, the fibration $f : S \to \mathbb{P}^1$ is mapped onto itself by the involution induced from the double cover $S' \to \Sigma'$.

Note that in this case, $f_*\omega_{S/\mathbb{P}^1}$ is semi-stable and $\lambda(f) = 7/2$. Namely, this case is the lower bound of [12, Lemma 2.5].

## 4.2 The case (II)

In this subsection, we argue the case that $S$ is in Remark 4 (II). Remember that there is a morphism $S \to \tilde{Q} := \mathbb{P}(\mathcal{O}_{\Sigma_1}(\Delta_0 + \Delta_\infty) \oplus \mathcal{O}_{\Sigma_1})$ which maps $S$ onto the image birationally. If $H_0$ is the tautological divisor of $\tilde{Q}$ and if $F$ is the fiber of $\tilde{Q} \to \mathbb{P}^1$, then we have $S \sim 3H_0 + 2F$. Let $\tilde{\pi} : \tilde{Q} \to \Sigma_1$ be the $\mathbb{P}^1$-bundle and $H_\infty$ the divisor of $\tilde{Q}$ satisfying $H_0 \sim H_\infty + \tilde{\pi}^*(\Delta_0 + \Delta_\infty)$.

We obtain the following birational transformation from $\tilde{Q}$ to $\mathbb{P}^3$.

Step 1 Note that $\tilde{\pi}^*\Delta_\infty \cong \Sigma_1$. Put $B_1 := H_0|_{\tilde{\pi}^*\Delta_\infty}$. If we consider $B_1$ as the divisor of $\tilde{\pi}^*\Delta_\infty \cong \Sigma_1$, then $B_1$ is the section of $\Sigma_1 \to \mathbb{P}^1$ with $B_1^2 = -1$. Hence, we have $\mathrm{Bs}\,|H_0| = B_1$. Let $\nu : X \to \tilde{Q}$ be the blow-up along $B_1$.

Step 2 Put $\mathbb{E} := \nu^{-1}(B_1)$. Let $\tilde{\Delta}$ be the proper transform of $\tilde{\pi}^*\Delta_\infty$ by $\nu$. Then we obtain the morphism

$$\nu_1 : X \to Q' := \mathbb{P}(\mathcal{O}_{\Sigma_1}(\Delta_0) \oplus \mathcal{O}_{\Sigma_1})$$

which maps $\tilde{\Delta}$ onto the section of $\tilde{\pi}' : Q' \to \mathbb{P}^1$. We may consider as $\mathbb{E} \subset Q'$. Then we have that $\mathbb{E} \sim \tilde{\pi}'^*\Delta_\infty$ and $\mathbb{E} \cong \mathbb{P}^1 \times \mathbb{P}^1$ hold.

Step 3 We obtain the following commutative diagram:

$$\begin{tikzcd} Q' \arrow[r, "\tilde{\zeta}"] \arrow[d, "\tilde{\pi}'"'] & \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}) \arrow[d, "\tilde{\pi}_0"] \\ \Sigma_1 \arrow[r, "\zeta"'] & \mathbb{P}^2, \end{tikzcd}$$

where $\zeta$ is the blow-up at a point of $\mathbb{P}^2$. Let $p_0$ be the point. Then $\tilde{\zeta}$ maps $\mathbb{E}$ onto $\tilde{\pi}_0^{-1}(p_0)\ (\cong \mathbb{P}^1)$.

Step 4 Put $W := \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2})$. Let $T_\infty$ be the section of $W \to \mathbb{P}^2$ with $\tilde{\pi}_{0*}\mathcal{O}_W(T_\infty) \cong \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}$. We obtain the morphism $W \to \mathbb{P}^3$ which maps $T_\infty$ to a point.

Assume $S$ is smooth. In the process of the transformation as above, $S$ is transformed as following:

If we consider $\tilde{\pi}^*\Delta_\infty = \Sigma_1$, we have $S_{|\tilde{\pi}^*\Delta_\infty} \sim \Delta_\infty + 2\Delta_0$, which leads us to $\mathrm{Bs}\,|3H_0 + 2F| = B_1$. Hence, $S$ is not transformed in Step 1. If we consider as $S \subset X$, then $S$ intersects with $\tilde{\Delta}\ (\cong \Sigma_1)$ along the divisor which is linearly equivalent to $2\Delta_0$, and intersects with $\mathbb{E}\ (\cong \mathbb{P}^1 \times \mathbb{P}^1)$ along the section of $\tilde{\pi}_{|\mathbb{E}} : \mathbb{E} \to \mathbb{P}^1$.

Let $S_1$ be the image of $S$ in Step 2. Then $S_1$ has the singular curve as the compound node. In particular, $S$ is non-normal.

Let $S_2$ be the image of $S_1$ in Step 3. The singular locus of $S_1$ is contracted to a rational double point of type $A_1$.

Let $S_3$ be the image of $S_2$ in Step 4. Then two rational curve going through the rational double point of $S_2$ are contracted to a point, and $S_3$ has the rational double point of type $A_3$.

Conversely, if $S_3 \subset \mathbb{P}^3$ is a quintic surface which has a rational double point of type $A_3$, then we obtain an irreducible and nonsingular member $S \in |3H_0 + 2F|$ in $\tilde{Q}$.

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