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Convergence of a numerical method for the compressible Navier–Stokes system on general domains

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Abstract We propose a mixed numerical method for solving the compressible Navier–Stokes system and study its convergence and stability with respect to the physical domain. The numerical solutions are shown to converge, up to a subsequence, to a weak solution of the problem posed on the limit domain.

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Contents

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1 Introduction

There is a great theoretical and evidently also practical interest in the problem of convergence of numerical methods used for simulation of fluids in continuum mechanics. Ignoring the influence of temperature changes we consider a mathematical model of a compressible, barotropic, viscous fluid occupying a bounded physical domain $\Omega \subset R^3$. In the Eulerian coordinate system, the time evolution of the fluid is described by means of the mass density $\varrho = \varrho(t,x)$ and the velocity field $\mathbf{u} = \mathbf{u}(t,x), t \in (0,T)$, $x \in \Omega$, governed by the *Navier–Stokes system* of equations:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{1.2}$$

where $p = p(\varrho)$ is the pressure, and the symbol $\mathbb{S}(\nabla_x \mathbf{u})$ denotes the viscous stress tensor, here determined by *Newton's rheological law*:

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u}\mathbb{I}, \quad \mu > 0, \quad \eta \geq 0. \tag{1.3}$$

The barotropic pressure $p = p(\varrho)$ is a continuously differentiable function of the density satisfying

$$p(0)=0, \quad p'(\varrho)>0 \quad \text{for all} \quad \varrho \geq 0, \quad \lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty > 0 \quad \text{for a certain} \quad \gamma > 3. \tag{1.4}$$

Remark 1.1 The condition $\gamma > 3$ is technical; the so-called adiabatic exponent for *real fluids* ranges in the interval $\gamma \in (1, 5/3]$, where the extremal value $\gamma = 1$ corresponds to the *isothermal* case, while $\gamma = 5/3$ characterizes the monoatomic gas.

Remark 1.2 Since the viscosity coefficients μ and η are constant, we may write

$$\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) = \mu \Delta \mathbf{u} + \lambda \nabla_x \mathrm{div}_x \mathbf{u}, \quad \lambda = \frac{\mu}{3} + \eta > 0. \tag{1.5}$$

The system is supplemented with the standard *no-slip* boundary condition

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{1.6}$$

and the initial conditions

$$\varrho(0, \cdot) = \varrho_0, \quad \mathbf{u}(0, \cdot) = \mathbf{u}_0, \quad \varrho_0 > 0 \ \text{ in } \overline{\Omega}. \tag{1.7}$$

Remark 1.3 We deliberately omitted the action of an external force to simplify the presentation. As will become clear in what follows, a *bounded* driving force can be incorporated in the system with only minor modifications of the proof of convergence.

1.1 Weak solutions

We adopt the standard *weak formulation* of the problem (1.1)–(1.7).

Definition 1.1 We say that $[\varrho, \mathbf{u}]$ is a *weak solution* to the problem (1.1)–(1.7) in $(0, T) \times \Omega$ if:

- $$\varrho \geq 0 \ \text{ a.a. in } (0, T) \times \Omega, \ \varrho \in L^\infty(0, T; L^\gamma(\Omega)), \ \mathbf{u} \in L^2(0, T; W_0^{1,2}(\Omega; R^3)), \tag{1.8}$$
  $$p(\varrho) \in L^1((0, T) \times \Omega), \ \varrho\mathbf{u} \in L^\infty(0, T; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^3)); \tag{1.9}$$

- $$\int_0^T \int_\Omega \Big[\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \Big] \,\mathrm{d}x \,\mathrm{d}t = - \int_\Omega \varrho_0 \varphi(0, \cdot) \,\mathrm{d}x \tag{1.10}$$

  for any $\varphi \in C_c^\infty([0, T) \times \overline{\Omega})$;

- $$\int_0^T \int_\Omega \Big[\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + p(\varrho) \mathrm{div}_x \varphi \Big] \,\mathrm{d}x \,\mathrm{d}t$$
  $$= \int_0^T \int_\Omega \Big[\mu \nabla_x \mathbf{u} : \nabla_x \varphi + \lambda \mathrm{div}_x \mathbf{u} \, \mathrm{div}_x \varphi \Big] \,\mathrm{d}x \,\mathrm{d}t - \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \varphi(0, \cdot) \,\mathrm{d}x \tag{1.11}$$

  for any $\varphi \in C_c^\infty([0, T) \times \Omega; R^3)$;

- the energy inequality

$$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right](\tau,\cdot)\,\mathrm{d}x + \int_0^\tau\int_\Omega \left[\mu|\nabla_x\mathbf{u}|^2 + \lambda|\mathrm{div}_x\mathbf{u}|^2\right]\mathrm{d}x\,\mathrm{d}t$$
$$\le \int_\Omega \left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + P(\varrho_0)\right]\mathrm{d}x, \quad \text{with } P(\varrho) = \varrho\int_1^\varrho \frac{p(z)}{z^2}\,\mathrm{d}z, \tag{1.12}$$

  holds for a.a. $\tau \in (0, T)$.

The *existence* of global-in-time weak solutions under the hypothesis $\gamma \ge 9/5$ in (1.4) was proved by Lions [20]. The result was later extended to the range $\gamma > 3/2$ in [13]. Unfortunately, the proof of existence in the "subcritical" range $\gamma \le 3$ consists of at least two steps performed at different level of approximations and as such therefore not directly transferable to the numerical setting.

1.2 Numerical method

Our goal is to propose a numerical method for solving the Navier–Stokes system (1.1–1.7) and to show its stability with respect to the underlying spatial domain and convergence towards a weak solution specified in Definition 1.1. To this end, we adapt the discontinuous Galerkin finite element scheme proposed in [17,18] for the compressible Navier–Stokes system.

Since we are interested in *smooth* spatial domains, we consider an *unfitted* mesh on a family of polyhedral domains $\{\Omega_h\}_{h>0}$ approximating the target physical space Ω in the following sense: For any compact $K_i \subset \Omega$ it holds

$$K_i \subset \Omega_h \quad \text{for all } h > 0 \quad \text{small enough}, \tag{1.13}$$

and, similarly, for any compact $K_e \subset R^3\backslash\overline{\Omega}$,

$$K_e \subset R^3\backslash\overline{\Omega}_h \quad \text{for all } h > 0 \quad \text{small enough}, \tag{1.14}$$

cf. Babuška and Aziz [2,3].

Besides the relatively straightforward modifications to accommodate the case of variable numerical domain, we also introduce a new "dissipative" discretization implemented in the upwind terms. In such a way, we eliminate completely the artificial viscosity regularization used by several authors (see e.g. Eymard et al. [9]) including the original scheme proposed in [18]. Very roughly indeed, this new approach may be compared to adding an artificial viscosity to both equations in (1.1, 1.2):

$$\partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{u}) \approx h^\alpha \mathrm{div}_x(g(|\mathbf{u}|)\nabla_x\varrho),$$
$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x p(\varrho) - \mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{u}) \approx h^\alpha \mathrm{div}_x(g(|\mathbf{u}|)\nabla_x(\varrho\mathbf{u})),$$

where the artificial viscosity is active only for small values of the velocity amplitude $|\mathbf{u}|$. The resulting "dissipative" upwind operator remains therefore much closer to the approximated convective terms in the continuous equations.

We note that the fact that the limit problem is defined on a possibly smooth domain may be of interest when establishing convergence of the scheme. The problem (1.1)–(1.7) is known to possess a local regular solution that can be extended to the full time interval $(0, T)$ as soon as we control the amplitude of the density, see Sun, Wang, and Zhang [21]. Moreover, any weak solution coincides with the strong solution as long as the latter exists, see [11]. Consequently, boundedness of the numerical densities implies unconditional convergence as long as the domain Ω is sufficiently smooth, see Sect. 8 for details.

The paper is organized as follows. In Sect. 2, we introduce the necessary numerical framework including the basic notation and several useful properties of the underlying function spaces. The numerical scheme is introduced in Sect. 3, where we also state our main result concerning convergence towards a weak solution of the Navier–Stokes system. In Sect. 4, we derive a renormalized version of the continuity equation as well as the discrete version of the total energy balance. Section 5 is devoted to the stability of the scheme, containing the uniform bounds necessary for the limit passage. In Sect. 6, we discuss the problem of consistency of the method rewriting finally the numerical scheme in terms of the standard weak formulation based on smooth test functions. Having established consistency, we show convergence of the scheme by adapting the steps of [10, Chapter 7]. Here, similarly to the existence theory, the key idea is the weak continuity property of the effective viscous flux discovered by Lions [20]. Finally, we discuss the implications of some recent results concerning the weak-strong uniqueness property and regularity of the weak solutions on the problem of unconditional convergence of the numerical scheme in Sect. 8.

2 Preliminaries

In this section, we collect the necessary material from numerical analysis. For two numerical quantities a, b, we shall write

$$a \lesssim b \quad \text{if } a \leq cb, \quad c > 0 \quad \text{a constant}, \quad a \approx b \quad \text{if } a \lesssim b \quad \text{and} \quad b \lesssim a.$$

Here, "constant" typically means a generic quantity independent of the size of the mesh and the time step used in the numerical scheme as well as other parameters as the case may be.

2.1 Mesh

We suppose that the numerical domains Ω_h admit a *tetrahedral* mesh E_h; the individual elements in the mesh will be denoted by $E \in E_h$. Faces in the mesh are denoted by Γ, whereas Γ_h is the set of all faces. Moreover, the set of faces $\Gamma \subset \partial\Omega_h$ is denoted by $\Gamma_{h,\mathrm{ext}}$, while $\Gamma_{h,\mathrm{int}} = \Gamma_h \backslash \Gamma_{h,\mathrm{ext}}$. The size (diameter of elements in the mesh) is proportional to a positive parameter h. For $E, F \in E_h$, $E \neq F$, the intersection $E \cap F$ is either a vertex, or an edge, or a face $\Gamma \in \Gamma_h$. The mesh is assumed to be *shape regular*, meaning that the radius of the circumsphere and the biggest ball inside each element are "$\approx$" proportional to h. Finally, the family $\{\Omega_h\}_{h>0}$ will approximate a limit domain $\Omega \subset R^3$ in the sense specified in (1.13, 1.14).

Each face $\Gamma \in \Gamma_h$ is associated with a fixed normal vector $\mathbf{n}$. On the other hand, we write Γ_E whenever a face $\Gamma_E \subset \partial E$ is considered as a part of the boundary of the element E. In such a case, the normal vector to Γ_E is always the *outer* normal vector with respect to E. Keeping this convention in mind we introduce for any function g, continuous on each element E,

$$\begin{aligned} g^{\mathrm{out}}|_\Gamma &= \lim_{\delta \to 0+} g(\cdot + \delta \mathbf{n}), \ \ g^{\mathrm{in}}|_\Gamma = \lim_{\delta \to 0+} g(\cdot - \delta \mathbf{n}), \\ [[g]]_\Gamma &= g^{\mathrm{out}} - g^{\mathrm{in}}, \{g\}_\Gamma = \frac{1}{2}\left(g^{\mathrm{out}} + g^{\mathrm{in}} \right). \end{aligned} \tag{2.1}$$

For $\Gamma_E \subset \partial E$ we simply write g for g^{in}. Occasionally, we also omit the subscript Γ if no confusion arises.

2.2 Piecewise constant finite elements

We introduce the space

$$Q_h(\Omega_h) = \left\{ v \in L^2(\Omega_h) \ \middle| \ v|_E = a_E \in R \quad \text{for any } E \in E_h \right\}$$

of piecewise constant functions along with the associated projection

$$\Pi_h^Q : L^1(\Omega_h) \to Q_h(\Omega_h), \quad \Pi_h^Q[v]|_E = \frac{1}{|E|} \int_E v \, \mathrm{d}x;$$

we will occasionally denote

$$\Pi_h^Q[v] \equiv \hat{v}.$$

Finally, we recall various forms of (scaled) Poincaré's inequality:

$$\int_E \left| v - \frac{1}{|E|} \int_E v \, \mathrm{d}x \right|^q \, \mathrm{d}x \lesssim h^q \int_E |\nabla_x v|^q \, \mathrm{d}x,$$

$$\int_E \left| v - \frac{1}{|\Gamma|} \int_\Gamma |v| \, \mathrm{dS}_x \right|^q \, \mathrm{d}x \lesssim h^q \int_E |\nabla_x v|^q \, \mathrm{d}x, \quad \text{for any } \Gamma \subset \partial E, \tag{2.2}$$

$$\int_\Gamma \left| v - \frac{1}{|\Gamma|} \int_\Gamma v \, \mathrm{dS}_x \right|^q \, \mathrm{dS}_x \lesssim h^{q-1} \int_E |\nabla_x v|^q \, \mathrm{d}x, \quad \text{for any } \Gamma \subset \partial E, \tag{2.3}$$

for any $1 \le q < \infty$, in particular,

$$\left\| v - \Pi_h^Q[v] \right\|_{L^q(\Omega_h)} \lesssim h \|\nabla_x v\|_{L^q(\Omega_h; R^3)}, \quad 1 \le q \le \infty \quad \text{for any } v \in W^{1,q}(\Omega_h). \tag{2.4}$$

2.3 Crouzeix–Raviart finite elements

A differential operator D acting on the x-variable will be discretized as

$$D_h v|_E = D(v|_E) \quad \text{for any } v \quad \text{differentiable on each element } E \in E_h.$$

The *Crouzeix–Raviart finite element spaces* (see Brezzi and Fortin [4], among others) are defined as

$$V_h(\Omega_h) = \Big\{ v \in L^2(\Omega_h) \;\Big|\; v|_E = \text{affine function},\ E \in E_h,$$
$$\int_\Gamma [[v]]\ \mathrm{dS}_x = 0 \quad \text{for any } \Gamma \in \Gamma_{h,\mathrm{int}} \Big\}, \tag{2.5}$$

together with

$$V_{h,0}(\Omega_h) = \left\{ v \in V_h \;\middle|\; \int_\Gamma v\ \mathrm{dS}_x = 0 \quad \text{for any } \Gamma \in \Gamma_{h,\mathrm{ext}} \right\}. \tag{2.6}$$

Next, we introduce the associated projection

$$\Pi_h^V : W^{1,q}(\Omega_h) \to V_h(\Omega_h)$$

requiring

$$\int_\Gamma \Pi_h^V[v]\ \mathrm{dS}_x = \int_\Gamma v\ \mathrm{dS}_x \quad \text{for any } \Gamma \in \Gamma_h.$$

It is easy to check that

$$\int_{\Omega_h} \mathrm{div}_h \Pi_h^V[\mathbf{u}]\ w\ \mathrm{d}x = \int_{\Omega_h} \mathrm{div}_h \mathbf{u}\ w\ \mathrm{d}x \quad \text{for any } w \in Q_h(\Omega_h), \tag{2.7}$$

and

$$\int_\Omega \nabla_h v \cdot \nabla_h \Pi_h^V[\varphi]\,\mathrm{d}x = \int_\Omega \nabla_h v \cdot \nabla_x \varphi\,\mathrm{d}x \quad \text{for all } v \in V_{h,0}(\Omega),\ \varphi \in W_0^{1,2}(\Omega), \tag{2.8}$$

see [19, Lemma 2.11].

We also recall the the error estimates

$$\left\| v - \Pi_h^V[v] \right\|_{L^q(\Omega_h)} + h \left\| \nabla_h \left(v - \Pi_h^V[v] \right) \right\|_{L^q(\Omega_h;R^3)}$$
$$\lesssim h^m \left\| \nabla^m v \right\|_{L^q(\Omega_h;R^{3m})},\ m = 1, 2,\ 1 < q < \infty, \tag{2.9}$$

for any $v \in W^{m,q}(\Omega_h)$, see Crouzeix and Raviart [6], and [19, Lemma 2.7].

2.4 "Dissipative" upwind operator

Denoting

$$[c]^+ = \max\{c, 0\}, \quad [c]^- = \min\{c, 0\}, \quad \tilde{v} = \frac{1}{|\Gamma|} \int_\Gamma v \, \mathrm{dS}_x,$$

we introduce a dissipative *upwind* operator $\mathrm{Up}[r, \mathbf{u}]$ on a face Γ in the form

$$\begin{aligned} \mathrm{Up}[r, \mathbf{u}] = & \frac{r^{\mathrm{in}}}{2} \left([\tilde{\mathbf{u}} \cdot \mathbf{n} + h^\alpha]^+ + [\tilde{\mathbf{u}} \cdot \mathbf{n} - h^\alpha]^+ \right) \\ & + \frac{r^{\mathrm{out}}}{2} \left([\tilde{\mathbf{u}} \cdot \mathbf{n} + h^\alpha]^- + [\tilde{\mathbf{u}} \cdot \mathbf{n} - h^\alpha]^- \right), \end{aligned} \tag{2.10}$$

with a positive exponent α determined below. Note that such a definition makes sense as soon as $r \in Q_h(\Omega_h)$, $\mathbf{u} \in V_h(\Omega_h; R^3)$ and $\Gamma \in \Gamma_{h,\mathrm{int}}$.

Setting, formally, $h^\alpha \approx 0$ in (2.10), we obtain the conventional definition of the upwind operator

$$r^{\mathrm{in}}[\tilde{\mathbf{u}} \cdot \mathbf{n}]^+ + r^{\mathrm{out}}[\tilde{\mathbf{u}} \cdot \mathbf{n}]^-.$$

To illuminate the dissipative character of the new upwind operator, we may also write

$$\mathrm{Up}[r, \mathbf{u}] = \underbrace{r^{\mathrm{in}}[\tilde{\mathbf{u}} \cdot \mathbf{n}]^+ + r^{\mathrm{out}}[\tilde{\mathbf{u}} \cdot \mathbf{n}]^-}_{\text{conventional upwind}} \underbrace{- [[r]]_\Gamma \, h^\alpha \chi \left(\frac{\tilde{\mathbf{u}} \cdot \mathbf{n}}{h^\alpha} \right)}_{\text{dissipative component}}, \tag{2.11}$$

where

$$\chi(z) = \begin{cases} 0 & \text{for } z < -1, \\ \frac{1}{2}(z+1) & \text{if } -1 \le z \le 0, \\ -\frac{1}{2}(z-1) & \text{if } 0 < z \le 1, \\ 0 & \text{for } z > 1. \end{cases}$$

Remark 2.1 The numerical diffusion supplied by the dissipative component is quite subtle; it acts only when $|\tilde{\mathbf{u}} \cdot \mathbf{n}| < h^\alpha$ and has amplitude h^α. Note that the conventional artificial diffusion used by Eymard et al. [9] and [19] corresponds to

$$-h^\alpha \, [[r]]_\Gamma \, .$$

For $r, F \in Q_h(\Omega_h)$, $\mathbf{u} \in V_h(\Omega_h, R^3)$, $\phi \in C^1(\overline{\Omega}_h)$, we may use Green's theorem to compute

$$\int_{\Omega_h} r\mathbf{u}\cdot\nabla_x\phi\,\mathrm{d}x = \sum_{E\in E_h}\int_E r\mathbf{u}\cdot\nabla_x(\phi-F)\,\mathrm{d}x = \sum_{E\in E_h}\int_{\partial E}(\phi-F)r\mathbf{u}\cdot\mathbf{n}\,\mathrm{dS}_x + \int_{\Omega_h}(F-\phi)r\mathrm{div}_h\mathbf{u}\,\mathrm{d}x. \tag{2.12}$$

Furthermore, going back to (2.11) we deduce that

$$\begin{aligned}
&\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma \mathrm{Up}[r,\mathbf{u}]\ [[g]]\ \mathrm{dS}_x \\
&= -\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E} g\left(r[\tilde{\mathbf{u}}\cdot\mathbf{n}]^+ + r^{\mathrm{out}}[\tilde{\mathbf{u}}\cdot\mathbf{n}]^-\right)\,\mathrm{dS}_x \\
&\quad -\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma [[r]]\ [[g]]\,h^\alpha\chi\left(\frac{\tilde{\mathbf{u}}\cdot\mathbf{n}}{h^\alpha}\right)\,\mathrm{dS}_x
\end{aligned} \tag{2.13}$$

for any $r, g \in Q_h(\Omega_h)$, $\mathbf{u} \in V_{h,0}(\Omega_h; R^3)$. Finally, using formula (2.13), we may compute the first integral on the right-hand side of (2.12) for $\mathbf{u} \in V_{h,0}(\Omega_h; R^3)$, specifically,

$$\begin{aligned}
&\sum_{E\in E_h}\int_{\partial E}(\phi-F)r\mathbf{u}\cdot\mathbf{n}\,\mathrm{dS}_x \\
&= \sum_{E\in E_h}\int_{\partial E}\phi r\mathbf{u}\cdot\mathbf{n}\ \mathrm{dS}_x - \sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E} Fr\left([\tilde{\mathbf{u}}\cdot\mathbf{n}]^+ + [\tilde{\mathbf{u}}\cdot\mathbf{n}]^-\right)\mathrm{dS}_x \\
&= \sum_{\Gamma\in\Gamma_h}\int_\Gamma \mathrm{Up}[r,\mathbf{u}]\,[[F]]\ \mathrm{dS}_x + h^\alpha\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma [[r]]\ [[F]]\,\chi\left(\frac{\tilde{\mathbf{u}}\cdot\mathbf{n}}{h^\alpha}\right)\,\mathrm{dS}_x \\
&\quad + \sum_{E\in E_h}\int_{\partial E}\phi r\mathbf{u}\cdot\mathbf{n}\ \mathrm{dS}_x - \sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E} F(r-r^{\mathrm{out}})[\tilde{\mathbf{u}}\cdot\mathbf{n}]^-\ \mathrm{dS}_x.
\end{aligned}$$

Thus, plugging the resulting expression in (2.12) we obtain a universal formula

$$\begin{aligned}
\int_{\Omega_h} r\mathbf{u}\cdot\nabla_x\phi\,\mathrm{d}x &= \sum_{\Gamma\in\Gamma_h}\int_\Gamma \mathrm{Up}[r,\mathbf{u}]\,[[F]]\ \mathrm{dS}_x + h^\alpha\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma [[r]]\ [[F]]\,\chi\left(\frac{\tilde{\mathbf{u}}\cdot\mathbf{n}}{h^\alpha}\right)\,\mathrm{dS}_x \\
&\quad + \sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}(F-\phi)\ [[r]]\ [\tilde{\mathbf{u}}\cdot\mathbf{n}]^-\ \mathrm{dS}_x \\
&\quad + \sum_{E\in E_h}\int_{\partial E}\phi r(\mathbf{u}-\tilde{\mathbf{u}})\cdot\mathbf{n}\,\mathrm{dS}_x + \int_{\Omega_h}(F-\phi)r\mathrm{div}_h\mathbf{u}\,\mathrm{d}x
\end{aligned} \tag{2.14}$$

for any $r, F \in Q_h(\Omega_h)$, $\mathbf{u} \in V_{h,0}(\Omega_h; R^3)$, $\phi \in C^1(\overline{\Omega}_h)$.

2.5 $L^p - L^q$ and trace estimates for finite elements

The estimates listed below are direct consequences of the assumed shape regularity of the mesh and follow by a scaling argument. We claim that

$$\|v\|^q_{L^q(\partial E)} \lesssim \frac{1}{h}\left(\|v\|^q_{L^q(E)} + h^q \|\nabla_x v\|^q_{L^q(E;R^3)}\right), \quad 1 \leq q < \infty \quad \text{for any } v \in C^1(E); \tag{2.15}$$

whence

$$\|w\|^q_{L^q(\partial E)} \lesssim \frac{1}{h}\|w\|^q_{L^q(E)} \quad \text{for any } 1 \leq q < \infty, \quad w \in P_m, \tag{2.16}$$

where P_m denotes the space of polynomials of degree m.

Similarly,

$$\|w\|_{L^p(E)} \lesssim h^{3\left(\frac{1}{p}-\frac{1}{q}\right)} \|w\|_{L^q(E)}, \quad 1 \leq q < p \leq \infty, \quad w \in P_m, \tag{2.17}$$

and, making use of the inequality

$$\left(\sum_i |a_i|^p\right)^{1/p} \leq \left(\sum_i |a_i|^q\right)^{1/q} \quad \text{whenever } p \geq q,$$

with the summation over a finite index set for i, we finally obtain

$$\|w\|_{L^p(\Omega_h)} \leq ch^{3\left(\frac{1}{p}-\frac{1}{q}\right)} \|w\|_{L^q(\Omega_h)}, \quad 1 \leq q < p \leq \infty, \quad \text{for any } w|_E \in P_m(E), \quad E \in E_h. \tag{2.18}$$

We will also need a variant of (2.17) and (2.18) for the functions of the time variable $t \in (0, T)$, where the discretization is of order Δt. Evidently,

$$\|w\|_{L^p([j\Delta t,(j+1)\Delta t])} \lesssim (\Delta t)^{\left(\frac{1}{p}-\frac{1}{q}\right)} \|w\|_{L^q([j\Delta t,(j+1)\Delta t])}, \quad 1 \leq q < p \leq \infty, \tag{2.19}$$

and, therefore

$$\|w\|_{L^p(0,T)} \lesssim (\Delta t)^{\left(\frac{1}{p}-\frac{1}{q}\right)} \|w\|_{L^q(0,T)}, \quad 1 \leq q < p \leq \infty \tag{2.20}$$

for any w that is constant on any time segment $[j\Delta t, (j+1)\Delta t]$ contained in $[0, T]$.

Finally, we recall the estimate

$$\sum_{\Gamma \in \Gamma_h} \int_\Gamma |v - \tilde{v}|^2 \, \mathrm{dS}_x \lesssim h \int_{\Omega_h} |\nabla_h v|^2 \, \mathrm{d}x \quad \text{for any} \quad v \in V_{h,0}(\Omega_h; R^3) \tag{2.21}$$

that follows directly from Poincarè's inequality (2.3).

2.6 Discrete Sobolev spaces

We introduce the discrete H^1-(semi)norm

$$\|v\|^2_{H^1_{Q_h}(\Omega_h)} = \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{[[v]]^2}{h} \,\mathrm{dS}_x$$

for $v \in Q_h(\Omega_h)$. We report the following estimate that may be seen as a discrete analogue of the well-known estimates for Sobolev functions in $W^{1,2}$:

$$\int_{K_i} |v(x) - v(x-\xi)|^2 \,\mathrm{d}x \lesssim \left(|\xi|^2 + h|\xi| \right) \|v\|^2_{H^1_{Q_h}(\Omega_h;R^3)}, \tag{2.22}$$

for any compact $K_i \subset \Omega$, $|\xi| < \mathrm{dist}[K_i, \partial\Omega_h]$, $v \in Q_h(\Omega_h)$, see Eymard et al. [8, Section 5].

Remark 2.2 In view of our hypothesis (1.13), the expression on the left is defined provided $h = h(K_i)$ is small enough.

Similarly, we may define a discrete H^1-norm on the space $V_{h,0}(\Omega_h)$ by setting

$$\|v\|^2_{H^1_{V_h}(\Omega_h)} = \int_{\Omega_h} |\nabla_h v|^2 \,\mathrm{d}x. \tag{2.23}$$

In view of the limit passage $\Omega_h \to \Omega$, it is convenient to extend a function $v \in V_{h,0}(\Omega)$ to be zero outside Ω. With this convention, we have

$$\|v\|_{L^6(R^3)} \lesssim \|v\|_{H^1_{V_h}(\Omega_h)}, \tag{2.24}$$

and

$$\int_{x\in R^3} |v(x) - v(x-\xi)|^2 \,\mathrm{d}x \lesssim \left(|\xi|^2 + h|\xi| \right) \|v\|^2_{H^1_{V_h}(\Omega_h)} \tag{2.25}$$

for any $v \in V_{h,0}(\Omega_h)$, see Gallouët et al. [16].

Finally, the following assertion follows from (2.22), (2.25) and can be seen as a special case of the results of Christiansen, Munthe-Kaas and Owren [5, Proposition 5.67]:

Lemma 2.1 *For any function $v \in V_{h,0}(\Omega_h)$ there exists an $R^V_h[v] \in C^\infty_c(R^3)$ such that*

$$\|\nabla_x R^V_h[v]\|_{L^2(R^3;R^3)} \lesssim \|v\|_{H^1_{V_h}(\Omega_h)}, \quad \|v - R^V_h[v]\|_{L^2(R^3;R^3)} \lesssim h\|v\|_{H^1_{V_h}(\Omega_h)}.$$

Moreover,

$$R_h^V|_{K_e} = 0 \ \ \textit{for any compact } K_e \subset R^3 \backslash \overline{\Omega} \ \ \textit{whenever } h > 0 \ \ \textit{is small enough.} \tag{2.26}$$

Similarly, for any $g \in Q_h(\Omega_h)$ *there is an* $R_h^Q[g] \in C^\infty(R^3)$ *such that*

$$\|\nabla_x R_h^Q[g]\|_{L^2(K_i;R^3)} \lesssim \|g\|_{H^1_{Q_h}(\Omega_h)}, \ \|g - R_h^Q[g]\|_{L^2(K_i;R^3)} \lesssim h\|g\|_{H^1_{Q_h}(\Omega_h)}$$

for any compact $K_i \subset \Omega$.

Remark 2.3 The regularizing operators $R_h^V[v]$, $R_h^Q[v]$ can be taken as a spatial convolution with a regularizing kernel, see [5] for details.

3 Numerical scheme, main result

Having collected the necessary material, we introduce the numerical scheme to solve the Navier–Stokes system (1.1)–(1.7).

3.1 Numerical scheme

We start by approximating the initial data by their projections onto the space $Q_h(\Omega_h)$. To this end, we assume that both ϱ_0 and $\mathbf{u}_0$ are functions defined on the whole space R^3 and set

$$\varrho_h^0 = \Pi_h^Q[\varrho_0] \in Q_h(\Omega_h), \quad \mathbf{u}_h^0 = \Pi_h^Q[\mathbf{u}_0] \in Q_h(\Omega_h; R^3). \tag{3.1}$$

Next, we introduce the discrete time derivative

$$D_t b_h^k = \frac{b_h^k - b_h^{k-1}}{\Delta t}, \quad \Delta t \approx h,$$

and define successively the sequence of numerical solutions $[\varrho_h^k, \mathbf{u}_h^k]_{h>0}, k = 1, 2, \ldots,$

$$\varrho_h^k \in Q_h(\Omega_h), \quad \mathbf{u}_h^k \in V_{h,0}(\Omega_h; R^3)$$

satisfying:

Continuity method

$$\int_{\Omega_h} D_t \varrho_h^k \phi \ \mathrm{dx} - \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \mathrm{Up}[\varrho_h^k, \mathbf{u}_h^k] \ [[\phi]] \ \mathrm{dS}_x = 0 \quad \text{for all } \phi \in Q_h(\Omega_h); \tag{3.2}$$

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$$\int_{\Omega_h} D_t(\varrho_h^k \widehat{\mathbf{u}}_h^k) \cdot \phi \, \mathrm{d}x - \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \mathrm{Up}[\varrho_h^k \widehat{\mathbf{u}}_h^k, \mathbf{u}_h^k] \cdot [[\widehat{\phi}]] \, \mathrm{dS}_x$$
$$+ \int_{\Omega_h} \left[\mu \nabla_h \mathbf{u}_h^k : \nabla_h \phi + \lambda \mathrm{div}_h \mathbf{u}_h^k \mathrm{div}_h \phi \right] \mathrm{d}x$$
$$- \int_{\Omega_h} p(\varrho_h^k) \mathrm{div}_h \phi \, \mathrm{d}x = 0 \quad \text{for all } \phi \in V_{h,0}(\Omega_h; R^3). \tag{3.3}$$

Remark 3.1 We recall that $\widehat{\mathbf{u}}_h^k = \Pi_h^Q[\mathbf{u}_h^k]$ denotes the projection onto the space Q_h of piecewise constant functions. As we will see, our discretization of the convective term in (3.3), taken over from [19], yields a numerical analogue of the energy inequality providing the necessary stability estimates.

3.2 Main result

Before stating our main result, it is convenient to extend the numerical solution to be defined for *any* $t \geq 0$. To this end, we set

$$\varrho_h(t, \cdot) = \varrho_h^0, \quad \mathbf{u}_h(t, \cdot) = \mathbf{u}_h^0 \quad \text{for } t \leq 0,$$
$$\varrho_h(t, \cdot) = \varrho_h^k, \quad \mathbf{u}_h(t, \cdot) = \mathbf{u}_h^k \text{ for } t \in [k\Delta t, (k+1)\Delta t), \quad k = 1, 2, \ldots.$$

Accordingly, we set

$$D_t v_h(t, \cdot) = \frac{v_h(t) - v_h(t - \Delta t)}{\Delta t}, \quad t > 0.$$

Besides, we also frequently use the already introduced convention that the functions in $V_{h,0}(\Omega_h)$ are defined on the whole space R^3, being extended to be zero outside Ω_h.

Our main result may be stated as follows:

Theorem 3.1 *Let $\Omega \subset R^3$ be a bounded Lipschitz domain approximated by a family of polyhedral domains $\{\Omega_h\}_{h>0}$ as in (1.13, 1.14), where each Ω_h admits a tetrahedral mesh satisfying the hypotheses specified in Sect. 2.1. Let $\mu > 0$, $\lambda \geq 0$, and let the pressure $p = p(\varrho)$ satisfy the hypothesis (1.4) with*

$$\gamma > 3.$$

Let $[\varrho_h, \mathbf{u}_h]_{h>0}$ be a family of numerical solutions constructed by means of the method (3.1)–(3.3) such that

$$\varrho_h > 0 \quad \textit{for all} \quad h > 0,$$

with

$$\Delta t \approx h, \quad 0 < \alpha < 1,$$

where α is the exponent in the dissipative upwinding (2.11).

Then, extending ϱ_h, $\mathbf{u}_h$ *to be zero outside* Ω_h *we have, at least for a suitable subsequence,*

$$\varrho_h \to \varrho \quad \text{weakly-}(*) \text{ in } L^\infty(0,T;L^\gamma(\Omega)) \text{ and strongly in } L^1((0,T)\times\Omega),$$
$$\mathbf{u}_h \to \mathbf{u} \quad \text{weakly in } L^2(0,T;L^6(\Omega;R^3)),$$
$$\nabla_h \mathbf{u}_h \to \nabla_x \mathbf{u} \quad \text{weakly in } L^2((0,T)\times\Omega;R^{3\times 3}),$$

where $[\varrho,\mathbf{u}]$ *is a weak solution of the problem* (1.1)–(1.7) *in* $(0,T)\times\Omega$ *in the sense of Definition 1.1.*

Remark 3.2 As a matter of fact, the assumption that Ω is Lipschitz is not really necessary and can be considerably relaxed, see [14]. It is enough to assume that the limit domain enjoys the so-called segment property, meaning that each point on the boundary $\partial\Omega$ is an endpoint of a segment of fixed length, the interior of which is contained in $R^3\setminus\overline{\Omega}$.

Remark 3.3 The *existence* of the numerical solutions $[\varrho_h,\mathbf{u}_h]$ can be shown by means of a fixed point argument exactly as in [19].

Remark 3.4 The assumption $p'(0)>0$ facilitates the analysis but can also be relaxed, see [19].

4 Renormalization and the total energy balance

We introduce a renormalized variant of the continuity method (3.2) and derive a discrete analogue of the total energy balance (1.12). In what follows we use the notation

$$\mathrm{co}\{A,B\} = [\min\{A,B\},\max\{A,B\}].$$

4.1 Renormalized equation of continuity

Take $\phi = b'(\varrho_h^k)$, where b is a smooth function, as a test function in the continuity method (3.2) to obtain

$$\begin{aligned}\int_{\Omega_h} b'(\varrho_h^k)D_t\varrho_h^k\,\mathrm{d}x &\equiv \int_{\Omega_h}\frac{\varrho_h^k-\varrho_h^{k-1}}{\Delta t}b'(\varrho_h^k)\,\mathrm{d}x\\ &= \int_{\Omega_h}\left[\frac{b(\varrho_h^k)-b(\varrho_h^{k-1})}{\Delta t} - \frac{\Delta t}{2}b''(\xi_h^k)\left(\frac{\varrho_h^k-\varrho_h^{k-1}}{\Delta t}\right)^2\right]\mathrm{d}x\\ &= \int_{\Omega_h} D_t b(\varrho_h^k)\,\mathrm{d}x - \int_{\Omega_h}\frac{\Delta t}{2}b''(\xi_h^k)\left(\frac{\varrho_h^k-\varrho_h^{k-1}}{\Delta t}\right)^2\mathrm{d}x\end{aligned}$$

for a certain $\xi_h^k\in\mathrm{co}\{\varrho_h^{k-1},\varrho_h^k\}$.

Similarly to (2.13), the upwind term can be written as

$$\begin{aligned}
&\sum_{\Gamma\in\Gamma_h}\int_\Gamma \mathrm{Up}[\varrho_h^k, u_h^k]\,\left[\left[b'(\varrho_h^k)\right]\right]\,\mathrm{dS}_x + h^\alpha \sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma \left[\left[\varrho_h^k\right]\right]\,\left[\left[b'(\varrho_h^k)\right]\right]\chi\left(\frac{\tilde{\mathbf{u}}\cdot\mathbf{n}}{h^\alpha}\right)\,\mathrm{dS}_x\\
&= -\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E} b'(\varrho_h^k)\left[\varrho_h^k[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^+ + \left(\varrho_h^k\right)^{\mathrm{out}}[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\right]\,\mathrm{dS}_x\\
&= -\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\left[b(\varrho_h^k)[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^+ + b\left(\left(\varrho_h^k\right)^{\mathrm{out}}\right)[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\right]\,\mathrm{dS}_x\\
&\quad+\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\left(b(\varrho_h^k)-b'(\varrho_h^k)\varrho_h^h\right)[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^+\,\mathrm{dS}_x\\
&\quad+\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\left(b\left(\left(\varrho_h^k\right)^{\mathrm{out}}\right)-b'(\varrho_h^k)\left(\varrho_h^k\right)^{\mathrm{out}}\right)[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\,\mathrm{dS}_x\\
&= \int_{\Omega_h}\left(b(\varrho_h^k)-b'(\varrho_h^k)\varrho_h^k\right)\mathrm{div}_h\mathbf{u}_h^k\,\mathrm{d}x\\
&\quad+\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\left(b\left(\left(\varrho_h^k\right)^{\mathrm{out}}\right)-b'(\varrho_h^k)\left(\varrho_h^k\right)^{\mathrm{out}}\right)[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\,\mathrm{dS}_x\\
&\quad-\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\left(b(\varrho_h^k)-b'(\varrho_h^k)\varrho_h^k\right)[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\,\mathrm{dS}_x\\
&= \int_{\Omega_h}\left(b(\varrho_h^k)-b'(\varrho_h^k)\varrho_h^k\right)\mathrm{div}_h\mathbf{u}_h^k\,\mathrm{d}x\\
&\quad+\frac{1}{2}\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E} b''(\eta_h^k)\left[\left[\varrho_h^k\right]\right]^2[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\,\mathrm{dS}_x.
\end{aligned}$$

Thus, summing up the previous estimates we obtain the integrated renormalized continuity method:

$$\begin{aligned}
&\int_{\Omega_h} D_t b(\varrho_h^k)\,\mathrm{d}x + \int_{\Omega_h}\left(b'(\varrho_h^k)\varrho_h^k - b(\varrho_h^k)\right)\mathrm{div}_h\mathbf{u}_h^k\,\mathrm{d}x\\
&= -\int_{\Omega_h}\frac{\Delta t}{2}b''(\xi_h^k)\left(\frac{\varrho_h^k-\varrho_h^{k-1}}{\Delta t}\right)^2\mathrm{d}x + \frac{1}{2}\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E} b''(\eta_h^k)\left[\left[\varrho_h^k\right]\right]^2[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\,\mathrm{dS}_x\\
&\quad-h^\alpha\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma b''(\omega_h^k)\left[\left[\varrho_h^k\right]\right]^2\chi\left(\frac{\tilde{\mathbf{u}}\cdot\mathbf{n}}{h^\alpha}\right)\mathrm{dS}_x
\end{aligned}\tag{4.1}$$

with

$$\xi_h^k\in\mathrm{co}\{\varrho_h^{k-1},\varrho_h^k\}\text{ on each element }E\in E_h,\ \ \eta_h^k,\omega_h^k\in\mathrm{co}\{\varrho_h^k,(\varrho_h^k)^{\mathrm{out}}\}\text{ on each face }\Gamma\in\Gamma_h.$$

4.2 Energy inequality

Our goal is to derive a discrete counterpart of the energy inequality (1.12). To this end, we take $\phi=\mathbf{u}_h^k$ as a test function in the momentum method (3.3). First, in accordance with the renormalized continuity method (4.1), we claim that

$$\begin{aligned}\int_{\Omega_h} p(\varrho_h^k)\mathrm{div}_h \mathbf{u}_h^k \,\mathrm{d}x = &-\int_{\Omega_h} D_t P(\varrho_h^k)\,\mathrm{d}x + \frac{1}{2}\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\\ &\times\int_{\Gamma_E} P''(\eta_h^k)\left(\left(\varrho_h^k\right)^{\mathrm{out}} - \varrho_k\right)^2 [\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\,\mathrm{dS}_x\\ &-h^\alpha\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma P''(\omega_h^k)\left[\left[\varrho_h^k\right]\right]^2\chi\left(\frac{\tilde{\mathbf{u}}\cdot\mathbf{n}}{h^\alpha}\right)\mathrm{dS}_x\\ &-\int_{\Omega_h}\frac{\Delta t}{2}P''(\xi_h^k)\left(\frac{\varrho_h^k-\varrho_h^{k-1}}{\Delta t}\right)^2\mathrm{d}x,\end{aligned}\tag{4.2}$$

where the pressure potential P has been introduced in (1.12).

Next, we compute

$$\begin{aligned}\int_{\Omega_h}\widehat{\mathbf{u}}_h^k\cdot D_t(\varrho_h^k\widehat{\mathbf{u}}_h^k)\,\mathrm{d}x &= \int_{\Omega_h}\widehat{\mathbf{u}}_h^k\cdot\frac{\varrho_h^k\widehat{\mathbf{u}}_h^k-\varrho_h^{k-1}\widehat{\mathbf{u}}_h^{k-1}}{\Delta t}\,\mathrm{d}x\\ &= \int_{\Omega_h}\left[\left|\widehat{\mathbf{u}}_h^k\right|^2\frac{\varrho_h^k-\varrho_h^{k-1}}{\Delta t} + \varrho_h^{k-1}\widehat{\mathbf{u}}_h^k\cdot\frac{\widehat{\mathbf{u}}_h^k-\widehat{\mathbf{u}}_h^{k-1}}{\Delta t}\right]\mathrm{d}x\\ &= \int_{\Omega_h}\left[\left|\widehat{\mathbf{u}}_h^k\right|^2 D_t\varrho_h^k + \varrho_h^{k-1}\frac{1}{2}D_t\left|\widehat{\mathbf{u}}_h^k\right|^2\right]\mathrm{d}x\\ &\quad+\int_{\Omega_h}\frac{\Delta t}{2}\varrho_h^{k-1}\left|\frac{\widehat{\mathbf{u}}_h^k-\widehat{\mathbf{u}}_h^{k-1}}{\Delta t}\right|^2\mathrm{d}x.\end{aligned}\tag{4.3}$$

The upwind term reads

$$\begin{aligned}&\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma \mathrm{Up}[\varrho_h^k\widehat{\mathbf{u}}_h^k,\mathbf{u}_h^k]\cdot\left[\left[\widehat{\mathbf{u}}_h^k\right]\right]\mathrm{dS}_x + h^\alpha\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma\left[\left[\varrho_h^k\widehat{\mathbf{u}}_h^k\right]\right]\cdot\left[\left[\widehat{\mathbf{u}}_h^k\right]\right]\chi\left(\frac{\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}}{h^\alpha}\right)\mathrm{dS}_x\\ &=-\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\widehat{\mathbf{u}}_h^k\cdot\left((\varrho_h^k\widehat{\mathbf{u}}_h^k)[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^+ + \left(\varrho_h^k\widehat{\mathbf{u}}_h^k\right)^{\mathrm{out}}[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\right)\mathrm{dS}_x\\ &=-\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\left|\widehat{\mathbf{u}}_h^k\right|^2\left(\varrho_h^k\,[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^+ + \left(\varrho_h^k\right)^{\mathrm{out}}[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\right)\mathrm{dS}_x\\ &\quad+\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\left(\varrho_h^k\right)^{\mathrm{out}}[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\widehat{\mathbf{u}}_h^k\cdot\left(\widehat{\mathbf{u}}_h^k-\left(\widehat{\mathbf{u}}_h^k\right)^{\mathrm{out}}\right)\mathrm{dS}_x\\ &=\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma\mathrm{Up}(\varrho_h^k,\mathbf{u}_h^k)\left[\left[\left|\widehat{\mathbf{u}}_h^k\right|^2\right]\right]\mathrm{dS}_x + h^\alpha\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma\left[\left[\varrho_h^k\right]\right]\cdot\left[\left[\left|\widehat{\mathbf{u}}_h^k\right|^2\right]\right]\chi\left(\frac{\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}}{h^\alpha}\right)\mathrm{dS}_x\\ &\quad+\frac{1}{2}\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\left(\varrho_h^k\right)^{\mathrm{out}}[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\left(\left|\widehat{\mathbf{u}}_h^k\right|^2-\left|\left(\widehat{\mathbf{u}}_h^k\right)^{\mathrm{out}}\right|^2\right)\mathrm{dS}_x\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\left|\widehat{\mathbf{u}}_h^k-\left(\widehat{\mathbf{u}}_h^k\right)^{\mathrm{out}}\right|^2\left(\varrho_h^k\right)^{\mathrm{out}}[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\,\mathrm{dS}_x\\
&=\frac{1}{2}\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma \mathrm{Up}(\varrho_h^k,\mathbf{u}_h^k)\left[\left[\left|\widehat{\mathbf{u}}_h^k\right|^2\right]\right]\mathrm{dS}_x+\frac{h^\alpha}{2}\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma\left[\left[\varrho_h^k\right]\right]\cdot\left[\left[\left|\widehat{\mathbf{u}}_h^k\right|^2\right]\right]\chi\left(\frac{\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}}{h^\alpha}\right)\mathrm{dS}_x\\
&+\frac{1}{2}\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\left|\widehat{\mathbf{u}}_h^k-\left(\widehat{\mathbf{u}}_h^k\right)^{\mathrm{out}}\right|^2\left(\varrho_h^k\right)^{\mathrm{out}}[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\,\mathrm{dS}_x. \qquad (4.4)
\end{aligned}
$$

Summing up (4.2)–(4.4) and making use of the continuity method (3.2) we deduce the energy inequality

$$
\begin{aligned}
&D_t\int_{\Omega_h}\left[\frac{1}{2}\varrho_h^k|\widehat{\mathbf{u}}_h^k|^2+P(\varrho_h^k)\right]\mathrm{d}x+\mu\int_{\Omega_h}|\nabla_h\mathbf{u}_h^k|^2\,\mathrm{d}x+\lambda\int_{\Omega_h}|\mathrm{div}_h\mathbf{u}_h^k|^2\,\mathrm{d}x\\
&\quad+\frac{\Delta t}{2}\int_{\Omega_h}\left(A\left|\frac{\varrho_h^k-\varrho_h^{k-1}}{\Delta t}\right|^2+\varrho_h^{k-1}\left|\frac{\widehat{\mathbf{u}}_h^k-\widehat{\mathbf{u}}_h^{k-1}}{\Delta t}\right|^2\right)\mathrm{d}x\\
&\quad-\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\left(\varrho_h^k\right)^{\mathrm{out}}[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\frac{\left|\widehat{\mathbf{u}}_h^k-\left(\widehat{\mathbf{u}}_h^k\right)^{\mathrm{out}}\right|^2}{2}\mathrm{dS}_x\\
&\quad+h^\alpha\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma\left\{\varrho_h^k\right\}\left|\widehat{\mathbf{u}}_h^k-\left(\widehat{\mathbf{u}}_h^k\right)^{\mathrm{out}}\right|^2\chi\left(\frac{\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}}{h^\alpha}\right)\mathrm{dS}_x\\
&\quad+\frac{A}{2}\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma\left(h^\alpha\chi\left(\frac{\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}}{h^\alpha}\right)+|\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}|\right)\left[\left[\varrho_h^k\right]\right]^2\mathrm{dS}_x\le 0 \qquad (4.5)
\end{aligned}
$$

with $A=\inf_{\varrho>0}\left\{P''(\varrho)\right\}$. Since $P''(\rho)=p'(\rho)/\rho$, we have $A>0$ according to (1.4).

5 Stability

In this section, we derive *uniform bounds* for the family $[\varrho_h,\mathbf{u}_h]_{h>0}$ independent of the time step $\Delta t\approx h$ and the element size h.

5.1 Mass conservation

Taking $\phi\equiv 1$ in the continuity method (3.2) we obtain

$$
\int_{\Omega_h}\varrho_h(t,\cdot)\,\mathrm{d}x=\int_{\Omega_h}\varrho_h^0\,\mathrm{d}x=\int_{\Omega_h}\varrho_0\,\mathrm{d}x\quad\text{for any}\quad h>0, \qquad (5.1)
$$

meaning that the total mass is conserved by the scheme.

5.2 Energy bounds

The energy inequality (4.5) yields

$$\int_{\Omega_h}\left[\frac{1}{2}\varrho_h|\widehat{\mathbf{u}}_h|^2+P(\varrho_h)\right](\tau,\cdot)\,\mathrm{d}x+\mu\int_0^\tau\int_{\Omega_h}|\nabla_h\mathbf{u}_h|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+\lambda\int_0^\tau\int_{\Omega_h}|\mathrm{div}_h\mathbf{u}_h|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq\int_{\Omega_h}\left[\frac{1}{2}\varrho_h^0|\widehat{\mathbf{u}}_h^0|^2+P(\varrho_h^0)\right]\mathrm{d}x\equiv E_{0,h},\ \ E_{0,h}\lesssim 1; \tag{5.2}$$

whence

$$\sup_{\tau\in(0,T)}\|\sqrt{\varrho_h}\widehat{\mathbf{u}}_h(\tau,\cdot)\|_{L^2(\Omega_h;R^3)}\lesssim 1, \tag{5.3}$$
$$\sup_{\tau\in(0,T)}\|\varrho_h(\tau,\cdot)\|_{L^\gamma(\Omega_h)}\lesssim 1, \tag{5.4}$$

and

$$\int_0^T\int_{\Omega_h}|\nabla_h\mathbf{u}_h|^2\,\mathrm{d}x\,\mathrm{d}t\lesssim 1; \tag{5.5}$$

whence, in accordance with (2.24),

$$\|\mathbf{u}_h\|_{L^2(0,T;L^6(R^3;R^3))}\lesssim 1, \tag{5.6}$$

where the bounds are uniform for $h\to 0$. We recall that $\mathbf{u}_h$ as well as other quantities, extended to be zero outside the numerical domain Ω_h, may be regarded as functions on the whole space R^3.

Finally, we record the bounds resulting from numerical dissipation :

$$\sum_{k\geq 0}\int_{\Omega_h}\left[\left|\varrho_h^k-\varrho_h^{k-1}\right|^2+\varrho_h^{k-1}\left|\widehat{\mathbf{u}}_h^k-\widehat{\mathbf{u}}_h^{k-1}\right|^2\right]\mathrm{d}x\lesssim 1, \tag{5.7}$$

$$-\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_0^T\int_{\Gamma_E}(\varrho_h)^{\mathrm{out}}\,[\tilde{\mathbf{u}}_h\cdot\mathbf{n}]^-\left|\widehat{\mathbf{u}}_h-(\widehat{\mathbf{u}}_h)^{\mathrm{out}}\right|^2\mathrm{dS}_x\,\mathrm{d}t\lesssim 1, \tag{5.8}$$

$$h^\alpha\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_0^T\int_\Gamma\{\varrho_h\}\left|\widehat{\mathbf{u}}_h-(\widehat{\mathbf{u}}_h)^{\mathrm{out}}\right|^2\chi\left(\frac{\tilde{\mathbf{u}}_h\cdot\mathbf{n}}{h^\alpha}\right)\mathrm{dS}_x\,\mathrm{d}t\lesssim 1, \tag{5.9}$$

and

$$\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_0^T\int_\Gamma\left(|\tilde{\mathbf{u}}_h\cdot\mathbf{n}|+h^\alpha\chi\left(\frac{\tilde{\mathbf{u}}_h\cdot\mathbf{n}}{h^\alpha}\right)\right)[[\varrho_h]]^2\,\mathrm{dS}_x\,\mathrm{d}t\lesssim 1. \tag{5.10}$$

6 Consistency formulation

Having collected all the available uniform bounds, our next task is to verify that our numerical method is *consistent* with the variational formulation of the original problem.

6.1 Continuity method

For $\phi \in C_c^\infty(R^3)$, take $\Pi_h^Q[\phi]$ as a test function in the continuity method (3.1). Using the formula (2.14) for $r = \varrho_h^k$, $\mathbf{u} = \mathbf{u}_h^k$, $F = \Pi_h^Q[\phi]$ we check without difficulty that

$$\begin{aligned}\int_{\Omega_h} \varrho_h^k \mathbf{u}_h^k \cdot \nabla_x \phi \, \mathrm{d}x = & \sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}[\varrho_h^k, \mathbf{u}_h^k] \left[\left[\Pi_h^Q[\phi] \right]\right] \mathrm{dS}_x \\ & - \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left(\phi - \Pi_h^Q[\phi]\right) \left[\left[\varrho_h^k\right]\right] [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \, \mathrm{dS}_x \\ & + \sum_{E \in E_h} \int_{\partial E} \phi \varrho_h^k (\mathbf{u}_h^k - \tilde{\mathbf{u}}_h^k) \cdot \mathbf{n} \, \mathrm{dS}_x \\ & + h^\alpha \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \left[\left[\varrho_h^k\right]\right] \left[\left[\Pi_h^Q[\phi] \right]\right] \chi \left(\frac{\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}}{h^\alpha} \right) \mathrm{dS}_x .\end{aligned}$$

Note that here

$$\int_{\Omega_h} (\Pi_h^Q[\phi] - \phi) \varrho_h^k \mathrm{div}_h \mathbf{u}_h^k \, \mathrm{d}x = \sum_{E \in E_h} \int_E (\Pi_h^Q[\phi] - \phi) \varrho_h^k \mathrm{div}_h \mathbf{u}_h^k \, \mathrm{d}x = 0$$

as $\mathrm{div}_h \mathbf{u}_h^k$ is constant on each element E.

Now, by Hölder's inequality,

$$\begin{aligned} & \left| \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left(\phi - \Pi_h^Q[\phi]\right) \left[\left[\varrho_h^k\right]\right] [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \, \mathrm{dS}_x \right| \\ & \lesssim \sum_{\Gamma \in \Gamma_h} \int_\Gamma \left|\phi - \Pi_h^Q[\phi]\right| \left| \left[\left[\varrho_h^k\right]\right] \right| \left| [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \right| \mathrm{dS}_x \\ & \lesssim \left(\sum_{\Gamma \in \Gamma_h} \int_\Gamma \left[\left[\varrho_h^k\right]\right]^2 \left| [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \right| \mathrm{dS}_x \right)^{1/2} \\ & \quad \times \left(\sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left(\phi - \Pi_h^Q[\phi]\right)^2 \left| \tilde{\mathbf{u}}_h^k \cdot \mathbf{n} \right| \mathrm{dS}_x \right)^{1/2}, \end{aligned}$$

where the first integral on the right-hand side is controlled in $L^2(0, T)$ by (5.10).

As for the second integral, we may apply Hölder's inequality, combined with Poincaré's inequality (2.4) and the trace estimates (2.15), (2.16) to obtain

$$
\begin{aligned}
&\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\left(\phi-\Pi_h^Q[\phi]\right)^2\left|\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}\right|\,\mathrm{dS}_x\\
&\lesssim\sum_{E\in E_h}\left(\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\left(\phi-\Pi_h^Q[\phi]\right)^{\frac{6\gamma}{\gamma+6}}\,\mathrm{dS}_x\right)^{\frac{\gamma+6}{3\gamma}}\left(\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\left|\tilde{\mathbf{u}}_h^k\right|^{\frac{3\gamma}{2\gamma-6}}\,\mathrm{dS}_x\right)^{\frac{2\gamma-6}{3\gamma}}\\
&\lesssim h\sum_{E\in E_h}\left\|\mathbf{u}_h^k\right\|_{L^{\frac{3\gamma}{2\gamma-6}}(E;R^3)}\|\nabla_x\phi\|^2_{L^{\frac{6\gamma}{\gamma+6}}(E;R^3)}\lesssim h\left\|\mathbf{u}_h^k\right\|_{L^{\frac{3\gamma}{2\gamma-6}}(\Omega_h;R^3)}\|\nabla_x\phi\|^2_{L^{\frac{6\gamma}{\gamma+6}}(\Omega_h;R^3)}.
\end{aligned}
$$

Finally, we use the interpolation L^p-L^q estimates (2.18, 2.20), and (5.6) to conclude

$$
\begin{aligned}
&h\left\|\mathbf{u}_h^k\right\|_{L^{\frac{3\gamma}{2\gamma-6}}(\Omega_h;R^3)}\|\nabla_x\phi\|^2_{L^{\frac{6\gamma}{\gamma+6}}(\Omega_h;R^3)}\\
&\quad\lesssim h^{\min\{1,\frac{5\gamma-12}{2\gamma}\}}\left\|\mathbf{u}_h^k\right\|_{L^6(\Omega_h;R^3)}\|\nabla_x\phi\|^2_{L^{\frac{6\gamma}{\gamma+6}}(\Omega_h;R^3)}\\
&\quad=h^{\min\{1,\frac{5\gamma-12}{2\gamma}\}}(\Delta t)^{-1/2}(\Delta t)^{1/2}\left\|\mathbf{u}_h^k\right\|_{L^6(\Omega_h;R^3)}\|\nabla_x\phi\|^2_{L^{\frac{6\gamma}{\gamma+6}}(\Omega_h;R^3)}\\
&\quad\lesssim h^{\min\{1,\frac{5\gamma-12}{2\gamma}\}}(\Delta t)^{-1/2}\|\nabla_x\phi\|^2_{L^{\frac{6\gamma}{\gamma+6}}(\Omega_h;R^3)}.
\end{aligned}
$$

The next step is to estimate

$$
\sum_{\Gamma\in\Gamma_h}\int_\Gamma\phi\varrho_n^k(\mathbf{u}_n^k-\tilde{\mathbf{u}}_n^k)\cdot\mathbf{n}\,\mathrm{dS}_x=\sum_{\Gamma\in\Gamma_h}\int_\Gamma(\phi-\tilde{\phi})\varrho_n^k(\mathbf{u}_n^k-\tilde{\mathbf{u}}_n^k)\cdot\mathbf{n}\,\mathrm{dS}_x,
$$

where, by Hölder's inequality, (2.21), and (2.15),

$$
\begin{aligned}
&\left|\sum_{\Gamma\in\Gamma_h}\int_\Gamma(\phi-\tilde{\phi})\varrho_n^k(\mathbf{u}_n^k-\tilde{\mathbf{u}}_n^k)\cdot\mathbf{n}\,\mathrm{dS}_x\right|\\
&\quad\le\sum_{\Gamma\in\Gamma_h}\left\|\phi-\tilde{\phi}\right\|_{L^2(\Gamma)}\left\|\varrho_h^k\right\|_{L^\gamma(\Gamma)}\|\mathbf{u}-\tilde{\mathbf{u}}\|_{L^{\frac{2\gamma}{\gamma-2}}(\Gamma;R^3)}\\
&\quad\lesssim h^{\frac{\gamma-3}{\gamma}}\|\varrho_h^k\|_{L^\gamma(\Omega_h)}\|\nabla_h\mathbf{u}_h^k\|_{L^2(\Omega_h;R^{3\times3})}\|\nabla_x\phi\|_{L^2(\Omega_h;R^3)}.
\end{aligned}
$$

The last step consists in controlling the numerical viscosity. To this end, we first claim that (5.10) gives rise to

$$
h^\alpha\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_0^T\int_\Gamma[[\varrho_h]]^2\,\mathrm{dS}_x\,\mathrm{d}t\lesssim1. \tag{6.1}
$$

Next, we get

$$\begin{aligned}&\left|\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma [[\varrho_h]]\left[\left[\Pi_h^Q[\phi]\right]\right]\,\mathrm{dS}_x\right|\\&\lesssim \sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E\cap\Gamma_{h,\mathrm{int}}}\int_{\Gamma_E}\left|[[\varrho_h]]\right|\left|\Pi_h^Q[\phi]-\phi\right|\,\mathrm{dS}_x;\end{aligned}$$

whence, by virtue of (6.1) combined with (2.4) and (2.15), we may infer that

$$\begin{aligned}&h^\alpha\left|\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma\left[\left[\varrho_h^k\right]\right]\left[\left[\Pi_h^Q[\phi]\right]\right]\chi\left(\frac{\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}}{h^\alpha}\right)\mathrm{dS}_x\right|\\&\lesssim h^{\frac{1+\alpha}{2}}r_h^1(t)\|\nabla_x\phi\|_{L^2(\Omega_h;R^3)},\quad \|r_h^1\|_{L^2(0,T)}\lesssim 1.\end{aligned}$$

Remark 6.1 Our estimates of the numerical viscosity are in fact considerably better than in [19, Section 5.3, Lemma 5.5]. This is due to the fact that the pressure considered here satisfies $p'(0)>0$ yielding (5.10).

Using the standard representation theorems for bounded linear forms on Sobolev spaces, we reformulate the continuity method as:

$$\int_{R^3}\left[D_t\varrho_h\phi-\varrho_h\mathbf{u}_h\cdot\nabla_x\phi\right]\quad\mathrm{d}x=\int_{R^3}\mathbf{R}_h^1(t,\cdot)\cdot\nabla_x\phi\,\mathrm{d}x \tag{6.2}$$

for any $\phi\in C_c^\infty(R^3)$, where $\mathbf{R}_h^1$ is a piecewise constant with respect to the time variable $t\in[0,T]$ such that

$$\left\|\mathbf{R}_h^1\right\|_{L^2(0,T;L^{\frac{6\gamma}{5\gamma-6}}(R^3;R^3))}\lesssim h^\beta\quad\text{for a certain}\quad\beta>0. \tag{6.3}$$

6.2 Momentum method

In order to derive a consistency formulation of the momentum method, we take

$$\Pi_h^V[\phi],\ \phi\in C_c^\infty(\Omega;R^3),$$

as a test function in the momentum method (3.3). Note that, in accordance with the hypothesis (1.13), $\phi\in C_c^\infty(\Omega_h;R^3)$ as soon as $h>0$ is small enough. By virtue of (2.7), (2.8), we have

$$\int_\Omega \left[\mu \nabla_h \mathbf{u}_h^k : \nabla_h \Pi_h^V[\phi] + \lambda \mathrm{div}_h \mathbf{u}_h^k \mathrm{div}_h \Pi_h^V[\phi] \right] \, \mathrm{d}x$$
$$= \int_\Omega \left[\mu \nabla_h \mathbf{u}_h^k : \nabla_x \phi + \lambda \mathrm{div}_h \mathbf{u}_h^k \mathrm{div}_x \phi \right] \, \mathrm{d}x,$$

and

$$\int_\Omega p(\varrho_h) \mathrm{div}_x \Pi_h^V[\phi] \, \mathrm{d}x = \int_\Omega p(\varrho_h) \mathrm{div}_x \phi \, \mathrm{d}x.$$

Consequently, we may rewrite (3.3) in the form

$$\begin{aligned}
&\int_\Omega D_t \varrho_h^k \widehat{\mathbf{u}}_h^k \cdot \phi \, \mathrm{d}x - \int_\Omega \varrho_h^k \mathbf{u}_h^k \otimes \widehat{\mathbf{u}_h^k} : \nabla_x \phi \, \mathrm{d}x \\
&\quad + \int_\Omega \left[\mu \nabla_h \mathbf{u}_h^k \cdot \nabla_x \phi + \lambda \mathrm{div}_h \mathbf{u}_h^k \mathrm{div}_x \phi \right] \mathrm{d}x - \int_\Omega p(\varrho_h^k, \vartheta_h^k) \mathrm{div}_x \phi \, \mathrm{d}x \\
&= \int_{\Omega_h} D_t \varrho_h^k \widehat{\mathbf{u}}_h^k \cdot \left(\phi - \Pi_h^V[\phi] \right) \, \mathrm{d}x \\
&\quad + \sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}[\varrho_h^k \widehat{\mathbf{u}}_h^k, \mathbf{u}_h^k] \cdot \left[\left[\widehat{\Pi_h^V[\phi]} \right]\right] \mathrm{dS}_x - \int_\Omega \varrho_h^k \mathbf{u}_h^k \otimes \widehat{\mathbf{u}_h^k} : \nabla_x \phi \, \mathrm{d}x \\
&= \int_{\Omega_h} D_t \varrho_h^k \widehat{\mathbf{u}}_h^k \cdot \left(\phi - \Pi_h^V[\phi] \right) \, \mathrm{d}x \\
&\quad + \sum_{\Gamma \in \Gamma_h} \int_\Gamma \left((\varrho_h \mathbf{u}_h)^{\mathrm{in}} [\tilde{\mathbf{u}} \cdot \mathbf{n}]^+ + (\varrho_h \mathbf{u}_h)^{\mathrm{out}} [\tilde{\mathbf{u}} \cdot \mathbf{n}]^- \right) \cdot \left[\left[\widehat{\Pi_h^V[\phi]} \right]\right] \mathrm{dS}_x \\
&\quad - \int_\Omega \varrho_h^k \mathbf{u}_h^k \otimes \widehat{\mathbf{u}_h^k} : \nabla_x \phi \, \mathrm{d}x - h^\alpha \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma [[\varrho_h \mathbf{u}_h]] \cdot \left[\left[\widehat{\Pi_h^V[\phi]} \right]\right] \chi \left(\frac{\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}}{h^\alpha} \right) \mathrm{dS}_x.
\end{aligned} \tag{6.4}$$

Our goal is to estimate the four integrals on the right-hand side of (6.4). We proceed in several steps.

6.2.1 Error in the discretized time derivative

We have

$$\int_{\Omega_h} D_t \varrho_h^k \widehat{\mathbf{u}}_h^k \cdot \left(\phi - \Pi_h^V[\phi] \right) \, \mathrm{d}x$$
$$\int_{\Omega_h} \sqrt{\varrho_h^{k-1}} \sqrt{\varrho_h^{k-1}} \frac{\mathbf{u}_h^k - \mathbf{u}_h^{k-1}}{\Delta t} \cdot \left(\phi - \Pi_h^V[\phi] \right) \, \mathrm{d}x + \int_{\Omega_h} \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \mathbf{u}_h^k \cdot \left(\phi - \Pi_h^V[\phi] \right) \, \mathrm{d}x,$$

where, by virtue of Hölder's inequality and the estimate (2.9),

$$\begin{aligned}
&\left| \int_{\Omega_h} \sqrt{\varrho_h^{k-1}} \sqrt{\varrho_h^{k-1}} \frac{\mathbf{u}_h^{k-1} - \mathbf{u}_h^{k-1}}{\Delta t} \cdot \left(\phi - \Pi_h^V[\phi] \right) \, \mathrm{d}x \right| \\
&\leq \| \varrho_h^{k-1} \|_{L^\gamma(\Omega_h)}^{1/2} \left(\int_\Omega \varrho_h^{k-1} \left(\frac{\mathbf{u}_h^{k-1} - \mathbf{u}_h^{k-1}}{\Delta t} \right)^2 \, \mathrm{d}x \right)^{1/2} \left\| \phi - \Pi_h^V[\phi] \right\|_{L^{\frac{2\gamma}{\gamma-1}}(\Omega_h)} \\
&\lesssim \| \varrho_h^{k-1} \|_{L^\gamma(\Omega_h)}^{1/2} \left(\Delta t \int_{\Omega_h} \varrho_h^{k-1} \left(\frac{\mathbf{u}_h^{k-1} - \mathbf{u}_h^{k-1}}{\Delta t} \right)^2 \, \mathrm{d}x \right)^{1/2} (\Delta t)^{-1/2} h \, \|\nabla_x \phi\|_{L^{\frac{2\gamma}{\gamma-1}}(\Omega; R^3)}.
\end{aligned}$$

In accordance with the energy estimates (5.7), we have

$$\sum_{k \geq 0} \Delta t \left(\Delta t \int_{\Omega_h} \varrho_h^{k-1} \left(\frac{\mathbf{u}_h^{k-1} - \mathbf{u}_h^{k-1}}{\Delta t} \right)^2 \, \mathrm{d}x \right) \lesssim 1. \tag{6.5}$$

Applying a similar treatment to the second integral we get

$$\begin{aligned}
&\left| \int_\Omega \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \mathbf{u}_h^k \cdot \left(\phi - \Pi_h^V[\phi] \right) \, \mathrm{d}x \right| \\
&\leq \left(\Delta t \int_{\Omega_h} \left(\frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \right)^2 \, \mathrm{d}x \right)^{1/2} \| \mathbf{u}_h^k \|_{L^6(\Omega_h; R^3)} (\Delta t)^{-1/2} h \| \nabla_x \phi \|_{L^3(\Omega)},
\end{aligned}$$

where the first integral on the right-hand side is controlled by means of (5.7).

Thus we may infer that

$$\left| \int_{\Omega_h} D_t (\varrho_h \widehat{\mathbf{u}}_h) \cdot \left(\phi - \Pi_h^V[\phi] \right) \, \mathrm{d}x \right| \lesssim \sqrt{h} \, r_h^2(t) \| \nabla_x \phi \|_{L^\gamma(\Omega)}, \;\; \| r_h^2 \|_{L^2(0,T)} \lesssim 1. \tag{6.6}$$

6.2.2 Error in the upwind term

Take $F = \widehat{\Pi_h^V[\phi]} = \Pi_h^Q \Pi_h^V[\phi]$ in (2.14) to obtain

$$\begin{aligned}
&\sum_{\Gamma \in \Gamma_h} \int_\Gamma \left((\varrho_h \mathbf{u}_h)^{\mathrm{in}} [\tilde{\mathbf{u}} \cdot \mathbf{n}]^+ + (\varrho_h \mathbf{u}_h)^{\mathrm{out}} [\tilde{\mathbf{u}} \cdot \mathbf{n}]^- \right) \\
&\quad \cdot \left[\left[\widehat{\Pi_h^V[\phi]} \right]\right] \mathrm{dS}_x - \int_\Omega \varrho_h^k \mathbf{u}_h^k \otimes \widehat{\mathbf{u}_h^k} : \nabla_x \phi \, \mathrm{d}x \\
&= \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left(\Pi_h^Q \Pi_h^V[\phi] - \phi \right) \cdot \left[\left[\varrho_h^k \widehat{\mathbf{u}_h^k} \right]\right] [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \, \mathrm{dS}_x
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{E \in E_h} \int_E \varrho_h^k \widehat{\mathbf{u}}_h^k (\phi - \Pi_h^Q \Pi_h^V[\phi]) \mathrm{div}_x \mathbf{u}_h^k \,\mathrm{d}x \\
&+ \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \phi \cdot \widehat{\mathbf{u}_h^k} \varrho_h^k (\tilde{\mathbf{u}}_n^k - \mathbf{u}_h^k) \cdot \mathbf{n} \,\mathrm{dS}_x \\
= &\sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} (\varrho_h^k)^+ \left(\phi - \Pi_h^Q \Pi_h^V[\phi]\right) [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \left[\left[\widehat{\mathbf{u}_h^k}\right]\right] \mathrm{dS}_x \\
&+ \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \widehat{\mathbf{u}}_h^k \cdot \left(\phi - \Pi_h^Q \Pi_h^V[\phi]\right) [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \left[\left[\varrho_h^k\right]\right] \mathrm{dS}_x \\
&+ \sum_{E \in E_h} \int_E \varrho_h^k \widehat{\mathbf{u}}_h^k (\phi - \Pi_h^Q \Pi_h^V[\phi]) \mathrm{div}_h \mathbf{u}_h^k \,\mathrm{d}x \\
&+ \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \phi \cdot \widehat{\mathbf{u}_h^k} \varrho_h^k (\tilde{\mathbf{u}}_n^k - \mathbf{u}_h^k) \cdot \mathbf{n} \,\mathrm{dS}_x \equiv I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

Step 1 Applying Hölder's inequality to I_1 we obtain

$$
\begin{aligned}
|I_1| = &\left| \sum_{E \in E_h} \int_{\partial E_h} (\varrho_h^k)^+ \left(\Pi_h^Q \Pi_h^V[\phi] - \phi\right) [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \left[\left[\widehat{\mathbf{u}_h^k}\right]\right] \mathrm{dS}_x \right| \\
\lesssim &\left(\sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} (\varrho_h^k)^+ \left|\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}\right| \left[\left[\widehat{\mathbf{u}_h^k}\right]\right]^2 \mathrm{dS}_x \right)^{1/2} \\
&\times \left(\sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} (\varrho_h^k)^+ \left|\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}\right| \left(\Pi_h^Q \Pi_h^V[\phi] - \phi\right)^2 \mathrm{dS}_x \right)^{1/2},
\end{aligned}
$$

where the first term is bounded in $L^2(0, T)$ in view of the energy estimates (5.8).

Next, as $\mathbf{u}_h^k$ are continuous on each element, we have

$$
\begin{aligned}
&\sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} (\varrho_h^k)^+ \left|\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}\right| \left(\Pi_h^Q \Pi_h^V[\phi] - \phi\right)^2 \mathrm{dS}_x \\
&\leq \sum_{E \in E_h} \|\varrho_h^k\|_{L^q(\partial E)} \|\mathbf{u}_h^k\|_{L^\infty(E, R^3)} \left\|\Pi_h^Q \Pi_h^V[\phi] - \phi\right\|^2_{L^\gamma(\partial E; R^3)}, \quad \frac{1}{q} + \frac{2}{\gamma} = 1,
\end{aligned}
$$

where, in accordance with the trace estimates (2.15, 2.16), and the $L^p - L^q$ estimates (2.18),

$$
\begin{aligned}
&\sum_{E \in E_h} \|\varrho_h^k\|_{L^q(\partial E)} \|\mathbf{u}_h^k\|_{L^\infty(E, R^3)} \left\|\Pi_h^Q \Pi_h^V[\phi] - \phi\right\|^2_{L^\gamma(\partial E; R^3)} \\
&\leq \frac{1}{h} \|\mathbf{u}_h^k\|_{L^\infty(\Omega_h, R^3)} \sum_{E \in E_h} \|\varrho_h^k\|_{L^q(E)} \left(\left\|\Pi_h^Q \Pi_h^V[\phi] - \phi\right\|^2_{L^\gamma(E; R^3)} + h^2 \|\nabla_x \phi\|^2_{L^\gamma(E; R^3)} \right)
\end{aligned}
$$

$$= \frac{1}{h^{3/2}} \|\mathbf{u}_h^k\|_{L^6(\Omega_h,R^3)} \sum_{E\in E_h} \|\varrho_h^k\|_{L^q(E)} \left(\left\| \Pi_h^Q \Pi_h^V[\phi] - \phi \right\|^2_{L^\gamma(E;R^3)} + h^2 \|\nabla_x \phi\|^2_{L^\gamma(E;R^3)} \right)$$

$$\leq \frac{1}{h^{3/2}} \|\mathbf{u}_h^k\|_{L^6(\Omega_h,R^3)} \|\varrho_h^k\|_{L^q(\Omega_h)} \left(\left\| \Pi_h^Q \Pi_h^V[\phi] - \phi \right\|^2_{L^\gamma(\Omega_h;R^3)} + h^2 \|\nabla_x \phi\|^2_{L^\gamma(\Omega;R^3)} \right).$$

Finally, by virtue of (2.4, 2.9),

$$\left\| \Pi_h^Q \Pi_h^V[\phi] - \phi \right\|_{L^\gamma(\Omega_h;R^3)} \leq \left\| \Pi_h^Q [\Pi_h^V[\phi] - \phi] \right\|_{L^\gamma(\Omega_h;R^3)} + \left\| \Pi_h^Q[\phi] - \phi \right\|_{L^\gamma(\Omega_h;R^3)}$$
$$\leq \left\| \Pi_h^V[\phi] - \phi \right\|_{L^\gamma(\Omega_h;R^3)} + \left\| \Pi_h^Q[\phi] - \phi \right\|_{L^\gamma(\Omega_h;R^3)} \leq ch \|\nabla_x \phi\|_{L^\gamma(\Omega;R^3)}.$$

As $\gamma > 3$, we conclude that

$$|I_1| = \left| \sum_{E\in E_h} \int_{\partial E_h} \left(\Pi_h^Q \Pi_h^V[\phi] - \phi \right) [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \left[\left[\varrho_h^k \widehat{\mathbf{u}_h^k} \right]\right] \, \mathrm{dS}_x \right|$$
$$\lesssim h^{\frac{1}{4}} r_h^3(t) \|\nabla_x \phi\|_{L^\gamma(\Omega)}, \ \|r_h^3\|_{L^1(0,T)} \lesssim 1. \tag{6.7}$$

Step 2 Next, we have

$$|I_2| = \left| \sum_{E\in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \widehat{\mathbf{u}}_h^k \cdot \left(\Pi_h^Q \Pi_h^V[\phi] - \phi \right) [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \left[\left[\varrho_h^k \right]\right] \, \mathrm{dS}_x \right|$$
$$\lesssim \left(- \sum_{\Gamma\in\Gamma_h} \int_\Gamma [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \left[\left[\varrho_h^k \right]\right]^2 \, \mathrm{dS}_x \right)^{1/2}$$
$$\times \left(\sum_{E\in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} |\widehat{\mathbf{u}}_h^k|^2 |\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}| \left| \Pi_h^Q \Pi_h^V[\phi] - \phi \right|^2 \, \mathrm{dS}_x \right)^{1/2}$$

where, in accordance with (5.10), the first integral is uniformly bounded in $L^2(0, T)$.

As for the second integral, we use Hölder's inequality to deduce that

$$\sum_{E\in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} |\widehat{\mathbf{u}}_h^k|^2 |\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}| \left| \Pi_h^Q \Pi_h^V[\phi] - \phi \right|^2 \, \mathrm{dS}_x$$
$$\leq \sum_{E\in E_h} \| \Pi_h^Q \Pi_h^V[\phi] - \phi \|^2_{L^\gamma(\partial E)} \left(\sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} |\widehat{\mathbf{u}}_h^k|^{\frac{2\gamma}{\gamma-2}} |\tilde{\mathbf{u}}|^{\frac{\gamma}{\gamma-2}} \, \mathrm{dS}_x \right)^{\frac{\gamma-2}{\gamma}}.$$

Next, by virtue of the trace estimate (2.15) and Hölder's inequality,

$$
\begin{aligned}
&\sum_{E\in E_h} \|\Pi_h^Q \Pi_h^V[\phi]-\phi\|^2_{L^\gamma(\partial E)} \left(\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E} |\widehat{\mathbf{u}}_h^k|^{\frac{2\gamma}{\gamma-2}} |\tilde{\mathbf{u}}|^{\frac{\gamma}{\gamma-2}}\,\mathrm{dS}_x\right)^{\frac{\gamma-2}{\gamma}}\\
&\lesssim \sum_{E\in E_h}\left(h^{-\frac{2}{\gamma}}\|\Pi_h^Q \Pi_h^V[\phi]-\phi\|^2_{L^\gamma(E)} + h^{2-\frac{2}{\gamma}}\|\nabla_x\phi\|^2_{L^\gamma(E)}\right)\\
&\quad\times\left(\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E} |\widehat{\mathbf{u}}_h^k|^{\frac{\gamma}{2\gamma-1}} |\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}|^{\frac{2\gamma}{2\gamma-1}}\,\mathrm{dS}_x\right)^{\frac{2\gamma-1}{2\gamma}}\\
&\lesssim \sum_{E\in E_h}\left(h^{-\frac{2}{\gamma}}\|\Pi_h^Q \Pi_h^V[\phi]-\phi\|^2_{L^\gamma(E)} + h^{2-\frac{2}{\gamma}}\|\nabla_x\phi\|^2_{L^\gamma(E)}\right)\\
&\quad\times\left(\int_{\partial E} |\widehat{\mathbf{u}}_h^k|^{\frac{3\gamma}{\gamma-2}}\,\mathrm{dS}_x\right)^{2\frac{\gamma-2}{3\gamma}}\left(\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}|\tilde{\mathbf{u}}_h^k|^{\frac{3\gamma}{\gamma-2}}\,\mathrm{dS}_x\right)^{\frac{\gamma-2}{3\gamma}}.
\end{aligned}
$$

Furthermore, by (2.16) and Hölder's inequality,

$$
\begin{aligned}
&\sum_{E\in E_h}\left(h^{-\frac{2}{\gamma}}\|\Pi_h^Q \Pi_h^V[\phi]-\phi\|^2_{L^\gamma(E)} + h^{2-\frac{2}{\gamma}}\|\nabla_x\phi\|^2_{L^\gamma(E)}\right)\left(\int_{\partial E} |\widehat{\mathbf{u}}_h^k|^{\frac{3\gamma}{\gamma-2}}\,\mathrm{dS}_x\right)^{2\frac{\gamma-2}{3\gamma}}\\
&\quad\times\left(\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}|\tilde{\mathbf{u}}_h^k|^{\frac{3\gamma}{\gamma-2}}\,\mathrm{dS}_x\right)^{\frac{\gamma-2}{3\gamma}}\\
&\lesssim \sum_{E\in E_h}\left(\frac{1}{h}\|\Pi_h^Q \Pi_h^V[\phi]-\phi\|^2_{L^\gamma(E)} + h\|\nabla_x\phi\|^2_{L^\gamma(E)}\right)\\
&\quad\times\left(\int_E |\widehat{\mathbf{u}}_h^k|^{\frac{3\gamma}{\gamma-2}}\,\mathrm{d}x\right)^{2\frac{\gamma-2}{3\gamma}}\left(\int_E |\mathbf{u}_h^k|^{\frac{3\gamma}{\gamma-2}}\,\mathrm{d}x\right)^{\frac{\gamma-2}{3\gamma}}\\
&\lesssim \frac{1}{h}\|\Pi_h^Q \Pi_h^V[\phi]-\phi\|^2_{L^\gamma(\Omega_h)}\left\|\widehat{\mathbf{u}}_h^k\right\|^2_{L^{\frac{3\gamma}{\gamma-2}}(\Omega_h)}\left\|\mathbf{u}_h^k\right\|_{L^{\frac{3\gamma}{\gamma-2}}(\Omega_h)}\\
&\quad+h\|\nabla_x\phi\|^2_{L^\gamma(\Omega)}\left\|\widehat{\mathbf{u}}_h^k\right\|^2_{L^{\frac{3\gamma}{\gamma-2}}(\Omega_h)}\left\|\mathbf{u}_h^k\right\|_{L^{\frac{3\gamma}{\gamma-2}}(\Omega_h)}\\
&\lesssim h\|\nabla_x\phi\|^2_{L^\gamma(\Omega)}\left\|\mathbf{u}_h^k\right\|^3_{L^{\frac{3\gamma}{\gamma-2}}(\Omega_h)}\\
&\lesssim h^{\frac{5}{2}-\max\{\frac{6}{\gamma};\frac{3}{2}\}}\|\nabla_x\phi\|^2_{L^\gamma(\Omega)}\left\|\mathbf{u}_h^k\right\|^3_{L^6(\Omega_h)}\ \text{ provided } \gamma>3,
\end{aligned}
$$

where we have used (2.17).

Finally, using the time estimates (2.20) we infer that

$$
\begin{aligned}
&h^{\frac{5}{2}-\max\{\frac{6}{\gamma};\frac{3}{2}\}}\|\nabla_x\phi\|^2_{L^\gamma(\Omega)}\left\|\mathbf{u}_h^k\right\|^3_{L^6(\Omega_h)}\\
&\quad\le(\Delta t)^{-\frac{1}{2}}h^{\frac{5}{2}-\max\{\frac{6}{\gamma};\frac{3}{2}\}}\|\nabla_x\phi\|^2_{L^\gamma(\Omega)}(\Delta t)^{\frac{1}{2}}\left\|\mathbf{u}_h^k\right\|^3_{L^6(\Omega_h)}.
\end{aligned}
$$

Summarizing we conclude that

$$
\begin{aligned}
|I_2|&=\left|\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\widehat{\mathbf{u}}_h^k\cdot\left(\Pi_h^Q\Pi_h^V[\phi]-\phi\right)[\mathbf{u}_h^k\cdot\mathbf{n}]^-\left[\left[\varrho_h^k\right]\right]\,\mathrm{dS}_x\right|\\
&\lesssim h^{2-\max\{\frac{3}{\gamma};\frac{3}{4}\}}r_h^4(t)\|\nabla_x\phi\|_{L^\gamma(\Omega)},\ \ \|r_h^4\|_{L^1(0,T)}\lesssim 1.
\end{aligned}
\tag{6.8}
$$

Step 3 Another application of Hölder's inequality gives rise to

$$
\begin{aligned}
|I_3|&=\left|\sum_{E\in E_h}\int_E\varrho_h^k\widehat{\mathbf{u}}_h^k(\phi-\Pi_h^Q\Pi_h^V[\phi])\mathrm{div}_h\mathbf{u}_h^k\,\mathrm{d}x\right|\\
&=\left|\sum_{E\in E_h}\int_K\varrho_h^k\widehat{\mathbf{u}}_h^k(\phi-\Pi_h^V[\phi])\mathrm{div}_h\mathbf{u}_h^k\,\mathrm{d}x\right|\\
&\le\|\varrho_h^k\|_{L^\infty(\Omega_h;R^3)}\sum_{E\in E_h}\|\mathrm{div}_h\mathbf{u}_h^k\|_{L^2(E)}\|\mathbf{u}_h^k\|_{L^6(E)}\left\|\phi-\Pi_h^V[\phi]\right\|_{L^3(E;R^3)}.
\end{aligned}
$$

Now, by virtue of (2.17) and (2.9),

$$
\begin{aligned}
&\|\varrho_h^k\|_{L^\infty(\Omega_h)}\sum_{E\in E_h}\|\mathrm{div}_h\mathbf{u}_h^k\|_{L^2(E)}\|\mathbf{u}_h^k\|_{L^6(E;R^3)}\left\|\phi-\Pi_h^V[\phi]\right\|_{L^3(E;R^3)}\\
&\lesssim\frac{1}{h^{3/\gamma}}\|\varrho_h^k\|_{L^\gamma(\Omega_h)}\|\mathrm{div}_h\mathbf{u}_h^k\|_{L^2(\Omega_h)}\|\mathbf{u}_h^k\|_{L^6(\Omega_h;R^3)}\left\|\phi-\Pi_h^V[\phi]\right\|_{L^3(\Omega_h;R^3)}\\
&\lesssim h^{1-\frac{3}{\gamma}}\|\varrho_h^k\|_{L^\gamma(\Omega_h)}\|\mathrm{div}_h\mathbf{u}_h^k\|_{L^2(\Omega_h)}\|\mathbf{u}_h^k\|_{L^6(\Omega_h;R^3)}\|\nabla_x\phi\|_{L^3(\Omega;R^3)}
\end{aligned}
$$

yielding the desired conclusion

$$
\begin{aligned}
|I_3|&=\left|\sum_{E\in K_h}\int_E\varrho_h^k\widehat{\mathbf{u}}_h^k(\phi-\Pi_h^Q\Pi_h^V[\phi])\mathrm{div}_x\mathbf{u}_h^k\,\mathrm{d}x\right|\\
&\lesssim\sqrt{h}r_h^5(t)\|\nabla_x\phi\|_{L^\gamma(\Omega)},\ \ \|r_h^5\|_{L^1(0,T)}\lesssim 1.
\end{aligned}
\tag{6.9}
$$

Step 4 The last integral

$$|I_4| = \left| \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \phi \cdot \widehat{\mathbf{u}_h^k} \varrho_h^k (\tilde{\mathbf{u}}_n^k - \mathbf{u}_h^k) \cdot \mathbf{n} \, \mathrm{dS}_x \right|$$

can be handled in the same way as its counterpart in the continuity method.

6.2.3 Bounds on numerical dissipation

Finally, the numerical viscosity

$$h^\alpha \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma [[\varrho_h \mathbf{u}_h]] \cdot \left[\left[\widehat{\Pi_h^V[\phi]} \right]\right] \chi \left(\frac{\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}}{h^\alpha} \right) \mathrm{dS}_x$$

can be estimated by means of (5.9, 5.10) in a similar way as in the continuity method

$$h^\alpha \left| \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma [[\varrho_h \mathbf{u}_h]] \cdot \left[\left[\widehat{\Pi_h^V[\phi]} \right]\right] \chi \left(\frac{\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}}{h^\alpha} \right) \mathrm{dS}_x \right| \lesssim h^{\alpha/2} r_h^6(t) \|\nabla_x \phi\|_{L^\gamma(\Omega; R^3)}. \tag{6.10}$$

Summing up (6.7)–(6.10) we obtain the consistency formulation of the momentum method:

$$\begin{aligned} &\int_\Omega D_t(\varrho_h \widehat{\mathbf{u}}_h) \cdot \phi \, \mathrm{d}x - \int_\Omega (\varrho_h \widehat{\mathbf{u}}_h \otimes \mathbf{u}_h) : \nabla_x \phi \, \mathrm{d}x \\ &\quad + \int_\Omega [\mu \nabla_h \mathbf{u}_h : \nabla_x \phi + \lambda \mathrm{div}_h \mathbf{u}_h \mathrm{div}_x \phi] \, \mathrm{d}x - \int_\Omega p(\varrho_h) \mathrm{div}_x \phi \, \mathrm{d}x \\ &= \int_\Omega \mathbb{R}_h^2(t, \cdot) : \nabla_x \phi \, \mathrm{d}x, \end{aligned} \tag{6.11}$$

for any $\phi \in C_c^\infty(\Omega; R^3)$, where $\mathbb{R}_h^2$ is piecewise constant in time,

$$\left\| \mathbb{R}_h^2 \right\|_{L^1(0,T; L^{\frac{\gamma}{\gamma-1}}(\Omega; R^3))} \lesssim h^\beta, \ \beta > 0. \tag{6.12}$$

7 Convergence of the numerical solutions

We are ready to establish convergence of solutions of our numerical method to a weak solution of the limit problem. We take advantage of the consistency formulation derived in the preceding section that converts the problem to the framework of the mathematical theory developed in [10] and [20]. The reader may also consult [1] for a complete existence proof based on the technique of time discretization very close to the numerical method applied in the present paper. Throughout the whole section we shall systematically use our convention that all quantities defined on Ω_h are extended to be zero outside Ω_h.

7.1 Local pressure estimates

The uniform bound (5.4) is not sufficient for passing to the limit in the pressure $p(\varrho)$, the latter being bounded only in the non-reflexive space $L^\infty(0,T;L^1(R^3))$. To get better integrability of the pressure, we use the quantities

$$\phi = \varphi \nabla_x \Delta^{-1}[\eta \varrho_h],$$

where

$$\begin{aligned}\varphi(t,x) = \psi(t)\omega(x),\ \psi \in C_c^\infty(0,T),\ \eta,\ \omega \in C_c^\infty(\Omega),\ -\Delta^{-1}[v] \\ \equiv \mathcal{F}^{-1}_{\xi\to x}\left[\frac{1}{|\xi|^2}\mathcal{F}_{x\to\xi}[v]\right],\end{aligned}$$

and $\mathcal{F}$ denotes the standard Fourier transform, as test functions in the consistency formulation (6.11) of the momentum method:

$$\begin{aligned}&\int_0^T\int_\Omega \varphi\eta\Big[p(\varrho_h)\varrho_h - \lambda\varrho_h \mathrm{div}_x \mathbf{u}_h\Big]\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T\int_\Omega \Big[\lambda \mathrm{div}_x \mathbf{u}_h - p(\varrho_h)\Big]\nabla_x\varphi\cdot\nabla_x(\Delta^{-1}[\eta\varrho_h])\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_0^T\int_\Omega \mathbb{R}_h^2 : \nabla_x(\varphi\nabla_x(\Delta^{-1}[\eta\varrho_h]))\,\mathrm{d}x\mathrm{d}t \\ &\quad + \int_0^T\int_\Omega \mu\nabla_h\mathbf{u}_h : \nabla_x\Big[\varphi\nabla_x(\Delta^{-1}[\eta\varrho_h])\Big]\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_0^T\int_\Omega (\varrho_h\widehat{\mathbf{u}}_h\otimes\mathbf{u}_h) : \nabla_x\Big(\varphi\nabla_x(\Delta^{-1}[\eta\varrho_h])\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^T\int_\Omega D_t(\varrho_h\widehat{\mathbf{u}}_h)\cdot\varphi\nabla_x(\Delta^{-1}[\eta\varrho_h])\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{7.1}$$

Furthermore, using a discretized version of the integration by parts formula and the consistency formulation of the momentum method (6.2), we deduce that

$$\begin{aligned}&\int_0^T\int_\Omega D_t(\varrho_h\widehat{\mathbf{u}}_h)\cdot\varphi\nabla_x(\Delta^{-1}[\eta\varrho_h])\,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^T\int_\Omega \frac{\varphi(t+\Delta t)-\varphi(t)}{\Delta t}\varrho_h\widehat{\mathbf{u}}_h\cdot\nabla_x(\Delta^{-1}[\eta\varrho_h])\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_0^T\int_\Omega \varphi\varrho_h(t-\Delta t)\widehat{\mathbf{u}}_h(t-\Delta t)\cdot\nabla_x\Delta^{-1}[\eta D_t\varrho_h]\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&= - \int_0^T \int_\Omega \frac{\varphi(t+\Delta t) - \varphi(t)}{\Delta t} \varrho_h \widehat{\mathbf{u}}_h \cdot \nabla_x (\Delta^{-1}[\eta \varrho_h]) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \varphi \varrho_h (t - \Delta t) \widehat{\mathbf{u}}_h (t - \Delta t) \cdot \nabla_x \Delta^{-1} \left[\eta \mathrm{div}_x (\varrho_h \mathbf{u}_h) \right] \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \varphi \varrho_h (t - \Delta t) \widehat{\mathbf{u}}_h (t - \Delta t) \cdot \nabla_x \Delta^{-1} \left[\eta \mathrm{div}_x \mathbf{R}_h^1 \right] \, \mathrm{d}x \, \mathrm{d}t. \qquad (7.2)
\end{aligned}
$$

We observe that the expression on the right-hand side of (7.1) is bounded uniformly for $h \to 0$. Indeed combining the estimates (5.3), (5.4) we have

$$
\sup_{\tau \in (0,T)} \| \varrho_h \mathbf{u}_h (\tau, \cdot) \|_{L^q(R^3, R^3)} \lesssim 1, \ q = \frac{2\gamma}{\gamma + 1}, \qquad (7.3)
$$

$$
\| \varrho_h \mathbf{u}_h \|_{L^2(0,T; L^s(R^3; R^3))} \lesssim 1, \ s = \frac{6\gamma}{\gamma + 6} > 2 \text{ if } \gamma > 3. \qquad (7.4)
$$

The integrals on the right-hand side of (7.1) can therefore be estimated in the same way as in [10, Chapter 5] and we may conclude that

$$
\| \varrho_h \|_{L^{\gamma+1}((0,T) \times K)} \leq 1 \text{ for any compact } K \subset \Omega. \qquad (7.5)
$$

7.2 Weak sequential compactness

In accordance with the uniform estimates (5.4)–(5.6), there is a subsequence of numerical solutions such that

$$
\varrho_h \to \varrho \text{ weakly-(*) in } L^\infty(0, T; L^\gamma(R^3)), \qquad (7.6)
$$

and

$$
\mathbf{u}_h \to \mathbf{u} \text{ weakly in } L^2(0, T; L^6(R^3; R^3)). \qquad (7.7)
$$

Moreover, we have $\varrho \geq 0$, and, by virtue of (5.1),

$$
\int_\Omega \varrho(\tau, \cdot) \, \mathrm{d}x = \int_\Omega \varrho_0 \, \mathrm{d}x \text{ for a.a. } t \in (0, T).
$$

Next, it follows from (2.9) that

$$
\| \widehat{\mathbf{u}}_h - \mathbf{u}_h \|_{L^2((0,T) \times \Omega_h; R^3)} \to 0, \qquad (7.8)
$$

in particular,

$$
\widehat{\mathbf{u}}_h \to \mathbf{u} \text{ weakly in } L^2(0, T; L^6(R^3; R^3)) \qquad (7.9)
$$

provided $\widehat{\mathbf{u}}_h$ is extended to be zero outside Ω_h.

Finally, we observe that (5.5) implies

$$
\nabla_h \mathbf{u}_h \to \nabla_x \mathbf{u} \text{ weakly in } L^2((0, T) \times R^3; R^{3\times 3}); \qquad (7.10)
$$

whence the limit velocity field satisfies

$$\mathbf{u} \in L^2(0, T; W^{1,2}(R^3; R^3)).$$

Remark 7.1 The fact that the weak limit of $\nabla_h \mathbf{u}_h$ coincides with $\nabla_x \mathbf{u}$ follows from the "density" of the spaces $V_{h,0}$ in $W_0^{1,2}$ stated in (2.9).

In addition, we may use Lemma 2.1 to construct the smooth approximations $R_h^V[\mathbf{u}_h]$,

$$R_h^V[\mathbf{u}_h] \to \mathbf{u} \text{ weakly in } L^2(0, T; W^{1,2}(R^3; R^3)).$$

It follows from (2.26) that the limit $\mathbf{u}$ vanishes on any compact $K_e \subset R^3 \setminus \overline{\Omega}$. Since Ω is Lipschitz, we conclude that

$$\mathbf{u} \in L^2(0, T; W_0^{1,2}(\Omega; R^3)).$$

Remark 7.2 Note that this is the only point, where certain regularity of $\partial\Omega$ is needed. As already pointed out, the assumption that Ω is Lipschitz can be considerably relaxed.

To establish the weak convergence of convective terms, we need the following result that can be seen as a variant of [18, Lemma 2.3].

Lemma 7.1 *Let $\{v_h\}_{h>0}$, $\{w_h\}_{h>0}$ be two sequences of functions in $(0, T) \times Q$, Q a domain in R^N, such that*

$$\begin{aligned}
&v_h, w_h \textit{ are constant functions in time on any interval } [k\Delta t, (k+1)\Delta t),\\
&\quad k = 0, 1, \dots, \ \Delta t \approx h,\\
&v_h \to v \textit{ weakly in } L^{p_1}(0, T; L^{q_1}(Q)),\\
&w_h \to w \textit{ weakly in } L^{p_2}(0, T; L^{q_2}(Q)), \ \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} = 1,
\end{aligned}$$

$$\left| \int_\Omega D_t v_h \phi \, \mathrm{d}x \right| \le r_h(t) \|\phi\|_{W^{k,p}(Q)} \textit{ for certain } k, p \ge 1, \ \|r^h\|_{L^1(0,T)} \lesssim 1, \tag{7.11}$$

$$\|w_h(t, x) - w_h(t, x - \xi)\|_{L^{p_2}(0,T;L^{q_2}(Q))} \to 0 \textit{ as } |\xi| \to 0 \textit{ uniformly in } h. \tag{7.12}$$

Then

$$v_h w_h \to vw \textit{ in the sense of distributions in } (0, T) \times Q.$$

In agreement with the gradient estimates (5.5) and the compactness properties of the space $H^1_{V_h}$ stated in (2.25), we observe that the sequence $\{\mathbf{u}_h\}_{h>0}$ satisfies the hypothesis (7.12) with $p_2 = q_2 = 2$ $Q = \Omega$, while the hypothesis (7.11) can be checked for ϱ_h, $\varrho_h \widehat{\mathbf{u}}_h$ with the help of the consistency formulations (6.2, 6.11). Thus successive application of Lemma 7.1 gives rise to the following limits:

$$\varrho_h \mathbf{u}_h \to \varrho \mathbf{u} \text{ weakly in } L^2(0,T; L^{\frac{6\gamma}{\gamma+6}}(\Omega; R^3)), \tag{7.13}$$

and

$$\varrho_h \widehat{\mathbf{u}}_h \otimes \mathbf{u}_h \to \varrho \mathbf{u} \otimes \mathbf{u} \text{ weakly in } L^q((0,T) \times \Omega; R^{3\times 3}) \text{ for some } q > 1, \tag{7.14}$$

Remark 7.3 As for the exponent q in (7.14), we recall that

$$\begin{aligned} \varrho_h \widehat{\mathbf{u}}_h \ &\in \ L^\infty(0,T; L^{\frac{2\gamma}{\gamma+2}}(\Omega; R^3)) \cap L^2(0,T; L^{\frac{6\gamma}{\gamma+6}}(\Omega; R^3)) \\ &\hookrightarrow L^r((0,T) \times \Omega; R^3) \text{ for a certain } r > 2 \end{aligned}$$

by interpolation.

7.3 Limit in the field equations

At this stage we are ready to pass to the limit in the consistency formulation of the numerical method. Letting $h \to 0$ in (6.2, 6.11) we obtain

$$\int_0^T \int_{R^3} \Big[\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \Big] \, \mathrm{d}x \, \mathrm{d}t = - \int_{R^3} \varrho_0 \varphi(0, \cdot) \, \mathrm{d}x \tag{7.15}$$

for any $\varphi \in C_c^\infty([0,T) \times R^3)$;

$$\begin{aligned} &\int_0^T \int_\Omega \Big[\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + \overline{p(\varrho)} \mathrm{div}_x \varphi \Big] \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_0^T \int_\Omega \Big[\mu \nabla_x \mathbf{u} : \nabla_x \varphi + \lambda \mathrm{div}_x \mathbf{u} \, \mathrm{div}_x \varphi \Big] \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \varphi(0, \cdot) \, \mathrm{d}x \end{aligned} \tag{7.16}$$

for any $\varphi \in C_c^\infty([0,T) \times \Omega; R^3)$.

Remark 7.4 In view of the local pressure estimate (7.5) we may assume that

$$p(\varrho_h^k) \to \overline{p(\varrho)} \text{ weakly in } L^{\frac{\gamma+1}{\gamma}}(K) \text{ for any compact } K \subset \Omega.$$

7.4 Strong convergence of the density

In order to finish the proof of convergence we have to show a.e. pointwise convergence of the numerical densities in order to replace $\overline{p(\varrho)}$ by $p(\varrho)$ in (7.16). To this end, we use the method of Lions [20] based on a "weak continuity" property of the effective viscous flux. Going back to (7.1), (7.2), we focus on the term

$$\int_0^T \int_\Omega \nabla_h \mathbf{u}_h : \nabla_x \left[\varphi \nabla_x (\Delta^{-1}[\eta \varrho_h]) \right] \, \mathrm{d}x \, \mathrm{d}t. \tag{7.17}$$

Following [19], we perform integration by parts to obtain

$$\begin{aligned}
\int_\Omega \nabla_h \mathbf{u}_h : \nabla_x \phi \, \mathrm{d}x &= \int_\Omega \nabla_h \mathbf{u}_h : \left(\nabla_x - \nabla_x^T \right) \phi \, \mathrm{d}x + \int_\Omega \nabla_h \mathbf{u}_h : \nabla_x^T \phi \, \mathrm{d}x \\
&= \int_\Omega \mathbf{curl}_h \mathbf{u}_h : \mathbf{curl}_x \phi \, \mathrm{d}x + \int_\Omega \nabla_h \mathbf{u}_h : \nabla_x^T \phi \, \mathrm{d}x \\
&= \int_\Omega \mathbf{curl}_h \mathbf{u}_h : \mathbf{curl}_x \phi \, \mathrm{d}x + \int_\Omega \mathrm{div}_h \mathbf{u}_h : \mathrm{div}_x \phi \, \mathrm{d}x + \text{ error term,}
\end{aligned}$$

where the error is estimated by means of [19, Lemma 8.2] as

$$\left| \int_\Omega \nabla_h \mathbf{u}_h : \nabla_x^T \phi \, \mathrm{d}x - \int_\Omega \mathrm{div}_h \mathbf{u}_h : \mathrm{div}_x \phi \, \mathrm{d}x \right| \lesssim h \| \nabla_h \mathbf{u}_h \|_{L^2(\Omega; R^{3\times 3})} \| \nabla^2 \phi \|_{L^2(\Omega; R^{27})}. \tag{7.18}$$

Returning to (7.17), we get

$$\begin{aligned}
&\int_0^T \int_\Omega \nabla_h \mathbf{u}_h : \nabla_x \left[\varphi \nabla_x (\Delta^{-1}[\eta \varrho_h]) \right] \, \mathrm{d}x \, \mathrm{d}t \\
&= \int_0^T \int_\Omega \mathbf{curl}_h \mathbf{u}_h : \mathbf{curl}_x \left[\varphi \nabla_x (\Delta^{-1}[\eta \varrho_h]) \right] \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \nabla_h \mathbf{u}_h : \nabla_x^T \left[\varphi \nabla_x (\Delta^{-1}[\eta \varrho_h]) \right] \, \mathrm{d}x \, \mathrm{d}t,
\end{aligned}$$

with

$$\begin{aligned}
&\int_0^T \int_\Omega \nabla_h \mathbf{u}_h : \nabla_x^T \left[\varphi \nabla_x (\Delta^{-1}[\eta \varrho_h]) \right] \, \mathrm{d}x \, \mathrm{d}t \\
&= \int_0^T \int_\Omega \nabla_h \mathbf{u}_h : \nabla_x^T \left[\varphi \nabla_x \left(\Delta^{-1} \left[\eta \varrho_h - \eta R_h^Q [\varrho_h] \right] \right) \right] \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \nabla_h \mathbf{u}_h : \nabla_x^T \left[\varphi \nabla_x \left(\Delta^{-1} \left[\eta R_h^Q [\varrho_h] \right] \right) \right] \, \mathrm{d}x \, \mathrm{d}t
\end{aligned}$$

where R_h^Q are the regularizing operators introduced in Lemma 2.1. Thus, recalling the bounds (5.10) and applying Lemma 2.1, we obtain

$$\int_0^T \| \varrho_h - R_h^Q [\varrho_h] \|_{L^2(K)}^2 \, \mathrm{d}t \lesssim h^2 \int_0^T \| \varrho_h \|_{H^1_{Q_h}(\Omega_h)}^2 \, \mathrm{d}t \lesssim h^{1-\alpha},$$

where $K \subset \Omega$ is a compact set containing the spatial support of the function η. In particular, the integral

$$\int_0^T \int_\Omega \nabla_h \mathbf{u}_h : \nabla_x^T \left[\varphi \nabla_x \left(\Delta^{-1} \left[\eta \varrho_h - \eta R_h^Q [\varrho_h] \right] \right) \right] \, \mathrm{d}x \, \mathrm{d}t$$

vanishes for $h \to 0$ and may be included in the error term on the right-hand side of (7.1).

Similarly, by the same token,

$$\| \nabla_x R_h^Q [\varrho_h] \|_{L^2(0,T;L^2(K;R^3))} \approx \left(\int_0^T \| \varrho_h \|^2_{H^1_{Q_h}(\Omega)} \, \mathrm{d}t \right)^{1/2} \lesssim h^{-\frac{1+\alpha}{2}};$$

whence, in accordance with (7.18) we may replace

$$\begin{aligned} &\int_0^T \int_\Omega \nabla_h \mathbf{u}_h : \nabla_x^T \left[\varphi \nabla_x \left(\Delta^{-1} \left[\eta R_h^Q [\varrho_h] \right] \right) \right] \, \mathrm{d}x \, \mathrm{d}t \\ &\quad \approx \int_0^T \int_\Omega \mathrm{div}_h \mathbf{u}_h : \mathrm{div}_x \left[\varphi \nabla_x \left(\Delta^{-1} \left[\eta R_h^Q [\varrho_h] \right] \right) \right] \, \mathrm{d}x \, \mathrm{d}t \\ &\quad \approx \int_0^T \int_\Omega \mathrm{div}_h \mathbf{u}_h : \mathrm{div}_x \left[\varphi \nabla_x \left(\Delta^{-1} \left[\eta \varrho_h \right] \right) \right] \, \mathrm{d}x \, \mathrm{d}t. \end{aligned}$$

Summing up the previous estimates and regrouping terms in (7.1) we obtain

$$\begin{aligned} &\int_0^T \int_\Omega \varphi \eta \Big[p(\varrho_h) \varrho_h - (\lambda + \mu) \varrho_h \mathrm{div}_x \mathbf{u}_h \Big] \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_0^T \int_\Omega \Big[(\lambda + \mu) \mathrm{div}_x \mathbf{u}_h - p(\varrho_h) \Big] \nabla_x \varphi \cdot \nabla_x (\Delta^{-1} [\eta \varrho_h]) \, \mathrm{d}x \, \mathrm{d}t \\ &\quad - \int_0^T \int_\Omega \frac{\varphi(t + \Delta t) - \varphi(t)}{\Delta t} \varrho_h \widehat{\mathbf{u}}_h \cdot \nabla_x (\Delta^{-1} [\eta \varrho_h]) \, \mathrm{d}x \, \mathrm{d}t \\ &\quad + \int_0^T \int_\Omega \mu \mathbf{curl}_h \mathbf{u}_h \cdot \mathbf{curl}_x \left[\varphi \nabla_x (\Delta^{-1} [\eta \varrho_h]) \right] \, \mathrm{d}x \, \mathrm{d}t \\ &\quad - \int_0^T \int_\Omega (\varrho_h \widehat{\mathbf{u}}_h \otimes \mathbf{u}_h) : \Big(\nabla_x \varphi \otimes \nabla_x (\Delta^{-1} [\eta \varrho_h]) \Big) \, \mathrm{d}x \, \mathrm{d}t \\ &\quad - \int_0^T \int_\Omega \varphi (\varrho_h \widehat{\mathbf{u}}_h \otimes \mathbf{u}_h) : (\nabla_x \otimes \nabla_x) (\Delta^{-1} [\eta \varrho_h]) \, \mathrm{d}x \, \mathrm{d}t \\ &\quad + \int_0^T \int_\Omega \varphi (\varrho_h \widehat{\mathbf{u}}_h)(t - \Delta t) \cdot \nabla_x \Delta^{-1} \left[\eta \mathrm{div}_x (\varrho_h \mathbf{u}_h) \right] \, \mathrm{d}x \, \mathrm{d}t + E_h(\varphi, \eta), \end{aligned} \tag{7.19}$$

with the error term $E_h(\varphi, \eta) \to 0$ as $h \to 0$ for any fixed φ, η.

Remark 7.5 It is worth noting that this is the only step in the proof, where we have used the artificial viscosity term included in the upwinding.

Now we apply a similar treatment to the limit Eq. (7.16), specifically, we use the test functions

$$\phi = \varphi \nabla_x \Delta^{-1}[\eta \varrho].$$

After a straightforward manipulation (cf. [10, Chapter 6]) we arrive at

$$\begin{aligned}
&\int_0^T \int_\Omega \varphi \eta \Big[\overline{p(\varrho)} \varrho - (\lambda + \mu) \varrho \mathrm{div}_x \mathbf{u} \Big] \,\mathrm{d}x \,\mathrm{d}t \\
&\quad = \int_0^T \int_\Omega \Big[\lambda \mathrm{div}_x \mathbf{u} - \overline{p(\varrho)} \Big] \nabla_x \varphi \cdot \nabla_x (\Delta^{-1}[\eta \varrho]) \,\mathrm{d}x \,\mathrm{d}t \\
&\qquad - \int_0^T \int_\Omega \partial_t \varphi \varrho \mathbf{u} \cdot \nabla_x (\Delta^{-1}[\eta \varrho]) \,\mathrm{d}x \,\mathrm{d}t \\
&\qquad + \int_0^T \int_\Omega \mu \mathbf{curl}_x \mathbf{u} \cdot \mathbf{curl}_x \Big[\varphi \nabla_x (\Delta^{-1}[\eta \varrho]) \Big] \,\mathrm{d}x \,\mathrm{d}t \\
&\qquad - \int_0^T \int_\Omega (\varrho \mathbf{u} \otimes \mathbf{u}) : \Big(\nabla_x \varphi \otimes \nabla_x (\Delta^{-1}[\eta \varrho]) \Big) \,\mathrm{d}x \,\mathrm{d}t \\
&\qquad - \int_0^T \int_\Omega \varphi (\varrho \mathbf{u} \otimes \mathbf{u}) : (\nabla_x \otimes \nabla_x)(\Delta^{-1}[\eta \varrho]) \,\mathrm{d}x \,\mathrm{d}t \\
&\qquad + \int_0^T \int_\Omega \varphi \varrho \mathbf{u} \cdot \nabla_x \Delta^{-1} \Big[\eta \mathrm{div}_x (\varrho \mathbf{u}) \Big] \,\mathrm{d}x \,\mathrm{d}t. \qquad (7.20)
\end{aligned}$$

The principal idea due to Lions [20] is that all terms on the right-hand side of (7.19) converge to their counterparts in (7.20). This has been proved in the continuous case in [20] and for the time discretization problem in [1, Section 3.3], Lions [20]. The same result at the level of numerical discretization was obtained by Karlsen and Karper [18], Karper [19]. Here, we recall that the error terms in (7.19) vanish for $h \to 0$; whence the most difficult task is to show that

$$\begin{aligned}
&- \int_0^T \int_\Omega \varphi (\varrho_h \widehat{\mathbf{u}}_h \otimes \mathbf{u}_h) : (\nabla_x \otimes \nabla_x)(\Delta^{-1}[\eta \varrho_h]) \,\mathrm{d}x \,\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \varphi (\varrho_h \widehat{\mathbf{u}}_h)(t - \Delta t) \cdot \nabla_x \Delta^{-1} \mathrm{div}_x (\eta \varrho_h \mathbf{u}_h) \,\mathrm{d}x \,\mathrm{d}t \\
&\quad \to \\
&\quad - \int_0^T \int_\Omega \varphi (\varrho \mathbf{u} \otimes \mathbf{u}) : (\nabla_x \otimes \nabla_x)(\Delta^{-1}[\eta \varrho]) \,\mathrm{d}x \,\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \varphi \varrho \mathbf{u} \cdot \nabla_x \Delta^{-1} \mathrm{div}_x (\eta \varrho \mathbf{u}) \,\mathrm{d}x \,\mathrm{d}t. \qquad (7.21)
\end{aligned}$$

Moreover, in view of the numerical dissipation estimates (5.7), we may replace $(\varrho_h \widehat{\mathbf{u}}_h)(t - \Delta t)$ by $\varrho_h \widehat{\mathbf{u}}_h$. Finally, we observe that the velocity field $\mathbf{u}_h$ can be approximated by its spatial regularization in the spirit of Lemma 2.1,

$$\|\mathbf{u}_h - R_h^V[\mathbf{u}_h]\|_{L^2(0,T;L^q(\Omega;R^3))} \lesssim h^\beta,\ \beta = \beta(q) > 0 \text{ for any } 2 \leq q < 6.$$

In particular, we may write $R_h^V[\mathbf{u}_h]$ in place of $\mathbf{u}_h$ in (7.21). Now, the limit (7.21) can be verified exactly as in [1, Section 3.3] or Karper [19, Lemma 9.3].

Thus we get the desired conclusion - the effective viscous flux identity due to Lions [20]:

$$\begin{aligned} &\int_0^T \int_\Omega \varphi\Big[p(\varrho_h)\varrho_h - (\lambda+\mu)\varrho_h \mathrm{div}_x \mathbf{u}_h\Big]\,\mathrm{d}x\,\mathrm{d}t \\ &\quad \to \int_0^T \int_\Omega \varphi\Big[\overline{p(\varrho)}\varrho - (\lambda+\mu)\varrho \mathrm{div}_x \mathbf{u}\Big]\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{7.22}$$

as $h \to 0$ for any $\varphi \in C_c^\infty((0,T)\times\Omega)$, which yields the crucial relation

$$\overline{\varrho \mathrm{div}_x \mathbf{u}} \geq \varrho \mathrm{div}_x \mathbf{u}. \tag{7.23}$$

The inequality (7.23) implies convergence $\varrho_h \to \varrho$ a.e. in $(0,T)\times\Omega$. Indeed the regularization procedure of DiPerna and Lions [7] can be applied to show that ϱ is a renormalized solution of the continuity equation, in particular,

$$\int_\Omega \varrho \log(\varrho)(\tau,\cdot)\,\mathrm{d}x + \int_0^\tau \int_\Omega \varrho \mathrm{div}_x \mathbf{u}\,\mathrm{d}x\,\mathrm{d}t \leq \int_\Omega \varrho_0 \log(\varrho_0)\,\mathrm{d}x \text{ for any } \tau \in [0,T], \tag{7.24}$$

cf. [10, Chapter 6]. On the other hand, passing to the limit in the renormalized continuity method (4.1) for $b(\varrho) = \varrho\log(\varrho)$ we obtain

$$\int_\Omega \overline{\varrho \log(\varrho)}(\tau,\cdot)\,\mathrm{d}x + \int_0^\tau \int_\Omega \overline{\varrho \mathrm{div}_x \mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t \leq \int_\Omega \varrho_0 \log(\varrho_0)\,\mathrm{d}x \text{ for a.a } \tau \in (0,T). \tag{7.25}$$

Combining (7.23)–(7.25) we get

$$\overline{\varrho\log(\varrho)} = \varrho\log(\varrho)$$

yielding the desired conclusion

$$\varrho_h \to \varrho \text{ in } L^1((0,T)\times\Omega). \tag{7.26}$$

Seeing that the energy inequality (1.12) follows from (4.5) we have completed the proof of Theorem 3.1.

8 Unconditional convergence

Our ultimate goal is to discuss the situation when both the data ϱ_0, $\mathbf{u}_0$ and the underlying physical domain Ω are regular. Specifically, we claim the following result concerning unconditional convergence of *bounded* numerical solutions.

Theorem 8.1 *In addition to the hypotheses of Theorem 3.1, suppose that Ω is a bounded domain of class $C^{2+\nu}$ and the initial data satisfy*

$$\varrho_0 \in W^{1,6}(\Omega), \quad \varrho_0 \geq \underline{\varrho} > 0 \quad in \quad \Omega, \quad \mathbf{u} \in W^{2,2}(\Omega; R^3),$$

and $\eta = 0$. Moreover, suppose that there exists a positive constant r such that

$$\varrho_h^k \leq r \quad for\ all \quad k = 1, 2, \ldots, \quad h \to 0. \tag{8.1}$$

Then the convergence claimed in Theorem 3.1 is unconditional, meaning the limit solution ϱ, $\mathbf{u}$ is regular, unique, and the whole family of numerical solutions converges to it.

Proof The hypothesis (8.1) implies that the density component of the limit solution is bounded. Using the conditional regularity result proved in [11, Theorem 2.4] and [15, Theorem 4.6] we conclude that the limit solution is regular whence unique. □

Under the condition described in Theorem 8.1, it is possible to obtain qualitative estimates on the rate of convergence of the numerical scheme, see [12].

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