Z. Physik C, Particles and Fields 3, 209–222 (1980)

Zeitschrift für Physik C Particles and Fields

The Effect of Instanton–Anti-Instanton Interactions on Current Correlation Functions in Massless QCD*

William F. Palmer and Stephen S. Pinsky
Department of Physics, The Ohio State University, Columbus, OH 43210, USA

Received 16 July 1979

Abstract. The leading order effects, away from the Dilute Gas Approximation, of instanton–anti-instanton interactions are studied in massless QCD. Propagation and current correlation functions are computed in the interacting dilute gas. While no mass is generated, power law corrections are achieved for the correlation function entering into e^+e^- annihilation.

I. Introduction

Since the first application of the Belevin et al. [1] instanton by 't Hooft [2] to the breaking of chiral U(1) in the QCD of massless quarks, much interest has focused on the physics of these new Euclidean solutions. Early on, after the γ_5 non-invariance of the instanton corrected vacuum had been demonstrated by Jackiw and Rebbi [3] and by Callan et al. [4], it was conjectured that instantons were further responsible for chiral SU(N_F) breakdown and even the confinement of colored quarks into hadrons.

In this spirit a number of instanton calculations have been made which extend the scope of ordinary perturbative asymptotically free QCD. They are consistently interpreted as expansions in momentum space in powers of (μ_0/p) where μ_0 is the intrinsic renormalization mass. These calculations, the model for which was 't Hooft's [2] vacuum-to-vacuum one-loop amplitude and his preliminary computation of the di-fermion propagation functions, have been performed for exact, one-instanton configurations and for non-interacting dilute gas (NIDG) of instantons and anti-instantons, approximate solutions with large entropy, a scheme pioneered by Callan et al. [5]. The procedures in these calculations are simple in principle: one evaluates the classical correlation function in an instanton field, obtaining, for example, a propagator which depends on the collective coordinates, the size, position, and group orientation of the instanton. Then one multiplies by the determinantal factors arising from the one-loop approximation to the functional integral calculated in [2]. Finally, one does the remaining integration over the classical field parameters, the collective coordinates, restoring manifest translation and color symmetry.

A problem common to calculations of this type, both those using exact instanton or multi-instanton background fields as well as NIDG approximation, is that these gauge configurations support normalized solutions of the Dirac equation for massless fermions which have zero eigenvalues. These zero modes enter into the fermion determinant and thus suppress vacuum tunneling and inhibit instanton amplitudes unless a sufficient number of external fermion lines are attached. In essence, the tunneling is not between two gauge field vacua, but between a gauge field vacuum and another vacuum of different winding number which contains a number of zero energy quark–anti-quark pairs. The presence and persistence of these zero energy fermion modes seems guaranteed by the Antiyah–Singer (A–S) index theorem [6] for sectors with non-vanishing topological charge.

More vexing is the proliferation of these zero modes in the NIDG($|N_+ + N_-|N_f$ in number, where $N_{+(-)}$ is the number of instantons (anti-instantons)). It is of interest to inquire whether this proliferation is a property of the NIDG approximation or rather a natural suppression of instanton effects associated with high density configurations. Clearly, one must, to address this question, go beyond NIDG and learn more about instanton interactions.

Recently [12] we have suggested a way in which to treat instanton–anti-instanton two-body interactions, which are the dominating effect away from the NIDG. A fermion transiting a region of instantons and anti-instantons sees to leading order a condensed phase of

* Work supported in part by the Department of Energy under contract number EY-76-C-02-1545. *000

0170-9739/80/0003/0203/$02.80

instanton–anti-instanton neutral dipole pairs. Of the $|N_+ + N_-|N_f$ fermionic zero modes in the NIDG of N_+ instantons and N_- anti-instantons, $|N_+ - N_-|N_f$ exact zero modes remain, as expected from the A–S index theorem.

It is by now well known that zero mode fermionic solutions dominate certain one-loop correlation functions about instantons and force others to vanish in a theory with massless quarks [14]. An example of the latter is the current–current correlation function when there is more than one flavor. The leading contribution to this important amplitude, whose absorptive part is measured in e^+e^- annihilation, is thus given by the interaction which lifts the zero modes. The primary goal of this paper is to calculate this leading contribution to the current–current correlation function in the massless theory.

Our results, a power correction to naive scaling, are to be comparted and contrasted with the library of related calculations, those of Carlitz and Lee [7], Andrei and Gross [8], Baublieu et al. [9], Suzuki [10], and Appelquist and Shankar [11]. Although these authors differ in their conclusions concerning the power and coefficient of the corrections, they share among each other the common appeal to massive quarks, at least in the one-loop determinant. While they may take the massless limit in their estimate of the classical propagator, their final result is proportional to these masses, and vanishes in the massless theory.

Our results, in contrast, survive in the massless limit. They are power corrections to $\pi_{\mu\nu}$ which remain in what should be the very physical limit of massless quarks. Our results do not depend on the insertion of phenomenological constituent or current quark masses. Our results are scaled directly by the renormalization mass.

It may be argued that this difference is not fundamental because the same instanton–anti-instanton interactions which we use to calculate the current–current correlation function can also be used to calculate an effective quark mass. This is not the case: we show that this interaction leaves quarks massless, its effect yielding only wave function renormalization. Thus what we have obtained are power law corrections to scaling in the *massless* theory. Discussion of the power behavior relative to the inserted-mass calculations of [7, 8, 9] or [11] will be commented upon in Sect. IV and V.

In the next Sect. II, we describe the dipolar interaction approximation (DIA) for lifting zero modes. The need for concentrating on the topologically neutral ($N_+ = N_-$) gas component is explained. The improved fermion determinantal factor is calculated.

In Sect. III we calculate the fermion propagator in DIA. The current–current correlation function is calculated in Sect. IV. In Sect. V we discuss our results. All calculations are restricted to SU(2) of color.

Details of our computations are reserved for the Appendices A through E.

II. Beyond the Non-Interacting Dilute Gas Approximation

In addition to exact minima of the action—the winding number n pseudoparticles with action $8\pi^2|n|/g^2$—non-abelian gauge theory has a host of approximate classical solutions constructed by superposing winding number $n = \pm 1$ pseudoparticles (instantons and anti-instantons). The sum of N^+ instantons and N^- anti-instantons will be a good approximate minimum, with action $8\pi^2|N_+ + N_-|/g^2$ and topological charge $N_+ - N_-$, if the centers are widely separated. It has been shown that the action of such a configuration deviates from an actual minima by large inverse powers of the separation if the individual instantons are in the singular gauge. We always use the singular gauge because of the more rapid fall-off of cross terms; for additional discussions, see [12].

The essence of the dilute gas approach is the localization of interacting centers. The order of the approximation is determined by the number of functional overlaps one keeps in products of localized functions. In lowest order or NIDGA we neglect products of the form $F(x-z_1)G(x-z_2)$ when $z_1 \neq z_2$ and F, G are localized functions. (Eventually, it is true, one must perform the collective coordinate integration over z_1 and z_2, so that it appears that $z_1 \approx z_2$ *somewhere* in the analysis, invalidating the DGA. However, we shall below integrate cleanly over $z_1 + z_2$, achieving translation invariance, but effectively cut off the integral of $z_1 - z_2$, avoiding the strong overlap region.)

In second order, one takes into account overlaps of the form FG if z_1 and z_2 are nearest neighbors in some set $\{z_i\}$; however, overlaps of the form $F(x-z_1)H(x-z_2)G(x-z_3)$ are neglected if $z_1 \neq z_2 \neq z_3$. This scheme can be systematically extended to higher orders taking into account all terms in a series of one-body, two-body, ... n-body correlations.

We have previously studied NIDGA and single instantons extensively [12] with particular focus on mass generation [13] and chiral SU$(N_f > 1)$ symmetry breaking. Finding none, we are persuaded here to consider these and related questions to second order in the DGA. In extending our calculations to this next order, we will be somewhat selective, using second order DGA only when this improvement has a chance of producing qualitatively different results from NIDGA. Elsewhere, we persist with the non-interacting approximation.

Our results are limited to one-loop expansion in the quantum amplitude. Higher order corrections in that series are not expected to change our conclusions regarding chiral symmetry breaking.

The relevant part of the QCD Lagrangian for our purposes here is

$$\mathscr{L} = i\bar{\psi}\gamma_\mu\left[\partial_\mu + \sum_{i=1}^{N} A_\mu^+(i) + A_\mu^-(i)\right]\psi \tag{2.1}$$

where $A^\pm(i)$ are instantons $(+)$ and anti-instantons $(-)$ paired in singular gauge with collective coordinates z_i, ρ_i, n_i,

$$A_\mu^\pm(i) = A_\mu^\pm(x - z_i^\pm, \rho_i^\pm, n_i^\pm) \tag{2.2}$$

where n_i parametrizes the global color orientation (see Appendix A for detailed formulas).

The zero eigenvalue classical fermion solutions of (2.1) play a central role in instanton calculations for a number of reasons. First, as is well known from the work of 't Hooft [2], they come in a single chirality and break the γ_5 invariance [chiral U(1)] of (2.1). For an explicit calculation of the symmetry breaking Green's function, see [14]. Second, they dominate the classical propagator in the zero or small mass theory. Third, they contribute zeros to the one-loop determinant which enter into the calculation of quantum amplitudes, N_f in number for an instanton or an anti-instanton. Thus, unless a single instanton or anti-instanton process has at least N_f quark singular massless propagators to cancel these zeros, it will vanish. If the process has exactly N_f quark propagators, then the zero modes above control the amplitude; see examples in [14] for the $N_f = 1$ two-point function and $N_f = 2$ four-point function instanton calculations.

The situation is similar if the zero modes are lifted slightly, either by inserting a small mass m or by perturbing the NIDGA levels by a mechanism we shall consider below. From an instanton–anti-instanton *pair* (a configuration we shall find most important), we have $2N_f$ zero or almost zero modes, yielding a factor m^{2N_f} in the determinant. If the process involves p propagators, we have the form $m^{2N_f}(\varphi_0\varphi_0^*/m + \sum_i \varphi_i\varphi_i^\dagger/\lambda_i)^p$ where the lowest (erstwhile zero) eigenvalue has been explicitly separated in the spectral decomposition of the propagator. For small m, the leading contribution is then

$$m^{2N_f - p}(\varphi_0\varphi_0^*)^p \tag{2.3}$$

and we see that zero modes dominate the process.

Insertion of mass terms here, while a practical route, is not one we will take because it spoils the manifest chiral $\mathrm{SU}(N_f > 1)$ symmetry of the Lagrangian and tampers with an interesting physical limit. We shall find a dynamical mechanism which lifts the zero mode even in the massless theory.

The nearest neighbor interaction which succeeds in this task perturbs the NIDG zero mode by an amount proportional to the inverse third power of the pair constituents' separation. We must keep the separation relatively large (2 or 3 GeV) so that the action is still an approximate minimum ($e^{-\delta A} \approx 1$); hence there is only a slight shift in the zero mode eigenvalue. However, the determinant and classical propagators are changed in a very significant way. The shift correlates instanton and anti-instanton parameters in the eigenvalue. The erstwhile zero mode part of the determinant no longer factors into single center parts, but rather as dipole pairs. Even the non-zero mode parts of the determinant and Green's functions, initially assumed uncorrelated, develop correlations because they share collective coordinates with the shifted zero mode.

Let us consider how these new dipolar correlations arise. The Dirac operator in a field of N^+ instantons and N^- anti-instantons is given by

$$H = i\gamma_\mu\left[\partial_\mu + \sum_{i=1}^{N} A_\mu^+(i) + A_\mu^-(i)\right] \tag{2.4}$$

and has well-known NIDG two-component zero modes $\varphi_{0,A}^\pm(i)$

$$i\gamma_\mu[\partial_\mu + A_\mu^\pm(i)]\,\varphi_{0,A}^\pm(i) \equiv H_0^\pm\,\varphi_{0,A}^\pm(i) = 0 \tag{2.5}$$

where A labels the flavor degeneracy (see Appendix A for explicit formulas).

This problem resembles that of a multiple array of localized wells. It is well known here that tunneling occurs and level shifts have a non-perturbative character. We use a variational principle to determine the best linear combination of NIDG zero modes which approximate an eigenstate of H, allowing only nearest neighbor interactions. We order the instantons, starting with an arbitrary one called $A_\mu^+(1)$. The instanton nearest to $A_\mu^+(1)$ we call $A_\mu^+(2)$, and so on to $A_\mu^+(N^+)$. The anti-instanton closest to $A_\mu^{(+)}(1)$ is called $A_\mu^{(-)}(1)$, and so on. The excess of instantons or anti-instantons may be labeled arbitrarily. This counting is possible when the density is small enough.

In calculating the matrix elements of H, we set to zero all except those between instanton and its paired nearest neighbor anti-instanton. Of course, because of the chiral structure of H and $\varphi^\pm$, there is no matrix element of H between instanton and instanton, or anti-instanton and anti-instanton.

The unpaired $|N_+ - N_-|$ pseudoparticles suffer no level shift; all others are shifted, with an equal and opposite eigenvalue for each pair. Thus there are $|N_+ - N_-|N_f$ persisting zero modes in a N_+, N_- configuration, consistent with topological charge $(N_+ - N_-)$ and the A–S index theorem [6] counting.

Let us consider the influence of these eigenvalue shifts in the improved DGA on the quark propagator S and the current correlation functions in the case $N_f = 2$. Then as $|N_+ - N_-|N_f$ determinantal zeros persist, only $N_+ = N_-$ configurations contribute to the propagation function. In the case of the current–current amplitude $\pi_{\mu\nu}$, in principle, one instanton, one anti-instanton, or $|N_+ - N_-| = 1$ DGA configurations could contribute since the two determinantal zeros are canceled by the zero mode part of the classical propagator; however, the chirality of the zero modes which dominate that amplitude support only scalar and pseudoscalar density correlation functions [14]. Thus again the leading effect is from the $N_+ = N_- \equiv N$ topologically neutral gas. We now consider that configuration in detail.

The $2N_f N \times 2N_f N$ square matrix $\langle \varphi^{\pm}_{0,A}(i) | H | \varphi^{\pm}_{0,A}(j) \rangle$ has the form

$$\langle H \rangle = \begin{array}{c} \\ N_f N \end{array}\!\!\left\{ \begin{array}{c} \overbrace{\quad}^{N_f N} \\ \left[\begin{array}{c|c} 0 & iA \\ \hline -iA^{\dagger} & 0 \end{array}\right]^{+}_{-} \\ \quad + \qquad\quad - \end{array}\right. \tag{2.6}$$

where according to our nearest neighbor approximation (for $N_f = 2$)

$$A = \begin{bmatrix} A_{1,1} & & & & \\ & A_{1,2} & & & \\ & & A_{2,1} & & \\ & & & A_{2,2} & \\ & & & & \ddots \end{bmatrix} \tag{2.7}$$

with

$$\begin{aligned} iA_{i,j} &= \langle \varphi^{+}_{0,j}(i) | H | \varphi^{-}_{0,j}(i) \rangle \\ -iA^{\dagger}_{i,j} &= \langle \varphi^{-}_{0,j}(i) | H | \varphi^{+}_{0,j}(i) \rangle \end{aligned} \tag{2.8}$$

The off-diagonal matrix elements of H are pure imaginary in our basis.

In Appendix B we show that the product of shifted eigenvalues is

$$\det H = (-1)^{N_f N} \prod_i^N A_i^{2N_f} \tag{2.9}$$

where $A_{i,1} = A_{i,2} \equiv A_i$, because of the flavor degeneracy, and the eigenvectors and eigenvalues are

$$V_i^{\pm} = \frac{1}{\sqrt{2}} \begin{bmatrix} \pm i \varphi_0^{-}(i) \\ \\ \\ \varphi_0^{+}(i) \end{bmatrix} \tag{2.10}$$

$$\lambda_i^{\pm} = \pm A_i$$

A_i is explicitly calculated in Appendix B. These states are pair states associated with two centers. In the next section we use them to calculate the quark propagation function in a background of instantons and anti-instantons.

In closing this section, let us remark that we have been concerned here with estimating the eigenvalues of an operator which did not connect instantons with instantons, hence the interactive pairing occurred between instanton and anti-instanton, and instanton–instanton interactions were neglected. In other applications, one may wish to take these into account. Then if an instanton pair of *like* topological charge were nearest neighbors, one could replace the dilute gas form for this pair by the exact $N = 2$ pseudoparticle solution which remains an exact minimum of the action for any separation; of course, the functional form is much more complicated than that in the NIDG. Correspondingly, one may use known zero mode fermionic wave functions in the pair region.

III. Quark Propagation Function

The quark propagation function serves as perhaps the simplest probe for chiral $SU(N_f > 1)$ breaking from instantons: one seeks a dynamically generated mass. For $N_f = 1$, a single instanton dominated term clearly generates a dynamical momentum dependent mass [14], asymptotically zero for high p, $\frac{m(p)}{\mu_0} \approx (\mu_0/p)^{17/3}$ when μ_0 is the invariant renormalization mass.

For $N_f > 1$ (in particular $N_f = 2$ is a good example) chiral $U(1)$ breaking is clearly evident in the difermion propagators (scalar–pseudoscalar density amplitude) in a single instanton background, but no mass generation is visible in this amplitude. The counting of zeros of the last section clearly points to the topologically neutral pseudoparticle configuration with $N = N^+ - N^-$ as most important for propagator modifications.

To begin, consider the generating functional for QCD with fermionic sources in a dilute background of N^+ instantons and N^- anti-instantons.

$$W(\eta, \eta^+) = \frac{\sum\limits_{N^+, N^- = 0}^{\infty} \frac{1}{N^+! N^-!} W^{N^+, N^-}(\eta, \eta^+)}{\sum\limits_{N^+, N^- = 0}^{\infty} \frac{1}{N^+! N^-!} W^{N^+, N^-}(0, 0)} \tag{3.1}$$

In our applications all W^{N^+, N^-} will already have been normalized by $W^{0,0}$, so that $W^{0,0}$ can be taken as unity.

It is important to note that, for any particular process in the DGA, one may have a subset of the fixed N_+, N_- centers participating in the interaction while the remainder are passive observers. One must consistently keep the correct number of interactions in numerator and denominator. In Appendix C, we discuss this point at some length.

As shown in the previous section, the quark propagator (a = flavor, i = color, α = Dirac indices),

$$S^{a,b}_{i\alpha, j\beta}(x, y) = \left. \frac{\delta}{\delta \eta^a_{i\alpha}(x)} \frac{\delta}{\delta \eta^{\dagger b}_{j\beta}(y)} W(\eta, \eta^{\dagger}) \right|_{\eta, \eta^{\dagger} = 0} \tag{3.2}$$

receives, for $N_f > 1$, contributions only from the $N^+ = N^- = N$ sector, $N = 0, 1, 2 \ldots$. The contribution from N-pairs is

$$\begin{aligned} S^{N,ab}_{i\alpha, j\beta}(x, y) &= \left. \frac{\delta^2 W^{N,N}(\eta, \eta^{\dagger})}{\delta \eta^a_{i\alpha}(x) \delta \eta^{\dagger b}_{j\beta}(y)} \right|_{\eta, \eta^+ = 0} \\ &= \int \prod_{i=0}^{N} dz_i^+ dz_i^- d\rho_i^+ d\rho_i^- dn_i^+ dn_i^- D^+(i) D^-(i) (-A_i^2)^{N_f} \end{aligned}$$

$$\cdot\left\{\left[i\gamma_\mu\left(\partial_\mu+\sum_{i=1}^{N}A_\mu^+(i)+A_\mu^-(i)\right)\right]^{-1}\right\}^{ab}_{i\alpha,j\beta} \quad (3.3)$$

where $S^{0ab}_{i\alpha,j\beta}\equiv(i\gamma_\mu\partial_\mu)^{-1}\cdot D^\pm(1)$ is the NIDG determinant, less zero mode eigenvalues, calculated by 't Hooft [2], augmented by the single pseudoparticle action and factors arising from the gauge field integration. (See Appendix A for details). $\pm A_i$ is the energy shift of the NIDG zero modes associated with the ith instanton–anti-instanton pair.

In Appendix C, we obtain the classical propagator expansion

$$\left[i\gamma_\mu\left(\partial_\mu+\sum_j A_\mu^+(j)+A_\mu^-(j)\right)\right]^{-1}$$
$$=\sum_{i=1}^{N}\frac{M(x,y,i)}{A(i)}+S_0\left[1+\sum_{i=1}^{N}\mathscr{S}(i)\right.$$
$$\left.+\sum_{i\neq 1}^{N}\mathscr{S}(i)\mathscr{S}(j)+\sum_{i\neq j\neq k}^{N}\mathscr{S}(i)\mathscr{S}(j)\mathscr{S}(k)+\dots\right] \quad (3.4)$$

where $M(x,y,i)$ is the lifted zero mode part of the propagator associated with the ith pair, obtained in Appendix B and explicitly calculated in Appendix C. The remaining expansion represents the non-zero mode contribution, where

$$\mathscr{S}(i)=\not\partial[(\not\partial+\not A^+(i)+\not A^-(i))^{-1}-\not\partial^{-1}] \quad (3.5)$$

is the part due to a particular pair. $\mathscr{S}(i)$ is expanded in Appendix C in terms of the known single instanton and single anti-instanton Green's functions. Suppressing indices, we find from (3.4) and (3.3),

$$S^N(x,y)=V^{N-1}N^2\left\langle\frac{M(x,y)}{A}\right\rangle$$
$$+\sum_{m=0}^{N}V^{N-m}\left(\frac{N!}{(N-m)!}\right)^2 S_0\langle\mathscr{S}\rangle^m \quad (3.6)$$

where

$$V=\int d^4z^+\,d^4z^-\,d\rho^+\,d\rho^-\,dn^+\,dn^-\,D^+D^-(-A)^{2N_f}$$

and the notation $\langle\ \rangle$ implies integration over collective coordinates weighted according to (3.7). The zero mode part has N identical contributions, one from each pair; then as the collective coordinates $z_i^\pm$ run over all space, the nearest neighbor pairings change so that, for example, instanton 1 will at some time be paired with each of the N anti-instantons, providing the other factor of N.

For the second set of terms, there are $N!/(N-m)!$ identical pieces in each sum and then there is an additional identical factor arising from the various pairings of instantons and anti-instantons.

We note that the factor $\prod_{i=1}^{N}[A(i)]^{2N_f}$ in (3.2) refers to the specific pairing described in the previous section; as the $z_i^\pm$ run about space, these pairings change and the energy calculation must take this into account.

Using (3.6) in (3.2), we find

$$S(x,y)=\frac{\left[\sum_{N=1}^{\infty}\frac{V^{N-1}}{[(N-1)!]^2}\left\langle\frac{M(x,y)}{A}\right\rangle+\sum_{m=0}^{\infty}\sum_{N=m}^{\infty}\frac{V^{N-m}}{[(N-m)!]^2}S_0\langle\mathscr{S}\rangle^m\right]}{\sum_{N=0}^{\infty}\frac{V^N}{(N!)^2}} \quad (3.7)$$

The $\sum_N$ cancel as they must on physical grounds, leaving a volume-independent result

$$S(x,y)=\frac{M(x,y)}{A}+\frac{1}{S_0^{-1}-\langle\mathscr{S}\rangle S_0^{-1}} \quad (3.8)$$

The term $\langle\mathscr{S}\rangle S_0^{-1}$, computed in Appendix C, has the form

$$S_0^{-1}\{\int d^4z^+\,d^4z^-\,d\rho^+\,d\rho^-\,dn^+\,dn^-\,D^+D^-A^4$$
$$\cdot[(S^+-S_0)+(S^--S_0)+(S^--S_0)S_0^{-1}(S^+-S_0)$$
$$+(S^+-S_0)S_0^{-1}(S^--S_0)]\}S_0^{-1} \quad (3.9)$$

where $S^\pm$ are the propagators in the field of a single instanton or anti-instanton, zero modes excluded [15]. At first sight one might expect to pick up a factor of V in the first two terms since they only involve a single instanton or anti-instanton. This is not so because the z^+ and z^- are paired nearest neighbors, their separation limited through the eigenvalue A. Then large $|z^- - x|$, where the troublesome apparent infinite volume tail of $\int d^4z^-(S^+-S_0)$ arises, means large $|z^+-x|$, where $S^+\to S_0$, regulating that integral in a volume independent way.

Now we calculate the leading contribution to $S(x,y)$, the zero mode (ZM) contribution. It dominates because it is of order $0(A^3)$ in the lifted eigenvalue, whereas $\langle\mathscr{S}\rangle$ is of order $0(A^4)$. Joining the explicit form for the determinant of Appendix A to the explicit form of $M(x,y)$ and the shifted eigenvalue A of Appendix B, we proceed to the collective coordinate integration implicit in the bracket notation,

$$[S^{ab}_{i\alpha,j\beta}(x,y)]_{ZM}=(-i)(K(2)[e^{-8\pi^2/g^2}]\mu^6g^{-8})^2\delta_{ab}$$
$$\cdot\int dz^+\,dz^-\,d\rho^+\,d\rho^-\,dn\,d\bar n\,A^3\begin{bmatrix}0 & \varphi^{(-)}_{i\alpha}\varphi^{(+)*}_{j\beta}\\ -\varphi^{(+)}_{i\alpha}\varphi^{(-)*}_{j\beta} & 0\end{bmatrix} \quad (3.10)$$

where $\bar n$ is the relative gauge orientation between $A^{(-)}$ and $A^{(+)}$ and n is an overall gauge orientation of the pair. (See Appendix A for definition of $K(2)$.) Upon gauge averaging we have, after inserting the explicit was functions of Appendix A,

$$S^{ab}_{i\alpha,j\beta}(x,y)=-i\delta_{ab}\delta_{ij}[K(2)[e^{-8\pi^2/g^2}]\mu^6g^{-8}]^2$$
$$\cdot\int dz^+\,dz^-\,d\rho^+\,d\rho^-(\rho^+\rho^-)^3$$
$$\cdot\left(\frac{8\rho^+\rho^-}{(z^+-z^-)^4}\right)^3\frac{2\rho^+\rho^-}{\pi^2}\left(\frac{\pi}{2}\right)^2(z^+-z^-)^2. \quad (3.11)$$

$$\begin{bmatrix} 0 & \left(\dfrac{[\alpha\cdot(z^+-z^-)\bar\alpha\cdot(x-z^-)\sigma_2]_{k\alpha}}{|x-z^-|[(x-z^-)^2+\rho^{-2}]^{3/2}}\cdot\dfrac{[\alpha\cdot(y-z^+)\sigma^2]_{k\beta}}{|y-z^+|[(y-z^+)^2+\rho^{+2}]^{3/2}}\right) \\ \dfrac{-(\alpha\cdot(x-z^+)\sigma^2)_{k\alpha}[\alpha\cdot(z^+-z^-)\bar\alpha\cdot(x-z^-)\sigma^2]_{k\beta}}{\left(|x-z^+|[(x-z^+)^2+\rho^{+2}]^{3/2}|y-z^-|\cdot[(y-z^-)^2+\rho^{-2}]^{3/2}\right)} & 0 \end{bmatrix}$$

It is advantageous to fourier transform $S(x,y)$ at this stage. We recall that $(\rho^\pm)^{-1}$ can play the role of a running renormalization mass. DGA requires $\rho^\pm$ to be small or moderate if the gas is to be dilute. Thus the running mass is moderate to large and the momentum must be moderate to large for the perturbation expansion to be reliable. We therefore restrict ourselves to large to moderate p.

Defining the transform

$$\frac{x_\mu}{|x|(x^2+\rho^2)^{3/2}}=\rho^2\int\frac{d^4p}{(2\pi)^4}ip_\mu f((p\rho)^2)e^{ipx} \tag{3.12a}$$

we have, in terms of modified Bessel functions [16]

$$f(x)=\frac{2\pi^2}{x^2}\left[I_1\left(\frac{x}{2}\right)K_1\left(\frac{x}{2}\right)+x\frac{d}{dx}\left(I_0\left(\frac{x}{2}\right)K_0\left(\frac{x}{2}\right)\right)\right]. \tag{3.12b}$$

The $|z^+-z^-|$ singularily in (3.11) is an artifact of the approximation of the eigenvalue, which is good only for moderate to large separation. We improve that form by smoothing the behavior near $z^+-z^-=z\approx 0$; namely we define the transform

$$\frac{z_\mu}{(z^2+a^2)^5}=-\frac{1}{8}\int\frac{d^4q}{(2\pi)^4}iq_\mu g(q^2)e^{iqz} \tag{3.12c}$$

with

$$g(q^2)=q^4C(qa)$$

where

$$C(Q)=\frac{\pi^2K_2(Q)}{12Q^2} \tag{3.12d}$$

which is explicitly calculated in Appendix D. We expect these forms to be reliable when $|z|\gtrsim\mu_0^{-1}\gg a$, or $q\sim\mu_0^{-1}$.

In terms of these transforms we have

$$\begin{aligned} S^{ab}_{i\alpha,j\beta}(x,y)&=[K(2)e^{-8\pi^2/g^2}\mu^6g^{-8}]^2\\ &\cdot\delta_{ab}\delta_{ij}32\int d^4z^+d^4z^-d\rho^+d\rho^-(\rho^+\rho^-)^9\int\frac{d^4p^+d^4p^-d^4q}{(2\pi)^{12}}\\ &e^{iq(z^+-z^-)}f((\rho^+p^+)^2)f((\rho^-p^-)^2)g(q^2)\\ &\cdot\begin{bmatrix}0, & e^{ip^+(y-z^+)+ip^-(x-z^-)}(\alpha\cdot q\bar\alpha\cdot p^-\sigma_2)_{k\alpha}(\alpha\cdot p^+\sigma_2)_{k\beta}\\ -e^{ip^+(x-z^+)+ip^-(y-z^-)}(\alpha\cdot p^+\sigma_2)_{k\alpha}(\alpha\cdot q\bar\alpha\cdot p^-\sigma_2)_{k\beta}, & 0\end{bmatrix} \end{aligned} \tag{3.13}$$

We retrieve translation invariance by changing variables to $z=z^+-z^-$ and $R=(z^++z^-)/2$, and integrating over R. The result for the propagator is

$$\begin{aligned}&[K(2)e^{-8\pi^2/g^2}\mu^6g^{-8}]^2\delta_{ab}\delta_{ij}32\int d^4z\,d\rho^+d\rho^-(\rho^+\rho^-)^9\\ &\cdot\frac{d^4p\,d^4q}{(2\pi)}e^{iz(q-p)}f((\rho^+p^+)^2)f((\rho^-p^-)^2)\\ &\cdot g(q^2)\begin{bmatrix}0, & -e^{ip(x-y)}(\alpha\cdot q\bar\alpha\cdot p\sigma_2)_{k\alpha}(\alpha\cdot p\sigma_2)^*_{k\beta}\\ e^{ip(x-y)}(\alpha\cdot p\sigma_2)_{k\alpha}(\alpha\cdot q\bar\alpha\cdot p\sigma_2)^*_{k\beta}, & 0\end{bmatrix}\end{aligned} \tag{3.14}$$

The restriction to large $|z|$ requires $q\sim p$ or the integral oscillates to zero. Then $\int d^4ze^{iz(q-p)}$ is almost a delta-function. [In principle, z really need only be moderate to maintain a decent approximate minima, and the form for the energy is probably reasonably accurate for moderate z. Then the fourier transformed energy is a good estimate for small to moderate q^2. With z large to moderate, $q-p$ is small or moderate, leaving a region of moderate momentum where the renormalization group restrictions and the dilute gas may coexist.] For purposes of algebraic simplicity we however continue with our large z and $\delta(q-p)$ approximation, obtaining

$$\begin{aligned}S(x,y)&=32[K(2)e^{-8\pi^2/g^2}\mu^6g^{-8}]^2\int d\rho^+d\rho^-(\rho^+\rho^-)^9\\ &\cdot\int d^4q\,g(q^2)f((\rho^+q)^2)f((\rho^-q)^2)\\ &\cdot q^2\begin{bmatrix}0 & -\alpha\cdot qe^{-iq(x-y)}\\ \bar\alpha\cdot qe^{iq(x-y)} & 0\end{bmatrix}\end{aligned} \tag{3.15}$$

Shifting to 4×4 Dirac notation and defining $\rho^\pm q=\xi$

$$\begin{aligned}S(x,y)&=32[K(2)e^{-8\pi^2/g^2}\mu^6g^{-8}]^2\\ &\cdot\int\frac{d^4q}{(2\pi)^4}e^{iq(x-y)}q^2\gamma\cdot q\frac{g(q^2)}{q^{20}}\left[\int_0^{\rho_c q}d\xi\,\xi^9f(\xi)\right]^2\end{aligned} \tag{3.16}$$

where the ρ integral has been cut off at $\rho_c\sim\mu_0^{-1}$, keeping q moderate at $\sim\mu_0$, so that max $\xi\approx1$. Then we have

$$S(q)\approx32[K(2)e^{-8\pi^2/g^2}\mu^6g^{-8}]^2B^2C\left(\frac{\mu}{q}\right)^{12}\frac{\gamma\cdot q}{q^2}$$

where

$$B=\int_0^{bq/\mu_0}d\alpha\,\alpha^9f(\alpha)$$

with $\rho_c=b\mu_0^{-1}$. The detailed dependence on q requires a knowledge of B and C and depends on cutoffs for small z^+-z^- and large ρ. For q large (see Appendix D) $B^2\sim q^{-12}$ and C (3.12d) is exponentially damped, resulting in the behavior $S(q)\sim\dfrac{\gamma\cdot q}{q^2}e^{-q/\mu_0}$.

Given our uneasy stance between the conflicting demands of the DGA and an asymptotically free coupling, it is difficult to defend the exact q dependence, but the lesson seems clear that the leading effect away from DGA does not generate mass—it is a wave function renormalization correction.

IV. Current–Current Correlation Function

The current–current correlation function,

$$\prod_{\mu\nu}(x,y)=\langle\bar{\psi}(x)\gamma_\mu\psi(x)\bar{\psi}(y)\gamma_\nu\psi(y)\rangle$$

receives no contribution from a single instanton or anti-instanton in *massless* QCD of two or more flavors because the amplitude is zero mode dominated *and* the zero mode solutions support only scalar and pseudoscalar densities. This is true also for all non-interacting dilute gas approximations.

If a quark mass m_q is inserted in the theory, then single instantons do contribute one-loop corrections to $\prod_{\mu\nu}$, and this is the basis of the NIDG results of [7, 8, 9, 10] and [11]. When the limit $m_q=0$ is taken, these corrections vanish. Here we pursue a different mechanism, which survives in the massless limit, arising from the dipolar interaction discussed in the last section. The chirality and tensor structure of this mechanism *does* support current corrections, whereas as just noted, it does not contribute to mass generation. Thus we separate the origin of mass generation from that of power corrections to $\prod_{\mu\nu}(x,y)$. (Even through one can express these corrections in terms of an effective momentum dependent mass, we find this slightly misleading in view of the results of the last section.)

To be specific, consider the electromagnetic current–current amplitude

$$\prod_{\mu\nu}(x,y)=\frac{\operatorname{Tr}Q^2\sum_{N=0}^{\infty}\left(\frac{1}{N!}\right)^2\langle\gamma^\mu_{\alpha\beta}S^N_{i\beta,k\gamma}(x,y)\gamma^\nu_{\gamma\delta}S^N_{k\delta,i\alpha}(y,x)\rangle}{\sum_{N=0}^{\infty}\frac{V^N}{(N!)^2}}\tag{4.1}$$

where S^N is given in (3.4). Understood in (4.1) is a collective coordinate integration which connects the propagators in a variety of ways. With the abbreviation

$$dz_i^+dz_i^-d\rho_i^+d\rho_i^-D^+(i)D^-(i)A_i^4=dC(i)$$

we have

$$\begin{aligned}\langle\gamma^\mu S^N(x,y)\gamma^\nu S^N(y,x)\rangle&=\int\prod_i^N dC(i)\gamma_\mu\Bigg[\sum_{i=1}^N\frac{M(x,y,i)}{A_i}\\&+S_0\Bigg\{1+\sum_{i=1}^{\infty}\mathscr{S}(i)+\sum_{i\neq j}\mathscr{S}(i)\mathscr{S}(j)+\dots\Bigg\}\Bigg]\\&\cdot\gamma_\nu\Bigg[\sum_{i=1}^N\frac{M(y,x,i)}{A_i}+S_0\Bigg\{1+\sum_{i=1}^N\mathscr{S}(i)+\dots\Bigg\}\Bigg]\end{aligned}\tag{4.2}$$

In leading order away from NIDG $[A_i\to 0]$ the zero mode terms dominate, yielding the term

$$\int\prod_i^N dz_i^+dz_i^-d\rho_i^+d\rho_i^-D^+(i)D^-(i)A_i^4\gamma^\mu\cdot\sum_{i=1}^N\frac{M(x,y,i)}{A_i}\gamma^\nu\sum_{j=1}^N\frac{M(y,x,j)}{A_j}\tag{4.3}$$

There are contributions here when, in the product of sums, $i=j$,

$$V^{N-1}\left[\frac{N!}{(N-1)!}\right]^2\int dC(1)\gamma^\mu\frac{M(x,y,1)}{A_1}\gamma^\nu\frac{M(y,x,1)}{A_1}\tag{4.4}$$

and when $i\neq j$

$$V^{N-2}\left(\frac{N!}{(N-2)!}\right)\int dC(1)dC(2)\gamma^\mu\frac{M(x,y,1)}{A_1}\gamma^\nu\frac{M(y,x,2)}{A_2}\tag{4.5}$$

where V is defined in (3.7).

The remaining terms are given by

$$\begin{aligned}&\int\prod_i^N dC(i)\Bigg\{\gamma^\mu\sum_{i=1}^N\frac{M(x,y,i)}{A_i}\gamma^\nu S_0\Bigg(1+\sum_{j=1}^N\mathscr{S}(1)+\dots\Bigg)\\&+\gamma^\mu S_0\Bigg(1+\sum_{j=1}^N\mathscr{S}(j)+\dots\Bigg)\gamma^\nu\sum_{i=1}^N\frac{M(y,x,i)}{A(i)}\Bigg\}\end{aligned}\tag{4.6}$$

and

$$\int\prod_{i=1}^N dC(i)\Bigg\{\gamma^\mu S_0\Bigg[1+\sum_{i=1}^N\mathscr{S}(i)+\dots\Bigg]\gamma^\nu S_0\cdot\Bigg[1+\sum_{i=1}^N\mathscr{S}(j)+\dots\Bigg]\tag{4.7}$$

In (4.6) we have $i=j$ and $i\neq j$ terms, and again we have two distinct contributions in which two or more collective coordinates are the same, indicating an interaction with two or more dipole pairs. In the spirit of DGA we shall neglect these terms. All other terms sum to the following two contributions,

$$[\prod_{\mu\nu}(x,y)]_{\mathrm{DISC}}=\operatorname{Tr}Q^2\gamma^\mu S(x,y)\gamma^\mu S(y,x)\tag{4.8}$$

involving disconnected *dressed* propagators, and

$$\begin{aligned}{}[\prod_{\mu\nu}(x,y)]_{\mathrm{CON}}&=\operatorname{Tr}Q^2\int dC(1)\\&\cdot\gamma^\mu\left[\frac{M(x,y,1)}{A_1}+\frac{S_0}{1-\langle\mathscr{S}\rangle}\mathscr{S}\langle 1\rangle S_0^{-1}\frac{S_0}{1-\langle\mathscr{S}\rangle}\right]\\&\cdot\gamma^\nu\left[\frac{M(y,x,1)}{A_1}+\frac{S_0}{1-\langle\mathscr{S}\rangle}\mathscr{S}(1)S_0^{-1}\frac{S_0}{1-\langle\mathscr{S}\rangle}\right]\end{aligned}\tag{4.9}$$

The dominant contribution is

$$\prod_{\mu\nu}(x,y)=\operatorname{Tr}Q^2\gamma^\mu\frac{M(x,y,1)}{A_1}\gamma^\nu\frac{M(y,x,1)}{A_1}\tag{4.10}$$

as $A\to 0$.

Details of this calculation, involving integration over the dipole centers and use of the fourier trans-

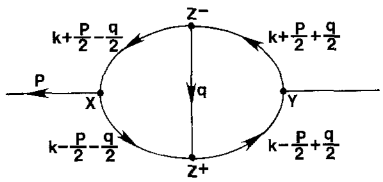

Fig. 1. Instanton–anti-instanton contribution to current correlation function

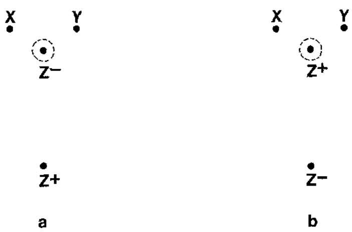

Fig. 2a and b. Dominant configurations for current correlation function

formed zero modes, are reported in Appendix E. The result, in terms of the dipole separation z, is

$$\prod_{\mu\nu}(x,y)=\frac{\operatorname{Tr}Q^2}{3}\left[\frac{K(2)e^{-8\pi^2/g^2}}{g^8}\mu^6\frac{16i}{\pi^2}\right]^2$$
$$\cdot\int d\rho^+ d\rho^-(\rho^+\rho^-)^{11}(g_{\rho\nu\sigma\delta}+\varepsilon_{\rho\nu\sigma\delta})(g_{\lambda\mu\xi\eta}+\varepsilon_{\lambda\mu\xi\eta})H_{\alpha\beta\delta\eta}$$
$$\cdot\int\frac{z_\alpha z_\beta}{(z^2+a^2)^4}dz\,e^{-iqz}\frac{d^4q}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}e^{ip(x-y)}$$
$$\cdot\int\frac{d^4k}{(2\pi)^4}\left(k-\frac{p}{2}+\frac{q}{2}\right)_\rho\left(k+\frac{p}{2}+\frac{q}{2}\right)_\sigma\left(k-\frac{p}{2}-\frac{q}{2}\right)_\lambda$$
$$\cdot\left(k+\frac{p}{2}-\frac{q}{2}\right)_\xi f\left(\left(\rho^-\left[k+\frac{p}{2}-\frac{q}{2}\right]\right)^2\right)$$
$$\cdot f\left(\left(\rho^+\left[k-\frac{p}{2}-\frac{q}{2}\right]\right)^2\right)f\left(\left(\rho^+\left[k-\frac{p}{2}-\frac{q}{2}\right]\right)^2\right)$$
$$f\left(\left(\rho^-\left[k+\frac{p}{2}+\frac{q}{2}\right]\right)^2\right)\tag{4.11}$$

where g and H are defined in Appendix D.

Figure 1 shows the appropriate momenta and conjugate coordinates for (4.11). Since $z=z^+-z^-$ is large, q is small. Here, however, the renormalization group and the dilute gas requirements are not in such sharp conflict as in the propagator calculation. The physical situations of importance are shown in Figs. 2a and 2b. In case (a), $(x-y)$ is small (so the p is large, as required for AF) and close to the *anti-instanton*. This contribution would be zero were it not for the nearest neighbor *instanton*, with which there is long range tunneling, lifting the zero mode. Figure 2b is similar, with role of instanton and anti-instanton interchanged. The configuration in Fig. 2a implies $k+p/2\pm q/2$ large and $k-p/2\pm q/2$ small, implying $k\approx p/2$. The criterion that the anti-instanton is close to x and y means $|x-z^-|$ or $|y-z^-|\lesssim\rho^-$.

In momentum space this means

$$k\approx\frac{p}{2}\pm\frac{1}{2\rho^-}$$

Since the variations involved are not large on this scale, we implement this restriction by taking $k\approx p/2$, yielding

$$\prod_{\mu\nu}(x,y)=-\frac{\operatorname{Tr}Q^2}{3}\left(\frac{16}{\pi^2}\right)^2\frac{1}{4}\left[\frac{K(2)}{g^8}\right]^2 A\int\frac{d^4p}{(2\pi)^4}e^{ip(x-y)}$$
$$\cdot p_\sigma p_\xi\left(\frac{\mu_0}{p}\right)^8(g_{\rho\nu\sigma\delta}+\varepsilon_{\rho\nu\sigma\delta})(g_{\lambda\mu\xi\eta}+\varepsilon_{\lambda\mu\xi\eta})H_{\alpha\beta}\delta_\eta$$
$$\cdot\left[\int\frac{d^4q}{(2\pi)^4}\frac{z^\alpha z^\beta}{(z^2+a^2)^4}\frac{q^\alpha q^\beta}{q^8}\left(\frac{\mu_0}{q}\right)^4 e^{-iqz}B\right]\tag{4.12}$$

where

$$A=\int_0^{\rho_c p}d(\rho^-p)f((\rho^-p)^2)(\rho^-p)^7$$
$$B=\int_0^{\rho_c q}d(\rho^+q)f((\rho^+p)^2)(\rho^+q)^{11}\tag{4.13}$$

and $\rho_c(\approx\mu_0^{-1})$ is the DGA upper limit for instanton size. Since $p\gg\mu_0$, $\rho_c p$ is large. The constant term

$$\int\frac{d^4q}{(2\pi)^4}d^4z\frac{z^\alpha z^\beta}{(z^2+a^2)^4}\frac{q^\rho q^\lambda}{q^8}\left(\frac{\mu_0}{q}\right)^4 B\left(\frac{q}{\mu_0}\right)e^{-iqz}\tag{4.14}$$

must be regulated in q; the net result is a number with tensor structure

$$d\{H_{\alpha\beta\rho\lambda}+e\delta_{\alpha\beta\delta\rho\lambda}\}\tag{4.15}$$

where the constants d and e depend on how the integrals are regulated. Implicitly, its evaluation depends also on previous approximations, most significantly on our use earlier of approximate (if improved) zero energy modes in the classical propagator. Rather than attempting to calculate these approximation-dependent constants, we choose $e=-3$ to implement electromagnetic gauge invariance, lost earlier when zero modes were used which did not exactly obey the Dirac equation for small instanton separation. We simply argue physically, therefore, that the correct regulation of the small separation singularity should insure gauge invariance.

The remaining contractions now result in the final form

$$\prod_{\mu\nu}(x,y)=\operatorname{Tr}Q^2\,48\left(\frac{8}{\pi^2}\right)^2 d\,\frac{K^2(2)}{g^{16}}$$
$$\cdot\int\frac{d^4p}{(2\pi)^4}A(p)e^{ipx}\left(\frac{\mu_0}{p}\right)^8(p^2\delta_{\mu\nu}-p_\mu p_\nu)\tag{4.16}$$

In Appendix E $A(p)$ is evaluated. For p moderate to large it behaves like $A(p)\sim(p/\mu_0)^4$. Then final power dependence is

$$\prod_{\mu\nu}(p)\approx\left(\frac{\mu_0}{p}\right)^4(p^2\delta_{\mu\nu}-p_\mu p_\nu)\tag{4.17}$$

which is a power correction to the free field theory propagator result, $\rho \to \infty$,

$$\left[\prod_{\mu\nu}\right]_{\mathrm{FFT}} = \frac{3}{4\pi^2} \operatorname{Tr} Q^2 \ln \frac{p^2}{\mu_0^2}(p_2 \delta_{\mu\nu} - p_\mu p_\nu)$$

The power dependence of (4.17) is the same as that found by Andrei and Gross [8]. It should be noted that the assumptions going into the two calculations, ours and theirs, are completely different.

V. Summary and Discussion

We have been concerned in this paper with QCD in the limit of vanishing current quark mass, where the Lagrangian exhibits full $U(N_f) \times U(N_f)$ symmetry. The vacuum and hence Green's functions in this limit exhibit a lower symmetry, $SU(N_f) \times SU(N_f)$, as has been evident since 't Hooft's classic paper on the quantum effects of instantons. Thus QCD begins more clearly to resemble the $SU(N_f) \times SU(N_f)$ σ model behavior, which incorporates the successful current algebra results of the last decade. These current algebra results indicate that this theory should exhibit a further breakdown to $SU(N_f)$ in the vacuum. This is evidenced by a vacuum expectation for the σ field in the σ model, with $\langle \sigma \rangle \sim F_\pi$, the PCAC constant, or, in the QCD, by a non-vanishing bi-quark scalar density, $\langle \bar{\psi}\psi \rangle$, and constituent quark mass generation. Our improved dilute gas approximation has not seen these effects.

We evidently are describing quark–instanton interactions at moderate and short distances in regions or phases where $SU(N_f) \times SU(N_f)$ is a good symmetry even of the "vacuum" or at least the local background, and where quarks are massless. Presumably that is what it is like inside a hadron, before the effects of the confining boundaries are felt. If this is the correct picture, mass generation and $SU(N_f) \times SU(N_f)$ breaking effects only come about when separations are large, i.e. they are one in the same with the confinement mechanism. Then we conclude that our calculations correctly describe short to moderate distance instanton effects such as those in e^+e^- annihilation, first corrections to the free field theory and asymptotic freedom calculations, valid at distances small compared to hadronic size, before the separating quarks feel the confining force. They stand midway between the free phase and the confined phase.

In the above discussion we have assumed a sensible massless current quark limit of the theory, as least for limited separations. Perhaps that is not the case. In combined theories of hadrons and leptons it is possible that the quarks acquire current quark mass from the Higgs mesons, $\langle \Phi^{\mathrm{Higgs}} \rangle$. Then the limit of zero current quark mass takes one to $\langle \Phi^{\mathrm{Higgs}} \rangle \to 0$, an unstable point of the weak/electromagnetic sector. In fact there are explicit models in which $\langle \sigma \rangle$ is proportional to $\langle \Phi \rangle$ and the chiral symmetry breaking of the hadrons arises from the same mechanism as the breaking of the $SU(2) \times U(1)$ lepton symmetry [17]. Then our calculational limit of zero current quark mass further requires a degenerate photon and W-meson, that theory at an unstable point of the Higgs potential. The last point need not be so disturbing if the Higgs potential is regarded as a phenomenological construct describing long distance dynamics; the effective Higgs potential could be modified within the hadron so that $\langle \Phi \rangle = 0$ is a stable point there. Indeed it has been suggested that [18] $\langle \Phi \rangle \sim \langle \bar{\psi}\psi \rangle$, and then $\langle \Phi \rangle = 0$ at short distances is consistent with our earlier discussion. Then again we have a sensible massless quark limit, that is one of stability with respect to the weak and electromagnetic interaction, at least in limited spatial domains. Again we conclude—if this picture is correct—that our calculations correctly describe short distance instanton effects of quarks within hadrons.

Are they observable in e^+e^- annihilation? This is a matter of estimating the size of the current–current correlation amplitude, which is subject to uncertainties of the cut-offs implicit in the nature of the approximation. Instanton separations must be limited from below, and sizes from above, both effects yielding low effective density. Moreover the $\prod_{\mu\nu}$ amplitude we have calculated is real and the absorptive part entering into e^+e^- annihilation must be estimated through a dispersion relation. We feel that these calculations are at an early stage when comparison with experiment seems somewhat premature; crudely speaking we guess the dimensionless coefficient of (4.17) is of order unity or smaller so that near moderate $p \approx 2\mu_0$ to $5\mu_0$ where our calculation is most reliable, the instanton corrections to $\prod_{\mu\nu}$ are at least several orders of magnitude less than the free field theory term. We will not comment further here on whether they are observable in e^+e^- annihilation, and to what extent they can compete with the asymptotic freedom (AF) term. Clearly, if they are there they will complicate fits to the logarithmic approach to scaling as given by AF. This is an interesting question which deserves further study.

Appendix A

General Notation and Formulas

Let us define notation and present a few useful formulas. The anti-hermitean matrix of gauge functions and field strengths is

$$A_\mu = \frac{\sigma^a}{2i} A_\mu a$$

where σ^a are 2×2 pauli matrices

$$\left[\frac{\sigma^a}{2}, \frac{\sigma^b}{2}\right] = i\varepsilon^{abc}\frac{\sigma^c}{2}.$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = \frac{\sigma^a}{2i} F_{\mu\nu} a$$

The Dirac operator is then

$$i\gamma^\mu(\partial_\mu + A_\mu) = i\gamma^\mu\left(\partial_\mu - \frac{i\sigma^a}{2} A_\mu^a\right)$$

with

$$\gamma^\mu = \begin{bmatrix} 0 & \alpha^\mu \\ \bar{\alpha}^\mu & 0 \end{bmatrix} \alpha_\mu = (1, -i\vec{\sigma})\alpha_\mu = \alpha_\mu^+$$

some useful identies are

$$\bar{\alpha}^\mu \alpha^\nu + \bar{\alpha}^\nu \alpha^\mu = 2\delta_{\mu\nu}$$
$$\alpha^\mu \bar{\alpha}^\nu + \alpha^\nu \bar{\alpha}^\mu = 2\delta_{\mu\nu}$$
$$\bar{\alpha}^\mu \alpha^\nu = \delta_{\mu\nu} + 2i\sigma^{\mu\nu}$$
$$\alpha^\mu \bar{\alpha}^\nu = \delta_{\mu\nu} + 2i\bar{\sigma}^{\mu\nu}$$
$$\sigma^{\mu\nu} = \frac{\sigma^a}{2}\eta^a_{\mu\nu} \quad \bar{\sigma}^{\mu\nu} = \frac{\sigma^2}{2}\eta^a_{\mu\nu}$$

where the η symbol is defined by 't Hooft [2].

The $\sigma^{\mu\nu}$, and $\bar{\sigma}^{\mu\nu}$ have the following properties under duality transformation,

$$*\sigma^{\mu\nu} = \sigma^{\mu\nu}$$
$$*\bar{\sigma}^{\mu\nu} = -\bar{\sigma}^{\mu\nu}$$

and obey the following identities:

$$\sigma^{i4} = -\bar{\sigma}^{i4} = \tfrac{1}{2}\sigma^i$$
$$\sigma^{ij} = \bar{\sigma}^{ij} = \tfrac{1}{2}\varepsilon_{ijk}\sigma^k$$
$$\bar{\sigma}^{\mu\nu}\alpha^\alpha = (\bar{\sigma}^{\mu\nu}\bar{\alpha}^\alpha)^\dagger = \frac{1}{2i}[\delta_{\nu\alpha}\alpha_\mu - \delta_{\mu\alpha}\alpha_\nu + \varepsilon_{\mu\nu\alpha\beta}\alpha_\beta]$$
$$\sigma^{\mu\nu}\bar{\alpha}^\alpha = (\alpha^\alpha\sigma^{\mu\nu})^\dagger = \frac{1}{2i}[\delta_{\nu\alpha}\bar{\alpha}_\mu - \delta_{\mu\alpha}\bar{\alpha}_\nu - \varepsilon_{\mu\nu\alpha\beta}\bar{\alpha}_\beta]$$
$$\bar{\alpha}_\lambda\bar{\sigma}_{\mu\nu}\alpha_\alpha = \frac{1}{2i}[\delta_{\nu\alpha}\delta_{\lambda\mu} - \delta_{\mu\alpha}\delta_{\nu\lambda} + \varepsilon_{\mu\nu\alpha\lambda}] + \sigma^{\lambda\mu}\delta_{\nu\alpha} - \sigma^{\lambda\nu}\delta_{\mu\alpha} + \varepsilon_{\mu\nu\alpha\beta}\sigma^{\lambda\beta}$$
$$\alpha_\lambda\sigma_{\mu\nu}\bar{\alpha}_\alpha = \frac{1}{2i}[\delta_{\nu\alpha}\delta_{\lambda\mu} - \delta_{\mu\alpha}\delta_{\lambda\mu} + \varepsilon_{\mu\nu\alpha\lambda}] + \bar{\sigma}^{\lambda\mu}\delta_{\nu\alpha} - \bar{\sigma}^{\lambda\nu}\delta_{\mu\alpha} + \varepsilon_{\mu\nu\alpha\beta}\bar{\sigma}^{\lambda\beta}$$
$$\sigma_{\lambda\mu}\sigma_{\mu\beta} = -\tfrac{3}{4}\delta_{\lambda\beta} - i\sigma_{\lambda\beta}$$
$$\bar{\sigma}_{\lambda\mu}\bar{\sigma}_{\mu\beta} = -\tfrac{3}{4}\delta_{\lambda\beta} - i\bar{\sigma}_{\lambda\beta}$$
$$i[\sigma^{\mu\alpha}, \sigma^{\nu\beta}] = \delta_{\alpha\nu}\sigma^{\mu\beta} - \delta_{\mu\nu}\sigma^{\alpha\beta} + \delta_{\alpha\beta}\sigma^{\mu\nu} - \delta_{\mu\beta}\sigma^{\nu\alpha}$$
$$i[\bar{\sigma}^{\mu\alpha}, \bar{\sigma}^{\nu\beta}] = \delta_{\alpha\nu}\bar{\sigma}^{\mu\beta} - \delta_{\mu\nu}\bar{\sigma}^{\alpha\beta} + \delta'_{\alpha\beta}\bar{\sigma}^{\mu\nu}\delta_{\mu\beta}\bar{\sigma}^{\nu\alpha}$$

In terms of these the BPST pseudoparticle in "singular" gauge is

$A_\mu^+ = i\bar{\sigma}^{\mu\nu}\partial_\nu \ln\rho$, positive winding number
$A_\mu^- = i\sigma^{\mu\nu}\partial_\nu \ln\rho$, negative winding number

where

$$\rho = 1 + \sum_{\substack{1 \\ i=1}}^{\nu} \frac{\lambda_i^2}{(x - z_i)^2}$$

For $\nu = \pm 1$

$$A_\mu^+ = -\frac{2i\bar{\sigma}^{\mu\nu}(x - z^+)^\nu\lambda^2}{(x - z^+)^2((x - z^+)^2 + \lambda^2)} = \frac{\lambda^2}{(x - z^+)^2 + \lambda^2}\Omega^{+1}(x - z^+)\partial_\mu\Omega^{-1}(x - z^+)$$

$$A_\nu^- = -\frac{2i\sigma^{\mu\nu}(x - z^+)^\nu\lambda^2}{(x - z^-)^2((x - z^+) + \lambda^2)} = \frac{\lambda^2}{(x - z^-)^2 + \lambda^2}\Omega^{-1}(x - z^-)\partial_\mu\Omega^{+1}(x - z^-)$$

where

$$\Omega^{+1}(x) = \frac{\alpha^\mu x^\mu}{x} \qquad \Omega^{-1}(x) = \frac{\bar{\alpha}^\mu x_\mu}{x}$$

In non singular gauge

$$\bar{A}^\pm(x - z, \lambda) = \frac{(x - z)^2}{(x - z)^2 + \lambda^2}\Omega^{\mp 1}(x - z)\partial_\mu\Omega^{\pm 1}(x - z)$$

and

$$\bar{A}^\pm(x, \lambda) = \Omega^\pm(x)(\partial_\mu + A_\mu^\pm(x, \lambda))\Omega(x)^\pm.$$

The Dirac equation is

$$i(\partial_\mu + A_\mu)_\mu\begin{bmatrix}\psi_U \\ \psi_L\end{bmatrix} = E\begin{bmatrix}\psi_U \\ \psi_L\end{bmatrix}$$

If the ψ_U and ψ_L are regarded as 2×2 matrices in color $\times$ dirac index space, we have in 2×2 matrix form

$$i(\partial_\mu + A_\mu)\psi^U\sigma^2\alpha^\mu\sigma^2 = E\psi_U$$
$$i(\partial_\mu + A_\mu)\psi^L\sigma^2\bar{\alpha}^\mu\sigma^2 = E\psi_U$$

Solutions for $A_\mu = A_\mu^+(x - z^+, \lambda)$ are

$$\psi^U = 0;\ \psi^L(x) = \varphi_0^+(x - z^+, \lambda) = \frac{\alpha_\mu(x - z^+)_\mu\sigma^2}{|x - z^+|((x - z^+)^2 + \lambda^2)^{3/2}}\frac{\sqrt{2}\lambda}{\pi}$$

and for $A_\mu = A_\mu^-(x - z^-, \lambda)$,

$$\psi^U_{(x)} = \varphi_0^- x + z^-, \lambda) = \frac{\bar{\alpha}_\mu(x - z^-)_\mu\sigma^2}{|x - z^-|((x - z^-)^2 + \lambda^2)^{3/2}}\frac{\sqrt{2}\lambda}{\pi};\ \psi_L = 0$$

We can further generalize $A_\mu^\pm$ by allowing for an arbitrary global gauge rotation

$$A_\mu^\pm(z - z_i^\pm, \lambda_i^\pm, n_i^\pm) = \alpha\cdot n_i^\pm A_\mu^\pm(x - z_i^\pm, \lambda_i^\pm)\bar{\alpha}\cdot n^\pm$$

The associated zero mode becomes

$$\varphi_0^\pm(x - z_i^\pm, \lambda_i^\pm, n_i^\pm) = \alpha\cdot n_i^\pm\varphi_0^\pm(x - z_i^\pm, \lambda_i^\pm)$$

The one loop determinant calculate by 't Hooft for the non zero modes is given by

$$D = K(2N_f)\frac{e^{-8\pi^2/g^2}}{g}\mu^{[22-2N_f]/3}\rho^{[7+N_f]/3}$$

where

$$K(N_f) = 2^{14+N_f}\pi^{6-2N_f}\exp[2N_f\alpha(\tfrac{1}{2}) - \alpha(1)]$$

$\alpha(\frac{1}{2})$ and $\alpha(1)$ are constants defined in [2].

Appendix B

Consider the Dirac operator

$$H = i\gamma_\mu\left[\partial_\mu + \sum_{i=1}^{N} A_\mu^+(i) + A_\mu^-(i)\right]$$

in a dilute gas background of N instantons and N anti-instantons. If there are N_f flavors there are $N_f N$ zero modes for single instanton and anti-instantons,

$$i\gamma_\mu[\partial_\mu + A_\mu^\pm(i)\psi_0^\pm] = 0$$

In the basis of these modes, H has the $2N\cdot N_f \times 2N\cdot N_f$ form

$$H = \begin{bmatrix} 0 & iA \\ -iA & 0 \end{bmatrix}$$

where the diagonal blocks vanish because ψ^+ and ψ^- have opposite chirality. The matrix A has N diagonal blocks, all proportional to the $N_f \times N_f$ unit matrix,

$$A = \begin{bmatrix} A_1 I & & 0 \\ & A_2 I & \\ 0 & & A_3 I \end{bmatrix}$$

where the off-diagonal blocks vanish because, as described in the text, it is assumed there is appreciable overlap only between nearest neighbor instantons and anti-instantons. A unitary transformation $\frac{1}{\sqrt{2}}\begin{pmatrix} I & iI \\ iI & I \end{pmatrix}$ takes H into the form

$$U^{-1}HU = \begin{bmatrix} -A & 0 \\ 0 & A \end{bmatrix}$$

with

$$\det H = \prod_{i=1}^{N} A_i^{2N_f}(-1)^{N\cdot N_f}$$

The eigenvectors $V_i^\pm$ of H belonging to the eigenvalues $\lambda = \pm A_i$ are, for example,

$$V_i^\pm = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ \hline \pm i \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

and similarly for other pairs.

The matrix element for a particular instanton–anti-instanton is given by

$$iA_i = \langle \varphi^+(x - z^+, \rho^+)|H|\varphi^-(x - z_i^-, \rho_i^-)\rangle$$

suppressing the index i this matrix element has the explicit form

$$= i\,\mathrm{Tr}\frac{2}{\pi^2}\rho^+\rho^-\int d^4x \frac{\sigma^2\bar{\alpha}\cdot(x - z^+)}{(x - z^+)((x - z^+)^2 + \rho^{+2})^{3/2}}$$
$$\cdot\frac{(-2i)\bar{\sigma}_{\mu\nu}(x - z^+)_\nu\rho^{+2}}{(x - z^+)((x - z^+)^2 + \rho^{+2})}$$
$$\cdot\frac{\alpha\cdot\hat{n}\bar{\alpha}\cdot(x - z^-)\alpha_\mu\sigma^2}{(x - z^-)((x - z^-)^2 + \rho^{-2})^{3/2}}$$

Taking the traces, on finds

$$\frac{12i}{\pi^2}(\rho^+\rho^-)\rho^{+2}\int d^4x$$
$$\cdot\frac{n\cdot(x - z^-)}{|x - z^+|((x - z^+)^2 + \rho^{+2})^{5/2}|x - z^-|}$$
$$\cdot((x - z^-)^2 + \rho^{-2})^{3/2}$$

Changing variables to $y = (x - z^+)$ yields

$$\frac{12}{\pi^2}(\rho^+\rho^-)(\rho^+)^2\int d^4y \frac{n\cdot(y + (z^+ - z^-))}{y(y^2 + \rho^{2+})^{5/2}|y + (z^+ - z^-)|}$$
$$\cdot((y + (z^+ - z^-))^2 + \rho^{-2})^{3/2}$$

Since this integral is strongly peaked at small y we find for large $|z^+ - z^-|$

$$A = 8(\rho^+\rho^-)\frac{\hat{n}\cdot(z^+ - z^-)}{(z^+ - z^-)4}$$

$\hat{n}$ gives the relative gauge orientation of the instanton–anti-instanton pair.

Appendix C

Some Remarks on Dilute Gas Approximations

Let W^N be the generating function when the background field contains N instantons. (We will consider paired instantons and anti-instantons later.) The complete generating function is then

$$W(\eta^+,\eta)=\frac{\sum_{N=0}^{\infty}\frac{W^{N(\eta^\dagger,\eta)}}{N!}}{\sum_{N=0}^{\infty}\frac{W^{N(0,0)}}{N!}}$$

A Green's function of n-external quark legs and n-external anti quark legs has the form

$$O^n\left(\frac{\delta}{\delta\eta^\dagger},\frac{\delta}{\delta\eta}\right)\Bigg|_{\eta=\eta^\dagger=0}=\sum_{m=0}^{N}A_m^N(n)$$

where $A_m^N(n)$ is the contribution when m of the N instantons interact and $N-m$ are disconnected [19]. It has the form

$$A_m^N(n)=V^{N-m}\sum_{\text{perm}}\int\prod_{i=1}^{m}d^4z_i d\rho_i D(i)\tilde{A}_m^n$$
$$\equiv V^{N-m}\sum_{\text{perm}}\langle\tilde{A}_m^n\rangle$$

where

$$V=\int d^4z\,d\rho\,D.$$

(Here $\langle\tilde{A}_0^n\rangle\equiv\tilde{A}_0^n$, with no collective coordinate average implied.) Note $\tilde{A}_m^n$ is independent of N, the total number of instantons. The permutation sums over sets of the m instantons of N which are interacting; as all such sets give the same contribution and there are $N!/(N-N)!$ sets we have

$$O^n\left(\frac{\delta}{\delta\eta^\dagger},\frac{\delta}{\delta\eta}\right)W^N=\sum_{m=0}^{N}V^{N-m}\frac{N!}{(N-m)!}\langle\tilde{A}_m^n\rangle$$

Then the full $2n$ legged Green's function $G(n)$ is

$$G(n)=\frac{\sum_{N=0}^{\infty}\sum_{m=0}^{N}\frac{V^{N-m}}{N!}\frac{N!}{(N-m)!}\langle\tilde{A}_m^n\rangle}{\sum_{N=0}^{\infty}\frac{V^N}{N!}}=\sum_{m=0}^{\infty}\langle\tilde{A}_m^n\rangle$$

We see that V—the volume of the world—drops out, as it must, on physical grounds.

To see the error of [8], consider the simple case when only $\tilde{A}_0^n$ and $\tilde{A}_1^n$ are non-zero, then

$$G(n)=\frac{\sum_{N=0}^{\infty}\frac{V^N}{N!}\langle\tilde{A}_0^n\rangle+\sum_{N=1}^{\infty}\frac{V^{N-1}}{(N-1)!}\langle\tilde{A}_1^n\rangle}{\sum_{N=0}^{\infty}\frac{V^N}{N!}}$$

If only lowest order terms in the numerator and first order terms in the denominator are kept, we have

$$G(n)\approx\frac{\tilde{A}_0^n+\langle A_1^n\rangle}{1+V}\approx\tilde{A}_0^n+\langle A_1^n-\tilde{A}_0^n\rangle$$

where the average is over the single collective coordinate corresponding to A_1. This result is in error because powers of the world volumes have not been consistently collected. The result depends on V through $\langle\tilde{A}_1^n-\tilde{A}_0^n\rangle=\langle\tilde{A}_1^n\rangle-V\tilde{A}_0^n$. The correct result is

$$G(n)=\tilde{A}_0^n+\langle A_1^n\rangle$$

which is V-independent.

Let us now consider the expansion of the propagator in dilute gas approximation. We develop $(\not\partial+\not V(1)+\not V(2))^{-1}$ where $V(i)$ represent instanton–anti-instanton pairs

$$V(i)=A^+(i)+A^-(i).$$

Beginning with

$$(\not\partial+\not V(1)+\not V(2))^{-1}=\not\partial^{-1}\sum_{n=0}^{\infty}\left[\frac{-(\not V(1)+\not V(2))}{\not\partial}\right]^n$$

we drop terms of the form $V(1)\,V(2)\,V(1)$ representing interactions in which the fermion interacts with one pair, then with another, and then *returns* to the original pair. Then we have

$$[\not\partial+\not V(1)+\not V(2)]^{-1}\approx\not\partial^{-1}\left[1+\sum_{n=1}^{\infty}\left(\frac{-\not V(1)}{\not\partial}\right)^n\right.$$
$$+\sum_{n=1}^{\infty}\left(\frac{-\not V(2)}{\not\partial}\right)^n+\sum_{n=1}^{\infty}\left(\frac{-\not V(1)}{\not\partial}\right)^n\sum_{m=1}^{\infty}\left(\frac{-\not V(2)}{\not\partial}\right)^m$$
$$\left.+\sum_{n=1}^{\infty}\left(\frac{-\not V(2)}{\not\partial}\right)^n\sum_{m=1}^{\infty}\left(\frac{-\not V(1)}{\not\partial}\right)^m\right]$$
$$=S^0[1+\mathcal{S}(1)+\mathcal{S}(2)+\mathcal{S}(1)\mathcal{S}(2)+\mathcal{S}(2)\mathcal{S}(1)]$$

where

$$S_0=\not\partial^{-1}$$

$$\mathcal{S}(i)=\left[\frac{1}{\not\partial+V(i)}-\frac{1}{\not\partial}\right]$$

When N pairs are present this generalizes to

$$\left[\not\partial+\sum_{i=1}^{N}(\not A^+(i)+\not A^-(i))\right]^{-1}$$
$$=S_0\left[1+\sum_{i=1}^{N}\mathcal{S}(i)+\sum_{i\neq j}^{N}\mathcal{S}(i)\mathcal{S}(j)\right.$$
$$\left.+\sum_{i\neq j\neq k}^{N}\mathcal{S}(i)\mathcal{S}(j)\mathcal{S}(k)+\dots\right]$$

To obtain our final form we expand $\mathcal{S}(i)$ itself using the above formalism with $V(1)=A^+$ and $V(2)=A^-$, resulting in

$$\mathcal{S}(i)=S_0^{-1}(S^+(i)-S_0)+S_0^{-1}(S^-(i)-S_0)$$
$$+S_0^{-1}(S^+(i)-S_0)S_0^{-1}(S^-(i)-S_0)$$
$$+S_0^{-1}(S^-(i)-S_0)S_0^{-1}(S^+(i)-S_0)$$

where

$$S^\pm(i)=[\not\partial+\not A^\pm(1)]^{-1}$$

is the propagation function in a simple instanton or anti-instanton field.

Since the Green's function that we are interested in will be dominated by what are the shifted zero modes of the NIDG, it is convenient to explicitly remove them from the propagator. It is also necessary if one wants to use the explicit form of $S^{\pm}$ given in [15] since they have the zero modes removed. Using the results of Appendix B, we find for the two point function

$$\frac{\delta_{ab}}{2}\left\{\frac{\begin{pmatrix} i\varphi_{i\alpha}^{(-1)} \\ \varphi_{i\alpha}^{(+)}\end{pmatrix}(i\varphi_{j\beta}^{(-)}\varphi_{j\beta}^{(+)})^*}{A_i}+\right.$$
$$\left.+\frac{\begin{pmatrix} -i\varphi_{i\alpha}^{(-)} \\ \varphi_{i\alpha}^{(+)}\end{pmatrix}(-i\varphi_{j\beta}^{(-)},\varphi_{j\beta}^{(+)})^*}{-A_i}\right\}$$
$$=\delta_{ab}\frac{\begin{pmatrix} 0 & i\varphi_{i\alpha}^{(-)}\varphi_{j\beta}^{(+)*} \\ -i\varphi_{i\alpha}^{(+)}\varphi_{j\beta}^{(-)*} & 0\end{pmatrix}}{A}=\frac{M(x,y)}{A}$$

where a,b are flavor, i,j are color and α,β are Dirac indices. The label of the instanton–anti-instanton pair is suppressed.

Appendix D

The transforms of (3.2) are worked out here. We have

$$(z^2+a^2)^{-4}\equiv\int\frac{d^4q}{(2\pi)^4}q(q)e^{iq\cdot z}$$

and

$$q(q)=4\pi^2\int x^3\,dx\,\frac{J_1(qx)}{qx}\frac{1}{(x^2+a^2)^4}$$
$$=\frac{4\pi^2q^2a^{-2}K_{+2}(qa)}{2^3\Gamma(4)}=q^4C(qa)$$

where

$$C(Q)=\frac{\pi^2K_{+2}(Q)}{12Q^2}$$

In (4.13) we require integrals of the form

$$\int_0^q dx\,x^kf(x)=G_k(Q)$$

where

$$f(x)=\frac{2\pi^2}{x^2}(I_1(x/2)K_1(x/2)+\frac{x}{2}\frac{d}{d(x/2)}(I_0(x/2)K_0(x/2))$$

They can be estimated for $Q\geqq 2$ as follows:

$$G_k(Q)=2\pi^2(2)^{k-1}\left\{\int_0^1+\int_1^0\right\}dy\,y^{k-2}$$
$$\cdot\left[I_1(y)\cdot K_1(y)+y\frac{d}{dy}I_0(y)K_0(y)\right]$$

for $y\geqq 1$. The integrand is well represented by its asymptotic form

$$y^{k-2}/(2y)^3$$

thus

$$G_k(Q)=A_k+\frac{2\pi^2}{k-4}q^{k-4}$$

where A_k for $k=7,9$, and 11 are -304.7, -922.7 and -2998 respectively.

Appendix E

Inserting the explicit form of M in (4.10) we have

$$\prod_{\mu\nu}(x,y)=\mathrm{Tr}\,Q^2\left[\frac{K(2)e^{-8\pi^2/g^2}}{g^8}\mu^6\right]^2$$
$$\cdot\int dp^+\,d\rho^-\,dz^+\,dz^-(\rho^+\rho^-)^3A^2(i)^2\,dn$$
$$\cdot\mathrm{tr}\begin{pmatrix}0, & \alpha_{\alpha\beta}^{\mu} \\ \bar{\alpha}_{\alpha\beta}^{\mu}, & 0\end{pmatrix}\begin{pmatrix}0, & \varphi_{i\beta}^{(-)}(x)\varphi_{k\gamma}^{(+)*}(y) \\ -\varphi_{i\beta}^{(+)}(x)\varphi_{k,\gamma}^{(-)*}(y), & 0\end{pmatrix}$$
$$\cdot\begin{pmatrix}0, & \alpha_{\gamma\delta}^{\nu} \\ \bar{\alpha}_{\gamma\delta}^{\nu}, & 0\end{pmatrix}\begin{pmatrix}0, & \varphi_{k,\delta}^{(-)}(y)\varphi_{i,\alpha}^{(+)*}(x) \\ -\varphi_{k,\delta}^{(+)}(y)\varphi_{i\alpha}^{(-)*}(x), & 0\end{pmatrix}$$

using the explicit form of the zero mode $\varphi^{(\pm)}$ from Appendix A, the eigenvalue A from Appendix B, and the following two identities

$$\mathrm{Tr}(\alpha^\rho\bar{\alpha}^\nu\alpha^\sigma\bar{\alpha}^\delta)=2(g_{\rho\nu\sigma\delta}+\varepsilon_{\rho\nu\sigma\delta})$$

where

$$g_{\rho\nu\sigma\delta}=\delta_{\rho\nu}\delta_{\sigma\delta}-\delta_{\nu\sigma}\delta_{\rho\delta}-\delta_{\rho\sigma}\delta_{\nu\delta}$$

and

$$\int dn\,n_\delta n_\eta n_\alpha n_\beta=\frac{1}{24}H_{\delta\eta\alpha\beta}$$

where

$$H_{\delta\eta\alpha\beta}=\delta_{d\eta}\delta_{\alpha\beta}+\delta_{\delta\alpha}\delta_{\eta\beta}+\delta_{\delta\beta}\delta_{\alpha\eta}$$

we find

$$\prod_{\mu\nu}(x,y)=\mathrm{Tr}\,Q^2\left[\frac{K(2)e^{-8\pi^2/g^2}}{g^8}\mu^6\right]^2$$
$$\cdot\int d^4z^\pm\,d\rho^\pm(\rho^+\rho^-)^7\left(\frac{2}{\pi^2}\right)^2(i)^2\,64$$
$$\cdot\frac{1}{24}H_{\delta\eta\alpha\beta}\frac{z_\alpha z_\beta}{z^8}8(g_{\rho\nu\sigma\delta}+\varepsilon_{\rho\nu\alpha\delta})(g_{\lambda\mu\xi\eta}+\varepsilon_{\lambda\mu\xi\eta})$$
$$\cdot\left[\frac{y_\rho^+}{y^+(y^{+2}+\rho^{+2})^{3/2}}\frac{y_\sigma^-}{y^-(y^{-2}+\rho^{-2})^{3/2}}\right.$$
$$\left.\cdot\frac{x_\lambda^+}{x^+(x^{+2}+\rho^{+2})^{3/2}}\frac{x_\xi^+}{x^-(x^{-2}+\rho^{-2})^{3/2}}\right]$$

where $x^\pm=(x-z^\pm)$; $y^\pm=(y-z^\pm)$. Using (3.12a) we can fourier transform the object in square brakets ([ ])

$$[\,]=(i)^4(\rho^+\rho^-)^4\int\frac{d^4p^+}{(2\pi)^4}\frac{d^4p^-}{(2\pi)^4}\frac{d^4k^+}{(2\pi)^4}\frac{d^4k^-}{(2\pi)^4}$$
$$\cdot p_\rho^+p_\sigma^-k_\lambda^+k_\xi^-f((p^+\rho^+)^2)$$
$$\cdot f((p^-\rho^-)^2)f((k^+\rho^+)^2)f((k^-\rho^-)^2)$$
$$\cdot\exp\{ip^+y^++ip^-y^-+ik^+x^++ik^-x^-\}$$

Changing variables to $z = z^+ - z$, $R = \frac{1}{2}(z^+ + z^-)$ we can do the d^4R integration because of translational invariance and obtain an energy momentum conserving delta function $(2\pi)^4\delta^4(k^+ + k^- + p^+ + p^-)$. Making the following change of variables,

$$k^- = (k + p/2 - q/2) \qquad p^- = -(k + p/2 + q/2)$$
$$k^+ = -(k - p/2 - q/2) \quad p^+ = (k - p/2 + q/2)$$

we then obtain (4.11).

References

1. A. Belavin, A. Polyakov, A. Schwarz, Y. Tyupkin: Phys. Lett. **59B**, 85 (1975)
2. G. 't Hooft: Phys. Rev. **D14**, 3432 (1976)
3. R. Jackiw, G. Rebbi: Phys. Rev. Lett. **37**, 172 (1976)
4. C.G. Callan, R. Dashen, D. Gross: Phys. Lett. **63B**, 334 (1976)
5. C.G. Callan, R. Dashen, D. Gross: Phys. Rev. **D17**, (1978)
6. M.F. Atiyah, I.M. Singer: Ann. of Math. **87**, 596 (1968)
7. R. Carlitz, C. Lee: Phys. Rev. **D17**, 3238 (1978)
8. N. Andrei, D. Gross: the effect of instants on the short distance structure of itadronic currents. Princeton preprint (1978)
9. L. Baulieu, J. Ellis, M.K. Gaillard, W.J. Zakyzewski: TH-2482-CERN (1978)
10. M. Suzuki: LBL PHY-74-08172-A02 (1978)
11. T. Appelquist, R. Shankar: A general approach to the computation of instanton effects. Yale Univ. preprint
12. J. Hietarinta, W.F. Palmer, S. Pinsky: Ohio State preprint COO-1545-227
13. J. Hietarinta, W.F. Palmer, S. Pinsky: Phys. Rev. Lett. **38**, 103 (1977)
14. J. Hietarinta, W.F. Palmer: Phys. Lett. **71B**, 415 (1977); Phys. Rev. **D16**, 3008 (1977)
15. L.S. Brown, R.D. Carlitz, D.B. Creamer, C. Lee: Phys. Lett. **70B**, 180 (1977); **71B**, 103 (1977); Phys. Rev. **D17**, 1583 (1978)
16. R.D. Carlitz: Phys. Rev. **D17**, 3225 (1978)
17. For an example see W.F. Palmer: Phys. Rev. **D6**, 1190 (1972)
18. L. Susskind: SLAC-PUB-2142
19. P. Carruthers, S.S. Pinsky, F. Zachariasen: Nucl. Phys. **B135**, 486 (1978)