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Developing a coordinated vendor–buyer model in two-stage supply chains with stochastic lead-times

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ABSTRACT

This paper develops an approach to determine the optimal production and shipment policy for an integrated vendor–buyer problem. The vendor manufactures the product in batches at a finite rate and ships the output to the buyer. All shipments to the buyer are equal-sized batches. Despite previous papers in the literature, we assume that the supply lead-time between vendor and buyer is stochastic and shortage is also allowed. The objective is to minimize the expected total cost of both buyer and vendor. We derive the expected annual integrated total cost function and propose an analytic solution procedure to determine the optimal policy. To illustrate the significance of cost-reduction of the integrated approach in comparison with independent decisions by buyer or vendor, some numerical examples are also presented.

1. Introduction

In today's competitive and dynamic market conditions, the effective collaboration of partners and coordination of all activities within the supply chain is prerequisite (Tarantilis [1]). One important benefit of this coordination is a more efficient management of inventories across the entire supply chain. In traditional inventory management, the optimal production and shipment policies for vendor and buyer in a two-echelon supply chain are managed independently. As a result, the optimal lot size for the buyer may not result in an optimal policy for the vendor, and vice versa. To overcome this difficulty, the integrated vendor–buyer model is developed, where the joint total relevant cost for the buyer as well as the vendor is minimized. Consequently, determining the production and shipment policies based on integrated total cost function, rather than buyer's or vendor's individual cost functions results in reduction of the total inventory cost of the system.

In the literature, integrated vendor–buyer problem is called the joint economic lot sizing (JELS) problem which can be considered as the building block for wider supply chain systems. The global supply chain can be very complex and link-by-link understanding of joint policies can be very useful (Ben-Daya et al. [2]).

Goyal [3] was the first who introduced the idea of a joint total cost for a single-vendor and a single-buyer scenario, under the assumption of having an infinite production rate for the vendor and lot-for-lot policy for the shipments from the vendor to the buyer. Banerjee [4] relaxed the infinite production rate assumption. Then, Goyal [5] contributed to the efforts of generalizing the problem by relaxing the assumption of lot for lot. He assumed that the production lot is shipped in a number of equal-size shipments. Later, Goyal [6] developed a model where the shipment size increases by a factor equal to the ratio of production rate to the demand rate. He formulated the problem and developed an optimal expression for the first shipment size as a function of the number of shipments. Hill [7] generalized the model of Goyal [6] by taking the geometric growth factor as a decision variable. He suggested a solution method based on an exhaustive search for both the growth factor and the number of shipments in certain ranges. Later, Hill [8] relaxed the assumptions of the shipment policy and developed an optimal solution of the problem. He showed that the structure of the optimal policy includes shipments increasing in size according to a geometric series followed by equal-sized shipments.

The basic JELS models have been extended in many different directions. It is beyond the scope of this paper to discuss all works in detail here. Broadly speaking, the existing literature on JELS may be divided into different categories such as "JELS and quality" (e.g., Huang [9]), "JELS and controllable lead-time" (e.g., Hoque and Goyal [10]), "JELS and multiple buyers" (e.g., Siajadi et al. [11]), "JELS and setup and order cost reduction" (e.g., Chang et al. [12]), "JELS and transportation" (e.g., Ertogral et al. [13]), "JELS and deteriorated item" (e.g., Yang and Wee [14], Wee et al. [15] and Chung and Wee [16]), "JELS and declining market" (e.g., Yang et al. [17]), "JELS and pricing strategy" (e.g., Wee and Yang [18]), "JELS and perishable delay in payment" (e.g., Yang and Wee [19]), and "JELS and three-level supply

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chain" (e.g., Lee [20]). Some papers in the literature may belong to more than one category. Readers are referred to Ben-Daya et al. [2] for a comprehensive review of the JELS problems.

It should be pointed out that, although there are considerable studies covering the different dimensions of JELS problem, most of them are limited to deterministic conditions, which bound their applicability. In recent years, a tremendous attention is given to the development of non-deterministic integrated vendor–buyer models. That is our motivation in this paper to develop an integrated inventory system and relax the assumption of deterministic lead-time in the hitherto existing JELS models and analyze the problem where the buyer lead-time is a stochastic variable. Moreover, the model is extended to the situation in which shortage is allowed.

This paper can be considered as one of the research efforts that deal with extensions aimed at relaxing assumptions that may not be realistic in practical situations such as the assumptions of deterministic parameters. In practice, there may be a lot of environmental causes which can affect lead-time components such as order processing time, transportation time of products from the vendor to the buyer, inspection, etc.; and therefore result lead-time uncertainty.

The paper is organized as follows. In Section 2, the problem is defined, and also notation and assumptions are introduced. Section 3 gives a discussion on independent policies for buyers and vendor as well as on the integrated model. In Section 4, an algorithm is developed to find the optimal solution of the integrated model. Section 5 presents some numerical examples. Finally, the paper findings and further research directions are summarized in Section 6.

2. Problem definition and notation

Consider a supply chain of a product which consists of a single vendor and single buyer. The final demand for this product is assumed to be deterministic. The lots delivered from the vendor to the buyer are equal-sized batches. As soon as the inventory position at buyer drops to r, then an order size of Q is released by the buyer. Then, the vendor manufactures this product at the production rate P and in sizes which are multiple of Q. The cost of system includes setup, shortage and holding costs. The objective is to determine the number of shipments of n, reorder point of r as well as the order size of Q such that the expected total cost of both vendor and buyer be minimized.

Unlike previous models in the literature, the supply lead-time between vendor and buyer is considered to be stochastic. This lead time is the sum of two elements, a deterministic length of time and a random one. The second element of lead time represents the total uncertain amount of time which is assumed to be an exponential random variable. Moreover, the model is also extended to the situation in which shortage is allowed.

The assumptions of the model are summarized as follows:

1. The integrated model deals with a single vendor and single buyer for a single product.
2. The demand rate is deterministic and constant.
3. A finite production rate for the vendor is considered which is greater than demand rate.
4. Inventory is continuously reviewed. The buyer orders a lot of size Q when the inventory position reaches the reorder point r.
5. A "non-delayed equal-sized shipment policy" is employed for the shipments from the vendor to the buyer. In this policy, the size of shipments is equal, and therefore the production batch is shipped to the buyer in n same-size shipments. In other words, the vendor manufactures nQ products at one setup and delivers the shipments of size Q to the buyer during and after the production phase. This policy is known as non-delayed because it allows the vendor to also deliver the shipments during production, which is the relaxation of the assumption about completing a lot before starting shipments. This policy is attractive in practice because the shipment size is constant, and therefore it is easy to implement.
6. Lead time to replenish the buyer's order consists of fixed and variable components, where the variable component is stochastic and follows an exponential distribution with parameter λ.
7. Shortages are allowed and completely backordered.
8. Time horizon is infinite.

2.1. Notation

D	demand rate
P	production rate of the vendor
Q	buyer's order quantity
r	buyer's reorder point
A_v	vendor's setup cost
A_b	buyer's ordering cost
h_v	inventory holding cost for the vendor per unit per unit time
h_b	inventory holding cost for the buyer per unit per unit time
π	shortage cost for the buyer per unit per unit time
n	number of shipments
T	buyer's cycle time
L	lead time to replenish the buyer's order

3. Model formulation

In this section, the optimal policy of the integrated system is derived. However, for comparison purposes, we first obtain the buyer and the vendor optimal policies, if each party optimizes his benefit only. Then, the policies and costs are compared with the case of integrated system when they cooperate, particularly in information sharing of their cost.

3.1. Non-integrated optimization model

3.1.1. Buyer's cost formulation

In order to find the optimal ordering policy, we first need to obtain the buyer expected total cost per unit time, $TC_b(r,Q)$. When lead times for the buyer are stochastic, orders may not be received in the same sequence they are placed. Although, the likelihood of order crossover would seem to be low, order crossover makes the analysis intractable. To circumvent this problem it is usually assumed that the orders do not cross in time (Hadley and Whitin [21]), or that no more than one order is outstanding at any point in time. Here, we assume that the probability of order crossover is negligible.

The fixed element of lead time just changes the reorder point and has no effect on optimization model. Thus, we eliminate it from our analysis and in rest of paper; the lead-time word is used for representing the variable part.

Different cases can occur during each cycle time for the buyer depending on the delivery time (Fig. 1). The lead time is assumed to be exponentially distributed, i.e., $L\sim\exp(\lambda)$. There is then a probability that the delivery is received after the cycle time (Case 3).

As can be seen in Fig. 1, we are calculating the cost function for one cycle time and extending it to the whole unit time. An implicit assumption of this calculation is that the selected cycle time should be renewable. The net stock at the start of each cycle time has been considered to be equal to the reorder level. This assumption will not be satisfied for the next cycle in Case 3, where the net stock after the last delivery is less than the reorder point. Thus, in order to create a renewable cycle, we include only the hatched part in the expected cost. In other words, the further parts of inventory and

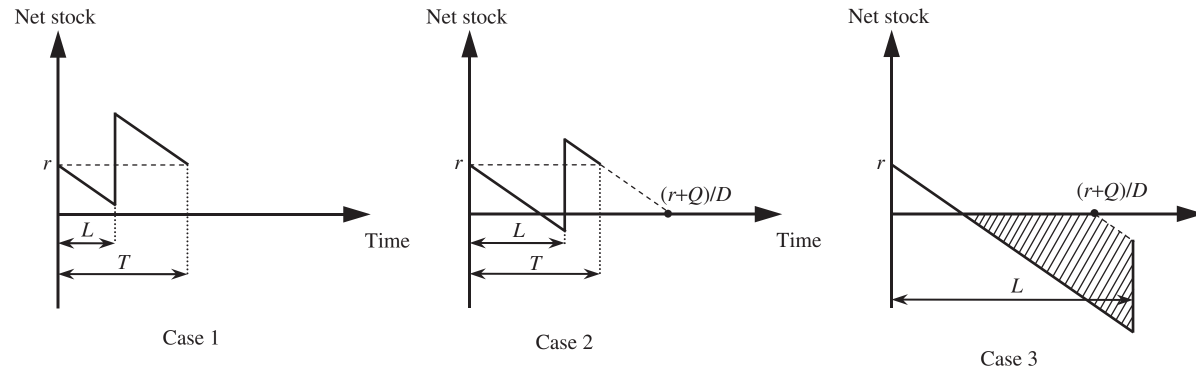

Fig. 1. Net stock vs. time for the buyer.

shortage costs are omitted in this case and will be considered in the next cycle time, on average.

Let $f_L(l)$ be the probability density function of L. Hence

$$f_L(l) = \lambda e^{-\lambda l}, \quad 0 \leqslant l < \infty \tag{1}$$

We calculate the buyer expected total cost during an order cycle by multiplying the inventory carried and shortages incurred during each cycle time and taking the expectation over the relevant limits for the lead time. We then obtain the buyer expected total cost per unit time, dividing these costs by the length of the buyer order cycle, Q/D. We then obtain the following expression

$$\begin{aligned} TC_b(r,Q) = \frac{DA_b}{Q} &+ h_b \int_0^{r/D} \left(\frac{Q}{2} + r - Dl\right) f_L(l) d_l \\ &+ \int_{r/D}^{(r+Q)/D} \left[\frac{\pi (Dl - r)^2}{2Q} + \frac{h_b (Q + r - Dl)^2}{2Q}\right] f_L(l) d_l \\ &+ \int_{(r+Q)/D}^{\infty} \pi \left(Dl - r - \frac{Q}{2}\right) f_L(l) d_l \end{aligned}$$

Substituting (1) into the above expression and simplifying, we obtain

$$\begin{aligned} TC_b(r,Q) = \frac{DA_b}{Q} &+ h_b \left(r + \frac{Q}{2} - \frac{D}{\lambda}\right) \\ &+ \frac{D^2(\pi + h_b)}{\lambda^2 Q} (e^{-r\lambda/D} - e^{-(r+Q)\lambda/D}) \end{aligned} \tag{2}$$

We have proved that $TC_b(r,Q)$ is strictly convex in r and Q (see Appendix A). Taking the partial derivative of $TC_b(r,Q)$ with respect to r, and equating it to zero, we have

$$\frac{\partial TC_b(r,Q)}{\partial r} = h_b - \frac{D(\pi + h_b)}{\lambda Q} (e^{-r\lambda/D} - e^{-(r+Q)\lambda/D}) = 0$$

Thus, we have

$$r = \frac{D}{\lambda} \ln[D(\pi + h_b)(1 - e^{-Q\lambda/D})/(h_b \lambda Q)] \tag{3}$$

Substituting (3) into (2) and simplifying, we get

$$\begin{aligned} TC_b(Q) = \frac{DA_b}{Q} &+ \frac{Dh_b}{\lambda} \ln[D(\pi + h_b)(1 - e^{-Q\lambda/D})/(h_b \lambda Q)] \\ &+ \frac{Qh_b}{2} \end{aligned} \tag{4}$$

Since $TC_b(r,Q)$ is strictly convex in r and Q, thus $TC_b(Q)$ is also convex in Q. However, no closed-form solution exists for the value of Q that minimizes $TC_b(Q)$. Since then, we need to use a one dimensional search algorithm (e.g., Newton, Fibonacci), which is also employed by most of optimization software. As $TC_b(Q)$ is convex in Q, the local solution found by these algorithms in the optimization software is surely the global solution. In this paper, the optimal solution of (4) is obtained using Maple 11.0.

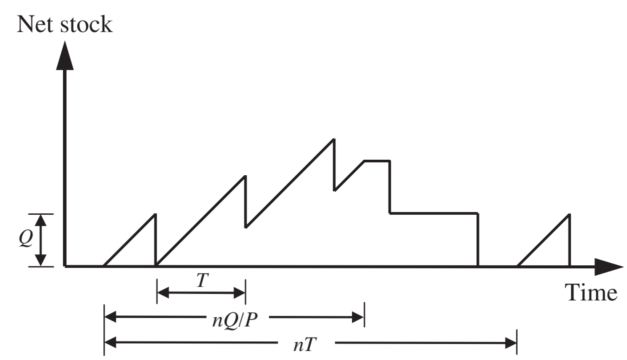

Fig. 2. Net stock vs. time for the vendor.

3.1.2. Vendor's cost formulation

When the buyer's order quantity is adopted, the orders are received by the vendor at known intervals T (Fig. 2). Vendor's average inventory can be obtained as follows:

$$\begin{aligned} AI_v &= \frac{D}{nQ} \left\{ \left[nQ \left(\frac{Q}{P} + (n-1)\frac{Q}{D}\right) - \frac{n^2 Q^2}{2P} \right] - \left[\frac{Q^2}{D} (1 + 2 + \cdots + (n-1)) \right] \right\} \\ &= \frac{Q}{2} \left((n-1)\left(1 - \frac{D}{P}\right) + \frac{D}{P} \right) \end{aligned} \tag{5}$$

Hence, the vendor's expected total cost is

$$TC_v(n) = \frac{DA_v}{nQ} + h_v \frac{Q}{2} \left((n-1)\left(1 - \frac{D}{P}\right) + \frac{D}{P} \right) \tag{6}$$

It can easily be shown that $TC_v(n)$ is convex in n. We then obtain the following optimality conditions on n^*:

$$n^*(n^* - 1) \leqslant \frac{2DA_v}{h_v Q^2 (1 - D/P)} \leqslant n^*(n^* + 1) \tag{7}$$

If the buyer is free to choose his own ordering policy (r,Q), and the vendor is free to choose his number of shipment n, then it is straightforward that the individually derived expected total cost $TC_I(r,Q,n)$ is equal to the summation of buyer's and vendor's expected costs, i.e., $TC_I(r,Q,n) = TC_b(r,Q) + TC_v(n)$.

3.2. Joint optimization model

Suppose that both parties decide to cooperate and agree to follow the jointly optimal integrated policy. Therefore, the joint expected total cost of the buyer and the vendor per unit time $TC_J(r,Q,n)$ is given by

$$TC_J(r,Q,n) = \frac{D(nA_b + A_v)}{nQ} + h_b\left(r + \frac{Q}{2} - \frac{D}{\lambda}\right) + \frac{D^2(\pi + h_b)}{\lambda^2 Q}(e^{-r\lambda/D} - e^{-(r+Q)\lambda/D}) + h_v\frac{Q}{2}\left((n-1)\left(1 - \frac{D}{P}\right) + \frac{D}{P}\right) \tag{8}$$

4. Solution procedure for joint model

It can be shown that $TC_J(r,Q,n)$ is not necessarily convex in r, Q and n. However, TC_J is convex in r and Q for a known value of n (see Appendix B). Moreover, TC_J is convex in n for known values of r and Q. Thus, the following optimality conditions on Q^* and n^* can be defined

$$n^*(n^*-1) \leqslant \frac{2DA_v}{h_v Q^{*2}(1 - D/P)} \leqslant n^*(n^*+1) \tag{9}$$

Additionally, it can be proved that $TC_J(r,Q,n)$ is convex in r for known values of Q and n. Taking the partial derivative of $TC_J(r,Q,n)$ with respect to r, and equating it to zero, we again obtain Eq. (3). Substituting (3) into (8) and simplifying, we get

$$TC_J(Q,n) = \frac{D(nA_b + A_v)}{nQ} + \frac{Dh_b}{\lambda}\ln[D(\pi + h_b)(1 - e^{-Q\lambda/D})/(h_b\lambda Q)] + \frac{Qh_b}{2} + h_v\frac{Q}{2}\left((n-1)\left(1 - \frac{D}{P}\right) + \frac{D}{P}\right) \tag{10}$$

Based on the following theorem, we develop a solution algorithm to find the optimal solution. We first obtain a corollary which is needed in the proof of the main theorem.

Corollary 1. *Consider Q^* as the optimal order quantity that minimizes TC_J for a known value of n. If the number of shipments increases to kn where $k > 1$, then the optimal order quantity will decrease to $Q^*/\beta k$, where $1/k < \beta < 1$. This corollary implies that Q and n vary inversely, and also that the decrease rate of order quantity is less than the increase rate of number of shipments.*

Proof. The joint expected total cost can be rewritten as $TC_J = TC_{J1} + TC_{J2}$, where $TC_{J1} = DA_v/nQ + h_v nQ(1 - D/P)/2$. If the number of shipments increases to kn, $k > 1$, the optimal order quantity that minimizes TC_{J1} will decreases to Q^*/k. Moreover, as TC_{J2} is independent of number of shipments, the optimal order quantity that minimizes TC_{J2} remains unchanged. Therefore, the new optimal order quantity for the sum of these two cost functions, Q_{new}, will be in the range of $Q^*/k < Q_{new} < Q^*/1$. In other words, $Q_{new} = Q^*/\beta k$, where $1/k < \beta < 1$. □

Theorem 1. *Consider that minimizing TC_J for a fixed value of $n = n^*$ results r^* and Q^*. Moreover, assume that the optimality conditions (9) are valid for $n^*$and Q^*. If there is a number of shipments, $n' > n^*$, where for given value of n', TC_J is minimized in r' and Q', but the optimality conditions (9) are not satisfied for $n'$and Q', then the optimal number of shipments is less than n'.*

Proof. Since the optimality conditions are satisfied in $n = n^*$, not satisfaction of these conditions for $n' > n^*$, based on the above corollary, means that the proposed number of shipments obtained employing these conditions, n'_p, is less than n', i.e., $n'_p < n'$. Moreover, for any $n'' > n'$ we can find $k > 1$ where $n'' = kn'$. Based on the above corollary, the optimal order quantity will then be $Q'' = Q'/\beta k$ for n''. Because $\beta < 1$, thus $Q'' > Q'/k$, and therefore the proposed number of shipments for given value of Q'' employing the optimality conditions, n_p'', cannot be more than kn'_p, i.e., $n''_p \leqslant kn'_p$. Then, we have

$$\left.\begin{matrix} n'_p < n' \\ n''_p \leqslant kn'_p \end{matrix}\right\} \Rightarrow \left.\begin{matrix} n''_p < kn' \\ n'' = kn' \end{matrix}\right\} \Rightarrow n''_p < n''$$

Since then, optimality conditions (9) will also not be satisfied for any $n'' > n'$. Therefore, the optimal number of shipments cannot be more than n'. □

4.1. Solution algorithm

We develop the following algorithm to determine the optimal values of the three decision variables for the integrated model. The algorithm proposed is based on the model attribute proved in Theorem 1.

1. Set $n = 1$ and $i = 0$. Set TC_J^{opt} an arbitrarily large number.
2. Find the optimal value of Q such that $TC_J(Q)$ is minimized for given value of n.
3. Compute the optimal value of r using Eq. (3).
4. If the optimality conditions (9) are satisfied for Q and n, then go to the next step. Otherwise, go to Step 7.
5. If $TC_J(r,Q,n) < TC_J^{opt}$, then set $i = 1$, $TC_J^{opt} = TC_J(r,Q,n)$, $n_{opt} = n$, $r_{opt} = r$, and $Q_{opt} = Q$.
6. Increment n by 1, and go to Step 2.
7. If $i = 0$, then go to Step 5. Otherwise, the current solution is globally optimal.

5. Numerical study

Referring to the existing literature, we consider an example with the following data: $D = 1000$/year, $P = 5000$/year, $A_v = \$400$/setup, $A_b = \$25$/order, $h_v = \$4$/unit/year, $h_b = \$5$/unit/year, $\pi = \$30$/unit/year and $1/\lambda = 20$ days.

The optimal values of r, Q, n, and expected total system cost for the individually optimized model are 46.4, 154.7, 3, and 2199.2, respectively. The corresponding values for joint optimization model are 21.9, 254.6, 2, and 2139.1. The saving in joint total cost over individually derived policies is 2.73%, which should be shared in some equitable manner through the mechanism of a side payment to the buyer from the vendor, or a price discount scheme in order to entice the buyer to change his/her lot size.

In order to analyze the effect of lead-time variability on the percentage saving in jointly optimized total cost over individually derived policies, nine levels for lead-time parameter have been defined: $\lambda \in [5, 10, 15, 20, 25, 30, 35, 40, 45]$. To represent savings in joint total cost over individually derived policies, we define percentage saving PS, as $(TC_I - TC_J)/TC_I \times 100$.

The total cost under joint optimization should be shared by both parties in some equitable fashion. In order to encourage the buyer to cooperate with the vendor, a judicious method is essential for allocating cost. A proposed way is that the joint total cost be allocated to the buyer and the vendor using the following approach (see Ouyang et al. [22], Wu and Ouyang [23], and Goyal [3]):

$$TC_{vJ} = \frac{TC_v(n)}{TC_I(r,Q,n)}TC_J(r,Q,n)$$

$$TC_{bJ} = \frac{TC_b(r,Q)}{TC_I(r,Q,n)}TC_J(r,Q,n)$$

where TC_{vJ} and TC_{bJ} are the cost of vendor and buyer under a coordinated supply chain, respectively.

As can be seen in Fig. 3, *PS* increases for the cases where lead times are more variable. On the other hand, it will be more beneficial for the buyer and the vendor to cooperate with each other in unpredictable purchasing environment, where lead times are stochastic.

Other fact can also be discerned from Fig. 3. Increase in *PS* seems to be faster for higher production rate cases. For instance, if the lead-time mean increases from 5 days to 45 days, this results 0.92% improvement in the saving percentage when $P = 3000$. While, the improvement is 3.91% for $P = 5000$ and 6.35% for $P = 7000$ (increasing from 1.87% to 8.21%). The reason is that the coordination between vendor and buyer lets them to lessen the lead-time variability effect by increasing the ordering batch size and decreasing the number of shipments as production rate increases. However, ordering batch size remains unchanged for different production rates when the buyer and the vendor do not cooperate with each other. Since then, the supply chain members cannot effectively react to lead-time variation.

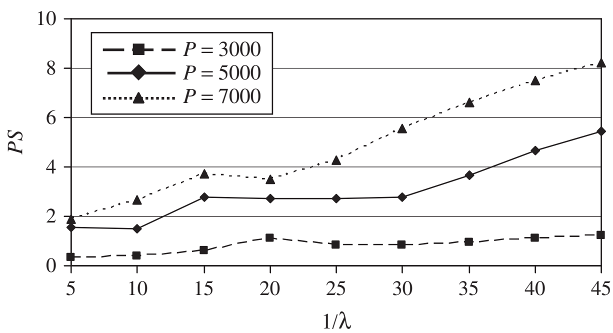

Fig. 3. Effect of lead-time variability on percentage saving.

The importance of taking into account the lead-time uncertainty in JELS can be more understood from the numerical results used in this article, where increasing the lead-time deviation from 5 to 15 days results the saving percentage for coordinated compared to non-coordinated strategy which is two times larger (increasing from 1.87 to 3.74).

Looking at the results in Table 1, we see that the optimal number of shipments decreases by lead-time mean $(1/\lambda)$. As $1/\lambda$ increases, the stock-out probability for the buyer increases, and therefore it will be more beneficial for the buyer to stock more amount of inventory and order less. Table 1 also shows that the optimal number of shipments in cooperative supply chain is less than that if the two parties do not choose to work together.

Moreover, as can be seen in Table 1, the cost of buyer under joint optimization shows sensitivity to the vendor's parameter. However, the buyer's cost under non-integrated system remains unchanged for different values of production rate.

6. Conclusions

This article presents an integrated production-inventory model for minimizing the joint expected total cost of the buyer and the vendor. Unlike the existing vendor–buyer integrated models, the presented JELS model assumes the buyer lead-time to be a stochastic variable. Furthermore, shortage is allowed and fully backordered. We formulated the integrated model as a nonlinear mixed integer programming problem to determine the optimal reorder point, order quantity, and number of shipments. We propose a solution method for finding the globally optimal production and shipment policies. Numerical examples were also used, providing that the cooperation between two supply chain partners is more critical in unpredictable purchasing environments in terms of stochastic lead-times. Consequently, the importance of profit sharing as a tool to entice

Table 1
Decision variables of non-integrated optimization vs. joint optimization

Parameters		Non-integrated optimization						Joint optimization						*PS*
P	$1/\lambda$	*r*	*Q*	*n*	TC_b	TC_v	TC_I	*r*	*Q*	*n*	TC_{bJ}	TC_{vJ}	TC_J	
3000	5	0.0	112.3	5	493.3	1386.2	1879.5	0.0	135.5	4	491.8	1381.8	1873.6	0.32
	10	10.5	129.6	4	569.4	1376.4	1945.7	2.2	176.8	3	567.0	1370.8	1937.8	0.41
	15	27.4	143.0	4	669.5	1366.6	2036.1	18.3	182.3	3	665.2	1358.0	2023.2	0.63
	20	46.4	154.7	4	780.4	1368.3	2148.7	37.6	186.8	3	771.8	1353.4	2125.2	1.09
	25	66.6	165.0	3	897.1	1358.1	2255.2	58.9	190.4	3	889.7	1346.9	2236.6	0.82
	30	87.7	174.2	3	1017.4	1346.1	2363.5	59.9	266.8	2	1008.8	1334.6	2343.4	0.85
	35	109.4	182.6	3	1140.3	1338.8	2479.1	80.9	271.7	2	1129.3	1326.0	2455.3	0.96
	40	131.5	190.4	3	1264.8	1335.0	2599.8	102.8	276.0	2	1251.0	1320.3	2571.3	1.09
	45	154.1	197.5	3	1390.7	1333.4	2724.1	125.4	279.8	2	1373.4	1316.9	2690.3	1.24
5000	5	0.0	112.3	4	493.3	1474.3	1967.7	0.0	164.4	3	485.8	1451.6	1937.4	1.54
	10	10.5	129.6	4	569.4	1445.5	2014.9	2.8	172.6	3	560.8	1423.6	1984.4	1.51
	15	27.4	143.0	4	669.5	1442.9	2112.4	6.1	247.2	2	650.7	1402.5	2053.2	2.80
	20	46.4	154.7	3	780.4	1418.8	2199.2	21.9	254.6	2	759.0	1380.1	2139.1	2.73
	25	66.6	165.0	3	897.1	1402.1	2299.2	40.1	261.1	2	872.9	1364.3	2237.2	2.70
	30	87.7	174.2	3	1017.4	1392.5	2409.9	59.9	266.8	2	989.3	1354.1	2343.4	2.76
	35	109.4	182.6	3	1140.3	1387.5	2527.8	33.3	470.7	1	1098.1	1336.4	2434.5	3.69
	40	131.5	190.4	3	1264.8	1385.7	2650.5	49.5	482.4	1	1206.1	1321.3	2527.4	4.65
	45	154.1	197.5	3	1390.7	1386.1	2776.8	66.6	493.6	1	1314.9	1310.6	2625.5	5.45
7000	5	0.0	112.3	4	493.3	1500.0	1993.3	0.0	162.9	3	484.1	1472.0	1956.1	1.87
	10	10.5	129.6	4	569.4	1475.1	2044.5	0.0	236.3	2	554.1	1435.6	1989.7	2.68
	15	27.4	143.0	3	669.5	1463.5	2133.0	6.1	247.2	2	644.4	1408.8	2053.2	3.74
	20	46.4	154.7	3	780.4	1436.4	2216.8	21.9	254.6	2	753.0	1386.1	2139.1	3.51
	25	66.6	165.0	3	897.1	1420.9	2318.0	3.3	456.3	1	858.7	1360.3	2219.0	4.27
	30	87.7	174.2	3	1017.4	1412.4	2429.8	16.4	469.6	1	961.1	1334.2	2295.3	5.54
	35	109.4	182.6	3	1140.3	1408.4	2548.7	31.0	482.6	1	1064.8	1315.3	2380.1	6.61
	40	131.5	190.4	3	1264.8	1407.5	2672.3	46.8	495.1	1	1169.8	1301.7	2471.5	7.51
	45	154.1	197.5	2	1390.7	1407.5	2798.2	63.5	507.0	1	1276.4	1291.9	2568.3	8.21

the buyer to joint in the collaborative system will increase in unstable situations.

The model can be further extended to some more practical situations, such as stochastic demand or stochastic price. Another extension that authors are working on is to consider more general distributions for lead times. Developing the model to the multi-vendor case is also proposed for future research.

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Appendix A.

Here, we prove that the Hessian matrix (H_M) as associated with TC_b is positive definite, and thus the objective function is strictly convex in r and Q. Consider H_M as follows:

$$H_M = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

where g_{ij} $(i,j=1,2)$ are the second partial derivations. Therefore, we have

$$g_{11} = \frac{\partial^2(TC_b)}{\partial r^2} = \frac{(h_b+\pi)}{Q}[e^{-r\lambda/D} - e^{-(r+Q)\lambda/D}]$$

$$g_{22} = \frac{\partial^2(TC_b)}{\partial Q^2} = \frac{2DA_b}{Q^3} + \frac{(h_b+\pi)e^{-r\lambda/D}}{\lambda^2 Q^3} \times [2D^2 - (D^2+(D+\lambda Q)^2)e^{-Q\lambda/D}]$$

$$g_{12} = g_{21} = \frac{\partial^2(TC_b)}{\partial r \partial Q} = \frac{(h_b+\pi)e^{-r\lambda/D}}{\lambda Q^2}[D-(D+\lambda Q)e^{-Q\lambda/D}]$$

Hessian determinants ($|H_1|$ and $|H|$) can be obtained as follows:

$$|H_1| = g_{11} > 0$$

$$|H| = g_{11}g_{22} - g_{12}g_{21} = \frac{2DA_b(h_b+\pi)}{Q^4}[e^{-r\lambda/D} - e^{-(r+Q)\lambda/D}] + \frac{(h_b+\pi)^2 e^{-2r\lambda/D}}{\lambda^2 Q^4}\chi$$

where $\chi = D^2(1+e^{-2Q\lambda/D} - 2e^{-Q\lambda/D}) - \lambda^2 Q^2 e^{-Q\lambda/D}$. Since the first expression of $|H|$ is positive, we just need to show that $\chi > 0$ to conclude that $|H|$ is positive definite.

Assuming $x = e^{-Q\lambda/D}$, we update χ as $\chi = D^2(1+x^2 - x\ln^2 x - 2x)$. Since $0 < Q < \infty$, thus $0 < x < 1$. Moreover, $(1-x)^2 - x\ln^2 x$ is a positive function of x in range of $0 < x < 1$. Consequently, χ is positive in range of $0 < Q < \infty$, and therefore TC_b is strictly convex.

Appendix B.

The Hessian matrix elements for $TC_J(r,Q)$ are the same as that found in Appendix A, except g_{22} that can be obtained as follows:

$$g_{22} = \frac{2D(A_b + A_v/n)}{Q^3} + \frac{(h_b+\pi)e^{-r\lambda/D}}{\lambda^2 Q^3} \times [2D^2 - (D^2+(D+\lambda Q)^2)e^{-Q\lambda/D}]$$

Thus, $|H_1|$ remains unchanged. Moreover, $|H|$ is updated as

$$|H| = \frac{2D(A_b + A_v/n)(h_b+\pi)}{Q^4}[e^{-r\lambda/D} - e^{-(r+Q)\lambda/D}] + \frac{(h_b+\pi)^2 e^{-2r\lambda/D}}{\lambda^2 Q^4}\chi$$

Again, it can be concluded that $|H|$ is positive definite, and therefore $TC_J(r,Q)$ is strictly convex.

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