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On a Boundary Value Problem in the Theory of Linear Water Waves

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A body Ω floating in a fluid is subjected to small periodic displacement. Under idealized conditions the resulting wave pattern can be described by a linear boundary value problem for the Laplacian in an unbounded domain with a non-coercive boundary condition on part of the boundary. Nevertheless uniqueness can be shown if Ω is confined to certain subsets of the fluid which can be described explicitly. This extends a result of V. G. Maz'ja saying that uniqueness holds provided that the exterior normal for $\partial\Omega$ avoids certain directions.

0. Introduction

Recently Angell *et al.* [1] considered a classical problem of John [3] from the viewpoint of 'optimal design': a body Ω floating in a fluid is subjected to small periodic displacement. Under idealized conditions the wave pattern created by this process can be described by a linear boundary value problem for the Laplacian. If the body is submerged completely (the case considered in [1]) John's classical uniqueness proof does not apply. However this case has been investigated by Maz'ja [4]. Roughly formulated Maz'ja's theorem states that uniqueness holds provided that a special vector field nowhere enters the body. This condition has been interpreted geometrically by Hulme [2] who also gave an interesting analysis of Maz'ja's method. It is the purpose of the present paper to enlarge the 'uniqueness class'. In fact, we shall show that uniqueness holds for bodies which are confined to certain subsets of the fluid. No other geometrical condition on Ω (or $\partial\Omega$) is needed. This will enlarge the class of geometries to which the 'optimal design' results of [1] can be applied.

Notation

(i) We use a summation convention: repeated indices in a product are to be summed from 1 to N.

(ii) $\langle , \rangle$ denotes the $\mathbb{C}^N$ scalar product:

$$\langle \mathbf{z}, \zeta \rangle := z_n \bar{\zeta}_n.$$

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(iii) $\mathbf{n}$ always denotes the *exterior* normal; thus the divergence theorem reads as follows:

$$\int_X \partial_i \mathbf{u}(\mathbf{x})\,d\mathbf{x} = \int_{\partial X} n_i(\boldsymbol{\xi})\mathbf{u}(\boldsymbol{\xi})\,d\boldsymbol{\xi}.$$

1. Problem formulation

Let us write

$$\mathbb{R}^N = \mathbb{R}^1 \oplus \mathbb{R}^{N-1}$$

and identify $\mathbb{R}^{N-1}$ with the $(N-1)$-dimensional subspace

$$F := \{\mathbf{x} \in \mathbb{R}^N | x_1 = 0\},$$

which represents the 'free surface' of the fluid. The fluid fills the strip

$$S := I \times \mathbb{R}^{N-1}, \qquad I := (-h, 0),$$

with the exception of the body Ω which is a bounded open subset of S. Thus our domain is $X := S \backslash \bar{\Omega}$, and ∂X has three components: the bottom $B := \{-h\} \times \mathbb{R}^{N-1}$, the free surface F and $\partial\Omega$. In [3] the physical background for the following boundary value problem is indicated:

$$\begin{aligned}
&\text{(a)} \quad \Delta \mathbf{u} &&= 0 \quad &&\text{in } X,\\
&\text{(b)} \quad N\mathbf{u} - k\mathbf{u} &&= 0 \quad &&\text{on } F,\\
&\text{(c)} \quad N\mathbf{u} &&= 0 \quad &&\text{on } B,\\
&\text{(d')} \quad N\mathbf{u} &&= \mathbf{g} \quad &&\text{on } \partial\Omega.
\end{aligned}$$

Here $N\mathbf{u} := \langle \mathbf{n}, \nabla \mathbf{u} \rangle$, where $\mathbf{n}$ is the exterior unit normal for X, i.e. $\mathbf{n}(0, \mathbf{x}') = \mathbf{e} := [1, 0, \ldots, 0]^T$, $\mathbf{n}(-h, \mathbf{x}') = -\mathbf{e}$ and $\mathbf{n}(\boldsymbol{\xi})$ is the *interior* normal for Ω if $\boldsymbol{\xi} \in \partial\Omega$. k is a given constant and $\mathbf{g}$ a given function. Of course, (a)–(d') is not a complete formulation of the boundary value problem. Two things are missing:

(i) *Formulation in suitable function spaces*: we shall not bother to write this down since any of the usual formulations—classical or variational—will serve our purpose.

(ii) *Description of the large-*$|\mathbf{x}|$ *asymptotics*: this has been given in considerable detail in [3]. For the convenience of the reader we give a short derivation of the correct 'radiation condition' by reduction to well known results for the Helmholtz equation: Choose R_0 such that

$$\Omega \subset Z := I \times K(0, R_0).$$

By regularity theory the restriction of $\mathbf{u}$ to $S \backslash Z$ is C^∞ (even analytic) and satisfies

$$\begin{aligned}
\partial_1 \mathbf{u}(0, \mathbf{x}') - k\mathbf{u}(0, \mathbf{x}') &= 0,\\
-\partial_1 \mathbf{u}(-h, \mathbf{x}') &= 0.
\end{aligned}$$

Therefore Sturm–Liouville theory shows

$$\mathbf{u}(x_1, \mathbf{x}') = \sum_{k=0}^{\infty} \mathbf{a}_k(\mathbf{x}') \cdot \psi_k(x_1) \tag{1}$$

where $\mathbf{a}_k(\mathbf{x}') := \int_I \mathbf{u}(t, x')\psi_k(t)\,dt$ and ψ_k is a complete orthonormal system of eigenfunctions for the operator

$$L\colon D(L) \subset L^2(I) \to L^2(I),$$
$$\psi \qquad \mapsto -\psi'',$$
$$D(L) := \{\psi \in H^2(I) | \psi'(-h) = 0 \wedge \psi'(0) - k\psi(0) = 0\}.$$

It happens that L has one negative eigenvalue $\lambda_0 := c_0^2$ where c_0 is the only real positive solution of

$$c\sinh(ch) = k\cosh(ch).$$

The other eigenvalues form a strictly increasing positive sequence

$$0 < \lambda_1 < \lambda_2 < \ldots$$

Therefore it seems natural to impose a Sommerfeld radiation condition as follows:

(e′) $\mathbf{Du} - ic_0\mathbf{u} \in L^2(S \setminus Z)$.

Here we introduce the following notation:

$$\mathbf{x} = [x_1, \mathbf{x}']^T, \quad x_1 \in \mathbb{R}, \quad \mathbf{x}' \in \mathbb{R}^{N-1},$$
$$r := r(\mathbf{x}) := |\mathbf{x}'| \qquad (!),$$
$$\boldsymbol{\omega} := \boldsymbol{\omega}(\mathbf{x}) := \nabla r(\mathbf{x}) = [0, \mathbf{x}'/r]^T,$$
$$\mathbf{Du}(\mathbf{x}) := \langle \boldsymbol{\omega}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x}) \rangle. \tag{2}$$

Using Fubini's theorem and the convergence of (1) both in $L^2(I)$ and in $D(L)$ we find

$$-\Delta'\mathbf{a}_k(\mathbf{x}') + \lambda_k\mathbf{a}_k(\mathbf{x}') = 0 \quad (\Delta' := \partial_2^2 + \cdots + \partial_N^2), \tag{3}$$
$$\mathbf{Da}_k - ic_0\mathbf{a}_k \in L^2(\mathbb{R}^{N-1} \setminus K(0, R_0)). \tag{4}$$

Thus $\mathbf{a}_0$ behaves like an outgoing solution of the Helmholtz equation. Now let us consider a solution to the homogeneous problem ($\mathbf{g} = 0$). We find for $R \geqslant R_0$:

$$\begin{aligned} 0 &= \int_{X \cap I \times K(0,R)} \Delta\mathbf{u}\cdot\bar{\mathbf{u}}\,d\mathbf{x} \\ &= \int_{\partial\Omega \cup B} N\mathbf{u}\cdot\bar{\mathbf{u}}\,d\mathbf{x} + \int_F N\mathbf{u}\cdot\bar{\mathbf{u}}\,d\mathbf{x} + \int_{I \times \partial K(0,R)} \mathbf{Du}\cdot\bar{\mathbf{u}}\,d\mathbf{x} \\ &= k\int_F |\mathbf{u}|^2\,d\mathbf{x} + ic_0\int_{I \times \partial K(0,R)} |\mathbf{u}|^2\,d\mathbf{x} + \int_{I \times \partial K(0,R)} (\mathbf{Du} - ic_0\mathbf{u})\bar{\mathbf{u}}\,d\mathbf{x}. \end{aligned}$$

From this

$$\int_{I \times \partial K(0,R)} |\mathbf{u}|^2\,d\mathbf{x} = -c_0^{-1}\,\mathrm{Im}\int_{I \times \partial K(0,R)} (\mathbf{Du} - ic_0\mathbf{u})\cdot\bar{\mathbf{u}}\,d\mathbf{x},$$

and therefore

$$\int_{\ldots} |\mathbf{u}|^2\,d\mathbf{x} \leqslant c_0^{-2}\int_{\ldots} |\mathbf{Du} - ic_0\mathbf{u}|^2\,d\mathbf{x}$$

which implies

$$\mathbf{u} \in L^2(S \backslash Z)$$

$$\mathbf{a}_k \in L^2(\mathbb{R}^{N-1} \backslash K(0, R_0)), \quad k = 0, 1, \ldots$$

Thus in particular (see [5], Theorem 4.2]) $a_0 = 0$ and by well-known asymptotics of the fundamental solution to $-\Delta + \lambda(\lambda \in \mathbb{R}^+) a_k$ decays exponentially for $k \geqslant 1$.

The previous discussion may be summarized as follows: we consider the problems

(P_N). Given $\mathbf{g}$ find $\mathbf{u}$ such that

(a) $\Delta \mathbf{u} = 0$ in $X \subset \mathbb{R}^N$

(b) $N\mathbf{u} - k\mathbf{u} = 0$ on F

(c) $N\mathbf{u} = 0$ on B

(d′) $N\mathbf{u} = \mathbf{g}$ on $\partial\Omega$

(e′) $\mathbf{D}\mathbf{u} - ic_0 \mathbf{u} \in L^2(X \backslash Z)$.

Lemma 1. *There exists* $\gamma \in \mathbb{R}^+$ *such that any solution to* (P_N) *with* $g = 0$, i.e.

(d) $N\mathbf{u} = 0$ *on* $\partial\Omega$

also satisfies

(e) $|\mathbf{u}(\mathbf{x})|, |\nabla \mathbf{u}(\mathbf{x})| = O(\exp(-\gamma|\mathbf{x}|)), \quad |\mathbf{x}| \to \infty.$

2. Maz'ja's technique

Consider a solution to (a)–(e). We want to show that $\mathbf{u} = 0$. By (e) the unboundedness of X is harmless. Rather we expect difficulties from the non-coercive boundary condition (b). These have been overcome by Maz'ja's technique, which is summarized in

Lemma 2 ([4], [2]). *Assumptions*:

(i) $\mathbf{u}$ *satisfies* (a)–(e)

(ii) $\mathbf{V} \in C^1(\bar{X}, \mathbb{R}^N)$

(iii) $V_1|_F = 0$

(iv) $\mathbf{V}$ *and* $\partial_i V_j$ *grow at most polynomially for* $|\mathbf{x}| \to \infty$.

Assertion:

$$\text{(v)} \int_X \langle \mathbf{Q}\nabla\mathbf{u}, \nabla\mathbf{u}\rangle \,\mathrm{d}\mathbf{x} - \int_F \left(\sum_{n=2}^{N} \partial_n V_n + 1\right) k|\mathbf{u}|^2 \,\mathrm{d}\mathbf{x} - \int_{B \cup \partial\Omega} \langle \mathbf{n}, \mathbf{V}\rangle |\nabla\mathbf{u}|^2 \,\mathrm{d}\mathbf{x} = 0,$$

where $\mathbf{Q}$ *is the real symmetric matrix with entries*

$$Q_{ij} := (\partial_k V_k + 1)\delta_{ij} - (\partial_i V_j + \partial_j V_i).$$

Proof. By a straightforward computation we have for any $\mathbf{V} \in C^1(X, \mathbb{R}^N)$ and any $\mathbf{u} \in C^2(X, \mathbb{C})$:

$$\partial_i\{\mathrm{Re}([2V_j\partial_j\mathbf{u} - \mathbf{u}]\partial_i\bar{\mathbf{u}}) - V_i|\nabla\mathbf{u}|^2\} = \mathrm{Re}([2V_j\partial_j\mathbf{u} - \mathbf{u}]\Delta\bar{\mathbf{u}}) - \langle Q\nabla\mathbf{u}, \nabla\mathbf{u}\rangle.$$

Thus using (i), (iii), (iv) and the divergence theorem we find

$$\int_X \langle \mathbf{Q}\nabla\mathbf{u}, \nabla\mathbf{u}\rangle\, d\mathbf{x} = -\int_{\partial X} \mathrm{Re}(2V_j\partial_j\mathbf{u} - \mathbf{u})N\bar{\mathbf{u}}\, d\xi + \int_{\partial X} \langle \mathbf{n}, \mathbf{V}\rangle |\nabla\mathbf{u}|^2\, d\xi$$

$$= \int_F - \mathrm{Re}(2V_j\partial_j\mathbf{u} - \mathbf{u})k\bar{\mathbf{u}}\, d\xi + \int_{B\cup\partial\Omega} \langle \mathbf{n}, \mathbf{V}\rangle\, |\nabla\mathbf{u}|^2\, d\xi.$$

Another partial integration on F—using (iii), (iv) and (e)—proves (v).

Lemma 2 is a powerful tool to prove uniqueness—all we have to do is to construct a vector field $\mathbf{V}$ such that $\mathbf{Q}_{|_X}$, $-(\partial_j V_j + 1)_{|_F}$ and $-\langle \mathbf{n}, \mathbf{V}\rangle_{B\cup\partial\Omega}$ are positive (semi-) definite. The construction will be simplified by the following:

Remark 1. Lemma 2 continues to hold if assumption (ii) is replaced by

(ii′) $\mathbf{V}\in C^0(\bar{X}, \mathbb{R}^N)$,
$\partial_j V_i$ is piecewise continuous,
$\mathbf{V}\in C^1$ near $\partial\Omega$.

This is easily seen by recalling that by regularity theory $\mathbf{u}\in C^\infty(\bar{X}\setminus\partial\Omega)$. Therefore (ii′) is enough to make the partial integrations which are needed in the proof of Lemma 2 possible.

3. The two-dimensional case

Owing to several simplifications, the main idea can be exhibited more clearly for problem (P_2). In this case

$$\mathbf{Q} = \mathbf{I} + \begin{bmatrix} -\alpha & -\beta \\ -\beta & \alpha \end{bmatrix},$$

where $\alpha := \partial_1 V_1 - \partial_2 V_2$, $\beta := \partial_1 V_2 + \partial_2 V_1$. Thus $1 \pm (\alpha^2 + \beta^2)^{1/2}$ are the eigenvalues of $\mathbf{Q}$ and we obtain

Lemma 3. $\mathbf{Q}$ *is positive semidefinite iff*

$$(\partial_1 V_2 + \partial_2 V_1)^2 + (\partial_1 V_1 - \partial_2 V_2)^2 \leqslant 1.$$

(*Side-*) *Remark* 2. Introduce $\mathbf{U} := [V_2, V_1]^{\mathrm{T}}$ and note that $\mathbf{Q}$ is positive semidefinite iff

$$|\mathrm{rot}\,\mathbf{U}|^2 + |\mathrm{div}\,\mathbf{U}|^2 \leqslant 1.$$

The special vector field used in [4] is homogeneous of degree 1. Let us make the same '*ansatz*'; noting that

$$(\partial_2 V_2 + 1)|_F \leqslant 0, \qquad V_1|_F = 0$$

suggests that we impose

$$\mathbf{V}(0, x_2) := -x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Thus our *ansatz* reads as follows:

$$V_i(\mathbf{x}) = \rho(\mathbf{x})v_i(\varphi(\mathbf{x})), \qquad i = 1, 2,$$

where

$$\rho(\mathbf{x}) := |\mathbf{x}|, \qquad \varphi(\mathbf{x}) := \arctan(x_1/x_2) \in [-\pi, 0].$$

Noting that

$$\begin{aligned} \partial_1 \rho &= \sin\varphi, & \partial_2 \rho &= \cos\varphi, \\ \partial_1 \varphi &= \rho^{-1}\cos\varphi, & \partial_2 \varphi &= -\rho^{-1}\sin\varphi, \end{aligned}$$

we find

$$\begin{aligned} \partial_1 V_i &= v_i \sin\varphi + v_i' \cos\varphi, \\ \partial_2 V_i &= v_i \cos\varphi - v_i' \sin\varphi, \\ \alpha &= (v_2' + v_1)\sin\varphi + (v_1' - v_2)\cos\varphi, \\ \beta &= (v_2' + v_1)\cos\varphi - (v_1' - v_2)\sin\varphi, \\ \alpha^2 + \beta^2 &= (v_2' + v_1)^2 + (v_1' - v_2)^2. \end{aligned}$$

We are thus led to the problem

(V). Find $v_i \in C^1(J, \mathbb{R})$, $J := [-\varphi_2, 0]$ satisfying

$$\begin{aligned} &v_1(0) = 0, \quad v_2(0) = -1, \\ &v_1(-\varphi_2) = v_2(-\varphi_2) = 0, \\ &(v_2' + v_1)^2 + (v_1' - v_2)^2 \leqslant 1. \end{aligned}$$

Our goal is to minimize φ_2. This is a problem of time-optimal control:

(V′) Find $\mathbf{u} \in U := \{\mathbf{u} \in L^2(J) \mid |\mathbf{u}|^2 \leqslant 1\}$ such that the solution to

$$\mathbf{v}(0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\mathbf{v}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{v} + \mathbf{u}$$

satisfies $\mathbf{v}(\varphi_2) = 0$ and φ_2 is minimized.

Pontryagin's maximum principle yields the following solution:

$$\mathbf{v}(\varphi) = -(\varphi + 1)\begin{bmatrix} \sin\varphi \\ \cos\varphi \end{bmatrix}. \tag{5}$$

(An elementary derivation of (5) can be obtained by observing that

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}|\mathbf{v}(\varphi)|^2 = 2\langle \mathbf{u}(\varphi), \mathbf{v}(\varphi)\rangle$$

will be maximized for $\mathbf{u} := \mathbf{v}/|\mathbf{v}|$. The corresponding ordinary differential equation may be solved by the *ansatz*

$$\mathbf{v}(\varphi) = R(\varphi)\begin{bmatrix} \sin\Phi(\varphi) \\ \cos\Phi(\varphi) \end{bmatrix}.\Bigg)$$

From (5) we find $\varphi_2 = 1$ and by symmetry there is an analogous solution in $[-\pi, -\pi+1]$. We define

$$C_2 := \{\mathbf{x}\in\mathbb{R}^2 \mid \varphi(\mathbf{x})\in(-\pi+1, -1)\}$$

and

$$\mathbf{V}(\mathbf{x}) := -|\mathbf{x}|\cdot \begin{cases} (\varphi(\mathbf{x})+1)\begin{bmatrix}\sin\varphi(\mathbf{x})\\ \cos\varphi(\mathbf{x})\end{bmatrix}, & \varphi(\mathbf{x})\in[-1,0] \\ 0, & \mathbf{x}\in C_2 \\ (-\pi+1-\varphi(\mathbf{x}))\begin{bmatrix}\sin(\varphi(\mathbf{x})+\pi)\\ \cos(\varphi(\mathbf{x})+\pi)\end{bmatrix}, & \varphi(\mathbf{x})\in[-\pi, -\pi+1] \end{cases}$$

$$= \begin{cases} -(\varphi(\mathbf{x})+1)\cdot\mathbf{x}, & \varphi(\mathbf{x})\in[-1,0] \\ 0, & \mathbf{x}\in C_2 \\ (\varphi(\mathbf{x})+\pi-1)\cdot\mathbf{x}, & \varphi(\mathbf{x})\in[-\pi, -\pi+1]. \end{cases}$$

By Remark 1 we may apply Lemma 2 to $\mathbf{V}$ and the corresponding $\mathbf{Q}$ is positive semidefinite by construction. Furthermore

$$\mathbf{V}(0, x_2) = \begin{bmatrix} 0 \\ -x_2 \end{bmatrix}$$

implies

$$(\partial_2 V_2 + 1)|_F = 0.$$

Finally

$$\langle \mathbf{V}, \mathbf{n}\rangle|_B = -V_1|_B \leqslant 0.$$

Thus each of the three integrals appearing in Lemma 2(v) must vanish. On $B\setminus\bar{C}_2$ we have

$$\langle \mathbf{n}(-h, x_2), V(-h, x_2)\rangle = -V_1(-h, x_2)$$
$$= -h\begin{Bmatrix} -\varphi(\mathbf{x})-1, & \varphi(\mathbf{x})\in(-1,0) \\ \varphi(\mathbf{x})+\pi-1, & \varphi(\mathbf{x})\in(-\pi, -\pi+1) \end{Bmatrix} < 0.$$

Therefore

$$\int_B \langle \mathbf{n}, \mathbf{V}\rangle |\nabla\mathbf{u}|^2\,d\mathbf{x} = 0$$

implies

$$\nabla\mathbf{u}|_{B\setminus\bar{C}_2} = 0, \tag{6}$$

hence

$$\mathbf{u}|_{B\setminus\bar{C}_2} = \text{const}.$$

From (e) we find $\mathbf{u}|_{B\setminus\bar{C}_2} = 0$, which (together with (6)) shows that the Cauchy data of $\mathbf{u}$ vanish at $B\setminus\bar{C}_2$. Therefore $\mathbf{u} = 0$ by unique continuation. We have proved

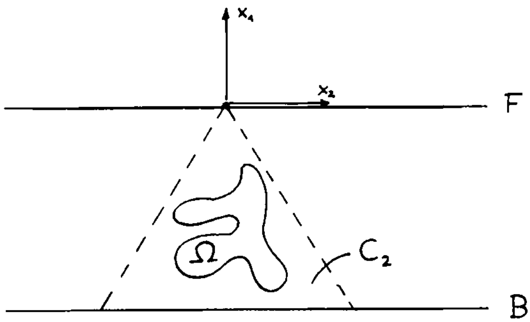

Fig. 1

Theorem 1. *Problem* (P_2) *has at most one solution if* Ω *is a subset of the cone* (Fig. 1)

$$C_2 := \{\mathbf{x} \in S \mid -\pi + 1 < \varphi(\mathbf{x}) < -1\}.$$

Remark 3.

$$C_2 = \left\{\mathbf{x} \,\middle|\, |x_2| < -\frac{x_1}{a}\right\}$$

where $a := \tan 1 \approx 1.6$.

Remark 4. $\mathbf{V}$ being identically zero on $\bar{\Omega}$, we do not need any regularity of $\mathbf{u}$ near $\partial\Omega$. Therefore Theorem 1 continues to hold if no regularity whatsoever is assumed for $\partial\Omega$, and Neumann's boundary condition (d′) is formulated in a weak sense (cf. [5], p. 53). In fact, the boundary condition on $\partial\Omega$ is needed in the proof of Lemma 1, only. The proof of Theorem 1 shows that any $\mathbf{u}$ satisfying (a)–(c) and (e) (no boundary condition on $\partial\Omega$) must vanish.

4. The N-dimensional case

Let us note to begin with that neglecting all but two dimensions and applying Theorem 1 gives the following non-trivial result:

Theorem 2. *Problem* (P_N) *has at most one solution if* Ω *is contained in some set* (Fig. 2)

$$W := \{\mathbf{x} \mid |\langle \mathbf{x} - \hat{\mathbf{x}}, \mathbf{z}\rangle| < -a^{-1}x_1\}$$

where $a := \tan 1$ *and* $\mathbf{z}, \hat{\mathbf{x}} \in F$, $|\mathbf{z}| = 1$.

We expect uniqueness for bodies confined to cones with angle of aperture greater than a^{-1} if we treat the $\mathbf{x}'$-variables symmetrically. Thus, let us make the '*ansatz*'

$$\mathbf{V}(\mathbf{x}) = W_1(x_1, r)\mathbf{e} + W_2(x_1, r)\boldsymbol{\omega}$$

using the notation (2) and $\mathbf{e} := [1, 0, \ldots, 0]^T$.

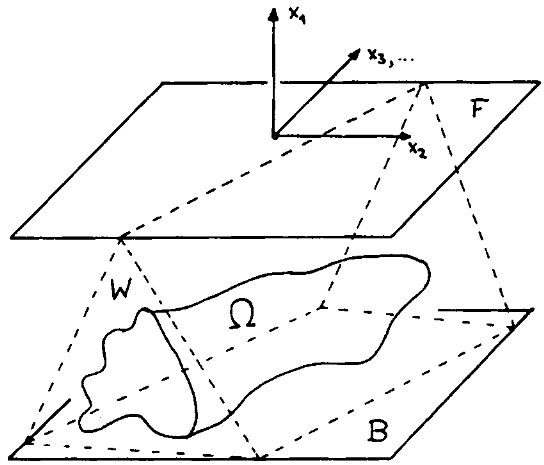

Fig. 2

For n, $m \geqslant 2$ we have

$$\partial_1 r = \partial_1 \omega_n = 0,$$

$$\partial_n r = \omega_n,$$

$$\partial_n \omega_m = r^{-1}(\delta_{nm} - \omega_n \omega_m),$$

$$\partial_1 V_1 = \partial_1 W_1,$$

$$\partial_n V_1 = (\partial_r W_1)\omega_n,$$

$$\partial_1 V_n = (\partial_1 W_2)\omega_n,$$

$$\partial_m V_n = \partial_r W_2 \omega_m \omega_n + r^{-1} W_2(\delta_{nm} - \omega_n \omega_m),$$

$$\operatorname{div} \mathbf{V} = \partial_1 W_1 + \partial_r W_2 + (N-2)r^{-1} W_2.$$

The symmetrized derivative

$$S := S(\mathbf{V}) := [\partial_i V_j + \partial_j V_i]_{i,j=1,\ldots,N}$$

(viewed as an operator in $\mathbb{R}^N$) is described completely by the following identities:

$$S\mathbf{e} = 2(\partial_1 W_1)\mathbf{e} + (\partial_1 W_2 + \partial_r W_1)\omega,$$

$$S\boldsymbol{\omega} = (\partial_1 W_2 + \partial_r W_1)\mathbf{e} + 2(\partial_r \mathbf{W})\boldsymbol{\omega},$$

$$S\boldsymbol{\tau} = 2(r^{-1} W_2)\boldsymbol{\tau} \quad \text{for } \boldsymbol{\tau} \in T := \{\mathbf{e}, \boldsymbol{\omega}\}^{\perp}.$$

Recalling that $\mathbf{Q} = (\operatorname{div} \mathbf{V} + 1)\mathbf{I} - S$ we obtain

Lemma 4.

(i) $\mathbf{Q}$ *is reduced by the orthogonal decomposition*

$$\mathbb{R}^N = U + T, \qquad U := T^{\perp}$$

(ii) $\mathbf{Q}|_T = (\partial_1 W_1 + \partial_r W_2 + 1 + (N-4)r^{-1} W_2)\mathbf{id}|_T.$

The matrix representation of $\mathbf{Q}|_U$ *for the basis* $\{\mathbf{e}, \boldsymbol{\omega}\}$ *is*

$$((N-2)r^{-1}W_2+1)\mathbf{I}+\begin{bmatrix} -\alpha & -\beta \\ -\beta & \alpha \end{bmatrix}$$

where $\alpha := \partial_1 W_1 - \partial_r W_2$, $\beta := \partial_1 W_2 + \partial_r W_1$.

(iii) $\mathbf{Q}$ *is positive semidefinite iff*

$$\partial_1 W_1 + \partial_r W_2 + 1 + (N-4)r^{-1}W_2 \geqslant 0$$

and

$$(N-2)r^{-1}W_2 + 1 - (\alpha^2+\beta^2)^{1/2} \geqslant 0.$$

Let us use the pattern which was successful in $\mathbb{R}^2$ and put

$$\varphi := \varphi(\mathbf{x}) := \arctan(x_1/r(\mathbf{x})) \in [-\pi, 0].$$

We make the '*ansatz*'

$$V(\mathbf{x}) = -f(\varphi(\mathbf{x}))(x_1\mathbf{e}+r(\mathbf{x})\boldsymbol{\omega}(\mathbf{x})) =: W_1\mathbf{e} + W_2\boldsymbol{\omega}.$$

We have

$$\sum_{n=2}^{N} \partial_n V_n(0, \mathbf{x}') = -f(0)(N-1)$$

which leads to the condition

$$f(0) = \frac{1}{N-1}.$$

Using the identities

$$x_1\partial_1\varphi = \sin\varphi\cos\varphi, \quad r\partial_1\varphi = \cos^2\varphi,$$
$$x_1\partial_r\varphi = \sin^2\varphi, \qquad r\partial_r\varphi = -\sin\varphi\cos\varphi,$$

we obtain

$$\partial_1 W_1 = -f'\sin\varphi\cos\varphi - f,$$
$$\partial_r W_1 = f'\sin^2\varphi,$$
$$\partial_1 W_2 = -f'\cos^2\varphi,$$
$$\partial_r W_2 = f'\sin\varphi\cos\varphi - f,$$
$$\alpha = \partial_1 W_1 - \partial_r W_2 = -f'\sin(2\varphi),$$
$$\beta = \partial_1 W_2 + \partial_r W_1 = -f'\cos(2\varphi).$$

Noting that $r^{-1}W_2 = -f$ we find by Lemma 4(iii) that $\mathbf{Q}$ will be positive semidefinite if

$$-2f + 1 + (N-4)(-f) = 1-(N-2)f \geqslant 0, \tag{7}$$

$$-(N-2)f + 1 - |f'| \geqslant 0. \tag{8}$$

Condition (8) will be satisfied if $f = f_N$, where f_N is the solution of the initial value problem

$$f_N' = 1 - (N - 2)f_N, \quad f_N(0) = \frac{1}{N-1}$$

in the interval $[-\varphi_N, 0]$, i.e.

$$f_N(\varphi) = \frac{1}{N-2}\left[1 - \frac{1}{N-1}\exp(-(N-2)\varphi)\right] \tag{9}$$

where φ_N is the only zero of f_N in $(-\infty, 0)$, namely

$$\varphi_N = -(N-2)^{-1}\ln(N-1). \tag{10}$$

We see that (7) is satisfied automatically because f_N is increasing in $[-\varphi_N, 0]$ and $f_N(0) = 1/N - 1$.

We obtain

Theorem 3. *Problem* $(\mathbf{P}_N)$ *has at most one solution if* Ω *is contained in the cone*

$$C_N := \left\{\mathbf{x} \,\middle|\, |\mathbf{x}'| < -\frac{x_1}{\tan\varphi_N}\right\}.$$

Proof. Using Remark 1 we put

$$\mathbf{V}(\mathbf{x}) := f_N(\varphi(\mathbf{x}))[-x_1\mathbf{e} - r(\mathbf{x})\boldsymbol{\omega}(\mathbf{x})]$$

outside C_N and extend it by zero to $\bar{X}$. By construction the corresponding $\mathbf{Q}$ is positive semidefinite.

Furthermore

$$-\sum_{n=2}^{N} \partial_n V_n(0, \mathbf{x}') - 1 = 0$$

and, for $\mathbf{x} = [-h, \mathbf{x}']^{\mathrm{T}} \in \mathbf{B}$,

$$\begin{aligned} -\langle \mathbf{n}(\mathbf{x}), \mathbf{V}(\mathbf{x})\rangle &= V_1(\mathbf{x}) \\ &= \begin{Bmatrix} f_N(\varphi(-h, \mathbf{x}'))h, & \mathbf{x} \notin C_N \\ 0, & \mathbf{x} \in C_N \end{Bmatrix} \geqslant 0. \end{aligned}$$

Noting that $V_1(\mathbf{x}) > 0$ for $\mathbf{x} \in B\backslash\bar{C}_N$ we obtain $\mathbf{u} = 0$ just as in the proof of Theorem 1.

As a final remark let us note that several generalizations (variable coefficients, perturbations of S, etc.) are possible.

Added in proof. By a different method Simon and Ursell [6] obtained a stronger result in the special case $N = 2$.

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