ON THE TRACE ANOMALY OF THE SUPERSYMMETRIC CP($N-1$) MODEL

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The trace of the effective energy–momentum tensor $T^{\rm eff}_{\mu\nu}$ of the supersymmetric (SUSY) CP($N-1$) model is calculated within the $1/N$ expansion. Because dimensional transmutation breaks the superconformal invariance of the classical theory $T^{\rm eff}_{\mu\nu}$ is not traceless. Additionally the calculation shows that there appears an anomalous term in the trace – the trace anomaly. Minkowski's conjecture, which states that the trace anomaly of the energy–momentum tensor is proportional to the lagrangian multiplied by the renormalization group function $\beta(\gamma)$, is verified.

In a massless classical field theory involving only dimensionless parameters it is possible to define an energy–momentum tensor with vanishing trace. In general this is not possible for the corresponding quantized theory. The non-vanishing trace, which cannot be eliminated by a redefinition of the energy–momentum tensor, is called the trace anomaly.

Now, quantization by $1/N$ expansion has proved to be a useful method for investigating a number of field theoretic models beyond the reach of ordinary perturbation theory. The incorporation of operator products and related objects is for many general discussions an unavoidable task. Particularly the understanding of spontaneous SUSY breaking by non-perturbative effects or the appearance of anomalies make indispensable a proper construction of such currents like the energy–momentum tensor in this approximation. Therefore an efficient method has been presented by the authors of ref. [1] to investigate operator product insertions within the $1/N$ expansion. It was successfully tested for the O(N) – Φ^4 model [2] and also for the CP($N-1$) model [3]. Applying this method to the SUSY CP($N-1$) model I expect

(i) No SUSY breaking, i.e. $\langle 0|T^{\rm eff}_{\mu\nu}|0\rangle = 0$ at least around the saddle point $\langle\Sigma\rangle = -\mathrm{i}N^{1/2}m$, $m^2 = \mu^2 \times \exp(-4\pi/\gamma)$, $\langle A_a\rangle = 0$;

(ii) The appearance of "hard terms" and eventually anomalous terms in the trace of $T^{\rm eff}_{\mu\nu}$ (for a complete discussion of anomalies in SUSY theories cf. refs. [4–6]). In passing I will remark that the vector field of the gauge superfield A_a has no degrees of freedom (because we are in two dimensions), yet it gives rise to a non-zero trace, if one does not use a classical gauge invariant and improved energy–momentum tensor. Doing so there is no contribution from the action of the gauge field to the anomalous trace.

Furthermore, I briefly discuss a conjecture made by Minkowski about the general form of the trace anomaly [7]. This conjecture states that in a massless model involving a single dimensionless coupling parameter the trace of the energy–momentum tensor should be proportional to the lagrangian multiplied by the renormalization group function $\beta(\gamma)$. This result is valid for gauge theories and – remarkably enough – carries over to SUSY quantum electrodynamics [8]. Thus Minkowski's conjecture seems to be valid for a large class of gauge field theories. For non-gauge theories, like the Φ^4 model [1,2,7] or the CP($N-1$) model [3], the situation is more intricate, but if one chooses the "right" improved energy–momentum tensor one finds $T_{\mu\mu} \propto \beta(\gamma)\,\partial\mathcal{L}/\partial\gamma$ + mass terms.

Let me start with a short description of the SUSY CP($N-1$) model (for details cf. refs. [9–11]). Formulated fully in terms of non-extended superfields one obtains as a classical gauge invariant action for the SUSY CP($N-1$) model

$$\Gamma^{(0)} = -\frac{1}{4}\int \mathrm{d}V\,\{\overline{\nabla\Phi}\gamma^5\nabla\Phi - (2\mathrm{i}\Sigma/\sqrt{N})(\bar{\Phi}\Phi - N/\gamma)\}. \tag{1}$$

Φ_k is an SU(N) matter superfield

$$\Phi_k = Z_k + i\theta\psi_k + \tfrac{1}{2}\,\theta\gamma^5\theta F_k, \tag{2}$$

subject to the constraint

$$\bar{\Phi}\Phi = N/\gamma. \tag{3}$$

∇_a is the gauge and SUSY covariant derivative, acting on Φ_k:

$$\nabla_a = \mathrm{D}_a - (1/\sqrt{N})A_a, \quad \bar{\nabla}_a = -\mathrm{D}_a - (1/\sqrt{N})A_a,$$

$$\mathrm{D}_a = \partial/\partial\theta_a + i(\not\partial\,\theta)_a, \tag{4}$$

and A_a is the gauge superfield (which transforms under SUSY as a scalar superfield!)

$$A_a = \chi_a + \theta_b(\gamma_\mu V_\mu + i\gamma^5 S + P)_{ba} + \tfrac{1}{2}\,\theta\gamma^5\theta\lambda_a. \tag{5}$$

The constraint (3) is built into the action via a Lagrange multiplier field Σ which is an SU(N) singlet:

$$\Sigma = \rho + i\theta\xi + \tfrac{1}{2}\,\theta\gamma^5\theta\sigma. \tag{6}$$

To set up a suitable perturbation scheme for this model – the $1/N$ expansion – I use the saddle-point method. Be $Z[j]$ (j stands collectively for all sources) the generating functional after integrating out the matter fields Φ and $\bar{\Phi}$

$$Z[j] = \exp(Z_c[j]) = \int [\mathrm{d}A_\alpha][\mathrm{d}\Sigma]\,\exp(-S_{\mathrm{eff}} + \bar{j}L_j^{-1} - \eta A - j_\Sigma\Sigma), \tag{7}$$

with

$$S_{\mathrm{eff}} = N\,\mathrm{Tr}\ln(1-\Delta M) - (i\sqrt{N}/2\gamma)\int \mathrm{d}V\,\Sigma,$$

$$L_{12} = -\tfrac{1}{4}[\mathrm{D}\gamma^5\mathrm{D} + (1/\sqrt{N})\,\mathrm{D}\gamma^5 A + (1/\sqrt{N})\,A\gamma^5\mathrm{D} + (1/N)\,A\gamma^5 A - 2i\Sigma/\sqrt{N}]_2\,\delta_{12},$$

$$L_{12} = K_{12} + M_{12} = K_{13}(\delta_{32} - \Delta_{34}M_{42}). \tag{8}$$

The vertex functional

$$\Gamma = Z_c[j] - j\Phi \tag{9}$$

reads

$$\Gamma = S_{\mathrm{eff}} + \bar{\Phi}L\Phi. \tag{10}$$

In this way one discusses the Green functions of ordinary fields. The next step is to extend the discussion to Green functions involving insertions of renormalizable operator products like the energy–momentum tensor $T_{\mu\nu}$. To this end one introduces a source $\lambda_{\mu\nu}$ for $T_{\mu\nu}$ and inserts $\int\lambda_{\mu\nu}T_{\mu\nu}$ into the path-integral. Consequently the generating functional (7) is replaced by

$$Z[j,\lambda_{\mu\nu}] = \int [\mathrm{d}A_a][\mathrm{d}\Sigma]\,\exp(-S^\lambda_{\mathrm{eff}} + \bar{j}L^{-1}j + \eta A + j_\Sigma\Sigma), \tag{11}$$

with

$$S^\lambda_{\mathrm{eff}} = N\,\mathrm{Tr}\ln[L - T(\lambda)] - (i\sqrt{N}/2\gamma)\int \mathrm{d}V\,(\Sigma - \mathrm{D}\gamma^5\mathrm{D}\Sigma\cdot\lambda). \tag{12}$$

$T_{\mu\nu}$ is the classical gauge invariant and improved energy–momentum tensor as constructed in ref. [12]:

$$T_{\mu\nu} = -2i(\gamma_\nu\mathrm{D})_a\,[\overline{\nabla_\mu\Phi}\nabla_a\Phi - \overline{\nabla_a\Phi}\nabla_\mu\Phi] - \delta_{\mu\nu}\mathrm{D}\gamma^5\mathrm{D}\mathcal{L},$$

$$\nabla_\mu = \partial_\mu + \tfrac{1}{2}i(\mathrm{D}\gamma_\mu A). \tag{13}$$

As already mentioned there is no principal need to use the improved $T_{\mu\nu}$ at the classical level; but switching on radiative corrections only the improved and gauge invariant $T_{\mu\nu}$ yields finite matrix elements. Properly performing the Legendre transformation one obtains for the effective action

$$\Gamma(\Phi,\bar{\Phi},A_\alpha,\Sigma) = \Phi[L - T(\lambda)]\,\Phi + S^\lambda_{\mathrm{eff}}. \tag{14}$$

Since the first term only describes effects of the tree-approximation I restrict myself to $\Phi = 0$. Hence $\delta\Gamma(\Phi=0, A_a, \Sigma, \lambda_{\mu\nu})\,|\,\delta\lambda_{\mu\nu}$ represents the generating functional with one $T_{\mu\nu}$ insertion. In more detail one finds:

$$\begin{aligned}
\delta\Gamma(\Phi=0,\Sigma,A_a)/\delta\lambda_{\mu\nu}|_{\lambda_{\mu\nu}=0} &:= T^{\mathrm{eff}}_{\mu\nu,1}\\
&= -\tfrac{1}{2}iN(\gamma_\nu\mathrm{D})^1_a\{[(\nabla^1_\mu\nabla^1_a + \nabla^1_a\nabla^1_\mu)L^{-1}_{12}]_{2=1}\\
&\quad - \partial^1_\mu(\nabla^1_a L^{-1}_{12})_{2=1} - \mathrm{D}^1_a(\nabla^1_\mu L^{-1}_{12})_{2=1}\}\\
&\quad - \tfrac{1}{4}N\delta_{\mu\nu}(\mathrm{D}\gamma^5\mathrm{D})_1\{[(\nabla\gamma^5\nabla)_1 L^{-1}_{12}]_{2=1}\\
&\quad - \mathrm{D}^1_a(\gamma^5_{ab}\nabla^1_b L^{-1}_{12})_{2=1} - L^{-1}_{11}\,2i\Sigma_1/\sqrt{N} - 2i\Sigma_1/\gamma\sqrt{N}\}
\end{aligned} \tag{15}$$

This effective energy–momentum tensor is conserved by virtue of the equations of motion [9]:

$$\partial^1_\mu T^{\mathrm{eff}}_{\mu\nu,1} = 0. \tag{16}$$

But $T^{\rm eff}_{\mu\nu}$ is not traceless, because superconformal invariance is broken by dimensional transmutation. Evaluating the vacuum expectation value (VEV) of (15) around the saddle point

$$\langle\Sigma\rangle = -\,\mathrm{i}m\sqrt{N}, \quad \langle A_a\rangle = 0, \tag{17}$$

I get

$$\langle 0|T^{\rm eff}_{\mu\nu}|0\rangle = 0, \tag{18}$$

and hence this is an explicitly SUSY phase of the model. The contributions to $T^{\rm eff}_{\mu\nu}$ with one exterior Σ- or A_a-leg are suppressed by a factor $N^{-1/2}$, but these are the only ones which contain divergent parts possibly yielding anomalous contributions to the trace. These diagrams can be obtained by calculating

$$\delta T^{\rm eff}_{\mu\nu,1}(A_a,\Sigma)/\delta\Sigma_5\Big|_{\substack{\Sigma=-\mathrm{i}m\sqrt{N}\\ A_a=0}} = \langle 0|\mathrm{T}\,T^1_{\mu\nu}\Sigma^5|0\rangle \tag{19}$$

and

$$\delta T^{\rm eff}_{\mu\nu,1}(A_a,\Sigma)/\delta A_{a,5}\Big|_{\substack{\Sigma=-\mathrm{i}m\sqrt{N}\\ A_a=0}} = \langle 0|\mathrm{T}\,T^1_{\mu\nu}A^5_a|0\rangle. \tag{20}$$

I will give the result for these two VEVs without quoting details of the tedious, but straightforward calculation; eq. (19) reads

$$\langle 0|\mathrm{T}\,T^1_{\mu\mu}\Sigma^5|0\rangle = \mathrm{i}\sqrt{N}\exp(\theta_1\not{p}\,\theta_5)$$
$$\times[1/4\pi - m^2\,\Pi(p^2) - \tfrac14\theta_{15}\gamma^5\theta_{15}mp^2\,\Pi(p^2)], \tag{21}$$

whilst (20) explicitly gives zero:

$$\langle 0|\mathrm{T}T^1_{\mu\mu}A^5_a|0\rangle = 0. \tag{22}$$

$\Pi(p^2)$ is the well-known integral

$$\Pi(p^2) = \frac{1}{16\pi^2}\int \mathrm{d}^2k\,\{[(p+k)^2+m^2](k^2+m^2)\}^{-1}$$
$$= 4\pi[(p^2+4m^2)p^2]^{-1/2}$$
$$\times\ln\{[(p^2)^{1/2}+(p^2+4m^2)^{1/2}]/2m^2\}. \tag{23}$$

As already mentioned before there is no contribution to a possible anomalous term from diagrams with two exterior legs, and hence (21) is indeed the final result. Rephrasing (21) in a perhaps more familiar form ($T_{\mu\mu}$ = contact terms + anomalous terms) I can write

$$T^{\rm eff}_{\mu\mu} = -\,\mathrm{D}\gamma^5\mathrm{D}(\Sigma\delta\Gamma/\delta\Sigma + \tfrac12 A_a\delta\Gamma/\delta A_a) - (\sqrt{N}/4\pi)\,\Sigma. \tag{24}$$

(24) includes an additional term which was not present in the classical theory. Since it is not feasible to compensate it by a redefinition of $T^{\rm eff}_{\mu\nu}$ it has to be identified with the trace anomaly. Moreover, recalling the $\beta(\gamma)$ function of the SUSY CP($N-1$) model,

$$\beta(\gamma) = -\,\gamma^2/2\pi, \tag{25}$$

one can write instead of (24)

$$T^{\rm eff}_{\mu\mu} = -\,\mathrm{D}\gamma^5\mathrm{D}(\Sigma\delta\Gamma/\delta\Sigma + \tfrac12 A_a\delta\Gamma/\delta A_a)$$
$$+\,\beta(\gamma)\,\partial\mathcal{L}_{\rm eff}/\partial\gamma, \tag{26}$$

and thus Minkowski's conjecture is also valid for the SUSY CP($N-1$) model. In my opinion further studies would be of interest to see whether it applies to all field models universally.

The method used in this paper to analyse the energy–momentum tensor can be easily generalized to any other (conformal) current of this model. Most interesting will be the analysis of the supercurrent $V_{a\rho}$, which can be written (classically)

$$V_{a\rho} = S_{a\rho} + 2\mathrm{i}(\gamma_\mu\theta)_a\,T_{\rho\mu}, \tag{27}$$

where $S_{a\rho}$ is the SUSY current of ref. [12], and the Adler–Bardeen theorem (cf. for instance ref. [6]).

A further step would be to discuss (3 + 1)-dimensional SUSY CP($N-1$) models within the framework given in this paper and to compare the results with the ones obtained in ref. [13].

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