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# A FE methodology for the static analysis of infinite periodic structures under general loading

**E. Moses, M. Ryvkin, M. B. Fuchs**



**Abstract** This paper presents a finite element methodology for the static analysis of infinite periodic structures under arbitrary loads. The technique hinges on the method of representative cell which through the discrete Fourier transform reduces the original problem to a boundary value problem defined over one module, the representative cell. Starting from the weak form of the transformed problem, or from the FE equations of the infinite structure, the equilibrium equations are written in terms of the complex-valued displacement transforms which are considered as the displacements in the representative cell. Having found the displacements in the transformed domain, the real displacements anywhere in the real structure are obtained by numerical integration of the inverse transform. The theory, which is valid for spatial structures with 1D up to 3D translational symmetry, is illustrated with examples of periodic structures having 1D translational symmetry under general static loading.

## 1 Introduction

The spatially periodic elastic continua or discrete systems considered in this paper are composed of a basic unit (bay, cell, module) which is duplicated translationally in one, two or three directions. For an infinite repetitive structure, if the loading is of the same periodicity as the structure then the stresses and displacements are also periodic and the analysis of the infinite structure can be reduced to a problem defined over a single cell with cyclic boundary conditions. The results are then identically repeated for the rest of the structure.

When structures are submitted to spatially repetitive loading it makes engineering sense to design them with the same periodicity. In practice, however, one encounters periodic structures even when they support non-periodic loading. This is mainly due to increased emphasis on the standardization of the manufacturing process and to the ease of the assembly of the repetitive modules which are produced in large series. Another reason for designing periodic structures is when the position of the loading is arbitrary and where there is consequently no apparent justification for having one part of the assembly different from another part.

This paper is concerned with the static analysis of infinite repetitive structures under general loading, also known as quasi-periodic problems. The structure is periodic but neither is the loading nor are the internal stresses and displacements. Such infinite models are applicable to the analysis of large finite repetitive structures with localized deformation fields, when the edge effects are negligible. In particular, this work proposes a general Finite Element (FE) methodology for dealing with such cases. As will be seen, the method uses the discrete Fourier transform (DFT) to reduce the infinite problem to a transformed boundary value problem defined on a single module, which is then solved by FEA (FE analysis).

The bulk of work in analysis of repetitive structures which takes into consideration the periodic nature of the structure was performed in dynamic loading. The review paper (Li and Benaroya, 1992) gives an extensive overview of techniques which solve such problems. It will be noticed, in particular, that by using Bloch wave theory the dynamic analysis of a non-periodic state of stress in periodic structures is performed by solving problems defined over a single cell. The Bloch waves method, originally developed for problems in solid state physics (Brillouin, 1953), was successfully employed for the solution of pure structural dynamic problems. In Langley (1996) and Langley et al. (1997) a 2D infinite grillage was subjected to a harmonic force. Due to damping the response was non-harmonic and in fact localized. By means of Bloch theory the structure was analyzed by considering a single period. In Allaire et al. (1998) Bloch wave theory in conjunction with a homogenization technique is used in the study of large array of spatially periodic immersed elastic rods. The reader is also referred to Conca et al. (1995). Bloch waves are similar to the DFT method from a mathematical point of view and provide a physical meaning to the formal Fourier transform. Each value of the transform parameter corresponds to a harmonic wave propagating in the periodic structure.

Several authors introduce the DFT explicitly such as in problems on fluid–solid interactions for repetitive structures in Eatwell and Willis (1982) and in the dynamic analysis of mechanical systems as applied to fracture mechanics (Slepyan, 1974, 1988). Examples of similar ideas can be found in other fields of mathematical physics as in Baklanov (1964) which deals with the radiation from a system of semi-infinite thin plates.

For the static analysis of repetitive structures the most developed procedure uses finite-difference equations

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(Renton, 1964a), which are elaborations of the three moments equation, whose inception can be traced to Clapeyron (see Gutkowski, 1974). The technique consists in reducing the infinite set of equations to characteristic equations which are then solved by an appropriate method. This enables analytical solutions for a host of reticulated structures, usually of finite dimension, or specific 'waffle' discrete-continuous elastic systems. A study of the stability of infinite translationally symmetric structures based on the finite difference method was performed in Renton (1964b), and an excellent review and examples of several finite difference-based techniques applied to regular elastic lattice systems can be found in Gutkowski (1974). The interested reader is also referred to monographs on different aspects of the subject (Wah and Calcote, 1970; Dean, 1976).

A more general approach to the analysis of symmetric systems is based on group representation theory. It allows to split the original problem into more simple ones as by block-diagonalization of the stiffness matrix. Most works considered finite groups related to rotational symmetry (Bossavit, 1986; Dinkevich, 1990) although translational symmetry was treated in Burishkin (1975). Additional references can be found in a comprehensive review on general methods for symmetric structures (Kangwai et al., 1999). As was noted in the same work, for structures with pure rotational and translational symmetry one can also use the DFT which provides a simple and convenient way for reducing the initial problem. In fact the DFT method can be derived from the group theory approach. DFT was used to obtain effective solutions for closed and open structures composed of a module repeated rotationally and translationally a finite number of times (Samartin, 1988).

Finally, the symmetry of repetitive truss-beams and double layered lattice grids was employed to build equivalent continuum beams and plates models (Noor et al., 1978; Noor and Andersen, 1979). Along a similar vein, beam theory is discussed and elaborated on the basis of the analysis of repetitive modular structures (Renton, 1970, 1996). An example of difference equation techniques for the solution of approximate models of periodically layered elastic media can also be found in Bolotin and Partsevskii (1968) where the stress distribution is examined in an elastic half-space consisting of alternating rigid and soft layers and acted upon by a concentrated normal force.

In the field of static analysis of periodic structures there are thus not many applications of the DFT (Bolotin, 1980), and few applications solve the infinite problem on a single cell. A noteworthy exception is an early solution to the load transfer problem in a periodically riveted sheet and stringer assembly (Budiansky and Wu, 1961).

Applying the DFT to the solution of infinite repetitive structures or elastic continua under general static loading is the subject matter of the present work. The emphasis is on presenting a general methodology based on the FEA of a single cell for solving the infinite problem. It should be noted that FEA was applied to single cell analysis methods in dynamic problems of repetitive structures (Mead et al., 1988; Langley et al., 1997) where the emphasis was on hierarchical FE. Herein we are advocating the use of FE for the general solution of quasi-periodic static problems.

We will be employing in particular the method of the representative cell (Nuller and Ryvkin, 1980). The special flavor of the representative cell method resides in applying the DFT directly to the formulation of the infinite problem rather then to the infinite set of equations which solve the problem, as is usually the case with DFT. Having obtained a finite, albeit complex-valued, boundary value problem formulated over a single (representative) cell the problem can be solved by any adequate method. As indicated we will be performing a FEA of the 'representative' boundary value problem. Such an FE approach was outlined and implemented in the particular case of antiplane deformation (Ryvkin and Nuller, 1997). An application to truss type structures can also be found in (Ryvkin et al., 1999) where the representative cell method was used in the analysis of infinite trusses in the framework of optimal design. As mentioned, a general FEA method for periodic 3D elasticity is treated herein. The method is valid for structures with 1D up to 3D translational symmetry.

In the following section the DFT will be applied to obtain the strong and weak forms of the boundary value problem defined for the representative cell. Next the weak form will be used for defining FEA equations to be solved. It will be shown that these equations can also be obtained directly in a systematic and simple manner. Numerical considerations and typical solutions are then shown in a subsequent section. The examples cover 2D and 3D structures with 1D symmetry. Finally, in a concluding section the main ingredients of the technique are summarized and discussed and some conclusions are drawn.

## 2
## Strong and weak formulations for quasi-periodic elasticity problems

In a first step we will convey in broad lines the gist of the representative cell method which is formulated for quasi-periodic problems of solid bodies. Quasi-periodicity means that the geometric and elastic properties of the body, including the location of the boundary domains where tractions or displacements are given, are periodic. The specified tractions and displacements, on the other hand, are arbitrary.

As will be shown the representative cell method essentially rewrites the infinite problem into a formulation defined over a single cell (Nuller and Ryvkin, 1980; Ryvkin and Nuller, 1997). This can then be solved by any appropriate method. We will confine the description to solids with one-dimensional translational symmetry. Such a body $\Omega_\infty$ may be viewed as an assemblage $(\cup)$ of cells $\Omega_k$ with identical geometry and elastic properties

$$\Omega_\infty = \bigcup_{k=-\infty}^{\infty} \Omega_k \ . \tag{1}$$

Note that the location of the artificial boundaries between the cells can be chosen arbitrarily, as long as periodicity is maintained, and do not influence the final result. The translational symmetry implies that the boundary of cell $\Omega_k$, denoted as $\Gamma_k$, has two congruent parts $\Gamma_k^-$ and $\Gamma_k^+$

where cell $\Omega_k$ contacts with cell $\Omega_{k-1}$ and cell $\Omega_{k+1}$ respectively. The position of any point in cell $\Omega_k$ is given by the position vector $\mathbf{r}$ in a local system of coordinates introduced in a similar manner in each cell.

In Fig. 1a we have a stylized depiction of three consecutive cells $\Omega_{k-1}, \Omega_k$ and $\Omega_{k+1}$. The boundary $\Gamma_k$ of cell $\Omega_k$ is seen to be composed of four subsets

$$\Gamma_k = \Gamma_k^- \cup \Gamma_k^u \cup \Gamma_k^+ \cup \Gamma_k^t \tag{2}$$

where $\Gamma_k^u$ is the boundary were the displacements are prescribed and $\Gamma_k^t$ is the boundary were the tractions are given.

The strong formulation of this boundary value problem on an infinite domain $\Omega_\infty$ can be written in non-component representation as

$$\begin{aligned} &\nabla \cdot \boldsymbol{\sigma}_k + \mathbf{f}_k = \mathbf{0} \\ &\boldsymbol{\varepsilon}_k = \tfrac{1}{2}\left(\nabla \mathbf{u}_k + \nabla \mathbf{u}_k^{\mathrm{T}}\right) \quad k = 0, \pm 1, \pm 2, \ldots \\ &\boldsymbol{\sigma}_k = \mathbf{C} \cdot\cdot\, \boldsymbol{\varepsilon}_k \end{aligned} \tag{3}$$

with the boundary conditions

$$\begin{aligned} &\mathbf{u}_k \,|_{\Gamma_k^u} = \hat{\mathbf{u}}_k \\ &\mathbf{n}_\Gamma \cdot \boldsymbol{\sigma}_k \,|_{\Gamma_k^t} = \hat{\mathbf{t}}_k \end{aligned} \tag{4}$$

and the continuity conditions between the neighboring cells

$$\mathbf{u}_k \,|_{\Gamma_k^+} = \mathbf{u}_{k+1} \,|_{\Gamma_{k+1}^-} \tag{5}$$

$$\mathbf{n}_\Gamma \cdot \boldsymbol{\sigma}_k \,|_{\Gamma_k^+} = \mathbf{n}_\Gamma \cdot \boldsymbol{\sigma}_{k+1} \,|_{\Gamma_{k+1}^-} \tag{6}$$

where $\mathbf{u}_k \equiv \mathbf{u}_k(\mathbf{r})$ denotes the vector of displacements in cell number $k$, $\mathbf{f}_k$ is the body forces in cell $k$, $\boldsymbol{\varepsilon}_k$ and $\boldsymbol{\sigma}_k$ are the corresponding strain and stress tensors related by the 4-th rank tensor of the elastic moduli $\mathbf{C}$ (independent on $k$), $\nabla$ is the Hamilton nabla operator, $\mathbf{n}_\Gamma$ is the external normal to the corresponding boundary and $\hat{\mathbf{u}}_k$ and $\hat{\mathbf{t}}_k$ are the prescribed displacements and tractions, respectively. Note that this form of the continuity conditions (5), (6) assumes that there are no external forces and displacements discontinuities at the interfaces between the cells. This apparent limitation does not infringe on the generality of the formulation. Indeed, since the division of the translationally symmetric domain $\Omega$ into repetitive cells is arbitrary it is always possible to choose the boundary between cells such that forces and displacement discontinuities are absent.

One will note that the above formulation is almost uncoupled, but for the continuity relations (5), (6) which tie the contiguous cells into one unit. In the representative cell method one transforms this infinite (on $k$) set of boundary value problems into a single boundary value problem defined over a single cell with Born–Von Karman type boundary conditions relating the opposite congruent boundaries. This is done by the DFT, i.e., by multiplying (3)–(6) by the complex exponent $\exp(\mathrm{i}k\varphi)$ and summing up the corresponding equations over $k$. This gives the transformed boundary value problem defined for the representative cell $\Omega$ (Fig. 1b)

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \mathbf{0} \tag{7}$$

$$\boldsymbol{\varepsilon} = \tfrac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{T}}\right) \tag{8}$$

$$\boldsymbol{\sigma} = \mathbf{C} \cdot\cdot\, \boldsymbol{\varepsilon} \tag{9}$$

with the boundary conditions

$$\mathbf{u} \,|_{\Gamma^u} = \hat{\mathbf{u}} \tag{10}$$

$$\mathbf{n}_\Gamma \cdot \boldsymbol{\sigma} \,|_{\Gamma^t} = \hat{\mathbf{t}} \tag{11}$$

and Born–Von Karman type boundary conditions

$$\mathbf{u}^+ = \gamma^{-1}\,\mathbf{u}^- \tag{12}$$

$$\mathbf{t}^+ = \gamma^{-1}\,\mathbf{t}^- \tag{13}$$

where

$$\gamma \equiv \exp(\mathrm{i}\varphi), \quad \text{and} \quad \mathbf{u}^\pm \equiv \mathbf{u}\,|_{\Gamma^\pm}, \quad \mathbf{t}^\pm \equiv (\mathbf{n}_\Gamma \cdot \boldsymbol{\sigma})\,|_{\Gamma^\pm} \ . \tag{14}$$

Unless specified to the contrary we are assuming that unsubscripted variables refer to the Fourier transforms and to the representative cell, namely,

$$\mathbf{u} = \sum_{k=-\infty}^{k=\infty} \mathbf{u}_k \exp(\mathrm{i}k\varphi) \tag{15}$$



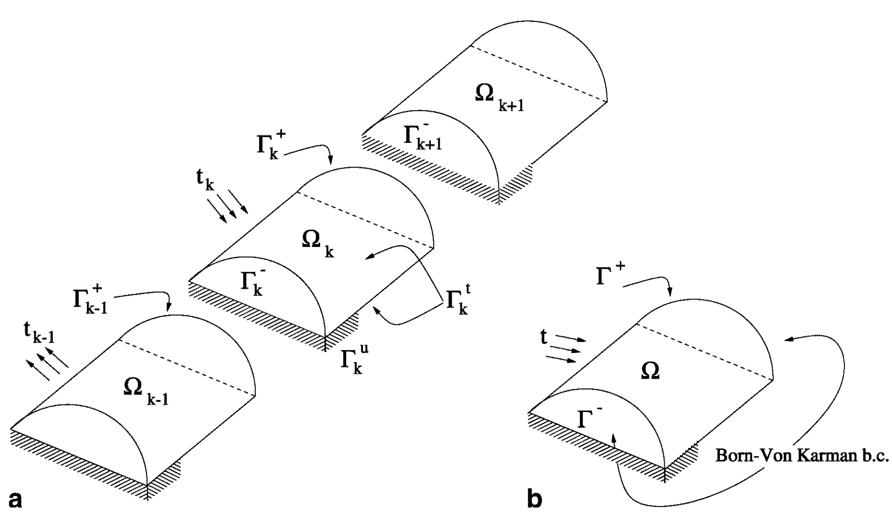


**Fig. 1.** **a** Three consecutive cells of the infinite domain. **b** The representative cell $\Omega$. Note the Born–Von Karman b.c. relating the $\Gamma^\pm$ boundaries. The loading is the transform of the loads applied to the real structure

This unusual nomenclature is adopted for the sake of clarity since we will be dealing most of the time with the transformed problem. Real quantities will always refer to some cell and will therefore carry the cell index in subscript.

Equations (7)–(13) are the strong formulation of the boundary value problem defined over one cell. The solution of this problem, which includes the non-standard boundary conditions (12) and (13) yields the stresses and displacements transforms[1] from which, by means of the inverse DFT, the real quantities can be obtained in any desired cell $k$, as for instance,

$$\mathbf{u}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{u} \exp(-\mathrm{i}k\varphi) \mathrm{d}\varphi \tag{16}$$

The weak formulation of the problem for the representative cell is derived from the strong one in a usual manner by multiplying the equilibrium equation (7) by weighting functions $\mathbf{w}$ and integrating over the cell domain $\Omega$ using the divergence theorem. The trial solutions $\mathbf{u}$ are assumed to be satisfying the essential boundary conditions (10) and the Born–Von Karman conditions on the displacements (12). The weighting functions fulfill the usual requirement

$$\mathbf{w}\,|_{\Gamma^u} = 0 \ . \tag{17}$$

All this leads to the following integral relation

$$\int_{\Omega} \boldsymbol{\varepsilon}_w \cdot\cdot \mathbf{C} \cdot\cdot \boldsymbol{\varepsilon}\, \mathrm{d}\Omega = \int_{\Omega} \mathbf{w} \cdot \mathbf{f}\, \mathrm{d}\Omega + \int_{\Gamma^\sigma} \mathbf{w} \cdot \hat{\mathbf{t}}\, \mathrm{d}\Gamma + \int_{\Gamma^-} (\mathbf{w}^- - \gamma^{-1}\mathbf{w}^+) \cdot \hat{\mathbf{t}}\, \mathrm{d}\Gamma \tag{18}$$

where the strain tensor $\boldsymbol{\varepsilon}_w$ is derived from the weighting functions $\mathbf{w}$ as in (8) and $\mathbf{w}^\pm$ is defined similar to (14). In this case, as in Ryvkin and Nuller (1997), it is effective to consider weighting functions which fulfill a condition which is conjugate to the boundary condition in (12)

$$\mathbf{w}^+ = \gamma \mathbf{w}^- \tag{19}$$

This removes the last term in the r.h.s. of (18) and yields the final weak form of the representative cell problem

$$\int_{\Omega} \boldsymbol{\varepsilon}_w \cdot\cdot \mathbf{C} \cdot\cdot \boldsymbol{\varepsilon}\, \mathrm{d}\Omega = \int_{\Omega} \mathbf{w} \cdot \mathbf{f}\, \mathrm{d}\Omega + \int_{\Gamma^\sigma} \mathbf{w} \cdot \hat{\mathbf{t}}\, \mathrm{d}\Gamma \tag{20}$$

A similar weak formulation for the Poisson problem may be found in the fundamental study of the group approach for the analysis of symmetric domains (Bossavit, 1986). Also, (20) is in line with the formulation obtained in a specific case of antiplane deformation (Ryvkin and Nuller, 1997). Additional examples of weak formulations for problems on fluid–solid interaction with Born–Von Karman type boundary conditions can also be found in Conca et al. (1995).

It will be remembered that both the strong and weak forms of the representative cell problem are expressed in terms of complex-valued transforms. Any adequate method can be applied to solve these equations. We will in the following employ a standard FEA approach based on the weak formulation.

## 3
## Discretization and solution by FEA

The conjugate conditions on the trial functions (12) and on the weighting functions (19) makes the weak formulation somewhat unusual. We will however show that by following the standard route of FE discretization a familiar form of the equilibrium equations can be obtained. We introduce in the representative cell an arbitrary FE mesh defined by a set of $M$ nodes. Let $M^+$, $M^-$ and $M^u$ be the set of nodes on the $\Gamma^+$, $\Gamma^-$, $\Gamma^u$ boundaries respectively and let $M^0$ be the set of remaining nodes

$$M = M^0 \cup M^+ \cup M^- \cup M^u \ . \tag{21}$$

The number of nodes in each set is denoted by the corresponding lower case letter

$$m = m^0 + m^+ + m^- + m^u \ . \tag{22}$$

Without loss of generality the $M^u$ nodes will be considered fixed. It is further assumed that the discretization of the congruent boundaries $\Gamma^+$ and $\Gamma^-$ is identical, i.e., $m^- = m^+$ and there is one-to-one correspondence between nodes $A^- \in M^-$ and nodes $A^+ \in M^+$ given by the relation

$$\mathbf{r}_{A^+} = \mathbf{r}_{A^-} + \mathbf{a} \tag{23}$$

where $\mathbf{a}$ is the vector defining the translational symmetry of the structure $\Omega_\infty$.

In a first step let us consider an auxiliary problem, similar to the one of the representative cell (7)–(11), but for the Born–Von Karman type conditions (12) and (13) which are replaced by conditions of zero traction on the periodic boundaries.

$$\mathbf{n}_\Gamma \cdot \boldsymbol{\sigma}\,|_{\Gamma^+} = \mathbf{n}_\Gamma \cdot \boldsymbol{\sigma}\,|_{\Gamma^-} = \mathbf{0} \tag{24}$$

The weak formulation for this auxiliary problem has the standard form and leads to a classical FEA matrix equation

$$\mathbf{K}\mathbf{u} = \mathbf{p} \tag{25}$$

where $\mathbf{K}$ is a real-valued stiffness matrix of order $n_{\mathrm{df}}(2m^+ + m^0)$ defined by the total number of free nodes and the number of degrees of freedom at each node $n_{\mathrm{df}}$, $\mathbf{u}$ is here the vector of nodal displacements (not to be confused with (15), the transformed displacement field) and $\mathbf{p}$ is a corresponding vector of loads originating in the prescribed tractions and displacements and body forces. In 3D elasticity, for instance, we have $n_{\mathrm{df}} = 3$. Note, $\mathbf{p}$ and consequently $\mathbf{u}$ are complex-valued quantities. For evaluating the elements of matrix $\mathbf{K}$ and the real and imaginary parts of vector $\mathbf{p}$ any usual real-valued shape functions $N_A(\mathbf{r})$ can be used. Herein piecewise bilinear "hat" functions will be considered. This is important since as we will see the final form of the equations of the original representative cell problem is based on submatrices of (25). Finally, in view of further use, (25) will be partitioned in accordance with the division of nodes (21)

[1] We will omit in the sequel the word 'transform' for terms related to the representative cell problem such as displacements, strains etc.

$$\begin{bmatrix} \mathbf{K}^{--} & \mathbf{K}^{-0} & \mathbf{K}^{-+} \\ \mathbf{K}^{0-} & \mathbf{K}^{00} & \mathbf{K}^{0+} \\ \mathbf{K}^{+-} & \mathbf{K}^{+0} & \mathbf{K}^{++} \end{bmatrix} \begin{Bmatrix} \mathbf{u}^- \\ \mathbf{u}^0 \\ \mathbf{u}^+ \end{Bmatrix} = \begin{Bmatrix} \mathbf{p}^- \\ \mathbf{p}^0 \\ \mathbf{p}^+ \end{Bmatrix} \tag{26}$$

It will be noted that the solution of this auxiliary problem can be split into two real-valued problems by solving once with the real components of $\mathbf{p}$ and then with the imaginary ones. For the representative cell formulation a similar division of the initial problem (7)–(11) into the real and imaginary ones is not possible due to the presence of the Born–Von Karman type boundary conditions. We will however make use of the real shape functions $N_A$ mentioned in the auxiliary problem to construct two sets of complex-valued shape functions for the trial and weighting functions of the weak formulation which satisfy the conditions (12) and (19) respectively. Such shape functions for the trial functions $\mathbf{u}(\mathbf{r})$ associated with the nodes $A \in M \backslash M^+$ are taken as

$$N_A^u(\mathbf{r}) = \begin{cases} N_A(\mathbf{r}) & \text{for } A \in (M^0 \cup M^u) \\ N_{A^-}(\mathbf{r}) + \gamma^{-1} N_{A^+}(\mathbf{r}) & \text{for } A \in M^- \end{cases} \tag{27}$$

and the weighting functions $\mathbf{w}(\mathbf{r})$ are approximated by using

$$N_A^w(\mathbf{r}) = \begin{cases} N_A(\mathbf{r}) & \text{for } A \in M^0 \\ N_{A^-}(\mathbf{r}) + \gamma N_{A^+}(\mathbf{r}) & \text{for } A \in M^- \end{cases} \tag{28}$$

Recall that the correspondence between nodes $A^-$ and $A^+$ was established by (23).

One can note that the Born–Von Karman type boundary conditions for the opposite congruent boundaries $\Gamma^-$ and $\Gamma^+$ reduce the number of "free nodes" of the representative cell from $m - m^u$ in the auxiliary problem to $m - (m^u + m^+)$ in the present one.

The approximating series for the trial and weighting functions have the usual form

$$\mathbf{u}(\mathbf{r}) = \sum_{i=1}^{n_{\mathrm{df}}} \left( \sum_{A \in M^0 \cup M^-} u_A^{(i)} N_A^u(\mathbf{r}) \mathbf{e}_i + \sum_{A \in M^u} \hat{u}_A^{(i)} N_A^u(\mathbf{r}) \mathbf{e}_i \right) \tag{29}$$

$$\mathbf{w}(\mathbf{r}) = \sum_{i=1}^{n_{\mathrm{df}}} \sum_{A \in M^0 \cup M^-} w_A^{(i)} N_A^w(\mathbf{r}) \mathbf{e}_i \tag{30}$$

where $u_A^{(i)}, \hat{u}_A^{(i)}, w_A^{(i)}$ denote the components of the nodal values for the corresponding vectors and $\mathbf{e}_i$ are the unit vectors. When substituting (29) and (30) in (20) one obtains after some manipulation the FEA equations for problem (7)–(13) defined for the representative cell

$$\mathbf{K}'\mathbf{u}' = \mathbf{p}' \tag{31}$$

where

$$\mathbf{K}' = \begin{bmatrix} (\mathbf{K}^{--} + \gamma \mathbf{K}^{-+} + \gamma^{-1} \mathbf{K}^{+-} + \mathbf{K}^{++}) & (\mathbf{K}^{-0} + \gamma \mathbf{K}^{+0}) \\ (\mathbf{K}^{0-} + \gamma^{-1} \mathbf{K}^{0+}) & \mathbf{K}^{00} \end{bmatrix}, \tag{32}$$

$$\mathbf{u}' = \begin{Bmatrix} \mathbf{u}^- \\ \mathbf{u}^0 \end{Bmatrix}, \tag{33}$$

and

$$\mathbf{p}' = \begin{Bmatrix} \mathbf{p}^- + \gamma \mathbf{p}^+ \\ \mathbf{p}^0 \end{Bmatrix}. \tag{34}$$

Hence matrix $\mathbf{K}'$ and vector $\mathbf{p}'$ are simply expressed in terms of $\mathbf{K}$ and $\mathbf{p}$ of the auxiliary problem. Note, similar expressions for matrix $\mathbf{K}'$ were obtained in a problem on eigenfrequencies of 2D periodic structures (Mead et al., 1988).



The nodal displacements $\mathbf{u}_k$ in any cell $\Omega_k$ of the real structure are obtained by the inverse transform (16) using numerical integration. Consequently one solves the system (31) for a number of values of $\varphi$ in order to carry out the numerical procedure. In view of the symmetry of matrix $\mathbf{K}$, matrix $\mathbf{K}'$ is Hermitian and the solution of system (31) presents no special problems.

## 4
## A short route to the representative cell

It should be noted that there exists an alternate shorter route to obtain the final system of equations (31). The derivation presented in the previous sections was based on first principles, which developed the weak formulation for the representative cell, the meshing of the cell into FE, and in particular the use of the special shape functions (27), (28). One can circumvent the development of FE theory for the representative cell by using FEA on the real infinite domain. In accordance with the representative cell approach we introduce an infinite periodic mesh, that is, we discretize all the cells in an identical manner. Then the FEA equations will consist of an infinite set of equilibrium equations similar to (25)

$$\begin{bmatrix} \mathbf{K}^{--} & \mathbf{K}^{-0} & \mathbf{K}^{-+} \\ \mathbf{K}^{0-} & \mathbf{K}^{00} & \mathbf{K}^{0+} \\ \mathbf{K}^{+-} & \mathbf{K}^{+0} & \mathbf{K}^{++} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_k^- \\ \mathbf{u}_k^0 \\ \mathbf{u}_k^+ \end{Bmatrix} = \begin{Bmatrix} \mathbf{p}_k^- \\ \mathbf{p}_k^0 \\ \mathbf{p}_k^+ \end{Bmatrix} + \begin{Bmatrix} \mathbf{f}_k^- \\ 0 \\ \mathbf{f}_k^+ \end{Bmatrix}, \quad k = 0, \pm 1, \pm 2, \ldots \tag{35}$$

completed by the continuity relation at the interfaces between the cells

$$\begin{Bmatrix} \mathbf{f}_k^+ \\ \mathbf{u}_k^+ \end{Bmatrix} = \begin{Bmatrix} -\mathbf{f}_{k+1}^- \\ \mathbf{u}_{k+1}^- \end{Bmatrix}, \quad k = 0, \pm 1, \pm 2, \ldots \; . \tag{36}$$

Here the vectors $\mathbf{u}_k^-, \mathbf{u}_k^0, \mathbf{u}_k^+$ denote the displacements of the inner and interface nodes of the $k$-th cell, $\mathbf{p}_k^-, \mathbf{p}_k^0, \mathbf{p}_k^+$ are the corresponding nodal forces generated by the generalized loading and $\mathbf{f}_k^-, \mathbf{f}_k^+$ are the contact forces which are applied by the adjacent cells to cell $k$. All loads are expressed in a common frame of reference, hence the negative sign of $\mathbf{f}_{k+1}^+$.

Applying the DFT to (35)–(36) one obtains the finite system of equations with respect to the unknown transforms $\mathbf{u}^\pm, \mathbf{u}^0, \mathbf{f}^\pm$

$$\begin{bmatrix} \mathbf{K}^{--} & \mathbf{K}^{-0} & \mathbf{K}^{-+} \\ \mathbf{K}^{0-} & \mathbf{K}^{00} & \mathbf{K}^{0+} \\ \mathbf{K}^{+-} & \mathbf{K}^{+0} & \mathbf{K}^{++} \end{bmatrix} \begin{Bmatrix} \mathbf{u}^- \\ \mathbf{u}^0 \\ \mathbf{u}^+ \end{Bmatrix} = \begin{Bmatrix} \mathbf{p}^- \\ \mathbf{p}^0 \\ \mathbf{p}^+ \end{Bmatrix} + \begin{Bmatrix} \mathbf{f}^- \\ 0 \\ \mathbf{f}^+ \end{Bmatrix} \tag{37}$$

$$\left\{\begin{array}{l}\mathbf{f}^{+}\\ \mathbf{u}^{+}\end{array}\right\} = \exp(-\mathrm{i}\phi)\left\{\begin{array}{c}-\mathbf{f}^{-}\\ \mathbf{u}^{-}\end{array}\right\} \tag{38}$$

The order of this system may be easily reduced, as was done in Mead et al. (1988). By multiplying the third row of (37) by $\exp(\mathrm{i}\phi)$ and taking into account the first of the Eqs. (38) one eliminates the contact forces $\mathbf{f}^{\pm}$ from the equilibrium equations. Substituting in these equations the values $u^{+}$ expressed from the second equation (38) one obtains the final form of the equilibrium equations of the representative cell (31)–(34).



One should realize that the above description is also valid for cases with 2D and 3D symmetry. The discrete transform in (15) must then be replaced by its multidimensional analog.

As was mentioned in the Introduction, the basic relation (38) connecting the unknown functions at the opposite sides of representative module is well known in the dynamic analysis of periodic structures. The suggested methodology can nevertheless provide a new and convenient notation for the FEA of a class of dynamic problems with spatial periodicity. Indeed, consider for instance the case of damped vibrations of a modular structure generated by a given system of external harmonic forces of a same frequency and with arbitrary phases. In the presence of damping the perturbed region will be localized in a neighborhood of the loaded region (which is assumed to be bounded) and for some range of parameters the perturbation will not reach the boundaries of the structure. Clearly, for linear damping the replacement of the static equilibrium equations (3) by the corresponding dynamic ones does not violate the linearity of the problem. Consequently, there are no apparent obstacles in applying the DFT similarly to what was done in the static case. It should be emphasized that only in the presence of damping will the solution for any excitation frequency be bounded and decaying at infinity, a condition which guarantees the existence of the DFT. Also, periodicity here implies not only elastic and geometric properties but also inertial and damping properties of the modules.

One could next examine near periodic structures where the periodicity in the geometry and/or the elastic properties are slightly off, as in the presence of some local imperfections. A possible approach to the analysis of such systems is to consider the pure periodic structures with some fictitious loading which models the system disorder. In dynamic problems a localization and increase of the system response usually takes place in the vicinity of the disorder for some excitation frequencies. We can probably expect a similar phenomenon in the static case for a spatially periodic loading which may be related to some propagating wave.

## 5 Numerical examples

The advantages of the proposed technique are evident. The loading is non-periodic and, consequently, the stress states in the geometrically identical modules are different. With the present method one deals however with one typical cell only. Summarizing the technique we find the following steps:

- Having selected the repeated module the transforms of the prescribed tractions and displacements, $\hat{\mathbf{t}}$, $\hat{\mathbf{u}}$, and body forces $\mathbf{f}$ are evaluated by the DFT (15).
- The repeated module is meshed into FE in any standard fashion thus producing the real-valued coefficients matrix $\mathbf{K}$ in (26) and (37). Care must be exercised by meshing the congruent boundaries in same fashion.
- On the basis of the discretization the first vector in the r.h.s. of (26) and (37) is computed.
- The components of $\mathbf{K}$ and $\mathbf{p}$ are then used to assemble $\mathbf{K}'$ and $\mathbf{p}'$ in (31) which is the system to be solved.
- In view of further numerical integration for the inverse transforms this system is solved for a set of values $\varphi_l$ yielding the corresponding set of displacements $\mathbf{u}(\mathbf{r}, \varphi_l)$.
- The real displacements in any cell $k$ are obtained via the inverse transform (16) which is evaluated by numerical integration.

The effectiveness of the method is thus mitigated by the inverse transform. In the case of cyclic symmetry the exact inverse is a finite sum

$$\mathbf{u}_k(\mathbf{r}) = \frac{1}{N}\sum_{l=0}^{N-1} \mathbf{u}(\mathbf{r}, \phi_l)\exp(-\mathrm{i}k\phi_l), \quad \phi_l = \frac{2\pi}{N} l \tag{39}$$

where $N$ is the number of cells (Samartin, 1988). For an infinite translationally symmetric domain the inverse is an integral (16) which is usually evaluated numerically by the approximation

$$\mathbf{u}_k(\mathbf{r}) \cong \sum_{l=0}^{M-1} w_l \mathbf{u}(\mathbf{r}, \varphi_l)\exp(-\mathrm{i}k\varphi_l) \tag{40}$$

Here $M$ is the number of integration points and $w_l$ is the weighting coefficients depending upon the particular integration formula that is used. The inverse is thus based on the evaluation of the transformed displacements for values $\varphi_l$, $l = 0, 1, \ldots, M-1$ from the interval $[0, 2\pi]$. In other words, one must solve (31) $M$ times in order to evaluate the integral. Moreover, it also requires an assembly of $\mathbf{K}'$ and $\mathbf{p}'$ for every value of $\varphi_l$. To this effect it is noteworthy that only small parts of $\mathbf{K}'$ are to be reassembled. Indeed, unless a single finite element has nodes belonging to both the $M^{-}$ and $M^{+}$ sets, we have $\mathbf{K}^{-+} = \mathbf{K}^{+-} = \mathbf{0}$. It will also be recognized that matrices $\mathbf{K}^{+0}$ and $\mathbf{K}^{-0}$ are relatively small.

All this has bearing on the numerical efficiency of the present method when it is compared to a standard FEA. In the latter case one must make an a priori decision on the number of modules to be included in the FE model which is the main factor influencing the numerical burden of the solution. With the present method the number of integration points in the inverse transform is the overriding concern. It is however clear that there is a large class of problems where using the DFT can significantly reduce the numerical cost of the solution. It is noteworthy that the solutions for different values of $\varphi$ are mutually independent and, as noted in Ikeda et al. (1992), one can consequently reduce the computational time by parallel computing.

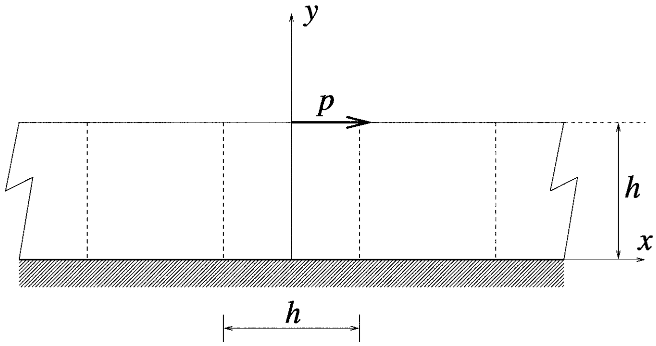


**Fig. 2.** Infinite strip with one clamped and one free boundary subjected to a concentrated tangential force. The repetitive cell was arbitrarily chosen as a square domain

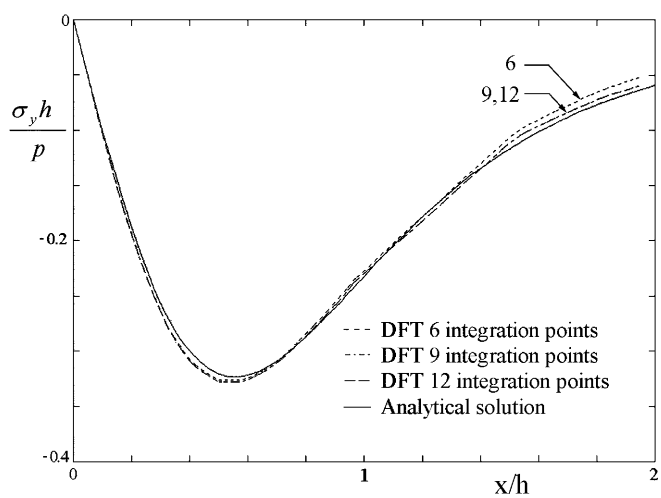


**Fig. 3.** An infinite strip under a point tangential load (Fig. 2). The normal stress at the boundary (full line) is compared with the FEA of the representative cell for 6, 9 and 12 integration points in the integral in the inverse transform. Curves 9 and 12 have converged to the classical FEA of a very long strip

The present methodology will be illustrated by several examples of FEA of translationally symmetric bodies. The first case is the analysis of an infinite elastic strip $-\infty < x < \infty$, $0 \leq y \leq a$, clamped along $y = 0$, and subjected to a tangential force $p$ applied at coordinate $(0, h)$ as shown in Fig. 2. The translational symmetry is in this case arbitrary and the magnitude of the translation vector was chosen equal to the width $h$ of the domain, thus creating square repetitive modules. This problem has an analytical closed form solution obtained by means of integral transforms (see, for example, Uflyiand, 1968). In Fig. 3 the computed normalized $\sigma_y$ stress at the boundary $y = 0$ is compared to the analytical results as a function of the distance $x/h$ from the origin for a plane-strain case. The positive $x$ half-space is depicted, where the stresses are compressive. In the opposite $x$-direction the stresses are the anti-symmetric image of the shown curves. The representative square domain had 1922 degrees of freedom and used 900 bilinear quadrilateral elements in plane stress with a Poisson's ratio $v = 0.3$. The emphasis in this example is on the influence of the number of integration

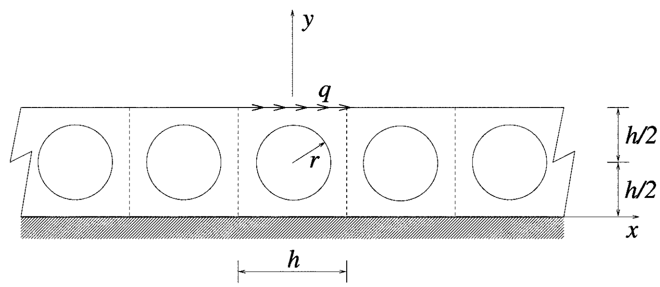


**Fig. 4.** Infinite strip, one clamped and one free boundary, with repetitive circular holes, subjected to a tangential force distributed along one cell

points (6, 9 and 12) on the accuracy of the results. One should bear in mind that every integration point entails a solution of (31). In fact only two numerical curves can be distinguished in the figure since curves 9 and 12 coincide (middle curve in region $1 < x/h < 2$). The discrepancy between the converged 9- and 12-curves and the analytical solution can be shown to be due to the FEA error with the employed mesh.

A second example compares a classical FEA approach for solving infinite problems with the present unicell solution. As shown in Fig. 4 the infinite strip of constant thickness and of constant width $h$ is perforated by equally spaced and centrally located circular holes of radius $r = 0.7\ h$, spaced at distances $h$, which determines the periodicity. The structure is subjected to a tangential distributed load of magnitude $q$ applied along part of the upper boundary $-h/2 \leq x \leq h/2$, $y = h$. Results for a plane-stress analysis are given in Fig. 5 where the work of the external forces using FEA on one cell calculated with the present method ($W_{\mathrm{DFT}}$) is compared with the work of the external forces using standard FEA ($W$) for an increasing number of cells to be included in the model. In the latter case, free boundaries were assumed at the end of the structure.

In the DFT approach the representative cell was meshed into 640 bilinear quadrilateral elements with 1440 degrees of freedom. Identical elements were used for the standard model, the number of elements and the number of degrees of freedom having been multiplied by the number of cells retained in the model. The required number of cells is evidently problem dependent. Indeed, the model must be long enough not to be affected by the boundary effects. The advantage of the DFT method is that one does not need to guess a number of cells before solving the problem. Moreover, at par (both are FEA) the DFT is seen to produce very good accuracy. One will note that in order to reach acceptable accuracy the number of degrees of freedom in the standard FEA model must be of the order of 10 times larger than the number of degrees of freedom with the present DFT approach.

A final case deals with an infinite I-beam on elastic foundation subjected to a tangential force $F_1$ and a lateral force $F_2$ of equal magnitude applied at two different locations along its span. The purpose of this example is descriptive the intention being to show results for a more intricate structure. Circular holes are periodically spaced

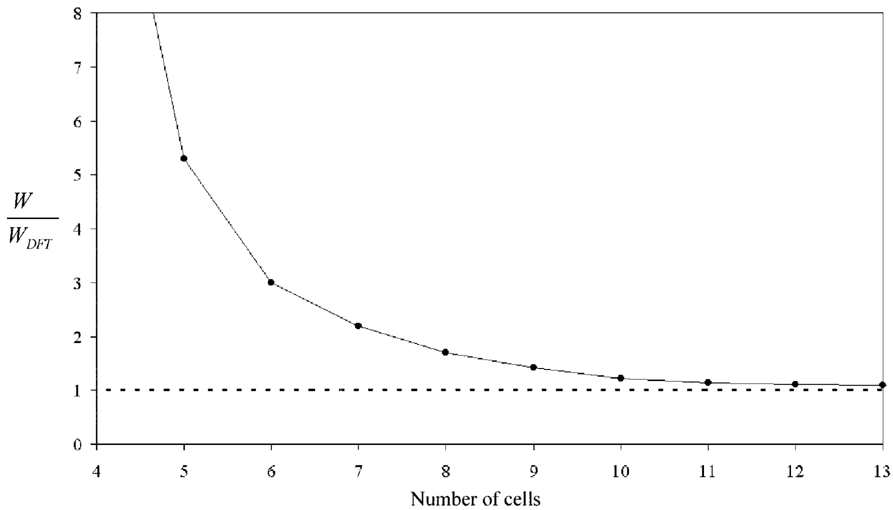


**Fig. 5.** An infinite strip with circular holes under a distributed tangential load along the central module (Fig. 4). The work of the external forces using FEA for the representative cell ($W_{\mathrm{DFT}}$) is compared with the work ($W$) obtained with classical FEA for increasing numbers of cells included in the FE model

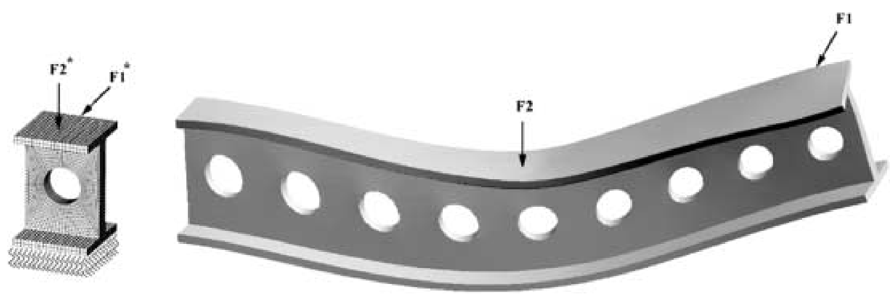


**Fig. 6.** An infinite I-beam on elastic foundation under general loading. The meshing of the representative cell with a stylized rendering of the transformed forces is shown to the left. The main picture is the deformed shape of the central part of the beam

along the midline of the web. To the left of Fig. 6 the representative cell is drawn with a stylized rendering of the forces transforms $F_1^*$ and $F_2^*$. These are in fact complex-valued quantities. The FEA of the transformed problem was based on 1504 trilinear hexahedral elements with 6468 degrees of freedom. The deformed shape of the central part of the structure, shown in Fig. 6 (the elastic foundation is omitted in the picture), exhibits, as expected, combined flexural and torsional displacements.

## 6
## Conclusions

In this paper we have presented a formulation for solving quasi-periodic structural static problems using FEA by considering a single cell only. The technique consists in applying the discrete Fourier transform to the infinite problem thus generating a boundary value problem defined over a representative cell.

Having obtained the weak formulation for the transformed problem the discretization of the domain is shown to produce the familiar FEA equations, after eliminating the cyclic boundary conditions from the formulation. A short rout to these equations is also indicated. The size of this system is smaller than the size of the stiffness matrix of the repeated module and it is shown that the solution can be obtained by a complex solver. The real displacements in any cell are then computed by the inverse transform. The integral appearing in the inverse transform is approximated by numerical evaluation and when based on $M$ sampling points it requires $M$ solutions of the above complex-valued system of equations.

Several examples illustrate the procedure. In a first case the closed form solution is compared to FEA results by the present method and the influence of the number of sampling points $M$ is stressed. It is shown that the method emulates the analytical solution with FE accuracy. In a next example, a perforated infinite strip, the present FEA is compared to classical FE results, for increasing number of cells of the standard model, a staple technique of the straightforward approach, such as to make allowance for the infinite nature of the problem. As indicated, for equal results the standard FE model requires in this case in the order of 10 times more degrees of freedom than the DFT-based solution. Finally, an infinite beam on elastic supports is presented where the flexural and torsional deformations along the structure are seen to emerge, as it were, from the analysis of the rather small representative cell FE model.

The main advantage of the technique is that it presents an exact and simple numerical procedure to the FEA of infinite quasi-periodic structures. Indeed, the accuracy of the solution can be enhanced by increasing the number of sampling point for the inverse transform. From a numerical efficiency point of view the technique has to be compared to classical FE solutions where large finite models must be used, large enough for the stress-fields to decay before reaching the boundaries. There remains however the issue of the number of sampling points in the inverse transform which controls the numerical burden. On the other hand, the technique should be very attractive in a parallel computing environment since the $M$ set of equations obtained for the $M$ sampling points can be solved separately. There is probably no room for overriding conclusions regarding the relative merits of the present technique, these are problem dependent, it is however felt that the proposed method can fill an important niche in the analysis of quasi-periodic structural problems.



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