Journal of Mathematical Sciences, Vol. 72, No. 2, 1994

SIMULATION ACCURACY OF GAUSSIAN STOCHASTIC PROCESSES IN $L^2(0, T)$

Yu. V. Kozachenko and L. F. Kozachenko UDC 519.21

We consider computer-implementable models that approximate Gaussian stochastic processes with given accuracy and reliability in $L^2(0, T)$.

Let $\xi(t)$ be a real Gaussian process defined on the interval $[0, T]$, $M\xi(t) = 0$, $B(t, s) = M\xi(t)\xi(s)$ is the covariance function of $\xi(t)$. We assume that $B(t, s)$ is continuous.

We construct a model $\hat{\xi}(t)$ that approximates the process $\xi(t)$ in $L^2(0, T)$ with prescribed accuracy and reliability. This means that for given $\varepsilon > 0$ and $p > 0$ a Gaussian stochastic process $\hat{\xi}(t)$ is constructed that allows exact simulation by computer. The simulated process satisfies the inequality

$$P\left\{\int_0^T (\xi(t) - \hat{\xi}(t))^2 dt > \varepsilon\right\} < p, \tag{1}$$

where ε is the accuracy and $(1 - p)$ is the reliability of the model. Note that the condition $M\xi(t) = 0$ does not detract from the generality of our analysis because for $M\xi(t) = m(t)$ we have $\xi(t) = \xi_1(t) + m(t)$, where $\xi_1(t) = \xi(t) - m(t)$, $M\xi_1(t) = 0$, and the mean $m(t)$ can be simulated without errors. Such a problem has been considered in [1]. Here we give more accurate bounds and consider a more general case [2, 3, 4].

LEMMA. Let $\vec{\xi}$ be a Gaussian random vector, $\vec{\xi}^T = (\xi_1, \xi_2, \ldots, \xi_n)$, $M\vec{\xi} = 0$, $B = \operatorname{cov}\vec{\xi} = M\vec{\xi}\vec{\xi}^T = \| r(k, l) \|\hat{k}, l = 1$. For any $z > 0$, we have the inequality

$$P\left\{\frac{\sum_{i=1}^n \xi_i^2 - \sum_{i=1}^n M\xi_i^2}{\left(\sum_{i=1}^n \sum_{j=1}^n r^2(i,j)\right)^{1/2}} > z\right\} \le \exp\left\{-\frac{z}{2}\right\}(z+1)^{1/2}. \tag{2}$$

Proof. Assume that ξ_i $(i = 1, \ldots, n)$ are independent. Lemma 2 [5] shows that for any $0 < s < 1$ we have the inequality

$$M\exp\left\{\left(\sum_{i=1}^n \xi_i^2 - \sum_{i=1}^n M\xi_i^2\right)/r\right\} \le \exp\left\{-\frac{s}{2}\right\}(1-s)^{-1/2}, \tag{3}$$

where $r = \dfrac{2\left(\sum_{k=1}^n \sigma^4(k)\right)^{1/2}}{s}$, $\sigma^2(k) = M\xi_k^2$. Let $B = \operatorname{cov}\vec{\xi}$ be positive definite. Hence there exists an orthogonal matrix P $(PP^T = I)$ that diagonalizes B: $P^TBP = R$, where R is a diagonal matrix with positive elements along the diagonal [6]. Let $\vec{\eta} = P^T\vec{\xi}$. Then $\vec{\eta}$ is a Gaussian vector with $M\vec{\eta} = P^TM\vec{\xi} = 0$, $\operatorname{cov}\vec{\eta} = P^TBP = R$. Thus, the components of $\vec{\eta}$ are independent normal random variables with positive variances. Hence, $\vec{\eta}$ satisfies inequality (3). Note that $\sum_{i=1}^n \eta_i^2 = \vec{\eta}^T\vec{\eta} = \vec{\xi}^T PP^T \vec{\xi} = \vec{\xi}^T\vec{\xi} = \sum_{i=1}^n \xi_i^2$. Seeing that $\sum_{i=1}^n (M\eta_i^2)^2 = \mathrm{Sp}RR = \mathrm{Sp}(P^TBBP) = \mathrm{Sp}BB = \sum_{i=1}^n \sum_{j=1}^n r^2(i,j)$, we obtain from (3)

Kiev University. Translated from Vychislitel'naya i Prikladnaya Matematika, No. 74, pp. 88-93, 1992; original article submitted July 27, 1989.

 1072-3374/94/7202-3040$12.50

$$M\exp\left\{\frac{\left(\sum_{i=1}^{n}\xi_i^2-\sum_{i=1}^{n}M\xi_i^2\right)s}{2\left(\sum_{i=1}^{n}\sum_{j=1}^{n}r^2(i,j)\right)^{1/2}}\right\}\exp\left\{-\frac{s}{2}\right\}(1-s)^{-1/2}. \tag{4}$$

Inequality (4) also holds when the matrix B is not positive definite. To prove this, replace B with $B+\varepsilon I$, $\varepsilon>0$. Write inequality (4) for the vector $\vec{\xi}(t)$ with the positive definite covariance matrix $B+\varepsilon I$ and then pass to the limit as $\varepsilon\to 0$. By the Chebyshev inequality $P\{\xi<z\}\le M\exp\{-\lambda z\}\exp\{\lambda\xi\}$, $\lambda>0$, $z>0$. For $0<s<1$ we obtain

$$P\left\{\frac{\sum_{i=1}^{n}\xi_i^2-\sum_{i=1}^{n}M\xi_i^2}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|r(i,j)|^2\right)^{1/2}}>z\right\}\le$$
$$\le M\exp\left\{\frac{\left(\sum_{i=1}^{n}\xi_i^2-\sum_{i=1}^{n}M\xi_i^2\right)s}{2\left(\sum_{i=1}^{n}\sum_{j=1}^{n}r^2(i,j)\right)^{1/2}}\right\}\exp\left\{-\frac{zs}{2}\right\}\le$$
$$\le\exp\left\{-\frac{s}{2}\right\}(1-s)^{-1/2}\exp\left\{-\frac{zs}{2}\right\}.$$

Taking the minimum of the right-hand side for $0<s<1$ ($s=z/(z+1)$), we obtain inequality (2). Q.E.D.

1. Modeling by Karhunen–Loève Expansions

By the Karhunen–Loève theorem [7], $\xi(t)$ is representable by a mean-square convergent series

$$\xi(t)=\sum_{n=1}^{\infty}\lambda_n\eta_n\psi_n(t), \tag{5}$$

where η_n is a sequence of independent identically distributed normal random variables with $M\eta_n=0$, $M\eta_n^2=1$, $\lambda_1^2>\lambda_2^2>\lambda_3^2>\ldots$ are the eigenvalues, $\psi_n(t)$ are the orthonormal eigenfunctions of the integral equation

$$\int_0^T B(t,s)\psi(s)ds=\lambda^2\psi(t). \tag{6}$$

Denote $\xi_N(t)=\sum_{n=1}^{N}\lambda_n\eta_n\psi_n(t)$. Then

$$\int_0^T(\xi(t)-\xi_N(t))^2dt=\int_0^T\left(\sum_{n=N+1}^{\infty}\lambda_n\eta_n\psi_n(t)\right)^2dt=$$
$$=\sum_{n=N+1}^{\infty}\sum_{k=N+1}^{\infty}\lambda_n\lambda_k\eta_n\eta_k\int_0^T\psi_n(t)\psi_k(t)dt=\sum_{k=N+1}^{\infty}\lambda_k^2\eta_k^2.$$

If in inequality (2) of the lemma we set $\xi_k=\lambda_k\eta_k$ and pass to the limit as $n\to\infty$, then noting that

$$M\xi_k^2=\lambda_k^2,\ r^2(k,l)=(M\xi_k\xi_l)^2=\begin{cases}0, & k\ne l,\\ \lambda_k^4, & k=l,\end{cases}$$

we obtain

$$P\left\{\frac{\sum_{k=N+1}^{\infty}\lambda_k^2\eta_k^2-\sum_{k=N+1}^{\infty}\lambda_k^2}{\left(\sum_{k=N+1}^{\infty}\lambda_k^4\right)^{1/2}}>z\right\}\le\exp\left\{-\frac{z}{2}\right\}(z+1)^{1/2}.$$

This leads to the inequality

$$P\left\{\int_0^T(\xi(t)-\xi_N(t))^2dt>z\left(\sum_{k=N+1}^{\infty}\lambda_k^4\right)^{1/2}+\sum_{k=N+1}^{\infty}\lambda_k^2\right\}\le$$
$$\le\exp\left\{-\frac{z}{2}\right\}(z+1)^{1/2}. \tag{7}$$

Using (7), we can construct the required model of the process $\xi(t)$. Given p, $0 < p < 1$, we can find z_p such that $\exp\left\{-\frac{z}{2}p\right\}$ $(z_p + 1)^{1/2} = p$. Then given $\varepsilon > 0$ we find the minimum N_ε such that

$$z_p(\sum_{k=N_\varepsilon+1}^{\infty} \lambda_k^4)^{1/2} + \sum_{k=N_\varepsilon+1}^{\infty} \lambda_k^2 < \varepsilon .$$

This N_ε must satisfy the inequality

$$P\left\{\int_0^T (\xi(t) - \hat{\xi}_{N_\varepsilon}(t))^2 dt > \varepsilon\right\} \leq p. \tag{8}$$

From (8) it follows that $\hat{\xi}N_\varepsilon(t) = \sum_{k=1}^{N_\varepsilon} \lambda_n \xi_n \psi_n(t)$ can be used as a model that approximates $\xi(t)$ in $L^2(0, T)$ with reliability $(1 - p)$ and accuracy ε. If λ_n and $\psi_n(t)$ are known, then the model $\hat{\xi}_{N_\varepsilon}(t)$ can be constructed without difficulty. It is sufficient to generate a series of independent normal variates ξ_k, $k = 1,\ldots,N_\varepsilon$, using a random number generator.

Remark. If we can find the exact eigenvalues and the exact eigenfunctions of the integral equation (7), then the proposed method is the best possible. However, in most cases, it is impossible to find the exact eigenvalues and eigenfunctions of Eq. (7). A loss of accuracy is incurred when approximate methods are used. In this case, it is sometimes advisable to apply the following method.

2. Modeling by Fourier Series Expansion

The covariance function B(t, s) of the process $\xi(t)$ is continuous. It can therefore be expanded in a Fourier series that converges in L^2 on $[0, T] \times [0, T]$:

$$B(t,s) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{mn} \cos\frac{\pi m t}{T} \cos\frac{\pi n s}{T}, \tag{9}$$

$$a_{mn} = \frac{4}{T^2}\rho_{mn} \int_0^T\int_0^T B(t,s)\cos\frac{\pi m t}{T}\cos\frac{\pi n s}{T} dt ds,$$

$$\rho_{mn} = \begin{cases} 1/4, & m=n=0, \\ 1/2, & m>0,\ n=0 \text{ or } m=0\ n>0, \\ 1, & m>0,\ n>0. \end{cases} \tag{10}$$

Note that by positive semidefiniteness of B(t, s) we always have $a_{mn} \geq 0$. Therefore, by Karhunen's theorem [8] $\xi(t)$ is representable on [0, T] by a mean-square convergent series

$$\xi(t) = \sum_{n=0}^{\infty} \eta_n \cos\frac{\pi n t}{T}, \tag{11}$$

where η_n, $n = 0,\ldots,\infty$, is a sequence of jointly normal random variables such that $M\eta_n = 0$, $M\eta_n\eta_m = a_{mn}$. Denote $\hat{\xi}N(t) = \sum_{n=1}^{N} \eta_n \cos\frac{\pi n t}{T}$. Then

$$\int_0^T (\xi(t) - \hat{\xi}_N(t))^2 dt = \int_0^T \left(\sum_{n=N+1}^{\infty} \eta_n \cos\frac{\pi n t}{T} dt = \right.$$
$$= \sum_{k=N+1}^{\infty} \eta_k^2 \int_0^T \cos^2\frac{\pi n t}{T} dt = \frac{T}{2}\sum_{k=N+1}^{\infty} \eta_k^2 .$$

If in inequality (2) of the lemma we now set $\xi_k = (T/2)^{1/2}\eta_k$ and pass to the limit as $n \to \infty$, then noting that

$$M\xi_k^2 = \frac{T}{2} a_{kk},\ r^2(k,l) = (M\xi_k\xi_l)^2 = \left(\frac{T}{2}\right)^2 a_{kl}^2 ,$$

we obtain for $z > 0$ the inequality

$$P\left\{\frac{\frac{T}{2}\sum_{k=N+1}^{\infty}\eta_k^2 - \frac{T}{2}\sum_{k=N+1}^{\infty}a_{kk}}{\frac{T}{2}\left(\sum_{k=N+1}^{\infty}\sum_{l=N+1}^{\infty}a_{kl}^2\right)^{1/2}} > z\right\} \le \exp\left\{-\frac{z}{2}\right\}(z+1)^{1/2}.$$

Hence it follows that for $z > 0$ we have the inequality

$$P\left\{\int_0^T(\xi(t) - \hat{\xi}_N(t))^2dt > z\left(\sum_{k=N+1}^{\infty}\sum_{l=N+1}^{\infty}a_{kl}^2\right)^{1/2}\frac{T}{2} + \right. \tag{12}$$
$$\left. + \frac{T}{2}\sum_{k=N+1}^{\infty}a_{kk}\right\} \le \exp\left\{-\frac{z}{2}\right\}(z+1)^{1/2}.$$

As in the previous case, application of inequality (12) produces the required model of the process $\xi(t)$. Given $0 < p < 1$, we find z_p such that $\exp\left\{-\frac{z_p}{2}\right\}(z_p+1)^{1/2} = p$. Then given ε we find the minimal N_ε such that

$$\left[z_p\left(\sum_{k=N_\varepsilon+1}^{\infty}\sum_{l=N_\varepsilon+1}^{\infty}a_{kl}^2\right)^{1/2} + \sum_{k=N_\varepsilon+1}^{\infty}a_{kk}\right]\frac{T}{2} < \varepsilon .$$

This N_ε satisfies inequality (1). Thus $\hat{\xi}N_\varepsilon(t) = \sum_{k=1}^{N_\varepsilon}\eta_k\cos\frac{\pi kt}{T}$ can be used as a model of $\xi(t)$ with reliability $(1 - p)$ and accuracy ε.

The model $\hat{\xi}_{N_\varepsilon}(t)$ is constructed in the following way. Let $A = \left|\left|a_{kl}\right|\right|_{k,l=1}^{N_\varepsilon}$ be the covariance matrix of the vector $\vec{\eta}^T = (\eta_1, \eta_2, \dots, \eta_{N_\varepsilon})$. Since A is positive semidefinite, we can find an orthogonal matrix P that diagonalizes A: $P^TAP = R$ [6]. Let $\vec{\theta}$ be a Gaussian vector, $M\vec{\theta} = 0$, such that $\vec{\theta} = P^T\vec{\eta}(\vec{\eta} = P\vec{\theta})$. Then $\operatorname{cov}\vec{\theta} = P^T \operatorname{cov}\vec{\eta}P = P^TAP = R$. Hence, the components of the vector $\vec{\theta}$ are independent normal variates with known variances. Therefore, using a random number generator, we first generate θ_i and then find the vector $\vec{\eta}$ from the formula $\vec{\eta} = P\vec{\theta}$. After that the model $\hat{\xi}_{N_\varepsilon}(t)$ is constructed.

Remark. The weakness of this method is that the determination of the matrix P and the computation of the vector $\vec{\eta}$ is not simple. Note that for each new N_ε we have to find a new matrix P. This method is therefore useful when N_ε are small and also when $a_{mn} = 0$ for $m \neq n$.

REFERENCES

1. I. N. Zelepugina and Yu. V. Kozachenko, "Estimating modeling accuracy of random variables in L^p spaces, $p > 1$," Issled. Oper. ASU, No. 21, 10-14 (1988).
2. S. M. Ermakov and G. A. Mikhailov, A Course in Simulation [in Russian], Moscow (1982).
3. Yu. V. Kozachenko and A. A. Pashko, "Simulation of Gaussian stationary stochastic processes representable by stochastic integrals," in: Theory and Applications of Simulation [in Russian], Novosibirsk (1988), pp. 10-24.
4. I. N. Zelepugina and Yu. V. Kozachenko, "On rate of convergence of Karhunen–Loève expansions of Gaussian stochastic processes," Teor. Veroyatn. Mat. Stat., No. 38, 41-51 (1988).
5. E. G. Besklinskaya and Yu. V. Kozachenko, "Convergence in Orlicz space norms and Levy–Baxter theorems," Teor. Veroyatn. Mat. Stat., No. 35, 3-6 (1986).
6. R. Bellman, Introduction to Matrix Analysis, McGraw-Hill, New York (1970).
7. M. Loève, Probability Theory [Russian translation], Moscow (1962).
8. I. I. Gikhman and A. V. Skorokhod, Theory of Stochastic Processes [in Russian], Vol. 1, Moscow (1971).