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The ill-posedness of the sampling problem and regularized sampling algorithm

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ABSTRACT

In this paper the ill-posedness of the sampling problem is discussed. The restoration algorithm in Shannon's Sampling Theorem is analyzed. A regularized sampling algorithm for band-limited signals is presented. The convergence of the regularized sampling algorithm is studied and compared with Shannon's Sampling Theorem by some examples.

1. Introduction

In this paper, the ill-posedness of the sampling problem is analyzed in theory and by examples in detail. A regularized sampling algorithm is presented with the proof of the convergence property and experimental results.

First, we describe the band-limited signals and the sampling theorem for function $f \in L^2(\mathbf{R})$. The details can be seen in [1].

Definition. A function $f \in L^2(\mathbf{R})$ is said to be Ω-band-limited if

$$\hat{f}(\omega) = 0, \quad \forall \omega \notin [-\Omega, \Omega].$$

Here $\hat{f}$ is the *Fourier transform* of f:

$$\mathcal{F}(f)(\omega) = \hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t) e^{i\omega t}\, dt, \quad \omega \in \mathbf{R}.$$

We then have the inversion formula:

$$\mathcal{F}^{-1}(\hat{f})(t) = f(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \hat{f}(\omega) e^{-i\omega t}\, d\omega, \quad \text{a.e. } t \in \mathbf{R}.$$

Remark. The definition of Fourier transform for a function $f \in L^2(\mathbf{R})$ can be seen in [1].

In this paper, we will consider the sampling problem:

$$\text{given } f(nh), \quad \text{find } f(t) \tag{1}$$

where h is the step size of sampling.

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There are more general sampling problems in which the samples are the standard inner product of the signal and the basis functions $\langle\varphi(t-n), f(t)\rangle$. Ideal sampling case, nonideal sampling case and non-band-limited case are discussed in [2]. We will just consider the ideal sampling, i.e., $\varphi(t)=\delta(t)$ for band-limited signals. However we will show the ill-posedness by examples in the noisy case and present a different algorithm to find the stable solution of the sampling problem.

For a band-limited signal $f \in L^2(\mathbf{R})$, we have the following sampling theorem [1,3].

Shannon Sampling Theorem. *The Ω_0-band-limited signal $f(t) \in L^2(\mathbf{R})$ can be exactly reconstructed from its samples $f(nh_0)$, and*

$$f(t)=\sum_{n=-\infty}^{\infty}\frac{\sin\Omega_0(t-nh_0)}{\Omega_0(t-nh_0)}f(nh_0)$$

where $h_0 := \pi/\Omega_0$.

Remark 1. To include the case of oversampling, let $h := \pi/\Omega$, where $\Omega \geqslant \Omega_0$. We have the formula

$$f(t)=\sum_{n=-\infty}^{\infty}\frac{\sin\Omega(t-nh)}{\Omega(t-nh)}f(nh)$$

where $h \leqslant h_0$.

Remark 2. According to the sampling theorem in [1], the series in the sampling theorem above converges uniformly on $\mathbf{R}$, so $f(t)$ is continuous.

In many practical problems, the samples $\{f(nh)\}$ are noisy:

$$f(nh)=f_E(nh)+\eta(nh) \tag{2}$$

where $\{\eta(nh)\}$ is the noise

$$\left|\eta(nh)\right| \leqslant \delta \tag{3}$$

and $f_E \in L^2$ is the exact band-limited signal. f_E is continuous as well by Remark 2 below the sampling theorem. The ill-posedness of the generalized sampling problem [4] is discussed in [5] and [6]. In [6–8] it was pointed out that the extrapolation problem and the generalized sampling problem are ill-posed, but the sampling problem and interpolation problems are well-posed. In [14], band-limited extrapolation methods are presented based on MSEF and Kalman filtering techniques when the observations contain noise.

If the samples are noisy however, the sampling problem is an ill-posed problem. We will present an example to negate the conclusion in [6–8]. So the solution is not reliable in the noisy case due to the ill-posedness. It is pointed out that the formula in the sampling theorem is not stable in [9], but there are no examples of noise that are given to show the ill-posedness. The regularization method is introduced to solve ill-posed problems [10]. In [11], Smale and Zhou presented a regularized solution by finding the minimum of a functional. This requires solving an Euler equation to find the minimum. In [2], the regularizing operator is also constructed by finding the minimum of a smoothing functional in which a regularization term is added to the data fitting error. In a restrictive choice such that the regularization term is a Sobolev norm according to [10], the regularized solution can be given by the regularized Fourier transform of [12]. In this paper, the regularized sampling algorithm will be given by the regularized Fourier transform of [12] directly. The process of finding the minimum of the smoothing functional is omitted. So the method of computation is more concise.

In this paper, we will study the properties of the sampling operator, the ill-posedness of the sampling problem, and we will find a reliable algorithm for this ill-posed problem by the regularized Fourier transform of [12]. In Section 2 we give the property of the sampling operator. In Section 3, the ill-posedness is analyzed. The regularized sampling algorithm and the proof of its convergence property are in Section 4. Finally, the experimental results of numerical examples are given in Section 5.

2. The property of the sampling operator

In this section, we discuss the property of the sampling operator from L^2 to l^2 where l^2 is the space $\{a(n)\colon n \in \mathbf{Z}\}$ of sequences such that

$$\sum_{n\in\mathbf{Z}}\left|a(n)\right|^2<\infty.$$

The norm of l^2 is defined by $\|\mathbf{a}\|_{l^2}^2=\sum_{n\in\mathbf{Z}}|a(n)|^2$.

Let

$$L_2^{BL}(\Omega) := \{f \in L^2 \mid f \text{ is } \Omega\text{-band-limited}\}$$

and the norm in $L_2^{BL}(\Omega)$ is defined to be L^2-norm. We can see that each function in $L_2^{BL}(\Omega)$ is continuous since it is band-limited. For $f \in L_2^{BL}$, we define the sampling operator $\mathcal{S}$:

$$\mathcal{S}f := \{\dots, f(-nh), \dots, f(-h), f(0), f(h), \dots, f(nh)\dots\}. \tag{4}$$

By the following theorem we will see that the sampling operator $\mathcal{S}$ is an operator from L_2^{BL} to l^2.

Theorem 1. *For band-limited signals, the operator $\mathcal{S}$ from $L_2^{BL}(\Omega)$ to l^2 is a continuous operator and*

$$\|\mathcal{S}f\| = \frac{1}{\sqrt{h}}\|f\|$$

where $h = \frac{\pi}{\Omega}$.

The proof is in Appendix A.

Remark. By the theorem, if η is band-limited, the error energy

$$\|\eta\|_{L^2}^2 = h \sum_{n=-\infty}^{\infty} |\eta(nh)|^2$$

so $\sum_{n=-\infty}^{\infty} |\eta(nh)|^2$ must be close to zero to guarantee the accuracy. But this is not easily satisfied in practice.

3. The ill-posedness of the sampling problem

The concept of ill-posed problems was introduced in [10]. Here we borrow the following definition from it:

Definition 3.1. Assume $\mathcal{A}: D \to U$ is an operator in which D and U are metric spaces with distances $\rho_D(*,*)$ and $\rho_U(*,*)$, respectively. The problem

$$\mathcal{A}z = u \tag{5}$$

of determining a solution z in the space D from the "initial data" u in the space U is said to be well-posed on the pair of metric spaces (D, U) in the sense of Hadamard if the following three conditions are satisfied:

i) For every element $u \in U$ there exists a solution z in the space D; in other words, the mapping $\mathcal{A}$ is surjective.
ii) The solution is unique; in other words, the mapping $\mathcal{A}$ is injective.
iii) The problem is stable in the spaces (D, U): $\forall \epsilon > 0$, $\exists \delta > 0$, such that

$$\rho_U(u_1, u_2) < \delta \quad \Rightarrow \quad \rho_D(z_1, z_2) < \epsilon.$$

In other words, the inverse mapping $\mathcal{A}^{-1}$ is uniformly continuous.

Problems that violate any of the three conditions are said to be ill-posed.

In this section, we discuss the ill-posedness of the sampling problem and the stability of the formula in Shannon's Sampling Theorem in the pair of spaces of (C_{BL}, l^∞) where

$$C_{BL} := \{f \in L^2\colon\ f \text{ is } \Omega\text{-band-limited}\}$$

and the norm of C_{BL} is defined by

$$\|f(t)\|_{C_{BL}} := \max_{t\in\mathbf{R}} |f(t)|.$$

The operator $\mathcal{A}$ in (5) is the operator $\mathcal{S}$ defined by (4) in Section 2. But the operator $\mathcal{S}$ is the sampling operator from C_{BL} to l^∞.

As usual, l^∞ is the space $\{a(n)\colon\ n \in \mathbf{Z}\}$ of bounded sequences. The norm of l^∞ is defined by

$$\|\mathbf{a}\|_{l^\infty} := \sup_{n\in\mathbf{Z}} |a(n)|.$$

By the sampling theorem in [1], since $f(t)$ is continuous, we can see that Shannon reconstruction formula is the inverse of the sampling operator. The ill-posedness can be seen next.

i) The existence condition is not satisfied.

We can choose the sampling $\{f(nh)\}$ to be the samples of a signal whose frequency distribution is not in $[-\Omega, \Omega]$. This is equivalent to the fact $\mathcal{S}(C_{BL}) \neq l^\infty$ where $\mathcal{S}(C_{BL})$ is the range of $\mathcal{S}$. We can prove this by choosing an element $\mathbf{a} \in l^\infty$ such that $a(n)$ does not converge to 0 as $n \to \infty$. Then $\mathbf{a} \notin \mathcal{S}(C_{BL})$.

ii) The uniqueness condition is satisfied.

This is obvious since $f(nh) \equiv 0$ implies $f(t) \equiv 0$ for band-limited functions in L^2. So $\mathcal{S}$ is injective.

iii) The stability condition is not satisfied. In other words, $\mathcal{S}^{-1}$ is not continuous from $\mathcal{S}(C_{BL})$ to C_{BL}.

This can be seen from the next example.

Example. Assume the noise is $\eta(nh) = \epsilon\, \mathrm{sign}\{\sin\Omega(t_0 - nh)/\Omega(t_0 - nh)\}$ where t_0 is a given point in the time domain and ϵ is close to zero. Then the noise signal by using Shannon's Sampling Theorem is

$$\eta(t) = \sum_{n=-\infty}^{\infty} \eta(nh) \frac{\sin\Omega(t - nh)}{\Omega(t - nh)} = \epsilon \sum_{n=-\infty}^{\infty} \frac{\sin\Omega(t - nh)}{\Omega(t - nh)} \,\mathrm{sign}\left[\frac{\sin\Omega(t_0 - nh)}{\Omega(t_0 - nh)}\right].$$

We can see that $|\eta(nh)| \leqslant \epsilon$. However, the noise at $t = t_0$ in the sampling theorem is

$$\eta(t_0) = \epsilon \sum_{n=-\infty}^{\infty} \left|\frac{\sin\Omega(t_0 - nh)}{\Omega(t_0 - nh)}\right| = \infty.$$

Also at any point $t = t_0 + k\pi/\Omega$, $k \in \mathbf{Z}$,

$$\begin{aligned}\eta(t_0 + k\pi/\Omega) &= \epsilon \sum_{n=-\infty}^{\infty} (-1)^k \frac{\sin\Omega(t_0 - nh)}{\Omega(t_0 + k\pi/\Omega - nh)} \,\mathrm{sign}\left[\frac{\sin\Omega(t_0 - nh)}{\Omega(t_0 - nh)}\right] \\ &= (-1)^k \epsilon \sum_{n=-\infty}^{\infty} \frac{\Omega(t_0 - nh)}{\Omega(t_0 + k\pi/\Omega - nh)} \left|\frac{\sin\Omega(t_0 - nh)}{\Omega(t_0 - nh)}\right| = \pm\infty.\end{aligned}$$

So this is an ill-posed problem.

Remark 1. The noise here is just for analysis in theory. In practical problems, the noise is generally random. We can see the case of white noise in the experimental section.

Remark 2. Since the sampling problem is ill-posed in the noise cases, the solution may not exist and the error can be very large due to the violation of existence condition and stability condition. In the next section, we will find a stable solution by the regularization.

4. The regularized sampling algorithm

First, we consider the regularized Fourier transform [12]:

$$\hat{f}_\alpha(\omega) = \int_{-\infty}^{\infty} \frac{f(t) e^{i\omega t}\, dt}{1 + 2\pi\alpha + 2\pi\alpha t^2},$$

where $\alpha > 0$ is the regularization parameter. Here $\hat{f}_\alpha(\omega)$ is the minimizer of a smoothing functional. We have proved $\hat{f}_\alpha(\omega)$ converges to the exact Fourier transform as the error of $f(t)$ approaches to zero. In [12], we have successfully used the regularized Fourier transform in extrapolation. So the weight function

$$\frac{1}{1 + 2\pi\alpha + 2\pi\alpha t^2}$$

is helpful to solve ill-posed problems. In this paper, we will use it to restore the signal from its sampling. In the sampling problem, the weight function becomes

$$\frac{1}{1 + 2\pi\alpha + 2\pi\alpha (nh)^2}.$$

Based on the regularized Fourier transform and the Shannon's Sampling Theorem we construct the regularized sampling formula:

$$f_\alpha(t) = \sum_{n=-\infty}^{\infty} \frac{\sin\Omega(t - nh)}{\Omega(t - nh)} \frac{f(nh)}{1 + 2\pi\alpha + 2\pi\alpha (nh)^2} \tag{6}$$

where $f(nh)$ is given in (1).

Since the exact signal f_E is in L^2, we can assume the signal is ϵ-concentrated in the interval $[-T, T]$:

$$\int_{|t|\geqslant T} \left|f_E(t)\right|^2 dt \leqslant \epsilon.$$

Since f_E is Ω-band-limited, it is easy to see that

$$f_E \in C_{BL}.$$

In order to prove the convergence property of this regularized sampling formula in $[-T, T]$, we need some lemmas. Then we construct a regularized sampling theorem.

Lemma 1.

$$\hat{K}(\omega) := \mathcal{F}\left[\frac{1}{1+2\pi\alpha+2\pi\alpha t^2}\right] = \frac{1}{2a\alpha}e^{-a|\omega|}$$

where $a = (\frac{1+2\pi\alpha}{2\pi\alpha})^{\frac{1}{2}}$.

The proof is in Appendix A.

Lemma 2.

$$\hat{f}_{E\alpha}(\omega) := \mathcal{F}\left[\frac{1}{1+2\pi\alpha+2\pi\alpha t^2} f_E(t)\right] = \frac{1}{4\pi a\alpha}\int_{-\Omega}^{\Omega} \hat{f}_E(u)e^{-a|u-\omega|}\,du.$$

The proof is in Appendix A.

Next, we need the following theorem in [13].

Theorem 2. *For a non-band-limited signal $f(t)$, if its Fourier transform $\hat{f}(\omega) \in L^1$, then*

$$\left|f(t) - \sum_{n=-\infty}^{\infty} \frac{\sin\Omega(t-nh)}{\Omega(t-nh)} f(nh)\right| \leqslant \frac{1}{\pi}\int_{|\omega|>\Omega} \left|\hat{f}(\omega)\right| d\omega. \tag{7}$$

Lemma 3.

$$d := \left|\sum_{n=-\infty}^{\infty} \frac{\sin\Omega(t-nh)}{\Omega(t-nh)} \frac{f_E(nh)}{1+2\pi\alpha+2\pi\alpha(nh)^2} - \frac{f_E(t)}{1+2\pi\alpha+2\pi\alpha t^2}\right| = O\left(\alpha^{\frac{1}{2}}\right).$$

The proof is in Appendix A.

Lemma 4.

$$\left|\sum_{n=-\infty}^{\infty} \frac{\sin\Omega(t-nh)}{\Omega(t-nh)} \frac{\eta(nh)}{1+2\pi\alpha+2\pi\alpha(nh)^2}\right| = O(\delta) + O(\delta/\sqrt{\alpha})$$

where η and δ are given in (2) *and* (3) *in Section* 1.

The proof is in Appendix A.

Now we have the convergence property of the regularized sampling formula.

Theorem 3. *If we choose $\alpha = \alpha(\delta)$ such that $\alpha(\delta) \to 0$ and $\delta/\sqrt{\alpha(\delta)} \to 0$ as $\delta \to 0$, then $f_\alpha(t) \to f_E(t)$ in $[-T, T]$ as $\delta \to 0$. Also we have the estimation*

$$\left\| f_\alpha(t) - f_E(t)\right\|_{C[-T,T]} \leqslant O\left(\alpha^{\frac{1}{2}}\right) + O(\delta) + O(\delta/\sqrt{\alpha}).$$

The proof is in Appendix A.

Remark. According to this theorem, α should be chosen by δ which is the error bound of the sampling. We can choose $\alpha = k\delta^{\mu}$ where $k > 0$ and $0 < \mu < 2$. Then $f_\alpha(t) \to f_E(t)$ in $[-T, T]$ as $\delta \to 0$.

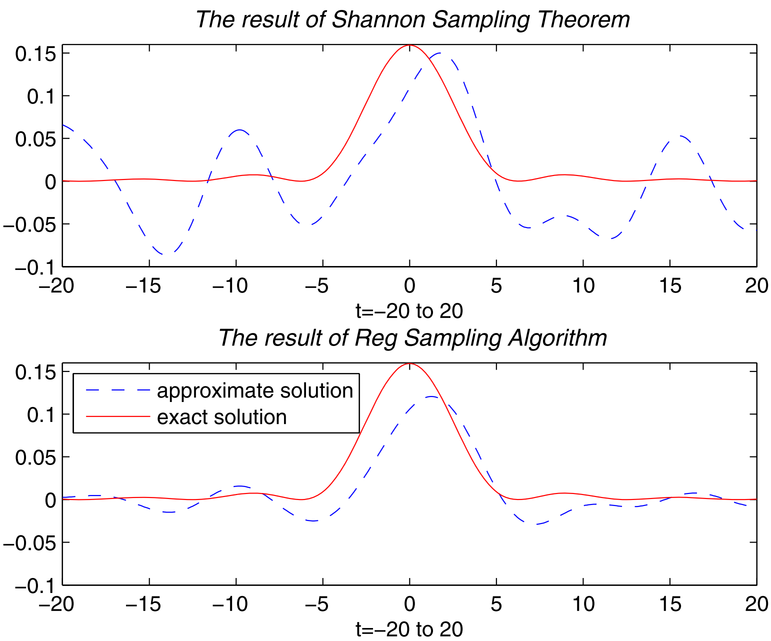

Fig. 1. The numerical results of Example 1.

5. Experimental results

In this section, we give some examples to show that the regularized sampling algorithm is more effective in controlling the noise than Shannon's Sampling Theorem.

We first describe the algorithm. In computation, only finite terms can be used in (6). So we choose a large integer N, and use next formula in computation:

$$f_\alpha(t) = \sum_{n=-N}^{N} \frac{\sin\Omega(t-nh)}{\Omega(t-nh)} \frac{f(nh)}{1+2\pi\alpha+2\pi\alpha(nh)^2} \tag{8}$$

where $f(nh)$ is the noisy sampling data given in (2). The regularization parameter α can be chosen according to the noise level in Theorem 3.

In each example,

$$\mathrm{SNR} := \frac{\sum_{n=-N}^{N}[f_E(nh)]^2}{\sum_{n=-N}^{N}[\eta(nh)]^2}$$

is given.

Suppose the exact signal in Examples 1 and 2 is

$$f_E(t) = \frac{1-\cos t}{\pi t^2}.$$

Then

$$\hat{f}_E(\omega) = \begin{cases} 1-|\omega|, & \omega \in [-\Omega,\Omega], \\ 0, & \omega \notin [-\Omega,\Omega], \end{cases}$$

where $\Omega = 1$.

Example 1. We consider the noise

$$\eta(nh) = \epsilon\,\mathrm{sign}\big\{\sin\Omega(t_0-nh)/\Omega(t_0-nh)\big\}$$

where $h = \pi/\Omega = \pi$, $t_0 = 20\pi$, and $\epsilon = 0.05$. This noise is used in the analysis of the stability of the algorithm in Shannon's Sampling Theorem. SNR = 13.5095.

The result of Shannon's Sampling Theorem and the result of the regularized sampling algorithm with $\alpha = 0.005$ are in Fig. 1. Their Fourier transforms are in Fig. 2. The error energies of the results are 0.9217 and 0.2415 respectively.

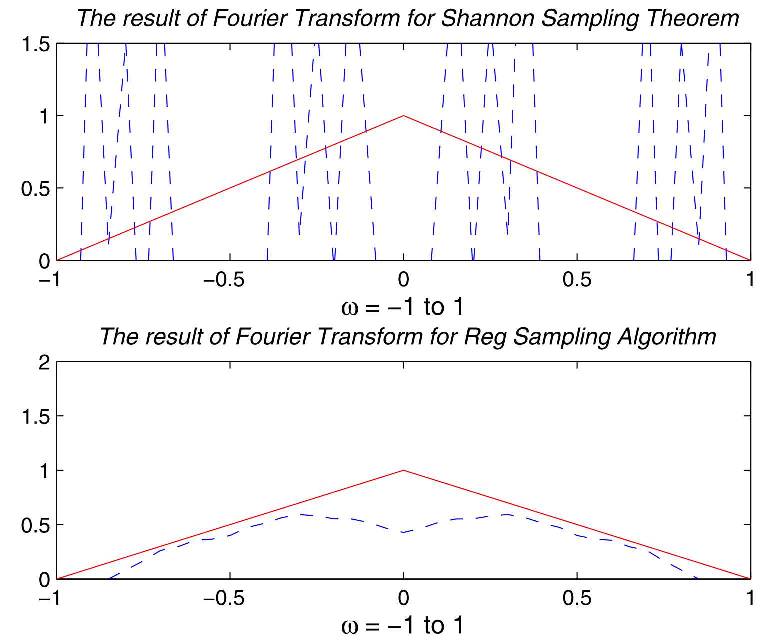

Fig. 2. The Fourier transform of Example 1.

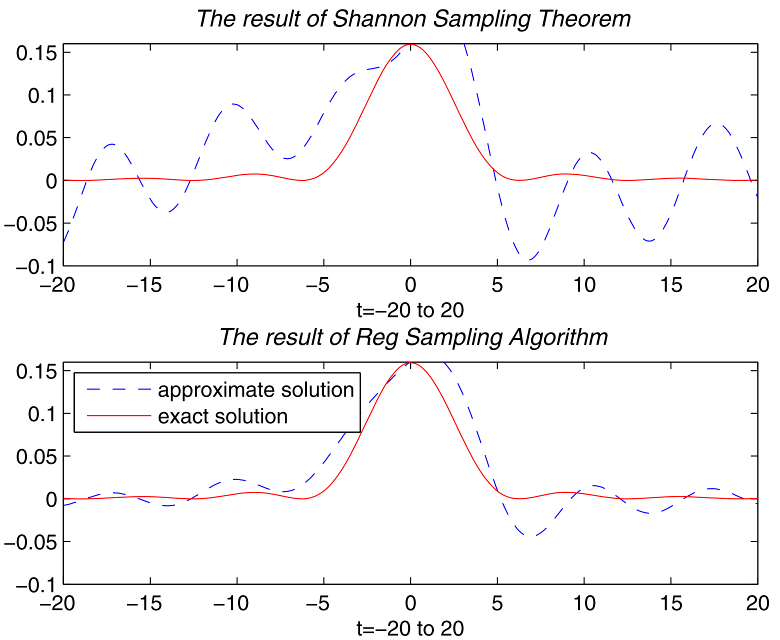

Fig. 3. The numerical results of Example 2.

Example 2. We consider the noise to be white noise that is uniformly distributed in $[-0.1, 0.1]$. SNR $= 11.1704$.

The result of Shannon's Sampling Theorem and the result of the regularized sampling algorithm with $\alpha = 0.005$ are in Fig. 3. Their Fourier transforms are in Fig. 4. The error energies of the results are 1.1974 and 0.1466 respectively.

Suppose the exact signal in Examples 3 and 4 is

$$f_E(t) = \frac{1}{\pi}\operatorname{sinc}(t).$$

Then

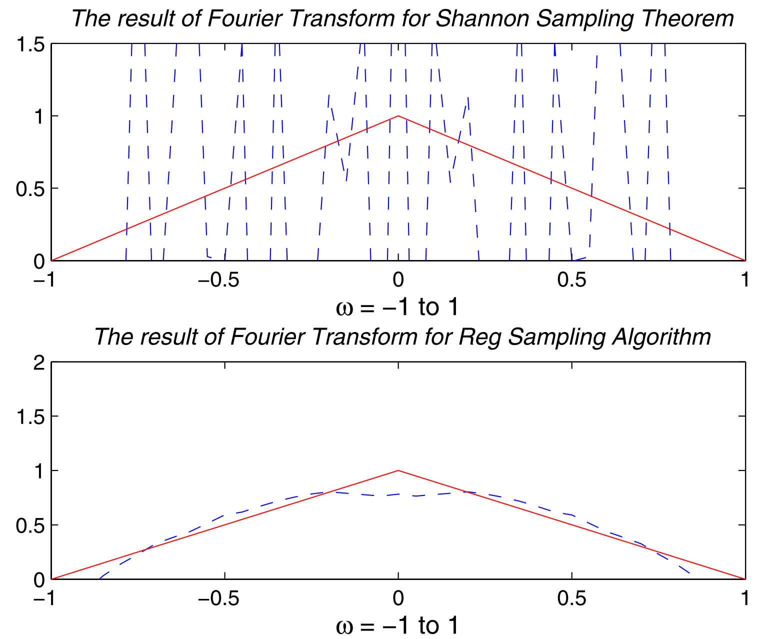

Fig. 4. The Fourier transform of Example 2.

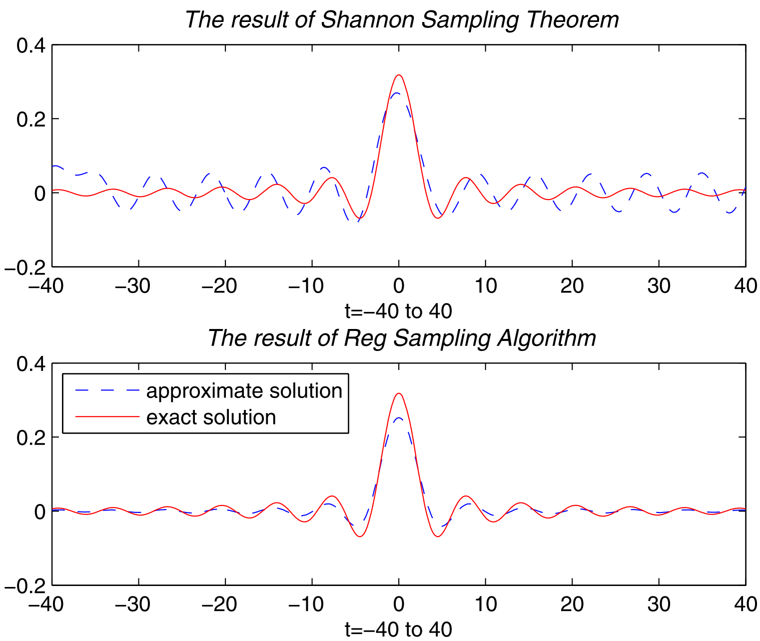

Fig. 5. The numerical results of Example 3.

$$\hat{f}_E(\omega) = \begin{cases} 1, & \omega \in [-\Omega, \Omega], \\ 0, & \omega \notin [-\Omega, \Omega], \end{cases}$$

where $\Omega = 1$.

Example 3. We consider the noise to be

$$\eta(nh) = \epsilon \operatorname{sign}\bigl\{\sin \Omega(t_0 - nh)/\Omega(t_0 - nh)\bigr\}$$

where $h = \pi/\Omega = \pi$, $t_0 = 30\pi$, and $\epsilon = 0.05$. SNR = 40.5285.

The result of Shannon's Sampling Theorem and the result of the regularized sampling algorithm with $\alpha = 0.01$ are in Fig. 5. Their Fourier transforms are in Fig. 6. The error energies of the results are 0.6147 and 0.0998 respectively.

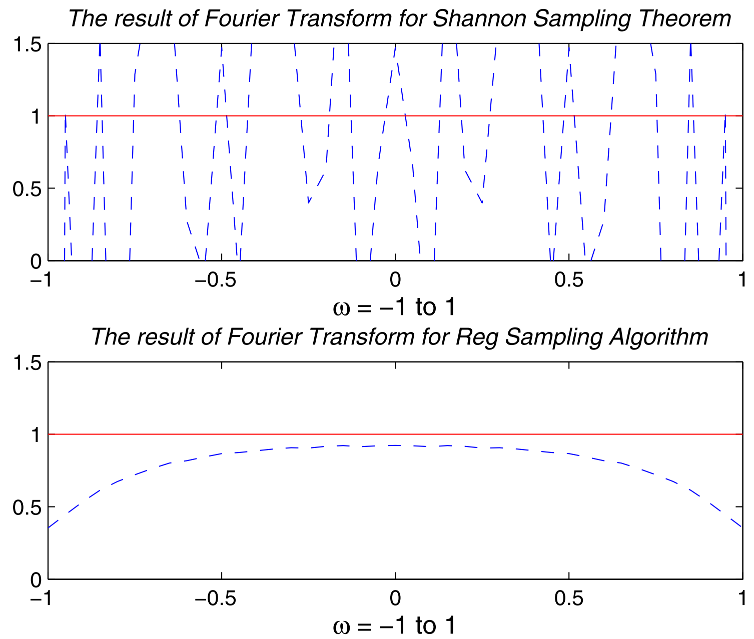

Fig. 6. The Fourier transform of Example 3.

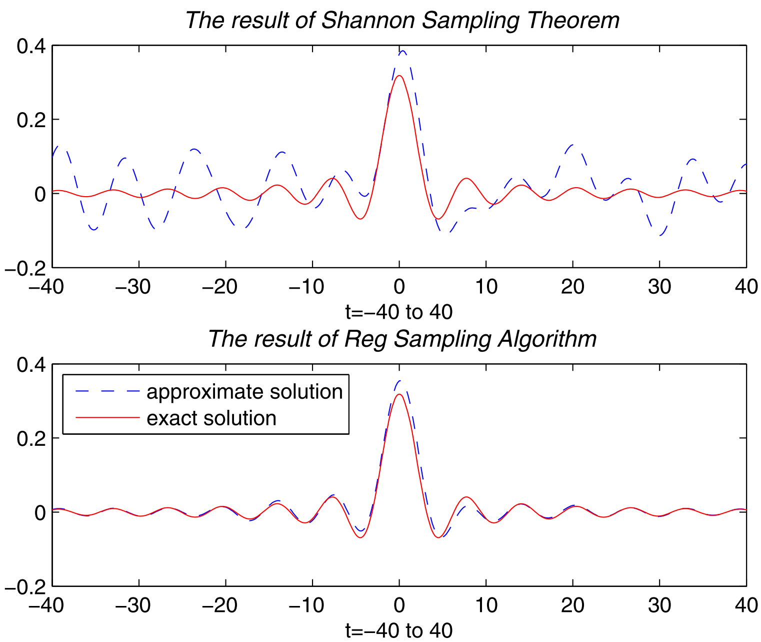

Fig. 7. The numerical results of Example 4.

Example 4. We consider the noise to be white noise that is uniformly distributed in $[-0.1, 0.1]$. SNR $= 74.0844$.

The result of Shannon's Sampling Theorem and the result of the regularized sampling algorithm with $\alpha = 0.01$ are in Fig. 7. Their Fourier transforms are in Fig. 8. The error energies of the results are 1.6428 and 0.0601 respectively.

Suppose the exact signal in Examples 5 and 6 is

$$f_E(t) = 2\frac{(t^2 - 4\pi^2)\cos t - 2t\sin t}{(t^2 - 4\pi^2)^2}.$$

Then

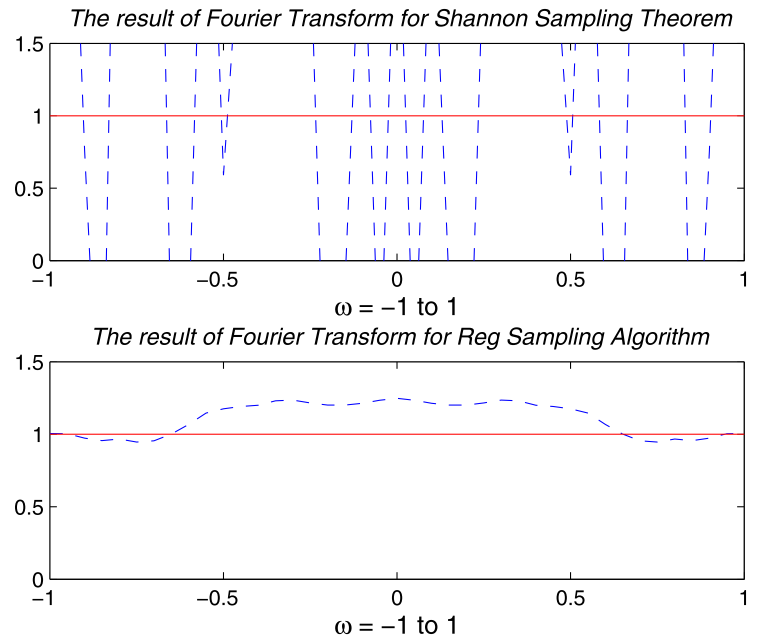

Fig. 8. The Fourier transform of Example 4.

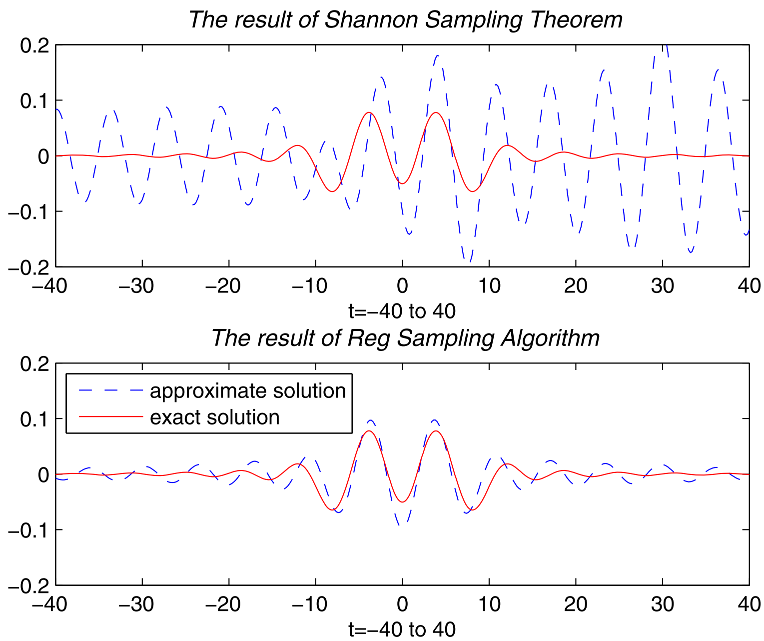

Fig. 9. The numerical results of Example 5.

$$\hat{f}_E(\omega) = \begin{cases} \omega \sin 2\pi\omega, & \omega \in [-\Omega, \Omega], \\ 0, & \omega \notin [-\Omega, \Omega], \end{cases}$$

where $\Omega = 1$.

Example 5. We consider the noise

$$\eta(nh) = \epsilon \operatorname{sign}\{\sin \Omega(t_0 - nh)/\Omega(t_0 - nh)\}$$

where $h = \pi/\Omega = \pi$, $t_0 = 30$, and $\epsilon = 0.05$. SNR = 7.6363.

The result of Shannon's Sampling Theorem and the result of the regularized sampling algorithm with $\alpha = 0.005$ are in Fig. 9. Their Fourier transforms are in Fig. 10. The error energies of the results are 3.3138 and 0.1374 respectively.

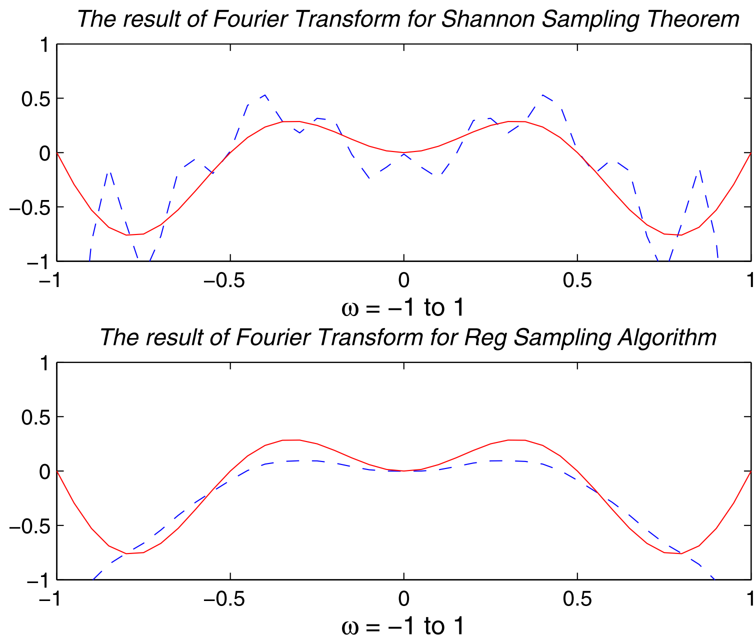

Fig. 10. The Fourier transform of Example 5.

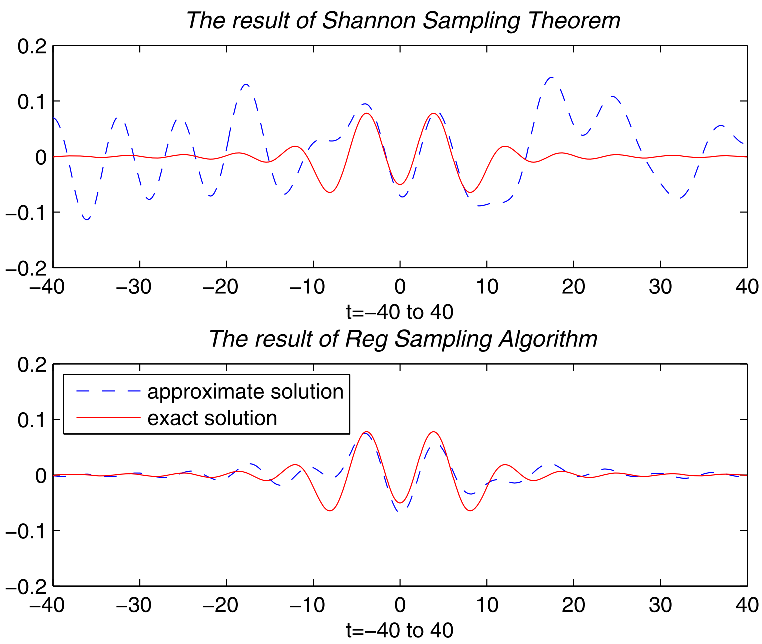

Fig. 11. The numerical results of Example 6.

Example 6. We consider the noise to be white noise that is uniformly distributed in $[-0.1, 0.1]$. SNR $= 11.3240$.

The result of Shannon's Sampling Theorem and the result of the regularized sampling algorithm with $\alpha = 0.005$ are in Fig. 11. Their Fourier transforms are in Fig. 12. The error energies of the results are 1.5452 and 0.1106 respectively.

Remark 1. In Example 5, if we choose $\epsilon = 0.01$, then SNR $= 190.9071$. The error energies of the results are 0.1658 and 0.0489 respectively. If we choose $\epsilon = 0.005$, then SNR $= 763.6285$. The error energies of the results are 0.0621 and 0.0467 respectively. So the SNR is so high as 763.6285 to make the performance improvement of regularization negligible.

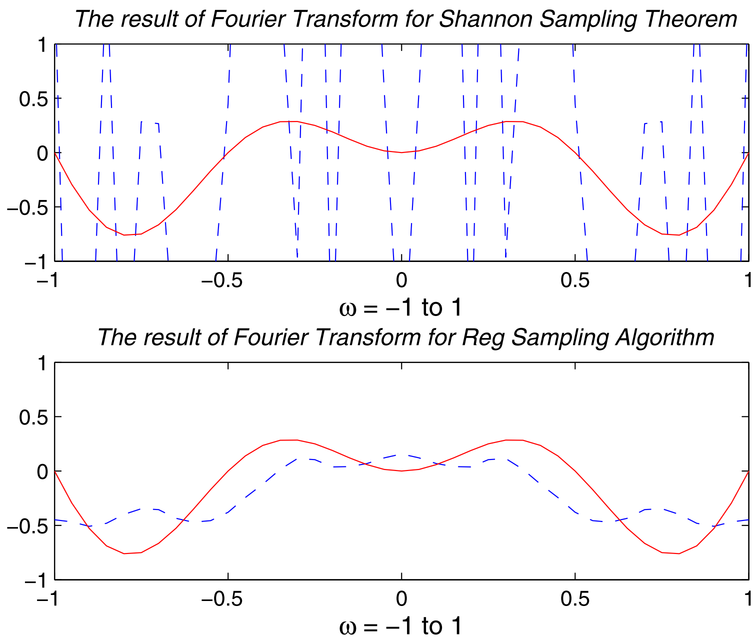

Fig. 12. The Fourier transform of Example 6.

Remark 2. Here we choose the regularization parameter by the experiment. A bad choice of α affects the algorithm performance seriously. In general α depends on the error energy. If the error energy is known, α can be determined by discrepancy principle [15]. The GCV and L-curve can be used [16,17] if the error energy is not known.

Now we consider more examples and compare the regularized sampling formula (6) with the Tikhonov regularization method [2,10]. We introduce the Tikhonov regularization method next.

First we consider the equation

$$\frac{1}{2\pi}\int_{-\Omega}^{\Omega}\hat{f}(\omega)e^{-i\omega kh}\,d\omega = f(kh), \quad k\in\mathbf{Z}.$$

Then we write it by the finite difference method

$$A\hat{f} = f$$

where

$$A = \left(\frac{1}{2\pi}e^{-i\omega_j t_k}h_w\right)_{(2N+1)\times(2M+1)}, \quad h_w = \Omega/M,$$

$$\omega_j = jh_w, \quad j=-M,\dots,M \quad \text{and} \quad t_k = kh, \quad k=-N,\dots,N,$$

and

$$\hat{f} = (\hat{f}_{-M},\dots,\hat{f}_0,\dots,\hat{f}_M),$$

$$f = (f_{-N},\dots,f_0,\dots,f_N).$$

Define the smoothing functional

$$M^{\alpha}[\hat{f}, f] = \|A\hat{f} - f\|^2 + \alpha\|\hat{f}\|^2.$$

Here the norms of $\hat{f}$ and f are defined by

$$\|\hat{f}\|^2 = \sum_{j=-M}^{M}|\hat{f}_j|^2, \qquad \|f\|^2 = \sum_{k=-N}^{N}|f_k|^2.$$

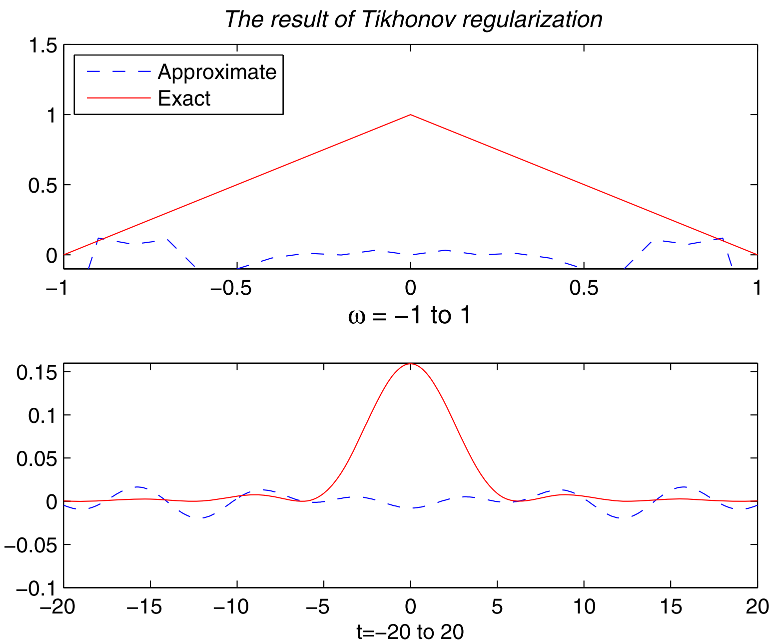

Fig. 13. The numerical results of Example 7 with $h = \pi$.

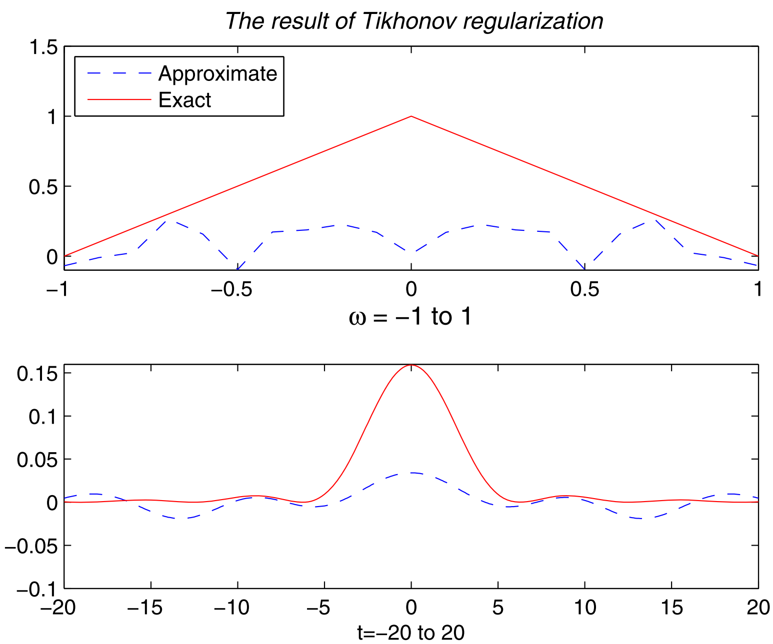

Fig. 14. The numerical results of Example 7 with $h = \pi/5$.

We can find the approximate Fourier transform by minimizing $M^\alpha[\hat{f}, f]$. And the minimizer is the solution of the Euler equation

$$\left(A^H A + \alpha I\right)\hat{f} = A^H f.$$

By using the inverse Fourier transform we can find the approximate signal in the time domain. The experiment result can be seen from the next example.

Example 7. We consider the signal and the noise in Example 1 and choose the same regularization parameter α. The results of the Tikhonov regularization method for the step size of sampling $h = \pi$ and $h = \pi/5$ are in Figs. 13 and 14 respectively.

Remark 3. We can see that the numerical result of $h=\pi$ is bad since the step size of sampling is large. In the case of the step size of sampling $h=\pi/5$ the numerical result is a little better. But it is not as good as the result by the regularized sampling algorithm in Example 1.

6. Conclusion

The sampling problem is ill-posed. The noise can give rise to the error that is infinite at infinitely many points if the formula in Shannon Sampling Theorem is used. The regularized sampling algorithm is presented. The convergence property is proved and tested by some examples. The convergence property and the numerical results show that the regularized sampling algorithm is excellent in computation in the noisy cases.

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Appendix A

Proof of Theorem 1.

$$\mathcal{S}f=\left\{\ldots,f(-nh),\ldots,f(-h),f(0),f(h),\ldots,f(nh),\ldots\right\}$$

where

$$f(nh)=\frac{1}{2\pi}\int_{-\Omega}^{\Omega}\hat{f}(\omega)e^{-i\omega nh}\,d\omega.$$

By the Parseval equality of Fourier series

$$\sum_{n=-\infty}^{\infty}\left|\frac{1}{\sqrt{2a}}\int_{-a}^{a}g(x)e^{in\frac{\pi}{a}x}\right|^2=\|g\|^2_{L^2[-a,a]}$$

for any $g\in L^2[-a,a]$, since $\hat{f}\in L^2[-\Omega,\Omega]$, we have

$$\sum_{n=-\infty}^{\infty}\left|f(nh)\right|^2=\frac{1}{2\pi h}\int_{-\Omega}^{\Omega}\left|\hat{f}(\omega)\right|^2d\omega.$$

By the Parseval equality of Fourier transform

$$\left\|\hat{f}(\omega)\right\|^2=2\pi\|f\|^2_{L^2},$$

we have

$$\sum_{n=-\infty}^{\infty}\left|f(nh)\right|^2=\frac{1}{2\pi h}\int_{-\Omega}^{\Omega}\left|\hat{f}(\omega)\right|^2d\omega=\frac{1}{h}\|f\|^2_{L^2}.$$

Therefore

$$\|\mathcal{S}f\|^2=\frac{1}{h}\|f\|^2.\qquad\square$$

Proof of Lemma 1.

$$\mathcal{F}\left[\frac{1}{1+2\pi\alpha+2\pi\alpha t^2}\right]=\frac{1}{2\pi\alpha}\mathcal{F}\left[\frac{1}{a^2+t^2}\right]=\frac{1}{2\pi\alpha}\frac{\pi}{a}e^{-a|\omega|}=\frac{1}{2a\alpha}e^{-a|\omega|}.\qquad\square$$

Proof of Lemma 2. In the equality

$$\mathcal{F}[fg]=\frac{1}{2\pi}\hat{f}*\hat{g}$$

let $f := f_E$ and $g(t) := \frac{1}{1+2\pi\alpha+2\pi\alpha t^2}$. We have the convolution form of $\hat{f}_{E\alpha}$

$$\hat{f}_{E\alpha} = \frac{1}{2\pi}\hat{f}_E * \hat{K}$$

where $\hat{K}$ is given in Lemma 1. □

Proof of Lemma 3. Assume $|\hat{f}_E(\omega)| \leqslant M$ where $M = const. > 0$.

By the error estimation (7)

$$d \leqslant \frac{1}{\pi}\int\limits_{|\omega|>\Omega} \left|\hat{f}_{E\alpha}(\omega)\right| d\omega.$$

By Lemma 2

$$\left|\hat{f}_{E\alpha}(\omega)\right| = \frac{1}{4\pi a\alpha}\left|\int\limits_{-\Omega}^{\Omega} \hat{f}_E(u)e^{-a|u-\omega|}\,du\right| \leqslant \frac{M}{4\pi a\alpha}\int\limits_{-\Omega}^{\Omega} e^{-a|\omega-u|}\,du.$$

Therefore we have

$$\begin{aligned}
d &\leqslant \frac{1}{\pi}\int\limits_{|\omega|>\Omega} \left|\hat{f}_{E\alpha}(\omega)\right| d\omega \leqslant \frac{M}{4\pi^2 a\alpha}\int\limits_{|\omega|>\Omega} d\omega \int\limits_{-\Omega}^{\Omega} e^{-a|\omega-u|}\,du \\
&= \frac{M}{2\pi^2 a\alpha}\int\limits_{\omega>\Omega} d\omega \int\limits_{-\Omega}^{\Omega} e^{a(u-\omega)}\,du = \frac{M}{2\pi^2 a^2\alpha}\left(e^{a\Omega} - e^{-a\Omega}\right)\int\limits_{\omega>\Omega} e^{-a\omega}\,d\omega \\
&= \frac{M}{2\pi^2 a^2\alpha}\left(e^{a\Omega} - e^{-a\Omega}\right)\frac{1}{a}e^{-a\Omega} = \frac{M}{2\pi^2 a^3\alpha}\left(1 - e^{-2a\Omega}\right) = O\left(\alpha^{\frac{1}{2}}\right). \qquad \square
\end{aligned}$$

Proof of Lemma 4.

$$\left|\sum_{n=-\infty}^{\infty} \frac{\sin\Omega(t-nh)}{\Omega(t-nh)}\frac{\eta(nh)}{1+2\pi\alpha+2\pi\alpha(nh)^2}\right| \leqslant \left|\frac{\sin\Omega t}{\Omega t}\frac{\eta(0)}{1+2\pi\alpha}\right| + \left|\sum_{n\neq 0} \frac{\sin\Omega(t-nh)}{\Omega(t-nh)}\frac{\eta(nh)}{1+2\pi\alpha+2\pi\alpha(nh)^2}\right|$$

where

$$\begin{aligned}
\left|\sum_{n\neq 0} \frac{\sin\Omega(t-nh)}{\Omega(t-nh)}\frac{\eta(nh)}{1+2\pi\alpha+2\pi\alpha(nh)^2}\right| &\leqslant 2\delta\sum_{n=1}^{\infty}\frac{1}{1+2\pi\alpha+2\pi\alpha(nh)^2} \\
&\leqslant \frac{2\delta}{h}\int\limits_0^{\infty}\frac{dx}{1+2\pi\alpha+2\pi\alpha x^2} = \frac{\delta}{h\sqrt{2\pi\alpha(1+2\pi\alpha)}} = O(\delta/\sqrt{\alpha}).
\end{aligned}$$

So

$$\left|\sum_{n=-\infty}^{\infty} \frac{\sin\Omega(t-nh)}{\Omega(t-nh)}\frac{\eta(nh)}{1+2\pi\alpha+2\pi\alpha(nh)^2}\right| = O(\delta) + O(\delta/\sqrt{\alpha}). \qquad \square$$

Proof of Theorem 3.

$$\begin{aligned}
f_\alpha(t) - f_E(t) &= \sum_{k=-\infty}^{\infty} \frac{\sin\Omega(t-nh)}{\Omega(t-nh)}\frac{f_E(nh)+\eta(nh)}{1+2\pi\alpha+2\pi\alpha(nh)^2} - f_E(t) \\
&= \sum_{n=-\infty}^{\infty} \frac{\sin\Omega(t-nh)}{\Omega(t-nh)}\frac{f_E(nh)}{1+2\pi\alpha+2\pi\alpha(nh)^2} - f_E(t) \\
&\quad + \sum_{n=-\infty}^{\infty} \frac{\sin\Omega(t-nh)}{\Omega(t-nh)}\frac{\eta(nh)}{1+2\pi\alpha+2\pi\alpha(nh)^2}
\end{aligned}$$

$$= \sum_{n=-\infty}^{\infty} \frac{\sin \Omega(t-nh)}{\Omega(t-nh)} \frac{f_E(nh)}{1+2\pi\alpha+2\pi\alpha(nh)^2} - \frac{f_E(t)}{1+2\pi\alpha+2\pi\alpha t^2}$$

$$+ \frac{f_E(t)}{1+2\pi\alpha+2\pi\alpha t^2} - f_E(t) + \sum_{n=-\infty}^{\infty} \frac{\sin \Omega(t-nh)}{\Omega(t-nh)} \frac{\eta(nh)}{1+2\pi\alpha+2\pi\alpha(nh)^2}$$

$$= \sum_{n=-\infty}^{\infty} \frac{\sin \Omega(t-nh)}{\Omega(t-nh)} \frac{f_E(nh)}{1+2\pi\alpha+2\pi\alpha(nh)^2} - \frac{f_E(t)}{1+2\pi\alpha+2\pi\alpha t^2}$$

$$+ \frac{(2\pi\alpha+2\pi\alpha t^2) f_E(t)}{1+2\pi\alpha+2\pi\alpha t^2} + \sum_{n=-\infty}^{\infty} \frac{\sin \Omega(t-nh)}{\Omega(t-nh)} \frac{\eta(nh)}{1+2\pi\alpha+2\pi\alpha(nh)^2}.$$

By Lemma 3,

$$\left| \sum_{n=-\infty}^{\infty} \frac{\sin \Omega(t-nh)}{\Omega(t-nh)} \frac{f_E(nh)}{1+2\pi\alpha+2\pi\alpha(nh)^2} - \frac{f_E(t)}{1+2\pi\alpha+2\pi\alpha t^2} \right| = O\left(\alpha^{\frac{1}{2}}\right).$$

By Lemma 4,

$$\left| \sum_{n=-\infty}^{\infty} \frac{\sin \Omega(t-nh)}{\Omega(t-nh)} \frac{\eta(nh)}{1+2\pi\alpha+2\pi\alpha(nh)^2} \right| = O(\delta) + O(\delta/\sqrt{\alpha}).$$

Since $t \in [-T, T]$,

$$\left| \frac{(2\pi\alpha+2\pi\alpha t^2) f_E(t)}{1+2\pi\alpha+2\pi\alpha t^2} \right| = O(\alpha).$$

Therefore

$$\| f_\alpha(t) - f_E(t) \|_{C[-T,T]} \leqslant O\left(\alpha^{\frac{1}{2}}\right) + O(\alpha) + O(\delta) + O(\delta/\sqrt{\alpha}) = O\left(\alpha^{\frac{1}{2}}\right) + O(\delta) + O(\delta/\sqrt{\alpha}). \qquad \square$$

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