Exact Results on the Random Potts Model

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We present a number of exact results on the random-bond, q-state Potts model. The quenched model on any finite planar graph or lattice is shown to obey a duality relation for general type of bond-randomness. In the annealed case, the solution of the model reduces to that of the regular (nonrandom) Potts model on the corresponding lattice. Explicit knowledge of the critical parameters of the q-state Potts model in two dimensions allows us to evaluate exactly the phase diagram of the annealed model on the square, triangular and honeycomb lattices. We discuss the behavior near the (random) critical point and comment on the relationship between the quenched and annealed systems. The exact phase diagram of the annealed system is obtained for the bond-diluted model and the spin-glass model with and without dilutions.

I. Introduction

In this paper we consider the random q-state Potts model with Hamiltonian $\mathcal{H}$ given by

$$\mathcal{H} = - \sum_{\langle ij \rangle} J_{ij}\, \delta_{kr}(\sigma_i, \sigma_j) \tag{1.1}$$

where $\sigma_i = 1, 2, \ldots, q$ is a spin variable specifying the spin state at the i-th site, J_{ij} is a random interaction governed by a probability distribution $P(J_{ij})$, independent of the interacting neighboring pairs $\langle i,j \rangle$ and subject to

$$\int dJ\, P(J) = 1. \tag{1.2}$$

For discrete distribution $P(J)$ takes the form

$$P(J) = \sum_{\alpha=1}^{n} p_\alpha \delta(J - J_\alpha), \tag{1.3}$$

subject to

$$\sum_{\alpha=1}^{n} p_\alpha = 1. \tag{1.4}$$

The specific forms of the probability distribution that have been considered in the past [1-9] are described by

$$P(J_{ij}) = p\,c\, \delta(J_{ij} - J) + p(1-c)\, \delta(J_{ij} + J) + (1-p)\, \delta(J_{ij}), \tag{1.5}$$

which reduces to a spin-glass model [1] with $p=1$, $q=2$, to the bond-diluted Potts model [2-5] with $c=1$ and the percolated (dilute) spin-glass [6, 7] including the random-bond Ising model [8, 9] with $q=2$.

The randomness can be either quenched (frozen in positions) or annealed (in thermal equilibrium with the surroundings). In the quenched case the free energy is given by

$$F^Q = \int \prod_{\langle ij \rangle} [P(J_{ij})\, dJ_{ij}] \ln \sum_{\{\sigma\}} e^{-\beta\mathcal{H}}, \tag{1.6a}$$

while in the annealed case the free energy can be written as

$$F^A = \ln \sum_{\{\sigma\}} \int \prod_{\langle ij \rangle} [P(J_{ij})\, dJ_{ij}]\, e^{-\beta\mathcal{H}}. \tag{1.6b}$$

The convexity of the logarithm then dictates the inequality

$$F^Q \leq F^A. \tag{1.7}$$

In a realistic model the randomness is quenched, but the annealed model is mathematically more tractable. Indeed, very few exact results on quenched systems are available. On the other hand the annealed models have not been generally discussed. In this paper we shall obtain exact results for both the quenched and the annealed models.

Quenched systems have often been studied with the use of the replica trick [1, 10], a device which often complicates the matter and requires great care in its application [11]. In Sect. II we shall derive an exact duality relation valid for the planar random Potts model without having recourse to the replica trick. This generalizes a duality relation obtained earlier by Schwartz [12] for the Ising model ($q=2$). The annealed $q=2$ model has been studied by Thorpe and Beeman [13]. In Sect. III we formulate the thermodynamics of the annealed model for general values of q along the lines of [13]. The formulation is subsequently applied to the particular discrete distribution (1.5) for which the exact phase diagram is obtained for all values of q. Comparisons of this phase diagram with previous results on the quenched model are made. Discussions and comments are given in Sect. IV.

II. Duality Relation for the Quenched System

For a graph or lattice G of N sites write the free energy (1.6a) of the quenched system as

$$F^Q = N f_G[q; P(K)], \qquad (2.1)$$

where $K = \beta J$ and $\beta = 1/k_B T$. Then we have the following theorem:

1. Duality Theorem

The free energy of the quenched model (1.1) on a planar graph G, subject to a probability distribution $P(K)$, with positive support, is related to that of a quenched model on D, the dual of G, subject to a probability distribution $P_D(K^*)$, through the relation

$$N f_G[q; P(K)] = (1 - N_D) \ln q + E \int dK \, P(K) \ln(e^K - 1)$$
$$+ N_D f_D[q; P_D(K^*)], \qquad (2.2)$$

with

$$(e^K - 1)(e^{K^*} - 1) = q,$$
$$P_D(K^*) |dK^*| = P(K) |dK|. \qquad (2.3)$$

Here N_D and E are respectively the numbers of sites and edges of D.

2. Proof

Consider first the discrete probability distribution (1.3) with $J_\alpha > 0$, and denote the corresponding free energy by $f_G(q; \{p\}, \{K\})$. Then, according to (1.6), we have

$$N f_G(q; \{p\}, \{K\}) = \sum_{M_\alpha}{}' [\prod_\alpha p_\alpha^{M_\alpha}] F_G^{(M)}(q; \{K\}), \qquad (2.4)$$

where $K_\alpha = \beta J_\alpha$ and the summation is taken over all values of $M_\alpha = 0, 1, 2, \ldots, E$, subject to $\sum_{\alpha=1}^n M_\alpha = E$. Also,

$$F_G^{(M)}(q; \{K\}) = \sum_{\{M\}} \ln Z_G^{\{M\}}(q; \{K\}), \qquad (2.5)$$

is a sum of the global free energies over all $E! / [\prod_\alpha M_\alpha!]$ configurations $\{M\}$ of the random distribution in which there are M_α interactions J_α, the function $Z_G^{\{M\}}(q; \{K\})$ being the partition function for a given configuration.

Now, each of the partition functions $Z_G^{\{M\}}(q; \{K\})$ satisfies the duality relation [14]

$$Z_G^{\{M\}}(q; \{K\}) = q^{1-N_D} [\prod_\alpha (e^{K_\alpha} - 1)^{M_\alpha}] Z_D^{\{M\}}(q; \{K^*\}), \qquad (2.6)$$

with

$$(e^{K_\alpha} - 1)(e^{K_\alpha^*} - 1) = q. \qquad (2.7)$$

Specifically, the duality (2.6) relates the partition functions of two dual models whose interactions K_α and K_α^* are $1-1$ corresponding. Substitution of (2.5) and (2.6) in (2.4) yields

$$N f_G(q; \{p\}, \{K\}) = (1 - N_D) \ln q + N_D f_D(q; \{p\}, \{K^*\})$$
$$+ \sum_{\gamma=1}^n \ln(e^{K_\gamma} - 1) \sum_{M_\alpha}{}' [\prod_\alpha p_\alpha^{M_\alpha}] M_\gamma \frac{E!}{\prod_\beta M_\beta!}. \qquad (2.8)$$

To evaluate the sum on the right-hand side of (2.8) note that

$$\sum_{M_\alpha}{}' [\prod_\alpha p_\alpha^{M_\alpha}] M_\gamma \frac{E!}{\prod_\beta M_\beta!} = \frac{\partial}{\partial x_\gamma} \left(\sum_{\alpha=1}^n p_\alpha x_\alpha \right)^E_{x_1 = \ldots = x_1 = 1}$$
$$= E p_\gamma. \qquad (2.9)$$

Then one can rewrite (2.8) as

$$N f_G(q; \{p\}, \{K\}) = (1 - N_D) \ln q + E \sum_{\alpha=1}^n p_\alpha \ln(e^{K_\alpha} - 1)$$
$$+ N_D f_D(q; \{p\}, \{K^*\}). \qquad (2.10)$$

The theorem is now established from (2.10) upon passing to a continuum limit. We mention in

passing that, as expected, the theorem can also be proven by a careful application of the replica trick, detail of which we shall not reproduce.

It is of interest to inquire whether the duality relation can be used to locate the critical point. Note, first, that the theorem is valid only for positive interactions. Consider the discrete form (2.10). On the square lattice the model is self-dual. But the duality relation reflects merely a special symmetry of the free energy which does not necessarily cover the whole parameter space. Consequently, the duality relation does not generally determine the critical point. An exception is the case of the two-valued discrete distribution, K_1 and K_2, with equal probabilities $p_1 = p_2 = \frac{1}{2}$. In this case the duality can be written as

$$(K_1, K_2) \to (K_1^*, K_2^*) \to (K_2^*, K_1^*), \qquad (2.11)$$

where the last step follows from the square lattice symmetry. The mapping now covers the entire parameter space. If a unique transition occurs, then it must occur at the self-dual point

$$(e^{K_1^c} - 1)(e^{K_2^c} - 1) = q. \qquad (2.12)$$

This result has been reported previously [4, 12, 15].

III. Annealed Model

In the annealed model (1.6b) the interactions are subject to the distribution $P(K_{ij})$ individually and, as a result, the partition function is given by a sum of the configurational partition functions weighted by the appropriate probabilities [as in (2.4)]. It can then be established quite easily that, in this case, the problem is equivalent to a regular (nonrandom) Potts model with an effective interaction K given by

$$e^K = \int P(K_{ij}) e^{K_{ij}} dK_{ij}. \qquad (3.1)$$

The situation is relatively uninteresting because of this straightforward connection.

To reflect more realistically the thermal equilibrium between the randomness and the spin degree of freedom, we consider a slightly different annealed model in which $P(K_{ij})$ specifies the global distribution of the interactions. The model is most conveniently treated in a grand canonical formulation. For the discrete distribution (1.3) one starts in such a treatment from the Hamiltonian

$$\mathscr{H} = \sum_{\langle ij \rangle} \mathscr{H}_{ij}, \qquad (3.2)$$

where the interaction $\mathscr{H}_{ij}$ between the neighboring sites i and j is given by

$$-\beta \mathscr{H}_{ij} = \sum_{\alpha=1}^{n} [K_\alpha \delta_{kr}(\sigma_i, \sigma_j) + \Delta_\alpha] \delta_{kr}(\xi_{ij}, \alpha). \qquad (3.3)$$

Here we have introduced a chemical potential $-\Delta_\alpha/\beta$ for the interaction K_α and assigned a positive integer $\xi_{ij} = 1, 2, \ldots, n$ to specify the interaction between sites i and j. In terms of these variables the grand partition function is

$$\Xi(q; \{K\}, \{\Delta\}) = \sum_{\{\xi\}} \sum_{\{\sigma\}} e^{-\beta \mathscr{H}}. \qquad (3.4)$$

The chemical potentials $\Delta_\alpha = \Delta_\alpha(q; \{p\}, \{K\})$ are to be determined by the constraints

$$p_\alpha = \frac{1}{E} \frac{\partial}{\partial \Delta_\alpha} \ln \Xi(q; \{K\}, \{\Delta\}), \qquad (3.5)$$

where $\alpha = 1, 2, \ldots, n$ and E is the total number of edges (bonds) of the lattice. The internal energy per bond is

$$U(q; \{p\}, \{K\}) = -\frac{1}{E} \sum_\alpha J_\alpha \frac{\partial}{\partial K_\alpha} \ln \Xi(q; \{K\}, \{\Delta\}), \qquad (3.6)$$

with the understanding that the variables Δ_α are to be eliminated by using (3.5) *after* taking the derivatives. The specific heat per bond is then given the usual expression

$$C(q; \{p\}, \{K\}) = -k\beta^2 \frac{\partial}{\partial \beta} U(q; \{p\}, \{K\}). \qquad (3.7)$$

The analysis of this annealed model is based on the observation that summations over the variables ξ_{ij} in (3.4) can be carried out explicitly. Writing

$$\sum_{\xi_{ij}} e^{-\beta \mathscr{H}_{ij}} = A(\{\Delta\}) \exp[K \delta_{kr}(\sigma_i, \sigma_j)], \qquad (3.8)$$

where $-\beta \mathscr{H}_{ij}$ is defined by (3.3), we obtain

$$A(\{\Delta\}) = \sum_{\alpha=1}^{n} e^{\Delta_\alpha},$$

$$e^K = \sum_{\alpha=1}^{n} e^{K_\alpha + \Delta_\alpha} \Big/ \sum_{\alpha=1}^{n} e^{\Delta_\alpha}, \qquad (3.9)$$

and

$$\Xi(q; \{K\}, \{\Delta\}) = A^E Z(q; K). \qquad (3.10)$$

Here

$$Z(q; K) = \sum_{\{\sigma\}} \prod_{\langle ij \rangle} \exp[K \delta_{kr}(\sigma_i, \sigma_j)], \qquad (3.11)$$

is the partition function of the underlying (non-random) Potts model.

Substituting (3.10) into (3.5) and making use of (3.9), we obtain

$$p_\alpha = (\sum_\beta e^{\Delta_\beta})^{-1} e^{\Delta_\alpha} [1 + (e^{K_\alpha - K} - 1) \varepsilon(K)], \qquad (3.12)$$

where

$$\varepsilon(K) \equiv \frac{1}{E} \frac{\partial}{\partial K} \ln Z(q; K), \qquad (3.13)$$

is essentially the internal energy per bond of the underlying Potts model. Eliminating the chemical potentials Δ_α in (3.12) gives

$$\sum_\alpha \frac{p_\alpha}{1 + (e^{K_\alpha - K} - 1) \varepsilon(K)} = 1. \qquad (3.14)$$

which determines $K = K(q; \{p\}, \{K\})$. An equivalent expression is

$$\sum_\alpha \frac{p_\alpha e^{K_\alpha - K}}{1 + (e^{K_\alpha - K} - 1) \varepsilon(K)} = 1. \qquad (3.15)$$

The equivalence between (3.14) and (3.15) can be established by combining (1.4) with (3.9).

It is clear from (3.9) that $K > 0$ if $K_\alpha \geq 0$ for all α. But K can have either sign if K_α includes positive as well as negative interactions. However, for $q = 2$ (Ising) and bipartite lattices we have the symmetry $\varepsilon(K) + \varepsilon(-K) = 1$. It can then be seen from (3.14) and (3.15) that if K is a solution for a given set of $\{p_\alpha, K_\alpha\}$, then $-K$ is a solution for a set in which the probabilities of K_α and $-K_\alpha$ are interchanged. Thus, if $K = K(p, c, K_1)$ is a solution for the distribution (1.5), then $K = -K(p, 1-c, K_1)$ is also a solution to (3.14). This symmetry proves useful in the considerations in Sect. IV.

It is now straightforward to follow our procedure and, using (3.6), obtain the internal energy

$$U(q; \{p\}, \{K\}) = -\varepsilon(K) \sum_{\alpha=1}^{n} \frac{p_\alpha J_\alpha e^{K_\alpha - K}}{1 + (e^{K_\alpha - K} - 1) \varepsilon(K)}. \qquad (3.16)$$

The entropy per bond is derived from the expression

$$S(q; \{p\}, \{K\}) = -\frac{1}{E} k \beta^2 \frac{\partial}{\partial \beta} \left[\frac{1}{\beta} \ln \Xi(q; \{K\}, \{\Delta\}) \right], \qquad (3.17)$$

which, after some manipulations, becomes

$$S(q; \{p\}, \{K\}) = S_{\text{Potts}}(q; K) + kK\varepsilon(K)$$
$$+ k\beta U(q; \{p\}, \{K\})$$
$$+ k\sum_\alpha p_\alpha \ln[1 + (e^{K_\alpha - K} - 1)\varepsilon(K)] - k\sum_\alpha p_\alpha \ln p_\alpha, \qquad (3.18)$$

where

$$S_{\text{Potts}}(q; K) = \frac{k}{E} \frac{\partial}{\partial K^{-1}} [K^{-1} \ln Z(q; K)]. \qquad (3.19)$$

In the case of a continuous distribution $P(J)$ the expressions (3.16) and (3.18) becomes

$$U[q; \beta, P(J)] = -\varepsilon(K) \int \frac{dJ P(J) J e^{\beta J - K}}{1 + (e^{\beta J - K} - 1) \varepsilon(K)}, \qquad (3.20)$$

and

$$S[q; \beta, P(J)] = S_{\text{Potts}}(q; K) + kK\varepsilon(K) + k\beta U[q; \beta, P(J)]$$
$$+ k\int dJ P(J) [1 + (e^{\beta J - K} - 1) \varepsilon(K)]$$
$$- k\int dJ P(J) \ln P(J). \qquad (3.21)$$

Here $K = K[q; \beta, P(J)]$ is to be determined from

$$\int \frac{dJ P(J)}{1 + (e^{\beta J - K} - 1) \varepsilon(K)} = 1, \qquad (3.22)$$

or, equivalently, from

$$\int \frac{dJ P(J) e^{\beta J - K}}{1 + (e^{\beta J - K} - 1)\varepsilon(K)} = 1. \qquad (3.23)$$

Equations (3.20)–(3.23) form the basis of the analysis of the annealed model (3.2). In particular, the specific heat per bond is obtained from (3.7) as

$$C[q; \beta, P(J)] = -k\beta^2 \varepsilon(1 - \varepsilon)$$
$$\cdot \frac{(I_1^2 - I_0 I_2) \varepsilon(1 - \varepsilon) - (I_1^2 - I_0 I_2 + I_2) \varepsilon'}{I_0 \varepsilon(1 - \varepsilon) + (1 - I_0) \varepsilon'} \qquad (3.24)$$

where $\varepsilon \equiv \varepsilon(K)$, $\varepsilon' \equiv \varepsilon'(K)$ and

$$I_m \equiv \int \frac{dJ P(J) J^m e^{\beta J - K}}{1 + (e^{\beta J - K} - 1) \varepsilon(K)}, \qquad m = 0, 1, 2. \qquad (3.25)$$

The above results can be used to construct the exact phase diagram of the annealed model, provided that the critical point of the underlying Potts model is exactly known. The situation parallels very much that of the dilute Potts model on a decorated lattice [16].

Define the phase boundary in the (J, q, T) or (J, q, β) space, where J denotes the set of parameters contained in the probability distribution $P(J)$, as the trajectory along which $U[q; \beta, P(J)]$ becomes nonanalytic in β. Observation of (3.10) then indicates that the phase boundary is given exactly by the trajectory along which $\varepsilon(K)$ is nonanalytic in β and this occurs at K_c, the critical point of the underlying Potts model.

We can, in particular, deduce the critical behavior of the specific heat $C(T)$ of the annealed model near the critical point T_c. Assume that for the underlying Potts model

$$\varepsilon(K) - \varepsilon(K_c) \sim |K - K_c|^{1-\alpha}, \qquad K \sim K_c, \qquad (3.26)$$

where $\alpha < 1$ ($\alpha' \leq 1$) is the specific heat exponent of the underlying Potts model. From (3.14) or (3.15) one finds also

$$\varepsilon(K) - \varepsilon(K_c) \sim |T - T_c|, \qquad T \sim T_c, \qquad (3.27a)$$

for $\alpha > 0$ while

$$\varepsilon(K) - \varepsilon(K_c) \sim |T - T_c|^{1-\alpha}, \qquad T \sim T_c, \qquad (3.27b)$$

for $\alpha < 0$. Expanding (3.24) about T_c and using (3.26) and (3.27) now yields

$$C(T) = C(T_c) + A_{\pm}|T - T_c|^{\alpha/(1-\alpha)} + B(T - T_c) + \dots, \quad (3.28a)$$

for $\alpha > 0$, while

$$C(T) = C(T_c) + A'_{\pm}|T - T_c|^{-\alpha} + \dots, \qquad (3.28b)$$

for $\alpha < 0$, where $A_{\pm}, A'_{\pm}$ and B are constants independent of T. The sign "$\pm$" differentiates the amplitudes above and below T_c respectively. Therefore, the specific heat is generally finite at T_c. When $\alpha < 0$, as expected for $q < 2$ [17], the specific heat anomaly of the annealed model is described by the critical exponent of the underlying system. For nonnegative values of α, corresponding to a diverging specific heat in the underlying model, the critical behavior of the annealed model is renormalized [18]. When $0 \leq \alpha \leq \frac{1}{2}$ the second term on the right-hand side of (3.28a) dominates and the specific heat has a cusp at T_c. The slope is infinite at the cusp. For $\alpha > \frac{1}{2}$ the leading correction to $C(T_c)$ is linear, and the specific heat and its derivative are continuous at T_c [19]. Nevertheless, in both cases, the singular part of the specific heat is described by the renormalized critical exponent

$$\alpha^A = -\alpha/(1-\alpha), \qquad (3.29)$$

in agreement with Fisher's theory of renormalized exponents [18].

IV. Phase Diagrams for the Annealed Model

The formulation developed in the preceeding section is now applied to the discrete distribution (1.5). In particular, the phase diagram of the annealed model (3.2) is obtained from (3.14) [or (3.15)] by setting $K = K_c$, where K_c is the critical point of the underlying Potts model.

Using (1.5) and (3.14) and after some manipulations, we find the following equation for determining the critical temperature $K_1^c = J/kT_c$ of the annealed model:
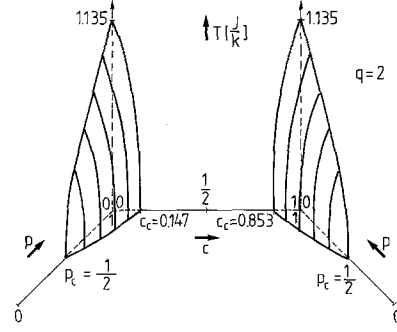
$$Ae^{K_1^c} + B + Ce^{-K_1^c} = 0. \qquad (4.1)$$

Fig. 1. Phase diagram of the $q = 2$ annealed model with the random distribution (1.5) on the square lattice. The two regions are ferro- and antiferromagnetically ordered; there is no spin glass phase
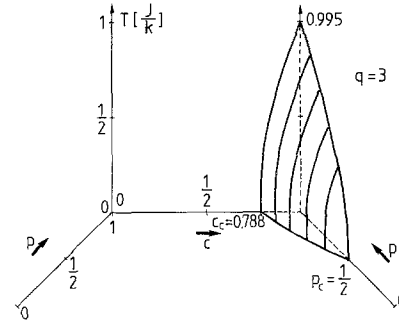

Fig. 2. Phase diagram of the $q = 3$ annealed model with the random distribution (1.5) on the square lattice. There is no antiferromagnetic ordering

The coefficients A, B and C are given explicitly by

$$A = u_c[u_c(c - \varepsilon_c)p + (1 - u_c)(1 - \varepsilon_c)(pc - \varepsilon_c)],$$
$$B = -(1 - u_c)[(1 - \varepsilon_c)^2 + \varepsilon_c^2 u_c^2] - u_c[(1 - \varepsilon_c) - \varepsilon_c u_c]p,$$
$$C = u_c\{u_c[(1 - c) - \varepsilon_c]p + (1 - u_c)(1 - \varepsilon_c)[p(1 - c) - \varepsilon_c]\},$$

$$(4.2)$$

where

$$\varepsilon_c = \varepsilon(K_c) \quad \text{and} \quad u_c = e^{-K_c}. \qquad (4.3)$$

Note that under the transformation $c \to 1 - c$ one has $A \to C$ and $C \to A$ so that $K_1^c \to -K_1^c$ in (4.1), reflecting the symmetry of the distribution (1.5).

The expression (4.1) gives the phase boundary of the annealed model (2.3) subject to the distribution (1.5), and is valid for arbitrary lattice G. The values of the critical parameters ε_c and u_c for various lattices are known [17, 20, 21]. Using these values, (4.1) gives the explicit dependence of T_c on p, c and q.

The phase diagram described by (4.1) is best visualized by plotting T_c as functions of p and c for fixed q. This is done for the square lattice in Figs. 1 and 2 for the $q = 2$ and $q = 3$ cases respectively.

For bipartite lattices the $(q=2)$ (Ising) phase diagram consists of two ordered regions (cf. Fig. 1) characterized by ferro- and antiferromagnetic orderings. There is no spinglass phase. The phase boundaries are determined from (4.1) with ε_c and u_c taking the respective values of the (ferro- and antiferromagnetic) Ising models. But since the two Ising models are related by spin symmetry, the two ordered regions are identical, being symmetric about the $c=1/2$ plane [cf. discussions after (3.15)].

When q changes continuously from the value $q=2$, we expect the boundaries of the two regions to change in a continuous fashion. However, the critical parameters of the antiferromagnetic Potts model are not known for $q \neq 2$. It is therefore not possible to follow the exact change of the antiferromagnetic branch of the phase boundary. But the antiferromagnetic region is expected to shrink in size as q increases, disappearing completely at some cutoff value of $q = q_0 < 3$. In particular since no antiferromagnetic transition is expected in two dimensions for $q = 3$ [17, 22], only the ferromagnetic branch of the phase boundary remains. This situation is shown in Fig. 2. The phase diagram for the triangular lattice, for both $q=2$ and $q=3$, is similar to that of Fig. 2 (without the antiferromagnetic region).

The intercepts of the phase boundary (4.1) with the coordinate planes (cf. Figs. 1 and 2) depict the various special models and considerations which we now describe.

1. Phase Diagram in the Ground State

The intercept of (4.1) with the plane $T_c = 0$ give the phase diagram of the ground state. This is obtained by setting $A = 0$ or

$$c = \varepsilon_c + (1 - \varepsilon_c) \frac{p_c(1-p)}{(1-p_c)p}. \tag{4.4}$$

This leads to the critical values

$$c_c = \varepsilon_c, \quad \text{for } p = 1, \tag{4.5}$$

$$p_c = \varepsilon_c(1 - u_c), \qquad \text{for } c = 1 \text{ (ferro)}, \tag{4.6a}$$

$$= (1 - \varepsilon_c^A)(1 - u_c^{A-1}), \quad \text{for } c = 0 \text{ (antiferro)}, \tag{4.6b}$$

beyond which an ordered state exists. Here p_c is also the critical probability beyond which a transition exists in the bond-diluted Potts model (see discussions below). We have also used ε_c^A and u_c^A to denote the critical parameters for the antiferromagnetic model. However, the only available data for the antiferromagnetic model are for $q=2$ bipartite lattices for which

$$\varepsilon_c^A = 1 - \varepsilon_c, \quad u_c^A = u_c^{-1}. \tag{4.7}$$

Table 1. Values of p_c

	Square	Triangular	HC	SC	BCC	FCC
$q=1$	0.5	0.347296	0.652704			
$q=2$	0.5	0.352208	0.647792	0.2383	0.1719	0.1152
$q=3$	0.5	0.355744	0.644256			
$q=4$	0.5	0.358601	0.641399			

Data for the three dimensional lattices are taken from Sykes et al. [23]

This implies that the two values of p_c in (4.6) are identical, as required by symmetry.

We list in Table 1 the values of p_c for various lattices. Data for the three-dimensional lattices are taken from Sykes et al. [23]. For two-dimensional lattices it can be readily verified using the duality relation of the Potts model [14] that

$$p_c + p_c^* = 1, \tag{4.8}$$

where $p_c^* = \varepsilon_c^*(1 - u_c^*)$ is the critical probability for the dual model. Equation (4.8) has been derived previously by Syozi and Miyazima [24] for $q=2$.

2. The Bond-Diluted Potts Model

The intercept of (4.1) with the plane $c=1$ $(c=0)$ describes the bond-diluted Potts model with ferromagnetic (antiferromagnetic) interactions. In this case (4.1) can be explicitly solved to give

$$e^{K_c^{\circ}(p)} = \frac{p + (1 - \varepsilon_c)(u_c^{-1} - 1)}{p - p_c} \tag{4.9}$$

which defines the transition temperature $T_c = T_c(p)$. Particularly, (4.9) implies that a transition exists only for $p > p_c$, where p_c is given by (4.6). Near $p = 1$ the transition temperature behaves as

$$T_c(p) = T_c(1) [1 - R(1-p) + O(1-p)^2], \tag{4.10}$$

where

$$R = \frac{1}{K_c} \frac{1 - e^{-K_c}}{1 - \varepsilon_c(1 - e^{-K_c})} = \frac{1}{K_c} \frac{p_c}{\varepsilon_c(1 - p_c)}, \tag{4.11}$$

is the initial slope of $T_c(p)$ at $p = 1$. Using the known values of ε_c, p_c and K_c [17, 20], we find

$$R = \frac{2}{\ln(1 + \sqrt{q})} \frac{\sqrt{q}}{1 + \sqrt{q}}, \quad (q \leq 4). \tag{4.12}$$

on the square lattice. This expression is the same as that found by Southern and Thorpe [2] using a perturbation approach. When $q=2$, moreover, (4.12) reduces to the expression obtained by Harris [25]

Table 2. Values of R

	Square	Triangular	HC	SC	BCC	FCC
$q=1$	1.4427	1.2472	1.7771			
$q=2$	1.3293	1.1878	1.5782	1.0605	1.0360	1.0223
$q=3$	1.2616	1.1511	1.4646			
$q=4$	1.2137	1.1246	1.3861			

Data for the three-dimensional lattices are taken from Sykes et al. [23]

Table 3. Values of S

	Square	Triangular	HC	SC	BCC	FCC
$q=1$	4.3281	3.1580	6.8938			
$q=2$	3.2091	2.7307	3.9455	2.1786	2.1123	2.0762
$q=3$	2.7183	2.4974	3.0478			
$q=4$	2.4273	2.3415	2.5898			

Data for the three-dimensional lattices are taken from Sykes et al. [23]

for the quenched Ising model. We argue that, in general, (4.11) and (4.12) gives exact values for the quenched dilute model as well, since the quenched and annealed systems are identical in the limit of small dilution [25, 26]. In Table 2 we present the numerical values of R obtained from (4.12).

The critical specific heat of the bond-diluted model can be computed from (3.24) in a straightforward fashion. For $q \leq 4$ [for which $\varepsilon'(K_c) = \infty$] we find after some algebra:

$$C(q; p, K_1^c) = k(K_1^c)^2$$
$$\cdot \frac{(p - p_c)[\varepsilon_c(1 - p_c) - (\varepsilon_c - p_c)(1 - p)]}{(1 - p_c)^2}$$
$$\frac{[\varepsilon_c(1 - p_c)^2 + (\varepsilon_c - p_c)(1 - p)]}{p_c^2(1 - p)} \quad (4.13)$$

where p_c is given by (4.6). Therefore the specific heat of the annealed bond-diluted model diverges only for $p = 1$.

3. The Spin-Glass Potts Model

The intercept of (4.1) with the plane $p = 1$ describes a "spin-glass" Potts model in which the interaction is a pure mixture of $+J_1$ and $-J_1$ strengths. In this case the coefficient (4.2) take a simpler form and (4.1) reduces to

$$u_c(c - \varepsilon_c) e^{K_1^c} - [1 - \varepsilon_c(1 + u_c^2)] +$$
$$[(1 - c) - \varepsilon_c] u_c e^{-K_1^c} = 0, \quad (4.14)$$

which for $c \geq \varepsilon_c$ has the positive solution

$$e^{K_1^c} = \frac{1 - \varepsilon_c(1 + u_c^2) + \{[1 - \varepsilon_c(1 - u_c^2)]^2 - 4u_c^2 c(1 - c)\}^{1/2}}{u_c(c - \varepsilon_c)}$$
$$\quad (4.15)$$

It is clear that there is no spin-glass phase. The transition temperature is defined by $kT_c(c) = JK_1^{-1}(c)$ and near $c = 1$ one has

$$T_c(c) = T_c(1)[1 - S(1 - c) + O(1 - c)^2], \quad (4.16)$$

where the slope S is given by

$$S = \frac{1}{K_c} \frac{1 - e^{-2K_c}}{1 - \varepsilon_c(1 - e^{-2K_c})}. \quad (4.17)$$

Using the known values of K_c and ε_c [17, 20] one obtains for the square lattice and $q < 4$

$$S = \frac{2(\sqrt{q} + 2)}{(\sqrt{q} + 1)\ln(\sqrt{q} + 1)}. \quad (4.18)$$

For the Ising model ($q = 2$) this reduces to the value obtained by Domany [27] using the replica method. The values of S for various lattices are listed in Table 3.

4. Tricritical Behavior of the Annealed Model

An interesting feature of the phase boundary (4.1) occurs when $q > 4$. In that case the underlying Potts model exhibits a first order transition with a non-zero latent heat

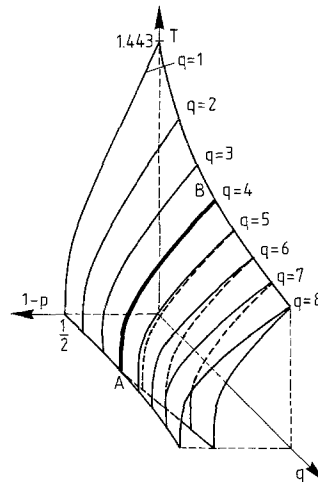$$\varepsilon(K_c +) - \varepsilon(K_c -) = \varepsilon_+ - \varepsilon_-. \quad (4.19)$$

Fig. 3. Phase diagram of the bond-diluted model ($c = 1$) in the (T, p, q) space. The surface of critical points ($q \leq 4$) splits into two coexistence surfaces along the line AB ($q = 4$). AB is a line of tricritical points

To each of the energies ε_+ and ε_- there corresponds a surface T_+ and T_- in the phase diagram, and the system separates into two coexisting phases between these two surfaces. The situation for the square lattice is shown in Fig. 3 for $c=1$. More generally if q is considered as a continuous parameter the two-phase region is bounded by two surfaces in the (T, p, c, q) space. These two surfaces coalesce at $q=4$ and for $q<4$ only one surface of critical points exists. The locus

$$T = T_c(q = 4; p, c), \tag{4.20}$$

is then a locus of tricritical points [28] (Fig. 3).

V. Discussions

In this paper we have derived exact results for both the quenched and the annealed Potts models. The validity of the duality relation (2.2) for the quenched model in two dimensions is quite general, but its usefulness in determining the exact position of the critical point of the quenched random system is limited. It might be of interest to find the probability distributions $P(K)$ obeying the condition (2.3).

The annealed model is more easily tractable and we have shown that its solution can be reduced to that of a regular (nonrandom) Potts model. If the critical parameters K_c and ε_c of the regular Potts model are known, as it is the case in one and two dimensions [17, 20, 21], the phase boundary and the relevant physical quantities along the phase boundary can be evaluated exactly. Using the conjectured values for the critical exponents in two dimensions [17, 29] one can deduce the critical behavior of the annealed model. As expected [18], the critical exponents of the annealed model are renormalized if the original system has a diverging specific heat anomaly $(\alpha > 0)$. For positive random interaction only, one may consider the annealed model as a good approximation for the quenched system. We note in passing that the "effective-interaction approximation" proposed by Turban [30] corresponds precisely to the annealed model [compare his (12) with our (3.15)].

For $q>4$, the existence of latent heat in the underlying Potts model produces a phase separation in the annealed model, a phenomenon which is not expected in the quenched system. In this case the phase

diagrams of the quenched and annealed systems differ qualitatively.

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