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Extensions of Group Representations over Arbitrary Fields*

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1. INTRODUCTION

Let G be a finite group with $N \lhd G$ and let $\mathbf{L}$ be any field. An $\mathbf{L}$-representation $\mathfrak{X}$ of N is said to be invariant in G if for every $g \in G$, the representation $\mathfrak{X}^g$ defined by $\mathfrak{X}^g(n) = \mathfrak{X}(gng^{-1})$ is similar to $\mathfrak{X}$. The main result of this paper is the following.

THEOREM A. *Let $N \lhd G$ be a Hall subgroup and let $\mathbf{L}$ be an arbitrary field. Then every invariant irreducible $\mathbf{L}$-representation of N is extendible to an $\mathbf{L}$-representation of G.*

An earlier, unpublished, version of this paper contained a weaker form of this result. In the case that $\mathrm{char}(\mathbf{L}) = 0$, that paper proved the theorem only when N is solvable, although the full result was obtained for fields of prime characteristic. After reading that version, E. C. Dade observed (in a private communication) that some of the machinery I developed to handle the prime characteristic case could be extended to work in characteristic zero. This provided a proof that was both far simpler that I had and also more general, since it did not require N to be solvable. In addition, Dade's letter provided me with a better understanding of the workings of this machinery. Because of this new insight, I decided to rewrite the paper completely, and with his permission I have included Dade's strengthened form of my theorem.

The machinery referred to above is contained in Sections 3 and 4 on "crossed representations" and their projective versions. I should add, however that Dade sees this material from a different point of view: that of his more or less parallel theory of "Clifford systems." Section 6 of this paper owes the greatest debt to Dade. He pointed out that the key idea in the proof of Theorem 6.1, which is standard for algebraically closed fields, works in this case also.

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In order to set the problem into historical perspective, we review some of the known facts concerning the subject of this paper. First consider the case where **L** is algebraically closed. If $\mathrm{char}(\mathbf{L}) = 0$, it is a well-known result of Gallagher [5] (see also Corollary 8.16 of [9]) that an invariant irreducible **L**-character α of the Hall subgroup $N \lhd G$ is extendible (and thus so is the corresponding representation). If **L** has prime characteristic, similar techniques can be used to show that α is extendible provided that $\deg \alpha$ and $|G:N|$ are coprime. This proviso does not necessarily hold, however, when $\mathrm{char}(\mathbf{L})$ divides $|N|$ since then one cannot conclude that $\deg \alpha$ divides $|N|$. Nevertheless, it was shown by Dade ([3], Theorem 7.1) that α is extendible in this case too. (A very short proof of Dade's result was obtained by Gow [6].) Another proof Dade's theorem was found by D. S. Passman (unpublished) and it is his idea which is the germ of the prime characteristic part of the proof of our main theorem.

The first attempt to solve the problem when **L** is not algebraically closed was apparently in the author's paper [8] where only characteristic zero was considered. The approach there was to consider an absolutely irreducible constituent $\alpha \in \mathrm{Irr}(N)$ of the character of the given invariant irreducible **L**-representation $\mathfrak{X}$. One can extend α (canonically) to $\beta \in \mathrm{Irr}(T)$, where T is the inertia group of α and then consider $\gamma = \beta^G \in \mathrm{Irr}(G)$. If one can show that the Schur index $m_{\mathbf{L}}(\gamma)$ is coprime to $|G:N|$, then an **L**-representation of G having γ as an absolutely irreducible constituent will extend $\mathfrak{X}$. This approach met with only limited success in [8] since control of $m_{\mathbf{L}}(\gamma)$ seemed difficult to obtain. The result was proved for nilpotent N, however, and some other very spevcial cases.

If $\mathrm{char}(\mathbf{L}) \neq 0$, Schur indices are always trivial and thus the main difficulty of the characteristic zero situation disappears. This observaztion underlies the note [4] of Fein. The problem now is that since $\deg \alpha$ and $|G:N|$ need not be coprime, it is not known how to obtain a "canonical" extension β of α to T. Fein thus found it necessary to assume that $(\deg \alpha, |G:N|) = 1$. Even Dade's (later) prime characteristic extension theorem does not eliminate the necessity to make the coprimeness assumption because without it, it is not clear how to choose a sufficiently well-behaved extension β. (In the case that G/N is a q-group for some prime q, we succeed, in Section 5 of this paper, in choosing an appropriate β.)

We give brief mention to the paper [1] of Becker. Becker weakens the hypothesis that N is Hall and succeeds in proving extendibility for all fields. Like Fein, however, he assumes that $(\deg \mathfrak{X}, |G:N|) = 1$.

A standard technique for studying extendibility problems over an algebraically closed field **L** is that of projective representations. This reduces the problem to deciding whether or not a particular element in the cohomology group $H^2(G/N, \mathbf{L}^{\times})$ (with trivial action) is the identity. A key idea in this paper is a generalization of this technique which works for arbitrary $\mathbf{L}$, provided that the Schur index $m_{\mathbf{L}}(\alpha) = 1$, where, as before, α is an absolutely irreducible constituent of the character afforded by $\mathfrak{X}$. In this case, it turns out (Corollary 4.4) that the obstruction to the extension of $\mathfrak{X}$ is a certain element of $H^2(G/N, \mathbf{L}(\alpha)^\times)$, where G/N does not necessarily act trivially on the field $\mathbf{L}(\alpha)$.

Throughout most of this paper, the assumption that N is Hall in G is irrelevant, and unless otherwise stated, it should be assumed that N is an arbitrary normal subgroup.

2. Semi-Invariant Characters

Let $\mathbf{E}$ be ban algebraically closed field. We write $\mathrm{Irr}_{\mathbf{E}}(G)$ to denote the set of irreducible $\mathbf{E}$-characters of the finite group G. (If $\mathbf{E} = \mathbf{C}$, the complex numbers, we omit the subscript and write $\mathrm{Irr}(G)$.) Suppose $\mathbf{L} \subseteq \mathbf{E}$ is an arbitrary subfield. Then $\mathrm{Gal}(\mathbf{E}/\mathbf{L})$ permutes $\mathrm{Irr}_{\mathbf{E}}(G)$ into finite orbits.

Let $N \lhd G$ and $\alpha \in \mathrm{Irr}_{\mathbf{E}}(N)$. Then (as in [8]) we say that α is $\mathbf{L}$-*semi-invariant* in G if its Galois orbit over $\mathbf{L}$ is G-invariant. The relevance of semi-invariance to the subject of this paper is as follows. Suppose $\mathfrak{X}$ is an irreducible $\mathbf{L}$-representation of N. Then its character $\mathrm{tr}\,\mathfrak{X}$ decomposes as

$$\mathrm{tr}\,\mathfrak{X} = m(\alpha_1 + \cdots + \alpha_r),$$

where the $\alpha_i \in \mathrm{Irr}_{\mathbf{E}}(N)$ are distinct and constitute an orbit under $\mathrm{Gal}(\mathbf{E}/\mathbf{L})$. The positive integer m is the Schur index $m_{\mathbf{L}}(\alpha_i)$ of each of the α_i over $\mathbf{L}$ and $m = 1$ if $\mathrm{char}(\mathbf{L}) > 0$. (See Theorem 9.21 of [9].) We shall refer to the characters α_i as the $\mathbf{E}$-*constituents* of $\mathfrak{X}$. They are uniquelly determined by the similarity class of $\mathfrak{X}$ and so if $\mathfrak{X}$ is invariant in G, the α_i are permuted by G and thus are $\mathbf{L}$-semi-invariant.

Lemma 2.1. *Let $N \lhd G$ and assume that $\alpha \in \mathrm{Irr}_{\mathbf{E}}(N)$ is $\mathbf{L}$-semi-invariant in G, where $\mathbf{E}$ is algebraically closed and $\mathbf{L} \subseteq \mathbf{E}$. Let $T = I_G(\alpha)$, the inertia group, and write $\mathbf{L}(\alpha)$ to denote the field generated over $\mathbf{L}$ by the values of α on N. Then*

(a) *The orbit of α under* $\mathrm{Gal}(\mathbf{E}/\mathbf{L})$ *is also its orbit under* $\mathrm{Gal}(\mathbf{L}(\alpha)/\mathbf{L})$ *and has size equal to $|\mathbf{L}(\alpha) : \mathbf{L}|$.*

(b) *For each $g \in G$, there is a unique $\sigma_g \in \mathrm{Gal}(\mathbf{L}(\alpha)/\mathbf{L})$ such that $(\alpha^g)^{\sigma_g} = \alpha$.*

(c) *The map $g \mapsto \sigma_g$ is a homomorphism with kernel equal to T.*

(d) *$T \lhd G$ and G/T is abelian.*

Proof. Since $\mathbf{L}(\alpha)$ is normal over $\mathbf{L}$, every $\sigma \in \mathrm{Gal}(\mathbf{E}/\mathbf{L})$ restricts to an element of $\mathrm{Gal}(\mathbf{L}(\alpha)/\mathbf{L})$. Since every automorphism of every subfield of the algebraically closed field $\mathbf{E}$ extends to $\mathbf{E}$, this restriction map is onto and thus $\mathrm{Gal}(\mathbf{L}(\alpha)/\mathbf{L})$ acts transitively on the orbit of α under $\mathrm{Gal}(\mathbf{E}/\mathbf{L})$. Since only the identity in $\mathrm{Gal}(\mathbf{L}(\alpha)/\mathbf{L})$ can fix α, part (a) follows. Also, (b) is now immediate.

We have

$$(\alpha^{gh})^{\sigma_g\sigma_h} = (((\alpha^g)^{\sigma_g})^h)^{\sigma_h} = (\alpha^h)^{\sigma_h} = \alpha$$

and thus $\sigma_{gh} = \sigma_g\sigma_h$ as desired. Also, if $g \in T$, then $\alpha = (\alpha^g)^{\sigma_g} = \alpha^{\sigma_g}$ and $\sigma_g = 1$. Conversely, if $\sigma_g = 1$, then $\alpha^g = \alpha$ and $g \in T$. This proves (c), and (d) follows since $\mathrm{Gal}(\mathbf{L}(\alpha)/\mathbf{L})$ is abelian. ∎

One strategy for extending the invariant irreducible $\mathbf{L}$-representation $\mathfrak{X}$ of N to G is to find a suitable extension of an $\mathbf{E}$-constituent $\alpha \in \mathrm{Irr}_{\mathbf{E}}(N)$ of $\mathfrak{X}$ to its inertia group in G.

Definition 2.2. Let $N \lhd G$ and let $\alpha \in \mathrm{Irr}_{\mathbf{E}}(N)$ be $\mathbf{L}$-semi-invariant in G, where $\mathbf{E}$ is algebraically closed and $\mathbf{L} \subseteq \mathbf{E}$. Let $T = I_G(\alpha)$. An extension $\beta \in \mathrm{Irr}_{\mathbf{E}}(T)$ of α is a *standard* extension provided that

(i) $\mathbf{L}(\beta) = \mathbf{L}(\alpha)$ and

(ii) β is $\mathbf{L}$-semi-invariant in G.

An important example of a standard extension occurs in characteristic zero. In this case, if the determinantal order $o(\alpha)$ and the degree $\alpha(1)$ are each relatively prime to $|T:N|$, then α has a *canonical* extension $\hat{\alpha} \in \mathrm{Irr}_{\mathbf{E}}(T)$. This is the unique extension such that $o(\hat{\alpha}) = o(\alpha)$. (See Corollary 8.16 of [9] for this result of Gallagher.) Note that the condition that $(|T:N|, o(\alpha)\,\alpha(1)) = 1$ is automatically satisfied (in characteristic zero) if N is a Hall subgroup of T.

In the above situation, such a canonical extension $\hat{\alpha}$ is necessarily standard. To see this, observe that $\mathbf{L}(\hat{\alpha}) \supseteq \mathbf{L}(\alpha)$ and each element σ of $\mathrm{Gal}(\mathbf{L}(\hat{\alpha})/\mathbf{L}(\alpha))$ carries $\hat{\alpha}$ to an extension of α. Since $o(\hat{\alpha}^\sigma) = o(\hat{\alpha})$, it follows from uniqueness that $\hat{\alpha}^\sigma = \hat{\alpha}$ and thus $\sigma = 1$ and $\mathbf{L}(\hat{\alpha}) = \mathbf{L}(\alpha)$. Furthermore, if $g \in G$, then $\hat{\alpha}^g$ is the canonical extension of α^g. However, $\alpha^g = \alpha^\tau$ for some $\tau \in \mathrm{Gal}(\mathbf{L}(\alpha)/\mathbf{L})$ and $\hat{\alpha}^\tau$ is the canonical extension of α^τ. It follows that $\hat{\alpha}^g = \hat{\alpha}^\tau$ and so $\hat{\alpha}$ is $\mathbf{L}$-semi-invariant in G. It thus is a standard extension as claimed.

Lemma 2.3. *In the situation of Definition* 2.2, *let β be a standard extension of α to T and let $\gamma = \beta^G$, the character of the induced representation. Then $\gamma \in \mathrm{Irr}_{\mathbf{E}}(G)$ and the Schur index $m_{\mathbf{L}}(\gamma)$ divides $m_{\mathbf{L}}(\alpha)\,|G:T|$.*

Proof. That γ is irreducible is well known, since T is the inertia group of α. If $\mathrm{char}(\mathbf{E}) > 0$ then $m_{\mathbf{L}}(\gamma) = 1$ and there is nothing further to prove. We assume, therefore, that $\mathrm{char}(\mathbf{E}) = 0$. We have $[\alpha^G, \gamma] = [\alpha, \gamma_N] = [\alpha, \beta_N] = 1$

and thus by Lemma 10.4 of [9], we see that $m_{\mathbf{L}}(\gamma)$ divides $m_{\mathbf{L}}(\alpha) \cdot |\mathbf{L}(\alpha, \gamma) : \mathbf{L}(\gamma)|$ and it suffices to show that $|\mathbf{L}(\alpha, \gamma) : \mathbf{L}(\gamma)|$ divides $|G : T|$.

The group $\mathrm{Gal}(\mathbf{L}(\alpha, \gamma)/\mathbf{L}(\gamma))$ permutes the set of irreducible constituents of γ_N semi-regularly. Since the cardinality of this set is equal to $|G : T|$ and the order of the Galois group is the degree of the field extension, the result follows. ∎

THEOREM 2.4. *Let $N \lhd G$ and let $\mathfrak{X}$ be an irreducible, G-invariant* **L**-*representation of N. Let $\mathbf{E} \supseteq \mathbf{L}$ be algebraically closed and let $\alpha \in \mathrm{Irr}_{\mathbf{E}}(N)$ be an* **E**-*constituent of $\mathfrak{X}$. Suppose that α has a standard extension $\beta \in \mathrm{Irr}_{\mathbf{E}}(T)$, where $T = I_G(\alpha)$ and suppose further that $m_{\mathbf{L}}(\gamma)$ divides $m_{\mathbf{L}}(\alpha)$, where $\gamma = \beta^G$. Then $\mathfrak{X}$ extends to an* **L**-*representation of G which has γ as an* **E**-*constituent.*

Note. Suppose N is a Hall subgroup of G. If $\mathrm{char}(\mathbf{L}) = 0$, then the existence of a standard extension is guaranteed but it is not obvious that the condition on Schur indices is satisfied. If $\mathrm{char}(\mathbf{L}) > 0$, the Schur indices cause no problem but it is no longer clear that a standard extension exists, although the existence of some extension is assured by Dade's Theorem.

Proof of Theorem 2.4. Write $m = m_{\mathbf{L}}(\alpha)$ and let $\gamma = \gamma_1, \gamma_2, \ldots, \gamma_s$ be the distinct conjugates of γ under $\mathrm{Gal}(\mathbf{E}/\mathbf{L})$. Since $m_{\mathbf{L}}(\gamma)$ divides m, there exists an **L**-representation $\mathfrak{Y}$ of G such that $\mathrm{tr}\, \mathfrak{Y} = m \sum \gamma_i$ and $\deg \mathfrak{Y} = ms \deg \gamma$.

We shall show that the restriction $\mathfrak{Y}_N$ is similar to $\mathfrak{X}$ and thus some representation similar to $\mathfrak{Y}$ extends $\mathfrak{X}$ to G as desired. Consider the representation $\mathfrak{Y}^{\mathbf{E}}$, which is $\mathfrak{Y}$ viewed as an **E**-representation. Then some irreducible constituent of $(\mathfrak{Y}^{\mathbf{E}})_N$ has the character α. It follows that $\mathfrak{Y}_N$ has an irreducible constituent $\mathfrak{Z}$ such that $\mathfrak{Z}^{\mathbf{E}}$ and $\mathfrak{X}^{\mathbf{E}}$ share a common constituent. By Corollary 9.7 of [9], it follows that $\mathfrak{Z}$ and $\mathfrak{X}$ are similar and so $\mathfrak{X}$ is a constituent of $\mathfrak{Y}_N$. It will now suffice to show that $\deg \mathfrak{Y} \leqslant \deg \mathfrak{X}$. In other words, we need that $ms \deg \gamma \leqslant mr \deg \alpha$, where r is the size of the orbit of α under $\mathrm{Gal}(\mathbf{E}/\mathbf{L})$.

Observe that $s = |\mathbf{L}(\gamma) : \mathbf{L}|$ and $r = |\mathbf{L}(\alpha) : \mathbf{L}|$. Also $\mathbf{L}(\gamma) = \mathbf{L}(\beta^G) \subseteq \mathbf{L}(\beta) = \mathbf{L}(\alpha)$, where the last equality is because β is a standard extension. This yields that $r = s\, |\mathbf{L}(\alpha) : \mathbf{L}(\gamma)|$ so that

$$r \deg \alpha = s\, |\mathbf{L}(\alpha) : \mathbf{L}(\gamma)| \deg \alpha.$$

Since $\deg \gamma = |G : T| \deg \alpha$, the desired inequality will follow when we show that $|G : T| \leqslant |\mathbf{L}(\alpha) : \mathbf{L}(\gamma)|$. Now the Galois group $\mathrm{Gal}(\mathbf{L}(\alpha)/\mathbf{L}(\gamma))$ permutes the irreducible constituents of γ_N which are the $|G : T|$ distinct G-conjugates of α. It therefore suffices to show that this action is transitive.

Let $g \in G$. Then $\alpha^g = \alpha^\sigma$ for some $\sigma \in \mathrm{Gal}(\mathbf{L}(\alpha)/\mathbf{L})$. We will be done when we show that $\gamma^\sigma = \gamma$ so that $\sigma \in \mathrm{Gal}(\mathbf{L}(\alpha)/\mathbf{L}(\gamma))$. Since β is a standard extension, it is **L**-semi-invariant in G and hence $\beta^g = \beta^\tau$ for some $\tau \in \mathrm{Gal}(\mathbf{L}(\beta)/\mathbf{L}) = \mathrm{Gal}(\mathbf{L}(\alpha)/\mathbf{L})$. Thus $\alpha^g = \alpha^\tau$ and so $\alpha^\tau = \alpha^\sigma$ and $\tau = \sigma$. Therefore, $\beta^g = \beta^\sigma$ and hence $\gamma = \gamma^g = \gamma^\sigma$ as desired, since $\gamma = \beta^G$. ∎

3. Crossed Representations

In this section we introduce a second technique for extending representations. This will be used in combination with the standard extensions in the previous section to prove Theorem A.

Let **F** be an arbitrary field and let G act on **F** via field automorphisms. (The example we have in mind is where $\alpha \in \mathrm{Irr}_{\mathbf{E}}(N)$ is **L**-semi-invariant in G and $\mathbf{F} = \mathbf{L}(\alpha)$.) This action induces an action of G on $GL(r, \mathbf{F})$ for positive integers r. We say that a map $\mathfrak{Z}: G \to GL(r, \mathbf{F})$ is an **F**-*crossed representation* of G provided

$$\mathfrak{Z}(gh) = \mathfrak{Z}(g)^h\, \mathfrak{Z}(h)$$

for all $g, h \in G$.

The relevance of crossed representations to the extension problem is as follows. Let $N \lhd G$. Let $\mathbf{E} \subseteq \mathbf{L}$ be algebraically closed and suppose $\alpha \in \mathrm{Irr}_{\mathbf{E}}(N)$ is an **E**-constituent of $\mathfrak{X}$ so that α is **L**-semi-invariant in G and thus G acts on $\mathbf{L}(\alpha)$. Let $\mathfrak{X}^{\mathbf{L}(\alpha)}$ denote the representation $\mathfrak{X}$ when viewed as an $\mathbf{L}(\alpha)$-representation of N and let $\mathfrak{Y}$ be an irreducible constituent of $\mathfrak{X}^{\mathbf{L}(\alpha)}$. It is no loss to assume that α is an **E**-constituent of $\mathfrak{Y}$.

THEOREM 3.1. *In the above situation, $\mathfrak{X}$ extends to an* **L**-*representation of G iff the $\mathbf{L}(\alpha)$-representation $\mathfrak{Y}$ of N extends to an $\mathbf{L}(\alpha)$-crossed representation of G with respect to the given action of G on $\mathbf{L}(\alpha)$.*

Before proceeding with the proof of Theorem 3.1 we review some (known) facts about irreducible representations over arbitrary fields.

Let N be a finite group and let $\mathfrak{X}$ be an irreducible **L**-representation of N. Let $\mathbf{E} \supseteq \mathbf{L}$ be algebraically closed and suppose $\alpha \in \mathrm{Irr}_{\mathbf{E}}(N)$ is an **E**-constituent of $\mathfrak{X}$. Write $\mathbf{F} = \mathbf{L}(\alpha)$ and let $\mathfrak{Y}$ be an irreducible constituent of $\mathfrak{X}^{\mathbf{F}}$. By Theorem 9.21 of [9] we know that $\mathfrak{X}^{\mathbf{E}}$ is completely reducible and $\mathfrak{Y}^{\mathbf{E}}$ has a unique irreducible constituent (which affords a character Galois conjugate to α). This constituent has multiplicity m $(= m_{\mathbf{L}}(\alpha)$, the Schur index) which is independent of the choice of $\mathfrak{Y}$. Furthermore, $\mathfrak{Y}$ has multiplicity 1 as a constituent of $\mathfrak{X}^{\mathbf{F}}$ and $\deg \mathfrak{X} = |\mathbf{F} : \mathbf{L}| \deg \mathfrak{Y}$.

We continue this review with the following known result.

LEMMA 3.2. *In he above situation, let V be a simple* **F***N-module corresponding to $\mathfrak{Y}$. Then*

(a) *V is simple as an* $\mathbf{L}N$*-module, and corresponds to* $\mathfrak{X}$.

(b) *Let* $\Delta = \mathrm{Hom}_{\mathbf{L}N}(V, V)$. *Then the center of* Δ *is the set of scalar multiplications by elements of* $\mathbf{F}$.

Proof. By Lemma 9.18 of [9], $\mathfrak{X}$ is a constituent of the $\mathbf{L}$-representation corresponding to the $\mathbf{L}N$-module V. However,

$$\deg \mathfrak{X} = |\mathbf{F} : \mathbf{L}| \deg \mathfrak{Y} = |\mathbf{F} : \mathbf{L}| \dim_{\mathbf{F}} V = \dim_{\mathbf{L}} V.$$

Part (a) now follows.

The $\mathbf{E}$-algebra $\mathfrak{X}(\mathbf{L}N) \otimes_{\mathbf{L}} \mathbf{E}$ is isomorphic to a direct sum of $|\mathbf{F}:\mathbf{L}|$ full matrix algebras, each of dimension $(\deg \alpha)^2$. This yields

$$\dim_{\mathbf{L}} \mathfrak{X}(\mathbf{L}N) = |\mathbf{F} : \mathbf{L}| (\deg \alpha)^2 = |\mathbf{F} : \mathbf{L}| (r/m)^2, \tag{1}$$

where $m = m_{\mathbf{L}}(\alpha)$ is the multiplicity of α as a constituent of $\mathfrak{Y}^{\mathbf{E}}$ and $r = \deg \mathfrak{Y}$.

Now Δ is a division algebra over $\mathbf{L}$ and we write $q = \dim_{\Delta} V$ and $d = \dim_{\mathbf{L}} \Delta$. Then $\mathfrak{X}(\mathbf{L}N)$ is $\mathbf{L}$-isomorphic to the full ring of $q \times q$ matrices over Δ and thus, using (1) we have

$$q^2 d = \dim_{\mathbf{L}} \mathfrak{X}(\mathbf{L}N) = |\mathbf{F} : \mathbf{L}| (r/m)^2$$

and $|\mathbf{F} : \mathbf{L}| r^2 = m^2 q^2 d$. However, $qd = \dim_{\mathbf{L}} V = |\mathbf{F} : \mathbf{L}| r$ and this yields

$$\dim_{\mathbf{L}} \Delta = d = m^2 |\mathbf{F} : \mathbf{L}|. \tag{2}$$

Note that in the case where $m = 1$ (which is the only case we really need in the proof of Theorem A) the result is now immediate since by (2) we conclude that $\Delta = \mathbf{F} \cdot 1$, the set of scalar multiplications by $\mathbf{F}$. For completeness, however, we finish the proof in the general case.

Certainly, $\mathbf{F} \cdot 1 \subseteq \Delta$. Let Δ_0 be the centralizer of $\mathbf{F} \cdot 1$ in Δ so that $\Delta_0 = \mathrm{Hom}_{\mathbf{F}N}(V, V)$ and we may apply (2) to $\mathfrak{Y}$ in place of $\mathfrak{X}$, $\mathbf{F}$ in place of $\mathbf{L}$ and Δ_0 in place of Δ. This yields $\dim_{\mathbf{F}} \Delta_0 = m^2 |\mathbf{F} : \mathbf{F}| = m^2$ and so $\dim_{\mathbf{L}} \Delta_0 = m^2 |\mathbf{F} : \mathbf{L}| = \dim_{\mathbf{L}} \Delta$. Thus $\Delta_0 = \Delta$ and $\mathbf{F} \cdot 1$ is central in Δ, as desired.

We shall now prove that $\mathbf{F} \cdot 1$ is exactly the center of Δ. Since $\mathfrak{Y}(\mathbf{F}N) \otimes_{\mathbf{F}} \mathbf{E}$ is isomorphic to a full matrix algebra over $\mathbf{E}$, it follows that $\mathfrak{Y}(\mathbf{F}N)$ is central-simple over $\mathbf{F}$ and thus $\mathfrak{Y}(\mathbf{F}N) \otimes_{\mathbf{F}} \Delta$ is simple. This algebra has a natural homomorphism into $\mathrm{Hom}_{\mathbf{F}}(V, V)$. Now $\dim_{\mathbf{F}} \mathfrak{Y}(\mathbf{F}N) = (r/m)^2$ by application of (1) to $\mathfrak{Y}$ in place of $\mathfrak{X}$. Since $\dim_{\mathbf{F}} \Delta = m^2$, we have

$$\dim_{\mathbf{F}}(\mathfrak{Y}(\mathbf{F}N) \otimes_{\mathbf{F}} \Delta) = r^2 = \dim_{\mathbf{F}} \mathrm{Hom}_{\mathbf{F}}(V, V).$$

The simplicity of the tensor product then yields that $\mathfrak{Y}(\mathbf{F}N) \otimes_{\mathbf{F}} \Delta \cong \mathrm{Hom}_{\mathbf{F}}(V, V)$ which has center of dimension 1. It follows that the center of Δ can be no larger than $\mathbf{F} \cdot 1$. ∎

Proof of Theorem 3.1. Write $\mathbf{F} = \mathbf{L}(\alpha)$ and let V be the $(\deg \mathfrak{Y})$-dimensional row space over $\mathbf{F}$. Then V is an $\mathbf{F}N$-module via the action $v \cdot n = v\mathfrak{Y}(n)$ for $v \in V$ and $n \in N$. By Lemma 3.2(a), when V is viewed as an $\mathbf{L}N$-module, it corresponds to the representation $\mathfrak{X}$, and so an extension of $\mathfrak{X}$ to G amounts to an $\mathbf{L}$-linear extension of the "dot" action of N on V to all of G.

Now assume that $\mathfrak{Z}$ is an $\mathbf{F}$-crossed representation of G which extends $\mathfrak{Y}$. For each $g \in D$ and $v \in V$, we define $v^g \in V$ by applying the field automorphism σ_g induced by g to each entry in the row vector v. Now define $v \cdot g = v^g \mathfrak{Z}(g)$ and observe that the map $v \mapsto v \cdot g$ is $\mathbf{L}$-linear since σ_g is trivial on $\mathbf{L}$. Also, since N acts trivially on $\mathbf{F}$ and $\mathfrak{Z}_N = \mathfrak{Y}$ we see that the two definitions of $v \cdot n$ for $n \in N$ agree. Now for $g, h \in G$ we have

$$(v \cdot g) \cdot h = (v^g \mathfrak{Z}(g)) \cdot h = (v^g \mathfrak{Z}(g))^h \, \mathfrak{Z}(h) = v^{gh} \mathfrak{Z}(g)^h \mathfrak{Z}(h)$$
$$= v^{gh} \mathfrak{Z}(gh) = v \cdot gh.$$

It follows that $\mathfrak{X}$ extends to G as desired.

Conversely, assume that $\mathfrak{X}$ extends to G so that we have an $\mathbf{L}$-linear action $v \mapsto v \cdot g$ such that $v \cdot n = v\mathfrak{Y}(n)$ for $n \in N$. Let $\Delta = \mathrm{Hom}_{\mathbf{L}N}(V, V)$. For $g \in G$ and $\delta \in \Delta$, observe that the map $v \mapsto ((v \cdot g^{-1})\,\delta) \cdot g$ is $\mathbf{L}$-linear and commutes with the action of N. It follows that this map, which we denote δ^{*g} lies in Δ and we see that $\delta \mapsto \delta^{*g}$ is an automorphism of Δ. This defines an action of G on Δ via automorphisms.

By Lemma 3.2(b), the center of Δ, which is necessarily G-invariant, is the set of scalar multiplications of $\mathbf{F}$ on V. Thus there is an action (which we continue to denote $*g$) of G on $\mathbf{F}$ such that

$$(v\lambda) \cdot g = (v \cdot g)\, \lambda^{*g}$$

for all $g \in G$, $v \in V$ and $\lambda \in \mathbf{F}$. Note that if $\lambda \in \mathbf{L}$, then $\lambda^{*g} = \lambda$ since the action of G is $\mathbf{L}$-linear. Thus $*g \in \mathrm{Gal}(\mathbf{F}/\mathbf{L})$. Also, since the action of N on V is $\mathbf{F}$-linear, we see that $*n$ is trivial for $n \in N$. The reader is cautioned, however, that we do not yet know that $*g = \sigma_g$.

Now let $s_1, \ldots, s_r$ be the standard $\mathbf{F}$-basis for the row space V. For $g \in G$, let $\mathfrak{Z}(g)$ be the matrix whose ith row is the vector $s_i \cdot g$ and note that $\mathfrak{Z}(n) = \mathfrak{Y}(n)$ for $n \in N$. In order to compute $\mathfrak{Z}(gh)$ for $g, h \in G$, we write $\mathfrak{Z}(g) = [x_{ij}]$ and $\mathfrak{Z}(h) = [y_{ij}]$, where all $x_{ij}, y_{ij} \in \mathbf{F}$. We have

$$\begin{aligned} s_i \cdot (gh) = (s_i \cdot g) \cdot h &= \Big(\sum_j s_j x_{ij}\Big) \cdot h \\ &= \sum_j (s_j \cdot h)(x_{ij})^{*h} \\ &= \sum_k \sum_j s_k\, y_{jk} (x_{ij})^{*h} \end{aligned}$$

and it follows that

$$\mathfrak{Z}(gh) = \mathfrak{Z}(g)^{*h}\mathfrak{Z}(h).$$

We have now constructed an $\mathbf{F}$-crossed representation of G which extends $\mathfrak{Y}$. What remains is to show that the $*$-action of G is the original action. Since σ_g is the unique $\sigma \in \mathrm{Gal}(\mathbf{F}/\mathbf{L})$ such that $(\alpha^g)^\sigma = \alpha$, the proof will be complete when we show that $\alpha(gng^{-1})^{*g} = \alpha(n)$ for all $g \in G$ and $n \in N$.

We have

$$\mathfrak{Z}(g)^{*n}\,\mathfrak{Z}(n) = \mathfrak{Z}(gn) = \mathfrak{Z}(gng^{-1} \cdot g) = \mathfrak{Z}(gng^{-1})^{*g}\,\mathfrak{Z}(g).$$

Since $*n$ is trivial and $\mathfrak{Z}_N = \mathfrak{Y}$, we obtain

$$\mathfrak{Z}(g)\,\mathfrak{Y}(n)\,\mathfrak{Z}(g)^{-1} = \mathfrak{Y}(gng^{-1})^{*g}$$

and taking traces yields

$$\mathrm{tr}\,\mathfrak{Y}(n) = \mathrm{tr}\,\mathfrak{Y}(gng^{-1})^{*g}.$$

Now $\mathrm{tr}\,\mathfrak{Y} = m\alpha$ where m is the Schur index and since $m = 1$ if $\mathrm{char}(\mathbf{F}) > 0$, we can cancel m and obtain $\alpha(n) = \alpha(gng^{-1})^{*g}$ as desired. The proof is complete. ∎

4. Projective Crossed Representation

Let $N \lhd G$ and let $\mathfrak{X}$ be an irreducible, G-invariant $\mathbf{L}$-representation of N. In the case that $\mathbf{L}$ is algebraically closed, the theory of projective representations is commonly used to investigate the extendibility of $\mathfrak{X}$ to G. In that situation, the theory yields a uniquely defined element f^* in the cohomology group $H^2(G/N, \mathbf{L}^\times)$ (where G/N acts trivially on $\mathbf{L}^\times$). It turns out that $\mathfrak{X}$ is extendible iff $f^* = 1$.

In this section we generalize the above to the case where $\mathbf{L}$ is arbitrary. We do need to assume, however, that the Schur index associated with $\mathfrak{X}$ is trivial. Let $\mathbf{E} \supseteq \mathbf{L}$ be algebraically closed and let $\alpha \in \mathrm{Irr}_{\mathbf{E}}(N)$ be an $\mathbf{E}$-constituent of $\mathfrak{X}$, as in the previous sections. Assuming $m_{\mathbf{L}}(\alpha) = 1$, we shall produce a uniquely defined cohomology element $f^* \in H^2(G/N, \mathbf{L}(\alpha)^\times)$ such that $\mathfrak{X}$ is extendible iff $f^* = 1$. In this case, the action of G/N on $\mathbf{L}(\alpha)$ is not necessarily trivial, but rather is the action associated with the fact that α is $\mathbf{L}$-semi-invariant as in Lemma 2.1.

Suppose G acts on some field $\mathbf{F}$ as in the previous section. Suppose $\mathfrak{Z}: G \to GL(r, \mathbf{F})$ and $f: G \times G \to \mathbf{F}^\times$ are maps such that

$$\mathfrak{Z}(gh) \doteq \mathfrak{Z}(g)^h\,\mathfrak{Z}(h)f(g, h)$$

for all $g, h \in G$. Then we say that $\mathfrak{Z}$ is a *projective* **F**-*crossed representation* of G and that f is the associated factor set. In this situation, it is routine to check that

$$f(g, hk)f(h, k) = f(gh, k)f(g, h)^k$$

and thus f really is a factor set, that is a 2-cocycle which defines an element of $H^2(G, \mathbf{F}^\times)$.

We need a technical definition.

DEFINITION 4.1. Let $\mathfrak{Z}$ be a projective $\mathbf{F}$-crossed representation of G and let $N \lhd G$. Then $\mathfrak{Z}$ *respects* N provided

(i) N acts trivially on $\mathbf{F}$.

(ii) $\mathfrak{Z}(gn) = \mathfrak{Z}(g)\,\mathfrak{Z}(n)$ for all $g \in G$, $n \in N$.

(iii) $\mathfrak{Z}(ng) = \mathfrak{Z}(n)^g\,\mathfrak{Z}(g)$ for all $g \in G$, $n \in N$.

LEMMA 4.2. *Let $\mathfrak{Z}$ be a projective* **F**-*crossed representation of G which respects $N \lhd G$, and let $f: G \times G \to \mathbf{F}^\times$ be the associated factor set. Then $f(gn, hm) = f(g, h)$ for all $g, h \in G$ and $m, n \in N$.*

Proof. Calculate $\mathfrak{Z}(gnhm) = \mathfrak{Z}(ghn^hm)$ using the three conditions in Definition 4.1. ∎

In the above situation, since f is constant on cosets of N, it defines a map $\bar{f}: G/N \times G/N \to \mathbf{F}^\times$. Because N acts trivially on $\mathbf{F}$, we have an action of G/N on $\mathbf{F}^\times$ and $\bar{f}$ defines an element $f^* \in H^2(G/N, \mathbf{F}^\times)$. We call f^* the *cohomology element* associated with the pair $(\mathfrak{Z}, N)$.

THEOREM 4.3. *Let $N \lhd G$ and let $\alpha \in \mathrm{Irr}_{\mathbf{E}}(N)$ be* **L**-*semi-invariant in G, where* **E** *is algebraically closed and $\mathbf{L} \subseteq \mathbf{E}$. Assume that $m_{\mathbf{L}}(\alpha) = 1$ so that α is afforded by some irreducible $\mathbf{L}(\alpha)$-representation $\mathfrak{Y}$ of N. Then*

(a) *$\mathfrak{Y}$ extends to a projective $\mathbf{L}(\alpha)$-crossed representation $\mathfrak{Z}$ of G which respects N.*

(b) *The associated cohomology element $f^* \in H^2(G/N, \mathbf{L}(\alpha)^\times)$ is uniquely determined by α.*

(c) *$\mathfrak{Y}$ extends to a (nonprojective) $\mathbf{L}(\alpha)$-crossed representation of G iff $f^* = 1$.*

In Theorem 4.3, the crossed representations are, of course, with respect to the natural action of G on $\mathbf{L}(\alpha)$ associated with the fact that α is **L**-semi-invariant.

Proof of Theorem 4.3. Fix a set of representatives U for the cosets of N in G and assume $1 \in U$. Write $\mathbf{F} = \mathbf{L}(\alpha)$. For each $u \in U$, the map

$n \mapsto \mathfrak{Y}(unu^{-1})^u$ is an irreducible **F**-representation of N which affords the character $(\alpha^u)^{\sigma_u}$, where $\sigma_u \in \mathrm{Gal}(\mathbf{F}/\mathbf{L})$ is the automorphism induced by u. By definition, $(\alpha^u)^{\sigma_u} = \alpha$, and thus this representation is similar to $\mathfrak{Y}$ and we can choose a nonsingular matrix P_u such that

$$P_u \mathfrak{Y}(n) P_u^{-1} = \mathfrak{Y}(unu^{-1})^u \tag{1}$$

for all $n \in N$. We can (and do) take $P_1 = I$.

Now define $\mathfrak{Z}$ on G by

$$\mathfrak{Z}(un) = P_u \mathfrak{Y}(n)$$

for $u \in U$ and $n \in N$. Note that $\mathfrak{Z}$ agrees with $\mathfrak{Y}$ on N and that $\mathfrak{Z}(gn) = \mathfrak{Z}(g)\,\mathfrak{Z}(n)$ for all $g \in G$ and $n \in N$.

Using (1) and the fact that the action of N on **F** is trivial, an easy calculation yields that

$$\mathfrak{Z}(g)\,\mathfrak{Y}(n)\,\mathfrak{Z}(g)^{-1} = \mathfrak{Y}(gng^{-1})^g \tag{2}$$

for all $g \in G$ and $n \in N$. For $m \in N$, we now obtain

$$\mathfrak{Z}(mg) = \mathfrak{Z}(gm^g) = \mathfrak{Z}(g)\,\mathfrak{Z}(m^g) = \mathfrak{Z}(m)^g\,\mathfrak{Z}(g),$$

where the last equality follows by putting $n = m^g$ in (2).

We have now shown that $\mathfrak{Z}$ satisfies conditions (i)–(iii) of Definition 4.1. To complete the proof of (a), we need to show that $\mathfrak{Z}$ is a projective **F**-crossed representation with respect to the given action. Using (2), we have

$$\begin{aligned}
\mathfrak{Z}(g)^h\,\mathfrak{Z}(h)\,\mathfrak{Y}(n)(\mathfrak{Z}(g)^h\,\mathfrak{Z}(h))^{-1} &= \mathfrak{Z}(g)^h\,\mathfrak{Y}(hnh^{-1})^h\,(\mathfrak{Z}(g)^h)^{-1} \\
&= (\mathfrak{Z}(g)\,\mathfrak{Y}(hnh^{-1})\mathfrak{Z}(g)^{-1})^h \\
&= \mathfrak{Y}(ghnh^{-1}g^{-1})^{gh} \\
&= \mathfrak{Z}(gh)\,\mathfrak{Y}(n)\,\mathfrak{Z}(gh)^{-1}
\end{aligned}$$

and thus $(\mathfrak{Z}(g)^h\,\mathfrak{Z}(h))^{-1}\,\mathfrak{Z}(gh)$ commutes with $\mathfrak{Y}(n)$ for all $g, h \in G$ and $n \in N$. Since $\mathfrak{Y}$ affords $\alpha \in \mathrm{Irr}_{\mathbf{E}}(N)$, it is absolutely irreducible and so $\mathfrak{Z}(gh) = \mathfrak{Z}(g)^h \mathfrak{Z}(h) f(g, h)$ for some scalar $f(g, h) \in \mathbf{F}^\times$.

To prove (b), suppose $\mathfrak{Z}_0$ is another projective **F**-crossed representation of G which respects N and which extends an **F**-representation of N that affords α. Let f_0 be its associated factor set. We can conjugate $\mathfrak{Z}_0$ by any nonsingular matrix without changing f_0 and so it is no loss to assume that $\mathfrak{Z}_0$ extends $\mathfrak{Y}$.

We have $\mathfrak{Z}_0(gn) = \mathfrak{Z}_0(g)\,\mathfrak{Z}_0(n) = \mathfrak{Z}_0(g)\,\mathfrak{Y}(n)$ and also

$$\mathfrak{Z}_0(gn) = \mathfrak{Z}_0(gng^{-1} \cdot g) = \mathfrak{Z}_0(gng^{-1})^g\,\mathfrak{Z}_0(g)$$

since $\mathfrak{Z}_0$ respects N. This yields

$$\mathfrak{Z}_0(g)\,\mathfrak{Y}(n)\,\mathfrak{Z}_0(g)^{-1} = \mathfrak{Y}(gng^{-1})^g.$$

Comparing this with (2), we obtain $\mathfrak{Z}_0(g) = \lambda(g)\,\mathfrak{Z}(g)$ for some scalar function $\lambda\colon G \to \mathbf{F}^\times$ since $\mathfrak{Y}$ is absolutely irreducible. A simple calculation now yields that

$$f(g, h) = f_0(g, h)\,\lambda(g)^h\,\lambda(h)/\lambda(gh)$$

and it follows that f and f_0 define the same element of $H^2(G, \mathbf{F}^\times)$. This is not sufficient to conclude that $f^* = f_0^*$ in $H^2(G/N, \mathbf{F}^\times)$, however, unless we show that λ is constant on cosets of N so that it defines a map $G/N \to \mathbf{F}^\times$.

We have

$$\lambda(gn)\,\mathfrak{Z}(gn) = \mathfrak{Z}_0(gn) = \mathfrak{Z}_0(g)\,\mathfrak{Z}_0(n) = \lambda(g)\,\mathfrak{Z}(g)\,\mathfrak{Y}(n)$$
$$= \lambda(g)\,\mathfrak{Z}(gn).$$

Thus $\lambda(gn) = \lambda(g)$ for all $g \in G$ and $n \in N$ and the proof of (b) is complete.

If $\mathfrak{Y}$ extends to a (nonprojective) $\mathbf{F}$-crossed representation of G, then the unique $f^* \in H^2(G/N, \mathbf{F}^\times)$ is clearly trivial since such a crossed representation certainly respects N. Conversely, suppose $f^* = 1$, where f is the factor set associated with a projective $\mathbf{F}$-crossed representation $\mathfrak{Z}$ of G which respects N and extends $\mathfrak{Y}$. Then

$$f(g, h) = \lambda(g)^h\,\lambda(h)/\lambda(gh)$$

for some function $\lambda\colon G \to \mathbf{F}^\times$ which is constant on cosets of N.

Define $\mathfrak{Z}_0(g) = \lambda(g)\,\mathfrak{Z}(g)$ for all $g \in G$. We have $f(1, 1) = 1$ and it follows that $\lambda(1) = 1$ and hence $\lambda(n) = 1$ for all $n \in N$. Thus $\mathfrak{Z}_0$ agrees with $\mathfrak{Y}$ on N. Now a routine calculation yields that $\mathfrak{Z}_0(gh) = \mathfrak{Z}_0(g)^h\,\mathfrak{Z}_0(h)$ and the proof is complete. ∎

In the situation of Theorem 4.3, we shall write $f^*(\alpha)$ to denote the unique element of $H^2(G/N, \mathbf{L}(\alpha)^\times)$ associated with α.

We can combine the results of this and the previous section.

COROLLARY 4.4. *Let $N \lhd G$ and let $\mathfrak{X}$ be an irreducible $\mathbf{L}$-representation of N which is invariant in G. Let $\mathbf{E} \supseteq \mathbf{L}$ be algebraically closed and let $\alpha \in \mathrm{Irr}_{\mathbf{E}}(N)$ be an $\mathbf{E}$-constituent of $\mathfrak{X}$. Assume that $m_{\mathbf{L}}(\alpha) = 1$. Then $\mathfrak{X}$ extends to an $\mathbf{L}$-representation of G iff the cohomology element $f^*(\alpha) \in H^2(G/N, \mathbf{L}(\alpha)^\times)$ is trivial.*

Proof. Let $\mathfrak{Y}$ be an irreducible constituent of $\mathfrak{X}^{\mathbf{L}(\alpha)}$ such that α is an $\mathbf{E}$-constituent of $\mathfrak{Y}$. By Theorem 3.1, $\mathfrak{X}$ extends to G iff $\mathfrak{Y}$ extends to an $\mathbf{L}(\alpha)$-

crossed representation of G with respect to the action of G on $\mathbf{L}(\alpha)$ induced by the fact that α is $\mathbf{L}$-semi-invariant in G. Since $m_{\mathbf{L}}(\alpha)=1$, $\mathfrak{Y}$ affords α and Theorem 4.3 applies. The result follows. ∎

COROLLARY 4.5. *In the situation of Corollary* 4.4, *suppose for each Sylow subgroup $P/N \subseteq G/N$ that $\mathfrak{X}$ extends to an* $\mathbf{L}$*-representation of P. Then $\mathfrak{X}$ extends to an* $\mathbf{L}$*-representation of G.*

Proof. From the definition of $f^*(\alpha)$, it is clear that the restriction map $H^2(G/N, \mathbf{L}(\alpha)^\times) \to H^2(P/N, \mathbf{L}(\alpha)^\times)$ carries $f^*(\alpha)$ to the cohomology element $f_P^*(\alpha)$ associated to α in the group P. Since $\mathfrak{X}$ extends to all P such that P/N is Sylow in G/N. We conclude that $f^*(\alpha)=1$. (This follows, for instance, from Hauptsatz I. 16.18 of [7].) By Corollary 4.4 again, $\mathfrak{X}$ extends to an $\mathbf{L}$-representation of G. ∎

5. PRIME CHARACTERISTIC

In this section we prove Theorem A in the case where $\operatorname{char}(\mathbf{L})=p>0$. Let $N \lhd G$ be a Hall subgroup and suppose $\mathfrak{X}$ is an irreducible invariant $\mathbf{L}$-representation of N. Let $\alpha \in \operatorname{Irr}_{\mathbf{E}}(N)$ be an $\mathbf{E}$-constituent of $\mathfrak{X}$, where $\mathbf{E} \supseteq \mathbf{L}$ is algebraically closed. Since $\operatorname{char}(\mathbf{L})>0$, $m_{\mathbf{L}}(\alpha)=1$ by Theorem 9.21(b) of [9], and Corollary 4.5 applies. It is therefore no loss to assume that G/N is a q-group for some prime q.

Let $T=I_G(\alpha)$. If we can find a standard extension β (Definition 2.2) of α to T, then by Theorem 2.4 we can extend $\mathfrak{X}$ to G. (To apply 2.4, we again use the fact that Schur indices are trivial in prime characteristic.) From these remarks we see that to obtain Theorem A in prime characteristic it suffices to prove the following result.

THEOREM 5.1. *Let $N \lhd G$ be a Hall subgroup and suppose that G/N is a q-group for some prime q. Let $\mathbf{E}$ be algebraically closed of characteristic $p>0$ and let $\mathbf{L} \subseteq \mathbf{E}$. Suppose $\alpha \in \operatorname{Irr}_{\mathbf{E}}(N)$ is* $\mathbf{L}$*-semi-invariant in G. Then α has a standard extension to $T=I_G(\alpha)$.*

Note that this is a two-hold generalization of Dade's Theorem 7.1 of [3] which, in the present notation, asserts that α is extendible to some $\beta \in \operatorname{Irr}_{\mathbf{E}}(T)$. Dade does not show that β can be chosen with values in the field $\mathbf{L}(\alpha)$ nor that it can be taken to be $\mathbf{L}$-semi-invariant in G. Both of these would follow from an appropriate uniqueness result such as exists in characteristic zero. Since no such result is available in prime characteristic, we follow a different route.

I learned of the next result from D. S. Passman. The proof is his (unpublished).

LEMMA 5.2. *Let* $C = [c_{ij}]$ *be an* $n \times n$ *matrix with rational integer entries. Let* q *be a prime not dividing* $\det(C)$ *and let* Q *be a* q*-group which permutes the index set* $N = \{1, 2, ..., n\}$. *Assume for every* $x \in Q$ *that* $c_{ij} = c_{i \cdot x, j \cdot x}$ *for all* $i, j \in N$. *If* Q *fixes* $a \in N$, *then there exists* Q*-fixed* $b \in N$ *(possibly equal to* a*) such that* $q \nmid c_{ab}$.

Proof. We may view $Q \subseteq S$, the symmetric group on N. For each $\pi \in S$ we write

$$e(\pi) = \prod_{i=1}^{n} c_{i, i \cdot \pi}$$

so that $\det(C) = \sum_{\pi} \operatorname{sgn}(\pi)\, e(\pi)$. For $x \in Q$, we have

$$e(\pi) = \prod_{i} c_{i \cdot x^{-1}, i \cdot x^{-1} \pi} = \prod_{i} c_{i, i \cdot x^{-1} \pi x} = e(\pi^x)$$

and also $\operatorname{sgn}(\pi) = \operatorname{sgn}(\pi^x)$. Therefore, the functions e and sgn are constant on each orbit Δ of the conjugation action of Q on S and we write $\operatorname{sgn}(\Delta)$ and $e(\Delta)$ for their constant values. Then

$$\det(C) = \sum_{\Delta} |\Delta| \operatorname{sgn}(\Delta)\, e(\Delta)$$

is not divisible by q.

Since Q is a q-group, there must exist a singleton orbit $\Delta = \{\pi\}$ with $q \nmid e(\pi)$. Now if Q fixes $a \in N$, let $b = a \cdot \pi$. Since Q centralizes π, we see that Q fixes b and since c_{ab} divides $e(\pi)$, we have $q \nmid c_{ab}$ as desired. ∎

Lemma 5.2 immediately yields the following, which generalizes part of Theorem 6.15 of [3].

LEMMA 5.3. *Let* $\mathrm{IBr}(N)$ *denote the set of irreducible Brauer characters for the prime* p *of the finite group* N, *computed with respect to some fixed lift of the modular roots of unity to the complex numbers. Let* Q *be a* q*-group (with* $q \neq p$*) which permutes* $\mathrm{Irr}(N)$ *and* $\mathrm{IBr}(N)$ *and preserves decomposition numbers. (That is, we assume that* $d_{\chi\varphi} = d_{\chi^x \varphi^x}$ *for all* $x \in Q$.*) Suppose* Q *fixes* $\alpha \in \mathrm{IBr}(N)$. *Then there exists* Q*-fixed* $\psi \in \mathrm{Irr}(N)$ *such that* $q \nmid d_{\psi\alpha}$.

Proof. Let C be the Cartan matrix for N with respect to p. Thus the rows and columns of C are indexed by $\mathrm{IBr}(N)$ and

$$c_{\varphi\theta} = \sum_{\chi \in \mathrm{Irr}(N)} d_{\chi\varphi}\, d_{\chi\theta}.$$

It follows that $c_{\varphi^x \theta^x} = c_{\varphi\theta}$ for all φ, $\theta \in \mathrm{IBr}(N)$ and all $x \in Q$.

Now $\det(C)$ is a power of p (see Theorem 84.17 of [2] or Problem 15.3 of [9]). Thus $q \nmid \det(C)$ and by Lemma 5.2, there exists Q-fixed $\beta \in \mathrm{IBr}(N)$ with $q \nmid c_{\alpha\beta}$. Since the decomposition numbers $d_{\psi\alpha}$ and $d_{\psi\beta}$ are each constant as ψ runs over a Q-orbit Δ of $\mathrm{Irr}(N)$, we may write $d_{\Delta\alpha}$ and $d_{\Delta\beta}$ for their constant values. Then

$$c_{\alpha\beta} = \sum_{\Delta} |\Delta|\, d_{\Delta\alpha}\, d_{\Delta\beta}$$

is not divisible by q and it follows that there exists some singleton orbit $\Delta = \{\psi\}$ with $q \nmid d_{\psi\alpha}$ as desired. ∎

We mention that Dade's extendibility theorem (which is the part of Theorem 5.1 which asserts that α is extendible to T) follows easily from Lemma 5.3. We shall not digress to give the proof here. The following lemma is quite standard.

LEMMA 5.4. *Let $N \lhd G$ and let $\mathbf{E}$ be algebraically closed. Let $\alpha \in \mathrm{Irr}_{\mathbf{E}}(N)$ and assume α extends to $\beta \in \mathrm{Irr}_{\mathbf{E}}(G)$. Let $\Lambda = \mathrm{Hom}(G/N, \mathbf{E}^{\times})$. Then $\lambda \mapsto \lambda\beta$ defines a bijection from Λ onto the set of all extensions of α in* $\mathrm{Irr}_{\mathbf{E}}(G)$.

Proof. Let $\mathfrak{X}$ be an $\mathbf{E}$-representation which affords β. Then for $\lambda \in \Lambda$, we have a representation $\lambda\mathfrak{X}$ which affords $\lambda\beta$, another extension of α. Since the matrices $\mathfrak{X}(g)$ are nonzero, distinct λ's yield distict $\lambda\mathfrak{X}$'s and the map $\lambda \mapsto \lambda\beta$ is one to one.

If $\gamma \in \mathrm{Irr}_{\mathbf{E}}(G)$ extends α, let $\mathfrak{Y}$ afford γ. Then the restriction $\mathfrak{Y}_N$ satisfies $P^{-1}\mathfrak{Y}_N P = \mathfrak{X}_N$ for some matrix P with entries in $\mathbf{E}$. We may replace $\mathfrak{Y}$ by $P^{-1}\mathfrak{Y}P$ and assume that $\mathfrak{Y}_N = \mathfrak{X}_N$. Now for $g \in G$ and $n \in N$, we have

$$\mathfrak{Y}(g)^{-1}\,\mathfrak{X}(n)\,\mathfrak{Y}(g) = \mathfrak{Y}(g^{-1}ng) = \mathfrak{X}(g)^{-1}\,\mathfrak{X}(n)\,\mathfrak{X}(g)$$

and thus $\mathfrak{Y}(g)\,\mathfrak{X}(g)^{-1}$ commutes with all $\mathfrak{X}(n)$ and hence is a scalar matrix $\lambda(g)I$ for some $\lambda(g) \in \mathbf{E}$. This gives $\mathfrak{Y} = \lambda\mathfrak{X}$ and it follows that $\lambda \in \Lambda$ and $\gamma = \lambda\beta$. ∎

We make some observations about field automorphisms and Brauer characters. Let G be a finite group and fix a prime p. Let $\mathbf{K}$ be the splitting field for $X^{|G|} - 1$ over the rationals $\mathbf{Q}$ in $\mathbf{C}$ and let $R = \mathrm{Int}(\mathbf{K})$, the ring of integers in $\mathbf{K}$. Pick a prime $P \subseteq R$ lying over (p), and let $\mathbf{F} = R/P$. Then $\mathbf{F}$ is a splitting field for G and all of its subgroups. For $r \in R$, we write r^* to denote the image of r in $\mathbf{F}$. Then for each irreducible $\mathbf{F}$-character α of G, we have a unique irreducible Brauer character $\varphi \in \mathrm{IBr}(G)$ with $\varphi(g)^* = \alpha(g)$ for p-regular $g \in G$. In general, a different choice of P will yield a different set of Brauer characters, and so we cannot expect $\mathrm{Gal}(\mathbf{K}/\mathbf{Q})$ to permute $\mathrm{IBr}(G)$.

(What is true in general, is that φ^σ is a rational integer linear combination of $\mathrm{IBr}(G)$ for $\sigma \in \mathrm{Gal}(\mathbf{K}/\mathbf{Q})$. We will not need this, however.)

Now let $\mathscr{G} = \{\sigma \in \mathrm{Gal}(\mathbf{K}/\mathbf{Q}) \mid P^\sigma = P\}$. Then $\mathscr{G}$ is a subgroup of the Galois group and $\mathscr{G}$ permutes $\mathrm{IBr}(H)$ for every subgroup $H \subseteq G$. Also, for $\varphi \in \mathrm{IBr}(H)$ and $\chi \in \mathrm{Irr}(H)$, we have $d_{\chi\varphi} = d_{\chi^\sigma\varphi^\sigma}$ for all $\sigma \in \mathscr{G}$. What makes this observation useful is that $\mathscr{G}$ is "big."

LEMMA 5.5. *Assume the above notation. For each $\sigma \in \mathscr{G}$, let $\bar{\sigma}$ be the automorphism of* $\mathbf{F}$ *defined by $\bar{\sigma}(r^*) = \sigma(r)^*$. Then $^{-}$ defines a homomorphism of $\mathscr{G}$ onto* $\mathrm{Aut}(\mathbf{F})$.

Proof. See $D(2)$ of Chapter 11 of [10]. ∎

Proof of Theorem 5.1. Let K be the splitting field of $X^{|G|} - 1$ over $\mathbf{Q}$ in $\mathbf{C}$ and let $R = \mathrm{Int}(\mathbf{K})$. Let $P \subseteq R$ be a prime lying over (p) and let $\mathbf{F} = R/P$.

It is no loss to assume that $\mathbf{F} \subseteq \mathbf{E}$ and we claim that we may also assume that $\mathbf{L} \subseteq \mathbf{F}$. The latter follows since if we set $\mathbf{L}_0 = \mathbf{L} \cap \mathbf{F}$, then restriction defines an isomorphism of $\mathrm{Gal}(\mathbf{L}(\alpha)/\mathbf{L})$ onto $\mathrm{Gal}(\mathbf{L}(\alpha)/\mathbf{L}_0)$ and therefore α is $\mathbf{L}_0$-semi-invariant in G. Any extension of α to T which is standard with respect to $\mathbf{L}_0$ will automatically be standard with respect to $\mathbf{L}$.

Let $\mathscr{G}$ be the subgroup of $\mathrm{Gal}(\mathbf{K}/\mathbf{Q})$ which stabilizes P as in Lemma 5.5 and het $\mathscr{H}$ be the inverse image in $\mathscr{G}$ of $\mathrm{Gal}(\mathbf{F}/\mathbf{L})$ so that $\mathscr{H}$ maps onto $\mathrm{Gal}(\mathbf{F}/\mathbf{L})$. Now $G \times \mathscr{H}$ acts on $\mathrm{IBr}(N)$ and we let $S \subseteq G \times \mathscr{H}$ be the stabilizer of φ in this action, where $\varphi \in \mathrm{IBr}(N)$ corresponds to α. Finally, let $Q \in \mathrm{Syl}_q(S)$.

Note that Q acts on $\mathrm{Irr}(N)$. We choose a Q-fixed $\chi \in \mathrm{Irr}(N)$ with $q \nmid d_{\chi\varphi}$. If $q \neq p$, such a character exists by Lemma 5.3 and if $q = p$, then $p \nmid |N|$ and so $\mathrm{Irr}(N) = \mathrm{IBr}(N)$ and we may take $\chi = \varphi$.

Since $T = I_G(\alpha)$, we see that T stabilizes φ and thus $T \subseteq S$. Because $T \lhd G$, we have $T \lhd S$ and since T/N is a q-group, it follows that $T = N(Q \cap T)$. We conclude that χ is invariant in T and we let $\hat{\chi}$ be the canonical extension of χ to T.

Now Q acts on $\mathrm{Irr}(T)$ and $\mathrm{IBr}(T)$ since $T \lhd G$. By the uniqueness of $\hat{\chi}$, it follows that Q fixes $\hat{\chi}$ and thus the decomposition number $d_{\hat{\chi}\theta}$ is constant as θ runs over a Q-orbit of $\mathrm{IBr}(T)$. Also constant over each Q-orbit is the function $a(\theta)$, the multiplicity of φ as a constituent of θ_N. Let $d(\Delta)$ and $a(\Delta)$ denote the constant values of $d_{\hat{\chi}\theta}$ and $a(\theta)$ for θ in the Q-orbit $\Delta \subseteq \mathrm{IBr}(T)$. Since $(\hat{\chi})_N = \chi$, it follows that

$$d_{\chi\varphi} = \sum_{\Delta} |\Delta|\, d(\Delta)\, a(\Delta)$$

and thus for some Δ we have $q \nmid |\Delta|$ and $q \nmid a(\Delta)$. It follows that $\Delta = \{\theta\}$ for some Q-fixed $\theta \in \mathrm{IBr}(T)$ with $\theta_N = a\varphi$, where $a = a(\Delta)$. Since T/N is a q-group, a must be a power of q and hence $a = 1$ and θ is an extension of φ.

Let $\beta \in \mathrm{Irr}_{\mathbf{E}}(T)$ be the (unique) character corresponding to θ. Since θ extends φ, we see that β extends α and in view of Lemma 5.4, we must find $\lambda \in \Lambda = \mathrm{Hom}(T/N, \mathbf{E}^{\times})$ such that $\gamma = \lambda\beta$ has values in $\mathbf{L}(\alpha)$ and is $\mathbf{L}$-semi-invariant in G.

Now $G \times \mathrm{Gal}(\mathbf{F}/\mathbf{L})$ acts on $\mathrm{Irr}_{\mathbf{E}}(N)$ and on $\mathrm{Irr}_{\mathbf{E}}(T)$. Let $\mathscr{S}$ be the stabilizer in this group of α and suppose we could find some extension γ of α to T which is $\mathscr{S}$-fixed. Since $\mathrm{Gal}(\mathbf{F}/\mathbf{L}(\alpha)) \subseteq \mathscr{S}$ it would follow that γ has values in $\mathbf{L}(\alpha)$. Also, for each $g \in G$, there exists $\sigma \in \mathrm{Gal}(\mathbf{F}/\mathbf{L})$ with $(\alpha^g)^\sigma = \alpha$. Thus $g\sigma \in \mathscr{S}$ and hence fixes γ and we have $(\gamma^g)^\sigma = \gamma$ and γ is $\mathbf{L}$-semi-invariant. We now proceed to find $\lambda \in \Lambda$ such that $\gamma = \lambda\beta$ is $\mathscr{S}$-fixed and this will complete the proof.

As a first step, we show that the size of the orbit of β under $\mathscr{S}$ is not divisible by q. Recall that we have a homomorphism $^-$ of $\mathscr{H}$ onto $\mathrm{Gal}(\mathbf{F}/\mathbf{L})$. We extend this to $\pi\colon G \times \mathscr{H} \to G \times \mathrm{Gal}(\mathbf{F}/\mathbf{L})$ by setting $\pi(g\sigma) = g\bar{\sigma}$. We claim that $\pi(S) = \mathscr{S}$, where S is the stabilizer of φ in $G \times \mathscr{H}$. If $g\sigma \in S$, then $(\varphi^g)^\sigma = \varphi$ and since $\alpha(h) = \varphi(h)^*$ for p-regular $h \in N$, it follows that α and $(\alpha^g)^{\bar{\sigma}}$ agree on such elements. Therefore $(\alpha^g)^{\bar{\sigma}} = \alpha$ and $\pi(g\sigma) = g\bar{\sigma} \in \mathscr{S}$. Conversely, every element of $\mathscr{S}$ can be written in the form $g\bar{\sigma}$ for some $\sigma \in \mathscr{H}$ and we have $(\varphi^g)^\sigma\,(h)^* = \varphi(h)^*$ for all p-regular $h \in N$. It follows by properties of Brauer characters, that $(\varphi^g)^\sigma = \varphi$ and $g\sigma \in S$. Thus π maps S onto $\mathscr{S}$ as claimed and it follows that $\pi(Q) \in \mathrm{Syl}_q(\mathscr{S})$. Since Q fixes θ, we see that $\pi(Q)$ fixes β and therefore the size of the $\mathscr{S}$-orbit of β is indeed not divisible by q.

Next, observe that working in $G \times \mathrm{Gal}(\mathbf{F}/\mathbf{L})$, we have $T = G \cap \mathscr{S}$ and thus $T \lhd \mathscr{S}$ and $\mathscr{S}/T$ is isomorphic to a subgroup of $\mathrm{Gal}(\mathbf{F}/\mathbf{L})$ which is cyclic. We may therefore write $\mathscr{S} = \langle T, s\rangle$ for some $s \in \mathscr{S}$ and the proof will be complete when we find $\lambda \in \Lambda$ such that $\gamma = \lambda\beta$ is fixed by s.

Write $s = uv$, where u and v are powers of s, $o(u)$ is a power of q and $q \nmid o(v)$. The $\mathscr{S}$-orbit of β, which has q'-size, is the orbit of β under $\langle s\rangle$ and it follows that $\beta^u = \beta$. Now $\beta^s = \beta^v$ is an extension of α and so $\beta^s = \mu\beta$ for some $\mu \in \Lambda$ by Lemma 5.4. Note that $G \times \mathrm{Gal}(\mathbf{F}/\mathbf{L})$ acts on Λ and we have

$$\mu^u\beta = \mu^u\beta^u = (\mu\beta)^u = \beta^{su} = \beta^s = \mu\beta.$$

Therefore $\mu^u = \mu$ since the map in Lemma 5.4 is one to one.

Now $v \in G \times \mathrm{Gal}(\mathbf{F}/\mathbf{L})$ and since v has q'-order and G/N is a q-group, it follows that $v = h\tau$ for some $h \in N$ and $\tau \in \mathrm{Gal}(\mathbf{F}/\mathbf{L})$. There exists an integer m such that $x^\tau = x^m$ for all $x \in \mathbf{F}$ and it follows that $\mu^s = \mu^v = \mu^\tau = \mu^m$.

Suppose $m \not\equiv 1 \bmod q$. Since Λ is a q-group, we can find an integer n such that $n(m-1) \equiv -1 \bmod |\Lambda|$ and thus

$$(\mu^n\beta)^s = \mu^{nm+1}\beta = \mu^n\beta$$

and $\gamma = \mu^n\beta$ is the desired character in this case.

Now assume $m \equiv 1 \bmod q$ so that $\mu^s = \mu^m = \mu\nu$ for some $\nu \in \langle \mu^q \rangle$. Since $\beta^s = \mu\beta$, it follows by induction that for each positive integer k, there exists $\nu_k \in \langle \mu^q \rangle$ such that

$$\beta^{s^k} = \mu^k \nu_k \beta.$$

Taking $k = o(v)$, we have $\beta^{s^k} = \beta^{u^k} = \beta$ and thus $\mu^k \nu_k = 1$ and $\mu^k \in \langle \mu^q \rangle$. Since $q \nmid k$, and μ has q-power order, it follows that $\mu = 1$ and hence $\beta^s = \beta$ in this case. The proof is complete. ∎

6. Characteristic Zero

Fix an algebraically closed field **E** of characteristic zero. All other fields considered in this section will be subfields of **E**. Our first main result is the following.

THEOREM 6.1. *Let $N \lhd G$ and suppose* $\mathbf{L} \subseteq \mathbf{E}$. *Let $\mathfrak{X}$ be an irreducible* **L***-representation of N which is invariant in G and let $\alpha \in \mathrm{Irr}_{\mathbf{E}}(G)$ be an* **E***-constituent of $\mathfrak{X}$. Assume that $m_{\mathbf{L}}(\alpha) = 1$ and that $(|G:N|, o(\alpha)\,\alpha(1)) = 1$. Then $\mathfrak{X}$ extends to an* **L***-representation of G.*

Recall that $o(\alpha)$ denotes the determinantal order of α. Theorem 6.1 generalizes Gallagher's Theorem (Corollary 8.16 of [9]), which is the case where $\mathbf{L} = \mathbf{E}$. Since the canonical extension $\hat{\alpha}$ of α to $T = I_G(\alpha)$ is standard (Definition 2.2), the theorem would follow by Theorem 2.4 if we could show that $m_{\mathbf{L}}(\hat{\alpha}^G) = 1$. Unfortunately, it is not apparent how to find a direct proof of this in general. This approach does work, however, in the special case where $\alpha(1) = 1$.

LEMMA 6.2. *Theorem* 6.1 *holds in the case where $\alpha(1) = 1$.*

Proof. It is no loss to assume that $\mathfrak{X}$ is faithful. Since all of the **E**-constituents of $\mathfrak{X}$ are Galois conjugate, they all have the same kernel and it follows that α is faithful and so N is cyclic of order equal to $o(\alpha)$. In particular, N is a Hall subgroup of G and so is complemented by some subgroup $C \subseteq G$. Let $K = T \cap C$ and note that N is central in T and thus $T = N \times K$.

Let $\beta = \alpha \times 1_K \in \mathrm{Irr}_{\mathbf{E}}(T)$ so that β is the canonical extension of α to T. Let $\gamma = \beta^G \in \mathrm{Irr}_{\mathbf{E}}(G)$. If we can show that $m_{\mathbf{L}}(\gamma) = 1$, then the result follows from Theorem 2.4. Since $(1_C)^G$ is the character of an **L**-representation of G, it follows from Corollary 10.2(c) of [9] that $m_{\mathbf{L}}(\gamma)$ divides $[\gamma, (1_C)^G]$. It therefore suffices to show that $[\gamma, (1_C)^G] = 1$.

We have $[\gamma, (1_C)^G] = [\beta^G, (1_C)^G] = [\beta, ((1_C)^G)_T]$. Since $CT = G$, it follows that $((1_C)^G)_T = (1_K)^T$. Therefore, $[\gamma, (1_C)^G] = [\beta, (1_K)^T] = [\beta_K, 1_K]$. However, $\beta_K = 1_K$ and the result follows. ∎

Proof of Theorem 6.1. Let $\mathfrak{Y}$ be an $\mathbf{L}(\alpha)$-representation of N which affords α. (This exists since $m_{\mathbf{L}}(\alpha) = 1$.) By Theorem 4.3, let $\mathfrak{Z}$ be a projective $\mathbf{L}(\alpha)$-crossed representation which extends $\mathfrak{Y}$ and respects N, where the action of G on $\mathbf{L}(\alpha)$ is that induced by the semi-invariance of α. Let f be the factor set associated with $\mathfrak{Z}$ so that

$$\mathfrak{Z}(gh) = \mathfrak{Z}(g)^h \mathfrak{Z}(h) f(g, h) \qquad \text{for } g, h \in G. \tag{1}$$

By Corollary 4.4, we will be done when we show that the cohomology element $f^* \in H^2(G/N, \mathbf{L}(\alpha)^\times)$ is trivial.

Write $\lambda = \det \alpha$ and $\mu(g) = \det \mathfrak{Z}(g)$ for $g \in G$ so that $\mu_N = \lambda$. Also, let $r = \alpha(1)$. Taking determinants in (1) we obtain

$$\mu(gh) = \mu(g)^h \mu(h) f(g, h)^r. \tag{2}$$

Note that it does not follow that $(f^*)^r = 1$ since we do not have that the function μ is constant on cosets of N and so it does not correspond to a function $G/N \to L(\alpha)^\times$.

Now $\lambda \in \mathrm{Irr}_{\mathbf{E}}(N)$ is $\mathbf{L}$-semi-invariant in G and $m_{\mathbf{L}}(\lambda)$ divides $\lambda(1) = 1$. Let $\mathfrak{X}_0$ be a (unique up to similarity) $\mathbf{L}$-representation of N having λ as an $\mathbf{E}$-constituent. The semi-invariance of λ yields that $\mathfrak{X}_0$ is G-invariant and Lemma 5.2 applies and proves that $\mathfrak{X}_0$ extends to an $\mathbf{L}$-representation of G. We may apply Theorem 3.1 and conclude that λ extends to an $\mathbf{L}(\lambda)$-crossed representation ν of G (of degree 1) with respect to the action of G on $\mathbf{L}(\lambda)$.

Since α is afforded by an $\mathbf{L}(\alpha)$-representation, we see that $\mathbf{L}(\lambda) \subseteq \mathbf{L}(\alpha)$. Because $\det(\alpha^g) = \lambda^g$ and $\det(\alpha^\sigma) = \lambda^\sigma$ for $\sigma \in \mathrm{Gal}(\mathbf{L}(\alpha)/\mathbf{L})$, it is clear that the action of G on $\mathbf{L}(\lambda)$ with respect to which ν is a crossed representation, is just the restriction of the original action on $\mathbf{L}(\alpha)$ to $\mathbf{L}(\lambda)$.

We now have a function $\nu\colon G \to \mathbf{L}(\alpha)^\times$ such that $\nu(gh) = \nu(g)^h \nu(h)$ for g, $h \in G$, and $\nu(n) = \lambda(n)$ for $n \in N$. Write $\omega(g) = \mu(g)\nu(g)^{-1}$. From (2) we obtain

$$\omega(gh) = \omega(g)^h \omega(h) f(g, h)^r \tag{3}$$

and we also have that $\omega(n) = 1$ for all $n \in N$. It follows that ω is constant on all cosets of N by setting $h = g^{-1}n$ in (3) and appealing to Lemma 4.2.

Since ω is constant on cosets of N, it defines a map $G/N \to \mathbf{L}(\alpha)^\times$ and it follows that $(f^*)^r = 1$. However, the exponent of $H^2(G/N, \mathbf{L}(\alpha)^\times)$ divides $|G/N|$. (See Satz I.16.9 of [7]) and so is relatively prime to $r = \alpha(1)$. It follows that $f^* = 1$ and the proof is complete. ∎

The following result enables us to drop the condition that $m_{\mathbf{L}}(\alpha) = 1$ in Theorem 6.1. Corollary 6.4, which is the strengthened version of 6.1, completes the proof of Theorem A.

THEOREM 6.3. *Let $T \lhd G$ and suppose $\beta \in \mathrm{Irr}_{\mathbf{E}}(T)$ is* **L***-semi-invariant in G, where $\mathbf{L} \subseteq \mathbf{E}$. Assume $\beta^G = \gamma \in \mathrm{Irr}_{\mathbf{E}}(G)$ and that $(|G:T|, \beta(1)\, o(\beta)) = 1$. Then $(m_{\mathbf{L}}(\gamma), |G:T|) = 1$ and $m_{\mathbf{L}}(\gamma) = m_{\mathbf{L}}(\beta)$.*

Proof. The fact that $m_{\mathbf{L}}(\beta)$ divides $m_{\mathbf{L}}(\gamma)$ is comparatively trivial and we dispose of it first. The character $m_{\mathbf{L}}(\gamma)\,\gamma$ is afforded by an $\mathbf{L}(\gamma) \subseteq \mathbf{L}(\beta)$ representation. Since $[\beta, (m_{\mathbf{L}}(\gamma)\,\gamma)_T] = m_{\mathbf{L}}(\gamma)$, it follows by Corollary 10.2(c) of [9] that $m_{\mathbf{L}(\beta)}(\beta)$ divides $m_{\mathbf{L}}(\gamma)$. However, $m_{\mathbf{L}(\beta)}(\beta) = m_{\mathbf{L}}(\beta)$ by part (a) of the same corollary.

By Lemma 2.3 (applied to the case $N = T$ and $\alpha = \beta$) we have that $m_{\mathbf{L}}(\gamma)$ divides $m_{\mathbf{L}}(\beta)\,|G:T|$ and thus the proof will be complete when we show that $(m_{\mathbf{L}}(\gamma), |G:T|) = 1$.

Now let ε be a primitive nth root of unity in $\mathbf{E}$, where n is the exponent of G. Since $\mathrm{Gal}(\mathbf{L}(\varepsilon)/\mathbf{L})$ is abelian, there is a unique minimal field $\mathbf{K}$, $\mathbf{L} \subseteq \mathbf{K} \subseteq \mathbf{L}(\varepsilon)$ such that $|\mathbf{L}(\varepsilon) : \mathbf{K}|$ involves no prime dividing $m_{\mathbf{L}}(\beta)$. We claim that $m_{\mathbf{K}}(\beta) = 1$, for suppose p is a prime divisor of $m_{\mathbf{K}}(\beta)$. Then p divides $m_{\mathbf{L}}(\beta)$ by Corollary 10.2(f) of [9] and hence $p \nmid |\mathbf{K}(\varepsilon) : \mathbf{K}|$. Thus the Sylow p-subgroup of $\mathrm{Gal}(\mathbf{K}(\varepsilon)/\mathbf{K}(\beta))$ is trivial and Theorem 10.12 of [9] yields that $p \nmid m_{\mathbf{K}}(\beta)$. This contradiction implies that $m_{\mathbf{K}}(\beta) = 1$ as claimed.

Now let $\mathfrak{X}$ be an irreducible **K**-representation of N having β as an **E**-constituent. Since β is certainly **K**-semi-invariant in G, Theorem 6.1 applies and $\mathfrak{X}$ extends to a **K**-representation $\mathfrak{Z}$ of G. Since γ is the only irreducible **E**-character whose restriction to T has β as a constituent, it follows that γ is an **E**-constituent of $\mathfrak{Z}$. Since the multiplicity of β in $\mathfrak{X}$ is 1, it follows that the multiplicity of γ in $\mathfrak{Z}$ is 1 and hence $m_{\mathbf{K}}(\gamma) = 1$. Therefore $m_{\mathbf{L}}(\gamma)$ divides $|\mathbf{K} : \mathbf{L}|$ by Corollary 10.2(g) of [9].

By the choice of **K**, it follows that all prime divisors of $|\mathbf{K} : \mathbf{L}|$ are divisors of $m_{\mathbf{L}}(\beta)$ and thus divide $\beta(1)$. These primes, therefore, do not divide $|G:T|$ and the result follows. ∎

COROLLARY 6.4. *Let $N \lhd G$ and let $\mathfrak{X}$ be an irreducible* **L***-representation of N, where $\mathbf{L} \subseteq \mathbf{E}$. Let α be an irreducible* **E***-constituent of $\mathfrak{X}$ and assume $(|G:N|, \alpha(1)\, o(\alpha)) = 1$. Then $\mathfrak{X}$ extends to an* **L***-representation of G which has γ as an* **E***-constituent, where $\gamma = (\hat{\alpha})^G$ and $\hat{\alpha}$ is the canonical extension of α to $T = I_G(\alpha)$.*

Proof. Since $\hat{\alpha}$ is a standard extension of α to T, it suffices by Theorem 2.4 to show that $m_{\mathbf{L}}(\gamma)$ divides $m_{\mathbf{L}}(\alpha)$. By Theorem 6.3, $m_{\mathbf{L}}(\gamma)$ is coprime with $|G:T|$ and the result follows by Lemma 2.3. ∎

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