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Spectral Decomposition of Invariant Differential Operators on Certain Nilpotent Homogeneous Spaces*

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If K is a connected subgroup of a nilpotent Lie group G, the irreducible decomposition of the action on $L^2(K\backslash G)$ has either pure infinite or boundedly finite multiplicities. In the finite case the authors recently proved that the algebra $D(K\backslash G)$ of G-invariant differential operators on $K\backslash G$ is commutative, even if the action is not multiplicity free, and produced evidence for the conjecture that $D(K\backslash G)$ is isomorphic to the algebra of all $Ad^*(K)$-invariant polynomials on the annihilator $\mathfrak{k}^\perp \subseteq \mathfrak{g}^*$, where $\mathfrak{k}$ is the Lie algebra of K. Here the conjecture is proved for a large class of data (K, G). For such pairs an explicit construction of the isomorphism can be found; it is a type of Fourier transform with some unusual nonlinear aspects. Furthermore the operators in $D(K\backslash G)$ have tempered fundamental solutions.

1. INTRODUCTION

Let K be a Lie subgroup of a connected, simply connected nilpotent Lie group G. We denote their Lie algebras by $\mathfrak{k}$, $\mathfrak{g}$, respectively. Let $\chi \in K^\wedge$ be a character, and set $\tau = \mathrm{Ind}(K \uparrow G, \chi)$. We regard τ as acting by right translation on the functions $\mathscr{L}^2(G, \tau) = \{f: G \to \mathbb{C} \mid f(kx) = \chi(k) f(x)$ for all $k \in K$, $x \in G$, and $\int_{K\backslash G} |f(x)|^2 d\dot{x} < \infty\}$. Similarly, we write $\mathscr{C}^\infty(G, \tau) = \{f \in \mathscr{C}^\infty(G): f(kx) = \chi(k) f(x)$ for all $k \in K$, $x \in G\}$, $\mathscr{C}_c^\infty(G, \tau) = \{f \in \mathscr{C}^\infty(G, \tau): f$ has compact support mod $K\}$, etc.

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We want to study the harmonic analysis of τ. At the most basic level, this means knowing a direct integral decomposition for τ. This problem was first solved in [9, 15]; see [7] or [21] for somewhat simpler treatments. We can describe the answer as follows: there is a Lie homomorphism $f \in \mathfrak{k}^*$ such that $\chi(\exp Y) = e^{2\pi i f(Y)}$, all $Y \in \mathfrak{k}$. Extend f to a functional on $\mathfrak{g}^*$ (which we also denote by f), and let $\Omega_\tau = f + \mathfrak{k}^\perp$. Then

$$\tau \cong \int_{\Omega_\tau/K}^{\oplus} \pi_l \, d\mu(l), \tag{1}$$

where K acts on Ω_τ by the coadjoint action, π_l corresponds to $l \in \Omega_\tau/K$ by the Kirillov correspondence, and μ is the push-forward of (a finite measure equivalent to) Lebesgue measure.

Beyond this, harmonic analysis can take various directions. Here we investigate some questions that appear to be related naturally to the study of differential operators on $\mathscr{C}^\infty(G, \tau)$. Let $\mathbb{D}_\tau(K\backslash G)$ be the algebra of operators $D \mid \mathscr{C}^\infty(G, \tau)$, where D is a differential operator on G taking $\mathscr{C}^\infty(G, \tau)$ into itself and commuting with the action of τ on that space. In [8], we showed (following Jacobsen [17]) that one can get a more concrete description of $\mathbb{D}_\tau(K\backslash G)$. Let $\mathfrak{a}_\tau$ be the subspace of $\mathfrak{u}(\mathfrak{g})$ generated by the elements $Y + 2\pi i f(Y) I$, $Y \in \mathfrak{k}$, and let $\mathfrak{u}(\mathfrak{g})\, \mathfrak{a}_\tau$ be the left ideal generated by $\mathfrak{a}_\tau$, $\mathfrak{u}(\mathfrak{g}, \tau) = \{A \in \mathfrak{u}(\mathfrak{g}) : [A, Y] \subseteq \mathfrak{u}(\mathfrak{g})\, \mathfrak{a}_\tau \text{ for all } Y \in \mathfrak{k}\}$. Elements of $\mathfrak{u}(\mathfrak{g})$ (regarded as right invariant operators) take $\mathscr{C}^\infty(G)$ to itself and commute with right translation, and those in $\mathfrak{u}(\mathfrak{g}, \tau)$ map $\mathscr{C}^\infty(G, \tau)$ to itself. Thus we get a natural map $\gamma : \mathfrak{u}(\mathfrak{g}, \tau) \to \mathbb{D}_\tau(K\backslash G)$. Theorem 4.1 of [8] says that

$$\gamma : \mathfrak{u}(\mathfrak{g}, \tau) \to \mathbb{D}_\tau(K\backslash G) \text{ is surjective, with kernel } \mathfrak{u}(\mathfrak{g})\, \mathfrak{a}_\tau.$$

We would like to understand $\mathbb{D}_\tau(K\backslash G)$ in terms of the orbit picture for the spectrum and multiplicities of τ. In particular, we would like to answer the following questions:

(a) When is $\mathbb{D}_\tau(K\backslash G)$ commutative?

(b) Do operators in the center $Z\mathbb{D}_\tau(K\backslash G)$ have tempered fundamental solutions (i.e., for $A \in Z\mathbb{D}_\tau(K\backslash G)$, is there a tempered distribution ξ such that $A\xi = \delta_{Ke}$, the point mass at the coset Ke)? (2)

We could then use these results to improve existing solvability criteria for more general operators on $K\backslash G$ (see Lion [18–20] for some known results and [8, Section 7] for some speculations).

The decomposition (1) for τ leads to a primary decomposition,

$$\tau \cong \int_{G^\wedge} m(\pi)\pi \, d\nu(\pi),$$

where ν is the push-forward of μ under the Kirillov orbit map and

$$m(\pi) = \text{number of } Ad^*(K)\text{-orbits in } \mathcal{O}_\pi \cap \Omega_\tau \qquad (\mathcal{O}_\pi = \text{Kirillov orbit for } \pi).$$

It is also known (from [7, 9, 21]) that either $m(\pi) = \infty$ ν-a.e. (the infinite case) or that $\exists M$ with $m(\pi) < M$ ν-a.e. (the finite case). As shown in [7, 9, 21], the condition

$$m(\pi) < \infty \qquad \text{for generic} \quad \pi \in \operatorname{spec}(\tau) = \operatorname{supp}(\nu) \tag{3}$$

holds precisely when

$$\dim G \cdot l = 2 \dim K \cdot l \qquad \text{for generic} \quad l \in \Omega_\tau = f + \mathfrak{k}^\perp. \tag{4}$$

Evidence has accumulated that there is a very nice description of $\mathbb{D}_\tau(K\backslash G)$ in the finite multiplicity case. We showed in [8] that $\mathbb{D}_\tau(K\backslash G)$ is always commutative in this case, while examples in low-dimensional cases suggest that $\mathbb{D}_\tau(K\backslash G)$ is never commutative in the case of infinite multiplicity. When $m(\pi) < \infty$, we can say more: in [8], commutativity was proved by showing that $\mathbb{D}_\tau(K\backslash G)$ is isomorphic to a generating subalgebra of the field $\mathbb{C}(\Omega_\tau)^K$ of $Ad^*(K)$-invariant rational functions on Ω_τ. There seems to be a close connection between $\mathbb{D}_\tau(K\backslash G)$ and the algebra $\mathbb{C}[\Omega_\tau]^K$ of $Ad^*(K)$-invariant polynomials, and we make the following conjecture:

Conjecture 1. If $m(\pi) < \infty$ for generic $\pi \in \operatorname{spec}(\tau)$, then $\mathbb{D}_\tau(K\backslash G) \cong \mathbb{C}[\Omega_\tau]^K$.

If this is true, it is also important to know how the isomorphism is realized. In [2, 3], Benoist gave a complete analysis of nilpotent symmetric spaces $K\backslash G$, i.e., the case where there is an involutive automorphism α of $\mathfrak{g}$ such that $\mathfrak{k} = \{X \in \mathfrak{g} : \alpha(X) = X\}$, and where $\chi \equiv 1$. For these he showed that

(a) $\mathbb{D}_\tau(K\backslash G) \cong \mathbb{C}[\mathfrak{k}^\perp]^K$, and in particular is commutative. (Here, $\Omega_\tau = \mathfrak{k}^\perp$.)

(b) Every operator in $\mathbb{D}_\tau(K\backslash G)$ has a tempered fundamental solution on $K\backslash G$. (5)

Using the special features of symmetric spaces, he also showed that the isomorphism is given by an explicit Fourier transform. Symmetric spaces are rather special; for instance, they have $m_\pi \equiv 1$, and then commutativity of $\mathbb{D}_\tau(K\backslash G)$ is not so surprising. The results in [8] are for arbitrary G, K, χ, but the methods intrinsically involve rational functions associated with orbit parametrizations and do not reveal the connection with K-invariant polynomials on Ω_τ. Also, the map implementing the embedding $\mathbb{D}_\tau(K\backslash G) \subseteq \mathbb{C}(\Omega_\tau)^K$ is constructed inductively, and in examples it is often difficult to express this map in closed form.

The present paper is concerned with finding a canonical, explicit realization of the decomposition (1) that reveals the validity of (5) for a reasonably large class of data $(\mathfrak{g}, \mathfrak{k}, \chi)$. Our work applies when the following conditions are met:

(i) $m_\pi < \infty$ for generic $\pi \in \operatorname{Spec}(\tau) = \operatorname{supp} \nu$.

(ii) $\mathfrak{g}$ contains a subalgebra $\mathfrak{b}$ that polarizes generic $l \in \Omega_\tau = f + \mathfrak{k}^\perp$ and is normalized by $\mathfrak{k}$. (6)

Our main result is the following:

1.1. THEOREM. *Let G be a connected, simply connected nilpotent Lie group, K a connected Lie subgroup, and $\chi = \chi_f$ a character on K associated with some $f \in \mathfrak{g}^*$ such that $f|_\mathfrak{k}$ is a Lie homomorphism. Set $\tau = \operatorname{Ind}(K \uparrow G, \chi_f)$, and assume that the conditions* (6) *are satisfied. Then*:

(a) *$\mathbb{D}_\tau(K\backslash G)$ is isomorphic to $\mathbb{C}[\Omega_\tau]^K$;*

(b) *every nonzero element of $\mathbb{D}_\tau(K\backslash G)$ has a tempered fundamental solution.*

We prove this by exhibiting an explicit isometry

$$\Psi_\tau : \mathscr{H}_\tau \to \mathscr{L}^2(\Omega_\tau, \mu_\Omega) = \int_{\Omega_\tau/K}^{\oplus} \mathscr{L}^2(\theta, \mu_\theta)\, d\nu(\theta),$$

where μ_Ω is Lebesgue measure on Ω_τ, μ_θ is $Ad^*(K)$-invariant on θ for generic $\theta \in \Omega_\tau/K$, and ν gives the disintegration $\mu_\Omega = \int_{\Omega_\tau/K}^{\oplus} \mu_\theta \, d\nu(\theta)$. This map has global smoothness properties which are revealed by examining a suitably defined "core" $\mathscr{S}_\tau \subseteq \mathscr{H}_\tau^\infty$ that is invariant under $\tau(\mathfrak{u}(\mathfrak{g}))$ and $\mathbb{D}_\tau(K\backslash G)$, and is Fréchet dense in $\mathscr{H}_\tau^\infty$. This core is defined as the set of functions in $\mathscr{C}^\infty(G, \tau)$ that are "Schwartz transverse to $K\backslash G$ cosets"; $\mathscr{S}_\tau$ is also the range of the averaging map $Q_\tau : \mathscr{S}(G) \to \mathscr{C}^\infty(G, \tau)$,

$$Q_\tau w(g) = \int_K w(kg)\, \overline{\chi(k)}\, dk, \qquad w \in \mathscr{S}(G),$$

which intertwines the right actions R_g, $R(A)$ for $A \in \mathfrak{u}(\mathfrak{g})$ with the actions τ_g, $\tau(A)$ on $\mathscr{H}_\tau^\infty$. Notice that $\mathbb{D}_\tau(K\backslash G)$ need not leave $\mathscr{H}_\tau^\infty$ invariant, and $\mathscr{S}_\tau$ is in some sense the largest subspace of $\mathscr{H}_\tau^\infty$ on which we may compare the actions of $\mathbb{D}_\tau(K\backslash G)$ and $\tau(\mathfrak{u}(\mathfrak{g}))$. Notice also that once we restrict attention to actions on $\mathscr{S}_\tau$ we are outside the realm of the usual theory of C^∞ vectors as given in, e.g., [25].

The first step in proving Theorem 1.1 is to construct a surjective isomorphism $\Psi_\tau : \mathscr{S}_\tau \to \mathscr{S}(\Omega_\tau)$ such that:

(i) Ψ_τ extends to an isometric isomorphism from $\mathscr{H}_\tau$ to $\mathscr{L}^2(\Omega_\tau, \mu_\Omega)$ for a suitably normalized choice of Lebesgue measure μ_Ω.

(ii) Each $A \in \mathfrak{u}(\mathfrak{g})$ corresponds to a polynomial coefficient differential operator $\tau(A)^\wedge$ on Ω_τ such that $\Psi_\tau(\tau(A)\varphi) = \tau(A)^\wedge(\Psi_\tau\varphi)$ for all $\varphi \in \mathscr{S}_\tau$.

(iii) Each $D \in \mathbb{D}_\tau(K\backslash G)$ corresponds to a polynomial coefficient differential operator $D^\wedge$ on Ω_τ such that $\Psi_\tau(D\varphi) = D^\wedge(\Psi_\tau\varphi)$ for all $\varphi \in \mathscr{S}_\tau$.

Next we show that the operators $\tau(A)^\wedge$, $D^\wedge$ all restrict to generic K-orbits $\theta \in \Omega_\tau/K$ (these are closed submanifolds in Ω_τ). This is the hard part of the proof; it uses the work of Fujiwara [13].

Under $\Psi_\tau: \mathscr{S}_\tau \to \mathscr{L}^2(\Omega_\tau, \mu_\Omega) \cong \int^\oplus_{\Omega_\tau/K} \mathscr{L}^2(\theta, \mu_\theta)\, d\nu(\theta)$, a vector $\varphi \in \mathscr{S}_\tau$ corresponds to a smooth function $\Psi_\tau\varphi \in \mathscr{S}(\Omega_\tau)$, which decomposes by restriction to K-orbits: $\Psi_\tau\varphi = \int^\oplus_{\Omega_\tau/K} \Psi_\tau\varphi|_\theta\, d\nu(\theta)$, with $\Psi_\tau\varphi|_\theta \in \mathscr{S}(\theta)$ for generic θ. The restriction properties of operators $\tau(A)^\wedge$, $D^\wedge$ can be used to show that τ decomposes as $\tau \cong \int^\oplus_{\Omega_\tau/K} \pi_\theta\, d\nu(\theta)$, where π_θ is modeled in $\mathscr{L}^2(\theta, \mu_\theta)$ with C^∞ vectors $\mathscr{H}^\infty_\pi = \mathscr{S}(\theta)$, $\pi_\theta \in G^\wedge$ corresponds to the G-orbit $G\cdot\theta$, and that

$$\tau(A) \cong \int^\oplus_{\Omega_\tau/K} \tau(A)^\wedge|_\theta\, d\nu(\theta) = \int^\oplus_{\Omega_\tau/K} \pi_\theta(A)\, d\nu(\theta)$$

is the C^∞ decomposition of $\tau(A)$ acting on $\mathscr{H}^\infty_\tau$, as in [14, 24]. The action of $D \in \mathbb{D}_\tau(K\backslash G)$ on vectors in $\mathscr{S}_\tau$ also decomposes:

$$D \cong \int^\oplus_{\Omega_\tau/K} D^\wedge|_\theta\, d\nu(\theta);$$

that is, if $\varphi \in \mathscr{S}_\tau$ then $\Psi_\tau\varphi \in \mathscr{S}(\Omega_\tau)$ has decomposition $\int^\oplus_{\Omega_\tau/K} \Psi_\tau\varphi|_\theta\, d\nu(\theta)$, while

$$\Psi_\tau(D\varphi) \cong \int^\oplus_{\Omega_\tau/K} (D^\wedge\Psi_\tau\varphi)|_\theta\, d\nu(\theta) \cong \int^\oplus_{\Omega_\tau/K} D^\wedge|_\theta(\Psi_\tau\varphi|_\theta)\, d\nu(\theta)$$

by definition of the restrictions $D^\wedge|_\theta$.

Once we have these decompositions, $D^\wedge|_\theta$ automatically commutes with $\{\tau(A)^\wedge|_\theta: A \in \mathfrak{u}(\mathfrak{g})\} = \pi_\theta(\mathfrak{u}(\mathfrak{g}))$, and since π_θ is irreducible on $\mathscr{H}^\infty_{\pi_\theta} = \mathscr{S}(\theta)$ we conclude that $D^\wedge|_\theta$ is scalar and is constant for generic θ. Since $D^\wedge$ is a polynomial coefficient differential operator on Ω_τ, $D^\wedge = p(l)I$ for some $Ad^*(K)$-invariant polynomial on Ω_τ. Conversely, we can use an inverse to the map Ψ_τ to associate an element of $\mathbb{D}_\tau(K\backslash G)$ with every element of $\mathbb{C}[\Omega_\tau]^K$. This leads directly to Theorem 1.1.

The construction of Ψ_τ can be outlined as follows: we choose a weak Malcev basis $X_1, \ldots, X_n$ for $\mathfrak{g}$ (so that $\mathfrak{g}_i = \mathbb{R}-\text{span}\{X_1, \ldots, X_i\}$ is always a subalgebra) passing through $\mathfrak{k}\cap\mathfrak{b}$, $\mathfrak{k}$, and $\mathfrak{k}+\mathfrak{b}$ (which is a subalgebra because (6) implies that $[\mathfrak{k},\mathfrak{b}]\subseteq\mathfrak{b}$). This basis lets us coordinatize G and $K\backslash G$, so that (for example) $K\backslash G\approx\mathbb{R}^{n-p}$, where $p=\dim\mathfrak{k}$. We then use a Fourier transform in these Malcev coordinates to transform the action of τ to one on $\mathscr{L}^2(\Omega_\tau)$. After some effort one can show that this transform has the desired properties.

The bulk of the work is in Sections 4 and 5. Section 2 is concerned with needed material on C^∞ vectors, and Section 3 with the enveloping algebra description of $\mathbb{D}_\tau(K\backslash G)$. Section 6 is devoted to examples.

2. C^∞ Vectors and $C^{-\infty}$ Vectors

Let ρ be a unitary representation of G, acting on $\mathscr{H}_\rho$. We assemble here some results about the space $\mathscr{H}_\rho^\infty$ of $\mathscr{C}^\infty$ vectors on ρ and about its antidual, $\mathscr{H}_\rho^{-\infty}$. The basic reference is [25]; however, that paper uses $\mathscr{C}_c^\infty(G)$ on arbitrary G, while we need $\mathscr{S}(G)$ for nilpotent G, so that we need to deal with some technical details. Let $\{X_1, ..., X_n\}$ be a basis for $\mathfrak{g}$. The (Fréchet) topology on $\mathscr{S}(G)$ is given by seminorms $\|p_\beta(g)\,R(X^\alpha)f\|_\infty$, or by the seminorms $\|p_\beta(g)\,L(X^\alpha)f\|_\infty$, or by the corresponding L^2-norms, where p_β is any polynomial function on G and $L(A), R(A)$ are the left, right actions of $A\in\mathfrak{u}(\mathfrak{g})$ on G as right-, left-invariant operators. (See [5, Appendix 2], for equivalence of these norms.) The (Fréchet) topology on $\mathscr{H}_\rho^\infty$ is given by seminorms

$$\|\rho(X^\alpha)\xi\|, \qquad X^\alpha=X_1^{\alpha_1}\cdots X_n^{\alpha_n};$$

$\mathscr{H}_\rho^{-\infty}$ is the space of all conjugate linear continuous functionals on $\mathscr{H}_\rho^\infty$ (in this topology). If $w\in\mathscr{S}(G)$, $\zeta\in\mathscr{H}_\rho$, then $\rho(w)\zeta=\int_G w(g)\,\rho(g)\zeta\,dg\in\mathscr{H}_\rho^\infty$, and

$$\rho(A)\,\rho(w)\zeta=\rho(L(A)w)\zeta, \qquad \rho(g)\,\rho(w)\zeta=\rho(L_g w)\zeta,$$
$$\forall A\in\mathfrak{u}(\mathfrak{g}), \quad g\in G, \quad w\in\mathscr{S}(G).$$

Also, $\mathbb{C}$-span$\{\rho(w)\xi\colon w\in\mathscr{S}(G), \xi\in\mathscr{H}_\rho\}$ is dense in $\mathscr{H}_\rho^\infty$ and invariant under $\rho(G)$ and $\rho(\mathfrak{u}(\mathfrak{g}))$. (From results in [12], this space is equal to $\mathscr{H}_\rho^\infty$.) We also have

The map $\rho\colon\mathscr{S}(G)\times\mathscr{H}_\rho^\infty\to\mathscr{H}_\rho^\infty$ is jointly (Fréchet) continuous. (7)

[By Banach–Steinhaus, we need only show separate continuity; that follows from the Closed Graph Theorem.]

On $\mathscr{H}_\rho^{-\infty}$ we have a contragredient action ρ' with $\langle \rho'(g)a, \zeta\rangle = \langle a, \rho(g^{-1})\zeta\rangle$. If $w \in \mathscr{C}_c^\infty(G)$, the compactness of the support lets us write

$$\begin{aligned}\langle \rho'(w)a, \zeta\rangle &= \int_G \langle w(g)\,\rho'(g)a, \zeta\rangle\, dg \\ &= \int_G w(g)\langle a, \rho(g^{-1})\zeta\rangle\, dg = \int_G \langle a, w^*(g)\,\rho(g)\zeta\rangle\, dg \\ &= \langle a, \rho(w^*)\zeta\rangle = \langle \rho(w^*)^t a, \zeta\rangle, \end{aligned} \tag{8}$$

where $w^*(x) = \overline{w(x^{-1})}$; the continuity of $\rho'(w)a$ follows from (7).

For $w \in \mathscr{S}(G)$, $\rho'(w)a$ is still defined as a transpose, $\rho'(w)a = \rho(w^*)^t a$; this is a continuous functional, and we get an algebraic action $\mathscr{S}(G) \times \mathscr{H}_\rho^{-\infty} \to \mathscr{H}_\rho^{-\infty}$. The integral formula (8) still holds; to see this we need to check absolute convergence, and this requires control of $g \mapsto \langle a, \rho(g^{-1})\zeta\rangle$. The continuity of the functional a means that there exist C, N with

$$|\langle a, \zeta\rangle|^2 \leqslant C \sum_{|\alpha| \leqslant N} \|\rho(X^\alpha)\zeta\|^2, \qquad \forall \zeta \in \mathscr{H}_\rho^{-\infty}.$$

Since $\rho(g^{-1})\,\rho(X^\alpha)\,\rho(g) = \rho(Ad(g^{-1})\,X^\alpha) = \sum_{|\beta| \leqslant N} p_{\alpha\beta}(g)\,\rho(X^\beta)$, where $p_{\alpha\beta}$ is a polynomial in g, we have (for any $|\alpha| \leqslant N$)

$$\|\rho(X^\alpha)\,\rho(g)\zeta\| = \left\|\rho(g) \sum_{|\beta| \leqslant N} p_{\alpha\beta}(g)\,\rho(X^\beta)\zeta\right\| = \left\|\sum_{|\beta| \leqslant N} p_{\alpha\beta}(g)\,\rho(X^\beta)\zeta\right\|.$$

Therefore $|\langle a, \rho(g)\zeta\rangle|$ is bounded by a polynomial, and the integral in (8) is absolutely convergent when $w \in \mathscr{S}(G)$. Now let $\{w_n\}$ be a sequence in $\mathscr{C}_c^\infty(G)$ converging to w in $\mathscr{S}(G)$. Then $w_n^* \to w^*$ and, from (7), $\rho(w_n^*)\zeta \to \rho(w^*)\zeta$ for $\zeta \in \mathscr{H}_\rho^\infty$. The calculation above gives

$$\langle \rho'(w_n)a, \zeta\rangle = \langle a, \rho(w_n^*)\zeta\rangle = \int_G \langle a, \rho(g^{-1})\zeta\rangle\, w_n(g)\, dg;$$

let $n \to \infty$ to see that (8) holds for $w \in \mathscr{S}(G)$.

Define $j: \mathscr{H}_\rho \to \mathscr{H}_\rho^{-\infty}$ by the obvious map $\langle j(\zeta), \xi\rangle = (\zeta, \xi)_{\mathscr{H}_\rho}$, where $(\cdot, \cdot)$ is the inner product in $\mathscr{H}_\rho$. Cartier [4] showed that $\rho'(\mathscr{C}_c^\infty(G))a \subseteq j(\mathscr{H}_\rho^\infty)$ if $a \in \mathscr{H}_\rho^{-\infty}$, and we will prove this for $\mathscr{S}(G)$ in a moment. There is a conjugate linear form $(\cdot, \cdot)$ on $\mathscr{H}_\rho^{-\infty} \times \mathscr{H}_\rho^{-\infty}$, defined when at least one entry is in $j(\mathscr{H}_\rho^\infty)$ and consistent with the inner product in $\mathscr{H}_\rho$ under j; just define

$$(a, j\xi) = \langle a, \xi\rangle, \qquad (j\xi, b) = \langle b, \xi\rangle^-, \; a, b \in \mathscr{H}_\rho^{-\infty}, \; \xi \in \mathscr{H}_\rho^\infty.$$

Clearly $(j\xi, j\zeta) = (\xi, \zeta)$. This bracketing and the result of Cartier allow us to form "matrix elements" $T = T_{(\rho,a,b)}$ as distributions in $\mathscr{S}'(G)$ by taking

$$T(w) = (\rho'(w)a, b), \qquad a, b \in \mathscr{H}_\rho^{-\infty}, w \in \mathscr{S}(G).$$

We will be particularly interested in *cyclic vectors* $a \in \mathscr{H}_\rho^{-\infty}$, those for which $\langle a, \rho(g)\xi\rangle = 0$ all $g \in G \Rightarrow \xi = 0$ $(\xi \in \mathscr{H}_\rho^\infty)$. The following result collects the information we need.

2.1. LEMMA. *Let G be a nilpotent Lie group, ρ a unitary representation, and fix a Haar measure $dg = m_G$ to determine the integrated form of ρ and ρ'.*

(a) *$\rho'(w)a \in j(\mathscr{H}_\rho^\infty)$ if $a \in \mathscr{H}_\rho^{-\infty}$ and $w \in \mathscr{S}(G)$.*

(b) *For fixed $a \in \mathscr{H}_\rho^{-\infty}$, define $Q_\rho = Q_{\rho,a}: \mathscr{S}(G) \to \mathscr{H}_\rho^\infty$ by $\langle j(Q_\rho(w)), \zeta\rangle = \langle \rho'(w^\sim)a, \zeta\rangle$, where $w^\sim(g) = w(g^{-1})$. Then*:

(i) *$\rho(g)\, Q_\rho w = Q_\rho(R_g w)$, where $R_g w(x) = w(xg)$.*

(ii) *$\rho(A)\, Q_\rho w = Q_\rho(R(A)w)$, where $A \in \mathfrak{u}(\mathfrak{g})$.*

(iii) *$w \mapsto Q_\rho w$ is Fréchet continuous.*

If a is also cyclic then $Q_\rho(\mathscr{S}(G))$ is Fréchet dense in $\mathscr{H}_\rho^\infty$.

(c) *Given $a, b \in \mathscr{H}_\rho^{-\infty}$ define $T(w) = (\rho'(w)a, b)$ as above. Then $T \in \mathscr{S}'(G)$ and*

$$(\rho'(w_1^* * w_2)a, b) = (Q_{\rho,a}w_2^\sim, Q_{\rho,b}w_1^\sim)_{\mathscr{H}_\rho}, \qquad \textit{all} \quad w_1, w_2 \in \mathscr{S}(G).$$

Proof. For $\zeta \in \mathscr{H}_\rho^\infty$ and $a \in \mathscr{H}_\rho^{-\infty}$ the continuity of a gives constants C, N such that

$$\begin{aligned} |\langle \rho'(w^*)a, \zeta\rangle|^2 = |\langle a, \rho(w)\zeta\rangle|^2 &\leqslant C \sum_{|\alpha| \leqslant N} \|\rho(X^\alpha)\,\rho(w)\zeta\|^2 \\ &= C \sum_{|\alpha| \leqslant N} \|\rho(L(X^\alpha)w)\zeta\|^2 \\ &\leqslant C \sum_{|\alpha| \leqslant N} \|L(X^\alpha)w\|_1^2\, \|\zeta\|^2; \end{aligned}$$

therefore $\rho'(w)a \in j(\mathscr{H}_\rho)$. Note that if $v \in \mathscr{H}_\rho$ and $w \in \mathscr{S}(G)$, then

$$\langle \rho'(w)\, j(v), \zeta\rangle = (v, \rho(w^*)\zeta) = (\rho(w)v, \zeta),$$

so that $\rho'(w)\, j(v) = j(\rho(w)v)$. In [12] it is remarked that any w can be written as a finite sum $\sum_{j=1}^r w_{0,j} * w_{1,j}$ with $w_{ij} \in \mathscr{S}(G)$. From this a proof of (a) is easy; however, there is a direct proof of (a) that helps with the proof of (b).

If $a \in \mathscr{H}_\rho^{-\infty}$ and we write $\rho'(w)a = j\psi$ for some $\psi \in \mathscr{H}_\rho$, we have $j(\rho(g)\psi) = \rho'(g)\, j\psi = \rho'(g)\,\rho'(w)a$ as above. If $X \in \mathfrak{g}$, we claim that

$$\rho(X)\psi \text{ exists and is equal to } \psi_1 = j^{-1}(\rho'(L(X)w)a).$$

In fact, for any $\xi \in \mathscr{H}_\rho^\infty$ we have

$$\begin{aligned}\left(\frac{\rho(\exp tX) - I}{t}\psi, \xi\right) &= \left\langle \rho'(w)a, \frac{\rho(\exp - tX) - I}{t}\xi\right\rangle. \\ &= \left\langle a, \rho(w^*)\frac{\rho(\exp(-tX)) - I}{t}\xi\right\rangle \\ &= \left\langle a, \rho\left(\left(\frac{L_{\exp tX} - I}{t}w\right)^*\right)\xi\right\rangle\end{aligned}$$

Writing $L_{X,t} = (L_{\exp tX} - I)/t$, we get

$$\begin{aligned}\left|\left(\frac{\rho(\exp tX) - I}{t}\psi - \psi_1, \xi\right)\right|^2 &= |\langle a, \rho((L_{X,t}w - L(X)w)^*)\xi\rangle|^2 \\ &\leqslant C\sum_{|\alpha| \leqslant N} \|\rho(X^\alpha)\,\rho((L_{X,t}w - L(X)w)^*)\xi\|^2 \\ &\leqslant C\sum_{|\alpha| \leqslant N} \|L(X^\alpha)(L_{X,t}w - L(X)w)^*\|_1^2\,\|\xi\|^2.\end{aligned}$$

But $L_{X,t}w - L(X)w \to 0$ in $\mathscr{S}(G)$ as $t \to 0$ (see [5, Appendix 2]), so this bound goes to zero uniformly for $\|\xi\|^2 \leqslant 1$, $\xi \in \mathscr{H}_\rho^\infty$. Induction now shows that for $A \in \mathfrak{u}(\mathfrak{g})$, $\rho(A)\psi$ exists in $\mathscr{H}_\rho$ and $j(\rho(A)\psi) = \rho'(L(A)w)a$. That proves (a), and (b(ii)) follows easily since $\psi = Q_\rho(w^\sim)$ and $L(A)(w^\sim) = (R(A)w)^\sim$. Similarly, we get (b(i)):

$$\begin{aligned}(Q_\rho(R_g w), \xi) &= \langle \rho'((R_g w)^\sim)a, \xi\rangle = \langle \rho'(L_g(w^\sim))a, \xi\rangle \\ &= \langle \rho'(g)\rho'(w^\sim)a, \xi\rangle = \langle \rho'(g)\, jQ_\rho(w), \xi\rangle \\ &= \langle j(\rho(g)\, Q_\rho w), \xi\rangle \\ &= (\rho(g)\, Q_\rho w, \xi) \qquad \text{for all} \quad \xi \in \mathscr{H}_\rho^\infty.\end{aligned}$$

The last of (b) is the Closed Graph Theorem. If $w_n \to 0$ in $\mathscr{S}(G)$ and $Q_\rho w_n \to \psi$ in $\mathscr{H}_\rho^\infty$, then

$$\langle a, \rho(\bar{w}_n)\zeta\rangle = \langle \rho'(w_n^\sim)a, \zeta\rangle = (Q_\rho w_n, \zeta) \to (\psi, \zeta), \qquad \forall \zeta \in \mathscr{H}_\rho^\infty.$$

But $w_n \to 0$, so that $\rho(\bar{w}_n)\zeta \to 0$ in $\mathscr{H}_\rho^\infty$, by (7). Hence $(\psi, \zeta) = 0$ and $\psi = 0$.

If a cyclic, then $Q_\rho(\mathscr{S}(G))$ not F-dense $\Rightarrow Q_\rho(\mathscr{S}(G))$ is not norm dense in $\mathscr{H}_\rho$ (it is a ρ-invariant subspace and Theorem 1.3 of [25] applies); thus there would be an $\eta \neq 0$ in $\mathscr{H}_\rho$ such that $(Q_\rho(\mathscr{S}), \eta) = 0$. But then

$(Q_\rho(\mathscr{S}), \rho(g)\eta)=0$ for all $g \in G$, and $\exists w_0 \in \mathscr{S}(G)$ such that $(Q_\rho(\mathscr{S}), \rho(w_0)\eta)=0$ while $\rho(w_0)\eta \neq 0$ in $\mathscr{H}_\rho^\infty$. Take $g \in G$ and $\{w_n\} \subseteq \mathscr{C}_c^\infty(G)$ such that $w_n \geqslant 0$ and $w_n \to \delta_g$ (i.e., $\{w_n\}$ is an approximate identity). Then $\rho(w_n)\rho(w_0)\eta \to \rho(g)\,\rho(w_0)\eta$ in the $\mathscr{C}^\infty$-topology of $\mathscr{H}_\rho^\infty$. (Standard approximate identity arguments apply once we know that the action $G \times \mathscr{H}_\rho^\infty \to \mathscr{H}_\rho^\infty$ is separately continuous; for this see [5, Appendix 2].) Thus

$$0=\langle \rho'(w_n^{\sim})a, \rho(w_0)\eta\rangle = \langle a, \rho(\overline{w_n})\,\rho(w_0)\eta\rangle \to \langle a, \rho(g)\,\rho(w_0)\eta\rangle,$$

and since a is cyclic we get $\rho(w_0)\eta=0$, a contradiction.

For (c) it is clear that $T \in \mathscr{S}'(G)$, and the rest is a calculation based on the definition $(\rho'(w)a, b)=\langle b, j^{-1}\rho'(w)a\rangle^{-}=\langle b, Q_\rho(w^{\sim})\rangle^{-}$, and the fact that $\rho'(w_1 * w_2)=\rho'(w_1)\,\rho'(w_2)$. ∎

Penney [24] and Fujiwara [13] examine cyclic vectors in $\mathscr{H}_\rho^{-\infty}$ and direct integral decompositions of $\mathscr{H}_\rho^\infty$. Two situations concern us here:

2.2. Example. Let $\rho=\tau=\operatorname{Ind}(K\uparrow G, \chi_f)$, where χ_f is a character on the Lie subgroup K of nilpotent Lie group G and $f \in \mathfrak{k}^*$ is a Lie homomorphism. Define $a_\tau \in \mathscr{H}_\tau^{-\infty}$ by $\langle a_\tau, \phi\rangle=\overline{\phi(e)}$, $\phi \in \mathscr{H}_\tau^\infty$ (in the standard realization of τ acting on left K-covariant functions on G; from [25], $\mathscr{H}_\tau^\infty \subseteq \mathscr{C}^\infty(G, \tau)$). A Sobolev-type argument [25; 5, Appendix 1], shows that a_τ is continuous, and it is easily seen to be cyclic. Furthermore, $\tau'(k)\,a_\tau=\chi(k)\,a_\tau$ for all $k \in K$. Let Haar measures m_G, m_K be specified. This determines the norm in $\mathscr{H}_\rho$ and the integrated form of ρ. Once this is done, the map $Q_\tau\colon \mathscr{S}(G) \to \mathscr{H}_\tau^\infty$ given by $\langle j(Q_\tau w), \phi\rangle=\langle \tau'(w^{\sim})\,a_\tau, \phi\rangle$ can be computed explicitly:

$$Q_\tau w(x)=\int_K w(kx)\,\overline{\chi_f(k)}\,dk, \tag{9}$$

where the integral converges absolutely. In fact, for $\phi \in \mathscr{H}_\tau^\infty$,

$$\begin{aligned}(Q_\tau w, \phi) &= \langle j(Q_\tau w), \phi\rangle = \langle \tau'(w^{\sim})\,a_\tau, \phi\rangle \\ &= \int_G \langle a_\tau, \tau(g)\phi\rangle\, w(g)\,dg \qquad \text{(from (8))} \\ &= \int_G \overline{\phi(g)}\, w(g)\,dg \\ &= \int_{K\backslash G}\int_K \overline{\phi(kx)}\, w(kx)\,dk\;\,d\dot{x} \\ &= \int_{K\backslash G} \overline{\phi(x)} \left(\int_K w(kx)\,\overline{\chi_f(k)}\,dk\right) d\dot{x}.\end{aligned}$$

The integral in (9) is absolutely convergent and gives $Q_\tau w \in \mathscr{C}^\infty(G, \tau)$. Therefore it is defined by its values on elements x in a cross-section for $K\backslash G$. Now (9) follows easily. Lemma 2.1 shows that Q_τ intertwines $R(A)$ on $\mathscr{S}(G)$ with $\tau(A)$ on $\mathscr{H}_\tau^\infty$.

2.3. EXAMPLE. In the above setting, extend f to an element of $\mathfrak{g}^*$ (which we also call f). Let $l \in \Omega_\tau = f + \mathfrak{k}^\perp$, let $\mathfrak{b}$ be any polarization of l, let $B = \exp \mathfrak{b}$, and set $\chi_l(\exp Y) = e^{2\pi i l(Y)}$ for $Y \in \mathfrak{b}$; let $\pi = \pi(l, \mathfrak{b}) = \operatorname{Ind}(B \uparrow G, \chi_l)$. Fujiwara [13] considered a certain distribution $a_\pi \in \mathscr{H}_{\pi(l,\mathfrak{b})}^{-\infty}$ constructed from these data: for the standard model of π (as left B-covariant functions on G) define

$$\langle a_\pi, \phi \rangle = \int_{(B \cap K)\backslash K} \overline{\phi(k)}\, \chi_f(k)\, d\dot{k}, \qquad \phi \in \mathscr{H}_{\pi(l,\mathfrak{b})}^\infty. \tag{10}$$

Since $\chi_l(u) = \chi_f(u)$ for $l \in f + \mathfrak{k}^\perp$ and $u \in B \cap K$, this integral makes sense formally. Certain normalizations of measures must be made here: Haar measures m_G, m_K, m_B, $m_{K \cap B}$ must be specified to determine the norm in $\mathscr{H}_\tau$ (via $m_{K\backslash G}$), the norm in $\mathscr{H}_\pi$ (via $m_{B\backslash G}$), the integrated forms of τ and π (via m_G), and finally the measure $m_{K \cap B\backslash K}$ to be used in (10).

The integral (10) is absolutely convergent, a_π is Fréchet continuous, and $a_\pi \neq 0$. Here is one proof: let $\{X_1, ..., X_n\}$ be a weak Malcev basis through $\mathfrak{b}$ (so that $\mathfrak{g}_i = \mathbb{R}\text{-span}\{X_1, ..., X_i\}$ is a subalgebra), with $\mathfrak{g}_q = \mathfrak{k} \cap \mathfrak{b}$, $\mathfrak{g}_r = \mathfrak{b}$, and X_i chosen to be in $\mathfrak{k}$ whenever possible. Let $J = \{j_1 < \cdots < j_k\}$ be the set of indices j with $X_j \in \mathfrak{k}$ and $j > q$, so that $j_1 > r$. Then $\mathfrak{k} \cap \mathfrak{g}_{j_i} = \mathbb{R}\text{-span}\{X_{j_1}, ..., X_{j_i}\} \oplus (\mathfrak{k} \cap \mathfrak{b})$, so that $\{X_1, ..., X_q, X_{j_1}, ..., X_{j_k}\}$ is a weak Malcev basis for $\mathfrak{k}$, and $\lambda(t_{j_1}, ..., t_{j_k}) = \exp t_{j_1} X_{j_1} \cdots \exp t_{j_k} X_{j_k}$ coordinatizes $K \cap B\backslash K$. Use this map to identify $\mathbb{R}^k$ with $K \cap B\backslash K$. Then the invariant measure for $K \cap B\backslash K$ is identified with Lebesgue measure on $\mathbb{R}^k$. Let $*$ denote the multiplication on G transferred to $\mathfrak{g}$ via exp, so that $X * Y = \log(\exp X \exp Y)$. Because f is a Lie homomorphism,

$$f(t_{j_1} X_{j_1} * \cdots * t_{j_k} X_{j_k}) = f\left(\sum_{i=1}^{k} t_{j_i} X_{j_i}\right) \quad \text{and} \quad \chi_f(\lambda(t')) = \operatorname{Exp} 2\pi i f\left(\sum t_{j_i} X_{j_i}\right).$$

Now let $\phi \in \mathscr{H}_\pi^\infty$. We may then rewrite (10) as

$$\langle a_\pi, \phi \rangle = \int_{\mathbb{R}^k} \overline{\phi(\lambda(t'))}\, \chi_f(\lambda(t'))\, dt',$$

and from here the convergence and continuity are clear.

As noted in [24], a_π is generalized cyclic (this is easy because π is irreducible). It is easily verified that $\pi'(k)\, a_\pi = \chi(k)\, a_\pi$ for $k \in K$. We define $Q_\pi = Q_{\pi(l,\mathfrak{b})}\colon \mathscr{S}(G) \to \mathscr{H}_\pi^\infty$ by

$$\langle j(Q_\pi w), \phi \rangle = \langle \pi'(w^\sim)\, a_\pi, \phi \rangle.$$

This operator intertwines $R(A)$ (on $\mathscr{S}(G)$) with $\pi(A)$ (on $\mathscr{H}_\pi^\infty$), $A \in \mathfrak{u}(\mathfrak{g})$. A more explicit formula for Q_π is given by absolutely convergent integrals

$$\begin{aligned} Q_\pi w(g) &= \int_K \int_{K \cap B\backslash B} \overline{\chi_f(k)}\, \overline{\chi_l(b)}\, w(kbg)\, d\dot{b}\, dk \\ &= \int_{K \cap B\backslash B} Q_\tau w(bg)\, \overline{\chi_l(b)}\, d\dot{b}, \end{aligned} \tag{11}$$

or

$$Q_\pi w(g) = \int_B \int_{K/K \cap B} \overline{\chi_f(k)}\, \overline{\chi_l(b)}\, w(kbg)\, d\dot{k}\, db$$

for $w \in \mathscr{S}(G)$. This is straightforward formally, but there does not seem to be a detailed justification in the literature that considers the convergence problems. Here is one. For $w \in \mathscr{S}(G)$ and $\phi \in \mathscr{H}_\pi^\infty$,

$$\begin{aligned} (Q_\pi w, \phi) &= \langle j(Q_\pi w), \phi \rangle = \langle \pi'(w^\sim)\, a_\pi, \phi \rangle \\ &= \int_G w(g) \langle a_\pi, \pi(g)\phi \rangle\, dg \qquad \text{(from (8))} \\ &= \int_G w(g) \left[\int_{K \cap B\backslash K} \overline{\phi(kg)}\, \chi_f(k)\, d\dot{k} \right] dg. \end{aligned} \tag{12}$$

We need the following estimate, which may be of more general use.

2.4. Lemma. *Let G be a nilpotent Lie group and B, K closed connected subgroups. If ϕ is continuous and constant on $B\backslash G$ cosets and has compact support* mod B, *then*

$$\int_{K \cap B\backslash K} |\phi(kg)|\, d\dot{k}$$

is polynomially bounded in g.

Proof. Form a weak Malcev basis $Y_1, \ldots, Y_q$ (through $\mathfrak{b} \cap \mathfrak{k}$), $Y_{q+1}, \ldots, Y_r$ (through $\mathfrak{b}$), $W_1, \ldots, W_{j_1}, \ldots, W_{j_k}, \ldots, W_m$ (through $\mathfrak{g}$; $m = n - r$). As above, we take $W_{j_1}, \ldots, W_{j_k} \in \mathfrak{k}$ whenever possible, so that $Y_1, \ldots, Y_q, W_{j_1}, \ldots, W_{j_k}$ is a weak Malcev basis for $\mathfrak{k}$. As before , a cross-section for $K \cap B \backslash K$ is given by the map $\lambda: \mathbb{R}^k \to K$, where $\lambda(t') = \exp(t'_1 W_{j_1}) \cdots \exp(t'_k W_{j_k})$. Write

$$\gamma(s, t) = \exp(s_1 Y_1) \cdots \exp(s_r Y_r) \exp(t_1 W_1) \cdots \exp(t_m W_m);$$

thus $\lambda(t') = \gamma(0, i(t'))$, where $i(t'_1, ..., t'_k) \in \mathbb{R}^m$ has t'_i at the j_ith entry $(1 \leqslant i \leqslant k)$ and zeros elsewhere.

Let us write $S_M = \{s \in \mathbb{R}^r: |s_i| \leqslant M \text{ for all } i\}$, $T_M = \{t \in \mathbb{R}^m: |t_j| \leqslant M \text{ for all } j\}$, $R_M = S_M \times T_M$, for $M > 0$. For our estimate we may assume that $\|\phi\|_\infty \leqslant 1$ and that supp ϕ has γ-coordinates within $\mathbb{R}^r \times T_M$ for some M. Fix $g \in G$; if $k \in K$ and $kg \in \operatorname{supp} \phi$, then $\exists b \in B$ such that $\gamma^{-1}(bkg) \in (0) \times T_M \subseteq R_M$. The multiplication law in G is polynomial in our coordinates, so there is a polynomial $P_M(g)$ such that $\gamma(R_M) g^{-1} \subseteq \gamma(R_{P_M(g)})$ for all g; we may write $bk = bkg \cdot g^{-1}$, hence $\gamma^{-1}(bk) \in R_{P_M(g)}$. Every $k \in K$ has a unique decomposition $k = u \cdot \lambda(t')$ with $u \in K \cap B$ and $t' \in \mathbb{R}^k$, and $k = u\lambda(t') \Leftrightarrow Bk = B \cdot \lambda(t')$. Thus $\gamma^{-1}(bk) \in R_{P_M(g)} \Rightarrow k \in B(bk)$ has coordinates in $\mathbb{R}^r \times T_{P_M(g)} \Rightarrow t' \in T_{P_M(g)}$ when we decompose $k = u \cdot \lambda(t')$. By constancy of ϕ on $B \backslash G$ cosets,

$$\begin{aligned}\int_{K \cap B \backslash K} |\phi(kg)| \, d\dot{k} &= \int_{\mathbb{R}^k} |\phi(\lambda(t) g)| \, dt \\ &\leqslant \operatorname{meas}\{t \in \mathbb{R}^k: \lambda(t) g \in \operatorname{supp} \phi\} \\ &= \operatorname{meas}\{t: |t_i| \leqslant P_M(g), 1 \leqslant i \leqslant k\}\end{aligned}$$

and this grows polynomially in g. ∎

Since $w \in \mathscr{S}(G)$, the integral in (12) converges absolutely, and we may use Fubini freely. Then

$$\begin{aligned}(Q_\pi w, \phi) &= \int_{K \cap B \backslash K} \left(\int_G w(g) \overline{\phi(kg)} \, dg \right) \chi_f(k) \, d\dot{k} \\ &= \int_{K \cap B \backslash K} \left(\int_G w(k^{-1}g) \overline{\phi(g)} \, dg \right) \chi_f(k) \, d\dot{k} \\ &= \int_{K/K \cap B} \left(\int_{K \cap B \backslash G} \int_{K \cap B} w(ku\dot{x}) \overline{\phi(u\dot{x})} \, du \, d\dot{x} \right) \overline{\chi_f(k)} \, d\dot{k}\end{aligned}$$

(by the measure preserving map $(K \cap B)k \to k^{-1}(K \cap B)$ of $K \cap B \backslash K$ to $K/K \cap B$)

$$= \int_{K \cap B \backslash G} \int_K w(k\dot{x}) \overline{\chi_f(k)} \, \overline{\phi(\dot{x})} \, dk \, d\dot{x}$$

(since $\chi_f = \chi_l$ on $K \cap B$)

$$= \int_{B\backslash G} \int_{K\cap B\backslash B} \overline{\phi(\dot{b}\dot{g})} \left(\int_K \overline{\chi_f(k)}\, w(k\dot{b}\dot{g})\, dk \right) d\dot{b}\, d\dot{g}$$

$$= \int_{B\backslash G} \overline{\phi(\dot{g})} \left(\int_{K\cap B\backslash B} \int_K \overline{\chi_f(k)}\, \overline{\chi_l(\dot{b})}\, w(k\dot{b}\dot{g})\, dk\, d\dot{b} \right) d\dot{g}.$$

Thus

$$(Q_\pi w, \phi) = \int_{B\backslash G} \overline{\phi(\dot{g})} \left(\int_{K\cap B\backslash B} \int_K \overline{\chi_f(k)}\, \overline{\chi_l(\dot{b})}\, w(k\dot{b}\dot{g})\, dk\, d\dot{b} \right) d\dot{g}$$

$$= \int_{B\backslash G} \overline{\phi(\dot{g})} \left(\int_{K\cap B\backslash B} Q_\tau w(\dot{b}\dot{g})\, \overline{\chi_l(\dot{b})}\, d\dot{b} \right) d\dot{g}. \tag{13}$$

Here, $\dot{g}$ runs through a cross-section of $B\backslash G$ (say $\gamma(\{0\} \times \mathbb{R}^m)$, with $d\dot{g} = dt$). Since (13) holds for a dense set of $\phi \in \mathscr{H}_\pi$, Fubini says that the inner double integral,

$$\int_{K\cap B\backslash B} \int_K \overline{\chi_f(k)}\, \overline{\chi_l(b)}\, w(kbg)\, dk\, d\dot{b},$$

is absolutely convergent, measurable (as a function of g), and equal a.e. to $Q_\pi w(g)$. This is half of what we need. For the other half, write the integral over K as $\int_{K/K\cap B} \int_{K\cap B}$ and rearrange, using Fubini. This completes the justification.

2.5. Lemma. *Let G, K, f be as above, let $l \in f + \mathfrak{k}^\perp$, and let $\mathfrak{b}$ be a polarization for l. Define a_π and Q_π as above. Then $Q_\pi : \mathscr{S}(G) \to \mathscr{H}^\infty_{\pi(l,\mathfrak{b})}$ is Fréchet continuous and surjective.*

Proof. Continuity was proved in Lemma 2.1(b). Take weak Malcev coordinates through $\mathfrak{b}$ as above. Then we may model $\mathscr{H}_\pi \cong \mathscr{L}^2(\mathbb{R}^m)$, with $\mathscr{H}^\infty_\pi \cong \mathscr{S}(\mathbb{R}^m)$ in the Fréchet topologies, if we scale basis vectors so that dt on $\mathbb{R}^m$ matches $d\dot{g} = m_{B\backslash G}$. As shown in Chapter 3 of [5], for every $w \in \mathscr{S}(G)$ there exists a kernel $K_w \in \mathscr{S}(\mathbb{R}^m \times \mathbb{R}^m)$ such that

$$(\pi(w)\phi)(t) = \int_{\mathbb{R}^m} K_w(t, t')\, \phi(t')\, dt', \qquad \forall \phi \in \mathscr{S}(\mathbb{R}^m).$$

Furthermore, $\{K_w : w \in \mathscr{S}(G)\} = \mathscr{S}(\mathbb{R}^m \times \mathbb{R}^m)$, by Theorem 3.4 of [16]. Choose w so that $K_{\bar{w}} = \phi_1 \otimes \overline{\phi_2}$, with $\phi_1, \phi_2 \in \mathscr{S}(\mathbb{R}^m)$. Then

$$(\pi(\bar{w})\phi)(t) = (\phi, \phi_2) \cdot \phi_1(t), \qquad \forall \phi \in \mathscr{S}(\mathbb{R}^m).$$

So if $\tilde{\phi} \in \mathscr{H}_\pi^\infty$ corresponds to $\phi \in \mathscr{S}(\mathbb{R}^m)$, then

$$\begin{aligned}(Q_\pi(w), \tilde{\phi})_{\mathscr{H}_\pi} &= \langle a_\pi, \pi(\bar{w})\phi \rangle = \langle a_\pi, (\phi, \phi_2)\, \phi_1 \rangle \\ &= (\phi_2, \phi)\langle a_\pi, \phi_1 \rangle = (\langle a_\pi, \phi_1 \rangle \phi_2, \phi).\end{aligned}$$

Choose $\phi_1 \in \mathscr{H}_\pi^\infty$ with $\langle a_\pi, \phi_1 \rangle = 1$. Then for any ϕ_2, choosing w as above gives $Q_\pi(w) = \phi_2$. That proves surjectivity. ∎

3. Orbits and Invariant Operators

In [8] we gave a survey of results about invariant operators on $K\backslash G$; here we summarize the most important facts we will need in this paper. Let $(\mathfrak{g}, \mathfrak{k}, \chi)$ be as in Section 1 and $\tau = \operatorname{Ind}(K \uparrow G, \chi)$. In the standard model, τ_g acts by right translation on certain functions f such that $f(kg) = \chi(k) f(g)$; the $\mathscr{C}^\infty$ vectors $\mathscr{H}_\tau^\infty$ lie in $\mathscr{C}^\infty(G, \tau)$ and $\tau(A) f = R(A) f$ for $f \in \mathscr{H}_\tau^\infty$, where $R(A)$ is the right action (left invariant on G) of $A \in \mathfrak{u}(\mathfrak{g})$. Given a polynomial cross-section $\sigma: \mathbb{R}^m \to G$ $(m = \dim K\backslash G)$ for $K\backslash G$ cosets associated with a weak Malcev basis $X_1, ..., X_p, ..., X_n$ through $\mathfrak{k}$,

$$\sigma(t) = \exp(t_1 X_{p+1}) \cdots \exp(t_m X_n) \qquad \text{if} \quad \mathfrak{k} = \mathbb{R}\text{-span}\{X_1, ..., X_p\},$$

we introduce polynomial coordinates into $K\backslash G$ and define Schwartz functions $\mathscr{S}(K\backslash G)$, polynomial coefficient differential operators $\mathscr{P}(K\backslash G)$. There is then a natural isomorphism $J: \mathscr{C}^\infty(K\backslash G) \to \mathscr{C}^\infty(G, \tau)$ such that $J\tilde{\phi}(\sigma(t)) = \tilde{\phi}(K \cdot \sigma(t))$. This allows us to identify τ_g with a cocycle action $\tilde{\tau}_g$ on $L^2(K\backslash G)$, and creates a homomorphism J^* from $\operatorname{Diff}(G, \tau) = (\mathscr{C}^\infty$ differential operators on G that leave $\mathscr{C}^\infty(G, \tau)$ invariant) onto $\operatorname{Diff}(K\backslash G)$ via $J^*L = J^{-1} \circ (L \mid \mathscr{C}^\infty(G, \tau)) \circ J$.

We define the τ-invariant operators on $K\backslash G$ to be those $D \in \operatorname{Diff}(K\backslash G)$ that commute with the operators $\tilde{\tau}_g$, $g \in G$. Under J^* these identify with $\mathbb{D}_\tau(K\backslash G) =$ restrictions $D \mid \mathscr{C}^\infty(G, \tau)$ of the $D \in \operatorname{Diff}(G, \tau)$ that commute with $R_g = \tau_g$ on the subspace $\mathscr{C}^\infty(G, \tau)$. In the enveloping algebra $\mathfrak{u}(\mathfrak{g})$ define

$$\begin{aligned}\mathfrak{u}(\mathfrak{g}, \tau) &= \{A \in \mathfrak{u}(\mathfrak{g}) \colon L(A) \text{ leaves } \mathscr{C}^\infty(G, \tau) \text{ invariant}\} \\ \mathfrak{a}_\tau &= \mathbb{C}\text{-span}\{Y + 2\pi i \langle f, Y \rangle I \colon Y \in \mathfrak{k}\} \\ \mathfrak{u}(\mathfrak{g})\, \mathfrak{a}_\tau &= \text{left ideal generated by } \mathfrak{a}_\tau.\end{aligned} \tag{14}$$

As explained in [8], $\mathfrak{u}(\mathfrak{g}, \tau)$ has a Lie algebra description,

$$\mathfrak{u}(\mathfrak{g}, \tau) = \{A \in \mathfrak{u}(\mathfrak{g}) \colon [A, Y] \in \mathfrak{u}(\mathfrak{g})\, \mathfrak{a}_\tau, \forall Y \in \mathfrak{k}\}.$$

There is a natural homomorphism $\gamma\colon \mathfrak{u}(\mathfrak{g},\tau)\to \mathbb{D}_\tau(K\backslash G)$ given by

$$\gamma(A)=J^{-1}\circ(L(A)\,|\,\mathscr{C}^\infty(G,\tau))\circ J,$$

and we showed that γ is surjective, with $\operatorname{Ker}\gamma=\mathfrak{u}(\mathfrak{g})\,\mathfrak{a}_\tau$; thus $\mathbb{D}_\tau(K\backslash G)\cong \mathfrak{u}(\mathfrak{g},\tau)/\mathfrak{u}(\mathfrak{g})\,\mathfrak{a}_\tau$ independent of the cross section used to define J. Operators in $\tilde{\tau}(\mathfrak{u}(\mathfrak{g}))$ or $\gamma(\mathfrak{u}(\mathfrak{g},\tau))$ all appear within $\mathscr{P}(K\backslash G)$ and so leave invariant $\mathscr{S}(K\backslash G)$, which is a Fréchet dense $\tilde{\tau}(G)$-invariant subspace of the $\mathscr{C}^\infty$ vectors $\mathscr{H}_{\tilde{\tau}}^\infty\subseteq\mathscr{H}_{\tilde{\tau}}\cap\mathscr{C}^\infty(K\backslash G)\subseteq\mathscr{H}_{\tilde{\tau}}=\mathscr{L}^2(K\backslash G)$. It is worth noting that operators in $\mathbb{D}_\tau(K\backslash G)$, unlike those in $\tilde{\tau}(\mathfrak{u}(\mathfrak{g}))$, need not leave $\mathscr{H}_{\tilde{\tau}}^\infty$ invariant; this is why the subspace $\mathscr{S}(K\backslash G)$, or $\mathscr{S}_\tau=J(\mathscr{S}(K\backslash G))\subseteq\mathscr{H}_\tau^\infty$, plays an important role for us even though it places us outside the comfortable realm of $\mathscr{C}^\infty$ theory.

We need to find convenient representatives modulo $\operatorname{Ker}\gamma=\mathfrak{u}(\mathfrak{g})\,\mathfrak{a}_\tau$. Here is a situation in which $\mathrm{D}_\tau(K\backslash G)$ is much more tractable. In Section 4 of [8], we showed the following:

3.1. PROPOSITION. *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra,* $\mathfrak{k}$ *a subalgebra, and* $f\in\mathfrak{g}^*$ *such that* $f|_{\mathfrak{k}}$ *is a Lie homomorphism. Define* $\mathfrak{u}(\mathfrak{g},\tau)$ *and* $\mathfrak{a}_\tau$ *as above; set* $\mathfrak{u}^K(\mathfrak{g})=\{A\in\mathfrak{g}\colon [A,\mathfrak{k}]=0\}$. *If* $(\mathfrak{g},\mathfrak{k})$ *is reductive in the sense that*

$$\textit{there is a subspace } \mathfrak{m}\subseteq\mathfrak{g} \textit{ such that } \mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m} \textit{ and } [\mathfrak{k},\mathfrak{m}]\subseteq\mathfrak{m}, \tag{15}$$

then we have $\mathfrak{u}(\mathfrak{g},\tau)=\mathfrak{u}^K(\mathfrak{g})+\mathfrak{u}(\mathfrak{g},\tau)\,\mathfrak{a}_\tau$.

3.2. COROLLARY. *Let* $(\mathfrak{g},\mathfrak{k},\chi)$ *be as above. If* $\dim\mathfrak{k}=1$, *then the pair* $(\mathfrak{g},\mathfrak{k})$ *is always reductive and* $\mathfrak{u}(\mathfrak{g},\tau)=\mathfrak{u}^K(\mathfrak{g})+\mathfrak{u}(\mathfrak{g},\tau)\,\mathfrak{a}_\tau$.

These results hold without any requirement that $\tau=\operatorname{ind}(K\uparrow G,\chi)$ have finite multiplicity, where $\chi(\exp Y)=e^{2\pi i\langle f,Y\rangle}$ for $Y\in\mathfrak{k}$. Benoist [2] has studied symmetric spaces $K\backslash G$ with $\chi\equiv 1$; for these $\mathfrak{k}=\{X\in\mathfrak{g}\colon\alpha(X)=X\}$ for some involutive automorphism α of $\mathfrak{g}$, and the pair $(\mathfrak{g},\mathfrak{k})$ is automatically reductive, with $\mathfrak{m}=\{X\colon\alpha(X)=-X\}$. In this special situation we have $m(\pi)\equiv 1$ for $\pi\in\operatorname{spec}(\tau)$; moreover, $\mathbb{D}_\tau(K\backslash G)\cong\mathbb{C}[\Omega_\tau]^K=$ the Ad^*K invariant polynomials on $\Omega_\tau=\mathfrak{k}^\perp$, and every $D\in\mathbb{D}_\tau(K\backslash G)$ has a tempered fundamental solution. The analysis in [2, 3] uses the special properties of symmetric spaces.

Our objective here is to deal with more general data $(\mathfrak{g},\mathfrak{k},\chi)$ such that τ has finite multiplicity. The case when $\dim\mathfrak{k}=1$ is already interesting because it includes many examples that are not symmetric spaces, and because these examples exhibit the full range of multiplicity behavior: $m(\pi)\equiv 1$, $m(\pi)$ mixed finite, and $m(\pi)\equiv+\infty$.

Now we review some basic facts and notation for K-orbits in Ω_τ. Let $(\mathfrak{g},\mathfrak{k},\chi)$ be as in Section 1 and $\Omega_\tau=f+\mathfrak{k}^\perp$. Let $X_1,\ldots,X_n$ be a weak Malcev basis for $\mathfrak{g}$ through $\mathfrak{k}$, so that $\mathfrak{g}_i = \mathbb{R}\text{-span}\{X_1, ..., X_i\}$ is a subalgebra for each i and $\mathfrak{g}_p = \mathfrak{k}$; assume also that $\{X_1, ..., X_p\}$ is strong Malcev basis for $\mathfrak{k}$, so that $\mathfrak{g}_i \lhd \mathfrak{k}$ for $i < p$. Then the dual basis $X_1^*, ..., X_n^*$ is Jordan–Hölder for the action of $Ad^*(K)$. Following [5, Chap. 3] or [26], we may define dimension indices for K-orbits, $d_i(l) = \dim(K \cdot p_i(l))$, where $p_i: \mathfrak{g}^* \to \mathfrak{g}_i^*$ is the natural projection. Let $\mathscr{D} = \{d = (d_1, ..., d_n): \exists l \text{ with } d_i = d_i(l), \forall i\}$. We then define layers $U_d(K) = \{l \in \mathfrak{g}^*: d_i(l) = d_i,\ 1 \leqslant i \leqslant n\}$ for $d \in \mathscr{D}$. We henceforth write d for the (unique) index such that $U_d(K) \cap \Omega_\tau$ is non-empty and Zariski-open in Ω_τ; this is the generic set of K-orbits in Ω_τ. Let $E_d = f + \mathbb{R}\text{-span}\{X_i^*: d_i = d_{i-1},\ p+1 \leqslant i \leqslant n\}$. Then $U_d(K) \cap E_d$ is Zariski-open in E_d and is a cross-section for generic orbits (see [7]); furthermore, there is a birational nonsingular parametrization map

$$P_d: (U_d(K) \cap E_d) \times V_S \to U_d(K) \cap \Omega_\tau \tag{16}$$

with inverse

$$P_d^{-1}(l) = (K \cdot l \cap E_d, \pi_S(l)). \tag{17}$$

Here, $V_T = \mathbb{R}\text{-span}\{X_i^*: d_i = d_{i-1}\}$, $V_S = \mathbb{R}\text{-span}\{X_i^*: d_i = d_{i-1} + 1\}$ $(d_0 = 0)$, $\mathfrak{g}^* = V_T \oplus V_S$, and π_T, π_S are the projections corresponding to this direct sum. For $l \in E_d \cap U_d(K)$, the map $s \to P_d(l, s)$ is polynomial and parametrizes $K \cdot l$. Thus generic K-orbits $\theta \in \Omega_\tau / K$ project diffeomorphically to V_S. This gives us coordinates in each generic θ and lets us define Schwartz functions $\mathscr{S}(\theta)$ and polynomial coefficient differential operators $\mathscr{P}(\theta)$. Furthermore, Lebesgue measure ds on V_S corresponds to a K-invariant measure μ_θ on generic θ (see [5, 7, 22]). Then there is a unique rational function $r(l)$ on E_d such that Lebesgue measure μ_Ω on Ω_τ decomposes as $\mu_\Omega = \int_{E_d} \mu_{K \cdot l} r(l)\, dl$. Other choices of polynomial coordinates on the orbits θ give equivalent definitions of $\mathscr{S}(\theta)$ and $\mathscr{P}(\theta)$, and change μ_Ω only by a scalar; see [7], and especially [22], for this.

We have obvious definitions of $\mathscr{S}(\Omega_\tau)$, $\mathscr{P}(\Omega_\tau)$, and it is easy to verify that

> For generic $\theta \in \Omega_\tau / K$ the restriction maps $\mathscr{S}(\mathfrak{g}^*) \xrightarrow{r_\Omega} \mathscr{S}(\Omega_\tau) \xrightarrow{r_\theta} \mathscr{S}(\theta)$ are surjective and continuous (in the usual Fréchet topologies). (18)

The decomposition of μ_Ω also gives a direct integral decomposition

$$\mathscr{L}^2(\Omega_\tau, \mu_\Omega) \cong \int_{E_d}^{\oplus} \mathscr{L}^2(K \cdot l, \mu_{K \cdot l})\, d\nu(l), \qquad d\nu(l) = r(l)\, dl,$$

under which a function $f \in \mathscr{L}^2(\Omega_\tau, \mu_\Omega)$ disintegrates into a field of vectors $r_{K \cdot l} f = f|_{K \cdot l}$. We have

$$\|f\|^2_{L^2(\Omega_\tau)} = \int_{E_d} \|r_{K \cdot l} f\|^2_{L^2(K \cdot l, \mu_{K \cdot l})} \, d\nu(l). \tag{19}$$

For $f \in \mathscr{S}(\Omega_\tau)$, we have $r_{K \cdot l} f \in \mathscr{S}(K \cdot l)$ for $l \in E_d \cap U_d(K)$.

If $X \subseteq \mathfrak{g}^*$ is a closed submanifold and $D \in \mathrm{Diff}(\mathfrak{g}^*)$ (= algebra of $\mathscr{C}^\infty$ differential operators on $\mathfrak{g}^*$), we say that D restricts to X, giving a $\mathscr{C}^\infty$ operator $D|_X \in \mathrm{Diff}(X)$, if

$$\phi \in \mathscr{C}_c^\infty(\mathfrak{g}^*), \; \phi|_X \equiv 0 \Rightarrow D\phi|_X \equiv 0.$$

We then define $(D|_X)(\phi|_X) = D\phi|_X$. If $X = K \cdot l$ ($l \in U_d(K) \cap E_d$) and if a polynomial coefficient operator $D \in \mathscr{P}(\Omega_\tau)$ restricts to X, then $D|_X \in \mathscr{P}(X)$. [Proof: Polynomials $\tilde{f} \in \mathbb{C}[V_S]$ identify with functions $\pi_S^* \tilde{f} = \tilde{f} \circ \pi_S$ on $X = K \cdot l$. Then $F = \tilde{f} \circ \pi_S$ on Ω_τ is in $\mathbb{C}[\Omega_\tau]$, $F|_X = \pi_s^* \tilde{f}$, and $(D|_X)(\pi_s^* \tilde{f}) = DF|_X$. Hence

$$P_d(l, \cdot)^* \circ D|_X \circ \pi_s^*(\tilde{f}) = (DF|_X) \circ P_d(l, s) = DF(P_d(l, s)) \in \mathbb{C}[V_S],$$

since $D \in \mathscr{P}(\Omega_\tau)$. Therefore $\tilde{D} = P_D(l, \cdot)^* \circ (D|_X) \circ \pi_s^* \in \mathrm{Diff}(V_S)$ maps polynomials to polynomials, so that $D|_X \in \mathscr{P}(K \cdot l)$.] It is also clear that if $D \in \mathscr{P}(\mathfrak{g}^*)$ restricts to Ω_τ, then $D|_{\Omega_\tau} \in \mathscr{P}(\Omega_\tau)$.

4. The Malcev–Fourier Transform

We now turn to the analysis of $\tau = \mathrm{Ind}(K \uparrow G, \chi)$, where K, G, $\chi = \chi_f$, f are as in Section 1. We let $\Omega_\tau = f + \mathfrak{k}^\perp$ and fix invariant measures on m_G, m_K (and hence $m_{K/G}$) once and for all. We impose the following conditions on the data $(\mathfrak{g}, \mathfrak{k}, \chi)$:

(i) Finite multiplicity: $m(\pi) < \infty$ for generic $\pi \in \mathrm{Spec}(\tau) \approx \Omega_\tau / K$.

(ii) There is a subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$ that polarizes generic points $l \in \Omega_\tau$ and is normalized by $\mathfrak{k}$. (20)

As we have noted, (i) holds $\Leftrightarrow$ $\dim G \cdot l = 2 \dim K \cdot l$ for generic $l \in \Omega_\tau$ $\Leftrightarrow$ for generic $l \in \Omega_\tau$ the connected components of $G \cdot l \cap \Omega_\tau$ are single K-orbits.

Now let $\{X_1, ..., X_q, ..., X_p, ..., X_m, ..., X_n\}$ be a weak Malcev basis for $\mathfrak{g}$ such that $\mathfrak{g}_q = \mathfrak{k} \cap \mathfrak{b}$, $\mathfrak{g}_p = \mathfrak{k}$, $\{X_1, ..., X_p\}$ is a strong Malcev basis for $\mathfrak{k}$, and $\mathfrak{g}_m = \mathfrak{k} + \mathfrak{b}$. (By (ii), $\mathfrak{k} \cap \mathfrak{b} \lhd \mathfrak{k}$ and $\mathfrak{k} + \mathfrak{b}$ is a subalgebra.) We also may assume that $X_i \in \mathfrak{b}$ for $p + 1 \leqslant i \leqslant m$. We call such a basis *admissible* for $(\mathfrak{g}, \mathfrak{k}, \mathfrak{b})$. We then define polynomial coordinates $\eta: \mathbb{R}^n \to G$ via $\eta(x) = \exp(x_1 X_1) \cdots \exp(x_n X_n)$ and use them to define a "Malcev–Fourier transform" $\mathscr{F}_\eta: \mathscr{S}(G) \to \mathscr{S}(\mathfrak{g}^*)$,

$$\mathscr{F}_\eta w(l) = \int_{\mathbb{R}^n} e^{2\pi i \langle l, \sum x_i X_i \rangle} w(\eta(x))\, dx, \qquad w \in \mathscr{S}(G), l \in \mathfrak{g}^*. \tag{21}$$

This map is bicontinuous for the standard topologies on Schwartz spaces, and polynomial coefficient differential operators $D \in \mathscr{P}(G)$ transform to polynomial coefficient operators $D^\wedge = \mathscr{F}_\eta \circ D \circ \mathscr{F}_\eta^{-1}$. Notice that this transform is different from the "Euclidean" Fourier transform that Kirillov employed in his character formula [5], which uses exponential coordinates on G. We use $\mathscr{F}_\eta$ to set up an explicit "smooth" realization of the decomposition (1). One major point of interest for us is whether the operators $L(A)^\wedge$, $R(A)^\wedge$ $(A \in \mathfrak{u}(\mathfrak{g}))$ restrict to $X = \Omega_\tau$ and to $X = K \cdot l$ for generic $l \in \Omega_\tau$. It is not hard to show that $R(A)^\wedge$ does restrict to these X; the situation for $L(A)^\wedge$ is more complicated.

We have set up out basis so that $G_m = B \cdot K$. We first see how this affects the structure of generic K-orbits in Ω_τ.

4.1. Lemma. *Let $(\mathfrak{g}, \mathfrak{k}, \mathfrak{b}, \chi)$ satisfy the conditions (20), let $X_1, ..., X_n$ be an admissible basis as above, with $G_m = B \cdot K$, and let $\tau = \mathrm{Ind}(K \uparrow G, \chi)$. Then for generic $l \in \Omega_\tau = f + \mathfrak{k}^\perp$, we have*

(a) $K \cdot l = K \cdot p_m l \oplus \mathbb{R}\text{-Span}\{X_{m+1}^*, ..., X_n^*\} = K \cdot p_m l \oplus \mathfrak{g}_m^\perp$ *(we regard $K \cdot p_m l$ as a subset in the space $\mathfrak{g}_m^* \subseteq \mathfrak{g}_n^*$, spanned by $X_1^*, ..., X_m^*$).*

(b) $K \cap B = \mathrm{Stab}_K(p_m l)$.

(c) $(K \cap B) \cdot l = l + \mathfrak{g}_m^\perp$.

Proof. By hypothesis $m(\pi) < \infty$, so that $\dim G_k \cdot p_k(l) = 2 \dim K \cdot p_k l$ for generic $l \in \Omega_\tau$ and all $k \geqslant p$ (recall that $G_p = K$). Let $l \in \Omega_\tau$ be generic, and let $l_k = p_k l$, $l_m = p_m l$. Since $\mathfrak{b} \subseteq \mathfrak{g}_m$ polarizes l, π_l is irreducible, and

$$\pi_l = \mathrm{Ind}(B \uparrow G, \chi_{l|\mathfrak{b}}) = \mathrm{Ind}(G_m \uparrow G, \pi_{l_m}), \qquad \pi_{l_m} = \mathrm{Ind}(B \uparrow G_m, \chi_{l_m|\mathfrak{b}}).$$

Therefore π_{l_m} is irreducible, which implies that $\mathfrak{b}$ polarizes l_m (see [5, Theorem 2.5.2 and the following remark]). Furthermore, π_{l_m} induces irreducibly at every step $G_m \subseteq G_{m+1} \subseteq \cdots \subseteq G_n = G$, so that $\dim(G_{k+1} \cdot l_{k+1}) = \dim(G_k \cdot l_k) + 2$ and $G_k \cdot l_k$ lies under a unique orbit $G_{k+1} \cdot l_{k+1}$. The condition $m(\pi) < \infty$ now implies that $\dim K \cdot l_k$ increases by one at each step from G_m to G. Since the projection from $\mathfrak{g}_{k+1}^*$ to $\mathfrak{g}_k^*$

is K-equivariant, this means that $K \cdot l_{k+1}$ is the pre-image of $K \cdot l_k$ under this projection. Induction gives

$$K \cdot l = p_m^{-1}(K \cdot l_m) \qquad \text{for generic} \quad l \in \Omega_\tau,$$

and this is (a).

Again for generic $l \in \Omega_\tau$, $\mathfrak{b}$ is a polarization, and therefore (Proposition 3.1.18 of [5]) $B \cdot l = l + \mathfrak{b}^\perp$. Hence

$$B \cdot l \cap \Omega_\tau = (l + \mathfrak{b}^\perp) \cap (l + \mathfrak{k}^\perp) = l + (\mathfrak{b} + \mathfrak{k})^\perp = l + \mathfrak{g}_m^\perp.$$

Since

$$(B \cap K) \cdot l \subseteq (B \cdot l) \cap (K \cdot l) \subseteq B \cdot l \cap \Omega_\tau = l + \mathfrak{g}_m^\perp,$$

we have

$$(B \cap K) \cdot l_m = \{l_m\},$$

which is $\subseteq$ of (b). Conversely, we know that $\dim(G_m \cdot p_m l) = 2 \dim(K \cdot p_m l)$ (because $m(\pi) < \infty$) and $\dim(G_m \cdot p_m l) = 2 \dim G_m/B$ (because $\mathfrak{b}$ polarizes $p_m l) = 2 \dim K/K \cap B$. That is, $\dim(K \cdot p_m l) = \dim(K/K \cap B)$; since the stability group of $p_m l$ is connected, it has dimension $= \dim(K \cap B)$. Now (b) follows.

For (c), we know that $(B \cap K) \cdot l \subseteq l + \mathfrak{g}_m^\perp$. It suffices to show that $\dim(B \cap K) \cdot l = \dim \mathfrak{g}_m^\perp$. We have

$$\begin{aligned}\dim \mathfrak{g}_m^\perp &= \dim \mathfrak{g} - \dim(\mathfrak{b} + \mathfrak{k}) = \dim \mathfrak{g} - \dim \mathfrak{b} - \dim \mathfrak{k} + \dim(\mathfrak{b} \cap \mathfrak{k}) \\ &= \dim G/B - \dim(K/B \cap K),\end{aligned}$$

and

$$\begin{aligned}\dim(B \cap K) \cdot l &= \dim(\mathfrak{b} \cap \mathfrak{k}) - \dim(\mathfrak{b} \cap \mathfrak{k} \cap \mathfrak{r}(l)) \\ &= \dim(\mathfrak{b} \cap \mathfrak{k}) - \dim(\mathfrak{r}(l) \cap \mathfrak{k}) \\ &= \dim(\mathfrak{b} \cap \mathfrak{k}) - \dim \mathfrak{k} + \dim(K/R(l) \cap K) \\ &= \dim(K \cdot l) - \dim(K/B \cap K).\end{aligned}$$

Since $\dim G/B = (1/2) \dim G \cdot l$ (because $\mathfrak{b}$ is a polarization) $= \dim K \cdot l$ (because $m(\pi) < \infty$, see above), (c) follows. ∎

For generic $l \in \Omega_\tau$ and $\pi = \pi_l$, we now produce a bijection $\Psi_\pi : \mathscr{H}_\pi^\infty \to \mathscr{S}(\theta)$, where $\theta = K \cdot l$ and $\mathscr{S}(\theta)$ is the space of Schwartz functions on θ, such that Ψ_π makes the following diagram commute:

$$\begin{array}{ccc} \mathscr{S}(G) & \xrightarrow{\mathscr{F}_\eta} & \mathscr{S}(\mathfrak{g}^*) \\ {\scriptstyle Q_\pi}\big\downarrow & & \big\downarrow{\scriptstyle r_\theta} \\ \mathscr{H}_\pi^\infty & \xrightarrow{\Psi_\pi} & \mathscr{S}(\theta) \end{array} \qquad (r_\theta = \text{restriction map}).$$

This map will intertwine $\pi(A)$ on $\mathscr{H}_\pi^\infty$ with $R(A)^\wedge \mid \theta$; part of our task is to show that $R(A)^\wedge = \mathscr{F}_\eta \circ R(A) \circ \mathscr{F}_\eta^{-1}$ does restrict to generic K-orbits $\theta \subseteq \Omega_\tau$. We start by assembling the ingredients; Fig. 1, given later, will show the relationships between them.

We write the Malcev coordinate map as $\eta\colon \mathbb{R}^p \times \mathbb{R}^{m-p} \times \mathbb{R}^{n-m} \to G$, and let a typical point in $\mathbb{R}^n$ be $x = (s, t) = (s', s'', t)$, $t \in \mathbb{R}^{n-m}$, $s \in \mathbb{R}^m$, $s' \in \mathbb{R}^p$, and $s'' \in \mathbb{R}^{m-p}$. Fixing a generic l in $E_d \approx \Omega_\tau/K$, we let $\pi = \pi(l, \mathfrak{b})$ be induced from χ_l on B. Fixing normalizations of m_B, $m_{K \cap B}$(hence of $m_{K/K \cap B}$, $m_{B/K \cap B}$) we then define $a_\pi \in (\mathscr{H}_\pi^{-\infty})^{(K,\chi)}$ and Fréchet continuous surjective map $Q_\pi\colon \mathscr{S}(G) \to \mathscr{H}_{\pi(l,\mathfrak{b})}^\infty$ as in Section 2 (see (11)). Functions $Q_\pi w$ are determined by their values on any set containing a cross-section of $B \backslash G$, and in particular on $\Sigma = \eta(\mathbb{R}^p \times \{0\} \times \mathbb{R}^{n-m}) = K\eta(\{0\} \times \mathbb{R}^{n-m})$. From (11), we have

$$Q_\pi w(g) = \int_{K/K \cap B} \int_B w(kbg)\, \overline{\chi_l(b)}\, \overline{\chi_f(k)}\, db\, d\dot{k}.$$

Write $g_t = \eta(0, 0, t)$, $g_y = \eta(y, 0, 0) \in K$ ($y \in \mathbb{R}^p$, $t \in \mathbb{R}^{n-m}$). For $g = g_y g_t = \eta(y, 0, t) \in \Sigma$, we have

$$\begin{aligned} Q_\pi w(g) &= Q_\pi w(\eta(y, 0, t)) \\ &= \int_B \int_{K/K \cap B} w(kg_y g_y^{-1} b g_y g_t)\, \overline{\chi_f(k)}\, \overline{\chi_{g_y^{-1} \cdot l}(g_y^{-1} b g_y)}\, db\, d\dot{k} \\ &= \int_B \int_{K/K \cap B} w(kg_y b g_t)\, \overline{\chi_f(k)}\, \overline{\chi_{g_y^{-1} \cdot l}(b)}\, db\, d\dot{k} \\ &= \int_{B \cap K \backslash B} \int_{K/K \cap B} \int_{K \cap B} w(kg_y u g_y^{-1} g_y b g_t) \\ &\quad \times \overline{\chi_f(kg_y u g_y^{-1})}\, \overline{\chi_{g_y^{-1} \cdot l}(b)}\, du\, d\dot{b}\, d\dot{k}. \end{aligned}$$

(Because $g_y^{-1} \cdot l \in f + \mathfrak{k}^\perp$, we may transfer $u \in K \cap B$ into χ_f.) Since

$K\cap B\lhd K$ and $Ad(g_y)$ acts unipotently on $\mathfrak{k}\cap\mathfrak{b}$, the map $u\to g_y u h_y^{-1}$ preserves $m_{K\cap B}$, so we may rewrite the integral as

$$\int_{B\cap K\backslash B}\int_K w(kg_y bg_t)\,\overline{\chi_f(k)}\,\overline{\chi_{g_y^{-1}\cdot l}(b)}\,dk\,d\dot{b}$$

$$=\chi_f(g_y)\int_{B\cap K\backslash B}\int_K w(kbg_t)\,\overline{\chi_f(k)}\,\overline{\chi_{g_y^{-1}\cdot l}(b)}\,dk\,d\dot{b}.$$

Switching the coordinates and noting that $K\cap B\backslash B\approx\eta(\{0\}\times\mathbb{R}^{m-p}\times\{0\})$, we get this last to be

$$\chi_f(g_y)\int_{\mathbb{R}^{m-p}}\int_{\mathbb{R}^p} w(\eta(s',s'',t))$$
$$\times e^{-2\pi i\langle f,\sum s_i'X_i\rangle}\,e^{-2\pi i\langle g_y^{-1}\cdot l,\sum s_i''X_i\rangle}ds'\,ds''.$$

(Because f is a Lie homomorphism on $\mathfrak{k}$, $f(s_1'X_1*\cdots*s_p'X_p)=\sum_{i=1}^p s_i'\langle f,X_i\rangle$, where $*$ denotes the Campbell–Baker–Hausdorff product $\exp(A*B)=\exp A\cdot\exp B$. Thus $\chi_f(\eta(s',0,0))=\chi_f(\exp\sum_{i=1}^p s_i'X_i)$. A similar remark applies to $g_y^{-1}\cdot l$ on $\mathfrak{b}$.) Since $g_y^{-1}\cdot l\in l+\mathfrak{k}^\perp=f+\mathfrak{k}^\perp=\Omega_\tau$ and $p_m\colon\mathfrak{g}^*\to\mathfrak{g}_m^*$ commutes with the action of K, we may write this as

$$Q_\pi w(\eta(y,0,t))=\chi_f(g_y)\int_{\mathbb{R}^m} w\circ\eta(s,t)\,e^{-2\pi i\langle g_y^{-1}\cdot p_m l,\sum_{i=1}^m s_iX_i\rangle}\,ds. \quad (22)$$

Now identify dual coordinates $(\dot{s}',\dot{s}'',\dot{t})\in\dot{\mathbb{R}}^p\times\dot{\mathbb{R}}^{m-p}\times\dot{\mathbb{R}}^{n-m}$ with elements of $\mathfrak{g}^*$ via $\lambda\colon l\mapsto(\dot{s}',\dot{s}'',\dot{t})=(\dot{s},\dot{t})$ if

$$l=\sum_{i=1}^p \dot{s}_i'X_i^*+\sum_{i=p+1}^m \dot{s}_i''X_i^*+\sum_{i=m+1}^n \dot{t}_iX_i^*.$$

We also identify $\mathfrak{g}^*=\mathfrak{g}_m^*\oplus\mathfrak{g}_m^\perp=\mathbb{R}\text{-span}\{X_1^*,...,X_m^*\}\oplus\mathbb{R}\text{-span}\{X_{m+1}^*,...,X_n^*\}$ and split $\lambda=\lambda_1\times\lambda_2$ correspondingly; we sometimes write $\lambda^{-1}(\dot{s},\dot{t})$ as $l(\dot{s},\dot{t})$ when convenient. The integral (22) is a Euclidean Fourier transform in the variables (s',s''):

$$Q_\pi w(\eta(y,0,t))=\chi_f(g_y)\,\mathscr{F}_{12}(w\circ\eta)(\lambda_1(g_y^{-1}\cdot p_m l),t).$$

Taking a Fourier transform in the t-variables, we get

$$(\mathscr{F}_3((Q_\pi w)\circ\eta))(y,\dot{t})=\chi_f(g_y)\,\mathscr{F}(w\circ\eta)(\lambda_1(g_y^{-1}\cdot p_m l),\dot{t})$$

(where on the left, $\mathscr{F}_3$ is the Fourier transform in the variables labeled by t).

Define $I\times\lambda_2\colon\mathbb{R}^p\times\mathfrak{g}_m^\perp\to\mathbb{R}^p\times\dot{\mathbb{R}}^{n-m}$ by $I\times\lambda_2(y,l')=(y,\lambda_2(l'))$ where λ_2

is defined as above, and use this to transfer functions; $(I\times\lambda_2)^*$: $\mathcal{S}_2(\mathbb{R}^p\times\dot{\mathbb{R}}^{n-m})\to\mathcal{S}_2(\mathbb{R}^p\times\mathfrak{g}_m^\perp)$ is given by $(I\times\lambda_2)^*\phi=\phi\circ(I\times\lambda_2)$, where $\mathcal{S}_2$ is the space of C^∞ functions that are Schwartz in the second variables. For $l_2\in\mathfrak{g}_m^\perp$ and $y\in\mathbb{R}^p$,

$$\begin{aligned}
&(I\times\lambda_2)^*\circ\mathcal{F}_3\circ\eta(\cdot,0,\cdot)^*\circ Q_\pi w(y,l_2)\\
&\quad=\mathcal{F}_3\{Q_\pi w(\eta(\cdot,0,\cdot))\}(y,\lambda_2(l_2))\\
&\quad=\chi_f(g_y)\,\mathcal{F}(w\circ\eta)(\lambda_1(g_y^{-1}\cdot p_m l),\lambda_2(l_2))\\
&\quad=\chi_f(g_y)\,\mathcal{F}(w\circ\eta)(\lambda(g_y^{-1}\cdot p_m l\oplus l_2))\\
&\quad=\chi_f(g_y)\,\lambda^*\mathcal{F}(w\circ\eta)(g_y^{-1}\cdot p_m l\oplus l_2)\\
&\quad=\chi_f(g_y)\,\mathcal{F}_\eta w(g_y^{-1}\cdot p_m l+l_2)=\chi_f(g_y)\,r_{K\cdot l}(\mathcal{F}_\eta w)(g_y^{-1}\cdot p_m l\oplus l_2),
\end{aligned}\tag{23}$$

by Lemma 4.1(a); here $\mathcal{F}_\eta$ is as defined in (21), and $r_{K\cdot l}$: $\mathcal{S}(\mathfrak{g}^*)\to\mathcal{S}(K\cdot l)$ is the restriction. We show the various maps in Fig. 1.

In Fig. 1, $\theta=K\cdot l$ is our (fixed) generic K-orbit in Ω_τ, and

(i) $r_\theta=$ restriction of $f\in\mathcal{S}(\mathfrak{g}^*)$ to θ;

(ii) $\eta(\cdot,0,0)$: $\mathbb{R}^p\to K$, with $\eta(y,0,0)=g_y$;

(iii) Φ: $\mathcal{S}(K\cdot l)\to\mathcal{S}_2(K\times\mathfrak{g}_m^\perp)$ is defined by

$$\Phi\psi(k,l_2)=\chi_f(k)\,\psi(k^{-1}\cdot p_m l\oplus l_2).$$

The map Φ is injective by Lemma 4.1(a). The maps $\eta(\cdot,0,0)^*$, $(I\times\lambda_2)^*$, and $\mathcal{F}_3$ are obviously bijections; $\eta(\cdot,0,\cdot)^*$ and Φ are injective; r_θ and Q_π

$$\begin{tikzcd}
R(A)\subsetneq\mathcal{S}(G) \arrow[rr, "\mathcal{F}_\eta"] \arrow[d, "Q_\pi"'] & & \mathcal{S}(\mathfrak{g}^*)\supsetneq R(A)^\wedge \arrow[d, "r_\theta"] \\
\pi(A)\subsetneq\mathcal{H}_\pi^\infty \arrow[rr, dashed, "\Psi_\pi", "\text{(induced injection)}"'] \arrow[d, "\eta(\cdot,0,\cdot)^*"'] & & \mathcal{S}(\theta)\supsetneq R(A)^\wedge|_\theta \arrow[dd, "\Phi"] \\
\mathcal{S}_2(\mathbb{R}^p\times\mathbb{R}^{n-m}) \arrow[d, "\mathcal{F}_3"'] & & \\
\mathcal{S}_2(\mathbb{R}^p\times\dot{\mathbb{R}}^{n-m}) \arrow[dr, "(I\times\lambda_2)^*"'] & & \mathcal{S}_2(K\times\mathfrak{g}_m^\perp) \arrow[dl, "\eta(\cdot,0,0)^*"] \\
& \mathcal{S}_2(\mathbb{R}^p\times\mathfrak{g}_m^\perp). &
\end{tikzcd}$$

FIG. 1. Maps involved in defining Ψ_π: $\mathcal{H}_\pi^\infty\to\mathcal{S}(\theta)$.

are surjective (Lemma 2.5), and $\mathscr{F}_\eta$ is bijective. The diagram commutes because

$$\begin{aligned}\eta(\cdot, 0, 0)^* \Phi r_\theta \mathscr{F}_\eta w(y, l_2) &= \Phi r_\theta \mathscr{F}_\eta w(g_y, l_2)\\ &= \chi_f(g_y)\, r_\theta \mathscr{F}_\eta w(g_y^{-1} \cdot p_m l \oplus l_2)\\ &= (I \times \lambda_2)^* \mathscr{F}_3 \eta(\cdot, 0, \cdot)^* Q_\pi w(y, l_2),\end{aligned}$$

from (23).

This induces a well defined map $\Psi_\pi : \mathscr{H}^\infty_{\pi(l,\mathfrak{b})} \to \mathscr{S}(\theta)$ if we take

$$\Psi_\pi(Q_\pi w) = r_\theta \mathscr{F}_\eta w, \qquad \text{all} \quad w \in \mathscr{S}(G). \tag{24}$$

Note first that if $Q_w w = 0$, then $Q_\pi w \equiv 0$ on Σ. Therefore $\eta(\cdot, 0, \cdot)^* Q_\pi w = 0$, or $r_\theta \mathscr{F}_\eta w = 0$. This shows that Ψ_π is well-defined. It is also injective on $Q_\pi(\mathscr{S}(G))$ because the same sort of argument shows that for all $w \in \mathscr{S}(G)$, $Q_\pi w = 0 \Leftrightarrow r_\theta \mathscr{F}_\eta w = 0$. Lemma 2.5 shows that Ψ_π is defined on all of $\mathscr{H}^\infty_{\pi(l,\mathfrak{b})}$; it is surjective because r_θ and $\mathscr{F}_\eta$ are surjective. We will prove below that Ψ_π is isometric. The diagram with Ψ_π adjoined obviously still commutes.

We know that Q_π intertwines R_g and $\pi(g)$ for $g \in G$, and hence also $R(A)$, $\pi(A)$ for $A \in \mathfrak{u}(\mathfrak{g})$. We define $R(A)^\wedge = \mathscr{F}_\eta \circ R(A) \circ \mathscr{F}_\eta^{-1}$; then $\mathscr{F}_\eta$ clearly intertwines $R(A)$ and $R(A)^\wedge \in \mathscr{P}(\mathfrak{g}^*)$.

4.2. Lemma. *For all $A \in \mathfrak{u}(\mathfrak{g})$, $R(A)^\wedge$ restricts to the generic K-orbits $K \cdot l \subseteq U_d(K) \cap \Omega_\tau$.*

Remark. As noted in Section 3, this implies that $R(A)^\wedge|_{K \cdot l} \in \mathscr{P}(K \cdot l)$.

Proof. Suppose that $\phi \in \mathscr{S}(\mathfrak{g}^*)$ and $\phi|_{K \cdot l} \equiv 0$. Let $w \in \mathscr{S}(G)$ satisfy $\phi = \mathscr{F}_\eta w$. Then

$$0 = r_{K \cdot l} \mathscr{F}_\eta w = \Psi_\pi Q_\pi w.$$

Since Ψ_π is injective, we have $Q_\pi w = 0$. Then $0 = \pi(A)\, Q_\pi w = Q_\pi R(A) w$, and retracing our steps, we get

$$0 = \Psi_\pi Q_\pi R(A) w = r_{K \cdot l} R(A)^\wedge \phi = R(A)^\wedge \phi \,|\, K \cdot l. \quad \blacksquare$$

We note as an immediate consequence that

$$R(A)^\wedge \text{ restricts to } \Omega_\tau, \text{ and } R(A)^\wedge|_{\Omega_\tau} \in \mathscr{P}(\Omega_\tau). \tag{25}$$

By definition, $r_{K \cdot l}$ intertwines $R(A)^\wedge$ with $R(A)^\wedge|_{K \cdot l}$. Therefore Ψ_π intertwines $\pi(A)$ and $R(A)^\wedge|_{K \cdot l}$.

So far we have fixed measures m_G, m_B, m_K, $m_{K \cap B}$, which determine the norm in $\mathscr{H}_\pi$, $a_\pi \in (\mathscr{H}_\pi^{-\infty})^{(K,\chi)}$, and the map Q_π. Since $B\backslash G_m \approx K \cap B\backslash K$, this allows us to fix a measure $m_{B\backslash G_m}$, and then $m_{G_m\backslash G}$. With respect to these normalizations we have:

4.3. LEMMA. *If $\theta = K \cdot l$ is a generic K-orbit in Ω_τ and $\pi = \pi(l, \mathfrak{b})$, there is a unique choice of K-invariant measure μ_θ on θ such that the map $\Psi_\pi : \mathscr{H}_\pi^\infty \to \mathscr{S}(\theta)$ defined in* (24) *is a surjectivity isometry* (*in the $\mathscr{L}^2$ norms*). *It is also bicontinuous between the $\mathscr{C}^\infty$ topology on $\mathscr{H}_\pi^\infty$ and the usual Fréchet topology on $\mathscr{S}(\theta)$.*

Proof. In $\mathscr{H}_\pi$ we have (all norms are Hilbert space norms)

$$\|Q_\pi w\|^2 = \int_{B\backslash G} |Q_\pi w(g)|^2 \, d\dot{g} = \int_{G_m\backslash G} \int_{B\backslash G_m} |Q_\pi w(g_m x)|^2 \, d\dot{g}_m \, d\dot{x}$$

$$= \int_{\mathbb{R}^{n-m}} \int_{B\backslash G_m} |Q_\pi w(g_m \eta(0, 0, t))|^2 \, dt \, d\dot{g}_m.$$

Now $B\backslash G_m = B\backslash KB \approx B \cap K\backslash K$, and the G_m-invariant measure for $B\backslash G_m$ is clearly the K-invariant measure for $B \cap K\backslash K$. Thus

$$\|Q_\pi w\|^2 = \int_{\mathbb{R}^{n-m}} \int_{K \cap B\backslash K} |Q_\pi w(k\eta(0, 0, t))|^2 \, d\dot{k} \, dt. \tag{26}$$

Meanwhile, take a K-invariant measure on $K \cdot p_m(l)$ corresponding to $d\dot{k} = m_{K\cap B\backslash K}$ (recall Lemma 4.1). Identifying $\mathfrak{g}_m^\perp = \dot{\mathbb{R}}^{n-m}$ via the coordinates $\dot{t}$ for $X_{m+1}^*, ..., X_n^*$, take Lebesgue measure $dl_2 \approx d\dot{t}$ on $\mathfrak{g}_m^\perp$. Then there is a unique $d\mu_\theta = d\mu_{K \cdot p_m(l)} \, dl_2$. We have

$$\|\Psi_\pi Q_\pi w\|^2 = \|r_{K\cdot l}\mathscr{F}_\eta w\|^2 = \int |r_{K\cdot l}\mathscr{F}_\eta w(l')|^2 d\mu_\theta(l')$$

$$= \int_{\mathfrak{g}_m^\perp} \int_{K\cdot p_m l} |r_{K\cdot l}\mathscr{F}_\eta w(l' \oplus l_2)|^2 \, d\mu_{K\cdot p_m l}(l') \, dl_2$$

$$= \int_{\mathfrak{g}_m^\perp} \int_{K\cap B\backslash K} |r_{K\cdot l}\mathscr{F}_\eta w(k^{-1} \cdot p_m l \oplus l_2)|^2 \, d\dot{k} \, dl_2. \tag{27}$$

Let $e \in \mathscr{C}^\infty(K)$ be a Bruhat function (of compact support on each coset $(K \cap B)k$, nonnegative, and with $\int_{K\cap B} e(k'x)\, dk' = 1$, $\forall x \in K$). Then for any $\psi \in \mathscr{C}^\infty(K)$ that is constant on $K \cap B\backslash K$ cosets, we have

$$\int_{K\cap B\backslash K} \psi(k) \, d\dot{k} = \int_K e(k) \, \psi(k) \, dk.$$

So from (26) and (27) we get

$$
\begin{aligned}
\|Q_\pi(w)\|^2 &= \int_{\mathbb{R}^{n-m}} \int_K |Q_\pi w(k\eta(0,0,t))|^2 \, e(k)\, dk\, dt \\
&= \int_{\mathbb{R}^{n-m}} \int_{\mathbb{R}^p} |\eta(\cdot,0,\cdot)^* Q_\pi w(y,t)|^2 \, e(g_y)\, dy\, dt \\
&= \int_{\hat{\mathbb{R}}^{n-m}} \int_{\mathbb{R}^p} |\mathscr{F}_3 \eta(\cdot,0,\cdot)^* \, Q_\pi w(y,\dot{t})|^2 \, e(g_y)\, d\dot{t}\, dy \\
&= \int_{\mathfrak{g}_m^\perp} \int_{\mathbb{R}^p} |(I\times\lambda_2)^* \mathscr{F}_3 \eta(\cdot,0,\cdot)^* Q_\pi w(y,l_2)|^2 \, e(g_y)\, dl_2\, dy \\
&= \int_{\mathfrak{g}_m^\perp} \int_{\mathbb{R}^p} |\eta(\cdot,0,0)^* \Phi r_{K\cdot l} \mathscr{F}_\eta w(y,l_2)|^2 \, e(g_y)\, dl_2\, dy \\
&= \int_{\mathfrak{g}_m^\perp} \int_K |\Phi r_{K\cdot l} \mathscr{F}_\eta w(k,l_2)|^2 \, e(k)\, dl_2\, dk \\
&= \int_{\mathfrak{g}_m^\perp} \int_K |r_{K\cdot l} \mathscr{F}_\eta w(k^{-1}\cdot p_m l \oplus l_2)|^2 \, e(k)\, dl_2\, dk \\
&= \int_{\mathfrak{g}_m^\perp} \int_{K\cap B\backslash K} |r_{K\cdot l} \mathscr{F}_\eta w(k^{-1}\cdot p_m l \oplus l_2)|^2 \, dl_2\, d\dot{k} = \|\Psi_\pi Q_\pi w\|^2.
\end{aligned}
$$

We verified surjectivity earlier. The Closed Graph Theorem implies that Ψ_π is a topological isomorphism. Obviously Ψ_π extends to an isometric isomorphism. ∎

We summarize our main results thus far as follows:

4.4. PROPOSITION. *Let* $(\mathfrak{g}, \mathfrak{k}, \mathfrak{b}, \chi)$ *satisfy the conditions* (20), *let* $f \in \mathfrak{g}^*$ *restrict to a Lie homomorphism on* $\mathfrak{k}$, *let* $\tau = \mathrm{Ind}(K \uparrow G, \chi_f)$, *and set* $\Omega_\tau = f + \mathfrak{k}^\perp$. *Define the Malcev–Fourier transform* $\mathscr{F}_\eta : \mathscr{S}(G) \to \mathscr{S}(\mathfrak{g}^*)$ *as in* (21). *For generic* $l \in \Omega_\tau$, *define* $\pi = \pi(l, \mathfrak{b})$ *and* $Q_\pi : \mathscr{S}(G) \to \mathscr{H}_\pi^\infty$ *as in* (11). *Then there is a topological isomorphism*

$$\Psi_\pi = \Psi_{\pi(l,\mathfrak{b})} : \mathscr{H}_\pi^\infty \to \mathscr{S}(K\cdot l)$$

such that the diagram of Fig. 2 *commutes. Furthermore,*

(a) $Q_\pi : \mathscr{S}(G) \to \mathscr{H}_\pi^\infty$ *is surjective and Fréchet continuous.*

(b) Ψ_π *is an isometry* (*in the* $\mathscr{L}^2$ *norm*) *for a suitably normalized* K-*invariant measure* $\mu_{K\cdot l}$ on $K\cdot l$.

$$\begin{tikzcd} R(A) \circlearrowright \mathscr{S}(G) \arrow[r, "\mathscr{F}_\eta"] \arrow[d, "Q_\pi"'] & \mathscr{S}(\mathfrak{g}^*) \circlearrowleft R(A)^\wedge \arrow[d, "r_\theta"] \\ \pi(A) \circlearrowright \mathscr{H}_\pi^\infty \arrow[r, dashed, "\Psi_\pi"'] & \mathscr{S}(\theta) \circlearrowleft R(A)^\wedge|_\theta \end{tikzcd}$$

FIG. 2. $\theta = K \cdot l$ a generic K-orbit in Ω_τ. Here $\Psi_\pi Q_\pi w = r_\theta \mathscr{F}_\eta w$.

(c) *For $A \in \mathfrak{u}(\mathfrak{g})$, $R(A)^\wedge = \mathscr{F}_\eta \circ R(A) \circ \mathscr{F}_\eta^{-1}$ is a polynomial coefficient differential operator that restricts to a polynomial coefficient operator $R(A)^\wedge|_{K \cdot l}$ on $K \cdot l$, and the maps in the diagram intertwine $R(A)$, $R(A)^\wedge|_{K \cdot l}$, $\pi(A)$ on the appropriate spaces.*

For τ, there is a similar, but easier result. The natural map Q_τ: $\mathscr{S}(G) \to \mathscr{H}_\tau^\infty$ corresponds to $a_\tau \in \mathscr{H}_\tau^{-\infty}$, with $\langle a_\tau, f \rangle = \overline{f(e)}$. As we saw in Section 2,

$$Q_\tau w(g) = \int_K w(kx) \overline{\chi_f(k)}\, dk, \qquad w \in \mathscr{S}(G).$$

Again, Q_τ is Fréchet continuous and intertwines $R(A)$, $\tau(A)$ for all $A \in \mathfrak{u}(\mathfrak{g})$. The space $\mathscr{S}_\tau = \text{range}\, Q_\tau$ is a $\tau(G)$-invariant and $\tau(\mathfrak{u}(\mathfrak{g}))$-invariant subspace in $\mathscr{H}_\tau^\infty \subseteq \mathscr{C}^\infty(G, \tau)$, and is even Fréchet dense in $\mathscr{H}_\tau^\infty$, but Q_τ is not surjective. (It is not hard to see that $\mathscr{S}_\tau = \text{all } \phi \in \mathscr{C}^\infty(G, \tau)$ that restrict to Schwartz functions on any polynomial cross-section for $K \backslash G$.)

4.5. PROPOSITION. *Let $(\mathfrak{g}, \mathfrak{k}, \mathfrak{b}, \chi)$ satisfy the hypotheses of Proposition 4.4, and define $Q_\tau : \mathscr{S}(G) \to \mathscr{S}_\tau \subseteq \mathscr{H}_\tau^\infty$ as in Section 2 (or as above). Then:*

(a) *For all $A \in \mathfrak{u}(\mathfrak{g})$, $R(A)^\wedge = \mathscr{F}_\eta \circ R(A) \circ \mathscr{F}_\eta^{-1}$ restricts to a polynomial coefficient operator on Ω_τ.*

(b) *There is an algebraic isomorphism $\Psi_\tau : \mathscr{S}_\tau \to \mathscr{S}(\Omega_\tau)$ that is also a norm isometry (for the $\mathscr{L}^2$ norms and a unique normalization of μ_Ω on Ω_τ) and makes the diagram (Fig. 3) commute. The maps also intertwine $R(A)$, $R(A)^\wedge|_{\Omega_\tau}$, and $\tau(A)$ on these spaces.*

$$\begin{tikzcd} R(A) \circlearrowright \mathscr{S}(G) \arrow[r, "\mathscr{F}_\eta"] \arrow[d, "Q_\tau"'] & \mathscr{S}(\mathfrak{g}^*) \circlearrowleft R(A)^\wedge \arrow[d, "r_\Omega"] \\ R(A) = \tau(A) \circlearrowright \mathscr{S}_\tau \arrow[r, dashed, "\Psi_\tau"'] & \mathscr{S}(\Omega_\tau) \circlearrowleft R(A)^\wedge|_{\Omega_\tau} \end{tikzcd}$$

FIG. 3. $\mathscr{S}_\tau = \text{range}(Q_\tau)$. Here $\Psi_\tau Q_\tau w = r_\Omega \mathscr{F}_\eta w$.

Note. We will also show that Ψ_τ is essentially a Euclidean Fourier transform in the Malcev coordinates $t \mapsto K \cdot \eta(0, t)$ on $K\backslash G$. Let $J: \mathscr{C}^\infty(K\backslash G) \to \mathscr{C}^\infty(G, \tau)$ be the identification map associated with the polynomial cross-section $t \mapsto \eta(0, t)$, $t \in \mathbb{R}^m$, for $K\backslash G$ cosets, and define identification maps $i_{K\backslash G}: \mathscr{S}(\mathbb{R}^m) \to \mathscr{S}(K\backslash G)$, $i_\Omega: \mathscr{S}(\dot{\mathbb{R}}^m) \to \mathscr{S}(\Omega_\tau)$ via

$$i_{K\backslash G}^{-1}\tilde{\varphi}(K \cdot \eta(0, t)) = \tilde{\varphi}(t), \qquad i_\Omega^{-1}\psi(\dot{t}) = \psi\left(\sum_{i=1}^{p} f_i X_i^* + \sum_{i=1}^{m} \dot{t}_i X_{p+i}^*\right).$$

Then we will show that

$$\Psi_\tau J i_{K\backslash G}\tilde{\varphi} = i_\Omega \mathscr{F}_2 \tilde{\varphi}, \qquad \forall \tilde{\varphi} \in \mathscr{S}(\mathbb{R}^m), \tag{28}$$

where

$$\mathscr{F}_2\varphi(\dot{t}) = \int_{\mathbb{R}^m} \varphi(t)\, e^{-2\pi i(\sum_{j=1}^m t_j \dot{t}_j)}\, dt, \qquad \dot{t} \in \dot{\mathbb{R}}^m.$$

Proof. The induced map Ψ_τ is well-defined if $Q_\tau w = 0 \Rightarrow r_\Omega \mathscr{F}_\eta w = 0$, injective if $Q_\tau w = 0 \Leftrightarrow r_\Omega \mathscr{F}_\eta w = 0$ for $w \in \mathscr{S}(G)$. Since J is a bijection from $\mathscr{S}(K\backslash G)$ to $\mathscr{S}_\tau = \operatorname{range}(Q_\tau)$, it suffices to show that

$$\mathscr{F}_2 i_{K\backslash G}^{-1} J^{-1} Q_\tau w = i_\Omega^{-1} r_\Omega \mathscr{F}_\eta w \qquad \text{on} \quad \dot{\mathbb{R}}^m, \quad \forall w \in \mathscr{S}(G).$$

But if $f = \sum_{i=1}^n f_i X_i^*$, we have

$$\begin{aligned} i_\Omega^{-1} r_\Omega \mathscr{F}_\eta w(\dot{t}) &= \mathscr{F}_\eta w\left(\sum_{i=1}^{p} f_i X_i^* + \sum_{i=1}^{m} \dot{t}_i X_{p+i}^*\right) \\ &= \iint w \circ \eta(s, t)\, e^{-2\pi i(\sum_{i=1}^p f_i s_i + \sum_{i=1}^m \dot{t}_i t_i)}\, ds\, dt \\ &= \int (Q_\tau w) \circ \eta(0, t)\, e^{-2\pi i(\sum_{i=1}^m \dot{t}_i t_i)}\, dt \end{aligned}$$

(because $\chi_f(\exp s_1 X_1 \cdots \exp s_p X_p) = \chi(\eta(s, 0)) = \exp[2\pi i \sum_{i=1}^p f_i s_i]$)

$$\begin{aligned} &= \int J^{-1} Q_\tau w(K \cdot \eta(0, t))\, e^{-2\pi i(\sum_{i=1}^m \dot{t}_i t_i)}\, dt \\ &= \mathscr{F}_2 i_{K\backslash G}^{-1} J^{-1} Q_\tau w(\dot{t}). \end{aligned}$$

It is immediate that Ψ_τ is a well-defined bijection and that (28) holds; Ψ_τ is an isometry for a unique choice of μ_Ω on Ω_τ because $J: \mathscr{S}(K\backslash G) \to \mathscr{H}_\tau^\infty$, $i_{K\backslash G}$, i_Ω, and $\mathscr{F}_2$ are isometries.

For part (a), suppose that $\varphi \in \mathscr{S}(\mathfrak{g}^*)$ satisfies $\varphi \mid \Omega_\tau = 0$. There is a $w \in \mathscr{S}(G)$ such that $\mathscr{F}_\eta w = \varphi$. Then $0 = r_\Omega \varphi = r_\Omega \mathscr{F}_\eta w \Rightarrow Q_\tau w = 0 \Rightarrow$

$R(A)\,Q_\tau w = Q_\tau R(A)w = 0 \Rightarrow 0 = r_\Omega \mathscr{F}_\eta(R(A)w) = r_\Omega(R(A)^\wedge \mathscr{F}_\eta w) = r_\Omega(R(A)^\wedge \varphi) \Rightarrow R(A)^\wedge \varphi|_{\Omega_\tau} = 0$, as required. ∎

Note. The above result is no longer true if we use exponential coordinates

$$\eta_0(s, t) = \exp\left(\sum_{i=1}^{p} s_i X_i + \sum_{i=1}^{m} t_i X_{p+i}\right)$$

in defining $\mathscr{F}_\eta$, because then $R(A)^\wedge = \mathscr{F}_{\eta_0} R(A)\, \mathscr{F}_{\eta_0}^{-1}$ will not restrict to Ω_τ or to K-orbits. For example, if $\mathfrak{g}=$ the 3-dimensional Heisenberg Lie algebra with basis X, Z, Y and $\mathfrak{k} = \mathbb{R}X$, $\chi \equiv 1$, then $\Omega_\tau = \mathbb{R}\text{-span}\{Z^*, Y^*\}$; $m(\pi) \equiv 1$. We have

$$\begin{aligned}&\eta_0(x_1, z_1, y_1)\,\eta_0(x_2, z_2, y_2)\\ &\qquad = \eta_0(x_1 + x_2, z_1 + z_2 + \tfrac{1}{2}[x_1 y_2 - x_2 y_1], y_1 + y_2)\end{aligned}$$

and

$$R(Z) = D_z, \qquad R(X) = D_x - \tfrac{1}{2} y D_z, \qquad R(Y) = D_y + \tfrac{1}{2} x D_z.$$

It is easily seen that

$$R(Y)^\wedge = 2\pi i \dot{y} - \tfrac{1}{2} \dot{z} D_{\dot{x}},$$

which does not restrict to Ω_τ. ∎

Remark. We do not show that $\Psi_\tau \colon \mathscr{S}_\tau \to \mathscr{S}(\Omega_\tau)$ is continuous in the topology of $\mathscr{S}_\tau$ as a subspace of $\mathscr{H}_\tau^\infty$. Since $\mathscr{S}_\tau$ is not closed there, we cannot use the Closed Graph Theorem. The true continuity properties are revealed if we augment Fig. 3, and we need these to discuss tempered fundamental solutions of operators in $\mathbb{D}_\tau(K\backslash G)$ in Section 6. If we use any polynomial cross-section, $\sigma\colon \mathbb{R}^{n-p} \to G$ for $K\backslash G$ cosets (recall Section 3) we get an identification map $J\colon \mathscr{C}^\infty(K\backslash G) \to \mathscr{C}^\infty(G, \tau)$ that carries $\mathscr{S}(K\backslash G)$ to $\mathscr{S}_\tau = \text{range}(Q_\tau)$. Thus we augment Fig. 3 as shown in Fig. 4.

$$\begin{tikzcd}
\vdots \arrow[d, "Q_\tau"'] & & \vdots \arrow[d, "r_\Omega"] \\
\tau(A) \circlearrowleft \mathscr{H}_\tau^\infty \supseteq \mathscr{S}_\tau \arrow[rr, "\Psi_\tau"] & & \mathscr{S}(\Omega_\tau) \leftarrow R(A)^\wedge|_{\Omega_\tau} \\
& \tilde{\tau}(A) \circlearrowleft \mathscr{S}(K\backslash G) \arrow[ul, "J"] \arrow[ur, "\Psi_\tau \circ J"'] &
\end{tikzcd}$$

FIG. 4. $\mathscr{S}_\tau = \text{range}(Q_\tau) = \text{range}(J)$.

4.6. COROLLARY. *Consider the situation described in Proposition* 4.5. *Let* $\sigma: \mathbb{R}^{n-p} \to G$ *be any polynomial cross-section for* $K\backslash G$ *cosets and let* $J: \mathscr{S}(K\backslash G) \to \mathscr{S}_\tau = \mathrm{range}(Q_\tau)$ *be the resulting identification map. Then*

(a) *The augmented diagram in Fig.* 4 *commutes and the operators* $\tau(A)$, $\tilde{\tau}(A)$, $R(A)^\wedge \mid \Omega_\tau$ *are intertwined for all* $A \in \mathfrak{u}(\mathfrak{g})$, *where* $\tilde{\tau}$ *is* τ *modeled on* $L^2(K\backslash G)$ *via* J.

(b) $J: \mathscr{S}(K\backslash G) \to \mathscr{H}_\tau^\infty$ *is Fréchet continuous.*

(c) $\Psi_\tau \circ J: \mathscr{S}(K\backslash G) \to \mathscr{S}(\Omega_\tau)$ *is a Fréchet bicontinuous isomorphism.*

(d) $J^{-1} \circ Q_\tau: \mathscr{S}(G) \to \mathscr{S}(K\backslash G)$ *is Fréchet continuous and surjective.*

In particular, $(\Psi_\tau \circ J)^t$: $\mathscr{S}'(K\backslash G) \to \mathscr{S}'(G)$ *is a linear isomorphism of tempered distributions.*

Proof. It is easily seen that $J(\mathscr{S}(K\backslash G)) = \mathrm{range}(Q_\tau) = \{\phi \in \mathscr{C}^\infty(G, \tau): \phi \circ \sigma \in \mathscr{S}(\mathbb{R}^{n-p})\}$. The operators $\tilde{\tau}_g$ $(g \in G)$ preserve $\mathscr{S}(K\backslash G)$ because τ_g preserves $\mathscr{S}_\tau$ (Q_τ intertwines R_g and τ_g), and $\tilde{\tau}_g$ has the general form $\tau_g \tilde{\phi}(\zeta) = e^{iQ(g,\zeta)}\tilde{\phi}(\zeta \cdot g)$ for $g \in G$, $\zeta \in K\backslash G$, where Q is a polynomial on $G \times K\backslash G$. Difference quotients converge to derivatives in L^2 norm since $\mathscr{S}_\tau \subseteq \mathscr{H}_\tau^\infty$, $\mathscr{S}(K\backslash G) \subseteq \mathscr{H}_{\tilde{\tau}}^\infty$, so we get

$$\tau(A)\, J\tilde{\phi} = J\tilde{\tau}(A)\tilde{\phi} \qquad \text{all} \quad A \in \mathfrak{u}(\mathfrak{g}),\, \tilde{\phi} \in \mathscr{S}(K\backslash G)$$
$$\tilde{\tau}(A) \in \mathscr{P}(K\backslash G),$$

which proves (a). For (b)–(d) we repeatedly use the Closed Graph Theorem or the fact that $\Psi_\tau \circ J$ is essentially a Fourier transform. ∎

We now combine these observations to show that the map Ψ_τ implements a concrete realization of the direct integral decomposition $\tau = \int_{\Omega_\tau/K}^{\oplus} \pi_\theta \, d\mu(\theta)$ over K-orbits in Ω_τ. See Fig. 5.

The maps are set up with the following considerations in mind. If $l \in E_d \cap U_d \approx$ generic orbits in Ω_τ/K, there is a unique K-invariant measure μ_θ on $\theta = K \cdot l$ such that $\Psi_{\pi(l,\mathfrak{b})}: \mathscr{H}_{\pi(l,\mathfrak{b})}^\infty \to \mathscr{S}(\theta) \subseteq L^2(\theta, \mu_\theta)$ extends to an isometric isomorphism, as in Proposition 4.4 (Fig. 2). There is also a unique Lebesgue measure μ_Ω on Ω_τ such that $\Psi_\tau: \mathscr{S}_\tau = Q_\tau(\mathscr{S}(G)) \to \mathscr{S}(\Omega_\tau)$ extends to an isometric isomorphism, as in Proposition 4.5 (Figs. 3, 4).

$$\begin{tikzcd}
\mathscr{H}_{\tilde{\tau}} = L^2(K\backslash G) \arrow[r, "J"] & \mathscr{H}_\tau \arrow[rr, "\Psi_\tau"] \arrow[d, dashed, "J_1"'] & & L^2(\Omega_\tau, \mu_\Omega) \arrow[d, "J_2"] \\
& \displaystyle\int_{E_d}^{\oplus} \mathscr{H}_{\pi(l,\mathfrak{b})}\, d\nu(l) \arrow[rr, "\Psi"'] & & \displaystyle\int_{\Omega_\tau/K}^{\oplus} L^2(\theta, \mu_\theta)\, d\nu(\theta)
\end{tikzcd}$$

FIGURE 5

Hence we uniquely determine ν on $E_d \approx \Omega_\tau/K$ such that $\mu_\Omega = \int_{\Omega_\tau/K}^{\oplus} \mu_\theta \, d\nu(\theta)$; ν is a rational function weighting dl on E_d. Then in Fig. 5:

(i) Ψ_τ is the isometric extension of Ψ_τ on $\mathscr{S}_\tau$;

(ii) J_2 is the obvious isometric isomorphism with

$$J_2 \phi = \int_{\Omega_\tau/K}^{\oplus} \phi|_\theta \, d\nu(\theta) \qquad \text{for} \quad \phi \in \mathscr{S}(\Omega_\tau);$$

(iii) $\Psi = \int_{E_d}^{\oplus} \Psi_{\pi(l, \mathfrak{b})} \, d\nu(l)$, obviously an isometric isomorphism (by Lemma 4.3 and the definition of ν);

(iv) J_1 is the induced isometric isomorphism;

(v) J is the identification map defined via the same Malcev coordinates used to define the Malcev–Fourier transform $\mathscr{F}_\eta$.

We can be quite explicit about the form of the induced map J_1 for vectors ϕ in $\mathscr{S}_\tau$; we may (nonuniquely) write $\phi = Q_\tau w$ for $w \in \mathscr{S}(G)$, and get

$$\Psi J_1(Q_\tau w) = J_2 \Psi_\tau(Q_\tau w) = \int_{\Omega_\tau/K}^{\oplus} r_\theta \mathscr{F}_\tau w \, d\nu(\theta). \tag{29}$$

Under Ψ^{-1} this corresponds to the field of vectors $\int_{E_d}^{\oplus} \Psi_\pi^{-1} r_{K \cdot l} \mathscr{F}_\eta w \, d\nu(l)$; but, as in Proposition 4.4, $\Psi_\pi Q_\pi w = r_{K \cdot l} \mathscr{F}_\eta w$ for generic $l \in E_d$, so that

$$J_1(Q_\tau w) = \int_{E_d}^{\oplus} Q_{\pi(l, \mathfrak{b})} w \, d\nu(l) \qquad \text{all} \quad w \in \mathscr{S}(G). \tag{30}$$

Note that the representations $\pi(l, \mathfrak{b})$ vary rationally with l. For vectors in $\mathscr{S}_\tau$, the components in the decomposition (30) vary continuously (even smoothly) on the generic set in Ω_τ/K.

This decomposition of $\mathscr{H}_\tau$ yields a decomposition of τ into irreducibles. For $l \in E_d \cap U_d \approx \Omega_\tau/K$,

$$\pi_\theta = \pi_{(l, \mathfrak{b})} \text{ with } \theta = K \cdot l, \qquad \text{and} \qquad \hat{\pi}_\theta = \Psi_\pi \circ \pi_\theta \circ \Psi_\pi^{-1} \text{ on } L^2(\theta, \mu_\theta)$$

are measurable fields of concrete unitary representations, and are intertwined by Ψ. The space of $\mathscr{C}^\infty$ vectors for $\hat{\pi}_\theta$ is $\mathscr{S}(\theta)$, by Proposition 4.4, and the $\mathscr{C}^\infty$ topology agrees with the Fréchet topology. The $\hat{\pi}_\theta$ vary smoothly on the generic set in Ω_τ/K, because we have

$$\hat{\pi}_\theta(g)(r_\theta \mathscr{F}_\eta w) = r_\theta \mathscr{F}_\eta(R_g w), \qquad \text{all} \quad w \in \mathscr{S}(G), \text{ generic } \theta \in \Omega_\tau/K,$$

as can be seen by chasing around Fig. 3. Similarly we see that $J_2\Psi_\tau=\Psi J_1$ intertwines τ and $\int_{\Omega_\tau/K}^{\oplus}\hat{\pi}_\theta\,d\nu(\theta)$; from Figs. 2, 3 we get

$$\begin{aligned}(J_2\Psi_\tau(\tau_g Q_\tau w))_\theta &= r_\theta\mathscr{F}_\eta(R_g w)=\Psi_\pi Q_\pi(R_g w)\\ &= \Psi_\pi\pi(g)\,Q_\pi w=\hat{\pi}_\theta(g)\,\Psi_\pi Q_\pi w=\hat{\pi}_\theta(g)[J_2\Psi_\tau(Q_\tau w)]_\theta\end{aligned}$$

for $\forall g\in G$, $\forall w\in\mathscr{S}(G)$, and generic θ. An easy L^2-norm argument then shows

$$(J_2\Psi_\tau(\tau_g\xi))_\theta=\hat{\pi}_\theta(g)(J_2\Psi_\tau\xi)_\theta \qquad \nu\text{-a.e. on } \Omega_\tau/K$$

for $\forall g\in G$, $\forall\xi\in\mathscr{H}_\tau$, as required.

For $A\in\mathfrak{u}(\mathfrak{g})$ the $\mathscr{C}^\infty$ decomposition of $\tau(A)$, as in [24], can be described in various ways. By the fact that $R(A)^\wedge$ restricts to generic orbits in Ω_τ/K, the commutativity of Fig. 2, and Proposition 4.4, we have

$$\hat{\pi}_\theta(A)=R(A)^\wedge|_\theta \qquad \text{on } \mathscr{H}^\infty_{\hat{\pi}_\theta}=\mathscr{S}(\theta),\ \forall A\in\mathfrak{u}(\mathfrak{g}),\ \text{generic } \theta\in\Omega_\tau/K. \tag{31}$$

In fact, any $\phi\in\mathscr{S}(\theta)$ is of the form $r_\theta\mathscr{F}_\eta w$ for some $w\in\mathscr{S}(G)$, and then

$$\begin{aligned}\hat{\pi}_\theta(A)\phi &= \Psi_\pi\pi_\theta(A)\,\Psi_\pi^{-1}r_\theta\mathscr{F}_\eta w=\Psi_\pi\pi_\theta(A)\,Q_\pi w\\ &= \Psi_\pi Q_\pi(R(A)w)=r_\theta\mathscr{F}_\eta(R(A)w)=R(A)^\wedge|_\theta\,(\phi).\end{aligned}$$

From (31) we see that the $\mathscr{C}^\infty$ decomposition of $\tau(A)$ under $J_2\Psi_\tau$ is just

$$\tau(A)\cong\int_{\Omega_\tau/K}^{\oplus}\hat{\pi}_\theta(A)\,d\nu(\theta)=\int_{\Omega_\tau/K}^{\oplus}R(A)^\wedge|_\theta\,d\nu(\theta). \tag{32}$$

Another version of this uses the fact that $V=\Psi_\tau\circ J\colon\mathscr{S}(K\backslash G)\to\mathscr{S}(\Omega_\tau)$ is bijective. If $\tilde{\tau}$ is τ realized in $L^2(K\backslash G)$ via J, then

$$V(\tilde{\tau}(A)\tilde{\phi})=R(A)^\wedge|_{\Omega_\tau}(V\tilde{\phi}), \qquad \forall\tilde{\phi}\in\mathscr{S}(K\backslash G),\ \forall A\in\mathfrak{u}(\mathfrak{g}). \tag{33}$$

Before we summarize these observations in the decomposition theorem below, one fact should be noted. If $X\in\mathfrak{g}$, the unitary group $\hat{\pi}_\theta(\exp tX)$ has infinitesimal form $\hat{\pi}_\theta(X)=R(X)^\wedge|_\theta$ on $\mathscr{H}^\infty_{\hat{\pi}_\theta}=\mathscr{S}(\theta)$ and $\hat{\pi}_\theta(X)$ is just a polynomial coefficient differential operator on θ. Although $R(X)$ on G is a vector field, $R(X)^\wedge$ on $\mathfrak{g}^*$ need not be first order, and likewise for $\hat{\pi}_\theta(X)\in\mathscr{P}(\theta)$. Thus the operators $\hat{\pi}_\theta(\exp tX)$ are rather unpleasant Fourier integral operators. We have avoided using them by working entirely with the action of $\mathfrak{u}(\mathfrak{g})$ on C^∞ vectors.

Here is our main result on the smooth decomposition of τ.

4.7. THEOREM. *Let* $(\mathfrak{g},\mathfrak{k},\mathfrak{b},\chi)$ *satisfy the conditions* (20) *and let* $\tau=\mathrm{ind}(K\uparrow G,\chi)$, $\Omega_\tau=f+\mathfrak{k}^\perp$ *where* $f\in\mathfrak{g}^*$ *determines the character* $\chi\in K^\wedge$.

Fix measures m_G, m_K, m_B, $m_{B\cap K}$ to establish norms in $\mathscr{H}_\tau$, $\mathscr{H}_{\pi(l,\mathfrak{b})}$ for generic $l\in\Omega_\tau$, and to define maps $Q_\tau\colon \mathscr{S}(G)\to\mathscr{H}_\tau^\infty$, $Q_{\pi(l,\mathfrak{b})}\colon \mathscr{S}(G)\to\mathscr{H}_{\pi(l,\mathfrak{b})}^\infty$ as in (9) and (11). Fixing a weak Malcev basis $X_1,\dots,X_n$ through $\mathfrak{b}\cap\mathfrak{k}$, $\mathfrak{k}$, $\mathfrak{b}+\mathfrak{k}$ compatible with these measures, define the Malcev–Fourier transform $\mathscr{F}_\eta\colon\mathscr{S}(G)\to\mathscr{S}(\mathfrak{g}^*)$ as in (21), and the identification map $J\colon\mathscr{L}^2(K\backslash G)\to\mathscr{H}_\tau$ as in Section 3. Define normalizations of Lebesgue measure μ_Ω on Ω_τ and K-invariant measures μ_θ for generic $\theta\in\Omega_\tau/K$ such that $\mathscr{L}^2(\Omega_\tau,\mu_\Omega)=\int_{\Omega_\tau/K}^{\oplus}\mathscr{L}^2(\theta,\mu_\theta)\,d\nu(\theta)$, as in Propositions 4.4 and 4.5.

If $\tilde\tau$ is τ realized on $\mathscr{L}^2(K\backslash G)$ via J, then $\mathscr{S}(K\backslash G)\subseteq\mathscr{H}_{\tilde\tau}^\infty$ and there exist

(i) *An isometry $V=\Psi_\tau\circ J\colon L^2(K\backslash G)\to L^2(\Omega_\tau,\mu_\Omega)$;*

(ii) *A measurable field of unitary representations $\hat\pi_\theta$ modeled in $\mathscr{L}^2(\theta,\mu_\theta)$ for generic θ;*

such that

(a) *V maps $\mathscr{S}(K\backslash G)$ to $\mathscr{S}(\Omega_\tau)$ and is Fréchet bicontinuous.*

(b) *$\tilde\tau$ decomposes under $J_2\circ V$, $\tilde\tau\cong\int_{\Omega_\tau/K}^{\oplus}\hat\pi_\theta\,d\nu(\theta)$.*

(c) *$\hat\pi_\theta\cong\pi_{(l,\mathfrak{b})}\in G^\wedge$ for generic θ, where l is the K-orbit representative in the flat cross-section $E_d\approx\Omega_\tau/K$ determined by our basis.*

(d) *$\mathscr{H}_{\hat\pi_\theta}^\infty=\mathscr{S}(\theta)$, with the $\mathscr{C}^\infty$ and Fréchet topologies in agreement, and $\hat\pi_\theta(A)=R(A)^\wedge|_\theta$ for all $A\in\mathfrak{u}(\mathfrak{g})$ and generic θ. Thus the $\mathscr{C}^\infty$ decomposition of $\tilde\tau(A)$ is $\int_{\Omega_\tau/K}^{\oplus}\hat\pi_\theta(A)\,d\nu(\theta)=\int_{\Omega_\tau/K}^{\oplus}R(A)^\wedge|_\theta\,d\nu(\theta)$.*

(e) *For any $\tilde\phi\in\mathscr{S}(K\backslash G)$ and $A\in\mathfrak{u}(\mathfrak{g})$ we have*

$$V(\tilde\tau(A)\tilde\phi)=R(A)^\wedge|_{\Omega_\tau}(V\tilde\phi).$$

Proof. We are just compiling previous results and giving an explicit form to V. We define J_2, Ψ_τ as explained in Fig. 5. For (a), V is bijective, and the rest is Corollary 4.6(c). For (b) see (30)–(31). The first part of (c) follows since $\Psi_\pi\colon\mathscr{H}_\pi^\infty\to\mathscr{S}(\theta)$ is Fréchet bicontinuous ($\mathscr{C}^\infty$ topology in $\mathscr{H}_\pi^\infty$), by Lemma 4.3, and since $\hat\pi_\theta=\Psi_\pi\pi_\theta\Psi_\pi^{-1}$; $\hat\pi_\theta(A)$ has been computed in (31). Part (e) is just (33). ▮

5. Decomposition of Operators in $\mathbb{D}_\tau(K\backslash G)$

We have shown that the map $V=\Psi_\tau\circ J\colon\mathscr{S}(K\backslash G)\to\mathscr{S}(\Omega_\tau)$ transforms operators $\tilde\tau(A)=J^*R(A)$, $A\in\mathfrak{u}(\mathfrak{g})$, to polynomial coefficient operators on Ω_τ,

$$V\tilde\tau(A)\,V^{-1}=R(A)^\wedge|_{\Omega_\tau},\qquad R(A)^\wedge=\mathscr{F}_\eta R(A)\,\mathscr{F}_\eta^{-1}\in\mathscr{P}(\mathfrak{g}^*).$$

The operators $R(A)^\wedge$ restrict to generic K-orbits and, upon identifying

$\mathscr{L}^2(\Omega_\tau, \mu_\Omega) = \int_{\Omega_\tau/K}^{\oplus} \mathscr{L}^2(\theta, \mu_\theta)\, dv(\theta)$, we get the $\mathscr{C}^\infty$ decomposition of $V\tilde{\tau}(A)\, V^{-1}$ corresponding to the decomposition $\tilde{\tau} \cong \int_{\Omega_\tau/K}^{\oplus} \pi_\theta\, dv(\theta)$.

We next investigate what happens to elements of $\mathbb{D}_\tau(K\backslash G)$ under this transformation. We know from [8] that $\mathbb{D}_\tau(K\backslash G)$ is Abelian; if τ is not multiplicity-free, however, there is no a priori reason why $\mathbb{D}_\tau(K\backslash G)$ should decompose diagonally.

Since $D \in \mathbb{D}_\tau(K\backslash G)$ leaves $\mathscr{S}(K\backslash G)$ invariant, we get a well-defined, Fréchet continuous transform VDV^{-1} acting on $\mathscr{S}(\Omega_\tau)$; see Corollary 4.6(c). We will show that this operator lies in $\mathscr{P}(\Omega_\tau)$, and then that it is scalar and constant on K-orbits.

5.1. LEMMA. *Let $\mathfrak{g}$ be a nilpotent Lie algebra, $\mathfrak{k}$ any subalgebra, and $\chi \in K^\wedge$ any character. Let $\mathfrak{a} \subseteq \mathrm{Diff}(G, \tau)$ be the subalgebra generated by operators $p(g)\, R(A)$, where $A \in \mathfrak{u}(\mathfrak{g})$ and p is a polynomial constant on $K\backslash G$ cosets. Then $\mathfrak{a}|_{\mathscr{C}^\infty(G,\tau)} = \mathscr{P}(G) \cap \mathrm{Diff}(G, \tau)|_{\mathscr{C}^\infty(G,\tau)}$, where $\mathscr{P}(G)$ is the algebra of polynomial coefficient differential operators on G.*

Note. Call two operators D, $D' \in \mathrm{Diff}(G, \tau)$ congruent if they are the same on $\mathscr{C}^\infty(G, \tau)$. Taking a weak Malcev basis $X_1, ..., X_p, ..., X_n$ through $\mathfrak{k}$ and coordinates $\eta(s, t)$ in G, $(s, t) \in \mathbb{R}^p \times \mathbb{R}^m$ with $m + p = n$, we may transfer the partial derivatives $D_t^\alpha D_s^\beta$ to operators on G. Functions $\phi \in \mathscr{C}^\infty(G, \tau)$ have the form $\phi \circ \eta(s, t) = e^{c \cdot s} F(t)$, where $c = (2\pi i f(X_1), ..., 2\pi i f(X_p))$ because $f|_{\mathfrak{k}}$ is a homomorphism. It is not hard to see that operators $p_\alpha(t)\, D_t^\alpha$ (where p_α is a polynomial not depending on s) are in $\mathrm{Diff}(G, \tau)$, and that any $D \in \mathrm{Diff}(G, \tau) \cap \mathscr{P}(G)$ has a representative of the form $\sum_{\alpha \in \mathbb{Z}_+^m} p_\alpha(t)\, D_t^\alpha$. If $J: \mathscr{C}^\infty(K\backslash G) \to \mathscr{C}^\infty(G, \tau)$ is the identification map in these coordinates, and we impose coordinates $K \cdot \eta(0, t)$ in $K\backslash G$, then $J^*(D_t^\alpha) = D_t^\alpha$ in an obvious sense; thus $J^* \cdot (\mathrm{Diff}(G, \tau) \cap \mathscr{P}(G)) = \mathscr{P}(K\backslash G)$ and the lemma says, in effect, that $J^*(\mathfrak{a}) = \mathscr{P}(K\backslash G)$. Furthermore, to prove the lemma it suffices to show that any operator of the form D_t^α ($\alpha \in \mathbb{Z}_+^m$) is congruent to one in $\mathfrak{a}$.

Proof of 5.1. We work by induction on $m = n - p$. Since $R(X_n) = \partial/\partial t_m$ in the given coordinates on G, the case $m = 1$ is easy. Assume $m > 1$ and let $\mathfrak{g}_0 = \mathbb{R}\text{-span}\{X_1, ..., X_{n-1}\}$; take coordinates $\eta_0(s, t') = \eta(s, t', 0)$ in $G_0 = \exp \mathfrak{g}_0$, and let $\tau_0 = \mathrm{Ind}(K \uparrow G_0, \chi)$, $R_0(Y) =$ right action of $Y \in \mathfrak{g}_0$ on G_0, etc. If $\alpha \in \mathbb{Z}_+^{m-1} \times \{0\}$ we write $\tilde{D}_t^\alpha$ for the operator corresponding to $D_{t_1}^{\alpha_1} \cdots D_{t_{m-1}}^{\alpha_{m-1}}$ in the coordinates $(s, t') = (s, t_1, ..., t_{m-1})$ in G_0, and write D_t^α for the corresponding operator on G; thus $\tilde{D}_t^\alpha \in \mathrm{Diff}(G_0, \tau_0)$, while $D_t^\alpha \in \mathrm{Diff}(G, \tau)$. The inductive hypothesis says that for any $\alpha \in \mathbb{Z}_+^{m-1} \times \{0\}$,

$$\tilde{D}_t^\alpha \text{ is congruent to an operator } \sum_{\beta \in \mathbb{Z}_+^{n-1}} f_{\alpha\beta}(g_0)\, R_0(X_0^\beta), \tag{34}$$

where $X_0^\beta = X_1^{\beta_1} \cdots X_{n-1}^{\beta_{n-1}}$ and the $f_{\alpha\beta}$ are polynomials on G_0 constant on $K\backslash G_0$ cosets.

Let $x(t_m) = \exp(t_m X_n)$, $g_0 \in G_0$. If $j < n$ and $\varphi \in \mathscr{C}^\infty(G)$, a straightforward calculation gives

$$R(X_j)\, \varphi(g_0 x(t_m)) = R_0(Ad_{x(t_m)} X_j)\, \varphi^{x(t_m)}(g_0), \qquad \forall t_m \in \mathbb{R},\ g_0 \in G_0,$$

where $\varphi^x(g_0) = \varphi(g_0 x)$. Repeated use of this gives

$$R(X_0^\alpha)\, \varphi(g_0 x(t_m)) = R_0(Ad_{x(t_m)} X_0^\alpha)\, \varphi^{x(t_m)}(g_0) \tag{35}$$

for $\alpha \in \mathbb{Z}_+^{n-1}$. Clearly $\varphi \in \mathscr{C}^\infty(G, \tau) \Rightarrow \varphi^{x(t_m)} \in \mathscr{C}^\infty(G_0, \tau_0)$. But $R_0(Ad_{x(t_m)} X_i) = \sum_{j=1}^{n-1} p_{ij}(t_m)\, R_0(X_j)$, where the p_{ij} are polynomials with $\operatorname{Det}[p_{ij}] = 1$, so there exist polynomials q_{ij} such that

$$R_0(X_j) = \sum_{i=1}^{n-1} q_{ij}(t_m)\, R_0(Ad_{x(t_m)} X_i), \qquad 1 \leqslant j \leqslant n-1, \quad t_m \in \mathbb{R}.$$

Similarly, if $\alpha \in \mathbb{Z}_+^{n-1}$ there are polynomials $q_{\alpha\beta}(t_m)$ such that

$$R_0(X_0^\alpha) = \sum_{\beta \in \mathbb{Z}_+^{n-1}} q_{\alpha\beta}(t_m)\, R_0(Ad_{x(t_m)} X_0^\beta). \tag{36}$$

Let $h_{\alpha\gamma}(g_0 x(t_m)) = \sum_{\beta \in \mathbb{Z}_+^{n-1}} f_{\alpha\beta}(g_0)\, q_{\beta\gamma}(t_m)$; these polynomials are constant on $K\backslash G$ cosets. If $\varphi \in \mathscr{C}^\infty(G, \tau)$, then

$$\begin{aligned}
&\sum_{\gamma \in \mathbb{Z}_+^{n-1}} h_{\alpha\gamma} R(X_0^\gamma)\, \varphi(g_0 x(t_m)) \\
&\quad = \sum_{\gamma,\beta} f_{\alpha\beta}(g_0)\, q_{\beta\gamma}(t_m)\, R(X_0^\gamma)\, \varphi(g_0 x(t_m)) \\
&\quad = \sum_{\alpha,\beta} f_{\alpha,\beta}(g_0)\, q_{\beta\gamma}(t_m)\, R_0(Ad_{x(t_m)} X_0^\gamma)\, \varphi^{x(t_m)}(g_0) \qquad \text{(by (35))} \\
&\quad = \sum_{\beta} f_{\alpha\beta}(g_0)\, R_0(X_0^\beta)\, \varphi^{x(t_m)}(g_0) \qquad \text{(by (36))} \\
&\quad = \tilde{D}_t^\alpha \varphi^{x(t_m)}(g_0) \qquad \text{(by (34))} \\
&\quad = D_t^\alpha \varphi(g_0 x(t_m)),
\end{aligned}$$

by a routine calculation. Thus D_t^α is congruent to $\sum h_{\alpha\gamma} R(X_0^\gamma) \in \mathfrak{a}$ for $\alpha \in \mathbb{Z}_+^{m-1} \times \{0\}$. As above, D_{t_m} is congruent to $R(X_n)$, so the above statement holds for all $\alpha \in \mathbb{Z}_+^n$ and the lemma is proved. ∎

This observation is the key to obtaining the spectral decomposition of elements in $\mathbb{D}_\tau(K\backslash G)$.

5.2. THEOREM. *Let* $(\mathfrak{g}, \mathfrak{k}, \mathfrak{b}, \chi)$ *satisfy* (20), *let* $X_1, ..., X_q, ..., X_p, ..., X_m, ..., X_n$ *be an admissible weak Malcev basis as in Section* 4, *let* $\tau = \operatorname{Ind}(K \uparrow G, \chi)$, *and let* $V = \Psi_\tau \circ J\colon \mathscr{S}(K\backslash G) \to \mathscr{S}(\Omega_\tau)$ *be the map defined in Proposition* 4.5 (*Fig.* 3), *where* $J\colon \mathscr{C}^\infty(K\backslash G) \to \mathscr{C}^\infty(G, \tau)$ *is the identification map corresponding to our basis. Then for any* $D \in \mathbb{D}_\tau(K\backslash G)$,

(a) VDV^{-1} *is an operator in* $\mathscr{P}(\Omega_\tau)$.

(b) *There is a K-invariant polynomial* $p_D \in \mathbb{C}[\Omega_\tau]^K$ *such that* $VDV^{-1} = p_D(l)I$.

(c) $\mathbb{D}_\tau(K\backslash G) \cong \mathbb{C}[\Omega_\tau]^K$ *under* V.

Under the decomposition map $J_2 \circ V\colon \mathscr{L}^2(K\backslash G) \to \mathscr{L}^2(\Omega_\tau, \mu_\Omega)$ *of Theorem* 4.7, *the action of* $D \in \mathbb{D}_\tau(K\backslash G)$ *on vectors in* $\mathscr{S}(K\backslash G) \subseteq \mathscr{H}_\tau^\infty$ *decomposes as*

$$D \cong \int_{\Omega_\tau/K}^{\oplus} p_D(\theta) I \, d\nu(\theta).$$

Proof. The last statement follows from (a) and (b). To prove (a), recall that $J(\mathscr{S}(K\backslash G)) = \mathscr{S}_\tau = Q_\tau(\mathscr{S}(G))$, where Q_τ is the averaging map (9). There is an $A \in \mathfrak{u}(\mathfrak{g}, \tau)$ such that $D = \gamma(A) = J^{-1}L(A)J = J^*L(A)$. Since $L(A) \in \mathscr{P}(G)$, Lemma 5.1 ensures that we can find polynomials p_α constant on $K\backslash G$ cosets such that

$$L(A) = \sum_\alpha p_\alpha(g)\, R(X^\alpha) \qquad \text{on} \quad \mathscr{C}^\infty(G, \tau).$$

Notice that $Q_\tau(p_\alpha w)(g) = p_\alpha(g)\, Q_\tau w(g)$ for all $w \in \mathscr{S}(G)$. Thus

$$L(A)\, Q_\tau w(g) = \sum_\alpha p_\alpha(g)\, R(X^\alpha)\, Q_\tau w(g) = Q_\tau\left(\sum_\alpha p_\alpha R(X^\alpha) w\right)(g), \tag{37}$$

because $R(X^\alpha)$ commutes with Q_τ (Lemma 2.1(b)). Under the coordinate map $\eta\colon \mathbb{R}^n \to G$, $p_\alpha \circ \eta(x_1, ..., x_n)$ does not depend on $x_1, ..., x_p$. Regard p_α as a pointwise multiplication operator; then $\hat{p}_\alpha = \mathscr{F}_\eta \circ p_\alpha \circ \mathscr{F}_\eta^{-1}$ is a constant coefficient differential operator on $\mathfrak{g}^*$ that does not involve derivatives in the $\mathfrak{k}^\perp$ direction; in particular, $\hat{p}_\alpha$ restricts to the variety Ω_τ. Now apply Ψ_τ (Fig. 3) to both sides of (37):

$$\begin{aligned}\Psi_\tau L(A)\,Q_\tau w &= \Psi_\tau Q_\tau\Big(\sum_\alpha p_\alpha R(X^\alpha)w\Big)\\ &= r_\Omega \mathscr{F}_\eta\Big(\sum_\alpha p_\alpha R(X^\alpha)w\Big) \qquad \text{(Fig. 3)}\\ &= r_\Omega\Big(\sum_\alpha \hat{p}_\alpha R(X^\alpha)^\wedge \mathscr{F}_\eta w\Big)\\ &= \sum_\alpha \hat{p}_\alpha R(X^\alpha)^\wedge r_\Omega \mathscr{F}_\eta w\\ &< \sum_\alpha \hat{p}_\alpha|_{\Omega_\tau} R(X^\alpha)^\wedge|_{\Omega_\tau} \Psi_\tau Q_\tau w. \end{aligned} \tag{38}$$

Thus $\Psi_\tau L(A)\,\Psi_\tau^{-1} \in \mathscr{P}(\Omega_\tau)$, and (a) follows.

On $\mathscr{C}^\infty(G,\tau)$, $L(A)$ commutes with $R(B)$, $\forall B\in \mathfrak{u}(\mathfrak{g})$, so $\Psi_\tau L(A)\,\Psi_\tau^{-1}$ commutes with $\Psi_\tau R(B)\,\Psi_\tau^{-1} = R(B)^\wedge|_{\Omega_\tau}$ (by Proposition 4.5). We now show that VDV^{-1} restricts to generic $\theta\in\Omega_\tau/K$, and then that these restrictions are scalar and constant on each θ; from this, we will prove (b).

Fix a strong Malcev basis $Y_1, ..., Y_n$ in $\mathfrak{g}$. In [8, Theorem 3.1], we described a partition of $\mathfrak{g}^*$ into $Ad^*(G)$-invariant layers $U_{e_1}, ..., U_{e_N}$ such that $\bigcup_{i=1}^k U_{e_i}$ is Zariski-open for each k (in particular , U_{e_1} is open). There is a unique index e such that U_e meets Ω_τ in a Zariski-open set. We showed that there are enough "e-central" elements in $\mathfrak{u}(\mathfrak{g})$ to separate $Ad^*(G)$-orbits in U_e—i.e., elements $A_1, ..., A_r$ such that

(i) A_i is e-central: $\pi_l(A_i)$ is scalar on $\mathscr{H}_{\pi_l}^\infty$ for all $l\in U_e$;

(ii) $\phi_i(l) = \pi_l(A_i)$ is rational nonsingular on all of U_e;

(iii) The functions $\phi_1, ..., \phi_r$ are G-invariant and separate G-orbits in U_e.

Obviously $\phi_i(l)$ is constant on each G-orbit since the condition $\pi_l(A) = cI$ on $\mathscr{H}_{\pi_l}^\infty$ is invariant under unitary equivalence $\pi_l \cong \pi_{l'}$. The ϕ_i are actually polynomials in $\mathbb{C}[\Omega_\tau]^K$: we have seen in Section 4 that $\tilde{\tau}(A_i)$ has C^∞ decomposition

$$\tilde{\tau}(A_i) = \int_{\Omega_\tau/K}^\oplus \hat{\pi}_\theta(A_i)\,d\nu(\theta) = \int_{\Omega_\tau/K}^\oplus \phi_i(\theta) I\,d\nu = \int_{\Omega_\tau/K}^\oplus R(A_i)^\wedge|_\theta\,d\nu,$$

and consequently $\phi_i = R(A_i)^\wedge|\Omega_\tau$ almost everywhere. Since ϕ_i is rational nonsingular and $R(A_i)^\wedge|_{\Omega_\tau} \in \mathscr{P}(\Omega_\tau)$, ϕ_i is a polynomial; it is actually $Ad^*(G)$-invariant, hence $Ad^*(K)$-invariant.

In (8) we showed that there is a K-invariant, open, dense, semialgebraic set $S\subseteq\Omega_\tau$ (automatically $\mu_\Omega(\Omega_\tau \sim S) = 0$, because of density) such that

> $G\cdot l\cap\Omega_\tau$ is a closed manifold in Ω_τ consisting of finitely many K-orbits

for all $l \in S$. Of course $U_e \cap S$ has the same properties, so we may assume $U_e \cap \Omega_\tau \supseteq S$. If we define layers for K-orbits in $\mathfrak{g}^*$ with respect to the admissible weak Malcev basis $X_1, ..., X_n$ through $\mathfrak{k}$, and if $U_d(K)$ is the layer that meets Ω_τ in a Zariski-open set, then $S \cap U_d(K)$ has the above properties and we may assume that $S \subseteq U_d(K) \cap U_e \cap \Omega_\tau$.

Let $\Phi = (\phi_1, ..., \phi_r)$: $S \to \mathbb{C}^r$ and

$$q = \dim \Omega_\tau - \text{generic} \dim\{K \cdot l : l \in \Omega_\tau\}.$$

The generic value of $\operatorname{rank}(d\Phi)_l$ on $U_e \cap \Omega_\tau$ is assumed on a Zariski-open set $\mathscr{Z} \subseteq U_e \cap \Omega_\tau$ and is $\leqslant q$ because Φ is constant on K-orbits in U_e. On the other hand, $\operatorname{rank}(d\Phi)_l$ is constant on $\mathscr{Z}$. Therefore $\mathscr{Z}$ foliates into submanifolds on which Φ is constant. If $l \in \mathscr{Z}$ and M_l is the leaf through l, M_l locally contains $K \cdot l$; since $K \cdot l$ is the component of $G \cdot l \cap \Omega_\tau$ through l and Φ separates G-orbits in U_e, M_l must locally coincide with $K \cdot l$ and their dimensions are equal. Thus $\operatorname{rank}(d\Phi)_l = q$ on $\mathscr{Z}$. Since K acts by diffeomorphisms on Ω_τ, $\mathscr{Z}$ is K-invariant and we may assume that $S \subseteq \mathscr{Z}$.

Thus if $l_0 \in S$ we can select $\{\phi_{i_1}, ..., \phi_{i_q}\} \subseteq \{\phi_i : 1 \leqslant i \leqslant r\}$ to get $\mathscr{C}^\infty$ coordinates on S near l_0. Assume for notational convenience that $i_j = j$ for $1 \leqslant j \leqslant q$. Then the coordinates rate $l \mapsto (s, t) \in \mathbb{R}^d \times \mathbb{R}^q$ (where $d = \dim K \cdot l$ for $l \in S$), where

(i) $t_i = \phi_i(l)$ for $1 \leqslant i \leqslant q$;

(ii) for any l near l_0 the orbit $K \cdot l$ is locally determined by setting $t = \text{constant}$.

We may assume that $s = t = 0$ at l_0.

By Proposition 4.5 we know that $R(A_i)^\wedge$ restricts to Ω_τ and to generic $\theta \in \Omega_\tau / K$. If we write A_0 for VDV^{-1}, we have seen that

$$0 = [A_0, R(A_i)^\wedge|_{\Omega_\tau}] = [A_0, \phi_i] = [A_0, t_i], \qquad 1 \leqslant i \leqslant q,$$

where we regard ϕ_i as a pointwise multiplication operator. Near l_0 we may express A_0 in the form

$$A_0 = \sum_{\alpha, \beta} c_{\alpha\beta} D_s^\alpha D_t^\beta, \qquad c_{\alpha,\beta} \in \mathscr{C}^\infty(\mathbb{R}^d \times \mathbb{R}^q),$$

and to say that A_0 restricts to $K \cdot l_0$ near l_0 means that $c_{\alpha\beta}(s, 0) \equiv 0$ if $\beta \neq 0$. But

$$[D_{t_i}^k, t_i] = k D_{t_i}^{k-1} \quad \text{for} \quad k \in \mathbb{Z}^+; \qquad [D_{t_j}^k, t_i] = 0 \quad \text{if} \quad i \neq j.$$

Hence

$$0 = [A_0, t_i] = \sum \beta_i c_{\alpha\beta} D_s^\alpha D_t^{\beta - e_i} \qquad (D_t^{e_i} = D_{t_i}).$$

Therefore $c_{\alpha\beta} \equiv 0$ near l_0 if $\beta_i \neq 0$. This applies to any $l_0 \in S$, so A_0 actually restricts to any K-orbit in S.

We have $A_0|_\theta \in \mathscr{P}(\theta)$, so $A_0|_\theta$ is continuous in the Fréchet topology of $\mathscr{S}(\theta)$, and

$$0 = [A_0|_\theta, R(\mathfrak{u}(\mathfrak{g}))^\wedge|_\theta] = [A_0, R(\mathfrak{u}(\mathfrak{g}))]^\wedge|_\theta \qquad \text{for generic } \theta.$$

By Proposition 4.4, $\Psi_\pi: \mathscr{H}_\pi^\infty \to \mathscr{S}(K \cdot l)$ is a topological isomorphism intertwining $\pi_{(l,\mathfrak{b})}(\mathfrak{u}(\mathfrak{g}))$ with $R\mathfrak{u}(\mathfrak{g}))^\wedge|_{K \cdot l}$. By Poulsen's irreducibility theorem (Theorem 3.5 of [25]), $A_0|_{K \cdot l}$ must be scalar for generic K-orbits in Ω_τ/K. Therefore there is a function p_D constant on K-orbits in Ω_τ such that $A_0|_\theta = p_D(\theta)$. Since A_0 is a polynomial coefficient differential operator, $p_D \in \mathbb{C}[\Omega_\tau]^K$, and this proves (b).

To prove (c) use the Malcev coordinates $t \mapsto K \cdot \eta(0, t)$ for our admissible weak Malcev basis to parametrize $K \backslash G$, and impose coordinates $\dot{t} \mapsto \Sigma_{i=1}^p f_i X_i^* + \sum_{i=1}^m \dot{t}_i X_{p+i}^*$, $\dot{t} \in \dot{\mathbb{R}}^m$, in Ω_τ. As noted after Proposition 4.5, the map $V = \Psi_\tau \circ J: \mathscr{S}(K \backslash G) \to \mathscr{S}(\Omega_\tau)$ then reduces to the ordinary Euclidean Fourier transform $\mathscr{F}_2: \mathscr{S}(\mathbb{R}^m) \to \mathscr{S}(\dot{\mathbb{R}}^m)$. Let $p \in \mathbb{C}[\Omega_\tau]^K$. In these coordinates pI transforms back to a constant coefficient differential operator $D = V^{-1} \circ pI \circ V$ on $K \backslash G$. But V also intertwines $\tilde{\tau}(A)$, $R(A)^\wedge|_{\Omega_\tau}$ for $A \in \mathfrak{u}(\mathfrak{g})$, and $R(A)^\wedge$ restricts to generic $Ad^*(K)$-orbits $\theta \in \Omega_\tau/K$, on which p is constant. Therefore $[pI, R(A)^\wedge|_{\Omega_\tau}] = 0$ for all A; hence $D \in \mathbb{D}_\tau(K \backslash G)$ and is an operator such that $VDV^{-1} = pI$. ▮

Although Ψ_τ is a Euclidean Fourier transform in suitably chosen coordinates, that is not the end of the story in computing examples. We have shown that $R(A)^\wedge = \mathscr{F}_\eta R(A) \mathscr{F}_\eta^{-1}$ restricts to Ω_τ; because $R(A)$ commutes with Q_τ, we get

$$\Psi_\tau \tau(A)\, \Psi_\tau^{-1} = \Psi_\tau R(A)\, \Psi_\tau^{-1} = R(A)^\wedge|_{\Omega_\tau}, \qquad \text{all} \quad A \in \mathfrak{u}(\mathfrak{g}). \tag{39}$$

If $D = \gamma(A) \in \mathbb{D}_\tau(K \backslash G)$ has a representative $A \in \mathfrak{u}(\mathfrak{g}, \tau)$ for which $L(A)$ commutes with Q_τ, a similar result holds for VDV^{-1}, and in fact this holds for all $D \in \mathbb{D}_\tau(K \backslash G)$ in all known examples.

5.3. Lemma. *Let* $(\mathfrak{g}, \mathfrak{k}, \mathfrak{b}, \chi)$ *satisfy* (20), *fix an admissible weak Malcev basis* $X_1, ..., X_p, ..., X_n$ *through* $\mathfrak{k}$, *and define* $V = \Psi_\tau \circ J: \mathscr{S}(K \backslash G) \to \mathscr{S}(\Omega_\tau)$, *where* Ψ_τ *is as in Proposition* 4.5 *and* $J: \mathscr{C}^\infty(K \backslash G) \to \mathscr{C}^\infty(G, \tau)$ *is the identification map for this basis. Then*:

(a) *For any* $A \in \mathfrak{u}(\mathfrak{g})$, $R(A)^\wedge = \mathscr{F}_\eta R(A) \mathscr{F}_\eta^{-1}$ *restricts to* Ω_τ, *and* $V\tilde{\tau}(A)\, V^{-1} = R(A)^\wedge|_{\Omega_\tau} \in \mathscr{P}(\Omega_\tau)$.

(b) *Let* $\mathfrak{u}(\mathfrak{g}, \tau)^0 = \{A \in \mathfrak{u}(\mathfrak{g}, \tau): L(A)$ *commutes with* $Q_\tau\}$, *where* $Q_\tau: \mathscr{S}(G) \to \mathscr{H}_\tau^\infty$ *is the averaging map* (9). *If* $D \in \mathbb{D}_\tau(K \backslash G)$ *has a representative* $A \in \mathfrak{u}(\mathfrak{g}, \tau)^0$, *so that* $D = \gamma(A)$, *then* $L(A)^\wedge = \mathscr{F}_\eta L(A) \mathscr{F}_\eta^{-1}$ *restricts to* Ω_τ *and* $VDV^{-1} = L(A)^\wedge|_{\Omega_\tau} \in \mathscr{P}(\Omega_\tau)$.

(c) *The subalgebra* $\mathfrak{u}(\mathfrak{g}, \tau)^0 \subseteq \mathfrak{u}(\mathfrak{g}, \tau)$ *contains* $\mathfrak{u}^K(\mathfrak{g}) = \{A \in \mathfrak{u}(\mathfrak{g})\colon [A, \mathfrak{k}] = (0)\}$. *If* $\mathfrak{u}(\mathfrak{g}, \tau)^0 + \mathfrak{u}(\mathfrak{g})\,\mathfrak{a}_\tau = \mathfrak{u}(\mathfrak{g}, \tau)$, *and in particular if* $\mathfrak{u}^K(\mathfrak{g}) + \mathfrak{u}(\mathfrak{g})\,\mathfrak{a}_\tau = \mathfrak{u}(\mathfrak{g}, \tau)$, *then every* $D \in \mathbb{D}_\tau(K\backslash G)$ *has a representative* $A \in \mathfrak{u}(\mathfrak{g}, \tau)^0$.

Proof. Part (a) follows easily from (39) and the definition of J. Part (c) follows from the remarks of Section 3 once we show that $\mathfrak{u}^K(\mathfrak{g}) \subseteq \mathfrak{u}(\mathfrak{g}, \tau)^0$. First note that if $A \in \mathfrak{u}^K(\mathfrak{g})$, then $L_k L(A) = L(A)\, L_k$ for $k \in K$, so if $\varphi \in \mathscr{C}^\infty(G)$ and $w \in \mathscr{S}(G)$ we get

$$\begin{aligned}
(Q_\tau L(A)w, \varphi)_{\mathscr{L}^2(G)} &= \int_G Q_\tau(L(A)w)(g)\,\overline{\varphi(g)}\,dg \\
&= \int_G \int_K L(A)\, w(kg)\,\overline{\chi(k)}\,dk\,\overline{\varphi(g)}\,dg \\
&= \int_K \left[\int_G L(A)\, L_k^{-1} w(g)\,\overline{\varphi(g)}\,dg\right] \overline{\chi(k)}\,dk \\
&= \int_K \left[\int_G L_k^{-1} w(g)\,\overline{L(A)^*\varphi(g)}\,dg\right] \overline{\chi(k)}\,dk \\
&= \int_G \int_K w(kg)\,\overline{\chi(k)}\,dk\,\overline{L(A)^*\varphi(g)}\,dg \\
&= (Q_\tau w, L(A)^*\varphi) = (L(A)\, Q_\tau w, \varphi).
\end{aligned}$$

Thus $A \in \mathfrak{u}(\mathfrak{g}, \tau)^0$.

For (b), assume that $A \in \mathfrak{u}(\mathfrak{g}, \tau)^0$. To see that $L(A)^\wedge$ restricts, suppose that $\psi \in \mathscr{S}(\mathfrak{g}^*)$ satisfies $\psi|_{\Omega_\tau} = 0$. There exists $w \in \mathscr{S}(G)$ with $\mathscr{F}_\eta w = \psi$ on $\mathfrak{g}^*$. Since Ψ_τ is an isometry and $\Psi_\tau \circ Q_\tau = r_\Omega \mathscr{F}_\eta$ (Fig. 3), we get

$$\|Q_\tau w\|_{\mathscr{H}_\tau} = \|\Psi_\tau Q_\tau w\|_{\mathscr{L}^2(\Omega_\tau)} = \|r_\Omega \mathscr{F}_\eta w\| = 0,$$

so that $Q_\tau w = 0$. Since $A \in \mathfrak{u}(\mathfrak{g}, \tau)^0$, $0 = L(A)\, Q_\tau w = Q_\tau L(A) w$. Reversing the argument gives

$$0 = r_\Omega \mathscr{F}_\eta L(A) w|_{\Omega_\tau} = L(A)^\wedge \psi|_{\Omega_\tau},$$

and $L(A)^\wedge$ restricts. The rest is calculation: let $\tilde{\varphi} \in \mathscr{S}(K\backslash G)$ and $w \in \mathscr{S}(G)$ such that $Q_\tau w = J\tilde{\varphi}$. Then we have

$$\begin{aligned}
VD\tilde{\varphi} = \Psi_\tau J\gamma(A)\tilde{\varphi} &= \Psi_\tau L(A)\, J\tilde{\varphi} \\
&= \Psi_\tau Q_\tau L(A) w \qquad (\text{since } A \in \mathfrak{u}(\mathfrak{g}, \tau)^0) \\
&= r_\Omega \mathscr{F}_\eta L(A) w = L(A)^\wedge|_{\Omega_\tau}(V\tilde{\varphi}),
\end{aligned}$$

since $L(A)^\wedge$ restricts. ∎

The conditions of (c) are satisfied whenever the pair $(\mathfrak{g}, \mathfrak{k})$ is reductive (i.e., when there is a space $\mathfrak{m}$ complementary to $\mathfrak{k}$ in $\mathfrak{g}$ with $[\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}$), and in particular whenever $\dim \mathfrak{k} = 1$; see [8]. In every known example where τ contains representations in general position, $\mathfrak{u}(\mathfrak{g}, \tau)^0 + \mathfrak{u}(\mathfrak{g})\mathfrak{a}_\tau = \mathfrak{u}(\mathfrak{g}, \tau)$.

Remark. One consequence of Theorem 5.2 is that $\mathbb{D}_\tau(K\backslash G)$ is commutative when τ has finite multiplicity. We already know that from [8], but in the present situation we can interpret this in a striking way. If $K\backslash G$ is given suitably chosen polynomial coordinates $t \mapsto K\eta(0, t)$, the element of $\mathbb{D}_\tau(K\backslash G)$ become constant coefficient differential operators on $\mathbb{R}^m$ and obviously commute! This follows because $\Psi_\tau \circ J$ is the Euclidean Fourier transform in these coordinates, as explained in the Note following the statement of Proposition 4.5.

6. Tempered Fundamental Solutions

Given $(\mathfrak{g}, \mathfrak{b}, \mathfrak{k}, \chi)$ satisfying conditions (20), let us fix Haar measures m_G, m_B, m_K, $m_{B\cap K}$ and an admissible weak Malcev basis $X_1, ..., X_n$ through $\mathfrak{k}$, and use the polynomial cross-section for $K\backslash G$ to get the corresponding map $J: \mathscr{S}(K\backslash G) \to \mathscr{S}_\tau \subseteq \mathscr{H}_\tau^\infty$. In this section we model $\tau = \operatorname{ind}(K\uparrow G, \chi)$ as an action $\tilde{\tau}$ on $L^2(K\backslash G)$ via J.

6.1. Theorem. *Let* $(\mathfrak{g}, \mathfrak{k}, \mathfrak{b}, \chi)$ *satisfy the conditions* (20). *Then every* $D \in \mathbb{D}_\tau(K\backslash G)$ *has a tempered fundamental solution* $u \in \mathscr{S}(K\backslash G)$: $Du = \xi_{Ke}$ *where* ξ_{Ke} *is the point mass at Ke in* $K\backslash G$.

Proof. We could apply the ideas of [1], following [2], once we have our realization of $\mathbb{D}_\tau(K\backslash G)$ as $\mathbb{C}[\Omega_\tau]^K$. However, we have seen that if we impose suitable coordinates on $K\backslash G$ and Ω_τ, then $V = \Psi_\tau \circ J: \mathscr{S}(K\backslash G) \to \mathscr{S}(\Omega_\tau)$ is just a Fourier transform, and $VDV^{-1} = p_D(l)I$ with $p_D \in \mathbb{C}[\Omega_\tau]^K$. Therefore D itself must be a constant coefficient differential operator in the coordinates on $K\backslash G$ and, as such, has a tempered fundamental solution. ∎

If $1d\mu_\Omega$ is the distribution $\varphi \mapsto \int \varphi(l)\, d\mu_\Omega(l)$ on $\mathscr{S}(\Omega_\tau)$, it is obvious that $V^t(1d\mu_\Omega) = \xi_{Ke}$ on $\mathscr{S}(K\backslash G)$. We thus get a decomposition

$$\xi_{Ke} = \int_{\Omega_\tau/K} V^t(1\, d\mu_\theta)\, d\nu(\theta) \tag{40}$$

corresponding to $\mu_\Omega = \int_{\Omega_\tau/K} \mu_\theta\, d\nu(\theta)$. Since generic K-orbits in Ω_τ need not be flat, we can expect the component Fourier transforms $V^t(1\, d\mu_\theta)$ to be fairly complicated elements in $\mathscr{S}'(K\backslash G)$; see [6]. If we define D^t on

$\xi \in \mathscr{S}'(K\backslash G)$ by $\langle D^t\xi, \bar{\phi}\rangle = \langle \xi, D\bar{\phi}\rangle$, then $\langle D^t\xi_{Ke}, \bar{\phi}\rangle = D\bar{\phi}(Ke)$ and the components in (40) are simultaneous eigendistributions for all $D \in \mathbb{D}_\tau(K\backslash G)$ because if $VDV^{-1} = p_D(l)I$ with $p_D \in \mathbb{C}[\Omega_\tau]^K$, we get

$$\langle D^t V^t(1d\mu_\theta), \bar{\phi}\rangle = \langle 1d\mu_\theta, p_D(l)\cdot V\bar{\phi}\rangle = \langle p_D(\theta)(1d\mu_\theta), \bar{\phi}\rangle. \quad (41)$$

For a generic set of θ these distributions are linearly independent: the K-invariant rational functions $\mathbb{C}(\Omega_\tau)^K$ separate generic K-orbits since there is a rational parametrization of these orbits. Hence the K-invariant polynomials separate them too because $\mathbb{C}(\Omega_\tau)^K$ consists of quotients of elements in $\mathbb{C}[\Omega_\tau]^K$, by standard arguments.

There is a close connection between the components $V^t(1d\mu_\theta)$ in (40) and the distribution matrix coefficients in $\mathscr{S}'(G)$ associated with the χ-covariant vectors a_τ, $a_{\pi(l,\mathfrak{b})} \in \mathscr{H}_\tau^{-\infty}$ defined in Section 2. Given any unitary representation ρ and $a \in \mathscr{H}_\rho^{-\infty}$ we defined the matrix element $T = T_{\rho,a} \in \mathscr{S}'(G)$ in Section 2,

$$\langle T, w\rangle = (\rho'(w)a, a) = \langle a, Q_{\rho,a}(\tilde{w})\rangle^-, \qquad \text{all} \quad w \in \mathscr{S}(G),$$

where $Q_{\rho,a}\colon \mathscr{S}(G) \to \mathscr{H}_\rho^\infty$ is given by $j(Q_{\rho,a}(\tilde{w})) = \rho'(w)a$. For $\rho = \tau$ and $a = a_\tau$, $a_\tau(\phi) = \overline{\phi(e)}$, as in Example 2.2, we have

$$\langle T_{\tau,a_\tau}, w\rangle = Q_\tau(\tilde{w})(e) = \int_K w(k)\,\chi(k)\,dk, \qquad \text{i.e.,} \quad T_\tau = \chi m_K. \quad (42)$$

If $l \in \Omega_\tau$ and $\mathfrak{m}$ is any polarization, and if we fix m_M, $m_{M\cap K}$ in addition to previous normalizations, we get $\pi = \pi_{(l,\mathfrak{m})} \in G^\wedge$, $a_\pi \in \mathscr{H}_\pi^{-\infty}$ such that $\pi'(k)\,a_\pi = \chi(k)\,a_\pi$, and a Fréchet continuous surjective map $Q_\pi\colon \mathscr{S}(G) \to \mathscr{H}_\pi^\infty$ via $Q_\pi(w) = j^{-1}(\pi'(\tilde{w})\,a_\pi)$, as in Example 2.3. Then

$$\langle T_{\pi,a_\pi}, w\rangle = \langle a_\pi, Q_\pi(\tilde{w})\rangle^-.$$

Following the discussion in [13], this matrix element only gets scaled by some $\lambda > 0$ if we

(i) Change the polarization $\mathfrak{m}$;

(ii) Pass from $(l, \mathfrak{m})$ to $\mathrm{Ad}^*(k)l$, $Ad(k)\mathfrak{m})$ for some $k \in K$.

Thus, up to a positive scalar, T_{π,a_π} depends only on the K-orbit $\theta = K\cdot l$ in Ω_τ. Fixing a flat cross-section $E_d \subseteq \Omega_\tau$ for generic K-orbits $\theta \in \Omega_\tau/K$, we take a representative l in $\theta \cap E_d$ and the particular polarization $\mathfrak{m} = \mathfrak{b}$; this fixes a normalization T_θ of T_{π,a_π}.

6.2. LEMMA. *Given* $(\mathfrak{g}, \mathfrak{k}, \mathfrak{b}, \chi)$ *satisfying conditions* (20), *fix a flat cross-section* E_d *for generic orbits in* Ω_τ/K, *and fix normalizations of* m_G, m_K, m_B, *and $m_{B\cap K}$. For each l in the generic set $U_d \cap E_d \approx \Omega_\tau/K$, let $\pi_\theta = \pi(l, \mathfrak{b})$ $a_\theta = a_{\pi(l,\mathfrak{b})}$ as in Example 2.3, and let $\mu_\Omega = \int_{E_d}^{\oplus} \mu_\theta \, d\nu(\theta)$ be the disintegration associated with the smooth decomposition of τ as in Section 3 (Fig. 5). Write $\tilde{w}(g) = w(g^{-1})$. Then the matrix elements $T_\tau = T_{\tau, a_\tau}$* and *$T_\theta = T_{\pi, a_\pi}$ satisfy*

(a) $\langle T_\theta, w\rangle = \int_\theta r_\theta \mathscr{F}_\eta \tilde{w} \, d\mu_\theta$ *for generic* $\theta \in \Omega_\tau/K$;

(b) $\langle T_\tau, w\rangle = \int_{\Omega_\tau} r_\Omega \mathscr{F}_\eta \tilde{w} \, d\mu_\Omega$;

(c) $T_\tau = \int_{\Omega_\tau/K}^{\oplus} T_\theta \, d\nu(\theta)$ *in the sense that* $\langle T_\tau, w\rangle = \int_{\Omega_\tau/K} \langle T_\theta, w\rangle \, d\nu(\theta)$ *(absolutely convergent), for all* $w \in \mathscr{S}(G)$.

Proof. From Section 2 we recall that for any representation ρ,

$$\rho'(g)\, j(\xi) = j(\rho(g)\xi), \qquad \forall \xi \in \mathscr{H}_\rho^\infty, \quad g \in G,$$

and hence $\rho'(w)\, j(\xi) = j(\rho(w)\xi)$ for any $w \in \mathscr{S}(G)$. We saw (Lemma 2.1) that $T_\theta(u^* * v) = (\pi'(u^* * v)\, a_\pi, a_\tau) = (Q_\pi(\tilde{v}), Q_\pi(\tilde{u}))_{\mathscr{H}_\pi}$; hence, as in Fig. 2,

$$T_\theta(u^* * v) = (\Psi_\pi Q_\pi \tilde{v}, \Psi_\pi Q_\pi \tilde{u}) = \int r_\theta \mathscr{F}_\eta \tilde{v} \cdot (r_\theta \mathscr{F}_\eta \tilde{u})^- \, d\mu_\theta.$$

If $u_n \geqslant 0$ is an $\mathscr{L}^1$ approximate identity with $u_n \in \mathscr{S}(G)$, $\mathrm{supp}(u_n)$ compact and decreasing to $\{e\}$, then the same is true of u_n^* and $\tilde{u}_n$; thus $u_n^* * v \to v$ and $v * u_n \to v$ in the Fréchet topology since the action $G \times \mathscr{S}(G) \to \mathscr{S}(G)$ is jointly continuous (see [5, Appendix 1]). Furthermore, $\mathscr{F}_\eta \tilde{u}_n$ is uniformly bounded on $\mathfrak{g}^*$ and converges pointwise to 1, so by dominated convergence and Fréchet continuity of T_θ we get (absolutely convergent integrals)

$$T_\theta(v) = \int r_\theta \mathscr{F}_\eta \tilde{v} \, d\mu_\theta;$$

$$T_\theta(v^*) = \int r_\theta \mathscr{F}_\eta \bar{v} \, d\mu_\theta \qquad \text{all} \quad v \in \mathscr{S}(G), \text{ generic } \theta = \Omega_\tau/K; \tag{43}$$

$$T_\theta(u^* * v) = \int r_\theta \mathscr{F}_\eta \tilde{v} \cdot \overline{r_\theta \mathscr{F}_\eta \tilde{u}} \, du_\theta.$$

Similarly,

$$T_\tau(v) = \int r_\Omega \mathscr{F}_\eta \tilde{v} \, d\mu_\Omega;$$

$$T_\tau(v^*) = \int r_\Omega \mathscr{F}_\eta \bar{v} \, d\mu_\Omega; \tag{44}$$

$$T_\tau(u^* * v) = \int r_\Omega \mathscr{F}_\eta \tilde{v} \cdot (r_\Omega \mathscr{F}_\eta \tilde{u})^- \, d\mu_\Omega.$$

Thus we get an absolutely convergent integral decomposition

$$T_\tau(w) = \int r_\theta \mathscr{F}_\eta \tilde{w}\, d\mu_\Omega = \int_{\Omega_\tau/K} \left[\int_\theta r_\theta \mathscr{F}_\eta \tilde{w}\, d\mu_\theta \right] d\nu(\theta)$$

$$= \int_{\Omega_\tau/K} T_\theta(w)\, d\nu(\theta). \quad \blacksquare$$

Note. There is no reason to expect that $\mathscr{F}_\eta(u^* * v) = \overline{\mathscr{F}_\eta u} \cdot \mathscr{F}_\eta v$ or $\mathscr{F}_\eta(u^*) = \overline{\mathscr{F}_\eta u}$ for our Malcev–Fourier transform, so the last integrals in (43) and (44) are particularly interesting.

Since we have

$$\begin{aligned} \langle T_\theta, w \rangle &= \langle 1 d\mu_\theta, r_\theta \mathscr{F}_\eta \tilde{w} \rangle = \langle 1 d\mu_\theta, r_\Omega \mathscr{F}_\eta \tilde{w} \rangle \\ &= \langle 1\, d\mu_\theta, \Psi_\tau Q_\tau \tilde{w} \rangle \\ &= \langle V_\tau(1 d\mu_\theta), J^{-1} Q_\tau \tilde{w} \rangle, \qquad \forall w \in \mathscr{S}(G), \end{aligned}$$

the results above show the T_θ are related to the Fourier transforms $V_\tau(1 d\mu_\theta)$ on $K\backslash G$ for generic θ. Recall (Corollary4.6) that $J^{-1}Q_\tau : \mathscr{S}(G) \mapsto \mathscr{S}(K\backslash G)$ is Fréchet continuous.

5. Examples

Examples in which $m(\pi) \not\equiv 1$ are particularly interesting in the way that distinct K-orbits lying in the same G-orbit are distinguished. In every example we have $\mathfrak{u}(\mathfrak{g}, \tau)^0 + \mathfrak{u}(\mathfrak{g})\, \mathfrak{a}_\tau = \mathfrak{u}(\mathfrak{g}, \tau)$, and in most we also have $\mathfrak{u}^K(\mathfrak{g}) + \mathfrak{u}(\mathfrak{g})\, \mathfrak{a}_\tau = \mathfrak{u}(\mathfrak{g}, \tau)$, but this can fail if Ω_τ is not in general position.

The first few examples involve the algebra $\mathfrak{g}$ with basis Z, Y, X, W and commutators $[W, X] = Y$, $[W, Y] = Z$. If $l = \dot{z}Z^* + \cdots + \dot{w}W^*$ in $\mathfrak{g}^*$, its Ad^*G orbit is given by

$$\begin{aligned} Ad^*_{\exp(zZ + \cdots + wW)} l = \dot{z}Z^* + (\dot{y} - w\dot{z})\, Y^* + \left(\dot{x} - w\dot{y} + \frac{w^2}{2} \dot{z} \right) X^* \\ + \left(\dot{w} + x\dot{y} + \left[y - \frac{wx}{2} \right] \dot{z} \right) W^*. \end{aligned} \tag{45}$$

In those examples with $\mathfrak{k} \supseteq \mathbb{R}X$ the weak Malcev coordinates can be read from those for $\mathfrak{k} = \mathbb{R}X$; in the latter case take basis $X_1, \ldots, X_4 = X, Z, Y, W$. Then $\eta(x, z, y, w) = \exp(xX) \cdots \exp(wW)$ and the group law in these coordinates is

$$\begin{aligned} &(x, z, y, w)(x', z', y', w') \\ &\quad = (x + x', z + z' + wy' + \tfrac{1}{2}w^2 x', y + y' + wx', w + w'). \end{aligned} \tag{46}$$

From this we get the left actions on functions $\phi\circ\eta$, $\phi\in\mathscr{S}(G)$,

$$\begin{aligned} L(Z) &= -D_z & L(Z)^\wedge &= -2\pi i\dot z\\ L(Y) &= -D_y & L(Y)^\wedge &= -2\pi i\dot y\\ L(X) &= -D_x & L(X)^\wedge &= -2\pi i\dot x\\ L(W) &= -D_w - xD_y - yD_z & L(W)^\wedge &= -2\pi i\dot w + \dot y D_{\dot x} + \dot z D_{\dot y}, \end{aligned} \tag{47}$$

where $L(A)^\wedge = \mathscr{F}_\eta\circ L(A)\circ\mathscr{F}_\eta^{-1}$ and the Malcev–Fourier transform $\mathscr{F}_\eta$ is

$$\mathscr{F}_\eta\phi(l) = \int_{\mathbb{R}^4} e^{-2\pi i\sum x_i\dot x_i}\phi\circ\eta(x)\,dx,$$

as in (21). Recall [11] that $\mathfrak{zu}(\mathfrak{g}) = \mathbb{C}[Z, Y^2-2XZ]$ and note that $L(Y^2-2XZ)^\wedge = -4\pi^2(\dot y^2 - 2\dot x\dot z)$. The ideal $\mathfrak{b} = \mathbb{R}\text{-span}\{X, Y, Z\}$ polarizes generic $l\in\mathfrak{g}^*$, those with $\dot z\neq 0$.

7.1. EXAMPLE. $\mathfrak{g}$ as above, $\mathfrak{k} = \mathbb{R}X$. Any $f = f_zZ^* + \cdots + f_wW^*$ satisfies $\langle f, [\mathfrak{k}, \mathfrak{k}]\rangle = 0$, and $\Omega_\tau = f_xX^* + \mathbb{R}\text{-span}\{Z^*, Y^*, W^*\}$. For $l\in\Omega_\tau$, $l\approx(\dot z, \dot y, f_x, \dot w)$, we get

$$\begin{aligned} &K\cdot l = (\dot z, \dot y, f_x, \mathbb{R}) \qquad \text{if } \dot z\neq 0,\ \dot y\neq 0 \text{ (generic in } \Omega_\tau),\\ &G\cdot l = \{(\dot z, t, q(t), \mathbb{R}) : t\in\mathbb{R}\} \qquad \text{if } \dot z\neq 0, \text{ where } q(t) = q(t;\dot z, \dot y)\\ &\qquad\text{is quadratic in } t,\\ &G\cdot l\cap\Omega_\tau = \{(\dot z, t, q(t), \mathbb{R}) : q(t) = f_x\}. \end{aligned} \tag{48}$$

Thus $G\cdot l\cap\Omega_\tau$ consists of two K-orbits, for generic $l\in\Omega_\tau$; $m(\pi)=2$ and (20(i)) is satisfied. From (48) it is clear that $\mathbb{C}[\Omega_\tau]^K = \mathbb{C}[\dot z, \dot y]$. We will show that $\mathfrak{u}(\mathfrak{g},\tau) = \mathbb{C}[Z, Y]\oplus\mathfrak{u}(\mathfrak{g})\,\mathfrak{a}_\tau$, and since $\mathbb{C}[Z, Y]\subseteq\mathfrak{u}^K(\mathfrak{g})\subseteq\mathfrak{u}(\mathfrak{g},\tau)^0$ we get $\mathbb{D}_\tau(K\backslash G)\cong L(\mathbb{C}[Z, Y])^\wedge|_{\Omega_\tau} = \mathbb{C}[\dot z, \dot y] = \mathbb{C}[\Omega_\tau]^K$ using (47) and Lemma 5.3. Notice that not all operators in $\mathbb{D}_\tau(K\backslash G)$ arise from $\mathfrak{zu}(\mathfrak{g})$. In our picture, $U = Y^2-2XZ$ has $\gamma(U)^\wedge = L(U)^\wedge|\Omega_\tau = -4\pi^2(\dot y^2 - 2f_x\dot z)$, the restriction of an $Ad^*(G)$-invariant polynomial on $\mathfrak{g}^*$. It cannot separate K-orbits in Ω_τ that lie in the same G-orbit; for that we need $\gamma(Y)^\wedge = L(Y)^\wedge|\Omega_\tau = -2\pi i\dot y$.

Since $\dim\mathfrak{k}=1$, we have $\mathfrak{u}(\mathfrak{g},\tau) = \mathfrak{u}^K(\mathfrak{g}) + \mathfrak{u}(\mathfrak{g})\,\mathfrak{a}_\tau$, by Corollary 4.5 of [8], so it suffices to show that $\mathfrak{u}^K(\mathfrak{g}) = \mathbb{C}[Z, Y, X]$. The inclusion $\supseteq$ is obvious, and if $A = \sum c_{\alpha m}Z^{\alpha_1}Y^{\alpha_2}X^{\alpha_3}W^m\in\mathfrak{u}^K(\mathfrak{g})$ we have

$$0 = [A, X] = \sum mc_{\alpha m}Z^{\alpha_1}Y^{\alpha_2+1}X^{\alpha_3}W^{m-1},$$

which implies that $c_{\alpha m} = 0$ if $m > 0$.

7.2. EXAMPLE. $\mathfrak{g}$ as above, $\mathfrak{k}=\mathbb{R}Z+\mathbb{Z}X$, $\chi=\chi_f$ with $f_z\neq 0$; then $\Omega_\tau=f_zZ^*+f_xX^*+\mathbb{R}\text{-span}\{Y^*, W^*\}$ is in general position. Here $K\cdot l=(f_z, \dot{y}, f_x, \mathbb{R})$ if $\dot{y}\neq 0$ (generic in Ω_τ), the multiplicities are $m(\pi)=2$, and $\mathbb{C}[\Omega_\tau]^K=\mathbb{C}[\dot{y}]$. In this case $\mathfrak{u}(\mathfrak{g},\tau)=\mathfrak{u}^K(\mathfrak{g})+\mathfrak{u}(\mathfrak{g})\,\mathfrak{a}_\tau=\mathbb{C}[Y]\oplus\mathfrak{u}(\mathfrak{g})\,\mathfrak{a}_\tau$ and

$$\mathbb{D}_\tau(K\backslash G)\cong\gamma(\mathfrak{u}^K(\mathfrak{g}))\cong L(\mathfrak{u}^K(\mathfrak{g}))^\wedge|_{\Omega_\tau}=\mathbb{C}[\dot{y}]=\mathbb{C}[\Omega_\tau]^K.$$

Once again, elements in $\gamma(\mathfrak{z}\mathfrak{u}(\mathfrak{g}))$ do not account for all of $\mathbb{D}_\tau(K\backslash G)$, or separate points in Ω_τ/K.

7.3. Example. $\mathfrak{g}$ as above, $\mathfrak{k}=\mathbb{R}Y+\mathbb{R}X$, $\chi=\chi_f$ with $f\in\mathfrak{g}^*$ arbitrary. Then $\Omega_\tau=f_yY^*+f_xX^*+\mathbb{R}\text{-span}\{Z^*, W^*\}$ is always in general position. For generic $l\in\Omega_\tau$ ($\dot{z}=\langle l, Z\rangle\neq 0$) we have $K\cdot l=(\dot{z}, f_y, f_x, \mathbb{R})$ and $m(\pi)\equiv 1$. Clearly $\mathbb{C}[\dot{z}]=\mathbb{C}[\Omega_\tau]^K$. A fairly easy calculation shows that $\mathfrak{u}(\mathfrak{g},\tau)=\mathbb{C}[Z]\oplus\mathfrak{u}(\mathfrak{g})\,\mathfrak{a}_\tau$, so $\mathfrak{u}(\mathfrak{g},\tau)=\mathfrak{u}^K(\mathfrak{g})+\mathfrak{u}(\mathfrak{g})\,\mathfrak{a}_\tau=\mathfrak{z}\mathfrak{u}(\mathfrak{g})+\mathfrak{u}(\mathfrak{g})\,\mathfrak{a}_\tau$ even though the pair $(\mathfrak{g},\mathfrak{k})$ is not reductive. Thus

$$\mathbb{D}_\tau(K\backslash G)=\mathbb{C}[\gamma(Z)]\cong L(\mathbb{C}[Z])^\wedge|_{\Omega_\tau}=\mathbb{C}[\dot{z}]=\mathbb{C}[\Omega_\tau]^K.$$

7.4. EXAMPLE. $\mathfrak{g}$ as above, $\mathfrak{k}=\mathbb{R}\text{-span}\{Z, Y, X\}$ with $f_z\neq 0$. Then $\Omega_\tau=f_zZ^*+f_yY^*+f_xX^*+\mathbb{R}W^*$ is in general position, $\mathfrak{k}=\mathfrak{b}$, $\tau\cong\pi_{(l,\mathfrak{b})}$ is irreducible for all $l\in\Omega_\tau$, and $K\cdot l=\Omega_\tau$. We have $\mathbb{C}[\Omega_\tau]^K=\mathbb{C}I$, $\mathfrak{u}(\mathfrak{g},\tau)=\mathbb{C}I+\mathfrak{u}(\mathfrak{g})\mathfrak{a}_\tau=\mathfrak{u}^K(\mathfrak{g})+\mathfrak{u}(\mathfrak{g})\,\mathfrak{a}_\tau$ and $\mathbb{D}(K\backslash G)\cong\mathbb{C}I$.

7.5. EXAMPLE. $\mathfrak{g}$ as above, $\mathfrak{k}=\mathbb{R}Y$. Here the Malcev coordinates in G are set up using the basis Y, Z, X, W. Except for a permutation of the x and y entries, the multiplication law is the same as (46), and (47) remains valid. Now $\Omega_\tau=f_yY^*+\mathbb{R}\text{-span}\{Z^*, X^*, W^*\}$ is always in general position and $K\cdot l=(\dot{z}, f_y, \dot{x}, \mathbb{R})=G\cdot l\cap\Omega_\tau$ if $\dot{z}\neq 0$, so $m(\pi_l)\equiv 1$ for generic $l\in\Omega_\tau$. Clearly $\mathbb{C}[\dot{z},\dot{x}]=\mathbb{C}[\Omega_\tau]^K$. Since $\dim\mathfrak{k}=1$ we know that $\mathfrak{u}(\mathfrak{g},\tau)=\mathfrak{u}^K(\mathfrak{g})+\mathfrak{u}(\mathfrak{g})\,\mathfrak{a}_\tau$, and it is easy to show that $\mathfrak{u}^K(\mathfrak{g})=\mathbb{C}[Z, Y, X]$. Thus we get

$$\mathbb{D}_\tau(K\backslash G)=\gamma(\mathbb{C}[Z, X])\cong L(\mathbb{C}[Z, X])^\wedge|_{\Omega_\tau}=\mathbb{C}[\dot{z},\dot{x}]=\mathbb{C}[\Omega_\tau]^K.$$

In this multiplicity-free example, $\mathbb{D}_\tau(K\backslash G)\neq\gamma(\mathfrak{z}\mathfrak{u}(\mathfrak{g}))\cong\mathbb{C}[\dot{z},\dot{z}\dot{x}]$.

In the next examples we have $\mathfrak{g}$ as above but $W\in\mathfrak{k}$; we need the group law in a Malcev basis that starts with W. If $X_1, ..., X_4=W, Z, Y, X$ the group law is

$$\begin{aligned}(w, z, y, x)&(w', z', y', x')\\ &=(w+w', z+z'-w'y+\tfrac{1}{2}(w')^2x, y+y'-w'x, x+x')\end{aligned}\tag{49}$$

so in these coordinates for G, and dual coordinates $l=\dot{z}Z^*+\cdots+\dot{w}W^*$ in $\mathfrak{g}^*$,

$$L(X)=-\mathrm{D}_x+wDy-\frac{1}{2}w^2D_z \qquad L(X)^\wedge=-2\pi i\dot{x}-\dot{y}D_{\dot{w}}-\frac{1}{2}\left(\frac{1}{2\pi i}\right)\dot{z}D_{\dot{w}}^2$$

$$L(Y)=-D_y+wD_z \qquad L(Y)^\wedge=-2\pi i\dot{y}-\dot{z}D_{\dot{w}}$$

$$L(X)=-D_z \qquad L(Z)^\wedge=-2\pi i\dot{z}$$

$$L(W)=-D_w \qquad L(W)^\wedge=-2\pi i\dot{w}$$

After considerable cancellation we get $L(Y^2-2XZ)^\wedge=-4\pi^2(\dot{y}^2-2\dot{x}\dot{z})$, $L(Z)^\wedge=-2\pi i\dot{z}$ for the generators of $\mathfrak{z}\mathfrak{u}(\mathfrak{g})$.

7.6. Example. $\mathfrak{g}$ as above, $\mathfrak{k}=\mathbb{R}W$. For any f, $\Omega_\tau=f_wW^*+\mathbb{R}\text{-span}\{Z^*, Y^*, X^*\}$ is in general position and

$$K\cdot l=\{(\dot{z}, t, q(t), f_w)\colon t\in\mathbb{R}\}=G\cdot l\cap\Omega_\tau \qquad \text{if} \quad \dot{z}=\langle l, Z\rangle\neq 0$$
$$(\text{generic in } \Omega_\tau),$$

where $q(t)$ is quadratic in t. Thus $m(\pi_l)\equiv 1$. The algebra $\mathbb{C}[\Omega_\tau]^K$ is best described by its restriction to the flat cross-section $E_d=f_wW^*+\mathbb{R}\text{-span}\{Z^*, X^*\}$ for generic K-orbits: $\mathbb{C}[\Omega_\tau]^K\cong$ the algebra of $\phi\in\mathbb{C}[E_d]$ such that ϕ extends to a K-invariant *polynomial* on Ω_τ (rather than a rational function) under the substitution $\phi\mapsto\phi\circ P_d(l)$, where

$$P_d(l)=P_d(\dot{z}, \dot{y}, \dot{x}, f_w)=\dot{z}Z+0\cdot Y^*+\left(\dot{x}-\frac{\dot{y}^2}{2\dot{z}}\right)X^*+f_wW^*\in E_d$$

for $l\in\Omega_\tau$. (Applying (45) we see that $\{P_d(l)\}=K\cdot l\cap E_d$ is the K-orbit representative in E_d if $\dot{z}=\langle l, Z\rangle\neq 0$.) The calculation is quite similar to one in [11, pp. 326–327], so we suppress the details. In the end we find that $\mathbb{C}[\Omega_\tau]^K\cong\mathbb{C}[\Omega_\tau]^K|_{E_d}=\mathbb{C}[\dot{z}, \dot{z}\dot{x}]$. In the next paragraph we will verify that $\mathfrak{u}(\mathfrak{g}, \tau)=\mathfrak{z}\mathfrak{u}(\mathfrak{g})+\mathfrak{u}(\mathfrak{g})\mathfrak{a}_\tau$ $(=\mathfrak{u}^K(\mathfrak{g})+\mathfrak{u}(\mathfrak{g})\,\mathfrak{a}_\tau$ since $\mathfrak{u}^K(\mathfrak{g})\supset\mathfrak{z}\mathfrak{u}(\mathfrak{g}))$, so by Lemma 5.3 we get $\mathbb{D}_\tau(K\backslash G)=\gamma(\mathfrak{z}\mathfrak{u}(\mathfrak{g}))\cong L(\mathfrak{z}\mathfrak{u}(\mathfrak{g}))^\wedge|_{\Omega_\tau}$. From what we know of the generators of the center restricted to E_d we see that the latter is all of $\mathbb{C}[\Omega_\tau]^K$.

If one tries to compute $\mathfrak{u}(\mathfrak{g}, \tau)$ using Lemma 4.2 of [8] and monomials in X, Y, Z, $U_W=W+2\pi i\langle f, W\rangle I$, the resulting recursion relations are a mess. It is easier to use the known structure of $\mathfrak{z}\mathfrak{u}(\mathfrak{g})=\mathbb{C}[Z, U]$ where $U=Y^2-2XZ$. It is easily seen that any $A\in\mathfrak{u}(\mathfrak{g})$ has a unique expansion in the form

$$A=\sum c_{\alpha n}Z^{\alpha_1}U^{\alpha_2}YX^{\alpha_3}U_W^n+\sum d_{\alpha n}Z^{\alpha_1}U^{\alpha_2}X^{\alpha_3}U_W^n.$$

If $A \in \mathfrak{u}(\mathfrak{g}, \tau)$, then by Lemma 4.2(b) of [8] we may assume that U_W^n does not appear in A with $n > 0$; then we must determine the choices of $c_\alpha = c_{\alpha 0}$, $d_\alpha = d_{\alpha 0}$ for which $[W, A] \equiv 0 \bmod \mathfrak{u}(\mathfrak{g})\, \mathfrak{a}_\tau$. Since $[W, Y^k] = kZY^{k-1}$, $[W, X^k] = kYX^{k-1}$ for $k \geqslant 1$ we get

$$\begin{aligned}[W, A] = {} & \sum_{\alpha_1 \geqslant 1} c_{\alpha_1 - 1, \alpha_2, \alpha_3} Z^{\alpha_1} U^{\alpha_2} X^{\alpha_3} \\ & + \sum_{\alpha_2 \geqslant 1} (\alpha_3 + 1)\, c_{\alpha_1, \alpha_2 - 1, \alpha_3 + 1} Z^{\alpha_1} U^{\alpha_2} X^{\alpha_3} \\ & + \sum_{\alpha_1 \geqslant 1} 2\alpha_3 c_{\alpha_1 - 1, \alpha_2, \alpha_3} Z^{\alpha_1} U^{\alpha_2} X^{\alpha_3} \\ & + \sum (\alpha_3 + 1)\, d_{\alpha_1, \alpha_2, \alpha_3} Z^{\alpha_1} U^{\alpha_2} Y X^{\alpha_3}\end{aligned}$$

(using $Y^2 = U + 2XZ$ to simplify certain terms). This whole sum lies in $\mathbb{C}\text{-span}\{Z^{\alpha_1} Y^{\alpha_2} X^{\alpha_3}\}$, which, by Lemma 4.2 of [8], is a vector space cross-section for $\mathfrak{u}(\mathfrak{g})\, \mathfrak{a}_\tau$, so $[W, A] \equiv 0 \Rightarrow [W, A] = 0$ in $\mathfrak{u}(\mathfrak{g})$ and all coefficients are zero when like monomials are combined. For the d-coefficients, $d_{\alpha_1, \alpha_2, \alpha_3 + 1} = 0$, so the d-sum is in $\mathbb{C}[Z, U]$. After regrouping the c-coefficients one gets

$$\begin{gathered} c_{\alpha_1, 0, \alpha_3} = 0, \qquad \text{all} \quad \alpha_1, \alpha_3 \geqslant 0; \\ c_{0, \alpha_2, \alpha_3} = 0, \qquad \text{all} \quad \alpha_2 \geqslant 0, \alpha_3 \geqslant 1; \\ (2\alpha_3 + 1)\, c_{\alpha_1, \alpha_2, \alpha_3} + (\alpha_3 + 1)\, c_{\alpha_1 + 1, \alpha_2 - 1, \alpha_3} = 0, \\ \text{all} \quad \alpha_1 \geqslant 0, \quad \alpha_2 \geqslant 1, \quad \alpha_3 \geqslant 0. \end{gathered}$$

Some thought reveals that all c-coefficients are zero. Hence $A \in \mathbb{C}[Z, U]$ and $\mathfrak{u}(\mathfrak{g}, \tau) = \mathfrak{z}\mathfrak{u}(\mathfrak{g}) + \mathfrak{u}(\mathfrak{g})\, \mathfrak{a}_\tau$. (A more detailed analysis would show that $u^K(\mathfrak{g}) = \mathbb{C}[Z, U, W]$, but we don't really need this.)

7.7. Example. $\mathfrak{g}$ as above, $\mathfrak{k} = \mathbb{R}Z + \mathbb{R}W$ with $f_z = \langle f, Z \rangle = 0$. Now $\Omega_\tau = f_w W^* + \mathbb{R}\text{-span}\{X^*, Y^*\}$ is *not* in general position. We have $K \cdot l = (0, \dot{y}, \mathbb{R}, f_w) = G \cdot l \cap \Omega_\tau$ for $l \in \Omega_\tau$ such that $\dot{y} = \langle l, Y \rangle \neq 0$, and $m(\pi_l) \equiv 1$ generically. Clearly $\mathbb{C}[\Omega_\tau]^K = \mathbb{C}[\dot{y}]$.

In this example $Y \in \mathfrak{u}(\mathfrak{g}, \tau)$ since $[W, Y] = Z = Z + 2\pi i \langle f, Z \rangle I = U_Z$, but $Y \notin \mathfrak{u}^K(\mathfrak{g})$ because $[W, Y] \neq 0$. Thus we have $\mathfrak{u}(\mathfrak{g}, \tau) \supsetneq \mathfrak{u}^K(\mathfrak{g}) + \mathfrak{u}(\mathfrak{g})\, \mathfrak{a}_\tau$. We omit the calculations showing that $\mathfrak{u}(\mathfrak{g}, \tau) = \mathbb{C}[Y, U] + \mathfrak{u}(\mathfrak{g})\mathfrak{a}_\tau$. In order to apply Lemma 5.3 we show that $L(Y)$ commutes with the averaging map $Q_\tau : \mathscr{S}(G) \to \mathscr{H}_\tau^\infty$, which implies that $\mathfrak{u}(\mathfrak{g}, \tau) = \mathfrak{u}(\mathfrak{g}, \tau)^0 + \mathfrak{u}(\mathfrak{g})\, \mathfrak{a}_\tau$. We then conclude that $D_\tau(K\backslash G) = \gamma(\mathbb{C}[Y, U]) \cong L(\mathbb{C}[Y, U])^\wedge |_{\Omega_\tau} = \mathbb{C}[\dot{y}] = \mathbb{C}[\Omega_\tau]^K$.

To see that $Y \in \mathfrak{u}(\mathfrak{g}, \tau)^0$ we compute $L(Y)$ in the weak basis W, Z, Y, X

through $\mathfrak{k}$. Since $Y \in \mathfrak{u}(\mathfrak{g}, \tau)$ we have $L(Y)\, Q_\tau w \in \mathscr{C}^\infty(G, \tau)$ for any $w \in \mathscr{S}(G)$, and it suffices to show $L(Y)\, Q_\tau w = Q_\tau L(Y) w$ at points $g = \eta(0, 0, y, x) \approx K\backslash G$. We have

$$L(Y)\, Q_\tau w(\eta(0, 0, y, x)) = \frac{d}{dt} \{Q_\tau w(\eta(0, 0, y-t, x))\}|_{t=0}$$
$$= -D_y[(Q_\tau w) \circ \eta](0, 0, y, x).$$

On the other hand, since $f_z = \langle f, Z \rangle = 0$ we get

$$Q_\tau(L(Y)w)(\eta(0, 0, y, x))$$
$$= \iint e^{-2\pi i w f_w} [L(Y)w](\eta(w, z, y, x))\, dz\, dw$$
$$= \iint e^{-2\pi i w f_w} [-D_y + wD_z(w \circ \eta)](w, z, y, x)\, dz\, dw.$$

But $\int D_z(w \circ \eta)\, dz = 0$, so this becomes

$$\iint e^{-2\pi i w f_w}(-1)\, D_y(w \circ \eta)(w, z, y, x)\, dz\ dw = -D_y[(Q_\tau w) \circ \eta](0, 0, y, x)$$

as required.

One could perform calculations similar to those in 7.1 and 7.6 for the Lie algebra with generators Z, Y, X, W, V and nontrivial relations

$$[V, W] = X, \qquad [V, X] = Y, \qquad [V, Y] = Z,$$

taking $\mathfrak{k} = \mathbb{R}W$ and $\mathfrak{k} = \mathbb{R}V$, respectively, and a character determined by $f = l_0 | \mathfrak{k}$ such that $l_0(Z) = \dot{z} \neq 0$. The polarizing ideal is $\mathfrak{b} = \mathbb{R}\text{-span}\{Z, Y, X, W\}$. This example is interesting because the center $\mathfrak{z}\mathfrak{u}(\mathfrak{g})$ consists of polynomials in the generators

$$Z, \qquad F_1 = Y^2 - 2XZ, \qquad F_2 = Y^3 - 3XYZ + 3WZ^2$$
$$F_3 = 9W^2Z^2 - 18WXYZ + 6WY^3 + 8X^3Z - 3X^2Y^2;$$

the generators are not free, since $Z^2F_3 = F_2^2 - F_1^3$, see [11]. When $\mathfrak{k} = \mathbb{R}W$ generic orbit intersections $G \cdot l \cap \Omega_\tau$ are cubic and the multiplicities are mixed, $m(\pi_l) = 1$ or 3. We find that $\mathfrak{u}(\mathfrak{g}, \tau) = \mathbb{C}[X, Y, Z] \oplus \mathfrak{u}(\mathfrak{g})\, \mathfrak{a}_\tau$, with $X, Y, Z \in \mathfrak{u}(\mathfrak{g})^K$, and that

$$\mathbb{D}_\tau(K\backslash G) = \gamma(\mathbb{C}[X, Y, Z]) \cong L(\mathbb{C}[X, Y, Z])^\wedge|_{\Omega_\tau} = \mathbb{C}[\Omega_\tau]^K.$$

Clearly $\gamma(\mathfrak{zu}(\mathfrak{g})) \neq \mathbb{D}_\tau(K\backslash G)$, although every element in $\mathbb{D}_\tau$ is algebraic over the fraction field generated by $\gamma(\mathfrak{zu}(\mathfrak{g}))$. When $\mathfrak{k} = \mathbb{R}V$, $G \cdot l \cap \Omega_\tau = K \cdot l$ and $m(\pi_l) \equiv 1$ for generic $l \in \Omega_\tau$; it is interesting to check that the operators $R(V)^\wedge, ..., R(Z)^\wedge$ do restrict to the (generic) cubic K-orbits in Ω_τ. In this example

$$\mathbb{D}_\tau(K\backslash G) \cong \gamma(\mathfrak{zu}(\mathfrak{g})) \cong L(\mathfrak{zu}(\mathfrak{g}))^\wedge|_{\Omega_\tau} = \mathbb{C}[\Omega_\tau]^K.$$

We omit the details in order to give another example, in which $\mathfrak{g}$ is not an extension of an abelian $\mathfrak{g}_0$ by $\mathbb{R}$.

7.8. EXAMPLE. Let $\mathfrak{g}$ have basis $Z, Y_3, Y_2, Y_1, X_3, X_2, X_1, W_2, W_1$ with

$$\begin{aligned} &[W_2, Y_3] = Z, \qquad [W_2, Y_2] = Y_3, \qquad [W_3, Y_1] = Y_2, \qquad [X_1, Y_1] = Z, \\ &[W_1, X_3] = Z, \qquad [W_1, X_2] = X_3, \qquad [W_1, X_1] = X_2, \end{aligned} \tag{50}$$

and take $\mathfrak{k} = \mathbb{R}\text{-span}\{Y_2, X_2, X_1\}$, $f = l_0|\mathfrak{k}$ with $l_0(Z) \neq 0$. The ideal $\mathfrak{b} = \mathbb{R}\text{-span}\{Z, Y_3, Y_2, X_3, X_2, X_1\}$ is polarizing for generic $l \in \mathfrak{g}^*$; for any $l \in \mathfrak{g}^*$,

$$\begin{aligned} &Ad^*_{\exp(zZ + \cdots + w_1 W_1)}(\dot{z}Z^* + \cdots + \dot{w}_1 W_1^*) \\ &\quad = \dot{z}Z^* + (\dot{y}_3 + w_2\dot{z})\, Y_3^* + \left(\dot{y}_2 + w_2\dot{y}_3 + \frac{w_2^2}{2}\dot{z}\right) Y_2^* \\ &\quad + \left(\dot{y}_1 + w_2\dot{y}_2 + \frac{w_2^2}{2}\dot{y}_3 + \left[\frac{w_2^3}{6} + x_1\right]\dot{z}\right) Y_1^* + (\dot{x}_3 + w_1\dot{z})\, X_3^* \\ &\quad + \left(\dot{x}_2 + w_1\dot{x}_3 + \frac{w_1^2}{2}\dot{z}\right) X_2^* \\ &\quad + \left(\dot{x}_1 + w_1\dot{x}_2 + \frac{w_1^2}{2}\dot{x}_3 + \left[\frac{w_1^3}{6} - y_1\right]\dot{z}\right) X_1^* \\ &\quad + \left(\dot{w}_2 - y_1\dot{y}_2 - \left[y_2 + \frac{w_2 y_1}{2}\right]\dot{y}_3 + \left[\frac{w_2 y_2}{2} - \frac{w_2^2 y_1}{6} - y_3\right]\dot{z}\right) W_2^* \\ &\quad + \left(\dot{w}_1 - x_1\dot{x}_2 - \left[x_2 + \frac{w_1 x_1}{2}\right]\dot{x}_3 + \left[\frac{w_1 x_2}{2} - \frac{w_1^2 x_1}{6} - x_3\right]\dot{z}\right) W_1^*. \end{aligned} \tag{51}$$

The weak Malcev basis $Y_2, X_2, X_1, Z, Y_3, Y_1, X_3, W_2, W_1$ passes through $\mathfrak{b} \cap \mathfrak{k} = \mathfrak{k}$ and $\mathfrak{b} + \mathfrak{k} = \mathfrak{b}$; in the Malcev coordinates $\eta(y_2, ..., w_1) = \exp(y_2, Y_2) \cdot \cdots \cdot \exp(w_1 W_1)$ a Campbell–Hausdorff computation yields

$$\begin{aligned}
&\eta(y_2, \ldots, w_1)\,\eta(y_2', \ldots, w_1') \\
&\quad = \eta\Bigg(y_2 + y_2' + w_2 y_1', x_1 + x_1', x_2 + x_2' + w_1 x_1', \\
&\qquad z + z' - x_1' y_1 + w_2 y_3' + w_1 x_3' + \frac{w_2^2 y_2'}{2} + \frac{w_1^2 x_2'}{2} + \frac{w_2^3 y_1'}{6} + \frac{w_1^3 x_1'}{6}, \\
&\qquad y_3 + y_3' + w_2 y_2' + \frac{w_2^2 y_1'}{2}, y_1 + y_1', x_3 + x_3' + w_1 x_2' + \frac{w_1^2 x_1'}{2}, \\
&\qquad w_2 + w_2', w_1 + w_1' \Bigg).
\end{aligned} \tag{52}$$

The generic layer of $Ad^*(G)$ orbits in $\mathfrak{g}^*$ (determined by strong Malcev basis $Z, Y_3, \ldots, X_3, \ldots, W_2, W_1$) is $U_e = \{l : \dot{z} \neq 0\}$; U_e meets Ω_τ. Generic G orbits are 6-dimensional, the non-jump vectors are Z, Y_2, X_2, and $V_{T(e)} = \mathbb{R}\text{-span}\{Z^*, Y_2^*, X_2^*\}$ is a cross-section for G-orbits in U_e. For $l \in U_e$ the orbit-representative map $\varphi: U_e \to V_{T(e)}$, $\varphi(l) = \varphi(\dot{z}, \dot{y}_3, \ldots, \dot{w}_1) = \dot{z}' Z^* + \dot{y}_2' Y_2^* + \dot{x}_2' X_2^*$ is given by the rational subsitutions

$$\dot{z}' = \dot{z}, \qquad \dot{x}_2' = \dot{x}_2 - \frac{\dot{x}_3^2}{2\dot{z}}, \qquad \dot{y}_2' = \dot{y}_2 - \frac{\dot{y}_3^2}{2\dot{z}},$$

which are determined directly from (51). From the numerators of these rational $Ad^*(G)$-invariant functions on $\mathfrak{g}^*$ we obtain independent generators of the fraction field $\mathrm{Frac}(\mathfrak{z}\mathfrak{u}(\mathfrak{g}))$ by symmetrization:

$$Z, \qquad U_1 = 2Y_2 Z - Y_3^2, \qquad U_2 = 2X_2 Z - X_3^2.$$

As in [11] it is not hard to see that these actually generate $\mathfrak{z}\mathfrak{u}(\mathfrak{g}) = \mathbb{C}[Z, U_1, U_2] \cong \mathbb{C}[\mathfrak{g}^*]^G$.

Let $f = f_{y_2} Y_2^* + f_{x_2} X_2^* + f_{x_1} X_1^* \in \mathfrak{k}^*$ determine the character $\chi \in \hat{K}$ that induces to obtain τ. Then points $l = \dot{z} Z^* + \cdots + \dot{w}_2 W_2^*$ in the Zariski-open subset $U = \Omega_\tau \cap \{l : \dot{z} \neq 0, \dot{y}_3 \neq 0, \dot{x}_3 \neq 0\} \subseteq U_e \cap \Omega_\tau$ of Ω_τ have

$$\begin{aligned}
K \cdot l &= \dot{z} Z^* + \dot{y}_3 Y_3^* + f_{y_2} Y_2^* + \mathbb{R} Y_1^* + \dot{x}_3 X_3^* \\
&\quad + f_{x_2} X_2^* + f_{x_1} X_1^* + \mathbb{R} W_2^* + \mathbb{R} W_1^*, \\
G \cdot l \cap \Omega_\tau &= \dot{z} Z^* + (\pm 1)\, \dot{y}_3 Y_3^* + f_{y_2} Y_2^* + \mathbb{R} Y_1^* \\
&\quad + (\pm 1)\, \dot{x}_3 X_3^* + f_{x_2} X_2^* + f_{x_1} X_1^* + \mathbb{R} W_2^* + \mathbb{R} W_1^* \\
&= (\text{union of four } K\text{-orbits}).
\end{aligned}$$

Thus $m(\pi_l) \equiv 4$.

As for $\mathbb{D}_\tau(K \backslash G)$, the elements Z and Y_3, X_3 (which are algebraic over $\mathrm{Frac}(\mathfrak{z}\mathfrak{u}(\mathfrak{g}))$) are in $\mathfrak{u}(\mathfrak{g})^K$; the algebra $\mathfrak{h} = \mathbb{R}\text{-span}\{Z, Y_3, X_3\}$ is abelian and $\mathfrak{u}(\mathfrak{h}) \cap \mathfrak{u}(\mathfrak{g})\,\mathfrak{a}_\tau = (0)$, so $\gamma: \mathfrak{u}(\mathfrak{h}) \to \mathbb{D}_\tau$ is injective. When we compute the

Malcev–Fourier transform $D \to D^\wedge \in \mathbb{C}[\Omega_\tau]^K$ we will see that $\gamma(\mathfrak{u}(\mathfrak{h}))^\wedge = \mathbb{C}[\Omega_\tau]^K$. Thus we must have $\mathbb{D}_\tau(K\backslash G) \cong \gamma(\mathfrak{u}(\mathfrak{h})) \cong \mathbb{C}[Z, Y_3, X_3]$; this could be shown directly, but such calculations are tedious.

From (52) we may compute the form of $L(Z), ..., L(W_2)$, $R(Z), ..., R(W_2)$ in Malcev coordinates on G, and then take the Malcev–Fourier transform $D \to D^\wedge = \mathscr{F}_\eta \circ D \circ \mathscr{F}_\eta^{-1} \in \mathscr{P}(\mathfrak{g}^*)$. The final outcome is

$$\begin{aligned}
L(Z)^\wedge &= -2\pi i\dot{z} & R(Z)^\wedge &= 2\pi i\dot{z}\\
L(Y_3)^\wedge &= -2\pi i\dot{y}_3 & R(Y_3)^\wedge &= 2\pi i\dot{y}_3 - \dot{z}D_{\dot{w}_2}\\
L(Y_2)^\wedge &= -2\pi i\dot{y}_2 & R(Y_2)^\wedge &= 2\pi i\dot{y}_2 + \frac{\dot{z}D^2_{\dot{w}_2}}{2(2\pi i)} - \dot{y}_3 D_{\dot{w}_2}\\
L(Y_1)^\wedge &= -2\pi i\dot{y}_1 - \dot{z}D_{\dot{x}_1} & R(Y_1)^\wedge &= 2\pi i\dot{y}_1 - \dot{y}_2 D_{\dot{w}_2} - \frac{\dot{z}D^3_{\dot{w}_2}}{6(2\pi i)^2} + \frac{\dot{y}_3 D^2_{\dot{w}_2}}{2(2\pi i)}\\
L(X_3)^\wedge &= -2\pi i\dot{x}_3 & R(X_3)^\wedge &= 2\pi i\dot{x}_3 - \dot{z}D_{\dot{w}_1}\\
L(X_2)^\wedge &= -2\pi i\dot{x}_2 & R(X_2)^\wedge &= 2\pi i\dot{x}_2 + \frac{\dot{z}D^2_{\dot{w}_1}}{2(2\pi i)} - \dot{x}_3 D_{\dot{w}_1}\\
L(X_1)^\wedge &= -2\pi i\dot{x}_1 & R(X_1)^\wedge &= 2\pi i\dot{x}_1 - \dot{x}D_{\dot{w}_1} + \frac{\dot{x}_3 D^2_{\dot{w}_1}}{2(2\pi i)}\\
& & &\quad + \dot{z}D_{\dot{y}_1} - \frac{\dot{z}D^3_{\dot{w}_1}}{6(2\pi i)^2}\\
L(W_2)^\wedge &= -2\pi i\dot{w}_2 + \dot{y}_3 D_{\dot{y}_2} & R(W_2)^\wedge &= 2\pi i\dot{w}_2\\
&\quad + \dot{z}D_{\dot{y}_3} + \dot{y}_2 D_{\dot{y}_1} & &\\
L(W_1)^\wedge &= -2\pi i\dot{w}_1 + \dot{x}_3 D_{\dot{x}_2} & R(W_1)^\wedge &= 2\pi i\dot{w}_1.\\
&\quad + \dot{z}D_{\dot{x}_3} + \dot{x}_2 D_{\dot{x}_1} & &
\end{aligned}$$

The operators $R(X)^\wedge$ all restrict to generic K-orbits in Ω_τ, as do $L(Z)^\wedge$, $L(Y_3)^\wedge$, $L(X_3)^\wedge$. Since $\mathfrak{u}(\mathfrak{h}) \subseteq \mathfrak{u}(\mathfrak{g})^K$, the transform $\mathbb{D}_\tau(K\backslash G) \to \mathbb{C}[\Omega_\tau]^K$ developed in this paper has the form $\gamma(A)^\wedge = L(A)^\wedge \mid \Omega_\tau$ for $A \in \mathfrak{u}(\mathfrak{h})$. Obviously $\gamma(\mathfrak{u}(\mathfrak{h}))^\wedge = \mathbb{C}[\Omega_\tau]^K$, so we must have $\mathfrak{u}(\mathfrak{g}, \tau) = \mathfrak{u}(\mathfrak{h}) \oplus \mathfrak{u}(\mathfrak{g})\,\mathfrak{a}_\tau$.

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