CALCULATING COORDINATE VARIANCES OF MULTIMASS SYSTEMS IN TRANSIENT CONDITIONS FOR STATIONARY RANDOM DISTURBANCES

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When carrying out design calculations for stationary random disturbances, which in engineering practice are used to model loads arising in transportation, earthquakes, acoustic excitation, and wind gusts, one often has to consider transient conditions of motion as well as steady-state conditions. This is connected with the fact that if a stationary random process begins to act on a system which is at rest or executing stationary random vibrations, the coordinate variances in the transient regime can exceed the corresponding steady-state values.

The basic subject dealt with in [1, 2, 3, 7, 8] was the behavior of a single-mass system subject to the action of a stationary random process for zero boundary conditions. Even in this very simple case it is not quite clear whether the transient regime is more dangerous for the system than the steady state. Determining coordinate variances for a multimass system taking transient processes into account involves carrying out laborious calculations. In what follows we consider a method of calculation based on the formulation of a matrix differential equation for the required correlation matrix of the coordinate moments. When the number of degrees of freedom is not large, this approach is preferable to the usual approach of integration in the frequency domain. Moreover, it turns out to be possible in this method to explain some qualitative features of changes in the coordinate variances for a multimass system in the transient regime.

The transient regime of a system, resulting from the action of a stationary random process, can be investigated for various assumptions regarding the initial state of the system. The simplest case occurs when the initial conditions are zero for all the system coordinates. In a more complicated case the initial conditions for the system coordinates are random with zero mathematical expectation and a known correlation matrix for the moments, determined from considerations of the foregoing stationary regime. If the mathematical expectations of the initial coordinate values or the mathematical expectations of the input random processes are also nonzero, then this is reflected only in the mathematical expectations of the coordinates in the transient regime, and the magnitudes of the coordinate variances are unchanged. In what follows, we shall assume that the random initial values of the system coordinates and the perturbations acting on the system are statistically independent, since the initial state of the system results from the action of perturbations which precede, and not of those which follow.

Assuming that the problem of constructing a shaping filter which transforms "white noise" into the input random process has been solved, we write the differential equation of the expanded system in its normal form

$$\dot{y} = Ay + \eta\xi(t), \tag{1}$$

where $y' = \|x', \dot{x}'\|_1^\nu$ is the expanded coordinate vector, η is a vector whose components characterize the intensity of the random influence, and $\xi(t)$ is "white noise."

Let $y(0)$ be the vector of random coordinate values for the system when $t = 0$; then Eq. (1) has the solution

$$y(t) = W(t)\,y(0) + \int_0^t W(t-\tau)\,\eta\xi(\tau)\,d\tau, \tag{2}$$

where $W(t)$ is the square matrix of transient functions satisfying the relationships

$$\dot{W}(t) = AW(t);\; W(0) = E; \tag{3}$$

E is a unitary matrix.

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We transpose expression (2), multiply on the right by the initial relation, and average the result over an ensemble of realizations of the random disturbances and initial conditions for the system coordinates. Further, when the fact that the vectors y(0) and $\eta\xi(t)$ are uncorrelated is taken into account, we obtain

$$I(t) = W(t) I(0) W'(t) + \int_0^t W(t-\tau)\, d\tau \int_0^t K(\tau_1 - \tau) W'(t-\tau_1)\, d\tau_1. \tag{4}$$

Here $I(t) = M[y(t)y'(t)]$ is the moment correlation matrix of the system coordinates, M indicates an average over an ensemble of realizations, $K(\tau_1 - \tau) = \delta(\tau_1 - \tau)\eta\eta'$ is the correlation matrix of the random disturbance, and $\delta(\tau_1 - \tau)$ is the Dirac delta function.

Setting $\eta\eta' = T$, we reduce relation (4) to the form

$$I(t) = W(t) I(0) W'(t) + \int_0^t W(t) T W'(t)\, dt. \tag{5}$$

We denote the first term of this expression by $I_1(t)$, and the second by $I_2(t)$. We then find the time derivatives

$$\begin{gathered} \frac{d}{dt} I_1(t) = A I_1(t) + I_1(t) A'; \\ \frac{d}{dt}[W(t) T W'(t)] = A W(t) T W'(t) + W(t) T W'(t) A' \end{gathered} \tag{6}$$

and integrate both sides of the last equation within the limits from 0 to t. Hence, it follows that the components $I_1(t)$ and $I_2(t)$ satisfy the relations

$$\dot{I}_1(t) = A I_1(t) + I_1(t) A', \quad I_1(0) = I(0); \tag{7}$$

$$\dot{I}_2(t) = A I_2(t) + I_2(t) A' + T; \quad I_2(0) = 0, \tag{8}$$

while their sum I(t) is the solution of the matrix equation

$$\dot{I}(t) = A I(t) + I(t) A' + T \tag{9}$$

for the given initial conditions I(0). This can easily be demonstrated by a direct substitution of expression (5) in Eq. (9), taking Eq. (3) into account.

We shall now consider the case when the same perturbation enters different inputs of the system with different time delays τ_i. This situation can arise, in particular, in the movement of transport vehicles over an uneven road or in the seismic disturbance of an extended structure. Using the method of calculation outlined above, we can show that in the transition regime the moment correlation matrix of the system coordinates

$$I(t) = W(t) I(0) W'(t) + \int_0^t \left[\sum_{i=1}^r H(t-\tau_i) W(t-\tau_i) \eta_i\right] \left[\sum_{i=1}^r H(t-\tau_i) W(t-\tau_i) \eta_i\right]' dt \tag{10}$$

satisfies the matrix equation

$$\dot{I}(t) = A I(t) + I(t) A' + \sum_{i=1}^r H(t-\tau_i) [\eta_i x'(\tau_i) + x(\tau_i) \eta_i'], \tag{11}$$

where $H(t-\tau_i)$ is the Heaviside unit function, r is the number of inputs, and $x(\tau_i) = 0.5\,[x(\tau_i - 0) + x(\tau_i + 0)]$ is a vector whose components, which are equal to the values of the system coordinates at time $t = \tau_i$, are calculated by solving the equation

$$\dot{x}(t) = A x(t) + \sum_{i=1}^r H(t-\tau_i) \eta_i. \tag{12}$$

In determining the matrix I(t) by numerical integration of the system of equations (9) or (11), the following circumstance must be borne in mind. For a "white noise" disturbance, the random process observed at the output of the shaping filter is stationary only after the damping of transient processes arising in the shaping filter. Thus, when investigating the transient regime, integration of the equations related to the mechanical system can commence only some time after integration of the shaping filter equations has begun [5]. The same result can be obtained by simultaneous integration of all the equations, specifying initial conditions for the shaping filter coordinates such that a stationary random process is immediately obtained at its output. This means that the part of the matrix I(0) corresponding to the shaping filter

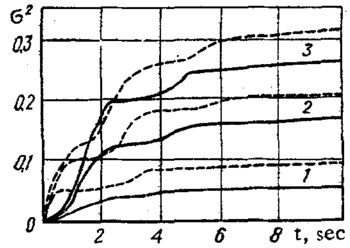

Fig. 1

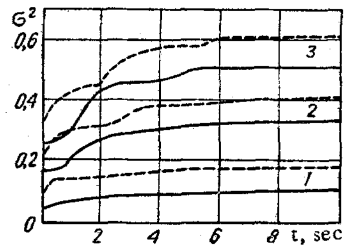

Fig. 2

must be determined beforehand by considering the shaping filter equations in the steady-state regime. It will be nonzero, even if the initial conditions for the coordinates of the original mechanical system are zero.

The solutions given above enable us to establish some interesting features in the changes of coordinate variances for a multimass mechanical system in the transient regime.

We shall first of all consider the particular case of zero initial conditions, which occurs if a stationary random process begins to act on an undisturbed mechanical system. We shall assume that the stationary random process is "white noise," so that there is no need for a shaping filter, and we can take $I(0) = 0$. Since the vector $V(t) = W(t)\eta$ is the solution of the homogeneous equation (3) for the initial conditions $V(0) = \eta$, expression (5) can be written in the form

$$I(t) = \int_0^t V(t)V'(t)\,dt \tag{13}$$

and, consequently, the diagonal elements of the matrix I(t) are integrals of squared components of the vector V(t).

Thus, the displacement and velocity variances of a multimass damped system under the action of "white noise" and with zero initial conditions are nondecaying functions of time which vary in the transient regime from zero to the maximum values corresponding to the steady-state regime. This conclusion is also valid in the case when "white noise" begins to act with time delay on different inputs of a previously undisturbed system, which is clear from Eq. (10). The curves for the change of variances with time are comparatively smooth, independently of which hypothesis is used to allow for inelastic resistance in the system – that of Voigt or that of Sorokin.

If a stationary random process different from "white noise" begins to act on an unperturbed system, then, for the reason given above, both components in Eqs. (5) and (10) must be taken into account. In this case, the variance of any coordinate is the sum of an oscillatory solution, damped in time, resulting from the first component, and a nondecaying function of time corresponding to the second component. It is very important from the practical point of view to know whether the coordinate variances in the transient regime will exceed the steady-state quantities. To a large extent the answer to this question depends on the parameter values of the system and disturbance.

In particular, for very narrow-band disturbances it is quite possible for the coordinate variances in the transient regime to exceed the steady-state values. This can be demonstrated without difficulty if we consider the limiting case of a narrow-band input disturbance, whose spectral matrix is equal to $S_{\xi\xi}(\omega) = (2\pi)^{-1}T\delta(\omega-\omega_0)$. With zero initial boundary conditions for the system coordinates, and taking the frequency approach, we have

$$I(t) = (2\pi)^{-1} W(i\omega_0, t)\, TW^*(i\omega_0, t),$$

where $W(i\omega_0, t)$ is the matrix of frequency characteristics satisfying the condition $W(i\omega_0, 0) = 0$, and the asterisk denotes a complex conjugate.

As a result of the well-known phenomenon of beating caused by the superposition of characteristic anf forced oscillations, the diagonal elements of the matrix I(t) can considerably exceed the corresponding values for $t \to \infty$, particularly if the frequency ω_0 lies above the eigenfrequency spectrum for system oscillations. As the effective width of the spectral density of the disturbance increases, the effect of the first term in Eq. (5) decreases compared with the second term when calculating coordinate variances. In the limiting case when "white noise" acts, the first term in Eq. (5) vanishes. The ideas outlined above are

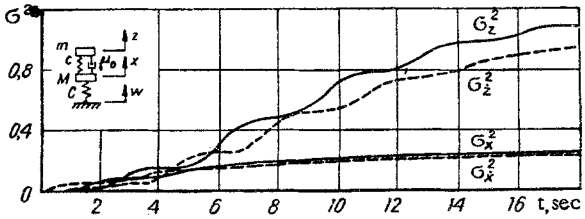

Fig. 3

confirmed by the calculations for a single-mass system [1, 2, 3] and a reservoir filled with liquid [6].

We shall consider the transient regime for motion of a system on the removal of the external disturbance. Let I(0) be the matrix found from calculations of the preceding steady-state regime; then the matrix $I_1(t)$ is determined by the relations (7). If we take $I_1(t) = I(0) - I_2(t)$, then $I_2(t)$ is the solution of Eq. (8) for $T = -AI(0) - I(0)A'$. In the absence of a disturbance the shaping filter is not required, and only those elements in matrix A related to the mechanical system are taken into account.

It can be shown that the variances of velocities and displacements consist of sums of functions which do not increase, but decrease from their steady-state values to zero, and oscillating functions whose amplitudes increase at first and then decrease. If the steady-state oscillations were caused by a narrow-band disturbance, then in some cases after the disturbance ceased the steady-state values of the variances could be exceeded in the transient regime. This turns out to be impossible after the cessation of a wide-band disturbance. In particular, after the cessation of "white noise" the variances of the system coordinates are strictly nonincreasing functions.

In the more general case $I(0) \neq 0$, $T \neq 0$, when a second stationary process introduced by form filtering "white noise" begins to act on the system instead of a single stationary random process. In this case it turns out to be possible, under certain conditions, for the coordinate variances to exceed their steady-state values, both in the transient regime as well as for $I(0) = 0$. Here the nature of change of the variances essentially depends on the spectral density bandwidth of the perturbing process and can be determined by numerical calculation for each specific system.

In conclusion, we give the results of some examples solved on a model "Minsk-22" computer using a program written in accordance with the algorithm given above.

We shall consider the random vibrations of a cable under the action of vertical wind gusts. The parameters taken from the data of [4] are as follows: length, 200 m; sag, 10 m; and diameter, 25.5 mm. For purposes of calculation the cable was treated as a gently sloping filament with six masses spaced at equal distances.

Curves showing the change in the variances of displacements in m^2 (solid curves) and velocities in $m^2 \cdot sec^{-2}$ (dashed curves) are given in Fig. 1 for three point masses under the action of "white noise" when the system is unperturbed at the initial moment of time. The numbers by the curves correspond to the number of the point mass taken in order from the point of support. The steady-state values of the variances are $\sigma_x^2 = (0.0564; 0.1713; 0.2644)\ m^2$, $\sigma_{\dot{x}}^2 = (0.0963; 0.2085; 0.3156)\ m^2 \cdot sec^{-2}$.

Curves for the change in the dispersions and velocities of the same masses are given in Fig. 2 for the case when the intensity of the "white noise" is increased by a factor of two compared with the initial state. It is clear from these graphs that in both cases the variances increase monotonically to their steady-state values.

Motion of an initially unperturbed two-mass system is investigated in the transient regime (Fig. 3) for kinematic displacement of the support:

$$\ddot{x} + p^2 x - \nu\mu(\dot{z} - \dot{x}) - \nu f^2 (z - x) = p^2 H(t)\, w(t);$$

$$\ddot{z} + \mu(\dot{z} - \dot{x}) + f^2 (z - x) = 0.$$

Here $\nu = m/M$; $f^2 = c/m$, $p^2 = C/M$, $\mu = \mu_0/m$, and w(t) is a stationary random function whose normalized spectral density is taken to have the form

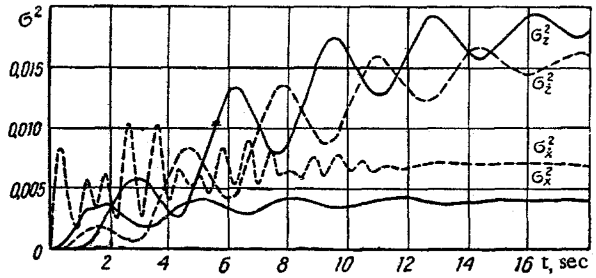

Fig. 4

$$S_w = \frac{2\alpha(\alpha^2+\beta^2)}{\pi[\omega^4 + 2(\alpha^2-\beta^2)\omega^2 + (\alpha^2+\beta^2)^2]},$$

and with the respective variances $\sigma_W^2 = 1$, $\sigma_{\dot{W}}^2 = \alpha^2 + \beta^2$.

The equation for the shaping filter is written as

$$\ddot{w} + 2\alpha\dot{w} + (\alpha^2+\beta^2)w = \sqrt{4\alpha(\alpha^2+\beta^2)}\,\xi(t).$$

With the system parameters p = 1 sec^{-1}, f = 0.95 sec^{-1}, $\nu = 0.1$, and $\mu = 0.4$ sec^{-1}, two variants of the perturbation parameters were considered: $\alpha = 7$ sec^{-1}, $\beta = 18$ sec^{-1} – the values usually taken for seismic disturbances [6] – and $\alpha = 0.1$ sec^{-1}, $\beta = 18$ sec^{-1}, which correspond to a narrow-band process.

Variances of the displacements and velocities of both masses are shown as functions of time in Fig. 3 for the first variant, and in Fig. 4 for the second. Steady-state values of the variances for the two variants of perturbation are, respectively, $\sigma_X^2 = 0.2595$, 0.00414, $\sigma_Z^2 = 1.215$, 0.01933, $\sigma_{\dot{X}}^2$ (0.2462; 0.007) sec^{-2}, $\sigma_{\dot{Z}}^2$ (1.025; 0.0163) sec^{-2}. Solutions for the second variant show that for a very narrow-band disturbance the change of the coordinate variances with time has a distinct oscillatory nature, while in the transient regime the magnitudes of the variances can exceed the corresponding steady-state values.

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