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Hypersurfaces with Constant Mean Curvature in Hyperbolic Space Form*

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Abstract. In this article, we prove the following theorem: A complete hypersurface of the hyperbolic space form, which has constant mean curvature and non-negative Ricci curvature Q, has non-negative sectional curvature. Moreover, if it is compact, it is a geodesic distance sphere; if its soul is not reduced to a point, it is a geodesic hypercylinder; if its soul is reduced to a point p, its curvature satisfies $\|\nabla Q\| < \infty$, and the geodesic spheres centered at p are convex, then it is a horosphere.

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1. Introduction

In 1899, H. Liebmann proved that spheres are the only surfaces with constant Gaussian curvature. In 1900, he also proved that spheres are the only ovaloids with constant mean curvature in Euclidean space (see [Hop83] for example). Generalizations of these classical rigidity results were made by many authors (D. Hilbert, H. Hopf, S. S. Chern, among others). K. Nomizu and B. Smyth [NS69] proved in 1969 that a non-negatively curved compact hypersurface with constant mean curvature of Euclidean space or a sphere, is a standard sphere, or a product of two spheres. In 1975, S. T. Yau improved this result when the hypersurface of Euclidean space has non-negative Ricci curvature, showing that, in this case, the hypersurface is a sphere [Yau74]. Recently, R. Walter [Wal85] gave the classification of non-negatively curved compact hypersurfaces in space form, with constant r-mean curvature. In this paper, we shall deal with the same problem in the hyperbolic space, under the weaker assumptions that the hypersurface is only complete, and its Ricci curvature is non-negative. Our conjecture is as follows (we denote by H^{n+1} the simply connected space form of constant sectional curvature-1):

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CONJECTURE. *Let M be a complete hypersurface of the hyperbolic space form H^{n+1}, with non-negative Ricci curvature, and constant mean curvature. Then, M is a geodesic distance sphere, a horosphere, or a geodesic hypercylinder*

(By a geodesic distance sphere in H^{n+1}, we mean the submanifold of points which are at a fixed distance of a fixed point. Such a hypersurface is totally umbilical. A hypercylinder is isometric with $R \times S^{n-2}$, a product of a line with a sphere, and embedded in H^{n+1} as the normal sphere bundle of a geodesic. Finally, to define a horosphere, we begin to remember that H^{n+1} has a standard embedding in Minkowsky space E^{n+2}. A horosphere of H^{n+1} is a flat (umbilical) hypersurface of H^{n+1} obtained as the intersection of a hyperplane of E^{n+2} with H^{n+1}.)

We cannot prove the conjecture in general. Our main observation is that the assumptions imply that the hypersurface M has non-negative curvature and we deal with its soul. If the soul is not a point, we can conclude that its dimension is 1 or $(n - 1)$, and we show that M is isoparametric. When its soul is reduced to a point p we cannot conclude. In this case, to prove that M is isoparametric, we must add the following geometric conditions:

(a) $\|\nabla Q\| < \infty$,

(b) *the geodesic spheres centered at p are convex.*

Then, we use the classical classification of isoparametric hypersurfaces of space forms (see [CR85] for instance). The way to prove that M is isoparametric consists in applying the Hopf lemma and its generalization to complete manifolds (due to S. T. Yau [Yau75b]), to suitable functions. We have no problem when this function is a square of the norm of the second fundamental form: it is a smooth function, and we can apply standard methods to compute its Laplacian and apply the Hopf lemma. However, we need to compute the Laplacian of the first eigenvalue of the second fundamental form, which is not smooth in general. That is why we consider a sequence of smooth functions, which approach locally the first principal curvature function, and work with the Laplacian of this sequence. The method used here may be regarded as an attempt to solve the classifying problems under completeness conditions.

2. Local Study of Hypersurfaces of the Hyperbolic Space Form

First of all, we derive from Gauss equation the local behavior of hypersurface of H^{n+1}, with non-negative Ricci curvature. we can summarize the results in the following

PROPOSITION 1. *Let M be a hypersurface with non-negative Ricci curvature, of a space of constant sectional curvature -1. Then, at p.*

1. *The second fundamental form h of M is positive semi-definite, of negative semi-definite.*
2. *$H \geq 1$, and if H is equal to 1 at p, then p is an umbilical and flat point.*
3. *Let $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ be the principal curvatures at p, and $(e_1, \ldots, e_n)$ be a corresponding local frame of principal vectors, that we extend on a neighborhood of p. The sectional curvature tensor $K_{ij} = K(e_i, e_j)$ satisfies at p, in this frame:*

   $K_{ij} = \lambda_i \lambda_j - 1$ *if $i \neq j$. In particular,*
   $K_{ij} \geq K_{(n-1)n}$ *if $i \neq j$.*
4. *The Ricci curvature tensor $Q_i = Q(e_i)$ satisfies in this frame:*

   $$Q_i = -(n-1) + \lambda_i(nH - \lambda_i).$$

   Moreover,

   $$Q_1 \geq \cdots \geq Q_i \geq \cdots \geq Q_n,$$

   and

   $$Q_n = \inf_{\{v, \|v\|=1\}} Q(v)$$
5. *If $Q_n = 0$, then $K_{1n} = \cdots = K_{(n-1)n} = 0$, and $\lambda_1 = \cdots = \lambda_{n-1}, \lambda_n = \lambda_1^{-1}$.*
6. *$\Sigma_{i,j}(\lambda_i - \lambda_j)^2 K_{ij} \geq 0$.*

Remark. In the following, we shall always assume that h is positive semi-definite. (If not, we replace the normal vector field of the hypersurface by its opposite.)

Proof of Proposition 1. Consider the second fundamental form h as a $(1-1)$-tensor. From the Gauss equation, we deduce immediately:

$$nH\langle h(v), v\rangle = Q(v) + (n-1)\langle v, v\rangle + \langle h(v), h(v)\rangle,$$

for every vector v, tangent to the hypersurface. (1)–(5) are consequences of this equality. (6) is a consequence of the following

LEMMA 1. *Let*

$$a_1 \geq \cdots \geq a_n,$$

$$b_1 \geq \cdots \geq b_n,$$

be two sequences of real numbers. Then,

$$\sum_i a_i b_i \geq \sum_i a_i b_{j_i} \geq \sum_i a_i b_{n-i}$$

for any permutation $j_1, \ldots, j_n$.

PROPOSITION 2. *Let M be a hypersurface with non-negative sectional curvature, of a space of constant sectional curvature -1. Let p be a point of M and let $u, v, \in ST_pM$ be two unit vectors such that:*

$$\frac{\pi}{4} \leq \sphericalangle(u, v) \leq \frac{3\pi}{4}.$$

Then,

$$\mathrm{Max}(Q(u), Q(v))_p \geq \frac{H_p - 1}{8(n-1)}.$$

Proof. Take any point p of M. At p, we have:

$$\lambda_{n-1}\lambda_n - 1 \geq 0,$$
$$\lambda_{n-1} \geq \lambda_n,$$

hence $\lambda_{n-1} \geq 1$, and moreover

$$\lambda_1 \geq H.$$

Then,

$$Q_{n-1} = \lambda_{n-1}\lambda_1 - 1 + \sum_{i=2}^{n} K(e_{n-1}, e_i) \geq H - 1.$$

Let u and v be two unit tangent vectors at p such that:

$$\frac{\pi}{4} \leq \sphericalangle(u, v) \leq \frac{3\pi}{4}.$$

We put:

$$u = \sum_{i=1}^{n} u^i e_i, \quad v = \sum_{i=1}^{n} v^i e_i.$$

We shall prove that:

$$\sum_{i=1}^{n-1} (u^i)^2 \geq \frac{1}{8(n-1)}, \quad \text{or} \quad \sum_{i=1}^{n-1} (v^i)^2 \geq \frac{1}{8(n-1)}.$$

In fact, if

$$\sum_{i=1}^{n-1} (u^i)^2 < \frac{1}{8(n-1)}, \quad \text{and} \quad \sum_{i=1}^{n-1} (v^i)^2 < \frac{1}{8(n-1)},$$

then,

$$(u^n)^2 > 1 - \frac{1}{8(n-1)}, \quad (v^n)^2 > 1 - \frac{1}{8(n-1)}.$$

This would imply:

$$\begin{aligned}
|\cos(\sphericalangle(u,v))| &\geq |u^n v^n| - \left|\sum_{i=1}^{n-1} u^i v^i\right| \\
&\geq |u^n||v^n| - \left(\sum_{i=1}^{n-1}(u^i)^2(v^i)^2\right)^{\frac{1}{2}} \\
&> 1 - \frac{1}{4(n-1)} > \frac{\sqrt{2}}{2}.
\end{aligned}$$

This leads to a contradiction. Finally,

$$\begin{aligned}
\mathrm{Max}(Q(u), Q(v)) &\geq \mathrm{Max}\left(\sum_{i=1}^{n-1}(u^i)^2 Q_i, \sum_{i=1}^{n-1}(v^i)^2 Q_i\right) \\
&\geq \mathrm{Max}\left(\sum_{i=1}^{n-1}(u^i)^2 Q_{n-1}, \sum_{i=1}^{n-1}(v^i)^2 Q_{n-1}\right) \\
&\geq \frac{H-1}{8(n-1)}.
\end{aligned}$$

The following Proposition is well known ([Che73], for example):

PROPOSITION A. *Let M be a hypersurface with non-negative Ricci curvature, of a space of constant sectional curvature -1. Suppose that M has constant mean curvature. Let De denote the Laplace–Beltrami operator. Then, we have:*

$$\Delta h_{ij} = nH\delta_{ij} - nh_{ij} + nH\sum_l h_{il}h_{lj} - \|h\|^2 h_{ij},$$

$$\Delta\|h\|^2 = 2\|\nabla h\|^2 + \sum_{i,j}(\lambda_i - \lambda_j)^2 K_{ij}.$$

3. Complete Hypersurfaces with Constant Mean Curvature in the Hyperbolic Space Form

3.1. THE COMPACT CASE

In this section, we begin by the simplest case: we assume that the hypersurface is compact, and prove the following

THEOREM 1. *Let M be a compact hypersurface of H^{n+1} with non-negative Ricci curvature, and constant mean curvature. Then M is a (totally umbilic) geodesic distance sphere.*

Proof. The proof of Theorem 1 is standard: Assume that M is compact. Applying Proposition 1(6) and the Hopf lemma, we deduce from Proposition A that M has a parallel second fundamental form. Then M is isoparametric ([Law69]). Now, since M is compact in H^{n+1}, we know from [Wal85] that M is a geodesic sphere. (Remark that this result is related to a special case of the theorem of Alexandrof, which says that any compact hypersurface embedded in a Euclidean or hyperbolic space with constant mean curvature, is a round sphere [Esc89]).

3.2. THE COMPLETE CASE

We continue the discussion and give results on complete hypersurfaces in a hyperbolic space form with constant mean curvature, and non-negative Ricci curvature. Although we cannot get a general complete theorem of classification, we are able to solve completely the problem when the soul of the hypersurface is not a point, or when the gradient of the Ricci curvature is finite at infinity. We state our main theorems:

THEOREM 2. *Let M be a complete (non-compact) hypersurface of H^{n+1} with non-negative Ricci curvature Q, and constant mean curvature. Then, M has non-negative sectional curvature. Moreover, suppose that one of the two following conditions hold:*

1. *The soul of M is not reduced to a point.*
2. *The soul of M is reduced to a point p, and*

$$\|\nabla Q\| < \infty,$$

and the geodesic spheres centered at p are convex.

Then, M is a geodesic hypercylinder, or a horosphere.

The proof of this theorem will be done in many steps. It will be a consequence of Theorems 3 and 4 below (Sections 3.7 and 3.8).

3.3. BEHAVIOR AT INFINITY

3.3.1. *Study of the First Principal Curvature of a Complete Hypersurface of H^{n+1}*

On any hypersurface M of any manifold (with oriented normal bundle), we can define the function λ_1 which associates to each point p of M the largest principal curvature at p. Of course, this function is not of class C^2 in general, but it is continuous. We shall say that *the point p is λ_1-regular if λ_1 is C^2 on a neighborhood* *of p.* It is well known that the measure of the set of points q which are not λ_1-regular is zero in M. We shall say that M is λ_1-regular if λ_1 *is of class* C^2 *in* M.

LEMMA 2. *Let M be a hypersurface of M^{n+1} with non-negative Ricci curvature, and constant mean curvature. Let p be a λ_1-regular point of M. Then at p:*

$$\Delta\lambda_1 \geq \frac{1}{n-1}\sum_i(\lambda_1 - \lambda_i)Q_i \geq 0.$$

Proof. From Proposition A, we have:

$$\begin{aligned}\Delta h_{11} &= nH - n\lambda_1 + nH\lambda_1^2 - \lambda_1\sum_i \lambda_i^2,\\ &= \sum_i(\lambda_1 - \lambda_i)(\lambda_1\lambda_i - 1).\end{aligned}$$

Using Proposition 1(4), and the fact that the Ricci curvature of M is non-negative, we have:

$$Q_i = (n-1)(\lambda_1\lambda_i - 1) \geq 0.$$

From this, we deduce immediately that

$$\Delta\lambda_1 \geq \frac{1}{n-1}\sum_i(\lambda_1 - \lambda_i)Q_i \geq 0.$$

Now, we need the following:

DEFINITION 1. We put

$$\Lambda = \sup_{p\in M} \lambda_1(p),$$

$$S = \sup_{p\in M} \|h\|^2(p).$$

Since H is constant and the principal curvatures are non-negative (see Proposition 1.3), Λ and S are finite. In particular, if M is λ_1-regular, we can apply the generalized maximum principle of S. T. Yau [Yau75a] and immediately get the following

PROPOSITION 3. *Let M be a complete (non-compact) λ_1-regular hypersurface of H^{n+1} with non-negative Ricci curvature and constant mean curvature. Then there exists a sequence (x_k) such that:*

1. $\lim_{k\to\infty}\lambda_1(x_k) = \Lambda$,

2. $\lim_{k\to\infty} \nabla\lambda_1(x_k) = 0$,
3. $\lim_{k\to\infty} \Delta\lambda_1(x_k) = 0$.

If one has no information on the λ_1-regularity of M, this lemma can be refined. For this goal, we need the following general result, which is a improvement of the classical Yau's maximum principle.

3.3.2. *A Technical Result*

LEMMA 3. *Let M be a complete (non-compact) Riemannian manifold whose Ricci curvature is bounded from below. Let*

$$f : M \to R^+,$$

be a non-negative function defined on M, which is bounded from above and almost everywhere differentiable. Suppose that, for any point $q \in M$, there exists a neighborhood U_q and q and a smooth function b_q defined on U_q such that:

1. $0 \leq b_q \leq f$,
2. $b_q(q) = f(q)$,
3. $\Delta b_q(q) \geq 0$.

Then, there exists a sequence (x_k) of points of M, and a sequence (l_k) of smooth functions on M such that:

1. *for any x_k, there exists a neighborhood $\tilde{U}_{x_k}$ of x_k in U_{x_k}, on which $l_k = b_{x_k}$,*
2. $\lim_{k\to\infty} l_k(x_k) = \lim_{k\to\infty} f(x_k) = \sup f$,
3. $\lim_{k\to\infty} \nabla l_k(x_k) = 0$,
4. $\lim_{k\to\infty} \Delta l_k(x_k) = \lim_{k\to\infty} \Delta b_{x_k}(x_k) = 0$.

Proof. If f is constant function, $f = A$, then, by the classical maximum principle, the only possibility for each b_q is to be the same constant A. Consequently, we can take $l_k = A$, and any sequence x_k. Now, we assume that f is not constant. The proof can be done in many steps:

1. Let $A = \sup f$. Let Ω be the domain where f is regular, and $\Omega' = M - \Omega$. Let $p \in \Omega$, such that $f(p) < A$. We define the following sequence of functions:

$$f_k(x) = \frac{f(x)}{(\rho(x) + 2)^{1/k}},$$

where ρ denotes the geodesic distance between p and x. Since

$$f_k(p) = \frac{f(p)}{2^{1/k}} > 0 \quad \text{and} \quad \lim_{\rho(x)\to\infty} f_k(x) = 0$$

for each k, there exists a point x_k such that

$$f_k(x_k) = \max f_k(x).$$

(Note that the problem will come from the fact that the function ρ is not differentiable on the cut locus of p.) It is only a Lipschitz function.

2. First of all, we prove that there exists a subsequence of (x_k) (that we still denote by (x_k)), such that

$$\lim_{k\to\infty} f(x_k) = A.$$

If this was not true, then, there would exist $\delta > 0$ and a point y in M such that

$$f(y) > f(x_k) + \delta,$$

for k large enough. Suppose that the sequnce $\rho(x_k)$ goes to infinity with k. Then

$$f_k(y) = \frac{f(y)}{(\rho(y)+2)^{1/k}} > \frac{f(x_k)}{(\rho(x_k)+2)^{1/k}} = f_k(x_k),$$

when $\rho(x_k) > \rho(y)$, which is a contradiction. Consequently, there exists a subsequence of (x_k), (that we still denote by (x_k)), such that

$$\lim_{k\to\infty} k_k = z \in M.$$

We have:

$$f(y) \geq f(z) + \delta,$$

for k large enough. Then, we get:

$$f_k(y) = \frac{f(y)}{(\rho(y)+2)^{1/k}} > \frac{f(x_k)+\delta}{(\rho(x_k)+2)^{1/k}} \cdot \frac{(\rho(x_k)+2)^{1/k}}{(\rho(y)+2)^{1/k}} > f_k(x_k),$$

for k large enough, which is again a contradiction. Consequently, we obtain that:

$$\lim_{k\to\infty} f(x_k) = A.$$

3. Now, remark that, since the measure of Ω' is null, each x_k belongs to $\bar{\Omega}$. Since $f(p) < A$, and $\lim_{k\to\infty} f(x_k) = A$, we can build for each k a small neighborhood V_k of x_k, defined in U_{x_k}, and a small neighborhood W_k of p in Ω such that:

$$V_k \cap W_k = \emptyset.$$

4. We define a sequence of functions on M by:

$$\begin{aligned} h_k(x) &= b_{x_k}(x),\ \forall x \in V_k, \\ h_k(x) &= f(x),\ \forall x \in W_k, \\ h_k(x) &= 0,\ \forall x \bar{\in} V_k \cup V_k. \end{aligned}$$

Hence, we have:

$$h_k(x) \leq f(x),$$

$$h_k(x_k) = b_{x_k}(x_k) = f(x_k).$$

5. We reduce V_k and W_k to V'_k and W'_k respectively. Using a partition of unity, we build a sequence of smooth functions i_k as follows:

$$\begin{aligned} i_k(x) &= 1,\ \forall x \in V'_k \cup W'_k, \\ i_k(x) &= 0,\ \forall x \bar{\in} V'_k \cup W'_k. \end{aligned}$$

We now define the sequence of smooth functions

$$l_k = i_k.h_k.$$

In such a way, l_k is a sequence of smooth functions which are equal at b_{x_k} on a neighborhood of x_k and p. In particular,

$$l_k(x_k) = f(x_k) \quad \text{and} \quad l_k(p) = f(p).$$

Moreover,

$$l_k(x) \leq h_k(x) \leq f(x),\ \forall x \in M.$$

By taking $\tilde{U}_{x_k} = V'_k$ we get sequences of points, smooth functions and neighborhoods which satisfy (i) and (ii) of the proposition.

6. We prove now (iii) and (iv) of the proposition. We must consider two cases:

(a) Suppose that there exists a subsequence of (x_k) (that we still denote by (x_k)), such that each x_k does not belong to the cut-locus of p. In this case, we define the sequence of functions

$$j_k(x) = \frac{l_k(x)}{(\rho(x) + 2)^{1/k}}.$$

Clearly, we have:

$$j_k(x) \leq f_k(x),\ \forall x \in M,$$

and

$$j_k(x_k) = \frac{l_k(x_k)}{(\rho(x_k)+2)^{1/k}} = f_k(x_k) = \max f_x(x).$$

Hence,

$$j_k(x_k) = \max j_k(x).$$

Then, we have:

$$\nabla j_k(x_k) = \frac{(\rho+2)^{1/k}\nabla l_k - (1/k) l_k (\rho+2)^{1/k-1}\nabla\rho}{(\rho+2)^{2/k}}(x_k) = 0.$$

Using the fact that:

$$\|\nabla\rho\|^2 = 1,$$

and that (l_k) is bounded, we deduce that:

$$\lim_{k\to\infty} \nabla l_k(x_k) = \lim_{k\to\infty} \frac{l_k.\nabla\rho}{k(\rho+2)}(x_k) = 0.$$

So, (iii) is proved. We need to prove (iv). For any k, we have: $l_k = b_{x_k}$ on $\tilde{U}_k$. Hence,

$$\Delta l_k(x_k) = \Delta b_{x_k}(x_k) \geq 0.$$

Since x_k is a maximum for j_k, we have:

$$\begin{aligned}\Delta j_k(x_k) \;=\; & \left(\frac{\Delta l_k}{(\rho+2)^{1/k}} - \frac{2 l_k \|\nabla\rho\|^2}{k^2(\rho+2)^{1/k+2}}\right.\\ & \left. - \frac{l_k.\Delta\rho}{k(\rho+2)^{1/k+1}} + \frac{(k+1) l_k}{k^2(\rho+2)^{1/k+2}}\right)(x_k) \leq 0.\end{aligned}$$

With the same argument as before, we deduce immediately that:

$$0 \leq \Delta l_k(x_k) \leq \frac{l_k.\Delta\rho}{k(\rho+2)^{1/k+1}}(x_k).$$

Finally, since the Ricci curvature of M is bounded from below, we can write:

$$Q \geq -(n-1)c^2, \quad c \geq 0,$$

from which we get, by a well-known result ([YS88]):

$$\Delta\rho \leq \frac{n-1}{\rho}(1+c\rho).$$

This implies:

$$\Delta l_k(x_k) \leq \frac{(n-1)(1+c\rho)l_k}{k\rho(\rho+2)}(x_k).$$

We consider two cases: If $\lim_{k\to\infty}\rho(x_k)=\infty$, then, it is clear that:

$$\lim_{k\to\infty}\Delta l_k(x_k)=\lim_{k\to\infty}\Delta b_{x_k}(x_k)=0.$$

Conversely, if there exists a subsequence of (x_k), (that we still denote by (x_k)), which converges to a point z in M, then $z\neq p$ (see 1), and $\rho(z)\neq 0$, from which the same conclusion holds.

(b) Suppose that all the (x_k) belong to the cut locus of p for k large enough. We can get same conclusion by a standard method, due to E. Calabi ([Yau75a]). We only sketch the proof in this case: For each x_k we consider a minimal geodesic σ_k joining p and x_k. We fix any point q in the interior of σ_k. q is not conjugate to x_k. Let N_q be a neighborhood of the geodesic segment joining q and x_k, defined such that, for any $x\in N_q$, there is at most one minimizing geodesic joining q and x. We denote by π_q the distance function from q to x taken in N_q. It is easy to see that the functions:

$$L_{q,k}(x)=\frac{f(x)}{(\pi_q(x)+\rho(x)+2)^{1/k}},$$

has a local maximum at the point x_k. The next step consists on building a sequence of smooth functions $l_{q,k}$ corresponding to $L_{q,k}$, exactly like in the regular case. From smoothness of $l_{q,k}$, we get:

$$\lim_{q\to p} l_{q,k}=l_k,$$

$$\lim_{q\to p}\Delta l_{q,k}=\Delta l_k.$$

The proof follows immediately.

3.3.3. *Geometric Properties at Infinity*

Here, we give a geometric application of the previous technical lemma.

PROPOSITION 4. *Let M be a complete (non-compact) hypersurface of H^{n+1} with non-negative Ricci curvature Q, and constant mean curvature. Then,*

(1) *there exists a sequence* (x_k), *and a sequence of smooth functions* (l_k) *such that*

(a) $\lim_{k\to\infty} l_k(x_k) = \Lambda$.

(b) $\lim_{k\to\infty} \Delta l_k(x_k) = (1/(n-1)) \lim_{k\to\infty} \sum_{i=1}^{i=n} (\lambda_1 - \lambda_n) Q_i(x_k) = 0$,

(c) $\lim_{k\to\infty} Q_n(x_k) = 0$.

(2) $nH = (n-1)\Lambda + \Lambda^{-1}$,

(3) *the principal curvatures satisfy at each point the following inequality:*

$$\lambda_{n-1} + \lambda_n \geq \Lambda + \Lambda^{-1},$$
$$\Lambda \geq \lambda_1 \geq \cdots \geq \lambda_n \geq \Lambda^{-1}.$$

Proof. We shall use Lemma 3 by considering the function λ_1 for f. Let Ω be the set of point where λ_1 is differentiable. Let Ω' be the complementary set of Ω in M. If q is a point of Ω, then, using the notations of Lemma 3, we can take $b_q = \lambda_1$ on any neighborhood U_q of q. Using Lemma 2, we see that all the assumptions of Lemma 3 are satisfied. If q is a point of Ω', we use the fact that λ_1 is almost everywhere differentiable to take a (small) open set U in Ω such that q belongs to the boundary of U. Let e_1 be a principal vector field defined on U associated to λ_1. We can extend U to an open neighborhood U_q of q, and extend e_1 to a smooth unit vector field E on U_q. On U_q, we put:

$$b_q = \langle h(E,E), e_{n+1} \rangle.$$

It is clear that the first two assumptions of Lemma 3 are satisfied. In order to compute $\Delta b_q(q)$, we consider a sequence of points y_l defined on U, whose limit is q. We get:

$$\begin{aligned}
\Delta b_q(q) &= \Delta \langle h(E,E), e_{n+1} \rangle (q) \\
&= \Delta \langle h(E,E), e_{n+1} \rangle \left(\lim_{l\to\infty} (y_l) \right) \\
&= \lim_{l\to\infty} \Delta \langle h(E,E), e_{n+1} \rangle (y_l) \\
&= \lim_{l\to\infty} \Delta h_{11}(y_l) \geq \lim_{l\to\infty} \frac{1}{n-1} \sum_i (\lambda_1 - \lambda_i) Q_i(y_l) \geq 0.
\end{aligned}$$

Consequently:

1. (a) We can apply the conclusion of Lemma 3, and find a sequence (x_k) of λ_1-regular points of M and a sequence of smooth functions l_k such that:

$$\lim_{k\to\infty} l_k(x_k) = \Lambda.$$

(b) Then, applying Lemmas 2 and 3, we deduce by continuity that:

$$0 = \lim_{k\to\infty} \Delta_k(x_k) \geq \lim_{k\to\infty} \frac{1}{n-1} \sum_i (\lambda_1 - \lambda_i) Q_i(x_k) \geq 0.$$

Then,

$$\lim_{k\to\infty} \frac{1}{n-1} \sum_i (\lambda_1 - \lambda_i) Q_i(x_k) = 0.$$

In particular, since M has non-negative Ricci curvature,

$$\lim_{k\to\infty} (\lambda_1 - \lambda_i) Q_i(x_k) = 0, \forall i \in (2, \dots, n).$$

(c) Now, since $Q_i(x_k)$ is bounded, it is always possible to find a subsequence of (x_k), that we still denote by (x_k) such that

$$\lim_{k\to\infty} Q_i(x_k) \quad \text{and} \quad \lim_{k\to\infty} (\lambda_1 - \lambda_i)(x_k)$$

exist, for all i.

Suppose that

$$\lim_{k\to\infty} Q_n(x_k) \neq 0.$$

Then, from Proposition 1, we deduce that

$$\lim_{k\to\infty} Q_i(x_k) \neq 0, \forall i.$$

Still from Proposition 1, we have:

$$\Lambda = \lim_{k\to\infty} \lambda_1(x_k) = \cdots = \lim_{k\to\infty} \lambda_i(x_k) = \cdots = \lim_{k\to\infty} \lambda_n(x_k).$$

(We can say that one has umbilicity at infinity.)

Now, let p be any point of M. We have:

$$nH(p) = \lambda_1(p) + \cdots + \lambda_i(p) + \cdots + \lambda_n(p) = nH = n\Lambda,$$

from which we deduce easily that p is umbilic, and

$$\lambda_1(p) = \cdots = \lambda_i(p) = \cdots = \lambda_n(p) = \Lambda.$$

Consequently, every point of M is umbilic, and $H > 1$. This means that M is a geodesic sphere, which is impossible since M is non-compact. Finally, we get:

$$\lim_{k\to\infty} Q_n(x_k) = 0.$$

2. From Proposition 1, we deduce that

$$\Lambda = \lim_{k\to\infty} \lambda_1(x_k) = \cdots = \lim_{k\to\infty} \lambda_{n-1}(x_k); \lim_{k\to\infty} \lambda_n(x_k) = \Lambda^{-1}.$$

Consequently,

$$nH = (n-1)\Lambda + \Lambda^{-1}.$$

3. On the other hand, at every point p, we have:

$$\begin{aligned}\lambda_{n-1}(p)+\lambda_n(p) &= nH-(\lambda_1(p)+\cdots+\lambda_{n-2}(p))\\ &\geq nh-(n-2)\Lambda=\Lambda+\Lambda^{-1}.\end{aligned}$$

In particular, $\lambda_n \geq \Lambda^{-1}$, and 3 is proved.

3.4. CURVATURE PROPERTIES

The goal of this section is to deduce from the previous results, curvature properties of the hypersurface.

PROPOSITION 5. *Let M be a complete (non-compact) hypersurface of H^{n+1} with non-negative Ricci curvature Q, and constant mean curvature. Then,*

(1) *M has non-negative sectional curvature,*
(2) *if M is not flat, then its sectional curvature satisfies the following property: $\forall p \in M, \forall u_i, v_j \in ST_pM (1 \leq i,j \leq 2)$ such that $\langle u_i, v_j\rangle = 0, u_1 \neq \pm u_2, v_1 \neq \pm v_2$, then*

$$\sum_{i,j} K(u_i, v_j) > 0.$$

Proof. 1. To prove that M has non-negative sectional curvature, it is enough to prove that $K_{(n-1)n} \geq 0$ at any point p. From Proposition 4, we have:

$$nH = (n-1)\Lambda + \Lambda^{-1},$$

and

$$\begin{aligned}K_{(n-1)n}(p) &= -1+\lambda_{n-1}\lambda_n\\ &\geq -1+\lambda_n(\Lambda+\Lambda^{-1}-\lambda_n)\\ &\geq -1+\Lambda^{-1}(\Lambda+\Lambda^{-1}-\Lambda^{-1}) \geq 0.\end{aligned}$$

Then, M has non-negative sectional curvature.

2. Since M is not flat, $\Lambda \neq 1$. We shall prove first of all that

$$K_{(n-2)(n-1)} > 0.$$

In fact, $K_{(n-1)n} \geq 0$ implies as usual $\lambda_{n-1} \geq 1$. If

$$K_{(n-2)(n-1)} = 0,$$

then,

$$\lambda_{n-1} = \frac{1}{\lambda_{n-2}}.$$

Since $\lambda_{n-2} \geq \lambda_{n-1}$, we get $\lambda_{n-2} = \lambda_{n-1} = 1$, and then $\lambda_n = 1$. On the other hand,

$$\lambda_{n-1} + \lambda_n \leq \Lambda + \Lambda^{-1},$$

from which we deduce:

$$nH \leq (n-3)\Lambda + 1 + \Lambda + \Lambda^{-1}.$$

Since M is not flat, $\Lambda > 1$, and

$$(n-3)\Lambda + 1 + \Lambda + \Lambda^{-1} < (n-1)\Lambda + \Lambda^{-1} = nH.$$

This leads to a contradiction. Finally, if $i \neq j, i \leq (n-1), j \leq (n-1)$, we have:

$$K_{ij} = -1 + \lambda_i\lambda_j \geq \lambda_{n-1}\lambda_{n-2} > 0,$$

The conclusion follows.

PROPOSITION 6. *Let M be a complete (non-compact) hypersurface of H^{n+1} with non-negative Ricci curvature Q and constant mean curvature. If there exists a point q such that* $\operatorname{Ker} Q_q \neq 0$, *then M is isoparametric.*

Proof. From Proposition 3, we know that M has non-negative sectional curvature. Then, from Proposition A, we conclude that

$$\Delta\|h\|^2 \geq 0,$$

at every point p. Since $\|h\|^2$ is finite, we can apply Yau's maximum principle and find a sequence (y_k) such that

$$\lim_{k\to\infty} \|h\|^2(y_k) = \sup_M \|h\|^2,$$

$$\lim_{k\to\infty} \Delta\|h\|^2(y_k) = 0.$$

From Proposition A, we deduce that

$$\lim_{k\to\infty} (\lambda_i - \lambda_j)^2 K_{ij}(y_k) = 0, \ \forall i, j.$$

Suppose that

$$\forall j, \lim_{k\to\infty} K_{1j}(y_k) > 0.$$

Then,

$$\lim_{k\to\infty} (\lambda_1)(y_k) = \cdots = \lim_{k\to\infty} (\lambda_i)(y_k) = \cdots = \lim_{k\to\infty} (\lambda_n)(y_k) = \lambda.$$

Then, $H = \lambda$, and

$$\lim_{k\to\infty} \|h\|^2(y_k) = n\lambda^2.$$

Now, let q be any point of M. We have:

$$\begin{aligned}\|h\|^2(q) &= \lambda_1^2(q) + \cdots + \lambda_n^2(q) \\ &\geq \frac{1}{n}(\lambda_1 + \cdots + \lambda_n)^2(q) = n\lambda^2 \\ &= \lim_{k\to\infty}(\lambda_1^2 + \cdots + \lambda_n^2)(y_k) = \lim_{k\to\infty} \|h\|^2(y_k).\end{aligned}$$

Then,

$$\|h\|^2(q) \geq \lim_{k\to\infty} \|h\|^2(y_k),$$

and, consequently,

$$\|h\|^2(q) = \lim_{k\to\infty} \|h\|^2(y_k).$$

Finally, the function $\|h\|^2$ is constant, and M is isoparametric (i.e. M is a geodesic sphere). Since M is not compact, this is impossible.

Hence, there exists j such that

$$\lim_{k\to\infty} K_{1j}(y_k) = 0,$$

and choose j as small as possible, that is:

$$\lim_{k\to\infty} K_{1l}(y_k) > 0, \quad l < j.$$

From the Gauss equation, we deduce that:

$$\lim_{k\to\infty} \lambda_1(y_k) = \cdots = \lim_{k\to\infty} \lambda_{j-1}(y_k),$$

and

$$\lim_{k\to\infty} \lambda_j(y_k) = \cdots = \lim_{k\to\infty} \lambda_n(y_k) = \lim_{k\to\infty} \lambda_1^{-1}(y_k).$$

If $H = 1$, then M is a (isoparametric) horosphere. If $H > 1$, then,

$$\lim_{k\to\infty} \lambda_1(y_k) > 1 \quad \text{and} \quad \lim_{k\to\infty} \lambda_j(y_k) < 1.$$

If $j \neq n$, this implies:

$$\lim_{k\to\infty} K_{(n-1)n}(y_k) = \lim_{k\to\infty} (\lambda_{n-1}\lambda_n - 1)(y_k) < 0,$$

which is impossible. Then, $j = n$ and we have:

$$\lim_{k\to\infty} \lambda_1(y_k) = \cdots = \lim_{k\to\infty} \lambda_{n-1}(y_k) = \lambda > 1,$$

$$\lim_{k\to\infty} \lambda_n = \lambda^{-1}.$$

Then,

$$nH = (n-1)\lambda + \lambda^{-1} = (n-1)\Lambda + \Lambda^{-1},$$

from which we deduce easily that $\lambda = \Lambda$. Finally, by assumption, we know that there exists a point q such that $Q_n(q) = 0$. We deduce that

$$\lambda_1(q) = \cdots = \lambda_{n-1}(q), \lambda_n(q) = \lambda_1^{-1}(q).$$

Then,

$$\begin{aligned} nH &= (n-1)\lambda_1(q) + \lambda_1^{-1}(q) \\ &= (n-1)\lim_{k\to\infty} \lambda_1(y_k) + \lim_{k\to\infty} \lambda_1^{-1}(y_k) \\ &= (n-1)\lambda + \lambda^{-1}. \end{aligned}$$

Then,

$$\lambda_1(q) = \lambda = \Lambda \quad \text{and} \quad ||h||^2(q) = \lim_{k\to\infty} ||h||^2(y_k).$$

This means that the function $||h||^2$ has a maximum at q, and then is constant. Consequently, M is isoparametric ([Law69]).

3.5. BASIC DESCRIPTION OF NON NEGATIVELY CURVED HYPERSURFACES. THE NOTION OF P-MANIFOLD

3.5.1. *Non-negatively Curved Manifolds*

Since we have proved that the hypersurfaces that we study have non-negative sectional curvature, we know by a classical result that they are homeomorphic to a vector bundle whose basis (its soul) is a compact manifold with non-negative sectional curvature. We recall now its geometric description ([CE75]):

PROPOSITION B. *Let M be a n-dimensional complete (non-compact) Riemannian manifold, with non-negative sectional curvature. Let S be a soul of M.*

Suppose that $\dim S = m$. *Let* $p \in S$. *Then, there exist* n-m *rays* γ_i, *orthogonal at p, and orthogonal to* S, *emanating form p, there exist* m *geodesics* σ_α, *staying in* S, *emanating from p, orthogonal at p, such that the surfaces defined by*

$$\varphi_{\alpha,j}(t,s) = \exp_{\sigma_\alpha(s)} tV_j(s)$$

are flat and totally geodesic. (V_j *denotes the parallel field along* σ_α *generated by* $\gamma_j'(0)$.)

For further use, we need the following result, essentially due to A. Avez [Ave72]:

LEMMA 4. *Let* M *be a complete non-compact Riemannian manifold, with non-negative Ricci curvature. Suppose that*

$$\|\nabla Q\| < \infty.$$

Let p be a point of M *and* γ *be any ray emanating from p:*

$$\gamma : [0, \infty[\to M,$$

$$\gamma(0) = p.$$

Then

$$\lim_{t\to\infty} Q(\gamma'(t)) = 0.$$

Proof. We use a formula of A. Avez [Ave72]: Let p be any point of M and let $\gamma(t)$ be any ray of M, emanating from p. Let ρ be the distance function on γ from p. Let

$$X = \Delta\rho.$$

Since γ has no conjugate points, we have:

$$Q(\gamma'(t)) + A + \frac{X^2}{n-1} + \frac{\mathrm{d}X}{\mathrm{d}t} = 0,$$

$$0 \le X(t) \le \frac{n-1}{t}, (t > 0),$$

where A is defined by:

$$A = \|\theta\|^2,$$

with

$$\theta_{ij} = \nabla_j \gamma_i' - \frac{1}{n-1}(g_{ij} - \gamma_i'\gamma_j')\nabla_k \gamma_k'.$$

Let $t_0 > 0$. We have:

$$\begin{aligned}
&\int_{t_0}^{t_0+(1/\sqrt{t_0})} (Q(\gamma'(t)) + A)t^{1/2}\,\mathrm{d}t \\
&= -\frac{1}{n-1}\int_{t_0}^{t_0+(1/\sqrt{t_0})} X^2 t^{1/2}\,\mathrm{d}t \\
&\quad - \int_{t_0}^{t_0+(1/\sqrt{t_0})} X'(t)t^{1/2}\,\mathrm{d}t \\
&= -\frac{1}{n-1}\int_{t_0}^{t_0+(1/\sqrt{t_0})} X^2 t^{1/2}\,\mathrm{d}t - [Xt^{1/2}]_{t_0}^{t_0+(1/\sqrt{t_0})} \\
&\quad +\frac{1}{2}\int_{t_0}^{t_0+(1/\sqrt{t_0})} t^{-(1/2)}X\,\mathrm{d}t \\
&\leq X(t_0)t_0^{1/2} + \frac{n-1}{2}\int_{t_0}^{t_0+(1/\sqrt{t_0})} t^{-(3/2)}X\,\mathrm{d}t \\
&\leq 2(n-1)t_0^{-1/2} - (n-1)\left(t_0 + \frac{1}{\sqrt{t_0}}\right)^{-(1/2)}
\end{aligned}$$

Let ξ be such that $t_0 < \xi < t_0 + (1/\sqrt{t_0})$. We have:

$$\begin{aligned}
\frac{1}{\sqrt{t_0}}Q(\gamma'(\xi))\sqrt{\xi} &= \int_{t_0}^{t_0+(1/\sqrt{t_0})} \sqrt{t}\,Q(\gamma'(t))\,\mathrm{d}t \\
&\leq \int_{t_0}^{t_0+(1/\sqrt{t_0})} (Q(\gamma'(t)) + A)\sqrt{t}\,\mathrm{d}t.
\end{aligned}$$

We deduce that:

$$Q(\gamma'(\xi)) \leq \frac{2(n-1)}{\sqrt{t_0}}.$$

By assumption there exists a constant D such that

$$\|\nabla Q\| < D.$$

Then,

$$\|Q(\gamma'(t_0)) - Q(\gamma'(\xi))\| \leq \frac{1}{\sqrt{t_0}}D,$$

and

$$\begin{aligned}
Q(\gamma'(t_0)) &\leq Q(\gamma'(\xi)) + \frac{1}{\sqrt{t_0}}D, \\
&\leq \frac{E}{\sqrt{t_0}}.
\end{aligned}$$

This implies that when t_0 goes to infinity, $Q(\gamma'(t_0))$ converges to 0.

3.5.2. *The Notion of P-Manifolds*

We introduce now a class of non-negatively curved manifolds which contains the hypersurfaces that we study. We give the following:

DEFINITION 2. A Riemannian manifold M is said to be a P-manifold, if it satisfies the following properties (P):

1. The sectional curvature K of M is non negative:

   $$K \geq 0.$$

2. $\forall \alpha \in [\pi/4, \pi/2], \exists Q_0 > 0$ such that, $\forall p \in M, \forall u, v \in ST_pM$, with

   $$0 < \alpha \leq \sphericalangle(u, v) \leq \pi - \alpha,$$

   then

   $$\mathrm{Max}(Q(u), Q(v)) \geq Q_0.$$

3. $\forall p \in M, \forall u_i, v_j \in ST_pM$ $(1 \leq i, j \leq 2)$ such that $\langle u_i, v_j \rangle = 0$, $u_1 \neq \pm u_2$, $v_1 \neq \pm v_2$, then

   $$\sum_{i,j} K(u_i, v_j) > 0.$$

The main properties of P-manifolds that we shall use, are given by the following

PROPOSITION 7. *Let M be a complete (non-compact) P-manifold. If*

(1) $\dim S = 0$,
(2) *the geodesic spheres centered at a soul p are convex, and*
(3) $\|\nabla Q\| < \infty$,

then, the family of geodesic balls $B_p(R)$, and their boundaries $\partial B_p(R)$ centered at p and of radius R, satisfies the following properties:

1. *There exists a constant A (independent of R), such that*

   $$\mathrm{Vol}\, \partial B_p(R) < A.$$

2. $\mathrm{Vol}\, B_p(R) \sim \mathrm{O}(R)$.

Proof. Let p be a soul of M. Consider the geodesic balls $B_p(R)$, centered at p and of radius R. Their boundaries $\partial B_p(R)$ are geodesic spheres, which are convex. Then, their second fundamental forms are semi-positive definite. Let Q^S be the Ricci tensor of such a sphere. Let q be a point of $\partial B_p(R)$. Let $(e_1, \ldots, e_n)$ be a local orthornormal frame at q, such that $(e_1, \ldots, e_{n-1})$ is tangent to $\partial B_p(R)$, and e_n is normal to $B_p(r)$. If γ is the ray from p to q, we have $\gamma'(q) = -e_n$.

1. Using the Gauss equation, we obtain:

$$Q^S(e_i) \geq Q(e_i) - K_{inin}.$$

Now, using Lemma 4, we see that if R is large enough, $Q(e_n)$ is smaller that $\frac{1}{2}Q_0$. The second property of P-manifold gives:

$$\mathrm{Max}(Q(e_i), Q(e_n)) = Q(e_i) > \tfrac{1}{2}Q_0, \quad \forall i \in \{1, \dots, (n-1)\}.$$

Since a P-manifold has non-negative sectional curvature, we deduce that:

$$Q^S(e_i) > \tfrac{1}{2}Q_0.$$

Since Q_0 is strictly positive, we deduce from standard results of Bishop [BC64] that Vol $\partial B_p(R)$ is finite. Moreover, since Q_0 is independent of R, there exists a constant A, (independent of R), such that

$$\mathrm{Vol}\, \partial B_p(R) < A.$$

2. We use now a classical inequality (see [BZ80] for instance):

$$\mathrm{Vol}\, B_p(R) \leq R\, \mathrm{Vol}(\partial B_p(R)).$$

We deduce immediately,

$$\mathrm{Vol}\, B_p(R) \sim \mathrm{O}(R).$$

3.6. IMMERSED P-MANIFOLDS

The crucial point is the following:

PROPOSITION 8. *Let M be a complete (non-compact) regular hypersurface of H^{n+1} with non-negative Ricci curvature Q and constant mean curvature $H \neq 1$. Then,*

(1) *M is a P-manifold,*
(2) *the dimension of the soul of M can only be* $0, 1, (n-1)$.
(3) *in particular, if the soul of M is not reduced to a point, then, at each point p of the soul,* $\mathrm{Ker}\, Q_p \neq 0$.

Proof. 1. Using Proposition 2 and Proposition 5, and the fact that H is constant and $\neq 1$, we see immediately that M is a P-manifold.

2. We compute now the possible dimensions of a soul S of a P-manifold: Suppose that $\dim S \geq 2$, and $\leq (n-1)$. In this case, one can build from any point p of S, two flat totally geodesic planes

$$\varphi_{\alpha,i}(t,s) = \exp_{\sigma_\alpha(s)} tV_i(s),$$

$$\varphi_{\beta,j}(t,s) = \exp_{\sigma_\beta(s)} tV_j(s),$$

using two geodesics emanating from p, orthogonal at p, staying in s, and two rays emanating from p, orthogonal at p, going out of p. This implies that condition (3) for P-manifolds is not satisfied. Then, we get a contradiction. Finally, the only possibility for the dimension of the soul is $0, 1, (n-1)$.

3. If $\dim S \neq 0$, then it is 1 or $(n-1)$. Suppose that $\dim S = 1$. Using Proposition B, we know that it is possible to find, at any point p of S, a unit tangent vector to S, v_1, and $(n-1)$ unit vectors, orthogonal to S, $v_s, \ldots, v_n$, such that the sectional curvatures $K_{12}, \ldots, K_{1n}$, are null at p. This implies that $Q_p(v_1) = 0$. If $\dim S = (n-1)$, the same proof works.

3.7. HYPERSURFACES WHOSE SOUL IS NOT REDUCED TO A POINT

As an immediate consequence of Proposition 6, Proposition 8(3), and the classification of isoparametric hypersurfaces (see [CR85]), we get the following:

THEOREM 3. *Let M be a complete (non-compact) hypersurface of H^{n+1} with non-negative Ricci curvature Q and constant mean curvature. Then, M has non-negative sectional curvature. Moreover, if the soul of M is not reduced to a point, then M is a geodesic hypercylinder.*

3.8. HYPERSURFACES WHOSE SOUL IS REDUCED TO A POINT

We shall prove the following:

THEOREM 4. *Let M be a complete (non-compact) hypersurface of H^{n+1} with non-negative Ricci curvature Q and constant mean curvature. Then, M has non-negative sectional curvature. Moreover, suppose that the two following conditions are satisfied:*

1. *The soul of M is reduced to a point p, and the geodesic spheres centered at p are convex.*
2. $\|\nabla Q\| < \infty$.

Then, M is a horosphere.

We need the following general

PROPOSITION 2. *Let M be a hypersurface with non-negative sectional curvature, of a space of constant sectional curvature -1. Let p be a point of M and let $u, v, \in ST_pM$ be two unit vectors such that:*

$$\frac{\pi}{4} \leq \prec(u, v) \leq \frac{3\pi}{4}.$$

Then,

$$\mathrm{Max}(Q(u), Q(v))_p \geq \frac{H_p - 1}{8(n-1)}.$$

Proof. Take any point p of M. At p, we have:

$$\lambda_{n-1}\lambda_n - 1 \geq 0,$$
$$\lambda_{n-1} \geq \lambda_n,$$

hence $\lambda_{n-1} \geq 1$, and moreover

$$\lambda_1 \geq H.$$

Then,

$$Q_{n-1} = \lambda_{n-1}\lambda_1 - 1 + \sum_{i=2}^{n} K(e_{n-1}, e_i) \geq H - 1.$$

Let u and v be two unit tangent vectors at p such that:

$$\frac{\pi}{4} \leq \prec(u, v) \leq \frac{3\pi}{4}.$$

We put:

$$u = \sum_{i=1}^{n} u^i e_i, \quad v = \sum_{i=1}^{n} v^i e_i.$$

We shall prove that:

$$\sum_{i=1}^{n-1} (u^i)^2 \geq \frac{1}{8(n-1)}, \quad \text{or} \quad \sum_{i=1}^{n-1} (v^i)^2 \geq \frac{1}{8(n-1)}.$$

In fact, if

$$\sum_{i=1}^{n-1} (u^i)^2 < \frac{1}{8(n-1)}, \quad \text{and} \quad \sum_{i=1}^{n-1} (v^i)^2 < \frac{1}{8(n-1)},$$

The r-mean curvature H_r of M is the function on M defined by:

$$C_n^r H_r = E_r$$

(In particular, H_1 is the usual mean curvature H of M.)

We can prove the following

THEOREM 5. *Let M be a compact hypersurface of H^{n+1} with non-negative Ricci curvature. Assume that there exists $r \in \{1, \ldots, n\}$ such that the r-mean curvature H_r is constant. Then M is a (totally umbilic) geodesic distance sphere.*

We shall not prove this result completely. It is an easy consequence of the theory of elliptic operators. We only sketch the main steps, which follow [Wal85]: First of all, we define the following tensors L_r on M by induction:

$$\begin{aligned} L_1 &= h \\ L_r &= E_{r-1} Id - h.L_{r-2} \end{aligned}$$

We can now introduce the following operator:

DEFINITION 4. Let M be an hypersurface of a manifold of constant sectional curvature -1. We put

$$\Delta_r = \langle L_r, \mathrm{Hess} \rangle$$

that is

$$\Delta_r f = L_r \nabla \,\mathrm{Grad}\, f$$

An easy computation shows that this operator is elliptic. Moreover, we have the following

PROPOSITION 9. *Let M be an hypersurface of a manifold of constant sectional curvature -1. Then,*

$$\begin{aligned} \Delta E_r &= \Delta_r E_1 - \frac{1}{2} \sum_{i,j} E_{r-2}(\lambda_1, \ldots, \hat{\lambda}_i \ldots \hat{\lambda}_j \ldots \lambda_n)(\lambda_i - \lambda_j)^2 K_{ij} \\ &\quad + \sum_{i,j,k} E_{r-2}(\lambda_1, \ldots, \hat{\lambda}_i, \ldots, \hat{\lambda}_j, \ldots, \lambda_n)(h_{ii,k} h_{jj,k} - h_{ij,k}^2). \end{aligned}$$

The proof of the theorem is a classical consequence of the previous results: From Proposition 7, Lemma 1, and using the fact that E_r is constant, we get:

$$\Delta_r E_1 \geq 0.$$

From the Hopf lemma, we deduce that E_1 is constant. Then, Theorem 5 is a consequence of Theorem 1.

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References

[Ave75] Avez, A.: Riemannian manifolds with non-negative Ricci curvature, *Duke Math. J.* **39** (1972), 55–64.

[BC64] Bishop, R. L. and Crittenden, R. J.: *Geometry of Manifolds*, Academic Press, 1964.

[BZ80] Burago, Y. D. and Zalgaller, V. A.: *Geometric Inequalities*, Springer-Verlag, 1980.

[CE75] Cheeger, J. and Ebin, D.: *Comparison Theorems in Riemannian Geometry*, North-Holland Publ. Co., 1975.

[Che73] Chen, B. Y.: *Geometry of Submanifolds*, Marcel Dekker, 1973.

[CR85] Cecil, T. and Ryan, P.: Tight and taut immersions of manifolds, *Res. Notes Math.* **107** (1985), Pitman, London.

[Esc89] Eschenburg, J. H.: Maximum principle for hypersurfaces, *Manuscripta Math.* **64** (1989), 55–75.

[Hop83] Hopf, H.: *Differential Geometry in the Large*, Lecture Notes in Math. 1000, Springer-Verlag, 1983.

[Law69] Lawson, H. B.: Local rigidity theorems for minimal hypersurfaces, *Ann. Math.* **89** (1969), 187–197.

[NS69] Nomizu, K. and Smyth, B.: A formula of Simons' type and hypersurfaces with constant mean curvature, *J. Differential Geom.* **3** (1969), 367–377.

[Wal85] Walter, R.: Compact hypersurfaces with a constant higher mean curvature function, *Math. Ann.* **270** (1985), 125–145.

[Yau74] Yau, S. T.: Submanifolds with constant mean curvature 1, *Amer. J. Math.* **96** (1974), 346–366.

[Yau75a] Yau, S. T.: Harmonic functions on complete Riemannian manifolds, *Comm. Pure Appl. Math.* **28** (1975), 201–228.

[Yau75b] Yau, S. T.: Submanifolds with constant mean curvature 2, *Amer. J. Math.* **97** (1975), 76–100.

[YS88] Yau, S. T. and Schoen, R.: *Differential Geometry*, Scientific Publ. House, China, 1988.