$$w_*(0,\alpha^j) < w_*(T,\alpha^i) + \pi < v^*(T,\alpha^i) + \pi < v^*(0,\alpha^j) < w_*(0,\alpha^j) + \pi. \quad (42)$$

Since for $t > 0$ and for all $\alpha \in J_i$ the exact solutions of Eq. (39) with initial conditions $u_1(0, \alpha^i) \pm 0.07$ lie between w_* and v^*, the exact solutions then satisfy an inequality similar to inequality (42). The intervals J_i partially overlap; therefore, piecewise-constant initial conditions can be replaced by conditions continuous in α, and Condition 1 is satisfied (with $\ell = -1$). Consequently, the conclusions of Theorem 2 hold for system (38).

To estimate λ from above and from below, we used on each interval J_i estimates $\sin\sqrt{2} \times (t + \alpha)$ in terms of m and M and estimates $v^*(t, \alpha^i)$, $w_*(t, \alpha^i)$ of the solution $u_1(t, \alpha)$. On J_i we obtained estimates of the function $V(t, \alpha, u_1(t, \alpha))$ from above, $V^*(t, \alpha)$, and from below, $V_*(t, \alpha)$, for $t = 0, h, 2h, \ldots$. The error made by replacing the integral

$$s(T,\alpha) - s(0,\alpha) = \int_0^T V(t,\alpha,u_1(t,\alpha))\,dt$$

by an integral sum S using the trapezoid method with step h does not exceed $Th^2 \max|\ddot{V}|/12$. Since $|\dot{u}| < 1.42$, $|\ddot{u}| < 3.8$, it follows that $|\ddot{V}| < 40.9$ and the error is less than 0.022. The sum S lies between the integral sums for V_* and V^*, which may be calculated. The estimates obtained are summed with respect to i, are divided in accordance with Eq. (12) by $2 \cdot 70T$ (since $W = V/2$), and, upon taking into account the error $0.022/2T$, an estimate for λ is obtained from above and from below. Instead of $[0, T]$ it is more accurate to use the interval $[T, 2T]$. This gives the estimate $0.444 < \lambda < 0.514$.

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ASYMPTOTIC OF NONCLASSICAL RAPIDLY OSCILLATING INTEGRALS

A. M. Chebotarev

1. Integrals of the form

$$I_N = \int \prod_{k=1}^{N} (1 + iS(x)/v_k)^{-1} f(x)\,dx, \quad (1)$$

where f is an absolutely integrable function, S is a real-valued function, and $\{v_k\}_1^\infty$ is a sequence of nonnegative numbers such that

$$\sum v_k^{-1} = \infty, \quad \sum v_k^{-2} < \infty \quad (2)$$

will be called nonclassical rapidly oscillating integrals. The case Σv_k^{-2} is less interesting, since the product under the sign of integral converges to zero in the strong sense as $N \to \infty$, which makes the convergence of I_N to zero obvious. In case (2) the absolute value of the product is bounded below by a positive constant and the convergence of the product to zero is possible only in the weak sense in analogy with the convergence of rapidly oscillating integrals to zero. As shown below, the sum $S_N = \sum_{k=1}^{N} v_k^{-1}$ plays the role of a large parameter for the nonclassical rapidly oscillating integrals.

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Integrals of form (1) arise in probability theory (see [1]). Thus, e.g., for a generalized Poisson process x_t in R^n, beginning at the moment $t = 0$ at a point $q \in R^n$ and making with intensity $v(x_t)$ random jumps whose magnitudes belong to a set $B \subseteq R^n$ with probability $m(x_t, B)$, the probability of nonaccumulation of an infinite number of jumps in finite time T is estimated with the help of the mathematical expectation of the one-dimensional integrals

$$P_q\{t^* > T\} \leqslant \inf_N M_q \int \prod_{k=1}^N (1 + is/v(p_k)T)^{-1} f(s)\,ds,$$
$$s \in R^1,$$

where $M_q F = \int \ldots \int \prod_k m(p_{k-1}, dq_k) F(p_1, p_2, \ldots)$; $p_k = q + q_1 + \ldots + q_k$, f is a finite smooth function, and t^* is the so-called moment of break. We are interested in the conditions on v and m, under which the integral converges to zero. These conditions ensure the regularity of the spasmodic process: $P_q\{t^* > T\} = 1, \forall T > 0$.

2. In this note we restrict ourselves to the study of the one-dimensional integrals (1). The following lemma (see [2]) plays the main technical role.

LEMMA 1. Let f(t) be a finite continuous real-valued function on R^1. Then

$$\int f(t) \exp(ict)\,dt = O(c^{-1} \operatorname{var} f).$$

Proof. For the finite continuous real-valued function f(t) we can indicate a sequence of simple functions $\{f_N(t)\}$ that take the discrete values kN^{-1} and satisfy the conditions

$$\operatorname{supp} f_N \subseteq \operatorname{supp} f,\ 0 \leqslant f - f_N \leqslant N^{-1},$$
$$\operatorname{var}(f - f_N) = O(N^{-1}).$$

In addition,

$$\int (f(t) - f_N(t)) \exp(ict)dt = O(N^{-1}). \tag{3}$$

It is convenient to represent the simple function f_N as the sum of a finite numbers $M = M(N, f)$ of horizontal steps of height N^{-1}. The number of these steps, multiplied by $2N^{-1}$, gives the variation of the function f_N, and the sum, multiplied by N^{-1}, of the integrals over their bases gives the integral of f_N:

$$\int f_N(t) e^{ict}\,dt = N^{-1} \sum_{m=1}^M \int_{t_m}^{T_m} e^{ict}\,dt = N^{-1} \sum_{m=1}^M (ic)^{-1} (e^{icT_m} - e^{ict_m}).$$

It is easily seen that

$$(CN)^{-1} \left| \sum_m (\exp icT_m - \exp ict_m) \right| \leqslant (cN)^{-1} \sum_m 2 = c^{-1} \operatorname{var} f_N.$$

Hence the assertion of the lemma follows from (3):

$$\left| \int f e^{ict}\,dt \right| \leqslant \left| \int f_N e^{ict}\,dt \right| + O(N^{-1}) \leqslant c^{-1} \operatorname{var} f_N + O(N^{-1}) \leqslant c^{-1} \operatorname{var} f + O(N^{-1}),$$

where $O(N^{-1})$ can be made arbitrarily small as $N \to \infty$. The lemma is proved.

Two estimates of nonclassical rapidly oscillating integrals are immediate consequences of Lemma 1.

THEOREM 1. Let f(x) be a finite continuous function on R^1, $S(x) = x$, and $\{v_k\}$ be a sequence that satisfies conditions (2). Then

$$I_N = \int f(x) \prod_{k=1}^N \left(1 + \frac{ix}{v_k}\right)^{-1} dx = O(S_N^{-1}), \tag{4}$$
$$S_N = \sum_{k=1}^N v_k^{-1}.$$

Proof. Let us consider the product in integral (4):

$$\prod_{k=1}^N \left(1 + \frac{ix}{v_k}\right)^{-1} = r_N(x) \exp iz_N(x);$$
$$r_N(x) = \prod_{k=1}^N \left(1 + \frac{x^2}{v_k^2}\right)^{-1/2};\quad z_N(x) = \sum_{k=1}^N \operatorname{arctg} \frac{x}{v_k}.$$

The functions $r_N(x)$ are monotonically decreasing with respect to N and x. Therefore, the variation of the product $f(x)r_N(x)$ is uniformly bounded with respect to N. We introduce a new variable $t = t(x)$:

$$t(x)=S_N^{-1}(z_N(x)),\ \frac{dt}{dx}=S_N^{-1}\sum_{k=1}^{N} v_k^{-1}\left(1+\left(\frac{x}{v_k}\right)^2\right)^{-1}; \quad \frac{dt}{dx}(0)=1, \tag{5}$$

with respect to which integral (4) is rapidly oscillating;

$$I_N=\int F_N(x(t))\,e^{iS_N t}\,dt;\quad F_N(x)=f(x)\,r_N(x)\left|\frac{dt}{dx}(x)\right|^{-1}.$$

Let us verify that change of variables (5) is nonsingular. Indeed, it follows from the convergence of the series Σv_k^{-2} that

$$0<\sup_n v_n^{-1}=c<\infty \tag{6}$$

and that

$$1\geqslant(1+(x/v_R)^2)^{-1}\geqslant(1+(cD)^2)^{-1}=R>0,\quad D=\sup\{|x|,\ x\in\operatorname{supp}f\} \tag{7}$$

on the support of f. The nonsingularity of (5) follows from (6) and (7): $0<R\leqslant\frac{dt}{dx}\leqslant 1$. It also follows from (7) that the second Jacobian derivative approaches to zero as $N\to\infty$:

$$0\geqslant\frac{d^2t}{dx^2}\geqslant-2S_N^{-1}x\sum_{k=1}^{N}v_k^{-3}=O(S_N^{-1}x). \tag{8}$$

Therefore, $|dt/dx|^{-1}$ is a bounded smooth function and the variation of the finite continuous function F_N is uniformly bounded with respect to N. Now the assertion of the theorem follows from the estimate of Lemma 1 and the independence of variation from the choice of variable:

$$I_N=\int F_N(x(t))\exp(iS_Nt)\,dt=O(S_N^{-1}\operatorname{var}F_N)=O(S_N^{-1}).$$

The theorem is proved.

Use of the special function spaces, described in [2], enables us to drop the condition of continuity of f in the deduction of estimates of type (4).

In the case $S(x) = x^2$ the integrals of type (1) reduce to integrals analogous to the integrals, studied in the method of stationary phase [2, 3].

THEOREM 2. Let conditions (2) be fulfilled and f be a function of bounded variation that is Lipschitz-continuous at the origin. Then the nonclassical analogue of the method of stationary phase is valid:

$$I_N=\sqrt{\frac{S_N}{\pi}}\int f(x)\prod_{k=1}^{N}\left(1+\frac{ix^2}{v_k}\right)^{-1}dx=f(0)+O(S_N^{-1/6}). \tag{9}$$

Proof. We introduce a new variable $t = t(x)$:

$$t=\left(S_N^{-1}\left(\sum_{k=1}^{N}\operatorname{arctg}\left(\frac{x^2}{v_k}\right)\right)\right)^{1/2}\operatorname{sign}x,$$

$$\frac{dt}{dx}=S_N^{-1/2}x\sum_{k=1}^{N}v_k^{-1}\left(1+\frac{x^4}{v_k^2}\right)^{-1}\Big/\left(\sum_{k=1}^{N}\operatorname{arctg}\frac{x^2}{v_k}\right)^{1/2}, \tag{10}$$

$$\frac{dt}{dx}(0)=1,$$

with respect to which integral (9) is a "standard" integral of the method of stationary phase. Using the inequalities

$$0<R=(1+(cD^2)^2)^{-1}\leqslant S_N^{-1}\sum_{k=1}^{N}v_k^{-1}\left(1+\frac{x^4}{v_k^2}\right)^{-1}\leqslant 1,$$

$$0<r\leqslant\frac{\operatorname{arctg}cD^2}{cD^2}\leqslant S_N^{-1}\sum_{k=1}^{N}v_k^{-1}\operatorname{arctg}\left(\frac{x^2}{v_k}\right)\frac{v_k}{x}\leqslant 1,$$

where c and D are defined by Eqs. (6) and (7), we get an estimate of the Jacobian:

$$0<R\leqslant\frac{dt}{dx}\leqslant r^{-1/2},\quad x\in\operatorname{supp}f,$$

ensuring the nonsingularity of (10). The estimate of the second derivative is carried out in the same manner as (8). Omitting the cumbersome computations, we give the final result:

$$\frac{d^2t}{dx^2}=O(S_N^{-1}x^2)+O(S_N^{-2}x),\quad x\in\operatorname{supp}f. \tag{11}$$

Thus, all three functions f, r_N, and dt/dx have variations, uniformly bounded with respect to N. Since the set of functions of bounded variation is an algebra, the functions

$$F_N(t)=f(x)\,r_N(x)\left|\frac{dt}{dx}\right|^{-1}_{x=x(t)},\quad F_N(0)-F_N(t)=f(0)-F_N(t)$$

are also functions of uniformly bounded variation.

It follows from (11) and the nonsingularity of (10) that the Jacobians dx(t)/dt and dt(x)/dx are uniformly (with respect to N) Lipschitz-continuous. In the sequel we will use the Lipschitz-continuity of the first Jacobian at the origin.

The Lipschitz-continuity of the function $r_N(x)=\prod_{k=1}^{N}(1+x^4/v_k^2)^{-1/2}$ at the origin follows from the estimate

$$0\leqslant r_N(0)-r_N(x)\leqslant\frac{1}{2}x^4\sum_{k=1}^{N}v_k^{-2}=O(x^4);$$
$$r_N(0)-r_N(x(t))=O(t^4).$$

Thus, the function $F_N(t)$ is Lipschitz-continuous at the origin uniformly with respect to N. Since $r_N(0)=1$, $(dt/dx)(0)=1$, and $F_N(0)=f(0)$, we can represent integral (9) in the form of a sum: $I_N=f(0)+J_1+J_2$, where

$$J_1=\sqrt{\frac{S_N}{\pi}}\int\limits_{|t|\leqslant S_N^{-1/3}}(F_N(t)-F_N(0))\,e^{iS_Nt^2}\,dt;$$
$$J_2=\sqrt{\frac{S_N}{\pi}}\int\limits_{|t|\geqslant S_N^{-1/3}}\frac{F_N(t)-F_N(0)}{t}\,e^{iS_Nt^2}\,dt^2.$$

In the estimation of the integral over the domain $|t|\leqslant S_N^{-1/3}$ we use the uniform Lipschitz-continuity of the function $F_N(t)$ at the origin

$$J_1=\sqrt{\frac{S_N}{\pi}}\,O(S_N^{-2/3})=O(S_N^{-1/6}),$$

and in the estimation of the integral over the domain $|t|\geqslant S_N^{-1/3}$ we use the result of Lemma 1 and the estimate of variation

$$\operatorname*{var}_{|t|\geqslant S_N^{-1/3}}(F_N(t)-F_N(0))\,t^{-1}\leqslant S_N^{1/3}\operatorname{var}F_N$$
$$J_2=\sqrt{\frac{S_N}{\pi}}\,O(S_N^{1/3}S_N^{-1}\operatorname{var}F_N)=O(S_N^{-1/6}).$$

Thus, $I_N=f(0)+O(S_N^{-1/6})$. The theorem is proved.

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