The Conjugacy Problem for a Free Product with Amalgamation

By

C. R. J. Clapham

1. Introduction. Let $\{A, B; H = K\}$ denote the free product of two finitely generated groups, amalgamating the subgroup H of A and the subgroup K of B. The word problem is solvable under simple conditions (see page 211 of [4] or Lemma 3.5 of [2]), but the conjugacy problem is much more complicated.

We find conditions under which the conjugacy problem is solvable. It emerges that the problem divides clearly into two parts: dealing with words of length 1, and dealing with words of length > 1. The latter is investigated in section 3 and the notion of a subgroup satisfying a *finite coset intersection property* is used.

Closely allied with free products is the concept of a HNN group (or 'Britton extension'), and conditions that a HNN group have solvable conjugacy problem are found in section 4.

The terms "word problem" and "conjugacy problem" refer in the first place to group presentations. But we deal throughout this paper with finitely generated groups. If the word problem or conjugacy problem is solvable for one finitely generated presentation of a group, then it is solvable for any other finitely generated presentation of that group. So we shall freely speak of the solvability of the word problem or conjugacy problem for a finitely generated group.

We use the term "*generalised word problem for H in A*" to describe the problem of determining, for arbitrary w in A, whether or not $w \in H$. And we use "*generalised conjugacy problem for H in A*" for the problem of determining, for arbitrary w in A, whether or not w is conjugate to some element of H.

2. The Conjugacy Problem for a Free Product. The following is an immediate consequence of Solitar's Theorem (Theorem 4.6 of [4]), and gives sufficient conditions that a free product (where, as always, we mean "with amalgamation") should have solvable conjugacy problem:

Theorem. *The conjugacy problem for $\{A, B; H = K\}$ is solvable if*

(i) *the generalised word problem for H in A is solvable, and the generalised word problem for K in B is solvable, and the isomorphism between H and K is recursive,*

(ii) *the generalised conjugacy problem for H in A is solvable, and the generalised conjugacy problem for K in B is solvable,*

(iii) *the conjugacy problems for A and B are solvable,*

(iv) (elements of length 1) *given w and w' in H, we can determine if there is a finite sequence $h_1, h_2, \ldots, h_s$ of elements of H with $w = h_1$ and $w' = h_s$, such that, for $i = 1, \ldots, s-1$, h_i and h_{i+1} are conjugate in one of the factors,*

(v) (elements of length > 1) *for arbitrary words $w_1, \ldots, w_r$, $w_1', \ldots, w_r'$ (r even), with w_i, w_i' in $A \setminus H$ for i odd, and w_i, w_i' in $B \setminus K$ for i even, we can determine if there is a sequence $h_1, \ldots, h_r$ of elements of H $(=K)$ such that*

$$h_1^{-1} w_1 h_2 = w_1', \quad h_2^{-1} w_2 h_3 = w_2', \ldots, h_r^{-1} w_r h_1 = w_r'.$$

We may record the following:

Note 1. *If the isomorphism between H and K can be extended to an isomorphism between A and B,* then (iv), the case of elements of length 1, is easily determined. For, in that case, h_i and h_{i+1} are conjugate in one factor if and only if they are conjugate in the other factor; thus if there is such a sequence, w and w' are simply conjugate in one of the factors.

Note 2. *If H is a direct factor of A* (or similarly if K is a direct factor of B), then again case (iv) is easily considered. For if $A = H \times L$, and h_i and h_{i+1} are conjugate in A, then for some h in H and l in L,

$$(h, l)^{-1} (h_i, 1) (h, l) = (h_{i+1}, 1),$$

i.e. $h^{-1} h_i h = h_{i+1}$. Thus h_i and h_{i+1} are conjugate by an element of H and hence conjugate also in the other factor B. So w and w' are conjugate in the free product if and only if they are conjugate in B.

3. The Finite Coset Intersection Property. We shall investigate further the case (v), which is concerned with elements of length > 1.

Given two elements w_1, w_1' of $A \setminus H$, we want to determine if there are elements h_1 and h_2 of H such that

$$h_1^{-1} w_1 h_2 = w_1',$$

i.e. such that

$$w_1 h_2 = h_1 w_1'.$$

Suppose now, that the number of elements in the intersection $w_1 H \cap H w_1'$ is finite and known, then the number of possible choices for h_1 is known. The elements of A can be effectively enumerated without repetitions, by listing all the words in the generators and rejecting those that are found (since the word problem is solvable) to be equal to an earlier word. We can test if each in turn belongs to H. With each element h (say) of H that we get, we test whether $w_1^{-1} h w_1'$ also belongs to H. If so, then this h is a possible choice for h_1, and we can continue until we have found the full number. For each h_1, we find the corresponding $h_2, \ldots, h_r$ (if possible) and finally test if $h_r^{-1} w_r h_1 = w_r'$. Thus we can determine if such a sequence as is required exists.

Definition. *If H is a subgroup of a finitely generated group G, let $n_H(g_1, g_2) =$ the number of elements in $g_1 H \cap H g_2$.*

H has the finite coset intersection property (f. c. i. p.) *in G, if for all g_1 and g_2 in $G \setminus H$, $n_H(g_1, g_2)$ is finite and n_H is a recursive function of g_1 and g_2.*

Our conclusion is that we can determine whether a sequence as described in (v) exists or not, if *either* H has f. c. i. p. in A *or* K has f. c. i. p. in B.

Examples of subgroups with the finite coset intersection property:

(1) *If G has solvable word problem, and the generalised word problem for H in G is solvable, and H is finite of known order, then H has f.c.i.p. in G.*

(2) *G has f.c.i.p. in G.*

(3) *If A and B have solvable word problems, and the generalised word problem for H in A is solvable, and H has f.c.i.p. in A, then H has f.c.i.p. in the ordinary free product $A * B$.*

Proof. Given elements w and w' of the free product, not in H, write them as

$$w = a_1 b_1 \cdots a_r b_r a_{r+1}$$

(where $a_i \in A$, $b_i \in B$, for all i, and a_1, a_{r+1} are possibly empty, and possibly $r = 0$); and

$$w' = a_1' b_1' \cdots a_s' b_s' a_{s+1}',$$

(similarly).

Suppose that $w h_2 = h_1 w'$, with h_1, h_2 in H. Then

$$a_1 b_1 \cdots a_r b_r (a_{r+1} h_2) = (h_1 a_1') b_1' \cdots b_s' a_{s+1}'.$$

By the normal form theorem, $r = s$, and if $r \neq 0$, $a_1 = h_1 a_1'$, $b_1 = b_1', \ldots, b_r = b_s'$, $a_{r+1} h_2 = a_{s+1}'$.

Thus, if $b_i = b_i'$ $(i = 1, \ldots, r)$ and $a_i = a_i'$ $(i = 2, \ldots, r)$ and if $a_{r+1}^{-1} a_{s+1}' \in H$ and $a_1 a_1'^{-1} \in H$, then $wH \cap Hw'$ contains exactly one element. If not, $wH \cap Hw'$ is empty.

If $r = s = 0$, $a_1 h_2 = h_1 a_1'$. Since w and w' are not in H, a_1 and a_1' are not in H, and we see that $n_H(w, w') = n_H(a_1, a_1')$; and we know that H has f.c.i.p. in A.

Hence, H has f.c.i.p. in $A * B$.

(4) *If A and B have solvable word problems, then each factor A and B has f.c.i.p. in the ordinary free product $A * B$.*

This follows from (2) and (3).

(5) *If A and B have solvable word problems, and the generalised word problems for H in A and for K in B are solvable, and the isomorphism between H and K is recursive, then the factor B has f.c.i.p. in the free product with amalgamation $\{A, B; H = K\}$ if H has f.c.i.p. in A.*

(The previous example is the special case of this one, with the amalgamated subgroup equal to $\{1\}$, which, being finite, has f.c.i.p. in either factor, by (1).)

Proof. Given elements w and w' of the free product, not in B, write them as

$$w = a_1 b_1 \cdots a_r b_r a_{r+1}, \quad w' = a_1' b_1' \cdots a_s' b_s' a_{s+1}',$$

where $a_i, a_i' \in A \setminus H$, and $b_i, b_i' \in B \setminus K$, for all i.

Suppose that $w\beta_2 = \beta_1 w'$, where β_1 and β_2 are in B. Then

$$a_1 b_1 \cdots a_r b_r a_{r+1} \beta_2 = \beta_1 a_1' b_1' \cdots a_s' b_s' a_{s+1}'.$$

(We may now assume that a_1, a_{r+1}, a_1', a_{s+1}' are not empty, for if a_1', for example, is empty, the last equation holds if and only if

$$a_1 b_1 \cdots a_r b_r a_{r+1} \beta_2 = \beta_1^* a_2' b_2' \cdots a_s' b_s' a_{s+1}',$$

where $\beta_1^* = \beta_1 b_1'$. The number of solutions for β_1 is the same as the number of solutions for β_1^*.) Hence

$$\beta_2 = a_{r+1}^{-1} b_r^{-1} a_r^{-1} \cdots b_1^{-1} a_1^{-1} \beta_1 a_1' b_1' \cdots a_s' b_s' a_{s+1}'.$$

Since the LHS has length 1, there must be cancellation on the RHS, so β_1 must be in the amalgamated subgroup, i.e. $\beta_1 = h$, say, and $a_1^{-1} h a_1'$ also in H. Thus $a_1 h' = h a_1'$, where a_1 and $a_1' \in A \setminus H$.

So we effectively enumerate the elements of A without repetitions, test if each belongs to H, and with each element h of H that we get, test whether $a_1^{-1} h a_1'$ also belongs to H. Since H has f.c.i.p. in A, we shall know when we have found the full number $n_H(a_1, a_1')$ of possible choices for h. Now test each of these to see if it is a possible choice for β_1, by finding out if $w^{-1}\beta_1 w'$ is also in B. This involves just the generalised word problem for B in $\{A,\ B;\ H = K\}$ and this is easily seen to be solvable under the conditions we have (see, if necessary, Cor. 3.5.1 of [2]). In this way, the number of solutions for β_1 can be found systematically, so that $n_B(w, w')$ is recursive as required.

4. Application to a HNN group. If A is a finitely generated subgroup of a finitely generated group G, then it is well-known that $\{G, t;\ t^{-1}\, at = a, \text{ for all } a \text{ in } A\}$ is the presentation of a group containing G as a subgroup and is a special case of a HNN group (or "Britton extension" of G), see [3] and [1]. Indeed, let A' be a copy of the subgroup A, and form the direct product $A' \times \{t\}$, where $\{t\}$ is an infinite cyclic group; then the above is a presentation of the free product with amalgamation $\{G, (A' \times \{t\});\ A = A'\}$.

Thus, the conjugacy problem for $\{G,\ t;\ t^{-1}at = a, \text{ for all } a \text{ in } A\}$ is solvable if we can satisfy the appropriate requirements of the Theorem of section 2:

(i) The generalised word problem for A in G is solvable. (The generalised word problem for A' in $A' \times \{t\}$ is clearly solvable.)

(ii) The generalised conjugacy problem for A in G is solvable. (The generalised conjugacy problem for A' in $A' \times \{t\}$ is easily seen to be solvable.)

(iii) The conjugacy problems for G and A are solvable. (The solvability of the conjugacy problem for A does not, of course, follow from that for G, and is necessary in order that A' and hence $A' \times \{t\}$ have solvable conjugacy problem.)

(iv) This part is dealt with by Note 2 of section 2, since A' is a direct factor of $A' \times \{t\}$.

(v) This part is dealt with by the work of section 3 if A has f.c.i.p. in G.

We can sum up therefore:

Theorem. *The HNN group* $\{G, t;\ t^{-1}at = a,$ *for all* a *in* $A\}$ *has solvable conjugacy problem if the generalised word problem and the generalised conjugacy problem for* A *in* G *are solvable, if* G *and* A *have solvable conjugacy problems, and if* A *has the finite coset intersection property in* G.

References

[1] J. L. BRITTON, The word problem. Ann. of Math. **77**, 16–32 (1963).
[2] C. R. J. CLAPHAM, Finitely presented groups with word problems of arbitrary degrees of insolubility. Proc. London Math. Soc. (3) **14**, 633–676 (1964).
[3] G. HIGMAN, B. H. NEUMANN and H. NEUMANN, Embedding theorems for groups. J. London Math. Soc. **24**, 247–254 (1949).
[4] W. MAGNUS, A. KARRASS and D. SOLITAR, Combinatorial Group Theory. New York 1966.

Eingegangen am 13. 10. 1970

Anschrift des Autors:
C. R. J. Clapham
Department of Mathematics
University of Aberdeen
Aberdeen, Scotland