FURTHER SCALING PROPERTIES FOR CIRCLE MAPS

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ABSTRACT. It is shown that certain iterations of $(k-1)$tuples of commuting invertible circle maps whose rotation numbers are algebraic of degree k, show very similar scaling properties to those found by Feigenbaum *et al.* in the case $k = 2$.

Several groups [1–3] have recently found some quite remarkable universal scaling properties for invertible maps of a circle whose rotation number has a periodic continued fraction expansion. They can be used to model the transition in two-parameter families of systems from quasiperiodicity to some kind of chaos. These scaling properties can be understood by a renormalization group transformation $\mathscr{R}$ acting on a space of a pair of functions (ξ, η) which, for rotation number $\omega = [n, n, n, \ldots]$, is defined as

$$\mathscr{R}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \frac{1}{\lambda}\begin{pmatrix} \xi^n \circ \eta(\lambda x) \\ \xi(\lambda x) \end{pmatrix}, \quad \lambda = -\xi(0). \tag{1}$$

The symbol $[n, n, \ldots]$ denotes, as usual, the continued fraction expansion of the number ω. The above transformation $\mathscr{R}$ is expected to have exactly two fixed points in an appropriate space. One of them is trivial and corresponds to a pure rotation of the circle by the angle ω. The other nontrivial fixed point whose existence, up to now, follows only from computer calculations, describes a circle map with a critical point and can be conjugated to a pure rotation only via a nondifferentiable circle homeomorphism. It is, therefore, expected to describe some kind of chaos.

In this letter we will show that this theory allows for a generalization to the $(k-1)$ tuples of circle maps whose rotation numbers are algebraic of degree k. In the case $k = 2$, we will recover the results just described.

The scaling relations, respectively the renormalization group transformation, have been derived in the last case by iterating a circle map f with rotation number ω for a certain number of times given by the rational approximants of this number. These approximants (p_n, q_n) are defined by truncating the continued fraction expansion of ω. They can be calculated by an iterated application of the 2×2 matrix A_n defined as

$$A_n = \begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix} \tag{2}$$

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to the initial vector $(p_0, q_0) = (0, 1)$.

To extend the theory to $(k-1)$tuples of commuting circle maps, we now need the analogous rational approximants for a $(k-1)$tuple of rotation numbers. They can be described by an integer k vector $(q_{1,m}, \dots, q_{k,m})$ with the property that for all $1 \leqslant i \leqslant k-1$ $\lim_{m\to\infty} q_{i,m}/q_{k,m} = \omega_i$, where ω_i denotes the rotation numbers of a $k-1$ tuple of the circle maps.

An algorithm providing such approximation vectors was established by Jacobi and Perron [4]. The matrix A_n in (2) is thereby replaced by the following $k \times k$ integer matrix $A_{\mathbf{n}}$ depending on the integer vector $\mathbf{n} = (n_1, \dots, n_{k-1})$:

$$A_{\mathbf{n}} = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 1 \\ 1 & 0 & \cdots & \cdots & n_1 \\ 0 & 1 & 0 & \cdots & n_2 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 1 & n_{k-1} \end{pmatrix} \tag{3}$$

where the integers n_i fulfill the relations $0 \leqslant n_i \leqslant n_{k-1}$, $n_{k-1} \geqslant 1$ and $(n_{k-1}, n_{k-2}, \dots, n_{k-1-j}) \geqslant (n_j, n_{j-1}, \dots, n_1, 1)$ in the lexicographic order for all $1 \leqslant j \leqslant k-2$. If we then set

$$\begin{pmatrix} q_{1,m} \\ \vdots \\ q_{k,m} \end{pmatrix} = A_{\mathbf{n}}^m \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{4}$$

it follows from the Perron–Frobenius Theorem applied to the matrix $A_{\mathbf{n}}$ that the vectors $(q_{1,m}, \dots, q_{k,m})$ are indeed rational approximation vectors for the $(k-1)$tuple of rotation numbers $(\omega_1, \dots, \omega_{k-1})$, where ω_i is given as

$$\omega_i = \omega_*^i + n_1 \omega_*^{i-1} + \dots + n_{i-1}\omega_*.$$

The number ω_* itself is the positive solution of the polynomial equation

$$\omega^k + n_1 \omega^{k-1} + \dots + n_{k-1}\omega - 1 = 0 \tag{5}$$

and, therefore, an algebraic number. If we choose the vector $\mathbf{n}$ in such a way that all other roots of (5) except ω_* are strictly larger than one in absolute value, then ω_* and, therefore, all ω_i are algebraic of degree k and the following approximation property holds:

$$\lim_{m\to\infty} |q_{k,m}\omega_i - q_{i,m}| = 0. \tag{6}$$

It should be remarked that this property does not generally hold for the approximants determined by the Jacobi–Perron algorithm [4].

To now define the scaling relations for $(k-1)$ tuples of the circle maps, we need a second integer $k \times k$ matrix $B_{\mathbf{n}}$ which is closely related to the matrix $A_{\mathbf{n}}$ in (3):

$$B_{\mathbf{n}} = \begin{pmatrix} n_1 & -1 & 0 \ldots\ldots\ldots 0 \\ n_2 & 0 & -1 \quad 0 \ldots\ldots 0 \\ \vdots & \vdots & \vdots \\ n_{k-2} & 0 & \ldots\ldots\ldots -1 \quad 0 \\ n_{k-1} & 0 & \ldots\ldots\ldots\ 0 \quad 1 \\ 1 & 0 & \ldots\ldots\ldots\ldots 0 \end{pmatrix}. \tag{7}$$

It is easy to see that its eigenvalues ρ_i are given simply in terms of the eigenvalues λ_i of the matrix $A_{\mathbf{n}}$ as $\rho_i = -1/\lambda_i$.

Define then for all $1 \leqslant i \leqslant k$ and all $1 \leqslant j \leqslant k-1$ integers $a_{i,j}^{(m)}$ and $c_i^{(m)}$ as

$$\begin{pmatrix} a_{1,j}^{(m)} \\ \vdots \\ a_{k,j}^{(m)} \end{pmatrix} = B_{\mathbf{n}}^m \begin{pmatrix} a_{1,j}^{(0)} \\ \vdots \\ a_{k,j}^{(0)} \end{pmatrix}; \quad \begin{pmatrix} c_1^{(m)} \\ \vdots \\ c_k^{(m)} \end{pmatrix} = B_{\mathbf{n}}^m \begin{pmatrix} c_1^{(0)} \\ \vdots \\ c_k^{(0)} \end{pmatrix} \tag{8}$$

with $a_{i,j}^{(0)} = \delta_{i,j}$, $c_j^{(0)} = 0$ and $c_k^{(0)} = 1$.

A straightforward calculation then shows that

$$\lim_{m \to \infty} \left| \sum_{j=1}^{k-1} a_{i,j}^{(m)} \omega_j - c_i^{(m)} \right| = 0, \quad \text{for all } 1 \leqslant i \leqslant k.$$

Consider now smooth invertible circle maps $f_1, \ldots, f_{k-1}$ which all commute and have rotation numbers $\rho(f_i) = \omega_i$ as defined in (5). It follows from the results of Yoccoz [5] and Kopell [6] that there exists a smooth circle homeomorphism h such that $f_i = h \circ R_{\omega_i} \circ h^{-1}$ for all i. The $(k-1)$ tuple in fact defines what mathematicians call a differentiable action of the abelian group $\mathbb{Z}^{k-1}$ on the circle. R_{ω_i} in the above expression denotes the pure rotation of the circle by the angle ω_i. Now we can write down the scaling relations: define for $m \geqslant 1$ and all $1 \leqslant i \leqslant k$ functions $\xi_{i,m}$ as follows

$$\xi_{i,m}(x) = 1/\lambda_m (f_1^{a_{i,1}^{(m)}} \circ \cdots \circ f_{k-1}^{a_{i,k-1}^{(m)}}(\lambda_m x) - c_i^{(m)}) \tag{9}$$

where

$$\lambda_m = -(f_1^{a_{k,1}^{(m)}} \circ \cdots \circ f_{k-1}^{a_{k,k-1}^{(m)}}(0) - c_k^{(m)}).$$

One then shows in complete analogy to the procedure in [3] that

$$\lim_{m \to \infty} \xi_{i,m}(x) = R_{\omega_i}(x), \quad \text{for } 1 \leqslant i \leqslant k-1 \tag{10}$$

$$\lim_{m\to\infty} \xi_{k,m}(x) = x - 1 \quad \text{and} \lim_{m\to\infty} \lambda_{m+1}/\lambda_m = -\omega_1 .$$

The above limits exist and do not depend on the conjugating homeomorphism h. The scaling relations (10) correspond in the case $k = 2$ exactly to the ones found in [1–3] as one can see by looking at the matrices A_n and B_n for this case.

It is also very easy to derive in a next step a renormalization group transformation explaining these scaling relations (10). The procedure is well known so that we can immediately obtain the result. In the general case of arbitrary k, the transformation $\mathscr{R}$ acts in a space of k-tuples $\xi = (\xi_1, \ldots, \xi_k)$ of functions ξ_i on the real line as follows:

$$\mathscr{R}\xi = \frac{1}{\lambda}\begin{pmatrix} \xi_1^{n_i} \circ \xi_{i+1}^{-1}(\lambda x) \quad 1 \leqslant i \leqslant k-2 \\ \xi_1^{n_{k-1}} \circ \xi_k(\lambda x) \\ \xi_1(\lambda x) \end{pmatrix} \tag{11}$$

with $\lambda = -\xi_1(0)$. For $k = 2$, this transformation is exactly the one studied in [3].

The transformation $\mathscr{R}$ in (11) again has a trivial fixed point ξ^* with $\xi_i^* = R_{\omega_i}$ for $1 \leqslant i \leqslant k - 1$ and $\xi_k^*(x) = x - 1$ corresponding to a $k - 1$ tuple of pure rotations of the circle. The structure of this fixed point can be completely analysed as for $k = 2$, and also turns out to be hyperbolic with, for general k, an unstable manifold which has the dimension $k - 1$.

We see, therefore, that at least as far as the trivial side of the diffeomorphisms is concerned, our theory presented above is in complete analogy to the one for $k = 2$, as developed in [1–3]. There remains, however, the much more difficult problem concerning the nontrivial side of this whole theory, namely the possible existence of a second nontrivial fixed point for our renormalization group transformation. As in the case $k = 2$, however, one is then forced to do explicit computer calculations. This is planned for the future.

REFERENCES

1. Feigenbaum, M., Kadanoff, L., and Shenker, S., *Physica* **5D**, 370–386 (1982).
2. Shenker, S., *Physica* **5D**, 405–411 (1982).
3. Ostlund, S., Rand, D., Sethna, J., and Siggia, E., *Physica* **8D**, 303–342 (1983).
4. Perron, O., *Math. Ann.* 64, 1–76 (1907).
5. Yoccoz, J., in J. Palis (ed.), *Geometric Dynamics*, Lecture Notes in Math., vol. 1007, Springer-Verlag, Berlin, 1983.
6. Kopell, N., in S. Chern and S. Smale (eds.) *Global Analysis*, Proc. Symp. Pure Math. XIV, Am. Math. Soc., Providence, RI, 1970.

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