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Inversion of the Pieri formula for Macdonald polynomials

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Abstract

We give the explicit analytic development of Macdonald polynomials in terms of "modified complete" and elementary symmetric functions. These expansions are obtained by inverting the Pieri formula. Specialization yields similar developments for monomial, Jack and Hall–Littlewood symmetric functions.

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1. Introduction

Fifty years ago, Hua [3] introduced a new family of polynomials defined on the space of complex symmetric matrices, and set the problem of finding their explicit analytic expansion in terms of Schur functions [3, p. 132].

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These polynomials were further investigated by James [4], who named them "zonal polynomials", studied their connection with symmetric group algebra, and gave a method to compute them. A large literature followed, mostly due to statisticians, but no explicit analytic formula was found for the zonal polynomials.

Hua's problem is now better understood in the more general framework of Macdonald polynomials (of type A_n) [19]. Zonal polynomials are indeed a special case of Jack polynomials, which in turn are obtained from Macdonald polynomials by taking a particular limit.

Macdonald polynomials are indexed by partitions, i.e. finite decreasing sequences of positive integers. These polynomials form a basis of the algebra of symmetric functions with rational coefficients in two parameters q, t. They generalize many classical bases of this algebra, including monomial, elementary, Schur, Hall–Littlewood, and Jack symmetric functions. These particular cases correspond to various specializations of the indeterminates q and t.

Two combinatorial expansions were known for Macdonald polynomials. The first one gives them as a sum of monomials associated with tableaux [19, p. 346]. The second one writes their development in terms of Schur functions as a determinant [12]. However, in general both methods do not lead to an analytic formula, since they involve combinatorial quantities which cannot be written in analytic terms.

Thus Hua's problem kept open for Macdonald polynomials. Their analytic expansion was explicitly known only when the indexing partition is a hook [6], has length two [5] or three [14], and in the dual cases corresponding to parts at most equal to three.

The aim of this paper is to present a general solution to this problem and to provide two explicit analytic developments for Macdonald polynomials. One of them is made in terms of elementary symmetric functions. The other one is made in terms of "modified complete" symmetric functions, which have themselves a known development in terms of any classical basis [15].

In the special case $q = t$, these two developments coincide with the classical Jacobi–Trudi formulas for Schur functions. Thus, our results appear as generalized Jacobi–Trudi expansions for Macdonald polynomials.

Our method relies on two ingredients, firstly, the Pieri formula for Macdonald polynomials, secondly, a method developed by Krattenthaler [9,10] for inverting infinite multidimensional matrices.

The Pieri formula has been obtained independently by Koornwinder [8] and Macdonald [18]. Most of the time, it is stated in combinatorial terms. We formulate it in analytic terms, hereby defining an infinite multidimensional matrix. We invert this "Pieri matrix" by adapting Krattenthaler's operator method to the multivariate case, as already done elsewhere [21] by the second author.

This article is organized as follows. Section 2 is devoted to the inversion of infinite multidimensional matrices and may be read independently of the rest of the paper. In Section 3, we introduce our notation and recall general facts about Macdonald polynomials. In particular we give the analytic form of the Pieri formula. The infinite multidimensional matrix thus defined is inverted in Section 4. The generalized Jacobi–Trudi expansions for Macdonald polynomials are derived in Section 5. Sections 6–8 are devoted to various specializations of our results, in particular for Schur, monomial, Hall–Littlewood and Jack symmetric functions. Most of the expansions obtained there are new. The example of hook partitions, already studied by Kerov [6,7], is then considered in Section 9. We conclude with a brief discussion of a possible extension of Macdonald polynomials.

Our results were announced in [17]. An alternative proof of our main theorem has subsequently been given in [16] (but requires the explicit form of the result here obtained). It is an open question whether our method can be generalized to Macdonald polynomials associated with other root systems than A_n.

2. New multidimensional matrix inversions

2.1. Krattenthaler's method

Let Z be the set of integers, n some positive integer and Z^n the set of multi-integers $\mathbf{m} = (m_1, \ldots, m_n)$. We write $\mathbf{m} \geqslant \mathbf{k}$ for $m_i \geqslant k_i$ $(1 \leqslant i \leqslant n)$, and for any set of indeterminates $\mathbf{z} = (z_1, \ldots, z_n)$, we put $\mathbf{z}^{\mathbf{m}} = z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$.

Let $\mathcal{L}$ be the algebra of formal Laurent series, defined as series of the form $a(\mathbf{z}) = \sum_{\mathbf{m} \geqslant \mathbf{k}} a_{\mathbf{m}} \mathbf{z}^{\mathbf{m}}$, for some $\mathbf{k} \in \mathsf{Z}^n$. On $\mathcal{L}$ we introduce the bilinear form $\langle\, , \rangle$ defined by

$$\langle a(\mathbf{z}), b(\mathbf{z}) \rangle = [a(\mathbf{z})\, b(\mathbf{z})]_1,$$

where $[c(\mathbf{z})]_1$ denotes the constant term in $c(\mathbf{z})$. Given any linear operator L on $\mathcal{L}$, we write $L \in \mathrm{End}(\mathcal{L})$ and denote L^* its adjoint with respect to $\langle\, , \rangle$, i.e. $\langle L^* a(\mathbf{z}), b(\mathbf{z}) \rangle = \langle a(\mathbf{z}), L b(\mathbf{z}) \rangle$.

We say that an infinite n-dimensional matrix $F = (f_{\mathbf{mk}})_{\mathbf{m},\mathbf{k} \in \mathsf{Z}^n}$ is lower-triangular if $f_{\mathbf{mk}} = 0$ unless $\mathbf{m} \geqslant \mathbf{k}$. When all $f_{\mathbf{kk}} \neq 0$, there exists a unique lower-triangular matrix $G = (g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l} \in \mathsf{Z}^n}$, called the inverse of F, such that

$$\sum_{\mathbf{m} \geqslant \mathbf{k} \geqslant \mathbf{l}} f_{\mathbf{mk}}\, g_{\mathbf{kl}} = \delta_{\mathbf{ml}}$$

for all $\mathbf{m}, \mathbf{l} \in \mathsf{Z}^n$, where $\delta_{\mathbf{ml}}$ is the usual Kronecker symbol.

In [9] Krattenthaler gave a method for solving Lagrange inversion problems, which are closely connected with inversion of lower-triangular matrices. We shall need the following special case of [9, Theorem 1].

Let $F = (f_{\mathbf{mk}})_{\mathbf{m},\mathbf{k} \in \mathsf{Z}^n}$ be an infinite lower-triangular matrix with all $f_{\mathbf{kk}} \neq 0$, and $G = (g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l} \in \mathsf{Z}^n}$ its inverse matrix. Define the formal Laurent series

$$f_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{m} \geqslant \mathbf{k}} f_{\mathbf{mk}}\, \mathbf{z}^{\mathbf{m}}, \quad g_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l} \leqslant \mathbf{k}} g_{\mathbf{kl}}\, \mathbf{z}^{-\mathbf{l}}.$$

Assume that

(i) there exist operators $U_i, V \in \mathrm{End}(\mathcal{L})$, V being bijective, such that for all $\mathbf{k} \in \mathsf{Z}^n$, one has

$$U_i f_{\mathbf{k}}(\mathbf{z}) = c_i(\mathbf{k}) V f_{\mathbf{k}}(\mathbf{z}), \quad 1 \leqslant i \leqslant n \tag{2.1}$$

with $c_i(\mathbf{k})$ some sequences of constants;

(ii) for all $\mathbf{m} \neq \mathbf{k} \in \mathsf{Z}^n$, there exists $j \in \{1, \ldots, n\}$ such that $c_j(\mathbf{m}) \neq c_j(\mathbf{k})$. (2.2)

Lemma 2.1. *If for all* $\mathbf{k} \in \mathsf{Z}^n$ *there exists* $h_{\mathbf{k}}(\mathbf{z}) \in \mathcal{L}$, $h_{\mathbf{k}}(\mathbf{z}) \neq 0$, *such that*

$$U_i^* h_{\mathbf{k}}(\mathbf{z}) = c_i(\mathbf{k}) V^* h_{\mathbf{k}}(\mathbf{z}), \quad 1 \leqslant i \leqslant n,$$

we have

$$g_{\mathbf{k}}(\mathbf{z}) = \frac{1}{\langle f_{\mathbf{k}}(\mathbf{z}), V^* h_{\mathbf{k}}(\mathbf{z}) \rangle} V^* h_{\mathbf{k}}(\mathbf{z}).$$

We shall apply this result of Krattenthaler through the following corollary, which is a special case of [21, Corollary 2.14], already used in [11, Corollary 2.2]. Assume that

(i) there exist operators U_i, $V_{ij} \in \mathrm{End}(\mathcal{L})$, satisfying the commutation relations

$$\begin{aligned} U_{i_1} V_{i_2 j} &= V_{i_2 j} U_{i_1}, && i_1 \neq i_2, \quad 1 \leqslant i_1, i_2, j \leqslant n, \\ V_{i_1 j_1} V_{i_2 j_2} &= V_{i_2 j_2} V_{i_1 j_1}, && i_1 \neq i_2, \quad 1 \leqslant i_1, i_2, j_1, j_2 \leqslant n; \end{aligned} \tag{2.3}$$

(ii) the operator $\det_{1 \leqslant i,j \leqslant n}(V_{ij})$ is invertible;

(iii) for all $\mathbf{k} \in \mathsf{Z}^n$ one has

$$U_i f_{\mathbf{k}}(\mathbf{z}) = \sum_{j=1}^{n} c_j(\mathbf{k}) V_{ij} f_{\mathbf{k}}(\mathbf{z}), \quad 1 \leqslant i \leqslant n \tag{2.4}$$

with $c_i(\mathbf{k})$ sequences of constants satisfying (2.2).

Lemma 2.2. *If for all* $\mathbf{k} \in \mathsf{Z}^n$ *there exists* $h_{\mathbf{k}}(\mathbf{z}) \in \mathcal{L}$, $h_{\mathbf{k}}(\mathbf{z}) \neq 0$, *such that*

$$U_i^* h_{\mathbf{k}}(\mathbf{z}) = \sum_{j=1}^{n} c_j(\mathbf{k}) V_{ij}^* h_{\mathbf{k}}(\mathbf{z}), \quad 1 \leqslant i \leqslant n, \tag{2.5}$$

we have

$$g_{\mathbf{k}}(\mathbf{z}) = \frac{1}{\langle f_{\mathbf{k}}(\mathbf{z}), \det(V_{ij}^*)h_{\mathbf{k}}(\mathbf{z})\rangle} \det(V_{ij}^*)h_{\mathbf{k}}(\mathbf{z}). \tag{2.6}$$

In general, for any pair of inverse matrices $(f_{\mathbf{mk}})_{\mathbf{m},\mathbf{k}\in Z^n}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in Z^n}$, and any sequence $(d_{\mathbf{k}})_{\mathbf{k}\in Z^n}$ with $d_{\mathbf{k}} \neq 0$, a new pair of inverse matrices is given by $(f_{\mathbf{mk}}d_{\mathbf{m}}/d_{\mathbf{k}})$ and $(g_{\mathbf{kl}}d_{\mathbf{k}}/d_{\mathbf{l}})$. In such case, we shall say that we "transfer" the factor $d_{\mathbf{k}}$ from one matrix to the other. This procedure will be applied several times below.

2.2. *Extensions of Krattenthaler's matrix inverse*

Let a_k, c_k $(k \in Z)$ be arbitrary sequences of indeterminates. In [10], Krattenthaler proved that the two matrices

$$f_{mk} = \frac{\prod_{y=k}^{m-1}(a_y - c_k)}{\prod_{y=k+1}^{m}(c_y - c_k)}, \quad g_{kl} = \frac{(a_l - c_l)}{(a_k - c_k)} \frac{\prod_{y=l+1}^{k}(a_y - c_k)}{\prod_{y=l}^{k-1}(c_y - c_k)} \tag{2.7}$$

are inverses of each other. By using the method described above, we shall derive two new multidimensional extensions of this result. Other generalizations have been obtained in [21, Theorem 3.1], [11,22].

Theorem 2.3. *Let b be an indeterminate and $a_i(k), c_i(k)$ $(k \in Z,\ 1 \leqslant i \leqslant n)$ be arbitrary sequences of indeterminates. Define*

$$f_{\mathbf{mk}} = \prod_{i=1}^{n} \frac{\prod_{y_i=k_i}^{m_i-1}\left[\left(a_i(y_i) - b/\prod_{j=1}^{n} c_j(k_j)\right)\prod_{j=1}^{n}\left(a_i(y_i) - c_j(k_j)\right)\right]}{\prod_{y_i=k_i+1}^{m_i}\left[\left(c_i(y_i) - b/\prod_{j=1}^{n} c_j(k_j)\right)\prod_{j=1}^{n}\left(c_i(y_i) - c_j(k_j)\right)\right]} \tag{2.8a}$$

and

$$\begin{aligned} g_{\mathbf{kl}} = {} & \prod_{i=1}^{n} c_i(k_i)^{-1} \prod_{1\leqslant i<j\leqslant n}\left(c_i(k_i) - c_j(k_j)\right)^{-1} \\ & \times \prod_{i=1}^{n}\prod_{y_i=l_i}^{k_i-1}\left[\frac{\left(a_i(y_i) - b/\prod_{j=1}^{n} c_j(k_j)\right)}{\left(c_i(y_i) - b/\prod_{j=1}^{n} c_j(k_j)\right)}\prod_{j=1}^{n}\frac{\left(a_i(y_i) - c_j(k_j)\right)}{\left(c_i(y_i) - c_j(k_j)\right)}\right] \end{aligned}$$

$$\times \det_{1\leqslant i,j\leqslant n}\left[c_i(l_i)^{n-j+1} - a_i(l_i)^{n-j+1}\frac{\left(c_i(l_i) - b/\prod_{s=1}^{n} c_s(k_s)\right)}{\left(a_i(l_i) - b/\prod_{s=1}^{n} c_s(k_s)\right)}\prod_{s=1}^{n}\frac{\left(c_i(l_i) - c_s(k_s)\right)}{\left(a_i(l_i) - c_s(k_s)\right)}\right]. \tag{2.8b}$$

Then the infinite-lower-triangular n-dimensional matrices $(f_{\mathbf{mk}})_{\mathbf{m},\mathbf{k}\in\mathsf{Z}^n}$ *and* $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathsf{Z}^n}$ *are inverses of each other.*

Remark 2.4. Krattenthaler's result [10] is recovered for $n = 1$. Indeed the determinant in (2.8b) then reduces (after relabeling) to

$$c_l - a_l \frac{(c_l - b/c_k)}{(a_l - b/c_k)} \frac{(c_l - c_k)}{(a_l - c_k)} = c_l \frac{(a_l - b/c_l)}{(a_l - b/c_k)} \frac{(a_l - c_l)}{(a_l - c_k)}$$

and the matrices in (2.8) (after relabeling) become

$$f_{mk} = \frac{\prod_{y=k}^{m-1}(a_y - b/c_k)(a_y - c_k)}{\prod_{y=k+1}^{m}(c_y - b/c_k)(c_y - c_k)}, \tag{2.9a}$$

$$g_{kl} = \frac{(b - a_l c_l)(a_l - c_l)}{(b - a_k c_k)(a_k - c_k)} \frac{\prod_{y=l+1}^{k}(a_y - b/c_k)(a_y - c_k)}{\prod_{y=l}^{k-1}(c_y - b/c_k)(c_y - c_k)}. \tag{2.9b}$$

It is not difficult to see that this matrix inverse is actually equivalent to its $b \to \infty$ special case, which is (2.7). To recover (2.9) from (2.7), do the substitutions $a_y \mapsto a_y + b/a_y$, $c_y \mapsto c_y + b/c_y$, transfer some factors from one matrix to the other, and simplify.

Remark 2.5. In case $a_i(k) = a$ for some constant a (for all $k \in \mathsf{Z}$, $1 \leqslant i \leqslant n$), the determinant appearing in (2.8b) factors, due to the evaluation

$$\det_{1\leqslant i,j\leqslant n}\left[c_i(l_i)^{n-j+1} - a^{n-j+1}\frac{\left(c_i(l_i) - b/\prod_{s=1}^n c_s(k_s)\right)}{\left(a - b/\prod_{s=1}^n c_s(k_s)\right)}\prod_{s=1}^n \frac{\left(c_i(l_i) - c_s(k_s)\right)}{\left(a - c_s(k_s)\right)}\right]$$
$$= \frac{\left(a - b/\prod_{j=1}^n c_j(l_j)\right)}{\left(a - b/\prod_{j=1}^n c_j(k_j)\right)}\prod_{i=1}^n c_i(l_i)\frac{\left(a - c_i(l_i)\right)}{\left(a - c_i(k_i)\right)}\prod_{1\leqslant i<j\leqslant n}\left(c_i(l_i) - c_j(l_j)\right),$$

which was first proved in [21, Lemma A.1]. A slightly more general evaluation and a much quicker proof may be found in [22, Lemma A.1]. The resulting multidimensional matrix inversion is the special case $a_t = a$ (for all $t \in \mathsf{Z}$) of [21, Theorem 3.1].

Proof of Theorem 2.3. We apply the operator method of Section 2.1. From (2.8a), for all $\mathbf{m} \geqslant \mathbf{k}$ and $1 \leqslant i \leqslant n$, we deduce the recurrence

$$\left(c_i(m_i) - b/\textstyle\prod_{j=1}^n c_j(k_j)\right)\displaystyle\prod_{s=1}^n \left(c_i(m_i) - c_s(k_s)\right) f_{\mathbf{m}-\mathbf{e}_i,\mathbf{k}}$$
$$= \left(a_i(m_i - 1) - b/\textstyle\prod_{j=1}^n c_j(k_j)\right)\displaystyle\prod_{s=1}^n \left(a_i(m_i - 1) - c_s(k_s)\right) f_{\mathbf{m},\mathbf{k}}, \tag{2.10}$$

where $\mathbf{e}_i \in \mathsf{Z}^n$ has all components zero but its ith component equal to 1. We write

$$\begin{aligned} f_{\mathbf{k}}(\mathbf{z}) &= \sum_{\mathbf{m}\geqslant\mathbf{k}} f_{\mathbf{mk}}\mathbf{z}^{\mathbf{m}} \\ &= \sum_{\mathbf{m}\geqslant\mathbf{k}} \prod_{i=1}^{n} \frac{\prod_{y_i=k_i}^{m_i-1}\left[\left(a_i(y_i)-b/\prod_{j=1}^{n}c_j(k_j)\right)\prod_{j=1}^{n}\left(a_i(y_i)-c_j(k_j)\right)\right]}{\prod_{y_i=k_i+1}^{m_i}\left[\left(c_i(y_i)-b/\prod_{j=1}^{n}c_j(k_j)\right)\prod_{j=1}^{n}\left(c_i(y_i)-c_j(k_j)\right)\right]}\mathbf{z}^{\mathbf{m}}. \end{aligned}$$

Define linear operators $\mathcal{A}_i$ and $\mathcal{C}_i$ by $\mathcal{A}_i\mathbf{z}^{\mathbf{m}} = a_i(m_i)\mathbf{z}^{\mathbf{m}}$, $\mathcal{C}_i\mathbf{z}^{\mathbf{m}} = c_i(m_i)\mathbf{z}^{\mathbf{m}}$, $(1\leqslant i\leqslant n)$. Then we may write (2.10) in the form

$$\begin{aligned} &\left(\mathcal{C}_i - b/\textstyle\prod_{j=1}^{n}c_j(k_j)\right)\displaystyle\prod_{s=1}^{n}\left(\mathcal{C}_i - c_s(k_s)\right)f_{\mathbf{k}}(\mathbf{z}) \\ &\quad = z_i\left(\mathcal{A}_i - b/\textstyle\prod_{j=1}^{n}c_j(k_j)\right)\displaystyle\prod_{s=1}^{n}\left(\mathcal{A}_i - c_s(k_s)\right)f_{\mathbf{k}}(\mathbf{z}), \end{aligned} \tag{2.11}$$

valid for all $1\leqslant i\leqslant n$ and $\mathbf{k} \in \mathsf{Z}^n$.

In order to write this system of equations in a way such that Lemma 2.2 may be applied, we expand the products on both sides in terms of the elementary symmetric functions of order j,

$$e_j\left(c_1(k_1), c_2(k_2), \ldots, c_n(k_n), b/\textstyle\prod_{s=1}^{n}c_s(k_s)\right),$$

which we denote $e_j(\mathbf{c}(\mathbf{k}))$ for short. The recurrence system (2.11) then reads, using $e_{n+1}(\mathbf{c}(\mathbf{k})) = b$,

$$\begin{aligned} &\sum_{j=1}^{n} e_j(\mathbf{c}(\mathbf{k}))\left[(-\mathcal{C}_i)^{n-j+1} - z_i(-\mathcal{A}_i)^{n-j+1}\right]f_{\mathbf{k}}(\mathbf{z}) \\ &\quad = \left[z_i(-\mathcal{A}_i)^{n+1} + bz_i - (-\mathcal{C}_i)^{n+1} - b\right]f_{\mathbf{k}}(\mathbf{z}). \end{aligned}$$

This is a system of type (2.4) with

$$\begin{aligned} U_i &= [z_i(-\mathcal{A}_i)^{n+1} + bz_i - (-\mathcal{C}_i)^{n+1} - b], \\ V_{ij} &= [(-\mathcal{C}_i)^{n-j+1} - z_i(-\mathcal{A}_i)^{n-j+1}], \\ c_j(\mathbf{k}) &= e_j(\mathbf{c}(\mathbf{k})). \end{aligned}$$

Conditions (2.2) and (2.3) are satisfied. The dual system (2.5) for $h_{\mathbf{k}}(\mathbf{z})$ writes as

$$\sum_{j=1}^{n} e_j(\mathbf{c}(\mathbf{k}))\big[(-\mathcal{C}_i^*)^{n-j+1} - (-\mathcal{A}_i^*)^{n-j+1} z_i\big] h_{\mathbf{k}}(\mathbf{z}) \\ = \big[(-\mathcal{A}_i^*)^{n+1} z_i + b z_i - (-\mathcal{C}_i^*)^{n+1} - b\big] h_{\mathbf{k}}(\mathbf{z}).$$

Equivalently we have

$$\Big(\mathcal{C}_i^* - b/\textstyle\prod_{j=1}^n c_j(k_j)\Big) \displaystyle\prod_{s=1}^{n} \big(\mathcal{C}_i^* - c_s(k_s)\big) h_{\mathbf{k}}(\mathbf{z}) \\ = \Big(\mathcal{A}_i^* - b/\textstyle\prod_{j=1}^n c_j(k_j)\Big) \displaystyle\prod_{s=1}^{n} \big(\mathcal{A}_i^* - c_s(k_s)\big) z_i h_{\mathbf{k}}(\mathbf{z}), \tag{2.12}$$

valid for all $1 \leqslant i \leqslant n$ and $\mathbf{k} \in \mathsf{Z}^n$. We have easily $\mathcal{A}_i^* \mathbf{z}^{-\mathbf{l}} = a_i(l_i) \mathbf{z}^{-\mathbf{l}}$ and $\mathcal{C}_i^* \mathbf{z}^{-\mathbf{l}} = c_i(l_i) \mathbf{z}^{-\mathbf{l}}$. Thus, writing $h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l} \leqslant \mathbf{k}} h_{\mathbf{kl}}\ \mathbf{z}^{-\mathbf{l}}$ and comparing coefficients of $\mathbf{z}^{-\mathbf{l}}$ in (2.12), we obtain

$$\Big(c_i(l_i) - b/\textstyle\prod_{j=1}^n c_j(k_j)\Big) \displaystyle\prod_{s=1}^{n} \big(c_i(l_i) - c_s(k_s)\big) h_{\mathbf{kl}} \\ = \Big(a_i(l_i) - b/\textstyle\prod_{j=1}^n c_j(k_j)\Big) \displaystyle\prod_{s=1}^{n} \big(a_i(l_i) - c_s(k_s)\big) h_{\mathbf{k},\mathbf{l}+\mathbf{e}_i}.$$

We may assume $h_{\mathbf{kk}} = 1$. Then we get

$$h_{\mathbf{kl}} = \prod_{i=1}^{n} \prod_{y_i=l_i}^{k_i-1} \left[\frac{\big(a_i(y_i) - b/\prod_{j=1}^n c_j(k_j)\big)}{\big(c_i(y_i) - b/\prod_{j=1}^n c_j(k_j)\big)} \prod_{j=1}^{n} \frac{\big(a_i(y_i) - c_j(k_j)\big)}{\big(c_i(y_i) - c_j(k_j)\big)} \right].$$

Now let us apply Lemma 2.2. Taking into account (2.6), we must compute the action of

$$\det_{1 \leqslant i,j \leqslant n} (V_{ij}^*) = \det_{1 \leqslant i,j \leqslant n} \big[(-\mathcal{C}_i^*)^{n-j+1} - (-\mathcal{A}_i^*)^{n-j+1} z_i\big]$$

when applied to

$$h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l} \leqslant \mathbf{k}} \prod_{i=1}^{n} \prod_{y_i=l_i}^{k_i-1} \left[\frac{\big(a_i(y_i) - b/\prod_{j=1}^n c_j(k_j)\big)}{\big(c_i(y_i) - b/\prod_{j=1}^n c_j(k_j)\big)} \prod_{j=1}^{n} \frac{\big(a_i(y_i) - c_j(k_j)\big)}{\big(c_i(y_i) - c_j(k_j)\big)} \right] \mathbf{z}^{-\mathbf{l}}.$$

Since

$$z_i h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l} \leqslant \mathbf{k}} h_{\mathbf{kl}} \ \mathbf{z}^{-\mathbf{l}} \frac{\big(c_i(l_i) - b/\prod_{j=1}^n c_j(k_j)\big)}{\big(a_i(l_i) - b/\prod_{j=1}^n c_j(k_j)\big)} \prod_{j=1}^n \frac{\big(c_i(l_i) - c_j(k_j)\big)}{\big(a_i(l_i) - c_j(k_j)\big)},$$

we obtain

$$\det_{1 \leqslant i,j \leqslant n} (V_{ij}^*) h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l} \leqslant \mathbf{k}} h_{\mathbf{kl}} \ \mathbf{z}^{-\mathbf{l}} \det_{1 \leqslant i,j \leqslant n} \Bigg[(-c_i(l_i))^{n-j+1} - (-a_i(l_i))^{n-j+1} \times \frac{\big(c_i(l_i) - b/\prod_{j=1}^n c_j(k_j)\big)}{\big(a_i(l_i) - b/\prod_{j=1}^n c_j(k_j)\big)} \prod_{j=1}^n \frac{\big(c_i(l_i) - c_j(k_j)\big)}{\big(a_i(l_i) - c_j(k_j)\big)} \Bigg].$$

Note that since $f_{\mathbf{kk}} = 1$, the pairing $\langle f_{\mathbf{k}}(\mathbf{z}), \det(V_{ij}^*) h_{\mathbf{k}}(\mathbf{z}) \rangle$ is simply the coefficient of $\mathbf{z}^{-\mathbf{k}}$ in the above expression, i.e. $\det_{1 \leqslant i,j \leqslant n} [(-c_i(k_i))^{n-j+1}]$. Thus Eq. (2.6) writes as

$$g_{\mathbf{k}}(\mathbf{z}) = \prod_{1 \leqslant i < j \leqslant n} \big(c_j(k_j) - c_i(k_i)\big)^{-1} \prod_{i=1}^n \big(- c_i(k_i)\big)^{-1} \det_{1 \leqslant i,j \leqslant n} (V_{ij}^*) h_{\mathbf{k}}(\mathbf{z}).$$

Since $g_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l} \leqslant \mathbf{k}} g_{\mathbf{kl}} \ \mathbf{z}^{-\mathbf{l}}$, we conclude by extracting the coefficient of $\mathbf{z}^{-\mathbf{l}}$ in $g_{\mathbf{k}}(\mathbf{z})$. □

The following theorem is another multidimensional generalization of Krattenthaler's result [10].

Theorem 2.6. *Let b be an indeterminate and $a_i(k), c_i(k)$ ($k \in \mathsf{Z}$, $1 \leqslant i \leqslant n$) be arbitrary sequences of indeterminates. Define*

$$f_{\mathbf{mk}} = \prod_{i=1}^n c_i(k_i)^{-1} \prod_{1 \leqslant i < j \leqslant n} \big(c_i(k_i) - c_j(k_j)\big)^{-1} \times \prod_{i=1}^n \prod_{y_i = k_i+1}^{m_i} \left[\frac{\big(a_i(y_i) - b/\prod_{j=1}^n c_j(k_j)\big)}{\big(c_i(y_i) - b/\prod_{j=1}^n c_j(k_j)\big)} \prod_{j=1}^n \frac{\big(a_i(y_i) - c_j(k_j)\big)}{\big(c_i(y_i) - c_j(k_j)\big)} \right]$$

$$\times \det_{1 \leqslant i,j \leqslant n} \left[c_i(m_i)^{n-j+1} - a_i(m_i)^{n-j+1} \frac{\big(c_i(m_i) - b/\prod_{s=1}^n c_s(k_s)\big)}{\big(a_i(m_i) - b/\prod_{s=1}^n c_s(k_s)\big)} \prod_{s=1}^n \frac{\big(c_i(m_i) - c_s(k_s)\big)}{\big(a_i(m_i) - c_s(k_s)\big)} \right]$$

and

$$g_{\mathbf{kl}} = \prod_{i=1}^{n} \frac{\prod_{y_i=l_i+1}^{k_i} \left[\left(a_i(y_i) - b/\prod_{j=1}^{n} c_j(k_j) \right) \prod_{j=1}^{n} \left(a_i(y_i) - c_j(k_j) \right) \right]}{\prod_{y_i=l_i}^{k_i-1} \left[\left(c_i(y_i) - b/\prod_{j=1}^{n} c_j(k_j) \right) \prod_{j=1}^{n} \left(c_i(y_i) - c_j(k_j) \right) \right]}.$$

Then the infinite lower-triangular n-dimensional matrices $(f_{\mathbf{mk}})_{\mathbf{m},\mathbf{k} \in \mathsf{Z}^n}$ *and* $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l} \in \mathsf{Z}^n}$ *are inverses of each other.*

Proof. For any multi-integer $\mathbf{k} = (k_1, \ldots, k_n)$, denote $-\mathbf{k} = (-k_1, \ldots, -k_n)$. Define two-multidimensional matrices $\tilde{g}_{\mathbf{mk}}$ and $\tilde{f}_{\mathbf{kl}}$ by $\tilde{g}_{\mathbf{mk}} = f_{-\mathbf{k},-\mathbf{m}}$ and $\tilde{f}_{\mathbf{kl}} = g_{-\mathbf{l},-\mathbf{k}}$. For $1 \leqslant i \leqslant n$ write $\tilde{a}_i(y_i) = a_i(-y_i)$ and $\tilde{c}_i(y_i) = c_i(-y_i)$. Then the matrices $\tilde{g}_{\mathbf{mk}}$ and $\tilde{f}_{\mathbf{kl}}$ are those considered in Theorem 2.3, associated to the sequences $\tilde{a}_i(k), \tilde{c}_i(k)$. Thus for all $\mathbf{m}, \mathbf{l} \in \mathsf{Z}^n$, we have

$$\sum_{\mathbf{m} \geqslant \mathbf{k} \geqslant \mathbf{l}} f_{\mathbf{mk}}\, g_{\mathbf{kl}} = \sum_{\mathbf{m} \geqslant \mathbf{k} \geqslant \mathbf{l}} \tilde{f}_{-\mathbf{l},-\mathbf{k}}\, \tilde{g}_{-\mathbf{k},-\mathbf{m}} = \delta_{\mathbf{ml}}. \qquad \square$$

2.3. An extension of Bressoud's matrix inverse

Let q be an indeterminate. For any integer k, the classical q-shifted factorial $(a;q)_k$ is defined by

$$(a;q)_\infty = \prod_{j \geqslant 0} (1 - aq^j), \quad (a;q)_k = \frac{(a;q)_\infty}{(aq^k;q)_\infty}.$$

Theorem 2.7. *Let t and $u_1, \ldots, u_n$ be indeterminates. Define*

$$f_{\mathbf{mk}} = \prod_{1 \leqslant i < j \leqslant n} \left(q^{m_i} u_i - q^{m_j} u_j \right)^{-1} \prod_{i=1}^{n} t^{m_i - k_i} \frac{(q/t;q)_{m_i-k_i}}{(q;q)_{m_i-k_i}} \frac{(q^{k_i+|\mathbf{k}|+1} u_i;q)_{m_i-k_i}}{(q^{k_i+|\mathbf{k}|+1} t u_i;q)_{m_i-k_i}}$$

$$\times \prod_{1 \leqslant i < j \leqslant n} \frac{(q^{k_i-k_j+1} u_i/t u_j;q)_{m_i-k_i}}{(q^{k_i-k_j+1} u_i/u_j;q)_{m_i-k_i}} \frac{(q^{k_i-m_j} t u_i/u_j;q)_{m_i-k_i}}{(q^{k_i-m_j} u_i/u_j;q)_{m_i-k_i}}$$

$$\times \det_{1 \leqslant i,j \leqslant n} \left[\left(q^{m_i} u_i \right)^{n-j} \left(1 - t^{j-1} \frac{\left(1 - q^{m_i+|\mathbf{k}|} t u_i\right)}{\left(1 - q^{m_i+|\mathbf{k}|} u_i\right)} \prod_{s=1}^{n} \frac{\left(q^{m_i} u_i - q^{k_s} u_s\right)}{\left(q^{m_i} u_i - q^{k_s} t u_s\right)} \right) \right]$$

and

$$g_{\mathbf{kl}} = \prod_{i=1}^{n} \frac{(t;q)_{k_i-l_i}}{(q;q)_{k_i-l_i}} \frac{(q^{l_i+|\mathbf{k}|+1}u_i;q)_{k_i-l_i}}{(q^{l_i+|\mathbf{k}|}tu_i;q)_{k_i-l_i}}$$

$$\times \prod_{1\leqslant i<j\leqslant n} \frac{(q^{l_i-l_j}tu_i/u_j;q)_{k_i-l_i}}{(q^{l_i-l_j+1}u_i/u_j;q)_{k_i-l_i}} \frac{(q^{l_i-k_j+1}u_i/tu_j;q)_{k_i-l_i}}{(q^{l_i-k_j}u_i/u_j;q)_{k_i-l_i}}.$$

Then the infinite lower-triangular n-dimensional matrices $(f_{\mathbf{mk}})_{\mathbf{m},\mathbf{k}\in\mathbb{Z}^n}$ *and* $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^n}$ *are inverses of each other.*

Remark 2.8. For $n = 1$ this matrix inversion reduces to Bressoud's result [1], which he derived from the terminating very-well-poised ${}_6\phi_5$ summation [2, Eq. (II.21)].

Proof. We specialize Theorem 2.6 by letting $b \mapsto t^{-1}\prod_{j=1}^{n} u_j$, $a_i(y_i) \mapsto q^{y_i}u_i/t$, and $c_i(y_i) \mapsto q^{y_i}u_i$, $(1\leqslant i\leqslant n)$, and rewrite the expressions using q-shifted factorials. After this first step, we obtain the inverse pair

$$f_{\mathbf{mk}} = q^{-|\mathbf{k}|}\prod_{i=1}^{n} u_i^{-1} \prod_{1\leqslant i<j\leqslant n} \left(q^{k_i}u_i - q^{k_j}u_j\right)^{-1}$$

$$\times \prod_{i=1}^{n} \frac{(q^{k_i+|\mathbf{k}|+1}u_i;q)_{m_i-k_i}}{(q^{k_i+|\mathbf{k}|+1}tu_i;q)_{m_i-k_i}} \prod_{i,j=1}^{n} \frac{(q^{k_i-k_j+1}u_i/tu_j;q)_{m_i-k_i}}{(q^{k_i-k_j+1}u_i/u_j;q)_{m_i-k_i}}$$

$$\times \det_{1\leqslant i,j\leqslant n}\left[\left(q^{m_i}u_i\right)^{n-j}\left(1 - t^{j-1}\frac{\left(1-q^{m_i+|\mathbf{k}|}tu_i\right)}{\left(1-q^{m_i+|\mathbf{k}|}u_i\right)}\prod_{s=1}^{n}\frac{\left(q^{m_i}u_i - q^{k_s}u_s\right)}{\left(q^{m_i}u_i - q^{k_s}tu_s\right)}\right)\right],$$

$$g_{\mathbf{kl}} = \prod_{i=1}^{n} \frac{(q^{l_i+|\mathbf{k}|+1}u_i;q)_{k_i-l_i}}{(q^{l_i+|\mathbf{k}|}tu_i;q)_{k_i-l_i}} \prod_{i,j=1}^{n} \frac{(q^{l_i-k_j+1}u_i/tu_j;q)_{k_i-l_i}}{(q^{l_i-k_j}u_i/u_j;q)_{k_i-l_i}}.$$

Now note that $f_{\mathbf{mk}}$ contains the factors

$$\prod_{i,j=1}^{n} \frac{(q^{k_i-k_j+1}u_i/tu_j;q)_{m_i-k_i}}{(q^{k_i-k_j+1}u_i/u_j;q)_{m_i-k_i}}$$

$$= \prod_{i=1}^{n} \frac{(q/t;q)_{m_i-k_i}}{(q;q)_{m_i-k_i}} \prod_{1\leqslant i<j\leqslant n} \frac{(q^{k_i-k_j+1}u_i/tu_j;q)_{m_i-k_i}}{(q^{k_i-k_j+1}u_i/u_j;q)_{m_i-k_i}} \frac{(q^{k_j-k_i+1}u_j/tu_i;q)_{m_j-k_j}}{(q^{k_j-k_i+1}u_j/u_i;q)_{m_j-k_j}}$$

$$
\begin{aligned}
&= \prod_{i=1}^{n} \frac{(q/t;q)_{m_i-k_i}}{(q;q)_{m_i-k_i}} \\
&\times \prod_{1\leqslant i<j\leqslant n} t^{k_j-m_j} \frac{(q^{k_i-k_j+1}u_i/tu_j;q)_{m_i-k_i}}{(q^{k_i-k_j+1}u_i/u_j;q)_{m_i-k_i}} \frac{(q^{k_i-m_j}u_it/u_j;q)_{m_j-k_j}}{(q^{k_i-m_j}u_i/u_j;q)_{m_j-k_j}}.
\end{aligned}
$$

Similarly $g_{\mathbf{kl}}$ contains

$$
\begin{aligned}
&\prod_{i,j=1}^{n} \frac{(q^{l_i-k_j+1}u_i/tu_j;q)_{k_i-l_i}}{(q^{l_i-k_j}u_i/u_j;q)_{k_i-l_i}} \\
&\quad = \prod_{i=1}^{n} \left(\frac{q}{t}\right)^{k_i-l_i} \frac{(t;q)_{k_i-l_i}}{(q;q)_{k_i-l_i}} \\
&\quad \prod_{1\leqslant i<j\leqslant n} \left(\frac{q}{t}\right)^{k_j-l_j} \frac{(q^{l_i-k_j+1}u_i/tu_j;q)_{k_i-l_i}}{(q^{l_i-k_j}u_i/u_j;q)_{k_i-l_i}} \frac{(q^{k_i-k_j}u_i/tu_j;q)_{k_j-l_j}}{(q^{k_i-k_j+1}u_i/u_j;q)_{k_j-l_j}}.
\end{aligned}
$$

To conclude, we transfer the factor

$$
d_{\mathbf{k}} = (t/q)^{|\mathbf{k}|} \prod_{1\leqslant i<j\leqslant n} \left(q^{k_i}u_i - q^{k_j}u_j\right)^{-1} t^{k_j} \frac{(tu_i/u_j;q)_{k_i-k_j}}{(u_i/u_j;q)_{k_i-k_j}},
$$

from one matrix to the other, and simplify the resulting expressions. □

3. Macdonald polynomials

The standard reference for Macdonald polynomials is Chapter 6 of [19].

3.1. Symmetric functions

Let $X = \{x_1, x_2, x_3, \ldots\}$ be an infinite set of indeterminates, and $\mathcal{S}$ the corresponding algebra of symmetric functions with coefficients in Q. Let $\mathsf{Q}[q,t]$ be the field of rational functions in two indeterminates q, t, and $\mathsf{Sym} = \mathcal{S} \otimes \mathsf{Q}[q,t]$ the algebra of symmetric functions with coefficients in $\mathsf{Q}[q,t]$.

The power sum symmetric functions are defined by $p_k(X) = \sum_{i\geqslant 1} x_i^k$. Elementary and complete symmetric functions $e_k(X)$ and $h_k(X)$ are defined by their generating functions

$$
\prod_{i\geqslant 1}(1+ux_i) = \sum_{k\geqslant 0} u^k\, e_k(X), \quad \prod_{i\geqslant 1} \frac{1}{1-ux_i} = \sum_{k\geqslant 0} u^k\, h_k(X).
$$

Each of these three sets form an algebraic basis of Sym, which can thus be viewed as an abstract algebra over $\mathbb{Q}[q,t]$ generated by functions e_k, h_k or p_k.

A partition $\lambda=(\lambda_1,\dots,\lambda_n)$ is a finite weakly decreasing sequence of nonnegative integers, called parts. The number $l(\lambda)$ of positive parts is called the length of λ, and $|\lambda|=\sum_{i=1}^n \lambda_i$ the weight of λ. For any integer $i\geqslant 1$, $m_i(\lambda)=\mathrm{card}\{j:\lambda_j=i\}$ is the multiplicity of the part i in λ. Clearly $l(\lambda)=\sum_{i\geqslant 1} m_i(\lambda)$ and $|\lambda|=\sum_{i\geqslant 1} i m_i(\lambda)$. We shall also write $\lambda=(1^{m_1},2^{m_2},3^{m_3},\dots)$. We set

$$z_\lambda=\prod_{i\geqslant 1} i^{m_i(\lambda)} m_i(\lambda)!.$$

We denote λ' the partition conjugate to λ, whose parts are given by $m_i(\lambda')=\lambda_i-\lambda_{i+1}$. We have $\lambda'_i=\sum_{j\geqslant i} m_j(\lambda)$.

For any partition λ, the symmetric functions e_λ, h_λ and p_λ defined by

$$f_\lambda=\prod_{i=1}^{l(\lambda)} f_{\lambda_i}=\prod_{i\geqslant 1}(f_i)^{m_i(\lambda)}, \tag{3.1}$$

where f_i stands for e_i, h_i or p_i, respectively, form a linear basis of Sym. Another classical basis is formed by the monomial symmetric functions m_λ, defined as the sum of all distinct monomials whose exponent is a permutation of λ.

For all $k\geqslant 0$, the “modified complete” symmetric function $g_k(X;q,t)$ is defined by the generating series

$$\prod_{i\geqslant 1}\frac{(tux_i;q)_\infty}{(ux_i;q)_\infty}=\sum_{k\geqslant 0} u^k g_k(X;q,t).$$

It is often written in λ-ring notation [15, p. 223], that is

$$g_k(X;q,t)=h_k\left[\frac{1-t}{1-q}X\right].$$

The symmetric functions $g_k(q,t)$ form an algebraic basis of Sym. They may be expanded in terms of any classical basis. This development is explicitly given in [19, p. 311, 314] in terms of power sums and monomial symmetric functions, and in [15, Section 10, p. 237] in terms of other classical bases. The functions $g_\lambda(q,t)$, defined as in (3.1), form a linear basis of Sym.

3.2. Macdonald operators

We now restrict to the case of a finite set of indeterminates $X = \{x_1, \ldots, x_n\}$. Let T_{q,x_i} denote the q-deformation operator defined by

$$T_{q,x_i} f(x_1, \ldots, x_n) = f(x_1, \ldots, qx_i, \ldots, x_n)$$

and for all $1 \leqslant i \leqslant n$,

$$A_i(X; t) = \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{tx_i - x_j}{x_i - x_j}.$$

Macdonald polynomials $P_\lambda(X; q, t)$, with λ a partition such that $l(\lambda) \leqslant n$, are defined as the eigenvectors of the following difference operator:

$$E(X; q, t) = \sum_{i=1}^{n} A_i(X; t)\, T_{q,x_i}.$$

One has

$$E(X; q, t)\, P_\lambda(X; q, t) = \left(\sum_{i=1}^{n} q^{\lambda_i}\, t^{n-i} \right) P_\lambda(X; q, t).$$

Let $\Delta(X)$ be the Vandermonde determinant $\prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)$. More generally Macdonald polynomials $P_\lambda(X; q, t)$ are eigenvectors of the difference operator

$$D(u; q, t) = \frac{1}{\Delta(X)} \det_{1 \leqslant i,j \leqslant n} \left[x_i^{n-j} \left(1 + ut^{n-j} T_{q,x_i} \right) \right],$$

where u is some indeterminate. One has

$$D(u; q, t)\, P_\lambda(X; q, t) = \prod_{i=1}^{n} \left(1 + u\, q^{\lambda_i}\, t^{n-i} \right) P_\lambda(X; q, t).$$

The polynomials $P_\lambda(X; q, t)$ define symmetric functions, which form an orthogonal basis of Sym with respect to the scalar product $<, >_{q,t}$ defined by

$$< p_\lambda, p_\mu >_{q,t} = \delta_{\lambda\mu}\, z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}.$$

Equivalently if $Y = \{y_1, \ldots, y_m\}$ is another set of m indeterminates, and

$$\Pi(X, Y; q, t) = \prod_{i=1}^{n} \prod_{j=1}^{m} \frac{(tx_i y_j; q)_\infty}{(x_i y_j; q)_\infty},$$

we have

$$\Pi(X, Y; q, t) = \sum_\lambda P_\lambda(X; q, t) Q_\lambda(Y; q, t),$$

where $Q_\lambda(X; q, t)$ denotes the dual basis of $P_\lambda(X; q, t)$ for the scalar product $<, >_{q,t}$. One has

$$Q_\lambda(X; q, t) = b_\lambda(q, t)\, P_\lambda(X; q, t) \tag{3.2}$$

with $b_\lambda(q, t) = < P_\lambda(q, t), P_\lambda(q, t) >_{q,t}{}^{-1}$ given by

$$b_\lambda(q, t) = \prod_{1 \leqslant i \leqslant j \leqslant l(\lambda)} \frac{(q^{\lambda_i - \lambda_j} t^{j-i+1}; q)_{\lambda_j - \lambda_{j+1}}}{(q^{\lambda_i - \lambda_j + 1} t^{j-i}; q)_{\lambda_j - \lambda_{j+1}}}.$$

As shown in [19, p. 315], we have

$$D(u; q, t) = \sum_{K \subset \{1,\ldots,n\}} u^{|K|} t^{\binom{|K|}{2}} \prod_{\substack{k \in K \\ j \notin K}} \frac{tx_k - x_j}{x_k - x_j} \prod_{k \in K} T_{q,x_k}.$$

This yields

$$\begin{aligned} &\frac{1}{\Delta(X)} \det_{1 \leqslant i,j \leqslant n} \left[x_i^{n-j} \left(1 + ut^{n-j} \prod_{k=1}^{m} \frac{1 - x_i y_k}{1 - tx_i y_k} \right) \right] \\ &= \sum_{K \subset \{1,\ldots,n\}} u^{|K|} t^{\binom{|K|}{2}} \prod_{\substack{k \in K \\ j \notin K}} \frac{tx_k - x_j}{x_k - x_j} \prod_{i=1}^{m} \prod_{k \in K} \frac{1 - x_k y_i}{1 - tx_k y_i}. \end{aligned} \tag{3.3}$$

Indeed since

$$\Pi^{-1}\, T_{q,x_i}\, \Pi = \prod_{k=1}^{m} \frac{1 - x_i y_k}{1 - tx_i y_k}$$

both terms are obviously $\Pi^{-1}\, D(u; q, t)_{(X)}\, \Pi$, where the suffix (X) indicates operation on the X variables.

There exists an automorphism $\omega_{q,t} = {\omega_{t,q}}^{-1}$ of Sym such that

$$\omega_{q,t}(Q_\lambda(q,t)) = P_{\lambda'}(t,q), \quad \omega_{q,t}(g_k(q,t)) = e_k. \tag{3.4}$$

In particular, the Macdonald symmetric functions associated with a row or a column partition are given by

$$P_{1^k}(q,t) = e_k, \quad Q_{1^k}(q,t) = \frac{(t;t)_k}{(q;t)_k}\, e_k,$$

$$P_{(k)}(q,t) = \frac{(q;q)_k}{(t;q)_k}\, g_k(q,t), \quad Q_{(k)}(q,t) = g_k(q,t).$$

The parameters q, t being kept fixed, we shall often write P_μ or Q_μ for $P_\mu(q,t)$ or $Q_\mu(q,t)$.

3.3. Pieri formula

Let $u_1, \dots, u_n$ be n indeterminates and N the set of nonnegative integers. For $\theta = (\theta_1, \dots, \theta_n) \in \mathsf{N}^n$, let $|\theta| = \sum_{i=1}^n \theta_i$ and define

$$\begin{aligned} &d_{\theta_1,\dots,\theta_n}(u_1,\dots,u_n) \\ &\quad = \prod_{k=1}^n \frac{(t;q)_{\theta_k}}{(q;q)_{\theta_k}} \frac{(q^{|\theta|+1}u_k;q)_{\theta_k}}{(q^{|\theta|}tu_k;q)_{\theta_k}} \prod_{1 \leqslant i<j \leqslant n} \frac{(tu_i/u_j;q)_{\theta_i}}{(qu_i/u_j;q)_{\theta_i}} \frac{(q^{-\theta_j+1}u_i/tu_j;q)_{\theta_i}}{(q^{-\theta_j}u_i/u_j;q)_{\theta_i}}. \end{aligned}$$

If we set $u_{n+1} = 1/t$, $\theta_{n+1} = -|\theta|$, and $v_k = q^{\theta_k} u_k$ $(1 \leqslant k \leqslant n+1)$, we may write

$$d_{\theta_1,\dots,\theta_n}(u_1,\dots,u_n) = \prod_{1 \leqslant i \leqslant j \leqslant n} \frac{(tu_i/u_j;q)_{\theta_i}}{(qu_i/u_j;q)_{\theta_i}} \prod_{1 \leqslant i<j \leqslant n+1} \frac{(qu_i/tv_j;q)_{\theta_i}}{(u_i/v_j;q)_{\theta_i}}.$$

Macdonald symmetric functions satisfy a Pieri formula generalizing the classical Pieri formula for Schur functions. This generalization was obtained independently by Koornwinder [8] and Macdonald [18]. Most of the time this Pieri formula is stated in combinatorial terms. Its analytic form is less popular, but will be crucial for our purposes.

Theorem 3.1. *Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be an arbitrary partition with length n and $\lambda_{n+1} \in \mathsf{N}$. For any $1 \leqslant k \leqslant n+1$ define $u_k = q^{\lambda_k - \lambda_{n+1}} t^{n-k}$. We have*

$$Q_{(\lambda_1,\dots,\lambda_n)}\, Q_{(\lambda_{n+1})} = \sum_{\theta \in \mathsf{N}^n} d_{\theta_1,\dots,\theta_n}(u_1,\dots,u_n)\, Q_{(\lambda_1+\theta_1,\dots,\lambda_n+\theta_n,\lambda_{n+1}-|\theta|)}.$$

Proof. We make use of the expressions given in [19, p. 340, Eq. (6.24)(ii)] and [19, p. 342, Example 2(b)]. Specifically, we write

$$Q_\lambda \, Q_{(\lambda_{n+1})} = \sum_{\kappa \supset \lambda} \psi_{\kappa/\lambda} \, Q_\kappa,$$

where the skew-diagram $\kappa - \lambda$ is a horizontal λ_{n+1}-strip, i.e. has at most one square in each column, and $\psi_{\kappa/\lambda}$ is given by

$$\psi_{\kappa/\lambda} = \prod_{1 \leqslant i \leqslant j \leqslant n} \frac{f(q^{\lambda_i - \lambda_j} t^{j-i})}{f(q^{\kappa_i - \lambda_j} t^{j-i})} \frac{f(q^{\kappa_i - \kappa_{j+1}} t^{j-i})}{f(q^{\lambda_i - \kappa_{j+1}} t^{j-i})} = \prod_{1 \leqslant i \leqslant j \leqslant n} \frac{w_{\kappa_i - \lambda_i}(q^{\lambda_i - \lambda_j} t^{j-i})}{w_{\kappa_i - \lambda_i}(q^{\lambda_i - \kappa_{j+1}} t^{j-i})}$$

with $f(u) = (tu;q)_\infty/(qu;q)_\infty$ and $w_r(u) = (tu;q)_r/(qu;q)_r$. Since $\kappa - \lambda$ is a horizontal strip, the length of κ is at most equal to $n+1$, so we can write $\kappa = (\lambda_1 + \theta_1, \ldots, \lambda_n + \theta_n, \lambda_{n+1} - |\theta|)$. Then

$$\psi_{\kappa/\lambda} = \prod_{1 \leqslant i \leqslant j \leqslant n} w_{\theta_i}(q^{\lambda_i - \lambda_j} t^{j-i}) \prod_{1 \leqslant i < j \leqslant n+1} \left(w_{\theta_i}(q^{\lambda_i - \kappa_j} t^{j-i-1}) \right)^{-1},$$

which is the statement. □

The Pieri formula defines an infinite transition matrix. Indeed, the Macdonald symmetric functions $\{Q_\lambda\}$ form a basis of Sym, and so do the products $\{Q_\mu Q_{(r)}\}$. We shall now compute the inverse of this matrix explicitly.

4. Main result

Let $u = (u_1, \ldots, u_n)$ be n indeterminates and $\theta = (\theta_1, \ldots, \theta_n) \in \mathsf{N}^n$. For clarity of display, we introduce n auxiliary variables $v = (v_1, \ldots, v_n)$ defined by $v_k = q^{\theta_k} u_k$. We write

$$\begin{aligned} &C^{(q,t)}_{\theta_1,\ldots,\theta_n}(u_1, \ldots, u_n) \\ &\quad = \prod_{k=1}^{n} t^{\theta_k} \frac{(q/t;q)_{\theta_k}}{(q;q)_{\theta_k}} \frac{(qu_k;q)_{\theta_k}}{(qtu_k;q)_{\theta_k}} \prod_{1 \leqslant i < j \leqslant n} \frac{(qu_i/tu_j;q)_{\theta_i}}{(qu_i/u_j;q)_{\theta_i}} \frac{(tu_i/v_j;q)_{\theta_i}}{(u_i/v_j;q)_{\theta_i}} \\ &\qquad \times \frac{1}{\Delta(v)} \det_{1 \leqslant i,j \leqslant n} \left[v_i^{n-j} \left(1 - t^{j-1} \frac{1 - tv_i}{1 - v_i} \prod_{k=1}^{n} \frac{u_k - v_i}{tu_k - v_i} \right) \right]. \end{aligned}$$

Setting $u_{n+1}=1/t$ we have

$$C^{(q,t)}_{\theta_1,\ldots,\theta_n}(u_1,\ldots,u_n)=\prod_{1\leqslant i<j\leqslant n+1}\frac{(qu_i/tu_j;q)_{\theta_i}}{(qu_i/u_j;q)_{\theta_i}}\prod_{1\leqslant i\leqslant j\leqslant n}\frac{(tu_i/v_j;q)_{\theta_i}}{(u_i/v_j;q)_{\theta_i}}$$
$$\times\frac{1}{\Delta(v)}\det_{1\leqslant i,j\leqslant n}\left[v_i^{n-j}\left(1-t^j\prod_{k=1}^{n+1}\frac{u_k-v_i}{tu_k-v_i}\right)\right].$$

We are now in a position to prove our main result.

Theorem 4.1. *Let $\lambda=(\lambda_1,\ldots,\lambda_{n+1})$ be an arbitrary partition with length $n+1$. For any $1\leqslant k\leqslant n+1$ define $u_k=q^{\lambda_k-\lambda_{n+1}}t^{n-k}$. We have*

$$Q_{(\lambda_1,\ldots,\lambda_{n+1})}=\sum_{\theta\in\mathsf{N}^n}C^{(q,t)}_{\theta_1,\ldots,\theta_n}(u_1,\ldots,u_n)\,Q_{(\lambda_{n+1}-|\theta|)}\,Q_{(\lambda_1+\theta_1,\ldots,\lambda_n+\theta_n)}.$$

Proof. Let $\beta=(\beta_1,\ldots,\beta_n)$, $\kappa=(\kappa_1,\ldots,\kappa_n)$, $\gamma=(\gamma_1,\ldots,\gamma_n)\in\mathsf{Z}^n$. Defining

$$f_{\beta\kappa}=C^{(q,t)}_{\beta_1-\kappa_1,\ldots,\beta_n-\kappa_n}\big(q^{\kappa_1+|\kappa|}u_1,\ldots,q^{\kappa_n+|\kappa|}u_n\big),$$
$$g_{\kappa\gamma}=d_{\kappa_1-\gamma_1,\ldots,\kappa_n-\gamma_n}\big(q^{\gamma_1+|\gamma|}u_1,\ldots,q^{\gamma_n+|\gamma|}u_n\big),$$

these infinite lower-triangular n-dimensional matrices are inverses of each other, by application of Theorem 2.7. Now if in Theorem 3.1, we replace λ_{n+1} by $\lambda_{n+1}-|\gamma|$ and λ_i by $\lambda_i+\gamma_i$, u_i by $q^{\gamma_i+|\gamma|}u_i$, $(1\leqslant i\leqslant n)$, we obtain (after shifting the summation indices)

$$\sum_{\kappa\in\mathsf{Z}^n}g_{\kappa\gamma}y_\kappa=w_\gamma\quad(\gamma\in\mathsf{Z}^n)$$

with

$$y_\kappa=Q_{(\lambda_1+\kappa_1,\ldots,\lambda_n+\kappa_n,\lambda_{n+1}-|\kappa|)},$$
$$w_\gamma=Q_{(\lambda_1+\gamma_1,\ldots,\lambda_n+\gamma_n)}Q_{(\lambda_{n+1}-|\gamma|)}.$$

This immediately yields

$$\sum_{\beta\in\mathsf{Z}^n}f_{\beta\kappa}w_\beta=y_\kappa\quad(\kappa\in\mathsf{Z}^n).$$

We conclude by setting $\kappa_i=0$ for all $1\leqslant i\leqslant n$. $\square$

In the case $n=1$, i.e. for partitions of length 2, Theorem 4.1 reads

$$Q_{(\lambda_1,\lambda_2)} = \sum_{\theta\in\mathrm{N}} C_\theta^{(q,t)}(u)\, Q_{(\lambda_2-\theta)}\, Q_{(\lambda_1+\theta)} \tag{4.1}$$

with $u=q^{\lambda_1-\lambda_2}$ and

$$\begin{aligned} C_\theta^{(q,t)}(u) &= t^\theta \frac{(q/t;q)_\theta}{(q;q)_\theta} \frac{(qu;q)_\theta}{(qtu;q)_\theta} \Big(1-\frac{1-q^\theta tu}{1-q^\theta u}\frac{u-q^\theta u}{tu-q^\theta u}\Big) \\ &= t^\theta \frac{(q/t;q)_\theta}{(q;q)_\theta} \frac{(qu;q)_\theta}{(qtu;q)_\theta} \frac{t-1}{t-q^\theta}\frac{1-q^{2\theta}u}{1-q^\theta u} \\ &= t^\theta \frac{(1/t;q)_\theta}{(q;q)_\theta} \frac{(u;q)_\theta}{(qtu;q)_\theta} \frac{1-q^{2\theta}u}{1-u}. \end{aligned}$$

We thus recover Jing and Józefiak's result [5], which appears as a consequence of Bressoud's matrix inverse [1].

The reader may also verify that for $n=2$, i.e. for partitions of length 3, our result gives the formula stated in an earlier note by the first author [14].

Applying the automorphism $\omega_{q,t}$ to Theorem 5.1, and taking into account (3.4), we obtain the following equivalent result.

Theorem 4.2. *Let $\lambda=(1^{m_1},2^{m_2},\ldots,(n+1)^{m_{n+1}})$ be an arbitrary partition consisting of parts at most equal to $n+1$. For any $1\leqslant k\leqslant n+1$ define $u_k=q^{n-k}t^{\sum_{j=k}^n m_j}$. We have*

$$P_\lambda = \sum_{\theta\in\mathrm{N}^n} C^{(t,q)}_{\theta_1,\ldots,\theta_n}(u_1,\ldots,u_n)\, e_{m_{n+1}-|\theta|}\, P_{(1^{m_1+\theta_1-\theta_2},\ldots,(n-1)^{m_{n-1}+\theta_{n-1}-\theta_n},n^{m_n+m_{n+1}+\theta_n})}.$$

Remark 4.3. Our proof of Theorem 4.1 looks somewhat external to Macdonald theory, and does not explain the particular form of $C^{(q,t)}_{\theta_1,\ldots,\theta_n}(u_1,\ldots,u_n)$. Observe that its last factor may be written

$$\begin{aligned} &\frac{1}{\Delta(v)} \det_{1\leqslant i,j\leqslant n}\left[v_i^{n-j}\left(1-t^j\prod_{k=1}^{n+1}\frac{u_k-v_i}{tu_k-v_i}\right)\right] \\ &\quad= \sum_{K\subset\{1,\ldots,n\}} (-1)^{|K|}(1/t)^{\binom{|K|+1}{2}} \prod_{\substack{k\in K\\ j\notin K}} \frac{v_j-v_k/t}{v_j-v_k} \prod_{i=1}^{n+1}\prod_{k\in K}\frac{u_i-v_k}{u_i-v_k/t}. \end{aligned} \tag{4.2}$$

This expression may be obtained from (3.3) by replacing t by $1/t$, u by $-1/t$, X by $V=(v_1,\ldots,v_n)$, and Y by $U=(1/u_1,\ldots,1/u_{n+1})$. If we write Π for

$\Pi(U, V, 1/q, 1/t)$, both sides of (4.2) are $\Pi^{-1} D(-1/t; 1/q, 1/t)_{(V)} \Pi$, where the suffix (V) indicates operation on the V variables. Unfortunately, our proof of Theorem 4.1 does not provide any explanation for the mysterious occurrence of this Macdonald operator.

5. Analytic expansions

Theorems 4.1 and 4.2 immediately generate the analytic development of Macdonald polynomials in terms of the symmetric functions g_k or e_k, which form two algebraic basis of Sym.

Let $\mathsf{M}^{(n)}$ denote the set of upper triangular $n \times n$ matrices with nonnegative integers, and 0 on the diagonal. Throughout this paper, for any $\theta = (\theta_{i,j}) \in \mathsf{M}^{(n)}$ we shall write

$$\zeta_k(\theta) = \sum_{j=k+1}^{n} \theta_{k,j} - \sum_{j=1}^{k-1} \theta_{j,k}, \quad 1 \leqslant k \leqslant n,$$

$$\xi_{ik}(\theta) = \sum_{j=k+2}^{n} (\theta_{i,j} - \theta_{k+1,j}), \quad 1 \leqslant i \leqslant k \leqslant n-1.$$

Obviously we have $\xi_{i,n-1}(\theta) = 0$, $\xi_{i,n-2}(\theta) = \theta_{i,n} - \theta_{n-1,n}$, and

$$\xi_{ik}(\theta) = \sum_{j=i}^{k} \left(\sum_{l=k+2}^{n} (\theta_{j,l} - \theta_{j+1,l}) \right). \tag{5.1}$$

By a straightforward iteration of Theorem 4.1 we obtain the analytic development of Macdonald polynomials in terms of the symmetric functions g_k.

Theorem 5.1. *Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be an arbitrary partition with length n. We have*

$$Q_\lambda(q, t) = \sum_{\theta \in \mathsf{M}^{(n)}} \prod_{k=1}^{n-1} C^{(q,t)}_{\theta_{1,k+1}, \ldots, \theta_{k,k+1}}(\{u_i = q^{\lambda_i - \lambda_{k+1} + \xi_{ik}(\theta)} t^{k-i}; 1 \leqslant i \leqslant k\}) \prod_{k=1}^{n} g_{\lambda_k + \zeta_k(\theta)}.$$

Proof. By induction on n. The property is trivial for $n = 1$. Let us assume it is true when n is replaced by $n - 1$. We may write Theorem 4.1 in the form

$$Q_\lambda = \sum_{(\theta_{1,n}, \ldots, \theta_{n-1,n}) \in \mathsf{N}^{n-1}} C^{(q,t)}_{\theta_{1,n}, \ldots, \theta_{n-1,n}}(\{u_i = q^{\lambda_i - \lambda_n} t^{n-i-1}; 1 \leqslant i \leqslant n-1\})$$

$$\times g_{\lambda_n - \sum_{j=1}^{n-1} \theta_{j,n}} \; Q_{(\lambda_1 + \theta_{1,n}, \ldots, \lambda_{n-1} + \theta_{n-1,n})}.$$

The partition $\rho = (\lambda_1 + \theta_{1,n}, \dots, \lambda_{n-1} + \theta_{n-1,n})$ has length $n-1$, and by the inductive hypothesis we have

$$Q_\rho = \sum_{\Theta \in \mathsf{M}^{(n-1)}} \prod_{k=1}^{n-2} C^{(q,t)}_{\Theta_{1,k+1},\dots,\Theta_{k,k+1}}(\{u_i = q^{\rho_i - \rho_{k+1} + \xi_{ik}(\Theta)}\, t^{k-i};\, 1 \leqslant i \leqslant k\}) \prod_{k=1}^{n-1} g_{\rho_k + \zeta_k(\Theta)}.$$

Now if $\Theta \in \mathsf{M}^{(n-1)}$ is obtained from $\theta \in \mathsf{M}^{(n)}$ by deleting the last column, we have

$$\prod_{k=1}^{n} g_{\lambda_k + \zeta_k(\theta)} = g_{\lambda_n - \sum_{j=1}^{n-1} \theta_{j,n}} \prod_{k=1}^{n-1} g_{\lambda_k + \theta_{k,n} + \zeta_k(\Theta)},$$
$$\lambda_i - \lambda_{k+1} + \xi_{ik}(\theta) = \rho_i - \rho_{k+1} + \xi_{ik}(\Theta)$$

and the theorem follows immediately. $\square$

This result may be stated in terms of "raising operators" [19, p. 9]. For each pair of integers $1 \leqslant i < j \leqslant n$, define an operator R_{ij} acting on multi-integers $a = (a_1, \dots, a_n)$ by $R_{ij}(a) = (a_1, \dots, a_i + 1, \dots, a_j - 1, \dots, a_n)$. Any product $R = \prod_{i<j} R_{ij}^{\theta_{ij}}$, with $\theta = (\theta_{ij}) \in \mathsf{M}^{(n)}$ is called a raising operator. Its action may be extended to any product $g_a = \prod_{k=1}^{n} g_{a_k}$, with $a = (a_1, \dots, a_n) \in \mathsf{N}^n$, by setting $R g_a = g_{Ra}$. Note that composing the action of raising operators on functions g_a should be avoided, since $R_1 R_2 g_a$ may be different from $R_1(R_2 g_a)$.

Then the last quantity appearing in the right-hand side of Theorem 5.1 may be written

$$\prod_{k=1}^{n} g_{\lambda_k + \zeta_k(\theta)} = \left(\prod_{1 \leqslant i < j \leqslant n} R_{ij}^{\theta_{ij}} \right) g_\lambda.$$

Applying $\omega_{q,t}$, we immediately deduce the following analytic expansion of Macdonald polynomials in terms of elementary symmetric functions e_k.

Theorem 5.2. *Let $\lambda = (1^{m_1}, 2^{m_2}, \dots, n^{m_n})$ be an arbitrary partition consisting of parts at most equal to n. For any $1 \leqslant k \leqslant n$, define $M_k = \sum_{j=k}^{n} m_j$. We have*

$$P_\lambda(q,t) = \sum_{\theta \in \mathsf{M}^{(n)}} \prod_{k=1}^{n-1} C^{(t,q)}_{\theta_{1,k+1},\dots,\theta_{k,k+1}}(\{u_i = q^{k-i}\, t^{M_i - M_{k+1} + \xi_{ik}(\theta)};\, 1 \leqslant i \leqslant k\}) \prod_{k=1}^{n} e_{M_k + \zeta_k(\theta)}.$$

It is clear that the analytic developments given by Theorems 5.1 and 5.2 are fully explicit. Two analogous formulas may be also obtained by using (3.2).

6. Some special cases

It is worth considering our results in some particular cases [19, p. 324], for instance $q = t$ (Schur functions), or $q = 1$ (elementary symmetric functions). Section 7 will be devoted to $q = 0$ (Hall–Littlewood symmetric functions) and $t = 1$ (monomial symmetric functions). Section 8 will be devoted to the $q = t^\alpha, t \to 1$ limit (Jack symmetric functions).

Let us first give a general property of the development (4.2). Since $v_k = q^{\theta_k} u_k$, we have

$$\frac{u_k - v_k}{u_k - v_k/t} = t\,\frac{1-q^{\theta_k}}{t-q^{\theta_k}}.$$

Obviously the summation on the right-hand side is therefore restricted to $K \subset T = \{k \in \{1,\dots,n\},\ \theta_k \neq 0\}$, and we have

$$\frac{1}{\Delta(v)} \det_{1\leqslant i,j\leqslant n}\left[v_i^{n-j}\left(1 - t^j \prod_{k=1}^{n+1} \frac{u_k - v_i}{tu_k - v_i}\right)\right] = \prod_{k\in T}\Big(\frac{1-q^{\theta_k}}{t-q^{\theta_k}}\Big) F_\theta,$$

where F_θ may be easily written

$$\sum_{K\subset T} (-1)^{|K|} (1/t)^{\binom{|K|}{2}} \prod_{j\in T-K} \frac{t-q^{\theta_j}}{1-q^{\theta_j}} \prod_{\substack{k\in K\\ j\in T-K}} \frac{v_j - v_k/t}{v_j - v_k} \prod_{k\in K}\left(\frac{1-tv_k}{1-v_k}\prod_{\substack{i\in T\\ i\neq k}} \frac{u_i - v_k}{u_i - v_k/t}\right). \tag{6.1}$$

Since for $\theta_k \neq 0$ we have

$$t\,\frac{1-q^{\theta_k}}{t-q^{\theta_k}}\,\frac{(q/t;q)_{\theta_k}}{(q;q)_{\theta_k}} = \frac{(q/t;q)_{\theta_k-1}}{(q;q)_{\theta_k-1}},$$

we conclude that

$$C_\theta^{(q,t)}(u) = \prod_{k\in T} t^{\theta_k-1} \frac{(q/t;q)_{\theta_k-1}}{(q;q)_{\theta_k-1}}\,\frac{(qu_k;q)_{\theta_k}}{(qtu_k;q)_{\theta_k}} \prod_{1\leqslant i<j\leqslant n} \frac{(qu_i/tu_j;q)_{\theta_i}}{(qu_i/u_j;q)_{\theta_i}}\,\frac{(tu_i/v_j;q)_{\theta_i}}{(u_i/v_j;q)_{\theta_i}} F_\theta. \tag{6.2}$$

The specialization $q = t$ corresponds to the case of Schur functions. Then $g_k(t,t) = h_k$ and $P_\lambda(t,t) = Q_\lambda(t,t) = s_\lambda$.

Lemma 6.1. *For $q = t$, we have $C_\theta^{(t,t)}(u) = 0$, except if $\theta_k \in \{0, 1\}$ for $1 \leqslant k \leqslant n$, in which case $C_\theta^{(t,t)}(u)$ is equal to $(-1)^{|\theta|}$.*

Proof. Using (6.2) it is clear that $C_\theta^{(t,t)}(u) = 0$, except if all $\theta_k \in \{0, 1\}$. It remains to compute the value of $C_\theta^{(t,t)}(u)$ in this case. Then $T = \{k \in \{1, \dots, n\},\ \theta_k = 1\}$, so that $v_k = u_k$ for $k \notin T$ and $v_k = tu_k$ for $k \in T$. We have only to prove

$$F_\theta = (-1)^{|T|} \prod_{k \in T} \frac{1 - t^2 u_k}{1 - tu_k} \prod_{\substack{i,j \in T \\ i<j}} \frac{1 - tu_i/u_j}{1 - u_i/u_j} \, \frac{1 - u_i/tu_j}{1 - u_i/u_j}.$$

But in (6.1) we see that, when $q = t$ and $\theta_k = 1$ for $k \in T$, the only nonzero contribution comes from $K = T$. Hence the result. $\square$

Thus for $q = t$, Theorem 4.1 reads

$$s_{(\lambda_1,\dots,\lambda_{n+1})} = \sum_{\theta \in \{0,1\}^n} (-1)^{|\theta|}\, h_{\lambda_{n+1}-|\theta|}\, s_{(\lambda_1+\theta_1,\dots,\lambda_n+\theta_n)}, \tag{6.3}$$

The following lemma shows that this result is a variant of the classical Jacobi–Trudi formula [19, p. 41, Eq. (3.4)]

$$s_{(\lambda_1,\dots,\lambda_{n+1})} = \det_{1 \leqslant i,j \leqslant n+1} [h_{\lambda_i - i + j}].$$

Lemma 6.2. *The right-hand side of* (6.3) *is the development of the Jacobi–Trudi determinant along its last row.*

Proof. For $0 \leqslant j \leqslant n$, let M_j denote the minor obtained by deleting the $(n+1)$th row and the $(n+1-j)$th column of the Jacobi–Trudi determinant. We have

$$s_{(\lambda_1,\dots,\lambda_{n+1})} = \sum_{j=0}^{n} (-1)^j M_j\, h_{\lambda_{n+1}-j}.$$

Let $\Lambda = (\lambda_1+1, \dots, \lambda_n+1)$. Using the Jacobi–Trudi expansion for skew Schur functions [19, p. 70, Eq. (5.4)], it is clear that M_j is exactly the skew Schur function $s_{\Lambda/(1^{n-j})}$. This skew Schur function can be expanded in terms of Schur functions by using [19, p. 70, Eq. (5.3)]. The classical Pieri rule yields

$$M_j = s_{\Lambda/(1^{n-j})} = \sum_\mu s_\mu$$

with μ such that $\Lambda - \mu$ is a vertical $(n-j)$-strip. In other words, μ is obtained from Λ by subtracting $(n-j)$ nodes (at most one from each row), or equivalently from $(\lambda_1, \ldots, \lambda_n)$ by adding j nodes (at most one to each row). $\square$

For $q = 1$ we readily obtain $C_\theta^{(t,1)}(u) = 0$ except if $\theta = (0, \ldots, 0)$. Theorem 4.2 thus reads

$$P_{(1^{m_1},\ldots,(n+1)^{m_{n+1}})}(1,t) = e_{m_{n+1}}\, P_{(1^{m_1},\ldots,(n-1)^{m_{n-1}},n^{m_n+m_{n+1}})}(1,t),$$

from which we deduce

$$P_\lambda(1,t) = \prod_{i=1}^{n+1} e_{\sum_{k=i}^{n+1} m_k(\lambda)} = e_{\lambda'}.$$

7. Hall–Littlewood polynomials

In this section we consider the case $q = 0$, which is known [19, p. 324] to correspond to the Hall–Littlewood symmetric functions. We have $P_\lambda(0,t) = P_\lambda(t)$ and $Q_\lambda(0,t) = Q_\lambda(t)$, these functions being defined in [19, pp. 208–210]. We shall follow the notation of Macdonald [19], writing $q_k(t)$ for $g_k(0,t) = Q_{(k)}(t)$ and $q_\mu(t)$ for $g_\mu(0,t)$. The parameter t being kept fixed, we shall also write P_λ, Q_λ, q_k and q_μ for short.

The following expansion of Hall–Littlewood polynomials is well-known [19, p. 213]. If λ is any partition with length $n+1$, one has

$$\begin{aligned} Q_\lambda &= \left(\prod_{1 \leqslant i < j \leqslant n+1} \frac{1 - R_{ij}}{1 - tR_{ij}} \right) q_\lambda \\ &= \left(\prod_{1 \leqslant i < j \leqslant n+1} \left(1 + (1 - 1/t) \sum_{\theta_{ij} \geqslant 1} t^{\theta_{ij}}\, R_{ij}^{\theta_{ij}} \right) \right) q_\lambda. \end{aligned}$$

This property seems difficult to recover as the $q = 0$ limit of Theorem 5.1 (see however Section 10). We shall give the $q = 0$ specialization of Theorem 4.2 instead.

Let $\left[{r \atop s} \right]_t$ denote the t-binomial coefficient $(t^{r-s+1};t)_s/(t;t)_s$. The Pieri formula for Hall–Littlewood polynomials [19, p. 215, Eq. (3.2)] writes as

$$\begin{aligned} &e_{m_{n+1}}\, P_{(1^{m_1},\ldots,n^{m_n})} \\ &\quad = \sum_{\theta \in \mathsf{N}^n} \prod_{k=1}^{n} \left[{m_k + \theta_k - \theta_{k+1} \atop \theta_k} \right]_t P_{(1^{m_1+\theta_1-\theta_2},\ldots,(n-1)^{m_{n-1}+\theta_{n-1}-\theta_n},n^{m_n+\theta_n-\theta_{n+1}},(n+1)^{\theta_{n+1}})} \end{aligned}$$

with $\theta_{n+1} = m_{n+1} - |\theta|$. This formula cannot be directly inverted by applying the method of Section 2.1 to the matrix thus defined. Actually the corresponding system of equations turns out to be *nonlinear*. We shall obtain the inverse relation as the $q = 0$ limit of Theorem 4.2.

Theorem 7.1. *Let $\lambda = (1^{m_1}, 2^{m_2}, \ldots, (n+1)^{m_{n+1}})$ be an arbitrary partition consisting of parts at most equal to $n+1$. We have*

$$P_\lambda = \sum_{\theta \in \mathbb{N}^n} C^{(t)}_{\theta_1,\ldots,\theta_n}(m_1, \ldots, m_n)\, e_{m_{n+1}-|\theta|}\, P_{(1^{m_1+\theta_1-\theta_2}, \ldots, (n-1)^{m_{n-1}+\theta_{n-1}-\theta_n}, n^{m_n+m_{n+1}+\theta_n})}$$

with $C^{(t)}_{\theta_1,\ldots,\theta_n}(m_1, \ldots, m_n)$ defined by

$$C^{(t)}_{\theta_1,\ldots,\theta_n}(m_1, \ldots, m_n) = (-1)^{|\theta|} \sum_{k=1}^{n+1} \left(\prod_{i=1}^{k-1} t^{\binom{\theta_i}{2}} \begin{bmatrix} m_i + \theta_i \\ \theta_i \end{bmatrix}_t \right) \left(\prod_{j=k}^{n} t^{m_j + \binom{\theta_j+1}{2}} \begin{bmatrix} m_j + \theta_j - 1 \\ \theta_j - 1 \end{bmatrix}_t \right). \tag{7.1}$$

Remark 7.2. This result is new. It has no direct connection with Morris' recurrence formula [20], although in both cases induction is done by removing the largest part of λ. Note that here all largest parts are simultaneously removed, whereas in [20] one part is removed at a time.

Proof. Let us define

$$C^{(t)}_{\theta_1,\ldots,\theta_n}(m_1, \ldots, m_n) = \lim_{q \to 0} C^{(t,q)}_{\theta_1,\ldots,\theta_n}(u_1, \ldots, u_n) \tag{7.2}$$

with $u_i = q^{n-i} t^{M_i}$ and $M_i = \sum_{j=i}^n m_j$. Using the auxiliary variables $v_k = t^{\theta_k} u_k$, we first compute

$$\lim_{q\to 0} \prod_{1 \leqslant i < j \leqslant n+1} \frac{(tu_i/qu_j;t)_{\theta_i}}{(tu_i/u_j;t)_{\theta_i}} \prod_{1 \leqslant i \leqslant j \leqslant n} \frac{(qu_i/v_j;t)_{\theta_i}}{(u_i/v_j;t)_{\theta_i}}$$
$$= \lim_{q\to 0} \prod_{1 \leqslant i < j \leqslant n+1} \frac{(q^{j-i-1} t^{M_i - M_j + 1};t)_{\theta_i}}{(q^{j-i} t^{M_i - M_j + 1};t)_{\theta_i}} \prod_{1 \leqslant i \leqslant j \leqslant n} \frac{(q^{j-i+1} t^{M_i - M_j - \theta_j};t)_{\theta_i}}{(q^{j-i} t^{M_i - M_j - \theta_j};t)_{\theta_i}}.$$

When $q \to 0$, all limits are 1 but

$$\prod_{i=1}^n \frac{(t^{M_i - M_{i+1} + 1};t)_{\theta_i}}{(t^{-\theta_i};t)_{\theta_i}} = (-1)^{|\theta|} \prod_{i=1}^n t^{\binom{\theta_i+1}{2}} \begin{bmatrix} m_i + \theta_i \\ \theta_i \end{bmatrix}_t.$$

Hence it remains to prove that

$$\lim_{q\to 0}\frac{1}{\Delta(v)}\det_{1\leqslant i,j\leqslant n}\left[v_i^{n-j}\left(1-q^{j-1}\frac{1-qv_i}{1-v_i}\prod_{k=1}^{n}\frac{u_k-v_i}{qu_k-v_i}\right)\right]$$
$$=\sum_{k=1}^{n+1}\left(\prod_{i=1}^{k-1}t^{-\theta_i}\right)\left(\prod_{j=k}^{n}t^{m_j}\frac{1-t^{\theta_j}}{1-t^{m_j+\theta_j}}\right).$$

Since $M_{j+1}-M_j=-m_j$, this is a direct consequence of the following more general result, applied for $a_k=t^{M_k}$, and $b_k=t^{M_k+\theta_k}$, i.e. $u_k=q^{n-k}a_k$, and $v_k=q^{n-k}b_k$. □

Lemma 7.3. *Let* $a=(a_1,\dots,a_n,a_{n+1})$ *and* $b=(b_1,\dots,b_n)$ *be* $2n+1$ *indeterminates. Define*

$$F_n(q)=\prod_{1\leqslant i<j\leqslant n}(q^{n-i}b_i-q^{n-j}b_j)^{-1}$$
$$\times\det_{1\leqslant i,j\leqslant n}\left[(q^{n-i}b_i)^{n-j}\left(1-q^j\prod_{k=1}^{n+1}\frac{b_i-q^{i-k}a_k}{b_i-q^{i-k+1}a_k}\right)\right]$$

and

$$G_n=\sum_{k=1}^{n+1}\prod_{i=1}^{k-1}\frac{a_i}{b_i}\prod_{j=k}^{n}\frac{a_j-b_j}{a_{j+1}-b_j}.$$

Then we have $\lim_{q\to 0}F_n(q)=G_n$.

Proof. Substituting q for t, $q^{n-i}b_i$ for v_i, and $q^{n-i}a_i$ for u_i in (4.2), we have

$$F_n(q)=\sum_{K\subset\{1,\dots,n\}}(-1)^{|K|}(1/q)^{\binom{|K|+1}{2}}\prod_{\substack{k\in K\\ j\notin K}}\frac{b_j-q^{j-k-1}b_k}{b_j-q^{j-k}b_k}\prod_{k\in K}\prod_{i=1}^{n+1}\frac{a_i-q^{i-k}b_k}{a_i-q^{i-k-1}b_k}.$$

The contribution of K can be written as

$$(-1)^{|K|}(1/q)^{\binom{|K|+1}{2}}\prod_{\substack{k\in K\\ j\notin K}}\left(\frac{b_j-q^{j-k-1}b_k}{b_j-q^{j-k}b_k}\,\frac{a_j-q^{j-k}b_k}{a_j-q^{j-k-1}b_k}\right)$$
$$\times\prod_{k\in K}\left(\frac{a_{n+1}-q^{n-k+1}b_k}{a_{n+1}-q^{n-k}b_k}\right)\prod_{\substack{i\in K\\ k\in K}}\left(\frac{a_i-q^{i-k}b_k}{a_i-q^{i-k-1}b_k}\right).$$

When $q \to 0$, the limit of the various factors are

$$\lim_{q\to 0} \prod_{\substack{k\in K\\ j\notin K}} \left(\frac{b_j - q^{j-k-1} b_k}{b_j - q^{j-k} b_k} \frac{a_j - q^{j-k} b_k}{a_j - q^{j-k-1} b_k} \right) = \prod_{\substack{k\in K, k\neq n\\ k+1\notin K}} \left(\frac{a_{k+1}}{b_{k+1}} \frac{b_{k+1} - b_k}{a_{k+1} - b_k} \right),$$

$$\lim_{q\to 0} \prod_{k\in K} \frac{a_{n+1} - q^{n-k+1} b_k}{a_{n+1} - q^{n-k} b_k} = \frac{a_{n+1}}{a_{n+1} - b_n} \quad \text{if } n \in K,$$

$$\lim_{q\to 0} (-1/q)^{|K|} \prod_{k\in K} \frac{a_k - b_k}{a_k - b_k/q} = \prod_{i\in K} \frac{a_i - b_i}{b_i},$$

$$\lim_{q\to 0} (1/q)^{\binom{|K|}{2}} \prod_{\substack{i,j\in K\\ i<j}} \left(\frac{a_i - q^{i-j} b_j}{a_i - q^{i-j-1} b_j} \frac{a_j - q^{j-i} b_i}{a_j - q^{j-i-1} b_i} \right) = \prod_{\substack{i\in K, i\neq n\\ i+1\in K}} \frac{a_{i+1}}{a_{i+1} - b_i}.$$

Putting these limits together, we have

$$\lim_{q\to 0} F_n(q) = \sum_{K\subset\{1,\ldots,n\}} \prod_{\substack{k\in K, k\neq n\\ k+1\notin K}} \frac{b_{k+1} - b_k}{b_{k+1}} \prod_{i\in K} \left(\frac{a_{i+1}}{b_i} \frac{a_i - b_i}{a_{i+1} - b_i} \right).$$

We are done once we have shown the following lemma. □

Lemma 7.4. *Let $a = (a_1, \ldots, a_n, a_{n+1})$ and $b = (b_1, \ldots, b_n)$ be $2n+1$ indeterminates. Define*

$$F_n = \sum_{K\subset\{1,\ldots,n\}} \prod_{\substack{k\in K, k\neq n\\ k+1\notin K}} \frac{b_{k+1} - b_k}{b_{k+1}} \prod_{i\in K} \left(\frac{a_{i+1}}{b_i} \frac{a_i - b_i}{a_{i+1} - b_i} \right).$$

Then $F_n = G_n$.

Proof. Obviously G_n satisfies the recurrence relation

$$G_n = \prod_{i=1}^{n} \frac{a_i}{b_i} + \frac{a_n - b_n}{a_{n+1} - b_n}\, G_{n-1},$$

which yields

$$G_n = \left(\frac{a_n}{b_n} + \frac{a_n - b_n}{a_{n+1} - b_n} \right) G_{n-1} - \frac{a_n}{b_n} \frac{a_{n-1} - b_{n-1}}{a_n - b_{n-1}}\, G_{n-2}.$$

We have $F_0 = G_0 = 1$ and

$$F_1 = 1 + \frac{a_2}{b_1}\frac{a_1 - b_1}{a_2 - b_1} = \frac{a_1}{b_1} + \frac{a_1 - b_1}{a_2 - b_1} = G_1.$$

Thus, we have only to prove that F_n satisfies the second recurrence relation. Summing the contributions of sets $K = L \cup \{n\}$, with $L \subset \{1, \ldots, n-1\}$ possibly empty, we find

$$F_n = H_n + \frac{a_{n+1}}{b_n}\frac{a_n - b_n}{a_{n+1} - b_n} F_{n-1}$$

with

$$H_n = \sum_{L \subset \{1,\ldots,n-1\}} \prod_{\substack{k \in L \\ k+1 \notin L}} \frac{b_{k+1} - b_k}{b_{k+1}} \prod_{i \in L} \left(\frac{a_{i+1}}{b_i} \frac{a_i - b_i}{a_{i+1} - b_i} \right).$$

Summing separately sets with $n-1 \notin L$ and $n-1 \in L$, we have

$$H_n = H_{n-1} + \frac{b_n - b_{n-1}}{b_n} \frac{a_n}{b_{n-1}} \frac{a_{n-1} - b_{n-1}}{a_n - b_{n-1}} F_{n-2}$$

or equivalently

$$F_n - \frac{a_{n+1}}{b_n}\frac{a_n - b_n}{a_{n+1} - b_n} F_{n-1} = F_{n-1} + \frac{a_n}{b_{n-1}} \frac{a_{n-1} - b_{n-1}}{a_n - b_{n-1}} \left(\frac{b_n - b_{n-1}}{b_n} - 1 \right) F_{n-2}.$$

Hence the result. $\square$

Taking the $q = 0$ limit of Theorem 5.2, and using the definitions (5.1) and (7.2), we deduce the following (new) expansion of Hall–Littlewood polynomials in terms of elementary symmetric functions.

Theorem 7.5. *Let $\lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$ be an arbitrary partition consisting of parts at most equal to n. For any $1 \leqslant k \leqslant n$, define $M_k = \sum_{j=k}^{n} m_j$. We have*

$$P_\lambda(t)$$
$$= \sum_{\theta \in \mathsf{M}^{(n)}} \prod_{k=1}^{n-1} C^{(t)}_{\theta_{1,k+1},\ldots,\theta_{k,k+1}}(\{m_i + \sum_{j=k+2}^{n} (\theta_{i,j} - \theta_{i+1,j}); 1 \leqslant i \leqslant k\}) \prod_{k=1}^{n} e_{M_k + \zeta_k(\theta)}$$

with $C^{(t)}_{\theta_1,\ldots,\theta_k}(m_1, \ldots, m_k)$ defined by Eq. (7.1).

It is known [19, p. 208] that monomial symmetric functions are the specialization of Hall–Littlewood symmetric functions for $t = 1$. One has $P_\lambda(1) = m_\lambda$, and in this situation Theorem 7.1 reads as follows.

Theorem 7.6. *Let $\lambda = (1^{r_1}, 2^{r_2}, \ldots, (n+1)^{r_{n+1}})$ be an arbitrary partition consisting of parts at most equal to $n+1$. We have*

$$m_\lambda = \sum_{\theta\in\mathsf{N}^n} C_{\theta_1,\ldots,\theta_n}(r_1,\ldots,r_n)\; e_{r_{n+1}-|\theta|}\; m_{(1^{r_1+\theta_1-\theta_2},\ldots,(n-1)^{r_{n-1}+\theta_{n-1}-\theta_n},n^{r_n+r_{n+1}+\theta_n})}$$

with $C_{\theta_1,\ldots,\theta_n}(r_1,\ldots,r_n)$ defined by

$$C_{\theta_1,\ldots,\theta_n}(r_1,\ldots,r_n) = (-1)^{|\theta|} \sum_{k=1}^{n+1} \prod_{i=1}^{k-1} \binom{r_i+\theta_i}{\theta_i} \prod_{j=k}^{n} \binom{r_j+\theta_j-1}{\theta_j-1}. \tag{7.3}$$

This gives the expansion of monomial symmetric functions in terms of elementary symmetric functions, a problem which was studied by Waring [25] as early as 1762. Some years later, Vandermonde [24] computed tables up to weight 10. Alain Lascoux mentions that these tables contain no mistake [13, Section 1.3]. The case of power sums had been considered by Girard as early as 1629.

Theorem 7.7. *Let $\lambda = (1^{r_1}, 2^{r_2}, \ldots, n^{r_n})$ be an arbitrary partition consisting of parts at most equal to n. For any $1\leqslant k\leqslant n$, define $M_k = \sum_{j=k}^n r_j$. We have*

$$m_\lambda = \sum_{\theta\in\mathsf{M}^{(n)}} \prod_{k=1}^{n-1} C_{\theta_{1,k+1},\ldots,\theta_{k,k+1}}(\{r_i + \sum_{j=k+2}^{n} (\theta_{i,j}-\theta_{i+1,j});\, 1\leqslant i\leqslant k\}) \prod_{k=1}^{n} e_{M_k+\zeta_k(\theta)}$$

with $C_{\theta_1,\ldots,\theta_k}(r_1,\ldots,r_k)$ defined by Eq. (7.3).

It would be interesting to get a combinatorial interpretation of this result.

Alain Lascoux pointed out that Theorems 7.1 and 7.6 have a direct proof using Hammond operators. For any symmetric function f, let D_f be the adjoint of the multiplication by f with respect to the standard scalar product of $\mathcal{S}$. The operators D_{s_λ} and D_{h_λ} are, respectively, called Foulkes and Hammond operators.

Using the properties [13, Section 1.7]

$$D_{h_k} m_\mu = \begin{cases} m_{\mu-k} & \text{if } k \text{ is a part of } \mu, \\ 0 & \text{otherwise,} \end{cases}$$

$$D_{h_k}(e_r f) = e_r D_{h_k} f + e_{r-1} D_{h_{k-1}} f$$

and applying $D_{h_{n+1}}$ and D_{h_n} to the expansion of Theorem 7.6, we obtain

$$C_{\theta_1,\dots,\theta_n}(r_1,\dots,r_n)+C_{\theta_1,\dots,\theta_n-1}(r_1,\dots,r_n)=C_{\theta_1,\dots,\theta_n}(r_1,\dots,r_n-1).$$

Starting from $C_{0,\dots,0}(r_1,\dots,r_n)=1$, this recurrence determines $C_{\theta_1,\dots,\theta_n}(r_1,\dots,r_n)$ completely. A similar but lengthier argument may be used for Hall–Littlewood polynomials, leading to the recurrence formula

$$\begin{aligned}&C^{(t)}_{\theta_1,\dots,\theta_n}(m_1,\dots,m_n)+C^{(t)}_{\theta_1,\dots,\theta_n-1}(m_1,\dots,m_n)\\&\quad=\frac{1-t^{m_n+\theta_n}+t(1-t^{-\theta_n})}{1-t^{m_n}}\,C^{(t)}_{\theta_1,\dots,\theta_n}(m_1,\dots,m_n-1).\end{aligned}$$

8. Jack polynomials

Jack polynomials are the limit of Macdonald polynomials when $t\to 1$, with $q=t^\alpha$. The indeterminates q,t are then considered as real variables, and α is some positive real number [19, p. 376]. We define

$$P_\lambda(\alpha)=\lim_{t\to 1}P_\lambda(t^\alpha,t),\qquad Q_\lambda(\alpha)=\lim_{t\to 1}Q_\lambda(t^\alpha,t).$$

The parameter α being kept fixed, we shall also write P_λ, Q_λ for short.

These polynomials are normalized differently from their "integral form" $J_\lambda(\alpha)$ studied in [23]. We have $J_\lambda(\alpha)=c_\lambda(\alpha)P_\lambda(\alpha)=c'_\lambda(\alpha)Q_\lambda(\alpha)$, with $c_\lambda(\alpha)$ and $c'_\lambda(\alpha)$ given in [19, p. 381, Eq. (10.21)].

The Jack polynomials $Q_{(k)}$ associated to row partitions (k) have the generating series

$$\prod_{i\geqslant 1}(1-ux_i)^{-1/\alpha}=\sum_{k\geqslant 0}u^kQ_{(k)}(\alpha).$$

Their development in terms of any classical basis is given in [23, p. 80, Proposition 2.2].

We now fix some positive real number a. We denote by $(u)_k$ the classical rising factorial, defined by $(u)_0=1$ and $(u)_k=\prod_{i=1}^k(u+i-1)$ for $k\neq 0$.

Let $u=(u_1,\dots,u_n)$ be n indeterminates and $\theta=(\theta_1,\dots,\theta_n)\in\mathsf{N}^n$. For clarity of display, we introduce n auxiliary variables $v=(v_1,\dots,v_n)$ defined by $v_k=u_k+\theta_k$. We write

$$C^{(a)}_{\theta_1,\dots,\theta_n}(u_1,\dots,u_n)$$

$$= \prod_{k=1}^{n} \frac{(1-a)_{\theta_k}}{\theta_k!} \frac{(u_k+1)_{\theta_k}}{(u_k+1+a)_{\theta_k}} \prod_{1\leqslant i<j\leqslant n} \frac{(u_i-u_j+1-a)_{\theta_i}}{(u_i-u_j+1)_{\theta_i}} \frac{(u_i-v_j+a)_{\theta_i}}{(u_i-v_j)_{\theta_i}}$$
$$\times \frac{1}{\Delta(v)} \det_{1\leqslant i,j\leqslant n}\left[v_i^{n-j} - (v_i-a)^{n-j} \frac{v_i+a}{v_i} \prod_{k=1}^{n} \frac{v_i-u_k}{v_i-u_k-a}\right].$$

Setting $u_{n+1} = -a$, this may be written as

$$C^{(a)}_{\theta_1,\ldots,\theta_n}(u_1,\ldots,u_n) = \prod_{1\leqslant i<j\leqslant n+1} \frac{(u_i-u_j+1-a)_{\theta_i}}{(u_i-u_j+1)_{\theta_i}} \prod_{1\leqslant i\leqslant j\leqslant n} \frac{(u_i-v_j+a)_{\theta_i}}{(u_i-v_j)_{\theta_i}}$$
$$\times \frac{1}{\Delta(v)} \det_{1\leqslant i,j\leqslant n}\left[v_i^{n-j} - (v_i-a)^{n-j} \prod_{k=1}^{n+1} \frac{v_i-u_k}{v_i-u_k-a}\right].$$

Lemma 8.1. *With $U=(q^{u_1},\ldots,q^{u_n})$, we have*

$$C^{(a)}_{\theta}(u) = \lim_{q\to 1} c^{(q,q^a)}_{\theta}(U).$$

Proof. Define $U_{n+1} = q^{u_{n+1}}$, so that the condition $U_{n+1} = 1/t$ is satisfied for $t = q^a$. Introduce the auxiliary variables $V = (q^{v_1},\ldots,q^{v_n})$, so that $V_k = q^{\theta_k} U_k$. Then we only have to prove

$$\frac{1}{\Delta(v)} \det_{1\leqslant i,j\leqslant n}\left[v_i^{n-j} - (v_i-a)^{n-j} \prod_{k=1}^{n+1} \frac{v_i-u_k}{v_i-u_k-a}\right]$$
$$= \lim_{\substack{t=q^a\\ q\to 1}} \frac{1}{\Delta(V)} \det_{1\leqslant i,j\leqslant n}\left[V_i^{n-j}\left(1 - t^j \prod_{k=1}^{n+1} \frac{U_k-V_i}{tU_k-V_i}\right)\right].$$

Consider the following difference operator:

$$D(z;a) = \frac{1}{\Delta(v)} \det_{1\leqslant i,j\leqslant n}\left[v_i^{n-j} + z\,(v_i-a)^{n-j} T_{a,v_i}\right],$$

acting on polynomials in v, where z is some indeterminate and T_{a,v_i} is the a-translation operator defined by

$$T_{a,v_i} f(v_1,\ldots,v_n) = f(v_1,\ldots,v_i+a,\ldots,v_n).$$

Then in a strictly parallel way to the proof given in [19, p. 315], we have

$$D(z;a) = \sum_{K\subset\{1,\dots,n\}} z^{|K|} \prod_{\substack{k\in K\\ j\notin K}} \frac{v_k - v_j - a}{v_k - v_j} \prod_{k\in K} T_{a,v_k}.$$

Applying this result to $\prod_{i=1}^{n+1}\prod_{j=1}^{n}(v_j - u_i - a)$, with $z=-1$, we get

$$\frac{1}{\Delta(v)} \det_{1\leqslant i,j\leqslant n}\left[v_i^{n-j} - (v_i - a)^{n-j}\prod_{k=1}^{n+1}\frac{v_i - u_k}{v_i - u_k - a}\right]$$
$$= \sum_{K\subset\{1,\dots,n\}} (-1)^{|K|} \prod_{\substack{k\in K\\ j\notin K}} \frac{v_k - v_j - a}{v_k - v_j} \prod_{i=1}^{n+1}\prod_{k\in K}\frac{v_k - u_i}{v_k - u_i - a}.$$

On the other hand (4.2), written for $t=q^a$, yields

$$\frac{1}{\Delta(V)} \det_{1\leqslant i,j\leqslant n}\left[V_i^{n-j}\left(1 - t^j\prod_{k=1}^{n+1}\frac{U_k - V_i}{tU_k - V_i}\right)\right]$$
$$= \sum_{K\subset\{1,\dots,n\}} (-1)^{|K|} q^{-a\binom{|K|+1}{2}} \prod_{\substack{k\in K\\ j\notin K}} \frac{V_j - q^{-a}V_k}{V_j - V_k} \prod_{i=1}^{n+1}\prod_{k\in K}\frac{U_i - V_k}{U_i - q^{-a}V_k}.$$

Hence the statement in the limit $q\to 1$. $\square$

The two following results are straightforward consequences of Theorems 5.1 and 5.2.

Theorem 8.2. *Let $\lambda = (\lambda_1,\dots,\lambda_{n+1})$ be an arbitrary partition with length $n+1$. For any $1\leqslant k\leqslant n+1$ define $u_k = \lambda_k - \lambda_{n+1} + (n-k)/\alpha$. We have*

$$Q_{(\lambda_1,\dots,\lambda_{n+1})} = \sum_{\theta\in\mathbb{N}^n} C^{(1/\alpha)}_{\theta_1,\dots,\theta_n}(u_1,\dots,u_n)\, Q_{(\lambda_{n+1}-|\theta|)}\, Q_{(\lambda_1+\theta_1,\dots,\lambda_n+\theta_n)}.$$

Theorem 8.3. *Let $\lambda = (1^{m_1}, 2^{m_2},\dots,(n+1)^{m_{n+1}})$ be an arbitrary partition consisting of parts at most equal to $n+1$. For any $1\leqslant k\leqslant n+1$ define $u_k = \sum_{j=k}^n m_j + (n-k)\alpha$. We have*

$$P_\lambda = \sum_{\theta\in\mathbb{N}^n} C^{(\alpha)}_{\theta_1,\dots,\theta_n}(u_1,\dots,u_n)\, e_{m_{n+1}-|\theta|}\, P_{(1^{m_1+\theta_1-\theta_2},\dots,(n-1)^{m_{n-1}+\theta_{n-1}-\theta_n},n^{m_n+m_{n+1}+\theta_n})}.$$

As in Section 5 these formulas generate the explicit analytic developments of Jack polynomials in terms of the classical bases $Q_{(k)}$ and e_k. These expansions are easily written by replacing $C^{(q,t)}$ by $C^{(1/\alpha)}$, and $C^{(t,q)}$ by $C^{(\alpha)}$, in the corresponding statements for Macdonald polynomials. They are left to the reader.

9. The hook case

The explicit development of Macdonald polynomials in terms of the classical bases g_k and e_k was already known when the partition λ is a hook. This result had been given by Kerov [6, Theorem 6.3] (see also [7]). For $\lambda = (r, 1^s)$ Kerov's result writes elegantly as

$$Q_\lambda = \det_{1\leqslant i,j\leqslant s+1}\left[\frac{1-q^{\lambda_i-i+j}t^{s-j+1}}{1-q^{\lambda_i}t^{s-i+1}}\, g_{\lambda_i-i+j}\right].$$

It was derived by using the Pieri formula

$$Q_{1^s}\, Q_{(r)} = \frac{1-t^s}{1-qt^{s-1}}\,\frac{1-q^{r+1}t^{s-1}}{1-q^r t^s}\, Q_{(r+1,1^{s-1})} + Q_{(r,1^s)}, \tag{9.1}$$

which is readily obtained from Theorem 3.1, the two contributions on the right-hand side corresponding to $\theta_1 = r, \theta_2 = \ldots = \theta_s = 0$ and $\theta_1 = r-1, \theta_2 = \ldots = \theta_s = 0$, respectively.

Since the expansion of Theorem 4.1 involves the partition $(r, 2, 1^{s-2})$, it cannot provide a method to compute $Q_{(r,1^s)}$ through a recursion on r and/or s. However, we have obtained the following development, which may be worth giving here since its equivalence with Kerov's result is not trivial.

Let n be a positive integer and $C(n)$ denote the set of positive multi-integers $c = (c_1, \ldots, c_l)$, $c_i > 0$, with weight $|c| = \sum_{i=1}^{l} c_i = n$. The integer $l = l(c)$ is called the length of c. For any $c = (c_1, \ldots, c_l)$ we write $[c_i] = \sum_{1\leqslant k\leqslant i} c_k$ for the ith partial sum.

In [14, p. 241] the expansion of the column Macdonald polynomial Q_{1^n} in terms of the modified complete symmetric functions g_k was written as

$$Q_{1^n} = (-1)^n \frac{(t;t)_n}{(q;t)_n} \sum_{c\in C(n)} \prod_{i=1}^{l(c)} \frac{q^{c_i}t^{[c_{i-1}]}-1}{1-t^{[c_i]}}\, g_{c_i}.$$

The following result gives the development of Kerov's determinant along its first row.

Theorem 9.1. *We have*

$$Q_{(r,1^s)}(q,t) = (-1)^s \frac{(t;t)_s}{(q;t)_s} \sum_{c\in C(s+1)} \left(\prod_{i=1}^{l(c)-1} \frac{q^{c_i} t^{[c_{i-1}]}-1}{1-t^{[c_i]}}\, g_{c_i} \right) \frac{1-q^{r+c_{l(c)}-1} t^{s-c_{l(c)}+1}}{1-q^r t^s}\, g_{r+c_{l(c)}-1}.$$

Proof. Since Q_{1^s} is known, the Pieri formula (9.1) defines $Q_{(r,1^s)}$ through induction on the integer r. We have $[c_{l(c)-1}] = |c| - c_{l(c)}$ and the property is true for $r = 1$. Assume it to be true for $Q_{(r,1^s)}$. In (9.1) we look for the contributions common to $Q_{(r,1^s)}$ and $Q_{(r+1,1^{s-1})}$. Equivalently we subtract from $Q_{(r,1^s)}$ the contributions coming from $Q_{1^s}\,Q_{(r)}$. These have the form

$$(-1)^s \frac{(t;t)_s}{(q;t)_s} \prod_{i=1}^{l-1} \frac{q^{c_i} t^{[c_{i-1}]}-1}{1-t^{[c_i]}}\, g_{c_i}\, g_r$$

with $c = (c_1, \ldots, c_{l-1}) \in C(s)$. Such contributions can be rewritten as

$$(-1)^s \frac{(t;t)_s}{(q;t)_s} \prod_{i=1}^{l(c)-1} \left(\frac{q^{c_i} t^{[c_{i-1}]}-1}{1-t^{[c_i]}}\, g_{c_i} \right) \frac{1-q^{r+c_{l(c)}-1} t^{s-c_{l(c)}+1}}{1-q^r t^s}\, g_{r+c_{l(c)}-1}$$

with $c \in C(s+1)$ having its last term $c_{l(c)} = 1$. Therefore the contributions to $Q_{(r+1,1^{s-1})}$ correspond to vectors $c \in C(s+1)$ having their last term $c_{l(c)} > 1$. Substracting 1 from the last component of $c \in C(s+1)$, we obtain an element of $C(s)$ having the same length. Simplifying some factors, we are done. □

Applying the automorphism $\omega_{q,t}$ we obtain the equivalent result

$$P_{(r,1^s)}(q,t) = (-1)^{r-1} \frac{(q;q)_{r-1}}{(t;q)_{r-1}} \sum_{c\in C(r)} \left(\prod_{i=1}^{l(c)-1} \frac{q^{[c_{i-1}]} t^{c_i}-1}{1-q^{[c_i]}}\, e_{c_i} \right) \frac{1-q^{r-c_{l(c)}} t^{s+c_{l(c)}}}{1-q^{r-1} t^{s+1}}\, e_{s+c_{l(c)}}.$$

10. Extension of Macdonald polynomials

In the Hall–Littlewood case, it is well known that the expansion

$$Q_\lambda = \left(\prod_{1\leqslant i<j\leqslant n+1} \frac{1-R_{ij}}{1-tR_{ij}} \right) q_\lambda, \tag{10.1}$$

may be used to *define* Hall–Littlewood polynomials Q_λ when $\lambda = (\lambda_1, \ldots, \lambda_{n+1})$ is *any* sequence of integers, positive or negative, not necessarily in descending order [19, p. 213, Example 2], see also [19, pp. 236–238, Example 8].

One may wonder whether Theorem 5.1 might be similarly used as a definition of Macdonald polynomials associated with any sequence of integers. Or equivalently, whether Theorem 4.1 might be inductively used to define Q_λ in that case.

This can indeed be done but leads to a trivial result: one obtains $Q_\lambda = 0$ when λ is not a partition. This fact shows a big difference between the general case (Macdonald) and its $q = 0$ limit (Hall–Littlewood).

Let us make this remark more precise through an elementary example. In the length 2 general case, as a consequence of (4.1), we have

$$\begin{aligned} Q_{(2,1)} &= Q_{(2)}Q_{(1)} + C_1^{(q,t)}(q)\; Q_{(3)}, \\ Q_{(1,2)} &= Q_{(1)}Q_{(2)} + C_1^{(q,t)}(1/q)\; Q_{(2)}Q_{(1)} + C_2^{(q,t)}(1/q)\; Q_{(3)}, \end{aligned}$$

the second equation being taken as a definition. Now

$$\begin{aligned} C_1^{(q,t)}(u) &= \frac{t-1}{1-q}\,\frac{1-q^2u}{1-qtu}, \\ C_2^{(q,t)}(u) &= \frac{t-1}{1-q}\,\frac{t-q}{1-q^2}\,\frac{1-qu}{1-qtu}\,\frac{1-q^4u}{1-q^2tu}, \end{aligned}$$

so that $Q_{(1,2)} = 0$.

However in the Hall–Littlewood case, (10.1) writes as

$$\begin{aligned} Q_{(2,1)} &= Q_{(2)}Q_{(1)} + (t-1)Q_{(3)}, \\ Q_{(1,2)} &= Q_{(1)}Q_{(2)} + (t-1)Q_{(2)}Q_{(1)} + t(t-1)Q_{(3)}, \end{aligned}$$

so that $Q_{(1,2)} = t\,Q_{(2,1)}$, as is well known.

In the Macdonald case, the fact that Theorem 4.1 inductively gives $Q_\lambda = 0$ when λ is not a partition may be easily explained as follows. Since Theorems 4.1 and 3.1 are equivalent by our matrix inversion, they must yield the same value for any Q_λ. But Theorem 3.1 has implicitly assumed that $Q_\lambda = 0$ when λ is not a partition. Actually, the right-hand side of the Pieri formula is restricted to those values of θ for which $(\lambda_1+\theta_1, \ldots, \lambda_n+\theta_n, \lambda_{n+1}-|\theta|)$ is a partition. But $d_{\theta_1,\ldots,\theta_n}(u_1, \ldots, u_n)$ does *not* vanish when $(\lambda_1+\theta_1, \ldots, \lambda_n+\theta_n, \lambda_{n+1}-|\theta|)$ is not a partition. So we have implicitly assumed $Q_{(\lambda_1+\theta_1,\ldots,\lambda_n+\theta_n,\lambda_{n+1}-|\theta|)} = 0$ in that case.

In the Hall–Littlewood situation, a specific structure does exist. Actually the definition (10.1) is equivalent to the following recurrence property:

$$Q_{(\lambda_1,\ldots,\lambda_{n+1})} = \sum_{\theta\in\mathsf{N}^n} t^{|\theta|}(1-1/t)^{n(\theta)}\; Q_{(\lambda_{n+1}-|\theta|)}\; Q_{(\lambda_1+\theta_1,\ldots,\lambda_n+\theta_n)}$$

with $n(\theta) = \text{card}\{j : \theta_j \neq 0\}$. We emphasize that the sum on the right-hand side is taken over *all* $\theta \in \mathsf{N}^n$, even over those θ for which $(\lambda_1 + \theta_1, \ldots, \lambda_n + \theta_n)$ is *not* a partition.

It is easily shown that this relation may be inverted by writing the Pieri formula

$$Q_{(\lambda_1,\ldots,\lambda_n)}\, Q_{(\lambda_{n+1})} = \sum_{\theta \in \mathsf{N}^n} (1-t)^{n(\theta)}\, Q_{(\lambda_1+\theta_1,\ldots,\lambda_n+\theta_n,\lambda_{n+1}-|\theta|)}.$$

Here again we emphasize that the sum is taken over *all* $\theta \in \mathsf{N}^n$, even over those θ for which $(\lambda_1 + \theta_1, \ldots, \lambda_n + \theta_n, \lambda_{n+1} - |\theta|)$ is *not* a partition.

Apparently this "analytic" Pieri formula had kept unnoticed. It is very different from the classical combinatorial one [19, p. 229, Eq. (5.7′)]. Of course the latter may be recovered once all the Q_μ, where μ is not a partition, are reduced to a linear combination of Q_ν, where ν are partitions.

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