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THE ACOUSTIC FIELD OF A HIGH-FREQUENCY SOURCE MOVING IN A WAVEGUIDE

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The acoustic field of a source moving at a subsonic velocity in a regular waveguide with perfectly reflecting boundaries is considered. The acceleration of the source is assumed to be small. In a moving coordinate system, the asymptotics of the wave field is obtained. This asymptotics is inapplicable near the critical cross sections, for which the Doppler frequency of the source coincides with the frequency of the waveguide mode under consideration. It is demonstrated that, in this case, the wave field can be represented locally by a special type of integral, which is analyzed by the saddle-point method. Bibliography: 6 *titles.*

1. In a Cartesian coordinate system (x, y, z) connected with an immovable media, the problem on a moving point source of oscillations is considered. Let $S : \vec{r} = \vec{r}_0(t)$, $\vec{r}_0 = (x_0(t), y_0(t), z_0(t))$, be the trajectory of the source and let t be time. Our aim is to find the solution of the equation

$$\Delta u - \frac{\partial^2 u}{\partial t^2} = -A(t) e^{-i\zeta(t)} \delta(\vec{r} - \vec{r}_0(t)), \quad 0 < z < \pi, \tag{1}$$

subject to the radiation condition and the boundary conditions

$$u\big|_{z=0} = u\big|_{z=\pi} = 0. \tag{2}$$

Here, u is the acoustic potential; $A(t)$ and $\zeta(t)$ are smooth functions, characterizing the dependences of the amplitude and phase of the source on time. The velocity of sound is assumed constant and equal to 1. We choose the unit of length in such a way that the width of the waveguide is equal to π.

It is assumed that, for $t \le 0$, the source moves uniformly with a subsonic velocity $v < 1$, and its trajectory is parallel to the x axis; the frequency of the source is constant, and its amplitude is equal to 1, i.e., $A(t) = 1$, $\zeta(t) = \omega t$, $\vec{r}_0(t) = (vt, 0, z_0)$. For such values of time, the solution of the problem (1), (2) can be found by the Fourier method and is of the form [1]

$$u = \frac{i}{2\pi\sqrt{1-v^2}} \exp\left(i\omega \frac{vx - t}{1 - v^2} \right) \sum_{m=1}^{\infty} H_0^{(1)}\left(\varkappa_m \sqrt{(x - vt)^2 + y^2(1 - v^2)} \right) \sin mz \sin mz_0, \quad t \le 0, \tag{3}$$

$\varkappa_m = \sqrt{\omega^2 - m^2(1 - v^2)}(1 - v^2)^{-1}$, $\operatorname{Re} \varkappa_m > 0$ for $\omega > m\sqrt{1 - v^2}$, and $\operatorname{Im} \varkappa_m > 0$ for $0 < \omega < m\sqrt{1 - v^2}$. For the critical frequencies, which correspond to $\varkappa_m = 0$, the related Hankel function in the series (3) must be replaced with $\ln((x - vt)^2 + y^2(1 - v^2))$.

Conditions (1)–(3) define the acoustic potential uniquely.

We also assume that, for $t \ge 0$, the instantaneous frequency of the source $\zeta'(t)$, its velocity $\vec{r}_0{}'(t)$, and amplitude are all slowly varying functions of time, i.e., they depend on the variable εt, where $\varepsilon \ll 1$ is a small parameter. In this case, for $t \ge 0$, the phase function and the coordinates of the source can be written in the form $\varepsilon^{-1}\widetilde{\zeta}(\varepsilon t)$ and $\varepsilon^{-1}\widetilde{\vec{r}}_0(\varepsilon t)$, respectively.

A similar problem on the wave field of a moving source was previously considered in [2, 3]. In contrast to this work, in [2, 3] the vertical component of the velocity was assumed to be small, and, in addition, the phenomenon of wave transformation at critical frequencies was not considered. The investigation of this phenomenon is the main concern of the present paper.

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2. In order to solve the problem formulated above, we first introduce the "slow" time $\sigma = \varepsilon t$ and expand the potential into the Fourier series with respect to the vertical coordinate

$$u(\vec{r}, \vec{r}_0, t) = \sum_{m=1}^{\infty} u_m(x, y, \sigma) \sin mz.$$

Then the Fourier coefficient u_m satisfies the equation

$$\frac{\partial^2 u_m}{\partial x^2} + \frac{\partial^2 u_m}{\partial y^2} - \varepsilon^2 \frac{\partial^2 u_m}{\partial \sigma^2} - m^2 u_m = \frac{iA(\sigma)}{\pi} \left(e^{-i\varepsilon^{-1}\zeta_1(\sigma)} - e^{-i\varepsilon^{-1}\zeta_2(\sigma)}\right) \delta\left(x - \varepsilon^{-1}\widetilde{x}_0(\sigma)\right) \delta\left(y - \varepsilon^{-1}\widetilde{y}_0(\sigma)\right), \quad (4)$$

where $\zeta_{1,2}(\sigma) = \widetilde{\zeta}(\sigma) \pm m\widetilde{z}_0(\sigma)$. Thus, in the case where the velocity of the source has a nonzero vertical component, the instantaneous frequency splits into two parts. In accordance with this fact, we represent u_m as the difference $u_m = u_{m1} - u_{m2}$, where every term u_{mi}, $i = 1, 2$, is a solution of Eq. (4), the right-hand side of which involves only the exponent with argument $-i\zeta_i(\sigma)$. In what follows, the subscript i will be omitted.

3. In the three-dimensional space (t, x, y), we introduce the Minkowski metric

$$dT^2 = \varepsilon^{-2} d\sigma^2 - dx^2 - dy^2.$$

Let s be the length of the curve S in this metric counted from the point $t = 0$, i.e.,

$$s = \int_0^{\sigma} \sqrt{1 - v^2(a)}\, da,$$

where $v(\sigma) = \sqrt{(x_0'(\sigma))^2 + (y_0'(\sigma))^2} < 1$ is the absolute value of the horizontal velocity. Further, let $\vec{n}$ and $\vec{b}$ be the unit vectors of the principal normal and binormal, respectively, and let $\varepsilon k(s)$ and $\varepsilon^2 l(s)$ be the curvature and torsion of S. Note that the assumption that the velocity is a slowly varying function of time implies that the curvature and torsion of S are small in the vicinity of S, and the radius vector $\vec{R}(\mu)$ of an arbitrary point μ is given by the equality

$$\vec{R}(\mu) = \vec{R}(s) + n_1 \vec{e}_1(s) + n_2 \vec{e}_2(s).$$

Here, $\vec{e}_1(s)$ and $\vec{e}_2(s)$ are the unit vectors, orthogonal to S and to each other, that are interrelated with the normal ($\vec{n}$) and binormal ($\vec{b}$) vectors in the following way:

$$\vec{e}_1 = \vec{n}\cos\theta - \vec{b}\sin\theta, \quad \vec{e}_2 = \vec{n}\sin\theta + \vec{b}\cos\theta.$$

The angle θ is defined by the formula $\theta = \varepsilon \int_0^s l(s) ds$.

For the uniformly moving source (for $t < 0$), the relationship between the old (x, t, y) and new (n_1, n_2, s) coordinates is of the form

$$s = \frac{t - vx}{\sqrt{1 - v^2}}, \quad n_1 = \frac{x - vt}{\sqrt{1 - v^2}}, \quad n_2 = y, \quad x = \frac{n + vs}{\sqrt{1 - v^2}}, \quad t = \frac{s + vn}{\sqrt{1 - v^2}}.$$

Now we introduce the cylindrical coordinates (n, φ) via the relations

$$n = \sqrt{n_1^2 + n_2^2}, \quad n_1 = n\cos\varphi, \quad n_2 = n\sin\varphi.$$

The coordinate system (s, n, φ) is orthogonal, and for the element of length we have the formula (cf. [5, Chap. 9])

$$dT^2 = \varepsilon^{-2} H^2 ds^2 - dn^2 - n^2 d\varphi^2,$$

where the notation $H(s, n, \varphi) = 1 - \varepsilon k(s) n \cos(\theta - \varphi)$ is used.

In the coordinates (s, n, φ), Eq. (4) takes the form

$$\frac{1}{H}\left(\frac{1}{n}\frac{\partial}{\partial n} Hn \frac{\partial u_m}{\partial n} + \frac{1}{n^2}\frac{\partial}{\partial \varphi} H \frac{\partial u_m}{\partial \varphi} - \varepsilon^2 \frac{\partial}{\partial s}\frac{1}{H}\frac{\partial u_m}{\partial s}\right) - m^2 u_m = B(s)\exp(-i\varepsilon^{-1}\zeta(s))\delta(n_1)\delta(n_2). \quad (5)$$

Here, $B(s) = iA(s)\sqrt{1 - v^2(s)}/\pi$ is the wave amplitude in the coordinate system moving with the source.

4. Since the coefficients in Eq. (5) depend only on the variables s and εn, it is convenient to introduce the stretched coodinate $\nu = \varepsilon n$. In the coordinates (σ, ν, φ), the solution of Eq. (5) will be sought in the form

$$u_m = \sum_{j=0}^{\infty} \left(a_j\, g_j(\varepsilon, P) + i b_j \varepsilon^{-1} g_{j+1}(\varepsilon, P)\right) e^{i\varepsilon^{-1}Q},$$

suggested in [6], where a_j, b_j, P, and Q are the unknown functions. The functions $g_j(\varepsilon, P)$ are defined by the relation

$$g_j(\varepsilon, P) = i\frac{\sqrt{\pi}}{2}(-2\varepsilon P)^j H_j^{(1)}(P/\varepsilon).$$

The method for constructing a_j, b_j, P, and Q was described in [6]. For this reason, we confine ourselves to the leading terms. Let the functions $\tau^{\pm} = P \pm Q$ be the solutions of the Cauchy problems for the eikonal equation

$$\frac{\partial \tau^{\pm}}{\partial \nu} = \pm\sqrt{\frac{1}{H}\left(\frac{\partial \tau^{\pm}}{\partial s}\right)^2 - m^2 - \frac{1}{\nu}\left(\frac{\partial \tau^{\pm}}{\partial \varphi}\right)^2}, \quad \tau^{\pm}\big|_{\nu=0} = -\zeta(s). \tag{6}$$

They are 2π-periodic functions of the angle φ. Equation (6) is written in Kowalewskaya's form, and the solution of it can be found in the form of an expansion in powers of ν.

Simple calculations yield

$$\tau^{\pm} = -\zeta(s) \pm \nu \varkappa_m(s) \pm \frac{R(s)\cos(\theta - \varphi)\zeta'(s)}{\varkappa_m(s)}\nu^2 + O\left(\frac{\nu^3}{\varkappa_m^2}\right),$$

where $\varkappa_m(s) = \sqrt{(\zeta'(s))^2 - m^2}$.

In a similar way, the coefficients a_j and b_j in the form of expansions in powers of ν can be found as solutions of the transport equation. For finite values of the coordinate n, these expansions can be regarded as series in powers of the small parameter ε. For $n = O(1)$, taking into account the leading terms only, we obtain

$$u_m = B(s) H_0^{(1)}\left(\varepsilon^{-1}\left(\varkappa_m \nu + O(\nu^2)\right)\right) \exp\left[-i\varepsilon^{-1}\left(\zeta(s) + O(\nu^3)\right)\right] \cdot \left[1 + O(\nu) + O(\varepsilon)\right]. \tag{7}$$

If the coefficient $\varkappa_m$ vanishes, then the expansion obtained does not exist, and it must be replaced by another expansion, which will be obtained in the next section.

5. Assume that the instantaneous frequency $\zeta'(s)$ decreases, so that, at some point of the trajectory $s = s_0$, the relations $\zeta'(s_0) = m$ and $\varkappa_m(s_0) = 0$ are satisfied. Thus, in the vicinity of the point $s = s_0$, Taylor's expansion of $\varkappa_m^2(s)$ begins with the linear term, i.e., $\varkappa_m^2(s) = \varkappa_{m0}(s - s_0) + \dots$, where $\varkappa_{m0} < 0$. The plane $s = s_0$ will be called critical. Far away from the critical plane, where $s < s_0$, the argument of Hankel's function is real, and the expression (7) describes a propagating wave. For $s > s_0$, the imaginary part of the function $\varkappa_m(s)$ is positive, and the acoustic field exponentially decays away from the trajectory. The main purpose of this paper is to describe the wave field in the transient region of small $s - s_0$. The case of the "birth" of a propagating wave from an attenuating one as a consequence of the increase of the instantaneous frequency can be considered in a similar way.

We introduce the stretched coordinate system $q = (-\varepsilon^{-1} m \varkappa_{m0})^{1/4} n$, $r = (-\varepsilon \varkappa_{m0}/m)^{1/2}(s - s_0)$ and single out the term $\exp(-i\zeta(s)\varepsilon^{-1})$, most oscillating in the vicinity of $s = s_0$, as a separate factor. In other words, the solution of Eq. (5) is sought in the form

$$u_m = \exp\left(-i\zeta(s)\varepsilon^{-1}\right) \sum_{j=0}^{\infty} v_j(r, q, \varphi) \varepsilon^{j/4}. \tag{8}$$

By inserting the ansatz (8) into Eq. (5) and equating the coefficients at the same powers of $\varepsilon^{1/4}$, we obtain a recurrent sequence of equations, beginning with the equation

$$2i\frac{\partial v_0}{\partial r} + \frac{1}{q}\frac{\partial}{\partial q} q \frac{\partial v_0}{\partial q} - r v_0 = \widetilde{B}(s_0)\frac{\delta(q)}{q}, \tag{9}$$

where

$$\widetilde{B}(s) = B(s)(1-v^2)^{-1/2}.$$

Equation (9) takes into account the variations of the field (in the leading approximation) along the trajectory and in the direction transverse to it, and the scales of stretching are chosen in such a way that all the terms in (9) are of the same order in ε.

The solution of Eq. (9), which can easily be found by the Fourier method, is of the form

$$v_0(q,r) = \frac{-\widetilde{B}(s_0)}{2\pi} \int_0^\infty \exp\left[\frac{i}{2}\left(\frac{p^2}{2} - rp + \frac{q^2}{p}\right)\right] \frac{dp}{p}. \tag{10}$$

Thus, in the vicinity of the critical cross section, the acoustic field is expressed by an integral of special type, which coincides with the integral

$$H_0^{(1)}(q\sqrt{-r}) = -\frac{i}{\pi} \int_0^\infty \exp\left[\frac{i}{2}\left(-rp + \frac{q^2}{p}\right)\right] \frac{dp}{p}, \quad r < 0,$$

representing the Hankel function, up to an additional term in the exponent of the integrand. This term provides for the boundedness of the integral (10) for $r = 0$, $q \neq 0$.

6. Now we examine the asymptotics of the integral (10) far away from the critical point $r = q = 0$. First we consider the case where the point of observation is located not far from the trajectory, i.e., the coordinate $r > 0$ is regarded as a large parameter, whereas the transverse coordinate q is finite and $q = O(r^{1/2})$. Under these assumptions, the exponent in the integrand in (10) has a unique singular point $p = 0$ and a unique stationary point $p = r$, which are separated from each other. In the complex plane p, we deform the contour of integration from the real axis to a new contour consisting of a segment of the imaginary axis and of a saddle-point contour crossing the real axis at the point $p = r$ at an angle of $-\pi/4$. In accordance with this deformation, the asymptotics of the integral (10) will involve, first, Macdonald's function $K_0(\sqrt{r}q) = \pi i H_0^{(1)}(i\sqrt{r}q)/2$, $r > 0$, which arises in integrating along the imaginary axis (this term coincides with (7) if $\operatorname{Im} \varkappa_m > 0$), and, second, the term

$$\frac{\widetilde{B}(s_0)}{\pi^{1/2}} \cdot \frac{1}{r} \exp\left[i\left(-\frac{\pi}{4} - \frac{r^2}{4} + \frac{q^2}{2r}\right)\right], \tag{11}$$

which arises in integrating along the saddle-point contour and corresponds to the "spherical" wave

$$\widetilde{B}(s_0) \exp i\left(-\frac{\pi}{4} - mR + \zeta(s_0)\right)/(2\pi^{3/2}R), \quad R = \sqrt{(s-s_0)^2 - n^2},$$

propagating from the critical point $r = q = 0$. This assertion can easily be verified by returning to the original coordinates n and s in (11) and by using the expansions

$$R(s) = s - s_0 - n^2/(2(s-s_0)) + \dots,$$
$$\zeta(s) = \zeta(s_0) + m(s-s_0) + \varkappa_{m0}(s-s_0)^2/(2m) + \dots.$$

Now we focus on the investigation of the region of large values of the coordinate q and small values $r = O(q^{-2/3})$ of r. In this case, the argument of the exponent in the integrand of (10) has only one stationary point $p = q^{2/3}$, and the contribution of the integral over the vicinity of the origin is exponentially small. The asymptotics of the integral (10) is of the form

$$\frac{\widetilde{B}(s_0)}{\sqrt{5\pi}} \exp i\left(-\frac{\pi}{4} + \frac{3}{2}q^{4/3} - rq^{2/3}\right)\frac{1}{q^{2/3}},$$

i.e., the acoustic field decays as $q^{-2/3}$ as the distance increases. It should be mentioned that, in the direction orthogonal to the trajectory, the acoustic field decays as $n^{-1/2}$ before the critical cross section is reached and as R^{-1} when moving away from it.

If q and r are both large and $q = O(r^{3/2})$, then all the terms in the argument of the exponent in (10) are large quantities, and the equation for the stationary points is of the form

$$p^3 - rp^2 - q^2 = 0. \tag{12}$$

Equation (12) has three roots, one of which (p_1) is real in the domain under consideration, whereas the other two are complex conjugate numbers. Standard calculations in accordance with the saddle-point method yield the following asymptotics of the acoustic field:

$$u_m(q,r) \sim \frac{\widetilde{B}(s_0)}{2\pi^{3/2}}\left[\frac{\exp(ih(p_1) - i\pi/4)}{p_1\sqrt{h''(p_1)}} + \frac{\exp(ih(p_2) - i\pi/4)}{p_2\sqrt{h''(p_2)}} \cdot \eta\big(r - cq^{2/3}\big)\right].$$

Here, $h(p) = \frac{1}{2}(\frac{p^2}{2} - rp + \frac{q^2}{p})$ denotes the argument of the exponent in the integrand of (10); $\eta(a) = 1$ for $a > 0$ and $\eta(a) = 0$ for $a < 0$; p_2 is the root of Eq. (12) for which $\operatorname{Im} h(p_2) > 0$, and c is a constant.

Thus, the asymptotic formulas obtained describe the acoustic field of a moving source both in the vicinity of a critical point and far away from it.

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