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On the Spectrum of Two-Dimensional Schrödinger Operators with Spherically Symmetric, Radially Periodic Magnetic Fields ⋆

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Abstract: We investigate the spectrum of the two-dimensional Schrödinger operator $H = -\left(\frac{\partial}{\partial x} - \mathrm{i}a_1(x,y)\right)^2 - \left(\frac{\partial}{\partial y} - \mathrm{i}a_2(x,y)\right)^2 + V(x,y)$, where the magnetic field $B(x,y) = \frac{\partial}{\partial x}a_2 - \frac{\partial}{\partial y}a_1$ and the electric potential V are spherically symmetric, i.e., $B(x,y) = b(r)$, $r = \sqrt{x^2+y^2}$, and b is p-periodic, similarly for V. By considering two different gauges we get the following results: In case $\int_0^p b(s)\,ds = 0$ the spectrum contains a semi-axis that consists alternately of intervals of absolutely continuous and dense point spectrum. In case $\int_0^p b(s)\,ds \neq 0$ the essential spectrum is purely dense point spectrum and possibly there are spectral gaps.

1. Introduction and Results

The fact that the spectrum of a Schrödinger operator with spherically symmetric electric potential (without magnetic field) contains a semi-axis is proved in [HHK], not by separation in spherical coordinates but by separation in rectangular coordinates. In [HHHK] the result is applied in the case of potentials which additionally are periodic with respect to the radius, and combined with results obtained by separation in spherical coordinates to get a more detailed description of the spectrum.

Motivated by these methods we will get our results concerning spherically symmetric, radially periodic magnetic fields. Whereas the electric potential arising in the Schrödinger operator is uniquely determined up to a constant, we can choose very different magnetic potentials which all describe the same magnetic field. In fact we will use different gauges in our examination.

Now we give the results in detail:
Let $b : \mathbb{R} \to \mathbb{R}$ be continuous and p-periodic ($p > 0$) with piecewise continuous

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and bounded derivative a.e., and let $a_1,\ a_2 \in L^4_{\mathrm{loc}}(\mathbb{R}^2)$, $\frac{\partial}{\partial x}\, a_1 + \frac{\partial}{\partial y}\, a_2 \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ (in distributional sense), satisfying

$$\frac{\partial}{\partial x}\, a_2(x,y) - \frac{\partial}{\partial y}\, a_1(x,y) = b(\sqrt{x^2+y^2})\ .$$

Let $v : \mathbb{R} \to \mathbb{R}$ be p-periodic, piecewise continuous and bounded, and let $V(x,y) = v(\sqrt{x^2+y^2})$.

Let H be the two-dimensional Schrödinger operator

$$H = -\left(\frac{\partial}{\partial x} - \mathrm{i}a_1(x,y)\right)^2 - \left(\frac{\partial}{\partial y} - \mathrm{i}a_2(x,y)\right)^2 + V(x,y)$$

acting on $L^2(\mathbb{R}^2)$. Due to [LS, Theorem 3] the operator H is essentially self-adjoint with core $C_0^\infty(\mathbb{R}^2)$ (the set of all infinitely differentiable functions with compact support in $\mathbb{R}^2$).

We define

$$\beta = \int_0^p b(s)\, ds\ ,\quad d(x) = \int_0^x b(s)\, ds\ ,\quad \delta = \frac{1}{p}\int_0^p d(s)\, ds$$

and the one-dimensional Schrödinger operator

$$H_\eta = -\frac{\partial^2}{\partial r^2} + (d(r)-\eta)^2 + v(r) \quad (\eta \in \mathbb{R})$$

acting on $L^2(\mathbb{R})$. The structure of the spectrum of H depends on whether $\beta = 0$ or $\beta \neq 0$.

In case $\beta = 0$ obviously the function d (and hence $(d(r)-\eta)^2 + v(r)$) is p-periodic. Hence $\sigma(H_\eta)$ consists of bands ([RS, Theorem XIII.90]; the theorem holds for any period, cf. [E]). Let

$$\mu = \inf_{\eta\in\mathbb{R}} \inf \sigma(H_\eta)$$

and $[\alpha_n, \beta_n]$, $n \in \mathbb{N} = \{1,2,\ldots\}$, be the spectral bands of H_δ as defined in [RS, Theorem XIII.90].

Theorem 1. *Under the definitions above in case $\beta = 0$ we have:*

(a) $[\mu,\infty) \subseteq \sigma(H)$.
(b) *H has no continuous spectrum in $(-\infty,\alpha_1)$ and in the gaps (β_n,α_{n+1}) of $\sigma(H_\delta)$.*
(c) *Eigenvalues of H are dense in (μ,α_1) and in the gaps (β_n,α_{n+1}) of $\sigma(H_\delta)$.*
(d) *The spectrum of H is purely absolutely continuous in the interior (α_n,β_n) of the spectral bands of H_δ.*

Remark 1. In general it remains open whether

$$\mu = \inf_{\eta\in\mathbb{R}} \inf \sigma(H_\eta) < \inf \sigma(H_\delta) = \alpha_1$$

or not. Below we will give a class of examples where $\mu < \alpha_1$. Then, due to Theorem 1, the lowest part of the essential spectrum is purely dense point spectrum.

In case $\beta \neq 0$ it is easy to see that $|d(r)| \to \infty$ for $|r| \to \infty$. Thus we have $(d(r) - \eta)^2 + v(r) \to \infty$ $(|r| \to \infty)$ and H_η has compact resolvent ([RS, Theorem XIII.67]), i.e.,

$$\sigma(H_\eta) = \{\lambda_1(\eta), \lambda_2(\eta), \ldots\}, \quad \lambda_1(\eta) \leq \lambda_2(\eta) \leq \ldots \quad \text{and } \lambda_n(\eta) \to \infty \ (n \to \infty),$$

where the eigenvalues λ_n are counted according to their multiplicity.

Theorem 2. *Under the definitions above in case $\beta \neq 0$ we have:*

(a) The essential spectrum $\sigma_{\text{ess}}(H)$ of H is purely dense point spectrum, i.e., it contains no continuous spectrum and eigenvalues are dense in $\sigma_{\text{ess}}(H)$.
(b) The maps $\eta \mapsto \lambda_n(\eta)$ are β-periodic and

$$\bigcup_{n=1}^{\infty} \lambda_n([0, \beta)) \subseteq \sigma_{\text{ess}}(H) \, .$$

Remark 2. A special example here, which we will examine below, is $b \equiv \text{const} \neq 0$ and $v \equiv 0$, i.e., a homogeneous magnetic field. Then all λ_n turn out to be constant, $\lambda_n \equiv (2n-1)b$ and $\sigma(H) = \sigma_{\text{ess}}(H) = \bigcup_{n=1}^{\infty} \{(2n-1)b\}$, i.e., the well-known spectral characterisation for the homogeneous case (see, e.g., [I90, Theorem 4.1]).

Essential for the proof is [L, Theorem 1.3]: We may choose an arbitrary L^4_{loc}-gauge with L^2_{loc}-divergence for the magnetic potential without varying the spectrum. We will choose two different gauges (Sects. 2 and 3 below) and in each case we will draw conclusions for $\beta = 0$ and $\beta \neq 0$. In Sect. 4 we analyse the two examples mentioned above.

2. The First Gauge

A well-known suitable gauge for the spherically symmetric magnetic field is the following (cf. [MS]): Let

$$a(r) = \frac{1}{r} \int_0^r s\, b(s)\, ds \, , \quad r = \sqrt{x^2 + y^2}$$

and

$$a_{1,\text{rad}}(x, y) = -\frac{y}{r} a(r), \quad a_{2,\text{rad}}(x, y) = \frac{x}{r} a(r) \, .$$

Then an easy computation yields

$$\frac{\partial}{\partial x} a_{2,\text{rad}}(x, y) - \frac{\partial}{\partial y} a_{1,\text{rad}}(x, y) = b(r) \, .$$

Thus we can apply [L, Theorem 1.3] to

$$H_{\text{rad}} = -\left(\frac{\partial}{\partial x} - \mathrm{i} a_{1,\text{rad}}(x, y) \right)^2 - \left(\frac{\partial}{\partial y} - \mathrm{i} a_{2,\text{rad}}(x, y) \right)^2 + V(x, y)$$

and obtain that H is unitarily equivalent to H_{rad} (formally: $H \cong H_{\text{rad}}$), in particular

$$\sigma(H) = \sigma(H_{\text{rad}}) \text{ and the same with } \sigma \text{ replaced by } \sigma_{\text{ess}}, \sigma_{\text{p}}, \sigma_{\text{ac}} \text{ and } \sigma_{\text{sc}} , \tag{1}$$

i.e., by the essential, pure point, absolutely continuous and singular continuous spectrum, respectively. By the standard transformation into polar coordinates (cf. [MS]) we get

$$H_{\text{rad}} \cong \bigoplus_{m\in\mathbb{Z}} H_{m,\text{rad}} ,$$

where

$$H_{m,\text{rad}} = -\frac{\partial^2}{\partial r^2} + \frac{m^2 - \frac{1}{4}}{r^2} + 2\,\frac{m}{r}\,a(r) + a^2(r) + v(r)$$

acts on $L^2((0,\infty))$. In particular we have

$$\begin{aligned} \sigma(H_{\text{rad}}) = \overline{\bigcup_{m\in\mathbb{Z}} \sigma(H_{m,\text{rad}})} & \text{ and the same with } \sigma \text{ replaced by } \sigma_{\text{ac}} \text{ and } \sigma_{\text{sc}} \\ \text{and } \sigma_{\text{p}}(H_{\text{rad}}) &= \bigcup_{m\in\mathbb{Z}} \sigma_{\text{p}}(H_{m,\text{rad}}) \end{aligned} \tag{2}$$

(see [Sch, Lemma 7]; the assertion concerning σ_{sc} can be proved similarly).

Now we shall determine $\sigma(H_{m,\text{rad}})$:

Lemma 1. *If $\beta = 0$ then, for all $m \in \mathbb{Z}$,*

$$\sigma_{\text{ess}}(H_{m,\text{rad}}) = \sigma_{\text{ac}}(H_{m,\text{rad}}) = \sigma(H_\delta) = \bigcup_{n\in\mathbb{N}} [\alpha_n, \beta_n] .$$

Further $\sigma_{\text{sc}}(H_{m,\text{rad}}) = \emptyset$ and $\sigma_{\text{p}}(H_{m,\text{rad}}) \cap (\alpha_n, \beta_n) = \emptyset$, i.e., $H_{m,\text{rad}}$ has no embedded eigenvalues in the interior of the spectral bands of H_δ.

Using Eqs. (1), (2) and Lemma 1, the claims (b) and (d) of Theorem 1 follow immediately and then (c) is a consequence of (a) and (b). Thus, after proving Lemma 1, it remains to show that $[\mu, \infty) \subseteq \sigma(H)$, which we will do in the third section.

Proof (of Lemma 1). We first rewrite $H_{m,\text{rad}}$: By partial integration we have

$$a(r) = \frac{1}{r}\left(r\,d(r) - \int_0^r d(s)\,ds\right) = d(r) - \frac{1}{r}\int_0^r d(s)\,ds \tag{3}$$

and, defining

$$v_m(r) = 2m\frac{a(r)}{r} + 2(d(r) - \delta)\left(\delta - \frac{1}{r}\int_0^r d(s)\,ds\right) + \left(\delta - \frac{1}{r}\int_0^r d(s)\,ds\right)^2 ,$$

it is easy to see that

$$H_{m,\text{rad}} = -\frac{\partial^2}{\partial r^2} + \frac{m^2 - \frac{1}{4}}{r^2} + v_m(r) + (d(r) - \delta)^2 + v(r) .$$

We want to apply [St, Theorem 2(b)] to $q_0(r) = (d(r) - \delta)^2 + v(r)$ (i.e., $\tau_0 = H_\delta$) and $q(r) = \frac{m^2 - \frac{1}{4}}{r^2} + v_m(r)$, that states that the assertions of Lemma 1 are valid if we can show that

(i) $H_{m,\text{rad}}|_{(0,1)}$ has purely discrete spectrum,

(ii) $\int_1^\infty |q(s+p) - q(s)|\, ds < \infty$,
(iii) $\lim_{r\to\infty} \int_r^{r+1} |q(s)|\, ds = 0$.

Obviously (i) is satisfied ([DSch, Theorem XIII 7.17]).
Since $\int_0^p \delta - d(s)\, ds = 0$ by the definition of δ, the p-periodicity of d yields

$$\left|\delta - \frac{1}{r}\int_0^r d(s)\, ds\right| = \frac{1}{r}\left|\int_0^r \delta - d(s)\, ds\right| \leq \frac{1}{r}\max_{t\in[0,p)}\left|\int_0^t \delta - d(s)\, ds\right| . \tag{4}$$

Thus it is easy to see that $a(r)$ is bounded (using Eq. (3)) and that $v_m(r) \overset{r\to\infty}{\longrightarrow} 0$. Hence (iii) is fulfilled.

To prove (ii), we first consider $\frac{a(r)}{r}$ and $(d(r) - \delta)(\delta - \frac{1}{r}\int_0^r d(s)\, ds)$:

- $$\left|\frac{a(r+p)}{r+p} - \frac{a(r)}{r}\right|$$

$$\overset{(3)}{=} \left|\frac{1}{r+p}\left(d(r) - \frac{1}{r+p}\left(\int_0^r d(s)\, ds + p\delta\right)\right) - \frac{1}{r}\left(d(r) - \frac{1}{r}\int_0^r d(s)\, ds\right)\right|$$

$$\leq \frac{p}{r(r+p)}|d(r)| + \frac{2pr+p^2}{r(r+p)^2}\left|\frac{1}{r}\int_0^r d(s)\, ds\right| + \frac{p}{(r+p)^2}|\delta|.$$

- $$\left|(d(r+p) - \delta)\left(\delta - \frac{1}{r+p}\int_0^{r+p} d(s)\, ds\right) - (d(r) - \delta)\left(\delta - \frac{1}{r}\int_0^r d(s)\, ds\right)\right|$$

$$= \left|(d(r) - \delta)\left[\frac{1}{r}\int_0^r d(s)\, ds - \frac{1}{r+p}\left(\int_0^r d(s)\, ds + p\delta\right)\right]\right|$$

$$= |d(r) - \delta|\frac{p}{r+p}\left|\frac{1}{r}\int_0^r d(s)\, ds - \delta\right|$$

$$\overset{(4)}{\leq} \frac{p}{r(r+p)}|d(r) - \delta|\max_{t\in[0,p)}\left|\int_0^t \delta - d(s)\, ds\right| .$$

These estimates, the boundness of $d(r)$ and (4) yield

$$|q(r+p) - q(r)| \leq \frac{1}{r^2}\cdot \text{const.}$$

and (ii) follows. $\square$

Lemma 2. *If* $\beta \neq 0$ *then* $\sigma_{\text{ess}}(H_{m,\text{rad}}) = \emptyset$ *for all* $m \in \mathbb{Z}$.

Using Eqs. (1), (2) and Lemma 2, the assertion (a) of Theorem 2 follows immediately.

Proof (of Lemma 2). We first show that $|a(r)| \overset{r\to\infty}{\longrightarrow} \infty$: Let $r = np + t$, $t \in (0,p]$, $n \in \{0,1,2,\ldots\}$. Then

$$a(r) = \frac{1}{r}\left(\sum_{j=0}^{n-1}\int_0^p (jp+s)b(s)\, ds + \int_0^t (np+s)b(s)\, ds\right)$$
$$= \frac{n(n-1)}{2r}p\beta + \frac{n}{r}\int_0^p sb(s)\, ds + \frac{np}{r}\int_0^t b(s)\, ds + \frac{1}{r}\int_0^t sb(s)\, ds .$$

Obviously the last three summands are bounded by a constant that depends only on b. Since $\beta \neq 0$ and $\frac{n(n-1)}{2r} \to \infty$ for $r \to \infty$ and corresponding decompositions $r = np + t$, we have $|a(r)| \stackrel{r\to\infty}{\longrightarrow} \infty$.

Now it is easy to see that

$$\frac{m^2 - \frac{1}{4}}{r^2} + 2\,\frac{m}{r}\,a(r) + a^2(r) + v(r) \stackrel{r\to\infty}{\longrightarrow} \infty\ .$$

Thus, due to [DSch, Theorems XIII 7.4, 7.16 and 7.17], the assertion follows. □

3. The Second Gauge

Let

$$a_{\mathrm{rec}}(x,y) = -\int_0^y b(\sqrt{x^2+s^2})\,ds\ ,$$

$$H_{\mathrm{rec}} = -\left(\frac{\partial}{\partial x} - \mathrm{i}a_{\mathrm{rec}}(x,y)\right)^2 - \frac{\partial^2}{\partial y^2} + V(x,y)\ .$$

Then $-\frac{\partial}{\partial y}\,a_{\mathrm{rec}}(x,y) = b(r)$ and again [L, Theorem 1.3], yields

$$H \cong H_{\mathrm{rec}},\ \text{in part. } \sigma(H) = \sigma(H_{\mathrm{rec}}) \ \text{ and }\ \sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(H_{\mathrm{rec}})\ . \tag{5}$$

Intuitively by this kind of definition we have $a_{\mathrm{rec}}(x,y) \sim -y\cdot b(x)$ for $|y| << |x|$, more precisely:

Lemma 3. *Let $w : \mathbb{R} \to \mathbb{R}$ be p-periodic ($p > 0$), $u(x,y) = -\int_0^y w(\sqrt{x^2+s^2})\,ds$. Then for all y_0 and each continuity point x_0 of w we have*

$$\lim_{k\to\infty} |u(x_0+kp, y_0) + y_0 w(x_0+kp)| = 0\ .$$

Proof. For $|s| \le |y_0|$ and large integers k (such that $x_0 + kp \ge \frac{kp}{2}$) we have

$$|\sqrt{(x_0+kp)^2+s^2} - kp - x_0| = \left|\frac{s^2}{\sqrt{(x_0+kp)^2+s^2}+kp+x_0}\right| \le \frac{{y_0}^2}{kp}\ . \tag{6}$$

Because of the continuity of w in x_0, for $\varepsilon > 0$ fixed there is a $\varepsilon_1 > 0$ such that $|w(x) - w(x_0)| < \varepsilon$ for all $|x - x_0| < \varepsilon_1$. Thus, for large k (so that $\frac{{y_0}^2}{kp} < \varepsilon_1$), using the periodicity of w and (6), we have

$$|-w(\sqrt{(x_0+kp)^2+s^2}) + w(x_0+kp)| = |w(\sqrt{(x_0+kp)^2+s^2} - kp) - w(x_0)| < \varepsilon\ ,$$

hence

$$|u(x_0+kp,y_0)+y_0w(x_0+kp)| = \left|\int_0^{y_0} -w(\sqrt{(x_0+kp)^2+s^2}) + w(x_0+kp)\,ds\right| \le |y_0|\varepsilon$$

and the assertion follows. □

Bearing this in mind, it is reasonable to consider

$$\hat{H} = -\left(\frac{\partial}{\partial x} + \mathrm{i}yb(x)\right)^2 - \frac{\partial^2}{\partial y^2} + v(x)\ .$$

Indeed there is a connection between the spectra of $\hat{H}$ and H_{rec}:

Lemma 4. $\sigma(\hat{H}) \subseteq \sigma_{\text{ess}}(H_{\text{rec}})$.

Proof. We use the following spectral characterisation (see, e.g., [W, Theorems 7.22 and 7.24]:

($*$) $\lambda \in \sigma(\hat{H})$ iff there exists a sequence $f_n \in C_0^\infty(\mathbb{R}^2)$, $\|f_n\|_2 = 1$ satisfying $\|\hat{H}f_n - \lambda f_n\|_2 \leq \frac{1}{n}$,

($**$) $\lambda \in \sigma_{\text{ess}}(H_{\text{rec}})$ iff there exists a sequence $g_n \in C_0^\infty(\mathbb{R}^2)$, $\|g_n\|_2 = 1$ satisfying $g_n \to 0$ weakly and $\lim_{n\to\infty} \|H_{\text{rec}}g_n - \lambda g_n\|_2 = 0$.

Now, fix $\lambda \in \sigma(\hat{H})$ and choose f_n corresponding to ($*$). The idea is to shift the f_n to the right so that $a_{\text{rec}}(x,y) \sim -yb(x)$ on the support of the shifted f_n. To specify this, fix an integer $n > 0$. Then there exists a constant R such that $\text{supp} f_n \subseteq [-R,R] \times [-R,R]$. Let $f_{n,k}(x,y) = f_n(x-kp,y)$ for integers k. Obviously $\|f_{n,k}\|_2 = \|f_n\|_2 = 1$ and because of the p-periodicity of b and v,

$$\|\hat{H}f_{n,k} - \lambda f_{n,k}\|_2 = \|\hat{H}f_n - \lambda f_n\|_2 \leq \frac{1}{n}\ .$$

By a short simple calculation we get

$$\begin{aligned}
&\|H_{\text{rec}}f_{n,k} - \hat{H}f_{n,k}\|_2 \\
=\ &\|2\mathrm{i}\big(a_{\text{rec}}(x,y) + yb(x)\big)\tfrac{\partial}{\partial x} f_{n,k}(x,y) \\
&+ \Big[\mathrm{i}\left(\tfrac{\partial}{\partial x} a_{\text{rec}}(x,y) + yb'(x)\right) + (a_{\text{rec}}(x,y))^2 - (yb(x))^2 \\
&\quad + V(x,y) - v(x)\Big] f_{n,k}(x,y)\|_2 \\
\leq\ &\|2\big(a_{\text{rec}}(x+kp,y) + yb(x+kp)\big)\tfrac{\partial}{\partial x} f_n(x,y)\|_2 \\
&+ \|\left(\tfrac{\partial}{\partial x} a_{\text{rec}}(x+kp,y) + yb'(x+kp)\right) f_n(x,y)\|_2 \\
&+ \|\left((a_{\text{rec}}(x+kp,y))^2 - (yb(x+kp))^2\right) f_n(x,y)\|_2 \\
&+ \|\big(V(x+kp,y) - v(x+kp)\big) f_n(x,y)\|_2\ .
\end{aligned}$$

We claim that the four summands tend to zero for $k \to \infty$. To show this we use the dominated convergence theorem. Due to the conditions on b there is a constant C such that $|b(x)| \leq C$ and $|b'(x)| \leq C$ a.e. on $\mathbb{R}$. Using this we have $|a_{\text{rec}}(x,y)| \leq |y|C$ a.e. and

$$\left|\frac{\partial}{\partial x} a_{\text{rec}}(x,y)\right| = \left|\int_0^y \frac{x}{\sqrt{x^2+s^2}}\, b'(\sqrt{x^2+s^2})\, ds\right| \leq |y|C\ .$$

Now, since f_n has compact support, it is easy to see that the functions arising in the summands above are bounded a.e. by constants independent of k and that we can find integrable majorants. Thus it remains to show that the integrands tend to zero pointwise a.e.:

– $\lim_{k\to\infty} \left|2\big(a_{\text{rec}}(x+kp,y)+yb(x+kp)\big)\right| \to 0$ due to Lemma 3. (7)
– We have

$$\begin{aligned}
&\left|\tfrac{\partial}{\partial x}\, a_{\text{rec}}(x+kp,y)+yb'(x+kp)\right| \\
\le &\left|yb'(x+kp)-\int_0^y b'(\sqrt{(x+kp)^2+s^2})\,ds\right| \\
&+\left|\int_0^y \left(1-\tfrac{x+kp}{\sqrt{(x+kp)^2+s^2}}\right)b'(\sqrt{(x+kp)^2+s^2})\,ds\right| .
\end{aligned}$$

The first summand tends to zero for each continuity point x of b' due to Lemma 3 applied to b'. The second summand tends to zero, because b' is bounded and, for $|s|\le|y|$ and large k, similarly to (6)

$$\begin{aligned}
\left|1-\frac{x+kp}{\sqrt{(x+kp)^2+s^2}}\right| &= \frac{1}{\sqrt{(x+kp)^2+s^2}}\left|\sqrt{(x+kp)^2+s^2}-(x+kp)\right| \\
&\le \frac{1}{\frac{1}{2}kp}\cdot\frac{y^2}{kp} .
\end{aligned}$$

– Because of (7) and the boundness independent of k we have

$$\begin{aligned}
&\left|(a_{\text{rec}}(x+kp,y))^2-(yb(x+kp))^2\right| \\
&= |a_{\text{rec}}(x+kp,y)+yb(x+kp)|\cdot|a_{\text{rec}}(x+kp,y)-yb(x+kp)| \overset{k\to\infty}{\longrightarrow} 0 .
\end{aligned}$$

– Using (6) with $s=y$ and the definition of V it is obvious that in each continuity point x of v,

$$|V(x+kp,y)-v(x+kp)| \overset{k\to\infty}{\longrightarrow} 0 .$$

Thus we have shown $\lim_{k\to\infty}\|H_{\text{rec}}f_{n,k}-\hat{H}f_{n,k}\|_2=0$, hence we can choose k_n such that $\|H_{\text{rec}}f_{n,k_n}-\hat{H}f_{n,k_n}\|_2\le\frac{1}{n}$ and $\operatorname{supp} f_{n,k_n}\cap[-n,n]^2=\emptyset$. Then

$$\|H_{\text{rec}}f_{n,k_n}-\lambda f_{n,k_n}\|_2\le\|H_{\text{rec}}f_{n,k_n}-\hat{H}f_{n,k_n}\|_2+\|\hat{H}f_{n,k_n}-\lambda f_{n,k_n}\|_2\le\frac{2}{n},$$

and obviously $f_{n,k_n}\to 0$ weakly. Using the spectral characterisation $(**)$, we obtain $\lambda\in\sigma_{\text{ess}}(H_{\text{rec}})$. □

To determine the spectrum of $\hat{H}$ we use partial Fourier-transformation like in [I85, §2]:

Defining $(Ug)(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}ty}g(x,y)\,dy$ for $g\in C_0^\infty(\mathbb{R}^2)$ and continuing to a unitary isomorphism of $L^2(\mathbb{R}^2)$ we obtain in the usual way

$$\hat{H}\cong U\hat{H}U^{-1}=-\left(\frac{\partial}{\partial x}-b(x)\frac{\partial}{\partial t}\right)^2+t^2+v(x)=:\tilde{H} .$$

Now, the unitary transformation of the variables $\xi=x$, $\eta=d(x)+t$, i.e.,

$$T:L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2),\quad T(w(x,t))=\big((\xi,\eta)\mapsto w(\xi,\eta-d(\xi))\big) ,$$

yields

$$\tilde{H} \cong T\tilde{H}T^{-1} = -\frac{\partial^2}{\partial\xi^2} + (d(\xi) - \eta)^2 + v(\xi) =: \tilde{H}_{\rm tra} \ .$$

If we decompose $L^2(\mathbb{R}^2)$ as a direct integral over $L^2(\mathbb{R})$ (the two-dimensional function $(\xi,\eta) \mapsto w(\xi,\eta)$ is considered as a collection $(\xi \mapsto w(\xi,\eta))_{\eta\in\mathbb{R}}$ of one dimensional functions) we get a related decomposition $\tilde{H}_{\rm tra} = \int_{\mathbb{R}}^{\oplus} H_\eta \, d\eta$, where

$$H_\eta = -\frac{\partial^2}{\partial\xi^2} + (d(\xi) - \eta)^2 + v(\xi)$$

is an operator acting on $L^2(\mathbb{R})$ (cf. [RS, Sect. XIII.16] and [I85, §2]). Due to [RS, Theorem XIII.85] we have

$$\begin{aligned} \sigma(\hat{H}) = \sigma(\tilde{H}) &= \sigma(\tilde{H}_{\rm tra}) \\ &= \{\lambda : \forall\varepsilon > 0 : |\{\eta : \sigma(H_\eta) \cap (\lambda - \varepsilon, \lambda + \varepsilon) \neq \emptyset\}| > 0\} \ , \end{aligned} \tag{8}$$

where $|M|$ indicates the Lebesgues measure of the set M.

Case $\beta = 0$: As mentioned above, in this case d is p-periodic. It is remarkable that H_δ is exactly the operator that appeared in the second section and induced the bands in the spectrum mentioned there. Here we do not need the precise structure of the spectrum of H_η but only the behaviour of

$$\mu(\eta) = \inf \sigma(H_\eta) \ .$$

Recalling the results of the second section, to prove Theorem 1 it remains to show $[\mu, \infty) \subseteq \sigma(H)$, where $\mu = \inf_{\eta\in\mathbb{R}} \mu(\eta)$.

Obviously $H_\eta = H_{\eta_0} + A_{\eta,\eta_0}$, where the operator of multiplication

$$A_{\eta,\eta_0} = 2(\eta_0 - \eta)(d(\xi) - \eta_0) + (\eta_0 - \eta)^2$$

is symmetric and bounded, satisfying $\lim_{\eta\to\eta_0} \|A_{\eta,\eta_0}\| = 0$. Thus, due to [K, Theorem V.4.10], the map $\eta \mapsto \mu(\eta)$ is continuous. Now, using (8), it is easy to see, that

$$\{\mu(\eta) : \eta \in \mathbb{R}\} \subseteq \sigma(\hat{H}) \ . \tag{9}$$

Since d and v are bounded (say by the constants d_0 and v_0 respectively), for $|\eta| > d_0$ and all $\xi \in \mathbb{R}$, we have $(d(\xi) - \eta)^2 + v(\xi) \geq (|\eta| - d_0)^2 - v_0$, hence

$$\mu(\eta) \geq (|\eta| - d_0)^2 - v_0 \ .$$

Thus $\mu(\eta) \to \infty$ for $|\eta| \to \infty$ and by (9) and the continuity of $\mu(\eta)$ we get

$$[\mu, \infty) = [\inf_{\eta\in\mathbb{R}} \mu(\eta), \infty) \subseteq \sigma(\hat{H}) \ .$$

(In fact equality holds since, if $\lambda < \inf_{\eta\in\mathbb{R}} \mu(\eta)$, Eq. (8) yields $\lambda \notin \sigma(\hat{H})$.) Now, Lemma 4 and Eq. (5) yield $[\mu, \infty) \subseteq \sigma(H)$ and the proof of Theorem 1 is complete.

Case $\beta \neq 0$: As noticed before Theorem 2, in this case $|d(\xi)| \to \infty$ for $|\xi| \to \infty$ and H_η has compact resolvent,

$$\sigma(H_\eta) = \{\lambda_1(\eta), \lambda_2(\eta), \dots\}, \quad \lambda_1(\eta) \leq \lambda_2(\eta) \leq \dots \text{ and } \lambda_n(\eta) \to \infty \ (n \to \infty) \ ,$$

where the eigenvalues λ_n are counted according to their multiplicity. The $\lambda_n(\eta)$ depend continuously (even analytically) on η (see [I85, Lemma 2.3.(iii)]; the proof there goes through under our conditions), thus Eq. (8) yields

$$\sigma(\hat{H}) = \bigcup_{n\in\mathbb{N}} \lambda_n(\mathbb{R}) \ .$$

Since a shift of the independent variable is a unitary isomorphism in $L^2(\mathbb{R})$ we get

$$\begin{aligned} H_\eta &\cong -\frac{\partial^2}{\partial\xi^2} + (d(\xi+p)-\eta)^2 + v(\xi+p) \\ &= -\frac{\partial^2}{\partial\xi^2} + (d(\xi)+\beta-\eta)^2 + v(\xi) \\ &= H_{\eta-\beta} \ , \end{aligned}$$

hence $\lambda_n(\eta) = \lambda_n(\eta-\beta)$, i.e., the λ_n are β-periodic and, by Lemma 4 and Eq. (5), Theorem 2(b) is proved.

4. Examples

First let us consider a homogeneous magnetic field which is of course a spherically symmetric, radially periodic field with arbitrary period $p > 0$. Then $b(r) \equiv b$ = const $\neq 0$ and $\beta = bp \neq 0$, $d(x) = xb$ and

$$H_\eta = -\frac{\partial^2}{\partial x^2} + (xb-\eta)^2 = -\frac{\partial^2}{\partial x^2} + b^2\left(x-\tfrac{\eta}{b}\right)^2 \ .$$

A shift in x of $\frac{\eta}{b}$ gives $H_\eta \cong H_0 = -\frac{\partial^2}{\partial x^2} + b^2x^2$ and a well-known result gives $\lambda_n(\eta) \equiv (2n-1)b$. (The fact that the $\lambda_n(\eta)$ are constant is not surprising because Theorem 2(b) states that they are β- hence bp-periodic, where p is arbitrary here). Since now $a_{\text{rec}}(x,y) = -yb$ we have even $H_{\text{rec}} = \hat{H}$, hence the analysis of Sect. 3 gives

$$\sigma(\hat{H}) \overset{\text{Lemma 4}}{\subseteq} \sigma_{\text{ess}}(H) \subseteq \sigma(H) \overset{(5)}{=} \sigma(H_{\text{rec}}) = \sigma(\hat{H}) \ .$$

Thus we have $\sigma(H) = \sigma_{\text{ess}}(H) = \{(2n-1)b : n \geq 1\}$.

By this result we can guess that there are also other cases where gaps in $\bigcup_{n\geq 1}\lambda_n([0,\beta))$ are spectral gaps of $\sigma(H)$.

Finally we give a class of examples for the case $\beta = 0$, satisfying

$$\inf_{\eta\in\mathbb{R}} \inf \sigma(H_\eta) < \inf \sigma(H_\delta) \ . \tag{10}$$

Then, as mentioned in Remark 1, the lowest part of the essential spectrum is purely dense point spectrum.

Let $v \equiv 0$, $\varepsilon \in (0, \frac{1}{2})$ and $b = b_\varepsilon$ be 2-periodic consisting of peaks like in Fig. 1 below. Then $\beta = 0$ and the conditions of Theorem 1 are fulfilled. Now, $d(x)$ looks like in Fig. 2 and obviously $\delta = 0$. [E, Theorem 5.5.1] applied to the potential $d^2(x)$ yields

$$\mu(0) \geq c - \frac{1}{16}\left(\int_0^2 |d^2(x) - c|\, dx\right)^2 \text{, where } c = \int_0^2 d^2(x)\,\frac{dx}{2} \ .$$

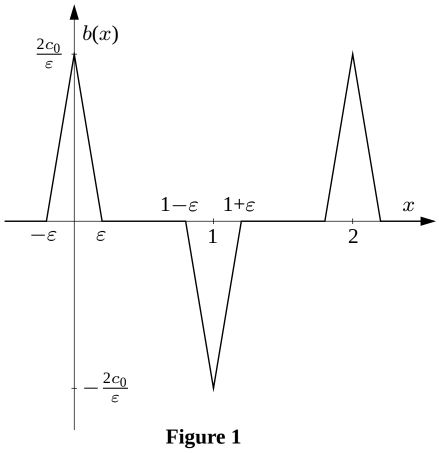

Figure 1

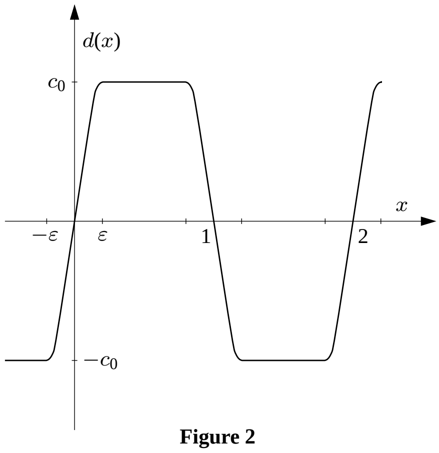

Figure 2

It is easy to see that

$$c_0{}^2 \geq c \geq (2-4\varepsilon)c_0{}^2 \cdot \tfrac{1}{2} = c_0{}^2 - 2\varepsilon c_0{}^2,$$

and hence

$$\int_0^2 |d^{\,2}(x) - c|\, dx \leq 4\varepsilon c_0{}^2 + (2-4\varepsilon)\left|c_0{}^2 - c\right| \\ \leq 4\varepsilon c_0{}^2 + (2-4\varepsilon)\cdot 2\varepsilon c_0{}^2 \leq 8\varepsilon c_0{}^2 \ ,$$

thus

$$\mu(0) \geq c_0{}^2 - 2\varepsilon c_0{}^2 - 4\varepsilon^2 c_0{}^4 \ . \tag{11}$$

Due to [E, (2.2.10), p. 23], we have $\mu(\eta) \leq \int_0^2 |f'(\xi)|^2 + (d(\xi)-\eta)^2 |f(\xi)|^2\, d\xi$ for every 2-periodic C^∞-function f satisfying $\int_0^2 |f(\xi)|^2\, d\xi = 1$. Choosing $f(\xi) = \frac{1}{2} - \frac{1}{\sqrt{2}}\sin(\pi\xi)$ and, using $\int_0^2 |f|^2 = 1$ and the symmetries of d and sin, we get

$$\begin{aligned} \mu(\eta) &\leq \int_0^2 \frac{\pi^2}{2}\cos^2(\pi\xi) + \left(c_0{}^2 - 2\eta d(\xi) + \eta^2\right)|f(\xi)|^2\, d\xi \\ &= \frac{\pi^2}{2} + c_0{}^2 + \eta^2 - 2\eta\int_0^2 d(\xi)\left(\frac{1}{4} - \frac{1}{\sqrt{2}}\sin(\pi\xi) + \frac{1}{2}\sin^2(\pi\xi)\right)\, d\xi \\ &= \frac{\pi^2}{2} + c_0{}^2 + \eta^2 + \sqrt{2}\eta\int_0^2 d(\xi)\sin(\pi\xi)\, d\xi \ . \end{aligned}$$

Obviously, for $\varepsilon < \frac{1}{4}$,

$$\int_0^2 d(\xi)\sin(\pi\xi)\, d\xi \geq 2\int_{\frac{1}{4}}^{\frac{3}{4}} d(\xi)\sin(\pi\xi)\, d\xi \geq 2\int_{\frac{1}{4}}^{\frac{3}{4}} c_0\frac{1}{\sqrt{2}}\, d\xi = \frac{c_0}{\sqrt{2}}.$$

Thus, for $\eta < 0$, $\mu(\eta) \leq c_0{}^2 + \frac{\pi^2}{2} + \eta^2 + c_0\eta$ and choosing $\eta = -\frac{c_0}{2}$ we get for large c_0 and $\varepsilon < \min\left\{\frac{1}{10}, \frac{1}{c_0{}^2}\right\}$,

$$\mu(-\tfrac{c_0}{2}) \leq {c_0}^2 - \tfrac{1}{4}{c_0}^2 + \tfrac{\pi^2}{2} \overset{c_0 \text{ large}}{<} {c_0}^2 - \tfrac{1}{5}{c_0}^2 - 4 \leq {c_0}^2 - 2\varepsilon {c_0}^2 - 4\varepsilon^2 {c_0}^4 \overset{(11)}{\leq} \mu(0),$$

and hence (10) holds.

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