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An explicit iteration for zeros of accretive operators

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ABSTRACT

In this paper, for Lipschitz accretive operator A, an iteration scheme is defined as follows:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(u - \beta_n A x_n).$$

Its strong convergence is established for finding some zero of A whenever $\alpha_n, \beta_n \in (0,1)$ satisfying conditions:

$$\lim_{n\to\infty}\alpha_n = 0,\quad \sum_{n=1}^{+\infty}\alpha_n = +\infty,\quad \lim_{n\to\infty}\beta_n = 0.$$

Furthermore, some applications for equilibrium problems are given also. In particular, the iteration coefficient is simpler and more general.

1. Introduction

Throughout this paper, a Banach space E will always be over the real scalar field. We denote its norm by $\|\cdot\|$ and its dual space by E^*. The value of $x^* \in E^*$ at $y \in E$ is denoted by $\langle y, x\rangle$ and the *normalized duality mapping* J from E into 2^{E^*} is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \|x\| = \|f\|\},\quad \forall x \in E.$$

It is well known (see, for example, [28, Theorems 4.3.1, 4.3.2]) that E is smooth (equivalently, E^* is strict convex [28, p. 113, Problem 3]) if and only if J is single-valued. In the sequel, we shall denote the single-valued normalized duality map by j. Let $A^{-1}0 = \{x \in D(A); 0 \in Ax\}$, the set of zeros of an operator A and $\mathbb{N}$ denote the set of all positive integer. We write $x_n \rightharpoonup x$ (respectively $x_n \overset{*}{\rightharpoonup} x$) to indicate that the sequence x_n weakly (respectively weak*) converges to x; as usual $x_n \to x$ will symbolize strong convergence.

Recall a mapping A with domain $D(A)$ and range $R(A)$ in E is called *Lipschitzian* if for all $x, y \in D(A)$, there exists $L > 0$ such that

$$\|Ax - Ay\| \leqslant L\|x - y\|.$$

In particular, A is called *nonexpansive* whenever $L = 1$ and *contractive* when $0 \leqslant L < 1$. A mapping $A : D(A) \subset E \to 2^E$ is called to be *accretive* if for all $x, y \in D(A)$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle u - v, j(x-y)\rangle \geqslant 0,\quad \text{for}\ \ u \in Ax\ \ \text{and}\ \ v \in Ay;$$

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If E is a Hilbert space, accretive operators are also called monotone. An operator A is called $m-accretive$ if it is accretive and $R(I+rA)$, range of $(I+rA)$, is E for all $r>0$; A is said to *satisfy the range condition* if $\overline{D(A)} \subset R(I+rA), \forall r>0$, where I is the identity operator of E and $\overline{D(A)}$ denotes the closure of the domain of A.

Interest in accretive operators stems mainly from their firm connection with equations of evolution. It is well known (see [33]) that many physically significant problems can be modeled by the initial-value problems of the form

$$\begin{cases} x'(t) + Ax(t) = 0, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where A is an accretive operator in an appropriate Banach space. Typical examples where such evolution equations occur can be found in the heat and wave equations or Schrodinger equations. Especially, one of the fundamental results in the theory of accretive operators, which is due to Browder [3], states that, if A is locally Lipschitzian and accretive, then A is m-accretive. This result was subsequently generalized by Martin [14] to the continuous accretive operators. If, in (1.1), $x(t)$ is independent of t, then (1.1) reduces to $Au=0$ whose solutions corresponds to the equilibrium points of the system (1.1). Consequently, considerable research effects have been devoted, especially, within the past 20 years or so, to the iterative methods for approximating these equilibrium points.

One popular method of solving $0 \in Ax$ is *Rockafellar proximal point algorithm* [20] which is recognized as a powerful and successful algorithm in finding a zero of accretive operators. Starting from any initial guess $x_0 \in E$, this proximal point algorithm generates a sequence $\{x_k\}$ according to the inclusion:

$$x_k + e_k \in x_{k+1} + c_k A(x_{k+1}),$$

where $\{e_k\}$ is a sequence of errors and $\{c_k\}$ is a sequence of positive regularization parameters. Note that the above algorithm can be rewritten as

$$x_{k+1} = J^A_{c_k}(x_k + e_k), \tag{1.2}$$

where $J^A_r = (I+rA)^{-1}$ for all $r>0$, is the resolvent of A with I being the identity map on the space E. Rockafellar [20] proved the weak convergence of his algorithm (1.2) provided the regularization sequence $\{c_k\}$ remains bounded away from zero and the error sequence $\{e_k\}$ satisfies the condition $\sum_{k=0}^{+\infty}\|e_k\| < \infty$. So to have strong convergence, one has to modify the algorithm (1.2). Several authors proposed modifications of Rochafellar's proximal point algorithm (1.2) to have strong convergence. Solodov–Svaiter [27] initiated such investigation followed by Kamimura–Takahashi [11] (in which the work of [11] is extended to the framework of uniformly convex and uniformly smooth Banach spaces). Bruck [1] introduced an iteration process and proved, in Hilbert space setting, the convergence of the process to a zero of a maximal monotone operator. In 1979, Reich [18] extended this result to uniformly smooth Banach spaces provided that the operator is m-accretive. Reich [19] unified the above results and proved the following (also see Takahashi and Ueda [29 Theorem 1]):

Theorem R [19, Theorem 1, Remark]. *Let E be a reflexive Banach space whose norm is uniformly Gâteaux differentiable. Suppose that every weakly compact convex subset of E has the fixed point property for non-expansive mappings. Assumed that $A : D(A) \subset E \to 2^E$ be an accretive operator with resolvent J^A_r for $r>0$ and $A^{-1}0 \neq \emptyset$, and K is a nonempty closed convex subset of E such that $\overline{D(A)} \subset K \subset \bigcap_{r>0} R(I+rA)$. If $0 \in R(A)$, then the strong $\lim_{r\to\infty} J^A_r u$ exists and belongs to $A^{-1}0$ for each $u \in K$. Further, if $Pu = \lim_{r\to\infty} J^A_r u$ for each $u \in K$, then P is the unique sunny nonexpansive retraction of K onto $A^{-1}0$.*

By the inspiration for the Rockafellar proximal point algorithm and the iterative methods of Halpern [10], Dominguez Benavides et al. [4] studied the Halpern type iteration process (1.2) to find a zero of an $m-$accretive operator A in a uniformly smooth Banach space with a weakly continuous duality mapping J_φ with gauge φ in virtue of the resolvent J^A_r of A:

$$x_{n+1} = \alpha_n u + (1-\alpha_n) J^A_{r_n} x_n, \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\} \subset (0,1)$ and $\{r_n\} \subset (0,+\infty)$ satisfying the conditions:

$$\lim_{n\to\infty}\alpha_n = 0, \ \sum_{n=1}^{+\infty}\alpha_n = +\infty, \ \lim_{n\to\infty} r_n = +\infty.$$

Xu [32] and Marino and Xu [13] also researched the above iteration process in a uniformly smooth Banach space.

Recently, Xu [30] and Song and Yang [26] studied the strong convergence of the regularization method for Rockafellar's proximal point algorithm of the resolvent J_r in a Hilbert space:

$$x_{n+1} = J^A_{r_n}(\alpha_n u + (1-\alpha_n)x_n), \tag{1.4}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0,+\infty)$ satisfying the conditions:

$$\lim_{n\to\infty}\alpha_n = 0, \ \sum_{n=1}^{+\infty}\alpha_n = +\infty, \ \sum_{n=1}^{+\infty}|\alpha_{n+1}-\alpha_n| < +\infty, \ \sum_{n=1}^{+\infty}|r_{n+1}-r_n| < +\infty.$$

Song [21] established the strong convergence for two explicit iteration schemes for approaching a zero of an accretive operator A:

$$x_{n+1} = \alpha_n u + \beta_n x_n + (1-\alpha_n-\beta_n) J^A_{r_n} x_n, \tag{1.5}$$

$$x_{n+1} = \beta_n x_n + (1-\beta_n) J^A_{r_n}(\alpha_n u + (1-\alpha_n)x_n), \tag{1.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $\{r_n\} \subset (0,+\infty)$ satisfying the following conditions:

$\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{+\infty}\alpha_n = +\infty$, $0 < \liminf_{n\to\infty}\beta_n \leqslant \limsup_{n\to\infty}\beta_n < 1$,
$\liminf_{n\to\infty} r_n > 0$ and $\lim_{n\to\infty}\frac{r_n}{r_{n+1}} = 1$.

The main purposes of this paper is, for general accretive operator A, to slightly modified the iteration (1.7) and to study its strong convergence under the simpler coefficient. The iteration scheme is defined as follows: for an anchor point $u \in E$ and an initial value $x_1 \in E$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(u - \beta_n A x_n). \tag{1.8}$$

Its strong convergence will be established for finding some zero of an accretive operator A whenever $\alpha_n, \beta_n \in (0,1)$ satisfying conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$,
(C2) $\sum_{n=1}^{+\infty}\alpha_n = +\infty$,
(C3) $\lim_{n\to\infty}\beta_n = 0$.

2. Preliminaries and basic results

Let $S(E) := \{x \in E; \|x\| = 1\}$ denote the unit sphere of a Banach space E. The space E is said to have.

(i) *a Gâteaux differentiable norm* (we also say that E is *smooth*), if the limit

$$\lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in S(E)$;
(ii) *a uniformly Gâteaux differentiable norm*, if for each y in $S(E)$, the limit (2.1) is uniformly attained for $x \in S(E)$;
(iii) *a Fréchet differentiable norm*, if for each $x \in S(E)$, the limit (2.1) is attained uniformly for $y \in S(E)$;
(iv) *a uniformly Fréchet differentiable norm* (we also say that E is *uniformly smooth*), if the limit (2.1) is attained uniformly for $(x,y) \in S(E) \times S(E)$;
(v) *fixed point property* for nonexpansive self-mappings, if each nonexpansive self-mapping defined on any bounded closed convex subset K of E has at least a fixed point.

A Banach space E is said to be (vi) *strictly convex* if $\|x\| = \|y\| = 1,\ x \neq y$ implies $\frac{\|x+y\|}{2} < 1$; (vii) *uniformly convex* if for all $\varepsilon \in [0,2], \exists \delta_\varepsilon > 0$ such that $\|x\| = \|y\| = 1$ implies $\frac{\|x+y\|}{2} < 1 - \delta_\varepsilon$ whenever $\|x - y\| \geqslant \varepsilon$.

Recall a mapping $A : D(A) \subset E \to E$ is called *strongly accretive* if for all $x, y \in D(A)$, there exists $k \in (0,1)$ and $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y)\rangle \geqslant k\|x-y\|^2;$$

If $k \equiv 0$, then A is said to be *accretive*. Let $A^{-1}0 = \{x \in D(A); 0 = Ax\}$. We use J_r^A to denote the resolvent of A. Namely,

$$J_r^A = (I + rA)^{-1},\ r > 0.$$

If C and D are nonempty subsets of a Banach space E such that C is nonempty closed convex and $D \subset C$, then a mapping $P : C \to D$ is called a *retraction* from C to D if P is continuous with $F(P) = D$. A mapping $P : C \to D$ is called *sunny* if $P(Px + t(x - Px)) = Px,\ \forall x \in C$ whenever $Px + t(x - Px) \in C$ and $t > 0$. A subset D of $C$is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction of C onto D. The term "sunny nonexpansive retraction" was coined by Reich in [17] and its following property is well known. For more details, see [12,17,18].

Lemma 2.1. *Let C be nonempty convex subset of a smooth Banach space $E, \emptyset \neq D \subset C$, and $P : C \to D$ a retraction. Then P is both sunny and nonexpansive if and only if there holds the inequality:*

$$\langle x - Px, J(y - Px)\rangle \leq 0 \quad \text{for all } x \in C \text{ and } y \in D, \tag{2.2}$$

where J is the normalized duality mapping. Hence there is at most one sunny non-expansive retraction from C onto D.

In the proof of our main theorems, we also need the following lemma.

Lemma 2.2 [4, Lemma 2.3] or [32]. *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property:*

$$a_{n+1} \leqslant (1-\gamma_n)a_n + \gamma_n\beta_n, \quad \forall n \geqslant 0,$$

where $\{\gamma_n\} \subset (0,1)$ and $\{\beta_n\} \subset \mathbb{R}$ such that (i) $\sum_{n=0}^{\infty}\gamma_n = \infty$, (ii) $\limsup_{n\to\infty}\beta_n \leq 0$. Then $\{a_n\}$ converges to zero as $n \to \infty$.

3. Strong convergence of the explicit iterations

In this section, for general Lipschitz accretive operator A, we shall study the strong convergence of the iteration scheme for finding some zero of A.

Theorem 3.1. *Let E be a reflexive Banach space which has a uniformly Gâteaux differentiable norm and has fixed point property for nonexpansive self-mappings. Suppose that $A : E \to E$ is a Lipschitz accretive operator with Lipschitz constant $L > 0$ and $A^{-1}0 \neq \emptyset$. For an anchor point $u \in E$ and an initial value $x_1 \in E$, let $\{x_n\}$ be a sequence given by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(u - \beta_n A x_n). \tag{3.1}$$

Assumed that $\{\alpha_n\}$ and $\{\beta_n\}$ is two real sequence in $(0,1)$ satisfying the conditions:

$$(C1)\ \lim_{n\to\infty}\alpha_n = 0,\ (C2)\ \sum_{n=1}^{+\infty}\alpha_n = +\infty,\ (C3)\ \lim_{n\to\infty}\beta_n = 0.$$

Then $\{x_n\}$ converges strongly to $Pu \in A^{-1}0$, where P is the unique sunny nonexpansive retraction from E onto $A^{-1}0$.

Proof. The proof consists of the following steps:

Step 1. The sequence $\{x_n\}$ is bounded. In fact, since $\lim_{n\to\infty}\beta_n = 0$, there exists $N \in \mathbb{N}$ sufficiently large such that

$$\beta_n < \frac{1}{2L} \quad \text{for all } n \geqslant N.$$

Take $p \in A^{-1}0$ and choose $M > 0$ sufficiently large such that

$$\sup_{1\leqslant i\leqslant N}\|x_i - p\| \leqslant M, \quad \|u - p\| \leqslant \frac{M}{2}.$$

Then, from the iteration process (3.1) and $Ap = 0$, we estimate the following: for all $n \geqslant N$

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1-\alpha_n)(x_n - p) + \alpha_n(u - p - \beta_n A x_n)\| \\
&\leqslant (1-\alpha_n)\|x_n - p\| + \alpha_n\|u - p\| + \alpha_n\beta_n\|Ax_n - 0\| \\
&\leqslant (1-\alpha_n)\|x_n - p\| + \alpha_n\|u - p\| + \alpha_n\beta_n\|Ax_n - Ap\| \\
&\leqslant (1-\alpha_n)\|x_n - p\| + \alpha_n\tfrac{M}{2} + \alpha_n\beta_n L\|x_n - p\| \\
&< (1-\alpha_n)\|x_n - p\| + \alpha_n\tfrac{M}{2} + \tfrac{\alpha_n}{2}\|x_n - p\| \\
&= (1-\tfrac{\alpha_n}{2})\|x_n - p\| + \tfrac{\alpha_n}{2}M \\
&\leqslant \max\{\|x_n - p\|, M\} \\
&\leqslant \max\{\|x_{n-1} - p\|, M\} \\
&\ \ \vdots \\
&\leqslant \max\{\|x_N - y\|, M\} \leqslant M.
\end{aligned}$$

Thus, the sequence $\{x_n\}$ is bounded, and hence so is $\{Ax_n\}$ since

$$\|Ax_n\| = \|Ax_n - Ap\| \leqslant L\|x_n - p\| \leqslant LM.$$

It follows from Theorem R that $A^{-1}0$ is a sunny nonexpansive retract of E. Let P is the unique sunny nonexpansive retraction from E onto $A^{-1}0$.

Step 2. $\limsup_{n\to\infty}\langle u - Pu, J(x_{n+1} - Pu)\rangle \leq 0$.

In fact, let A_m be defined by $A_m x := \alpha_m A x$ for each $x \in E$ and $m \in \mathbb{N}$. Clearly, for each m, A_m is an accretive operator and $A^{-1}0 = A_m^{-1}0$. By the boundedness of the sequence $\{Ax_n\}$ and $\lim_{m\to\infty}\alpha_m = 0$, we have

$$\lim_{m\to\infty}\|A_m(x_n)\| = \lim_{m\to\infty}\alpha_m\|Ax_n\| = 0. \tag{3.2}$$

Setting $J_r^m := J_r^{A_m}$ for each $m \in \mathbb{N}$ and $r > 0$. Then for each m, it follows from Theorem R that

$$\lim_{r\to\infty} J_r^m u = \lim_{r\to\infty} J_r^{A_m} u = P_m u,$$

where P_m is the unique sunny nonexpansive retraction from E onto $A_m^{-1}0 = A^{-1}0$. Then $P_m \equiv P$ by the uniqueness, and hence

$$\lim_{r\to\infty} J_r^m u = Pu \quad \text{for each } m. \tag{3.3}$$

Since $J_r^m = (I + rA_m)^{-1}$ for each m, then we have

$$(I + rA_m)(I + rA_m)^{-1} = I = (I + rA_m)(J_r^m) = J_r^m + rA_m(J_r^m).$$

So,

$$\frac{I-J_r^m}{r}=A_m(J_r^m).$$

By the accretion of A_m, we have

$$\langle A_m(x_n)-A_m(J_r^m u),J(x_n-J_r^m u)\rangle \geqslant 0.$$

Then

$$\left\langle \frac{u-J_r^m u}{r}-A_m(x_n),J(x_n-J_r^m u)\right\rangle \le 0.$$

Since for $p\in Z$,

$$\|x_n-J_r^m u\|\leqslant \|x_n-p\|+\|p-J_r^m u\|=\|x_n-p\|+\|J_r^m p-J_r^m u\|\leqslant \|x_n-p\|+\|p-u\|,$$

then we may take some sufficiently large positive constant C such that

$$\|x_n-J_r^m u\|\leqslant \|x_n-p\|+\|p-u\|<C.$$

Therefore we have

$$\begin{aligned}\frac{1}{r}\langle u-J_r^m u,J(x_n-J_r^m u)\rangle &\leqslant \langle A_m(x_n),J(x_n-J_r^m u)\rangle\\ &\leqslant \|A_m(x_n)\|\|x_n-J_r^m u\|\\ &\leqslant \|A_m(x_n)\|C.\end{aligned}$$

Noticing C does not depend on m, it follows from (3.2) that

$$\limsup_{m\to\infty}\langle u-J_r^m u,J(x_n-J_r^m u)\rangle \le 0,\quad \text{for each } r>0 \text{ and all } n\in\mathbb{N}.$$

Therefore, for any $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that for all $m\geqslant N$,

$$\langle u-J_r^m u,J(x_n-J_r^m u)\rangle<\frac{\varepsilon}{2} \text{ for each } r>0 \text{ and all } n\in\mathbb{N}. \tag{3.4}$$

On the other hand, since J is uniformly continuous on bounded set from norm topology to weak star topology and $\lim_{r\to\infty}J_r^N u=Pu$, then we have that

$$\begin{aligned}&|\langle u-Pu,J(x_n-Pu)\rangle-\langle u-J_r^N u,J(x_n-J_r^N u)\rangle|\\ &=|\langle u-Pu,J(x_n-Pu)-J(x_n-J_r^N u)\rangle+\langle J_r^N u-Pu,J(x_n-J_r^N u)\rangle|\\ &\leqslant |\langle u-Pu,J(x_n-Pu)-J(x_n-J_r^N u)\rangle|+\|J_r^N u-Pu\|C\to 0(r\to\infty).\end{aligned}$$

Hence, for the above $\varepsilon>0,\quad \exists M'>0,\forall r>M'$, for all n, we have

$$\langle u-Pu,J(x_n-Pu)\rangle<\langle u-J_r^N u,J(x_n-J_r^N u)\rangle+\frac{\varepsilon}{2}.$$

By (3.4), we obtain that

$$\langle u-Pu,J(x_n-Pu)\rangle<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Since ε is arbitrary, then

$$\limsup_{n\to\infty}\langle u-Pu,J(x_{n+1}-Pu)\rangle\le 0.$$

Step 3. $\lim_{n\to\infty}x_n=Pu$. Using (3.1) and $A(Pu)=0$, we make the following estimates:

$$\begin{aligned}\|x_{n+1}-Pu\|^2 &=(1-\alpha_n)\langle x_n-Pu,J(x_{n+1}-Pu)\rangle+\alpha_n\langle u-Pu,J(x_{n+1}-Pu)\rangle-\alpha_n\beta_n\langle Ax_n,J(x_{n+1}-Pu)\rangle\\ &\leqslant (1-\alpha_n)\|x_n-Pu\|\|x_{n+1}-Pu\|+\alpha_n\langle u-Pu,J(x_{n+1}-Pu)\rangle+\alpha_n\beta_n\|Ax_n-0\|\|x_{n+1}-Pu\|\\ &\leqslant (1-\alpha_n)\|x_n-Pu\|\|x_{n+1}-Pu\|+\alpha_n\langle u-Pu,J(x_{n+1}-Pu)\rangle+\alpha_n\beta_n L\|x_n-Pu\|\|x_{n+1}-Pu\|\\ &\leqslant [1-\alpha_n(1-\beta_n L)]\tfrac{\|x_n-Pu\|^2+\|x_{n+1}-Pu\|^2}{2}+\alpha_n\langle u-Pu,J(x_{n+1}-Pu)\rangle,\end{aligned}$$

and so

$$\|x_{n+1}-Pu\|^2\leqslant[1-\alpha_n(1-\beta_n L)]\|x_n-Pu\|^2+2\alpha_n\langle u-Pu,J(x_{n+1}-Pu)\rangle. \tag{3.5}$$

Since $\lim_{n\to\infty}\beta_n=0$, there exists $N\in\mathbb{N}$ sufficiently large such that

$$\beta_n<\frac{1}{2L}\quad \text{for all } n\geqslant N, \text{and hence} \frac{1}{2}\alpha_n<(1-\beta_n L)\alpha_n.$$

So, the condition (C2) implies

$$\sum_{n=1}^{+\infty}\alpha_n(1-\beta_n L)=+\infty.$$

Therefore, an application of Lemma 2.2 yields the desired result. This completes the proof. □

Recall a mapping A is called $\alpha-$*inverse strongly accretive* if for all $x,y\in D(A)$, there exists $\alpha\in(0,1)$ and $j(x-y)\in J(x-y)$ such that

$$\langle Ax-Ay,j(x-y)\rangle\geqslant\alpha\|Ax-Ay\|^2.$$

Clearly, $\alpha-$inverse strongly accretive mapping is Lipschitzian accretive mapping with Lipschitz constant $L=\frac{1}{\alpha}$. Then, the following corollary is obvious.

Corollary 3.2. *Let E be a reflexive Banach space which has a uniformly Gâteaux differentiable norm and has fixed point property for nonexpansive self-mappings. Suppose that $A:E\to E$ is a $\alpha-$ inverse strongly accretive operator with $A^{-1}0\neq\emptyset$. For an anchor point $u\in E$ and an initial value $x_1\in E$, let $\{x_n\}$ be a sequence defined by (3.1). Assumed that $\{\alpha_n\}$ and $\{\beta_n\}$ is two real sequence in $(0,1)$ satisfying the conditions (C1), (C2) and (C3). Then $\{x_n\}$ converges strongly to $Pu\in A^{-1}0$.*

Since every uniformly convex Banach space has fixed point property for nonexpansive self-mappings, then, the following corollary is obviously obtained.

Corollary 3.3. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Suppose that $A:E\to E$ is a Lipschitzian accretive operator with $A^{-1}0\neq\emptyset$. For an anchor point $u\in E$ and an initial value $x_1\in E$, let $\{x_n\}$ be a sequence defined by (3.1). Assumed that $\{\alpha_n\}$ and $\{\beta_n\}$ is two real sequence in $(0,1)$ satisfying the conditions (C1), (C2) and (C3). Then $\{x_n\}$ converges strongly to $Pu\in A^{-1}0$.*

Since each uniformly smooth Banach space has fixed point property for nonexpansive self-mappings and has a uniformly Gâteaux differentiable norm, then we have the following clearly.

Corollary 3.4. *Let E be a uniformly smooth Banach space. Suppose that $A:E\to E$ is a Lipschitzian accretive operator with $A^{-1}0\neq\emptyset$. For an anchor point $u\in E$ and an initial value $x_1\in E$, let $\{x_n\}$ be a sequence defined by (3.1). Assumed that $\{\alpha_n\}$ and $\{\beta_n\}$ is two real sequence in $(0,1)$ satisfying the conditions (C1), (C2) and (C3). Then $\{x_n\}$ converges strongly to $Pu\in A^{-1}0$.*

Recall a mapping $T:E\to E$ is called *pseudocontractive* if, for all $x,y\in E$, there exists $j(x-y)\in J(x-y)$ such that

$$\langle Tx-Ty,j(x-y)\rangle\leqslant\|x-y\|^2,$$

or equivalently,

$$\langle(I-T)x-(I-T)y,j(x-y)\rangle\leqslant 0.$$

Let $F(T)=\{x\in E;x=Tx\}$. It is now well known that the fixed point problem of pseudocontractive mappings closely corresponds to zeros of accretive operators. Consequently, considerable research works have been devoted to the approximation of fixed point of pseudocontractive mappings (see, e.g. [8,7,9,15,16,22–25,29,31] and the references contained therein). We also can obtain the following.

Theorem 3.5. *Let E be a reflexive Banach space which has a uniformly Gâteaux differentiable norm and has fixed point property for nonexpansive self-mappings. Suppose that $T:E\to E$ is a Lipschitz pseudocontractive mapping with Lipschitz constant $L>0$ and $F(T)\neq\emptyset$. For an anchor point $u\in E$ and an initial value $x_1\in E$, let $\{x_n\}$ be a sequence given by*

$$x_{n+1}=(1-\alpha_n(1+\beta_n))x_n+\alpha_n u+\alpha_n\beta_n Tx_n. \tag{3.6}$$

Assumed that $\{\alpha_n\}$ and $\{\beta_n\}$ is two real sequence in $(0,1)$ satisfying the conditions:

$$(C1)\ \lim_{n\to\infty}\alpha_n=0,\ \ (C2)\ \sum_{n=1}^{+\infty}\alpha_n=+\infty,\ \ (C3)\ \lim_{n\to\infty}\beta_n=0.$$

Then $\{x_n\}$ converges strongly to $Pu\in F(T)$, where P is the uniqueness of sunny nonexpansive retraction from E onto $F(T)$.

Proof. It follows from the definition of pseudocontractive mapping T that $I-T$ is accretive if and only if T pseudocontractive. Let $A=I-T$. Then $A^{-1}0=F(T)$ and the iteration (3.6) can be rewritten as follows:

$$x_{n+1}=(1-\alpha_n)x_n+\alpha_n(u-\beta_n(I-T)x_n)=(1-\alpha_n)x_n+\alpha_n(u-\beta_n Ax_n). \tag{3.7}$$

Following Theorem 3.1, the desired result is reached. □

Remark 3.6. Theorems 3.5 also provide a convergence sequence to a common fixed point of countably infinite family of strictly pseudocontractive mappings in the sense of Browder and Petryshyn [2] since the fact that T is a strictly pseudocontractive in the sense of Browder and Petryshyn if and only if $A = I - T$ is $\alpha-$inverse strongly accretive.

4. Applications for equilibrium problems

Let K be a nonempty closed convex subset of a real Hilbert space H and F be a bifunction of $K \times K$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem is to find $x \in K$ such that

$$F(x,y) \geqslant 0, \quad \forall y \in K. \tag{4.1}$$

The set of solutions of the equilibrium problem (4.1) is denoted by $EP(F)$. Given a mapping $T : K \to H$, let $F(x,y) = \langle Tx, y - x\rangle$ for all $x, y \in K$. Then $z \in EP(F)$ if and only if $\langle Tx, y - x\rangle \geqslant 0$ for all $y \in K$, i.e., z is a solution of the variational inequality.

Numerous problems in physics, optimization and economics reduce to find a solution of the equilibrium problem (4.1). Some methods have been proposed to solve the equilibrium problems (see, for instance, Blum-Oettli [5] and Combettes-Hirstoaga [6]).

For the purpose of solving the equilibrium problem, let us assume that a bifunction F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in K$;
(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leqslant 0$ for all $x, y \in K$;
(A3) for each $x, y, z \in K, \lim_{t\downarrow 0} F(tz + (1-t)x, y) \leqslant F(x,y)$;
(A4) for each $x \in K, y \mapsto F(x,y)$ is convex and lower semicontinuous.

The following lemmas were also given in [5,6], respectively.

Lemma 4.1 (Blum–Oettli [5, Corollary 1]). *Let K be a nonempty closed convex subset of H and let F be a bifunction of $K \times K$ into $\mathbb{R}$ satisfying the conditions (A1)–(A4). Let $r > 0$ and $x \in H$. Then there exists $z \in K$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geqslant 0, \quad \forall y \in K.$$

Lemma 4.2 (Combettes–Hirstoaga [6, Lemma 2.12]). *Assume that $F : K \times K \to \mathbb{R}$ satisfies the conditions (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to K$ as follows:*

$$T_r(x) = \{z \in K : \ F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geqslant 0, \ \forall y \in K\}, \quad \forall x \in H.$$

Then the following hold:

(1) *T_r is single-valued.*
(2) *for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leqslant \langle T_r x - T_r y, x - y\rangle.$$

(3) $F(T_r) = EP(F)$.

Theorem 4.3. *Let F be a bifunction from $E \times E$ into $\mathbb{R}$ satisfies the conditions (A1)–(A4) and $EP(F) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1, u \in E, r > 0$ and*

$$\begin{cases} F(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geqslant 0, & \forall y \in E, \\ x_{n+1} = (1 - \alpha_n(1+\beta_n))x_n + \alpha_n(u - \beta_n u_n), & n \geqslant 1. \end{cases} \tag{4.2}$$

Assume that $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ satisfy the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{+\infty} \alpha_n = +\infty$;
(C3) $\lim_{n\to\infty} \beta_n = 0$.

Then $\{x_n\}$ converge strongly to $Pu \in EP(F)$, where P is projection operator from E into $EP(F)$.

Proof. Following Lemma 4.2 ($K = E$), we have $u_n = T_r x_n$ and T_r is a nonexpansive mapping, and hence, T_r is a Lipschitz pseudocontractive mappings with Lipschitz constant 1. Then the iteration (4.2) can be rewritten as follows:

$$x_{n+1} = (1 - \alpha_n(1 + \beta_n))x_n + \alpha_n(u - \beta_n T_r x_n), \quad n \geqslant 1. \tag{4.3}$$

Following Theorem 3.1 or 3.5, the desired result is reached. □

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