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The simplified modified nucleolus of a cooperative TU-game

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Abstract In the present paper, we introduce a new solution concept for TU-games, the simplified modified nucleolus or the SM-nucleolus. It is based on the idea of the modified nucleolus (the modiclus) and takes into account both the constructive power and the blocking power of a coalition. The SM-nucleolus inherits this convenient property from the modified nucleolus, but it avoids its high computational complexity. We prove that the SM-nucleolus of an arbitrary n-person TU-game coincides with the prenucleolus of a certain n-person constant-sum game, which is constructed as the average of the game and its dual. Some properties of the new solution are discussed. We show that the SM-nucleolus coincides with the Shapley value for three-person games. However, this does not hold for general n-person cooperative TU-games. To confirm this fact, a counter example is presented in the paper. On top of this, we give several examples that illustrate similarities and differences between the SM-nucleolus and well-known solution concepts for TU-games. Finally, the SM-nucleolus is applied to the weighted voting games.

Keywords Cooperative TU-game · Solution concept · Modiclus · SM-nucleolus · Weighted voting games

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1 Introduction

In this paper, we consider a class of transferable utility games (TU-games) with a finite set of players. It is well known that there is a number of solution concepts in

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these games and all of them have some pros and cons. We propose a new solution concept which has an interesting interpretation and convenient properties.

Based on the idea of nucleolus (Schmeidler 1969), a solution concept was introduced which is known as the modified nucleolus or the modiclus (Sudhölter 1993). According to Sudhölter (1997), this concept takes into account "the blocking power" of a coalition—the amount by which the coalition cannot be prevented from by the complement coalition. Following these approaches, we introduce a new solution, the simplified modified nucleolus or the *SM*-nucleolus, which inherits some excellent characteristics from both. First, similar to the pre- and modified nucleoli, the new solution is a singleton. Second, as well as the modiclus, the *SM*-nucleolus takes into account the blocking power of coalitions in a game. Moreover, going further it clarifies the question in what ratio the constructive and the blocking powers are taken into account in the solution. Third, in contrast to the modiclus, the *SM*-nucleolus demonstrates construction simplicity.

In our interpretation, the blocking power of S is the amount the coalition brings to the grand coalition—its contribution to N. Following this spirit, we interpret the *SM*-nucleolus as a solution that minimizes the maximal segregation between any complementary sets of the whole society. The notions of the constructive and blocking power will be explained more thoroughly in Sect. 3.

Finally, we address the problem of poverty that provides a striking illustration of the new solution concept. The poverty problem can be characterized from different points of view and quantitatively described by various indices such as the poverty index and the Gini index.[1] They both reflect two different aspects of the poverty problem. To indicate the well-being of the poor is the essence of the poverty index. The heart of the Gini index is to show the difference between poor and wealthy people. As a matter of fact, the nucleolus is like the poverty coefficient, and it does not capture the other side of the problem. The *SM*-nucleolus captures both sides—it solves the problem comprehensively.

2 Basic definitions and notations

In this paper, we deal with *cooperative games with transferable utility*, or simply TU-games.

A cooperative TU-game is a pair (N, v), where $N = \{1, 2, \dots, n\}$ is the set of players and $v : 2^N \mapsto \mathbb{R}$ is a *characteristic function* with $v(\emptyset) = 0$. Here $2^N = \{S \subseteq N\}$ is the set of coalitions of (N, v).

Since the game (N, v) is completely determined by the characteristic function v, we shall sometimes represent a TU-game by its characteristic function v.

Due to the classical cooperative approach, we shall look for the ways to distribute the amount $v(N)$ over the members of the grand coalition.

[1] The Gini coefficient (index) was developed by Corrado Gini in 1912 in his paper "Variability and Mutability". The Gini index is a measure of inequality of income distribution.

Denote by $X^*(N, v)$ the set of feasible payoff vectors of a game (N, v), i.e.,

$$X^*(N, v) = \{x \in \mathbb{R}^N : x(N) \le v(N)\}.^2$$

Let G^N be the set of TU-games.

Definition 2.1 A solution on G^N is a mapping $f : G^N \mapsto \mathbb{R}^N$ obeying $f(N, v) \subseteq X^*(N, v)$.

Definition 2.2 An imputation in the game (N, v) is a payoff vector $x = (x_1, x_2, \dots, x_n)$ satisfying

$$x(N) = v(N) \quad \text{(group rationality)}, \tag{2.1}$$

$$x_i \ge v(i) \quad \text{for all } i \in N \quad \text{(individual rationality (IR))}. \tag{2.2}$$

Definition 2.3 A preimputation in (N, v) is a group rational payoff vector, i.e., Condition (2.2) is not required.

We denote by $X(v)$ and $X^0(v)$ the imputation and preimputation spaces of the game (N, v), respectively.

A solution f on G^N should satisfy the following convenient and well-known properties:

AN. f is anonymous if for each $(N, v) \in G^N$ and each bijective mapping $\pi : N \mapsto N$ $f(N, \pi v) = \pi(f(N, v))$ holds (where $\pi v(\pi(S)) = v(S),\ S \subseteq N$).
COV. f is covariant under strategic equivalence if for any two games (N, v), $(N, w) \in G^N$ with $w = \alpha v + \beta$ for some $\alpha > 0$, $\beta \in \mathbb{R}^N$, $f(N, w) = \alpha f(N, v) + \beta$ holds.
SIVA. f is single valued if $|f(N, v)| = 1$ for any $(N, v) \in G^N$.
NE. f satisfies nonemptiness if $f(N, v) \ne \emptyset$ for any $(N, v) \in G^N$.
PO. f is Pareto optimal if $f(N, v) \subseteq X^0(v)$ for any $(N, v) \in G^N$.

Here we concentrate the reader's attention on single-valued solution concepts.

The most known and widely used among the single valued solutions is the Shapley value. It should be noticed that the Shapley value satisfies all properties given above (Shapley 1953). One more well-known and meaningful solution concept is the nucleolus introduced by Schmeidler (1969).

The solution concept proposed in the paper is based on the nucleolus approach, therefore, some precise definitions are needed.

Let x be a preimputation in a game (N, v).

Definition 2.4 The excess $e(x, v, S)$ of a coalition S at x is

$$e(x, v, S) = v(S) - x(S).$$

[2] By $x(S)$ we mean $\sum_{i \in S} x_i$ $(S \subseteq N)$.

Let X be an arbitrary nonempty closed set in $\mathbb{R}^N$. For each $x \in \mathbb{R}^N$, define a mapping θ such that $\theta(x) = y \in \mathbb{R}^N$, where y is a vector that arises from x by arranging its components in a non-increasing order.

Definition 2.5 [3] The nucleolus of X denoted by $\mathcal{N}(X)$, or $\mathcal{N}(N, v, X)$, is the set of vectors in X whose $\theta(e(x, v, S)_{S\subseteq N})$'s are lexicographically least, i.e.,

$$\mathcal{N}(X) = \Big\{x \in X : \theta\big(e(x, v, S)_{S\subseteq N}\big) \preceq_{\text{lex}} \theta\big(e(y, v, S)_{S\subseteq N}\big) \\ \text{for all } y \in X\Big\}.$$

If $X = X(v)$, it is called the nucleolus of the game.

If $X = X^0(v)$, it is called the prenucleolus of the game.

We denote the last two cases by $\mathcal{N}$ and $\mathcal{PN}$, respectively.

The following assertions proved by Schmeidler (1969) are needed for the further consideration.

(i) Let X be a non-empty and compact set. Then $\mathcal{N}(N, v, X) \neq \emptyset$ for every game (N, v).
(ii) If, in addition, X is convex, then the nucleolus $\mathcal{N}(N, v, X)$ consists of a unique point.
(iii) If X is a non-empty closed convex subset of $X^*(N, v)$, the $\mathcal{N}(N, v, X)$ is a singleton.

By assertion (iii), the prenucleolus of a game is a singleton, and it is Pareto optimal. The unique element $\nu(v)$ of $\mathcal{PN}$ is called the prenucleolus of (N, v).

A convincing intuitive meaning of the prenucleolus was given by Maschler et al. (1979). They propose to regard the excess of a coalition as a measure of dissatisfaction, which should be minimized.

Despite some attractive properties the prenucleolus possesses some authors try to search for a different solution concept. One of the reasons may be that the prenucleolus does not obey coalitional monotonicity.

Definition 2.6 A single valued solution f is called *coalitionally monotonic* if for every pair of games (N, v) and (N, w), satisfying

$$v(T) > w(T) \quad \text{for some subset } T \text{ of } N,$$
$$v(S) = w(S) \quad \text{for all } S,\ S \neq T,$$

it follows that

$$f_i[v] \geq f_i[w] \quad \text{for all } i \in T.$$

Young (1985) proved that for the class of TU-games with nonempty core there does not exist a single valued coalitionally monotonic solution concept that always

[3] The definition has been taken from Maschler (1992).

lies in the core. Observe that the Shapley value is a coalitionally monotonic solution concept.

In continuation of the subject, we apply a different approach to defining a solution for cooperative TU-games.

3 Solution concepts taking into account the blocking power of a coalition

Let us address the solution concept of the modiclus introduced by Sudhölter (1997).

In contrast to the nucleolus, the modiclus takes into account both the constructive power and the blocking (preventive) power of a coalition in a game. We find this idea quite interesting.

Let us start with the definition of the modiclus.

Definition 3.1 [4] For each $x \in \mathbb{R}^N$ define

$$\widetilde{\theta}(x, v) = \theta\Big(\big(e(x, v, S) - e(x, v, T)\big)_{(S,T)\in 2^N \times 2^N}\Big) \in \mathbb{R}^{2^{2N}}.$$

The modiclus of (N, v) is the set

$$\Psi(v) = \big\{x \in X^0(v) : \widetilde{\theta}(x, v) \preceq_{\text{lex}} \widetilde{\theta}(y, v) \text{ for all } y \in X^0(v)\big\}.$$

The modiclus lexicographically minimizes the differences of excesses of the pairs of coalitions within the set of preimputations. This differences, according to Peleg and Sudhölter (2003), may be regarded as the envy of S against T at x.

The modified nucleolus takes into account the blocking power of a coalition, however, it does not answer the question, in what ratio the constructive and blocking powers are taken into account in the solution.

At the same time, its computational complexity is higher than computational complexity of the prenucleolus. In fact, for an arbitrary n-person game the computation complexity of the modiclus is comparable with that of the prenucleolus for a $2n$-person game.

In order to take into account the blocking power of a coalition and at the same time to overcome computational complexity, we introduce a new solution.

Let (N, v) be a TU-game, then the dual game (N, v^*) of (N, v) is defined by

$$v^*(S) = v(N) - v(N \backslash S) \tag{3.1}$$

for all coalitions S.

First, we need to clarify the notion of the constructive and the blocking power. The meaning of the constructive power of S is clear. It is the worth of the coalition, or exactly what S can reach by cooperation. Let us try to provide an intuitive meaning for the blocking power.

Indeed, by the blocking power of coalition S we understand the amount $v^*(S)$ that this coalition would bring to N if the last were to be formed—its contribution

[4] Sudhölter (1997).

to the grand coalition. The value $v^*(S)$ is the difference between the amounts that can be received by the compliment $N\backslash S$ when it cooperates with S (in this case, the grand coalition forms) and does not cooperate with S. In fact, if S leaves the grand coalition, then the complement coalition gets $v(N\backslash S)$. However, acting together they can receive $v(N)$. Thus, the difference between $v(N)$ and $v(N\backslash S)$ is a subject which should be taken into account in a solution of a game. In our opinion, the blocking power can be judged as a measure of necessity of S for N—how much S contributes to N. So, each coalition S is estimated by N in this spirit.

In order to define the *SM*-nucleolus, we consider the sum-excess of a coalition $S \subseteq N$ at each $x \in \mathbb{R}^N$ as follows:

$$\overline{e}(x, v, S) = e(x, v, S) + e(x, v^*, S). \tag{3.2}$$

Definition 3.2 The simplified modified nucleolus, or shortly—the *SM*-nucleolus, of a game (N, v) is the set

$$\mu(v) = \big\{x \in X^0(v) : \theta\big(\overline{e}(x, v, S)_{S\subseteq N}\big) \preceq_{\text{lex}} \theta\big(\overline{e}(y, v, S)_{S\subseteq N}\big) \\ \text{for all } y \in X^0(v)\big\},$$

where $\theta(\overline{e}(x, v, S)_{S\subseteq N})$ is a vector whose components are the sum-excesses arranged in non-increasing order.

Formula (3.2) contains two summands—the excess $e(x, v, S)$ w.r.t. v and the excess $e(x, v^*, S)$ w.r.t. v^*. It follows from Definition 3.2 that both the constructive and the blocking powers are taken into account equally.

Remark 3.1 If x is a preimputation of (N, v), the following equality holds

$$\begin{aligned} e(x, v^*, S) &= v^*(S) - x(S) \\ &= v(N) - v(N\backslash S) - x(N) + x(N\backslash S) \\ &\overset{\text{by PO of } x}{=} x(N\backslash S) - v(N\backslash S) = -e(x, v, N\backslash S). \end{aligned}$$

Let (N, v) be a game. At each $x \in \mathbb{R}^N$ let us denote

$$\overline{\overline{e}}(x, v, S) = e(x, v, S) - e(x, v, N\backslash S) \tag{3.3}$$

for all $S \subseteq N$. Now consider a set

$$\big\{x \in X^0(v) : \theta\big(\overline{\overline{e}}(x, v, S)_{S\subseteq N}\big) \preceq_{\text{lex}} \theta\big(\overline{\overline{e}}(y, v, S)_{S\subseteq N}\big) \\ \text{for all } y \in X^0(v)\big\}. \tag{3.4}$$

Proposition 3.1 *Let (N, v) be a TU-game. The set defined by Formula (3.4) coincides with the SM-nucleolus of (N, v) for $x \in X^0(v)$.*

Proof By Remark 3.1, the sum-excesses $e(x, v, S) + e(x, v^*, S)$ in Definition 3.2 can be replaced by the differences $e(x, v, S) - e(x, v, N\backslash S)$ for every $S \subseteq N$ at $x \in X^0(v)$, i.e., the excesses $\overline{e}(x, v, S)$ and $\overline{\overline{e}}(x, v, S)$ coincide for $x \in X^0(v)$ and all coalitions $S \subseteq N$. Hence, $\theta(\overline{e}(x, v, S)_{S\subseteq N}) = \theta(\overline{\overline{e}}(x, v, S)_{S\subseteq N})$ holds for $x \in X^0(v)$. This completes the proof. □

In contrast to the modiclus, where all pairs of coalitions $(S, T) \in 2^N \times 2^N$ are considered, the set addressed in Proposition 3.1 takes into account only pairs $(S, N\backslash S) \in 2^N$—a coalition and its complement.

Proposition 3.1 provides a new intuitive meaning of the *SM*-nucleolus. According to it, the *SM*-nucleolus approach proposes as a solution of a game a vector that minimizes the discrepancy between the satisfactions of any pairs of complementary sets of the whole society. Coalitions S and $N\backslash S$ are compared with respect to x.

Let us recall the definitions of a symmetric game and a constant-sum game.

Definition 3.3 The game (N, v) is a symmetric game if its characteristic function has the form

$$v(S) = c_s \quad \text{where } s = |S| \text{ for all } S \subset N.$$

Definition 3.4 The game (N, v) is a constant-sum game if

$$v(S) + v(N\backslash S) = v(N) \quad \text{for all } S \subset N.$$

The following proposition is useful.

Proposition 3.2 [5] *Let (N, v) be a TU-game. The SM-nucleolus of (N, v) coincides with the prenucleolus of the constant-sum game (N, w), where*

$$w(S) = \frac{1}{2}v(S) + \frac{1}{2}v^*(S) = \frac{1}{2}\big(v(S) + v(N) - v(N\backslash S)\big).$$

Proof At first, we shall prove that the *SM*-nucleolus of (N, v) coincides with the prenucleolus of (N, w). Then, it will be shown that (N, w) is a constant-sum game.

Consider the game (N, w), where $w(S) = \frac{1}{2}v(S) + \frac{1}{2}v^*(S)$. The excess $e(x, w, S)$ of a coalition S at x is

$$\begin{aligned} e(x, w, S) = w(S) - x(S) &= \frac{1}{2}v(S) + \frac{1}{2}v^*(S) - x(S) \\ &= \frac{1}{2}\big(v(S) - x(S)\big) + \frac{1}{2}\big(v^*(S) - x(S)\big) \\ &= \frac{1}{2}\big(e(x, v, S) + e(x, v^*, S)\big) = \frac{1}{2}\,\overline{e}(x, v, S). \end{aligned}$$

Thereby, $e(x, w, S) = \frac{1}{2}\,\overline{e}(x, v, S)$ for all $S \subset N$ and $x \in X^0(v)$. Thus, the *SM*-nucleolus of (N, v) coincides with the prenucleolus of (N, w).

[5] This property of the *SM*-nucleolus was noticed by my student N. Smirnova.

Note that

$$w(S) + w(N\backslash S) = \frac{1}{2}\big(v(S) + v(N) - v(N\backslash S)\big) + \frac{1}{2}\big(v(N\backslash S) + v(N) - v(S)\big) = v(N).$$

By equality $v(N) = w(N)$, we have

$$w(S) + w(N\backslash S) = w(N) \quad \text{for all } S \subset N.$$

Hence, (N, w) is a constant-sum game. □

Proposition 3.2 states a direct relationship between the *SM*-nucleolus of a game (N, v) and the prenucleolus of a corresponding constant-sum game (N, w).

Remark 3.2 Let (N, v) be a constant-sum game, then $\mu(v) = \nu(v)$.

Proof Since (N, v) is a constant-sum game, we see that $v^*(S) = v(S)$ for all $S \subseteq N$. Then, for all $S \subseteq N$ $w(S) = v(S)$. By Proposition 3.2, we have $\mu(v) = \nu(v)$. □

The *SM*-nucleolus has many properties in common with the prenucleolus: it satisfies AN, COV, PO, SIVA, and NE. These properties follow from the assertions (i)–(iii) of the Schmeidler results and Proposition 3.2. So, the *SM*-nucleolus is a singleton.

Definition 3.5 The set $G \subseteq G^N$ is closed under duality, if $(N, v^*) \in G$ for every $(N, v) \in G$. A solution f on a set $G \subseteq G^N$ closed under duality is self dual (SD), if $f(N, v) = f(N, v^*)$ for every (N, v).

It follows from Definition 3.2 that the *SM*-nucleolus satisfies self-duality, i.e., $\mu(v) = \mu(v^*)$ holds. Observe that the Shapley value also satisfies self-duality.

It is important to consider the following result.

Proposition 3.3 *For three person games* (N, v), *the* 0*-normalization of the corresponding constant-sum game* (N, w) *is symmetric.*

Proof Let (N, w) be a constant-sum game defined by Proposition 3.2. The 0-normalization of (N, w) denote by (N, w_0). It can be seen that for any zero-normalized constant-sum three-person game, $w_0(\{i, j\}) = w(N)$ for any pairs of $\{i, j\}$. Hence, the game (N, w_0) is symmetric. □

It is clear that the *SM*-nucleolus fails individual rationality on G^N. However, we can show that the *SM*-nucleolus is *reasonable*.

Let (N, v) be a game and $i \in N$. We denote

$$b^i_{\max}(N, v) = \max_{S \subseteq N\backslash\{i\}} \big(v\big(S \cup \{i\}\big) - v(S)\big),$$

$$b^i_{\min}(N, v) = \min_{S \subseteq N\backslash\{i\}} \big(v\big(S \cup \{i\}\big) - v(S)\big).$$

The amounts $b^i_{\max}(N, v)$ and $b^i_{\min}(N, v)$ are, respectively, the maximal and minimal incremental contributions of player i to a coalition with respect to (N, v).

Definition 3.6 A solution f on G^N is

- Reasonable from above (REAB) if

$$\big((N, v) \in G^N \text{ and } x \in f(N, v)\big) \quad \Rightarrow \quad x_i \le b^i_{\max}(N, v) \text{ for all } i \in N;$$

- Reasonable from below (REBE) if

$$\big((N, v) \in G^N \text{ and } x \in f(N, v)\big) \quad \Rightarrow \quad x_i \ge b^i_{\min}(N, v) \text{ for all } i \in N;$$

- Reasonable from both sides (RE) if it satisfies REAB and REBE.

Now we are going to prove that the *SM*-nucleolus is reasonable from both sides. First, we need the following result.

Lemma 1 *Let (N, v) be a game and (N, v^*) is its dual. Then*

$$b^i_{\max}(N, v) = b^i_{\max}(N, v^*) \quad \textit{and} \quad b^i_{\min}(N, v) = b^i_{\min}(N, v^*) \quad \textit{for all } i \in N.$$

Proof For every $i \in N$ and every $S \subseteq N \backslash \{i\}$, we have

$$v^*\big(S \cup \{i\}\big) - v^*(S) = v(N \backslash S) - v\big((N \backslash S) \backslash \{i\}\big).$$

The proof is complete. □

Proposition 3.4 *The SM-nucleolus satisfies RE on every set of games.*

Proof Let (N, v) be a game and (N, w) is a corresponding constant-sum game with $w = \frac{1}{2}(v + v^*)$. Let $\mu(v)$ be the *SM*-nucleolus. By Proposition 3.2, $\mu(v) = \nu(w)$. Using Lemma 1 and $w = \frac{1}{2}(v + v^*)$, we have $b^i_{\max}(N, w) = b^i_{\max}(N, v)$ and $b^i_{\min}(N, w) = b^i_{\min}(N, v)$ for all $i \in N$. Hence, reasonability of the prenucleolus and equality $\mu(v) = \nu(w)$ complete the proof. □

At the same time, the *SM*-nucleolus satisfies individual rationality (IR) for a remarkable class of 0-monotonic games.

Definition 3.7 A game (N, v) is said to be 0-monotonic if for all $S, T \subset N$ with $S \subset T$ we have

$$v(S) - \sum_{i \in S} v\big(\{i\}\big) \le v(T) - \sum_{i \in T} v\big(\{i\}\big).$$

Let G^N_0 be the set of all 0-monotonic games.

Proposition 3.5 *The SM-nucleolus satisfies IR on $G^N_0 \subset G^N$.*

Proof Let $(N, v) \in G_0^N$ and $x = \mu(N, v)$. It follows from Proposition 3.4 that $\mu(v)$ satisfies REBE, i.e., $x_i \geq b^i_{\min}(N, v)$. Note that $b^i_{\min}(N, v) = v(\{i\})$ for $i \in N$ and every 0-monotonic game (N, v). Thus, $x_i \geq v(\{i\})$. This completes the proof. □

Remark 3.3 Computational complexity of the *SM*-nucleolus is reduced as compared with that of the modiclus.

It can be seen from Definitions 3.1 and 3.2.

Remark 3.4 Computational complexity of the *SM*-nucleolus is comparable with that of the prenucleolus due to Proposition 3.2.

Therefore, the *SM*-nucleolus can be computed by each of the presently known algorithms for the calculation of the prenucleolus (Kopelowitz 1967; Sankaran 1991). In fact, any prenucleolus algorithm for TU-games can be adapted to calculate the *SM*-nucleolus. Our conjecture is that computational complexity of the *SM*-nucleolus can be reduced due to the following property

$$\overline{e}(x, v, S) = -\overline{e}(x, v, N\backslash S) \quad \text{for } x \in X^0(v) \text{ and } S \subset N.$$

By now, this problem has not been solved.

Remark 3.5 The *SM*-nucleolus can be found by two ways

$$\begin{aligned} \mu(v) &= \arg\min_{x \in X^0(v)} \max_{S \subset N} \overline{e}(x, v, S) \\ &= \arg\min_{x \in X^0(v)} \max_{S \subset N} \big(-\overline{e}(x, v, S)\big). \end{aligned}$$

By two ways we mean that there is no difference in what order the excess vectors are arranged.

In order to illustrate the introduced solution, let us look at the following example (see Sudhölter 1997).

Example 3.1 Consider the glove-market game with three players, one of them (player 1) possesses a unique right-hand glove whereas the other players (2 and 3) possess one single left-hand glove each. The worth of a coalition is the number of pairs of gloves the coalition has (i.e., one or zero). Thus, the characteristic function has the form

$$\begin{gathered} v\big(\{1\}\big) = v\big(\{2\}\big) = v\big(\{3\}\big) = v\big(\{23\}\big) = 0, \\ v\big(\{12\}\big) = v\big(\{13\}\big) = v(N) = 1. \end{gathered}$$

The different solutions of this game are as follows:

The core	$C(v) = (1, 0, 0)$
The nucleolus	$\nu(v) = (1, 0, 0)$
The Shapley value	$\varphi(v) = (\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$
The modiclus	$\psi(v) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$
The *SM*-nucleolus	$\mu(v) = (\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$

The nucleolus assigns one to player 1 and zero to the other players. On the other hand, players 2 and 3 together can prevent player 1 from getting 1 by forming a coalition against him. Therefore, together they have the same blocking power as player 1. The modiclus takes care of this fact and assigns $\frac{1}{2}$ to player 1 and $\frac{1}{4}$ to each of the other players.

The *SM*-nucleolus also takes into account the blocking power of a coalition but to a lesser extent. It assigns $\frac{2}{3}$ to player 1 and $\frac{1}{6}$ to players 2 and 3 each. Note that for this game the *SM*-nucleolus coincides with the Shapley value.

4 The *SM*-nucleolus and the Shapley value

In Example 3.1, the *SM*-nucleolus has happened to coincide with the Shapley value. At the same time, we have come across a rather interesting and unexpected result which indicates that the pointed above coincidence is not accidental.

Let us consider three-person TU-games. The following proposition is true.

Proposition 4.1 *For three person games, the SM-nucleolus and the Shapley value coincide.*

Proof Let (N, w) be a constant-sum game defined by Proposition 3.2 and (N, w_0) be its zero-normalization. By Proposition 3.3, (N, w_0) is symmetric. The prenucleolus of a symmetric game is the equal-split vector. Hence, $\mu_i(v) = \nu_i(w) = w(\{i\}) + \frac{1}{3}w_0(N)$ for all $i \in N$. On the other hand, since the Shapley value obeys linearity, self-duality, covariance, and symmetry, we have $\varphi_i(v) = \varphi_i(w) = w(\{i\}) + \varphi_i(w_0) = w(\{i\}) + \frac{1}{3}w_0(N)$ for all $i \in N$. This completes the proof. □

The previous result brings up the question whether the *SM*-nucleolus coincides with the Shapley value for a general case of n-person games. The positive answer to this question would tell us about new interpretation and new properties of the Shapley value, whereas the negative one would prove the existence of a new solution concept with properties similar to the Shapley value.

Remark 4.1 For a general cooperative TU-game (N, v) $(|N| \geq 4)$, the *SM*-nucleolus is not congruent to the Shapley value.

Example 5.2 in Sect. 5 verifies Remark 4.1.

5 Glove-market games

To illustrate some additional properties and to give the intuition of the introduced solution concept, let us look at two significant examples.

First, we consider a symmetric glove-market game with four players and then we go to a nonsymmetric version of this game. It is interesting to observe the changes, which are going on with different solutions when we transform one game into another.

Example 5.1 (Symmetric four-person glove-market game)[6] There are four players participating in a game: two of them (1 and 2) possess one right-hand glove each, and the others (3 and 4) possess one left-hand glove each. The worth of a coalition is the number of pairs of gloves the coalition gets.

This situation can be described by the following TU-game

$$\begin{gathered} v\big(\{i\}\big)=0 \quad \text{for all } i\in N,\\ v\big(\{1,2\}\big)=v\big(\{3,4\}\big)=0,\\ v\big(\{1,3\}\big)=v\big(\{1,4\}\big)=v\big(\{2,3\}\big)=v\big(\{2,4\}\big)=1,\\ v(S)=1 \quad \text{if } |S|=3,\\ v(N)=2. \end{gathered}$$

It is known that the nucleolus in this game is the point $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$.

In order to compute the *SM*-nucleolus, first let us construct the corresponding constant-sum game (N,w), where

$$w(S)=\frac{1}{2}\big(v(S)+v(N)-v(N\backslash S)\big),\quad S\subset N \quad \text{(see Proposition 3.2)}.$$

Thus,

$$\begin{gathered} w\big(\{i\}\big)=\frac{1}{2} \quad \text{for all } i\in N,\\ w(S)=1 \quad \text{if } |S|=2,\\ w(S)=\frac{3}{2} \quad \text{if } |S|=3,\\ w(N)=2. \end{gathered}$$

Clearly, (N,w) is a symmetric game. It implies (by anonymity)

$$\nu(w)=\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right).$$

According to Proposition 3.1, we have $\mu(v)$ equal to $\nu(w)$, hence,

$$\mu(v)=\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right).$$

[6] This example has been taken from Moulin (1988).

Table 1 Comparison of the solutions for two glove-market games

Symmetric four-person glove-market game	Nonsymmetric four-person glove-market game
$\nu(v)=(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$\nu(v')=(0,0,1,1)$
$\mu(v)=(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$\mu(v')=(\frac{36}{60},\frac{12}{60},\frac{36}{60},\frac{36}{60})$
$\varphi(v)=(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$\varphi(v')=(\frac{35}{60},\frac{15}{60},\frac{35}{60},\frac{35}{60})$

As a matter of fact, all players in the game are treated equally and all considered solution concepts propose the same payoff vector.

Example 5.2 (Nonsymmetric four-person glove-market game)[7] Let (N,v') be the game described in Example 5.1 except that player 1 is given one more right-hand glove (all together he possesses two right-hand gloves).

This leads to getting the maximal payoff 2 by coalition $\{1,3,4\}$. So, we get the following game (N,v'), where

$$v'(\{1,3,4\})=2,$$
$$v'(S)=v(S) \quad \text{for all } S\subseteq N,\ S\neq\{1,3,4\}.$$

In fact, it is quite natural to expect that this kind of change of game (N,v) would bring some profit to player 1.

It can be easily verified that the solution concepts considered throughout the paper assign the following payoffs to the players

$$\nu(v')=(0,0,1,1), \qquad \varphi(v')=\left(\frac{35}{60},\frac{15}{60},\frac{35}{60},\frac{35}{60}\right),$$
$$\mu(v')=\left(\frac{36}{60},\frac{12}{60},\frac{36}{60},\frac{36}{60}\right).$$

As a result, we give Table 1 and compare the pointed solutions.

Remark 5.1 Note that, when going from game (N,v) to game (N,v'), the core transforms from the interval to one point $(0,0,1,1)$ which is the nucleolus as well.

The nucleolus here completely punishes players 1 and 2, whereas the Shapley value and the *SM*-nucleolus punishes only player 2. Obviously, the vector $(0,0,1,1)$ is never accepted by player 1—an additive glove does not make him better off, moreover, it deprives him any positive payoff. Clearly, coalitional monotonicity fails.

There is an interesting detail about the structure of this game: since coalition $\{1,3,4\}$ gets the maximal value 2, one can propose to divide the value between players 1, 3, and 4, leaving player 2 without any positive payoff. However, such reasoning

[7] Moulin (1988).

can lead to the situation when player 1 becomes a dictator. So, one can see that existence of player 2 helps to avoid a dictatorship in this game. Hence, player 2 should not be given a value of 0, it would be unfair. This fact is taken into account by both the Shapley value and the *SM*-nucleolus. The difference is what payoff each of these solutions assigns to player 2. The *SM*-nucleolus gives him less than the Shapley value does.

6 The weighted voting games

The *SM*-nucleolus has an important application in modeling the power of players in voting games. There are some well-known power indices such as the Shapley–Shubik power index (Shapley and Shubik 1954) and the Banzhaf index (Banzhaf 1965).

In such games, a proposed bill or decision is either passed or rejected. In a voting body, the voting rule specifies which subsets of players are large enough to pass bills, and which are not. Those subsets that can pass bills without outside help are called *winning coalitions*, while those that cannot are called *losing coalitions*.

In such a case, we can take the worth of a winning coalition to be 1 and the worth of a losing coalition to be 0. The resulting game, in which all coalitions have a value of either 1 or 0, is called a *simple game*.

A simple game is completely specified once its winning coalitions are known, and it is traditional to require it to satisfy some reasonable conditions.

Definition 6.1 [8] A simple game is a pair $(N, \mathcal{W})$, where N is the set of players and $\mathcal{W}$ is the collection of winning coalitions, such that

- $\emptyset \notin \mathcal{W}$ (the empty set is a losing coalition);
- $N \in \mathcal{W}$ (the grand coalition is winning);
- $S \in \mathcal{W}$ and $S \subseteq T$ imply $T \in \mathcal{W}$ (if S is a winning coalition, so is any coalition that contains S).

One common type of a simple game is a weighted voting game, which is usually represented by $[q; \omega_1, \dots, \omega_n]$. Such games are defined by a characteristic function of the form

$$v(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} \omega_i > q, \\ 0 & \text{if } \sum_{i \in S} \omega_i \leq q, \end{cases}$$

for some non-negative numbers ω_i, called the *weights*, and some positive number q, called the *quota*. If $q = \frac{1}{2} \sum_{i \in N} \omega_i$, we deal with a *weighted majority game*.

Example 6.1 As an example, consider a game with players 1, 2, 3, and 4, having 10, 20, 30, and 40 shares of stock, respectively, in a corporation. A decision requires approval by a majority (more than 50%) of the shares. This is a weighted majority game with weights $\omega_1 = 10,\ \omega_2 = 20,\ \omega_3 = 30,\ \omega_4 = 40$ and with quota $q = 50$.

[8] Straffin (1993).

Table 2 The worths of the coalitions in the game [6; 5, 2, 2, 2]

S	$v(S)$	$v^*(S)$	$w(S)$
1	0	1	0.5
2	0	0	0
3	0	0	0
4	0	0	0
1, 2	1	1	1
1, 3	1	1	1
1, 4	1	1	1
2, 3	0	0	0
2, 4	0	0	0
3, 4	0	0	0
1, 2, 3	1	1	1
1, 2, 4	1	1	1
1, 3, 4	1	1	1
2, 3, 4	0	1	0.5
1, 2, 3, 4	1	1	1
The nucleolus		$\nu(v) = (1, 0, 0, 0)$	
The *SM*-nucleolus		$\mu(v) = (\frac{7}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10})$	
The Shapley–Shubik power index		$\varphi(v) = (\frac{3}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12})$	

The Shapley–Shubik power index in this game is $\varphi(v) = (\frac{1}{12}, \frac{3}{12}, \frac{3}{12}, \frac{5}{12})$. Note that the power of players 2 is the same as the one of player 3, although player 3 has more shares. In fact, players 2 and 3 play symmetric roles in the game.

Let us find now the *SM*-nucleolus of this game. The winning coalitions are $\{2, 4\}$, $\{3, 4\}$, $\{1, 2, 3\}$, and all supersets (sets containing one of these). The *SM*-nucleolus is the following vector $\mu(v) = (0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Example 6.2 Consider two versions of a voting game with four players in which player 1 has 5 shares and players 2 to 4 have 2 shares each. The quota in the first case is 6, and in the second case is 5.

So, there are the following games [6; 5, 2, 2, 2] and [5; 5, 2, 2, 2] under consideration and we propose to look at the behavior of different solutions in these games.

The list of all coalitions and their worths are presented in Tables 2 and 3.

Case 1. The game [6; 5, 2, 2, 2].

Note that the core in this game consists of the unique point $(1, 0, 0, 0)$ and, clearly, the nucleolus is the same point. Comparing the *SM*-nucleolus and the Shapley–Shubik power index one can see that the *SM*-nucleolus assigns to the weak players payoff $\frac{1}{10}$ which is more than the Shapley–Shubik power index does. So, the new solution treats the weak players higher likely because the *SM*-nucleolus takes into account the blocking power of coalition $\{2, 3, 4\}$.

Case 2. The game [5; 5, 2, 2, 2].

This case differs from the previous one by the status of coalition $\{2, 3, 4\}$. Here coalition $\{2, 3, 4\}$ belongs to the set of winning coalitions of the game. This is a

Table 3 The worths of the coalitions in the game [5; 5, 2, 2, 2]

S	$v(S)$	$v^*(S)$	$w(S)$
1	0	0	0
2	0	0	0
3	0	0	0
4	0	0	0
1, 2	1	1	1
1, 3	1	1	1
1, 4	1	1	1
2, 3	0	0	0
2, 4	0	0	0
3, 4	0	0	0
1, 2, 3	1	1	1
1, 2, 4	1	1	1
1, 3, 4	1	1	1
2, 3, 4	1	1	1
1, 2, 3, 4	1	1	1
The nucleolus			$\nu(v) = (\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$
The *SM*-nucleolus			$\mu(v) = (\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$
The Shapley–Shubik power index			$\varphi(v) = (\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$

constant-sum simple game, and it is known that the core in these games is empty. By Proposition 3.2, we get $\mu(v) = \nu(v) = (\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$. The difference between the *SM*-nucleolus payoffs and the Shapley–Shubik payoffs are interpreted as in the previous case: the weak players get more in the *SM*-nucleolus than in the Shapley–Shubik power index.

However, the fact that coalition $\{2, 3, 4\}$ is winning plays a positive role increasing the power of the weak players in two times; this cannot be said about player 1, whose power decreases.

In Case 2, coalition $\{2, 3, 4\}$ becomes winning which increases the power of the weak players twice compared with Case 1 ($\frac{1}{5}$ in Case 2 versus $\frac{1}{10}$ in Case 1).

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