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Constraints on $\pi\pi$ Partial Waves from Positivity and Analyticity.

F. J. YNDURAIN
CERN - Geneva

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Summary. — We consider the constraints on the partial waves for $\pi^0\pi^0$ or $\pi\pi$ in the isospin-zero channel, that follow from analyticity and positivity. We show that, if these two requirements are met, then a set of inequalities on the partial waves with $l \geqslant 2$ and energy below threshold must be satisfied. The converse is also proved showing that, if these constraints are fulfilled, then an analytic scattering amplitude with positive absorptive part can be constructed. The practical importance of these conditions is discussed showing that, for example, they imply the inequalities (pion mass $= 1$) $a_{l+2} < (1/16)\big((l+2)(l+1)/(l+\frac{5}{2})(l+\frac{3}{2})\big)a_l$; here a_l are the scattering lengths, $a_l = \lim f_l(s)/q^{2l}$. Furthermore, these inequalities (which had essentially been also found by Martin) are saturated very accurately when $l \geqslant 4$. The relevance of our analysis for models of $\pi\pi$ scattering is discussed.

1. – Introduction.

In recent papers (1,2), MARTIN and collaborators have shown that, by using positivity and dispersion relations, one can obtain a set of conditions on the partial waves of the $\pi\pi$ scattering amplitude which ensure full crossing symmetry. As remarked by MARTIN himself, these constraints are incomplete in the sense that extra consistency conditions, following again from positivity and analyticity, must also be imposed for higher waves ($l \geqslant 2$). It is precisely these conditions that we want to investigate in the present article.

To be definite, let $T(s, t)$ be the scattering amplitude for $\pi^0\pi^0$ scattering

(1) A. MARTIN: CERN preprint TH. 1008 (1969).

(2) G. AUBERSON, O. BRANDER, G. MAHOUX and A. MARTIN: CERN preprint TH. 1032 (1969), to appear in the *Proceedings of the Argonne-Purdue Conference on $\pi\pi$ Interactions*.

or $\pi\pi$ scattering with isospin zero in the s-channel (*). Then it is known that from *e.g.* axiomatic field theory, one can show that a Froissart-Gribov representation

$$(1) \qquad f_l(s) = \int_{z_0(s)}^{\infty} \mathrm{d}z\, A_t(s, z) Q_l(z) \,; \qquad z_0(s) = \frac{4+s}{4-s}$$

holds for the even waves, $l \geqslant 2$, whenever s is in the range $0 \leqslant s \leqslant 4$ (we are consistently taking units such that the pion mass $= 1$). Furthermore, from unitarity, it follows that $A_t(s, z)$, which is the absorptive part of the scattering amplitude in the t-channel, is positive. Now the question we want to study is the derivation of a set of constraints of the f's such that, if they are satisfied then there exists a unique scattering amplitude T solving (1) and with the properties of positivity and analyticity, and conversely. In this context we are able to solve completely the problem, deriving a set of necessary and sufficient conditions. Each of these conditions involves an infinite number of waves; however, by suitable majorizations, we are able to reformulate them to be valid when only a finite number of waves is known. As a particular instance of these relations we study them at threshold, obtaining, in particular, that the scattering lengths, defined by

$$(2) \qquad a_l = \lim_{q \to 0} f_l(s) q^{-2l} \,; \qquad q = \frac{\sqrt{|s-4|}}{2} \,,$$

satisfy the bounds

$$(3) \qquad a_{l+2} \leqslant \frac{1}{16} \frac{(l+2)(l+1)}{(l+\frac{5}{2})(l+\frac{3}{2})} a_l$$

and that, when $l \geqslant 4$, these bounds are saturated to within 6%. For, *e.g.*, the lowest waves, this gives $a_4 \leqslant 0.05 \times a_2$, $a_6 < 0.003 \times a_2$, ..., $a_6 \leqslant 0.06 a_4$, ..., $|a_6 - 0.06 a_4| \leqslant 0.06 \times |a_4 - 0.05 a_2|$, ... etc.

The relevance of these results is then discussed for fashionable parametrizations of the $\pi\pi$ scattering amplitude. Finally, a few words are devoted to the problem (1) for other isospin states or above threshold where it reads

$$(4a) \qquad \operatorname{Re} f_l(s) = \int \mathrm{d}z\, \operatorname{Re} A_t(s, z) Q_l(z) \,,$$

$$(4b) \qquad \operatorname{Im} f_l(s) = \int \mathrm{d}z\, \varrho(s, z) Q_l(z) \,,$$

(*) Or, more generally, any combination of the form $(1 + \lambda/3)\, T_s^{(0)}(s, t) + (2\lambda/3) \cdot T_s^{(2)}(s, t)$ with $\lambda \geqslant -1$.

for l even $\geqslant l_0$; ϱ is the double spectral function and we are assuming now validity of the Mandelstam representation with l_0 subtractions. Of course, the problem is not so definite now as $\mathrm{Re}\,A_t$ will not be positive any more and also there is no reason to suppose ϱ to be so (in fact, there are strong indications ([3]) that ϱ actually oscillates).

Section **2** is devoted to presenting some useful mathematical results which are then used to derive, in Sect. **3**, the announced conditions which are discussed there. The application to scattering lengths and then to $\pi\pi$ parametrizations is given in Sect. **4**, concluding then the article with some comments about the conditions obtained, the possible ways to improve them, and the fact that our constraints are obtained for unphysical energies.

2. – Some mathematical results.

2·1. *The general problem of moments.* – The techniques we shall use are generalizations of those of the so-called Hausdorff moment problem. For detailed information on the subject we refer to the treatises of AKHIEZER and KREIN ([4]) and of SHOHAT and TAMARSKIN ([5]). Some of the relevant results shall be listed here for convenience.

Theorem 1 [ref. ([4])]. Let $\alpha_n(x)$ be a decreasing sequence of positive bounded functions. Then, a necessary and sufficient condition for a solution to the moment problem

$$\eta_n = \int_a^b \mathrm{d}x\, \Phi(x)\, \alpha_n(x)\,, \qquad b > a \geqslant 0\,,$$

with Φ positive to exist is that whenever a finite combination

$$p(x) = \sum_{n=0}^{n_0} c_n \alpha_n(x)$$

is positive in the interval (a, b), then also the functional

$$\eta(p) \equiv \sum_{n=0}^{n_0} c_n \eta_n$$

is positive.

([3]) A. W. MARTIN: *Phys. Lett.*, **28** B, 679 (1969); F. J. YNDURAIN: *Phys. Lett.*, **29** B, 125 (1969); G. GOEBEL: *Proc. Int. Conf. High-Energy Phenomena*, CERN 61-22 (1961), p. 353.

([4]) N. I. AKHIEZER and M. KREIN: *Some questions in the theory of moments*, Amer. Math. Soc., translations (1962).

([5]) J. A. SHOHAT and J. D. TAMARSKIN: *The problem of moments*, Amer. Math. Soc., monographs (1943).

Remarks. The necessity is obvious, as one may write

$$\eta(p) = \int_a^b \mathrm{d}x\,\Phi(x)\,p(x)\,.$$

Sufficiency is proved by using Riesz's theorem ([4-6]) which allows us to extend $\eta(p)$, as defined on the manifold of p's to a suitable larger space containing that manifold. Exact descriptions of what is meant by « suitable » shall be given in each explicit case.

Theorem 2. A necessary condition for a solution to the (modified ([7])) Hausdorff moment problem

$$(5)\qquad \mu_l = \int_{z_0}^{\infty} \mathrm{d}z\,A(z)z^{-l-1}\,; \qquad l = \text{even} \geqslant l_0,\; A \geqslant 0,\; z_0 > 0$$

to exist is that, for all $l \geqslant l_0$,

$$(6)\qquad \begin{cases} \delta^0\mu_l \equiv \mu_l \geqslant 0\,, \\ \delta^1\mu_l \equiv z_0^{-2}\mu_l - \mu_{l+2} \geqslant 0\,, \\ \dots\dots\dots\dots\dots \\ \delta^k\mu_l \equiv \sum_{j=0}^{k} \binom{k}{j} (-1)^j \mu_{l+2j} z_0^{2(j-k)} \geqslant 0\,. \\ \dots\dots\dots\dots\dots \end{cases}$$

Conversely, if (6) is satisfied, then there exists a positive A such that it solves (5) and, furthermore, the integral

$$\int_{z_0}^{\infty} \mathrm{d}z\,A(z)\varphi(z)$$

exists for all functions (*) $\varphi(z)$ which, in the interval (z_0, ∞), admit the bound

$$(7)\qquad |\varphi(z)| \leqslant \text{const} \times z^{-l_0-1}\,.$$

([6]) M. E. MUNROE: *Introduction to the theory of measure and integration* (Cambridge, Mass., 1953).

([7]) The usual Hausdorff moment problem is related to the equation $\mu_n = \int_0^1 \mathrm{d}u\,\Psi(u)\,u^n$, $n = 0, 1, \dots$. It can be reduced to (5), and conversely, by a simple change of variables.

(*) This does not specify completely the domain of definition of the integral, as one should specify that φ's discontinuities should not coincide with A's. We leave aside these questions (see however, ref. ([4-6])) which are irrelevant for our problems here.

To prove this we need the following result:

Lemma 1. If a « polynomial »

$$p(z^{-2}) = \sum_{\nu=0}^{n_0} \gamma_\nu \, z^{-l_0-2\nu-1}$$

is positive in (z_0, ∞), then it can be approximated as

(8) $$p(z^{-2}) = \sum_{\substack{n\text{-even}\\ n\geqslant l_0}} \sum_k C_{n,k} z^{-n-1}(z_0^{-2} - z^{-2})^k + \varepsilon z^{-l_0-1},$$

where the constants $C_{n,k}$ are positive. This lemma is proved by reducing it, through a change of variables, to the standard result for polynomials positive in $(0, 1)$ (ref. [5], p. 8 and ref. [8]).

The proof of the theorem is now easy: conditions (6) are nothing but

$$\delta^k \mu_l = \int_{z_0}^{\infty} dz \, A(z) z^{-l-1}(z_0^{-2} - z^{-2})^k \geqslant 0 \; ,$$

which of course must be satisfied as $z^{-l-1}(z_0^{-2} - z^{-2})^k$ is always positive. Conversely, if (6) are satisfied and p is positive then it follows from (8) that

$$\mu(p) \equiv \sum_\nu \gamma_\nu \mu_{l_0-2\nu} = \sum_{n,k} C_{n,k} \delta^k \mu_n + \varepsilon \mu_{l_0} \, ,$$

and this is positive if ε is small enough. Hence, the functional $\mu(p)$ is positive and we are in the situation of Theorem 1. *q.e.d.*

Theorem 3. If a solution exists to (5) then it is unique, in the sense that, if A_1, A_2 are two solutions of (5), then $\int dz [A_1 - A_2] \varphi = 0$ for all φ admitting the bound (7).

Proof. In ref. [5], pp. 10, 11, it is shown that a necessary and sufficient condition for the solution to (5) to be unique is that, if $\varphi(z)$ is a continuous function admitting the bound (7), then

(9) $$\begin{cases} \inf\limits_{p(z^{-2}),\, p\geqslant\varphi} \mu(p) = \sup\limits_{p(z^{-2}),\, p\leqslant\varphi} \mu(p) \, , \\ \mu(\varphi) \equiv \int dz \, A(z) \varphi(z) \, . \end{cases}$$

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[8] T. H. Hildebrand and I. J. Schoenberg: *Ann. Math.*, **34**, 317 (1933).

As is well known, any continuous function $\psi(u)$ in the interval $(0, 1)$ can be approximated by two polynomials $p_i^\varepsilon(u)$

$$p_1^\varepsilon(u) < \psi(u) < p_2^\varepsilon(u) , \qquad |p_2^\varepsilon(u) - p_1^\varepsilon(u)| < \varepsilon \qquad \text{on } (0, 1);$$

hence, by change of variables, we can find two « polynomials » $p_1^\varepsilon(z^{-2})$, $p_2^\varepsilon(z^{-2})$ with

$$p_1^\varepsilon(z^{-2}) < \varphi(z) < p_2^\varepsilon(z^{-2}) , \qquad |p_2^\varepsilon(z^{-2}) - p_1^\varepsilon(z^{-2})| < \varepsilon z^{-l_0-1} ,$$

on (z_0, ∞). Therefore, we get

$$|\inf \mu(p) - \sup \mu(p)| \leqslant \varepsilon \mu_{l_0} ,$$

and letting $\varepsilon \to 0$ the desired result, (9), follows.

2·2. *The « Froissart-Gribov » moment problem.* – For convenience we shall rewrite (1) dropping the s dependence,

$$f_l = \int_{z_0}^{\infty} dz A_t(z) Q_l(z) ; \qquad z_0 > 1, \; l \geqslant l_0; \; A_t \geqslant 0 . \tag{10}$$

Clearly, the problem associated with (10), that we shall call Froissart-Gribov moment problem, is a special case of (4) and, therefore, the result applies:

Corollary 1. A necessary and sufficient condition for (10) to hold is that whenever $p_Q(z) \geqslant 0$ on (z_0, ∞),

$$p_Q(z) \equiv \sum_{\substack{n=l_0 \\ n=\text{even}}}^{n_0} c_n Q_n(z) ,$$

then

$$f_Q(p) \equiv \sum c_n f_n = \int dz A_t(z) p_Q(z) \geqslant 0 .$$

The generalization of Theorems 2 and 3 is less simple; nevertheless, we shall be able to prove the following results:

Corollary 2. Form the quantities μ_l,

$$\mu_l \equiv \frac{2^{l+1}}{l! \sqrt{\pi}} \sum_{\nu=0}^{\infty} (-1)^\nu \frac{\Gamma(l + \nu + \frac{3}{2})}{\Gamma(\nu + 1)} f_{l+2\nu} , \qquad l \geqslant l_0 . \tag{11}$$

Then a solution to (10) exists if, and only if, these μ_l exist and satisfy the inequalities (6). The functional

$$f(\varphi) \equiv \int_{z_0}^{\infty} \mathrm{d}z\, A_t(z)\varphi(z)$$

is then defined for all φ's which, in the interval (z_0, ∞) admit the estimate (7) or, equivalently (as $z_0 > 1$), the estimate

$$|\varphi(z)| \leqslant \text{const} \times Q_{l_0}(z) .$$

Proof. A) Necessity. We have the representation [9]

$$z^{-l-1} = \frac{2^{l+1}}{l!\sqrt{\pi}} \sum_{\nu=0}^{\infty} (-1)^\nu \frac{\Gamma(l+\nu+\frac{3}{2})}{\Gamma(\nu+1)} Q_{l+2\nu}(z) , \tag{12}$$

which is convergent because $z_0 > 1$.

If A_t exists, then, because of Corollary 1

$$f(\varphi) \geqslant 0 \qquad \text{if} \qquad \varphi \geqslant 0 .$$

Since [10]

$$Q_l(z)/Q_l(z_0) - Q_{l+n}(z)/Q_{l+n}(z_0) \geqslant 0 ,$$

it then follows that

$$f_{l+n} \leqslant \frac{Q_{l+n}(z_0)}{Q_l(z_0)} f_l . \tag{13}$$

But, for large n,

$$Q_{l+n}(z_0)/Q_l(z_0) = O(1/z_0^n) . \tag{14}$$

Hence, the expression (11) for the μ's is convergent. It is then sufficient to plug in the representation (12) to get that

$$\delta^k \mu_l = f(b_{l,k}) ,$$
$$b_{l,k}(z) = z^{-l-1}(z_0^{-2} - z^{-2})^k .$$

As $b_{l,k} \geqslant 0$ on (z_0, ∞), Corollary 1 tells us that $f(b_{l,k})$ must be so and (6) is satisfied.

B) Sufficiency. If (6) is satisfied, Theorem 2 tell us that there exists an A_t solving eq. (5) with the desired properties. To show that this A_t also

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([9]) W. MAGNUS, F. OBERHETTINGER and R. P. SONI: *Formulas and Theorems for the Special Functions of Mathematical Physics* (Berlin, 1966).

([10]) A. MARTIN: *Nuovo Cimento*, **61** A, 56 (1969), Appendix.

solves (10), we insert the expansion ([9]), inverse to (12),

$$Q_l(z)=\frac{\sqrt{\pi}}{2^{l+1}}\frac{l!\,z^{-l-1}}{\Gamma(l/2+1)\Gamma(l/2+\frac{1}{2})}\sum_{\mu=0}^{\infty}\frac{\Gamma(l/2+\mu+1)\Gamma(l/2+\mu+\frac{1}{2})}{\mu!\,\Gamma(l+\mu+\frac{3}{2})}z^{-2\mu}, \tag{15}$$

to get that the f_l can be recovered as

$$f_l=\frac{\sqrt{\pi}}{2^{l+1}}\frac{l!}{\Gamma(l/2+1)\Gamma(l/2+\frac{1}{2})}\sum_{\mu=0}^{\infty}\frac{\Gamma(l/2+\mu+1)\Gamma(l/2+\mu+\frac{1}{2})}{\mu!\,\Gamma(l+\mu+\frac{3}{2})}\mu_{l+2\mu}, \tag{16}$$

where this expression is again convergent because, since $z_0^{-2n}-z_0^{-2n}\geqslant 0$, we have that

$$\mu_{l+2n}\leqslant O(1/z_0^{2n})\mu_l .$$

Finally, we have:

Corollary 3. If a solution to (10) exists then it is unique. The proof is a simple consequence of Theorem 3 and the fact that we have just shown that the problems (10), (5) are equivalent.

2·3. *The associated Hausdorff problem.* – Given a Froissart-Gribov problem, (10), we construct its *associated Hausdorff problem*,

$$f_l=\int_{u_0}^{\infty}\mathrm{d}z B(z)z^{-l-1}\,; \qquad l\geqslant l_0,\; u_0=z_0+\sqrt{z_0^2-1}>1\,. \tag{17}$$

Of course, $B)$ will be different from $A)$. We have

Theorem 4. The associated Hausdroff moment problem has a solution if and only if

$$\left\{\begin{array}{l}\delta^0 f_l\equiv f_l>0\,,\\ \delta^1 f_l\equiv u_0^{-2}f_l-f_{l+2}\geqslant 0\,,\\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\\ \delta^k f_l\equiv\displaystyle\sum_{j=0}^{k}\binom{k}{j}(-1)^j u_0^{2(j-k)}f_{l+2j}\geqslant 0\,,\\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\end{array}\right. \tag{18}$$

In particular, these conditions are satisfied if the original Froissart-Gribov problem had a solution ([11]), so that (18) is a necessary (but not sufficient!) condition for (10) to be true ([*]).

([11]) A result essentially equivalent to this has been obtained independently by A. K. COMMON (CERN preprint, TH. 1034 (1969)). We thank Dr. COMMON for communication of his results prior to publication.

([*]) It is, however, interesting to remark that eqs. (18) are good as approximate constraints, in the sense that they are asymptotically sufficient (for large z_0, l).

Proof. From the representation

$$Q_l(z) = \int_0^\infty \mathrm{d}t[u(z,t)]^{-l-1}\,, \qquad u(z,t) = z + (z^2-1)^{\frac{1}{2}}\cosh t\,,$$

and the fact that, when $z \geqslant z_0$, $u(z,t) \geqslant u_0$, it follows that the functions

$$\beta_{l,k}(z) \equiv \sum_{j=0}^{k}\binom{k}{j}(-1)^j u_0^{2(j-k)} Q_{l+2j}(z) = \int_0^\infty \mathrm{d}t[\mu(z,t)]^{-l-1}\{u_0^{-2} - [\mu(z,t)]^{-2}\}^k$$

are positive. Furthermore, they clearly satisfy the bound (7), so that, by Corollary 1, we should have

$$f(\beta_{l,k}) = \delta^k f_l \geqslant 0\,,$$

and the second part of the Theorem is proved. The first part is obvious, as (Theorem 2) eqs. (18) are nothing but the necessary and sufficient conditions for (17) to have a solution.

3. – Conditions on the partial waves from positivity. Analyticity.

3'1. *Positivity.* – From the preceding analysis is it quite clear that the results of Corollaries 1 to 3 will apply to (1) with $l_0 = 2$, when $0 < s \leqslant 4$, providing the desired set of necessary and suffcient constraints, namely, eqs. (6) with the μ_l's given by (11). To what extent do eqs. (6) imply positivity of $T(s,t)$? (The converse, of course, is well known; cf., *e.g.*, ref. [1]). Let us use the fact, to be proved below, Subsect. **3'3**, that, if (6) is satisfied, then the scattering amplitude

$$(19) \qquad \begin{cases} T(s,t) = f_0(s) + \sum\limits_{l=\text{even}\geqslant 2}^{\infty}(2l+1)f_l(s)P_l(s)P_l(\cos\theta_s)\,; \\ \cos\theta_s \;= 1 + t/2(s-4)\,, \end{cases}$$

is defined and can be extended to be analytic in the cut $\cos\theta_s$-plane. If we then make the *assumption* [12] of crossing symmetry, we will get that the partial-

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[12] That crossing symmetry is a separate assumption is obvious as there is no guarantee that the conditions of ref. [1] be fulfilled; thus, *e.g.*, our constraints say nothing about the S-wave.

wave expansion of A_t in the t-channel,

$$(20)\qquad \begin{cases} A_t(s,t) = \sum\limits_{l=\text{even}\geqslant 0} (2l+1)\,\text{Im}\, f_l(t) P_l(\cos\theta_t)\,, \\ \cos\theta_t \;\; = 1 + s/2(t-4)\,, \end{cases}$$

is convergent for $\cos\theta_t$ in the Martin ellipse [13]. Furthermore, it is positive for all t and $0\leqslant s\leqslant 4$. Now, we see that the available information is *not* sufficient to conclude that T has positivity, as this necessitates to have not only $A_t\geqslant 0$, but all the $\text{Im}\, f_l(t)\geqslant 0$ which is clearly a stronger requirement [14]. Of course what happens is that (1) does not incorporate the full content of positivity which also demands, for example, that $\partial^n A_t/\partial s^n \geqslant 0$. This problem is currently under investigation.

3'2. *Analysis of the conditions.* – At first sight conditions (6) do not appear very convenient because (11) makes an infinite number of waves involved in each inequality. Fortunately, however, it is possible, using again the positivity of A_t, to terminate effectively the sums (11) after a finite number of terms giving a precise estimate of the error committed. This error, is in the region of physical interest (s near threshold), very small so that the fact that only a finite number of waves is retained is not worse than, say, the uncertainty produced by experimental errors in the known waves.

To be precise, we consider the physical case $l_0 = 2$, suppose that only the first waves up to $l = N$ are known, and that we want to use inequalities (6) for $l+k\leqslant L$, $L\leqslant N$, *i.e.* involving μ_l's up to $l = L$. Let us then define, from (11),

$$(21a)\qquad \mu_l^N(s) \equiv \frac{2^{l+1}}{l!\sqrt{\pi}} \sum_{\nu=0}^{(N-l)/2} (-1)^\nu \frac{\Gamma(l+\nu+\frac{3}{2})}{\Gamma(\nu+1)} f_{l+2\nu}(s)\,, \qquad l\geqslant 2\,,$$

$$(21b)\qquad R_l^{N,L}(s) \equiv \mu_l(s) - \mu_l^N(s) = \frac{2^{l+1}}{l!\sqrt{\pi}} \sum_{\nu=1+(N-l)/2}^{\infty} (-1)^\nu \frac{\Gamma(l+\nu+\frac{3}{2})}{\Gamma(\nu+1)} f_{l+2\nu}(s)\,.$$

Then, as the series defining R is alternate, it is clear that a good estimate will be given for it as soon as its terms become decreasing in absolute magnitude, since then,

$$(22)\qquad \begin{cases} R_l^{N,L}(s) = (-1)^{1+(N-l)/2} B_l^{N,L}(s)\,, \\ 0 < B_l^{N,L}(s) < \dfrac{2^{l+1}}{l!\sqrt{\pi}} \dfrac{\Gamma([N+l]/2+\frac{5}{2})}{\Gamma([N-l]/2+2)} f_{N+2}(s)\,, \end{cases}$$

([13]) A. MARTIN: *Nuovo Cimento*, **42** A, 930 (1966).

([14]) As shown, *e.g.*, by the counter example $\text{Im}\, f_{2n}(t_1) = (-e^{-1})^n$.

i.e. the sign and a simple bound for R will be known. This situation will be encountered as soon as

$$\frac{\Gamma([N+l]/2+\frac{5}{2})}{\Gamma([N-l]/2+2)}f_{N+2}(s)\leqslant\frac{\Gamma([N+l]/2+\frac{3}{2})}{\Gamma([N-l]/2+1)}f_N(s)\,.$$

We want this to be valid for all $l\leqslant L$. The most unfavourable situation is when $l=L$ so that the above inequality will be satisfied if

$$\frac{f_{N+2}(s)}{f_N(s)}\leqslant\frac{N-L+2}{N+L+3}\,. \tag{23}$$

We now need a bound for f_{N+2}/f_N. From (13), we have

$$f_{N+2}(s)/f_N(s)\leqslant Q_{N+2}(z_0)/Q_N(z_0)\,,$$

and, on the other hand, using the formula

$$z_0Q_n(z_0)-Q_{n+1}(z_0)=\frac{z_0^2-1}{n+1}\sum_{n+1}^{\infty}(2\nu+1)Q_\nu(z_0)\,,$$

we see that

$$z_0Q_n(z_0)/Q_{n+1}(z_0)=1+\frac{2n+3}{n+1}(z_2^2-1)+\text{rest}\,;$$

therefore, since the rest is positive [and $O(1/z_0)$],

$$\frac{Q_{n+1}(z_0)}{Q_n(z_0)}<\frac{z_0}{1+(z_0^2-1)(2+[n+1]^{-1})}<\frac{z_0}{2z_0^2-1}\,.$$

The bound is good for large values of n and z_0 (its accuracy is $\sim 1/(n+1)+1/z_0$) which, of course, are the interesting ones. Then, we get

$$\frac{f_{n+2}(s)}{f_n(s)}<\left[\frac{z_0}{2z_0^2-1}\right]^2, \tag{24}$$

so that to satisfy (23) it is sufficient to have

$$\frac{z_2^2}{(2z_0^2-1)^2}\leqslant\frac{N-L+2}{N+L+3}\,. \tag{25}$$

Next, we compute $B^{N,L}$. Of course, the interesting quantity is its maximum contribution relative to the last nonneglected term in (21), *viz.*, the quantity

$$\varepsilon_{N,L}(s)\equiv\max_{l\leqslant L}\frac{B_l^{N,L}(s)}{2^{l+1}\Gamma([N+l]/2+\frac{3}{2})\,f_N(s)/l!\sqrt{\pi}\,\Gamma([N-l]/2+1)}\,,$$

because, then, one will have $\mu_l = \mu_l'^N$ where $\mu_l'^N$ is given by (21a) with $f_N(s)$ substituted by

$$f_N(s) + (-1)^{1+(N-l)/2}\,\varepsilon_{N,L,l}(s)\,, \qquad 0 \leqslant \varepsilon_{N,L,l}(s) < \varepsilon_{N,L}(s)\,,$$

that is to say, we have only increased the « experimental » error in the last known wave by the amount $\varepsilon_{N,L,l}$, $|\varepsilon_{N,L,l}| \leqslant \varepsilon_{N,L}$. An explicit bound on $\varepsilon_{N,L}$ is then obtained directly from (22) and (24):

$$\varepsilon_{N,L}(s) \leqslant \frac{N+L+3}{N-L+2}\left[\frac{z_0}{2z_2^0-1}\right]^2. \tag{26}$$

To get an idea of the practical meaning of these results, suppose that we want to use as many of the constraints (6) as possible, *i.e.* $L = N$, and take $s \geqslant 2$. Then, (25) is satisfied if $L \leqslant 30$. If, furthermore, $s \geqslant 3$, the relative error is $\varepsilon_{L,L}(s) < 17\,\%$. As seen in this example, in the upper half of the range $0 \leqslant s \leqslant 4$, full use can be made of the conditions (6) up to $l + k \simeq 30$, and the approximation made by replacing μ_l by μ_l^N is not worse than an error of $\leqslant 17\%$ in the last wave, even if we consider the extreme situation where the first thirty waves are known and one wants to use of all of them. Clearly, and at least in the physically interesting region, eqs. (6) in the approximation (21a) are as good for practical purposes as if they only involved the first N waves.

3·3. *Analyticity.* – Let us consider the physical situation $l_0 = 2$, for definiteness.

Define the function k_2 as

$$\begin{cases} k_2(u,\zeta) \equiv \left(\mu^2 \dfrac{\partial}{\partial u} + \dfrac{u}{2}\right)[k(u,\zeta) + (-u,\zeta) - 2]\,, \\[2ex] k(u,\zeta) \equiv (1 - 2u\zeta + u^2)^{-\frac{1}{2}}\,. \end{cases} \tag{27}$$

By direct inspection we may check that $k_2(1/z, \cos\theta_s)$ is analytic in $\cos\theta_s$, and satisfies the bound (7) with $l_0 = 2$, when $\cos\theta_s$ is in the cut complex plane. From this we shall prove:

Theorem 5. If the f_l are solutions of a Froissart-Gribov moment problem, *i.e.*. if eqs. (6) are satisfied, then the amplitude $T(s,t)$, as given by (19), exists and can be analytically continued into the cut t-plane; its absorptive part is A_t, which is positive. More generally, analyticity (but perhaps not positivity) will still be obtained even if the f_l are not solutions of a Froissart-Gribov moment problem, but only the constraints (18) are satisfied.

Remarks:

A) The result of the theorem shows that, as was to be expected, analyticity is weaker than positivity, even in its rudimentary form $A_t \geqslant 0$.

B) Conditions (18) do not constitute a unique set, although they seem to us the most convenient. Thus, COMMON ([11]), has obtained a set of determinantal inequalities that also provide a sufficient set of conditions.

C) That the f_l's coming from (1) generate an analytic amplitude is not surprising since analyticity is one of the ingredients used to prove the Froissart-Gribov representation (*). Actually, a direct proof of the first part of the Theorem could be given, essentially by replacing, in the proof below, the identity

$$\sum_0^\infty u^n P_n(\zeta) = k(u, \zeta)$$

by

$$\sum_0^\infty (2n+1) Q_n(u) P_n(\zeta) = 1/(u-\zeta)\,.$$

However, it is interesting to find weaker « phenomenological » conditions for analyticity, *viz.*, eq. (18), and to prove the converse to the Froissart-Gribov analysis (see Subsect. **3**.4 *B*)) in this connection.

Proof of Theorem 5. We first prove the second half. Assume (18) to hold; due to Theorem 2, we get that the function

$$T_2(s, \cos\theta_s) = \int \mathrm{d}z B(z) k_2(1/z, \cos\theta_s)\,, \tag{28a}$$

is defined and analytic in the cut $\cos\theta_s$-plane. By inserting in (27), (28*a*) the identity

$$\sum_{n=0}^\infty u^n P_n(\cos\theta_s) = k(u, \cos\theta_s)\,,$$

we see that, formally,

$$f_0(1) + T_2(s, \cos\theta_s) = \sum_{l-\text{even}\geqslant 0} (2l+1) f_l(s) P_l(\cos\theta_s) \equiv T(s, t)\,. \tag{28b}$$

Since the expansion of k_2 is convergent when $\cos\theta_s$ is inside the Martin ellipse (in the t-plane), the expansion (28*b*) is also convergent there, *i.e.* $f_0(s) + T_2(s, \cos\theta_s)$ coincides with $T(s, t)$, as given by (19), in that region, and it thus follows that it is its (unique) analytic continuation to the cut t-plane.

The proof of the first part of the Theorem is now very simple: by what has just been said, T is analytic in the cut t-plane; performing the usual Frois-

(*) In this connection, see also T. KINOSHITA, J. J. LOEFFEL and A. MARTIN: *Phys. Rev.*, **135**, B 1464 (1964).

sart-Gribov analysis for it, we get that

$$f_l(s) = \int \mathrm{d}z\, A'_t(s, z)\, Q_l(z)\,,$$

where A'_t is the absorptive part of T. That A'_t coincides with A_t then follows from Corollary 3. *q.e.d.*

3·4. *Oscillating functions.* – If we had taken the « wrong » isospin combination (which we shall denote by the superscript w) or if we are above threshold then, as we may expect oscillations of A^w_t, $\mathrm{Re}\, A_t$, ϱ, we shall not be able to obtain clear-cut results. For the sake of completeness, however, we shall comment briefly on the problem.

If we write generically f^w_l, $\mathrm{Re}\, f_l$, $\mathrm{Im}\, f_l$ as h_l, and A^w_t, $\mathrm{Re}\, A_t$, ϱ, as H, then we can decompose ([15]) $H = \Phi_1 - \Phi_2$, $\Phi_i \geqslant 0$. Consequently, one can generalize the results obtained before just by substituting the sentence « that f_l satisfy such requirements » by « that h_l be the difference of two quantities satisfying such requirements ». More definite statements may, however, be made about three important questions.

A) Corollary 3 remains unchanged, because the argument of ref. ([5]), p.97, Theorem 3.9 still holds.

B) If the h_l are given by a Froissar-Gribov representation or, more generally, by a Hausdorff representation, then one can, in a way that is unique (up to, of course, the lowest waves) construct an analytic scattering amplitude. This is because the proof of Theorem 5 also holds, *mutatis mutandis*, for oscillating H's. In particular, we can give a simple proof of the fact that analyticity in l implies analyticity in t:

Theorem 6. If h_λ is analytic in λ for $\mathrm{Re}\,\lambda \geqslant l_0$, and $O(\exp[-\alpha\lambda])$, $\alpha > 0$ there, then one can construct a scattering aamplitude, analytic in the cut t plane and bounded by t_0.

Proof. It is sufficient to show that there exists H with

$$h_l = \int \mathrm{d}z\, z^{-l-1} H(z)\,, \qquad l \geqslant l_0\,.$$

But this is so: because of the analyticity and boundedness of h_λ, it follows ([16])

([15]) Subject, eventually, to continuity assumptions, as A_t is a distribution in general.

([16]) G. Doetsch: *Handbuch der Laplace Transformation* (Basel, 1950); E. J. Beltrami and M. R. Wohlers: *Distributions and the Boundary Values of Analytic Functions* (New York, 1966).

that we can write

$$h_\lambda = \int \mathrm{d}\xi \exp[-\lambda\xi] H_1(\xi) ,$$

i.e. h_λ is the Laplace transform of H_1, and it suffices to change variables, $z = e^\xi$. *q.e.d.* Thus, we see that analyticity in l implies full analyticity in t.

C) If one has detailed information on the continuity properties of H, then more definite results might be obtained, as the methods of ref. (5), p. 97 ff. could be applied. We shall not pursue this problem further, though.

4. – Applications and comments.

4·1. *Scattering lengths.* – As follows from the analysis of Subsect. **3·2**, the most favourable situation is when $q = 0$, as then we can neglect all but the first term in (11) without committing any error. In fact, if the scattering lengths are given by (2), eqs. (11), (6) give the constraints

$$(29)\qquad \begin{cases} \delta^0 a_l \equiv a_l \geqslant 0 , \\ \delta^1 a_l \equiv \dfrac{2^{l+1}\Gamma(l+\frac{3}{2})}{\Gamma(l+1)}\, a_l - 4\, \dfrac{2^{l+3}\Gamma(l+2+\frac{3}{2})}{\Gamma(l+3)}\, a_{l+2} \geqslant 0 , \\ \dots\dots\dots\dots\dots\dots \\ \delta^k a_l \equiv \displaystyle\sum_{j=0}^{k} \binom{k}{j} (-1)^j 2^{l+2k+4+1} \dfrac{\Gamma(l+2j+\frac{3}{2})}{\Gamma(l+2j+1)}\, a_{l+2} \geqslant 0 . \\ \dots\dots\dots\dots\dots\dots \end{cases}$$

For $\delta^1 a_l$ we get that (17)

$$a_{l+2} - \frac{1}{16}\,\frac{(l+2)(l+1)}{(l+\frac{3}{2})(l+\frac{5}{2})}\, a_l \geqslant 0 ,$$

i.e. the result announced in the Introduction, eq. (3). (Intuitively, such a small value can be understood by remarking that, as pions are pseudoscalars, the nearest singularity (that should dominate the forces) is at a mass of $2m_\pi$, so that $a_{l+2}/a_l \sim (\text{typical mass})^{-4} \sim 1/16 m_\pi^4$. This is exactly the situation if we saturate the amplitudes by a pole due to the exchange of the lowest lying

(17) This result has also been obtained, from eq. (13), by A. MARTIN: *Nuovo Cimento*, **47**, 265 (1967). It is to be remarked that the corresponding estimate, eq. (39) of Martin's paper contains a numerical error so that with his normalization $\alpha_l = \lim f_l(s)/(4-s)^l$, eq. (39) should read $\alpha_{l+2} < (1/256 m_\pi^2)((l+2)(l+1)/(l+\frac{3}{2})(l+\frac{5}{2}))\,\alpha_l$.

state.) Similarly, decomposing $\delta^2 a_l \propto \delta^1 a_l - \delta_1 a_{l+2}$, (28) gives that

$$\left|a_{l+2} - 16\,\frac{(l+2+\frac{3}{2})(l+3+\frac{3}{2})}{(l+3)(l+4)}\,a_{l+4}\right| \leqslant$$

$$\leqslant \frac{1}{16}\,\frac{(l+1)(l+2)}{(l+\frac{3}{2})(l+\frac{5}{2})}\left|a_l - 16\,\frac{(l+\frac{3}{2})(l+\frac{5}{2})}{(l+1)(l+2)}\,a_{l+2}\right|,$$

i.e. the constraints (3) are practically saturated when $l \geqslant 4$.

4·2. *Comments on ππ parametrizations.* – The results we have just obtained indicate why the low-energy ππ scattering amplitude is sensitive only to a few parameters, something that had already been made clear by some independent work [1,2,18]: once the S and D waves are fixed the remaining waves are very small and can be estimated with little uncertainty. Since, on the other hand, it has been shown [1,2] that the S-wave fixes with good accuracy the D wave, this may be taken as an explanation of the fact that any model that gives a_0 and neglects terms above quadratic is likely to represent a good approximation to the low-energy ππ scattering amplitude. These conditions are obviously met by Weinberg's linear parametrization [19], and also by the one-term Veneziano model [20] (in fact, for the last, direct computation shows that $a_{l+2} < (2\cdot 10^{-4})a_l$) and thus our results may be considered as a justification of these approximations.

4·3. *Discussion.* – There are two points to which we would like to devote a few words. First, one may question the usefulness of our constraints on the grounds that they are only valid (except at $s=4$) for unphysical energies. This is, of course, not the case because, due to the analyticity properties of the f_l, their values for $s<4$ are determined by their value on the physical region. As a matter of fact, actual partial-wave analyses are based on dispersion relations which determine the f_l's below as well as above threshold and on which our constraints could be imposed as consistency conditions [21]. Second, although we have shown that the problems caused by the fact that (11) contains all waves can be tackled, it would still be nice to have conditions involving

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[18] G. Auberson, O. Piguet and G. Wanders: *Phys. Lett.*, **28** B, 141 (1968); O. Piguet and G. Wanders: University of Lausanne preprint and *Nuovo Cimento*, in press.

[19] S. Weinberg: *Phys. Rev. Lett.*, **17**, 616 (1966).

[20] G. Veneziano: *Nuovo Cimento*, **57** A, 190 (1968); C. Lovelace: *Phys. Lett.*, **28** B, 265 (1968).

[21] We thank several of our colleagues for pointing out the convenience of this discussion.

only finite numbers of waves. This might be done by generalizing the representation of Lemma 1. An obvious candidate for such generalization is the set of « binomials »

$$\sum_{j=0}^{k} \binom{k}{j} (-1)^j Q_{l+2j}(z)/Q_{l+j2}(z_0) .$$

In this context, if would be necessary to show that they are positive if $z \geqslant z_0$, something which is true for $k = 0, 1$ and for l or z_0 large and when z_0 is close to 1. To prove it in general one would have to show, for example, that $Q_\lambda(z)/Q_\lambda(z_0)$ is « totally convex », *i.e.* to show that

$$(-1)^n \frac{\partial^n}{\partial \lambda^n} \frac{Q_\lambda(z)}{Q_\lambda(z_0)} \geqslant 0 ,$$

a fact that is again true for $n = 0, 1$, on the range λ or z_0 large, when $z_0 \simeq 1$, and that we have checked numerically for $0 \leqslant \lambda \leqslant 30$, $1.2 \leqslant z_0 \leqslant z \leqslant 20$, but we have not been able to prove in general.

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RIASSUNTO (*)

Si considerano i vincoli sulle onde parziali per $\pi^0\pi^0$ o $\pi\pi$ nel canale di isospin zero, che seguono dall'analiticità e positività. Si dimostra che, se si verificano queste due condizioni, allora si deve soddisfare un insieme di diseguaglianze sulle onde parziali con $l \geqslant 2$ ed energia al di sotto della soglia. Si prova anche l'inverso mostrando che, se questi vincoli sono soddisfatti, si può costruire un'ampiezza di scattering analitica con parte assorbitiva positiva. Si discute l'importanza pratica di queste condizioni mostrando che, ad esempio, esse implicano le diseguaglianze (massa del pione = 1) $a_{l+2} < (1/16)\big((l+2)(l+1)/(l+\frac{5}{2})(l+\frac{3}{2})\big)a_l$ dove gli a_l sono le lunghezze di scattering, $a_l = \lim f_l(s)/q^{2l}$. Inoltre queste disuguaglianze (che sono state essenzialmente trovate anche da Martin) sono saturate molto accuratamente quando $l \geqslant 4$. Si discute l'importanza della nostra analisi per modelli dello scattering $\pi\pi$.

(*) *Traduzione a cura della Redazione.*

Ограничения на $\pi\pi$ парциальные волны из положительности и аналитичности.

Резюме (*). — Мы рассматриваем ограничения на парциальные волны для $\pi^0\pi^0$ или $\pi\pi$, в канале с нулевым изоспином, которые следуют из аналитичности и положительности. Мы показываем, что если встречаются эти два требования, то тогда должна удовлетворяться система неравенств на парциальные волны с $l \geqslant 2$ и энергией ниже порога. Также доказывается и обратное, показывающее, что если эти ограничения выполняются, то тогда может быть сконструирована амплитуда рассеяния с положительной абсорбционной частью. Обсуждается практическая важность этих условий, отмечая, что они, например, предполагают неравенства (пионная масса=1) $a_{l+2} < (1/16)\big((l+2)(l+1)/(l+\frac{5}{2})(l+\frac{3}{2})\big)a_l$; здесь a_l представляют длины рассеяния, $a_l = \lim f_l(s)/q^{2l}$. Кроме того, эти неравенства (которые, по существу, также были найдены Мартиным) насыщаются очень аккуратно, когда $l \geqslant 4$. Обсуждается уместность нашего анализа для моделей $\pi\pi$ рассеяния.

(*) *Переведено редакцией.*