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A direct boundary integral method for the three-dimensional lifting flow

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Abstract

A regularized integral formulation in velocity vector terms for three-dimensional incompressible non-lifting and lifting flow is developed. In the case of lifting flow, the circulatory flow around the body is generated by a discrete distribution of inner vortices, and a vectorial equality in velocity terms is used as an equivalent of the equal-pressure Kutta condition. The numerical comparison with the panel method, in the case of non-lifting flow, shows the superiority of the present method. Also, a two-dimensional lifting flow is simulated (using a high spanwise wing) and comparison with the analytic solution shows a very good concordance even at high incidence.

1. Introduction

The three-dimensional flow is mainly studied today by indirect integral methods, via the potential function, the velocity field being obtained ulterior by differentiation. Such a method is, for instance, the panel method [1, 2]. An alternative indirect method is also presented in [3].

The method herein is a direct one, the integral equation being formulated in velocity vector terms. This is mainly favorable in numerical implementation, saving computing effort. The integral formulation is regularized, i.e. it is free of singular integrals, so that boundary limit problems are eliminated.

It is known that in two-dimensional cases, direct methods lead to better results than indirect ones [4]. Similarly, in three-dimensional cases, the present direct method leads to better results than the indirect panel method, at least for non-lifting flow [1, 2].

In the case of lifting flow, an inner discrete vortex distribution generates a circulatory flow around the body and the equal-pressure Kutta condition is applied at the trailing edge. Indirect methods use the Kutta condition by prescribing an arbitrary direction of flow at the trailing edge. Only Hess [2] uses the physical meaning of the Kutta condition (i.e. equal pressure at the trailing edge) but this generates a non-linear problem. The present method expresses the equal-pressure condition in vectorial circulation terms [5] and the linear character of the problem is kept.

The integral velocity formulation for two- and three-dimensional flow in [6] is not followed by any numerical results in the three-dimensional case. Numerical implementations for 3D kinetic methods are given in [7, 8] but only for non-lifting bodies. One can also notice that both methods need a parametric (not always available) representation of the surface.

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2. Statement of problem

The incompressible flow of an inviscid fluid in a three-dimensional domain D exterior to a given body, is governed by the system of differential equations

$$\operatorname{div} \boldsymbol{V} = 0\,, \qquad \operatorname{rot} \boldsymbol{V} = \boldsymbol{0} \tag{2.1}$$

and by the boundary condition of tangential flow along the body surface Σ

$$\boldsymbol{V} \cdot \boldsymbol{n} = 0 \tag{2.2}$$

Here, $\boldsymbol{V}$ has physical dimension.

We assume that the motion is a superposition of the onset flow and of a perturbatory flow, generated by the body. In that case, the global velocity $\boldsymbol{V}$ may be written as

$$\boldsymbol{V} = U_\infty(\boldsymbol{i} + \boldsymbol{v}) \tag{2.3}$$

where $U_\infty \boldsymbol{i}$ is the velocity of the onset flow and $\boldsymbol{v}$ is the non-dimensional velocity of the perturbatory flow. In terms of perturbation, the system (2.1) becomes

$$\operatorname{div} \boldsymbol{v} = 0\,, \qquad \operatorname{rot} \boldsymbol{v} = \boldsymbol{0} \tag{2.4}$$

which is valid even when the function $\boldsymbol{v}$ are translated by any constant vector $\boldsymbol{c}$, i.e.

$$\operatorname{div}(\boldsymbol{v} - \boldsymbol{c}) = 0\,, \qquad \operatorname{rot}(\boldsymbol{v} - \boldsymbol{c}) = \boldsymbol{0} \tag{2.5}$$

The boundary condition (2.2) is written in terms of perturbations as

$$\boldsymbol{v} \cdot \boldsymbol{n} = -n_x \tag{2.6}$$

It is also required that the perturbation velocity vanishes at infinity

$$\lim_{\boldsymbol{x} \to \infty} \boldsymbol{v}(\boldsymbol{x}) = 0 \tag{2.7}$$

Thus, the problem is to solve the differential equations system (2.4) with the supplementary conditions (2.6) and (2.7).

3. Non-lifting flow

3.1. Integral representation formula

In this section we will give a regularized representation formula for the velocity at any point of $D + \Sigma$.

Let $\boldsymbol{v}^* = (u^*, v^*, w^*)$ be the fundamental solution determined by the system

$$\operatorname{div} \boldsymbol{v}^* = \delta_{\boldsymbol{x}_0}(\boldsymbol{x})\,, \qquad \operatorname{rot} \boldsymbol{v}^* = \boldsymbol{0}\,, \quad \forall\, \boldsymbol{x}_0 \in D \tag{3.1.1}$$

($\delta_{\boldsymbol{x}_0}$ is the Dirac's function) which is a source-type solution and is given by

$$\boldsymbol{v}^* = \operatorname{grad} \mathscr{E}_{\boldsymbol{x}_0}\,, \qquad \mathscr{E}_{\boldsymbol{x}_0}(\boldsymbol{x}) \equiv -\frac{1}{4\pi}\frac{1}{R}\,, \qquad R = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} \tag{3.1.2}$$

For any functions f, $\boldsymbol{g}$ we have the identity (due to Eqs. (2.5))

$$\int_D [f \operatorname{div}(\boldsymbol{v} - \boldsymbol{c}) + \boldsymbol{g} \cdot \operatorname{rot}(\boldsymbol{v} - \boldsymbol{c})]\, \mathrm{d}V = 0 \tag{3.1.3}$$

In order to apply the Gauss type formulas

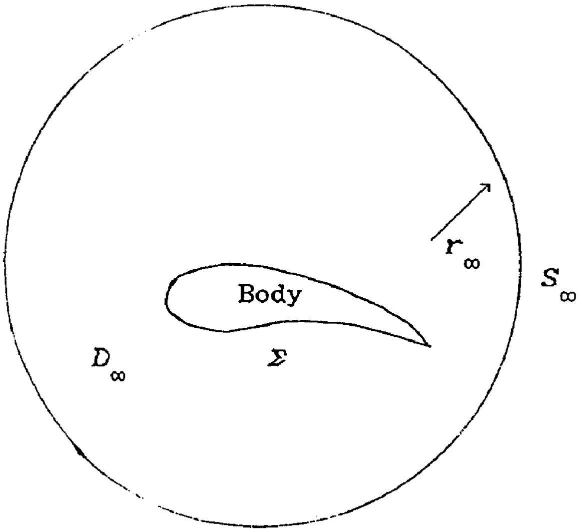


Fig. 1. The domain of integration.

$$\int_D f \operatorname{div}(\boldsymbol{v}-\boldsymbol{c})\, \mathrm{d}V = -\int_D (\boldsymbol{v}-\boldsymbol{c}) \cdot \operatorname{grad} f \, \mathrm{d}V + \int_\Sigma f\boldsymbol{n} \cdot (\boldsymbol{v}-\boldsymbol{c})\, \mathrm{d}a$$

$$\int_D \boldsymbol{g} \cdot \operatorname{rot}(\boldsymbol{v}-\boldsymbol{c}) = \int_D (\boldsymbol{v}-\boldsymbol{c}) \cdot \operatorname{rot} \boldsymbol{g}\, \mathrm{d}V - \int_\Sigma (\boldsymbol{v}-\boldsymbol{c}) \cdot (\boldsymbol{n} \times \boldsymbol{g})\, \mathrm{d}a$$

on the unbounded domain D, we use a supplementary remote sphere S_∞ of radius r_∞ (Fig. 1) and define D_∞ the domain bounded by Σ and S_∞. We are able now to apply the above Gauss formulas on the domain D_∞, that gives

$$\int_{D_\infty} (\boldsymbol{v}-\boldsymbol{c}) \cdot (\operatorname{grad} f - \operatorname{rot} \boldsymbol{g})\, \mathrm{d}V = \int_{\Sigma + S_\infty} (\boldsymbol{v}-\boldsymbol{c}) \cdot (f\boldsymbol{n} - (\boldsymbol{n} \times \boldsymbol{g}))\, \mathrm{d}a \tag{3.1.4}$$

where $\boldsymbol{n}$ is the normal unit vector outward the fluid (inward the body). For the following sets of functions

$$\begin{aligned}
&(f, \boldsymbol{g}) \to (\boldsymbol{i} \cdot \operatorname{grad} \mathscr{E}_{x_0}(\boldsymbol{x}), -\boldsymbol{i} \times \operatorname{grad} \mathscr{E}_{x_0}(\boldsymbol{x}))\\
&(f, \boldsymbol{g}) \to (\boldsymbol{j} \cdot \operatorname{grad} \mathscr{E}_{x_0}(\boldsymbol{x}), -\boldsymbol{j} \times \operatorname{grad} \mathscr{E}_{x_0}(\boldsymbol{x}))\\
&(f, \boldsymbol{g}) \to (\boldsymbol{k} \cdot \operatorname{grad} \mathscr{E}_{x_0}(\boldsymbol{x}), -\boldsymbol{k} \times \operatorname{grad} \mathscr{E}_{x_0}(\boldsymbol{x}))\,, \quad \boldsymbol{x}_0 \in D_\infty + \Sigma
\end{aligned} \tag{3.1.5}$$

from (3.1.4), we get the equation

$$\int_{D_\infty} (\boldsymbol{v}-\boldsymbol{c}) \cdot \operatorname{div} \boldsymbol{v}^*\, \mathrm{d}V = \int_{\Sigma + S_\infty} [\boldsymbol{n} \cdot (\boldsymbol{v}-\boldsymbol{c}) \operatorname{grad} \mathscr{E}_{x_0} + \boldsymbol{n} \times (\boldsymbol{v}-\boldsymbol{c}) \times \operatorname{grad} \mathscr{E}_{x_0}]\, \mathrm{d}a\,, \quad \boldsymbol{x}_0 \in D_\infty + \Sigma \tag{3.1.6}$$

Now, we get back to the unbounded domain D by making $r_\infty \to \infty$.

By using (3.1.1), the left-side term in (3.1.6) becomes

$$\lim_{r_\infty \to \infty} \int_{D_\infty} (\boldsymbol{v}-\boldsymbol{c}) \cdot \operatorname{div} \boldsymbol{v}^*\, \mathrm{d}V = \int_D (\boldsymbol{v}-\boldsymbol{c})(\boldsymbol{x}) \cdot \delta_{x_0}(\boldsymbol{x})\, \mathrm{d}V = \boldsymbol{v}(\boldsymbol{x}_0) - \boldsymbol{c} \tag{3.1.6'}$$

By splitting the integral on S_∞ in

$$\int_{S_\infty} [(\boldsymbol{n} \cdot \boldsymbol{v}) \operatorname{grad} \mathscr{E}_{x_0} + \boldsymbol{n} \times \boldsymbol{v} \times \operatorname{grad} \mathscr{E}_{x_0}]\, \mathrm{d}a - \int_{S_\infty} [(\boldsymbol{n} \cdot \boldsymbol{c}) \operatorname{grad} \mathscr{E}_{x_0} + \boldsymbol{n} \times \boldsymbol{c} \times \operatorname{grad} \mathscr{E}_{x_0}]\, \mathrm{d}a$$

we observe that first term vanishes when $r_\infty \to \infty$ due to the perturbation velocity vanishing condition (2.7). The second term has to be evaluated by passing to spherical coordinates $x = x(r_\infty, \theta, \phi)$. Then, when $r_\infty \to \infty$ we have

$$\lim_{r_\infty \to \infty} \int_{S_\infty} [(\boldsymbol{n} \cdot \boldsymbol{c}) \operatorname{grad} \mathscr{E}_{x_0} + \boldsymbol{n} \times \boldsymbol{c} \times \operatorname{grad} \mathscr{E}_{x_0}] \, da = \frac{\boldsymbol{c}}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta \cdot d\theta \, d\phi = \boldsymbol{c}$$

Thus, the right-side term in (3.1.6) becomes

$$\lim_{r_\infty \to \infty} \int_{\Sigma + S_\infty} [\boldsymbol{n} \cdot (\boldsymbol{v} - \boldsymbol{c}) \operatorname{grad} \mathscr{E}_{x_0} + \boldsymbol{n} \times (\boldsymbol{v} - \boldsymbol{c}) \times \operatorname{grad} \mathscr{E}_{x_0}] \, da$$

$$= -\boldsymbol{c} + \int_{\Sigma} [\boldsymbol{n} \cdot (\boldsymbol{v} - \boldsymbol{c}) \operatorname{grad} \mathscr{E}_{x_0} + \boldsymbol{n} \times (\boldsymbol{v} - \boldsymbol{c}) \times \operatorname{grad} \mathscr{E}_{x_0}] \, da \tag{3.1.6''}$$

Equating (3.1.6′) and (3.1.6″) we get

$$\boldsymbol{v}(\boldsymbol{x}_0) = \int_{\Sigma} [\boldsymbol{n} \cdot (\boldsymbol{v} - \boldsymbol{c}) \operatorname{grad} \mathscr{E}_{x_0}(\boldsymbol{x}) + \boldsymbol{n} \times (\boldsymbol{v} - \boldsymbol{c}) \times \operatorname{grad} \mathscr{E}_{x_0}(\boldsymbol{x})] \, da \,, \quad \boldsymbol{x}_0 \in D + \Sigma$$

By setting $\boldsymbol{c} = \boldsymbol{v}(\boldsymbol{x}_0)$ one obtains the representation formula

$$\boldsymbol{v}(\boldsymbol{x}_0) = \overset{\mathrm{R}}{\int_{\Sigma}} \{\boldsymbol{n} \cdot [\boldsymbol{v} - \boldsymbol{v}(\boldsymbol{x}_0)] \operatorname{grad} \mathscr{E}_{x_0}(\boldsymbol{x}) + \boldsymbol{n} \times [\boldsymbol{v} - \boldsymbol{v}(\boldsymbol{x}_0)] \times \operatorname{grad} \mathscr{E}_{x_0}(\boldsymbol{x})\} \, da \,, \quad \boldsymbol{x}_0 \in D + \Sigma \tag{3.1.7}$$

This is a *regularized* integral representation formula, which is valid both in the domain D and on the surface Σ. Although the function $\mathscr{E}$ has a singularity at $\boldsymbol{x} = \boldsymbol{x}_0 \in \Sigma$, the integral is an ordinary type, not a Cauchy type, due to the fact that the term $\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{v}(\boldsymbol{x}_0)$ become zero when $\boldsymbol{x} \to \boldsymbol{x}_0 \in \Sigma$.

This form of the representation formula is an important achievement because we do not have to deal with the difficult problem of three-dimensional surface discontinuities [9–11].

In the following, the letter R on the integral sign will denote a regularized integral.

When applying the boundary condition $\boldsymbol{n} \cdot \boldsymbol{v} = -n_x$, Eq. (3.1.7) becomes

$$\boldsymbol{v}_0 = \overset{\mathrm{R}}{\int_{\Sigma}} [-(n_x + \boldsymbol{n} \cdot \boldsymbol{v}_0) \operatorname{grad} \mathscr{E}_{x_0} + \boldsymbol{n} \times (\boldsymbol{v} - \boldsymbol{v}_0) \times \operatorname{grad} \mathscr{E}_{x_0}] \, da \,, \quad \boldsymbol{x}_0 \in D + \Sigma \tag{3.1.8}$$

where the notation $\boldsymbol{v}_0 \equiv \boldsymbol{v}(\boldsymbol{x}_0)$ was used.

3.2. Integral boundary equation

A new vectorial unknown function $\boldsymbol{F}$ is introduced instead of $\boldsymbol{v}$,

$$\boldsymbol{F} \equiv \boldsymbol{n} \times \boldsymbol{v} \tag{3.2.1}$$

which is a natural choice from Eq. (3.1.7) (see also [6]). Added to the boundary condition $\boldsymbol{n} \cdot \boldsymbol{v} = -n_x$, Eq. (3.2.1) leads to

$$\boldsymbol{v} = -n_x \boldsymbol{n} - \boldsymbol{n} \times \boldsymbol{F} \tag{3.2.2}$$

Thus, Eq. (3.1.8) may be written as

$$\boldsymbol{v}_0 = \overset{\mathrm{R}}{\int_{\Sigma}} [((\boldsymbol{n} \times \boldsymbol{n}_0) \cdot \boldsymbol{F}_0) \operatorname{grad} \mathscr{E}_{x_0} + \boldsymbol{F} \times \operatorname{grad} \mathscr{E}_{x_0} + \boldsymbol{n} \times (\boldsymbol{n}_0 \times \boldsymbol{F}_0) \times \operatorname{grad} \mathscr{E}_{x_0}] \, da$$

$$+ \overset{\mathrm{R}}{\int_{\Sigma}} [n_x^0 \boldsymbol{n} \times \boldsymbol{n}_0 \times \operatorname{grad} \mathscr{E}_{x_0} + (n_x^0 \boldsymbol{n} \cdot \boldsymbol{n}_0 - n_x) \operatorname{grad} \mathscr{E}_{x_0}] \, da \tag{3.2.3}$$

In order to get $\boldsymbol{F}$ outside the integral, we apply the cross product $\boldsymbol{n}_0 \times$ (Eq. (3.2.3)) and we use the vectorial formulas

$$(\boldsymbol{U} \times \boldsymbol{V}) \times \boldsymbol{W} = (\boldsymbol{U} \cdot \boldsymbol{W})\boldsymbol{V} - (\boldsymbol{V} \cdot \boldsymbol{W})\boldsymbol{U}$$

$$(\boldsymbol{U} \times \boldsymbol{V}) \times \boldsymbol{W} + (\boldsymbol{V} \times \boldsymbol{W}) \times \boldsymbol{U} + (\boldsymbol{W} \times \boldsymbol{U}) \times \boldsymbol{V} = \boldsymbol{0}$$

Calculating, we get the following expression for the boundary integral equation

$$\boldsymbol{F}_0 + {}^{\mathrm{R}}\!\!\int_\Sigma \left[(\boldsymbol{n}_0 \cdot \boldsymbol{F}) \operatorname{grad} \mathscr{E}_{x_0} + \left(\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}n} \boldsymbol{F}_0 + \frac{\mathrm{d}\mathscr{E}}{\mathrm{d}n_0} \boldsymbol{F} \right) \right] \mathrm{d}a + (\boldsymbol{n}_0 \times \boldsymbol{F}_0)\ {}^{\mathrm{R}}\!\!\int_\Sigma (\boldsymbol{n} \times \boldsymbol{n}_0) \cdot \operatorname{grad} \mathscr{E}_{x_0}\, \mathrm{d}a$$
$$= n_x^0\ {}^{\mathrm{R}}\!\!\int_\Sigma \left[\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}n_0} (\boldsymbol{n} \times \boldsymbol{n}_0) + (\boldsymbol{n} \cdot \boldsymbol{n}_0 - 1)(\boldsymbol{n}_0 \times \operatorname{grad} \mathscr{E}_{x_0}) \right] \mathrm{d}a \tag{3.2.4}$$

$\boldsymbol{F}$ on Σ is obtained and it easily furnishes $\boldsymbol{v}$ on Σ (via Eq. (3.2.2)). Further, the velocity field on the D domain may be determined by Eq. (3.1.8).

4. The lifting flow

In the following we consider the flow around a non-smooth body under the hypotheses of non-rotational motion (rot $\boldsymbol{v} = \boldsymbol{0}$). Unlike other methods that consider a sub-domain of a given shape where rot $\boldsymbol{v} \neq \boldsymbol{0}$ (a vortex wake surface behind the body, for instance), the present method uses a discrete distribution of vortices inside the body at a given location. By that, although the condition rot $\boldsymbol{v} = \boldsymbol{0}$ is kept on the entire flow domain, non-zero values for circulation are obtained on different curves.

From a geometrical point of view, a discrete distribution of points is easier to handle than a three-dimensional wake surface when the problem of the most realistic flow is considered.

The Kutta condition is expressed in a vectorial relation. It gives a two-dimensional pattern flow on the smooth curves of the trailing edge, and a pure three-dimensional pattern flow at the corners.

4.1. Vectorial equal-pressure Kutta condition

We have chosen among various properties related to the Kutta condition [2], the most physically meaningful one: as the trailing edge (T.E.) is approached, the velocity magnitudes (pressures) on the upper and lower surface have a common limit, i.e.

$$\lim_{\substack{P \to P_{\mathrm{T.E.}} \\ P \in \text{upper} \\ \text{surface}}} |V(P)| = \lim_{\substack{P \to P_{\mathrm{T.E.}} \\ P \in \text{lower} \\ \text{surface}}} |V(P)| = |V(P_{\mathrm{T.E.}})| \tag{4.1.1}$$

We recall that V (the velocity) has physical dimension.

For a two-dimensional flow, equal-pressure condition at T.E. may be written as

$$\boldsymbol{V}(P_{\mathrm{up}}) \cdot \boldsymbol{S} + \boldsymbol{V}(P_{\mathrm{low}}) \cdot \boldsymbol{S} \xrightarrow[P_{\mathrm{up}}, P_{\mathrm{low}} \to P_{\mathrm{T.E.}}]{} 0 \tag{4.1.2}$$

because the dot products have the same magnitude ($\boldsymbol{S}$ is a unit vector) but opposite signs (Fig. 2).

We observe that $\boldsymbol{n} \times \boldsymbol{V} = (0, 0, \boldsymbol{V} \cdot \boldsymbol{S})$ thus, Eq. (4.1.2) is equivalent to

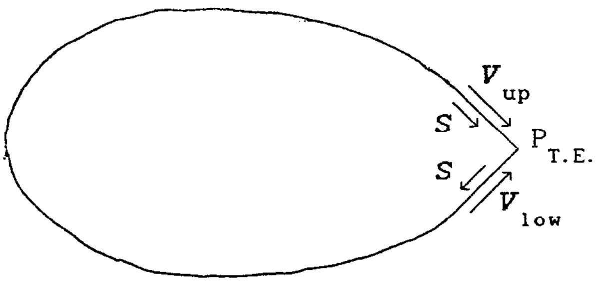


Fig. 2. Velocity direction at the trailing edge in 2D case.

$$\boldsymbol{n} \times \boldsymbol{V}(P_{\mathrm{up}}) + \boldsymbol{n} \times \boldsymbol{V}(P_{\mathrm{low}}) \xrightarrow[P_{\mathrm{up}}, P_{\mathrm{low}} \to P_{\mathrm{T.E.}}]{} \boldsymbol{0} \tag{4.1.3}$$

We extend this equation to the three-dimensional case and name it *the vectorial Kutta condition*. This extension is natural: for a two-dimensional case, the terms in Eq. (4.1.2) represent the integrand of the circulation formula

$$\int_C \boldsymbol{V} \cdot \boldsymbol{S} \, \mathrm{d}S$$

Analogously, in the three-dimensional case, the terms in Eq. (4.1.3) represent the integrand of the surface circulation defined by [5]

$$\int_\Sigma \boldsymbol{n} \times \boldsymbol{V} \, \mathrm{d}a \, .$$

The vectorial Kutta condition is physically meaningful on the straight lines of T.E. The relation (4.1.3) shows that the vectors $\boldsymbol{n} \times \boldsymbol{V}_{\mathrm{up}}$ and $\boldsymbol{n} \times \boldsymbol{V}_{\mathrm{low}}$ are parallel and have opposite signs. They have, of course, the same magnitude. Because $\boldsymbol{n} \times \boldsymbol{V}_{\mathrm{up}}$ and $\boldsymbol{n} \times \boldsymbol{V}_{\mathrm{low}}$ stands in non-parallel plane (Fig. 3), the relation (4.1.3) holds only if both vectors are parallel to the panels common line (i.e. to T.E.). That implies $\boldsymbol{V} \perp$T.E. The direction along which the fluid leaves the wing is contained into the perpendicular to T.E. plane (as for two-dimensional flow). That is the main achievement over other methods using Kutta condition in terms of imposing an arbitrary flow direction on the straight lines of T.E.

When a corner is considered at T.E., the condition (4.1.3) is written for three or more terms as

$$\sum_{i \geq 3} \boldsymbol{n} \times \boldsymbol{V}(\boldsymbol{x}_i) \xrightarrow[x_i \to \mathrm{T.E.}]{} 0$$

and gives distinct directions and magnitudes for the vectors $\boldsymbol{V}(\boldsymbol{x}_i)$ at each panel of the corner.

4.2. Representation formulae and integral boundary equation

As shown in the previous paragraph, Kutta condition gives as many supplementary vectorial relations as pairs of panels we have at T.E. Thus, we have an over-determined problem to solve. In order to overpass that inconvenience, we add a supplementary vectorial unknown for each supplementary vectorial relation. We do that by taking an onset flow as

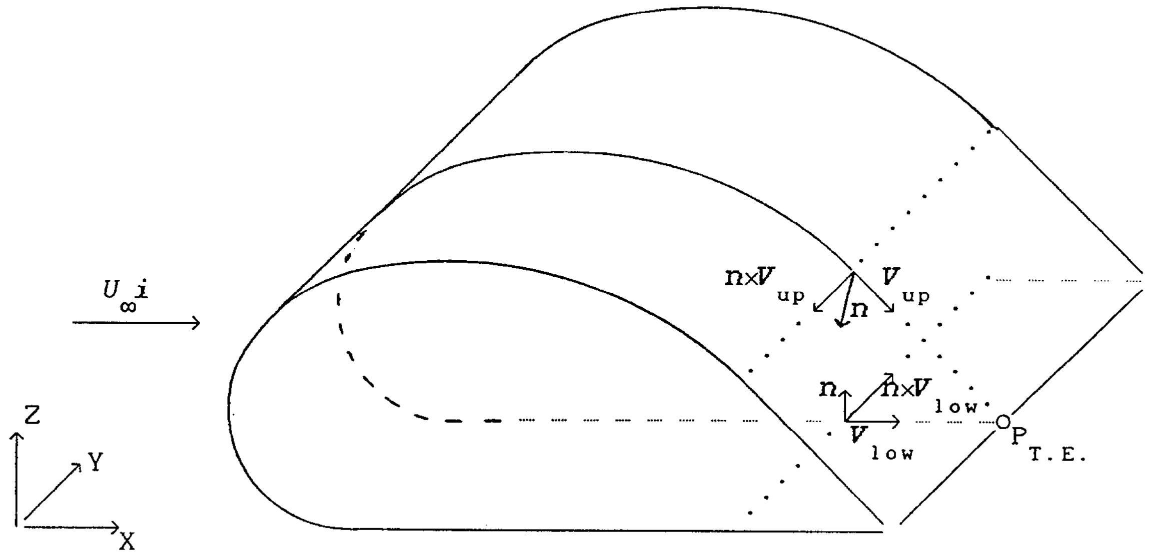


Fig. 3. Vectorial Kutta condition at the trailing edge: velocity direction on upper and lower surface.

$$\boldsymbol{U}_\infty = U_\infty\Big[\boldsymbol{i} + \sum_k \boldsymbol{\Gamma}_k \times \operatorname{grad}\mathscr{E}(\boldsymbol{x} - \boldsymbol{x}_k)\Big]$$

where $\boldsymbol{x}_k$ is an inner point to the body, and $\boldsymbol{\Gamma}_k$ is the supplementary unknown.

The flow generated by the term $\boldsymbol{\Gamma} \times \operatorname{grad}\mathscr{E}$ is circulatory about the axis through the point $\boldsymbol{x}_k$, its direction being given by $\boldsymbol{\Gamma}$. That could be considered a limit case of the flow generated by a vortex region $\mathscr{V}$ (volume $(\mathscr{V}) \to 0$). Thus, the point $\boldsymbol{x}_k$ may be called a 'point vortex', analogously to the two-dimensional flow which is closely related as it follows.

In order to simplify calculus we take $\boldsymbol{\Gamma} = (0, 0, \Gamma)$. The velocity induced by the 'point vortex' $\boldsymbol{x}_k \equiv \boldsymbol{0}$ is

$$\boldsymbol{w}(\boldsymbol{x}) = \frac{\Gamma}{4\pi}\left(-\frac{y}{r^3}, \frac{x}{r^3}, 0\right), \qquad r = \sqrt{x^2 + y^2 + z^2}$$

A continuous distribution of 'point vortices' along the $0z$-axis (that is $\boldsymbol{x}_k = (0, 0, z_k)$ and $r = \sqrt{x^2 + y^2 + (z - z_k)^2}$) induces the velocity

$$w_x(\boldsymbol{x}) = \frac{-\Gamma}{4\pi}\int_{z_k=-\infty}^{z_k=+\infty} \frac{y}{r^3}\,\mathrm{d}z_k = -\frac{\Gamma y}{4\pi}\frac{z - z_k}{r(x^2 + y^2)}\Bigg|_{z_k=+\infty}^{z_k=-\infty} = \frac{\Gamma}{2\pi}\frac{-y}{x^2 + y^2}$$

$$w_y(\boldsymbol{x}) = \frac{\Gamma}{2\pi}\frac{x}{x^2 + y^2}, \qquad w_z(\boldsymbol{x}) = 0$$

which is just the two-dimensional vortex situation.

As a property of the 'point vortex', we mention that circulation is depending on the curve. As an example, for $\boldsymbol{x}_k \equiv \boldsymbol{0}$ and $\boldsymbol{\Gamma} = \Gamma\boldsymbol{k}$, the circulation around the circle of radius R standing in the plane $z = h$ (Fig. 4) has the value

$$\int_{\text{circle}} \boldsymbol{w} \cdot \boldsymbol{s}\,\mathrm{d}s = \frac{\Gamma}{2}\frac{R^2}{(R^2 + h^2)^{3/2}}$$

We observe that $\lim_\infty \boldsymbol{w} = 0$ and the circulation vanishes at infinity as well.

Returning to the main problem, we first consider an onset flow with a single 'point vortex' centered at $\boldsymbol{x}_v$ inside the body

$$\boldsymbol{V} = U_\infty(\boldsymbol{i} + \boldsymbol{w} + \boldsymbol{v}), \qquad \boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{\Gamma} \times \operatorname{grad}\mathscr{E}_v, \qquad \mathscr{E}_v \equiv \mathscr{E}(\boldsymbol{x} - \boldsymbol{x}_v) \tag{4.2.1}$$

where $\boldsymbol{\Gamma} \equiv (\Gamma_x, \Gamma_y, \Gamma_z)$ is the a priori unknown vortex intensity; it will be determined such that the vectorial Kutta condition be fulfilled. Due to the properties

$$\operatorname{div}\boldsymbol{w} = 0, \qquad \operatorname{rot}\boldsymbol{w} = \boldsymbol{\Gamma}\delta_v$$

the system (2.5) is changed to

$$\operatorname{div}(\boldsymbol{v} - \boldsymbol{c}) = 0, \qquad \operatorname{rot}(v - \boldsymbol{c}) = -\boldsymbol{\Gamma}\delta_v \tag{4.2.2}$$

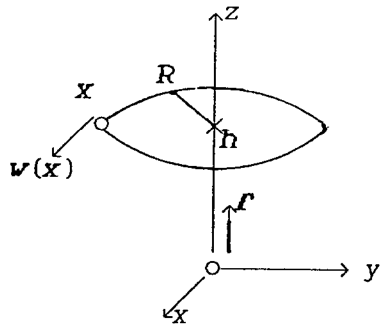


Fig. 4. Circulatory flow generated by a 'point vortex' pointing in the z-direction.

In order to obtain a regularized representation formula for the perturbatory velocity $\boldsymbol{v}$, we repeat the previous section's procedure. Starting with the relation (3.1.3) we have

$$\int_D [f\,\mathrm{div}(\boldsymbol{v}-\boldsymbol{c})+\boldsymbol{g}\cdot\mathrm{rot}(\boldsymbol{v}-\boldsymbol{c})]\,\mathrm{d}V=\begin{cases}-\boldsymbol{g}(\boldsymbol{x}_v)\cdot\boldsymbol{\Gamma} & \text{when } \boldsymbol{x}_v\in D\\ 0 & \text{when } \boldsymbol{x}_v\not\in D\end{cases}$$

As the point $\boldsymbol{x}_v$ stands outside the flow domain D, the entire previous procedure is valid but replacing the boundary condition (2.6) by

$$\boldsymbol{v}\cdot\boldsymbol{n}=-n_x-w_n\,,\qquad w_n\equiv\boldsymbol{w}\cdot\boldsymbol{n} \tag{4.2.3}$$

Finally, the representation formula (3.1.8) is changed to

$$\boldsymbol{v}_0={}^{\mathrm{R}}\!\!\int_\Sigma[-(n_x+w_n+\boldsymbol{n}\cdot\boldsymbol{v}_0)\,\mathrm{grad}\,\mathscr{E}_{\boldsymbol{x}_0}+\boldsymbol{n}\times(\boldsymbol{v}-\boldsymbol{v}_0)\times\mathrm{grad}\,\mathscr{E}_{\boldsymbol{x}_0}]\,\mathrm{d}a\,,\quad \boldsymbol{x}_0\in D+\Sigma \tag{4.2.4}$$

The relation between $\boldsymbol{v}$ and $\boldsymbol{F}$ being now

$$\boldsymbol{v}=-(n_x+w_n)\boldsymbol{n}-\boldsymbol{n}\times\boldsymbol{F}\,, \tag{4.2.5}$$

the integral equation (3.2.4) changes to

$$\begin{aligned}\boldsymbol{F}_0&+{}^{\mathrm{R}}\!\!\int_\Sigma\left[(\boldsymbol{n}_0\cdot\boldsymbol{F})\,\mathrm{grad}\,\mathscr{E}_{\boldsymbol{x}_0}+\left(\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}n}\boldsymbol{F}_0+\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}n_0}\boldsymbol{F}\right)\right]\mathrm{d}a+(\boldsymbol{n}_0\times\boldsymbol{F}_0)\,{}^{\mathrm{R}}\!\!\int_\Sigma(\boldsymbol{n}\times\boldsymbol{n}_0)\cdot\mathrm{grad}\,\mathscr{E}_{\boldsymbol{x}_0}\,\mathrm{d}a\\&-w_n^0(\boldsymbol{\Gamma})\,{}^{\mathrm{R}}\!\!\int_\Sigma\left[\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}n_0}(\boldsymbol{n}\times\boldsymbol{n}_0)+(\boldsymbol{n}\cdot\boldsymbol{n}_0-1)(\boldsymbol{n}_0\times\mathrm{grad}\,\mathscr{E}_{\boldsymbol{x}_0})\right]\mathrm{d}a\\&=n_x^0\,{}^{\mathrm{R}}\!\!\int_\Sigma\left[\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}n_0}(\boldsymbol{n}\times\boldsymbol{n}_0)+(\boldsymbol{n}\cdot\boldsymbol{n}_0-1)(\boldsymbol{n}_0\times\mathrm{grad}\,\mathscr{E}_{\boldsymbol{x}_0})\right]\mathrm{d}a\end{aligned} \tag{4.2.6}$$

the vectorial unknowns are $\boldsymbol{F}$ and $\boldsymbol{\Gamma}$, and the expression of w_n^0 is

$$w_n^0(\boldsymbol{\Gamma})=-\boldsymbol{\Gamma}\cdot[\boldsymbol{n}_0\times\mathrm{grad}\,\mathscr{E}(\boldsymbol{x}_0-\boldsymbol{x}_v)] \tag{4.2.7}$$

In the general case, when we have L vortices inside the body, i.e.

$$\boldsymbol{w}(\boldsymbol{x})=\sum_{i=1}^{L}\boldsymbol{\Gamma}_i\times\mathrm{grad}\,\mathscr{E}(\boldsymbol{x}-\boldsymbol{x}_v^i)\,,$$

there are L supplementary unknowns $(\boldsymbol{\Gamma}_i)_{i=1,L}$.

4.3. Connecting Kutta condition to the integral formulation

First, considering a discrete (numerical) constant-elements formulation of the problem, Eq. (4.1.3) should be used in the form

$$\boldsymbol{n}\times\boldsymbol{V}(P_{\mathrm{up}})+\boldsymbol{n}\times\boldsymbol{V}(P_{\mathrm{low}})=\boldsymbol{0} \tag{4.3.1}$$

The cross product $\boldsymbol{n}\times\boldsymbol{V}$ is written in $\boldsymbol{F}$ terms ($\boldsymbol{F}$ is the main unknown in the integral equation (4.2.6)) as

$$\frac{1}{U_\infty}\boldsymbol{n}\times\boldsymbol{V}=\boldsymbol{F}-\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{F})+\begin{pmatrix}0\\ n_z\\ -n_y\end{pmatrix}=\boldsymbol{F}+\begin{pmatrix}0\\ n_z\\ -n_y\end{pmatrix} \tag{4.3.2}$$

resulted from Eqs. (4.2.1) and (4.2.3). We write this equation L times for the case of L pairs of panels at the trailing edge.

Note, Kutta condition's appliance at small distance to the trailing edge (for the representing points of the adjacent panels). This shows that the problems (4.2.6) and (4.3.2) may give a good but not quite satisfactory solution. It can be improved by using convenient vortices intensities that will 'move' Kutta condition as close to the trailing edge as possible.

Thus, an accurate Kutta condition implementation appears to be the following:

(a) the vortex intensities *determination* by solving the problems (4.2.6) and (4.3.2)
(b) solving the problem (4.2.6) for *imposed* ($w_n(\boldsymbol{\Gamma})$ terms is moving to the right side of Eq. (4.2.6)) intensities, whose values have been determined at the precedent step but slightly modified according to the designer's experience. Numerical examples in this paper were obtained by applying the first step of the procedure only.

REMARK. The other methods using the equal-pressure Kutta condition lead to a *non-linear* problem having to be solved by an iterative procedure and yield the question of multiple roots [2]. The present method leads to a *linear* problem and that is another of its advantages.

5. Discrete equation

Herein, we treat the lifting flow problem only, the non-lifting flow being immediately deduced as a particular case. As in the panel method, we solve the integral equation (4.2.6) by the collocation method. The body surface is divided into N panels (quadrilaterals and/or triangles) on which we assume that F has a constant value, equal to that at the panel centroid. We write the discrete equation for all centroids $\boldsymbol{x}_i (i = 1, N)$ and obtain an algebraic system. Recalling that the integrals in (4.2.6) have regularized form, the omission of calculus on the singular panel comes from the constant-value assumption, i.e.

$$^{\mathrm{R}}\int_{\Sigma=\sum_{j=1}^{N}\text{Panel } j} = \int_{\sum_{j=1,\, j\neq i}^{N}\text{Panel } j}$$

In vectorial equation (4.2.6), all the quantities under the integral sign are panel-constant values, except the term grad $\mathscr{E}$. Thus, we have to calculate the integral

$$\int_{\text{Panel } j} \operatorname{grad} \mathscr{E}(\boldsymbol{x}_j - \boldsymbol{x}_i)\, \mathrm{d}a(\boldsymbol{x}_j)$$

It can be computed either by quadrature formula or by using analytical expressions ([12, p. 184])

$$\int_{\text{Panel } j} \frac{\partial}{\partial x} \mathscr{E}(\boldsymbol{x}_j - \boldsymbol{x}_i)\, \mathrm{d}a(\boldsymbol{x}_j) = \frac{1}{4\pi} \sum_{k=1}^{3(4)} \frac{y^j_{k+1} - y^j_k}{d_{k,k+1}} \ln \frac{r_k + r_{k+1} - d_{k,k+1}}{r_k + r_{k+1} + d_{k,k+1}}$$

$$\int_{\text{Panel } j} \frac{\partial}{\partial y} \mathscr{E}(\boldsymbol{x}_j - \boldsymbol{x}_i)\, \mathrm{d}a(\boldsymbol{x}_j) = -\frac{1}{4\pi} \sum_{k=1}^{3(4)} \frac{x^j_{k+1} - x^j_k}{d_{k,k+1}} \ln \frac{r_k + r_{k+1} - d_{k,k+1}}{r_k + r_{k+1} + d_{k,k+1}}$$

$$\int_{\text{Panel } j} \frac{\partial}{\partial z} \mathscr{E}(\boldsymbol{x}_j - \boldsymbol{x}_i)\, \mathrm{d}a(\boldsymbol{x}_j) = -\frac{1}{4\pi} \sum_{k=1}^{3(4)} \left(\arctan \frac{m_{k,k+1} e_k - h_k}{(z_i - z^j_k) r_k} - \arctan \frac{m_{k,k+1} e_{k+1} - h_{k+1}}{(z_i - z^j_k) r_{k+1}} \right)$$

$$d_{k,k+1} = \sqrt{(x^j_{k+1} - x^j_k)^2 + (y^j_{k+1} - y^j_k)^2}\,, \qquad m_{k,k+1} = \frac{y^j_{k+1} - y^j_k}{x^j_{k+1} - x^j_k}$$

$$r_k = \sqrt{(x_i - x^j_k)^2 + (y_i - y^j_k)^2 + (z_i - z^j_k)^2}\,, \quad e_k = (x_i - x^j_k)^2 + (z_i - z^j_k)^2\,, \quad h_k = (x_i - x^j_k)(y_i - y^j_k)$$

where the range of k is $\overline{1,3}$ if the panel is a triangle, or $\overline{1,4}$, if the panel is a quadrilateral; the panel j has the corner points denoted by $\boldsymbol{x}^j_1, \boldsymbol{x}^j_2, \boldsymbol{x}^j_3 (, \boldsymbol{x}^j_4)$. These formulas were deduced considering that the panel stands in a plane parallel to $\{z = 0\}$.

The vectoral algebraic system that comes from the discretization of Eq. (4.2.6) is

$$\sum_{j=1}^{N} A_{ij} \boldsymbol{F}_j + \sum_{k=1}^{L} B_k \boldsymbol{\Gamma}_k = C_i\,, \quad i = 1, \ldots, N \tag{5.1}$$

where

$$A_{ii}\boldsymbol{F}_i = \boldsymbol{F}_i + \boldsymbol{F}_i \sum_{\substack{j=1\\ j\neq i}}^{N} \int_{\text{Panel } j} \frac{\mathrm{d}\mathscr{E}}{\mathrm{d}n}\,\mathrm{d}a + (\boldsymbol{n}_i \times \boldsymbol{F}_i) \sum_{\substack{j=1\\ j\neq i}}^{N} \int_{\text{Panel } j} (\boldsymbol{n}_j \times \boldsymbol{n}_i)\cdot \operatorname{grad}\mathscr{E}\cdot \mathrm{d}a$$

$$A_{ij}\boldsymbol{F}_j = \sum_{\substack{j=1\\ j\neq i}}^{N} \boldsymbol{F}_j \int_{\text{Panel } j} \frac{\mathrm{d}\mathscr{E}}{\mathrm{d}n_i}\,\mathrm{d}a + \sum_{\substack{j=1\\ j\neq i}}^{N} \int_{\text{Panel } j} (\boldsymbol{n}_i \cdot \boldsymbol{F}_j) \operatorname{grad}\mathscr{E}\cdot \mathrm{d}a$$

$$B_k\boldsymbol{\Gamma}_k = -\boldsymbol{\Gamma}_k \cdot [\boldsymbol{n}_i \times \operatorname{grad}\mathscr{E}(\boldsymbol{x}_i - \boldsymbol{x}_k)] \sum_{\substack{j=1\\ j\neq i}}^{N} \int_{\text{Panel } j} \left[\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}n_i}(\boldsymbol{n}_j \times \boldsymbol{n}_i) + (\boldsymbol{n}_j \cdot \boldsymbol{n}_i - 1)(\boldsymbol{n}_i \times \operatorname{grad}\mathscr{E})\right] \mathrm{d}a$$

$$C_i = n_x^i \sum_{\substack{j=1\\ j\neq i}}^{N} \int_{\text{Panel } j} \left[\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}n_i}(\boldsymbol{n}_j \times \boldsymbol{n}_i) + (\boldsymbol{n}_j \cdot \boldsymbol{n}_i - 1)(\boldsymbol{n}_i \times \operatorname{grad}\mathscr{E})\right] \mathrm{d}a$$

The Kutta condition (4.2.3) for a single pair of panels (p, q) generates the additional vectorial equation

$$\boldsymbol{F}_p + \boldsymbol{F}_q = -\begin{pmatrix} 0 \\ n_z^p \\ -n_y^p \end{pmatrix} - \begin{pmatrix} 0 \\ n_z^q \\ -n_y^q \end{pmatrix} \tag{5.2}$$

Analogously, a multiple panels formula can be written as well.

The matrix form of the system (5.1) and (5.2) is

$$\left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{K} & \boldsymbol{0} \end{array}\right] \cdot \left[\begin{array}{c} \boldsymbol{F} \\ \hline \boldsymbol{\Gamma} \end{array}\right] = \left[\begin{array}{c} \boldsymbol{C} \\ \hline \boldsymbol{C}_{\Gamma} \end{array}\right] \tag{5.3a}$$

or

$$\begin{cases} \boldsymbol{AF} + \boldsymbol{B\Gamma} = \boldsymbol{C} \\ \boldsymbol{KF} = \boldsymbol{C}_{\Gamma} \end{cases} \tag{5.3b}$$

where $\boldsymbol{K}$ and $\boldsymbol{C}_{\Gamma}$ comes from the supplementary Kutta conditions (5.2).

It is clear that for the lifting case, an iterative method fails on system (5.3) because there are null elements on the main diagonal.

When a non-lifting case is considered, the system $\boldsymbol{AF} = \boldsymbol{C}$ has to be solved and an iterative method can be applied: as the numerical tests showed, $\boldsymbol{A}$ is a full and diagonal-dominant matrix.

6. Testing the method

In order to test the method we have simulated a 2D flow, using a rectangular high spanwise wing; thus, in the middle section, the velocity distribution is a 2D one. The comparison was made by the 2D analytic solution and also by another numerical method, the panel method.

The wing profile is the Van de Vooren symmetric profile (Fig. 5). The complex coordinate of a point on the profile is [12]

$$z = 1 + \frac{[a(\cos\theta - 1) + ia\sin\theta]^k}{[a(\cos\theta - \varepsilon) + ia\sin\theta]^{k-1}}, \qquad a = 2(1+\varepsilon)^{k-1}2^{-k}, \qquad \tau = \pi(2-k)$$

where ε is a thickness parameter and k controls the trailing-edge angle, τ. The exact velocity distribution on the profile at incidence α is

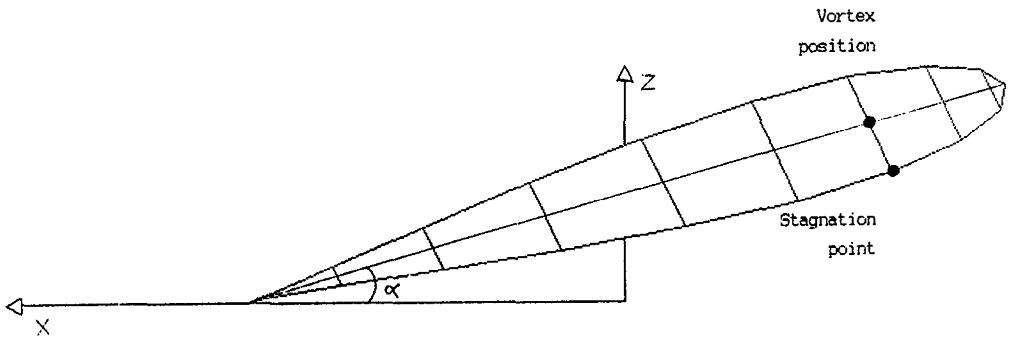


Fig. 5. Van de Vooren profile.

$$U = 2U_\infty \frac{r_2^k}{r_1^{k-1}} \frac{\sin\alpha - \sin(\alpha - \theta)}{D_1^2 + D_2^2} (D_1 \sin\theta + D_2 \cos\theta)$$

$$W = -2U_\infty \frac{r_2^k}{r_1^{k-1}} \frac{\sin\alpha - \sin(\alpha - \theta)}{D_1^2 + D_2^2} (D_1 \cos\theta - D_2 \sin\theta)$$

where

$$r_1 = a\sqrt{(\cos\theta - 1)^2 + \sin^2\theta}\ , \qquad r_2 = a\sqrt{(\cos\theta - \varepsilon)^2 + \sin^2\theta}$$

$$\theta_1 = \arctan\frac{\sin\theta}{\cos\theta - 1} + \pi\ , \qquad \theta_2 = \arctan\frac{\sin\theta}{\cos\theta - \varepsilon} + n\pi$$

$$A = \cos(k-1)\theta_1 \cos k\theta_2 + \sin(k-1)\theta_1 \sin k\theta_2\ , \qquad B = \sin(k-1)\theta_1 \cos k\theta_2 - \cos(k-1)\theta_1 \sin k\theta_2$$

$$D_0 = a(1 - k + k\varepsilon)\ , \qquad D_1 = A(a\cos\theta - D_0) - Ba\sin\theta\ , \qquad D_2 = B(a\cos\theta - D_0) + Aa\sin\theta$$

Here, n depends on the quadrant where θ_2 is being evaluated: $n = 0$ in the first quadrant, $n = 1$ in the second and third quadrants and $n = 2$ in the fourth quadrant. For our example we have chosen $\varepsilon = 0.1$ and $\tau = 10^0$.

In the case of non-lifting flow, the present method proves its superiority over the panel method (Fig. 7) at the same discretization of the wing in 20 panels per strip (Fig. 6). In order to make this comparison, we have implemented the panel method [1].

In the lifting flow case we have compared our results to the analytical solution and they appeared to be suitable (Fig. 8). On a strip, the 'point vortex' has been located at the same place as for the two-dimensional case [13]: on the camber line, near the stagnation point (Fig. 5). For a wing tip, the 'point vortex' has been located near the corner. Both the present method's solution and the analytical one are displayed for a 20° incidence in Fig. 8. Each strip has been divided into 40 panels. Fig. 9 shows the stream surface starting from T.E.

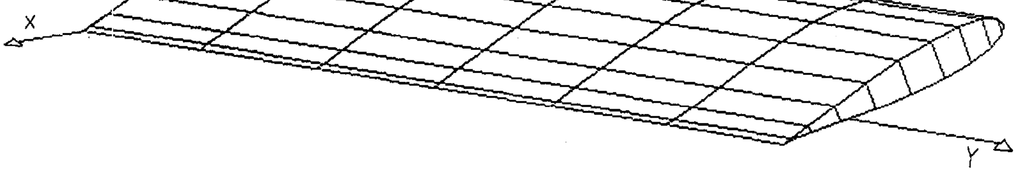


Fig. 6. Rectangular high-spanwise wing: discretization into 130 panels/half-wing.

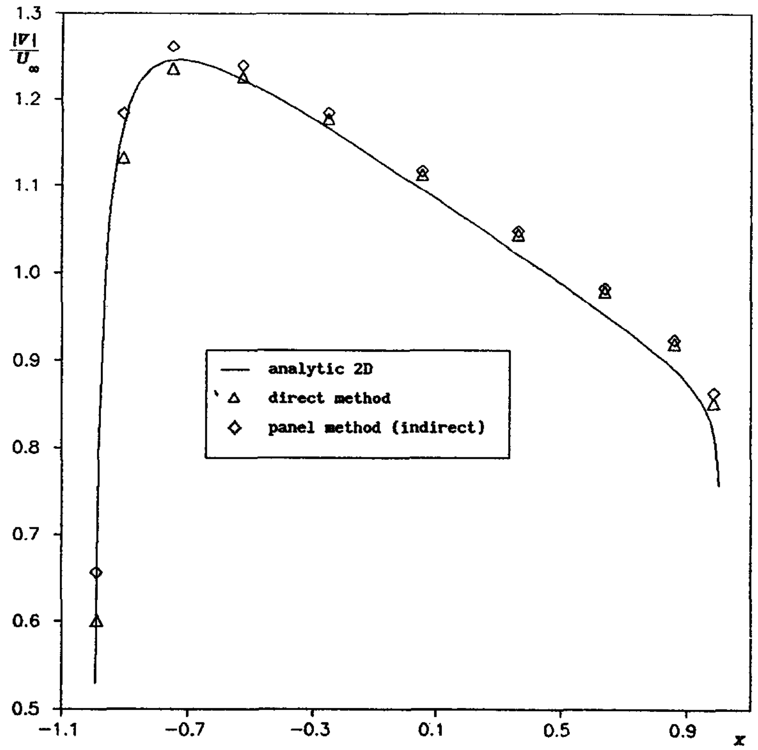
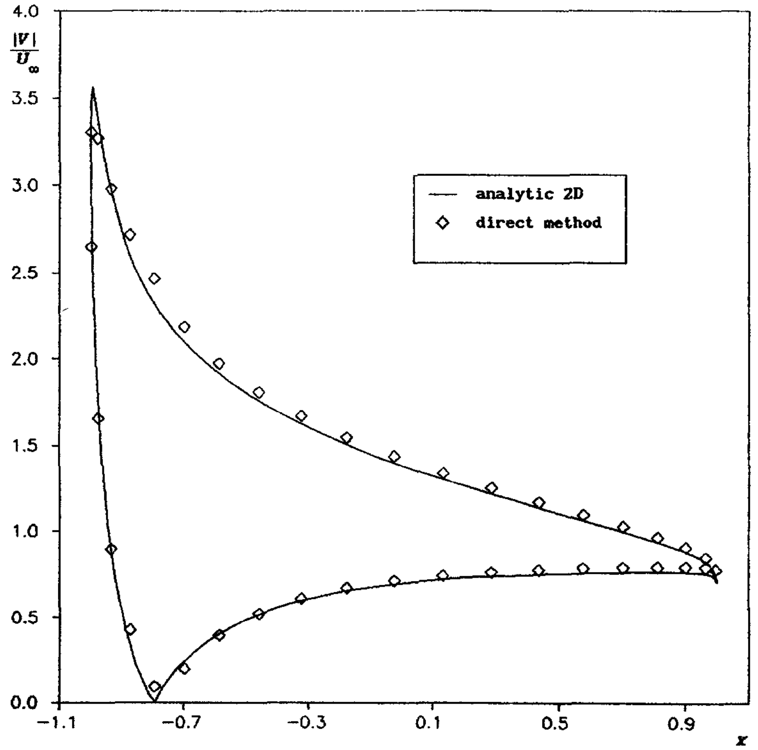


Fig. 7. Velocity distribution on the middle section of the rectangular high spanwise non-lifting wing



ig. 8. Velocity distribution on the middle section of the rectangular high spanwise lifting wing at 20° incidence.

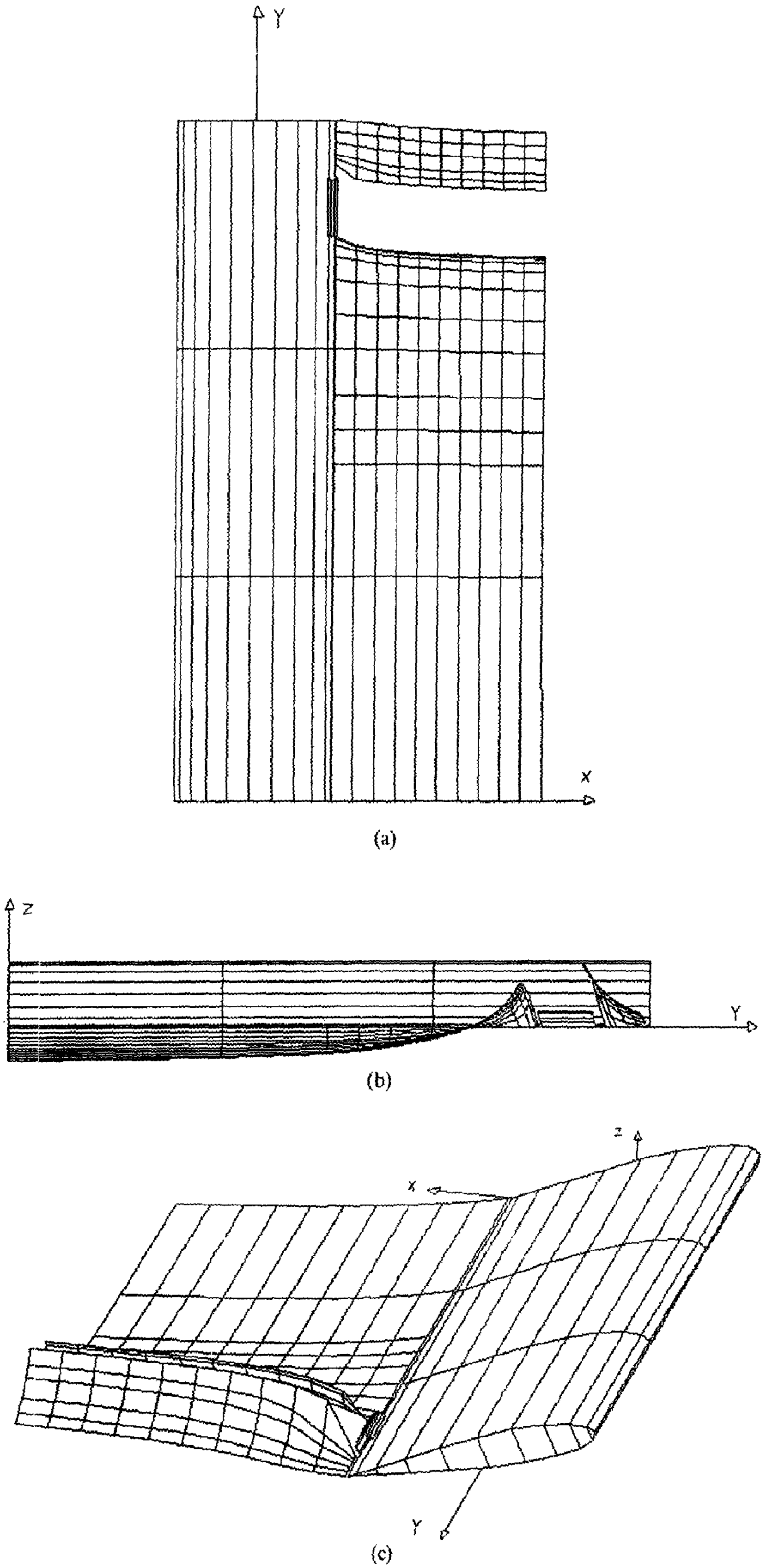


Fig. 9. Half stream-surface starting from the T.E. of the lifting wing. (a) Top view; (b) back side view; (c) extrados view.

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