Acta Mechanica 42, 3—10 (1982)

ACTA MECHANICA
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On the Slow Steady Motion of a Viscous Fluid Due to Two Rotating Tori

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With 3 Figures

(*Received June 9, 1980, revised January 7, 1981*)

Summary

An approximate solution, correct to order three in a small parameter, is established for the velocity field induced in a viscous fluid by the slow rotation of two tori. The tori possess a common axis and have concentric circular sections. The small parameter is the ratio of the radius of the outer circular section to the distance of the section centre from the axis of rotation. The torque acting on each toroidal surface is also obtained.

1. Introduction

The problem of determining the velocity field induced in a viscous fluid by the slow steady rotation of axially symmetric bodies has received the attention of many authors in the literature. Exact solutions of the linearized equations of motion, which apply when inertial effects are neglected, have been obtained for a number of body shapes. Many of the solutions which have been found recently (e.g. see [1], [2]) involved the use of a dipolar or toroidal coordinate system for their determination. Using these coordinate systems it was possible to express the solutions for the velocity fields in terms of well known potential functions. However, when the slow steady motion is induced by the rotation of two tori which have a common axis and concentric circular sections it is not possible to use toroidal coordinates since these surfaces do not coincide with two of the coordinate surfaces of the toroidal system.

In the solution of this problem an orthogonal curvilinear coordinate system is employed which is a special case of the coordinate system established recently for a general twisted tube [3]. The boundary value problem governing the fluid motion is formulated and it is shown how a solution scheme can be constructed which involves a small parameter. This small parameter is the ratio of the radius of the outer circular section to the distance of the section centre from the axis of rotation. An approximate solution is obtained for the velocity field which is correct to order three in the small parameter and an expression for the torque exerted by the fluid on each toroidal surface is also obtained which is correct to order three in the small parameter.

0001-5970/82/0042/0003/$01.60

2. The Boundary Value Problem — Solution Scheme

We consider the slow steady motion of a viscous incompressible fluid between two tori which are rotating about the z-axis (see Fig. 1). The tori have concentric circular sections with radii b and c and rotate about the z-axis with angular velocies ω_i and ω_0 respectively. If a is the distance of a section centre from the z-axis then the coordinates of a point in the region bounded by the tori are given by

$$x = (a - r\cos\theta)\cos\phi\,, \tag{2.1}$$

$$y = (a - r\cos\theta)\sin\phi\,, \tag{2.2}$$

$$z = r\sin\theta\,, \tag{2.3}$$

where ϕ is the angle between the plane section containing the point and the xz plane. The scaling factors associated with the orthogonal curvilinear coordinate system (r, θ, ϕ) are $h_1 = 1$, $h_2 = r$ and $h_3 = a - r\cos\theta$ respectively.

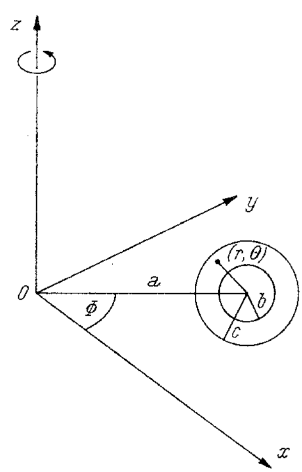

Fig. 1

If the components of velocity in the plane $\phi = \text{const.}$ are zero and $w(R, z)$ is the component normal to $\phi = \text{const.}$ where $R = (x^2 + y^2)^{1/2}$, and the fluid pressure p is constant then (e.g. see [1]) the Stokes equations governing the fluid motion reduce to

$$\frac{\partial^2 w}{\partial R^2} + \frac{1}{R}\frac{\partial w}{\partial R} + \frac{\partial^2 w}{\partial z^2} - \frac{w}{R^2} = 0\,. \tag{2.4}$$

Equation (2.4) must be satisfied subject to the boundary conditions

$$w = \omega_i(a - b\cos\theta)\,, \qquad r = b\,, \qquad 0 \leqq \theta \leqq 2\pi\,, \tag{2.5a}$$

and

$$w = \omega_0(a - c\cos\theta)\,, \qquad r = c\,, \qquad 0 \leqq \theta \leqq 2\pi\,. \tag{2.5b}$$

This completes the formulation of the boundary value problem.

In the (r, θ, ϕ) coordinate system Eq. (2.4) has the form

$$\frac{1}{r(a - r\cos\theta)}\left(\frac{\partial}{\partial r}\left(r(a - r\cos\theta)\frac{\partial w}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{a - r\cos\theta}{r}\frac{\partial w}{\partial\theta}\right)\right) - \frac{w}{(a - r\cos\theta)^2} = 0. \tag{2.6}$$

Throughout the rest of the analysis we will assume that $c/a (\equiv \varepsilon) \ll 1$ so that if $w = a\omega(r, \theta)$ and $r = cr'$ the equation governing ω will be, after dropping the prime for convenience,

$$\frac{\partial^2\omega}{\partial r^2} + \frac{1}{r}\frac{\partial\omega}{\partial r} + \frac{1}{r^2}\frac{\partial^2\omega}{\partial\theta^2} = \varepsilon\left(r\cos\theta\left(2\frac{\partial^2\omega}{\partial r^2} + \frac{3}{r}\frac{\partial\omega}{\partial r} + \frac{2}{r^2}\frac{\partial^2\omega}{\partial\theta^2}\right) - \frac{\sin\theta}{r}\frac{\partial\omega}{\partial\theta}\right)$$
$$+ \varepsilon^2\left(\omega + \sin\theta\cos\theta\frac{\partial\omega}{\partial\theta} - r^2\cos^2\theta\left(\frac{\partial^2\omega}{\partial r^2} + \frac{2}{r}\frac{\partial\omega}{\partial r} + \frac{1}{r^2}\frac{\partial^2\omega}{\partial\theta^2}\right)\right). \tag{2.7}$$

The boundary conditions are

$$\omega = \omega_i(1 - \varepsilon r_i\cos\theta), \qquad r = r_i, \qquad 0 \leqq \theta \leqq 2\pi, \tag{2.8}$$

and

$$\omega = \omega_0(1 - \varepsilon\cos\theta), \qquad r = 1, \qquad 0 \leqq \theta \leqq 2\pi, \tag{2.9}$$

where

$$r_i = b/c.$$

We will seek a series solution of the boundary value problem posed by Eqs. (2.7) to (2.9) in the form

$$\omega = \sum_{n=0}^{\infty} \varepsilon^n f^{(n)}(r, \theta). \tag{2.10}$$

On substitution in (2.7)—(2.9) the resulting system of boundary value problems is

$$\frac{\partial^2 f^{(0)}}{\partial r^2} + \frac{1}{r}\frac{\partial f^{(0)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f^{(0)}}{\partial\theta^2} = 0, \tag{2.11}$$

$$f^{(0)} = \omega_i, \qquad r = r_i, \qquad 0 \leqq \theta \leqq 2\pi, \tag{2.12}$$

$$f^{(0)} = \omega_0, \qquad r = 1, \qquad 0 \leqq \theta \leqq 2\pi, \tag{2.13}$$

and

$$\frac{\partial^2 f^{(1)}}{\partial r^2} + \frac{1}{r}\frac{\partial f^{(1)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f^{(1)}}{\partial\theta^2} = \cos\theta\frac{\partial f^{(0)}}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f^{(0)}}{\partial\theta}, \tag{2.14}$$

$$f^{(1)} = -\omega_i r_i\cos\theta, \qquad r = r_i, \qquad 0 \leqq \theta \leqq 2\pi, \tag{2.15}$$

$$f^{(1)} = -\omega_0\cos\theta, \qquad r = 1, \qquad 0 \leqq \theta \leqq 2\pi, \tag{2.16}$$

together with

$$\frac{\partial^2 f^{(n)}}{\partial r^2} + \frac{1}{r}\frac{\partial f^{(n)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f^{(n)}}{\partial\theta^2} = r\cos\theta\left(2\frac{\partial^2 f^{(n-1)}}{\partial r^2} + \frac{3}{r}\frac{\partial f^{(n-1)}}{\partial r} + \frac{2}{r^2}\frac{\partial^2 f^{(n-1)}}{\partial\theta^2}\right)$$
$$- \frac{\sin\theta}{r}\frac{\partial f^{(n-1)}}{\partial\theta} + f^{(n-2)} + \sin\theta\cos\theta\frac{\partial f^{(n-2)}}{\partial\theta} - r^2\cos^2\theta$$
$$\cdot\left(\frac{\partial^2 f^{(n-2)}}{\partial r^2} + \frac{2}{r}\frac{\partial f^{(n-2)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f^{(n-2)}}{\partial\theta^2}\right), \qquad n \geqq 2, \tag{2.17}$$

$$f^{(n)} = 0, \qquad r = r_i, 1, \qquad 0 \leqq \theta \leqq 2\pi, \qquad n \geqq 2. \tag{2.18}$$

In the following section the first four terms of the solution (2.10) will be obtained.

3. Solutions for $f^{(k)}$, $k = 0, 1, 2, 3$

It can readily be shown that $f^{(0)}$ is given by

$$f^{(0)} = (\omega_i - \omega_0)\frac{\ln r}{\ln r_i} + \omega_0. \tag{3.1}$$

The equation governing $f^{(1)}$ is then

$$\frac{\partial^2 f^{(1)}}{\partial r^2} + \frac{1}{r}\frac{\partial f^{(1)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f^{(1)}}{\partial \theta^2} = \frac{\omega_i - \omega_0}{\ln r_i}\cdot\frac{\cos\theta}{r}. \tag{3.2}$$

We set $f^{(1)}(r, \theta) = g^{(1)}(r)\cos\theta$ so that $g^{(1)}(r)$ satisfies

$$\frac{d^2 g^{(1)}}{dr^2} + \frac{1}{r}\frac{dg^{(1)}}{dr} - \frac{g^{(1)}}{r^2} = \frac{\omega_i - \omega_0}{\ln r_i}\,\frac{1}{r}, \qquad r_i < r < 1, \tag{3.3}$$

subject to the boundary conditions

$$g^{(1)} = -\omega_i r_i, \qquad r = r_i, \tag{3.4}$$

$$g^{(1)} = -\omega_0, \qquad r = 1. \tag{3.5}$$

The boundary value problem (3.3)—(3.5) admits the solution

$$g^{(1)}(r) = \frac{3}{2}\,\frac{(\omega_i - \omega_0)}{1 - r_i^2}\,r_i^2\left(r - \frac{1}{r}\right) - \omega_0 r + \frac{(\omega_i - \omega_0)\,r\ln r}{2\ln r_i}. \tag{3.6}$$

Using (2.17) together with the solutions for $f^{(0)}$ and $f^{(1)}$ we find that $f^{(2)}$ satisfies

$$\frac{\partial^2 f^{(2)}}{\partial r^2} + \frac{1}{r}\frac{\partial f^{(2)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f^{(2)}}{\partial \theta^2}$$

$$= \frac{3}{2}(\omega_i - \omega_0)\left(\frac{r_i^2}{1 - r_i^2} + \frac{1 + 2\ln r}{2\ln r_i} + \frac{\cos 2\theta}{2\ln r_i} + \frac{r_i^2}{(1 - r_i^2)}\,\frac{\cos 2\theta}{r^2}\right). \tag{3.7}$$

We will seek $f^{(2)}$ in the form

$$f^{(2)} = \frac{3}{2}\,\frac{(\omega_i - \omega_0)\,r_i^2}{1 - r_i^2}\,g_1^{(2)}(r) + \frac{3}{2}\,\frac{(\omega_i - \omega_0)}{\ln r_i}\,g_2^{(2)}(r)$$

$$+ \frac{3}{4}\,\frac{(\omega_i - \omega_0)}{\ln r_i}\,g_3^{(2)}(r)\cos 2\theta + \frac{3}{2}\,\frac{(\omega_i - \omega_0)}{1 - r_i^2}\,r_i^2 g_4^{(2)}(r)\cos 2\theta, \tag{3.8}$$

where

$$\frac{d^2 g_1^{(2)}}{dr^2} + \frac{1}{r}\frac{dg_1^{(2)}}{dr} = 1, \qquad r_i < r < 1, \tag{3.9}$$

$$\frac{d^2 g_2^{(2)}}{dr^2} + \frac{1}{r}\frac{dg_2^{(2)}}{dr} = \frac{1}{2} + \ln r, \qquad r_i < r < 1, \tag{3.10}$$

$$\frac{d^2 g_3^{(2)}}{dr^2} + \frac{1}{r}\frac{dg_3^{(2)}}{dr} - \frac{4g_3^{(2)}}{r^2} = 1, \qquad r_i < r < 1, \tag{3.11}$$

$$\frac{d^2 g_4^{(2)}}{dr^2} + \frac{1}{r}\frac{dg_4^{(2)}}{dr} - \frac{4g_4^{(2)}}{r^2} = \frac{1}{r^2}, \qquad r_i < r < 1, \tag{3.12}$$

and

$$g_j^{(2)} = 0, \qquad r = r_i, 1, \qquad j = 1, 2, 3, 4. \tag{3.13}$$

Employing standard techniques we obtain the solutions of (3.9)—(3.13) in the form

$$g_1^{(2)} = \frac{1}{4}(r^2 - 1) + \frac{1}{4}\frac{1 - r_i^2}{\ln r_i}\ln r, \tag{3.14}$$

$$g_2^{(2)} = \frac{1}{8}(1 - r^2) + \frac{1}{4}r^2 \ln r + \left(\frac{1}{8}\frac{r_i^2 - 1}{\ln r_i} - \frac{1}{4}r_i^2\right)\ln r, \tag{3.15}$$

$$g_3^{(2)} = \frac{r_i^4 \ln r_i}{4(1 - r_i^4)}\left(r^2 - \frac{1}{r^2}\right) + \frac{1}{4}r^2 \ln r, \tag{3.16}$$

$$g_4^{(2)} = \frac{1}{4(1 + r_i^2)}\left(r^2 + \frac{r_i^2}{r^2}\right) - \frac{1}{4}. \tag{3.17}$$

Finally, with the aid of the solutions for $f^{(0)}$, $f^{(1)}$ and $f^{(2)}$ we find that $f^{(3)}$ satisfies

$$\begin{aligned}\frac{\partial^2 f^{(3)}}{\partial r^2} + \frac{1}{r}\frac{\partial f^{(3)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f^{(3)}}{\partial \theta^2} &= \frac{3}{8}(\omega_i - \omega_0)\left(\left(\frac{r_i^2(11r_i^2 + 12)}{1 - r_i^4} + \frac{13}{4\ln r_i}\right) r\cos\theta\right.\\ &+ \frac{11r\ln r}{\ln r_i}\cos\theta - \left(\frac{3r_i^2}{1 - r_i^2} + \frac{1 - r_i^2}{2(\ln r_i)^2}\right)\frac{\cos\theta}{r}\\ &\left.+ \frac{5r\cos 3\theta}{4\ln r_i} + \frac{3r_i^2}{1 - r_i^2}\cdot\frac{\cos 3\theta}{r} - \frac{r_i^4}{1 - r_i^4}\cdot\frac{\cos 3\theta}{r^3}\right).\end{aligned} \tag{3.18}$$

The solution for $f^{(3)}$ can be written in the form

$$\begin{aligned}f^{(3)} = \frac{3}{8}(\omega_i - \omega_0)&\left(\left(\frac{(11r_i^2 + 12)r_i^2}{1 - r_i^4} + \frac{13}{4\ln r_i}\right) g_1^{(3)}(r)\cos\theta\right.\\ &+ \frac{11}{\ln r_i} g_2^{(3)}(r)\cos\theta + \left(\frac{3r_i^2}{1 - r_i^2} + \frac{1 - r_i^2}{2(\ln r_i)^2}\right) g_3^{(3)}\cos\theta\\ &+ \frac{5}{4\ln r_i} g_4^{(3)}(r)\cos 3\theta + \frac{3r_i^2}{1 - r_i^2} g_5^{(3)}(r)\cos 3\theta\\ &\left.+ \frac{r_i^4}{1 - r_i^4} g_6^{(3)}(r)\cos 3\theta\right),\end{aligned} \tag{3.19}$$

where

$$\frac{d^2 g_1^{(3)}}{dr^2} + \frac{1}{r}\frac{dg_1^{(3)}}{dr} - \frac{g_1^{(3)}}{r^2} = r, \qquad r_i < r < 1, \tag{3.20}$$

$$\frac{d^2 g_2^{(3)}}{dr^2} + \frac{1}{r}\frac{dg_2^{(3)}}{dr} - \frac{g_2^{(3)}}{r^2} = r\ln r, \qquad r_i < r < 1, \tag{3.21}$$

$$\frac{d^2 g_3^{(3)}}{dr^2} + \frac{1}{r}\frac{dg_3^{(3)}}{dr} - \frac{g_3^{(3)}}{r^2} = -\frac{1}{r}, \qquad r_i < r < 1, \tag{3.22}$$

$$\frac{d^2 g_4^{(3)}}{dr^2} + \frac{1}{r}\frac{dg_4^{(3)}}{dr} - \frac{9g_4^{(3)}}{r^2} = r, \qquad r_i < r < 1, \tag{3.23}$$

$$\frac{d^2 g_5^{(3)}}{dr^2} + \frac{1}{r}\frac{dg_5^{(3)}}{dr} - \frac{9g_5^{(3)}}{r^2} = \frac{1}{r}, \qquad r_i < r < 1, \tag{3.24}$$

$$\frac{d^2 g_6^{(3)}}{dr^2} + \frac{1}{r}\frac{dg_6^{(3)}}{dr} - \frac{9g_6^{(3)}}{r^2} = -\frac{1}{r^3}, \qquad r_i < r < 1, \tag{3.25}$$

and

$$g_j^{(3)} = 0, \qquad r = r_i, 1, \qquad j = 1, 2, 3, 4, 5, 6. \tag{3.26}$$

Again, by standard techniques, we find

$$g_1^{(3)} = \frac{1}{8}\left(-(1 + r_i^2)\, r + \frac{r_i^2}{r} + r^3\right), \tag{3.27}$$

$$g_2^{(3)} = \frac{1}{8}\left(\left(\frac{3}{4}(1 + r_i^2) + \frac{r_i^4 \ln r_i}{1 - r_i^2}\right) r - \left(\frac{3r_i^2}{4} + \frac{r_i^4 \ln r_i}{1 - r_i^2}\right)\frac{1}{r} + r^3 \ln r - \frac{3r^3}{4}\right), \tag{3.28}$$

$$g_3^{(3)} = -\frac{1}{2}\left(\frac{r_i^2 \ln r_i}{1 - r_i^2}\left(r - \frac{1}{r}\right) + r \ln r\right), \tag{3.29}$$

$$g_4^{(3)} = \frac{1}{6}\left(\frac{r_i^6 \ln r_i}{1 - r_i^6}\left(r^3 - \frac{1}{r^3}\right) + r^3 \ln r\right), \tag{3.30}$$

$$g_5^{(3)} = \frac{1}{8}\left(\frac{1 - r_i^4}{1 - r_i^6} \cdot r^3 + \frac{r_i^4(1 - r_i^2)}{1 - r_i^6} \cdot \frac{1}{r^3} - \frac{r}{8}\right), \tag{3.31}$$

$$g_6^{(3)} = -\frac{1}{8}\left(\frac{1 - r_i^2}{1 - r_i^6} \cdot r^3 + \frac{r_i^2(1 - r_i^4)}{1 - r_i^6} \cdot \frac{1}{r^3} - \frac{1}{r}\right). \tag{3.32}$$

This completes the determination of the functions $f^{(k)}$, $k = 0, 1, 2, 3$ and provides us with a solution for the velocity field which is correct to order three in ε. In the next and final section we will obtain expressions for the torque acting on the two toroidal surfaces.

4. The Torque

The torque T_i exerted by the fluid on the inner torus is given by

$$T_i = \left[\mu \int_0^{2\pi}\int_0^{2\pi} h_2 h_3^2 \frac{\partial}{\partial r}\left(\frac{w}{h_3}\right) d\theta\, d\phi\right]_{r=r_i}, \tag{4.1}$$

so that

$$T_i = 2\pi\mu a^3 \int_0^{2\pi} r_i \left((1 - \varepsilon r_i \cos\theta)^2 \left[\frac{\partial\omega}{\partial r}\right]_{r=r_i} + (1 - \varepsilon r_i \cos\theta)\, \varepsilon \cos\theta [\omega]_{r=r_i}\right) d\theta, \tag{4.2}$$

where

$$[\omega]_{r=r_i} = \omega_i(1 - \varepsilon r_i \cos\theta), \tag{4.3}$$

and

$$\left[\frac{\partial\omega}{\partial r}\right]_{r=r_i} = \sum_{n=0}^{3} \varepsilon^n \left[\frac{\partial f^{(n)}}{\partial r}\right]_{r=r_i}. \tag{4.4}$$

Using the solutions for $f^{(n)}$, $n = 0, 1, 2, 3$ obtained in the last section we find

$$T_i = \frac{4\pi^2 \mu a^3(\omega_i - \omega_0)}{\ln r_i} - 3\pi^2 \mu a^3(\omega_i - \omega_0)\left(\frac{3r_i^2}{1 - r_i^2} + \frac{1 - r_i^2}{4(\ln r_i)^2}\right)\varepsilon^2 + 0(\varepsilon^4). \quad (4.5)$$

Similarly the torque T_0 exerted by the fluid on the outer torus is given by

$$T_0 = -\left[\mu \int_0^{2\pi}\int_0^{2\pi} h_2 h_3^2 \frac{\partial}{\partial r}\left(\frac{w}{h_3}\right) d\theta\, d\phi\right]_{r=1}, \quad (4.6)$$

and we find $T_0 = -T_i$.

We note that the expression for the torque does not contain terms in ε or ε^3. Also the contribution from $f^{(3)}$ in the computation of the torque is $0(\varepsilon^4)$. We can write the expression for T_i $(= -T_0 \equiv T)$ in the form

$$T = \pi^2 \mu a^3(\omega_0 - \omega_i)\,(T^{(0)} + \varepsilon^2 T^{(2)}) + 0(\varepsilon^4), \quad (4.7)$$

where the dimensionless quantities $T^{(0)}$ and $T^{(2)}$ are given by

$$T^{(0)} = -\frac{4}{\ln r_i}, \quad (4.8)$$

and

$$T^{(2)} = 3\left(\frac{3r_i^2}{1 - r_i^2} + \frac{1 - r_i^2}{4(\ln r_i)^2}\right). \quad (4.9)$$

In Figs. 2 and 3 $10^{-2}T^{(0)}$ and $10^{-2}T^{(2)}$ are shown plotted against r_i.

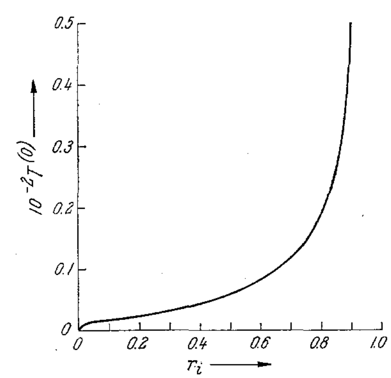

Fig. 2

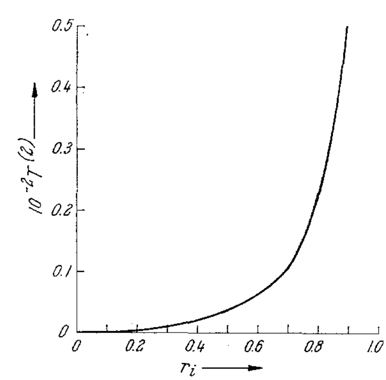

Fig. 3

In conclusion we note that the results obtained above are valid for $0 \leqq r_i < 1$. When $r_i \to 0$ it can be shown that $f^{(0)} = \omega_0$, $f^{(1)} = -\omega_0 r \cos\theta$, $f^{(k)} = 0$, $k \geqq 2$. Therefore when the inner torus is absent the fluid rotates like a rigid body and the torque is zero. Also when $\omega_i = \omega_0$, $f^{(0)} = \omega_0$, $f^{(1)} = -\omega_0 r \cos\theta$, $f^{(k)} = 0$, $k \geqq 2$ and again the fluid rotates like a rigid body and the torque is zero.

The author wishes to thank the reviewer for many helpful comments on the paper.

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