Discrete Applied Mathematics 158 (2010) 110–122

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Characterising (k, ℓ)-leaf powers

Andreas Brandstädt*, Peter Wagner

Institut für Informatik, Universität Rostock, D-18051 Rostock, Germany

ARTICLE INFO

Article history:
Received 18 December 2008
Received in revised form 18 August 2009
Accepted 2 September 2009
Available online 29 September 2009

Keywords:
(k, ℓ)-leaf powers
k-leaf powers
Trees
Block graphs
Strictly chordal graphs
Characterisations
Recognition algorithms

ABSTRACT

We say that, for $k \geq 2$ and $\ell > k$, a tree T with distance function $d_T(x, y)$ is a *(k, ℓ)-leaf root of a finite simple graph* $G = (V, E)$ if V is the set of leaves of T, for all edges $xy \in E$, $d_T(x, y) \leq k$, and for all non-edges $xy \notin E$, $d_T(x, y) \geq \ell$. A graph is a *(k, ℓ)-leaf power* if it has a (k, ℓ)-leaf root. This new notion modifies the concept of k-leaf powers (which are, in our terminology, the $(k, k+1)$-leaf powers) introduced and studied by Nishimura, Ragde and Thilikos; k-leaf powers are motivated by the search for underlying phylogenetic trees. Recently, a lot of work has been done on k-leaf powers and roots as well as on their variants phylogenetic roots and Steiner roots. Many problems, however, remain open.

We give the structural characterisations of (k, ℓ)-leaf powers, for some k and ℓ, which also imply an efficient recognition of these classes, and in this way we improve and extend a recent paper by Kennedy, Lin and Yan on strictly chordal graphs; one of our motivations for studying (k, ℓ)-leaf powers is the fact that strictly chordal graphs are precisely the $(4, 6)$-leaf powers.

© 2009 Elsevier B.V. All rights reserved.

1. Introduction

Motivated by the background in phylogenetics, i.e., the study of evolutionary history, Nishimura, Ragde and Thilikos [28] introduced the notion of *k-leaf power* and *k-leaf root* as follows: Let $G = (V, E)$ be a finite simple graph, i.e., a finite, undirected graph without loops and parallel edges. For $k \geq 2$, a tree T is a *k-leaf root* of G if V is the set of leaves of T and, for any two distinct vertices $x, y \in V$, x and y are adjacent in G if and only if their distance $d_T(x, y)$ in T is at most k; that is, $xy \in E \iff d_T(x, y) \leq k$. A graph is a *$k$-leaf power* if it has a k-leaf root. Obviously, a graph is a 2-leaf power if and only if it is the disjoint union of cliques; that is, it contains no induced path P_3 with three vertices and two edges.

See [4–8,10–12,15,17,18,25,22–24,29] for recent work on k-leaf powers and their variants (including characterisations of 3-leaf powers [5,17,29] as well as of 4-leaf powers [10,29] and a linear time recognition of 5-leaf powers [15]). For $k \geq 6$, no characterisation of k-leaf powers and no efficient recognition is known.

While the restriction to the leaf set of a tree might seem somewhat artificial, Lin, Kearney and Jiang [25] introduced the highly related and more general notion of *Steiner root*, where any subset of the vertex set of a tree is allowed. Let $G = (V, E)$ be a finite simple graph. For any $k \geq 1$, a tree T is a kth *Steiner root* of G if V can be identified as a subset of the vertex set of T, called the set of *real nodes*, and, for any two distinct vertices $x, y \in V$, $xy \in E \iff d_T(x, y) \leq k$. The vertices of T that are not real nodes are called *Steiner nodes*. G is a kth *Steiner power* if it has a kth Steiner root. Clearly, by definition, for any $k \geq 2$, every k-leaf power is a kth Steiner power. However, the key relation we shall make extensive use of is the equality of certain $(k+2)$-leaf powers and kth Steiner powers (see Proposition 9).

In [11], we defined the following natural modification of k-leaf powers and k-leaf roots: Let $G = (V, E)$ be a finite simple graph. For any pair (k, ℓ) of integers with $2 \leq k < \ell$, a tree T is a *(k, ℓ)-leaf root* of G if V is the set of leaves of T, for all edges

* Corresponding author.
E-mail addresses: ab@informatik.uni-rostock.de (A. Brandstädt), Wagner@demogr.mpg.de (P. Wagner).

0166-218X/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.dam.2009.09.008

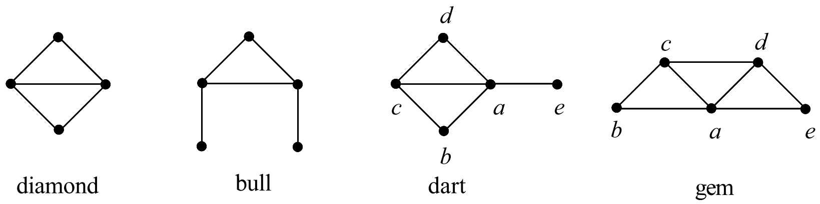

Fig. 1. Diamond, bull, dart and gem.

$xy \in E$, we have $d_T(x, y) \leq k$, and, for all non-edges $xy \notin E$, $d_T(x, y) \geq \ell$ holds. G is a (k, ℓ)-*leaf power* if it has a (k, ℓ)-leaf root.

Thus, every k-leaf power is a $(k, k+1)$-leaf power, and every (k, ℓ)-leaf power is an (i, j)-leaf power, for all pairs (i, j) with $k \leq i < j \leq \ell$. In particular, every (k, ℓ)-leaf power is a k'-leaf power, for all k' with $k \leq k' \leq \ell - 1$. In a similar way, Steiner roots and powers can be modified.

A graph is *chordal* if it contains no induced cycles of length at least four. See e.g. [9] for the importance and the many facets of chordal graphs.

In [24], Kennedy, Lin and Yan study so-called *strictly chordal* graphs which were originally defined in [24] via (rather complicated) hypergraph properties but turn out to be exactly the (dart, gem)-free chordal graphs [22] (see Fig. 1 for dart and gem).

It is known (see, for example, [6]) that a connected graph is (dart, gem)-free chordal if and only if it results from substituting cliques into the vertices of a block graph, that is, a (connected) graph whose blocks are cliques.

Being our main motivation for this paper, we will show in Theorem 3 that strictly chordal graphs are exactly the $(4, 6)$-leaf powers, which explains various of their properties. For another characterisation of strictly chordal graphs in terms of 2-simplicial powers of block graphs, see [6]. Moreover, we give a structural characterisation of the important class of $(6, 8)$-leaf powers, which also implies an efficient recognition of this class, and we characterise various other classes such as $(8, 11)$-leaf powers.

The $(4, 6)$-leaf powers are related to block graphs, and the $(6, 8)$-leaf powers are related to squares of block graphs. However, in forthcoming work, we will show that the $(8, 10)$-leaf powers properly contain all induced subgraphs of cubes of block graphs. Parts of this paper are contained in the extended abstract [11].

2. Basic notions and results

Throughout this paper, let $G = (V, E)$ be a finite simple graph with vertex set V and edge set E. For a vertex $v \in V$, let $N_G(v) = N(v) = \{w \in V \mid vw \in E\}$ denote the (*open*) *neighbourhood* of v in G, and let $N_G[v] = N[v] = N(v) \cup \{v\}$ denote the *closed neighbourhood* of v in G. For vertices $x, y \in V$, x *sees* y if $xy \in E$ and x *misses* y if $xy \notin E$. A *clique* is a set of mutually adjacent vertices. A *stable set* is a set of mutually non-adjacent vertices.

A vertex $z \in V \setminus \{x, y\}$ *distinguishes* two distinct vertices $x, y \in V$ if z is adjacent to exactly one of them, say $zx \in E$ and $zy \notin E$. A vertex subset $U \subseteq V$ is a *module* in G if no vertex from $V \setminus U$ distinguishes two vertices in U. A *clique module* in G is a module which induces a clique in G. Two distinct vertices $x, y \in V$ are *true twins* in G if $N[x] = N[y]$. Since inclusion-maximal clique modules are exactly the equivalence classes of relation R, where vRw if and only if $N[v] = N[w]$, the subsequent Proposition 1 is a well-known fact and goes back to a paper [31] by Roberts on indifference graphs.

Proposition 1. *The inclusion-maximal clique modules of a graph are pairwise disjoint.*

In [25], Lin, Kearney and Jiang call the inclusion-maximal clique modules of $G = (V, E)$ *critical cliques* of G, and they define the *critical clique graph* $\mathcal{CC}(G)$ of G as the graph having the critical cliques of G as its nodes, and two distinct nodes Q and Q' are adjacent in $\mathcal{CC}(G)$ if there are vertices $x \in Q$ and $y \in Q'$ such that $xy \in E$. Note that $\mathcal{CC}(G)$ has no true twins.

A graph $H = (V_H, E_H)$ results from a graph $G = (V, E)$ by *substituting* a clique Q into a vertex $v \in V$ (or *substituting a vertex* v *by a clique* Q), if V_H is the union of $V \setminus \{v\}$ and the vertices in Q, and E_H results from E by removing all edges containing v, adding all clique edges in Q and adding all edges between vertices in Q and in $N_G(v)$.

Let $d_G(x, y)$ (or $d(x, y)$ for short if G is understood) be the length, i.e., number of edges, of a shortest path in G between x and y. Let $G^k = (V, E_{G^k})$ with $xy \in E_{G^k}$ if and only if $d_G(x, y) \leq k$ denote the kth *power of* G.

For $U \subseteq V$, let $G[U]$ denote the subgraph of G induced by U. Throughout this paper, all subgraphs are understood to be induced subgraphs. Let $\mathcal{F}$ denote a set of graphs. A graph is $\mathcal{F}$-*free* if none of its induced subgraphs is in $\mathcal{F}$.

For $k \geq 1$, let P_k denote a chordless path with k vertices and $k - 1$ edges, and, for $k \geq 3$, let C_k denote a chordless cycle with k vertices and k edges.

For $k \geq 3$, let S_k denote the (*complete*) *sun* with $2k$ vertices $u_1, \ldots, u_k$ and $w_1, \ldots, w_k$ such that $u_1, \ldots, u_k$ is a clique, $w_1, \ldots, w_k$ is a stable set and, for $i \in \{1, \ldots, k\}$, w_i is adjacent to exactly u_i and u_{i+1} (index arithmetic modulo k).

A 2-*connected component* (or *block*) of G and a *cut vertex* of G are defined in the usual way. If $xy \notin E$ then a vertex set S is an x–y-separator if x and y are in different connected components of $G[V \setminus S]$. S is a *minimal* x–y-*separator* if it is an x–y-separator and minimal with respect to set inclusion. S is a (*minimal*) *separator* if it is a (minimal) x–y-separator for some x and y.

As already mentioned, a graph is *chordal* if it contains no induced C_k, $k \geq 4$. The maximal cliques of a chordal graph G can be arranged as nodes of a tree T_G (a so-called *clique tree* of G) such that, for every vertex v, the maximal cliques containing v form a subtree of T_G (see, e.g., [32]). Let $\mathcal{C}(G)$ denote the *clique hypergraph of* G, i.e., the vertex set of $\mathcal{C}(G)$ is the same as that of G, and the hyperedges of $\mathcal{C}(G)$ are the inclusion-maximal cliques of G.

A graph is *strongly chordal* if it is chordal and sun-free, i.e., S_k-free for all $k \geq 3$ (see, e.g., [9] for various characterisations of chordal and strongly chordal graphs).

In [16,26,30], it is shown that the class of strongly chordal graphs is closed under powers. Let T be a k-leaf root of a graph G. Then, by definition, G is isomorphic to the subgraph of T^k induced by the leaves of T. Since trees are strongly chordal, powers of strongly chordal graphs are strongly chordal, and induced subgraphs of strongly chordal graphs are strongly chordal, Proposition 2 [5] follows immediately.

Proposition 2. *For all $k \geq 2$, every k-leaf power is strongly chordal.*

This strengthens the fact that k-leaf powers are chordal, which is observed in some previous papers dealing with k-leaf powers, and this also implies that (k, ℓ)-leaf powers are strongly chordal. The converse implication is not true as mentioned in [4] (based on [3,13]): There are strongly chordal graphs which are not a k-leaf power for any $k \geq 2$.

In [24], the notion of *strictly chordal graphs* is defined in the following way:

Let $H = (V, \mathcal{E})$ be a hypergraph with $\mathcal{E} = \{E_1, \ldots, E_m\}$. A hyperedge $E_t \in \mathcal{E}$ is a *twig* if there is another hyperedge E_b (called *branch*) such that $E_t \cap (\bigcup_{E \in \mathcal{E} - E_t} E) = E_t \cap E_b$.

Hypergraph $H = (V, \mathcal{E})$ is a *hypertree* if its hyperedges can be ordered, say $(E_1, \ldots, E_m)$ such that for all $i \in \{2, 3, \ldots, m\}$, E_i is a twig in the sub-hypergraph $H_i = (V, \mathcal{E}_i)$ where $\mathcal{E}_i = \{E_1, E_2, \ldots, E_i\}$.

Let $\mathcal{E}' = \{E_{i_1}, \ldots, E_{i_l}\}$ $(l \geq 2)$ be a subset of hyperedges with nonempty intersection, that is, $I = \bigcap_{j=1}^{l} E_{i_j} \neq \emptyset$. For simplicity, I is called the *intersection of* $\mathcal{E}'$. I is a *strict intersection of* $\mathcal{E}'$ if for every pair of hyperedges $E', E'' \in \mathcal{E}'$, $E' \cap E'' = I$ and for every other hyperedge $E''' \in \mathcal{E} - \mathcal{E}'$, $E''' \cap I = \emptyset$. A hypertree is *strict* if all its intersections are strict. A graph is *strictly chordal* if it is chordal and its clique hypergraph $\mathcal{C}(G)$ is a strict hypertree.

The following characterisation of strictly chordal graphs was given by Kennedy [22] (see Fig. 1 for dart and gem).

Proposition 3 ([22]). *A graph is strictly chordal if and only if it is (dart, gem)-free chordal.*

The following notion is of central importance in this paper. A graph is a *block graph* if it is connected and its 2-connected components are cliques. A *diamond* (or $K_4 - e$, see Fig. 1) consists of four vertices and five edges. Proposition 4 is well known (see, for example, the proof of Proposition 1 in [1]):

Proposition 4. *A connected graph is a block graph if and only if it is diamond-free and chordal.*

Buneman's Four-Point Condition $(*)$ for distances in connected graphs requires that for every four vertices u, v, x and y the following inequality holds:

$$d(u, v) + d(x, y) \leq \max\{d(u, x) + d(v, y), d(u, y) + d(v, x)\}. \qquad (*)$$

This condition will be subsequently called *condition* $(*)$ throughout this paper. Theorem 1 highlights the metric similarity between trees and block graphs in terms of condition $(*)$. While we shall mainly be using part (i) of Theorem 1 subsequently, part (ii) sheds some light on the fact that block graphs arise in this paper.

Theorem 1. *Let G be a connected graph.*

(i) *Buneman* [14]*: G is a tree if and only if G contains no triangles and satisfies condition* $(*)$.
(ii) *Howorka* [21]*: G is a block graph if and only if G satisfies condition* $(*)$.

In a tree, an *internal vertex* is a vertex which is not a leaf, and an edge is called *pendant edge* if it contains a leaf and *internal edge* otherwise.

The following simple facts are well known for k-leaf powers (see, e.g., [5]) and can easily be shown for (k, ℓ)-leaf powers.

Proposition 5. (i) *Every induced subgraph of a (k, ℓ)-leaf power is a (k, ℓ)-leaf power.*
(ii) *A graph is a (k, ℓ)-leaf power if and only if each of its connected components is a (k, ℓ)-leaf power.*

Proposition 6. (i) *For $2 \leq k \leq k' < \ell' \leq \ell$, if G is a (k, ℓ)-leaf power then it is a (k', ℓ')-leaf power. In particular, every (k, ℓ)-leaf power is a k'-leaf power, for all k' with $k \leq k' \leq \ell - 1$.*
(ii) *If G is a (k, ℓ)-leaf power then it is a $(k + 2i, \ell + 2i)$-leaf power, for all $i \geq 1$.*
(iii) *If G is a (k, ℓ)-leaf power then it is a $(k + i(k - 2), \ell + i(\ell - 2))$-leaf power, for all $i \geq 1$.*

Proof. Fact (i) holds by definition. Fact (ii) is shown by subdividing each pendant edge of a (k, ℓ)-leaf root T of G. Fact (iii) is shown by subdividing each internal edge of T. □

Thus, by Proposition 6(iii), every 4-leaf power is a $(6, 8)$-leaf power, and also every 4-leaf power is an $(8, 11)$-leaf power. Hence $(6, 8)$-leaf powers as well as $(8, 11)$-leaf powers are natural extensions of the important class of 4-leaf powers. We will characterise both of them.

Due to its high relevance in this paper, we state an immediate consequence of parts (i) and (ii) of Proposition 6 in Proposition 7.

Proposition 7. *For all $k \geq 2$, every $(k, k+2)$-leaf power is a k'-leaf power, for all $k' \geq k$. In particular, every $(4, 6)$-leaf power is a k-leaf power, for all $k \geq 4$.*

By Proposition 6(ii), every k-leaf power is also a $(k+2)$-leaf power; in [19], an example of a 4-leaf power is given which is not a 5-leaf power (for arbitrary $k \geq 4$, see [12] for examples of k-leaf powers that are not $(k+1)$-leaf powers). However, it is noted in [5] (and also follows from Proposition 6(iii) with $i = 1$ and Proposition 7) that every 3-leaf power is a k-leaf power, for all $k \geq 3$.

In [10], a graph G is called *basic k-leaf power* if G has a k-leaf root T such that no two leaves of T are attached to the same parent vertex in T (a so-called *basic k-leaf root*). Obviously, for $k \geq 2$, the set of leaves having the same parent node in T form a clique, and in [10], the following is shown:

Proposition 8. *For every graph G and for every $k \geq 2$, G is a k-leaf power if and only if G results from a basic k-leaf power by substituting nonempty cliques into its vertices.*

If T is a basic k-leaf root of G then T minus its leaves is a $(k-2)$th Steiner root of G (where the set of real nodes is the set of parent nodes of leaves of T). Summarising, the following equivalence holds:

Proposition 9. *For a graph G, the following conditions are equivalent for all $k \geq 2$:*

(i) *G has a kth Steiner root.*
(ii) *G is an induced subgraph of the kth power of a tree.*
(iii) *G is a basic $(k+2)$-leaf power.*

Analogously to basic k-leaf roots, we say that a (k, ℓ)-leaf root T is *basic* if the distance $d_T(x, y)$ between any two distinct leaves x and y of T satisfies $d_T(x, y) \geq \ell - k + 2$. A (k, ℓ)-leaf power is called *basic* if it has a basic (k, ℓ)-leaf root.

Proposition 10. *For every graph G and for every pair (k, ℓ) of integers with $2 \leq k < \ell$, G is a (k, ℓ)-leaf power if and only if G results from a basic (k, ℓ)-leaf power by substituting nonempty cliques into its vertices.*

Proof. "$\Longrightarrow$": Let G be a (k, ℓ)-leaf power, and let $G' := \mathcal{CC}(G)$. Since G' is isomorphic to an induced subgraph of G, it is a (k, ℓ)-leaf power. Let T' be a (k, ℓ)-leaf root of G'. Note that $\mathcal{CC}(G)$ has no true twins. Suppose that T' is not a basic (k, ℓ)-leaf root of G', i.e., for some distinct leaves x, y of T', $d_{T'}(x, y) < \ell - k + 2$. Then every other leaf z of T' has either a distance of at most k to both of x and y or has a distance of at least ℓ to both of x and y since in (k, ℓ)-leaf root T', no distance of two leaves is between $k+1$ and $\ell - 1$, but now x and y are true twins in G', a contradiction. Thus G' is a basic (k, ℓ)-leaf power and G results from G' by substituting cliques into G'.

"$\Longleftarrow$": In general, if T is a (k, ℓ)-leaf root for the (k, ℓ)-leaf power $G = (V, E)$, and G' is the result of substituting a clique Q into a vertex $u \in V$, then attach all vertices in Q at the same parent in T as u and skip u; the resulting tree T' is a (k, ℓ)-leaf root for G'. □

Proposition 10 shows that we can restrict our attention to basic (k, ℓ)-leaf powers. This fact is used repeatedly in forthcoming arguments.

From now on, let G be a (k, ℓ)-leaf power with (k, ℓ)-leaf root T. We apply condition $(*)$ with respect to distances in T to various induced subgraphs of G such as diamond, dart and gem (see Fig. 1).

Proposition 11. *If dart or gem is a (k, ℓ)-leaf power, then $2\ell \leq 3k - 2$.*

Proof. Suppose that dart or gem is a (k, ℓ)-leaf power with (k, ℓ)-leaf root T. Then, as in Fig. 1, there are five vertices a, b, c, d and e with $\{a, b, c, d\}$ inducing a diamond with $bd \notin E$, and e distinguishing a and c, say $ae \in E$ and $ce \notin E$. According to condition $(*)$ in Theorem 1(i) applied to T,

$$d_T(a, c) + d_T(b, d) \leq \max\{d_T(a, b) + d_T(c, d), d_T(a, d) + d_T(b, c)\} \leq 2k$$

holds since $ab, ad, bc, cd \in E$. As $bd \notin E$, we have $d_T(b, d) \geq \ell$. Thus $d_T(a, c) \leq 2k - \ell$. Furthermore, $d_T(a, e) \leq k$ and $d_T(c, e) \geq \ell$. Hence

$$\ell \leq d_T(c, e) \leq d_T(c, a) + d_T(a, e) - 2 \leq 3k - \ell - 2$$

implying $2\ell \leq 3k - 2$, and we are done. □

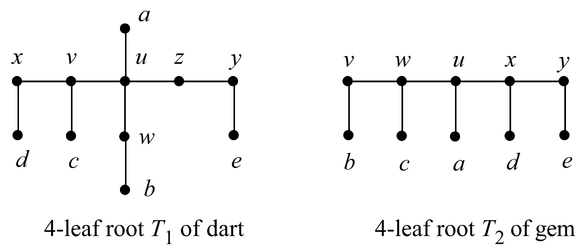

Fig. 2. 4-leaf roots of dart and gem.

Unlike Proposition 11 which is used in forthcoming proofs, Proposition 12 is merely given for the sake of completeness as it complements Proposition 11.

Proposition 12. *If $2\ell \le 3k - 2$, then dart and gem are (k, ℓ)-leaf powers.*

Proof. For $k = 2$ and $k = 3$, there is no integer ℓ satisfying $2k < 2\ell \le 3k-2$ and, hence, nothing to prove. By Proposition 6, it suffices to prove the claim for the largest possible ℓ, for all $k \ge 4$ satisfying $2k < 2\ell \le 3k - 2$, i.e., $\ell = \lfloor 3k/2 - 1 \rfloor$.

Case 1. $k = 4 + 2i$: Then $\ell = \lfloor 3k/2 - 1 \rfloor = 5 + 3i$. Thus, it suffices to show that dart and gem have $(4 + 2i, 5 + 3i)$-leaf roots for all $i \ge 0$. By Proposition 6(iii), it is sufficient to show that dart and gem have 4-leaf roots.

Case 2. $k = 5 + 2i$: Then $\ell = \lfloor 3k/2 - 1 \rfloor = 6 + 3i$. Thus, it suffices to show that dart and gem have $(5 + 2i, 6 + 3i)$-leaf roots for all $i \ge 0$.

Let D be an induced dart in G as in Fig. 1. Let T_1 be the tree as in Fig. 2.

It is a simple exercise to check that T_1 is a (4, 5)-leaf root for D. For all $i \ge 0$, we obtain a $(4+2i, 5+3i)$-leaf root for D by subdividing each internal edge of T_1 by exactly i nodes. If we further subdivide bw and ey by exactly one vertex, we obtain a $(5 + 2i, 6 + 3i)$-leaf root for D. This holds because, for every edge of D, the length of the corresponding path in the tree increases by at most 1 and, for every non-edge of D, the length of the corresponding path in the tree increases by at least 1.

Let M be an induced gem in G as in Fig. 1. Let T_2 be the tree as in Fig. 2. Again, it is a simple exercise to check that T_2 is a (4, 5)-leaf root for M. As in the case of the dart, for all $i \ge 0$, we obtain a $(4+2i, 5+3i)$-leaf root for M by subdividing each internal edge of T_2 by exactly i vertices. If we further subdivide bv and ey by exactly one vertex, we obtain a $(5+2i, 6+3i)$-leaf root for M, for the same reason as in the case of the dart. □

3. Strictly chordal graphs are exactly (4, 6)-leaf powers

The characterisation of (4, 6)-leaf powers given in this section is very similar to the following one for 3-leaf powers (for bull, dart and gem see Fig. 1):

Theorem 2 (*[17]*). *For a connected graph G, the following conditions are equivalent:*

(i) *G is a* 3*-leaf power.*
(ii) *G is* (*bull*, *dart*, *gem*)*-free chordal.*
(iii) *G results from substituting cliques into the vertices of a tree.*
(iv) *The critical clique graph* $\mathcal{CC}(G)$ *of G is a tree.*

See [5,29] for more details and other equivalent conditions, and in particular for the equivalence of conditions (i), (ii) and (iv) to condition (iii).

Now we consider the class of (4, 6)-leaf powers. Recall from Proposition 7 that every (4, 6)-leaf power is a k-leaf power, for all $k \ge 4$.

Recall the definition of strictly chordal graphs given in Section 2 and their characterisation as (dart, gem)-free chordal graphs in Proposition 3.

The subsequent Theorem 3 has been our motivation for defining and investigating the notion of (k, ℓ)-leaf powers in [11]. The equivalence of conditions (ii) and (iii) in Theorem 3 was shown already in [6]. To make this paper self-contained, we repeat parts of its proof here.

Theorem 3. *For a connected graph $G = (V, E)$, the following conditions are equivalent:*

(i) *G is a* (4, 6)*-leaf power.*
(ii) *G is* (*dart*, *gem*)*-free chordal (i.e., strictly chordal).*
(iii) *G results from substituting cliques into the vertices of a block graph.*
(iv) *The critical clique graph* $\mathcal{CC}(G)$ *of G is a block graph.*
(v) *G is chordal, and the pairwise intersections of maximal cliques in G are pairwise disjoint or equal.*
(vi) *G is chordal, and the pairwise intersections of maximal cliques in G are clique modules in G.*

The equivalence of (ii) and (iv) is implicitly mentioned in [24]: Lemma 2.4 of [24] says that G is strictly chordal if and only if in the critical clique graph $\mathcal{CC}(G)$ of G the nodes of every simple cycle form a clique. Note that the diamond is a simple cycle which is not a clique and thus $\mathcal{CC}(G)$ is diamond-free chordal and thus, by Proposition 4, is a block graph.

Proof of Theorem 3. (i) ⇒ (ii): Let G be a $(4, 6)$-leaf power with $(4, 6)$-leaf root T. Suppose that G contains dart or gem (see Fig. 1) as an induced subgraph. Then there are five vertices a, b, c, d and e in G such that $\{a, b, c, d\}$ induce a diamond with $bd \notin E$ and e distinguishing a and c, say $ae \in E$ and $ce \notin E$. According to condition $(*)$ in Theorem 1(i) applied to T,

$$d_T(a, c) + d_T(b, d) \le \max\{d_T(a, b) + d_T(c, d), d_T(a, d) + d_T(b, c)\} \le 8$$

holds since $ab, ad, bc, cd \in E$. As $bd \notin E$, we have $d_T(b, d) \ge 6$. Thus $d_T(a, c) \le 2$, i.e., the leaves a and c have the same parent node a' in T. Furthermore, $d_T(a, e) \le 4$ and $d_T(c, e) \ge 6$. Then

$$6 \le d_T(c, e) \le d_T(c, a') + d_T(a', e) \le 1 + 3$$

- a contradiction. Hence G is (dart, gem)-free. Moreover, G is chordal since, by Proposition 2, every (k, ℓ)-leaf power is chordal.

(ii) ⇒ (iv): Let G be (dart, gem)-free chordal. We claim that the critical clique graph $G' = \mathcal{CC}(G)$ is a block graph. As an induced subgraph of G, $G' = (V', E')$ is (dart, gem)-free chordal. We claim that G' is diamond-free. Suppose to the contrary that the vertices a, b, c and d induce a diamond in G' with $bd \notin E'$. As G' has no non-trivial clique modules (implying that $\{a, c\}$ is not a module in G'), there is a vertex z in G' distinguishing a and c, say $az \in E'$ and $cz \notin E'$. Since G' is dart-free, z is adjacent to b or d, and since G' is gem-free, z is adjacent to both of them, but now z, b, c and d induce a C_4 in the chordal graph G', a contradiction. Thus, G' is diamond-free chordal, i.e., by Proposition 4, a block graph.

(iv) ⇒ (iii): By definition of its critical clique graph $\mathcal{CC}(G)$, G results from $\mathcal{CC}(G)$ by substituting the corresponding clique modules into the vertices of $\mathcal{CC}(G)$ which is supposed in (iv) to be a block graph.

(iii) ⇒ (i): Let G result from substituting cliques into the vertices of a block graph $G' = (V', E')$. By Proposition 10, it suffices to show that G' is a $(4, 6)$-leaf power. We construct a $(4, 6)$-leaf root T of G' as follows: For every block B of G', we construct a subtree T_B by first taking a star whose center is a new node c_B and whose leaf set is the set of vertices of B. Then each pendant edge is subdivided by exactly one node. Now if B and B' are two blocks of G' such that B and B' have a cut vertex v in common, then identify the two copies of v and also identify its two parent nodes in T_B and $T_{B'}$. This construction guarantees that the T-distance of vertices $x, y \in V'$ with $xy \in E'$ is at most 4 and the T-distance of vertices $x, y \in V'$ with $xy \notin E'$ is at least 6, so that T is a $(4, 6)$-leaf root of G'.

(iii) ⇒ (vi): Let G result from substituting cliques into the vertices of a block graph B. Clearly, G is chordal. Substituting a clique Q into a vertex v of B which is not a cut vertex of B only enlarges the block containing v while substituting Q into a cut vertex v creates a clique module which is the pairwise intersection of blocks containing cut vertex v. Now it is easy to see that every pairwise intersection of maximal cliques in G is a clique module in G.

(vi) ⇒ (v): Suppose that all pairwise intersections of maximal cliques in G are clique modules but there are intersections $Q \cap Q'$, $R \cap R'$ of maximal cliques in G which are neither equal nor disjoint. Let without loss of generality, $x \in (Q \cap Q') \setminus (R \cap R')$ and $y \in (Q \cap Q') \cap (R \cap R')$. Then, without loss of generality, $x \notin R$ and $y \in R$ which means that there is a third vertex $z \in R$ with $yz \in E$ and $xz \notin E$, thereby distinguishing x and y contradicting that $Q \cap Q'$ is a module.

(v) ⇒ (iii): Suppose that G is connected and chordal such that the pairwise intersections of maximal cliques in G are pairwise disjoint or equal. We claim that the pairwise intersections of maximal cliques in G are modules. Suppose to the contrary that, for the maximal cliques Q and Q', $Q \cap Q'$ is not a module, i.e., there are vertices $x, y \in Q \cap Q'$ and a vertex z distinguishing x and y, say $zx \in E$ and $zy \notin E$. Then $z \notin Q \cup Q'$, and there is a maximal clique Q'' containing x and z. Since $y \notin Q''$, $Q \cap Q' \neq Q \cap Q''$ but $Q \cap Q'$ intersects $Q \cap Q''$, a contradiction which shows the claim.

Now contract each nonempty intersection of maximal cliques to one vertex and denote the resulting graph by G'. Being isomorphic to a subgraph of G, G' is chordal. We claim that G' is diamond-free. Suppose to the contrary that G' contains a diamond with vertices a, b, c, d such that a and d are non-adjacent. Then in G there are maximal cliques Q containing a, b, c and Q' containing b, c, d such that $b, c \in Q \cap Q'$ but in G', the intersection was contracted to one vertex, a contradiction.

It follows from Proposition 4 that G' is a block graph, and substituting the nonempty intersections of maximal cliques in G into the vertices of G' gives G. □

Now Theorem 3 together with Proposition 7 implies:

Corollary 1. *Strictly chordal graphs are k-leaf powers for all $k \ge 4$.*

Corollary 1 is one of the main results (namely Theorem 4.1) in [24]. It has also been mentioned in Theorem 2.5 of [24] that strictly chordal graphs can be recognised in linear time; the proof in [24] is based on a linear time algorithm for constructing the critical clique graph $\mathcal{CC}(G)$ for a given chordal graph G; the existence of such an algorithm was claimed in [25] and is shown, e.g., in [2,27] where the maximal clique modules of a (not necessarily chordal) graph are constructed in linear time.

By Theorem 3, the linear time recognition of strictly chordal graphs given in [24] can be simplified in the following way: (1) construct $\mathcal{CC}(G)$; (2) check whether $\mathcal{CC}(G)$ is a block graph; according to Theorem 3 (equivalence of (ii) and (iv)), this recognises strictly chordal graphs.

Below we give another, conceptually very simple, linear time algorithm for recognising $(4, 6)$-leaf powers without constructing $\mathcal{CC}(G)$.

Corollary 2. *$(4, 6)$-leaf powers (and thus also strictly chordal graphs) can be recognised in linear time.*

Proof. Let G be a chordal graph (otherwise, G is not a $(4, 6)$-leaf power). By Theorem 3 (equivalence of (i) and (vi)), a connected graph G is a $(4, 6)$-leaf power if and only if G is chordal and the pairwise intersections of maximal cliques in G are clique modules. This condition can be checked in the following way:

Determine a clique tree T of G (which can be done in linear time, see e.g. [20,32]) and check for all consecutive cliques Q and Q' in T whether their intersection $Q \cap Q'$ is a clique module (in the usual way by checking whether pairs of vertices in such intersections are true twins). If this is fulfilled, then also all other intersections of maximal cliques are clique modules. □

4. Further results

Very similar to the fact that block graphs are $(4, 6)$-leaf powers (see Theorem 3), we obtain:

Proposition 13. *Every block graph is a $(5, 7)$-leaf power, and a $(5, 7)$-leaf root of a given block graph can be determined in linear time.*

Proof. Let G be a block graph, and let $B_1, \dots, B_b$, $b \geq 1$, be its blocks. Since G is connected, we may assume for the order of the blocks that each B_i, $1 < i \leq b$, has a cut vertex in common with some previous B_j, $1 \leq j < i$. Hence, let G_i, $1 \leq i \leq b$, be the block graph consisting of the first i blocks. In particular, $G_1 = B_1$ and $G_b = G$. For each G_i, $1 \leq i \leq b$, we will inductively construct a $(5, 7)$-leaf root T_i with the additional property that the distance between any two leaves is 4 or greater. Finally obtaining T_b, we will be done.

First, for each block B, if $x_1, \dots, x_k$, $k \geq 2$, are the vertices of B, let T_B be the tree with $2k + 1$ vertices defined as follows: T_B has the k leaves $x_1, \dots, x_k$, the k vertices $x'_1, \dots, x'_k$ of degree 2 and the vertex c_B of degree k, such that, for each $1 \leq i \leq k$, x'_i is adjacent to x_i and c_B. For $i = 1$, clearly, T_{B_1} is a $(5, 7)$-leaf root T_1 of G_1 with the additional property. Suppose that, for some $1 < i \leq b$, T_{i-1} is a $(5, 7)$-leaf root of G_{i-1} with the additional property. Let x be the cut vertex B_i and G_{i-1} have in common. Note that x is a leaf in both T_{B_i} and T_{i-1}. Now form T_i, by first identifying the leaf x and its neighbour in both T_{B_i} and T_{i-1} and then subdividing the edge $x'c_{B_i}$ in T_{B_i} by exactly one vertex. This construction guarantees that the T_i-distance between adjacent vertices of G_i is 4 or 5 and that the T_i-distance between non-adjacent vertices of G_i is at least 7. Obviously, T_b can be constructed in linear time. □

Theorem 4. (i) *For all pairs (k, ℓ) with $k \geq 2$ and $\ell > 2k - 2$, the class of (k, ℓ)-leaf powers is the class of P_3-free graphs.*
(ii) *For all pairs (k, ℓ) of the form $(k, \ell) = (2i + 1, 4i)$ with $i \geq 1$, the class of (k, ℓ)-leaf powers is the class of 3-leaf powers.*
(iii) *For all pairs (k, ℓ) with $k \geq 4$ and $3k/2 - 1 < \ell \leq 2k - 2$ and not the situation of* (ii)*, the class of (k, ℓ)-leaf powers is the class of $(4, 6)$-leaf powers.*

Proof. (i): Suppose that $k \geq 2$ and $\ell > 2k - 2$. Let G be any (k, ℓ)-leaf power with (k, ℓ)-leaf root T. If P_3 was an induced subgraph of G, say with edges ab and bc, then $\ell \leq d_T(a, c) \leq d_T(a, b) + d_T(b, c) - 2 \leq 2k - 2$. Thus G is P_3-free. Conversely, by Proposition 5(ii), it is an obvious fact that disjoint unions of cliques have (k, ℓ)-leaf roots with $k \geq 2$ and $\ell > 2k - 2$.

(ii): The case $k = 3$ and thus $i = 1$ is the case of 3-leaf powers. For odd $k = 2i + 1 > 3$, by Proposition 6(iii), every (3,4)-leaf power is a $(2i + 1, 4i)$-leaf power. Conversely, $(2i + 1, 4i)$-leaf powers are (dart, gem)-free, by Proposition 11, since, for $k = 2i + 1$ and $\ell = 4i$, the inequality $2\ell \leq 3k - 2$ in Proposition 11 is not fulfilled. We claim that $(2i + 1, 4i)$-leaf powers are also bull-free: Let G be any $(2i + 1, 4i)$-leaf power with $(2i + 1, 4i)$-leaf root T. First note that if P_3 is an induced subgraph of G, say with edges xy and yz, then $4i \leq d_T(x, z) \leq d_T(x, y) + d_T(y, z) - 2 \leq 4i$ and hence $d_T(x, z) = 4i$ and $d_T(x, y) = d_T(y, z) = 2i + 1$. Let a, b, c, d and e induce a bull with the $P_4 abcd$ with edges ab, bc and cd and vertex e adjacent to b and c. Each of $\{a, b, c\}$, $\{a, b, e\}$ and $\{c, d, e\}$ induces a P_3, and we can deduce $d_T(b, c) = d_T(b, e) = d_T(c, e) = 2i + 1$, contradicting the fact that the sum of the three distances between b, c and e must be even. Thus, $(2i + 1, 4i)$-leaf powers are (bull, dart, gem)-free chordal, and, by Theorem 2, they are 3-leaf powers.

(iii): By Proposition 11, for $2\ell > 3k - 2$, (k, ℓ)-leaf powers are (dart, gem)-free. Then, by Theorem 3, they are $(4, 6)$-leaf powers. For the other direction, again by Theorem 3, it suffices to show that every block graph has a (k, ℓ)-leaf root. By Proposition 6, it suffices to show this for the largest possible $\ell \leq 2k - 2$, for every k, i.e., for the (k, ℓ)-pairs $(4, 6)$, $(5, 7)$, $(6, 10)$, $(7, 11)$ and so on, i.e., for the (k, ℓ)-pairs $(4 + 2i, 6 + 4i)$, $(5 + 2i, 7 + 4i)$, for all $i \geq 0$.

Theorem 3 and Proposition 13 deal with the case $i = 0$; in that case, we start with block roots which are stars whose edges are subdivided exactly once. For a general $i \geq 0$, we use a similar construction with stars whose edges are subdivided exactly $i + 1$ times. □

Fig. 3 illustrates (k, ℓ)-leaf powers up to $k = 10$ and $\ell = 16$; diagonally (horizontally, vertically, respectively) dashed squares represent exactly disjoint unions of cliques (the 3-leaf powers, $(4, 6)$-leaf powers, respectively).

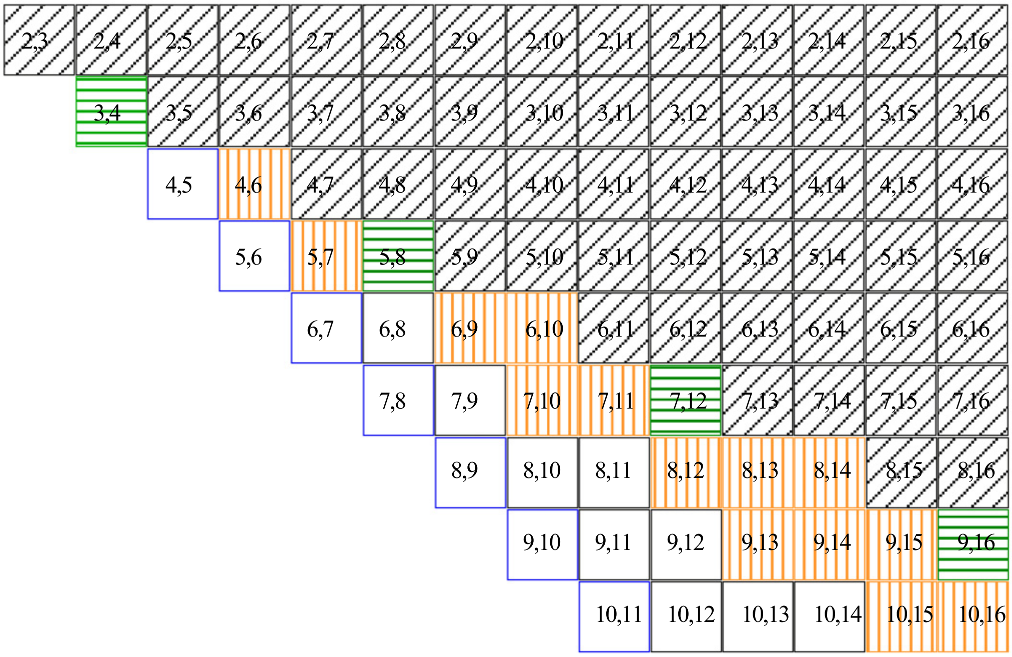

Fig. 3. (k, ℓ)-leaf powers up to $k = 10$ and $\ell = 16$; diagonally (horizontally, vertically, respectively) dashed squares represent exactly disjoint unions of cliques (the 3-leaf powers, (4, 6)-leaf powers, respectively).

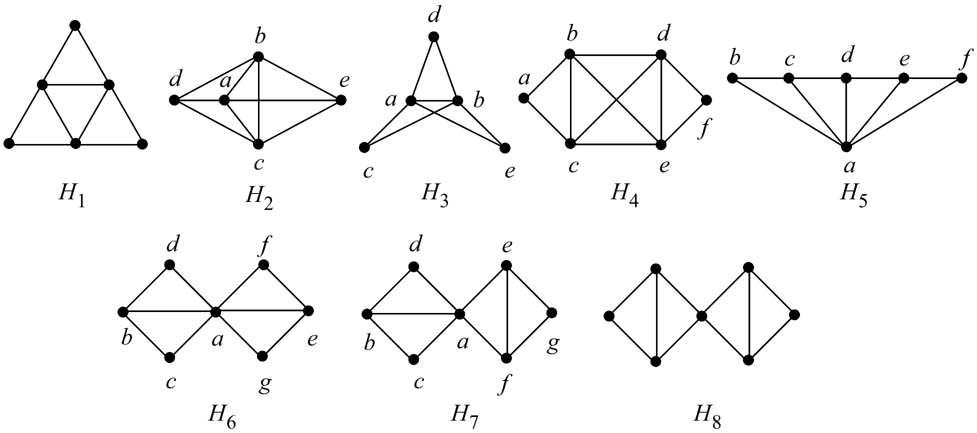

Fig. 4. Forbidden subgraphs $H_1, \ldots, H_8$.

5. Characterising (6, 8)-leaf powers

Recall that by definition, every (4, 6)-leaf power is a 4-leaf power. By Proposition 6(iii) with $k = 4$, $\ell = 5$ and $i = 1$, it follows that every 4-leaf power is a (6, 8)-leaf power. The characterisation of basic 4-leaf powers in the subsequent Theorem 5 inspired our corresponding characterisation of basic (6, 8)-leaf powers given in Theorem 6 which is the main result of this section.

For the graphs $H_1, \ldots, H_8$ in Theorem 5 see Fig. 4. Rautenbach [29] showed that a graph without true twins is a 4-leaf power if and only if it is $(H_1, \ldots, H_8)$-free chordal. In [10], the following more detailed characterisation is shown:

Theorem 5 (*[10]*). *G is a basic* 4*-leaf power if and only if G is* $(H_1, \ldots, H_8)$*-free chordal. Moreover, G is the square of a tree if and only if G is* 2*-connected and* $(H_1, \ldots, H_5)$*-free chordal.*

In fact, the forbidden subgraphs $H_1, \ldots, H_5$ are responsible for the blocks of a basic 4-leaf power, and H_6, H_7 and H_8 represent the gluing conditions of blocks; see [10] for further details and linear time recognition of 4-leaf powers.

While Theorem 5 has to do with squares of trees, in this paper, block graphs are of great significance, and whereas the (4, 6)-leaf powers result from substituting cliques into block graphs, it turns out that the basic (6, 8)-leaf powers are the

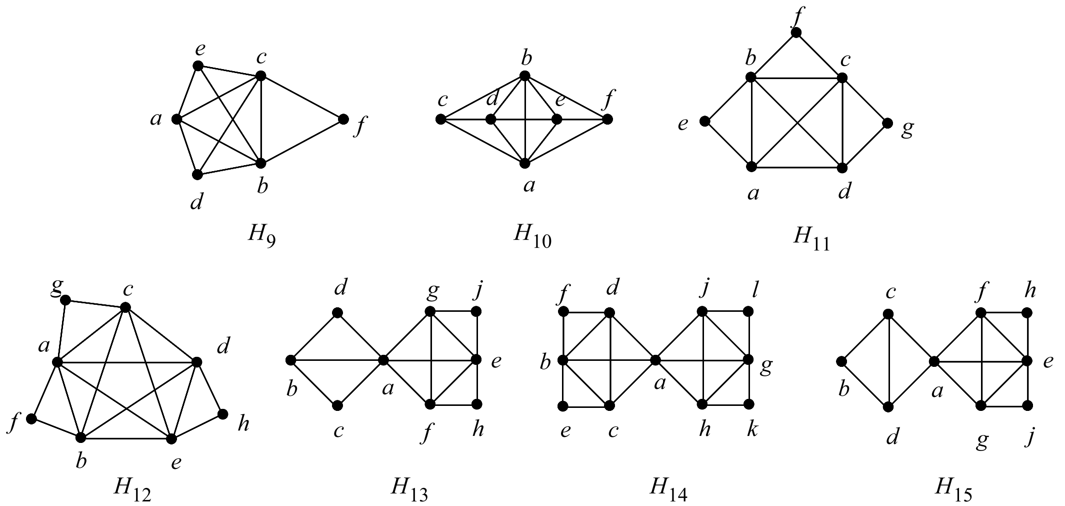

Fig. 5. Forbidden subgraphs $H_9, \ldots, H_{15}$.

induced subgraphs of squares of block graphs. It is straightforward to show that the induced subgraphs of the kth power of a block graph are basic $(2k+2, 2k+4)$-leaf powers; in forthcoming work it will be shown that the converse does not hold.

For the proof of the subsequent Theorem 6, we need the following:

Lemma 1. *Let G be a 2-connected (H_5, H_9, H_{10})-free chordal graph. If M and M' are any two distinct maximal cliques in G such that $|M \cap M'| \geq 2$, then $M \cap M'$ is an inclusion-minimal separator in G.*

Proof. We have to show that if two maximal cliques M and M' intersect in at least two vertices, then their intersection is a minimal separator.

To see this we will first show that the intersection is a separator. As M and M' are maximal cliques, we can choose four vertices x, y, u and v, such that $x \in M \setminus M'$, $y \in M' \setminus M$, $u, v \in M \cap M'$ and $xy \notin E$. Now suppose that $M \cap M'$ is not a separator. Then, after deleting $M \cap M'$, there must be a path from x to y. Let us consider a shortest such path. As it is shortest, it is a chordless path. As G is chordal, both u and v must be adjacent to every vertex on that path. If the path has length 4 or greater, then, together with, say, u, we have an H_5, a contradiction. If the path has length 3, then, together with u and v, we have an H_{10}, a contradiction. Hence there must be a vertex $w \notin M \cap M'$ being adjacent to x, y, u and v. Without loss of generality, we may assume $w \notin M$. As M is a maximal clique, there must be a vertex $z \in M$ such that $wz \notin E$ but then, if $yz \in E$ then $\{x, y, w, z\}$ induce a C_4, and if $yz \notin E$ then $\{x, y, u, v, w, z\}$ induce an H_{10}, both yielding a contradiction.

Suppose that $M \cap M'$ is not a minimal separator. Then $M \cap M'$ must properly contain a minimal separator S, and we can choose six vertices x, y, u, v, w and z, such that $x \in M \setminus M'$, $y \in M' \setminus M$, $u \in (M \cap M') \setminus S$, $v, w \in S$, $z \notin M \cup M'$, and $xy \notin E$.

We claim that z is only adjacent to v and w:

Let z and z' be vertices which are separated by S. At least one of them is not in $M \cup M'$, say $z \notin M \cup M'$, and if $z' \in M \cup M'$ then $uz \notin E$. If $z, z' \notin M \cup M'$ then at least one of them is non-adjacent to u, say $uz \notin E$. Now if $xz \in E$ then $yz \notin E$, else x, y, u, z induce a C_4 in G, but now x, y, u, v, w, z induce an H_{10}, a contradiction, and similarly for $xz \notin E$ and $yz \in E$. Thus, z is only adjacent to v and w but now the six vertices induce an H_9, a contradiction. □

Theorem 6. *The following conditions are equivalent:*

(i) *G is a basic $(6, 8)$-leaf power.*
(ii) *G is $(H_1, H_5, H_6, H_9, \ldots, H_{14})$-free chordal.*
(iii) *G is an induced subgraph of the square of some block graph.*

Proof. (i) ⇒ (ii): Suppose that G is a basic $(6, 8)$-leaf power. Since G is strongly chordal, G is H_1-free. By deleting the leaves of a basic $(6, 8)$-leaf root T' of G, we obtain a $(4, 6)$-Steiner root T of G with the property that no two real nodes have a distance of 1, since leaf distances of 2 and 3 are forbidden in T'. Suppose further that the four vertices u, v, w and x induce a diamond D in G with $ux \notin E$. By condition $(*)$ in Theorem 1(i), we have $d_T(u, x) + d_T(v, w) \leq \max\{d_T(u, v) + d_T(w, x), d_T(u, w) + d_T(v, x)\} \leq 8$. Since we also have $d_T(u, x) \geq 6$ and $d_T(v, w) \geq 2$, we can deduce $d_T(u, x) = 6$, $d_T(v, w) = 2$ and that at least one of $d_T(u, v) = d_T(w, x) = 4$ and $d_T(u, w) = d_T(v, x) = 4$ holds.

Suppose that G contains an H_5 as in Fig. 4. Then in the two diamonds induced by $\{a, b, c, d\}$ and $\{a, d, e, f\}$ we can deduce $d_T(a, c) = 2$ and $d_T(a, e) = 2$, respectively, which implies $d_T(c, e) \leq 4$ and that ce is an edge in G, a contradiction.

Suppose that G contains an H_6 as in Fig. 4. Then in the two diamonds induced by $\{a, b, c, d\}$ and $\{a, e, f, g\}$ we can deduce $d_T(a, b) = 2$ and $d_T(a, e) = 2$, respectively, which implies $d_T(b, e) \leq 4$ and that be is an edge in G, a contradiction.

Suppose that G contains an H_9 as in Fig. 5. Then in the two diamonds induced by $\{a, b, d, e\}$ and $\{a, c, d, e\}$ we can deduce $d_T(a, b) = 2$ and $d_T(a, c) = 2$, respectively, which contradicts the fact that in the diamond induced by $\{a, b, c, f\}$ at least one of $d_T(a, b)$ and $d_T(a, c)$ should be 4.

Suppose that G contains an H_{10} as in Fig. 5. Then in the four diamonds induced by $\{a, c, d, e\}$, $\{a, d, e, f\}$, $\{b, c, d, e\}$ and $\{b, d, e, f\}$ we can deduce $d_T(a, d) = 2$, $d_T(a, e) = 2$, $d_T(b, d) = 2$ and $d_T(b, e) = 2$, respectively. Then, in each of the four diamonds, we can also deduce $d_T(d, e) = 4$. Condition $(*)$, applied to a, b, d and e, then implies $d_T(a, b) + 4 \le 4$, a contradiction.

H_{11} up to H_{14} all contain a special induced subgraph H, which is obtained from a gem as in Fig. 1 by adding a sixth vertex f and the three edges af, cf and df. By considering the two diamonds of the gem, we can deduce $d_T(a, c) = 2$, $d_T(a, d) = 2$ and $d_T(c, d) = 4$. Furthermore, by applying condition $(*)$ to a, c, d and f, we obtain that $d_T(a, f) = 2$ must hold.

Now suppose that G contains an H_{11} as in Fig. 5. From the H induced by all vertices but g we can deduce $d_T(a, c) = 4$, but from the H induced by all vertices but e we can deduce $d_T(a, c) = 2$, a contradiction.

Suppose that G contains an H_{12} as in Fig. 5. From the subgraph H induced by all vertices but e and h we can deduce $d_T(a, d) = 2$, and from the subgraph H induced by all vertices but d and h we obtain $d_T(a, e) = 2$. This contradicts the fact that in the diamond induced by $\{a, d, e, h\}$ at least one of $d_T(a, d)$ and $d_T(a, e)$ must be 4.

Suppose that G contains an H_{13} as in Fig. 5. From the subgraph H induced by all vertices but c, b and d we can deduce $d_T(a, e) = 2$, and from the diamond induced by $\{a, b, c, d\}$ we obtain $d_T(a, b) = 2$. But then $d_T(b, e) \le d_T(a, b) + d_T(a, e) = 4$, so that be is an edge in G, a contradiction.

Finally, suppose that G contains an H_{14} as in Fig. 5. From the two obvious subgraphs H we can deduce $d_T(a, b) = 2$ and $d_T(a, g) = 2$. But then $d_T(b, g) \le d_T(a, b) + d_T(a, g) = 4$, so that bg is an edge in G, a contradiction.

(ii) $\Rightarrow$ (iii): Let G be $(H_1, H_5, H_6, H_9, \ldots, H_{14})$-free chordal. For the moment, let us assume that G is 2-connected, i.e., a block, and discuss later how to perform when G has more than one block. We need to construct a block graph $\mathcal{B}$ and specify for each of its vertices whether it is a real node or a Steiner node, such that G is obtained from squaring $\mathcal{B}$ and deleting the Steiner nodes. The idea is to construct, for each maximal clique M of G, a corresponding block graph $\mathcal{B}_M$, depending on how minimal separators of G therein can lie with respect to each other, and then joining all the block graphs $\mathcal{B}_M$ together.

By assumption, every minimal separator has at least two vertices. Note that every minimal separator in a chordal graph is a clique. By Lemma 1, if two maximal cliques M and M' intersect in at least two vertices then their intersection is a minimal separator.

Now we consider how minimal separators can lie within a maximal clique M. Suppose two minimal separators S and S' intersect in at least two vertices. Then we can choose six vertices u, v, w, x, y and z, such that $u \in S \setminus S'$, $v \in S' \setminus S$, $w, x \in S \cap S'$, $y, z \notin M$, y sees only u, w and x, and z sees only v, w and x. The six vertices induce an H_{10}, a contradiction. So any two minimal separators intersect in at most one vertex.

Suppose that there is a vertex v which constitutes the intersection of two minimal separators S and S' in M. Suppose that there is a third minimal separator S'' in M that does not contain v. If it intersects both of S and S', then we can deduce the existence of an H_1, if it intersects exactly one of S and S', then we can deduce an H_{11}, and if it intersects none of S and S', then we can deduce an H_{12}, always giving a contradiction. Thus, for any maximal clique, either there is a vertex, which constitutes the pairwise intersection of all minimal separators, or all minimal separators are pairwise disjoint. In the former case we call the maximal clique to be of *type* 1 and in the latter to be of *type* 2.

For a maximal clique M of type 1, let the block graph $\mathcal{B}_M$ be defined as follows. All vertices of M are real nodes. The minimal separators are blocks intersecting at v. The remaining vertices see v. For a maximal clique M of type 2, again all vertices of M are real nodes, but we add one Steiner node w to which each minimal separator is joined to form a block. All remaining vertices see w.

Whenever two maximal cliques M and M' intersect in at least two vertices, their intersection is a minimal separator, and, for the corresponding $\mathcal{B}_M$ and $\mathcal{B}_{M'}$, we can identify the respective blocks. In order to keep it a block graph, in case both maximal cliques are of type 2, we need to add an extra edge between the two respective Steiner nodes. This way we build up a desired block graph with specified real and Steiner nodes.

If G has more than one block, we construct, as above, a block graph for each block, noting that a minimal separator of a block need not be a minimal separator of G, and identify the block graphs at the corresponding cut vertex. This can be done with the help of the gluing conditions given by the forbidden induced subgraphs H_6, H_{13} and H_{14}. Let a candidate cut vertex be c, and let M and M' be two maximal cliques in the two blocks containing c. If c lies in a minimal separator in both of them, then we could deduce an H_6. If c lies in a minimal separator in precisely one of them, say M, and M' is of type 1, then we could deduce an H_{13}. And if c lies in a minimal separator in none of them, and both of them are of type 1, then we could deduce an H_{14}. It follows that one of them, say M, must be of type 2 and that c does not lie in a minimal separator of M. Now c sees only the Steiner node in $\mathcal{B}_M$ and thus has a distance of 2 to any other real node in $\mathcal{B}_M$, so that the distance of any two real nodes distinct from c and corresponding to two vertices in the two blocks indeed exceeds 2.

(iii) $\Rightarrow$ (i): Let G be an induced subgraph of the square of some block graph $\mathcal{B}$. It is sufficient to show that G has a (4, 6)-Steiner root T with the property that no two real nodes are at distance 1. We construct such a Steiner root (which is, in fact, a star) for each block of $\mathcal{B}$ and then join them together to form a required Steiner root for G. For any given block B of $\mathcal{B}$, mark all vertices that correspond to a vertex of G as a real node and all other vertices as a Steiner node. Add an extra Steiner node and join them to all the vertices to form a star T_B of size $|B|$. Whenever two blocks B and B' of $\mathcal{B}$ have a cut vertex in common, then we identify T_B and $T_{B'}$ at the corresponding leaf. It is straightforward to see that, for the obtained Steiner tree T, any two adjacent vertices in G correspond to two real nodes with distance not exceeding 4 and any two non-adjacent vertices in G correspond to two real nodes with distance 6 or greater. Furthermore, no two real nodes are at distance 1, which finishes the proof. □

By Theorem 6, it follows that (6, 8)-leaf powers can be recognised in polynomial time. In [6], another characterisation of basic (6, 8)-leaf powers in terms of 3-simplicial powers of block graphs and a faster polynomial time recognition algorithm of basic (6, 8)-leaf powers is given.

The proof of Theorem 6(i) ⇒ (ii) straightforwardly generalises to the proof of Lemma 2.

Lemma 2. *For every integer $i \geq 1$, the basic $(2 + 4i, 2 + 6i)$-leaf powers are $(H_1, H_5, H_6, H_9, \ldots, H_{14})$-free.*

As seen before in Theorem 4, here Theorem 7 says that the basic (6, 8)-leaf powers occur as the basic (k, ℓ)-leaf powers, for infinitely many pairs (k, ℓ), and hence, by Proposition 10, that the (6, 8)-leaf powers occur as the (k, ℓ)-leaf powers, for infinitely many pairs (k, ℓ).

Theorem 7. *For every positive integer i, the basic $(2 + 4i, 2 + 6i)$-leaf powers are precisely the basic (6, 8)-leaf powers.*

Proof. Let G be a basic $(2 + 4i, 2 + 6i)$-leaf power. By Lemma 2, G is $(H_1, H_5, H_6, H_9, \ldots, H_{14})$-free and therefore, by Theorem 6, a basic (6, 8)-leaf power.

Conversely, let G be a basic (6, 8)-leaf power, and let T be a basic (6, 8)-leaf root of G. It is straightforward to check that, by subdividing each internal edge of T by exactly $i-1$ new vertices, we obtain a basic $(2+4i, 2+6i)$-leaf root T' of G. Hence G is a basic $(2 + 4i, 2 + 6i)$-leaf power. □

Corollary 3. *For every positive integer i, the $(2 + 4i, 2 + 6i)$-leaf powers can be recognised in polynomial time.*

6. Characterising (8, 11)-leaf powers

Recall that, by Proposition 6(iii), the class of (8, 11)-leaf powers is a natural extension of 4-leaf powers. Incidentally, for characterising the (8, 11)-leaf powers we additionally need H_{15}, which is given in Fig. 5 and replaces H_8 as a gluing condition.

As in Theorems 4 and 7 before, here Theorem 8 (apart from characterising the basic (8, 11)-leaf powers) says that the basic (8, 11)-leaf powers occur as the basic (k, ℓ)-leaf powers, for infinitely many pairs (k, ℓ), and hence, by Proposition 10, that the (8, 11)-leaf powers occur as the (k, ℓ)-leaf powers, for infinitely many pairs (k, ℓ).

Theorem 8. *For every integer $i \geq 1$, the basic $(4+4i, 5+6i)$-leaf powers are exactly the $(H_1, \ldots, H_7, H_{15})$-free chordal graphs.*

Proof. Suppose that $G = (V, E)$ is a basic $(4+4i, 5+6i)$-leaf power. Since G is strongly chordal, G is chordal and H_1-free. By deleting the leaves of a basic $(4+4i, 5+6i)$-leaf root T' of G we obtain a $(2+4i, 3+6i)$-Steiner root T of G with the property that no two real nodes have a distance of 1 up to $2i$, since leaf distances of 2 up to $2 + 2i$ are forbidden in T'. Suppose further that the four vertices u, v, w and x induce a diamond in G with $ux \notin E$.

By condition $(*)$ we have

$$d_T(u, x) + d_T(v, w) \leq \max\{d_T(u, v) + d_T(w, x), d_T(u, w) + d_T(v, x)\} \leq 4 + 8i.$$

Since we also have $d_T(u, x) \geq 3 + 6i$ and $d_T(v, w) \geq 1 + 2i$, we can deduce $d_T(u, x) = 3 + 6i$, $d_T(v, w) = 1 + 2i$ and that at least one of $d_T(u, v) = d_T(w, x) = 2 + 4i$ and $d_T(u, w) = d_T(v, x) = 2 + 4i$ holds.

Suppose that G contains an H_2 as in Fig. 4. Then in the three diamonds containing d and e we must have $d_T(a, b) = d_T(a, c) = d_T(b, c) = 1 + 2i$, which is a contradiction as the three distances should add up to an even number.

Suppose that G contains an H_3 as in Fig. 4. Then in the three diamonds containing a and b we must have $d_T(c, d) = d_T(c, e) = d_T(d, e) = 3 + 6i$, which is a contradiction as the three distances should add up to an even number.

Suppose that G contains an H_4 as in Fig. 4. Then in the three diamonds induced by $\{a, b, c, d\}$, $\{a, b, c, e\}$ and $\{b, d, e, f\}$ we can deduce $d_T(a, d) = 3 + 6i$, $d_T(a, e) = 3 + 6i$ and $d_T(d, e) = 1 + 2i$, respectively, which is a contradiction as the three distances should add up to an even number.

Suppose that G contains an H_5 as in Fig. 4. Then in the two diamonds induced by $\{a, b, c, d\}$ and $\{a, d, e, f\}$ we can deduce $d_T(a, c) = 1 + 2i$ and $d_T(a, e) = 1 + 2i$, respectively, which implies $d_T(c, e) \leq 2 + 4i$ and that ce is an edge in G, a contradiction.

Suppose that G contains an H_6 as in Fig. 4. Then in the two diamonds induced by $\{a, b, c, d\}$ and $\{a, e, f, g\}$ we can deduce $d_T(a, b) = 1 + 2i$ and $d_T(a, e) = 1 + 2i$, respectively, which implies $d_T(b, e) \leq 2 + 4i$ and that be is an edge in G, a contradiction.

Suppose that G contains an H_7 as in Fig. 4. Then in the two diamonds induced by $\{a, b, c, d\}$ and $\{a, e, f, g\}$ we can deduce $d_T(a, b) = 1+2i$ and $d_T(e, f) = 1+2i$, respectively. Furthermore, we have $3+6i \leq d_T(b, e) \leq d_T(a, b)+d_T(a, e) \leq 3+6i$, so that $d_T(b, e) = 3 + 6i$. By symmetry, we also have $d_T(b, f) = 3 + 6i$, so that $d_T(b, e)$, $d_T(b, f)$ and $d_T(e, f)$ sum up to an odd number, a contradiction.

Suppose that G contains an H_{15} as in Fig. 5. Then in the two diamonds induced by $\{e, f, g, h\}$ and $\{e, f, g, j\}$ we can deduce $d_T(e, f) = 1 + 2i$ and $d_T(e, g) = 1 + 2i$, respectively. Furthermore, for example in the former diamond, since at least one of

$d_T(e, g) = d_T(f, h) = 2 + 4i$ and $d_T(f, g) = d_T(e, h) = 2 + 4i$ holds, we have $d_T(f, g) = 2 + 4i$. Then condition ($*$), applied to a, e, f and g, implies $d_T(a, e) + d_T(f, g) \leq 3 + 6i$ and hence $d_T(a, e) \leq 1 + 2i$, so that we have $d_T(a, e) = 1 + 2i$. In the diamond induced by $\{a, b, c, d\}$ we can deduce $d_T(c, d) = 1 + 2i$. Now $d_T(a, c) = 2 + 4i$ and $d_T(a, d) = 2 + 4i$ cannot both hold, as otherwise the three distances between the three vertices a, c and d would add up to an odd number, a contradiction. Hence, without loss of generality, we may assume that $d_T(a, c) < 2 + 4i$. But then $d_T(c, e) \leq d_T(a, c) + d_T(a, e) < 3 + 6i$, which contradicts the fact that $ce \notin E$.

For the other direction, let G be an $(H_1, \ldots, H_7, H_{15})$-free chordal graph. Note that by Theorem 5, G being $(H_1, \ldots, H_5)$-free chordal implies that its blocks are squares of trees. The forbidden subgraphs H_6, H_7 and H_{15} are the gluing conditions for the blocks. Our aim is to construct a $(2 + 4i, 3 + 6i)$-Steiner root for G with the property that no two real nodes have a distance of 1 up to $2i$.

In the following we will construct such a Steiner root for each block and gradually join all together to obtain a desired Steiner root for G. For blocks that are cliques, we take stars and subdivide their edges by exactly $2i$ vertices. The real nodes are the leaves. For blocks that are not cliques, the construction is slightly more complicated. As they are squares of trees, we first consider a square root and subdivide its edges by exactly $2i$ vertices. The real nodes are the original vertices of the square root. Now if v is an internal vertex (non-leaf) of the square root with the property that it has exactly one internal vertex w as a neighbour, then we adjust the Steiner root as follows. Let the path of length $1 + 2i$ between v and w be $vx_1 \ldots x_{2i}w$. Add the path $x_iy_1 \ldots y_i$ of length i between x_i and y_i. Let y_1 up to y_{i-1} be Steiner nodes, turn v into a Steiner node, and let y_i take over the role of v as a real node. This adjustment is done for all such internal vertices of the square root simultaneously.

Whenever joining two blocks with one being a clique, we can just identify their constructed Steiner roots at the cut vertex and obtain a required $(2 + 4i, 3 + 6i)$-Steiner root. It is crucial that non-clique blocks can only be joined at a vertex, which, in the respective square roots, is a leaf and adjacent to an internal vertex with exactly one internal vertex as a neighbour. The above construction then ensures that identifying the respective Steiner roots at the cut vertex yields a required $(2 + 4i, 3 + 6i)$-Steiner root.

It remains to justify the claim about non-clique blocks. For any block that is not a clique and for any vertex, there is a diamond containing that vertex. So the forbidden H_6 and H_7 imply that when gluing two non-clique blocks together at a vertex v, then v cannot be a mid-vertex of a diamond on either side. Thus, v must be a leaf in both respective square roots, as it cannot be a midpoint of a P_4 (whose square is a diamond). So we may assume that $abcv$ and $vxyz$ are two paths P_4 in the two roots, respectively. Suppose that in one of the two roots v is not adjacent to an internal vertex with exactly one internal vertex as a neighbour. Without loss of generality, let this be the case in the latter root. Then it contains two further vertices u and w such that xuw is a P_3. But then the square of $abcv$ and the square of the subtree induced by $\{u, v, w, x, y, z\}$ together form an H_{15}, the final contradiction. □

Corollary 4. *For every integer $i \geq 1$, the $(4 + 4i, 5 + 6i)$-leaf powers can be recognised in polynomial time.*

7. Conclusion

In this paper, we give structural characterisations of the classes of (4, 6)-leaf powers and (6, 8)-leaf powers, which imply efficient recognition of these classes, and in this way we improve and extend a recent paper [24] by Kennedy, Lin and Yan on strictly chordal graphs. The following inclusions between corresponding leaf power classes hold:

3 − leaf powers $\subset$ (4, 6)-leaf powers $\subset$ 4-leaf powers $\subset$ (6, 8)-leaf powers

Our main results are presented in Theorems 3 and 6. We show that the strictly chordal graphs are precisely the (4, 6)-leaf powers, which implies several of their properties and leads to simpler proofs of related results. The connected (4, 6)-leaf powers result from substituting cliques into block graphs, and the basic (6, 8)-leaf powers are precisely the induced subgraphs of squares of block graphs. However, in a forthcoming work, we will show that the basic (8, 10)-leaf powers properly contain all induced subgraphs of cubes of block graphs. In Theorems 4, 7 and 8 we give further results related to (k, ℓ)-leaf powers, including a characterisation of the (8, 11)-leaf powers (Theorem 8). Their common theme is that certain (k', ℓ')-leaf power classes occur as the (k, ℓ)-leaf powers for infinitely many pairs (k, ℓ). It might be an interesting open problem to analyse whether our respective theorems have captured all possible repetitions.

Acknowledgement

The second author was supported by DFG research grant BR 2479/7-1.

References

[1] H.-J. Bandelt, H.M. Mulder, Distance-hereditary graphs, J. Combin. Theory B 41 (1986) 182–208.
[2] A. Berry, A. Sigayret, Representing a concept lattice by a graph, Discrete Appl. Math. 144 (2004) 27–42.
[3] E. Bibelnieks, P.M. Dearing, Neighborhood subtree tolerance graphs, Discrete Appl. Math. 43 (1993) 13–26.
[4] A. Brandstädt, C. Hundt, Ptolemaic graphs and interval graphs are leaf powers, in: Proceedings of LATIN 2008, in: LNCS, vol. 4957, 2008, pp. 479–491. (Extended abstract).
[5] A. Brandstädt, V.B. Le, Structure and linear time recognition of 3-leaf powers, Inform. Process. Lett. 98 (2006) 133–138.

[6] A. Brandstädt, V.B. Le, Simplicial powers of graphs, in: Proceedings of COCOA 2008, in: LNCS, vol. 5165, 2008, pp. 160–170 (Extended abstract), Full version electronically available in Theor. Computer Science.
[7] A. Brandstädt, V.B. Le, Dieter Rautenbach, Exact leaf powers, manuscript 2006 (submitted for publication).
[8] A. Brandstädt, V.B. Le, Dieter Rautenbach, Distance-hereditary 5-leaf powers, Discrete Math. 309 (2009) 3843–3852.
[9] A. Brandstädt, V.B. Le, J.P. Spinrad, Graph classes: A survey, in: SIAM Monographs on Discrete Mathematics and Applications, vol. 3, Philadelphia, 1999.
[10] A. Brandstädt, V.B. Le, R. Sritharan, Structure and linear time recognition of 4-leaf powers, ACM Trans. Algorithms 5 (1) (2008) available online.
[11] A. Brandstädt, P. Wagner, On (k, ℓ)-leaf powers, in: L. Kučera, A. Kučera (Eds.), Proceedings of MFCS 2007, in: LNCS, vol. 4708, 2007, pp. 525–535 (Extended abstract).
[12] A. Brandstädt, P. Wagner, On k- versus $(k+1)$-leaf powers, in: Proceedings of COCOA 2008, in: LNCS, vol. 5165, 2008, pp. 171–179 (Extended abstract). Full version electronically available in Theor. Computer Science.
[13] M.W. Broin, T.J. Lowe, A dynamic programming algorithm for covering problems with (greedy) totally balanced constraint matrices, SIAM J. Alg. Disc. Meth. 7 (1986) 348–357.
[14] P. Buneman, A note on the metric properties of trees, J. Combin. Theory B 1 (1974) 48–50.
[15] M.-S. Chang, M.-T. Ko, The 3-Steiner root problem, in: Proceedings of WG 2007, in: LNCS, vol. 4769, 2007, pp. 109–120 (Extended abstract).
[16] E. Dahlhaus, P. Duchet, On strongly chordal graphs, Ars Combin. 24 B (1987) 23–30.
[17] M. Dom, J. Guo, F. Hüffner, R. Niedermeier, Error compensation in leaf root problems, in: Proceedings of ISAAC 2004, in: LNCS, vol. 3341, 2004, pp. 389–401 (Extended abstract); Algorithmica 44 (4) (2006) 363–381.
[18] M. Dom, J. Guo, F. Hüffner, R. Niedermeier, Extending the tractability border for closest leaf powers, in: Proceedings of 31st Workshop on Graph-Theoretic Concepts in Computer Science WG 2005, in: LNCS, vol. 3787, 2005, pp. 397–408 (Extended abstract). Appeared under the title Closest 4-leaf power is fixed-parameter tractable, Discrete Appl. Math., 156 (2008) 3345–3361.
[19] M. Fellows, D. Meister, F. Rosamond, R. Sritharan, J.A. Telle, Leaf powers and their properties: Using the trees, in: Proceedings of ISAAC 2008, in: LNCS, vol. 5369, 2008, pp. 402–413 (Extended abstract).
[20] P. Galinier, M. Habib, C. Paul, Chordal graphs and their clique graphs, in: Proceedings of the Workshop on Graph-Theoretic Concepts in Computer Science WG, in: LNCS, vol. 1017, 1995, pp. 358–371 (Extended abstract).
[21] E. Howorka, On metric properties of certain clique graphs, J. Combin. Theory B 27 (1979) 67–74.
[22] W. Kennedy, Strictly chordal graphs and phylogenetic roots, Master's Thesis, University of Alberta, 2005.
[23] W. Kennedy, G. Lin, 5-th phylogenetic root construction for strictly chordal graphs, in: Proceedings ISAAC, in: LNCS, vol. 3827, 2005, pp. 738–747 (Extended abstract).
[24] W. Kennedy, G. Lin, G. Yan, Strictly chordal graphs are leaf powers, J. Discrete Algorithms 4 (2006) 511–525.
[25] G.-H. Lin, P.E. Kearney, T. Jiang, Phylogenetic k-root and Steiner k-root, in: Proceedings of 11th Annual International Symposium on Algorithms and Computation ISAAC, in: LNCS, vol. 1969, 2000, pp. 539–551 (Extended abstract).
[26] A. Lubiw, Γ-free matrices, Master's Thesis, Dept. of Combinatorics and Optimization, University of Waterloo, Canada, 1982.
[27] R.M. McConnell, Linear time recognition of circular-arc graphs, Algorithmica 37 (2003) 93–147.
[28] N. Nishimura, P. Ragde, D.M. Thilikos, On graph powers for leaf-labeled trees, J. Algorithms 42 (2002) 69–108.
[29] D. Rautenbach, Some remarks about leaf roots, Discrete Math. 306 (13) (2006) 1456–1461.
[30] A. Raychaudhuri, On powers of strongly chordal and circular arc graphs, Ars Combin. 34 (1992) 147–160.
[31] F.S. Roberts, Indifference graphs, in: F. Harary (Ed.), Proof Techniques in Graph Theory, Academic Press, 1969, pp. 139–146.
[32] J.P. Spinrad, Efficient Graph Representations, Fields Institute Monographs, American Mathematical Society, Providence, Rhode Island, 2003.