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On the ordered anti-Weber problem for any norm in $\mathbb{R}^2$

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ABSTRACT

In this paper, a family of single-obnoxious-facility location problems is modelled by considering the same objective function as is used in the ordered median location problem. This function involves distances defined with any arbitrary norm and hence it can be used in a general framework. We prove that the solutions to these obnoxious location problems, restricted to a polygonal region with m vertices and considering n existing population centers, can be found in a set defined in terms of the weighted equidistant points. For many usual norms, this dominating set is finite and can be constructed in $\mathcal{O}(mn^2 + n^4)$.

1. Introduction

The problem of placing an undesirable but necessary facility has received significant attention in the last few decades, due to the increasing environmental and social impact of facilities such as chemical industrial plants, radioactive waste storage sites, etc. Ideally, these facilities should be located as far as possible from residential areas, ecosystems, etc., in order to minimize the undesirable effects or environmental and social impacts introduced by their presence. These zones or areas are called affected sites and the set of areas where the facility can be installed is called the feasible region.

Most of the decisions for site selection are eventually made on the basis of economic and political considerations. This may be due to the lack of quantitative data regarding the risk posed by the facility. There are two issues in the selection of such an ideal site: the problem of risk or impact evaluation and the subsequent problem of finding points that minimize that risk or impact.

The determination of the environmental and social impact of a given industrial activity as a function of its location offers a valuable source of information for the planner. Thus, a dispersion mechanism of the undesirable effects associated with the facility and the environmental impact of a given industrial activity can be studied using a dispersion model. In this paper this dispersion model or environmental impact function is given by a norm. If the undesirable effects are propagated equally in all directions then the Euclidean norm could be used as an impact dispersion function and the resulting norm level lines are circumferences. However, there are facilities where factors such as the wind affect the dispersion of pollutants (e.g. sulphur dioxide SO_2, nitrogen monoxide, etc.) and the Euclidean norm is not adequate.

In this paper, we are concerned with the problem of finding in a region a minimum risk or impact location for the affected sites. In such models, it is unnatural to use the metaphor of "clients or customers" for given locations; thus we prefer the term "sites".

Erkut and Neuman [6] gave a detailed survey of the literature and the analytical models for locating undesirable facilities and a synthesis of solution procedures with emphasis on similarities and differences is presented. They concluded that "the final selection of a site is a complex problem and should be approached using multicriteria tools", and that "there has been little research on multicriteria optimization models".

The general maxmin location problem was first addressed by Dasarathy and White [4] and algorithms were presented for solving the problem in both two-dimensional and three-dimensional spaces. Drezner and Wesolowsky [5] formulated the problem of locating multiple waste disposal sites in an unrestricted space with point population centers. The maxmin criterion has been traditionally used for undesirable facility location and generates optimal solutions that are often at the boundary of the feasible region (see [10], one of the earliest obnoxious facility papers). This is not desirable if on the other side of the boundary there are population centers belonging to a different administrative region.

Melachrinoudis [13] has proposed a maxmin–minisum bicriteria location model with rectilinear distances and has developed an algorithm that constructs the entire nondominated and efficient sets. Melachrinoudis [14] describes the set of efficient solutions for a location problem under two conflicting criteria: maximize the

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distance from population centers and minimize total transportation costs, and proposes an efficient algorithm for its solution. A very interesting property found is that most of these efficient solutions are on edges of a Voronoi diagram. Also, Ohsawa (see [18] or [17]) obtains similar results related to Voronoi diagrams using the Euclidean distance or squared Euclidean distance.

Romero et al. [22] propose a semi-obnoxious facility model which takes into account both the environmental impact and the transportation costs; they have proposed a solution method based on the well-known Big Square–Small Square Method, also used in [10] and [9]. Brimberg and Juel [1] have described a trajectory method for constructing an efficient frontier of points for a bicriteria model for locating a semi-desirable facility in the plane where the first criterion is based on transportation costs and the second one is based on social or environmental cost. Plastria and Carrizosa [20] have proposed a bicriteria model seeking the least effect on the population at the highest level of protection. They have developed fast polynomial algorithms for constructing the complete trade-off curve for the two objectives together with corresponding efficient solutions. Fernández et al. [7] have presented a model that minimizes the global repulsion of the inhabitants of a region while taking into account environmental concerns which make some areas suitable for the location of the facility. Recently, Tamir [24] presented two algorithms for solving the weighted two-obnoxious-facility location problem in the case where the feasible region is a convex compact polygon for the rectilinear and Euclidean distances.

These location problems have been studied in a variety of metric spaces. Due to their potential applications, the most popular models deal with planar and network settings. In this paper we propose a general model where the objective function is given by a general norm in the plane. An algorithm is developed for constructing a set that contains at least one optimal solution. Location planners can then choose a solution that provides the best compromise for the conflicting objectives.

2. The problem statement and properties

Let the feasible region S be a closed and bounded subset of the plane, let $\|\cdot\|$ be a norm in $\mathbb{R}^2$, let $\{P_1, P_2, \dots, P_n\}$ be a collection of points of the plane corresponding to the n sites or population centers and let $\{w_1, w_2, \dots, w_n\}$ be positive weights associated with each population center such that w_i represents the importance, relative incompatibility or level of repulsion assigned to the site P_i. A discussion on the interpretation of the weights can be seen in [23] and, for simplicity, we will denote by $\mathcal{I}_k$ the set $\{1, 2, \dots, k\}$, where k is a natural number.

We call the *ordered anti-Weber problem* (OAWP) the problem of maximizing, in the region S, the following objective function:

$$f(x) = \sum_{i=1}^{n} k_i w_{\sigma(i)} \|x - P_{\sigma(i)}\| = \sum_{i=1}^{n} k_i D_{\sigma(i)}(x)$$

where $D_k(x) = w_k\|x - P_k\|$, σ is the permutation of $\mathcal{I}_n$ such that $w_{\sigma(i)}\|x - P_{\sigma(i)}\| \le w_{\sigma(i+1)}\|x - P_{\sigma(i+1)}\|$ for all $i \in \mathcal{I}_{n-1}$, and $\{k_1, k_2, \dots, k_n\}$ are n non-negative constants determined by the selected criteria. Several choices of the constants lead to some well-known obnoxious location problems. For example:

- $k_1 = 1, k_2 = k_3 = \cdots = k_n = 0$; we obtain the "maxmin" location problem.
- $k_1 = k_2 = k_3 = \cdots = k_n = 1$; we obtain the "maxisum" location problem or the "anti-Weber" problem.
- With fixed $j \in \mathcal{I}_n$, taking $k_j = 1$ and $k_i = 0$ for all $i \neq j$ leads us to the problem of maximizing the distance between the undesirable facility and the j-th closest affected center. It is an extension of the "maxmin" location problem called the j-th quantile location problem.
- With fixed $j \in \mathcal{I}_n$, taking $k_i = 1$ for all $i \le j$ and $k_i = 0$ for all $i > j$ leads us to the problem of maximizing the sum of the distances between the undesirable facility and the j closest affected centers. It is an extension of the "maxisum" facility location problem called the j-anticentrum problem (see [12]).
- If $k_1 = 1$ and $k_i = \lambda \in [0, 1]$ for all $i > 1$, we obtain the λ-anticentdian facility location problem. It is the continuous version of the problem stated in [15] or [3] in which the objective is the maximization of the objective function obtained combining in the following convex way the objective functions corresponding to the "maxmin" and "maxisum" facility location problems: $(1-\lambda)\min_{i\in\mathcal{I}_n} D_i(x) + \lambda \sum_{i=1}^{n} D_i(x)$.

This function has been used in [21] for modelling a generalization of the Weber problem that the authors call the *ordered Weber problem*. They study finding the minimum of this function and obtain some properties that they use to suggest an algorithm and other applications (for further details see [16] and references therein).

Our OAWP is the natural extension to an arbitrary norm of the function used in [23]. The following results that can be found in that paper are still true when an arbitrary norm is used. Because the proofs therein are only based on the norm properties of the Euclidean norm, the proofs for any other norm can be constructed simply by replacing the Euclidean norm by an arbitrary norm:

Proposition 1. *$f(x)$ is an isotone function.*

Proposition 2. *Let X be the convex hull of $P_1, P_2, \dots, P_n$, and T the boundary of S. The set $(S \cap X) \cup (T - X)$ contains at least one optimal solution to the OAWP.*

Theorem 1. *$f(x)$ is a Lipschitz function (and so, is a continuous function).*

Corollary 1. *The OAWP always has a solution, because the function $f(x)$ is continuous and S is a closed and bounded set.*

Usual regions of interest for the restricted optimization problems are polygonal regions because the installation of the obnoxious facility must lie in a geographical zone that can be delimited by a polygon. In that context, [23] states and proves the following main result for the Euclidean OAWP:

Theorem 2. *If $x_0 \in S$ is an optimal solution for the Euclidean OAWP, then x_0 belongs to the set $C = V \cup W \cup I$, where:*

- *V is the set of vertices of S.*
- *W is the set of points x belonging to the boundary of S and such that there exists some pair (P_i, P_j) of different points of $\{P_1, P_2, \dots, P_n\}$ with $w_i \cdot \|x - P_i\|_2 = w_j \cdot \|x - P_j\|_2$.*
- *I is the set of points x belonging to the interior of S such that there exist two different pairs (P_p, P_q) and (P_r, P_s) of different points of $\{P_1, P_2, \dots, P_n\}$ with $\{p, q\} \neq \{r, s\}$ satisfying that $w_p \cdot \|x - P_p\|_2 = w_q \cdot \|x - P_q\|_2$ and $w_r \cdot \|x - P_r\|_2 = w_s \cdot \|x - P_s\|_2$.*

Considering the above equalities we can state the following definition: With $\|\cdot\|$ a norm in $\mathbb{R}^2$ and P and Q two points of $\mathbb{R}^2$ associated with the positive values w_P and w_Q respectively, the set

$$WB(P, Q, w_P, w_Q) = \{x \in \mathbb{R}^2 \mid w_P \cdot \|x - P\| = w_Q \cdot \|x - Q\|\}$$

is called the *weighted bisector* (see [11,16] or [21]).

If $\|\cdot\|$ is an arbitrary norm in $\mathbb{R}^2$, S is a closed and bounded polygon of $\mathbb{R}^2$ with nonempty interior, and $\{P_1, P_2, \dots, P_n\}$ is a collection of points of the plane associated with the positive weights $\{w_1, w_2, \dots, w_n\}$, respectively, we can construct the set $C = V \cup W \cup I$, where:

- V is the set of vertices of S.
- W is the set of points x belonging to the boundary of S and such that there exists some pair (P_i, P_j) of different points of $\{P_1, P_2, \dots, P_n\}$ with $x \in WB(P_i, P_j, w_i, w_j)$.

- I is the set of points x belonging to the interior of S such that there exist two different pairs (P_p, P_q) and (P_r, P_s) of different points of $\{P_1, P_2, \dots, P_n\}$ with $\{p, q\} \neq \{r, s\}$ satisfying that $x \in WB(P_p, P_q, w_p, w_q) \cap WB(P_r, P_s, w_r, w_s)$.

The results in the following section are oriented to proving that the maximum value of f, restricted to a closed and bounded polygon S with nonempty interior, is always attained in the set C defined above (i.e., C is a dominating set).

Note that, for non-strict norms, alternative optimal solutions may exist outside C, as can be seen in the following example:

Example 1. Consider the rectilinear norm defined as $\|(x_1, x_2)\|_1 = |x_1| + |x_2|$, the polygonal set S determined by the points $(-12, 0)$, $(0, -12)$, $(12, 0)$ and $(0, 12)$ and a collection of four sites at the points $P_1 = (-4, 0)$, $P_2 = (2, 6)$, $P_3 = (6, -2)$ and $P_4 = (0, -12)$ with equal weights. If $\|\cdot\|$ is an arbitrary norm, the unweighted "maxmin" facility location problem is equivalent (see [4]) to the determination of the smallest r that $S \subset \bigcup_{i=1}^{n} B(P_i, r)$, where $B(P_i, r)$ is the closed ball with radius r and centered at P_i; that is $B(P_i, r) = \left\{x \in \mathbb{R}^2 \mid \|x - P_i\| \leq r\right\}$. As $S \subset \bigcup_{i=1}^{4} B(P_i, 8)$, we can assert that $\max_{x \in S} f(x) \leq 8$ where $f(x) = \min\{\|x - P_1\|_1, \|x - P_2\|_1, \|x - P_3\|_1, \|x - P_4\|_1\}$. On the other hand, $Q = (10, 2) \in S$ and satisfies that $f(Q) = 8$ and so the maximum value of the function f is 8. Therefore, Q is an optimal solution to the "maxmin" problem, and Q does not belong to $C = V \cup W \cup I$. Nevertheless, there are points in C (for example, the point $(0, 12)$) that are as good a solution as Q and so the maximum value of f is attained in C.

3. The dominating set of candidate points

In the following, we will consider that the feasible region S is a closed and bounded polygon with nonempty interior and we will denote by $Bd(S)$ its boundary. Under this assumption, and taking into account Proposition 2, the search for optimal solutions of OAWP may be restricted to $(S \cap X) \cup (Bd(S) - X)$, with X being the convex hull of $P_1, P_2, \dots, P_n$.

Also, we will consider the sets S_σ defined as the points $x \in S$ such that $D_{\sigma(i)}(x) \leq D_{\sigma(i+1)}(x)$, for all $i \in \mathcal{I}_{n-1}$, where σ is a permutation of $\mathcal{I}_n$.

Note that $f(x)$ is a convex function over each S_σ. Also, the following lemma will be used:

Lemma 1. *Let $f(x)$ be a convex function over a set D and suppose that $x_0 \in D$ satisfies that $f(x_0) \geq f(x)$, for all $x \in D$ (i.e., x_0 is a maximum of f in D). If there exist $0 < \lambda < 1$ and a and b in D such that $x_0 = \lambda \cdot a + (1 - \lambda) \cdot b$, then $f(a) = f(x_0) = f(b)$.*

Proof. As x_0 is the maximum of $f(x)$ in D and $f(x)$ is a convex function, we can assert that

$$f(a) \leq f(x_0) = f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda) f(b)$$

and so $f(a) \leq f(b)$. On the other hand,

$$f(b) \leq f(x_0) = f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda) f(b)$$

and so $f(b) \leq f(a)$, it following that $f(a) = f(b)$. This condition and the inequalities above lead us to the result. □

Now, we are going to determine a dominating set of candidate points for each of the possibilities. Let us start with the boundary of S:

Theorem 3. *If x_0 is a maximum of $f(x)$ in the region S and $x_0 \in Bd(S) \cap S_\sigma$, then there exists $x^* \in Bd(S) \cap S_\sigma$ such that $f(x^*) = f(x_0)$ where x^* is a vertex of S or there exists $i \in \mathcal{I}_{n-1}$ such that $x^* \in WB(P_{\sigma(i)}, P_{\sigma(i+1)}, w_{\sigma(i)}, w_{\sigma(i+1)})$.*

Proof. Suppose that, on the contrary, x_0 is not a vertex and $D_{\sigma(i)}(x_0) < D_{\sigma(i+1)}(x_0)$, for all $i \in \mathcal{I}_{n-1}$. Because any norm is a continuous function, there exists $\delta > 0$ such that any $x \in B(x_0, \delta)$ satisfies that $D_{\sigma(i)}(x) \leq D_{\sigma(i+1)}(x)$, for all $i \in \mathcal{I}_{n-1}$. Let E be the edge of S containing x_0. Two cases may occur:

- $WB(P_{\sigma(i)}, P_{\sigma(i+1)}, w_{\sigma(i)}, w_{\sigma(i+1)}) \cap E = \varnothing$, for all $i \in \mathcal{I}_{n-1}$. In this case, take x^* as the extreme point of E nearest to x_0 and consider the set $A = \{x \in S \mid x \in B(x_0, \delta) \cap L\}$. This set is not empty because $x_0 \in A$. Also, $A \subset S_\sigma$. Take $1 < \lambda < 1 + \frac{\delta}{\|x^* - x_0\|}$ and consider the point $y = \lambda \cdot x_0 + (1 - \lambda) \cdot x^*$. The point y belongs to A because it is in E and y is collinear with x^* and x_0, and satisfies that

  $$\|y - x_0\| = (\lambda - 1) \cdot \|x^* - x_0\| < \delta.$$

  On the other hand, we also have that $x_0 = \frac{1}{\lambda} \cdot y + \frac{\lambda - 1}{\lambda} \cdot x^*$ and by Lemma 1 this first case is proved.
- There exists $i \in \mathcal{I}_{n-1}$ such that $WB(P_{\sigma(i)}, P_{\sigma(i+1)}, w_{\sigma(i)}, w_{\sigma(i+1)}) \cap E \neq \varnothing$. In this case, consider the set

  $$B = \{x \in E \mid \exists i \in \mathcal{I}_{n-1} \text{ with } x \in WB(P_{\sigma(i)}, P_{\sigma(i+1)}, w_{\sigma(i)}, w_{\sigma(i+1)})\}$$

  and take x^* the point of B nearest to x_0 and take $\delta^* = min\{\delta, \|x^* - x_0\|\}$. The same procedure as above, over the set $A = \{x \in S \mid x \in B(x_0, \delta^*) \cap E\}$, concludes the proof. □

Note that this result allows us, in our search for optimal solutions of OAWP in the boundary of S, to reduce this set to the vertices of S and the points of its boundary that belong to a weighted bisector associated with a pair of population centers.

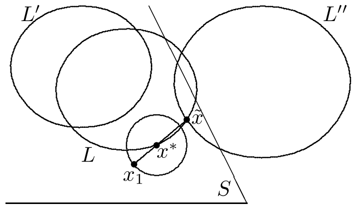
In the same way, our next step consists in the reduction of the set of candidate optimal solutions in the interior of the region S.

Theorem 4. *If $x_0 \in S_\sigma$ is a maximum of $f(x)$ belonging to the interior of S, then there exists $x^* \in S_\sigma$ such that $f(x^*) = f(x_0)$ where $x^* \in Bd(S)$ or there exists $i \in \mathcal{I}_{n-1}$ such that $x^* \in WB(P_{\sigma(i)}, P_{\sigma(i+1)}, w_{\sigma(i)}, w_{\sigma(i+1)})$.*

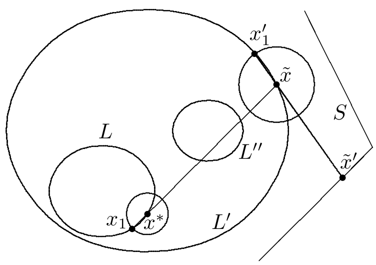
Proof. Let σ be the permutation of the set $\mathcal{I}_n$ such that $x_0 \in S_\sigma$. If there exists $i \in \mathcal{I}_{n-1}$ such that $x_0 \in WB(P_{\sigma(i)}, P_{\sigma(i+1)}, w_{\sigma(i)}, w_{\sigma(i+1)})$, then the proof is over. So, suppose that $D_{\sigma(i)}(x_0) < D_{\sigma(i+1)}(x_0)$ for all $i \in \mathcal{I}_{n-1}$

- If $WB(P_{\sigma(i)}, P_{\sigma(i+1)}, w_{\sigma(i)}, w_{\sigma(i+1)}) \cap S = \varnothing$, for all $i \in \mathcal{I}_{n-1}$, then the permutation σ for the evaluation of the objective function is the same for all $x \in S$. In this case, take x_1 any point of S such that $x_1 \neq x_0$ and take $\lambda^* = max\{\lambda \in \mathbb{R} \mid \lambda \cdot x_0 + (1 - \lambda) \cdot x_1 \in S\}$. Note that $\lambda^* > 1$, because x_0 belongs to the interior of S. Also, $x^* = \lambda^* \cdot x_0 + (1 - \lambda^*) \cdot x_1 \in Bd(S)$ and so we have that $x_0 = \frac{1}{\lambda^*} \cdot x^* + \frac{\lambda^* - 1}{\lambda^*} \cdot x_1$. By Lemma 1, we have that $f(x_0) = f(x^*)$ and x^* belongs to the boundary of S.
- Otherwise, if there exists $i \in \mathcal{I}_{n-1}$ such that $WB(P_{\sigma(i)}, P_{\sigma(i+1)}, w_{\sigma(i)}, w_{\sigma(i+1)}) \cap S \neq \varnothing$, note that all the inequalities in the construction of S_σ are strict for the point x_0 and so x_0 belongs to the interior of S_σ. Because any norm is a continuous function, we know that some $\delta > 0$ exists such that $B(x_0, \delta)$ is included in the interior of S_σ. Let x_1 be any point of $B(x_0, \delta)$ such that $x_1 \neq x_0$ and take $\lambda^* = max\{\lambda \in \mathbb{R} \mid \lambda \cdot x_0 + (1 - \lambda) \cdot x_1 \in S_\sigma\}$.

  Again, we can assert that $\lambda^* > 1$, because $x_0 \in B(x_0, \delta)$. Taking $x^* = \lambda^* \cdot x_0 + (1 - \lambda^*) \cdot x_1$, we have the following two possibilities:
  - $x^* \in Bd(S)$, and the same reasoning as above leads us to $f(x^*) = f(x_0)$ and the result follows.
  - $x^* \in Bd(S_\sigma)$, which implies the existence of $i \in \mathcal{I}_{n-1}$ such that $x^* \in WB(P_{\sigma(i)}, P_{\sigma(i+1)}, w_{\sigma(i)}, w_{\sigma(i+1)})$ and the same reasoning completes the proof. □



The weighted bisector L intersects with $Bd(S)$ and/or with another weighted bisector.



The weighted bisector L doesn't intersect with $Bd(S)$ and with any other weighted bisector.

Fig. 1. Diagrams for the proof of Theorem 5.

On the basis of the previous result, we can reduce the set of optimal solutions in the interior of the region S, taking into account the following result:

Theorem 5. *If x_0 is a maximum of $f(x)$ belonging to the interior of S, then there exists $\tilde{x} \in S$ such that $f(x_0) = f(\tilde{x})$ satisfying that $\tilde{x} \in Bd(S)$ or there exists a permutation σ of the set $\mathcal{I}_n$ and $i, j \in \mathcal{I}_{n-1}$ with $i \neq j$ such that*

$$\tilde{x} \in WB(P_{\sigma(i)}, P_{\sigma(i+1)}, w_{\sigma(i)}, w_{\sigma(i+1)}) \cap WB(P_{\sigma(j)}, P_{\sigma(j+1)}, w_{\sigma(j)}, w_{\sigma(j+1)}).$$

Proof. Let $x_0 \in S_\tau$ be an interior point of S that is a maximum of $f(x)$. From Theorem 4, we assert that there exists a point $x^* \in S_\tau$ such that $f(x_0) = f(x^*)$ with $x^* \in Bd(S)$, or there exists $i \in \mathcal{I}_{n-1}$ such that $x^* \in WB(P_{\tau(i)}, P_{\tau(i+1)}, w_{\tau(i)}, w_{\tau(i+1)})$.

If $x^* \in Bd(S)$, the result is proved. So, suppose that $x^* \notin Bd(S)$ and there exists $i \in \mathcal{I}_{n-1}$ such that $x^* \in WB(P_{\tau(i)}, P_{\tau(i+1)}, w_{\tau(i)}, w_{\tau(i+1)})$, and consider the following sets:

- $L = WB(P_{\tau(i)}, P_{\tau(i+1)}, w_{\tau(i)}, w_{\tau(i+1)})$,
- $T_1 = \{x \in S \mid D_{\tau(1)}(x) \leq \cdots \leq D_{\tau(i-1)}(x) \leq D_{\tau(i)}(x) \leq D_{\tau(i+1)}(x) \leq D_{\tau(i+2)}(x) \leq \cdots \leq D_{\tau(n)}(x)\}$,
- $T_2 = \{x \in S \mid D_{\tau(1)}(x) \leq \cdots \leq D_{\tau(i-1)}(x) \leq D_{\tau(i+1)}(x) \leq D_{\tau(i)}(x) \leq D_{\tau(i+2)}(x) \leq \cdots \leq D_{\tau(n)}(x)\}$,
- $T = T_1 \cup T_2$.

If $x^* \in Bd(T)$, the theorem is proved. Otherwise, there exists $\delta > 0$ such that $B(x^*, \delta)$ is a subset of the interior of T. Then, we have one of the following two possibilities (see diagrams in Fig. 1):

- The weighted bisector L intersects with $Bd(S)$ and/or with another weighted bisector. In this case, let $\tilde{x}$ be the point nearest to x^* belonging to any of such intersections. Now, we take

  $$x_1 = \left(\frac{\delta}{\|x^* - \tilde{x}\|} + 1\right) \cdot x^* - \frac{\delta}{\|x^* - \tilde{x}\|} \cdot \tilde{x}.$$

  We have that x^* and $\tilde{x}$ belongs to $T_1 \cap T_2$. On the other hand, x_1 belongs to T_1 or belongs to T_2 because $\|x_1 - x^*\| = \delta$ and $B(x^*, \delta) \subset T$. So, there exists a permutation ρ such that x^*, $\tilde{x}$ and x_1 belongs to S_ρ. Since the point x^* can be rewritten as

  $$x^* = \frac{\|x^* - \tilde{x}\|}{\|x^* - \tilde{x}\| + \delta} \cdot x_1 + \frac{\delta}{\|x^* - \tilde{x}\| + \delta} \cdot \tilde{x},$$

  Lemma 1 proves the claim.
- The weighted bisector L does not intersect with $Bd(S)$ and with any other weighted bisector. In this case, take $x_1 \in \big(L \cap B(x^*, \delta)\big)$ such that $x_1 \neq x^*$. Note that x_1 and x^* belongs to T_1 and belongs to T_2. So, $\lambda^* = \max\{\lambda \in \mathbb{R} \mid \lambda x^* + (1-\lambda)x_1 \in T\}$ verifies that $\lambda^* > 1$ because $x^* \in T$. Also, the point $\tilde{x} = \lambda^* \cdot x^* + (1-\lambda^*) \cdot x_1$ belongs to T_1 or belongs to T_2, but, in any case, there exists a permutation ρ such that x^*, $\tilde{x}$ and x_1 belongs to S_ρ. Since $x^* = \frac{1}{\lambda^*} \cdot \tilde{x} + \frac{\lambda^*-1}{\lambda^*} \cdot x_1$, Lemma 1 completes the proof. □

Note that the same reasoning as was used in the proof of Theorem 3 allows us to reduce the set V to the extreme vertices of S (i.e., the extreme points of the convex hull of S).

4. Two particular cases: Elliptical and rectilinear norms

The results in [23] could be extended to the elliptical norms (introduced and studied mainly in [19], and for minimax problems in [2]) defined as

$$\|(x_1, x_2)\| = \sqrt{\alpha x_1^2 + \beta x_2^2 + 2\gamma x_1 x_2}$$

with α, β and $\alpha\beta - \gamma^2 > 0$, taking into account that the invertible matrix

$$\begin{pmatrix} \sqrt{\alpha} & \dfrac{\gamma}{\sqrt{\alpha}} \\ 0 & \dfrac{\sqrt{\alpha\beta - \gamma^2}}{\sqrt{\alpha}} \end{pmatrix}$$

transforms the ellipse $\alpha x_1^2 + \beta x_2^2 + 2\gamma x_1 x_2 = 1$ into the circumference $x_1^2 + x_2^2 = 1$, weighted bisectors of the corresponding elliptical norm into weighted bisectors of the Euclidean norm and the feasible region S with m vertices into another polygon with the same number of vertices. Thus, a finite dominating set C could be constructed in $\mathcal{O}(mn^2 + n^4)$ and the solutions of the OAWP belong to C.

If the rectilinear norm $\|\cdot\|_1$ is used then a weighted bisector could intersect with another one or with the border of S in an infinite set of points (see Example 1) and so the set C will be infinite too. Nevertheless, the set C could be reduced to a finite one C^* such that if a point x_0 is a solution for the OAWP then there exists another point in C^* such that it is a solution as good as x_0. The set C^* is defined as $C^* = V \cup W^* \cup I^*$, where:

- V is the set of vertices of S.
- $W^* = WB(P_i, P_j, w_i, w_j) \cap Bd(S)$ where (P_i, P_j) is a pair of different points of $\{P_1, P_2, \ldots, P_n\}$. If any of these intersections is a finite number of line segments, we only consider the extreme points of these line segments, according with Lemma 1.
- $I^* = WB(P_p, P_q, w_p, w_q) \cap WB(P_r, P_s, w_r, w_s)$ where (P_p, P_q) and (P_r, P_s) are two different pairs of different points of $\{P_1, P_2, \ldots, P_n\}$ with $\{p, q\} \neq \{r, s\}$. Again, if any of these intersections is a finite number of line segments or polygons, we only consider the extreme points of these line segments or these polygons.

Again, note that the set V can be reduced to the extreme points of the convex hull of S.

5. The algorithm

The above results allow us to implement an algorithm to obtain an optimum location point in the feasible region:

Step 1 Calculate the weighted bisectors.
Step 2 Determine the set C (or C^*) defined above.
Step 3 Evaluate the function $f(x)$ at the remaining points to obtain an optimum location.

Table 1
Solutions in C^* for several criteria.

Criterion	Solutions
(0 0 0 1)	Q_j, $j \in \{1, 4, 5, 7, 11, 17, 19\}$
(0 0 1 0)	Q_1, Q_2, Q_4, Q_5, Q_7
(0 0 1 1)	Q_1, Q_4, Q_5, Q_7
(0 1 0 0)	Q_1
(0 1 0 1)	Q_1
(0 1 1 0)	Q_1
(0 1 1 1)	Q_1
(1 0 0 0)	Q_j, $j \in \{1, 2, 3, 4, 5, 6, 7, 9, 14, 16, 17, 19\}$
(1 0 0 1)	Q_1, Q_4, Q_5, Q_7, Q_{17}, Q_{19}
(1 0 1 0)	Q_1, Q_2, Q_4, Q_5, Q_7
(1 0 1 1)	Q_1, Q_4, Q_5, Q_7
(1 1 0 0)	Q_1
(1 1 0 1)	Q_1
(1 1 1 0)	Q_1
(1 1 1 1)	Q_1

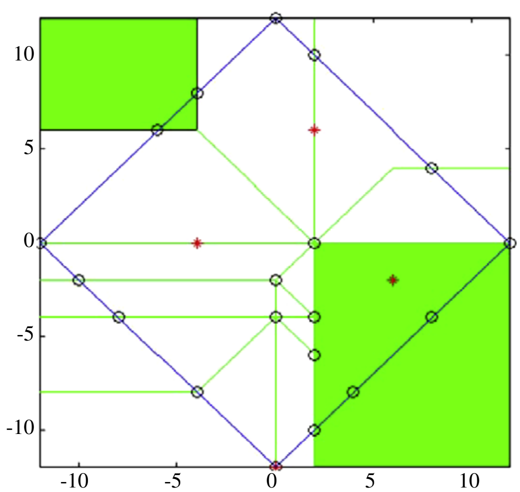


Fig. 2. The set C^* for Example 1.

6. An example and computational results

The algorithm coded in MATLAB 7.1, and running under Windows XP on a PC with a Pentium 4 at 3 GHz, has been applied to Example 1.

Our implementation finds the coordinates of the points Q_j of the set C^* shown in Table 1 (see Fig. 2) in 0.47 seconds and all the solutions in C^* for the $2^4 - 1$ OAWP associated with each selection of the constants $k_i \in \{0, 1\}$ (see Table 1): $Q_1 = (-12, 0)$, $Q_2 = (-10, -2)$, $Q_3 = (-8, -4)$, $Q_4 = (-6, 6)$, $Q_5 = (-4, 8)$, $Q_6 = (-4, -8)$, $Q_7 = (0, 12)$, $Q_8 = (0, -2)$, $Q_9 = (0, -4)$, $Q_{10} = (0, -12)$, $Q_{11} = (2, 10)$, $Q_{12} = (2, 0)$, $Q_{13} = (2, -4)$, $Q_{14} = (2, -6)$, $Q_{15} = (2, -10)$, $Q_{16} = (4, -8)$, $Q_{17} = (8, 4)$, $Q_{18} = (8, -4)$ and $Q_{19} = (12, 0)$.

Computational results are shown in Table 2. For each n (number of facilities) and m (number of edges of the polygonal region), the algorithm has been launched 10 times, and the mean time (in seconds) spent in the generation of the set C defined at the end of Section 2 has been calculated.

Note that the increase in the number of edges of the polygonal region does not lead to a significant increase in the average running time. Nevertheless, the increase in the number of facilities does result in a large increase in the running time. Thus, the usage of aggregation models for such problems will be useful. A good starting point in this sense could be [8].

Table 2
Mean CPU time spent in the generation of C.

$m \setminus n$	5	10	20	30
4	0.03125	0.99375	57.75625	1199.778125
8	0.0328125	1.05	59.7734375	1216.2671875
12	0.0375	1.1046875	61.7390625	1224.5765625
16	0.0390625	1.1921875	60.915625	1209.2703125
20	0.040625	1.234375	61.6796875	1214.340625
24	0.046875	1.3203125	62.1765625	1222.4046875
28	0.05	1.2921875	63.334375	1218.1515625
32	0.0515625	1.265625	64.3296875	1229.525
36	0.0546875	1.3203125	65.53125	1239.3796875
40	0.0609375	1.44375	66.4015625	1222.8328125
44	0.0578125	1.4875	65.425	1235.5796875
48	0.0625	1.440625	65.696875	1244.503125
52	0.0671875	1.4828125	66.4546875	1246.696875
56	0.06875	1.453125	67.0359375	1242.99375
60	0.0765625	1.5171875	67.5953125	1247.26875

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