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Vector spaces of delta-plurisubharmonic functions and extensions of the complex Monge–Ampère operator

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ABSTRACT

In this paper we shall consider two types of vector ordering on the vector space of differences of negative plurisubharmonic functions, and the problem whether it is possible to construct supremum and infimum. Then we consider two different approaches to define the complex Monge–Ampère operator on these vector spaces, and we solve some Dirichlet problems. We end this paper by stating and discussing some open problems.

1. Introduction

Extending objects is one of the most fundamental interest in analysis. When it comes to extending linear functionals, this question has its origin in the works of Helly and Riesz [29,38,39]. Their work were the predecessor of one of the crown jewels of functional analysis, namely, the Hahn–Banach theorem [8,28]. There are numerous extensions and application of this celebrated theorem (see e.g. [36]). The extension theorem related to this article is the following seminal theorem by L.V. Kantorovich [32]:

Theorem K. *Suppose that E and F are two Riesz spaces with F Archimedean. Assume also that $T : E^+ \to F^+$ is an additive mapping, i.e.*

$$T(x + y) = T(x) + T(y) \quad \textit{holds for all } x, y \in E^+.$$

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Here E^+ and F^+ are the positive cones of E and F, respectively. *Then T has a unique extension to a positive operator from E to F. Moreover, the extension (denoted by T again) is given by*

$$T(x) = T\left(x^+\right) - T\left(x^-\right), \tag{1.1}$$

for all $x \in E$. Here $x^+ = x \vee 0$ is the positive part of x, and $x^- = (-x) \vee 0$ is the negative part. (See Sections 2 and 3 for the definitions.)

For a proof we refer to [6, Theorem 1.10, pp. 9–10]. An important and well known consequence of Theorem K is that the mapping $T : E^+ \to F^+$ extends to a (unique) positive operator from E to F if, and only if, T is additive on E^+. Furthermore, Theorem K is not valid if we drop the assumption that F is Archimedean.

In this article we shall consider the complex Monge–Ampère operator defined on certain cones of plurisubharmonic functions. This operator is not additive, so as mentioned above there is no unique extension. We shall here propose and study two possible extensions. This yields a bridge between the theory of ordered vector spaces that is not locally convex, with the theory of pluripotential theory. We now continue with a brief discussion about the setting, and we refer the reader to Section 2 and Section 3 for more detailed background and definitions.

Let Ω be an open set in $\mathbb{C}^n$. With $\mathcal{PSH}(\Omega)$ we denote the family of plurisubharmonic functions defined on Ω. Throughout this article we always assume that Ω is a so called hyperconvex domain. This assumption is made to ensure a satisfying amount of plurisubharmonic functions with certain properties. Set

$$\mathcal{E}_0 = \mathcal{E}_0(\Omega) = \left\{\varphi \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega) : \lim_{z\to\partial\Omega} \varphi(z) = 0,\ \int_\Omega \left(dd^c\varphi\right)^n < +\infty\right\},$$

and let $\mathcal{E}$ be the cone of negative plurisubharmonic functions (see e.g. page 964 for details). The families $\mathcal{K}$ that we are interested in are cones of negative plurisubharmonic functions with $\mathcal{E}_0 \subseteq \mathcal{K} \subseteq \mathcal{E}$ such that if $\mathcal{K} \ni u \leq v \leq 0$ and v is a negative plurisubharmonic function, then $v \in \mathcal{K}$. Examples of interesting cones $\mathcal{K}$ are $\mathcal{E}_0$, $\mathcal{E}_p$, $\mathcal{F}$, $\mathcal{E}$ and $\mathcal{E}_\chi$. There is a wide range of literature on these cones and their applications. We mention only few of them here [9–12,27,33,35]. It should be emphasized that it is not only within pluripotential theory these cones have been proven useful, but also as a tool in for example dynamical systems and birational geometry (see e.g. [1,17,18]). The space of delta-plurisubharmonic functions $\mathcal{K} - \mathcal{K}$ is a vector space under pointwise addition and usual scalar multiplication. To simplify the notations we set $\delta\mathcal{K} = \mathcal{K} - \mathcal{K}$. In this article we shall consider $\delta\mathcal{K}$ with two vector orders; the order induced by the positive cone $\succcurlyeq$, and the classical pointwise ordering $\geq$.

Delta-plurisubharmonic function $u \in \delta\mathcal{K}$ is a difference of two plurisubharmonic functions $u = u_1 - u_2$, where $u_1, u_2 \in \mathcal{K}$. There are two standard ways handling the situation $-\infty - (-\infty)$. We can simply say that it is undefined or we can implement the convention that $-\infty - (-\infty) = -\infty$. Here we choose the second option. The order relations $\succcurlyeq$ and $\geq$ on $\delta\mathcal{K}$ are related as follows: if $u \succcurlyeq v$, then $u \leq v$. There are functions u, v in $(\delta\mathcal{K}, \geq)$ with $u \geq v$, such that u and v are not comparable w.r.t. $\succcurlyeq$ (Example 3.1). Our first result in this article is that the ordered vector space $(\delta\mathcal{K}, \geq)$ is a Riesz space (Theorem 3.2). It should be noted that in dimension $n = 1$, this is also possible for $(\delta\mathcal{K}, \succcurlyeq)$, but as Example 3.3 shows this is not always possible in higher dimensions.

In Section 4, we shall make a short introduction to functions which are differences of two radially symmetric plurisubharmonic functions defined on the unit ball in $\mathbb{C}^n$, since they will play a crucial role in Section 5. We shall denote this family by $\delta\mathcal{PSH}^R(\mathbb{B})$. Without loss of generality we shall assume that functions in this cone have boundary values zero.

Let ∂, $\bar\partial$ be the usual differential operators, $d = (\partial + \bar\partial)$ and $d^c = i(\bar\partial - \partial)$. Consider now the complex Monge–Ampère operator defined by

$$\mathrm{MA} : (\mathcal{K}, \succcurlyeq) \ni u \to \mathrm{MA}(u) = \left(dd^c u\right)^n \in (\mathcal{M}_{\mathcal{K}}, \succcurlyeq),$$

where

$$\mathcal{M}_{\mathcal{K}} = \left\{\mu : \mu \textit{ is a positive Radon measure such that } \left(dd^c u\right)^n = \mu \textit{ for some } u \in \mathcal{K}\right\}.$$

The operator MA is non-additive, bijective, and monotone. Note also that in $\delta\mathcal{M}_{\mathcal{K}}$ the classical order, and the order generated by cone $\mathcal{M}_{\mathcal{K}}$, coincide.

The first extension $\mathrm{MA}_1 : (\delta\mathcal{K}, \succcurlyeq) \to (\delta\mathcal{M}_{\mathcal{K}}, \succcurlyeq)$ we shall study is defined by

$$\mathrm{MA}_1(u) = \mathrm{MA}_1(u_1 - u_2) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \left(dd^c u_1\right)^k \wedge \left(dd^c u_2\right)^{n-k}.$$

This extension was introduced in [14, p. 210] and extensively studied in [34]. From the definition we see that the n-linearity is preserved, but the operator is not injective nor surjective (Example 5.2 and Proposition 5.4).

Let $\mathcal{M}^R$ denote the class of positive radial Radon measures μ defined on the unit ball $\mathbb{B} \subset \mathbb{C}^n$ for which

$$\int_{\frac{1}{2}}^{1} F(t)^{\frac{1}{n}}\, dt < \infty,$$

where $F(t) = \frac{1}{(2\pi)^n}\mu(\mathbb{B}_t)$ for $t \in [0,1]$. The main theorem related to the extension MA_1 is the following:

Theorem 5.5. *Let $\mu \in \delta\mathcal{M}^R$ be defined on the unit ball in $\mathbb{C}^n$ and assume that $F^{\frac{1}{n}}$ is of bounded variation on* $[0,1]$.

(1) *Let $n = 2k+1$. Then there exists $u \in \delta\mathcal{PSH}^R(\mathbb{B})$ such that $\mathrm{MA}_1(u) = \mu$.*
(2) *Let $n = 2k$ and assume additionally that $F(t) \geq 0$ on $[0,1]$. Then there exists $u \in \delta\mathcal{PSH}^R(\mathbb{B})$ such that $\mathrm{MA}_1(u) = \mu$.*

We shall next proceed with the second extension. Let $u = u_1 - u_2 \in \delta\mathcal{K}$, and then we follow [2] introducing u^+ and u^- through

$$u^- = \left(\sup\{\beta \in \mathcal{K} : \text{there exists } \alpha \in \mathcal{K} \text{ such that } u_1 + \beta = u_2 + \alpha\}\right)^*,$$

and

$$u^+ = \left(\sup\{\alpha \in \mathcal{K} : \text{there exists } \beta \in \mathcal{K} \text{ such that } u_1 + \beta = u_2 + \alpha\}\right)^*,$$

where $(w)^*$ denotes the upper semicontinuous regularization of the function w. These functions are in general very difficult to explicitly compute, but in the special case when u_1 and u_2 are radially symmetric it is possible (the computations start at page 975).

We can now define the second extension $\mathrm{MA}_2 : (\delta\mathcal{K}, \succcurlyeq) \to (\delta\mathcal{M}_{\mathcal{K}}, \succcurlyeq)$ by

$$\mathrm{MA}_2(u) = \mathrm{MA}_2\left(u^+ - u^-\right) = \mathrm{MA}\left(u^+\right) - \mathrm{MA}\left(u^-\right).$$

Example 5.6 shows that the extensions MA_1, and MA_2, do not coincide. The following Dirichlet problem is our main result about the extension MA_2:

Theorem 5.10. *For any signed measure $\mu \in \delta\mathcal{M}^R$ there exists a unique solution $u \in \delta\mathcal{PSH}^R(\mathbb{B})$ of the following Dirichlet problem*

$$\mathrm{MA}_2(u) = \mu.$$

Furthermore, $\mathrm{MA}_2 : \delta\mathcal{PSH}^R(\mathbb{B}) \to \delta\mathcal{M}^R$ is monotone and bijective. In particular, if $u \succcurlyeq v$, $u, v \in \delta\mathcal{PSH}^R(\mathbb{B})$, then $\mathrm{MA}_2(u) \geq \mathrm{MA}_2(v)$.

We end in Section 6 by stating and discussing open problems concerning the two different extension of the complex Monge–Ampère operator, and delta-plurisubharmonic functions in general.

2. Preliminaries

Let us start with giving some background on ordered vector spaces. We say that $\mathcal{K}$ is a *cone* in a vector space X over $\mathbb{R}$ if it is a non-empty subset of X that satisfies:

(1) $\mathcal{K} + \mathcal{K} \subseteq \mathcal{K}$,
(2) $\alpha\mathcal{K} \subseteq \mathcal{K}$ for all $\alpha \geq 0$, and
(3) $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$.

A binary relation $\succcurlyeq$ on a set X is said to be an *order relation* on X if it satisfies the following three properties:

(1) $x \succcurlyeq x$ for all $x \in X$ (*reflexivity*),
(2) for all $x, y \in X$ such that $x \succcurlyeq y$ and $y \succcurlyeq x$, imply that $x = y$ (*antisymmetry*),
(3) for all $x, y \in X$ such that $x \succcurlyeq y$ and $y \succcurlyeq z$, imply that $x \succcurlyeq z$ (*transitivity*).

It should be noted that some authors use the terminology *partial ordering* instead of order relation that we use here. An order relation $\succcurlyeq_L$ on a vector space L is said to be a *vector ordering* if, in addition to being reflexive, antisymmetric and transitive, $\succcurlyeq_L$ is also compatible with the algebraic structure of L in the sense that if $x \succcurlyeq_L y$, then:

(a) $x + z \succcurlyeq_L y + z$ for each $z \in L$ and
(b) $\alpha x \succcurlyeq_L \alpha y$ for all $\alpha \geq 0$.

An *ordered vector space* $(L, \succcurlyeq_L)$ is a vector space L with a vector ordering $\succcurlyeq_L$. Let $L^+ = \{x \in L : x \succcurlyeq_L 0\}$ be the *positive cone* of L. On the other hand, any cone in a vector space X generates a vector ordering $\succcurlyeq_X$ defined on X by letting $x \succcurlyeq_X y$ whenever $x - y$ is in the given cone.

Definition 2.1. A vector u in an ordered vector space $(L, \succcurlyeq)$ is called the *supremum* of a nonempty subset A of L if:

$a)$ u is an upper bound of A, i.e. $u \succcurlyeq a$ holds for all $a \in A$, and
$b)$ u is the least upper bound of A in the sense that for any upper bound v of A we have $v \succcurlyeq u$.

The definition of the *infimum* of a nonempty set A is introduced analogously.

It should be emphasized that infimum and supremum of a nonempty set A in an ordered vector space $(L, \succcurlyeq)$ do not always exist, therefore we make the following definition.

Definition 2.2. An ordered vector space L is said to be a *Riesz space* if every nonempty finite subset of L has a supremum, and an infimum in the sense of Definition 2.1.

Remark. An ordered vector space L is a Riesz space if, and only if, every two vectors of L have a supremum and an infimum in L.

Definition 2.3. A vector space $(L, \succcurlyeq)$ is called *Archimedean* if $y \in L$, $x \in L^+$, and $ny \preccurlyeq x$, for all $n \in \mathbb{N}$, implies that $y \preccurlyeq 0$.

In the rest of this note assume that $\Omega \subseteq \mathbb{C}^n$, $n \geq 1$, is a bounded, connected, open, and hyperconvex set. Recall that a bounded, connected, and open set $\Omega \subseteq \mathbb{C}^n$ is called *hyperconvex* if there exists a bounded plurisubharmonic function $\varphi : \Omega \to (-\infty, 0)$ such that the closure of the set $\{z \in \Omega : \varphi(z) < c\}$ is compact in Ω, for every $c \in (-\infty, 0)$. As in [11], we set

$$\mathcal{E}_0 = \mathcal{E}_0(\Omega) = \left\{\varphi \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega) : \lim_{z \to \partial\Omega} \varphi(z) = 0,\ \int_\Omega \left(dd^c\varphi\right)^n < +\infty\right\}.$$

We now define $\mathcal{E}$ $(= \mathcal{E}(\Omega))$ to be the class of plurisubharmonic functions φ defined on Ω, such that for each $z_0 \in \Omega$ there exists a neighborhood ω of z_0 in Ω and a decreasing sequence $[\varphi_j]_{j=1}^\infty$, $\varphi_j \in \mathcal{E}_0$, which converges pointwise to φ on ω and

$$\sup_j \int_\Omega \left(dd^c\varphi_j\right)^n < +\infty.$$

We have that $\mathcal{E}_0 \subset \mathcal{E}$.

Let us recall briefly the definition of the complex Monge–Ampère operator. Let ∂, $\bar\partial$ be the usual differential operators, and let $d = (\partial + \bar\partial)$ and $d^c = i(\bar\partial - \partial)$. The *complex Monge–Ampère operator* is then defined by

$$\mathrm{MA}(u_1, \ldots, u_n) = dd^c u_1 \wedge \cdots \wedge dd^c u_n,$$

where $u_1, \ldots, u_n$ are plurisubharmonic function of class $\mathcal{C}^2$. Note that the operator MA is n-linear. If $u = u_1 = \ldots = u_n$, then

$$\mathrm{MA}(u, \ldots, u) = \mathrm{MA}(u) = 4^n n! \det\left(\frac{\partial^2 u}{\partial z_j \partial \bar z_k}\right) dV_{2n},$$

where dV_{2n} is the Lebesgue measure in $\mathbb{C}^n$. In [12] the second-named author proved that the complex Monge–Ampère operator can be extended to the cone $\mathcal{E}$, and that it is the natural domain of definition for the complex Monge–Ampère operator (Theorem 4.5 in [12]).

For further information on ordered vector spaces theory, and pluripotential theory see e.g. [5,6,16,33].

3. Riesz spaces

Let $\mathcal{K}$ be a cone of negative plurisubharmonic functions with $\mathcal{E}_0 \subseteq \mathcal{K} \subseteq \mathcal{E}$ such that if $\mathcal{K} \ni u \leq v \leq 0$ and v is a negative plurisubharmonic function, then $v \in \mathcal{K}$. With pointwise addition of functions, and usual scalar multiplication $\delta\mathcal{K}$ is a vector space over $\mathbb{R}$. We first consider $\delta\mathcal{K}$ with the vector ordering induced by its positive cone, i.e. for $u, v \in \delta\mathcal{K}$ we say that $u \succcurlyeq_{\delta\mathcal{K}} v$ if $u - v \in \mathcal{K}$. It should be noted that $u \succcurlyeq_{\delta\mathcal{K}} 0$ for all $u \in \mathcal{K}$, even if $u(x) \leq 0$ for every $x \in \Omega$. One advantage with this vector ordering is that $(\delta\mathcal{K})^+ = \mathcal{K}$, and therefore it holds that $\delta\mathcal{K} = (\delta\mathcal{K})^+ - (\delta\mathcal{K})^+$. To simplify the notation we shall from here on use $\succcurlyeq$ instead of $\succcurlyeq_{\delta\mathcal{K}}$.

Recall that u is a plurisubharmonic function if and only if the Hessian matrix $H(u) = [\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}]_{j,k=1}^n$ is positive definite in a sense of distributions. Therefore for $u, v \in \mathcal{K}$ we have $u \succcurlyeq v$ if and only if the Hessian matrix $H(u - v)$ is positive definite in a sense of distributions.

The pointwise vector ordering $\geq$ is defined as $u \geq v$ if $u(x) \geq v(x)$ for every $x \in \Omega$. The order relations $\succcurlyeq$ and $\geq$ on $\delta\mathcal{K}$ are related as follows: if $u \succcurlyeq v$, then $u - v \in \mathcal{K}$. Hence, $u \leq v$. But as the following simple example shows there are functions u, v in $(\delta\mathcal{K}, \geq)$ with $u \geq v$, but u and v are not comparable with respect to $\succcurlyeq$.

Example 3.1. (See [2].) Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. Let g_w be the pluricomplex Green function with pole at $w \in \Omega$, and define $u_a = \max(g_w, a)$. Then for $a < b < 0$ we have that $u_a, u_b \in \mathcal{E}_0$, and $u_b \geq u_a$. But u_b and u_a are not comparable with respect to $\succcurlyeq$.

Theorem 3.2. *The ordered vector space $(\delta\mathcal{K}, \geq)$ is a Riesz space with the supremum, and infimum defined as*

$$u \vee_{\geq} v = \sup(u, v) = \max(u, v) = \max(u_1 - u_2, v_1 - v_2) \quad \textit{and}$$
$$u \wedge_{\geq} v = \inf(u, v) = \min(u, v) = \min(u_1 - u_2, v_1 - v_2),$$

where $u, v \in \delta\mathcal{K}$, $u = u_1 - u_2$, and $v = v_1 - v_2$ for some $u_j, v_j \in \mathcal{K}$, $j = 1, 2$.

Proof. Let $u, v \in \delta\mathcal{K}$, then $u = u_1 - u_2$, and $v = v_1 - v_2$ for some $u_j, v_j \in \mathcal{K}$, $j = 1, 2$. Then by a classical lattice identity we have that

$$\begin{aligned} \sup(u, v) &= \max(u, v) = \max(u_1 - u_2, v_1 - v_2) \\ &= \max(u_1 + v_2, u_2 + v_1) - (u_2 + v_2). \end{aligned} \tag{3.1}$$

It follows from the assumption on the cone $\mathcal{K}$ that $\max(u_1 + v_2, u_2 + v_1) \in \mathcal{K}$, hence $\sup(u, v) \in \delta\mathcal{K}$. By using (3.1) we get that

$$\inf(u, v) = -\max(-u, -v) = \big((u_1 + v_1) - \max(u_2 + v_1, u_1 + v_2)\big) \in \delta\mathcal{K}. \qquad \square$$

For the case $n = 1$, we shall now define a supremum, and an infimum of two vectors in $(\delta\mathcal{K}, \succcurlyeq)$. For the moment assume that $u, v \in \mathcal{K}$, and define

$$u \vee_{\succcurlyeq} v = \sup(u, v) := w$$

to be the unique function $w \in \mathcal{K}$ that satisfies $\Delta w = \sup(\Delta u, \Delta v)$ (see e.g. [21]). Here Δ denotes the Laplace operator. Recall that for two measures μ, ν one can define their supremum as

$$\sup(\mu, \nu)(A) = \sup\big\{\mu(B) + \nu(A \setminus B) : B \subset A\big\}.$$

Now we are going to show that w is actually a supremum of u and v. This construction implies that $\Delta(w - u) \geq 0$, which means that $w \succcurlyeq u$. In a similar manner we get that $w \succcurlyeq v$. Furthermore, if there exist ϕ such that $\phi \succcurlyeq u$, and $\phi \succcurlyeq v$, then $\Delta(\phi - u) \geq 0$. Hence, $\Delta\phi \geq \sup(\Delta u, \Delta v)$. Therefore we have that w is the supremum of u and v, since $\Delta\phi \geq \Delta w$ and $\phi \succcurlyeq w$.

The infimum $u \wedge_{\succcurlyeq} v := t$ of $u, v \in \mathcal{K}$ is defined by the unique solution $t \in \mathcal{K}$ to the equation $\Delta t = \inf(\Delta u, \Delta v)$ (see e.g. [21]). The infimum of the two measures μ, ν is defined by

$$\inf(\mu, \nu)(A) = \inf\big\{\mu(B) + \nu(A \setminus B) : B \subset A\big\}.$$

That our definition of infimum is compliant with Definition 2.1 follows as for the supremum case. In the general case when $u, v \in \delta\mathcal{K}$, $u = u_1 - u_2$, $v = v_1 - v_2$ we define

$$u \vee_{\succcurlyeq} v = \sup(u, v) = \sup(u_1 - u_2, v_1 - v_2) = \sup(u_1 + v_2, u_2 + v_1) - (u_2 + v_2),$$
$$u \wedge_{\succcurlyeq} v = \inf(u, v) = \inf(u_1 - u_2, v_1 - v_2) = \inf(u_1 + v_2, u_2 + v_1) - (u_2 + v_2).$$

Remark. Observe that for $u, v \in \mathcal{K}(\Omega)$, $n = 1$, in general the classical order $\geq$ and the order defined by the cone $\mathcal{K}$ do not coincide. To see this take $u, v \in \mathcal{K}$ such that $\operatorname{supp}\Delta u \cap \operatorname{supp}\Delta v = \emptyset$. Then $\Delta(u \vee_{\succcurlyeq} v) = \Delta u + \Delta v$, so $u \vee_{\succcurlyeq} v = u + v$, $u \wedge_{\succcurlyeq} v = 0$ and $\max(u, v) = u \vee_{\geq} v$, $\min(u, v) = u \wedge_{\geq} v$.

Example 3.3 shows that there are functions u, v in $\mathbb{C}^2$ such that $u \vee_{\succcurlyeq} v$ does not exist. This example is due to Phạm Hoàng Hiệp [37].

Example 3.3. Let $\mathbb{B}$ be the unit ball in $\mathbb{C}^2$. In this example we prove that there are functions $u, v \in \mathcal{E}_0 \cap C^\infty(\mathbb{B})$ such that $u \vee_{\succcurlyeq} v$ does not exist. Set

$$\varphi(z_1, z_2) = |z_1|^2 + 2|z_2|^2 - 4,$$
$$\psi(z_1, z_2) = 2|z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2 - 6,$$
$$\eta(z_1, z_2) = 10\big(|z_1|^2 + |z_2|^2 - 1\big).$$

Then $\varphi, \psi \in \mathcal{PSH}(\mathbb{B}) \cap C^\infty(\mathbb{B})$, $\varphi < 0$, $\psi < 0$, and $\eta \in \mathcal{E}_0 \cap C^\infty(\mathbb{B})$. Furthermore, $-4 \leq \varphi \leq -1$, $-6 \leq \psi \leq -1$. Choose $r > 0$ such that

$$\mathbb{B}_r \subset \{\eta < \varphi\} \Subset \mathbb{B} \quad \text{and} \quad \mathbb{B}_r \subset \{\eta < \psi\} \Subset \mathbb{B}.$$

By [13] we can find $u, v \in \mathcal{PSH}(\mathbb{B}) \cap C^\infty(\mathbb{B})$ such that

$$u \geq \max(\varphi, \eta), \qquad v \geq \max(\psi, \eta),$$
$$u = \max(\varphi, \eta) \quad \text{on } \Omega \setminus \omega_1, \quad \text{and} \quad v = \max(\psi, \eta) \quad \text{on } \Omega \setminus \omega_2.$$

Here ω_1 is a neighborhood of $\{\varphi = \eta\}$, and ω_2 is a neighborhood of $\{\psi = \eta\}$. Hence, $u, v \in \mathcal{E}_0 \cap C^\infty(\mathbb{B})$, and

$$u = v = 10\big(|z_1|^2 + |z_2|^2 - 1\big) \quad \text{on } \mathbb{B} \setminus K \text{ for some } K \Subset \mathbb{B}.$$

Furthermore, $u = \varphi$, and $v = \psi$ on $\mathbb{B}_r$. Choose $A > 0$ such that on $\mathbb{B}$ we have that

$$dd^c u \leq A\, dd^c|z|^2 \quad \text{and} \quad dd^c v \leq A\, dd^c|z|^2.$$

Set $f = \sup\{\phi \in \mathcal{PSH}^-(\mathbb{B}) : \phi \leq |z_1|^2 - 2 \text{ on } \mathbb{B}_r\}$. Then $f = |z_1|^2 - 2$ on $\mathbb{B}_r$ and $f \in \mathcal{E}_0$. Assume now that the function $u \vee_{\succcurlyeq} v$ exists, and call this function w. Note that w is not necessarily smooth, and therefore we shall use distributional derivatives when needed. We have that

$$dd^c w \geq dd^c u = dd^c \varphi \quad \text{on } \mathbb{B}_r$$
$$dd^c w \geq dd^c v = dd^c \psi \quad \text{on } \mathbb{B}_r.$$

Hence,

$$\frac{\partial^2 w}{\partial z_1 \partial \bar{z}_1} \geq 2, \quad \text{and} \quad \frac{\partial^2 w}{\partial z_2 \partial \bar{z}_2} \geq 2 \quad \text{on } \mathbb{B}_r.$$

We shall next prove that

$$\frac{\partial^2 w}{\partial z_1 \partial \bar{z}_1} \le 2, \quad \text{and} \quad \frac{\partial^2 w}{\partial z_2 \partial \bar{z}_2} \le 2 \quad \text{on } \mathbb{B}_{\frac{r}{2}}.$$

Choose $B > 0$ such that

$$dd^c\left(u + Bf + A\max\left(|z|^2 - 1, -1 + \frac{r^2}{4}\right)\right) \ge dd^c v,$$

then $dd^c w \le dd^c(u + Bf + A\max(|z|^2 - 1, -1 + \frac{r^2}{4}))$. Hence

$$\frac{\partial^2 w}{\partial z_2 \partial \bar{z}_2} \le 2 \quad \text{on } \mathbb{B}_{\frac{r}{2}}.$$

In a similar manner we get that $\frac{\partial^2 w}{\partial z_1 \partial \bar{z}_1} \le 2$ on $\mathbb{B}_{\frac{r}{2}}$. Thus,

$$\frac{\partial^2 w}{\partial z_1 \partial \bar{z}_1} = \frac{\partial^2 w}{\partial z_2 \partial \bar{z}_2} = 2 \quad \text{on } \mathbb{B}_{\frac{r}{2}}.$$

Since, $dd^c w \ge dd^c \varphi$ on $\mathbb{B}_{\frac{r}{2}}$, we get that

$$\frac{\partial^2 w}{\partial z_1 \partial \bar{z}_2} = \frac{\partial^2 w}{\partial z_2 \partial \bar{z}_1} = 0 \quad \text{on } \mathbb{B}_{\frac{r}{2}}.$$

Furthermore, since $dd^c w \ge dd^c \psi$ on $\mathbb{B}_{\frac{r}{2}}$, we have that

$$\frac{\partial^2 w}{\partial z_1 \partial \bar{z}_2} = \frac{\partial^2 w}{\partial z_2 \partial \bar{z}_1} = 1 \quad \text{on } \mathbb{B}_{\frac{r}{2}}.$$

This is not possible. Thus, $u \vee_{\succcurlyeq} v$ does not exist.

Definition 3.4. For $u = u_1 - u_2 \in \delta\mathcal{K}$ we define

$$u^- = \left(\sup\{\beta \in \mathcal{K} : \text{there exists } \alpha \in \mathcal{K} \text{ such that } u_1 + \beta = u_2 + \alpha\}\right)^*,$$

and

$$u^+ = \left(\sup\{\alpha \in \mathcal{K} : \text{there exists } \beta \in \mathcal{K} \text{ such that } u_1 + \beta = u_2 + \alpha\}\right)^*,$$

where $(w)^*$ denotes the upper semicontinuous regularization of the function w.

Remark. If $u \in \delta\mathcal{K}$, then from the assumption on $\mathcal{K}$ we have that $u^+, u^- \in \mathcal{K}$, and $u = u^+ - u^-$ (see also [2]).

We shall end this section by giving some necessary conditions that guarantee the existence of a supremum $u \vee_{\succcurlyeq} v$, $u, v \in \delta\mathcal{K}$ (Proposition 3.6). Recall that in a Riesz space $(X, \succcurlyeq)$ we always have that

$$x_+ \wedge_{\succcurlyeq} x_- = 0, \quad \text{and} \quad x_+ \vee_{\succcurlyeq} x_- = x_+ + x_-, \tag{3.2}$$

where $x \vee_{\succcurlyeq} 0 = x_+$, and $x \wedge_{\succcurlyeq} 0 = -x_-$.

It should be pointed out that Example 3.3 yields that $u \wedge_{\succcurlyeq} 0$, and $u \vee_{\succcurlyeq} 0$ generally do not exist. On the other hand, the functions u^+, and u^- are always well-defined.

Lemma 3.5. *Let $u \in (\delta\mathcal{K}, \succcurlyeq)$. If $u \wedge_{\succcurlyeq} 0$, and $u \vee_{\succcurlyeq} 0$ exist, then we have that $u \vee_{\succcurlyeq} 0 = u^+$ and $u \wedge_{\succcurlyeq} 0 = -u^-$.*

Proof. Assume that $u \vee_{\succcurlyeq} 0 = \phi$, and $u \wedge_{\succcurlyeq} 0 = \psi$ exist. Then we have that $u = \phi + \psi = u^+ - u^-$, which yields that $u^+ \geq \phi$, and $u^- \geq -\psi$. Furthermore, $\phi \geq u^+$, since the fact that $u^+ \succcurlyeq u$ implies that $u^+ \succcurlyeq \phi$. Thus, $\phi = u^+$. Finally, $\psi = -u^-$, since $u \wedge_{\succcurlyeq} 0 + u \vee_{\succcurlyeq} 0 = u = u^+ - u^-$. □

Proposition 3.6. *Let $u, v \in (\delta\mathcal{K}, \succcurlyeq)$. If $u \vee_{\succcurlyeq} v$ exists, then*

$$u \vee_{\succcurlyeq} v = u + (u - v)^- = v + (u - v)^+$$
$$u \wedge_{\succcurlyeq} v = u - (u - v)^+ = v - (u - v)^-.$$

In particular, the function $u^+ + v^- - (u - v)^+$ is plurisubharmonic.

Proof. Assume that $u \vee_{\succcurlyeq} v$ exists. We have that $u + (u - v)^- = v + (u - v)^+$, which implies that

$$u + (u - v)^- \succcurlyeq u \vee_{\succcurlyeq} v, \quad \text{and} \quad v + (u - v)^+ \succcurlyeq u \vee_{\succcurlyeq} v.$$

Therefore we have that $u + (u - v)^- \leq u \vee_{\succcurlyeq} v$, and $v + (u - v)^+ \leq u \vee_{\succcurlyeq} v$. On the other hand, there exist $\alpha, \beta \in \mathcal{K}$ such that

$$u \vee_{\succcurlyeq} v = u + \alpha, \quad \text{and} \quad u \vee_{\succcurlyeq} v = v + \beta,$$

since we have assumed that $u \vee_{\succcurlyeq} v$ exists. Hence, $\alpha \leq (u - v)^-$, and $\beta \leq (u - v)^+$. Thus,

$$u \vee_{\succcurlyeq} v = u + \alpha \leq u + (u - v)^-.$$

This gives us that $u \vee_{\succcurlyeq} v = u + (u - v)^- = v + (u - v)^+$. Since $u \vee_{\succcurlyeq} v + u \wedge_{\succcurlyeq} v = u + v$, we finally get that $u \wedge_{\succcurlyeq} v = u - (u - v)^+ = v - (u - v)^-$. Thus, $u \wedge_{\succcurlyeq} v = u - (u - v)^+ = v - (u - v)^- \succcurlyeq -u^- - v^-$, which means that $u - (u - v)^+ + u^- + v^- \in \mathcal{K}$. Hence, $u^+ + v^- - (u - v)^+$ is a plurisubharmonic function. □

Proposition 3.7. *Assume that $u, v \in \delta\mathcal{K}$ are such that $u \wedge_{\succcurlyeq} v$ exists. Then $u \wedge_{\succcurlyeq} v = 0$ if, and only if, $(u - v)^+ = u$ and $(u - v)^- = v$.*

Proof. Assume that $u \wedge_{\succcurlyeq} v = 0$. Then we have that

$$(u - v) \wedge_{\succcurlyeq} 0 = u \wedge_{\succcurlyeq} v - v = -v,$$

and therefore Lemma 3.5 yields that

$$-(u - v)^- = (u - v) \wedge_{\succcurlyeq} 0 = u \wedge_{\succcurlyeq} v - v = -v.$$

Similarly, we get that $(u - v)^+ = u$. On the other hand, assume that $(u - v)^+ = u$, and $(u - v)^- = v$. The underlying assumption that $u \wedge_{\succcurlyeq} v$ exists implies that

$$(u - v) \wedge_{\succcurlyeq} 0 = u \wedge_{\succcurlyeq} v - v$$

exists, and therefore Lemma 3.5 implies that

$$(u - v) \wedge_{\succcurlyeq} 0 = -(u - v)^- = -v, \quad \text{and}$$
$$(u - v) \vee_{\succcurlyeq} 0 = (u - v)^+ = u.$$

To complete the proof, note that from (3.2) we obtain that

$$u \wedge_{\succcurlyeq} v = (u-v)^+ \wedge_{\succcurlyeq} (v-v)^- = 0. \qquad \square$$

4. Delta-radially symmetric plurisubharmonic functions

In this section let us recall some basic facts on radially symmetric functions and convex analysis. A function $u : \mathbb{B} \to [-\infty, \infty)$ is said to be *radially symmetric* if we have that

$$u(z) = u\big(|z|\big) \quad \text{for all } z \in \mathbb{B}.$$

For each radially symmetric function $u : \mathbb{B} \to [-\infty, +\infty)$ we define the function $\tilde{u} : [-\infty, 0) \to [-\infty, +\infty)$ by

$$\tilde{u}(t) = u\big(|z|\big), \quad \text{where } t = \ln|z|. \tag{4.1}$$

On the other hand, to every function $\tilde{u} : [-\infty, 0) \to [-\infty, +\infty)$ we can construct a radially symmetric function u through (4.1).

Let $U(n)$ denote the unitary group of degree n. A function u defined on $\mathbb{B}$ is radially symmetric if, and only if, it is unitary invariant, i.e. $u \circ T = u$ for all $T \in U(n)$. It is well known fact (see e.g. [3]) that u is a radially symmetric plurisubharmonic function, if, and only if, $\tilde{u}$ is an increasing and convex function. Moreover u is a continuous function and the limit $\lim_{t\to 0^-} \tilde{u}(t)$ exists. Let $\mathcal{PSH}^R(\mathbb{B})$ denote the set of functions defined on $\mathbb{B}$ that are *non-positive*, radially symmetric, plurisubharmonic and

$$\lim_{\substack{z\to\xi \\ z\in\mathbb{B}}} u(z) = 0 \quad \text{for all } \xi \in \partial\mathbb{B}.$$

Furthermore, if $u \in \mathcal{PSH}^R(\mathbb{B})$, then $\tilde{u}''$ (derivatives are taken in distributional sense) is a positive measure.

Proposition 4.1. *The vector space $\delta\mathcal{PSH}^R(\mathbb{B})$ is a Riesz space.*

Proof. Observe that it is enough to prove that for any $u, v \in \mathcal{PSH}^R(\mathbb{B})$ there exists $u \vee_{\succcurlyeq} v$. Let

$$\mu = \sup\big(\tilde{u}'', \tilde{v}''\big)$$

and let $\tilde{w}$ be a convex increasing function such that $\tilde{w}'' = \mu$ and $\tilde{w}(0) = 0$. We claim that

$$w(z) = \tilde{w}\big(\ln|z|\big) = u \vee_{\succcurlyeq} v.$$

To see this note that since $\tilde{w}'' \geq \tilde{u}''$, then $\tilde{w} - \tilde{u} \in \mathcal{PSH}^R(\mathbb{B})$, so $w \succcurlyeq u$. And similarly $w \succcurlyeq v$, so $w \succcurlyeq u \wedge_{\succcurlyeq} v$. Now if there exists $h \in \mathcal{PSH}^R(\mathbb{B})$ such that $h \succcurlyeq u$ and $h \succcurlyeq v$, then $\tilde{h}'' \geq \tilde{u}''$ and $\tilde{h}'' \geq \tilde{v}''$, so $\tilde{h}'' \geq \tilde{w}''$, and therefore $h \succcurlyeq w$. $\square$

Remark. Note that if $u(z) = \tilde{u}(\ln|z|)$ is a smooth function, then

$$MA(u) = \big(dd^c u\big)^n = 4^n n! \det(\tilde{u}_{j\bar{k}}) dV_{2n} = 2^{n-1} n! e^{-2nt} \big(\tilde{u}'(t)\big)^{n-1} \tilde{u}''(t) dV_{2n},$$

where $t = \ln|z|$. We have also (see e.g. [3])

$$\big(\tilde{u}'(t)\big)^n = \frac{1}{(2\pi)^n} \big(dd^c u\big)^n (\mathbb{B}_{e^t}) = F\big(e^t\big)$$

and

$$\tilde{u}(t) = -\int_t^0 F\left(e^s\right)^{\frac{1}{n}} ds.$$

Remark. Let $\mathcal{M}^R$ denote the class of positive radial Radon measures μ for which

$$\int_{\frac{1}{2}}^{1} F(t)^{\frac{1}{n}} dt < \infty,$$

where F is the function defined on $[0,1]$ by

$$F(t) = \frac{1}{(2\pi)^n}\mu(\mathbb{B}_t). \tag{4.2}$$

It was proved in [3] that for any $\mu \in \mathcal{M}^R$ there exists a unique $u \in \mathcal{PSH}^R(\mathbb{B})$ such that $\mathrm{MA}(u) = \mu$, where $u(|z|) = \tilde{u}(t)$, $t = \ln|z|$, and

$$\tilde{u}(t) = -\int_t^0 F\left(e^s\right)^{\frac{1}{n}} ds, \tag{4.3}$$

where F is defined by (4.2).

Furthermore, for any non-decreasing left-continuous function F such that (4.2) holds, the function u defined by the formula (4.3) satisfies $\frac{1}{(2\pi)^n}\,\mathrm{MA}(u)(\mathbb{B}_t) = F(t)$.

5. The extensions of the Monge–Ampère operator

In this section we shall consider two extensions of the Monge–Ampère operator

$$\mathrm{MA} : (\mathcal{K}, \succcurlyeq) \ni u \to \mathrm{MA}(u) = \left(dd^c u\right)^n \in (\mathcal{M}_{\mathcal{K}}, \succcurlyeq),$$

where

$$\mathcal{M}_{\mathcal{K}} = \left\{\mu : \mu \textit{ is a positive Radon measure such that } \left(dd^c u\right)^n = \mu \textit{ for some } u \in \mathcal{K}\right\}.$$

Here we shall discuss the possibilities of extending MA to $\delta\mathcal{K}$. First note that in $\delta\mathcal{M}_{\mathcal{K}}$ the classical order, and the order generated by the cone $\mathcal{M}_{\mathcal{K}}$, coincide. The operator MA is non-additive, bijective, and monotone. The third of these assertions follows from the fact that: if $u \succcurlyeq v$, then $u = v + \phi$ for some $\phi \in \mathcal{K}$, and therefore it holds that $(dd^c u)^n = (dd^c(v + \phi))^n \geq (dd^c v)^n$. Thus, $\mathrm{MA}(u) \succcurlyeq \mathrm{MA}(v)$.

Even though the operator MA is monotone, its inverse is not. To see this consider the following example.

Example 5.1. Let $\Omega \subset \mathbb{C}^2$, and fix $w \in \Omega$. For $a < 0$ we define as before

$$u_a(z) = \max\left(g_w(z), a\right),$$

where g_w is the pluricomplex Green function with pole at w. For $a < b < 0$, we then have that

$$\mu = \left(dd^c(u_a + u_b)\right)^2 = d\sigma_{\{g_w = a\}} + 3d\sigma_{\{g_w = b\}} \quad \text{and} \quad \nu = \left(dd^c(\sqrt{3}u_b)\right)^2 = 3d\sigma_{\{g_w = b\}}.$$

Here $d\sigma_A$ is the surface measure defined on the set A. Thus, $\mu \succcurlyeq \nu$, but $u_a + u_b$ and $\sqrt{3}u_b$ are not comparable.

If we consider the classical order instead of the order induced by the cone $\mathcal{K}$, then the operator MA is no longer monotone, but instead its inverse is.

Let us now go back to our original question. First let us consider the case when $n = 1$. Then there exists a unique extension of the Monge–Ampére operator (in this case the Laplace operator and the Monge–Ampére operator are one and the same) to $\delta\mathcal{K}$. In other words, we have that

$$\mathrm{MA}(u - v) = \mathrm{MA}(u) - \mathrm{MA}(v) \quad u, v \in \mathcal{K}.$$

It follows from Theorem K since MA is linear, and $\delta\mathcal{M}_{\mathcal{K}}$ is so called *Archimedean* (see Definition 2.3). Furthermore, there exists a correspondence between the following decompositions:

$$u = u^+ - u^- \quad \text{and} \quad \mu = \mu^+ - \mu^-.$$

Here u^+, u^- are as in Definition 3.4, and

$$\mu^+ = \frac{1}{2}\big(|\mu| + \mu\big) \quad \text{and} \quad \mu^- = \frac{1}{2}\big(|\mu| - \mu\big),$$

where $|\mu|$ is the total variation of the measure μ. In particular, this implies that MA is a bijection. The inverse is given by the following: for arbitrary $\mu \in \delta\mathcal{M}_{\mathcal{K}}$, $\mu = \mu^+ - \mu^-$, there exist unique functions $u, v \in \mathcal{K}$ such that

$$\mathrm{MA}(u) = \mu^+ \quad \text{and} \quad \mathrm{MA}(v) = \mu^-.$$

Therefore, we have that

$$\mathrm{MA}(u - v) = \mu \quad \text{and} \quad u = (u - v)^+,\ v = (u - v)^-.$$

Finally, note that the constructions of $u \vee v$, and $u \wedge v$ yield that

$$\mathrm{MA}(u \vee v) = \mathrm{MA}(u) \vee \mathrm{MA}(v) \quad \text{and} \quad \mathrm{MA}(u \wedge v) = \mathrm{MA}(u) \wedge \mathrm{MA}(v).$$

Thus, MA is a lattice isomorphism.

We would like to define a similar extension of the Monge–Ampére operator in the case when $n \geq 2$ that shares at least some of the properties from the case $n = 1$.

5.1. *Multilinear extension* MA_1

One direction could be to follow the second-named author and Wiklund in [14], and for $u \in \delta\mathcal{K}$ make the following definition

$$\mathrm{MA}_1(u) = \mathrm{MA}_1(u_1 - u_2) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \big(dd^c u_1\big)^k \wedge \big(dd^c u_2\big)^{n-k}. \tag{5.1}$$

By Lemma 5.1 in [34] we have that (5.1) is independent of representation of u. This extension is n-linear in the sense that if $v_1, v_2, \ldots, v_n \in \delta\mathcal{K}$, then the operator

$$\mathrm{MA}_1(v_1, v_2, \ldots, v_n) = dd^c v_1 \wedge \cdots \wedge dd^c v_n,$$

is n-linear. Then by taking $v_1, v_2, \ldots, v_n = u$ we obtain equation (5.1), and get n-linearity.

The operator MA_1 is not injective, as the following example shows. It should be pointed out that in pluripotential theory injectivity of MA_1 means that we have uniqueness of the solution to the corresponding Dirichlet problem.

Example 5.2. Let $n = 2$, and $\Omega = \mathbb{B}$ be the unit ball in $\mathbb{C}^2$. For $a < b < 0$ define the following functions on Ω

$$u_1 = \max\bigl(\log|z|, a\bigr), \qquad u_2 = 2\max\bigl(\log|z|, b\bigr).$$

These constructions imply that

$$\begin{aligned}\mathrm{MA}_1(u_1 - u_2) &= \bigl(dd^c u_1\bigr)^2 + \bigl(dd^c u_2\bigr)^2 - 2dd^c u_1 \wedge dd^c u_2 \\ &= (2\pi)^2 d\sigma_a + 4(2\pi)^2 d\sigma_b - 4(2\pi)^2 d\sigma_b = (2\pi)^2 d\sigma_a = \mathrm{MA}_1(u_1).\end{aligned}$$

Thus, $\mathrm{MA}_1(u_1 - u_2) = \mathrm{MA}_1(u_1)$, but $u_1 - u_2 \neq u_1$.

As we shall see in Proposition 5.4 the operator MA_1 is not surjective, at least for the even dimension case. At the point of writing we do not have any counterexample for the odd dimension case. In pluripotential theory, surjectivity of MA_1 means the existence of the solution to the appropriate Dirichlet problem.

Definition 5.3. Let μ and ν be signed measures defined on a hyperconvex domain $\Omega \subset \mathbb{C}^n$. Then we say that $\mu \succcurlyeq_{PSH} \nu$ if for all $h \in \mathcal{E}_0(\Omega) \cap \mathcal{C}(\overline{\Omega})$ we have that

$$\int_\Omega (-h) d\mu \geq \int_\Omega (-h) d\nu.$$

We call $\succcurlyeq_{PSH}$ the plurisubharmonic ordering in the space of signed measures.

Proposition 5.4. *Let $u = u_1 - u_2 \in \delta\mathcal{K}$ then for all closed positive currents T of bidegree $(n - 2k, n - 2k)$ and all $h \in \mathcal{E}_0$*

$$\int_\Omega (-h)\bigl(dd^c u\bigr)^{2k} \wedge T \geq 0.$$

In particular in $\mathbb{C}^{2n}$ for all $u \in \delta\mathcal{K}$ we have

$$\mathrm{MA}_1(u) \succcurlyeq_{PSH} 0.$$

Proof. Fix $h \in \mathcal{E}_0$. By using Theorem 2.1 in [12], we can assume without losing generality that $u = u_1 - u_2 \in \delta\mathcal{E}_0$, and that $\int_\Omega (-h)(dd^c u)^{2k} \wedge T$ is finite.

We shall prove this proposition by induction. For $k = 1$, Theorem 5.5 in [12] yields that

$$\begin{aligned}\int_\Omega (-h) dd^c u_1 \wedge dd^c u_2 \wedge T &\leq \left(\int_\Omega (-h)\bigl(dd^c u_1\bigr)^2 \wedge T\right)^{\frac{1}{2}} \left(\int_\Omega (-h)\bigl(dd^c u_2\bigr)^2 \wedge T\right)^{\frac{1}{2}} \\ &\leq \frac{1}{2}\left(\int_\Omega (-h)\bigl(dd^c u_1\bigr)^2 \wedge T + \int_\Omega (-h)\bigl(dd^c u_2\bigr)^2 \wedge T\right),\end{aligned}$$

which gives the desired inequality.

Assume now that our statement holds for some $k \geq 1$, for all positive closed current T of bidegree $(n-2k, n-2k)$, and all $h \in \mathcal{E}_0(\Omega) \cap \mathcal{C}(\overline{\Omega})$. To complete the proof we shall prove that it holds also for $k+1$. For $\alpha, \beta \in \mathcal{E}_0$, it follows from the inductive assumption that the following bilinear form

$$(\alpha, \beta) \to \int_\Omega (-h) dd^c\alpha \wedge dd^c\beta \wedge \left(dd^c(u_1 - u_2)\right)^{2k} \wedge \widetilde{T}$$

is positive definite for all closed positive currents $\widetilde{T}$ of bidegree $(n-2k-2, n-2k-2)$. Therefore, by Cauchy–Schwarz inequality we have that

$$\begin{aligned}\int_\Omega (-h) dd^c\alpha \wedge dd^c\beta \wedge \left(dd^c(u_1 - u_2)\right)^{2k} \wedge \widetilde{T} \leq &\left(\int_\Omega (-h)\left(dd^c\alpha\right)^2 \wedge \left(dd^c(u_1 - u_2)\right)^{2k} \wedge \widetilde{T}\right)^{\frac{1}{2}} \\ &\times \left(\int_\Omega (-h)\left(dd^c\beta\right)^2 \wedge \left(dd^c(u_1 - u_2)\right)^{2k} \wedge \widetilde{T}\right)^{\frac{1}{2}},\end{aligned}$$

which implies that the following polynomial

$$w(t) = \int_\Omega (-h)\left(dd^c(\alpha + t\beta)\right)^2 \wedge \left(dd^c(u_1 - u_2)\right)^{2k} \wedge \widetilde{T}$$

is positive. In particular, taking $\alpha = u_1$, $\beta = u_2$, we get that

$$\int_\Omega (-h)\left(dd^c(u_1 - u_2)\right)^{2k+2} \wedge \widetilde{T} = w(-1) \geq 0. \qquad \square$$

Next, we shall prove that it is possible to solve the Dirichlet problem for the operator MA_1. From Proposition 5.4 it follows that the condition $\mu \succcurlyeq_{PSH} 0$ is necessary for solving the Dirichlet problem for even n. But here we are able to prove it with a stronger condition.

Theorem 5.5. *Let $\mu \in \delta\mathcal{M}^R$ be defined on the unit ball in $\mathbb{C}^n$ and define*

$$F(t) = \frac{1}{(2\pi)^n}\mu(\mathbb{B}_t) \quad \textit{for } t \in [0, 1].$$

Assume that $F^{\frac{1}{n}}$ is of bounded variation on $[0, 1]$.

(1) *Let $n = 2k+1$. Then there exists $u \in \delta\mathcal{PSH}^R(\mathbb{B})$ such that $\mathrm{MA}_1(u) = \mu$.*
(2) *Let $n = 2k$ and assume additionally that $F(t) \geq 0$ on $[0, 1]$. Then there exists $u \in \delta\mathcal{PSH}^R(\mathbb{B})$ such that $\mathrm{MA}_1(u) = \mu$.*

Proof. Part (1): Because $F^{\frac{1}{n}}$ is of bounded variation on $[0, 1]$ there exist bounded functions G_1 and G_2 which are nondecreasing such that

$$F^{\frac{1}{n}} = G_1 - G_2 \quad \text{on } [0, 1].$$

Furthermore, we have that G_1 and G_2 are both non-negative and left continuous. Next, define $F_1 = G_1^n$ and $F_2 = G_2^n$. Then we continue by letting

$$\tilde{u}_1(t) = -\int_t^0 F_1^{\frac{1}{n}}\left(e^s\right) ds \quad \text{and} \quad \tilde{u}_2(t) = -\int_t^0 F_2^{\frac{1}{n}}\left(e^s\right) ds.$$

Then

$$u_1(z) = \tilde{u}_1\bigl(\ln|z|\bigr), \quad \text{and} \quad u_2(z) = \tilde{u}_2\bigl(\ln|z|\bigr)$$

are radially symmetric plurisubharmonic functions. We can now follow as in Lemma 3.6 in [3] and get that $\mathrm{MA}_1(u_1 - u_2) = \mu$ (see also the last remark in Section 4 on page 970).

Part (2): From the additional assumption that $F \geq 0$ it follows that $F^{\frac{1}{n}}$ exists and we can proceed as in the previous part. $\square$

Remark. Assume that μ is a signed unitary invariant measure in $\mathbb{B} \subset \mathbb{C}^n$, such that $\mu(\mathbb{B}_t) \geq 0$, then $\mu \succcurlyeq_{PSH^R} 0$. To prove it let $h \in \mathcal{PSH}^R(\mathbb{B})$, and $F(t) = \frac{1}{(2\pi)^n}\mu(\mathbb{B}_t) \geq 0$, then $\{h < -t\} = \mathbb{B}_s$, where $s = h^{-1}(-t)$ and therefore

$$\int_{\mathcal{B}} (-h)\,d\mu = \int_0^\infty \mu\bigl(\{h < -t\}\bigr)\,dt = \int_0^\infty F\bigl(h^{-1}(-t)\bigr)\,dt \geq 0.$$

We believe that the condition $\mu \succcurlyeq_{PSH^R} 0$ is not only necessary but also sufficient for solvability of the Dirichlet problem in the case of even dimension.

Remark. We suspect that the assumption that $F^{\frac{1}{n}}$ is of bounded variation in the statement of Theorem 5.5 is superfluous.

5.2. Maximal extension MA_2

Another way to extend the complex Monge–Ampère operator could be as follows: Let $u \in \delta\mathcal{K}$, and u^+ and u^- be as in Definition 3.4. Then we can define the complex Monge–Ampère operator as

$$\mathrm{MA}_2(u) = \mathrm{MA}_2\bigl(u^+ - u^-\bigr) = \mathrm{MA}\bigl(u^+\bigr) - \mathrm{MA}\bigl(u^-\bigr).$$

Example 5.6 shows that the definitions of MA_1, and MA_2, do not coincide.

Example 5.6. For $a < b < 0$, define $u = \max(\log|z|, b) - \max(\log|z|, a)$ on the unit ball in $\mathbb{C}^2$. It was proved in [2] that $u^+ = \max(\log|z|, b)$, $u^- = \max(\log|z|, a)$. Therefore we have that

$$\begin{aligned}\mathrm{MA}_1(u) &= \bigl(dd^c \max\bigl(\log|z|, a\bigr)\bigr)^2 + \bigl(dd^c \max\bigl(\log|z|, b\bigr)\bigr)^2 - 2dd^c \max\bigl(\log|z|, a\bigr) \wedge dd^c \max\bigl(\log|z|, b\bigr) \\ &= \bigl(dd^c \max\bigl(\log|z|, a\bigr)\bigr)^2 - \bigl(dd^c \max\bigl(\log|z|, b\bigr)\bigr)^2 = -\mathrm{MA}_2(u).\end{aligned}$$

Given a function $u \in \delta\mathcal{K}$ it is in general difficult to explicitly calculate the functions u^+ and u^- in Definition 3.4. In this section, we give a formula for these functions under the condition that u is radially symmetric. Then we end this section by proving that under the same assumption we have that MA_2 is a bijection.

Lemma 5.7. *If $u \in \delta\mathcal{PSH}^R(\mathbb{B})$, then $u^+, u^- \in \mathcal{PSH}^R(\mathbb{B})$.*

Proof. To see this note that by the definition of u^+ we have that for all $T \in U(n)$, where $U(n)$ denotes the unitary group in $\mathbb{C}^n$,

$$u^+(z) \geq u^+\bigl(T(z)\bigr),$$

so

$$u^+(z) = \left(\sup\left\{u^+\left(T(z)\right); T \in U(n)\right\}\right)^*.$$

Note that the function $(\sup\{u^+(T(z)); T \in U(n)\})^*$ is radially symmetric, so also $u^+ \in \mathcal{PSH}^R(\mathbb{B})$. Similarly one can prove that $u^- \in \mathcal{PSH}^R(\mathbb{B})$. $\square$

Remark. Note that since $\delta\mathcal{PSH}^R(\mathbb{B})$ is a Riesz space, then for any $u = u_1 - u_2$ we have

$$u_1 \succcurlyeq u^+, \qquad u_2 \succcurlyeq u^-,$$

so there exists $v \in \mathcal{PSH}^R(\mathbb{B})$ such that

$$u_1 = u^+ + v, \qquad u_2 = u^- + v.$$

The functions u^+ and u^- are in general very difficult to explicitly compute. Now we shall show that it is at least possible when $u(z) = u_1(z) - u_2(z) \in \delta\mathcal{PSH}^R(\mathbb{B})$. Then $\tilde{u}_1''$ and $\tilde{u}_2''$ are positive measures. Let us define

$$\mu^+ = \sup\left(\tilde{u}_1'' - \tilde{u}_2'', 0\right)$$

and

$$\mu^- = -\inf\left(\tilde{u}_1'' - \tilde{u}_2'', 0\right).$$

There exist v, w increasing and convex functions, such that $v(0) = w(0) = 0$ and

$$v'' = \mu^+, \qquad w'' = \mu^-.$$

Note also that since the following decomposition $\tilde{u}_1'' - \tilde{u}_2'' = \mu^+ - \mu^-$ is minimal (e.g. the measures μ^+, μ^- are minimal) then the decomposition $\tilde{u}_1 - \tilde{u}_2 = v - w$ is maximal. Now we can define

$$u^+(z) = v\left(\ln|z|\right), \qquad u^-(z) = w\left(\ln|z|\right),$$

and then

$$u(z) = u^+(z) - u^-(z).$$

Let us now use this for a concrete example.

Example 5.8. Let $u(z) = |z|^2 - |z|^4$ be defined on $\mathbb{B} \subset \mathbb{C}^n$. Then we have that $\tilde{u}(t) = e^{2t} - e^{4t}$. Therefore we have

$$v''(t) = \sup\left(4e^{2t} - 16e^{4t}, 0\right),$$

and

$$v'(t) = \begin{cases} 2e^{2t} - 4e^{4t}, & t \in (-\infty, -\ln 2] \\ \frac{1}{4}, & t \in (-\ln 2, 0], \end{cases}$$

and finally

$$v(t) = \begin{cases} e^{2t} - e^{4t} - \frac{3}{16} - \frac{1}{4}\ln 2, & t \in (-\infty, -\ln 2] \\ \frac{1}{4}t, & t \in (-\ln 2, 0]. \end{cases}$$

Hence,

$$u^+(z) = \begin{cases} |z|^2 - |z|^4 - \frac{3}{16} - \frac{1}{4}\ln 2, & |z| \le \frac{1}{2} \\ \frac{1}{4}\ln|z|, & |z| > \frac{1}{2}, \end{cases}$$

and similarly

$$u^-(z) = \begin{cases} -\frac{3}{16} - \frac{1}{4}\ln 2, & |z| \le \frac{1}{2} \\ |z|^2 - |z|^4 + \frac{1}{4}\ln|z|, & |z| > \frac{1}{2}. \end{cases}$$

Thus,

$$\begin{aligned} \mathrm{MA}_2(u) &= \mathrm{MA}_2\big(|z|^2 - |z|^4\big) = \big(dd^c u^+\big)^n - \big(dd^c u^-\big)^n \\ &= \chi_{\{|z|\le\frac{1}{2}\}} 4^n n! \big(1 - 2|z|^2\big)^{n-1}\big(1 - 4|z|^2\big) dV_{2n} \\ &\quad - \chi_{\{|z|>\frac{1}{2}\}} 4^n n! 2^{n-1}\left(|z| - \frac{1}{4|z|}\right)^{2n-2}\big(4|z|^2 - 1\big) dV_{2n}, \end{aligned}$$

and by direct calculation

$$\begin{aligned} \mathrm{MA}_1(u) &= \mathrm{MA}_1\big(|z|^2 - |z|^4\big) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \big(dd^c|z|^2\big)^k \wedge \big(dd^c|z|^4\big)^{n-k} \\ &= 4^n n! \big(1 - 2|z|^2\big)^{n-1}\big(1 - 4|z|^2\big) dV_{2n}. \end{aligned}$$

Note that $\mathrm{MA}_2(u)|_{\mathbb{B}_{1/2}} = \mathrm{MA}_1(u)|_{\mathbb{B}_{1/2}}$, but $\mathrm{MA}_2(u) \neq \mathrm{MA}_1(u)$.

Next, we shall prepare for the main theorem concerning the extension MA_2.

Lemma 5.9. *Let $\mu = \mu_1 - \mu_2 \in \delta\mathcal{M}^R$ and let $v_j'(t) = \frac{1}{(2\pi)^n}\mu_j(\mathbb{B}_{e^t})$, $j = 1, 2$. Then*

$$\operatorname{supp}\mu_1 \cap \operatorname{supp}\mu_2 = \emptyset \quad \Leftrightarrow \quad \operatorname{supp} v_1'' \cap \operatorname{supp} v_2'' = \emptyset.$$

Proof. Let $[0,1]\backslash(\operatorname{supp}\mu_1 \cap [0,1]) = \bigcup_{s=1}^{\infty} I_s$, where I_s are connected components. Then $v_1'(t) = \frac{1}{(2\pi)^n}\mu_1(\mathbb{B}_{e^t})$ is constant on the interval $J_s = \ln(I_s)$, and therefore $v_1'(t)$ is constant on J_s, so $v_1'' = 0$ on J_s. So we have proved that if $\operatorname{supp}\mu_1 \cap \operatorname{supp}\mu_2 = \emptyset$ then $\operatorname{supp} v_1'' \cap \operatorname{supp} v_2'' = \emptyset$.

The proof in the other direction is similar. Let $[0,1] \setminus \operatorname{supp} v_1'' = \bigcup_{s=1}^{\infty} A_s$, where A_s are connected components. Then $v_1'(t)$ is constant on the interval A_s. And therefore $\mu_1(\mathbb{B}_t) = (2\pi)^n v_1'(\ln t)$ is constant on the interval $B_s = \exp(A_s)$, which means that $\mu_1(\{z \in \mathbb{B} : |z| \in B_s\}) = 0$. So we have proved that if $\operatorname{supp} v_1'' \cap \operatorname{supp} v_2'' = \emptyset$ then $\operatorname{supp}\mu_1 \cap \operatorname{supp}\mu_2 = \emptyset$. □

Before stating the main theorem in this section, let us recall the classical Jordan decomposition of measures in our setting. Take a signed measure $\mu \in \delta\mathcal{M}^R$. Then we can take the Jordan decomposition of μ

$$\mu = \mu^+ - \mu^-,$$

where $\mu^+ = \frac{1}{2}(|\mu| + \mu)$, $\mu^- = \frac{1}{2}(|\mu| - \mu)$. This decomposition is minimal and $\operatorname{supp}\mu^+ \cap \operatorname{supp}\mu^- = \emptyset$.

Theorem 5.10. *For any signed measure $\mu \in \delta\mathcal{M}^R$ there exists a unique solution $u \in \delta\mathcal{PSH}^R(\mathbb{B})$ of the following Dirichlet problem*

$$\mathrm{MA}_2(u) = \mu.$$

Furthermore, $\mathrm{MA}_2 : \delta\mathcal{PSH}^R(\mathbb{B}) \to \delta\mathcal{M}^R$ *is monotone and bijective. In particular, if* $u \succcurlyeq v$, $u, v \in \delta\mathcal{PSH}^R(\mathbb{B})$, *then* $\mathrm{MA}_2(u) \geq \mathrm{MA}_2(v)$.

Proof. Let $\mu \in \delta\mathcal{M}^R$, and let $\mu = \mu^+ - \mu^-$ be its Jordan decomposition. Define

$$f(t) = \frac{1}{(2\pi)^n}\mu^+(\mathbb{B}_{e^t}), \qquad g(t) = \frac{1}{(2\pi)^n}\mu^-(\mathbb{B}_{e^t}).$$

Let

$$u_1(t) = \int_{-\infty}^{t} f(s)^{\frac{1}{n}}\,ds \quad \text{and} \quad u_2(t) = \int_{-\infty}^{t} g(s)^{\frac{1}{n}}\,ds,$$

then

$$u_1'(t) = f(t)^{\frac{1}{n}} \quad \text{and} \quad u_2'(t) = g(t)^{\frac{1}{n}}.$$

Define $u(z) = u_1(\ln|z|) - u_2(\ln|z|)$. We will prove that $u^+(z) = u_1(\ln|z|)$, and $u^-(z) = u_2(\ln|z|)$. Recall that $u^+(z) = \varphi(\ln|z|)$, where φ is a non-positive, increasing and convex function such that $\varphi(0) = 0$ and $\varphi'' = \sup(u_1'' - u_2'', 0)$. Since $\operatorname{supp}\mu^+ \cap \operatorname{supp}\mu^- = \emptyset$ then by Lemma 5.9 $\operatorname{supp} u_1'' \cap \operatorname{supp} u_2'' = \emptyset$, therefore $\varphi'' = u_1''$ and $\varphi = u_1$. In a similar manner one can prove that $u_2(\ln|z|) = u^-(z)$. Now it is easy to check that u is a solution of the Dirichlet problem, namely

$$\mathrm{MA}_2(u) = \mathrm{MA}\big(u^+\big) - \mathrm{MA}\big(u^-\big) = \mathrm{MA}(u_1) - \mathrm{MA}(u_2) = \mu^+ - \mu^- = \mu.$$

Uniqueness of the solution also follows from Lemma 5.9. If $\mathrm{MA}_2(u) = \mathrm{MA}_2(v)$, then

$$\big(dd^cu^+\big)^n - \big(dd^cu^-\big)^n = \big(dd^cv^+\big)^n - \big(dd^cv^+\big)^n = \mu.$$

From Lemma 5.9 we know that

$$\operatorname{supp}\big(dd^cu^+\big)^n \cap \operatorname{supp}\big(dd^cu^-\big)^n = \emptyset \quad \text{and} \quad \operatorname{supp}\big(dd^cv^+\big)^n \cap \operatorname{supp}\big(dd^cv^-\big)^n = \emptyset.$$

But from the uniqueness of the Jordan decomposition of the measure μ it follows that $(dd^cu^+)^n = (dd^cv^+)^n$ and $(dd^cu^-)^n = (dd^cv^-)^n$. Moreover the uniqueness of the solution of Dirichlet problem in the class $\mathcal{PSH}^R(\mathbb{B})$ (see [3]) implies $u^+ = v^+$, and $u^- = v^-$, so $u = v$.

To prove the last part of the theorem assume that $u \succcurlyeq v$. Then there exists $w \in \mathcal{PSH}^R(\mathbb{B})$ such that $u - v = w$. Then

$$u = u^+ - u^- = v^+ + w - v^-,$$

so by the remark before Theorem 5.10 there exist α, β such that

$$u^+ + \alpha = v^+ + w, \qquad u^- + \alpha = v^-,$$

and similarly

$$v^+ + \beta = u^+, \qquad v^- + \beta = u^- + w.$$

Therefore we have

$$\begin{aligned}\mathrm{MA}_2(u) &= \big(dd^cu^+\big)^n - \big(dd^cu^-\big)^n = \big(dd^c\big(v^+ + \beta\big)\big)^n - \big(dd^cu^-\big)^n \\ &\geq \big(dd^cv^+\big)^n - \big(dd^c\big(u^- + \alpha\big)\big)^n = \big(dd^cv^+\big)^n - \big(dd^cv^-\big)^n = \mathrm{MA}_2(v). \qquad \square\end{aligned}$$

6. Open questions

We end this article by stating some interesting open questions. We start with some detailed questions concerning this paper.

(1) Proposition 5.4 yields that the extension MA_1 is not surjective for even dimension. What happens in odd dimensions?
(2) Is Theorem 5.5 true without the assumption that $F^{\frac{1}{n}}$ is of bounded variation?
(3) This question also concerns Theorem 5.5. Is the condition that $\mu \succcurlyeq_{PSH} 0$ sufficient for the solvability of the Dirichlet problem in the case of even dimension?
(4) Can we generalize Theorem 5.5 and Theorem 5.10 for more general types of measures defined on more arbitrary domains?

To end this article we state some questions of a more general nature.

(5) To study and use spaces of differences of functions has a long and colorful history. For example, in Theorem 5.5 we used the classical Jordan's decomposition theorem which states that a function is of bounded variation if, and only if, it is the difference of two increasing functions. Let now $I \subseteq \mathbb{R}$ be an open interval. Another very classical result, this time concerning convex functions goes back to Aleksandrov. In [4], he proved that a function f defined on some open interval $I \subseteq \mathbb{R}$ is the difference of two convex functions if, and only if, it has left and right derivatives and these derivatives are of bounded variation on every closed bounded interval interior to I. Furthermore, in [7], Arsove studied the characterization of the space of delta-subharmonic functions, and in [12] it was proved that

$$\mathcal{C}_0^\infty \subset \mathcal{E}_0 \cap \mathcal{C}(\bar{\Omega}) - \mathcal{E}_0 \cap \mathcal{C}(\bar{\Omega}).$$

Now to the question: can one give a transparent characterization of the delta-plurisubharmonic functions? Before leaving this question let us present a partial result. We start by taking a radially symmetric function $f : \mathbb{B} \to [-\infty, \infty)$, and then we construct a new function $\tilde{f} : [-\infty, \infty) \to [-\infty, \infty]$ by

$$\tilde{f}\big(\ln |z|\big) = f(z).$$

Now we need to impose some restrictions on $\tilde{f}$. Assume that the left derivative of $\tilde{f}$, $\tilde{f}_l'$, exists and is left continuous, $\tilde{f}_l'$ is of bounded variation, and $\tilde{f}(0) = 0$. Then one can proceed in a similar manner as in the proof of Theorem 5.5 to get that f is in $\delta\mathcal{PSH}^R(\mathbb{B})$. On the other hand, let f be in $\delta\mathcal{PSH}^R(\mathbb{B})$, i.e. $f = u - v$ with $u, v \in \mathcal{PSH}^R(\mathbb{B})$. Recall that in the definition of $\mathcal{PSH}^R(\mathbb{B})$ we assumed that $\tilde{u}(0) = \tilde{v}(0) = 0$. We also know that $\tilde{u}$ and $\tilde{v}$ are increasing and convex. Then by basic convex analysis (see e.g. Chapter 1.1 in [30]) we get that $\tilde{u}_l'$ and $\tilde{v}_l'$ exist and are left continuous. Furthermore, they are increasing. Thus, $\tilde{u}_l' - \tilde{v}_l'$ is of bounded variation.
(6) During the recent years the attention to the so called *k-plurisubharmonic functions*, $1 \leq k \leq n$, has increased, see e.g. [15,20,19,31,41]. Recall that $k = 1$ yields the ordinary subharmonic functions and $k = n$ gives us the plurisubharmonic functions. It would be interesting to understand the spaces of differences of k-plurisubharmonic functions for $1 < k < n$.
(7) In 1986, Fuglede introduced the plurifine topology on open subsets of $\mathbb{C}^n$ as the weakest topology in which all plurisubharmonic functions are continuous [26]. The idea goes back to H. Cartan who introduced the fine topology on $\mathbb{R}^n$ as the weakest topology in which all subharmonic functions are continuous. As it turns out there is a strong connection between the so called *plurifine plurisubharmonic functions* and delta-plurisubharmonic functions, since a plurifine plurisubharmonic functions can

be locally written as a delta-plurisubharmonic function (cf. [22,25]). To our knowledge this strong connection has not yet been fully used. For further reading about the plurifine topology and the plurifine pluripotential theory we refer to [23,24,40] and the references therein.

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