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COMPLETION OF LINEARLY ORDERED GROUPS

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To my teacher A. I. Kokorin

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We give examples of linearly ordered groups that are not embeddable in divisible orderable. In the first example, the group does not embed in any divisible group with strictly isolated unity. In the second example, the group in question is an O^*-group, and in the third, it is a group with a central system of convex subgroups.

INTRODUCTION

A question whether linearly ordered groups (l.o. groups), and in particular, orderable groups, are embedded in divisible linearly ordered (orderable) ones — known as the Mal'tsev–Neumann problem —, has remained open over half a century. A divisible group, here, is conceived of as one that is complete under divisibility, that is, such that for every element $g \in G$ and every natural $n > 1$, there exists an $x \in G$ satisfying $x^n = g$. In this event we also say that roots of any degree n are extracted of each element of the group. The following questions concerning completions of l.o. groups and their related problems are well known.

Q.1 [1, Problem 1; 2, Chap. 11, Problem 1]. Can any l.o. group be embedded in a divisible l.o. group?

Q.2 [3, Problem 10; 2, Chap. 11, Problem 2]. Can an orderable (soluble) group be embedded in a divisible orderable group?

Q.3 [3, Problem 11]. A group G is said to be *G-divisible* if it satisfies the equation $x^{g_1+\dots+g_n} = a$, for any $g_i, a \in G$. Can any orderable group G be embedded in a G-divisible orderable group G^*? (A direct check shows that a divisible locally nilpotent group G is always G-divisible.)

Q.4 [2, Chap. 11, Problem 3; 4, Question 5.4 and 11.68]. Is any l.o. group embeddable in an l.o. group in which every two strictly positive elements are conjugate?

Q.5 [3, Problem 16]. Is an l.o. division ring embeddable in an l.o. division ring whose multiplicative group of positive elements is divisible?

Q.6 [3, Problem 12]. Let G be an orderable group, with $a \in G$. Under which conditions can G be embedded in an orderable group H containing an element x such that $x^2 = a$? In which situation can the linear order inherent in G be extended to H?

Q.7 [3, Problem 15]. Can an analog of the Artin–Schreier theory of formally real fields be constructed for l.o. division rings?

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According to [5], every abstract group embeds in a divisible group (see also [6, Thm. 3.4]). The first step in solving the completion problem for l.o. groups was to prove an embedding theorem for torsion-free nilpotent groups in divisible torsion-free nilpotent (cf. [7]). The next step was obtaining a theorem on completions for the center of an ordered group (cf. [8]). In [9], then, an embedding theorem was proved for orderable metabelian groups in complete orderable metabelian. For all that, note, a question remained open whether any order of an initial group can be extended to its completion. An embedding theorem for an l.o. group in an l.o. group with a divisible normal Abelian subgroup was obtained in [10]. In [11], this theorem was generalized to the case of normal locally nilpotent subgroups. Lastly, in [12], an embedding theorem was proved for linearly ordered metabelian groups in divisible linearly ordered metabelian. Thereby the questions posed in [9] were done away with.

In the present account we give negative answers to Q1-Q3 (Thms. 3.2, 3.4, Cor. 3.5). We construct three examples of soluble (centrally metabelian) l.o. groups that do not embed in any divisible orderable groups (Examples 2.1-2.3). This points to bounded possibilities for completions of l.o. groups in classes that are wider than the class of metabelian or nilpotent groups. Yet a possibility is not excluded for arriving at a positive solution to the completion problem for l.o. groups in the variety $\mathfrak{N}_k\mathfrak{A} \bigcap \mathfrak{A}\mathfrak{N}_c$, an intersection of the variety of groups with nilpotent commutant and the variety of nilpotent extensions of Abelian groups. And a question remains open whether l.o. groups in $\mathfrak{A}\mathfrak{N}_c$ admit completions.

Note also that examples given in the paper are modifications of the example announced in [13, 14], and refine the situation with completions of linearly ordered groups. In Example 2.2, we point out an O^*-group that is not embeddable in any divisible orderable groups, which is of interest because nilpotent and metabelian groups, for which completion problems were given positive solutions, are O^*-groups. In Example 2.3, we construct an l.o. group with a central system of convex subgroups that is not embeddable in divisible orderable groups. A question as to completions of such groups was invoked by [15].

Negative solutions to completion problems for l.o. groups are responsible for negative answers to associated issues: Corollary 3.7 and Theorem 3.6 give negative answers to Q.4 and Q.5, respectively. The examples furnished in the paper show that l.o. groups defy extracting of square roots in general. Necessary and sufficient conditions under which taking square roots in l.o. groups is possible are unknown. In Proposition 3.3, however, we specify a necessary condition for an l.o. group to be embedded in an l.o. group containing a solution for $x^n = g$, which gives a partial answer to Q.6. With the question if l.o. division rings can be embedded in l.o. division rings every positive element of which admits of taking the root of two having been answered in the negative, we find it impossible to construct a theory of linearly ordered division rings that will simulate the Artin–Schreier theory for formally real fields. This is associated with the fact that a possibility for embedding l.o. fields in l.o. fields every positive element of which allows extracting of a square root is a principal point in the Artin–Schreier theory [16]; see also [17, Sec. 81]. In this way Q.7 is also answered in the negative.

1. PRELIMINARY INFORMATION AND THE NOTATION

In our reasoning we use the standard notation and definitions, which can be found in [2, 3, 6, 17-19]. We recall some of these. $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Z}^+$ stand for the sets of natural numbers, integers, and positive integers, respectively.

A subgroup H of G is said to be *isolated* if $g^n \in H$, $n \in \mathbb{N}$, and $g \in G$ imply $g \in H$. A least isolated subgroup of G containing H is called an *isolator* of H in G and is denoted by $\mathrm{I}(H)$. The isolator of H

is an intersection of all isolated subgroups containing H. Also $\mathrm{I}(H) = \bigcup_{k=0}^{\infty} \mathrm{I}_k(H)$, where $\mathrm{I}_0(H) = H$ and $\mathrm{I}_{k+1}(H)$ is a subgroup generated by $\mathrm{I}_k(H)$ and by all elements $g \in G$ such that $g^n \in \mathrm{I}_k(H)$, for $n \neq 0$. A generalization for an isolated subgroup is a *strictly isolated* subgroup H in which $g^{1+x_1+\ldots+x_n} \in H$ and $g, x_1, \ldots, x_n \in G$ imply $g \in H$.

A group with unique extraction of roots is called an *R-group*, and a group in which $g^{1+h_1+\ldots+h_n} = 1$ implies $g = 1$ is called an R^*-*group*, or a *group with strictly isolated unity*. Every orderable group is an R^*-group. The converse is not true in general, but remains valid for centrally metabelian groups (see [19]).

Every element g of an l.o. group G determines a *jump* $H_g \prec H(g)$ in a system of convex subgroups. Here, H_g is a union of all convex subgroups not containing g, and $H(g)$ is an intersection of all convex subgroups containing g. If H_g is a normal subgroup of G then the jump $H_g \prec H(g)$ is said to be *normal*. In this case the subgroup $H(g)$ is normal as well. A normal jump $H_g \prec H(g)$ is said to be *central* if $H(g)/H_g$ lies in the center of G/H_g. The last-mentioned condition is equivalent to the fact that $|[g,x]| \ll \min\{|g|, |x|\}$, for any $x \in G$. A system of convex subgroups of an l.o. group is said to be *central* if all of its jumps are central. An *order homomorphism* of l.o. groups is a group homomorphism preserving linear ordering on the groups.

In the lemma below, G^* can be constructed as an HNN-extension of an initial group by an order monomorphism φ (see [20; 6, Chap. IV]), or using the embedding construction for ordered modules in [3, Chap. V, Sec. 3, Thm. 1]. In general, however, HNN-extensions fail to respect linear orders. Therefore, here, we present a proof of the lemma in which ultraproducts are made use of, which — from our standpoint — is more exemplifying.

LEMMA 1.1. If $\varphi : G \to G$ is an order monomorphism of an l.o. group G then there exists an order embedding ε_0 of G in an l.o. group G^*, admitting an order automorphism ψ, whose restriction to G^{ε_0} coincides with the action of φ in G. Moreover, G^* is the union of an increasing sequence of l.o. groups $G^{\varepsilon_0\psi^{-n}}$, $n \in \mathbb{Z}^+$, which are order isomorphic to G.

Proof. Consider an ultrapower $U = \prod_{i\in\mathbb{N}} G_i/\mathcal{U}$, where G_i are isomorphic copies of G and $\mathcal{U}$ is a non-principal ultrafilter on the set of natural numbers. The liner order on G is extended to the ultrapower U, that is, $\{g_i\}_{i\in\mathbb{N}}/\mathcal{U} \geq 1 \Leftrightarrow \{i \mid g_i \geq 1\} \in \mathcal{U}$. Note also that the order monomorphism φ of G, applied coordinatewise to the elements of U, determines an order monomorphism ψ of the ultrapower, that is, if $\bar{g} = \{g_i\}_{i\in\mathbb{N}}/\mathcal{U} \in U$ then $\bar{g}^{\psi} = \{g_i^{\varphi}\}_{i\in\mathbb{N}}/\mathcal{U}$. The fact that φ is an order monomorphism of G implies that for every $k \in \mathbb{Z}^+$, the maps $\varepsilon_k : G \to U$ given by the rule $g^{\varepsilon_k} = \{h_i\}_{i\in\mathbb{N}}/\mathcal{U}$, where $h_i = g^{\varphi^{i-k}}$, for $i \geqslant k$, and $h_i = 1$ for $i < k$, are order preserving embeddings of the l.o. group G in the l.o. group U. Put $G_k = G^{\varepsilon_k}$. In this instance $G_k \leq G_{k+1}$, for all $k \geqslant 0$. For $k \geqslant 1$, $G_k^{\psi} = G_{k-1}$ by construction, and the properties of ultraproducts imply that ψ is an order automorphism of the group $G^* = \bigcup_{k=0}^{\infty} G_k$. Moreover, $G^{\varepsilon_0\psi} = G_0^{\psi} = G^{\varphi\varepsilon_0}$. $\square$

2. EXAMPLES

In this section we give examples of l.o. groups for which there exist no embeddings in divisible orderable groups. A group in Example 2.1 does not embed in any divisible R^*-group, a group in Example 2.2 is an O^*-group, and one in Example 2.3 is a group with a central system of convex subgroups.

Example 2.1. We fix a natural $n \geqslant 2$ and denote by H_n a nilpotent group of class two, defined by generators, h_i, $i \in \mathbb{Z}$, and by the relations

$$[h_i, h_j, h_k] = 1, \quad i, j, k \in \mathbb{Z}, \tag{1}$$

$$[h_{i+k}, h_{j+k}] = [h_i, h_j], \quad i, j, k \in \mathbb{Z}, \tag{2}$$

$$\left[\prod_{k=0}^{n} h_{i+k}^{\binom{n}{k}}, \prod_{k=0}^{n} h_{k}^{\binom{n}{k}}\right] = [h_i, h_0], \quad i \geqslant 1, \tag{3}$$

$$[h_i, h_0] = 1, \quad 2 \leqslant i \leqslant n. \tag{4}$$

In (3), $\binom{n}{k}$ stand for binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

It follows from (1)-(4) that H_n is the factor group of a free nilpotent group N of class two on a set of free generators h_i, $i \in \mathbb{Z}$, w.r.t. the central subgroup, as mentioned in (2)-(4).

In N, the elements $[h_i, h_j]$, for $i > j$, form a basis for the center $\zeta(N)$ treated as a free Abelian group. Relations (2) allow $[h_i, h_j]$, for $j \neq 0$, to be expressed via elements $c_{i-j} = [h_{i-j}, h_0]$. We drop $[h_i, h_j]$ from the basis, replacing these elements by c_{i-j}, and drop, in so doing, relations (2) from the representation of the subgroup $\zeta(N)$. Then we eliminate relations (4), simultaneously excluding elements c_k, $2 \leqslant k \leqslant n$, from the basis. Lastly, we eliminate relations (3), dropping c_k, for $k \geqslant n+1$, from the basis. This is possible since the left parts of the relations in (3), for each i fixed, are a product of commutators c_j, where j ranges from zero to $n+i$ and c_{n+i} occurs in the relation with exponent one. Using combinatorial formulas, we may express c_i explicitly via commutators with smaller numbers, that is,

$$c_i = c_{i-n} \cdot \left(\prod_{j=0}^{2n-1} c_{i-2n+j}^{\binom{2n}{j}}\right)^{-1}, \quad i \geqslant n+1.$$

Here, $c_t = [h_t, h_0]$ for any $t \in \mathbb{Z}$. Moreover, $c_0 = 1$ and $c_{-t} = c_t^{-1}$. Thus $\zeta(H_n) = C_n = \langle c_1 \rangle$ is a free Abelian group of rank one and H_n/C_n is a free Abelian group of infinite rank, with basis h_i, $i \in \mathbb{Z}$. The group N being free nilpotent admits an automorphism $d : N \to N$, given on generators by the rule

$$h_i^d = h_{i+1}, \ i \in \mathbb{Z}. \tag{5}$$

We verify that d induces an automorphism of H_n, which is denoted by the same symbol — d. It suffices to check whether (1)-(4) are preserved under the action of a map d. Relations (1) are respected because H_n is nilpotent of class three. We verify (2)-(4). We have

$$[h_i, h_j]^d = [h_{i+1}, h_{j+1}] = [h_i, h_j], \ i, j \in \mathbb{Z},$$

from which it follows that all elements $[h_i, h_j]$, $i, j \in \mathbb{Z}$, are fixed under the map d. Therefore all relations expressed via $[h_i, h_j]$, and in particular, those in (2)-(4), are preserved under the action of d; hence, d is an automorphism of H_n.

Denote by K_n a semidirect product $H_n \rtimes \langle d \rangle$, with d defined as in (5). Applying d to relations (3) m times, we arrive at

$$\left[\prod_{k=0}^{n} h_{i+k+m}^{\binom{n}{k}}, \prod_{k=0}^{n} h_{k+m}^{\binom{n}{k}}\right] = [h_{i+m}, h_m] \ \ i, m \in \mathbb{Z}, \ i \geqslant 1. \tag{6}$$

By the definition of d, the factor group K_n/C_n satisfies the equality

$$\left(\prod_{k=0}^{n} h_{i+k}^{\binom{n}{k}}\right) C_n = h_i^{(1+d)^n} C_n, \ i \in \mathbb{Z}. \tag{7}$$

Put

$$h_i^\alpha = h_i^{1+d}, \ i \in \mathbb{Z}.$$

By induction on k, from (3), (6), and (7), we derive

$$[h_i^{\alpha^{n+k}}, h_0^{\alpha^{n+k}}] = [h_i^{\alpha^k}, h_0^{\alpha^k}], \ i, k \geqslant 1. \tag{8}$$

For $k = 0$, (8) assumes the form of (3). Further,

$$\begin{aligned}
[h_i^{\alpha^{n+k+1}}, h_0^{\alpha^{n+k+1}}] &= [h_i^{\alpha^{n+k}d} h_i^{\alpha^{n+k}}, h_0^{\alpha^{n+k}d} h_0^{\alpha^{n+k}}] \\
&= [h_{i+1}^{\alpha^{n+k}} h_i^{\alpha^{n+k}}, h_1^{\alpha^{n+k}} h_0^{\alpha^{n+k}}] \\
&= [h_{i+1}^{\alpha^{n+k}}, h_1^{\alpha^{n+k}}] \cdot [h_{i+1}^{\alpha^{n+k}}, h_0^{\alpha^{n+k}}] \cdot [h_i^{\alpha^{n+k}}, h_1^{\alpha^{n+k}}] \cdot [h_i^{\alpha^{n+k}}, h_0^{\alpha^{n+k}}] \\
&= [h_{i+1}^{\alpha^k}, h_1^{\alpha^k}] \cdot [h_{i+1}^{\alpha^k}, h_0^{\alpha^k}] \cdot [h_i^{\alpha^k}, h_1^{\alpha^k}] \cdot [h_i^{\alpha^k}, h_0^{\alpha^k}].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
[h_i^{\alpha^{k+1}}, h_0^{\alpha^{k+1}}] &= [h_i^{\alpha^k d} h_i^{\alpha^k}, h_0^{\alpha^k d} h_0^{\alpha^k}] = [h_{i+1}^{\alpha^k} h_i^{\alpha^k}, h_1^{\alpha^k} h_0^{\alpha^k}] \\
&= [h_{i+1}^{\alpha^k}, h_1^{\alpha^k}] \cdot [h_{i+1}^{\alpha^k}, h_0^{\alpha^k}] \cdot [h_i^{\alpha^k}, h_1^{\alpha^k}] \cdot [h_i^{\alpha^k}, h_0^{\alpha^k}],
\end{aligned}$$

yielding (8).

In view of (5), $K_n/C_n \cong \langle h_0 \rangle \wr \langle d \rangle$, that is, the group K_n is orderable. By (6), we have $C_n = \zeta(K_n)$, and hence K_n is a centrally metabelian group with cyclic center.

On the generators for K_n, we define a map a setting

$$d^a = d, \ h_i^a = h_i^{\alpha^n}, \ i \in \mathbb{Z}. \tag{9}$$

We verify that a extends to an endomorphism of K_n. It suffices to verify that the defining relations in (1)-(5) are stable under a. Relations (1) and (5) are obviously valid. We check (2)-(4). We have

$$\begin{aligned}
[h_{i+k}, h_{q+k}]^a &= [h_{i+k}^a, h_{q+k}^a] = [h_{i+k}^{\alpha^n}, h_{q+k}^{\alpha^n}] = [h_i^{d^k \alpha^n}, h_q^{d^k \alpha^n}] \\
&= [h_i^{\alpha^n}, h_q^{\alpha^n}]^{d^k} = [h_i^{\alpha^n}, h_q^{\alpha^n}] = [h_i^a, h_q^a] = [h_i, h_q]^a,
\end{aligned}$$

and so relations (2) are preserved. To verify (3) and (4), we make use of the relations in (8), for $k = n$. We obtain

$$[h_i^{\alpha^n}, h_0^{\alpha^n}]^a = [h_i^{\alpha^{2n}}, h_0^{\alpha^{2n}}] = [h_i, h_0], \quad [h_i, h_0]^a = [h_i^{\alpha^n}, h_0^{\alpha^n}] = [h_i, h_0] = 1,$$

where $2 \leqslant i \leqslant n$. Thus a exerts no impact on the form of the relations in (3) and (4); hence, a extends to an endomorphism of K_n. From (9), it follows that g and g^a have the same leading exponent; so, a is an order monomorphism of the group K_n. Now, (3), (7), and (9) can be combined to yield

$$c_1^a = c_1. \tag{10}$$

We fix a linear order on K_n. By construction, the endomorphism a is one-to-one and respects any linear order on K_n. Using Lemma 1.1, we extend a to an order automorphism of K_n^*, that is, embed K_n in an l.o. group K_n^*, which admits a and is a union of an ascending chain of groups $K_n^{a^{-t}}$, $t \in \mathbb{Z}^+$. Since all the groups $K_n^{a^{-t}}$ are centrally metabelian, K_n^* is as well.

We put $A = \langle a \rangle$ and consider a semidirect product $G_1(n) = K_n^* \rtimes A$, defined as in (9). Since a is an order automorphism of the l.o. group K_n^*, we turn $G_1(n)$ into an l.o. group by setting $ga^s \geq 1$ if (1) $s > 0$, or (2) $s = 0$ and $g \geq 1$ in K^*. In view of (10), the center of K_n^* is also the center in $G_1(n)$. Combining (5) and (9) yields $[h_i, d] = h_i^{-1} h_{i+1}$, $[a, d] = 1$, and $[h_i, a] = h_i^{-1} h_i^{\alpha^n}$, whence $[G_1(n), G_1(n)] \leq H_n^A$. An image of the subgroup H_n^A in the factor $G_1(n)/C_n$ is Abelian; so, the group $G_1(n)$ is also centrally metabelian.

In the next example, we embed the group $G_1(n)$ specified in Example 2.1 in an O^*-group $G_2(n)$ using the construction from [21].

Example 2.2. Let H_n and C_n be as in Example 2.1. Prior to constructing a group K_n, we extend H_n to a group L_n via automorphisms $u_{j,k}$, where $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^+$, setting

$$h_i^{u_{j,0}} = \begin{cases} h_i c_1 & \text{if } i = j; \\ h_i & \text{otherwise.} \end{cases} \tag{11}$$

The action of $u_{j,k}$ on h_i is defined inductively so as to satisfy

$$[h_i^{(1+d)^n}, u_{j,k+1}] = [h_i, u_{j,k}], \ k \geqslant 0. \tag{12}$$

It suffices to put

$$h_i^{u_{j,k+1}} = \begin{cases} h_i & \text{if } i < j + (k+1)n; \\ h_i c_1 & \text{if } i = j + (k+1)n; \\ h_i \left[\prod_{q=0}^{n-1} h_{i-n+q}^{\binom{n}{q}}, u_{j,k+1} \right]^{-1} [h_{i-n}, u_{j,k}] & \text{if } i > j + (k+1)n, \end{cases} \tag{13}$$

for $k \geqslant 0$. Formulas (11) and (13) completely define the action of the elements $u_{j,k}$ on the generators h_i for the group H_n. Since all relations on H_n are given in the commutator form, and acts of $u_{j,k}$ reduce to multiplication by central elements, it follows that all the defining relations for H_n are preserved under the action of $u_{j,k}$, and hence the acts of $u_{j,k}$ extend to automorphisms of H_n.

Denote by L_n a semidirect product of the groups H_n and $U = \langle u_{j,k} \mid j \in \mathbb{Z}, k \in \mathbb{Z}^+ \rangle$ in accordance with (11) and (13). Note that these formulas imply that automorphisms $u_{j,k}$ are pairwise commuting, and so U can be conceived of as an Abelian group. Thus L_n is nilpotent of class two and $\bar{L}_n = L_n/C_n$ is a free Abelian group with basis $\bar{h}_i, \bar{u}_{i,j}$, $i \in \mathbb{Z}$, $j \in \mathbb{Z}^+$. The relations for L_n are stable under the map $d : L_n \to L_n$, given on generators by the rule

$$h_i^d = h_{i+1}, \quad u_{i,j}^d = u_{i+1,j}, \quad \text{where } i \in \mathbb{Z}, \ j \in \mathbb{Z}^+;$$

so, d extends to an automorphism of L_n.

Denote by K_n a semidirect product of the group L_n with the automorphism d specified above. Note that K_n/C_n is isomorphic to a wreath product $\langle \bar{h}_0, \bar{u}_{0,j} \mid j \in \mathbb{Z}^+ \rangle \wr \langle d \rangle$, and so K_n is an orderable group. Using (11), for every $h \in H_n \setminus C_n$, we can find a $u \in U$ such that

$$h^u = h c_1^m, \tag{14}$$

where $m \neq 0$ depends on u and h. Let V be the centralizer of H_n in U. If $V \neq \{1\}$ then V is a normal strictly isolated subgroup in K_n, and K_n/V is centrally metabelian and, hence, is an orderable R^*-group (see [19]). Taking U/V in place of U, L_n/V in place of L_n, and K_n/V in place of K_n, we see that for any $1 \neq u \in U$, there is an $h \in H_n$ such that

$$u^h = u c_1^m, \tag{15}$$

where $m \neq 0$ depends on h and u. Using (14) and (15), for every $g \in L_n \setminus C_n$, we can find a $v \in L_n$ such that

$$g^v = g c_1^m, \tag{16}$$

where $m \neq 0$ depends on g and v.

We fix a linear order on K_n and consider a map $a : K_n \to K_n$, given on generators by the rule

$$d^a = d, \quad h_i^a = h_i^{(1+d)^n}, \quad u_{i,j}^a = u_{i,j+1}, \quad i \in \mathbb{Z}, \ j \in \mathbb{Z}^+.$$

As shown in Example 2.1, the map a checks with the defining relations on the subgroup H_n, and with conjugation by an element d. In virtue of (12), a agrees with conjugations by elements $u_{i,j}$ and, hence, extends to an endomorphism of K_n. As in Example 2.1, we extend the order monomorphism a of a group K_n to an order automorphism of the group K_n^*, and construct a semidirect product $G_2(n) = K_n^* \rtimes \langle a \rangle$. By construction, $G_2(n)$ is an orderable centrally metabelian group, containing an isomorpic copy of the group $G_1(n)$ from Example 2.1.

We claim that $G_2(n)$ is an O^*-group. Denote by T the normal closure of a subgroup L_n in $G_2(n)$. By the construction of $G_2(n)$, its factor group w.r.t. T is isomorphic to a free Abelian group $\langle a, d \rangle$. Therefore we need only show that the subgroup T is a G-preordered group (see [3, Chap. VI, Sec. 1, Cor. 5]). It suffices to verify that, for any elements x_1 and x_2 in $S(x)$, the intersection of semigroups $S(x_1)$ and $S(x_2)$ is not empty, $x \in T$, and $S(x)$ is a semigroup generated by elements conjugate to x in $G_2(n)$ (see [3, Chap. II, Sec. 1, Cor. 3]). If $x \in C_n$ then $x_1 = x^{n_1}$, $x_2 = x^{n_2}$, and $x_1^{n_2} = x_2^{n_1}$. Let $x \in T \setminus C_n$. The factor group $G_2(n)/C_n$ is metabelian and, hence, orderable (see [3, Chap. VI, Sec. 1, Prop. 2]). Therefore $S(x_1)C_n \bigcap S(x_2)C_n \neq \varnothing$. Hence there are $y_1 \in S(x_1)$ and $y_2 \in S(x_2)$ such that

$$y_1 = y_2 c_1^q. \tag{17}$$

Suppose $q \neq 0$. Interchanging y_1 and y_2 if necessary, we may assume that $q \geqslant 1$. Recall that $T < K_n^* = \bigcup\limits_{i \in \mathbb{Z}^+} K_n^{a^{-i}}$. Conjugating (17) by a suitable degree of a yields $y_2 \in T \bigcap K_n = L_n$. In view of (16), in this instance we can find a $v \in L_n$ such that $y_2^v = y_2 c_1^m$. There is no loss of generality in assuming that $m > 0$. (Conjugating y_2 by v^{-1} yields $y_2^{v^{-1}} = y_2 c_1^{-m}$.) Then

$$y_2^{m-1} y_2^{v^q} = y_2^m c_1^{mq} = y_1^m,$$

and $S(x_1) \bigcap S(x_2) \neq \varnothing$.

In these examples, note, since $h_0 = [h_0, a][d, h_0]$, the groups $G_1(n)$ and $G_2(n)$ admit no orderings with a central system of convex subgroups. In the next example, the construction of $G_1(n)$ will be slightly modified so as to produce an l.o. group with a central system of convex subgroups.

Example 2.3. Let K_n, H_n, and C_n be as in Example 2.1. As noted, the factor K_n/C_n is isomorphic to a wreath product $\langle h_0 \rangle \wr \langle d \rangle$, that is, it is approximated by torsion-free nilpotent groups. Thus K_n/C_n, and so also K_n, will admit a linear order with a central system of convex subgroups. We fix such ordering on the group K_n.

Before defining an endomorphism b of K_n, we embed this group (extending its order) in the l.o. group $\tilde{K}_n$ with a divisible subgroup $\tilde{H}_n$, so that $\tilde{H}_n$ will be a minimal completion of H_n, and $\tilde{K}_n/\tilde{H}_n \cong K_n/H_n \cong \langle d \rangle$. Since H_n is a convex normal subgroup of K_n, this can be done appealing to a theorem of Gurchenkov in [11], or the Mal'tsev theorem on completions for torsion-free nilpotent groups in [7].

We define a map $\beta : \tilde{H}_n \to \tilde{H}_n$ by setting

$$g^\beta = \left(g^{\frac{1}{2}}\right)^{1+a}, \quad g \in H_n^*. \tag{18}$$

The group $\tilde{K}_n$ is generated by elements d and $h_i^{\frac{1}{k}}$, $i \in \mathbb{Z}$, $k \in \mathbb{N}$. On these generators, we define a map b by setting

$$d^b = d, \ \left(h_i^{\frac{1}{k}}\right)^b = \left(h_i^{\beta^n}\right)^{\frac{1}{k}}, \quad i \in \mathbb{Z}, \ k \in \mathbb{N},$$

and show that b extends to an order endomorphism of $\tilde{K}_n$. It suffices to verify that b preserves the defining relations for $\tilde{K}_n$.

We have $\left(h_i^{\frac{1}{kt}}\right)^t = h_i^{\frac{1}{k}}$. In this instance $\left(\left(h_i^{\frac{1}{kt}}\right)^t\right)^b = \left(\left(h_i^{\frac{1}{kt}}\right)^b\right)^t = \left(h_i^{\beta^n}\right)^{\frac{1}{kt}\cdot t} = \left(h_i^{\beta^n}\right)^{\frac{1}{k}} = \left(h_i^{\frac{1}{k}}\right)^b$. Furthermore, $\left(h_i^{\frac{1}{k}}\right)^{d^t} = h_{i+t}^{\frac{1}{k}}$. Since d and b are commuting, $\left(h_i^{\frac{1}{k}}\right)^{d^t b} = \left(h_i^{\frac{1}{k}}\right)^{bd^t} = \left(\left(h_i^{\beta^n}\right)^{\frac{1}{k}}\right)^{d^t} = \left(h_{i+t}^{\beta^n}\right)^{\frac{1}{k}} = \left(h_{i+t}^{\frac{1}{k}}\right)^b$.

Relations (1)-(4) are defined on commutators, in which case the action of b differs from that of a by the rational exponent. In nilpotent groups of class two, $[x^r, y^q] = [x, y]^{rq}$ for any elements x, y and any rational r, q. Therefore verification of the relations in question in this case is exactly the same as in Example 2.1.

Again, as in Example 2.1, we extend the endomorphism b of $\tilde{K}_n$ to an order automorphism of the centrally metabelian group K_n^*, and consider a semidirect product $G_3(n) = K_n^* \rtimes \langle b \rangle$, defined as in (18). On $G_3(n)$, we induce an order by setting $gb^s \geq 1$ if (1) $s > 0$, or (2) $s = 0$ and $g \geq 1$ in K^*. The system of convex subgroups in K_n^* was central. We claim that the same is true of the group $G_3(n)$. It suffices to verify that $|[g, b]| \ll |g|$, for any $g \in K_n^*$. Indeed, $[b, d] = 1$, $[b, c_1] = 1$, $[b, g] = g^{-1} g^{\beta^n} v$, $|v| \ll |g|$ for $|c_1| \ll |g| \ll |d|$, and $g^\beta = g^{\frac{1}{2}} (g^{\frac{1}{2}})^d = gw$, where $|w| \ll |g|$, modulo the center.

3. MAIN RESULTS

LEMMA 3.1. For any $n > 1$, the groups $G_1(n)$ and $G_2(n)$ specified in Examples 2.1 and 2.2 do not embed in any R^*-groups in which the root of degree n is extracted of an element $a \in G_1(n), G_2(n)$.

Proof. Consider a group $G = G_1(n)$. Assume, to the contrary, that G^* is an R^*-group containing G and an element x such that $x^n = a$. From (9) and (10), it follows that $[x^n, d] = [x^n, c_1] = 1$, and since G^* is an R^*-group,

$$[x, d] = [x, c_1] = 1 \tag{19}$$

in the group G^*. Put $f = h_1^{-1} h_0^{-1} h_0^x$. Then $h_0^x = h_0 h_1 f = h_0^\alpha f$. Since x and d are commuting, for any $k \in \mathbb{N}$ we have $h_0^{\alpha^k x} = h_0^{x \alpha^k} = h_0^{\alpha^{k+1}} f_k$, where f_k is a product of elements conjugate to f. We use induction to prove that $h_0^{x^k} = h_0^{\alpha^k} \bar{f}_k$, where $\bar{f}_k$ is a product of elements conjugate to f. This is true for $k = 1$. For $k + 1$, we have

$$h_0^{x^{k+1}} = (h_0^{\alpha^k} \bar{f}_k)^x = h_0^{\alpha^k x} \bar{f}_k^x = h_0^{\alpha^{k+1}} f_k \bar{f}_k^x,$$

and the inductive assumption is thus verified.

Therefore $h_0^{x^n} = h_0^{\alpha^n} \bar{f}_n$. On the other hand, $h_0^{x^n} = h_0^a = h_0^{\alpha^n}$. Hence $\bar{f}_n = 1$. Since $\bar{G}$ is an R^*-group, we have $f = 1$, and so $h_0^x = h_0^\alpha$ and $h_1^x = h_1^\alpha$. In view of (19), $c_1^x = c_1$. On the other hand, $c_1^x = [h_1, h_0]^x = [h_1^\alpha, h_0^\alpha] = [h_1 h_2, h_1 h_0] = [h_1, h_0][h_2, h_1][h_2, h_0] = c_1^2$. Contradiction. Since $G_1(n) < G_2(n)$, it follows that $G_2(n)$, too, does not embed in any R^*-group in which the root of degree n is extracted of a. □

Lemma 3.1 implies the following:

THEOREM 3.2. There exist linearly ordered groups (O^*-groups) which do not embed in any divisible orderable groups. Moreover, such exist in the variety of central extensions of metabelian groups.

In order to refine this theorem, we appeal to Example 2.3. First, we specify a necessary condition for l.o. groups to admit completions.

Proposition 3.3. Let $H = \langle h_i \mid i \in I \rangle$ be a subgroup of an l.o. group G, and let $\varphi : \{h_i \mid i \in I\} \to G$ be a map given on generators of H by the rule

$$h_i^{\varphi} = h_i^{\frac{k_1}{m_1}a_1 + \ldots + \frac{k_s}{m_s}a_s},\ i \in I,$$

$a_1, \ldots, a_s \in G$, and $\frac{k_1}{m_1}, \ldots, \frac{k_s}{m_s}$ be positive rational numbers. Suppose $b \in G$ commutes with $a_1, \ldots, a_s$, and $h_i^b = h_i^{\varphi^n}$ for all $i \in I$. If G embeds in a divisible orderable group G^*, then the map φ extends to an isomorphic embedding of H in G.

Proof. We fix a liner order on the group G^*, and extend φ to a map φ^* of the group G^* into itself, setting

$$g^{\varphi^*} = g^{\frac{k_1}{m_1}a_1 + \ldots + \frac{k_s}{m_s}a_s}.$$

If $g_1 < g_2$ then $g_1^{\frac{k_j}{m_j}} < g_2^{\frac{k_j}{m_j}}$, $j = 1, \ldots, s$, and $g_1^{\varphi^*} < g_2^{\varphi^*}$. Thus φ^* preserves the strict order on G^*. Let $x \in G^*$ and $x^n = b$; then $[x^n, a_j] = [b, a_j] = 1$, $j = 1, \ldots, s$, which implies that $[x, a_j] = 1$ in an orderable group. It follows that φ^* commutes with conjugation by an element x, and $g^{\varphi^* x} = g^{x\varphi^*}$. The map $g \mapsto g^{\varphi^* x^{-1}}$ also respects the strict order on G^*. If $h_i^{\varphi^* x^{-1}} > h_i$ for some $i \in I$, then $h_i = h_i^{\varphi^n b^{-1}} = h_i^{(\varphi^*)^n x^{-n}} = h_i^{(\varphi^* x^{-1})^n} > h_i$, a contradiction. That $h_i^{\varphi^* x^{-1}} < h_i$ is also impossible; so, $h_i^x = h_i^{\varphi^*} = h_i^{\varphi}$. Hence $H^x < G$ and the map φ extends to an isomorphic embedding of H in G given by the rule $H \to H^x$. □

THEOREM 3.4. There exist l.o. groups with a central system of convex subgroups which do not embed in any divisible orderable groups. Moreover, such exist in the variety of central extensions of metabelian groups.

Proof. The map β defined on generators of the subgroup $\tilde{H}_n$ of $G_3(n)$ satisfies the hypotheses of Proposition 3.3, but it does not extend to an isomorphic embedding of $\tilde{H}_n$ in $G_3(n)$, since otherwise $[h_1, h_0]^{\beta^n} = [h_1, h_0]^b = [h_1, h_0]$, whence $[h_1, h_0] = [h_1, h_0]^{\beta} = [h_1^{\beta}, h_0^{\beta}] = [h_1^{\frac{1}{2}} h_2^{\frac{1}{2}}, h_0^{\frac{1}{2}} h_1^{\frac{1}{2}}] = [h_1, h_0]^{\frac{1}{2}}$. □

COROLLARY 3.5. There exists an orderable group G which embeds in a divisible orderable group, and moreover, G admits orderings that do not extend to any divisible orderable groups containing an isomorphic copy of G.

Proof. Let F be a free (or free soluble) group, admitting a homomorphism φ onto the group $G_1(n)$ with kernel T. Then the linear order with a convex subgroup T exists on F. If F embeds in a divisible l.o. group F^* so that the specified order extends to F^*, then the isolator $\mathrm{I}(F)$ in F^* is also a divisible group. Since $\mathrm{I}(F) = \bigcup_{k=0}^{\infty} \mathrm{I}_k(F)$, and every $\mathrm{I}_k(F)$, $k > 0$, is generated by $\mathrm{I}_{k-1}(F)$ and by elements x falling in some degree into $\mathrm{I}_{k-1}(F)$, then the subgroup T is normal in $\mathrm{I}_k(F)$ and, hence, in $\mathrm{I}(F)$. It follows that $\mathrm{I}(F)/T$ is a divisible orderable group containing $G_1(n)$, which is a contradiction with Lemma 3.1. On the other hand, F is approximated by torsion-free nilpotent groups, and so on F, there exists a linear order with convex subgroups, which are members of the lower central series $\gamma_i(F)$. Nilpotent groups $N_i = F/\gamma_i(F)$ embed in divisible l.o. groups N_i^*. Now, F embeds in the complete direct product of groups N_i^*, which is a divisible l.o. group. □

THEOREM 3.6. For any natural $n > 1$, there exist l.o. division rings that are not embeddable in any l.o. division rings all of whose positive elements admit the extraction of roots of degree n.

Proof. Let R be a group algebra of an l.o. group $G_1(n)$ over $\mathbb{Q}$. By the Mal'tsev–Neumann theorem in [22, 23], the algebra R embeds in an l.o. division ring $\hat{R}$ (see also [3, Chap. V, Sec. 1, Thm. 1]). If $\hat{R}$

embeds in an l.o. division ring R^* in which the root of degree n is extracted of every positive element of the ring, then the multiplicative subgroup R^+ of positive elements of the division ring R^* contains $G_1(n)$, and every element of R^+ admits of taking a root of degree n, a contradiction with Lemma 3.1. □

COROLLARY 3.7. There exist l.o. groups that are not embeddable in l.o. groups whose positive elements are conjugate.

Proof. Suppose an l.o. group G does not embed in any divisible l.o. group (one of the groups $G_i(n)$, $i = 1, 2, 3$, for instance). If G embeds in an l.o. group G^* whose positive elements are conjugate, then $g = x^{-1}g^n x$, and the root of degree n is taken of an element g in G^*, that is, G^* is a divisible l.o. group, a contradiction with the assumption on G. □

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