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# On the number of triple points of an immersed surface with boundary

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Given a generic immersion $f: S^1 \to S^2$ of a circle into the sphere, we find the best possible lower estimation for the number of triple points of a generic immersion $F : (M, S^1) \to (B^3, S^2)$ extending $f$, where $M$ is an oriented surface with boundary $\partial M = S^1$, $B^3$ is the 3-dimensional ball with boundary $S^2$.

**Keywords.** immersion – double point – triple point

## Introduction

Let $M_g$ be an oriented surface of genus $g$ with one hole and let us identify its boundary $\partial M_g$ with the standard circle $S^1$. Assume we are given a generic selftransversal immersion $f: S^1 \to S^2 = \partial B^3$. By genericity, $f$ has no triple points and the number of selftransverse double points is finite. The following theorem can be proved easily.

**Theorem 1** *The map $f: S^1 \to S^2 = \partial B^3$ extends to a selftransversal immersion $F: M_g \to B^3$, which is also transversal to the sphere $S^2$, if and only if the number of double points of $f$ is even.*

Assume now that $f$ has $2k$ double points. Introduce the notation

$$\tau_g(f) = \min_F \#\text{triple points of } F,$$

where the minimum is taken for all extensions $F$ of $f$ described in the theorem above.

It is clear that

$$\tau_0(f) \geq \tau_1(f) \geq \tau_2(f) \dots,$$

since one can attach a handle to any immersed surface without increasing the number of triple points.

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The theorem below can be proved applying Theorem 2 of **[6]**.

**Theorem 2** *The numbers $\tau_i(f)$, $i = 1, 2, \ldots$ are all of the same parity.*

*If one colors the connected components of $S^2 \setminus f(S^1)$ with black and white in the usual chessboard-like way (see Fig. 1), then*

$$\tau_i(f) \equiv \#\text{black domains } + 1 \equiv \#\text{white domains } + 1 \, (\text{mod } 2).$$

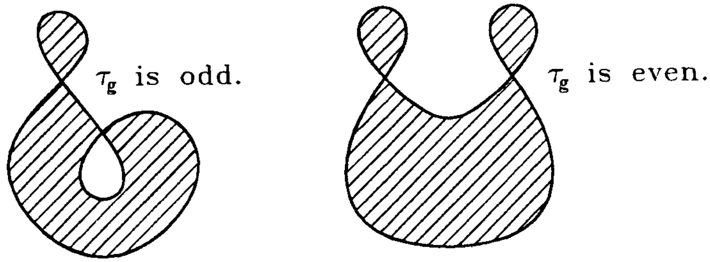



**Fig. 1.**

*Remark .* Theorem 2 can be extended to unorientable surfaces as follows. Let $f$ be as above and $M^2$ be any compact surface with one boundary component. Let $F : (M^2, \partial M^2) \to (D^3, S^2)$ be a generic immersion such that $F|_{\partial M^2} = f$, and let $\tau(F)$ be the number of triple points of $F$. By Banchoff's theorem ([1]), the parity of the sum $\chi(M^2) + \tau(F)$ depends only on $f$. By Theorem 2, it coincides with the parity of the number of black regions.

*Proof .* Suppose $F$ is a self-transverse immersion of an oriented surface with boundary $S^1$ into $B^3$, which extends $f$. The union of the image of $F$ and the black domains is topologically equivalent to the image of a closed surface $M$ under a generic smooth map $\tilde{F}$ having only Whitney umbrella singularities at the double points of $f$ and triple point singularities at the triple points of $F$. The Euler characteristic of $M$ is congruent to 1+ the number of black domains (mod 2). The linking number $l(\tilde{F})$ defined in [6] is equal to 0 in our case. Applying Theorem 2 of **[6]** to the map $\tilde{F}$ we obtain Theorem 2 above.

*Remark .* The referee called our attention to the fact that Theorem 2 can be given an alternative proof relying on work of Carter and Ko [4] (superseded by Izumiya and Marar [5] and Banchoff [1].

It is natural to ask how to compute $\tau_g(f)$ if $f$ is given. This question has not been solved completely. In **[2]**, Carter asked the natural question how to compute $\tau_g(f)$ if $f$ is given. In **[3]**, he gave a method to determine $\tau_0(f)$ and indicated further work needs to be done in these regards.

Our goal is to find the minimum

$$\tau(f) = \min_g \tau_g(f).$$

Now we introduce those easily computable quantities in terms of which we shall express $\tau(f)$. Every sufficiently small neighborhood of a double point of $f$ is cut into four quadrants by $f(S^1)$. The orientation of $M_g$ defines an orientation of $S^1 = \partial M_g$ thus we obtain an orientation of the arcs of $f(S^1)$ passing through the double point. This orientation allows us to distinguish the positive quadrant (see Fig. 2). Let us take a point near to every double point in the positive quadrant. We obtain $2k$ points $P_1, P_2, \ldots, P_{2k}$. Let us denote by $l_{ij}$ the linking number of the oriented 0-dimensional cycle $P_j - P_i$ with the 1-dimensional cycle $f(S^1)$. Let us define a pre-ordering $\preceq$ on the points $P_1, P_2, \ldots, P_{2k}$ by setting

$$P_i \preceq P_j \iff l_{ij} \geq 0.$$

We may assume that the points $P_i$ are indexed so that

$$P_1 \preceq P_2 \preceq \ldots \preceq P_{2k}.$$

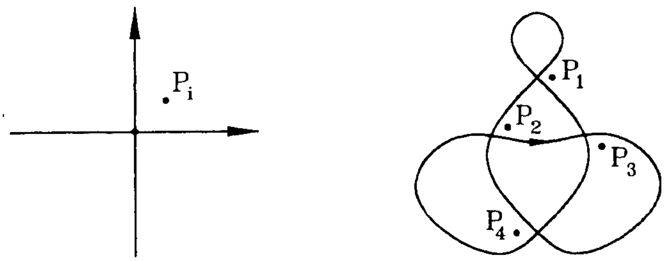



**Fig. 2.**

**Theorem 3** *Using the notation above we have*

$$\tau(f) = \sum_{i=1}^{k} l_{2i-1,2i}.$$

This theorem implies the following

**Corollary 4** *If $f$ has $2k$ double points, then*

$$\tau(f) \leq k$$

To prove the corollary, observe first, that if $P_i$ and $P_j$ belong to two neighboring (along $f(S^1)$) double points, then $|l_{ij}| \leq 1$. If passing from the double point of $P_i$ to the double point of $P_{i+1}$ along $f(S^1)$ we are passing by the points $P_i = P_{i_0}, P_{i_1}, \ldots, P_{i_s} = P_{i+1}$, then the difference between the consecutive members of the sequence $l_{i,i_0}, l_{i,i_1}, \ldots, l_{i,i_s}$, is at most 1. Therefore, if $l_{i,i+1} > 0$, then for one of the members of this sequence we have $l_{i,i_j} = 1$. In view of the definition of the pre-ordering, this implies $|l_{i,i+1}| \leq 1$ for $i = 1, \ldots, 2k-1$.

*Remark .* One may also pose analogous questions for non-orientable surfaces. Denote by $\tau_g^-(f)$ the minimum of the number of triple points of an arbitrary selftransversal extension $F: M_g^- \to B^3$, where $M_g^-$ is the connected sum of a sphere and $g$ projective planes, with an open disc removed, $\partial M_g^- = S^1$. Though the computation of $\tau_g^-(f)$ seems as difficult as that of $\tau_g(f)$, the unorientable analogue of $\tau(f)$ is not interesting. Indeed, we can remove triple points "plugging in" a Boy surface. What we do is that we remove a small neighborhood of the triple point from the immersed surface, we do the same with Boy's surface, and then we glue together the resulting surfaces. (For illustration and more detailed description of this operation we refer to [1].) Thus,

$$\tau^-(f) = \min_g \tau_g^-(f) = 0.$$

Plugging a Boy surface in the surface increases the number of Möbius bands in the surface by five and generally this is a large price for the elimination of one triple point. For example, if a double line goes through two different triple points, then we can eliminate two consecutive triple points by gluing to the surface a tube obtained as the boundary of a tubular neighborhood of the segment of the double line connecting the two triple points. If none of the double lines contains two different triple points, but we have at least two triple points then let us take a simple curve $\alpha$ in the surface such that the endpoints of $\alpha$ lie on different double lines but they are not triple points and except for these endpoints $\alpha$ contains no more double points. Attaching to our surface the boundary of a small tubular neighborhood of $\alpha$ we modify the picture of double lines in such a way that the modified double lines contain two different triple points. Therefore, gluing two handles (four Möbius bands) to the surface we can always decrease the number of triple points by two provided there are at least two triple points. This implies the inequalities

$$\max\{\tau_g^-(f) - 2, 1\} \geq \tau_{g+4}^-(f), \quad \tau_{2\tau_0(f)}^-(f) \leq 1.$$

## Proof of Theorem 3

*Part A. Lower estimation for the number of triple points*

Here we show

$$\tau(f) \geq \sum_{i=1}^{k} l_{2i-1,2i}.$$

Let $F\colon M_g \to B^3$ be any selftransversal immersion such that $f = F\,|_{\partial M_g}\colon S^1 \to S^2$. Let $t = t(F)$ be the number of triple points of $F$.

$1^{\text{st}}$ *step.* First we perform a surgery on the map $F$ in order to simplify the picture of double lines. Namely, we do the following. Locally, in the neighborhood of a triple point, the immersed surface looks like three intersecting planes, like the coordinate planes of a 3-dimensional Cartesian coordinate system (see Fig. 3.a). Identifying the surface locally with the latter picture, consider a small torus of rotation centered at the origin with axis of symmetry the $x$-axis (see Fig. 3.b). This torus is cut into four tubes $T_1, T_2, T_3, T_4$ by the coordinate planes $xy$ and $xz$. Let us remove the discs bounded by the four circles of $\partial T_1 \cup \partial T_3 = \partial T_2 \cup \partial T_4$ from the surface. Glue to the surface along the boundaries of these holes either $T_1 \cup T_3$ or $T_2 \cup T_4$ choosing the right pair of opposite handles in such a way that we could extend the orientation of the surface to the handles. By this procedure the genus of $M_g$ increases by 2.

Executing such a surgery at all the $t = t(F)$ triple points we obtain finally an immersion $\tilde{F}: M_{g+2t} \to B^3$, for which $\tilde{F}\,|_{S^1} = F\,|_{S^1} = f$ and the number of triple points of $\tilde{F}$ is the same as that of $F$.

Though these surgeries do not have any effect on the boundary $f(S^1)$ of the immersed surface and the number of triple points, they simplify the picture of double lines in the following sense. Two of the three double lines meeting at a triple point are driven off from the triple point by the surgery, while their arcs through the triple point are connected to form a "bow tie" (see Fig. 3.c). Thus, the system of double lines of $\tilde{F}$ consists of (i) bow ties, (ii) a disjoint union of embedded arcs having endpoints at the double points of $f$ which we call "long double lines", and (iii) embedded closed curves.

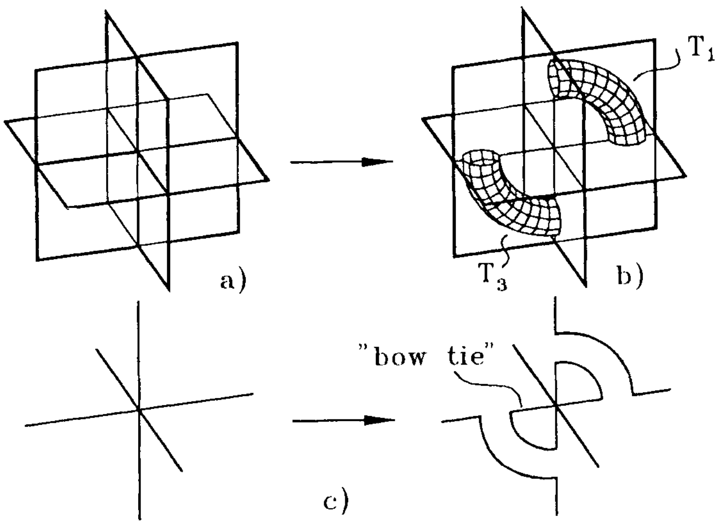



**Fig. 3.**

$2^{\text{nd}}$ *step.* We give an estimation for the number of triple points of $\tilde{F}$ lying along a given long double line $\delta$. Let us fix an orientation of $\delta$ and suppose that the initial point of $\delta$ is the double point of $f$ near to $P_i$, while the endpoint of $\delta$ is the double point of $f$ near to $P_j$. At every point $P$ of $\delta$ we can define uniquely the following three vectors:

- the unit tangent vector $\mathbf{a}$ of $\delta$ pointing in the positive direction with respect to the orientation of $\delta$;
- the unit vectors $\mathbf{b}$ and $\mathbf{c}$, for which $(\mathbf{a}, \mathbf{b})$ and $(\mathbf{a}, \mathbf{c})$ are positively oriented orthonormal bases of the tangent planes to the two sheets of the surface meeting at $P$.

Move $\delta$ a little in the direction of $\mathbf{b} + \mathbf{c}$ and denote by $\delta'$ the translate. The initial point $P_i'$ of $\delta'$ is in the corner opposite to the corner containing $P_i$. For the endpoint of $\delta'$ we may choose $P_j$. Since the 0-dimensional cycle $P_i - P_i'$ has linking number 0 with $f(S^1)$, the linking numbers of the cycles $P_j - P_i$ and

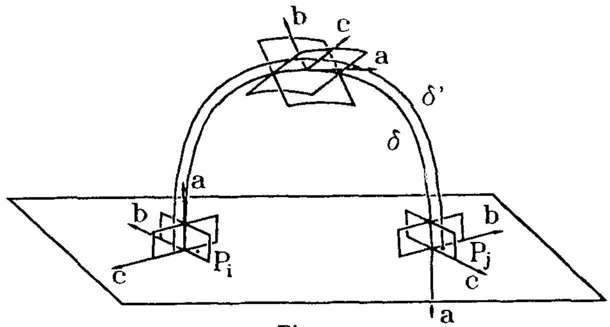



**Fig. 4.**

$P_j - P_i'$ with $f(S^1)$ are the same, i.e.

$$l_{ij} = l(P_j - P_i', f(S^1)).$$

But the linking number $l(P_j - P_i', f(S^1))$ coincides with the algebraic intersection number of $\delta'$ and $\tilde{F}(M_{g+2t})$. Since the algebraic intersection number counts the geometric intersection points with signs, the algebraic intersection number is less then or equal to the number of geometric intersection points. Since $\delta'$ intersects the immersed surface $\tilde{F}(M_{g+2t})$ only around the triple points, we obtain the following inequality

$$|l_{ij}| \leq \#\text{triple points of } \tilde{F} \text{ lying along } \delta.$$

3rd *step.* $\tilde{F}$ has $k$ long double lines, say $\delta_1, \ldots, \delta_k$, that connect double points of $f$. Assume $\delta_i$ connects the double points at $P_{\alpha_i}$ and $P_{\beta_i}$. Then, by the previous step,

$$\#\text{triple points of } F \; = \#\text{triple points of } \tilde{F} \; \geq \sum_{i=1}^{k} |l_{\alpha_i\beta_i}|.$$

The right hand side is at least $\sum_{i=1}^{k} l_{2i-1,2i}$. This follows from the elementary observation that if for the integers $1 \leq i, j, k, l \leq 2k$ the intervals $[i, j]$ and $[k, l]$ overlap, then $l_{ij} + l_{kl} > |l_{ik}| + |l_{jl}|$. Since the immersion $F$ was arbitrary, these estimations yield

$$\tau(f) \geq \sum_{i=1}^{k} l_{2i-1,2i}.$$

*Part B. Construction of an immersed surface with minimal number of triple points*

Now we construct an immersed surface with a given boundary $f(S^1)$ with

$$\sum_{i=1}^{k} l_{2i-1,2i}$$

triple points. The construction is split into three steps.

1st *step*. Consider the immersed oriented 1-manifold

$$f_1 = f_{11} \amalg \ldots \amalg f_{1s}\colon S^1 \amalg \ldots \amalg S^1 \to S^2$$

obtained from $f(S^1)$ by surgeries around the double points of $f$, as depicted in Fig. 5.a.

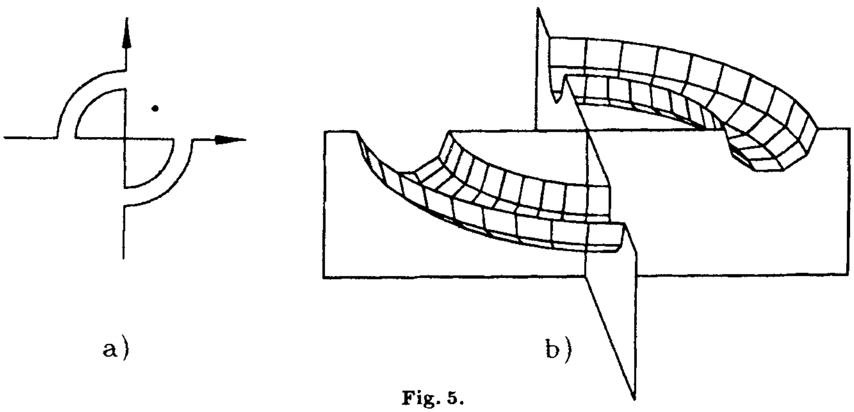



Fig. 5.

Connected components of the image of $f_1$ are disjoint embedded closed curves and bow ties. The union of $f(S^1) \times \{0\}$ and $\operatorname{im} f_1 \times \{1/2\}$ with the opposite orientation bounds an immersed oriented surface in $S^2 \times [0, 1/2]$ having no triple points (see Fig. 5.b).

2nd *step*. Assume that two components $f_{1i}(S^1)$ and $f_{1j}(S^1)$ of $f_1$ are embedded circles (not bow ties) and their orientations are induced from an orientation of the annulus in $S^2$ lying between them. Suppose furthermore that $f_{1i}(S^1)$ and $f_{1j}(S^1)$ can be connected by an arc $\alpha$ not crossing any other component of $f_1$. Then replace the components $f_{1i}(S^1)$ and $f_{1j}(S^1)$ by their connected sum as shown in Fig. 6.a.

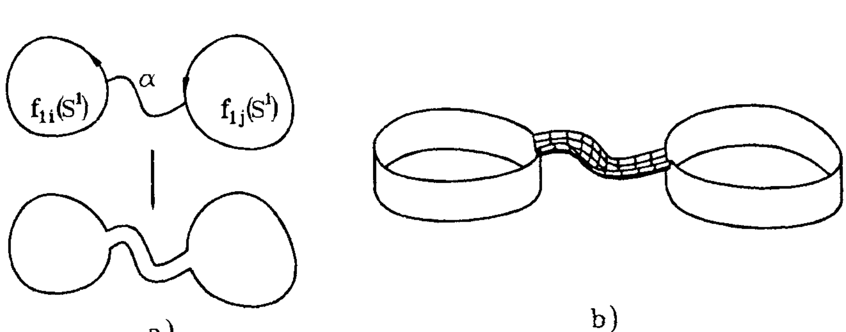



Fig. 6.

Repeat this transformation as long as we can find suitable circles to join. At the end we obtain an immersion

$$f_2 = f_{21} \amalg \ldots \amalg f_{2r} : S^1 \amalg \ldots \amalg S^1 \to S^2$$

whose image is a disjoint collection of embedded closed curves together with the untouched bow ties of $f_1$. It is clear that im $f_1 \times \{1/2\}$ and im$f_2 \times \{1\}$ are cobordant to one another in the category of immersed oriented manifolds without triple points (see Fig. 6.b).

The advantage of $f_2$ over $f_1$ is that there is a path from $P_i$ to $P_j$ crossing exactly $|l_{ij}|$ circles of $f_2$. Therefore the "geometric linking" of $P_i - P_j$ with im $f_2$ coincides with the algebraic one.

$3^{\text{rd}}$ *step.* The curves im $f \times \{0\}$ and im $f_2 \times \{1\}$ bound an immersed oriented surface in $S^2 \times [0,1]$ having no triple points. Representing the ball $B^3$ as the union of $S^2 \times [0,1]$ and a ball $B^3_*$ glued to $S^2 \times [0,1]$ along $S^2 \times \{1\}$ we close this surface within $B^3_*$ by gluing disjoint disks to the embedded components of $f_2$ and connecting the bow tie at $P_{2i-1}$ to the bow tie at $P_{2i}$ by a cylinder over the bow tie, the double line of which crosses the disks we have just glued in $l_{2i-1,2i}$ times (see Fig. 7).

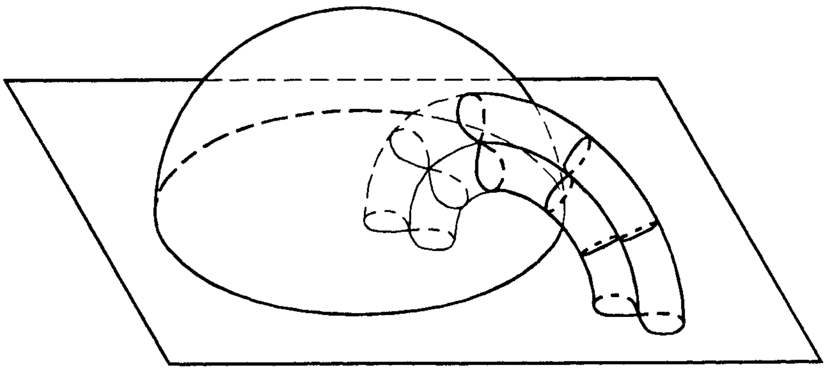


**Fig. 7.**

The surface we have obtained has exactly $\sum_{i=1}^{k} l_{2i-1,2i}$ triple points, thus

$$\tau(f) \leq \sum_{i=1}^{k} l_{2i-1,2i},$$

and, as a matter of fact, we have equality.

## Remarks and examples

It would be interesting to find the smallest genus $g$ a punctured surface $M_g$ can have if it admits an immersion in $D^3$ with the minimal number of triple points (and bounding a given immersed curve $f : S^1 \looparrowright S^2$).

In other words, we are asking for

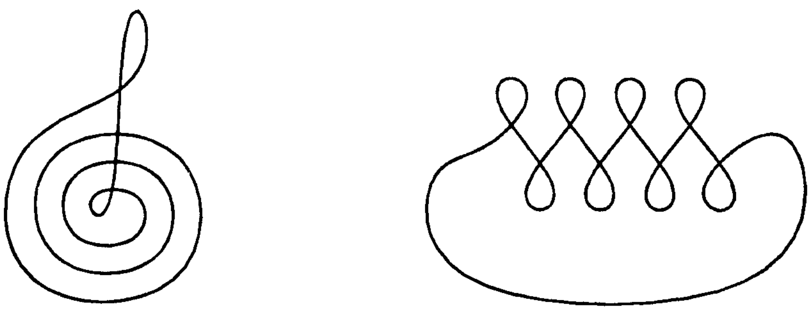
$$g_0(f) = \min\{g | \tau_g(f) = \tau(f)\}.$$

Unfortunately, our procedure certainly does not provide this minimal genus. As an example one can consider the curve with Gauss word $aa^{-1}bb^{-1}$. Our procedure gives a genus 3 surface for it while obviously a disc would suit.

Concerning the surface $M_g$ arising from our procedure one can show the following estimates: $k \le g \le 3k$.

The "treble clef" shown in Fig. 8.a is an immersed curve with $2k$ double points, for which $\tau = k$. Thus, the inequality of Corollary 4 is the best possible. ($k = 2$ in the figure.)

The immersed curve described in Fig. 8.b is a member of a sequence of curves with $4k$ double points ($k = 2$ in the figure) for which $\tau_0 = k$ but $\tau = 0$.



a) b)

**Fig. 8.**

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