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A conjecture on 3-anti-quasi-transitive digraphs[☆]

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ABSTRACT

A digraph D is a 3-anti-quasi-transitive digraph, if for any four distinct vertices $x_0, x_1, x_2, x_3 \in V(D)$ such that $x_0 \to x_1 \leftarrow x_2 \to x_3$, x_0 and x_3 are adjacent. Bang-Jensen conjectured (Bang-Jensen, 2004) that a 3-anti-quasi-transitive digraph is Hamiltonian if and only if it is strong and has a cycle factor. In this paper, we shall prove that this conjecture is true.

1. Terminology and introduction

We only consider finite digraphs without loops and multiple arcs. Let D be a digraph with a vertex set $V(D)$ and an arc set $A(D)$. For any $x, y \in V(D)$, we will write $x \to y$ if $xy \in A(D)$, and also, we will write $\overline{xy}$ if x and y are adjacent. A digraph H is called a *subdigraph* of D, if $V(H) \subseteq V(D)$ and $A(H) \subseteq A(D)$. If $V(H) = V(D)$, we say that H is a *spanning subdigraph* of D. The subdigraph induced by a subset S of $V(D)$ is denoted by $D[S]$. By $D - S$ we denote the digraph $D[V(D) - S]$. If $S = \{x\}$ is a single vertex, we write $D - x$ instead of $D - \{x\}$. For disjoint subsets X and Y of $V(D)$ or subdigraphs of D, $X \to Y$ means that every vertex of X dominates every vertex of Y and $X \Rightarrow Y$ means that there is no arc from Y to X. For a pair X, Y of vertex sets of D, define $(X, Y) = \{xy \in A(D) : x \in X, y \in Y\}$. Let D' and D'' be two vertex-disjoint subdigraphs of D. We say that D' and D'' are adjacent if some vertex of D' and some vertex of D'' are adjacent.

A *path* is a finite sequence of distinct vertices $P = x_0x_1 \cdots x_n$ such that $x_{i-1} \to x_i$ for every $1 \le i \le n$ and its length is n. A digraph D is said to be strongly connected or just strong, if for every pair x, y of vertices of D, there is an (x, y)-path.

A *cycle* is a finite sequence of distinct vertices $C = x_0x_1 \cdots x_nx_0$ such that $x_{i-1} \to x_i$ for every $1 \le i \le n$ and $x_n \to x_0$, whose length is $n + 1$. We denote the subpath of C from x_i to x_j by $C[x_i, x_j] = x_ix_{i+1} \cdots x_j$. A *k-cycle factor* (or a cycle factor) of D is a spanning subdigraph D' of D that consists of k vertex-disjoint cycles. A cycle of D with order $|V(D)|$ is called a Hamiltonian cycle and D is called a *Hamiltonian digraph*. A path is *anti-directed* if the orientation of each arc on the path is opposite to that of its predecessor. An anti-directed path of order $k + 1$ is called a k anti-directed path. The concepts not defined here we refer the reader to [3].

A digraph is *arc-locally in-semicomplete (arc-locally out-semicomplete)*, if for any four vertices x, y, z, w such that $x \to y \to z \leftarrow w$ ($x \leftarrow y \to z \to w$), x and w are adjacent. A digraph is *arc-locally semicomplete*, if it is arc-locally in-semicomplete and arc-locally out-semicomplete. Arc-locally semicomplete (in-semicomplete) digraphs have been studied by several authors (see [1,2,4,6–8]). A digraph is 3-*quasi-transitive*, if for any four vertices x, y, z, w such that $x \to y \to z \to w$, x and w are adjacent. 3-quasi-transitive digraphs have been studied by several authors (see [5,8,9]). A digraph is 3-*anti-quasi-transitive*, if for any four vertices x, y, z, w such that $x \to y \leftarrow z \to w$, x and w are adjacent. The results on 3-anti-quasi-transitive digraphs are still very few. In particular, H. Galeana-Sánchez and R. Gómez in [6] proved that there exists an

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independent set meeting every longest path in 3-anti-quasi-transitive digraphs. In [2], Bang-Jensen proposed the following conjecture on the Hamiltonicity of 3-anti-quasi-transitive digraphs.

Conjecture 1.1. *A 3-anti-quasi-transitive digraph is Hamiltonian if and only if it is strong and has a cycle factor.*

In Section 2, we will prove that the conjecture is true.

2. Main result

We start with the following several useful lemmas.

Lemma 2.1. *Let D be a 3-anti-quasi-transitive digraph, $C_1 = x_0x_1\cdots x_{m-1}x_0$ and $C_2 = y_0y_1\cdots y_{n-1}y_0$ be two vertex-disjoint cycles of D. Suppose that D has no cycles with the vertex set $V(C_1)\cup V(C_2)$. For any $x_i \in V(C_1)$ and $y_j \in V(C_2)$, if $x_i \to y_j$, then $x_{i+k} \to y_{j-k}$ and $x_{i-k} \to y_{j+k}$, for any integer k, where all the subscripts of x_i are taken modulo m and all the subscripts of y_i are taken modulo n.*

Proof. If $x_i \to y_j$, then since $y_{j-1} \to y_j \leftarrow x_i \to x_{i+1}$ and D is a 3-anti-quasi-transitive digraph, we have that $\overline{x_{i+1}y_{j-1}}$. If $y_{j-1} \to x_{i+1}$, then $y_{j-1}C_1[x_{i+1}, x_i]C_2[y_j, y_{j-1}]$ is a cycle with the vertex set $V(C_1)\cup V(C_2)$, a contradiction. Hence $x_{i+1} \to y_{j-1}$. A similar argument can be applied to show that $x_{i+k} \to y_{j-k}$, for any integer k. In particular, for any integer k, $x_{i+(am-k)} \to y_{j-(am-k)}$, where $ma = nb$. It is easy to see that $x_{i-k} \to y_{j+k}$. □

The following useful fact is an easy consequence of Lemma 2.1.

Corollary 2.2. *Let D be a 3-anti-quasi-transitive digraph, $C_1 = x_0x_1\cdots x_{m-1}x_0$ and $C_2 = y_0y_1\cdots y_{n-1}y_0$ be two vertex-disjoint cycles of D. Suppose that D has no cycles with the vertex set $V(C_1)\cup V(C_2)$. For any $x_i \in V(C_1)$ and $y_j \in V(C_2)$, if $\overline{x_iy_j}$, then $\overline{x_{i+k}y_{j-k}}$ and $\overline{x_{i-k}y_{j+k}}$, for any integer k, where all the subscripts of x_i are taken modulo m and all the subscripts of y_i are taken modulo n.*

Lemma 2.1 also implies the following (below gcd means the greatest common divisor. For example, $\gcd(12, 8) = 4$).

Lemma 2.3. *Let D be a 3-anti-quasi-transitive digraph, $C_1 = x_0x_1\cdots x_{m-1}x_0$ and $C_2 = y_0y_1\cdots y_{n-1}y_0$ be two vertex-disjoint cycles of D. Suppose that D has no cycles with the vertex set $V(C_1)\cup V(C_2)$. For any $x_i \in V(C_1)$ and $y_j \in V(C_2)$, if $x_i \to y_j$, then $x_i \to y_{j+d}$ and $x_i \to y_{j-d}$, where $d = \gcd(m, n)$ and all the subscripts of x_i are taken modulo m and all the subscripts of y_i are taken modulo n.*

Proof. For convenience, without loss of generality, assume that $i = 0$ and $j = 0$. From $x_0 \to y_0$ and Lemma 2.1, we conclude that $x_0 \to y_{km}$, for the integer $k \geq 0$. Let $W = \{km \in Z_n \mid k \in Z\}$. It is easy to show that $W = \{km \in Z_n \mid k \in Z\} = \{kd \mid k \in \{0, 1, \ldots, \frac{n}{d} - 1\}\}$. Therefore, the lemma yields. □

Lemma 2.4. *Let D be a strong 3-anti-quasi-transitive digraph containing a cycle factor $C_1 \cup C_2$. Let $C_1 = x_0x_1\cdots x_{m-1}x_0$ and $C_2 = y_0y_1\cdots y_{n-1}y_0$. Suppose that D has no Hamiltonian cycles. For any $x_i \in V(C_1)$ and $y_j \in V(C_2)$, if $\overline{x_iy_j}$, then x_i is not adjacent to y_{j+1} and y_{j-1}, where all the subscripts of x_i are taken modulo m and all the subscripts of y_i are taken modulo n.*

Proof. By symmetry, we, without loss of generality, assume that $x_i \to y_j$ and for convenience, assume that $i = 0$ and $j = n-1$. We prove the lemma by contradiction. If $y_{n-2} \to x_0$, then by Lemma 2.1, $y_{n-2-k} \to x_k$, for any $k \in Z$. Since $x_0 \to y_{n-1}$, Lemma 2.1 implies $x_k \to y_{n-1-k}$ for any $k \in Z$.

If $m \geq n$, then

$$x_0y_{n-1}C_1[x_{n-1}, x_{m-1}]y_0x_{n-2}y_1x_{n-3}\cdots y_lx_{n-2-l}\cdots y_{n-3}x_1y_{n-2}x_0$$

is a Hamiltonian cycle of D, a contradiction.

If $m < n$, then

$$C_2[y_0, y_{n-m-1}]x_{m-1}y_{n-m}x_{m-2}y_{n-m+1}\cdots x_ky_{n-1-k}\cdots x_1y_{n-2}x_0y_{n-1}y_0$$

is a Hamiltonian cycle of D, a contradiction.

If $y_0 \to x_0$, then by Lemma 2.1, $y_{n-1} \to x_1$. Note that $y_{n-1} \to x_1$ and $x_0 \to y_{n-1}$. Similar to the above argument, we can get a contradiction.

If $x_0 \to y_{n-2}$, then by Lemma 2.1, we have $x_1 \to y_{n-2}$. By $x_1 \to y_{n-2} \leftarrow x_0 \to y_{n-1}$, we have $\overline{x_1y_{n-1}}$. From this with the above argument, we have $x_1 \to y_{n-1}$, which also implies $x_0 \to y_0$ using Lemma 2.1. Repeating this way around the cycle C_2, we can obtain that $x_0 \to V(C_2)$. Since D is strong, using Lemma 2.1, there exists $y_j \in V(C_2)$ such that $y_j \to x_1$. Then $y_jC_1[x_1, x_0]C_2[y_{j+1}, y_j]$ is a Hamiltonian cycle of D, a contradiction. Similarly, we can conclude that x_0 does not dominate y_0. □

Lemma 2.5. *Let D be a strong 3-anti-quasi-transitive digraph containing a cycle factor $C_1 \cup C_2$. Then D is a Hamiltonian digraph.*

Proof. Let $C_1 = x_0x_1\cdots x_{m-1}x_0$ and $C_2 = y_0y_1\cdots y_{n-1}y_0$. From now on, all the subscripts of x_i are taken modulo m and all the subscripts of y_i are taken modulo n.

Suppose, on the contrary, that D is not a Hamiltonian digraph. Since D is strong, $(C_1, C_2) \neq \emptyset$ and $(C_2, C_1) \neq \emptyset$. This with Lemma 2.1 implies that, for any $x_i \in V(C_1)$, $(x_i, C_2) \neq \emptyset$ and $(C_2, x_i) \neq \emptyset$. In particular, $(x_0, C_2) \neq \emptyset$ and $(C_2, x_0) \neq \emptyset$.

Assume, without loss of generality, that $x_0 \to y_{n-1}$. This with Lemma 2.1 implies that $x_{m-1} \to y_0$. If $y_{n-1} \to x_0$, then $y_{n-1}C_1[x_0, x_{m-1}]C_2[y_0, y_{n-1}]$ is a Hamiltonian cycle, a contradiction. Hence $x_0 \mapsto y_{n-1}$. We may assume that for any $x_p \in V(C_1)$ and $y_q \in V(C_2)$, if $x_p \to y_q$, then $x_p \mapsto y_q$. Let $d = \gcd(m, n)$. Applying $x_0 \to y_{n-1}$ to Lemma 2.3, we obtain that $x_0 \to y_{n-1-d}$, $x_0 \to y_{n-1-2d}, \dots, x_0 \to y_{n-1-(\frac{n}{d}-1)d}$. By $(C_2, x_0) \neq \emptyset$ and Lemma 2.3, there must exist some vertex $y_{n-i} \in V(C_2)$ with $n-d \le n-i \le n-2$ such that $y_{n-i} \to x_0$. Without loss of generality, assume that $(n-1)-(n-i) = i-1 = \min\{k | x_0 \to y_s, y_{s-k} \to x_0\}$. By the choice of i, we have the following.

($\star$) x_0 and every vertex of $C_2[y_{n-i+1}, y_{n-2}]$ are not adjacent.

From this with Lemma 2.4, we next assume that $n-d+1 \le n-i \le n-3$, that is $3 \le i \le d-1$.

Next we first give a claim.

Claim 1. *There exists no vertex $x_j \in V(C_1)$ such that $x_j \to y_{n-i+1}$ and $y_{n-1} \to x_{j+1}$, where $0 \le j \le m-2$.*

Suppose, on the contrary, that there exists a vertex $x_j \in V(C_1)$ such that $x_j \to y_{n-i+1}$ and $y_{n-1} \to x_{j+1}$, where $0 \le j \le m-2$. By $x_0 \to y_{n-1}$ and Lemma 2.1, we have $x_{m-1} \to y_0$. Note that

$$y_{n-i}C_1[x_0, x_j]C_2[y_{n-i+1}, y_{n-1}]C_1[x_{j+1}, x_{m-1}]C_2[y_0, y_{n-i}]$$

is a Hamiltonian cycle of D, a contradiction. The proof of Claim 1 is complete.

By $y_{n-i} \to x_0$ and Lemma 2.1, $y_{n-i-1} \to x_1$. Since $y_{n-i-1} \to x_1 \leftarrow x_0 \to y_{n-1}$ and D is a 3-anti-quasi-transitive digraph, we have

$$\overline{y_{n-i-1}y_{n-1}}. \tag{1}$$

To complete the proof, it suffices to consider the following three cases.

Case 1. $n-i = n-3$.

By (1), we have $\overline{y_{n-4}y_{n-1}}$. According to ($\star$), x_0 and y_{n-2} are not adjacent.

Claim 2. *There exists no vertex $x_j \in V(C_1)$ such that $x_j \to y_{n-1}$ and $y_{n-1} \to x_{j+2}$.*

Suppose, on the contrary, that there exists $x_j \in V(C_1)$ such that $x_j \to y_{n-1}$ and $y_{n-1} \to x_{j+2}$. By $x_j \to y_{n-1}$ and Lemma 2.1, $x_{j+1} \to y_{n-2}$, which is a contradiction to Claim 1. The proof of Claim 2 is complete.

Subcase 1.1. $y_{n-1} \to y_{n-4}$.

Claim 3. *If there exists a vertex $x_j \in V(C_1)$ such that $x_j \to y_{n-1}$ and $x_{j+2} \to y_{n-1}$, then $x_{j+4} \to y_{n-1}$.*

By $x_j \to y_{n-1}$ and Lemma 2.1, $x_{j+3} \to y_{n-4}$. By $y_{n-1} \to y_{n-4} \leftarrow x_{j+3} \to x_{j+4}$, we have $\overline{y_{n-1}x_{j+4}}$. Combining this with $x_{j+2} \to y_{n-1}$ and Claim 2, $x_{j+4} \to y_{n-1}$. The proof of Claim 3 is complete.

By $y_{n-3} \to x_0$ and Lemma 2.1, $y_{n-5} \to x_2$. Then $y_{n-1} \to y_{n-4} \leftarrow y_{n-5} \to x_2$ implies $\overline{y_{n-1}x_2}$ and $x_2 \to y_{n-1}$ from $x_0 \to y_{n-1}$ and Claim 2. By Claim 3, we have $x_4 \to y_{n-1}$. Continuing in this way, we can obtain that $x_{2i} \to y_{n-1}$ for $i = 0, 1, \dots$. If m is even, then $x_{m-2} \to y_{n-1}$. Combining this with Lemma 2.1, we have $x_0 \to y_{n-3}$, a contradiction to the fact that $y_{n-3} \to x_0$. If m is odd, then $x_{m-1} \to y_{n-1}$. Combining this with Lemma 2.1, we have $x_0 \to y_{n-2}$, a contradiction to the fact that x_0 and y_{n-2} are not adjacent.

Subcase 1.2. $y_{n-4} \to y_{n-1}$.

Similarly to Claim 2, we can obtain the following claim.

Claim 4. *There exists no vertex $x_j \in V(C_1)$ such that $y_{n-2} \to x_j$ and $x_j \to y_{n-4}$.*

Suppose, on the contrary, that there exists $x_j \in V(C_1)$ such that $y_{n-2} \to x_j$ and $x_j \to y_{n-4}$. Combining this with Lemma 2.1, we have $x_{j-3} \to y_{n-1} \to x_{j-1}$, a contradiction to Claim 2. The proof of Claim 4 is complete.

By $y_{n-3} \to x_0$ and Lemma 2.1, we have $y_{n-2} \to x_{m-1}$. Then $y_{n-4} \to y_{n-1} \leftarrow y_{n-2} \to x_{m-1}$ implies $\overline{y_{n-4}x_{m-1}}$. By Claim 4, $y_{n-4} \to x_{m-1}$. Combining this with Lemma 2.1, $y_{n-2} \to x_{m-3}$. Repeating this procedure results in $y_{n-2} \to x_1$ or $y_{n-2} \to x_0$ depending on the parity of m. By ($\star$), $y_{n-2} \to x_0$ is a contradiction. If $y_{n-2} \to x_1$, then by Lemma 2.1, $y_{n-1} \to x_0$, which is also a contradiction.

Case 2. $n-i = n-d+1$.

By the above argument, we may assume that $n-3 > n-d+1$, that is $d > 4$. Hence x_0 and y_{n-3} are not adjacent. By $y_{n-d+1} \to x_0$ and Lemma 2.1, we have that $y_{n-d-2} \to x_3$. Then $x_0 \to y_{n-d-1} \leftarrow y_{n-d-2} \to x_3$ implies $\overline{x_0x_3}$.

If $x_0 \to x_3$, then by $x_0 \to x_3 \leftarrow x_2 \to y_{n-3}$, we have $\overline{x_0y_{n-3}}$, a contradiction to the fact that x_0 and y_{n-3} are not adjacent.

If $x_3 \to x_0$, then by $x_3 \to x_0 \leftarrow x_{m-1} \to y_0$, we have $\overline{x_3y_0}$. Combining this with Corollary 2.2, we have $\overline{x_0y_3}$. This together with Lemma 2.3 and Corollary 2.2, we have $\overline{x_0y_{n-d+3}}$, which is a contradiction to ($\star$), because $n-1 > n-d+3 > n-d+1$.

Case 3. $n-d+2 \le n-i \le n-4$.

In this case, $4 \le i \le d-2$. By (1), we have $\overline{y_{n-1}y_{n-i-1}}$.

Subcase 3.1. $y_{n-1} \to y_{n-i-1}$.

By $y_{n-i} \to x_0$ and Lemma 2.1, we have that $y_{n-i-2} \to x_2$. Then $y_{n-1} \to y_{n-i-1} \leftarrow y_{n-i-2} \to x_2$ implies $\overline{y_{n-1}x_2}$.

If $x_2 \to y_{n-1}$, then by $x_0 \to y_{n-1} \leftarrow x_2 \to y_{n-3}$, we have $\overline{x_0y_{n-3}}$, a contradiction to ($\star$).

Next assume that $y_{n-1} \to x_2$. By Lemma 2.1, we have $y_{n+1} \to x_0$. This together with Lemma 2.3 implies that $y_{n-d+1} \to x_0$ and so $y_{n-d-2} \to x_3$. By $x_0 \to y_{n-d-1} \leftarrow y_{n-d-2} \to x_3$, we have $\overline{x_0x_3}$.

If $x_0 \to x_3$, then by $x_0 \to x_3 \leftarrow x_2 \to y_{n-3}$, we have $\overline{x_0y_{n-3}}$, a contradiction to ($\star$).

If $x_3 \to x_0$, then by $x_3 \to x_0 \leftarrow y_{n-i} \to y_{n-i+1}$, we have $\overline{x_3y_{n-i+1}}$. This together with Corollary 2.2 implies that $\overline{x_0y_{n-i+4}}$. Since x_0 and every vertex of $C_2[y_{n-i+1}, y_{n-2}]$ are not adjacent and $d \geq 6$, we have that $n-i+4 \geq n-1$, that is, $i \leq 5$. Recalling $i \geq 4$, it must be $i = 4$ or $i = 5$. Again since x_0 and y_0 are not adjacent, $i \neq 4$. Therefore $i = 5$. By $x_0 \to y_{n-1}$ and Lemma 2.1, we have $x_1 \to y_{n-2}$ and $x_3 \to y_{n-4}$. By $y_{n-1} \to x_2$ and Lemma 2.1, we have $y_{n-3} \to x_4$. Hence $y_{n-5}x_0x_1y_{n-2}y_{n-1}x_2x_3y_{n-4}y_{n-3}$ $x_4x_5 \cdots x_{m-1}y_0y_1 \cdots y_{n-5}$ is a Hamiltonian cycle, a contradiction.

Subcase 3.2. $y_{n-i-1} \to y_{n-1}$.

Claim 5. *For any $x_j \in V(C_1)$, if $y_{n-2} \to x_j$, then $y_{n-i-1} \to x_j$.*

By $y_{n-i-1} \to y_{n-1} \leftarrow y_{n-2} \to x_j$, we have $\overline{x_jy_{n-i-1}}$. If $x_j \to y_{n-i-1}$, then $x_{j-2} \to y_{n-i+1}$. By $y_{n-2} \to x_j$ and Lemma 2.1, we have $y_{n-1} \to x_{j-1}$. Then $x_{j-2} \to y_{n-i+1}$, $y_{n-1} \to x_{j-1}$ and Claim 1 implies a contradiction. Hence $y_{n-i-1} \to x_j$. The proof of Claim 5 is complete.

By $y_{n-i} \to x_0$ and Lemma 2.1, we have $y_{n-2} \to x_{m-i+2}$. By Lemma 2.3, we have $y_{n-2} \to x_{d-i+2}$. This together with Claim 5 implies that $y_{n-i-1} \to x_{d-i+2}$. From this with Lemma 2.1, we have $y_{n-(2i-d-1)} \to x_0$.

By ($\star$) and $x_0 \mapsto y_{n-1}$, we have $n-(2i-d-1) \leq n-i$ or $n-(2i-d-1) \geq n$, that is, $i \geq d+1$ or $2i-d-1 \leq 0$. Recall that $4 \leq i \leq d-2$. Hence $2i-d-1 \leq 0$. By Lemma 2.4, $2i-d-1 \neq 0$. Therefore $2i-d-1 < 0$, that is, $n-(2i-d-1) > n$. By $y_{n-i-1} \to x_{d-i+2}$ and Lemma 2.1, we have $y_{n-2} \to x_{d-i+2-(i-1)}$. By Claim 5, $y_{n-i-1} \to x_{d-i+2-(i-1)}$. Continuing in this way, we can get that, for any integer k, $y_{n-2} \to x_{(d-i+2)-k(i-1)}$ and $y_{n-i-1} \to x_{(d-i+2)-k(i-1)}$. Then there exists an integer k such that $y_{n-i-1} \to x_{(d-i+2)-k(i-1)}$, where $(d-i+2)-k(i-1) \geq 0$ and $(d-i+2)-(k+1)(i-1) < 0$.

Since x_0 and y_{n-2} are not adjacent, $(d-i+2)-k(i-1) \neq 0$. If $(d-i+2)-k(i-1) = 1$, then $y_{n-2} \to x_1$. From this with Lemma 2.1, we have $y_{n-1} \to x_0$, a contradiction. If $(d-i+2)-k(i-1) \geq 2$, then by $y_{n-i-1} \to x_{(d-i+2)-k(i-1)}$ and Lemma 2.1, $y_{(n-i-1)+(d-i+2)-k(i-1)} \to x_0$. Since $(d-i+2)-(k+1)(i-1) < 0$, we have $(d-i+2)-k(i-1) < i-1$. Hence, $n-i+1 = (n-i-1)+2 \leq (n-i-1)+(d-i+2)-k(i-1) < (n-i-1)+(i-1) = n-2$. Combining this with $y_{(n-i)+(d-i+2)-k(i-1)} \to x_0$, we get a contradiction to ($\star$). $\square$

The following is our main result.

Proof of Conjecture 1.1. The necessity is clear. Next we prove the sufficiency. Let $F = C_1 \cup C_2 \cup \cdots \cup C_t$ be a cycle factor. We may assume that F is chosen, such that t is minimum. If $t = 1$, then D is Hamiltonian. If $t = 2$, then by Lemma 2.5, D is Hamiltonian. Next assume that $t \geq 3$. If there exist two cycles C_i and C_j such that $(C_i, C_j) \neq \emptyset$ and $(C_j, C_i) \neq \emptyset$, then $D[V(C_i) \cup V(C_j)]$ is a strong 3-anti-quasi-transitive digraph and has a cycle factor $C_i \cup C_j$. By Lemma 2.5, D contains a cycle Z with the vertex set $V(C_i) \cup V(C_j)$, a contradiction with the choice of F. Hence we assume that for any two cycles C_i and C_j of F, $C_i \Rightarrow C_j$ or $C_j \Rightarrow C_i$.

Define a digraph $T(F)$ as follows: $\{C_1, C_2, \ldots, C_t\}$ forms the vertex set of $T(F)$ and $C_i \to C_j$ in $T(F)$ if and only if there exist arcs from C_i to C_j in D. Clearly, $T(F)$ has no cycles of length 2.

First we give a claim.

Claim A. *For any three distinct vertices $C_i, C_j, C_k \in V(T(F))$, if $C_i \to C_j \to C_k$ or $C_i \leftarrow C_j \to C_k$, then $\overline{C_iC_k}$ in $T(F)$.*

Let $C_i = x_0x_1 \cdots x_{n-1}x_0$, $C_j = y_0y_1 \cdots y_{m-1}y_0$ and $C_k = z_0z_1 \cdots z_{s-1}z_0$. If $C_i \to C_j \to C_k$ in $T(F)$, by the definition of $T(F)$, we, without loss of generality, assume that $x_0 \to y_0 \to z_0$. By the minimality of t and Lemma 2.1, we have $y_{m-1} \to z_1$. Then $x_0 \to y_0 \leftarrow y_{m-1} \to z_1$ implies that $\overline{x_0z_1}$ in D, that is $\overline{C_iC_k}$ in $T(F)$. If $C_i \leftarrow C_j \to C_k$ in $T(F)$, we, without loss of generality, assume that $x_0 \leftarrow y_0 \to z_0$. Then $x_{n-1} \to x_0 \leftarrow y_0 \to z_0$ implies that $\overline{x_{n-1}z_0}$ in D, that is, $\overline{C_iC_k}$ in $T(F)$. The proof of Claim A is complete.

Next, we show that $T(F)$ is a tournament. We only need to show that, for any $C_i, C_j \in V(T(F))$, C_i and C_j are adjacent. Let $P = Y_0Y_1 \cdots Y_{p-1}$ be a shortest path from C_i to C_j, where $Y_0 = C_i$, $Y_{p-1} = C_j$ and $p \geq 2$. It clearly holds for $n = 2$. If $n = 3$, then by Claim A, $\overline{Y_0Y_2}$ in $T(F)$. Now assume that $n \geq 4$. By Claim A, $\overline{Y_0Y_2}$ in $T(F)$ and $Y_2 \to Y_0$ from the minimality of P. By $Y_0 \leftarrow Y_2 \to Y_3$ and Claim A, we have $\overline{Y_3Y_0}$ and so $Y_3 \to Y_0$. As above, we can get that $Y_{p-1} \to Y_0$. Hence C_i and C_j are adjacent and so $T(F)$ is a tournament. Since D is strong, $T(F)$ is also strong. It is well known that in any strong tournament, there exists a 3-cycle. Hence there exist three vertices C_i, C_j, C_k in $T(F)$ such that $C_i \to C_j \to C_k \to C_i$, that is, there exist three cycles C_i, C_j and C_k in D such that $(C_i, C_j) \neq \emptyset$, $(C_j, C_k) \neq \emptyset$ and $(C_k, C_i) \neq \emptyset$. Let $C_i = x_0x_1 \cdots x_{m-1}x_0$, $C_j = y_0y_1 \cdots y_{n-1}y_0$ and $C_k = z_0z_1 \cdots z_{s-1}z_0$. Assume, without loss of generality, that $x_0 \to y_0$ and $y_{n-1} \to z_0$. By $x_0 \to y_0 \leftarrow y_{n-1} \to z_0$, we have $\overline{x_0z_0}$ and $z_0 \to x_0$ as $C_k \Rightarrow C_i$. From this with Lemma 2.1, we have $\overline{x_1z_{s-1}}$ and so $z_{s-1} \to x_1$. Then $x_0C_j[y_0, y_{n-1}]C_k[z_0, z_{s-1}]$ $C_i[x_1, x_{m-1}]x_0$ is a cycle with the vertex set $V(C_i) \cup V(C_j) \cup V(C_k)$, a contradiction with the choice of F. $\square$

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