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Synchronization in an array of linearly coupled networks with time-varying delay ☆

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Abstract

This paper studies the dynamics of a system of linearly coupled identical connected neural networks with time-varying delay. Some sufficient conditions for synchronization of such a system are obtained based on Lyapunov functional method and matrix inequality techniques, which can be checked numerically very efficiently by using the Matlab toolbox. Finally, an example is provided to demonstrate the effectiveness of the proposed results.

Keywords: Linearly coupled identical connected neural networks; Lyapunov functional; Synchronization; Time-varying delay; Matrix inequality; Neural networks

1. Introduction

In the past few years, the dynamical properties of artificial neural networks with or without time delays have been extensively investigated [1–6,29–31], many applications have been found in different areas, such as their extensive implementations in signal processing, moving image processing, optimization, speed detection of moving subjects [7,8], and other areas [9,10]. The introduction of the delays into neural networks is a necessary step for practical design and application of neural networks. A single time delay $\tau \geqslant 0$ was first introduced into the neural networks by Marcus and Westervelt [2]. However, in practice, delays in many electronic neural networks usually vary violently with time due to the finite switching speed of amplifiers and faults in the electrical circuit. So it is necessary to study the neural networks with variable delays [11].

As mentioned above, most of the previous studies have mainly concentrated on the stability analysis and periodic oscillations of this kind of networks [12,13,29–31]. However, synchronization in coupled dynamical has attracted much more attention of researchers in different areas. In application, there are many types of synchronization such as chaotic synchronization, phase synchronization, cluster synchronization, generalized

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synchronization and so on (see Refs. [14,15,32,33]). Specially, synchronization in chaotic system has been extensively investigated due to its potential in various fields including image processing [16], general complex network [17], biological system [18], secure communication [19]. In addition, arrays of coupled systems can show many interesting phenomena, such as auto waves, spiral waves and so on. Therefore, it is very important to study the synchronization of coupled neural networks for practical design and applications of delayed neural networks.

In many cases in the literature the authors have investigated the synchronization of linearly coupled networks assuming the coupling configuration matrix being irreducible, symmetric and satisfied the diffusive coupling connection, and provided local synchronization analysis via linearization techniques (see Refs. [20–22]). This method and these conditions are conservative and difficult to check. In Ref. [23], the authors defined a nonnegative real-valued function that measures the distance between the various cells, and gave sufficient conditions for an array of linearly coupled system to synchronize. In Ref. [24], a sufficient condition for global asymptotical synchronization of coupled identical delayed neural networks was given based on Hermitian matrix theory and Lyapunov functional method. A sufficient condition was derived for global exponential synchronization of coupled connected neural networks with *constant delays* using Lyapunov functional method and linear matrix inequality (LMI) technique in [25].

In this paper, motivated by Refs. [25,23], we will investigate some synchronization phenomena in a system consisting of N isolated linearly coupled identical connected neural networks with *time-varying delay*. In particular, we analyze when individual system in the array are synchronized in strong sense, i.e., each system's trajectory tracks the trajectories of all other system in the array. The conditions for the global exponential synchronization can be expressed in terms of matrix and linear matrix inequalities, which can be performed efficiently via algorithm such as the interior point algorithms [26]. It is shown that global exponential synchronization of coupled systems is ensured by a suitable design of the coupling matrix and the inner linking matrix. For this purpose, the theoretical results will be illustrated by computer simulations.

We shall consider an isolated neural network with *time-varying delay* described by the following state equation:

$$\frac{\mathrm{d}x_k(t)}{\mathrm{d}t} = -d_k x_k(t) + \sum_{l=1}^{n} a_{kl} g_l(x_l(t)) + \sum_{l=1}^{n} b_{kl} f_l(x_l(t-\tau(t))) + I_k(t), \quad k = 1, 2, \ldots, n \tag{1}$$

or

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -Dx(t) + Ag(x(t)) + Bf(x(t-\tau(t))) + I(t), \tag{2}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^{\mathrm{T}}$ is the state vector of the network at time t; $D = \mathrm{diag}\{d_1, d_2, \ldots, d_n\} > 0$ with $d_i > 0$ denotes the rate with which the cell i resets its potential to the resting state when isolated from other cells and inputs; $A = (a_{kl})_{n\times n}$, $B = (b_{kl})_{n\times n} \in \mathbb{R}^{n\times n}$ represent the connection weight matrix and the delayed connection weight matrix, respectively; a_{kl}, b_{kl} denote the strengths of connectivity between the cell k and l at time t and at time $t-\tau(t)$, respectively; f_i, g_j are activation functions, $g(x(t)) = [g_1(x_1(t)), \ldots, g_n(x_n(t))]^{\mathrm{T}}$, $f(x(t-\tau(t))) = [f_1(x_1(t-\tau(t))), \ldots, f_n(x_n(t-\tau(t)))]^{\mathrm{T}}$; $I(t) = [I_1(t), I_2(t), \ldots, I_n(t)]^{\mathrm{T}} \in \mathbb{R}^n$ is an external input vector and the variable time delay $\tau(t)$ is a bounded function, i.e., $0 \leqslant \tau(t) \leqslant \tau$.

We always assume system (1) or (2) satisfy the following initial conditions:

$$x_i(\theta) = \phi_i(\theta) \in C([-\tau, 0], \mathbb{R}), \quad \forall i = 1, 2, \ldots, n, \tag{3}$$

in which $\phi_i(\theta)$, $i = 1, 2, \ldots, n$, are continuous functions. By the theory of functional differential equations [27], for any $\phi \in C[-\tau, 0]$, system (1) or (2) has a unique solution $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^{\mathrm{T}}$ satisfying the initial condition (3).

Finally, a configuration of the isolated coupled delayed neural networks is formulated. We consider the dynamical system consisting of N linearly coupled identical delayed neural networks with each network being an n-dimensional dynamical system as (1) or (2). The state equation of this system is described by the following differential equations:

$$\begin{aligned}\frac{\mathrm{d}x_{i,k}(t)}{\mathrm{d}t} = & -d_k x_{i,k}(t) + \sum_{l=1}^{n} a_{kl} g_l(x_{i,l}(t)) + \sum_{l=1}^{n} b_{kl} f_l(x_{i,l}(t-\tau(t))) + I_i(t) \\ & + \sum_{j=1, j\neq i}^{N} c_{ij}\gamma_k (x_{j,k}(t) - x_{i,k}(t)), \quad i = 1,2,\ldots,N; \quad k = 1,2,\ldots,n \end{aligned} \tag{4}$$

or in vector form

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -Dx_i(t) + Ag(x_i(t)) + Bf(x_i(t-\tau(t)) + I(t) + \sum_{j=1, j\neq i}^{N} c_{ij}\Gamma(x_j(t) - x_i(t)), \quad i = 1,2,\ldots,N, \tag{5}$$

where $x_i(t) = [x_{i,1}(t), x_{i,2}(t), \ldots, x_{i,n}(t)]^{\mathrm{T}} \in \mathbb{R}^n$, $i = 1,2,\ldots,N$, is the state vector of the ith delayed neural network; $\Gamma = \mathrm{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \in \mathbb{R}^{n\times n}$ is a coupling link matrix between network i and network j $(i \neq j)$ for all $1 \leqslant i, j \leqslant N$; $C = (c_{ij})_{N\times N}$ is the coupling configuration matrix representing the coupling strength and the topological structure of the networks which satisfy the following conditions:

$$\begin{aligned} & c_{ij} \geqslant 0, \quad i \neq j, \\ & c_{ii} = -\sum_{j=1, j\neq i}^{N} c_{ij}. \end{aligned} \tag{6}$$

So system (4) or (5) can be rewritten as

$$\begin{aligned}\frac{\mathrm{d}x_{i,k}(t)}{\mathrm{d}t} = & -d_k x_{i,k}(t) + \sum_{l=1}^{n} a_{kl} g_l(x_{i,l}(t)) + \sum_{l=1}^{n} b_{kl} f_l(x_{i,l}(t-\tau(t))) + I_i(t) \\ & + \sum_{j=1}^{N} c_{ij}\gamma_k x_{j,k}(t), \quad i = 1,2,\ldots,N; \quad k = 1,2,\ldots,n, \end{aligned} \tag{7}$$

or

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -Dx_i(t) + Ag(x_i(t)) + Bf(x_i(t-\tau(t)) + I(t) + \sum_{j=1}^{N} c_{ij}\Gamma x_j(t), \quad i = 1,2,\ldots,N. \tag{8}$$

Throughout this paper, the activation functions $g_i(x)$, $f_j(x)$ and the delay $\tau(t)$ satisfy the following assumptions:

(A_1) The activation functions $f_j(x_j)$, $g_i(x_i)$, $(i,j = 1,2,\ldots,n)$ are Lipschitz continuous, that is, there exist constants $G_i > 0$, $F_j > 0$ such that

$$|g_i(x) - g_i(y)| \leqslant G_i|x-y|, \quad |f_j(x) - f_j(y)| \leqslant F_j|x-y|,$$

(A_2) The delay is a nonnegative, bounded and continuous differentiable function defined on $\mathbb{R}_+$ and $\dot{\tau}(t) \leqslant \eta < 1$.

Notations. Throughout this paper, the superscript "T" represents the transpose; $\|\cdot\|$ denotes the Euclidean norm on vectors in $\mathbb{R}^n$; for real symmetric matrix A, $A \geqslant 0$ $(A > 0)$ means that the matrix A is positive semi-definite (positive definite).

This paper is organized as follows: in Section 2, we introduce some related notations, definitions and lemmas which would be used later; in Section 3, several sufficient criteria are obtained for the global exponential synchronization of coupled system (7) or (8) with *time-varying delay*; in Section 4, an example is provided to demonstrate the effectiveness of the proposed results; finally, in Section 5, the concluding remarks are given.

2. Preliminaries

In this section, we give some definitions and lemmas which will be used later. Assume that each isolated network is a n-dimensional dynamical system described by (2) with initial condition (3). In the following we consider the synchronization of dynamical network (8).

Definition 1 (*Lu and Chen [25]*). The coupled system (7) or (8) is said to be *locally exponentially synchronized*, if for each $\varepsilon > 0$, there exist constants $\delta > 0$, $M > 0$, such that if $\sup_{s\in[-\tau,0]}\|\varphi_{i0}(s) - \varphi_{j0}(s)\| \leqslant \delta$, then for sufficient large $T > 0$, we have $\|x_i(t) - x_j(t)\| \leqslant M\mathrm{e}^{-\varepsilon t}$, for all $t > T$ and $i, j = 1, 2, \ldots, N$. And if for all $\varphi_{i0}(s) \in C([-\tau, 0], \mathbb{R}^n)$, there holds $\|x_i(t) - x_j(t)\| \leqslant M\mathrm{e}^{-\varepsilon t}$, we will say that system (8) is *globally exponentially synchronized*.

Obviously, the coupling condition (6) guarantees the synchronization state solution $x_1(t) = x_2(t) = \cdots = x_N(t)$ be a solution $s(t) \in \mathbb{R}^n$ of an individual delayed neural network (2), namely

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} = -Cs(t) + Ag(s(t)) + Bf(x(s(t - \tau(t)))) + I(t),$$

which $s(t)$ can be an equilibrium point, a periodic orbit, or a chaotic orbit. In the sequel, we only study the global exponential synchronization of system (8).

In order to obtain our main results, we use the following definitions introduced in Refs. [25,23].

Definition 2. Let $\hat{\mathbb{R}}$ denotes a ring, $\mathbf{M}_{n\times m}(\hat{\mathbb{R}})$ as the set of $n \times m$ matrix with entries in $\hat{\mathbb{R}}$. Define $T(\hat{\mathbb{R}}, B) =${matrices with entries in $\hat{\mathbb{R}}$ such that the sum of the entries in each row is equal to B for some $B \in \hat{\mathbb{R}}$}.

Definition 3. $\mathbf{M}_1^N(1)$ is composed of matrices with N columns, and each row of $\tilde{M} \in \mathbf{M}_1^N(1)$ contains zeros and exactly one entry α_i and one entry $-\alpha_i$, $\alpha_i \neq 0$.

Definition 4. $\mathbf{M}_1^N(n)$ and $\mathbf{M}_2^N(n)$: $\mathbf{M}_1^N(n) = \{\mathbf{M} = \tilde{M} \otimes I_n \mid \tilde{M} \in \mathbf{M}_1^N(1)\}$i.e., $\mathbf{M} \in \mathbf{M}_1^N(n)$ is obtained by replacing m_{ij} in $\tilde{M} \in \mathbf{M}_1^N(1)$ with $m_{ij}I_n$; $\mathbf{M}_2^N(n)$ are matrices $\mathbf{M}$ in $\mathbf{M}_1^N(n)$ such that for any pair of indexes i and j, there exist indexes $i_1, i_2, \ldots, i_l$ with $i_1 = i$ and $i_l = j$, and $p_1, p_2, \ldots, p_{l-1}$ such that $\mathbf{M}_{(p_q, i_q)} \neq 0$ and $\mathbf{M}_{(p_q, i_{q+1})} \neq 0$ for all $1 \leqslant q < l$.

In [23], the authors use a nonnegative real-valued function defined in the following to measure the distance between different networks.

Definition 5.

$$\mathrm{d}(x) = \|\mathbf{M}x\|^2 = x^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{M}x, \quad \mathbf{M} \in \mathbf{M}_2^N(n),$$

by virtue of the property of $\mathbf{M} \in \mathbf{M}_2^N(n)$, $\mathrm{d}(x) \to 0$ if and only if $\|x_i - x_j\| \to 0$ for all i and j, $1 \leqslant i, j \leqslant N$, so the dynamical system (8) achieve synchronization when $\mathrm{d}(x) \to 0$.

Specially, if we make

$$\mathbf{M}_1 = \begin{pmatrix} I_n & -I_n & & & \\ & I_n & -I_n & & \\ & & \ddots & & \\ & & & I_n & -I_n \end{pmatrix}_{(N-1)\times N},$$

which $\mathbf{M}_1 \in \mathbf{M}_2^N(n)$ is a $(N-1)n \times Nn$ real-valued matrix, and correspond to $d_1(x) = \sum_{i=1}^{N-1} \|x_i - x_{i+1}\|^2$.

For the proof of network synchronization, we need the following lemmas:

Lemma 1. *Given any vectors x, y of appropriate dimensions and a positive definite matrix $Q > 0$ with compatible dimensions, then the following inequality holds*:

$$2x^{\mathrm{T}}y \leqslant x^{\mathrm{T}}Qx + y^{\mathrm{T}}Q^{-1}y.$$

Lemma 2 (*Scular complement [28]*). *The following linear matrix inequality* (*LMI*):

$$\begin{pmatrix} Q(x) & S(x) \\ S^{\mathrm{T}}(x) & R(x) \end{pmatrix} < 0,$$

where $Q(x) = Q^{\mathrm{T}}(x)$, $R(x) = R^{\mathrm{T}}(x)$, *and* $S(x)$ *depend affinely on* x, *is equivalent to each of the following conditions*:

(1) $Q(x) < 0$, $R(x) - S^{\mathrm{T}}(x)Q^{-1}(x)S(x) < 0$,
(2) $R(x) < 0$, $Q(x) - S(x)R^{-1}(x)S^{\mathrm{T}}(x) < 0$.

Lemma 3 (*Wu and Chua [23]*). *Let* $\mathbf{D}$ *be a* $N \times N$ *matrix in* $T(\hat{\mathbb{R}}, K)$, *then the* $(N-1) \times (N-1)$ *matrix* $\mathbf{H}$ *defined by* $\mathbf{H} = \mathbf{MDG}$ *satisfies* $\mathbf{MD} = \mathbf{HM}$ *where* $\mathbf{M}$ *is the* $(N-1) \times N$ *matrix*

$$\mathbf{M} = \begin{pmatrix} \mathbf{1} & \mathbf{-1} & & & & \\ & \mathbf{1} & \mathbf{-1} & & & \\ & & & \ddots & & \\ & & & & \mathbf{1} & \mathbf{-1} \end{pmatrix}_{(N-1)\times N},$$

$\mathbf{G}$ *is* $N \times (N-1)$ *matrix*

$$\mathbf{G} = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} \\ & & \ddots & & \mathbf{1} \\ & & & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}_{N\times(N-1)},$$

and $\mathbf{1}$ *is the multiplicative identity of* $\hat{\mathbb{R}}$. *The matrix* $\mathbf{H}$ *can be written explicitly as* $\mathbf{H}_{(i,j)} = \sum_{k=1}^{j} [\mathbf{D}_{(i,k)} - \mathbf{D}_{(i+1,k)}]$ *for* i, $j \in \{1, 2, \ldots, N-1\}$. *The proof of* Lemma 2 *is similar to that given in* ([23, Lemma 1]).

Lemma 4 (*Wu and Chua [23]*). *A symmetric irreducible matrix* T *satisfies condition* (6) *if and only if there exists a* $p \times N$ *matrix* $M \in \mathbf{M}_2^N(1)$ *such that* $T = -M^{\mathrm{T}}M$.

3. Main results

In this section, we will use Lyapunov direct method to investigate the global synchronization of system (8). On the other hand, we use $\mathrm{d}(x) = \|\mathbf{M}x\|^2 = x^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{M}x$ to measure the distance between various networks and obtain several conditions for the synchronization of the dynamical system (8) by proving the distance $\mathrm{d}(x) \to 0$ with convergence rate ε.

Theorem 1. *Under assumptions* (A_1) *and* (A_2), *the coupling matrix* C *satisfies condition* (6). *The dynamical system* (8) *is globally exponentially synchronized, if there exist positive definite diagonal matrices* $P = \mathrm{diag}\{p_1, p_2, \ldots, p_n\} > 0$, $Q = \mathrm{diag}\{q_1, q_2, \ldots, q_n\} > 0$, $\Sigma = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\} > 0$, *a diagonal matrix* $\Delta = \mathrm{diag}\{\delta_1, \delta_2, \ldots, \delta_n\} \in \mathbb{R}^{n\times n}$ *and an irreducible symmetric matrix* $U = (u_{ij}) \in \mathbb{R}^{N\times N}$ *satisfying condition* (6), *such that*

$$\begin{pmatrix} -2P(D+\Delta) + FQF + G\Sigma G & PA & PB \\ A^{\mathrm{T}}P & -\Sigma & 0 \\ B^{\mathrm{T}}P & 0 & -(1-\dot{\tau}(t))Q \end{pmatrix} < 0, \tag{9}$$

and

$$\{U(\gamma_i C+\delta_i I_N)\}^s \geqslant 0, \tag{10}$$

hold for $i=1,2,\ldots,n$, *where* $G=\mathrm{diag}\{G_1,G_2,\ldots,G_n\}>0$, $F=\mathrm{diag}\{F_1,F_2,\ldots,F_n\}>0$, $A^s=\frac{1}{2}(A+A^{\mathrm{T}})$ *is the symmetric part of* A.

Proof. $U=(u_{ij})\in\mathbb{R}^{N\times N}$ is an irreducible symmetric matrix, so according to Lemma 4, there exists a $p\times N$ matrix $\tilde{M}\in\mathbf{M}_2^N(1)$ such that $U=-\tilde{M}^{\mathrm{T}}\tilde{M}$. Let $\mathbf{M}=\tilde{M}\otimes I_n$, then $\mathbf{M}\in\mathbf{M}_2^N(n)$ and let

$$\begin{aligned}
&\mathbf{D}=I_N\otimes D,\quad \mathbf{D}^1=I_p\otimes D,\quad \mathbf{A}=I_N\otimes A,\quad \mathbf{A}^1=I_p\otimes A,\\
&\mathbf{B}=I_N\otimes B,\quad \mathbf{B}^1=I_p\otimes B,\quad \varDelta=I_N\otimes\varDelta,\quad \varDelta^1=I_p\otimes\varDelta,\\
&\mathbf{P}=I_p\otimes P,\quad \mathbf{Q}=I_p\otimes Q,\quad \mathbf{C}=C\otimes\Gamma.
\end{aligned}$$

$$\begin{aligned}
&x_i(t)=[x_{i,1}(t),x_{i,2}(t),\ldots,x_{i,n}(t)]^{\mathrm{T}},\quad \forall i=1,2,\ldots,N,\\
&x(t)=[x_1^{\mathrm{T}}(t),x_2^{\mathrm{T}}(t),\ldots,x_N^{\mathrm{T}}(t)]^{\mathrm{T}},\\
&\mathbf{g}(x(t))=[g^{\mathrm{T}}(x_1(t)),g^{\mathrm{T}}(x_2(t)),\ldots,g^{\mathrm{T}}(x_N(t))]^{\mathrm{T}},\\
&\mathbf{f}(x(t-\tau(t)))=[f^{\mathrm{T}}(x_1(t-\tau(t))),f^{\mathrm{T}}(x_2(t-\tau(t))),\ldots,f^{\mathrm{T}}(x_N(t-\tau(t)))]^{\mathrm{T}},\\
&\mathbf{I}(t)=[I^{\mathrm{T}}(t),I^{\mathrm{T}}(t),\ldots,I^{\mathrm{T}}(t)]^{\mathrm{T}}.
\end{aligned}$$

Then Eq. (8) can be rewritten as

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t}=-(\mathbf{D}+\varDelta)x(t)+\mathbf{A}\mathbf{g}(x(t))+\mathbf{B}\mathbf{f}(x(t-\tau(t)))+\mathbf{I}(t)+(\mathbf{C}+\varDelta)x(t). \tag{11}$$

Let $y(t)=\mathbf{M}x(t)=[y_1^{\mathrm{T}}(t),y_2^{\mathrm{T}}(t),\ldots,y_p^{\mathrm{T}}(t)]^{\mathrm{T}}$, $y_i(t)=[y_{i,1}(t),y_{i,2}(t),\ldots,y_{i,n}(t)]^{\mathrm{T}}$, $i=1,2,\ldots,p$, where $y_i(t)$ is assumed to be $\alpha_i(x_{i_1}(t)-x_{i_2}(t))$ with $\alpha_i\neq 0$ and $\bar{x}_j(t)=[x_{1,j}(t),x_{2,j}(t),\ldots,x_{N,j}(t)]^{\mathrm{T}}$, then $\bar{y}_j(t)=\tilde{M}\bar{x}_j(t)$, for $j=1,2,\ldots,n$.

From Lemma 2, we know that condition (9) is equivalent to

$$-2P(D+\varDelta)+FQF+G\Sigma G+(1-\dot{\tau}(t))^{-1}PBQ^{-1}B^{\mathrm{T}}P+PA\Sigma^{-1}A^{\mathrm{T}}P<0,$$

then there exists a scalar $\varepsilon>0$ such that

$$-2P(D+\varDelta)+2\varepsilon P+FQF+G\Sigma G+(1-\dot{\tau}(t))^{-1}\mathrm{e}^{2\varepsilon\tau(t)}PBQ^{-1}B^{\mathrm{T}}P+PA\Sigma^{-1}A^{\mathrm{T}}P\leqslant 0. \tag{12}$$

We will use the following Lyapunov functional:

$$V(x(t))=\mathrm{e}^{2\varepsilon t}x^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{P}\mathbf{M}x(t)+\int_{t-\tau(t)}^{t}\mathbf{f}^{\mathrm{T}}(x(s))\mathbf{M}^{\mathrm{T}}\mathbf{Q}\mathbf{M}\mathbf{f}(x(s))\mathrm{e}^{2\varepsilon s}\,\mathrm{d}s.$$

Calculating the upper right Dini derivative of the functional $V(x(t))$ along the trajectories of (11)

$$\begin{aligned}
\frac{\mathrm{d}V(x(t))}{\mathrm{d}t}\Big|_{(11)}={}&2\varepsilon\mathrm{e}^{2\varepsilon t}x^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{P}\mathbf{M}x(t)+2\mathrm{e}^{2\varepsilon t}x^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{P}\mathbf{M}[-(\mathbf{D}+\varDelta)x(t)+\mathbf{A}\mathbf{g}(x(t))\\
&+\mathbf{B}\mathbf{f}x(t-\tau(t))+\mathbf{I}(t)]+2\mathrm{e}^{2\varepsilon t}x^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{P}\mathbf{M}(\mathbf{C}+\varDelta)x(t)\\
&+\mathrm{e}^{2\varepsilon t}\mathbf{f}^{\mathrm{T}}(x(t))\mathbf{M}^{\mathrm{T}}\mathbf{Q}\mathbf{M}\mathbf{f}(x(t))-(1-\dot{\tau}(t))\mathrm{e}^{2\varepsilon(t-\tau(t))}\mathbf{f}^{\mathrm{T}}(x(t-\tau(t)))\mathbf{M}^{\mathrm{T}}\mathbf{Q}\mathbf{M}\mathbf{f}(x(t-\tau(t))).
\end{aligned}$$

And by the structure of $\mathbf{M}$, the following equalities hold:

$$\begin{aligned}
&\mathbf{M}\mathbf{A}=\mathbf{A}^1\mathbf{M}\quad \mathbf{M}\mathbf{B}=\mathbf{B}^1\mathbf{M}\quad \mathbf{M}\mathbf{I}(t)=0,\\
&\mathbf{M}\varDelta=\varDelta^1\mathbf{M}\quad \mathbf{M}\mathbf{D}=\mathbf{D}^1\mathbf{M}.
\end{aligned}$$

So

$$
\begin{aligned}
\frac{\mathrm{d}V(x(t))}{\mathrm{d}t}\Big|_{(11)} = {} & 2\mathrm{e}^{2\varepsilon t}\{x^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{P}[-(\mathbf{D}^{1}+\mathbf{\Delta}^{1}-\varepsilon\mathbf{I})]\mathbf{M}x(t)+x^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{P}\mathbf{A}^{1}\mathbf{M}\mathbf{g}(x(t)) \\
& + x^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{P}\mathbf{B}^{1}\mathbf{M}\mathbf{f}(x(t-\tau(t)))\} + 2\mathrm{e}^{2\varepsilon t}x^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{P}\mathbf{M}(\mathbf{C}+\mathbf{\Delta})x(t) \\
& + \mathrm{e}^{2\varepsilon t}\mathbf{f}^{\mathrm{T}}(x(t))\mathbf{M}^{\mathrm{T}}\mathbf{Q}\mathbf{M}\mathbf{f}(x(t)-(1-\dot{\tau}(t))\mathrm{e}^{2\varepsilon(t-\tau(t))}\mathbf{f}^{\mathrm{T}}(x(t-\tau(t)))\mathbf{M}^{\mathrm{T}}\mathbf{Q}\mathbf{M}\mathbf{f}(x(t-\tau(t))).
\end{aligned}
$$

From Lemma 1, it easily follows that:

$$
\begin{aligned}
2x^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{P}\mathbf{A}^{1}\mathbf{M}\mathbf{g}(x(t)) &= 2\sum_{j=1}^{p}\alpha_j(x_{j_1}(t)-x_{j_2}(t))^{\mathrm{T}}PA\alpha_j[g(x_{j_1}(t))-g(x_{j_2}(t))] \\
&\leqslant \sum_{j=1}^{p}\alpha_j[g(x_{j_1}(t))-g(x_{j_2}(t))]^{\mathrm{T}}\Sigma\alpha_j[g(x_{j_1}(t))-g(x_{j_2}(t))]+\sum_{j=1}^{p}y_j^{\mathrm{T}}(t)PA\Sigma^{-1}A^{\mathrm{T}}Py_j(t) \\
&\leqslant \sum_{j=1}^{p}\alpha_j^2(x_{j_1}(t)-x_{j_2}(t))^{\mathrm{T}}G\Sigma G(x_{j_1}(t)-x_{j_2}(t))+\sum_{j=1}^{p}y_j^{\mathrm{T}}(t)PA\Sigma^{-1}A^{\mathrm{T}}Py_j(t) \\
&= \sum_{j=1}^{p}y_j^{\mathrm{T}}(t)[G\Sigma G+PA\Sigma^{-1}A^{\mathrm{T}}P]y_j(t), \qquad (13)
\end{aligned}
$$

It is noted that

$$
\begin{aligned}
& -(1-\dot{\tau}(t))e^{-2\varepsilon\tau(t)}\mathbf{f}^{\mathrm{T}}(x(t-\tau(t)))\mathbf{M}^{\mathrm{T}}\mathbf{Q}\mathbf{M}\mathbf{f}(x(t-\tau(t)))+2x^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{P}\mathbf{B}^{1}\mathbf{M}\mathbf{f}(x(t-\tau(t))) \\
&= 2\sum_{j=1}^{p}\alpha_j(x_{j_1}(t)-x_{j_2}(t))^{\mathrm{T}}PB\alpha_j[f(x_{j_1}(t-\tau(t)))-f(x_{j_2}(t-\tau(t)))]-(1-\dot{\tau}(t))e^{-2\varepsilon\tau(t)} \\
&\quad \times\sum_{j=1}^{p}\alpha_j^2[f(x_{j_1}(t-\tau(t)))-f(x_{j_2}(t-\tau(t)))]^{\mathrm{T}}Q[f(x_{j_1}(t-\tau(t)))-f(x_{j_2}(t-\tau(t)))] \\
&= \sum_{j=1}^{p}[-(1-\dot{\tau}(t))e^{-2\varepsilon\tau(t)}\varphi_j^{\mathrm{T}}(y_j(t-\tau(t)))Q\varphi_j(y_j(t-\tau(t)))+2y_j^{\mathrm{T}}(t)PB\varphi_j(y_j(t-\tau(t)))] \\
&= \sum_{j=1}^{p}\Big\{-[(1-\dot{\tau}(t))^{\frac{1}{2}}e^{-\varepsilon\tau(t)}Q^{\frac{1}{2}}\varphi_j(y_j(t-\tau(t)))-(1-\dot{\tau}(t))^{-\frac{1}{2}}e^{\varepsilon\tau(t)}Q^{-\frac{1}{2}}B^{\mathrm{T}}Py_j(t)]^{\mathrm{T}}\cdot[(1-\dot{\tau}(t))^{\frac{1}{2}} \\
&\quad e^{-\varepsilon\tau(t)}Q^{\frac{1}{2}}\varphi_j(y_j(t-\tau(t)))-(1-\dot{\tau}(t))^{-\frac{1}{2}}e^{\varepsilon\tau(t)}Q^{-\frac{1}{2}}B^{\mathrm{T}}Py_j(t)]+(1-\dot{\tau}(t))^{-1}e^{2\varepsilon\tau(t)} \\
&\quad y_j^{\mathrm{T}}(t)PBQ^{-1}B^{\mathrm{T}}Py_j(t)\Big\} \\
&\leqslant \sum_{j=1}^{p}(1-\dot{\tau}(t))^{-1}e^{2\varepsilon\tau(t)}y_j^{\mathrm{T}}(t)PBQ^{-1}B^{\mathrm{T}}Py_j(t) \qquad (14)
\end{aligned}
$$

where $\varphi_j(y_j(t-\tau(t)))=\alpha_j[f(x_{j1}(t-\tau(t)))-f(x_{j2}(t-\tau(t)))]$.

$$
\begin{aligned}
\mathbf{f}^{\mathrm{T}}(x(t))\mathbf{M}^{\mathrm{T}}\mathbf{M}\mathbf{Q}\mathbf{f}(x(t)) &= \sum_{j=1}^{p}\alpha_j^2[f(x_{j_1}(t))-f(x_{j_2}(t))]^{\mathrm{T}}Q[f(x_{j_1}(t))-f(x_{j_2}(t))] \\
&\leqslant \sum_{j=1}^{p}\alpha_j^2(x_{j_1}(t)-x_{j_2}(t))^{\mathrm{T}}FQF(x_{j_1}(t)-x_{j_2}(t)) \\
&= \sum_{j=1}^{p}y_j^{\mathrm{T}}(t)FQFy_j(t), \qquad (15)
\end{aligned}
$$

$$2x^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{P}[-(\mathbf{D}^1+\mathit{\Delta}^1-\varepsilon\mathbf{I})]\mathbf{M}x(t)=2\sum_{j=1}^{p}y_j^{\mathrm{T}}(t)P[-(D+\mathit{\Delta}-\varepsilon I)]y_j(t)$$
$$=\sum_{j=1}^{p}y_j^{\mathrm{T}}(t)[-2P(D+\mathit{\Delta})+2\varepsilon P]y_j(t). \quad (16)$$

So we can derive that

$$\frac{\mathrm{d}V(x(t))}{\mathrm{d}t}\Big|_{(11)}\leqslant \mathrm{e}^{2\varepsilon t}\sum_{j=1}^{p}y_j^{\mathrm{T}}(t)[-2P(D+\mathit{\Delta})+2\varepsilon P+FQF+G\Sigma G$$
$$+(1-\dot{\tau}(t))^{-1}\mathrm{e}^{2\varepsilon\tau(t)}PBQ^{-1}B^{\mathrm{T}}P+PA\Sigma^{-1}A^{\mathrm{T}}P]y_j(t)+2\mathrm{e}^{2\varepsilon t}x^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{PM}(\mathbf{C}+\mathit{\Delta})x(t)$$
$$=\mathrm{e}^{2\varepsilon t}\sum_{j=1}^{p}y_j^{\mathrm{T}}(t)\Omega y_j(t)+2\mathrm{e}^{2\varepsilon t}x^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{PM}(\mathbf{C}+\mathit{\Delta})x(t).$$

By condition (12) we have that

$$\Omega=-2P(D+\mathit{\Delta})+2\varepsilon P+FQF+G\Sigma G+(1-\dot{\tau}(t))^{-1}\mathrm{e}^{2\varepsilon\tau(t)}PBQ^{-1}B^{\mathrm{T}}P+PA\Sigma^{-1}A^{\mathrm{T}}P\leqslant 0.$$

And by condition (10), we have

$$x^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{PM}(\mathbf{C}+\mathit{\Delta})x(t)=\sum_{j=1}^{n}p_j\bar{x}_j^{\mathrm{T}}\tilde{M}^{\mathrm{T}}\tilde{M}(\gamma_jC+\delta_jI_N)\bar{x}_j$$
$$=-\sum_{j=1}^{n}p_j\bar{x}_j^{\mathrm{T}}U(\gamma_jC+\delta_jI_N)\bar{x}_j\leqslant 0.$$

So $\frac{\mathrm{d}V(x(t))}{\mathrm{d}t}|_{(11)}\leqslant 0$, and obtain $V(t)\leqslant V(0)$, namely $V(t)$ is a bounded function and $\mathrm{e}^{2\varepsilon t}x^{\mathrm{T}}(t)\mathbf{M}^{\mathrm{T}}\mathbf{PM}x(t)$ is also bounded, we can get $\|y(t)\|=O(\mathrm{e}^{-\varepsilon t})$, this completes the proof. □

In the special situation that $\tau(t)=\tau=$ constant, so $\dot{\tau}(t)=0$, we can derive the following criterion:

Corollary 1. *Under assumptions* (A_1) *and* (A_2), *the coupling matrix C satisfies condition* (6). *Then the dynamical system* (8) *with constant delay τ is globally exponentially synchronized if there exist positive definite diagonal matrices* $P=\mathrm{diag}\{p_1,p_2,\dots,p_n\}>0$, $Q=\mathrm{diag}\{q_1,q_2,\dots,q_n\}>0$, $\Sigma=\mathrm{diag}\{\sigma_1,\sigma_2,\dots,\sigma_n\}>0$, *a diagonal matrix* $\mathit{\Delta}=\mathrm{diag}\{\delta_1,\delta_2,\dots,\delta_n\}\in\mathbb{R}^{n\times n}$ *and an irreducible symmetric matrix* $U=(u_{ij})\in\mathbb{R}^{N\times N}$ *satisfying condition* (6), *such that*

$$\begin{pmatrix}-2P(D+\mathit{\Delta})+FQF+G\Sigma G & PA & PB\\ A^{\mathrm{T}}P & -\Sigma & 0\\ B^{\mathrm{T}}P & 0 & -Q\end{pmatrix}<0,$$

and

$$\{U(\gamma_iC+\delta_iI_N)\}^s\geqslant 0,$$

hold for $i=1,2,\dots,n$, *where* $G=\mathrm{diag}\{G_1,G_2,\dots,G_n\}>0$, $F=\mathrm{diag}\{F_1,F_2,\dots,F_n\}>0$. *The proof of Corollary* 1 *is the same as Theorem* 1, *and omitted here.*

If the coupling matrix C is symmetric and irreducible, *we also obtain the same Corollary as in* ([25, Corollary 1]).

Corollary 2. *Suppose that the coupling matrix C is symmetric*, *irreducible and satisfies condition* (6), *let the eigenvalues of C be ordered as* $0=\lambda_1>\lambda_2\geqslant\lambda_3\geqslant\cdots\geqslant\lambda_N$, *if there exists* δ_i,γ_i *satisfying the following conditions*:

$$\frac{\delta_i}{\gamma_i}\leqslant-\lambda_2,\quad \gamma_i>0,$$

$$\frac{\delta_i}{\gamma_i} \geqslant -\lambda_N, \quad \gamma_i < 0,$$

$$\delta_i \leqslant 0, \quad \gamma_i = 0, \tag{17}$$

then $C(C\gamma_i + \delta_i I_N) \geqslant 0$, *so condition* (10) *in Theorem* 1 *can be replaced by* (17), *and the proof of Corollary* 2 *is the same as* ([25, Corollary 1]) *and omitted here*.

In Corollary 2, let $P = I$, we can obtain the following result:

Corollary 3. *Under assumptions* (A_1) *and* (A_2), *and the coupling matrix C is symmetric*, *irreducible and satisfies condition* (6), *and let the eigenvalues of C be ordered as* $0 = \lambda_1 > \lambda_2 \geqslant \lambda_3 \geqslant \cdots \geqslant \lambda_N$. *Then the dynamical system* (8) *is globally exponentially synchronized if there exists scalar* δ_i, γ_i, *and positive definite diagonal matrices* $Q = \mathrm{diag}\{q_1, q_2, \ldots, q_n\} > 0$, $\Sigma = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\} > 0$, *a diagonal matrix* $\Delta = \mathrm{diag}\{\delta_1, \delta_2, \ldots, \delta_n\} \in \mathbb{R}^{n\times n}$, *such that the following LMI holds*:

$$\begin{pmatrix} -2(D+\Delta) + FQF + G\Sigma G & A & B \\ A^{\mathrm{T}} & -\Sigma & 0 \\ B^{\mathrm{T}} & 0 & -(1-\eta)Q \end{pmatrix} < 0, \tag{18}$$

and the following conditions are satisfied:

$$\frac{\delta_i}{\gamma_i} \leqslant -\lambda_2, \quad \gamma_i > 0,$$

$$\frac{\delta_i}{\gamma_i} \geqslant -\lambda_N, \quad \gamma_i < 0,$$

$$\delta_i \leqslant 0, \quad \gamma_i = 0, \tag{19}$$

where $G = \mathrm{diag}\{G_1, G_2, \ldots, G_n\} > 0$, $F = \mathrm{diag}\{F_1, F_2, \ldots, F_n\} > 0$.

Based on the theory of Refs. [25,23], we will use another method to deal with the coupling configuration matrix C in calculating the time derivative of Lyapouov functional along the trajectories of system (11). And obtain another new criterion.

Theorem 2. *Under assumptions* (A_1) *and* (A_2), *and the coupling matrix C satisfies condition* (6). *Then the dynamical system* (8) *is globally exponentially synchronized if there exist positive definite matrix* $P > 0$, *and positive definite diagonal matrices* $Q = \mathrm{diag}\{q_1, q_2, \ldots, q_n\} > 0$, $\Sigma = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\} > 0$, *a diagonal matrix* $\Delta = \mathrm{diag}\{\delta_1, \delta_2, \ldots, \delta_n\} \in \mathbb{R}^{n\times n}$ *such that*

$$\begin{pmatrix} -P(D+\Delta) - (D+\Delta)P + FQF + G\Sigma G & PA & PB \\ A^{\mathrm{T}}P & -\Sigma & 0 \\ B^{\mathrm{T}}P & 0 & -(1-\dot{\tau}(t))Q \end{pmatrix} < 0, \tag{20}$$

and satisfies the following condition:

$$\mathbf{PH} + \mathbf{H}^{\mathrm{T}}\mathbf{P} \leqslant 0, \tag{21}$$

where $G = \mathrm{diag}\{G_1, G_2, \ldots, G_n\} > 0$, $F = \mathrm{diag}\{F_1, F_2, \ldots, F_n\} > 0$, *and* $\mathbf{H}$ *and* $\mathbf{P}$ *are defined in the proof of Theorem*:

$$\mathbf{D} = I_N \otimes D, \quad \mathbf{D}_1 = I_{N-1} \otimes D, \quad \mathbf{A} = I_N \otimes A, \quad \mathbf{A}_1 = I_{N-1} \otimes A,$$
$$\mathbf{B} = I_N \otimes B, \quad \mathbf{B}_1 = I_{N-1} \otimes B, \quad \boldsymbol{\Delta} = I_N \otimes \Delta, \quad \boldsymbol{\Delta}_1 = I_{N-1} \otimes \Delta,$$
$$\mathbf{P} = I_{N-1} \otimes P, \quad \mathbf{Q} = I_{N-1} \otimes Q, \quad \mathbf{G} = I_{N-1} \otimes G, \quad \mathbf{F} = I_{N-1} \otimes F,$$
$$\mathbf{C} = C \otimes \Gamma, \quad \boldsymbol{\Sigma} = I_{N-1} \otimes \Sigma.$$

Proof. Let $\mathbf{M}_1 = \tilde{M} \otimes I_n$,

$$\tilde{M} = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & & \ddots & \\ & & & 1 & -1 \end{pmatrix}_{(N-1)\times N},$$

and $\tilde{M}$ is a $(N-1)\times N$ real-valued matrix, then $\mathrm{d}(x) = \sum_{i=1}^{N-1} \|x_i(t) - x_{i+1}(t)\|^2$.

Let $y(t) = \mathbf{M}_1 x(t) = [y_1^{\mathrm{T}}(t), y_2^{\mathrm{T}}(t), \ldots, y_{N-1}^{\mathrm{T}}(t)]^{\mathrm{T}}$, $y_i(t) = [y_{i,1}(t), y_{i,2}(t), \ldots, y_{i,n}(t)]^{\mathrm{T}}$, $i = 1, 2, \ldots, N-1$. By the structure of $\mathbf{M}_1$, we can get $y_i(t) = x_i(t) - x_{i+1}(t)$, $i = 1, 2, \ldots, N-1$. From Lemma 2, condition (20) is equivalent to

$$-P(D+\Delta) - (D+\Delta)P + FQF + G\Sigma G + (1-\dot{\tau}(t))^{-1} PBQ^{-1}B^{\mathrm{T}}P + PA\Sigma^{-1}A^{\mathrm{T}}P < 0,$$

then there exists $\varepsilon > 0$ such that

$$\begin{aligned} &- P(D+\Delta) - (D+\Delta)P + 2\varepsilon P + FQF + G\Sigma G + (1-\dot{\tau}(t))^{-1} \mathrm{e}^{2\varepsilon\tau(t)} PBQ^{-1}B^{\mathrm{T}}P \\ &+ PA\Sigma^{-1}A^{\mathrm{T}}P \leqslant 0. \end{aligned} \tag{22}$$

We will use the following Lyapunov functional:

$$V(x(t)) = \mathrm{e}^{2\varepsilon t} x^{\mathrm{T}}(t)\mathbf{M}_1^{\mathrm{T}}\mathbf{P}\mathbf{M}_1 x(t) + \int_{t-\tau(t)}^{t} \mathbf{f}^{\mathrm{T}}(x(s))\mathbf{M}_1^{\mathrm{T}}\mathbf{Q}\mathbf{M}_1\mathbf{f}(x(s))\mathrm{e}^{2\varepsilon s}\,\mathrm{d}s.$$

Calculating the upper right Dini derivative of the functional $V(x(t))$ along the trajectories of (11)

$$\begin{aligned} \frac{\mathrm{d}V(x(t))}{\mathrm{d}t}\Big|_{(11)} = {} & 2\varepsilon \mathrm{e}^{2\varepsilon t} x^{\mathrm{T}}(t)\mathbf{M}_1^{\mathrm{T}}\mathbf{P}\mathbf{M}_1 x(t) + 2\mathrm{e}^{2\varepsilon t} x^{\mathrm{T}}(t)\mathbf{M}_1^{\mathrm{T}}\mathbf{P}\mathbf{M}_1[-(\mathbf{D}+\mathbf{\Delta})x(t) + \mathbf{A}\mathbf{g}(x(t)) \\ & + \mathbf{B}\mathbf{f}(x(t-\tau(t)) + \mathbf{I}(t)] + 2\mathrm{e}^{2\varepsilon t} x^{\mathrm{T}}(t)\mathbf{M}_1^{\mathrm{T}}\mathbf{P}\mathbf{M}_1(\mathbf{C}+\mathbf{\Delta})x(t) \\ & + \mathrm{e}^{2\varepsilon t}\mathbf{f}^{\mathrm{T}}(x(t))\mathbf{M}_1^{\mathrm{T}}\mathbf{Q}\mathbf{M}_1\mathbf{f}(x(t)) \\ & - (1-\dot{\tau}(t))\mathrm{e}^{2\varepsilon(t-\tau(t))}\mathbf{f}^{\mathrm{T}}(x(t-\tau(t)))\mathbf{M}_1^{\mathrm{T}}\mathbf{Q}\mathbf{M}_1\mathbf{f}(x(t-\tau(t))), \end{aligned}$$

and by the structure of $\mathbf{M}_1$, the following equalities hold:

$$\begin{aligned} &\mathbf{M}_1\mathbf{A} = \mathbf{A}_1\mathbf{M}_1 \quad \mathbf{M}_1\mathbf{B} = \mathbf{B}_1\mathbf{M}_1 \quad \mathbf{M}_1\mathbf{I}(t) = 0, \\ &\mathbf{M}_1\mathbf{\Delta} = \mathbf{\Delta}_1\mathbf{M}_1 \quad \mathbf{M}_1\mathbf{D} = \mathbf{D}_1\mathbf{M}_1, \end{aligned}$$

and

$$\mathbf{C}+\mathbf{\Delta} = C\otimes\Gamma + I_N\otimes\Delta = \begin{pmatrix} c_{11}\Gamma+\Delta & c_{12}\Gamma & \cdots & c_{1N}\Gamma \\ c_{21}\Gamma & c_{22}\Gamma+\Delta & \cdots & c_{2N}\Gamma \\ \vdots & \vdots & \ddots & \vdots \\ c_{N1}\Gamma & c_{N2}\Gamma & \cdots & c_{NN}\Gamma+\Delta \end{pmatrix}_{N\times N}.$$

So $\mathbf{C}+\mathbf{\Delta} \in T(\hat{\mathbb{R}}, \Delta)$, let $\mathbf{L} = \mathbf{C}+\mathbf{\Delta}$, by Lemma 3, there exists $\mathbf{H} = \mathbf{M}_1\mathbf{L}\mathbf{G}$ ($\mathbf{G}$ is defined as in Lemma 3 satisfying $\mathbf{M}_1\mathbf{L} = \mathbf{H}\mathbf{M}_1$ and $\mathbf{H}_{(i,j)} = \sum_{k=1}^{j}[\mathbf{L}_{(i,k)} - \mathbf{L}_{(i+1,k)}]$. Therefore

$$\begin{aligned} \frac{\mathrm{d}V(x(t))}{\mathrm{d}t}\Big|_{(11)} = {} & \mathrm{e}^{2\varepsilon t}\{x^{\mathrm{T}}(t)\mathbf{M}_1^{\mathrm{T}}[-\mathbf{P}(\mathbf{D}_1+\mathbf{\Delta}_1) - (\mathbf{D}_1+\mathbf{\Delta}_1)\mathbf{P} + 2\varepsilon\mathbf{P}]\mathbf{M}_1 x(t) \\ & + 2x^{\mathrm{T}}(t)\mathbf{M}_1^{\mathrm{T}}\mathbf{P}\mathbf{A}_1\mathbf{M}_1\mathbf{g}(x(t)) + 2x^{\mathrm{T}}(t)\mathbf{M}_1^{\mathrm{T}}\mathbf{P}\mathbf{B}_1\mathbf{M}_1\mathbf{f}(x(t-\tau(t)))\} \\ & + \mathrm{e}^{2\varepsilon t}x^{\mathrm{T}}(t)\mathbf{M}_1^{\mathrm{T}}(\mathbf{P}\mathbf{H}+\mathbf{H}^{\mathrm{T}}\mathbf{P})\mathbf{M}_1 x(t) + \mathrm{e}^{2\varepsilon t}\mathbf{f}^{\mathrm{T}}(x(t))\mathbf{M}_1^{\mathrm{T}}\mathbf{Q}\mathbf{M}_1\mathbf{f}(x(t) \\ & - (1-\dot{\tau}(t))\mathrm{e}^{2\varepsilon(t-\tau(t))}\mathbf{f}^{\mathrm{T}}(x(t-\tau(t)))\mathbf{M}_1^{\mathrm{T}}\mathbf{Q}\mathbf{M}_1\mathbf{f}(x(t-\tau(t))). \end{aligned}$$

Let $\phi(y(t)) = \mathbf{M}_1\mathbf{f}(x(t))$, $\phi(y(t-\tau(t))) = \mathbf{M}_1\mathbf{f}(x(t-\tau(t)))$, and $\varphi(y(t)) = \mathbf{M}_1\mathbf{g}(x(t))$,

$$\begin{aligned}\frac{\mathrm{d}V(x(t))}{\mathrm{d}t}|_{(11)} = \mathrm{e}^{2\varepsilon t}\{&y^{\mathrm{T}}(t)[-\mathbf{P}(\mathbf{D}_1+\Delta_1)-(\mathbf{D}_1+\Delta_1)\mathbf{P}+2\varepsilon\mathbf{P}+\mathbf{PH}+\mathbf{H}^{\mathrm{T}}\mathbf{P}]y(t)\\ &+2y^{\mathrm{T}}(t)\mathbf{PA}_1\varphi(y(t))+2y^{\mathrm{T}}(t)\mathbf{PB}_1\phi(y(t-\tau(t)))+\phi^{\mathrm{T}}(y(t))\mathbf{Q}\phi(y(t))\\ &-(1-\dot{\tau}(t))\mathrm{e}^{-2\varepsilon\tau(t)}\phi^{\mathrm{T}}(y(t-\tau(t)))\mathbf{Q}\phi(y(t-\tau(t)))\}.\end{aligned}$$

By Lemma 1

$$\begin{aligned}2y^{\mathrm{T}}(t)\mathbf{PA}_1\varphi(y(t)) &= y^{\mathrm{T}}(t)\mathbf{PA}_1\varphi(y(t))+\varphi^{\mathrm{T}}(y(t))\mathbf{A}_1^{\mathrm{T}}\mathbf{P}y(t)\\ &\leqslant y^{\mathrm{T}}(t)\mathbf{PA}_1\boldsymbol{\Sigma}^{-1}\mathbf{A}_1^{\mathrm{T}}\mathbf{P}y(t)+\varphi^{\mathrm{T}}(y(t))\boldsymbol{\Sigma}\varphi(y(t))\\ &\leqslant y^{\mathrm{T}}(t)[\mathbf{PA}_1\boldsymbol{\Sigma}^{-1}\mathbf{A}_1^{\mathrm{T}}\mathbf{P}+\mathbf{G}\boldsymbol{\Sigma}\mathbf{G}]y(t),\end{aligned}$$

$$\begin{aligned}&2y^{\mathrm{T}}(t)\mathbf{PB}_1\phi(y(t-\tau(t)))-(1-\dot{\tau}(t))\mathrm{e}^{-2\varepsilon\tau(t)}\phi^{\mathrm{T}}(y(t-\tau(t)))\mathbf{Q}\phi(y(t-\tau(t)))\\ &\quad=-[(1-\dot{\tau}(t))^{1/2}\mathrm{e}^{-\varepsilon\tau(t)}\mathbf{Q}^{1/2}\phi(y(t-\tau(t)))-(1-\dot{\tau}(t))^{-1/2}\mathrm{e}^{\varepsilon\tau(t)}\mathbf{Q}^{-1/2}\mathbf{B}_1^{\mathrm{T}}\mathbf{P}y(t)]^{\mathrm{T}}\cdot[(1-\dot{\tau}(t))^{1/2}\\ &\qquad\times\mathrm{e}^{-\varepsilon\tau(t)}\mathbf{Q}^{1/2}\phi(y(t-\tau(t)))-(1-\dot{\tau}(t))^{-1/2}\mathrm{e}^{\varepsilon\tau(t)}\mathbf{Q}^{-1/2}\mathbf{B}_1^{\mathrm{T}}\mathbf{P}y(t)]+(1-\dot{\tau}(t))^{-1}\mathrm{e}^{2\varepsilon\tau(t)}\\ &\qquad\times y^{\mathrm{T}}(t)\mathbf{PB}_1\mathbf{Q}^{-1}\mathbf{B}_1^{\mathrm{T}}\mathbf{P}y(t)\\ &\quad\leqslant(1-\dot{\tau}(t))^{-1}\mathrm{e}^{2\varepsilon\tau(t)}y^{\mathrm{T}}(t)\mathbf{PB}_1\mathbf{Q}^{-1}\mathbf{B}_1^{\mathrm{T}}\mathbf{P}y(t),\end{aligned}$$

$$\phi^{\mathrm{T}}(y(t))\mathbf{Q}\phi(y(t))\leqslant y^{\mathrm{T}}(t)\mathbf{FQF}y(t).$$

So we can derive from the above inequalities that

$$\begin{aligned}\frac{\mathrm{d}V(x(t))}{\mathrm{d}t}|_{(11)}&\leqslant\mathrm{e}^{2\varepsilon t}y^{\mathrm{T}}(t)\{-\mathbf{P}(\mathbf{D}_1+\Delta_1)-(\mathbf{D}_1+\Delta_1)\mathbf{P}+2\varepsilon\mathbf{P}+\mathbf{PA}_1\boldsymbol{\Sigma}^{-1}\mathbf{A}_1^{\mathrm{T}}\mathbf{P}\\ &\quad+\mathbf{G}\boldsymbol{\Sigma}\mathbf{G}+\mathbf{FQF}+(1-\dot{\tau}(t))^{-1}\mathrm{e}^{2\varepsilon\tau(t)}\mathbf{PB}_1\mathbf{Q}^{-1}\mathbf{B}_1^{\mathrm{T}}\mathbf{P}\}y(t)+\mathrm{e}^{2\varepsilon t}y^{\mathrm{T}}(t)(\mathbf{PH}+\mathbf{H}^{\mathrm{T}}\mathbf{P})y(t)\\ &=\mathrm{e}^{2\varepsilon t}\sum_{j=1}^{N-1}y_j^{\mathrm{T}}(t)\Lambda y_j(t)+\mathrm{e}^{2\varepsilon t}y^{\mathrm{T}}(t)(\mathbf{PH}+\mathbf{H}^{\mathrm{T}}\mathbf{P})y(t).\end{aligned}$$

Using Scular complement and condition (21) implies that

$$\begin{aligned}\Lambda = &-P(D+\Delta)-(D+\Delta)P+2\varepsilon P+FQF+G\Sigma G+(1-\dot{\tau}(t))^{-1}\mathrm{e}^{2\varepsilon\tau(t)}PBQ^{-1}B^{\mathrm{T}}P\\ &+PA\Sigma^{-1}A^{\mathrm{T}}P\leqslant 0.\end{aligned}$$

and combine with condition (22)

$$\frac{\mathrm{d}V(x(t))}{\mathrm{d}t}|_{(11)}\leqslant 0,$$

thus $\|y(t)\| = O(\mathrm{e}^{-\varepsilon t})$, this complete the proof. □

From the proof of Theorem 2, we can obtain the following sufficient condition:

Corollary 4. *Under assumptions* (A_1) *and* (A_2), *and the coupling matrix C satisfies condition* (6). *Then the dynamical system* (8) *is globally exponentially synchronized if there exist positive definite matrix* $\mathbf{P} = \{p_{ij}\}_{(N-1)n\times(N-1)n} > 0$, *and positive definite diagonal matrices* $\mathbf{Q} = \mathrm{diag}\{q_i\} > 0$, $\Sigma = \mathrm{diag}\{\sigma_i\} > 0$,

$i = 1, 2, \ldots, (N-1)n \times (N-1)n$ and a diagonal matrix $\Delta = \mathrm{diag}\{\delta_1, \delta_2, \ldots, \delta_n\} \in \mathbb{R}^{n\times n}$ such that

$$\begin{pmatrix} -\mathbf{P}(\mathbf{D}_1 + \Delta_1) - (\mathbf{D}_1 + \Delta_1)\mathbf{P} + \mathbf{FQF} + \mathbf{G\Sigma G} & \mathbf{PA}_1 & \mathbf{PB}_1 \\ \mathbf{A}_1^{\mathrm{T}}\mathbf{P} & -\mathbf{\Sigma} & \mathbf{0} \\ \mathbf{B}_1^{\mathrm{T}}\mathbf{P} & \mathbf{0} & -(1-\dot{\tau}(t))\mathbf{Q} \end{pmatrix} < 0,$$

and satisfy the following condition:

$$\mathbf{PH} + \mathbf{H}^{\mathrm{T}}\mathbf{P} \leqslant 0,$$

where $G = \mathrm{diag}\{G_1, G_2, \ldots, G_n\} > 0$, $F = \mathrm{diag}\{F_1, F_2, \ldots, F_n\} > 0$.

In Theorem 2, if we let $P = I$, we can obtain the following result:

Corollary 5. *Under assumptions* (A_1) *and* (A_2), *and the coupling matrix C satisfies condition* (6). *Then the dynamical system* (8) *is globally exponentially synchronized if there exist positive definite diagonal matrices $Q = \mathrm{diag}\{q_1, q_2, \ldots, q_n\} > 0$, $\Sigma = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\} > 0$, a diagonal matrix $\Delta = \mathrm{diag}\{\delta_1, \delta_2, \ldots, \delta_n\} \in \mathbb{R}^{n\times n}$ such that*

$$\begin{pmatrix} -2(D+\Delta) + FQF + G\Sigma G & A & B \\ A^{\mathrm{T}} & -\Sigma & 0 \\ B^{\mathrm{T}} & 0 & -(1-\eta)Q \end{pmatrix} < 0, \tag{23}$$

and satisfying the following condition:

$$\mathbf{H} + \mathbf{H}^{\mathrm{T}} \leqslant 0, \tag{24}$$

where $G = \mathrm{diag}\{G_1, G_2, \ldots, G_n\} > 0$, $F = \mathrm{diag}\{F_1, F_2, \ldots, F_n\} > 0$.

Remark. The method we used in this paper can discuss the synchronization of an array of the linearly coupled system with *multiple time-varying delays* in the future. For example

$$\begin{aligned} \frac{\mathrm{d}x_{i,k}(t)}{\mathrm{d}t} = & -d_k x_{i,k}(t) + \sum_{l=1}^{n} a_{kl} g_l(x_{i,l}(t)) + \sum_{l=1}^{n} b_{kl} f_l(x_{i,l}(t - \tau_{kl}(t))) + I_i(t) \\ & + \sum_{j=1}^{N} c_{ij}\gamma_k x_{j,k}(t), \quad i = 1, 2, \ldots, N; \quad k = 1, 2, \ldots, n \end{aligned}$$

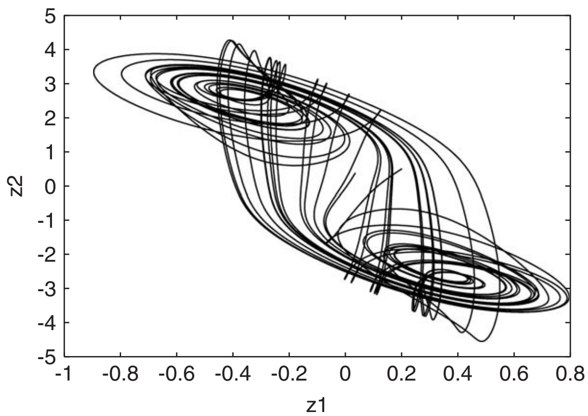

Fig. 1. Chaotic trajectory of the model (25).

or

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -Dx_i(t) + Ag(x_i(t)) + Bf(x_i(t - \tau_{kl}(t)) + I(t) + \sum_{j=1}^{N} c_{ij}\Gamma x_j(t), \quad i = 1, 2, \ldots, N.$$

4. Numerical examples

In this section, an example is given to show the effectiveness of the results obtained in the previous sections. Consider a two-dimensional neural network with *time-varying delay* of form (2), with

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 2.0 & -0.11 \\ -5.0 & 3.2 \end{pmatrix}, \quad B = \begin{pmatrix} -1.6 & -0.1 \\ -0.18 & -2.4 \end{pmatrix}.$$

$I(t) = (0, 0)^{\mathrm{T}}$ and $\tau(t) = \mathrm{e}^t/(1 + \mathrm{e}^t)$, so the state equations of the network are

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -Dx(t) + Ag(x(t)) + Bf(x(t - \tau(t))), \tag{25}$$

where $x(t) = [x_1(t), x_2(t)]^{\mathrm{T}}$ is the state vector of the network, and $f(x(t)) = g(x(t)) = [\tanh(x_1), \tanh(x_2)]^{\mathrm{T}}$, $0 < \tau(t) < 1$ and $\dot{\tau}(t) = \mathrm{e}^t/(1 + \mathrm{e}^t)^2 \leqslant \frac{1}{2} < 1$.

The initial condition associated with (25) is

$$x_1(s) = 0.2, \quad x_2(s) = 0.5, \quad \forall s \in [-1, 0]. \tag{26}$$

The model has a chaotic attractor as shown in Fig. 1.

Example. Consider a dynamical system consisting of three linearly coupled identical model (25), the state of the entire system is

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -Dx_i(t) + Ag(x_i(t)) + Bf(x_i(t - \tau(t))) + \sum_{j=1}^{3} c_{ij}\Gamma x_j(t), \quad i = 1, 2, 3, \tag{27}$$

where $x_i(t) = [x_{i,1}(t), x_{i,2}(t)]^{\mathrm{T}}$ are the state variables of the ith neural network, and the coupling matrix

$$C = \begin{pmatrix} -8 & 2 & 6 \\ 2 & -4 & 2 \\ 6 & 2 & -8 \end{pmatrix},$$

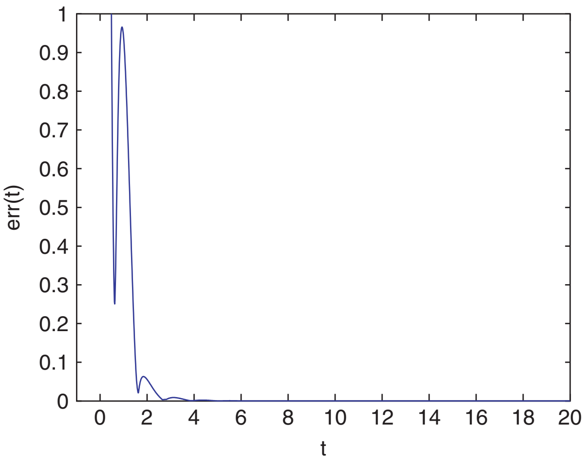

Fig. 2. Global synchronization of the coupled networks (27) with $\rho_1 = 1.1152$.

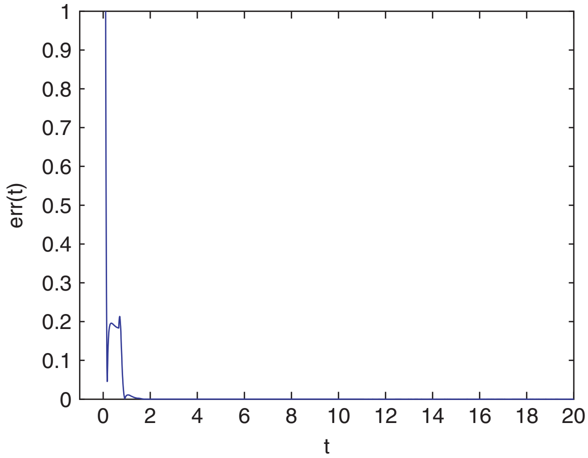

Fig. 3. Global synchronization of the coupled networks (27) with $\rho_1 = 5.0683$.

with the eigenvalues 0, -6 and -14. If the inner linking matrix $\Gamma = \rho_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then we apply Corollary 3, then the LMI (18) combined with (19) have the following feasible solutions:

$$\Delta = \begin{pmatrix} 4.4609 & 0 \\ 0 & 4.4609 \end{pmatrix}, \quad Q = \begin{pmatrix} 4.4609 & 0 \\ 0 & 4.4609 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 2.2305 & 0 \\ 0 & 2.2305 \end{pmatrix},$$

and $\rho_1 = 1.1152$, thus system (27) can be synchronized. The synchronization performance is illustrated by Fig. 2. Fig. 2 shows the distance among three trajectories by calculating

$$\mathrm{err}(t) = \sum_{i=1}^{2} \sqrt{[x_{1,i}(t) - x_{2,i}(t)]^2 + [x_{1,i}(t) - x_{3,i}(t)]^2}. \tag{28}$$

In the simulation, the initial conditions are

$$\begin{aligned} &x_{1,1}(s) = 0.2, \quad x_{1,2}(s) = 0.5, \quad \forall s \in [-1, 0], \\ &x_{2,1}(s) = 8.0, \quad x_{2,2}(s) = -9.0, \quad \forall s \in [-1, 0], \\ &x_{3,1}(s) = -6.0, \quad x_{3,2}(s) = 10.0, \quad \forall s \in [-1, 0]. \end{aligned}$$

We still consider the linearly coupled system (27) with the same initial conditions, using Corollary 5, LMI (23) combined with (24) have following feasible solutions:

$$\Delta = \begin{pmatrix} 25.6504 & 0 \\ 0 & 25.6504 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 17.8436 & 0 \\ 0 & 17.8436 \end{pmatrix}, \quad Q = \begin{pmatrix} 20.2252 & 0 \\ 0 & 20.2252 \end{pmatrix},$$

and $\rho_1 = 5.0638$, thus system (27) can be synchronized. The synchronization performance is illustrated by Fig. 3.

5. Conclusion

In this paper, the global exponential synchronization of an array of linearly coupled networks with *time-varying delay* have been obtained by using Lyapunov functional method and matrix inequality technique, the results obtained in this paper are given in term of the matrix inequalities. It is shown that the global synchronization of coupled delayed neural networks can be achieved by a suitable design of the coupling matrix and inner linking matrix, and the results can be applied to the delayed Hopfield neural networks. Finally, an illustrated example is provided to verify the effectiveness of the obtained results.

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