APPROXIMATION BY INFINITELY DIVISIBLE DISTRIBUTIONS IN THE MULTIDIMENSIONAL CASE

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Let $\mathfrak{A}$ be the collection of parallelepipeds in R^k with edges parallel with the coordinate axes and let $\mathfrak{B}$ be the collection of closed sets in R^k. Let $\pi(G,H)=\inf\{\varepsilon \mid G\{A\}\leqslant H\{A^\varepsilon\}+\varepsilon,\ H\{A\}\leqslant G\{A^\varepsilon\}+\varepsilon$ for any $A\in\mathfrak{B}\}$; $L(G,H)=\inf\{\varepsilon \mid G\{A\}\leqslant H\{A^\varepsilon\}+\varepsilon, H\{A\}\leqslant G\{A^\varepsilon\}+\varepsilon$ for any $A\in\mathfrak{A}\}$; $\rho(G,H)=\sup_{A\in\mathfrak{A}}|G\{A\}-H\{A\}|$, where G, H are distributions in R^k, $A^\varepsilon=\{x\in R^k \mid \inf_{y\in A}\|x-y\|<\varepsilon\}$. In the paper one gives the proofs of results announced earlier by the author (Dokl. Akad. Nauk SSSR, 253, No. 2, 277-279 (1980)). One considers the problem of the approximation of the distributions of sums of independent random vectors with the aid of infinitely divisible distributions. One obtains estimates for the distances $\pi(\cdot,\cdot), L(\cdot,\cdot)$, and $\rho(\cdot,\cdot)$. It is proved that

$$\rho\Big(\prod_{i=1}^{n}((1-p_i)E+p_iV_i),\prod_{i=1}^{n}\exp(p_i(V_i-E))\Big)\leqslant c\,k\sqrt{p\sum_{i=1}^{n}p_i^2},$$

where $0\leqslant p_i\leqslant 1$, $p=\max_{1\leqslant i\leqslant n}p_i$; E is the distribution concentrated at zero; V_i $(i=1,\ldots,n)$ are arbitrary distributions; the products and the exponentials are understood in the sense of convolution.

In this paper we investigate the accuracy of the approximation of distributions of sums of independent random vectors by infinitely divisible distributions. In the one-dimensional case this problem has been considered by Kolmogorov in [1]. Here we give the proofs of the results formulated (in a somewhat weaker and less general form) in [2].

We shall use the following notations and conventions. As the norm of the elements $x=(x_1,\ldots,x_k)\in R^k$ we take $\|x\|=\max_{1\leqslant j\leqslant k}|x_j|$. By virtue of the equivalence of the norms on R^k, all the subsequent estimates hold also for another choice of the norm, with a possible change of the constants. Assume, further, that $\mathfrak{A}_k$ is the collection of closed parallelepipeds in R^k with edges parallel to the coordinate axes; $\mathfrak{B}_k$ is the collection of Borel sets in R^k, $\mathfrak{F}_k$ is the set of k-dimensional probability distributions, $\mathfrak{D}_k$ is the set of the infinitely divisible distributions from $\mathfrak{F}_k$; E_a is the distribution concentrated at the point $a\in R^k$, $E=E_0$. For $G, H\in\mathfrak{F}_k$ we shall denote by $g(t), h(t)$ the corresponding characteristic functions; D_G, D_H are the covariance matrices; $G^{(j)}, H^{(j)}\in\mathfrak{F}_1$, $j=1,\ldots,k$, are the distributions of the j-coordinate of the vectors having distributions G and H;

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$$\pi(G,H)=\inf\{\varepsilon \mid G\{A\}\le H\{A^{\varepsilon}\}+\varepsilon,$$
$$H\{A\}\le G\{A^{\varepsilon}\}+\varepsilon \quad \text{for any} \quad A\in\mathfrak{B}_{K}\};$$
$$L(G,H)=\inf\{\varepsilon \mid G\{A\}\le H\{A^{\varepsilon}\}+\varepsilon,$$
$$H\{A\}\le G\{A^{\varepsilon}\}+\varepsilon \quad \text{for any} \quad A\in\mathfrak{A}_{K}\};$$
$$\rho_{var}(G,H)=\sup_{A\in\mathfrak{B}_K}|G\{A\}-H\{A\}|;\ \rho(G,H)=\sup_{A\in\mathfrak{A}_K}|G\{A\}-H\{A\}|.$$

Here A^{ε} is an ε -neighborhood of the set A in the sense of the norm $\|\cdot\|$, $\pi(\cdot,\cdot)$ is the Levy—Prokhorov metric, $\rho_{var}(\cdot,\cdot)$ is the variation distance. In the one-dimensional case the metric $\rho(\cdot,\cdot)$ is equivalent to the uniform distance between the distribution functions, while the metric $L(\cdot,\cdot)$ is equivalent to the Levy metric. We note, however, that there exists also other, possibly more natural generalizations of the uniform distance and of the Levy distance to the multidimensional case. We shall denote by $c(\cdot)$ positive constants (in general, different ones) depending only on the argument. The product and the power of measures will be understood in the sense of the convolution, in particular, we shall use the notation

$$\exp(\lambda(G-E))=e^{-\lambda}\sum_{s=0}^{\infty}\frac{\lambda^s}{s!}G^s \qquad (G^0=E),\ \lambda\ge 0,\ G\in\mathfrak{F}_K.$$

It is easy to see that $\exp(\lambda(G-E))\in\mathfrak{D}_K$. For $G\in\mathfrak{F}_1$ the concentration function is defined by the formula

$$Q(G,\lambda)=\sup_x G\{[x,x+\lambda]\},\ \lambda\ge 0.$$

Now we formulate the fundamental result.

THEOREM 1. Assume that the distributions $F_i\in\mathfrak{F}_K$, $i=1,\dots,n$, can be represented in the form $F_i=p_iV_i+(1-p_i)\mathcal{U}_i$, where $0\le p_i\le 1$; $\mathcal{U}_i, V_i\in\mathfrak{F}_K$ and the distributions $\mathcal{U}_i, V_i$ are concentrated on the sets

$$S_i=\{x\in R_K \mid x_j\in[\alpha_{ij},\alpha_{ij}+\tau_{ij}],\ j=1,\dots,K\}$$

and on the closures of the sets $R^K\setminus S_i$, respectively. Assume further, that

$$p=\max_{1\le i\le n}p_i;\qquad \tau_j=\max_{1\le i\le n}\tau_{ij};\qquad \tau=\max_{1\le j\le K}\tau_j;$$
$$a_i=(a_{i1},\dots,a_{iK})\in R^K, \text{where } a_{ij}=\int_{R^1}x\,\mathcal{U}_i^{(j)}\{dx\}$$
$$F=\prod_{i=1}^{n}F_i;\quad \mathfrak{D}=\prod_{i=1}^{n}(E_{-a_i}\exp(F_iE_{-a_i}-E)).$$

Then, we have the estimates

$$L(F,\mathfrak{D})\le c(K)\Big(\sqrt{p\sum_{i=1}^{n}p_i^2}+p(|\ln p|+1)^{K/4}+\tau^{2/3}(|\ln\tau|+1)^{\frac{K+8}{12}}\Big); \tag{1}$$
$$\pi(F,\mathfrak{D})\le c(K)\Big(\sum_{i=1}^{n}p_i^2+p(|\ln p|+1)^{K/4}+\tau^{2/3}(|\ln\tau|+1)^{\frac{K+8}{12}}\Big). \tag{2}$$

If $p_iV_i=p_1V_1$, $i=1,\dots,n$, then

$$\pi(F,\mathfrak{D})\le c(K)\big(p(|\ln p|+1)^{K/4}+\tau^{2/3}(|\ln\tau|+1)^{\frac{K+8}{12}}\big). \tag{3}$$

If instead of $p_iV_i = p_1V_1$, $i=1,\dots,n$, we assume that for the measures $p_i V_i^{(j)}$ one has the representation

$$p_i V_i^{(j)} = \gamma_{ij} p_{ij} V_{ij1} + (p_i - p_{ij}) V_{ij2} + (1-\gamma_{ij}) p_{ij} V_{ij3},$$

where $0 \le \gamma_{ij} \le 1$, $0 \le p_{ij} \le p_i$ and the distributions V_{ij1}, V_{ij2}, $V_{ij3} \in \mathfrak{F}_1$ are concentrated on the sets $(-\infty, \alpha_{ij}]$, $[\alpha_{ij}, \alpha_{ij}+\tau_{ij}]$ and $[\alpha_{ij}+\tau_{ij}, \infty)$, respectively, then

$$\rho(F,\mathfrak{D}) \le c(K)\Big(\sqrt{p\sum_{i=1}^{n} p_i^2} + p + \sum_{j=1}^{K}\Big(\sum_{i=1}^{n} \frac{\tau_{ij}^2}{\tau_j^2}\gamma_{ij}(1-\gamma_{ij})p_{ij}\Big)^{-1/2}\Big), \tag{4}$$

and there exist distributions $\mathfrak{D}_1, \mathfrak{D}_2 \in \mathfrak{D}_K$, such that

$$\rho(F,\mathfrak{D}_1) \le c(K)\Big(\frac{\sqrt{p}}{\gamma^{1/4}} + \sum_{j=1}^{K}\Big(\sum_{i=1}^{n}\frac{\tau_{ij}^2}{\tau_j^2}\gamma_{ij}(1-\gamma_{ij})p_{ij}\Big)^{-1/2}\Big), \tag{5}$$

$$\gamma = \min_{i,j} \frac{\tau_{ij}^2}{\tau_j^2}\gamma_{ij}(1-\gamma_{ij});$$

$$L(F,\mathfrak{D}_2) \le c(K)(\sqrt{p} + \tau^{2/3}(|\ln\tau|+1)^{\frac{K+2}{2K}}) \tag{6}$$

(if $\tau_j = 0$, then in (4), (5) one has to set $\tau_{ij}^2/\tau_j^2 = 1$, $i=1,\dots,K$).

<u>COROLLARY.</u> Let $L(F_i, E) \le \varepsilon$, $i=1,\dots,n$; $F=\prod_{i=1}^{n} F_i$. Then there exists a distribution $\mathfrak{D}_3 \in \mathfrak{D}_K$ such that $L(F,\mathfrak{D}_3) \le c(K)\varepsilon^{1/2}$.

In order to obtain this corrolary it is sufficient to note that from the conditions $L(F_i, E) \le \varepsilon$ there follows the conditions of Theorem 1 with parameters $p \le 2\varepsilon$, $\tau \le 2\varepsilon$.

<u>Remark 1.</u> In the one-dimensional case one has obtained essentially stronger estimates than Theorem 1 and its Corollary, since they are optimal with respect to the order (see [3]). It seems very likely that in the multidimensional case one must have estimates having the same orders with respect to p and τ as the corresponding unimprovable one-dimensional estimates.

<u>Remark 2.</u> The distribution $\mathfrak{D}$ is frequently called the accompanying infinitely divisible law for the distribution F.

The proof of Theorem 1 is carried out according to the scheme developed for the one-dimensional case in [1, 4-7].

First we obtain estimates (1)-(3). We introduce the auxiliary distributions

$$H_i = (1-p_i)\mathcal{U}_i E_{-a_i} + p_i E;\quad G_i = (1-p_i)E + p_i V_i E_{-a_i};$$

$$F^* = \prod_{i=1}^{n}(F_i E_{-a_i});\quad F' = \prod_{i=1}^{n}(G_i H_i);\quad F'' = \prod_{i=1}^{n}(G_i \exp(H_i - E));$$

$$\mathfrak{D}^* = \prod_{i=1}^{n}\exp(F_i E_{-a_i} - E) = \prod_{i=1}^{n}(\exp(H_i - E)\exp(G_i - E)).$$

It is easy to see that

$$R(F,\mathfrak{D}) = R(F^*,\mathfrak{D}^*), \tag{7}$$

where $R(\cdot,\cdot)$ is any of the three metrics: π, L, or ρ. In addition, by the triangle inequality, we have

$$R(F^*,\mathfrak{D}^*) \le R(F^*,F') + R(F',F'') + R(F'',\mathfrak{D}^*). \tag{8}$$

In the sequel we shall systematically make use of the property of weak regularity of these metrics (see [8]): for any $B_1, B_2, B_3 \in \mathfrak{F}_K$ we have

$$R(B_1 B_3, B_2 B_3) \leq R(B_1, B_2). \tag{9}$$

For the proof we shall also apply repeatedly the results of [9, 10].

Obviously, the distributions H_i are concentrated on the set $\{x \in R^K \mid \|x\| \leq 2\tau\}$ and have zero mean. Therefore, from Theorem 6 [10] there follows that

$$L(F^*, F') \leq \pi(F^*, F') \leq c(K)(p(|\ln p|+1)^{K/4} + \tau(|\ln \tau|+1)). \tag{10}$$

Then, from (9) and from Theorem 5 [10] there follows the estimate

$$L(F', F'') \leq \pi(F', F'') \leq \pi\left(\prod_{i=1}^{n} H_i, \prod_{i=1}^{n} \exp(H_i - E)\right) \leq c(K)\tau^{2/3}(|\ln \tau|+1)^{\frac{K+8}{12}}. \tag{11}$$

From Theorem 3 [9] we obtain

$$L(F'', \mathfrak{D}^*) \leq \rho(F'', \mathfrak{D}^*) \leq \rho\left(\prod_{i=1}^{n} G_i, \prod_{i=1}^{n} \exp(G_i - E)\right) \leq cK\sqrt{p\sum_{i=1}^{n} p_i^2}. \tag{12}$$

Theorem 3 has been given in [9] without proof. Since in the present paper we make use of it in an essential manner, we shall give its proof at the end of the paper. Theorem 1 from [6], obtained for the first time by A. Ya. Khinchin, leads us to the inequality

$$\pi(F'', \mathfrak{D}^*) \leq \rho_{var}(F'', \mathfrak{D}^*) \leq \rho_{var}\left(\prod_{i=1}^{n} G_i, \prod_{i=1}^{n} \exp(G_i - E)\right) \leq c\sum_{i=1}^{n} p_i^2. \tag{13}$$

If $p_i V_i = p_1 V_1$, $i = 1, \ldots, n$, then with the aid of an estimate due to Prokhorov [11], we obtain

$$\pi(F'', \mathfrak{D}^*) \leq \rho_{var}(F'', \mathfrak{D}^*) \leq \rho_{var}(G_1^n, \exp(n(G_1 - E))) \leq cp. \tag{14}$$

From (8), (10)-(14) there follow the inequalities (1)-(3).

We proceed to the proof of inequality (4). Without loss of generality, we shall assume that $p \leq 1/5$. First we establish estimate (4) in the case when the distributions F_i have continuous bounded densities which do not vanish anywhere. In this case we can assume that $\tau_j = \tau = 1/2\sqrt{K}$, since the metric $\rho(\cdot,\cdot)$ is invariant relative to the compression—extension transformations, applied to the corresponding distributions.

We denote by $H \in \mathfrak{F}_K$ the distribution with density $\varphi(x) = \prod_{j=1}^{K} \psi(x_j)$, where

$$\psi(x_j) = c(m)\alpha\left(\frac{\sin \alpha x_j}{\alpha x_j}\right)^{2m}, \tag{15}$$

where $m = [K/2] + 3$ ($[\cdot]$ is the integer part of the number), $\alpha = \frac{T}{2m}$, $T = 1/K$. It is easy to see that the characteristic function $h(t)$ vanishes outside the set $\{t \in R^K \mid \|t\| \leq T\}$.

Let Y be a unitary transformation of R^K such that $Y(\sum_{i=1}^{n} \mathfrak{D}_{u_i})Y^T$ is a diagonal matrix (T is the symbol for the transpose). Without loss of generality, we shall assume that its diagonal entries σ_j^2 are in nonincreasing order: $\sigma_1^2 \geq \sigma_2^2 \geq \ldots \geq \sigma_K^2 > 0$

We denote by W the distribution defined by the formula $W\{A\}=H\{YA\}$ for any $A\in\mathfrak{B}_K$ and we apply Lemma 2 from [12], according to which

$$(2W\{\{x\in R^K:\|x\|<h\}\}-1)\rho(F^*,\mathfrak{D}^*)\leqslant\rho(F^*W,\mathfrak{D}^*W)+W\{\{x\in R^K:\|x\|<h\}\}\sum_{j=1}^{K}Q(F^{*(j)},h). \tag{16}$$

We select $h=c(K)$ so that

$$2W\{\{x\in R^K:\|x\|<h\}\}-1\geqslant\frac{1}{2}. \tag{17}$$

Incidentally, we note that in [12] one has investigated the accuracy of the approximation of the distributions of sums of independent, identically distributed vectors with the aid of accompanying laws and for this case one has obtained estimates that are very close to (4). For the proof of inequalities (4)-(6) we make use of certain ideas from [12].

For the estimation of the concentration function we apply Theorem 5, Chap. III of [13], according to which we have

$$Q(F^{*(j)},h)\leqslant c(K)Q(F^{*(j)},\tau_j)\leqslant c(K)\Big(\sum_{i=1}^{n}\frac{\tau_{ij}^2}{\tau_j^2}\gamma_{ij}(1-\gamma_{ij})p_{ij}\Big)^{-1/2}. \tag{18}$$

In order to estimate $\rho(F^*W,\mathfrak{D}^*W)$, we apply the triangle inequality:

$$\rho(F^*W,\mathfrak{D}^*W)\leqslant\rho(F^*W,F'W)+\rho(F'W,F''W)+\rho(F''W,\mathfrak{D}^*W). \tag{19}$$

Let B be the transformation of R^K, defined by the diagonal matrix with the diagonal entries $b_{jj}=\sigma_j^{-1}$, $j=1,\ldots,K$. For an arbitrary distribution $\mathcal{U}\in\mathfrak{F}_K$ we shall denote by $\tilde{\mathcal{U}}$ the distribution defined by the formula

$$\tilde{\mathcal{U}}\{A\}=\mathcal{U}\{(BY)^{-1}A\},\quad A\in\mathfrak{B}_K.$$

We have selected the transformations Y and B in such a manner that $\sum_{i=1}^{n}\mathfrak{D}_{\tilde{\mathcal{U}}_i}$ be the unit diagonal matrix.

Now we make use of Lemma 5 [10], according to which

$$\rho(F^*W,F'W)\leqslant\rho_{var}(F^*W,F'W)=\rho_{var}(\tilde{F}^*\tilde{W},\tilde{F}'\tilde{W})\leqslant c(K)pm^{K/4}\leqslant c(K)p. \tag{20}$$

For the estimation of $\rho(F''W,D^*W)$ we make use of (9) and (12):

$$\rho(F''W,\mathfrak{D}^*W)\leqslant\rho(F'',\mathfrak{D}^*)\leqslant c\,K\sqrt{p\sum_{i=1}^{n}p_i^2}. \tag{21}$$

In order to estimate $\rho(F'W,F''W)$, we apply Theorem 1 [9], according to which

$$\begin{aligned}\rho(F'W,F''W)=\rho\Big(\big(W\prod_{i=1}^{n}H_i\big)\big(\prod_{i=1}^{n}G_i\big),\ \big(W\prod_{i=1}^{n}\exp(H_i-E)\big)\big(\prod_{i=1}^{n}G_i\big)\Big)\leqslant2\sum_{j=1}^{K}Q\Big(\prod_{i=1}^{n}G_i^{(j)},1\Big)\times\\ \times\Big\{K+\sum_{m=1}^{K}\int_0^\infty\sup_{A\in\mathfrak{B}_{m,x}}|P\{(\xi_1,\ldots,\xi_m)\in A\}-P\{(\eta_1,\ldots,\eta_m)\in A\}|dx\Big\},\end{aligned} \tag{22}$$

where $\xi=Y\xi'=(\xi_1,\ldots,\xi_K)$, $\eta=Y\eta'=(\eta_1,\ldots,\eta_K)$: the vectors ξ', η' have the distributions $W\sum_{i=1}^{n}H_i$ and $W\prod_{i=1}^{n}\exp(H_i-E)$, respectively, while

$$\mathfrak{B}_{m,x}=\{A\in\mathfrak{B}_m,\ A\subset\{y=(y_1,\ldots,y_m)\in R^m:|y_m|>x\}\}.$$

Obviously, $E\xi_j = E\eta_j = 0$, $\mathcal{D}\xi_j = \mathcal{D}\eta_j < \infty$, $j=1,\dots,K$. Now we prove that

$$\int_0^\infty \sup_{A\in\mathfrak{B}_{K,x}} |P\{(\xi_1,\dots,\xi_K)\in A\} - P\{(\eta_1,\dots,\eta_K)\in A\}|\,dx \leqslant C(K). \tag{23}$$

We consider separately two cases: a) $0<\sigma_K\leqslant 1$, b) $\sigma_K>1$.

Assume first that $0<\sigma_K\leqslant 1$ and $A\in\mathfrak{B}_{K,x}$, $x>0$. Then, by Chebyshev's inequality

$$|P\{\eta\in A\} - P\{\xi\in A\}| \leqslant P\{|\xi_K|>x\} + P\{|\eta_K|>x\} \leqslant 2\min\left\{1, \frac{E\eta_K^2}{x^2}\right\} \leqslant 2\min\left\{1, \frac{C(K)+\sigma_K^2}{x^2}\right\}.$$

From here we obtain (23) in the case a).

Assume now that $\sigma_K>1$. We note that

$$\sup_{A\in\mathfrak{B}_{K,x}} |P\{\xi\in A\} - P\{\eta\in A\}| = \sup_{A\in\mathfrak{B}_{K,y}} |(\tilde{W}\prod_{i=1}^n \tilde{H}_i)\{A\} - (\tilde{W}\prod_{i=1}^n \exp(\tilde{H}_i - E))\{A\}|, \tag{24}$$

where $y = x\sigma_K^{-1}$. Let $A\in\mathfrak{B}_{K,y}$. According to Lemma 1 of [14], we have

$$\begin{aligned} |(\tilde{W}\prod_{i=1}^n \tilde{H}_i)\{A\} - (\tilde{W}\prod_{i=1}^n \exp(\tilde{H}_i - E))\{A\}| \leqslant C(K)\Big\{\int_{\tilde{w}(t)\neq 0} |\tilde{w}(t)(\prod_{i=1}^n \tilde{h}_i(t) - \prod_{i=1}^n \exp(\tilde{h}_i(t)-1))|\,dt + \\ + \sup_\omega \int_{\tilde{w}(t)\neq 0} \left|\frac{\partial^{K+2}}{\partial t_\omega^{K+2}} (\tilde{w}(t)(\prod_{i=1}^n \tilde{h}_i(t) - \exp(\tilde{h}_i(t)-1))\right|dt\Big\}\frac{1}{1+y^2}, \end{aligned} \tag{25}$$

where $\frac{\partial^{K+2}}{\partial t_\omega^{K+2}}$ is the operator of $(K+2)$-fold partial differentiation in the direction ω, where $\omega\in R^K$ is a vector with unit Euclidean norm.

Performing similar calculations as those carried out for the proof of Theorem 5 [10], with the aid of (25) one can show that

$$|(\tilde{W}\prod_{i=1}^n \tilde{H}_i)\{A\} - (\tilde{W}\prod_{i=1}^n \exp(\tilde{H}_i - E))\{A\}| \leqslant \frac{C(K)}{(1+y^2)\sigma_K^2}. \tag{26}$$

From (24), (26) there follows (23) also in the case b).

Taking into account that $p<\frac{1}{5}<\frac{1}{2}$, $\tau_j = \frac{1}{2\sqrt{K}}$ and making use again of Theorem 5, Chap. III of [13], we obtain:

$$Q(\prod_{i=1}^n G_i^{(j)}, 1) \leqslant C(K)\, Q(\prod_{i=1}^n G_i^{(j)}, \tau_j) \leqslant C(K)\Big(\sum_{i=1}^n \frac{\tau_{ij}^2}{\tau_j^2} r_{ij}(1-r_{ij})p_{ij}\Big)^{-1/2}. \tag{27}$$

From (22), (23), (27) there follows that

$$\rho(F''W, F'W) \leqslant C(K)\sum_{j=1}^K \Big(\sum_{i=1}^n \frac{\tau_{ij}^2}{\tau_j^2} r_{ij}(1-r_{ij})p_{ij}\Big)^{-1/2}. \tag{28}$$

Now from (19)-(21), (28) there follows the estimate (4) for the considered case of absolutely continuous distributions.

In order to prove (4) in the general case, it is sufficient to construct $\varepsilon_{ijl}\to 0$ as $l\to\infty$; $i=1,\dots,n$; $j=1,\dots,K$, so that the distributions

$$F_{il} = F_i\Phi(\varepsilon_{i1l}^2, \varepsilon_{i2l}^2, \dots, \varepsilon_{iKl}^2)$$

should admit a representation, similar to the representation

$$F_i = p_i V_i + (1-p_i)\mathfrak{U}_i \tag{29}$$

and whose parameters should approach unboundedly the corresponding parameters of representation (29). Here $\Phi(\varepsilon^2_{i1\ell}, \varepsilon^2_{i2\ell}, \ldots, \varepsilon^2_{ik\ell})$ is a Gaussian distribution with a diagonal covariance matrix, whose diagonal elements are equal to $\varepsilon^2_{i1\ell}, \varepsilon^2_{i2\ell}, \ldots, \varepsilon^2_{ik\ell}$. Moreover, the distributions $\prod_{i=1}^n F_{i\ell}$ converge weakly to $\prod_{i=1}^n F_i$, and a similar convergence will take place for the corresponding accompanying laws $\mathfrak{D}_\ell$. In order to obtain (4) one has to make use of the relation

$$\rho\Big(\prod_{i=1}^n F_i, \mathfrak{D}\Big) \leq \varlimsup_{\ell\to\infty} \rho\Big(\prod_{i=1}^n F_{i\ell}, \mathfrak{D}_\ell\Big), \tag{30}$$

and for the estimation of the right-hand side of (30) one has to apply the already proved estimates.

The construction of the quantities $\varepsilon^2_{ij\ell} > 0$ does not require any effort if $\tau_j > 0$. If, however, $\tau_j = 0$, then one can construct $\varepsilon^2_{ij\ell} > 0$ in such a manner that the corresponding parameters $\tau_{ij\ell}$ of the representations for $F_{i\ell}$ be identical for all $i = 1, \ldots, n$. Therefore, we can assume that in (4) for such j we have $\tau_{ij}/\tau_j = 1$. We shall not carry out here the rigorous construction of the quantities $\varepsilon_{ij\ell}$ in view of its simultaneous awkwardness and simplicity.

We proceed to the proof of inequality (5). It is easy to show that under the conditions of the theorem one can construct for the distributions F_i a new representation $F_i = \bar{p}_i \bar{V}_i + (1-\bar{p}_i)\bar{\mathfrak{U}}_i$ whose parameters will be denoted in the same way as the parameters of the representation $F_i = p_i V_i + (1-p_i)\mathfrak{U}_i$ with a bar over the corresponding symbols. Moreover, the new representation can be selected so that

$$\bar{p}_i \bar{V}_i\{A\} \leq p_i V_i\{A\} \quad \text{for any} \quad A \in \mathfrak{B}_K;$$

$$\bar{p}_{ij} \leq p_{ij}; \quad \bar{\gamma}_{ij} = \gamma_{ij}; \quad \bar{\tau}_{ij} \geq \tau_{ij}; \quad \frac{\bar{\tau}_{ij}}{\bar{\tau}_j} = \frac{\tau_{ij}}{\tau_j},$$

$$i = 1, \ldots, n; \quad j = 1, \ldots, K;$$

$$\sum_{i=1}^n \frac{\bar{\tau}^2_{ij}}{\bar{\tau}^2_j} \bar{\gamma}_{ij}(1-\bar{\gamma}_{ij})\bar{p}_{ij} = \sum_{i=1}^n \frac{\tau^2_{ij_0}}{\tau^2_{j_0}} \gamma_{ij_0}(1-\gamma_{ij_0})p_{ij_0} = \min_j \sum_{i=1}^n \frac{\tau^2_{ij}}{\tau^2_j}\gamma_{ij}(1-\gamma_{ij})p_{ij}.$$

According to the already proved inequality (4), we have

$$\rho\Big(\prod_{i=1}^n F_i, \prod_{i=1}^n (E_{-\bar{a}_i} \exp(F_i E_{-\bar{a}_i} - E))\Big) \leq c(K)\Big(\sqrt{\bar{p}\sum_{i=1}^n \bar{p}_i^2} + \bar{p} + \sum_{j=1}^K \Big(\sum_{i=1}^n \frac{\bar{\tau}^2_{ij}}{\bar{\tau}^2_j}\bar{\gamma}_{ij}(1-\bar{\gamma}_{ij})\bar{p}_{ij}\Big)^{-1/2}\Big). \tag{31}$$

If, moreover, $\sum_{i=1}^n \bar{p}_i \leq \frac{1}{\bar{p}\sqrt{\bar{\gamma}}}$, then

$$\sqrt{\bar{p}\sum_{i=1}^n \bar{p}_i^2} \leq \frac{\sqrt{\bar{p}}}{\bar{\gamma}^{1/4}} \leq \frac{\sqrt{p}}{\gamma^{1/4}} \tag{32}$$

and thus, for $\mathfrak{D}_1$ one can take

$$\bar{\mathfrak{D}} = \prod_{i=1}^n (E_{-\bar{a}_i} \exp(F_i E_{-\bar{a}_i} - E)).$$

If, however, $\sum_{i=1}^{n} \bar{p}_i > \frac{1}{\bar{p}\sqrt{\bar{\gamma}}}$, then for the distributions F_i one can construct a new representation $F_i = \bar{\bar{p}}_i \bar{\bar{V}}_i + (1-\bar{\bar{p}}_i)\bar{\bar{\mathfrak{U}}}_i$ (its parameters will be provided with two bars) so that one should have the relations

$$\bar{\bar{p}}_i \bar{\bar{V}}_i\{A\} \leq \bar{p}_i \bar{V}_i\{A\} \quad \text{for any} \quad A \in \mathfrak{B}_K$$

$$\bar{\bar{p}}_{ij} \leq \bar{p}_{ij}\,; \quad \bar{\bar{\tau}}_{ij} \geq \bar{\tau}_{ij}\,; \quad \frac{\bar{\bar{\tau}}_{ij}}{\bar{\bar{\tau}}_j} = \frac{\bar{\tau}_{ij}}{\bar{\tau}_j}\,; \tag{33}$$

$$\sum_{i=1}^{n} \frac{\bar{\bar{\tau}}_{ij}^2}{\bar{\bar{\tau}}_j^2} \bar{\bar{\gamma}}_{ij}(1-\bar{\bar{\gamma}}_{ij})\bar{\bar{p}}_{ij} = \sum_{i=1}^{n} \frac{\bar{\bar{\tau}}_{i1}^2}{\bar{\bar{\tau}}_1^2} \bar{\bar{\gamma}}_{i1}(1-\bar{\bar{\gamma}}_{i1})\bar{\bar{p}}_{i1}\,; \quad \bar{\bar{\gamma}}_{ij} = \bar{\gamma}_{ij}\,;$$

$$\sum_{i=1}^{n} \bar{\bar{p}}_i = \frac{1}{\bar{\bar{p}}\sqrt{\bar{\bar{\gamma}}}}\,. \tag{34}$$

The fact is that we can vary the representation $F_i = \bar{p}_i \bar{V}_i + (1-\bar{p}_i)\bar{\mathfrak{U}}_i$ so that relations (33) should hold and the parameters $\bar{p}_i$ should vary in a continuous manner until equality (34) will hold. Moreover, one can also achieve that

$$\frac{1}{\bar{\bar{p}}\sqrt{\bar{\bar{\gamma}}}} = \sum_{i=1}^{n} \bar{\bar{p}}_i \leq \sum_{j=1}^{K}\sum_{i=1}^{n} \bar{\bar{p}}_{ij} \leq \frac{1}{\bar{\bar{\gamma}}} \sum_{j=1}^{K}\sum_{i=1}^{n} \frac{\bar{\bar{\tau}}_{ij}^2}{\bar{\bar{\tau}}_j^2} \bar{\bar{\gamma}}_{ij}(1-\bar{\bar{\gamma}}_{ij})\bar{\bar{p}}_{ij} \leq$$

$$\leq \frac{K}{\bar{\bar{\gamma}}} \sum_{i=1}^{n} \frac{\bar{\bar{\tau}}_{ij}^2}{\bar{\bar{\tau}}_j^2} \bar{\bar{\gamma}}_{ij}(1-\bar{\bar{\gamma}}_{ij})\bar{\bar{p}}_{ij}\,, \qquad j = 1,\ldots,K.$$

Consequently, according to inequality (4) we have

$$\rho\Big(\prod_{i=1}^{n} F_i, \prod_{i=1}^{n}(E_{\bar{\bar{a}}_i}\exp(F_i E_{-\bar{\bar{a}}_i} - E))\Big) \leq c(K)\Big(\bar{\bar{p}}\sqrt{\sum_{i=1}^{n}\bar{\bar{p}}_i} + \bar{\bar{p}} + \sum_{j=1}^{K}\Big(\sum_{i=1}^{n}\frac{\bar{\bar{\tau}}_{ij}^2}{\bar{\bar{\tau}}_j^2}\bar{\bar{\gamma}}_{ij}(1-\bar{\bar{\gamma}}_{ij})\bar{\bar{p}}_{ij}\Big)^{-1/2}\Big) \leq c(K)\frac{\sqrt{\bar{\bar{p}}}}{\bar{\bar{\gamma}}^{1/4}}. \tag{35}$$

From (32), (35) there follows (5) and in both cases the approximating infinitely divisible distribution is accompanying.

We obtain estimate (6). In general, the approximating distribution $\mathfrak{D}_2$ will not be accompanying for the districution $F = \prod_{i=1}^{n} F_i$. We construct $\mathfrak{D}_2$ in the form

$$\mathfrak{D}_2 = \mathfrak{D}_4 \prod_{i=1}^{n}(E_{a_i}\exp(H_i - E)),$$

$\mathfrak{D}_4 \in \mathcal{D}_K$ (obviously, then $\mathfrak{D}_2 \in \mathcal{D}_K$).

In the same way as for the proof of inequalities (1)-(3) we apply inequalities (10), (11), according to which,

$$L(F,\mathfrak{D}_2) = L(F^*, \mathfrak{D}_4\prod_{i=1}^{n}\exp(H_i - E)) \leq L(F^*, F') + L(F', F'') + L(F'', \mathfrak{D}_4\prod_{i=1}^{n}\exp(H_i - E)) \leq$$

$$\leq c(K)(p(|\ln p|+1)^{K/4} + \tau^{2/3}(|\ln\tau|+1)^{\frac{K+8}{12}} + L(\prod_{i=1}^{n} G_i, \mathfrak{D}_4)). \tag{36}$$

We consider again two cases.

If $\sum_{i=1}^{n} p_i \leq p^{-1}$ then we take

$$\mathfrak{D}_4 = \prod_{i=1}^{n}\exp(G_i - E)$$

and, according to Theorem 3 of [9], we have

$$L\Big(\prod_{i=1}^{n} G_i, \mathfrak{D}_4\Big) \leq \rho\Big(\prod_{i=1}^{n} G_i, \mathfrak{D}_4\Big) \leq c(K)\sqrt{p\sum_{i=1}^{n} p_i^2} \leq c(K)\sqrt{p}. \tag{37}$$

From (36), (37) there follows (6) for the case $\sum_{i=1}^{n} p_i \leq p^{-1}$. We note that in this case $\mathfrak{D}_2 = \mathfrak{D}$.

If, however, $\sum_{i=1}^{n} p_i > p^{-1}$, then, since we assume $p < 1/5 < 1/2$, the distributions G_i admit a representation similar to the representation $F_i = (1-p_i)\mathcal{U}_i + p_i V_i$:

$$G_i = (1-p_i^*)\mathcal{U}_i^* + p_i^* V_i^*$$

(the parameters of this representation will be provided with asterisks) and

$$p_i^* = 2p_i; \quad V_i^* = \frac{1}{2}E + \frac{1}{2}V_i E_{-a_i}; \quad \gamma_{ij}^* = \frac{1}{2};$$
$$\alpha_{ij}^* = \tau_{ij}^* = 0, \quad i = 1,\dots,n; \quad j = 1,\dots,K.$$

From the fact that (5) is already proved there follows that there exist $b_i \in R^K$ such that if

$$\mathfrak{D}_4 = \prod_{i=1}^{n} \big(E_{b_i} \exp(G_i E_{-b_i} - E)\big),$$

then

$$\rho\Big(\prod_{i=1}^{n} G_i, \mathfrak{D}_4\Big) \leq c(K)\Big(\sqrt{p^*} + \Big(\sum_{i=1}^{n} p_i^*\Big)^{-1/2}\Big) \leq c(K)\sqrt{p}. \tag{38}$$

From (36), (38), there follows (6) also in the case $\sum_{i=1}^{n} p_i > p^{-1}$.

In conclusion, we give the complete proof of Theorem 3, formulated without proof in [9].

<u>THEOREM 2.</u> Let $G_i = (1-p_i)E + p_i V_i$; $0 \leq p_i \leq 1$; $V_i, E \in \mathfrak{F}_K$, $i = 1,\dots,n$. Then

$$\rho\Big(\prod_{i=1}^{n} G_i, \prod_{i=1}^{n} \exp(G_i - E)\Big) \leq cK\Big(p\sum_{i=1}^{n} p_i^2\Big)^{1/2},$$

where $p = \max_{1 \leq i \leq n} p_i$.

<u>Proof.</u> Let $\Phi_\varepsilon = \Phi(\varepsilon^2, \varepsilon^2, \dots, \varepsilon^2) \in \mathfrak{F}_K$. First we assume that $V_i = V_i' \Phi_\varepsilon$, $\varepsilon > 0$, $V_i' \in \mathfrak{F}_K$, $i = 1,\dots,K$. Without loss of generality, we shall assume that $p \leq 1/2$. Let

$$G = \prod_{i=1}^{n} G_i; \quad \mathfrak{D} = \prod_{i=1}^{n} \exp(G_i - E), \quad \mathcal{U}_i = \prod_{l=1}^{i-1} G_l \prod_{l=i+1}^{n} \exp(G_l - E).$$

Then

$$G - \mathfrak{D} = \sum_{i=1}^{n} \mathcal{U}_i \Big((e^{-p_i} - (1-p_i))(V_i - E) + \sum_{m=2}^{\infty} e^{-p_i} \frac{p_i^m}{m!}(V_i - V_i^m)\Big).$$

Obviously, we have

$$e^{-p_i} - (1-p_i) + \sum_{m=2}^{\infty} e^{-p_i} \frac{p_i^m (m-1)}{m!} \leq c p_i^2.$$

Therefore, with the aid of Lemma 1 of [9] we obtain:

$$\rho(G,\mathfrak{D})\leqslant c\sum_{i=1}^{n}p_i^2\rho(\mathcal{U}_iV_i,\mathcal{U}_i)\leqslant c\sum_{j=1}^{K}\sum_{i=1}^{n}p_i^2\int_{R}Q(\mathcal{U}_i^{(j)},|x|)V_i^{(j)}\{dx\}. \tag{39}$$

We define the measures H_j with the aid of their "distribution functions":

$$H_j(x)=H_j\{(-\infty,x)\}=\begin{cases}0 & x\leqslant 0\\ \sum_{i=1}^{n}p_i^2V_i^{(j)}\{(-x,x)\}, & x>0.\end{cases}$$

We denote by $v=\sum_{i=1}^{n}p_i^2$ the total variation of the measure H_j. Assume, further that

$$\mathfrak{D}_{p,j}=\{x\in R^1: H_j\{(|x|,\infty)\}\geqslant 2p^2\}.$$

Then

$$\sum_{i=1}^{n}p_i^2\int_{R^1}Q(\mathcal{U}_i^{(j)},|x|)V_i^{(j)}\{dx\}\leqslant 2p^2+\sum_{i=1}^{n}p_i^2\int_{\mathfrak{D}_{p,j}}Q(\mathcal{U}_i^{(j)},|x|)V_i^{(j)}\{dx\}. \tag{40}$$

Taking into account that $p\leqslant 1/2$ and making use of the well-known estimates for the concentration function (see, for example, Theorems 1, 5, Chap. III [13]), we obtain that

$$Q(\mathcal{U}_i^{(j)},|x|)\leqslant c\Big(\sum_{l\neq i}p_lV_l^{(j)}\{R^1\setminus[-|x|,|x|]\}\Big)^{-1/2}. \tag{41}$$

In addition, for $x\in\mathfrak{D}_{p,j}$ we have

$$\sum_{l\neq i}p_l^2V_l^{(j)}\{R^1\setminus[-|x|,|x|]\}\geqslant\frac{1}{2}\sum_{l=1}^{n}p_l^2V_l^{(j)}\{R^1\setminus[-|x|,|x|]\}. \tag{42}$$

By virute of (41), (42) we have

$$\sum_{i=1}^{n}p_i^2\int_{\mathfrak{D}_{p,j}}Q(\mathcal{U}_i^{(j)},|x|)V_i^{(j)}\{dx\}\leqslant c\sum_{i=1}^{n}\int_{\mathfrak{D}_{p,j}}\frac{p_i^2V_i^{(j)}\{dx\}}{(\sum_{l\neq i}p_lV_l^{(j)}\{R\setminus[-|x|,|x|]\})^{1/2}}\leqslant c\sqrt{p}\sum_{i=1}^{n}\int_{\mathfrak{D}_{p,j}}\frac{p_i^2V_i^{(j)}\{dx\}}{(\sum_{l\neq i}p_l^2V_l^{(j)}\{R\setminus[-|x|,|x|]\})^{1/2}}\leqslant c\sqrt{2p}\int_{-\infty}^{\infty}\frac{dH_j(x)}{(v-H_j(x))^{1/2}}\leqslant 2c\sqrt{2p}\sqrt{v}.$$

From here and from (39), (40) there follows the assertion of the lemma for $V_i=V_i'\Phi_\varepsilon$.

We consider now the general case. For $\varepsilon>0$ we denote $G_{i,\varepsilon}=(1-p_i)E+p_iV_i\Phi_\varepsilon$. Then, taking into account what has been already proved, we obtain:

$$\rho\Big(\prod_{i=1}^{n}G_i,\prod_{i=1}^{n}(G_i-E)\Big)\leqslant\overline{\lim_{\varepsilon\to+0}}\,\rho\Big(\prod_{i=1}^{n}G_{i,\varepsilon},\prod_{i=1}^{n}\exp(G_{i,\varepsilon}-E)\Big)\leqslant cK\sqrt{p\sum_{i=1}^{n}p_i^2},$$

which is what we intended to prove.

Remark. After this paper has been written, the author has succeeded to prove that if the distributions $H_i\in\mathfrak{F}_K$, $i=1,\ldots,n$, have zero means and $H_i\{\{x:\|x\|\leqslant\tau\}\}=1$, then

$$\pi\Big(\prod_{i=1}^{n}H_i,\prod_{i=1}^{n}\exp(H_i-E)\Big)\leqslant c(K)\tau(|\ln\tau|+1). \tag{43}$$

In the one-dimensional case this result has been obtained in [3]. Strengthening in an essential manner Theorem 5 of [10], inequality (43) allows us to replace $\tau^{2/3}(|\ln\tau|+1)^{\frac{K+3}{12}}$ by

$\tau(|\ln\tau|+1)$ in formulas (1), (2), (3), (11) and thus, one approaches the unimprovable one-dimensional inequalities proved in [3] (see Remark 1).

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THE GB AND GC PROPERTIES OF GENERALIZED ELLIPOSIDS

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In the paper one investigates the conditions under which the subsets of the form $B\{a_n; p_n\} = \{x = \sum x_\kappa e_\kappa : \sum \left|\frac{x_\kappa}{a_\kappa}\right|^{p_\kappa} \leq 1\}$ of a Hilbert space, where $a_\kappa \downarrow 0$, $p_\kappa > 1$, $\kappa = 1, \ldots,$ possesses the GB or the GC properties.

Let $(\Omega, \mathfrak{A}, P)$ be a probability space. A sequence $\{X_n\}$ of random variables is said to be ortho-Gaussian if the random variables X_i are independent and $X_i \in N(0,1)$. Let H be a real, infinite-dimensional Hilbert space. A Gaussian process L on the space H is said to be isonormal if L is a linear mapping of the space H into the set of real Gaussian random variables while $EL(x) = 0$ and $EL(x)L(y) = (x,y)$ for all $x, y \in H$. In particular, if $\{e_n\}$ is some orthonormal basis in H, then for any $x \in H, x = \sum_\kappa x_\kappa e_\kappa$ one can set $L(x) = \sum_n x_n Y_n$, where $\{Y_n\}$ is an ortho-Gaussian sequence. We shall say

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