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Chaos, Solitons and Fractals 24 (2005) 191–196
CHAOS SOLITONS & FRACTALS
www.elsevier.com/locate/chaos

Wave packet propagating in an electrical transmission line

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Accepted 14 September 2004

Communicated by Prof. M. Wadati

Abstract

A nonlinear Schrödinger equation (NLSE) is derived for a nonlinear transmission line in which the nonlinear capacitance C is a function of voltage. For a linear long wavelength perturbations to a solution of a NLSE, the instability region is given in this paper.

1. Introduction

Distributed electrical transmission lines constructed by a large number of identical sections have been used to study the propagation of KdV solitons which satisfy the famous Korteweg de Vries equation. This equation was originally derived to model the shallow water wave experiments of John Scott Russel in the 19th century. It is also found both in plasmas and dusty plasmas to describe the propagation and interaction of acoustic solitons. Later, the KdV solitons have been found in almost all fields of physics [1–15]. Resonances of a two dimensional transmission line have also been examined [9]. By assuming that the nonlinear capacitance C to be of the form $C = \frac{C_0}{1+(V/V_0)^p}$, Duan et al. have studied the nonlinear waves on coupled nonlinear transmission line [3] where V is the voltage in the transmission line, C_0, V_0 and p are all constants. They found that, in the continuum limit, the voltage for transmission line is described by a modified Zakharov–Kuznetsov (ZK) equation. The exact cut-off frequencies of the growth rate of the solitary waves for the transverse perturbations have been obtained. Some instability for solitary travelling waves between two cut-off frequencies have been found.

However, few study [10–12] has been made to study the solitary waves described by a nonlinear Schrödinger equation (NLSE) by using the model of the electrical transmission line until now. The typical one is the works of Taniuti et al. [10] who have studied the NLS equation theoretically. Fukushima et al. [11,12] have also investigated the NLS-type solitons in an electric transmission line experimentally. It is noted, in this paper, that it is interesting to study the propagation and the instability of solitons obeying the nonlinear Schrödinger equation by using the model of distributed electrical transmission lines. As well known, this equation is also a famous one to study the envelope solitons. Therefore, in this paper, we study the coupled nonlinear transmission line. We assume that there are many identical

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0960-0779/$ - see front matter
doi:10.1016/j.chaos.2004.09.015

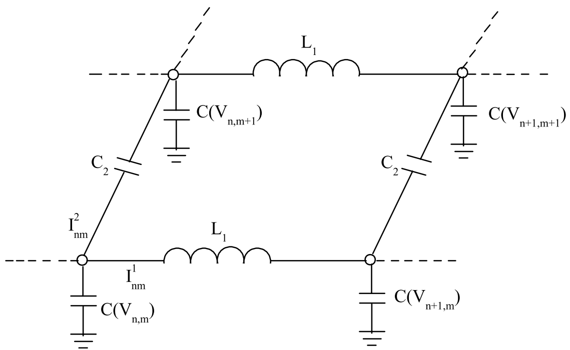

Fig. 1. Part of the system of the nonlinear transmission line coupled by a capacitor C_2.

lines which are coupled by means of capacitors C_2 at each node, see Fig. 1. Each section of line consists of a constant inductor L_1 in parallel with a nonlinear capacitance $C = C(V)$. The nodes in the system are labelled with two discrete coordinates n and m, where n specifies the nodes in the direction of propagation of the pulse, and m labels the lines in the transverse direction. We apply Kirchoff's law in the orthogonal loops and obtain the circuit equation for this system as follows: $L_1 \frac{\partial I^1_{n,m}}{\partial t} = V_{n,m} - V_{n+1,m}$, $I^2_{n,m} = C_2 \frac{\mathrm{d}}{\mathrm{d}t}(V_{n,m} - V_{n,m+1})$, $\frac{\partial Q_{n,m}}{\partial t} = I^1_{n-1,m} - I^1_{n,m} + I^2_{n,m-1} - I^2_{n,m}$. Therefore, we obtain: $\frac{\partial^2 Q_{nm}}{\partial t^2} = \frac{1}{L_1}(V_{n+1,m} + V_{n-1,m} - 2V_{n,m}) + C_2 \frac{\partial^2}{\partial t^2}(V_{n,m+1} + V_{n,m-1} - 2V_{n,m})$. We now use the continuous limit assuming that the wavelength is much larger than the lattice spacing. The perturbation voltage V is also small enough compared with equilibrium value. Continuum approximation has the merit that it is easier than discrete lattice to treat both analytically and numerically. Moreover, the results obtained can be related to those for the discrete version in most cases. The continuum approximation is obtained by assuming the validity of the Taylor expansion to rewrite the equation for nonlinear lattices. We assume that the waves are smooth enough, n and m are two continuous variables, the perturbation voltage $V_{n,m}$ is the function of two continuous variables of n and m. Based on these reasonable assumptions, we obtain that $V_{n\pm1,m} = V_{n,m} \pm \frac{\partial V_{n,m}}{\partial n} + \frac{1}{2}\frac{\partial^2 V_{n,m}}{\partial n^2} \pm \frac{1}{6}\frac{\partial^3 V_{n,m}}{\partial n^3} + \frac{1}{24}\frac{\partial^4 V_{n,m}}{\partial n^4} + \cdots$, and $V_{n,m\pm1} = V_{n,m} \pm \frac{\partial V_{n,m}}{\partial m} + \frac{1}{2}\frac{\partial^2 V_{n,m}}{\partial m^2} \pm \frac{1}{6}\frac{\partial^3 V_{n,m}}{\partial m^3} + \frac{1}{24}\frac{\partial^4 V_{n,m}}{\partial m^4} + \cdots$. Letting $V_{n,m}(t) = V(n,m,t)$, we obtain, in the continuum limit, the circuit equation for this system as follows [3,4]

$$L_1 C_0 \frac{\partial^2}{\partial t^2}\left[V - \frac{V^{p+1}}{(p+1)V_0^p}\right] = \frac{\partial^2}{\partial n^2}\left[V + \frac{1}{12}\frac{\partial^2 V}{\partial n^2}\right] + L_1 C_2 \frac{\partial^4 V}{\partial n^2 \partial t^2} \tag{1}$$

where we actually obtain an equation for a one dimensional nonlinear transmission line. The variations of voltage V with respect to transverse coordinates m are neglected.

2. Mathematical formalism

The linear dispersion relation for this system can be obtained by using standard linear perturbation analysis. For simplicity, we first study the one dimensional case. Assuming that the first order quantities vary as $\exp[\mathrm{i}(kn - \omega t)]$, we then obtain the following dispersion relation

$$\omega^2 = k^2 \frac{A - Bk^2}{1 + Ck^2} \tag{2}$$

where $A = \frac{1}{L_1 C_0}$, $B = \frac{1}{12 L_1 C_0}$ and $C = \frac{C_2}{C_0}$. Fig. 2 show the relationship between the ω^2 and the wave number k.

In order to obtain the envelop solitary waves solution for this system, we now use the traditional perturbation technique and expand V as follows

$$V = \sum_{m=1}^{\infty} \epsilon^m \sum_{l=-\infty}^{+\infty} V_l^{(m)}(\xi, \tau) \mathrm{e}^{\mathrm{i}l(kn - \omega t)} \tag{3}$$

where $\xi = \epsilon(n - v_s t)$ and $\tau = \epsilon^2 t$ are slow variables. Substituting Eq. (3) into Eq. (1), we obtain the same result as the linear case:

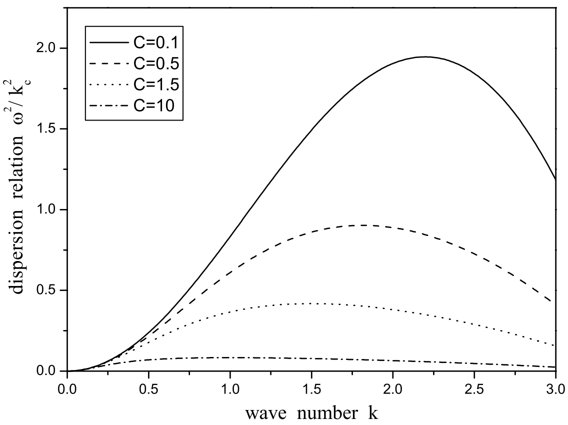

Fig. 2. The profiles of dimensionless quantity of ω^2 as a function of wave number k with different variable $C = 0.1, 0.5, 1.5, 10$ respectively.

$$\omega^2 = k^2 \frac{A - Bk^2}{1 + Ck^2} \tag{4}$$

and the group velocity:

$$v_s = \frac{\partial \omega}{\partial k} = \frac{k}{\omega} \frac{A - 2Bk^2 - C\omega^2}{1 + Ck^2} \tag{5}$$

the variations of v_s as a function of wave number k have been shown in Fig. 3 for different C. We finally obtain the NLSE as follows:

$$\mathrm{i}\frac{\partial V_1^{(1)}}{\partial \tau} + R_m V_1^{(1)} |V_1^{(1)}|^2 + Q_m \frac{\partial^2 V_1^{(1)}}{\partial \xi^2} = 0 \tag{6}$$

where $m = 1, 2$ which are, respectively, for $p = 1, 2$. For $p = 1$

$$R_1 = -\frac{3\omega}{2(1 + Ck^2)v_0^2(2\omega^2 - 2Ak^2 + 8Bk^4 + 8Ck^2\omega^2)} \tag{7}$$

$$Q_1 = -\frac{v_s^2 - A + 6Bk^2 + C(4k\omega v_s + \omega^2 + k^2 v_s^2)}{2\omega(1 + Ck^2)} \tag{8}$$

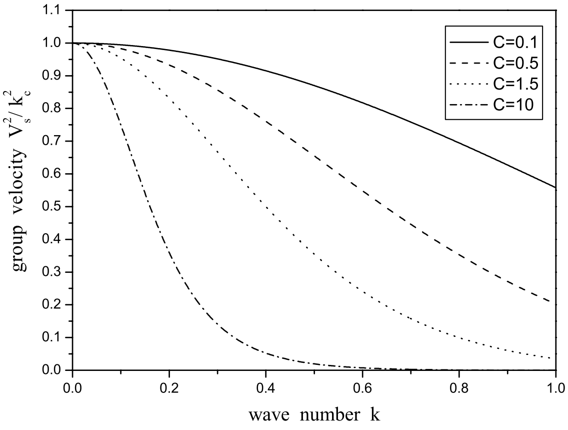

Fig. 3. The profiles of dimensionless quantity of group velocity, v_s, as a function of wave number k with different variable $C = 0.1, 0.5, 1.5, 10$ respectively.

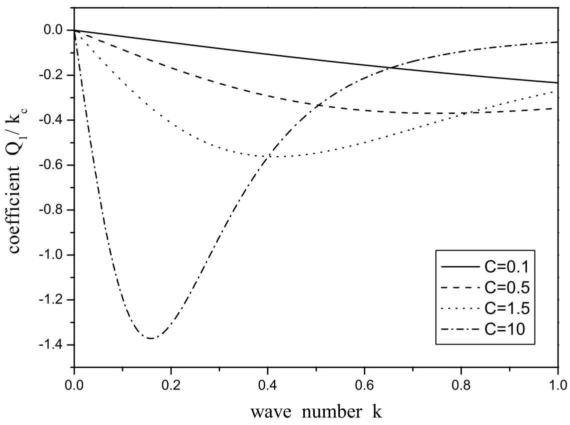

Fig. 4. The profiles of dimensionless quantity of $Q_1 = Q_2$ as a function of wave number k with different variable $C = 0.1, 0.5, 1.5, 10$ respectively.

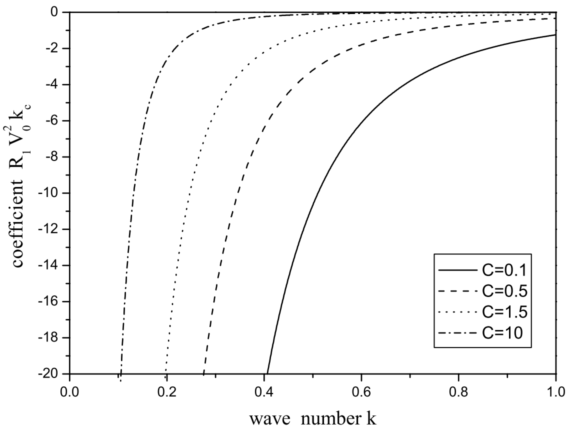

Fig. 5. The profiles of dimensionless quantity of R_1 as a function of wave number k with different variable $C = 0.1, 0.5, 1.5, 10$ respectively.

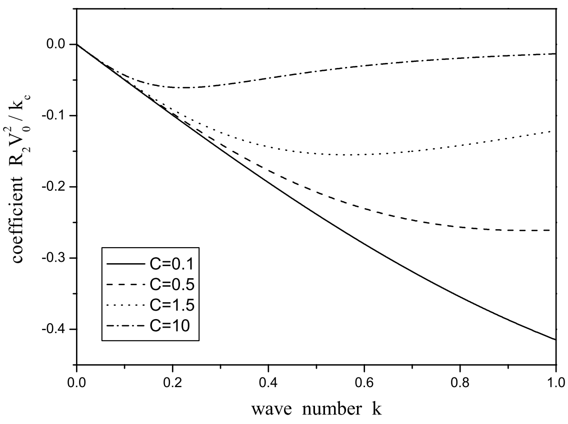

Fig. 6. The profiles of dimensionless quantity of R_2 as a function of wave number k with different variable $C = 0.1, 0.5, 1.5, 10$ respectively.

and for $p = 2$

$$R_2 = -\frac{\omega}{2(1+Ck^2)v_0^2} \tag{9}$$

$$Q_2 = -\frac{v_s^2 - A + 6Bk^2 + C(4k\omega v_s + \omega^2 + k^2 v_s^2)}{2\omega(1+Ck^2)} \tag{10}$$

It seems that $Q_1 = Q_2$. The variation of Q_1, R_1 and R_2 as the functions of both wave number k and variable of C have been shown in Figs. 4–6 respectively. It is noted from these figures that all the Q_1, R_1 and R_2 are negative. These results will help us to investigate their instability region for modulational instability.

3. Discussion

We now study the linear stability of Eq. (6) when modulation on the wave amplitude (packet) takes place in the propagation direction. Instead of stationary solution, here we consider the dynamic solution of Eq. (6). Accordingly, we separate the amplitude $V_1^{(1)}$ into two parts as follows

$$V_1^{(1)} = (V_\mathrm{C} + \delta V(\zeta))\mathrm{e}^{-\mathrm{i}\Delta\tau} \tag{11}$$

where $\zeta = K\xi - \Omega\tau$ is the modulation phase, $K \ll k$ and $\Omega \ll \omega$ are respectively the wave number and the frequency of the modulation. V_C is a constant (real) amplitude of the pump carrier wave, $\delta V \ll V_\mathrm{C}$ is the small amplitude perturbation, Δ is a nonlinear frequency shift. Assuming that the amplitude perturbation δV varies as $\exp[\mathrm{i}(K\xi - \Omega\tau)]$, we then obtain the following nonlinear dispersion relation for the amplitude modulation from Eq. (6) [15]

$$\Omega^2 = Q_m K^2 \left(Q_m K^2 - 2R_m |V_\mathrm{C}|^2\right) \tag{12}$$

It is found from Eq. (12) that $\Omega^2 > 0$ if $R_m Q_m < 0$. In this case Ω have real solutions, therefore the waves are stable. However, if $R_m Q_m > 0$, Ω may have no real solution, then the waves may be unstable. It is found from Eq. (12) that the waves are stable when (1) $Q_m > 0$, $R_m > 0$ and $K^2 > \frac{2R_m}{Q_m}(V_\mathrm{C})^2$. (2) $Q_m < 0$, $R_m < 0$ and $K^2 > \frac{2R_m}{Q_m}(V_\mathrm{C})^2$. Namely, the waves are modulational stable if $R_m Q_m > 0$ and $K^2 > \frac{2R_m}{Q_m}(V_\mathrm{C})^2$, otherwise, it is unstable.

Now we discuss the solitary wave solution. To obtain the profile, we let $V_1^{(1)} = V(\xi,\tau)\mathrm{e}^{[\mathrm{i}\sigma(\xi,\tau)]}$, where V and σ are two real variables. Substituting this into the NLSE (6), and separating the real and imaginary parts, we obtain the following equation when both R_m and Q_m have same sign

$$V(\xi,\tau) = V_{00}\,\mathrm{sech}\left(\sqrt{\frac{R_m}{2Q_m}}V_\mathrm{C}\xi\right) \tag{13}$$

where V_{00} is a constant and represents the nonlinear maximum amplitude. On the other hand, when R_m and Q_m have the opposite sign, we obtain

$$V(\xi,\tau) = V_1\left[1 - V_2^2\,\mathrm{sech}^2\left(\sqrt{\frac{V_1|R_m|}{2|Q_m|}}V_2\xi\right)\right]^{1/2} \tag{14}$$

where $V_2^2 = \frac{V_1^2 - V_\mathrm{C}^2}{V_1^2} \leqslant 1$, V_1 is a constant. The solution of (14) is referred to as an envelope hole sometimes called a dark soliton. The parameter V_2 in Eq. (14) determines the depth of modulation. For $V_2 = 1$, we have

$$V(\xi,\tau) = V_1 \tanh\left(\sqrt{\frac{V_1|R_m|}{2|Q_m|}}V_2\xi\right) \tag{15}$$

which is known as an envelope shock. It is noted from Figs. 4–6 that all the Q_1, R_1 and R_2 are negative. Therefore, the waves are modulational stable if $K^2 > \frac{2R_m}{Q_m}(V_\mathrm{C})^2$, otherwise it is unstable.

Acknowledgments

This is supported by the National Natural Science Foundation of China (10247008), the Natural Science Foundation of Gansu Province of China (YS021-A22-018), the Natural Science Foundation of Northwest Normal University (NWNU-KJCXGC-215) and the Royal Society K.C. Wong Fellowship.

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