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A high noncuppable Σ_2^0 *e*-degree

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Abstract We construct a Σ_2^0 *e*-degree which is both high and noncuppable. Thus demonstrating the existence of a high *e*-degree whose predecessors are all properly Σ_2^0.

1 Introduction

The concept of enumeration reducibility is that of relative enumerability between sets of natural numbers, where a set A is enumeration reducible to a set B if and only if there is a procedure that uniformly provides an enumeration of A when given any enumeration of B. Freidberg and Rogers [3] formalized this concept via the notion of an enumeration operator. An *enumeration operator* (or simply *e-operator*) is a mapping Φ: $2^\omega \rightarrow 2^\omega$ for which there exists a c.e. set W such that, for each $X \subseteq \omega$,

$$\Phi^X = \{x \mid (\exists u)[\langle x, u\rangle \in W \wedge F_u \subseteq X]\},$$

where F_u is the finite set with canonical index u. We say that a set A is *enumeration reducible* (or simply *e-reducible*) to a set B, written symbolically as $A \leq_e B$, if and only if $A = \Phi^B$ for some *e*-operator Φ. As is commonplace we write $\langle x, F\rangle$ instead of $\langle x, u\rangle$ where u is the canonical index of F, and speak of $\langle x, F\rangle \in \Phi$ when we in fact mean $\langle x, F\rangle \in W$ where W is the c.e. set determining the *e*-operator Φ.

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We denote by $\equiv_e$ the equivalence relation generated by the preordering relation $\leq_e$ and $\deg_e(X)$ denotes the equivalence class (or the *e-degree*) of X. The degree structure $\mathcal{D}_e$ of the e-degrees ordered by $\leq$, where $\deg_e(A) \leq \deg_e(B) \Leftrightarrow A \leq_e B$, is an upper semilattice with least element $\mathbf{0}_e$ (the e-degree of the c.e. sets).

What makes the e-degrees particularly interesting is that they extend the Turing degrees. There is a natural embedding $\iota\colon \mathcal{D}_T \to \mathcal{D}_e$ of the Turing degrees into the e-degrees (in fact onto the total e-degrees) which preserves joins and least element, defined by $\iota(\deg_T(A)) = \deg_e(\chi_A)$, where χ_A denotes the characteristic function of A.

Cooper [1] and McEvoy [5] defined a jump operation on the e-degrees where $\mathbf{a}'$ denotes the jump of the e-degree $\mathbf{a}$. Importantly, their definition of the jump agrees with the natural embedding of the Turing jump. The e-degree $\mathbf{0}'_e = \deg_e(\overline{K})$ and those e-degrees $\leq \mathbf{0}'_e$ are precisely the Σ^0_2 e-degrees, that is the e-degrees which contain a Σ^0_2 set (in fact, only Σ^0_2 sets). This jump operation also allows us to introduce the notions of *low* and *high* e-degrees. That is those e-degrees $\mathbf{a} \leq \mathbf{0}'_e$ whose jumps are as low as possible $\mathbf{a}' = \mathbf{0}'_e$ are called low, whilst those whose jumps are as high as possible $\mathbf{a}' = \mathbf{0}''_e$ are called high. Finally, we say that a Σ^0_2 e-degree $\mathbf{a}$ is *cuppable* if there exists a Σ^0_2 e-degree $\mathbf{b} < \mathbf{0}'_e$ such that $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'_e$. Naturally a Σ^0_2 e-degree for which there is no such $\mathbf{b}$ we call *noncuppable*.

We let CUP and NCUP denote the classes of cuppable and noncuppable e-degrees respectively, and similarly LOW and HIGH denote the classes of low and high e-degrees. What can one say about the relationships between these classes? It is known that every low e-degree is cuppable, i.e. LOW $\subseteq$ CUP. This follows from two facts, firstly that every low e-degree is Δ^0_2 (which follows from a useful characterization of the low e-degrees given by McEvoy and Cooper [6]: $\mathbf{a} \in$ LOW if and only if $\mathbf{a}$ contains a set A such that, for every $B \leq_e A, B \in \Delta^0_2$) and secondly that every $\Delta^0_2$$e$-degree is cuppable (Theorem 3.1 of Cooper et al. [2]). On the other hand, it is not true that every cuppable e-degree is low, i.e. CUP $\nsubseteq$ LOW, since any high c.e. Turing degree embeds to a high $\Pi^0_1$$e$-degree, and so is Δ^0_2 and hence cuppable. This also shows that not every high e-degree is noncuppable, i.e. HIGH $\nsubseteq$ NCUP. Which leaves us only to decide whether each noncuppable e-degree is also high, that is "Is NCUP $\subseteq$ HIGH?" Intuitively, this seems unlikely, since NCUP is an ideal and so closed downwards whilst HIGH is closed upwards, though at present we are unable to definitively answer this question. We are, however, able to show that there is an overlap in the two classes, that is NCUP $\cap$ HIGH $\neq \emptyset$.

Theorem 1 *There exists a high noncuppable Σ^0_2 e-degree.*

Let $\mathbf{h}$ be as in the theorem. We show that there can be no nonzero $\Delta^0_2$$e$-degree $\mathbf{a} < \mathbf{h}$. Supposing, to the contrary, that one such existed, it would be cuppable, since, as already noted, every nonzero Δ^0_2 e-degree is cuppable, but then $\mathbf{h}$, itself, would be cuppable. As a Σ^0_2 e-degree is *properly* Σ^0_2 if it does not contain a Δ^0_2 set (i.e. is not a Δ^0_2 e-degree), we have shown:

Corollary 2 *There exists an e-degree* $\mathbf{h}$ *whose nonzero predecessors* $\mathbf{a} \leq \mathbf{h}$ *are properly* Σ_2^0. *In particular,* $\mathbf{h}$ *does not bound a low e-degree.*

In order to prove the theorem we must construct a Σ_2^0 set which is of high *e*-degree and is such that the join of it with any other incomplete Σ_2^0 set is not e-equivalent ($\equiv_e$) to $\overline{K}$. First we consider how we can guarantee our constructed set has high *e*-degree. Recall, a Σ_2^0 approximation to a set A is a computable sequence $\{A_s\}_{s\in\omega}$ of sets such that, for every x,

$$x \in A \Leftrightarrow (\exists t)(\forall s \geq t)[x \in A_s].$$

It is known that one can find a computable sequence $\{X_{k,s}\}_{k,s\in\omega}$ of sets such that $\{X_k\}_{k\in\omega}$ is a listing of all Σ_2^0 sets, where $\{X_{k,s}\}_{s\in\omega}$ is a Σ_2^0 approximation to X_k.

Definition 3 (McEvoy [4], McEvoy and Cooper [6]) *A* Σ_2^0*-high approximation* $\{A_s\}_{s\in\omega}$ *to a set A is a* Σ_2^0 *approximation such that the function*

$$C_A(x) = \mu s\,[\, s > x \wedge A_s\lceil x \subseteq A\,],$$

(*called the computation function for A relative to the given approximation; here* $A_s\lceil x = \{y \mid y \in A_s \wedge y < x\}$) *is total and dominates every computable function. A set A is called* Σ_2^0*-high if it has a* Σ_2^0*-high approximation and, naturally, an e-degree is said to be* Σ_2^0*-high if it contains a* Σ_2^0*-high set.*

The coincidence of the Σ_2^0-high *e*-degrees with the high *e*-degrees was shown by Shore and Sorbi [7]. Therefore by constructing a set H which meets the following *highness* requirements

$$\mathcal{H}_i\colon C_H \text{ dominates } \varphi_i \text{ or } \varphi_i \text{ is not total.}$$

for each $i \in \omega$, we ensure that it has high *e*-degree. (Recall, a function f *dominates* g if $f(x) \geq g(x)$ for all but finitely many x.)

We guarantee that the degree of the set H is noncuppable by constructing a Σ_2^0 set B such that H and B satisfy the following *noncupping* requirements.

$$\mathcal{N}_{\langle j,k\rangle}\colon B = \Psi_j^{H\oplus X_k} \Rightarrow (\exists\Gamma)[\overline{K} = \Gamma^{X_k}],$$

for all $\langle j,k\rangle \in \omega$ ($\langle j,k\rangle$ is the standard pairing function) where $\{\Psi_j\}_{j\in\omega}$ and $\{X_k\}_{k\in\omega}$ are computable listings of all *e*-operators and all Σ_2^0 sets respectively, and Γ is an *e*-operator to be built by us. These noncupping requirements ensure that for no incomplete Σ_2^0 set X can we have $\overline{K} \leq_e H \oplus X$ for otherwise for some Ψ we would get $B = \Psi^{H\oplus X}$ since $B \leq_e \overline{K}$, and thus, by satisfaction of $\mathcal{N}_{\langle j,k\rangle}$ where $\Psi = \Psi_j$ and $X = X_k$, getting $\overline{K} \leq_e X$ contradicting X being an incomplete Σ_2^0 set.

We can give a natural priority ordering to the requirements as follows.

$$\mathcal{N}_{\langle 0,0\rangle} < \mathcal{H}_0 < \mathcal{N}_{\langle 0,1\rangle} < \mathcal{H}_1 < \mathcal{N}_{\langle 1,0\rangle} < \cdots.$$

2 The noncupping strategy in isolation

The basic strategy for satisfying a noncupping requirement is as follows. Consider a requirement $\mathcal{N}_{\langle j,k\rangle}$ and drop the subscripts j and k for simplicity.

Action is taken on a number z when all numbers less than z have been chosen and currently reside at step 5 or step 8 of the module below.

1. Choose $z \in \overline{K} - \Gamma^X$;
2. Choose a number b_z and define $b_z \in B$;
3. Wait for $b_z \searrow \Psi^{H\oplus X}$ (i.e. b_z gets enumerated into $\Psi^{H\oplus X}$) via, say, some axiom $\langle b_z, F_H \oplus F_X\rangle \in \Psi$;
4. Enumerate the axiom $\langle z, F_X\rangle \in \Gamma$;
5. While $z \in \overline{K}$, keep $b_z \in B$, and if $z \nearrow \Gamma^X$ (i.e. z gets extracted from Γ^X due to some X-change), then go to step 3;
6. If $z \notin \overline{K}$ then enumerate and restrain $F_H \subseteq H$ for each existing axiom $\langle b_z, F_H \oplus F_X\rangle \in \Psi$ used to define axioms $\langle z, F_X\rangle \in \Gamma$, and if not already extracted extract b_z from B;
7. Wait for $z \nearrow \Gamma^X$;
8. Remove H-restraints imposed at step 6, and wait for $z \searrow \Gamma^X$, following which return to step 6.

We may satisfy this strategy finitarily, which we denote by fin, either by waiting forever for some z at step 3 in which case $B \neq \Psi^{H\oplus X}$ because $b_z \in B - \Psi^{H\oplus X}$, or at step 7 in which case $B \neq \Psi^{H\oplus X}$ because $b_z \in \Psi^{H\oplus X} - B$. Otherwise, which we denote by ∞, the strategy acts infinitely often on each $z \in \omega$ and is satisfied because either for each z it stops at step 5, waits at step 8, or loops through step 8 infinitely often in which case $\overline{K}(z) = \Gamma^X(z)$ for all $z \in \omega$, or for some z it loops through step 5 infinitely often in which case $B \neq \Psi^{H\oplus X}$ because $b_z \in B - \Psi^{H\oplus X}$.

3 The highness strategy in isolation

Note when constructing H it is more convenient for the highness strategies if we start with $H = \omega$. Let $\{\xi_i\}_{i\in\omega}$ be a computable partition of ω into infinite sets, so each highness requirement $\mathcal{H}_i$ works with a unique set of witnesses ξ_i. We shall denote by $\varphi_i(\leq y)\downarrow^s$ that $\varphi_{i,s}(x)\downarrow$ for all $x \leq y$ but that there exists some $x \leq y$ such that $\varphi_{i,s-1}(x)\uparrow$. The strategy for satisfying the highness requirement $\mathcal{H}_i$ is as follows. At stage $s+1$ if $\varphi_i(\leq y)\downarrow^s$ for some $y \in \xi_i$ then extract from H all $x \in \xi_i$ such that $x < y$, otherwise do nothing.

If, for some $y \in \xi_i$, there is no stage s such that $\varphi_i(\leq y)\downarrow^s$ then φ_i is not total since there exists some $x \in \omega$ such that $x \leq y$ and $\varphi_i(x)\uparrow$. In this case the highness requirement $\mathcal{H}_i$ is satisfied finitely and we denote this outcome by 0. On the other hand, if we extract all of ξ_i from H then φ_i is total and the computation function C_H dominates φ_i. To see that φ_i is total, note that if $\varphi_i(x)\uparrow$ for some $x \in \omega$ then there would be some $y \in \xi_i$ such that $y < x$ and is never extracted from H. To see that C_H dominates φ_i note that for any pair of successive elements y_0 and y_1 of ξ_i we extract y_0 for the first time from H at stage $s+1$ when we see $\varphi_i(\leq y_1)\downarrow^s$, hence $C_H(x) \geq s+1$ for all $x \in [y_0, y_1]$ whilst $\varphi_i(x) \leq s$. (Of course we may assume that for every i, x and s, if $\varphi_{i,s}(x)\downarrow$ then $x \leq s$.) This outcome we shall denote by tot.

4 The tree of strategies

It will be easier to discuss the interaction of these strategies if first we describe the tree of strategies. We let

$$\Lambda = \{\infty <_\Lambda \text{fin}\} \cup \{\text{tot} <_\Lambda 0\}$$

Let $T = \Lambda^{<\omega}$ be our tree. If $\sigma \in T$ then $|\sigma|$ denotes the length of σ. We shall refer to the usual lexicographical order on T, denoted by $\preceq$. We write $\sigma \prec_{\text{L}} \tau$ to mean that $\sigma \prec \tau$ but σ is not an initial segment of τ. Each node of the tree T is assigned a strategy according to its length. We assign the strategy to satisfy $\mathcal{N}_{\langle j,k\rangle}$ to the nodes σ for which $|\sigma| = 2\langle j,k\rangle$ and assign the strategy to satisfy $\mathcal{H}_i$ to the nodes σ for which $|\sigma| = 2i+1$. Thus even nodes are assigned noncupping strategies whilst odd nodes are assigned highness strategies, which we call *noncupping* and *highness* nodes respectively, and we denote by $T_{\mathcal{N}}$ the noncupping nodes and by $T_{\mathcal{H}}$ the highness nodes on the tree.

A node $\sigma \in T$ may not act at every stage of the construction, in fact it may never act. Stages of the construction at which σ does act we call *σ-stages*. If a node σ is acting at a stage s we may refer to the previous σ-stage which we denote by s^-. After acting, if the node σ takes, say, the outcome $o \in \Lambda$ we *initialize* all nodes $\tau \in T$ such that $\sigma^\frown\langle o\rangle \prec_{\text{L}} \tau$, that is we essentially discard anything these nodes have done so far and force them to start their strategies again if they are called upon to act at some later stage.

5 Interaction of $\mathcal{N}$ strategies

The only interaction we need to be concerned with between noncupping strategies is if more than one were to choose the same number b_z, then the extraction of b_z by one strategy would injure the others. Let $\{\nu_\gamma\}_{\gamma\in T_{\mathcal{N}}}$ be a computable partition of ω into infinite sets, we then avoid the injury by insisting that each noncupping node γ only chooses numbers b_z from ν_γ.

A further complication is that a noncupping node σ may be waiting for $z \upharpoonright \Gamma^X$ but $z \in \Gamma^X_s$ at every σ-stage $s+1$ even though $z \notin \Gamma^X_t$ for infinitely many stages t. We can avoid this problem by instead of asking whether $z \notin \Gamma^X_s$ at a σ-stage $s+1$ we ask whether there exists a stage $t \in [s^-, s]$ such that $z \notin \Gamma^X_t$.

6 Interaction of $\mathcal{H}$ strategies

To avoid conflict between highness nodes assigned to satisfy the same requirement $\mathcal{H}_i$, we let $\{\xi_\tau\}_{\tau\in T_{\mathcal{H}}}$ be a computable partition of ω into infinite sets and insist that each highness node τ extract from H only numbers which belong to the set ξ_τ. In fact only if witnesses re-enter H may τ be injured, because the computation function $C_H(x)$ drops back to a value less than $\varphi_i(x)$, where $|\tau| = 2i + 1$, but since highness strategies only extract this kind of injury can only be caused by noncupping strategies and is dealt with below.

As a highness node τ only acts at τ-stages we modify our definition of $\varphi_i(\leq y)\downarrow^s$ in the following way so that τ is aware of changes in φ_i at each τ-stage. Let $y \in \xi_\tau$ and $s+1$ be a τ-stage then $\varphi_i(\leq y)\downarrow^s$ denotes that $\varphi_{i,s}(x)\downarrow$ for all $x \leq y$ but that there exists some $x \leq y$ such that $\varphi_{i,s^-}(x)\uparrow$.

7 Interaction of $\mathcal{N}$ and $\mathcal{H}$ strategies

A noncupping strategy will quite possibly enumerate and restrain finite sets of numbers in H and even want to permanently keep numbers in H, such numbers we will call "fixed" and those fixed by a particular noncupping node σ we denote by H_σ^{fix}. Where σ is a highness node H_σ^{fix} will represent all the numbers currently fixed by higher priority noncupping nodes. In what follows suppose σ is a noncupping node and τ is a highness node assigned to satisfy the highness requirement $\mathcal{H}_i$.

If $\sigma \prec \tau$ then the noncupping strategy σ may fix numbers in H which a lower priority highness strategy τ wishes to extract. Note if the "true" outcome of σ is ∞ then it does not fix any numbers in H. Suppose then that the "true" outcome of σ is fin then σ only ever acts on behalf of finitely many numbers z, because in order to choose a new number z to act on, all numbers less than z must be chosen and currently reside at step 5 or step 8 of the σ strategy. So for σ to act on infinitely many numbers σ must take the outcome ∞ infinitely often contradicting our supposition of fin being the true outcome. Thus let z_0 be the largest number σ acts on. Either there is at least one $z \leq z_0$ such that a finite subset of H is fixed permanently by σ after some stage s_0 (because the noncupping σ strategy has performed step 6 on behalf of the number z) or σ waits at step 3 forever on behalf of z_0, for otherwise σ would choose and act on $z_0 + 1$ contradicting z_0 being the largest number σ acts on. Thus the highness strategy τ is prevented from extracting at most only finitely many numbers from H. Since to show that C_H dominates φ_i it is sufficient to show that $C_H(x) \geq \varphi_i(x)$ for almost all x we see that the highness strategy still succeeds in the satisfaction of its requirment.

If $\tau \prec \sigma$ then only if $\sigma \supseteq \tau^\frown\langle\text{tot}\rangle$ will the lower priority noncupping strategy σ not be initialized and so may possibly be injured if the highness strategy τ extracts some number, which σ had fixed, from H. Since σ expects all numbers $x \in \xi_\tau$ which are not fixed to be extracted from H at some point, we shall not allow σ to believe in an axiom $\langle b, F_H \oplus F_X\rangle \in \Psi$ if F_H involves numbers from ξ_τ which are not fixed. Later a higher priority strategy may drop its restraint, which would allow τ to extract some previously fixed number from ξ_τ and so again possibly injuring σ, but in this case the action of the higher priority strategy in dropping its restraint also results in σ and τ being initialized.

Definition 4 *Let $\sigma \in T_{\mathcal{N}}$ and suppose we are at stage $s+1$ of the construction. An axiom $\langle b, F_H \oplus F_X\rangle \in \Psi_s$ is called σ-believable if*

$$F_H \cap \xi_\tau \subseteq \bigcup_{\gamma \prec \tau} H^{fix}_{\gamma,s+1}$$

for all $\tau \in T_{\mathcal{H}}$ for which $\sigma \supseteq \tau^\frown\langle \text{tot}\rangle$.

8 The construction

Besides $H^{\text{fix}}_{\sigma,s}$ the construction shall make use of several parameters. First of all at stage s we build a string $\delta_s \in T$ with $|\delta_s| = s$ and we define the approximations B_s and H_s to B and H respectively. Moreover, the finite set $Z_{\sigma,s}$ records which numbers z the noncupping strategy σ has already acted on or is ready to act on; $B_{\sigma,s}$ denotes the set of numbers b_z chosen from ν_σ corresponding to the elements of $Z_{\sigma,s}$; the parameter $\text{St}_s(\sigma,z)$ takes a value in $\{-1,3,5,7,8\}$ and records the state z is currently in relative to the module for the noncupping σ strategy: we let $\text{St}_s(\sigma,z) = -1$ denote that z is not as yet in the module; finally the parameter $\eta_{\sigma,z,s+1}$ counts how many different sets F_X for z have appeared during the construction for the noncupping strategy σ.

The parameters used during the construction retain their values from the previous stage unless otherwise specified.

If at a stage s we are asked to initialize a noncupping strategy σ we let $Z_{\sigma,s} = H^{\text{fix}}_{\sigma,s} = B_{\sigma,s} = \Gamma_{\sigma,s} = \emptyset$ and set $\text{St}_s(\sigma,z) = -1$ for all $z \in \omega$. Whereas if we are asked to initialize a highness strategy τ at stage s our only action is to set $H^{\text{fix}}_{\tau,s} = \emptyset$.

The construction proceeds by stages.

Stage $s = 0$. Let $H_0 = \omega, B_0 = \emptyset$, and $\delta_0 = \lambda$. For all $\sigma \in T$ let $Z_{\sigma,0} = H^{\text{fix}}_{\sigma,0} = B_{\sigma,0} = \emptyset$ and for all $\sigma \in T_{\mathcal{N}}$ let $\Gamma_{\sigma,0} = \emptyset$.

Stage $s+1$. Let $\delta_{s+1}\restriction 0 = \lambda$ and proceed by induction on $n < s+1$. Assume $\delta_{s+1}\restriction n = \sigma$ has been defined, we then carry out the applicable action described below depending on whether σ is even and hence an $\mathcal{N}$-node or odd and hence a $\mathcal{H}$-node.

σ-action for an $\mathcal{N}$-node. Assume the noncupping node σ is assigned the strategy to satisfy $\mathcal{N}_{\langle j,k\rangle}$, that is $|\sigma| = 2\langle j,k\rangle$. If $Z_{\sigma,s} = \emptyset$ let $Z_{\sigma,s+1} = \{0\}$ and define $\delta_{s+1}\restriction n+1 = \sigma^\frown\langle\text{fin}\rangle$. Otherwise $Z_{\sigma,s} = \{0,1,\ldots,m\}$ for some $m < s$ and we proceed inductively on the witnesses $z \in Z_{\sigma,s}$ performing the following σ-witness activity for each z.

σ-witness activity.

(a) If $z \in Z_{\sigma,s}$ is such that $z \in \overline{K}_s - \Gamma^X_{\sigma,s}$ and $\text{St}_s(\sigma,z) = -1$ then choose $b_z(\sigma)$ to be the least $b \in \nu_\sigma - B_{\sigma,s}$, let $B_{s+1} = B_s \cup \{b_z(\sigma)\}$ and $B_{\sigma,s+1} = B_{\sigma,s} \cup \{b_z(\sigma)\}$, and set $\text{St}_{s+1}(\sigma,z) = 3$ and $\eta_{\sigma,z,s+1} = 0$.

(b) If $z \in Z_{\sigma,s}$ is already in the module, attested by $\text{St}_s(\sigma,z) \neq -1$, we have a number of differring actions depending on where in the module z is. Assume in the following that $\eta = \eta_{\sigma,z,s}$ unless explicitly mentioned.

(i) $\mathrm{St}_s(\sigma, z) = 3$ and $b_z(\sigma) \in \Psi_{j,s}^{H_s \oplus X_{k,s}}$ via σ-believable axioms

$$\langle b_z(\sigma), F_H \oplus F_X \rangle \in \Psi_{j,s}.$$

If for any of these axioms, $F_X = F_X^i$ for any $i < \eta$ then set $\mathrm{St}_{s+1}(\sigma, z) = 5$, otherwise let $F_H^\eta(\sigma, z, s+1) = F_H, F_X^\eta(\sigma, z, s+1) = F_X$,

$$\Gamma_{\sigma,s+1} = \Gamma_{\sigma,s} \cup \left\{\langle z, F_X^\eta(\sigma, z, s+1)\rangle\right\},$$

where F_H and F_X are chosen from the σ-believable axiom for which

$$t_X = \mu t[(\forall u)(t \le u \le s \wedge F_X \subseteq X_{k,u})]$$

is minimal (if there is more than one then choose the one for which $F_H \oplus F_X$ has least canonical index), and set $\mathrm{St}_{s+1}(\sigma, z) = 5$ and $\eta_{\sigma,z,s+1} = \eta_{\sigma,z,s} + 1$.

(ii) $\mathrm{St}_s(\sigma, z) = 5$ and $z \in \overline{K}_s$ but there is some $t \in [s^-, s]$ such that $z \notin \Gamma_{\sigma,t}^{X_k}$. Set $\mathrm{St}_{s+1}(\sigma, z) = 3$.

(iii) $\mathrm{St}_s(\sigma, z) = 5$ and $z \notin \overline{K}_s$. Let

$$H_{s+1} = H_s \cup \bigcup_{i \le \eta} F_H^i(\sigma, z, s),$$
$$H_{\sigma,s+1}^{\mathrm{fix}} = H_{\sigma,s}^{\mathrm{fix}} \cup \bigcup_{i \le \eta} F_H^i(\sigma, z, s),$$

and $B_{s+1} = B_s - \{b_z(\sigma)\}$. Set $\mathrm{St}_{s+1}(\sigma, z) = 7$.

(iv) $\mathrm{St}_s(\sigma, z) = 7$ and there is some $t \in [s^-, s]$ such that $z \notin \Gamma_{\sigma,t}^{X_k}$. Let

$$H_{\sigma,s+1}^{\mathrm{fix}} = H_{\sigma,s}^{\mathrm{fix}} - \bigcup_{i \le \eta} F_H^i(\sigma, z, s)$$

and set $\mathrm{St}_{s+1}(\sigma, z) = 8$.

(v) $\mathrm{St}_s(\sigma, z) = 8$ and $z \in \Gamma_{\sigma,s}^{X_k}$. Let

$$H_{s+1} = H_s \cup \bigcup_{i \le \eta} F_H^i(\sigma, z, s)$$

and

$$H_{\sigma,s+1}^{\mathrm{fix}} = H_{\sigma,s}^{\mathrm{fix}} \cup \bigcup_{i \le \eta} F_H^i(\sigma, z, s).$$

Set $\mathrm{St}_{s+1}(\sigma, z) = 7$.

(c) Otherwise, we do nothing.

After the σ-witness activity for z has been completed, if $\mathrm{St}_{s+1}(\sigma, z) \in \{3,7\}$ define $\delta_{s+1} \upharpoonright n+1 = \sigma^\frown\langle\mathrm{fin}\rangle$ without performing σ-witness activity for any further witnesses $z \in Z_{\sigma,s}$. Otherwise, if $z < m$ let $z = z+1$ and perform the σ-witness activity on this new witness z, or, if $z = m$ set $Z_{\sigma,s+1} = Z_{\sigma,s} \cup \{m+1\}$ and define $\delta_{s+1} \upharpoonright n+1 = \sigma^\frown\langle\infty\rangle$.

σ-action for an $\mathcal{H}$-node. Assume the highness node σ is assigned the strategy to satisfy $\mathcal{H}_i$, that is $|\sigma| = 2i+1$. First set

$$H^{\mathrm{fix}}_{\sigma,s+1} = \bigcup_{\tau \prec \sigma} H^{\mathrm{fix}}_{\tau,s+1}.$$

Then, if this is the first σ-stage or if there is no $y \in \xi_\sigma$ such that $\varphi_i(\leq y)\downarrow^s$ then set $\delta_{s+1} \upharpoonright n+1 = \sigma^\frown\langle 0\rangle$, otherwise let

$$H_{s+1} = H_s - \left(\xi_\sigma \upharpoonright y - H^{\mathrm{fix}}_{\sigma,s+1}\right)$$

and set $\delta_{s+1} \upharpoonright n+1 = \sigma^\frown\langle\mathrm{tot}\rangle$.

At the end of stage $s+1$ when we have finished performing the action required by any of the applicable nodes and before proceeding to the next stage we initialize all nodes τ such that $\delta_{s+1} \upharpoonright s+1 \prec_{\mathrm{L}} \tau$.

9 The verification

Proof that the construction works.

Definition 5 *Let the true path* TP *be such that for every n*

$$\mathrm{TP} \upharpoonright n = \liminf_s \delta_s \upharpoonright n,$$

where the lim inf *is taken over the ordering $\preceq$ of nodes on the tree.*

Lemma 6 *For each $\langle j,k\rangle$ requirement $\mathcal{N}_{\langle j,k\rangle}$ is satisfied.*

Proof Let $\sigma = \mathrm{TP} \upharpoonright n \subset \mathrm{TP}$ be the node on the true path assigned the strategy to satisfy requirment $\mathcal{N}_{\langle j,k\rangle}$ (so $n = 2\langle j,k\rangle$). As $\sigma \subset \mathrm{TP}$ there is a stage s_0 such that $\sigma \preceq \delta_s$ for all $s \geq s_0$ and hence σ is not initialized at any stage $s \geq s_0$ since only if $\delta_s \prec_{\mathrm{L}} \sigma$ is σ initialized at stage s.

To verify that the requirement $\mathcal{N}_{\langle j,k\rangle}$ is satisfied it is enough to assume that $B = \Psi_j^{H \oplus X_k}$ and show there exists an e-operator Γ such that $\overline{K} = \Gamma^{X_k}$, for otherwise the requirement is satisfied trivially. Under this assumption $\sigma^\frown\langle\infty\rangle \subset$ TP, because if not $\lim_s \mathrm{St}_s(\sigma, z) \in \{3,7\}$ for some z and thus there is an associated number b_z such that $B(b_z) \neq \Psi_j^{H \oplus X_k}(b_z)$ contradicting our assumption. We show that the e-operator Γ_σ which σ builds is such that $\overline{K}(z) = \Gamma_\sigma^{X_k}(z)$ for all z.

Pick any $z \in \omega$. Note that if σ never performs σ-witness activity on z, then z has left $\overline{K}$ before σ defines any axiom putting z in $\Gamma_\sigma^{X_k}$. Thus $\overline{K}(z) = \Gamma_\sigma^{X_k}(z)$. Assuming then that σ-witness activity is performed on z there are two cases to consider.

Case I. If $z \in \overline{K}$ then $\liminf_s \mathrm{St}_s(\sigma, z) = 5$.

If further $\lim_s \mathrm{St}_s(\sigma, z) = 5$, this corresponds to waiting at step 5 of the basic noncupping strategy for $z \nearrow \Gamma_\sigma^{X_k}$, then there is some stage $s_1 \geq s_0$ such that $\mathrm{St}_s(\sigma, z) = 5$ for all $s \geq s_1$. Thus there can be no σ-stage $s+1 > s_1$ at which there exists some $t \in [s^-, s]$ such that $z \notin \Gamma_{\sigma,t}^{X_k}$ because at such a stage part b(ii) of the σ-witness activity would apply and σ would perform the associated action setting $\mathrm{St}_{s+1}(\sigma, z) = 3$ contrary to the choice of s_1. Therefore $z \in \Gamma_{\sigma,s}^{X_k}$ for all $s \geq s_1$ and hence $z \in \Gamma_\sigma^{X_k}$.

Otherwise, which corresponds to looping infinitely often through step 5 of the basic noncupping strategy, we are able to show contrary to our hypothesis that $B \neq \Psi_\sigma^{H \oplus X_k}$ via the witness b_z. Suppose for a contradiction that $B(b_z) = \Psi_\sigma^{H \oplus X_k}(b_z)$, then there exists at least one *true* σ-believable axiom $\langle b_z, F_H \oplus F_X \rangle \in \Psi_\sigma$, that is there is some stage $s_t \geq s_0$ for which $F_X \subseteq X_{k,s}$ for all $s \geq s_t$. Choose from these *true* σ-believable axioms the one for which such a stage s_t is least and, if necessary, for which $F_H \oplus F_X$ has least canonical index. Denote this choice by $\langle b_z, F_H^t \oplus F_X^t \rangle$, then if we have not already done so, there will be a σ-stage $s_2 \geq s_t$ at which we will define the axiom $\langle z, F_X^t \rangle \in \Gamma_{\sigma,s_2}$. Thus $z \in \Gamma_{\sigma,s}^{X_k}$ and hence $\mathrm{St}_s(\sigma, z) = 5$ for all $s \geq s_2$ contradicting our looping hypothesis. Thus there can be no such *true* σ-believable axioms $\langle b_z, F_H \oplus F_X \rangle \in \Psi_\sigma$ showing that $B(b_z) \neq \Psi_\sigma^{H \oplus X_k}(b_z)$ as required.

Case II. If $z \notin \overline{K}$ then $\liminf_s \mathrm{St}_s(\sigma, z) = 8$.

If further $\lim_s \mathrm{St}_s(\sigma, z) = 8$, which corresponds to waiting at step 8 of the basic noncupping strategy for $z \searrow \Gamma_\sigma^{X_k}$, there is some stage $s_1 \geq s_0$ such that $\mathrm{St}_s(\sigma, z) = 8$ for all $s \geq s_1$. Thus there can be no σ-stage $s+1 > s_1$ at which $z \in \Gamma_{\sigma,s}^{X_k}$ (although it is possible that $z \in \Gamma_{\sigma,t}^{X_k}$ for some $t \in [s^-, s)$) because at such a stage part b(v) of the σ-witness activity would apply and σ would perform the associated action setting $\mathrm{St}_{s+1}(\sigma, z) = 7$ contrary to the choice of s_1. Therefore $z \notin \Gamma_{\sigma,s}^{X_k}$ for infinitely many $s \geq s_1$ and hence $z \notin \Gamma_\sigma^{X_k}$.

Otherwise, which corresponds to looping through step 8 of the basic noncupping strategy, there exist infinitely many σ-stages $s+1 > s_0$ at which part b(v) of the σ-witness action applies, that is there are infinitely many stages $s \geq s_0$ such that $z \notin \Gamma_{\sigma,s}^{X_k}$. Therefore $z \notin \Gamma_\sigma^{X_k}$.

We have therefore shown that $B = \Psi_j^{H \oplus X_k} \Rightarrow \overline{K}(z) = \Gamma_\sigma^{X_k}(z)$ for all z as required. □

Lemma 7 *For each i requirement $\mathcal{H}_i$ is satisfied.*

Proof Let $\sigma = \mathrm{TP}\restriction n \subset \mathrm{TP}$ be the node on the true path assigned the strategy to satisfy the requirement $\mathcal{H}_i$ (so $n = 2i + 1$). As for the previous lemma there exists a stage s_0 such that $\sigma \preceq \delta_s$ for all $s \geq s_0$ and hence σ is not initialized at any stage $s \geq s_0$. A further consequence is that $H^{\text{fix}}_{\sigma,s} = H^{\text{fix}}_{\sigma,s_0}$ for all $s \geq s_0$.

We suppose that φ_i is total and show that C_H dominates it. This is sufficient to show that $\mathcal{H}_i$ is satisfied, for if φ_i is not total then $\mathcal{H}_i$ is satisfied vacuously. Choose any $x \in \omega$ such that $x \geq \mu y \in \xi_\sigma - H^{\text{fix}}_{\sigma,s_0}$. As φ_i is total there exists a stage s_x such that $\varphi_i(x)\downarrow^{s_x}$, that is $\varphi_{i,s}(x)\uparrow$ for all $s < s_x$ and $\varphi_{i,s}(x)\downarrow$ for all $s \geq s_x$. So $\varphi_i(x) < s_x$. Let $y_0, y_1 \in \xi_\sigma - H^{\text{fix}}_{\sigma,s_0}$ be the unique successive pair such that $x \in [y_0, y_1)$, then there exists as σ-stage $s + 1 \geq s_0$ such that $\varphi_i(\leq y_1)\downarrow^s$. Clearly, $s \geq s_x$. At stage $s + 1$ the node σ extracts y_0 from H for the first time, therefore $C_H(x) \geq s + 1$. This shows that $C_H(x) > \varphi_i(x)$ for all but those $x < \mu y \in \xi_\sigma - H^{\text{fix}}_{\sigma,s_0}$, hence C_H dominates φ_i. $\square$

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