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Invariance for Stochastic Differential Systems with Time-dependent Constraining Sets

Marius APETRII
Faculty of Mathematics, "Alexandru Ioan Cuza" University of Iaşi, Bd. Carol no. 9-11, 700506, *Iaşi, România*
E-mail: *mapetrii@uaic.ro*

Mihaela-Hanako MATCOVSCHI **Octavian PĂSTRĂVANU**
Department of Automatic Control and Industrial Informatics, Technical University "Gh. Asachi" of Iaşi, Blvd. Mangeron 53*A,* 700050*, Iaşi, România*
E-mail: *mhanako@ac.tuiasi.ro* *opastrav@ac.tuiasi.ro*

Eduard ROTENSTEIN[1)]
Faculty of Mathematics, "Alexandru Ioan Cuza" University of Iaşi, Bd. Carol no. 9-11, 700506, *Iaşi, România*
E-mail: *eduard.rotenstein@uaic.ro*

Abstract The first part of this article presents invariance criteria for a stochastic differential equation whose state evolution is constrained by time-dependent security tubes. The key results of this section are derived by considering an equivalent problem where the square of distance function represents a viscosity solution to an adequately defined partial differential equation. The second part of the paper analyzes the broader context when solutions are constrained by more general time-dependent convex domains. The approach relies on forward stochastic variational inequalities with oblique reflection, the generalized subgradients acting as a reacting process that operates only when the solution reaches the boundary of the domain.

Keywords Stochastic differential (in)equations, invariance, oblique reflection

MR(2010) Subject Classification 60H10, 60G17, 34G20, 93E15

1 Introduction

The problem of keeping the solution of a deterministic or stochastic differential system in a constraining constant or time-dependent set represents the interest of study for a vast community of researchers. Since early sixties, research focused on the study of reflected stochastic differential equations, the reflection process being considered in different ways. Skorohod, for instance, considered the problem of reflection for diffusion processes into a bounded domain (see, e.g., [21]). Tanaka focused on the problem of reflecting boundary conditions into convex sets for SDEs (see [23]). For practical applications, the study on reflected deterministic

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1) Corresponding author

and stochastic processes has an interesting and useful role in a variety of domains, such as control theory, game theory, financial mathematics, image processing, heavy traffic analysis of queueing networks or molecular dynamics. In the early studies, by imposing some admissibility conditions on the domain, there were proved the existence and uniqueness of the solution to the classical Skorohod problem in two different cases. In the first one, the authors consider the normal reflection on domains satisfying a uniform exterior ball condition. In the second case, the study is extended for smoothly varying directions of reflection on smooth domains. Saisho [20] (1987) proved that, in the situation of the normal reflection considered by Lions and Sznitman (see [13]), the admissibility condition can be weakened.

The multivalued SDEs with subdifferential operator, also called stochastic variational inequalities were introduced by Răşcanu [16] and consistent results were provided in 1997 by Asiminoaei and Răşcanu [1] and Cépa [6]. They proved the existence and uniqueness results for the case of stochastic variational differential systems involving subdifferential operators and they provide approximation and splitting-up schemes for this type of equations. A different approach for solving these type of equations was introduced in 2011 by Răşcanu and Rotenstein in [18]. Using the Fitzpatrick function, they reduced the existence problem for forward and backward stochastic variational inequalities to a minimizing problem of a convex lower semicontinuous function. The solutions of these equations are identified with the minimum points of some convex lower semicontinuous functionals, technique that became important for numerical approximations of the solution. In [12] the problem of keeping the solution in some constraining sets becomes more general allowing not only the reflection upon the normal direction but also by some oblique reflection to the frontier of the domain.

Our article presents new results that allow one to keep any trajectory of a stochastic differential system in a time-dependent set. Section 2 approaches the invariance of a time-dependent security tube for stochastic differential equations; we provide a necessary and sufficient condition by exploiting the square of the distance function as a viscosity solution to a partial differential equation with an appropriate construction. We also derive a sufficient condition for the case of the drift coefficient defined by a linear expression; this sufficient condition has a significantly simpler form that facilitates the concrete use in applications. For the considered invariance problem the last part of Section 2 deals with the construction of an absolutely continuous feedback. Section 3 investigates a broader context, where the solution of a deterministic or stochastic differential system is maintained in a time-dependent convex set via a bounded variation reacting process (that operates only when the solution reaches the boundary of the domain). The more general time dependence of the domain leads to the study of forward stochastic variational inequalities with oblique subgradients.

2 Security Tube Invariance for Some Specific SDEs

2.1 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ be a stochastic basis and $\{W_t : t \geq 0\}$ an $\mathbb{R}^k$-valued Brownian motion. Given a nonempty closed set $K \in \mathbb{R}^d$, a starting moment $t_0 \geq 0$ and a starting point $x_0 \in K$, it is known that, by adding a supplementary source (for example the convex indicator function of a given convex set K) on the stochastic equation

$$X_t^{t_0,x_0} = x_0 + \int_{t_0}^{t\vee t_0} f(r, X_r^{t_0,x_0})dr + \int_{t_0}^{t\vee t_0} g(r, X_r^{t_0,x_0})dW_r, \quad t \geq 0, \tag{2.1}$$

we can maintain the solution $X_t^{t_0,x_0} \in K$ for all $t \geq t_0$. It is natural to see what are the conditions on the drift and diffusion coefficients such that the evolution of the state satisfies the constraint $X_t^{t_0,x_0} \in K$ for all $t \geq t_0$. We will consider, as a particular case, the situation when the coefficients of (2.1) are characterized by polyhedral representations. This type of drift coefficients are often found in engineering applications, when the structure of the generator function can be represented as a convex combination of a finite number of time-dependent measurable data.

Let us consider now $\mathcal{K} = \{K(t) : t \geq 0\}$ a family of nonempty closed subsets $K(t) \subset \mathbb{R}^d$.

Definition 2.1 *We state that*

• *The family $\mathcal{K}$ is strongly invariant for* SDE (2.1) *if, for all $t_0 \geq 0$, $x_0 \in K(t_0)$ and, for all the solutions $\{X_t^{t_0,x_0} : t \geq t_0\}$ it follows that*

$$X_t^{t_0,x_0} \in K(t), \quad \mathbb{P}\text{- } a.s.\ \forall t \geq t_0.$$

• *The family $\mathcal{K}$ is weakly invariant (viable) for* SDE (2.1) *if, for every $t_0 \geq 0$ and $x_0 \in K(t_0)$, there exists a solution $\{X_t^{t_0,x_0} : t \geq t_0\}$ such that*

$$X_t^{t_0,x_0} \in K(t), \quad \mathbb{P}\text{- } a.s.\ \forall t \geq t_0.$$

Remark 2.2 If for every $(t_0, x_0) \in \mathbb{R}_+ \times K(t_0)$ equation (2.1) admits a unique solution, then the two notions coincide and in this case we shall say that $\mathcal{K}$ is invariant for SDE (2.1) (*or, alternatively, we can say that equation* (2.1) *is $\mathcal{K}$-invariant on* $[t_0, \infty)$). Of course, the invariance property can be defined for any bounded time interval $[t_0, T]$.

Our goal is to give a characterization of the invariance in the moving sets $K(t)$, $t \geq 0$. Consider the continuous functions $f : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ and $g : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^{d\times k}$ and assume that there exist $L, M > 0$ and $\mu \in \mathbb{R}$ such that, $\forall t \in [0, T]$, $\forall x, y \in \mathbb{R}^d$ we have

$$\begin{cases} \text{i) } \langle x - y, f(t, x) - f(t, y)\rangle \leq \mu|x - y|^2, \\ \text{ii) } |g(t, x) - g(t, y)| \leq L|x - y|, \\ \text{iii) } \sup\limits_{t\in[0,T]} |f(t, x)| \leq M(1 + |x|). \end{cases} \tag{2.2}$$

According to Friedman [11] it follows that, for every $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^d$, the equation (2.1) admits a unique solution X^{t_0,x_0} which verifies, for $p \geq 1$,

$$\begin{aligned} &\text{a) } \mathbb{E}\Big(\sup_{s\in[0,T]} |X_s^{t_0,x_0}|^p\Big) \leq C_{p,T}(1 + |x_0|^p), \\ &\text{b) } \mathbb{E}\Big(\sup_{s\in[0,T]} |X_s^{t_0,x_0} - X_s^{t_0',x_0'}|^p\Big) \leq C_{p,T}(1 + |x_0|^p)(|t_0 - t_0'|^{p/2} + |x_0 - x_0'|^p) \end{aligned} \tag{2.3}$$

for all $x_0, x_0' \in \mathbb{R}^d$ and $t_0, t_0' \in \mathbb{R}_+$.

Recall the notations

• the distance from x to the set $K(t)$:

$$d(t,x) = d_{K(t)}(x) = \inf\{|x-y| : y \in K(t)\},$$

• $\mathbb{S}^d \subset \mathbb{R}^{d\times d}$, the space of symmetric non-negative matrices,

• $C^{k,n}_{\rm pol}([0,T]\times\mathbb{R}^d)$, the set of functions $h:[0,T]\times\mathbb{R}^d \to \mathbb{R}$ of class $C^{k,n}$ such that the function h and its derivatives $D^i_t h(t,x)$, $j \in \overline{0,k}$ and $D^\alpha_x h(t,x)$, $\alpha = (\alpha_1,\ldots,\alpha_d)$, $0 \le \alpha_1 + \cdots + \alpha_d \le n$, $\alpha_i \in \mathbb{N}$ for every i, have polynomial increasing to infinity in the space variable, that is, there exist $C = C_T \ge 0$ and $p = p_T \in \mathbb{N}^*$ such that

$$\sum_{i,\alpha}[|D^i_t h(t,x)| + |D^\alpha_x h(t,x)|] \le C(1+|x|^p)$$

for all $(t,x) \in [0,T]\times\mathbb{R}^d$.

• the infinitesimal generator associated with $\{X^{t_0,x_0}_t : t \ge t_0\}$:

$$\begin{aligned}\mathcal{A}(t)\varphi(x) &= \frac{1}{2}\mathrm{Tr}[D^2_x\varphi(x)g(t,x)g^T(t,x)] + \langle f(t,x), \nabla_x\varphi(x)\rangle \\ &= \frac{1}{2}\sum_{j,l=1}^d (gg^*)_{j,l}(t,x)\frac{\partial^2\varphi(x)}{\partial x_j\partial x_l} + \sum_{j=1}^d f_j(t,x)\frac{\partial\varphi(x)}{\partial x_j}.\end{aligned}$$

Recall the definition of a viscosity solution for the following PDE:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} + \mathcal{A}(t)u(t,x) + G(t,x) = 0, \\ u(T,x) = \Theta(x), \quad (t,x) \in [0,T]\times\mathbb{R}^d, \end{cases} \tag{2.4}$$

where $G \in C^{0,0}_{\rm pol}([0,T]\times\mathbb{R}^d)$ and $\Theta \in C^{0,0}_{\rm pol}(\mathbb{R}^d)$.

Definition 2.3 *Let $u:[0,T]\times\mathbb{R}^d \to \mathbb{R}$ be an upper semicontinuous function and $(t,x) \in (0,T)\times\mathbb{R}^d$. We denote by $\mathcal{P}^{\,2,+}u(t,x)$ (the parabolic superjet of u at (t,x)) the set of triples $(p,q,S) \in \mathbb{R}\times\mathbb{R}^d\times S^d$ such that*

$$u(s,y) \le u(t,x) + p(s-t) + \langle q, y-x\rangle + \frac{1}{2}\langle S(y-x), y-x\rangle + o(|s-t| + |y-x|^2).$$

Let $u:[0,T]\times\mathbb{R}^d \to \mathbb{R}$ be a lower semicontinuous function and $(t,x) \in (0,T)\times\mathbb{R}^d$. We denote by $\mathcal{P}^{\,2,-}u(t,x)$ (the parabolic subjet of u at (t,x)) the set of triples $(p,q,S) \in \mathbb{R}\times\mathbb{R}^d\times S^d$ such that

$$u(s,y) \ge u(t,x) + p(s-t) + \langle q, y-x\rangle + \frac{1}{2}\langle S(y-x), y-x\rangle + o(|s-t| + |y-x|^2).$$

Definition 2.4 *We state*

(a) *A lower semicontinuous function $u:[0,T]\times\mathbb{R}^d \to \mathbb{R}$ is a viscosity supersolution of* (2.4) *if*

$$u(T,x) \ge \Theta(x), \quad \forall x \in \mathbb{R}^d$$

and, for any point $(t,x) \in (0,T)\times\mathbb{R}^d$ and for any $(p,q,S) \in \mathcal{P}^{2,-}u(t,x)$,

$$p + \frac{1}{2}\mathrm{Tr}(gg^*(t,x)S) + (f(t,x),q) + G(t,x) \le 0.$$

(b) *An upper semicontinuous function $u:[0,T]\times\mathbb{R}^d \to \mathbb{R}$ is a viscosity subsolution of* (2.4) *if*

$$u(T,x) \le \Theta(x), \quad \forall x \in \mathbb{R}^d$$

and, for any point $(t,x) \in (0,T) \times \mathbb{R}^d$ *and for any* $(p,q,S) \in \mathcal{P}^{2,+}u(t,x)$,

$$p + \frac{1}{2}\mathrm{Tr}((gg^*)(t,x)S) + (f(t,x),q) + G(t,x) \geq 0.$$

(c) $u \in C([0,T] \times \mathbb{R}^d)$ *is a viscosity solution of* (2.4) *if it is both a viscosity sub- and super-solution.*

According to Crandall et al. [9], we can write the definition of the viscosity solution for the parabolic PDE (2.4) under the equivalent form.

Definition 2.5 (a) *A lower semicontinuous function* $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ *is a viscosity supersolution of* (2.4) *if* $u(T,x) \geq \Theta(x)$, $\forall x \in \mathbb{R}^d$, *and for any* $\varphi \in C^2(\mathbb{R}^d)$ *and any* $(t,x) \in (0,T) \times \mathbb{R}^d$ *which is a local minimum of* $u - \varphi$,

$$\frac{\partial}{\partial t}\varphi(t,x) + \mathcal{A}(t)\varphi(t,x) + G(t,x) \leq 0.$$

(b) *An upper semicontinuous function* $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ *is a viscosity subsolution of* (2.4), *if* $u(T,x) \leq \Theta(x)$, $\forall x \in \mathbb{R}^d$, *and for any* $\varphi \in C^2(\mathbb{R}^d)$ *and any* $(t,x) \in (0,T) \times \mathbb{R}^d$ *which is a local maximum of* $u - \varphi$,

$$\frac{\partial}{\partial t}\varphi(t,x) + \mathcal{A}(t)\varphi(t,x) + G(t,x) \geq 0.$$

(c) *A continuous function* $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ *is a viscosity solution of* (2.4), *if it is a viscosity supersolution and a viscosity subsolution.*

The following characterization result takes place (see Buckdahn et al. [4, 5]).

Theorem 2.6 *Assume that, for every* $T > 0$, $x \in \mathbb{R}^d$ *the function* $t \mapsto d^2(t,x) : [0,T] \to \mathbb{R}$ *is lower semicontinuous and there exists* $b_T \geq 0$ *such that* $d^2(t,0) \leq b_T$, $\forall t \in [0,T]$. *Then the next assertions are equivalent*:

(I) *Equation* (2.1) *is* $\mathcal{K}$*-invariant on* $\mathbb{R}_+$.

(II) *For every* $T > 0$, *there exists a constant* $C = C_T \in \mathbb{R}$ *such that the square of the distance function* $h(t,x) = d^2(t,x)$ *is a viscosity supersolution of the equation*

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} + \mathcal{A}(t)u(t,x) - Cd^2(t,x) = 0, \quad (t,x) \in (0,T) \times \mathbb{R}^d, \\ u(T,x) = d^2(T,x), \ x \in \mathbb{R}^d, \end{cases} \tag{2.5}$$

that is,

$$p + \frac{1}{2}\mathrm{Tr}(Sgg^*(t,x)) + \langle f(t,x), q\rangle \leq Cd^2(t,x) \tag{2.6}$$

for all $(p,q,S) \in \mathcal{P}^{2,-}d^2(t,x)$, $(t,x) \in (0,T) \times \mathbb{R}^d$.

2.2 Main Development

We will analyze the case of a *control security tube.* We derive necessary and sufficient conditions that allow us to maintain the trajectory as a certain distance from a time-dependent point. If the drift coefficient has a linear form, given by a polyhedral representation, then the invariance conditions can be expressed using matrix measures.

Theorem 2.7 *Consider $\rho \in C^1([0,T];\mathbb{R}_+)$, $\rho > 0$, $a \in C^1([0,T];\mathbb{R}^d)$ and the time-dependent domain*

$$K(t) = \overline{B(a(t),\rho(t))} = \{x \in \mathbb{R}^d : |x-a(t)| \le \rho(t)\}. \tag{2.7}$$

Equation (2.1) *is $\overline{B(a(t),\rho(t))}$-invariant if and only if, $\forall (t,x) \in [0,T]\times\mathbb{R}^d$ with $|x-a(t)| = \rho(t)$, we have*

$$\begin{cases} \text{whenever } g^*(t,x)(x-a(t)) = 0 \text{ then} \\ 2\langle x-a(t), f(t,x)\rangle + |g(t,x)|^2 \le 2\langle x-a(t), a'(t)\rangle + 2\rho(t)\rho'(t). \end{cases} \tag{2.8}$$

Proof For $(t,x) \in (0,T)\times\mathbb{R}^d$,

$$h(t,x) = d^2(t,x) = ((|x-a(t)|-\rho(t))^+)^2$$

and, for $x \ne a(t)$,

$$p = \frac{\partial h}{\partial t}(t,x) = -2(|x-a(t)|-\rho(t))^+\left[\frac{1}{|x-a(t)|}\langle x-a(t), a'(t)\rangle + \rho'(t)\right],$$

$$q = \nabla_x h(t,x) = \begin{cases} 0, & \text{if } |x-a(t)| \le \rho(t), \\ 2\dfrac{|x-a(t)|-\rho(t)}{|x-a(t)|}(x-a(t)), & \text{if } |x-a(t)| > \rho(t), \end{cases}$$

$$S = \begin{cases} 0, \quad \text{if } |x-a(t)| \le \rho(t), \\ 2\dfrac{|x-a(t)|-\rho(t)}{|x-a(t)|}\mathbf{I}_{d\times d} + \dfrac{2\rho(t)}{|x-a(t)|^3}[x-a(t)]\otimes[x-a(t)], \text{ if } |x-a(t)| > \rho(t). \end{cases}$$

The SDE (2.1) is $\mathcal{K}$-invariant on $[0,T]$ iff the square of the distance function $d^2(t,x)$ is a viscosity supersolution of equation (2.5), or equivalently, for $(t,x) \in (0,T)\times\mathbb{R}^d$ with $|x-a(t)| > \rho(t)$,

$$\begin{aligned} &\frac{|x-a(t)|-\rho(t)}{|x-a(t)|}[2\langle x-a(t), f(t,x)\rangle + |g(t,x)|^2 - 2\langle x-a(t), a'(t)\rangle \\ &\quad - 2|x-a(t)|\rho'(t)] + \frac{\rho(t)}{|x-a(t)|^3}|g^*(t,x)(x-a(t))|^2 \\ &\le C(|x-a(t)|-\rho(t))^2. \end{aligned}$$

By taking the limit as $|x-a(t)| \searrow \rho(t)$, we obtain, for all $t \in [0,T]$ and for all $x \in \mathbb{R}^d$ with $|x-a(t)| = \rho(t)$, that, whenever $g^*(t,x)(x-a(t)) = 0$, we have

$$2\langle x-a(t), f(t,x)\rangle + |g(t,x)|^2 \le 2\langle x-a(t), a'(t)\rangle + 2\rho(t)\rho'(t).$$

This condition is also sufficient for the $\mathcal{K}$-invariance. Indeed, from (2.2), for all $x \in \mathbb{R}^d$ and $0 < \lambda < 1$,

$$\begin{aligned} |g(t,x)|^2 &\le [|g(t,\lambda(x-a(t))+a(t))| + L|x-\lambda(x-a(t))-a(t)|]^2 \\ &\le \frac{1}{\lambda}|g(t,\lambda(x-a(t))+a(t))|^2 + (1-\lambda)L^2|x-a(t)|^2 \end{aligned}$$

and

$$\begin{aligned} \langle x-a(t), f(t,x)\rangle &= \frac{1}{1-\lambda}\langle x-\lambda(x-a(t))-a(t), f(t,x)\rangle \\ &\le \frac{1}{1-\lambda}\langle x-\lambda(x-a(t))-a(t), f(t,\lambda(x-a(t))+a(t))\rangle \end{aligned}$$

$$+\frac{1}{1-\lambda}\mu^{+}|x-\lambda(x-a\,(t))-a\,(t)\,|^{2}$$
$$\leq\frac{1}{\lambda}\left\langle\lambda(x-a\,(t)),f(t,\lambda(x-a(t))+a(t))\right\rangle+(1-\lambda)\mu^{+}|x-a\,(t)\,|^{2}.$$

Therefore, for all $x\in\mathbb{R}^d$ with $|x-a(t)|>\rho(t)$ and $\lambda=\frac{\rho(t)}{|x-a(t)|}$,

$$\begin{aligned}
&2\left\langle x-a(t),f(t,x)\right\rangle+\left|g\,(t,x)\right|^{2}\\
&\quad\leq\frac{1}{\lambda}2\left\langle\lambda(x-a\,(t)),f(t,\lambda(x-a(t))+a(t))\right\rangle\\
&\qquad+\frac{1}{\lambda}\left|g\,(t,\lambda\,(x-a\,(t))+a\,(t))\right|^{2}+(1-\lambda)(2\mu^{+}+L^{2})|x-a\,(t)\,|^{2}\\
&\quad\leq\frac{2}{\lambda}\left[\left\langle\lambda(x-a\,(t)),a'(t)\right\rangle+\rho(t)\rho'\,(t)\right]+(1-\lambda)\left(2\mu^{+}+L^{2}\right)|x-a\,(t)\,|^{2}\\
&\quad=2\left\langle x-a\,(t)\,,a'(t)\right\rangle+2|x-a\,(t)\,|\rho'\,(t)+\left(2\mu^{+}+L^{2}\right)|x-a\,(t)\,|(|x-a\,(t)\,|-\rho\,(t)).
\end{aligned}$$

We also have

$$\begin{aligned}
&\left|g^{*}\,(t,x)\,(x-a(t))\right|^{2}\\
&\quad=\frac{|x-a\,(t)\,|^{2}}{\rho^{2}\,(t)}\left|g^{*}\,(t,x)\,\frac{\rho\,(t)\,(x-a\,(t))}{|x-a\,(t)\,|}-g^{*}\left(t,\frac{\rho\,(t)\,(x-a\,(t))}{|x-a\,(t)\,|}+a\,(t)\right)\frac{\rho\,(t)\,(x-a\,(t))}{|x-a\,(t)\,|}\right|^{2}\\
&\quad\leq\frac{|x-a\,(t)\,|^{2}}{\rho^{2}\,(t)}L^{2}\left|x-\frac{\rho\,(t)}{|x-a\,(t)\,|}(x-a\,(t))-a\,(t)\right|^{2}\left|\frac{\rho\,(t)\,(x-a\,(t))}{|x-a\,(t)\,|}\right|^{2}\\
&\quad=L^{2}|x-a\,(t)\,|^{2}(|x-a\,(t)\,|-\rho\,(t))^{2},
\end{aligned}$$

which leads to

$$\begin{aligned}
&\frac{|x-a(t)|-\rho(t)}{|x-a(t)|}[2\langle x-a(t),f(t,x)\rangle+|g(t,x)|^{2}-2\langle x-a(t),a'(t)\rangle\\
&\quad-2|x-a(t)|\rho'(t)]+\frac{\rho(t)}{|x-a(t)|^{3}}|g^{*}(t,x)(x-a(t))|^{2}\\
&\leq2(\mu^{+}+L^{2})(|x-a(t)|-\rho(t))^{2}.
\end{aligned}$$

Hence the equation (2.1) is $\overline{B(a(t),\rho(t))}$-invariant if and only if, $\forall(t,x)\in[0,T]\times\mathbb{R}^d$ with $|x-a(t)|=\rho(t)$, relation (2.8) takes place. Consequently, the one dimensional nondegenerate SDE cannot have the $\{\overline{B(a(t),\rho(t))}:t\in[0,T]\}$-invariance property. The proof is now complete. □

Comments on the linear case We consider a linear form for the drift coefficient in SDE (2.1) and let the set $K(t)$ be defined by (2.7), with $a(t)\equiv0$. Define $f(t,x)=A(t)x$, where $A:\mathbb{R}_+\to\mathbb{R}^{d\times d}$ is a continuous and bounded in norm matrix. The second condition from (2.8) can be rewritten as

$$2x^{T}A\,(t)\,x+\left|g\,(t,x)\right|^{2}\leq2\rho(t)\rho'(t)\quad\text{for }|x|=\rho(t),$$

where

$$2x^{T}A\,(t)\,x=x^{T}(A\,(t)+A^{T}(t))x=:x^{T}Q\,(t)\,x,$$

$Q\,(t)$ is a symmetric matrix. The Courant–Fischer theorem applied to $Q(t)$ gives us the dominant eigenvalue

$$\sigma_{\max}(Q\,(t))=\max_{x\neq0}\frac{x^{T}Q\,(t)\,x}{x^{T}x},$$

which implies

$$\max_{|x|=\rho(t)} x^T Q(t)\, x = \rho^2(t)\mu_{\max}(Q(t)).$$

On the other hand,

$$\sigma_{\max}(Q(t)) = 2\sigma_{\max}\left(\frac{1}{2}(A(t)+A^T(t))\right) = 2\mu_2(A(t)),$$

where

$$\mu_2(M) = \lim_{h\searrow 0}\frac{1}{h}\left[|I+hM|-1\right] = \frac{1}{2}\sigma_{\max}(M+M^T)$$

represents the measure of matrix M with respect to the matrix norm (e.g. Bernstein [2], Fact 11.15.7). Therefore, we have

$$\max_{\substack{|x|=\rho(t)\\ g^*(t,x)x=0}} 2x^T A(t)\, x \le \max_{|x|=\rho(t)} 2x^T A(t)\, x = 2\rho^2(t)\mu_2(A(t)).$$

Thus, from (2.8) we get the following sufficient condition for the invariance of the set $K(t)$:

$$\mu_2(A(t)) + \frac{1}{2}\rho^{-2}(t)\max_{\substack{|x|=\rho(t)\\ g^*(t,x)x=0}} |g(t,x)|^2 \le \rho^{-1}(t)\rho'(t), \quad \forall t\in[0,T]. \tag{2.9}$$

It is worth noticing that, for $g(t,x)\equiv 0$, $|x|=\rho(t)$, $\forall t\in[0,T]$, the necessary and sufficient condition (2.8) is equivalent to the inequality

$$\mu_2(A(t)) \le \rho^{-1}(t)\,\rho'(t), \quad \forall t\in[0,T]. \tag{2.10}$$

Inequality (2.10) is identical to the necessary and sufficient condition presented by paper [15] for the invariance of $K(t)$ with respect to deterministic linear dynamics. As expected, if the noise acts strictly inside the tube (i.e., the noise does not affect the boundary of $K(t)$), then we have the same characterization for the tube invariance as when the noise is completely absent within the whole tube.

Remark 2.8 Very often, in practical engineering applications, the matrix $A(t)$ is represented, at each moment, as an element from the convex hull of some measured data $A_1,\dots,A_n\in\mathbb{R}^{d\times d}$, which are non-singular matrices, bounded in norm. The form of the matrix $A(t)$ implies that the conditions for invariance can be rewritten by using the vertices of the convex hull. As consequence, we obtain that conditions (2.10) and (2.9) take place with $A(t)$ replaced by each vertex A_i for every $i=\overline{1,n}$.

2.3 Feedback-based Approach

Alternatively, keeping the above notations, we can reinterpret conditions (2.8) and the invariance problem from the perspective of finding a Lipschitz feedback law $U(t,x)$ which yields the time-dependent set $K(t)$ invariant for the SDE

$$X_t^{t_0,x_0} = x_0 + \int_{t_0}^{t\vee t_0} U(r,X_r^{t_0,x_0})dr + \int_{t_0}^{t\vee t_0} f(r,X_r^{t_0,x_0})dr + \int_{t_0}^{t\vee t_0} g(r,X_r^{t_0,x_0})dW_r, \quad t\ge 0. \tag{2.11}$$

The general case will be analyzed in the next section, by considering the problem in the framework of multivalued (normal and oblique reflected) stochastic variational inequalities. In this section we consider only the very simple case of $K(t)\equiv\overline{B(0,\rho)}$ for every t. According to

Răşcanu [17], there exists a continuous feedback law $K \in L^0(\Omega;\mathrm{BV}_{\mathrm{loc}}([0,\infty);\mathbb{R}^d))$ such that, for all $x \in \overline{B(0,\rho)}$, we have

$$\begin{cases} X_t^{t_0,x_0} = x_0, \quad \forall 0 \le t \le t_0, \\ X_t^{t_0,x_0} = x_0 - (K_t - K_{t_0}) + \displaystyle\int_{t_0}^t f(r, X_r^{t_0,x_0})dr + \int_{t_0}^t g(r, X_r^{t_0,x_0})dW_r, \quad t_0 \le t, \\ X_t^{t_0,x_0} \in \overline{B(0,\rho)} \quad \text{and} \quad dK_t \in \partial I_{\overline{B(0,\rho)}}(X_t^{t_0,x_0})(dt), \quad \forall t \ge 0. \end{cases}$$

By $L^0(\Omega;\mathrm{BV}_{\mathrm{loc}}([0,\infty);\mathbb{R}^d))$ we denoted the space of random processes with values in the space of local bounded variation functions $\mathrm{BV}_{\mathrm{loc}}([0,\infty);\mathbb{R}^d)$; the $\partial I_{\overline{B(0,\rho)}}$ represents the subdifferential operator of the convex indicator function $I_{\overline{B(0,\rho)}}(x) = 0$ if $x \in \overline{B(0,\rho)}$ and $+\infty$ otherwise. We have that $\partial I_{\overline{B(0,\rho)}}(x) = \{\hat{x} \in \mathbb{R}^d : \langle \hat{x}, y - x\rangle \le 0, \ \forall y \in \overline{B(0,\rho)}\} = N_{\overline{B(0,\rho)}}(x)$, where $N_{\overline{B(0,\rho)}}(x)$ represents the closed external cone normal to $\overline{B(0,\rho)}$ at x.

In this context, the problem consists in finding an absolutely continuous control

$$K_t = \int_0^t U(r, X_r^{t_0,x_0})dr. \tag{2.12}$$

The regularity (2.12) of the control will be lost if we consider general domains. Assuming that such a control exists and maintaining the particular linear form of the drift coefficient introduced in the previous sub-section, then by (2.8), $\overline{B(0,\rho)}$ is invariant if and only if, for all $t \ge 0$ and $|x| = \rho$,

$$\begin{cases} \text{each time when } g^*(t,x)\,x = 0, \text{ we obtain} \\ 2\langle x, U(t,x) + A(t)x\rangle + |g(t,x)|^2 \le 0. \end{cases}$$

Hence, in general, a $\overline{B(0,\rho)}$-invariance control of the form (2.12) does not exist, but if $g^*(t,x)x = 0$ for all $t \ge 0$ and $|x| = \rho$, then the feedback law

$$U(t,x) := -A(t)x - \frac{1}{2\rho^2}|g(t,x)|^2 x$$

yields the set $\overline{B(0,\rho)}$ invariant for the SDE (2.11). If we study the invariance only on the finite time interval $[0,T]$ and we want to use a linear feedback $U(t,x) = \theta(t)x$, then we can take

$$\theta(t) := -\frac{1}{2\rho^2}\sup_{|x|=\rho}|g(t,x)|^2 - |A(t)|$$

in order to obtain sufficient conditions of invariance for (2.11). Moreover, if the structure of $A(t)$ is the one introduced in Remark 2.8, then the linear feedback $U(t,x) = \theta(t)x$, with

$$\theta(t) := -\frac{1}{2\rho^2}\sup_{|x|=\rho}|g(t,x)|^2 - \max_{i=\overline{1,n}}|A_i|$$

assures the $\overline{B(0,\rho)}$-invariance for (2.11). Finally, it is important to mention that a time-independent feedback of the form $U(x) = \theta x$ can be used, where the constant θ is defined by

$$\theta := -\sup_{t\in[0,T]}\left\{\frac{1}{2\rho^2}\sup_{|x|=\rho}|g(t,x)|^2 + |A(t)|\right\}$$

and, respectively, by

$$\theta := -\frac{1}{2\rho^2}\sup_{\substack{|x|=\rho \\ t\in[0,T]}}|g(t,x)|^2 - \max_{i=\overline{1,n}}|A_i|.$$

3 Convex Domain Invariance for Generalized SVIs

The last part of the previous section was dedicated to the $\overline{B(0,\rho)}$-invariance of an SDE. In a more general setting we study the problem of keeping the solution in a convex time-dependent domain. When the solution reaches the frontier of the domain it is reflected, upon the normal direction, into the interior of the constraining time-dependent set. The feedback process is no more absolutely continuous as we previously obtained, but only a bounded variation one (see also Rozkosz, Słomiński [19]). The constant set given by the closed ball $\overline{B(0,\rho)}$ appears only as particular case of the sets studied in the current framework.

Consider the stochastic variational inequality (for short, SVI)

$$\begin{cases} dX_t + \partial I_{H(t)S}(X_t)(dt) \ni f(t,X_t)dt + g(t,X_t)\,dW_t, \quad t \geq 0, \\ X_0 = x_0, \end{cases} \tag{3.1}$$

where $S \subset \mathbb{R}^d$ is a nonempty closed convex set, $H : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function or $H = (h_{i,j})_{d\times d} \in C_b^2(\mathbb{R}_+;\mathbb{R}^{d\times d})$ is a symmetric matrix, such that for all $t \geq 0$,

$$\frac{1}{c}|u|^2 \leq \langle H(t)\,u, u\rangle \leq c\,|u|^2, \quad \forall u \in \mathbb{R}^d \text{ (for some } c \geq 1). \tag{3.2}$$

Let $[H(t)]^{-1}$ be the inverse matrix of $H(t)$. We have that $[H(t)]^{-1}$ has the same properties (3.2) as $H(t)$. Denote

$$b_H = \sup_{t,t'\in\mathbb{R}_+,\ t\neq t'} \frac{|H(t) - H(t')|}{|t-t'|} + \sup_{t,t'\in\mathbb{R}_+,\ t\neq t'} \frac{|\,[H(t)]^{-1} - [H(t')]^{-1}\,|}{|t-t'|},$$

where

$$|H(t)| := \left(\sum_{i,j=1}^{d} |h_{i,j}(t)|^2\right)^{1/2}.$$

Remark 3.1 It is important to mention that the set $H(t)\,S$, with H and S given as above is also a convex closed subset of $\mathbb{R}^d$. Therefore, the subdifferential operator of the lower semi-continuous convex indicator function $\partial I_{H(t)S}$ is well defined.

The coefficient functions $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $g(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times k}$ are supposed to be Carathéodory functions (i.e., measurable with respect to t and continuous with respect to x) and they must satisfy the following boundedness condition

$$M_{f,g} := \int_0^T \left[\sup_{x\in H(t)S} |f(t,x)|^2 + \sup_{x\in H(t)S} |g(t,x)|^4\right] dt < \infty. \tag{3.3}$$

Moreover, we also add Lipschitz continuity conditions on g: there exists $\ell \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$ such that, for every $x, y \in \mathbb{R}^d$, a.e. $t \geq 0$,

$$|g(t,x) - g(t,y)| \leq \ell(t)\,|x-y|\,. \tag{3.4}$$

Let $k : [t,T] \to \mathbb{R}^d$, where $0 \leq t \leq T$. We denote

$$\|k\|_{[t,T]} := \sup\{|k(s)| : t \leq s \leq T\},$$

and, for $t = 0$, $\|k\|_T := \|k\|_{[0,T]}$, which is the norm on the space of continuous functions $C([0,T];\mathbb{R}^d)$ Considering now $\mathcal{D}[t,T]$ the set of the partitions of the time interval $[t,T]$, of the form $\Delta = (t = t_0 < t_1 < \cdots < t_n = T)$, let

$$S_\Delta(k) = \sum_{i=0}^{n-1} |k(t_{i+1}) - k(t_i)|$$

and $\updownarrow k \updownarrow_{[t,T]} := \sup_{\Delta \in \mathcal{D}} S_\Delta(k)$; if $t = 0$, denote $\updownarrow k \updownarrow_T := \updownarrow k \updownarrow_{[0,T]}$. Let the space of bounded variation functions $\mathrm{BV}([0,T];\mathbb{R}^d) = \{k : [0,T] \to \mathbb{R}^d, \updownarrow k \updownarrow_T < \infty\}$. We will say that a function $k \in \mathrm{BV}_{\mathrm{loc}}([0,+\infty[;\mathbb{R}^d)$ if, for every $T > 0$, $k \in \mathrm{BV}([0,T];\mathbb{R}^d)$.

Definition 3.2 *Given two functions $x, k : \mathbb{R}_+ \to \mathbb{R}^d$, we say that $dk(t) \in \partial I_{H(t)S}(x(t))(dt)$ if*

$$\begin{aligned} &\text{(a) } x, k : \mathbb{R}_+ \to \mathbb{R}^d \text{ are continuous, } x(t) \in H(t)S, \\ &\text{(b) } k \in \mathrm{BV}_{\mathrm{loc}}\left([0,+\infty[;\mathbb{R}^d\right),\ k(0) = 0, \\ &\text{(c) } \int_s^t \langle y(r) - x(r), dk(r)\rangle \le 0, \text{ for all } 0 \le s \le t \le T \text{ and } y \in C\left([0,T];\mathbb{R}^d\right). \end{aligned} \tag{3.5}$$

Definition 3.3 *Given a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge 0})$, a pair of functions (X, K) is a strong solution of* (3.1) *if $X, K : \mathbb{R}_+ \to \mathbb{R}^d$ are two continuous functions and*

$$\begin{cases} \text{(i) } X_t + K_t = x_0 + \displaystyle\int_0^t f(r, X_r)dr + \int_0^t g(r, X_r)\, dW_r, \quad \forall t \ge 0, a.e.\ \omega \in \Omega \\ \text{(ii) } dK_r \in \partial I_{H(r)S}(X_r)(dr) \quad \text{for every } r. \end{cases} \tag{3.6}$$

Remark that, for the situation of a contracting time-dependent set characterized by the decreasing function $H(t) = \mathrm{e}^{-\alpha t}$ $(\alpha > 0)$, we have

$$\begin{aligned} \partial I_{\mathrm{e}^{-\alpha t}S}(x) &= \left\{\hat{x} \in \mathbb{R}^d : \langle \hat{x}, y - x\rangle \le 0,\ \forall y \in \mathrm{e}^{-\alpha t}S\right\} = \left\{\hat{x} \in \mathbb{R}^d : \left\langle \hat{x}, \mathrm{e}^{-\alpha t}z - x\right\rangle \le 0,\ \forall z \in S\right\} \\ &= \left\{\hat{x} \in \mathbb{R}^d : \left\langle \mathrm{e}^{-\alpha t}\hat{x}, z - \mathrm{e}^{\alpha t}x\right\rangle \le 0,\ \forall z \in S\right\} = \partial I_S(\mathrm{e}^{\alpha t}x). \end{aligned}$$

As consequence, denoting in (3.1) $Y_t := \mathrm{e}^{\alpha t}X_t$, we obtain that X is a solution for (3.1) if and only if Y is a solution for the following SVI

$$\begin{cases} dY_t + \partial I_S(Y_t)(dt) \ni (\alpha Y_t + \mathrm{e}^{\alpha t} f(t, \mathrm{e}^{-\alpha t}Y_t))\, dt + \mathrm{e}^{\alpha t} g(t, \mathrm{e}^{-\alpha t}Y_t)\, dW_t, \quad t \ge 0, \\ Y_0 = x_0. \end{cases} \tag{3.7}$$

According to Asiminoaei, Răşcanu [1], the equation (3.7) admits a unique solution (Y, K) in the sense of the above definition. The role of the bounded variation process K consists in pushing back the process Y into the constant set C when it reaches the frontier of the domain. The direction of the input is the normal direction at the frontier.

The situation changes when the function H is not a real valued one, but a time-dependent matrix, as we can see in the sequel. We have

$$H(t)\partial I_{H(t)S}(x) = H(t)\{\hat{x} \in \mathbb{R}^d : \langle H(t)\hat{x}, y - [H(t)]^{-1}x\rangle \le 0,\ \forall y \in S\} = \partial I_S([H(t)]^{-1}x)$$

Similar to the case of a real valued function H, we change the variable by denoting $Y_t = [H(t)]^{-1} X_t$ and we obtain, by the regularity of H, that Y must be a solution of the following SVI (the coefficient functions $\tilde{f}$ and $\tilde{g}$ are obtained in the same manner)

$$\begin{cases} dY_t + ([H(t)]^{-1})^2 \partial I_S(Y_t)(dt) \ni \tilde{f}(t, Y_t)dt + \tilde{g}(t, Y_t)\, dW_t, \quad t \ge 0, \\ Y_0 = [H(0)]^{-1} x_0, \end{cases} \tag{3.8}$$

where

$$\tilde{f}(t,Y_t) = [H(t)]^{-1}(f(t,H(t)Y_t) - H'(t)Y_t),$$
$$\tilde{g}(t,Y_t) = [H(t)]^{-1} g(t,H(t)Y_t).$$

As we can see the reflection is no more by the normal direction, the matrix $\tilde{H}(t) := ([H(t)]^{-1})^2$ transforming the normal reflecting direction into an oblique subgradient (see also Colombo, Goncharov [7], Constantini [8] or Dupuis, Ishii [10]). For the simplicity of the presentation, we will renounce the symbol tilde for H, f and g in the equation (3.8).

Using the same technique used in [12], we first consider the generalized Skorohod problem

$$\begin{cases} dy(t) + H(t)\partial I_S(y(t))(dt) \ni f(t,y(t))dt + dm_t, & t \in [0,T], \\ y(0) = y_0 = [H(0)]^{-1} x_0 \in S, \end{cases} \tag{3.9}$$

with the singular input $m \in C([0,T];\mathbb{R}^d)$, $m(0) = 0$; its modulus of continuity is given by $\mathbf{m}_m(\varepsilon) = \sup\{|m(u) - m(v)| : u,v \in [0,T],\ |u-v| \le \varepsilon\}$.

Theorem 3.4 *The (deterministic) differential equation* (3.9) *admits at least one solution (and we write* $(y,k) \in \mathcal{SP}(H\partial I_S; y_0, m)$*), i.e.,* $y,k : \mathbb{R}_+ \to \mathbb{R}^d$ *are two continuous functions and*

$$\begin{cases} y(t) + \int_0^t H(r)\,dk(r) = y_0 + \int_0^t f(r,y(r))\,dr + m(t), & \forall t \ge 0, \\ dk(r) \in \partial I_S(y(r))(dr) \quad \textit{for every } r. \end{cases} \tag{3.10}$$

Moreover, if $m \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^d)$*, then the generalized convex Skorohod problem with oblique subgradients* (3.9) *admits a unique solution* (y,k) *in the space*

$$C(\mathbb{R}_+;\mathbb{R}^d) \times [C(\mathbb{R}_+;\mathbb{R}^d) \cap \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^d)].$$

Proof First we will analyze the case of the smooth function m and, in the sequel, we consider the situation of the singular input $dm(t)$. For the first step, we use the Moreau–Yosida regularization of a convex function (see Brézis [3]) and we construct a penalized problem, whose solution converges to the solution of the equation. In the next step, we approximate the continuous function m by a sequence of smooth functions m_ε, we construct again a family of approximating equations and prove the convergence of their solutions to a solution of (3.10). We will sketch the main lines of the proof.

Step 1 Case $m \in C^1(\mathbb{R}_+;\mathbb{R}^d)$.

We consider the penalized problem

$$\begin{aligned} &y_\varepsilon(t) = y_0, \quad \text{if } t < 0, \\ &y_\varepsilon(t) + \int_0^t H(s)\,dk_\varepsilon(s) = x_0 + \int_0^t [f(s-\varepsilon, \pi_S(y_\varepsilon(s-\varepsilon))) + m'(s-\varepsilon)]\,ds, \quad t \in [0,T], \end{aligned}$$

or, equivalently,

$$\begin{aligned} &y_\varepsilon(t) = y_0, \quad \text{if } t < 0, \\ &y_\varepsilon(t) + \frac{1}{\varepsilon}\int_0^t H(s)(y_\varepsilon(s) - \pi_S(y_\varepsilon(s)))\,ds \end{aligned}$$

$$= y_0 + \int_{-\varepsilon}^{t-\varepsilon} \left[f\left(s, \pi_S\left(y_\varepsilon\left(s\right)\right)\right) + m'\left(s\right) \right] ds, \quad t \in [0,T], \tag{3.11}$$

where

$$k_\varepsilon(t) = \frac{1}{\varepsilon} \int_0^t \left(y_\varepsilon\left(s\right) - \pi_S\left(y_\varepsilon\left(s\right)\right) \right) ds$$

and $\pi_S(y)$ is the projection of y on the set

$$S = \overline{\mathrm{Dom}(I_S)} = \mathrm{Dom}(I_S),$$

uniquely defined by $\pi_S(x) \in C$ and

$$\mathrm{dist}(y, S) = |y - \pi_S(y)|.$$

The approximating equation (3.11) admits a unique solution $y_\varepsilon \in C\left([0,T]; \mathbb{R}^d\right)$. An easy differential equality applied to the class C^1 function

$$y \mapsto |y|^2 + \frac{1}{2\varepsilon} |y - \pi_S(y)|^2$$

leads to

$$\begin{aligned}
&|y_\varepsilon\left(t\right)|^2 + \frac{1}{2\varepsilon} |y_\varepsilon\left(t\right) - \pi_S(y_\varepsilon\left(t\right))|^2 \\
&\quad + \frac{1}{\varepsilon} \int_0^t \left\langle H\left(s\right)\left(y_\varepsilon\left(s\right) - \pi_S\left(y_\varepsilon\left(s\right)\right)\right), 2y_\varepsilon\left(s\right) + \frac{1}{\varepsilon}\left(y_\varepsilon\left(s\right) - \pi_C\left(y_\varepsilon\left(s\right)\right)\right) \right\rangle ds \\
&\ = |y_0|^2 + \frac{1}{2\varepsilon} |y - y_0|^2 \\
&\quad + \int_0^t \left\langle 2y_\varepsilon\left(s\right) + \frac{1}{\varepsilon}\left(y_\varepsilon\left(s\right) - \pi_C\left(y_\varepsilon\left(s\right)\right)\right), f\left(s-\varepsilon, \pi_S\left(y_\varepsilon\left(s-\varepsilon\right)\right)\right) + m'\left(s-\varepsilon\right) ds \right\rangle.
\end{aligned}$$

Taking now into consideration the hypothesis imposed on the matrix H and using standard inequalities of the form $xy \leq \frac{ax^2}{2} + \frac{y^2}{2a}$, with $a > 0$, one can follow the arguments used in [12] and we obtain, by taking the $\sup_{s \leq t}$, that there exists a positive constant C such that

$$\|y_\varepsilon\|_t^2 + \frac{1}{2\varepsilon} \sup_{s \leq t} |y_\varepsilon\left(s\right) - \pi_S(y_\varepsilon\left(s\right))|^2 + \frac{1}{\varepsilon^2} \int_0^t |(y_\varepsilon\left(r\right) - \pi_C\left(y_\varepsilon\left(r\right)\right))|^2 dr \leq C + C \int_0^t \|y_\varepsilon\|_r^2 \, dr.$$

We apply the Gronwall inequality and we finally get that there exists $C_T > 0$, independent of ε, such that

$$\sup_{t \in [0,T]} |y_\varepsilon\left(t\right)|^2 + \frac{1}{2\varepsilon} \sup_{t \in [0,T]} |y_\varepsilon\left(t\right) - \pi_S(y_\varepsilon\left(t\right))|^2 + \frac{1}{\varepsilon^2} \int_0^T |(y_\varepsilon\left(s\right) - \pi_S\left(y_\varepsilon\left(s\right)\right))|^2 ds \leq C_T. \tag{3.12}$$

Now, using again the equation, the Cauchy–Schwarz inequality and (3.12), we obtain, for all $0 \leq s \leq t \leq T$,

$$|y_\varepsilon\left(t\right) - y_\varepsilon\left(s\right)| \leq C_T[(t-s)^{1/2} + \mathbf{m}_m\left(t-s\right)],$$

where $\mathbf{m}_m$ is the modulus of continuity of m. Moreover,

$$\begin{aligned}
\updownarrow x_\varepsilon \updownarrow_{[s,t]} &\leq \frac{1}{\varepsilon} \int_s^t |H\left(r\right)\left(y_\varepsilon\left(r\right) - \pi_S\left(y_\varepsilon\left(r\right)\right)\right)| \, dr + \int_{s-\varepsilon}^{t-\varepsilon} |f\left(r, \pi_S\left(y_\varepsilon\left(r\right)\right)\right)| \, dr + \int_{s-\varepsilon}^{t-\varepsilon} |m'\left(r\right)| \, dr \\
&\leq C_T\left(t-s\right)^{1/2}.
\end{aligned}$$

Consequently, $\{y_\varepsilon : \varepsilon \in (0,1]\}$ is a bounded and uniformly equicontinuous subset of $C([0,T];\mathbb{R}^d)$. From Ascoli–Arzelà's theorem, it follows that there exists $\varepsilon_n \to 0$ and $y \in C\left([0,T];\mathbb{R}^d\right)$ such that

$$\lim_{n\to\infty}\Big[\sup_{t\in[0,T]}|y_{\varepsilon_n}(t)-y(t)|\Big]=0$$

and there exists $h \in L^2\left(0,T;\mathbb{R}^d\right)$ such that, on a subsequence, also denoted ε_n,

$$\pi_S\left(y_{\varepsilon_n}(s)\right) \to y \quad \text{in } L^2(0,T;\mathbb{R}^d) \text{ and a.e. in } [0,T], \quad \text{as } \varepsilon_n \to 0$$

and

$$\frac{1}{\varepsilon_n}\left(y_{\varepsilon_n}(s)-\pi_S\left(y_{\varepsilon_n}(s)\right)\right) \rightharpoonup h, \quad \text{weakly in } L^2(0,T;\mathbb{R}^d).$$

Therefore, for all $t\in[0,T]$,

$$\lim_{n\to\infty}\frac{1}{\varepsilon_n}\int_0^t H(s)\left(y_{\varepsilon_n}(s)-\pi_S\left(y_{\varepsilon_n}(s)\right)\right)ds=\int_0^t H(s))h(s)ds. \tag{3.13}$$

Passing to limit for $\varepsilon=\varepsilon_n\to 0$ in the approximating equation (3.11), via the Lebesgue dominated convergence theorem for the integral from the right-hand side, we get

$$y(t)+\int_0^t H(s)\,dk(s)=y_0+\int_0^t f(s,y(s))\,ds+m(t),$$

where

$$k(t)=\int_0^t h(s)\,ds.$$

Step 2 Case $m \in C\left([0,T];\mathbb{R}^d\right)$.

Extend the function $m(s)=0$ for $s\le 0$ and define, for $\varepsilon>0$,

$$m_\varepsilon(t)=\frac{1}{\varepsilon}\int_{t-\varepsilon}^t m(s)\,ds=\frac{1}{\varepsilon}\int_0^\varepsilon m(t+r-\varepsilon)\,dr,$$

We have $m_\varepsilon \in C^1([0,T];\mathbb{R}^d)$, $\|m_\varepsilon\|_T \le \|m\|_T$ and $\mathbf{m}_{m_\varepsilon}(\delta)\le \mathbf{m}_m(\delta)$. Let now $(y_\varepsilon,k_\varepsilon)$ be a solution of the approximating equation

$$\begin{cases} y_\varepsilon(t)+\displaystyle\int_0^t H(r)dk_\varepsilon(r)=y_0+\int_0^t f(r,y_\varepsilon(r))dr+m_\varepsilon(t), \quad t\ge 0,\\ dk_\varepsilon(r)\in\partial I_S(y_\varepsilon(r))(dr),\end{cases}$$

solution which exists according to the first step of the proof. We have

$$\int_s^t \langle z(r)-y_\varepsilon(r),dk_\varepsilon(r)\rangle \le 0, \quad \forall 0\le s\le t\le T,\ \forall z\in C\left([0,T];\mathbb{R}^d\right). \tag{3.14}$$

Denote by M_ε the continuous smooth function

$$M_\varepsilon(t):=\int_0^t f(r,y_\varepsilon(r))\,dr+m_\varepsilon(t).$$

Technical computations, which can be found in details in [14, Chapter 4] (and which are, in fact, inspired from [13] and [22]) yields that there exists $C_T=C_T\left(\|M_\varepsilon\|_T\right)>0$ such that, for all $0\le s\le t\le T$,

$$\begin{cases} \|y_\varepsilon\|_T+\updownarrow k_\varepsilon\updownarrow_T\le C_T\left(\|M_\varepsilon\|_T\right),\\ |y_\varepsilon(t)-y_\varepsilon(s)|+\updownarrow k_\varepsilon\updownarrow_t-\updownarrow k_\varepsilon\updownarrow_s\le C_T\left(\|M_\varepsilon\|_T\right)\sqrt{t-s+\mathbf{m}_{M_\varepsilon}(t-s)}.\end{cases} \tag{3.15}$$

Since

$$\|M_\varepsilon\|_T \leq \int_0^T \sup_{y\in S} |f(r,y)|dr + \|m\|_T$$

and

$$\mathbf{m}_{M_\varepsilon}(t-s) \leq (t-s)^{1/2} \int_0^T \sup_{y\in S} |f(r,y)|^2 dr + \mathbf{m}_m(t-s),$$

estimations (3.15) permit the use of Ascoli–Arzelà's theorem and it follows that there exists $\varepsilon_n \to 0$ and $y,k \in C([0,T];\mathbb{R}^d)$ such that

$$y_{\varepsilon_n} \to y \quad \text{and} \quad k_{\varepsilon_n} \to k \quad \text{in } C([0,T]\,;\mathbb{R}^d).$$

Moreover, since $\updownarrow\cdot\updownarrow : C\left([0,T]\,;\mathbb{R}^d\right) \to \mathbb{R}$ is a lower semicontinuous function, then

$$\updownarrow k \updownarrow_T \leq \liminf_{n\to+\infty} \updownarrow k_{\varepsilon_n} \updownarrow_T \leq C_{T,m}.$$

By the Helly–Bray theorem, we can pass to the limit and we have, for all $0 \leq s \leq t \leq T$,

$$\lim_{n\to\infty} \int_s^t \langle z(r) - y_{\varepsilon_n}(r), dk_{\varepsilon_n}(r)\rangle = \int_s^t \langle z(r) - y(r), dk(r)\rangle\,.$$

Passing now to $\liminf_{n\to+\infty}$ in (3.14) we infer

$$dk\,(r) \in \partial I_C\,(y\,(r))\,(dr)\,.$$

Finally, taking $\lim_{n\to\infty}$ in the approximating equation we obtain that (y,k) is a solution of the equation (3.10). The proof is now complete. □

We return now to the transformed oblique reflected stochastic variational inequality. First we introduce the notion of weak solution for an SVI; the existence of a weak solution and the pathwise uniqueness imply the existence of a strong solution for the equation.

Definition 3.5 *If there exists a stochastic basis* $(\Omega,\mathcal{F},\mathbb{P},\mathcal{F}_t)_{t\geq0}$*, an* $\mathbb{R}^k$*-valued* $\mathcal{F}_t$*-Brownian motion* $\{W_t : t\geq 0\}$ *and a pair* $(Y_\cdot, K_\cdot) : \Omega\times\mathbb{R}_+ \to \mathbb{R}^d\times\mathbb{R}^d$ *of* $\mathcal{F}_t$*-progressively measurable continuous stochastic processes such that*

$$(Y_\cdot\,(\omega)\,, K_\cdot\,(\omega)) \in \mathcal{SP}\,(H\partial I_S; Y_0, M_\cdot\,(\omega))\,, \quad \mathbb{P}\text{-}a.s.\ \omega\in\Omega,$$

with

$$M_t := \int_0^t f\,(s,Y_s)\,ds + \int_0^t g\,(s,Y_s)\,dW_s,$$

then the collection $(\Omega,\mathcal{F},\mathbb{P},\mathcal{F}_t,W_t,Y_t,K_t)_{t\geq0}$ *is called a weak solution of the* SVI (3.8)*.*

Theorem 3.6 *The* SVI *with oblique subgradients* (3.8) *admits at least one weak solution in the sense of Definition* 3.5*.*

Proof We will present only the main ideas of the proof, for more details the interested reader can consult [12]. First one constructs a sequence of approximating equations, whose unique sequence of solutions is tight in $C([0,T]\,;\mathbb{R}^{2d+1})$, fact which facilitate the use of the Prohorov and Skorokod theorems. Finally, we pass to the limit in order to obtain a weak solution for the transformed SVI.

We extend the functions $f(t,x)=0$ and $g(t,x)=0$ for $t<0$ and, for $l\in\mathbb{N}$, we consider the approximating equation

$$\begin{cases} Y_t^l = x_0, \quad \text{if } t<0, \quad Y_t^l + \int_0^t H(s)\, dK_s^l = x_0 + M_t^l, \quad t\geq 0, \\ dK_t^l \in \partial I_S\left(Y_t^l\right) dt, \end{cases} \tag{3.16}$$

where

$$\begin{aligned} M_t^l &= \int_0^t f(s,\pi_S(Y_{s-1/l}^l))ds + l\int_{t-1/l}^t \left[\int_0^s g(r,\pi_S(Y_{r-1/l}^l))dW_r\right] ds \\ &= \int_0^t f(s,\pi_S(Y_{s-1/l}^l))ds + \int_0^1 \left[\int_0^{t-\frac{1}{l}+\frac{1}{l}u} g(r,\pi_S(Y_{r-1/l}^l))dW_r\right] du \end{aligned}$$

and $\pi_S(y)$ is the orthogonal projection of y on $S=\overline{\mathrm{Dom}\,(I_S)}$. Since M^l is a C^1-continuous progressively measurable stochastic process, then the approximating equation (3.16) has a unique solution $\left(Y^l,K^l\right)$ of continuous progressively measurable stochastic processes. We show that our framework permits the use of some tightness results presented in [14] in order to prove that $U^l := \left(Y^l,K^l,\updownarrow K^l \updownarrow\right)$ is tight in $C\left([0,T];\mathbb{R}^{2d+1}\right)$. We estimate first the behavior of $\{M_t^l\}_l$ by using the convexity of the function $h(y)=y^4$ and the Cauchy–Schwarz inequality. Therefore, for $0\leq s<\tau\leq T$, $\tau-s\leq\varepsilon$,

$$\begin{aligned} &\mathbb{E}\left[\sup_{0\leq\theta\leq\varepsilon}\left|M_{t+\theta}^l - M_t^l\right|^4\right] \\ &\leq 8\varepsilon\left(\int_t^{t+\varepsilon}\sup_{y\in S}|f(r,y)|^2\,dr\right)^2 + C\int_0^1\left(\int_{t-\frac{1}{n}+\frac{1}{n}u}^{t+\varepsilon-\frac{1}{n}+\frac{1}{n}u}\sup_{y\in S}|g(r,y)|^2\,dr\right)^2 du \\ &\leq 8\varepsilon\left(\int_t^{t+\varepsilon}\sup_{y\in S}|f(r,y)|^2\,dr\right)^2 + C\varepsilon\int_0^1\left(\int_{t-\frac{1}{n}+\frac{1}{n}u}^{t+\varepsilon-\frac{1}{n}+\frac{1}{n}u}\sup_{y\in S}|g(r,y)|^4\,dr\right) du \\ &\leq C\varepsilon\times\sup_{0\leq s<\tau\leq T,\ \tau-s\leq\varepsilon}\left\{\left(\int_s^\tau\sup_{y\in S}|f(r,y)|^2\,dr\right)^2 + \int_s^\tau\sup_{y\in S}|g(r,y)|^4\,dr\right\}\leq\tilde{C}\varepsilon. \end{aligned}$$

Consequently, according to [14, Proposition 1.47], the family of laws of $\left\{M^l : l\geq 1\right\}$ is tight on $C\left([0,T];\mathbb{R}^d\right)$. Using the same results from [14, Chapter 4], that we used to derive (3.15) we also get

$$||U^l||_T \leq C_T\big(||M^l||_T\big) \quad \text{and} \quad \mathbf{m}_{U^l}(\varepsilon)\leq C_T\big(||M^l||_T\big)\sqrt{\varepsilon+\mathbf{m}_{M^l}(\varepsilon)}.$$

Finally, by evoking again Pardoux, Răşcanu [14, Proposition 1.48], it infers that the family of laws of the random variables $U^l = \left(Y^l,K^l,\updownarrow K^l\updownarrow\right)$ is tight on $C\left([0,T];\mathbb{R}^d\right)\times C\left([0,T];\mathbb{R}^d\right)\times C\left([0,T];\mathbb{R}\right)\left[\equiv C\left([0,T];\mathbb{R}^{2d+1}\right)\right]$. By the Prohorov theorem, there exists a subsequence such that, as $l\to\infty$,

$$\left(Y^l,K^l,\updownarrow K^l\updownarrow,W\right)\to(Y,K,V,W), \quad \text{in law}$$

on $C([0,T];\mathbb{R}^{2d+1+k})$ and, by the Skorohod theorem, we can choose a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and some random quadruples $(\bar{Y}^l,\bar{K}^l,\bar{V}^l,\bar{W}^l)$, $(\bar{Y},\bar{K},\bar{V},\bar{W})$ defined on $(\Omega,\mathcal{F},\mathbb{P})$, having the same laws as resp. $(Y^l,K^l,\updownarrow K^l\updownarrow,W)$ and (Y,K,V,W), such that, in $C([0,T];\mathbb{R}^{2d+1+k})$, as $l\to\infty$,

$$(\bar{Y}^l,\bar{K}^l,\bar{V}^l,\bar{W}^l)\to(\bar{Y},\bar{K},\bar{V},\bar{W}).$$

Passing to the limit we obtain that $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \mathcal{F}_t, \bar{Y}_t, \bar{K}_t, \bar{W}_t)_{t\geq 0}$ is a weak solution of SVI (3.8). □

Remark 3.7 According to [12], we have the pathwise uniqueness for (3.8) and due to the results of Ikeda and Watanabe we know that the existence of a weak solution and the pathwise uniqueness imply the existence of a strong one. As consequence, SVI (3.8) admits a unique strong solution.

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References

[1] Asiminoaei, I., Răşcanu, A.: Approximation and simulation of stochastic variational inequalities — splitting up method. *Numer. Funct. Anal. Optim.*, **18**(3–4), 251–282 (1997)

[2] Bernstein, D. S.: Matrix Mathematics. In: Theory, Facts, and Formulas, Princeton Univ. Press, Princeton, New Jersey, 2009

[3] Brézis, H.: Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert, North-Holland, Amsterdam, 1973

[4] Buckdahn, R., Quincampoix, M., Rainer, C., et al.: Viability of moving sets for stochastic differential equation. *Adv. Differential Equations*, **7**(9), 1045–1072 (2002)

[5] Buckdahn, R., Quincampoix, M., Răşcanu, A.: Viability property for a backward stochastic differential equation and applications to partial differential equations. *Probab. Theory Related Fields*, **116**(4), 485–504 (2000)

[6] Cépa, E.: Problème de Skorohod multivoque, [multivalued Skorohod problem] (in French). *Ann. Probab.*, **26**(2), 500–532 (1998)

[7] Colombo, G., Goncharov, V.: Variational inequalities and regularity properties of closed sets in Hilbert spaces. *J. Convex Anal.*, **8**(1), 197–221 (2001)

[8] Constantini, C.: The Skorohod oblique reflection problem in domains with corners and application to stochastic differential equations. *Probab. Theory Related Fields*, **91**, 43–70 (1992)

[9] Crandall, M. G., Ishii, H., Lions, P. L.: User's Guide to Viscosity Solutions of Second Order Partial Differential Equations. *Bull. Amer. Math. Soc.*, **27**(1), 1–67 (1992)

[10] Dupuis, P., Ishii, H.: SDEs with oblique reflection on nonsmooth domains. *Ann. Probab.*, **21**(1), 554–580 (1993)

[11] Friedman, A.: Stochastic Differential Equations and Applications I, Academic Press, New York, 1975

[12] Gassous, A., Răşcanu., A., Rotenstein, E.: Stochastic variational inequalities with oblique subgradients. *Stochastic Process. Appl.*, **122**(7), 2668–2700 (2012)

[13] Lions, P.-L., Sznitman, A.: Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math.*, **37**(4), 511–537 (1984)

[14] Pardoux, E., Răşcanu, A.: Stochastic differential equations, Backward SDEs, Partial differential equations, Stochastic Modelling and Applied Probability, Vol. 69, XVII, Springer, Cham, 2014

[15] Păstrăvanu, O., Matcovschi, M.-H.: Linear time-variant systems: Lyapunov functions and invariant sets defined by Hölder norms. *J. Franklin Inst.*, **347**, 627–640 (2010)

[16] Răşcanu, A.: Existence for a class of stochastic parabolic variational inequalities. *Stochastics* , **5**(3), 201–239 (1981)

[17] Răşcanu, A.: Deterministic and stochastic differential equations in Hilbert spaces involving multivalued maximal monotone operators. *Panamer. Math. J.*, **6**(3), 83–119 (1996)

[18] Răşcanu, A., Rotenstein, E.: The Fitzpatrick function — a bridge between convex analysis and multivalued stochastic differential equations. *J. Convex Anal.*, **18**(1), 105–138 (2011)

[19] Rozkosz, A., Słomiński, L.: On stability and existence of solutions of SDEs with reflection at the boundary. *Stochastic Process. Appl.*, **68**(2), 285–302 (1997)

[20] Saisho, Y.: Stochastic differential equations for multidimensional domain with reflecting boundary. *Probab. Theory Related Fields*, **74**, 455–477 (1987)

[21] Skorohod, A.: Stochastic equations for diffusion processes in a bounded region. *Teor. Veroyatn. Primen.*, **6**, 264–274 (1961); **7**, 3–23 (1962)

[22] Słomiński, L.: On existence, uniqueness and stability of solutions of multidimensional SDEs with reflecting boundary conditions. *Ann. Inst. H. Poincaré Probab. Statist.*, **29**(2), 163–198 (1993)

[23] Tanaka, H.: Stochastic differential equations with reflecting boundary condition in convex regions. *Hiroshima Math. J.*, **9**(1), 163–177 (1979)