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Convergence analysis of hybrid projection with Cesàro mean method for the split equilibrium and general system of finite variational inequalities[☆]

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ABSTRACT

In this paper, we present a new iterative algorithm for finding a common element of three solution sets; (i) split equilibrium problem; (ii) general system of finite variational inequality problem; and (iii) fixed point problem. This algorithm is modified by hybrid method based on Cesàro mean in real Hilbert spaces. Furthermore, a strong convergent theorem is established and this theorem is the generalization of many previously known results in this research area. Finally, we study the rate of convergence of the iterative algorithm and some illustrative numerical examples are presented.

1. Introduction

Let H_1 and H_2 be real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $\{x_n\}$ be a sequence in H_1, then $x_n \to x$ (respectively, $x_n \rightharpoonup x$) will denote strong (respectively, weak) convergence of the sequence $\{x_n\}$. We denote $Fix(T) := \{x \in C : Tx = x\}$, the fixed points set of a mapping T.

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A mapping $T : C \to C$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

A mapping $T : C \to C$ is called *L-Lipschitzian* if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

A mapping $A : C \to H_1$ is called *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

A mapping $A : C \to H_1$ is called *α-inverse-strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Let $A : C \to H_1$ be a nonlinear mapping. The *variational inequality problem* is to find a $x^* \in C$ such that

$$\langle Ax^*, y - x^*\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

In this paper, our main purpose is to study the split problem. First, we recall some background in the literature.

Problem 1 (*The Split Feasibility Problem (SFP)*)**.** Let C and Q be two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively and $A : H_1 \to H_2$ be a bounded linear operator. The *split feasibility problem* (SFP) is to find a point

$$x^* \in C \quad \text{such that } Ax^* \in Q, \tag{1.2}$$

which was first introduced by Censor and Elfving [1] in medical image reconstruction.

A special case of the SFP is the *convexly constrained linear inverse problem* (CLIP) in a finite dimensional real Hilbert space [2]:

$$\text{find } x^* \in C \quad \text{such that } Ax^* = b, \tag{1.3}$$

where C is a nonempty closed convex subset of a real Hilbert space H_1 and b is a given element of a real Hilbert space H_2, which has extensively been investigated by using the Landweber iterative method [3]:

$$x_{n+1} = x_n + \gamma A^T(b - Ax_n), \quad n \in \mathbb{N}.$$

Assume that the SFP (1.2) is consistent (i.e., (1.2) has a solution), it is not hard to see that $x^* \in C$ solves (1.2) if and only if it solves the following *fixed point equation*;

$$x^* = P_C(I - \gamma A^*(I - P_Q)A)x^*, \quad x^* \in C, \tag{1.4}$$

where P_C and P_Q are the (orthogonal) projections onto C and Q, respectively, $\gamma > 0$ is any positive constant and A^* denotes the adjoint of A. Moreover, for sufficiently small $\gamma > 0$, the operator $P_C(I - \gamma A^*(I - P_Q)A)$ which defines the fixed point equation in (1.4) is nonexpansive.

An iterative sequence for solving the SFP, called the CQ algorithm, has the following form:

$$x_{k+1} = P_C(x_k + \gamma A^T(P_Q - I)Ax_k). \tag{1.5}$$

The operator

$$T = P_C(I - \gamma A^T(I - P_Q)A), \tag{1.6}$$

is averaged whenever $\gamma \in (0, \frac{2}{L})$ with L is the largest eigenvalue of the matrix A^TA (T stand for matrix transposition), and so the CQ algorithm converges to a fixed point of T, whenever such fixed points exist.

When the SFP has a solution, the CQ algorithm converges to a solution; when it does not, the CQ algorithm converges to a minimizer, over C, of the proximity function $g(x) = \|P_QAx - Ax\|$, whenever such minimizer exists. The function $g(x)$ is convex and according to [4], its gradient is

$$\nabla g(x) = A^T(I - P_Q)Ax. \tag{1.7}$$

Problem 2 (*The Split Equilibrium Problem (SEP)*)**.** In 2011, Moudafi [5] introduced the following split equilibrium problem (SEP):

Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions and $A : H_1 \to H_2$ be a bounded linear operator, then the *split equilibrium problem* (SEP) is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \quad \forall x \in C, \tag{1.8}$$

and

$$y^* = Ax^* \in Q \quad \text{such that } F_2(y^*, y) \geq 0, \ \forall y \in Q. \tag{1.9}$$

As is known, (1.8) is the classical equilibrium problem (EP) (find $x \in C$ such that $F(x, y) \geq 0$, $\forall y \in C$), and we denoted its solution set by $EP(F_1)$. The SEP (1.8) and (1.9) constitutes a pair of equilibrium problems which have to be solved so that the image $y^* = Ax^*$ under a given bounded linear operator A, of the solution x^* of the EP (1.8) in H_1 is the solution of another EP (1.9) by $EP(F_2)$.

The solution set SEP (1.8) and (1.9) is denoted by $\Gamma = \{x^* \in EP(F_1) : Ax^* \in EP(F_2)\}$.

In 1975, Baillon [6] proved the first non-linear ergodic theorem.

Theorem 1.1 (*Baillon's Ergodic Theorem*)**.** *Suppose that C is a nonempty closed convex subset of Hilbert space H_1 and $T : C \to C$ is nonexpansive mapping such that $Fix(T) \neq \emptyset$ then $\forall x \in C$, the* ***Cesàro mean***

$$T_n x = \frac{1}{n+1}\sum_{i=0}^{n} T^i x, \tag{1.10}$$

weakly converges to a fixed point of T.

In 1997, Shimizu and Takahashi [7] studied the convergence of an iteration process sequence $\{x_n\}$ for a family of nonexpansive mappings in the framework of a real Hilbert space. this sequence $\{x_n\}$ is as follows:

$$x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j x_n, \quad \text{for } n = 0, 1, 2, \ldots, \tag{1.11}$$

where x_0 and x are all elements of C and α_n is an appropriate in $[0, 1]$. They proved that x_n converges strongly to an element of fixed point of T which is the nearest to x.

Let $A, B : C \to H_1$ be two mappings. Ceng et al. [8] consider the following problem which finds $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C. \end{cases} \tag{1.12}$$

The problem (1.12) is called a *general system of variational inequalities*, where $\lambda > 0$ and $\mu > 0$ are constants. In particular, if $A = B$, then the problem (1.12) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \mu Ax^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C. \end{cases} \tag{1.13}$$

Furthermore, if we add up the requirement that $x^* = y^*$, $\lambda = \mu = 1$, $A = B$, the problem (1.12) reduces to the classical variational inequality problem (1.1).

In order to find the common element of the solutions of problem (1.12) and the set of fixed points of one nonexpansive mapping T, Ceng et al. [8] studied the following algorithm: $x_1 = u \in C$ and

$$\begin{cases} y_n = P_C(x_n - \mu Bx_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda Ay_n). \end{cases} \tag{1.14}$$

Under appropriate conditions they obtained one strong convergence theorem.

In this paper, motivated by above facts, we introduce the following system of variational inequalities in a Hilbert space H_1. Let C be a nonempty closed convex subset of a real Hilbert space H_1. Let $\{B_i\}_i^N : C \to H_1$ be a family of mappings. We consider the following problem which find $(x_1^*, x_2^*, \ldots, x_N^*) \in C \times C \cdots \times C$ such that

$$\begin{cases} \langle \lambda_N B_N x_N^* + x_1^* - x_N^*, x - x_1^* \rangle \geq 0, & \forall x \in C, \\ \langle \lambda_{N-1} B_{N-1} x_{N-1}^* + x_N^* - x_{N-1}^*, x - x_N^* \rangle \geq 0, & \forall x \in C, \\ \vdots \\ \langle \lambda_2 B_2 x_2^* + x_3^* - x_2^*, x - x_3^* \rangle \geq 0, & \forall x \in C, \\ \langle \lambda_1 B_1 x_1^* + x_2^* - x_1^*, x - x_2^* \rangle \geq 0, & \forall x \in C. \end{cases} \tag{1.15}$$

This problem is called a more *general system of variational inequalities* in Hilbert spaces, where $\lambda_i > 0$ for all $i \in \{1, 2, \ldots, N\}$. The solution set of (1.15) is denoted by $GSVI(C, B_i)$. In particular, if $N = 2$, $B_1 = A$, $B_2 = B$, $\lambda_1 = \lambda$, $\lambda_2 = \mu$, $x_1^* = y^* = x_3^*$, $x_2^* = x^*$, the problem (1.15) reduces to problem (1.12).

In this paper, we present a new iterative algorithm for finding a common element of three solution sets; (i) split equilibrium problem; (ii) general system of finite variational inequality problem; and (iii) fixed point problem. This algorithm is modified by hybrid method based on Cesàro mean in real Hilbert spaces. Furthermore, a strong convergent theorem is established and this theorem is the generalization of many previously known results in this research area. Finally, we study the rate of convergence of the iterative algorithm and some illustrative numerical examples are presented.

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2. Preliminaries

Let H_1 be a real Hilbert space. Then

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle, \tag{2.1}$$

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \tag{2.2}$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \tag{2.3}$$

for all $x, y \in H_1$ and $y \in [0, 1]$. It is also known that H_1 satisfies the *Opial's condition* [9], i.e., for any sequence $\{x_n\} \subset H_1$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \tag{2.4}$$

holds for every $y \in H_1$ with $x \neq y$. Hilbert space H_1 satisfies the *Kadec–Klee property* [10] that is, for any sequence $\{x_n\}$ if $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ then $\|x_n - x\| \to 0$.

We recall some concepts and results which are needed in sequel. A mapping P_C is said to be *metric projection* of H_1 onto C if for every point $x \in H_1$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C. \tag{2.5}$$

It is well known that P_C is a nonexpansive mapping and is characterized by the following property:

$$\|P_C x - P_C y\|^2 \le \langle x-y, P_C x - P_C y\rangle, \quad \forall x, y \in H_1. \tag{2.6}$$

Moreover, $P_C x$ is characterized by the following properties:

$$\langle x - P_C x, y - P_C x\rangle \le 0, \tag{2.7}$$

$$\|x-y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in H_1, y \in C, \tag{2.8}$$

and

$$\|(x-y) - (P_C x - P_C y)\|^2 \ge \|x-y\|^2 - \|P_C x - P_C y\|^2, \quad \forall x, y \in H_1. \tag{2.9}$$

It is known that every nonexpansive operator $T : H_1 \to H_1$ satisfies the inequality;

$$\langle (x - T(x)) - (y - T(y)), T(y) - T(x)\rangle \le \frac{1}{2}\|(T(x) - x) - (T(y) - y)\|^2, \tag{2.10}$$

for all $(x, y) \in H_1 \times H_1$. Therefore, for all $(x, y) \in H_1 \times Fix(T)$, we get

$$\langle x - T(x), y - T(x)\rangle \le \frac{1}{2}\|T(x) - x\|^2, \tag{2.11}$$

(see, e.g., Theorem 3 in [11] and Theorem 1 in [12]).

In the context of the variational inequality problem implies the following:

$$u \in VI(C, B) \Leftrightarrow u = P_C(u - \lambda Bu), \quad \forall \lambda > 0.$$

If B is α-inverse-strongly monotone mapping of C into H_1, then, for all $u, v \in C$ and $\lambda > 0$, we have

$$\begin{aligned}
\|(I-\lambda B)u - (I-\lambda B)v\|^2 &= \|(u-v) - \lambda(Bu - Bv)\|^2 \\
&= \|u-v\|^2 - 2\lambda\langle Bu - Bv, u - v\rangle + \lambda^2\|Bu - Bv\|^2 \\
&\le \|u-v\|^2 + \lambda(\lambda - 2\alpha)\|Bu - Bv\|^2.
\end{aligned} \tag{2.12}$$

So, if $\lambda \le 2\alpha$, then $I - \lambda B$ is a nonexpansive mapping from C to H_1.

Lemma 2.1 *([13]). Let C be a nonempty closed convex subset of a real Hilbert space H_1. Let $B_i : C \to H_1$ be an α_i-inverse-strongly accretive mapping, where $i \in \{1, 2, \ldots, N\}$. Let $G : C \to C$ be a mapping defined by*

$$G(x) = P_C(I - \lambda_N B_N)P_C(I - \lambda_{N-1}B_{N-1}) \cdots P_C(I - \lambda_2 B_2)P_C(I - \lambda_1 B_1)x, \quad \forall x \in C.$$

If $0 < \lambda_i \le 2\alpha_i$, $i = 1, 2, \ldots, N$, then $G : C \to C$ is nonexpansive.

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Proof. Put $\Omega^i = P_C(I-\lambda_i B_i)P_C(I-\lambda_{i-1}B_{i-1})\cdots P_C(I-\lambda_2 B_2)P_C(I-\lambda_1 B_1)$, $i=1,2,\ldots,N$ and $\Omega^0 = I$, where I is identity mapping. Then $G=\Omega^N$. For all $x,y\in C$, it follows from (2.12) that

$$\begin{aligned}
\|Gx-Gy\| &= \|\Omega^N x - \Omega^N y\| \\
&= \|P_C(I-\lambda_N B_N)\Omega^{N-1}x - P_C(I-\lambda_N B_N)\Omega^{N-1}y\| \\
&\le \|(I-\lambda_N B_N)\Omega^{N-1}x - (I-\lambda_N B_N)\Omega^{N-1}y\| \\
&\le \|\Omega^{N-1}x - \Omega^{N-1}y\| \\
&\ \ \vdots \\
&\le \|\Omega^0 x - \Omega^0 y\| \\
&= \|x-y\|,
\end{aligned}$$

which implies G is nonexpansive. This completes the proof. □

Lemma 2.2 ([14]). *Let $F: C\times C\to\mathbb{R}$ be a bifunction satisfying the following assumptions:*

(i) $F(x,x)\ge 0$, $\forall x\in C$;
(ii) *F is monotone, i.e.,* $F(x,y)+F(y,x)\le 0$, $\forall x\in C$;
(iii) *F is upper hemicontinuous, i.e., for each $x,y,z\in C$,*

$$\limsup_{t\to 0} F(tz+(1-t)x,y)\le F(x,y); \tag{2.13}$$

(iv) *For each $x\in C$ fixed, the function $y\mapsto F(x,y)$ is convex and lower semicontinuous;*
(v) *Fixed $r>0$ and $z\in C$, there exists a nonempty compact convex subset K of H_1 and $x\in C\cap K$ such that*

$$F(y,x)+\frac{1}{r}\langle y-x, x-z\rangle < 0, \quad \forall y\in C\setminus K. \tag{2.14}$$

Lemma 2.3 ([15]). *Assume that the bifunctions $F_1: C\times C\to\mathbb{R}$ satisfying Lemma 2.2. For $r>0$ and for all $x\in H_1$, define a mapping $T_r^{F_1}: H_1\to C$ as follows:*

$$T_r^{F_1}(x)=\left\{z\in C: F_1(z,y)+\frac{1}{r}\langle y-z, z-x\rangle\ge 0,\ \ \forall y\in C\right\}. \tag{2.15}$$

Then, the followings hold:

(1) *$T_r^{F_1}$ is nonempty and single-valued.*
(2) *$T_r^{F_1}$ is firmly nonexpansive, i.e.,*

$$\|T_r^{F_1}x - T_r^{F_1}y\|^2 \le \langle T_r^{F_1}x - T_r^{F_1}y, x-y\rangle, \quad \forall x,y\in H_1. \tag{2.16}$$

(3) *$Fix(T_r^{F_1}) = EP(F_1)$.*
(4) *$EP(F_1)$ is closed and convex.*

Further, assume that $F_2: Q\times Q\to\mathbb{R}$ satisfies Lemma 2.2. For $s>0$ and for all $w\in H_2$, a mapping $T_s^{F_2}: H_2\to Q$ is defined as follows:

$$T_s^{F_2}(w)=\left\{d\in Q: F_2(d,e)+\frac{1}{s}\langle e-d, d-w\rangle\ge 0,\ \ \forall e\in Q\right\}. \tag{2.17}$$

Then, we easily observe that $T_s^{F_2}$ is nonempty, single-valued and firmly nonexpansive. Also, $EP(F_2,Q)$ is closed and convex, and $Fix(T_s^{F_2}) = EP(F_2,Q)$, where $EP(F_2,Q)$ is the solution set of the following equilibrium problem:

Find $y^*\in Q$ such that $F_2(y^*,y)\ge 0$, $\forall y\in Q$.

We observe that $EP(F_2)\subset EP(F_2,Q)$. Furthermore, it is easy to prove that Γ is a closed and convex set.

Lemma 2.4 ([16]). *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space and $T: C\to C$ be a nonexpansive mapping. For each $x\in C$ and the Cesàro means $T_n x = \frac{1}{n+1}\sum_{i=0}^{n} T^i x$, then $\limsup_{n\to\infty}\|T_n x - T(T_n x)\| = 0$.*

Lemma 2.5 ([17]). *Each Hilbert space H satisfies the Opial condition that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\longrightarrow\infty}\|x_n - x\| < \liminf_{n\longrightarrow\infty}\|x_n - y\|$, holds for every $y\in H$ with $y\ne x$.*

Lemma 2.6 ([18] Demiclosedness Principle). *Let C be a closed convex subset of a real Hilbert space H and let $T: C\longrightarrow C$ be a nonexpansive mapping. Then $I-T$ is demiclosed at zero, that is, $x_n\rightharpoonup x$, $x_n - Tx_n\to 0$ implies $x = Tx$.*

3. Main result

Theorem 3.1. *Let H_1 and H_2 be two real Hilbert spaces. Let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let B_i be η_i-inverse-strongly monotone, respectively, where $i \in \{1, 2, \dots, N\}$. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Lemma 2.2 and F_2 be upper semicontinuous. Let $\{S^i\}_{i=1}^n$ be a sequence of nonexpansive mappings from C into itself such that*

$$\mathcal{F} := \Gamma \cap Fix(G) \cap (\cap_{i=1}^n Fix(S^i)) \neq \emptyset,$$

where G defined by Lemma 2.1. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $\alpha_n \le a$ for all $n \ge 1$ and for some $0 < a < 1$. Pick any $x_0 \in H_1$ and set $C_1 = C$. Let $\{x_n\}$ be a sequence generated by $x_1 = P_{C_1}x_0$ and

$$\begin{cases} u_n = T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n), \\ y_n = P_C(I - \lambda_N B_N)P_C(I - \lambda_{N-1}B_{N-1}) \cdots P_C(I - \lambda_2 B_2)P_C(I - \lambda_1 B_1)u_n, \\ z_n = \alpha_n y_n + (1 - \alpha_n)\dfrac{1}{n+1}\displaystyle\sum_{i=1}^{n} S^i y_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad \forall n \ge 1, \end{cases} \tag{3.1}$$

where $\{\lambda_i\} \subset (0, 2\eta_i)$, $i \in \{1, 2, \dots, N\}$, $\{r_n\} \subset (0, \infty)$, $\liminf_{n\to\infty} r_n > 0$ and $\gamma \in (0, \frac{1}{L})$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A. Then the sequence $\{x_n\}$ converges strongly to $P_{\mathcal{F}}x_0$.

Proof. Step 1. We show that $\{x_n\}$ is well defined and C_n is closed and convex for any $n \in \mathbb{N}$.

From the assumption, we see that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \ge 1$. Now, we will show that C_{k+1} is closed and convex for some k. For any $p \in C_k$, we obtain that

$$\begin{aligned} \|z_k - p\| \le \|x_k - p\| &\Leftrightarrow \|z_k - p\|^2 \le \|x_k - p\|^2 \\ &\Leftrightarrow \|z_k - x_k + x_k - p\|^2 \le \|x_k - p\|^2 \\ &\Leftrightarrow \|z_k - p\|^2 + 2\langle z_k - x_k, x_k - p\rangle + \|x_k - p\|^2 \le \|x_k - p\|^2. \end{aligned}$$

This implies that $\|z_k - p\| \le \|x_k - p\|$ is equivalent to $\|z_k - x_k\|^2 + 2\langle z_k - x_k, x_k - p\rangle \le 0$. So, C_{k+1} is closed and convex. Then, for any $n \in \mathbb{N}$, C_n is closed and convex. This implies that $\{x_n\}$ is well-defined.

Step 2. We show by mathematics induction $\mathcal{F} \subset C_n$ for each $n \in \mathbb{N}$.

Putting

$$\Omega^i = P_C(I - \lambda_i B_i)P_C(I - \lambda_{i-1}B_{i-1}) \cdots P_C(I - \lambda_2 B_2)P_C(I - \lambda_1 B_1), \quad \forall i \in \{1, 2, \dots, N\},$$

$\Omega^0 = I$ where I is the identity mapping on H_1.

Since $p \in \mathcal{F}$, i.e., $p \in \Gamma$, and we have $p = T_{r_n}^{F_1}p$ and $Ap = T_{r_n}^{F_2}Ap$.

We estimate

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n) - p\|^2 \\ &= \|T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n) - T_{r_n}^{F_1}p\|^2 \\ &\le \|x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n - p\|^2 \\ &\le \|x_n - p\|^2 + \gamma^2\|A^*(T_{r_n}^{F_2} - I)Ax_n\|^2 + 2\gamma\langle x_n - p, A^*((T_{r_n}^{F_2}) - I)Ax_n\rangle. \end{aligned} \tag{3.2}$$

Thus, we have

$$\|u_n - p\|^2 \le \|x_n - p\|^2 + \gamma^2\langle (T_{r_n}^{F_2} - I)Ax_n, AA^*(T_{r_n}^{F_2} - I)Ax_n\rangle + 2\gamma\langle x_n - p, A^*(T_{r_n}^{F_2} - I)Ax_n\rangle. \tag{3.3}$$

Now, we have

$$\begin{aligned} \gamma^2\langle (T_{r_n}^{F_2} - I)Ax_n, A^*A(T_{r_n}^{F_2} - I)Ax_n\rangle &\le L\gamma^2\langle (T_{r_n}^{F_2} - I)Ax_n, AA^*(T_{r_n}^{F_2} - I)Ax_n\rangle \\ &= L\gamma^2\|(T_{r_n}^{F_2} - I)Ax_n\|^2. \end{aligned} \tag{3.4}$$

Denoting $\Lambda := 2\gamma\langle x_n - p, A^*(T_{r_n}^{F_2} - I)Ax_n\rangle$ and using (2.11), we have

$$\begin{aligned} \Lambda &= 2\gamma\langle x_n - p, A^*(T_{r_n}^{F_2} - I)Ax_n\rangle \\ &= 2\gamma\langle A(x_n - p), (T_{r_n}^{F_2} - I)Ax_n\rangle \\ &= 2\gamma\langle A(x_n - p) + (T_{r_n}^{F_2} - I)Ax_n - (T_{r_n}^{F_2} - I)Ax_n, (T_{r_n}^{F_2} - I)Ax_n\rangle \\ &= 2\gamma\{\langle T_{r_n}^{F_2}Ax_n - Ap, (T_{r_n}^{F_2} - I)Ax_n\rangle - \|(T_{r_n}^{F_2} - I)Ax_n\|^2\} \end{aligned}$$

$$\begin{aligned}&\leq 2\gamma\left\{\frac{1}{2}\|(T_{r_n}^{F_2}-I)Ax_n\|^2-\|(T_{r_n}^{F_2}-I)Ax_n\|^2\right\}\\&\leq -\gamma\|(T_{r_n}^{F_2}-I)Ax_n\|^2.\end{aligned}\tag{3.5}$$

Using (3.3)–(3.5), we obtain

$$\|u_n-p\|^2\leq\|x_n-p\|^2+\gamma(L\gamma-1)\|(T_{r_n}^{F_2}-I)Ax_n\|^2.\tag{3.6}$$

From the definition of γ, we obtain

$$\|u_n-p\|^2\leq\|x_n-p\|^2.\tag{3.7}$$

It follows from (3.1) and (3.7) that

$$\begin{aligned}\|y_n-p\|&=\|\Omega^N u_n-\Omega^N p\|\\&\leq\|u_n-p\|\\&\leq\|x_n-p\|.\end{aligned}\tag{3.8}$$

Let $S_n=\frac{1}{n+1}\sum_{i=0}^{n}S^i$, it follows that

$$\begin{aligned}\|S_nx-S_ny\|&=\left\|\frac{1}{n+1}\sum_{i=0}^{n}S^ix-\frac{1}{n+1}\sum_{i=0}^{n}S^iy\right\|\\&\leq\frac{1}{n+1}\sum_{i=0}^{n}\|S^ix-S^iy\|\\&\leq\frac{1}{n+1}\sum_{i=0}^{n}\|x-y\|\\&=\frac{n+1}{n+1}\|x-y\|\\&=\|x-y\|,\end{aligned}$$

which implies that S_n is nonexpansive. Since $p\in\mathcal{F}$, we have $S_np=\frac{1}{n+1}\sum_{i=0}^{n}S^ip=\frac{1}{n+1}\sum_{i=0}^{n}p=p$, for all $x,y\in C$.

It follows from (3.1) and (3.8), we have

$$\begin{aligned}\|z_n-p\|&=\|\alpha_n(y_n-p)+(1-\alpha_n)(S_ny_n-p)\|\\&\leq\alpha_n\|y_n-p\|+(1-\alpha_n)\|S_ny_n-p\|\\&\leq\alpha_n\|y_n-p\|+(1-\alpha_n)\|y_n-p\|\\&\leq\|y_n-p\|\\&\leq\|x_n-p\|.\end{aligned}\tag{3.9}$$

And hence $p\in C_{n+1}$. This implies that $\mathcal{F}\subset C_n$ for all $n\in\mathbb{N}$.

Step 3. We will show that $\lim_{n\to\infty}\|x_n-x_0\|$ exists.

From $x_n=P_{C_n}x_0$, we have

$$\langle x_0-x_n,z-x_n\rangle\leq 0,\tag{3.10}$$

then

$$\langle x_0-x_n,x_n-z\rangle\geq 0,\tag{3.11}$$

for each $z\in C_n$. Using $\mathcal{F}\subset C_n$, we also have

$$\langle x_0-x_n,x_n-p\rangle\geq 0\quad\text{for each } p\in\mathcal{F}\ \text{ and }\ n\in\mathbf{N}.\tag{3.12}$$

Then, for $p\in\mathcal{F}$, we obtain

$$\begin{aligned}0&\leq\langle x_0-x_n,x_n-p\rangle\\&=\langle x_0-x_n,x_n-x_0+x_0-p\rangle\\&=\langle x_0-x_n,x_n-x_0\rangle+\langle x_0-x_n,x_0-p\rangle\\&\leq-\langle x_0-x_n,x_0-x_n\rangle+\langle x_0-x_n,x_0-p\rangle\\&\leq-\|x_0-x_n\|^2+\langle x_0-x_n,x_0-p\rangle\\&\leq-\|x_0-x_n\|^2+\|x_0-x_n\|\|x_0-p\|.\end{aligned}$$

This implies that

$$\|x_0 - x_n\| \le \|x_0 - p\| \quad \text{for all } p \in \mathcal{F} \text{ and } n \in \mathbf{N}. \tag{3.13}$$

From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we have

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \ge 0, \tag{3.14}$$

and

$$\begin{aligned} 0 &\le \langle x_0 - x_n, x_n - x_{n+1}\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\ &= \langle x_0 - x_n, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\ &= -\langle x_0 - x_n, x_0 - x_n\rangle + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\ &\le -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\ &\le -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|. \end{aligned}$$

It follows that

$$\|x_0 - x_n\| \le \|x_0 - x_{n+1}\|, \quad \text{for all } n \in \mathbf{N}. \tag{3.15}$$

Thus, the sequence $\{\|x_n - x_0\|\}$ is a bounded and nondecreasing sequence. Hence, there exists m such that

$$\lim_{n\to\infty} \|x_n - x_0\| = m. \tag{3.16}$$

Step 4. We will show the following:

(i) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$,
(ii) $\lim_{n\to\infty} \|x_n - z_n\| = 0$,
(iii) $\lim_{n\to\infty} \|x_n - u_n\| = 0$,
(iv) $\lim_{n\to\infty} \|u_n - y_n\| = 0$,
(v) $\lim_{n\to\infty} \|y_n - x_n\| = 0$,
(vi) $\lim_{n\to\infty} \|S_n y_n - y_n\| = 0$.

From (3.14), we get

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|x_n - x_0 + x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n\rangle + 2\langle x_n - x_0, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &\le \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 - 2\langle x_n - x_0, x_n - x_0\rangle + \|x_0 - x_{n+1}\|^2 \\ &\le \|x_n - x_0\|^2 - 2\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2 \\ &= -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2. \end{aligned}$$

From (3.16), we obtain $\|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \to 0$. Therefore

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.17}$$

By $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we have

$$\|x_{n+1} - z_n\| \le \|x_{n+1} - x_n\|. \tag{3.18}$$

Furthermore, we also obtain

$$\|x_n - z_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \le \|x_n - x_{n+1}\| + \|x_n - x_{n+1}\| \le 2\|x_n - x_{n+1}\|. \tag{3.19}$$

From (3.17), we have

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.20}$$

From (2.3), (3.1), (3.6) and (3.8), we have

$$\begin{aligned}\|z_n-p\|^2 &= \|\alpha_n y_n+(1-\alpha_n)S_n y_n-p\|^2\\ &\le \alpha_n\|y_n-p\|^2+(1-\alpha_n)\|S_n y_n-p\|^2\\ &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|y_n-p\|^2\\ &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|u_n-p\|^2\\ &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)[\|x_n-p\|^2+\gamma(L\gamma-1)\|(T_{r_n}^{F_2}-I)Ax_n\|^2]\\ &\le \|x_n-p\|^2-(1-\alpha_n)\gamma(1-L\gamma)\|(T_{r_n}^{F_2}-I)Ax_n\|^2. \end{aligned} \tag{3.21}$$

Therefore,

$$\begin{aligned}(1-\alpha_n)\gamma(1-L\gamma)\|(T_{r_n}^{F_2}-I)Ax_n\|^2 &\le \|x_n-p\|^2-\|z_n-p\|^2\\ &= (\|x_n-p\|-\|z_n-p\|)(\|x_n-p\|+\|z_n-p\|)\\ &= (\|x_n-p-z_n+p\|)(\|x_n-p\|+\|z_n-p\|)\\ &= \|x_n-z_n\|(\|x_n-p\|+\|z_n-p\|).\end{aligned}$$

From $(1-\alpha_n)\gamma(1-L\gamma)>0$ and (3.20), we obtain

$$\lim_{n\to\infty}\|(T_{r_n}^{F_2}-I)Ax_n\|=0. \tag{3.22}$$

Next, we show that $\|x_n-u_n\|\to 0$ as $n\to\infty$. Since $p\in\Gamma$, we get

$$\begin{aligned}\|u_n-p\|^2 &= \|T_{r_n}^{F_1}(x_n+\gamma A^*(T_{r_n}^{F_2}-I)Ax_n)-p\|^2\\ &= \|T_{r_n}^{F_1}(x_n+\gamma A^*(T_{r_n}^{F_2}-I)Ax_n)-T_{r_n}^{F_1}p\|^2\\ &\le \langle u_n-p, x_n+\gamma A^*(T_{r_n}^{F_2}-I)Ax_n-p\rangle\\ &= \frac{1}{2}\left\{\|u_n-p\|^2+\|x_n+\gamma A^*(T_{r_n}^{F_2}-I)Ax_n-p\|^2-\|(u_n-p)-[x_n+\gamma A^*(T_{r_n}^{F_2}-I)Ax_n-p]\|^2\right\}\\ &= \frac{1}{2}\left\{\|u_n-p\|^2+\|x_n-p\|^2-\|u_n-x_n-\gamma A^*(T_{r_n}^{F_2}-I)Ax_n\|^2\right\}\\ &= \frac{1}{2}\left\{\|u_n-p\|^2+\|x_n-p\|^2-[\|u_n-x_n\|^2+\gamma^2\|A^*(T_{r_n}^{F_2}-I)Ax_n\|^2-2\gamma\langle u_n-x_n, A^*(T_{r_n}^{F_2}-I)Ax_n\rangle]\right\}.\end{aligned}$$

Hence, we obtain

$$\|u_n-p\|^2\le\|x_n-p\|^2-\|u_n-x_n\|^2+2\gamma\|A(u_n-x_n)\|\|(T_{r_n}^{F_2}-I)Ax_n\|. \tag{3.23}$$

It follows from (3.1), (3.8) and (3.23), we get

$$\begin{aligned}\|z_n-p\|^2 &= \|\alpha_n y_n+(1-\alpha_n)S_n y_n-p\|^2\\ &\le \alpha_n\|y_n-p\|^2+(1-\alpha_n)\|S_n y_n-p\|^2\\ &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|y_n-p\|^2\\ &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|u_n-p\|^2\\ &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)[\|x_n-p\|^2-\|u_n-x_n\|^2+2\gamma\|A(u_n-x_n)\|\|(T_{r_n}^{F_2}-I)Ax_n\|]\\ &= \|x_n-p\|^2-(1-\alpha_n)\|u_n-x_n\|^2+2(1-\alpha_n)\gamma\|A(u_n-x_n)\|\|(T_{r_n}^{F_2}-I)Ax_n\|.\end{aligned}$$

Therefore

$$\begin{aligned}(1-\alpha_n)\|u_n-x_n\|^2 &\le \|x_n-p\|^2-\|z_n-p\|^2+2(1-\alpha_n)\gamma\|A(u_n-x_n)\|\|(T_{r_n}^{F_2}-I)Ax_n\|\\ &= \|x_n-z_n\|(\|x_n-p\|+\|z_n-p\|)+2(1-\alpha_n)\gamma\|A(u_n-x_n)\|\|(T_{r_n}^{F_2}-I)Ax_n\|.\end{aligned}$$

From (3.20) and (3.22), we have

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.24}$$

Next, we show that

$$\lim_{n\to\infty}\|B_i\Omega^{i-1}u_n-B_i\Omega^{i-1}p\|=0,\quad i=1,2,\ldots,N. \tag{3.25}$$

By (2.12), we have

$$\begin{aligned}\|\Omega^N u_n-\Omega^N p\|^2 &= \|P_C(I-\lambda_N B_N)\Omega^{N-1}u_n-P_C(I-\lambda_N B_N)\Omega^{N-1}p\|^2\\ &\le \|(I-\lambda_N B_N)\Omega^{N-1}u_n-(I-\lambda_N B_N)\Omega^{N-1}p\|^2\\ &\le \|\Omega^{N-1}u_n-\Omega^{N-1}p\|^2+\lambda_N(\lambda_N-2\eta_N)\|B_N\Omega^{N-1}u_n-B_N\Omega^{N-1}p\|^2. \end{aligned}\tag{3.26}$$

By induction and (3.7), we get

$$\begin{aligned}\|\Omega^N u_n-\Omega^N p\|^2 &\le \|u_n-p\|^2+\Sigma_{i=1}^N\lambda_i(\lambda_i-2\eta_i)\|B_i\Omega^{i-1}u_n-B_i\Omega^{i-1}p\|^2\\ &\le \|x_n-p\|^2+\Sigma_{i=1}^N\lambda_i(\lambda_i-2\eta_i)\|B_i\Omega^{i-1}u_n-B_i\Omega^{i-1}p\|^2.\end{aligned}\tag{3.27}$$

From (3.9), we get

$$\begin{aligned}\|z_n-p\|^2 &\le \|y_n-p\|^2\\ &= \|\Omega^N u_n-\Omega^N p\|^2.\end{aligned}\tag{3.28}$$

Substituting (3.27) into (3.28), we have

$$\|z_n-p\|^2\le \|x_n-p\|^2+\Sigma_{i=1}^N\lambda_i(\lambda_i-2\eta_i)\|B_i\Omega^{i-1}u_n-B_i\Omega^{i-1}p\|^2,$$

which implies

$$\begin{aligned}\Sigma_{i=1}^N\lambda_i(2\eta_i-\lambda_i)\|B_i\Omega^{i-1}u_n-B_i\Omega^{i-1}p\|^2 &\le \|x_n-p\|^2-\|z_n-p\|^2\\ &\le \|x_n-z_n\|(\|x_n-p\|+\|z_n-p\|).\end{aligned}\tag{3.29}$$

Since $\{\lambda_i\}\subset(0,2\eta_i)$, where $i\in\{1,2,\dots,N\}$ and (3.20), we have (3.25) holds.

From (2.1) and (2.6), we obtain

$$\begin{aligned}\|\Omega^N u_n-\Omega^N p\|^2 &= \|P_C(I-\lambda_N B_N)\Omega^{N-1}u_n-P_C(I-\lambda_N B_N)\Omega^{N-1}p\|^2\\ &\le \langle(I-\lambda_N B_N)\Omega^{N-1}u_n-(I-\lambda_N B_N)\Omega^{N-1}p,\Omega^N u_n-\Omega^N p\rangle\\ &= \frac{1}{2}\Big(\|(I-\lambda_N B_N)\Omega^{N-1}u_n-(I-\lambda_N B_N)\Omega^{N-1}p\|^2+\|\Omega^N u_n-\Omega^N p\|^2\\ &\quad -\|(I-\lambda_N B_N)\Omega^{N-1}u_n-(I-\lambda_N B_N)\Omega^{N-1}p-(\Omega^N u_n-\Omega^N p)\|^2\Big)\\ &\le \frac{1}{2}\Big(\|\Omega^{N-1}u_n-\Omega^{N-1}p\|^2+\|\Omega^N u_n-\Omega^N p\|^2\\ &\quad -\|\Omega^{N-1}u_n-\Omega^N u_n+\Omega^N p-\Omega^{N-1}p-\lambda_N(B_N\Omega^{N-1}u_n-B_N\Omega^{N-1}p)\|^2\Big),\end{aligned}$$

which implies

$$\begin{aligned}\|\Omega^N u_n-\Omega^N p\|^2 &\le \|\Omega^{N-1}u_n-\Omega^{N-1}p\|^2\\ &\quad -\|\Omega^{N-1}u_n-\Omega^N u_n+\Omega^N p-\Omega^{N-1}p-\lambda_N(B_N\Omega^{N-1}u_n-B_N\Omega^{N-1}p)\|^2\\ &= \|\Omega^{N-1}u_n-\Omega^{N-1}p\|^2-\|\Omega^{N-1}u_n-\Omega^N u_n+\Omega^N p-\Omega^{N-1}p\|^2\\ &\quad -\lambda_N^2\|B_N\Omega^{N-1}u_n-B_N\Omega^{N-1}p\|^2\\ &\quad +2\lambda_N\langle\Omega^{N-1}u_n-\Omega^N u_n+\Omega^N p-\Omega^{N-1}p,B_N\Omega^{N-1}u_n-B_N\Omega^{N-1}p\rangle\\ &\le \|\Omega^{N-1}u_n-\Omega^{N-1}p\|^2-\|\Omega^{N-1}u_n-\Omega^N u_n+\Omega^N p-\Omega^{N-1}p\|^2\\ &\quad +2\lambda_N\|\Omega^{N-1}u_n-\Omega^N u_n+\Omega^N p-\Omega^{N-1}p\|\|B_N\Omega^{N-1}u_n-B_N\Omega^{N-1}p\|.\end{aligned}\tag{3.30}$$

By induction and (3.7), we have

$$\begin{aligned}\|\Omega^N u_n-\Omega^N p\|^2 &\le \|u_n-p\|^2-\sum_{i=1}^N\|\Omega^{i-1}u_n-\Omega^i u_n+\Omega^i p-\Omega^{i-1}p\|^2\\ &\quad +\sum_{i=1}^N 2\lambda_i\|\Omega^{i-1}u_n-\Omega^i u_n+\Omega^i p-\Omega^{i-1}p\|\|B_i\Omega^{i-1}u_n-B_i\Omega^{i-1}p\|\\ &\le \|x_n-p\|^2-\sum_{i=1}^N\|\Omega^{i-1}u_n-\Omega^i u_n+\Omega^i p-\Omega^{i-1}p\|^2\\ &\quad +\sum_{i=1}^N 2\lambda_i\|\Omega^{i-1}u_n-\Omega^i u_n+\Omega^i p-\Omega^{i-1}p\|\|B_i\Omega^{i-1}u_n-B_i\Omega^{i-1}p\|.\end{aligned}\tag{3.31}$$

Substituting (3.31) into (3.28), we have

$$\|z_n - p\|^2 \le \|x_n - p\|^2 - \sum_{i=1}^{N} \|\Omega^{i-1}u_n - \Omega^i u_n + \Omega^i p - \Omega^{i-1}p\|^2 + \sum_{i=1}^{N} 2\lambda_i \|\Omega^{i-1}u_n - \Omega^i u_n + \Omega^i p - \Omega^{i-1}p\| \|B_i\Omega^{i-1}u_n - B_i\Omega^{i-1}p\|,$$

which implies

$$\begin{aligned}\sum_{i=1}^{N} &\|\Omega^{i-1}u_n - \Omega^i u_n + \Omega^i p - \Omega^{i-1}p\|^2 \\ &\le \|x_n - p\|^2 - \|z_n - p\|^2 + \sum_{i=1}^{N} 2\lambda_i \|\Omega^{i-1}u_n - \Omega^i u_n + \Omega^i p - \Omega^{i-1}p\| \|B_i\Omega^{i-1}u_n - B_i\Omega^{i-1}p\| \\ &\le \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|) + \sum_{i=1}^{N} 2\lambda_i \|\Omega^{i-1}u_n - \Omega^i u_n + \Omega^i p - \Omega^{i-1}p\| \|B_i\Omega^{i-1}u_n - B_i\Omega^{i-1}p\|.\end{aligned}$$

It follows from (3.20) and (3.25), we have

$$\lim_{n\to\infty} \|\Omega^{i-1}u_n - \Omega^i u_n + \Omega^i p - \Omega^{i-1}p\| = 0, \quad i = 1, 2, \dots, N. \tag{3.32}$$

From (3.32), we obtain

$$\begin{aligned}\|u_n - y_n\| &= \|\Omega^0 u_n - \Omega^N u_n\| \\ &\le \sum_{i=1}^{N} \|\Omega^{i-1}u_n - \Omega^i u_n + \Omega^i p - \Omega^{i-1}p\| \\ &\to 0, \quad \text{as } n \to \infty.\end{aligned} \tag{3.33}$$

From (3.24) and (3.33), we have

$$\begin{aligned}\|x_n - y_n\| &\le \|x_n - u_n\| + \|u_n - y_n\| \\ &\to 0, \quad \text{as } n \to \infty.\end{aligned} \tag{3.34}$$

From (3.17) and (3.24), we have

$$\begin{aligned}\|u_n - x_{n+1}\| &\le \|u_n - x_n\| + \|x_n - x_{n+1}\| \\ &\to 0, \quad \text{as } n \to \infty.\end{aligned} \tag{3.35}$$

From (3.17), (3.20) and (3.35), we have

$$\begin{aligned}\|z_n - u_n\| &\le \|z_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - u_n\| \\ &\to 0, \quad \text{as } n \to \infty.\end{aligned} \tag{3.36}$$

Observe that from (3.1), (3.33), (3.36) and $\alpha_n \le a \le 1$ for all n, then

$$\begin{aligned}\|z_n - y_n\| &= \|\alpha_n y_n + (1-\alpha_n)S_n y_n - y_n\| \\ &= (1-\alpha_n)\|S_n y_n - y_n\|\end{aligned}$$

which implies

$$\begin{aligned}\|S_n y_n - y_n\| &= \frac{1}{(1-\alpha_n)}\|z_n - y_n\| \\ &\le \frac{1}{1-a}(\|z_n - u_n\| + \|u_n - y_n\|) \\ &\to 0, \quad \text{as } n \to \infty.\end{aligned} \tag{3.37}$$

By Lemma 2.4, we have

$$\limsup_{n\to\infty} \|S_n y_n - S(S_n y_n)\| = 0. \tag{3.38}$$

Step 5. We show that $w \in \mathcal{F}$.

(a) First, we show that $w \in Fix(S_n) = \frac{1}{n+1}\sum_{i=0}^{n} Fix(S^i)$. Assume that $w \notin \frac{1}{n+1}\sum_{i=0}^{n} Fix(S^i)$. Since $y_{n_i} \rightharpoonup w$ and $Sw \neq w$, by Lemma 2.5, we have

$$\begin{aligned}\liminf_{i\longrightarrow\infty}\|y_{n_i} - w\| &< \liminf_{i\longrightarrow\infty}\|y_{n_i} - Sw\|\\ &\leq \liminf_{i\longrightarrow\infty}(\|y_{n_i} - Sy_{n_i}\| + \|Sy_{n_i} - Sw\|)\\ &\leq \liminf_{i\longrightarrow\infty}\|y_{n_i} - w\|,\end{aligned}$$

which is a contradiction. Thus, we obtain $w \in Fix(S_n) = \frac{1}{n+1}\sum_{i=0}^{n} Fix(S^i)$.

(b) Next, we show that $w \in Fix(G)$.

From Lemma 2.1, we know $G = G^N$ is nonexpansive, it follows that

$$\|y_n - Gy_n\| = \|G^N u_n - G^N y_n\| \leq \|u_n - y_n\|.$$

From (3.33), thus

$$\lim_{n\to\infty}\|y_n - Gy_n\| = 0. \tag{3.39}$$

Since G is nonexpansive, (3.34) and (3.39), we get

$$\begin{aligned}\|x_n - Gx_n\| &\leq \|x_n - y_n\| + \|y_n - Gy_n\| + \|Gy_n - Gx_n\|\\ &\leq 2\|x_n - y_n\| + \|y_n - Gy_n\|,\end{aligned}$$

and so

$$\lim_{n\to\infty}\|x_n - Gx_n\| = 0. \tag{3.40}$$

By Lemma 2.6 and (3.40), we obtain $w \in Fix(G)$.

(C) Next, we show that $w \in \Gamma$.

Since $u_n = T_{r_n}^{F_1} x_n$, we have

$$F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

It follows from monotonicity of F_1 that

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F_1(y, u_n)$$

and hence

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \geq F_1(y, u_{n_i}).$$

Since $\|u_n - x_n\| \to 0$, we get $u_{n_i} \rightharpoonup w$ and $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$. It follows from Lemma 2.2(iv) that $0 \geq F_1(y, w), \forall w \in C$. For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$, $w \in C$, we get $y_t \in C$ and hence $F_1(y_t, w) \leq 0$. So, from Lemma 2.2(i) and (iv), we have

$$0 = F_1(y_t, y_t) \leq tF_1(y_t, y) + (1-t)F_1(y_t, w) \leq tF_1(y_t, y).$$

Therefore, $0 \leq F_1(y_t, y)$. From Lemma 2.2(iii), we have $0 \leq F_1(w, y_t)$. This implies that $w \in EP(F_1)$.

Next, we show that $Aw \in EP(F_2)$. Since $\|u_n - x_n\| \to 0$, $u_n \rightharpoonup w$ as $n \to \infty$ and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup w$. Since A is bounded linear operator, then we have $Ax_{n_i} \rightharpoonup Aw$.

Now, setting $v_{n_i} = Ax_{n_i} - T_{r_{n_i}}^{F_2} Ax_{n_i}$. It follows from (3.22) that $\lim_{i\to\infty} v_{n_i} = 0$ and $Ax_{n_i} - v_{n_i} = T_{r_{n_i}}^{F_2} Ax_{n_i}$.

Therefore from Lemma 2.3, we have

$$F_2(Ax_{n_i} - v_{n_i}, z) + \frac{1}{r_{n_i}}\langle z - (Ax_{n_i} - v_{n_i}), (Ax_{n_i} - v_{n_i}) - Ax_{n_i}\rangle \geq 0, \quad \forall z \in Q.$$

Since F_2 is upper semicontinuous in first argument, taking lim sup to above inequality as $i \to \infty$ and using $\liminf_{n\to\infty} r_n > 0$, we obtain

$$F_2(Aw, z) \geq 0, \quad \forall z \in Q,$$

which means that $Aw \in EP(F_2)$ and hence $w \in \mathcal{F}$.

Step 6. Finally, we show that $x_n \to v$, where $v = P_{\mathcal{F}} x_0$.
Suppose that $v = P_{\mathcal{F}} x_0$. Since $w \in F$ and

$$x_{n+1} = P_{C_{n+1}} x_0,\ v \in \Gamma \cap Fix(G) \cap (\cap_{i=1}^{n} Fix(S^i)) \subset C_{n+1}.$$

Then $\|x_{n+1} - x_0\| \leq \|v - x_0\|$. Since the norm is weakly lower semicontinuous, we have

$$\|v - x_0\| \leq \|w - x_0\| \leq \liminf_{i\to\infty} \|x_{n_i} - x_0\| \leq \limsup_{i\to\infty} \|x_{n_i} - x_0\| \leq \|v - x_0\|.$$

Thus, we obtain $w = v$ and $\lim_{i\to\infty} \|x_{n_i} - x_0\| = \|w - x_0\| = \|v - x_0\|$. From $x_{n_i} - x_0 \rightharpoonup w - x_0$ and the Kadec–Klee property of H, we have $x_{n_i} - x_0 \to w - x_0$, and hence $x_{n_i} \to w$. This implies that $x_n \to w = v$. This complete the proof. □

4. Consequently results

Corollary 4.1. *Let H_1 and H_2 be two real Hilbert spaces and $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let B be η-inverse-strongly monotone. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are bifunctions satisfying Lemma 2.2 and F_2 is upper semicontinuous. Let $\{S^i\}_{i=1}^{n}$ be a sequence of nonexpansive mappings from C into itself such that*

$$\mathcal{F} := \Gamma \cap VI(C, B) \cap (\cap_{i=1}^{n} Fix(S^i)) \neq \emptyset.$$

Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $\alpha_n \leq a$ for all n and for some $0 < a < 1$. Pick any $x_0 \in H_1$ and set $C_1 = C$. Let $\{x_n\}$ be a sequence generated by $x_1 = P_{C_1} x_0$ and

$$\begin{cases} u_n = T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n), \\ y_n = P_C(I - \lambda_n B)u_n, \\ z_n = \alpha_n y_n + (1 - \alpha_n)\dfrac{1}{n+1}\displaystyle\sum_{i=1}^{n} S^i y_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases} \tag{4.1}$$

where $\{\lambda_n\} \subset (0, 2\eta)$, $\{r_n\} \subset (0, \infty)$, $\liminf_{n\to\infty} r_n > 0$ and $\gamma \in (0, \frac{1}{L})$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A. Then the sequence $\{x_n\}$ converges strongly to $P_{\mathcal{F}} x_0$.

Proof. Put $N = 1$ in Theorem 3.1, we can conclude the desired conclusion easily. This completes the proof. □

Corollary 4.2. *Let H_1 and H_2 be two real Hilbert spaces and $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let B be η-inverse-strongly monotone. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are bifunctions satisfying Lemma 2.2 and F_2 is upper semicontinuous. Let $S : C \to C$ be a nonexpansive mapping such that*

$$\mathcal{F} := \Gamma \cap VI(C, B) \cap Fix(S) \neq \emptyset.$$

Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $\alpha_n \leq a$ for all n and for some $0 < a < 1$. Pick any $x_0 \in H_1$ and set $C_1 = C$. Let $\{x_n\}$ be a sequence generated by $x_1 = P_{C_1} x_0$ and

$$\begin{cases} u_n = T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n), \\ y_n = P_C(I - \lambda_n B)u_n, \\ z_n = \alpha_n y_n + (1 - \alpha_n)Sy_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases} \tag{4.2}$$

where $\{\lambda_n\} \subset (0, 2\eta)$, $\{r_n\} \subset (0, \infty)$, $\liminf_{n\to\infty} r_n > 0$ and $\gamma \in (0, \frac{1}{L})$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A. Then the sequence $\{x_n\}$ converges strongly to $P_{\mathcal{F}} x_0$.

Proof. Put $N = 1$ and take $S^i = S$ in Theorem 3.1, we can conclude the desired conclusion easily. This completes the proof. □

Corollary 4.3. *Let H_1 and H_2 be two real Hilbert spaces and $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let B_i be η_i-inverse-strongly monotone, respectively, where $i \in \{1, 2, \ldots, N\}$. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are bifunctions satisfying Lemma 2.2 and F_2 is upper semicontinuous. Let $S : C \to C$ be a nonexpansive mapping such that*

$$\mathcal{F} := \Gamma \cap Fix(G) \cap Fix(S) \neq \emptyset,$$

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where G defined by Lemma 2.1. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $\alpha_n \le a$ for all $n \ge 1$ and for some $0 < a < 1$. Pick any $x_0 \in H_1$ and set $C_1 = C$. Let $\{x_n\}$ be a sequence generated by $x_1 = P_{C_1}x_0$ and

$$\begin{cases} u_n = T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n), \\ y_n = P_C(I - \lambda_N B_N)P_C(I - \lambda_{N-1}B_{N-1})\cdots P_C(I - \lambda_2 B_2)P_C(I - \lambda_1 B_1)u_n, \\ z_n = \alpha_n y_n + (1-\alpha_n)Sy_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad \forall n \ge 1, \end{cases} \tag{4.3}$$

where $\{\lambda_i\} \subset (0, 2\eta_i)$, $i \in \{1,2,\ldots,N\}$, $\{r_n\} \subset (0,\infty)$, $\liminf_{n\to\infty} r_n > 0$ and $\gamma \in (0, \frac{1}{L})$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A. Then the sequence $\{x_n\}$ converges strongly to $P_{\mathcal{F}}x_0$.

Proof. Take $S^i = S$ in Theorem 3.1, we can conclude the desired conclusion easily. This completes the proof. □

5. Numerical examples and convergence analysis

Let us show a numerical example to demonstrate the performance and convergence of Corollary 4.1 as follows:

Example 5.1. Let $H_1 = H_2 = \mathbb{R}$, $C = [0, 1000]$ and $Q = [-1000, 0]$. Let $A, B : \mathbb{R} \to \mathbb{R}$ be operators defined by $A(x) = -x$ and $B(x) = \frac{4x-3}{5}$. For each $i = 1,2,3,\ldots,n$, we set $S_i(x) = \frac{x}{i}$. Define two bifunctions $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ by

$$F_1(z,y) = 3y^2 + 2zy - 5z^2 \quad \text{and} \quad F_2(z,y) = y^2 - z^2.$$

By the definition, it is not too hard to check that the operators A, B, F_1 and F_2 satisfy all of conditions in Corollary 4.1. So, by Lemma 2.2, we have $T_r^{F_1}(x)$ and $T_r^{F_2}(x)$ are a single-value mapping for each $x \in C$. Hence, for $r_n = r > 0$ there exists $z \in C$ such that

$$F_1(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in C.$$

This inequality is equivalent to the following inequality:

$$P(y) := 3ry^2 + (z - x + 2rz)y + (xz - 5rz^2 - z^2) \ge 0, \quad \forall y \in C.$$

We see that $P(y)$ is a quadratic function in variable y which has the form $P(y) = ay^2 + by + c$, so we have $a = 3r$, $b = (z - x + 2rz)$ and $c = (xz - 5rz^2 - z^2)$. Note that $b^2 - 4ac = (x - (8rz + z))^2 \ge 0$. Since $P(y) \ge 0$ for all $y \in C$ then $b^2 - 4ac = (x - (8rz + z))^2 \le 0$. Therefore, $(x - (8rz + z))^2 = 0$ and $z = \frac{x}{1+8r}$. That is $T_r^{F_1}(x) = \frac{x}{1+8r}$ for each $r > 0$. Similarly, we can find the formula $T_r^{F_2}(x) = \frac{x}{1+2r}$ for each $r > 0$.

To certify the convergent result of the sequence x_n in Corollary 4.1, we present the following algorithm:

Algorithm 5.2 (*The Split Equilibrium on Hybrid Projection with Cesàro Mean Method*).

Step 0. Choose the initial point $x_0 \in C$ a, $\lambda > 0$ arbitrarily and let $C_1 = C$.
Step 1. Find $x_1 = P_{C_1}x_0$ and let $n = 1$.
Step 2. Compute $x_{n+1} \in C$ as follows:

$$\begin{cases} u_n = T_{r_n}^{F_1}\left(x_n + (\frac{1}{2})A^*\left(T_{r_n}^{F_2} - I\right)Ax_n\right), \\ y_n = P_C\left(I - \frac{B}{4}\right)u_n, \\ z_n = 0.2y_n + 0.8\dfrac{1}{n+1}\displaystyle\sum_{i=1}^{n} S^i y_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0 \end{cases} \tag{5.1}$$

Step 3. Put $n := n + 1$ and go to step 2.

In this experiment, we set three random initial points x_0 for Algorithm 5.2 with $r = \frac{1}{8}$. This indicates that the sequence x_n with the different initial points converges to the same point which shows in Fig. 1. Corollary 4.1 guarantees that the sequence $\{x_n\}$ converges strongly to $P_{\mathcal{F}}x_0$.

Fig. 2 presents the behaviors of x_n for Algorithm 5.2 with a random initial point x_0 and the different $r = 1, 0.1, 0.01$. Also, the sequence $\{x_n\}$ converges to the same point; that is 0.

Moreover, we random a initial point x_0 and present the behaviors of C_n in Fig. 3. This figure shows that $C_n \subset C_{n-1} \subset \ldots \subset C_2 \subset C_1 = C_0$. We note that the iteration of C_n from Algorithm 5.2 will generate C_{n+1} and reduce the area of solution set. Therefore, the iteration of C_n will squeeze the area until we obtain the approximated solution.

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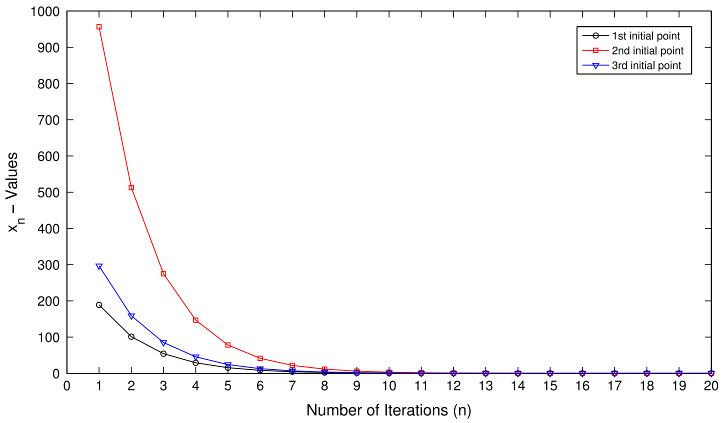

Fig. 1. Behaviors of x_n with three random initial points x_0 and $r = \frac{1}{8}$.

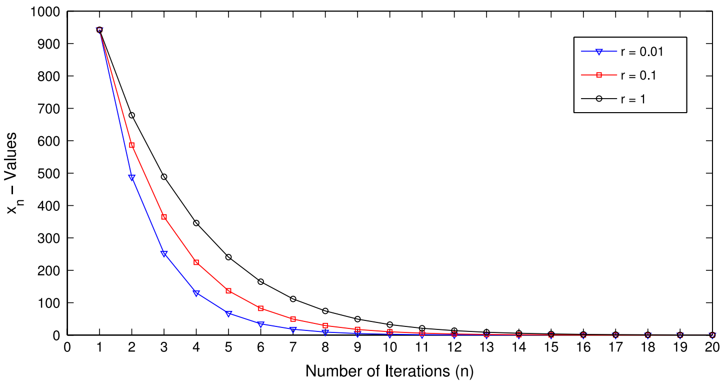

Fig. 2. Behaviors of x_n with a random initial point x_0 and the different $r = 1, 0.1, 0.01$.

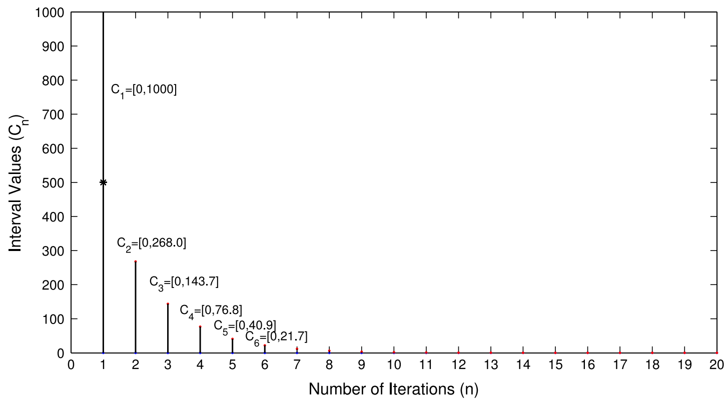

Fig. 3. Behaviors of the set C_n.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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