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Bounds for the generalized Marcum Q-function

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ABSTRACT

In this paper we consider the generalized Marcum Q-function of order $\nu > 0$ real, defined by

$$Q_\nu(a,b) = \frac{1}{a^{\nu-1}} \int_b^\infty t^\nu e^{-\frac{t^2+a^2}{2}} I_{\nu-1}(at)\mathrm{d}t,$$

where $a > 0$, $b \geqslant 0$ and I_ν stands for the modified Bessel function of the first kind. Our aim is to extend some results on the (first order) Marcum Q-function to the generalized Marcum Q-function in order to deduce some new lower and upper bounds. Moreover, we show that the proposed bounds are very tight for the generalized Marcum Q-function of integer order, and we deduce some new inequalities for the more general case of real order. The chief tools in our proofs are some monotonicity properties of certain functions involving the modified Bessel function of the first kind, which are based on a criterion for the monotonicity of the quotient of two Maclaurin series.

1. Introduction and preliminary results

Let $Q_\nu(a,b)$ be the generalized Marcum Q-function, defined by

$$Q_\nu(a,b) = \frac{1}{a^{\nu-1}} \int_b^\infty t^\nu e^{-\frac{t^2+a^2}{2}} I_{\nu-1}(at)\mathrm{d}t, \tag{1.1}$$

where $b \geqslant 0$, $a, \nu > 0$, and I_ν stands for the well-known modified Bessel function of the first kind of order ν. When $\nu = 1$, the function

$$Q_1(a,b) = \int_b^\infty t e^{-\frac{t^2+a^2}{2}} I_0(at)\mathrm{d}t$$

appears in literature as the Marcum Q-function. It is known that for all $b \geqslant 0$ and $a, \nu > 0$ the function $Q_\nu(a,b)$ can be rewritten as

$$Q_\nu(a,b) = 1 - \frac{1}{a^{\nu-1}} \int_0^b t^\nu e^{-\frac{t^2+a^2}{2}} I_{\nu-1}(at)\mathrm{d}t. \tag{1.2}$$

These functions have a long history and play an important role in communication studies, especially in problems on radar communications [9]. In this field ν is actually the number of independent samples of the output of a square-law detector,

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however, in our analysis ν is a not necessarily an integer number. The generalized Marcum Q-function has applications also in error performance analysis of multichannel dealing with partially coherent, differentially coherent, and non-coherent detections in digital communications [10]. Moreover, it was shown recently by Sun and Baricz [12] that not surprisingly this function satisfies the new-is-better-than-used property, which arises in economic theory.

Since, the precise computation of the Marcum Q-function, generalized Marcum Q-function, respectively is fairly difficult, in the last few decades several authors established approximation formulas and bounds for the function $Q_\nu(a,b)$. There are many type bounds for the Marcum Q-function, generalized Marcum Q-function, respectively, which were deduced by using completely different methods. The interested reader is referred to the papers [1,2,4,7,11,14] and to the references therein. An important contribution to the subject is the paper of Corazza and Ferrari [5], where extremely tight bounds were deduced for the Marcum Q-function. Recently, some of the results from [5] were improved by Wang [15] by using a similar method. Motivated by the papers [5,15], the authors in [2,4] deduced new lower and upper bounds for the generalized Marcum Q-function by using some classical tools from the theory of modified Bessel functions and by pointing out also some incompleteness in the proofs of the main results of [5,15]. More precisely, in [4] Baricz and Sun extended all the results from [5] to generalized Marcum Q-function. In [2] Baricz extended all the results from [15] and found the best possible upper bounds by using some known monotonicity properties of functions involving the modified Bessel functions. We note that the results from [2,4] were deduced with the aid of some results whose origins comes from the paper of Gronwall [6] published in 1932, and which have important applications in wave mechanics. Moreover, in [2] there is used a function with its monotonicity and convexity properties, that arises in finite elasticity in the problem of buckling and necking of nonlinearly elastic cylinders. This paper is a further contribution to the subject and is the direct continuation of the papers [2,4]. The detailed content is as follows: in this section we prove the preliminary results, concerning the monotonicity of some functions involving the modified Bessel function of the first kind, which will be used in the sequel. Some of these results were deduced recently by the first author [3] by using a frequently used criterion for the monotonicity of the quotient of two Maclaurin series. In Section 2 we present our main result concerning new tight lower and upper bounds for the generalized Marcum Q-function. However, our aim here is twofold: to present two new extensions of the lower bounds of Wang [15] to the generalized Marcum Q-function, which improves the results from [2], and to deduce a new improvement and its extension of the upper bounds of Corazza and Ferrari [5]. Moreover, we improve also the lower bounds deduced in [4], which were the natural extensions of the lower bounds of Corazza and Ferrari [5]. We note here that almost all the results of the papers [2,4,5,15] and of this paper are based on the monotonicity of the functions of the type $x \mapsto x^{\alpha\nu+\beta} I_\nu(x)/f(e^x, \sinh x, \cosh x)$, where α and β are real numbers. Due to the importance of the monotonicity of the above type functions in Section 3 we give also the sharpness of our main results, which guarantees that our new extensions of the lower bounds of Wang [15] cannot be further improved, and similarly the improvements and its extensions of the upper bounds of Corazza and Ferrari [5] cannot be further improved too. More precisely, our results are sharp in the sense that the power $\alpha\nu + \beta$, which appears in the expression of $x^{\alpha\nu+\beta} I_\nu(x)/f(e^x, \sinh x, \cosh x)$, in most of cases is shown to be the smallest or largest constant such that the corresponding monotonicity property holds. Note that in Section 3 a detailed discussion on the comparison of the new results with the main results of [2,4] is given. Finally, in Section 4 we compare our results from the Section 2 with the other known bounds from the literature. Our results have some direct applications in digital communications, such as bounding the outage probability of wireless communication systems with a minimum signal power constraint, evaluating the average error probability of digital communication systems operating over slow-fading channels and extracting the log-likelihood ratio for decoding turbo or low-density parity check (LDPC) codes with differential phase-shift keying (DPSK) modulation. The interested readers are referred to [13,14] and the references therein.

We would like to mention here that the results of Theorems 1–6 stated in Section 2 are stated in terms of some integrals, which cannot be evaluated easily for arbitrary ν. However, we have included them in this paper not only for the sake of completeness, but to point out precisely the connection between our new results and the results from [2,4]. We note that although the results of Theorems 1–6 are useful only for integer values of ν, they are interesting in their own right, because provide in a sense the best possible lower and upper bounds for the generalized Marcum Q-function, and this in turn implies that the results from [2,4] cannot be further improved. Moreover, we note that the bounds given in Corollaries 1 and 2 for the integer order generalized Marcum Q-function in some cases provide better bounds than the recently discovered very tight log-concavity based bounds [14]. See Section 4 for further details.

The following result, which is of independent interest, is the key tool in the proof of our main results of Section 2.

Lemma 1. *The following assertions are true:*

(a) *The function $x \mapsto xI_\nu(x)e^{-x}$ is strictly increasing on $(0,\infty)$ for all $\nu > -1$.*
(b) *The function $x \mapsto x^{1/2}I_\nu(x)e^{-x}$ is strictly increasing on $(0,\infty)$ for all $\nu \geqslant 1/2$. Moreover, the smallest constant α_ν for which the function $x \mapsto x^{\alpha_\nu}I_\nu(x)e^{-x}$ is strictly increasing $(0,\infty)$ for all $\nu \geqslant 1/2$ is $\alpha_\nu = 1/2$.*
(c) *The function $x \mapsto xI_\nu(x)/\sinh x$ is strictly increasing on $(0,\infty)$ for all $\nu \geqslant 0$.*
(d) *The function $x \mapsto x^{1/2}I_\nu(x)/\sinh x$ is increasing for all $\nu \geqslant 1/2$. Moreover, the smallest constant β_ν for which the function $x \mapsto x^{\beta_\nu}I_\nu(x)/\sinh x$ is increasing on $(0,\infty)$ for all $\nu \geqslant 1/2$ is $\beta_\nu = 1/2$.*
(e) *The function $x \mapsto x^{-\nu}I_\nu(x)/\cosh x$ is decreasing on $(0,\infty)$ for all $\nu \geqslant -1/2$. Moreover, the largest constant γ_ν for which the function $x \mapsto x^{\gamma_\nu}I_\nu(x)/\cosh x$ is decreasing on $(0,\infty)$ for all $\nu \geqslant -1/2$ is $\gamma_\nu = -\nu$.*
(f) *The function $x \mapsto x^{1-\nu}I_\nu(x)/\sinh x$ is decreasing on $(0,\infty)$ for all $\nu \geqslant 1/2$. Moreover, the largest constant δ_ν for which the function $x \mapsto x^{\delta_\nu}I_\nu(x)/\sinh x$ is decreasing on $(0,\infty)$ for all $\nu \geqslant 1/2$ is $\delta_\nu = 1 - \nu$.*

(g) *The function* $x \mapsto x^{1-\nu} I_\nu(x)/\sinh x$ *is increasing on* $(0,\infty)$ *for all* $-1 < \nu \leqslant 1/2$. *Moreover, the smallest constant* ϵ_ν *for which the function* $x \mapsto x^{\epsilon_\nu} I_\nu(x)/\sinh x$ *is increasing on* $(0,\infty)$ *for all* $-1 < \nu \leqslant 1/2$ *is* $\epsilon_\nu = 1 - \nu$.

Proof. First we prove the assertions stated in Lemma 1 without sharpness. Parts (a), (b), (e), (f) and (g) has been proved in [3], by using among other things a criterion for the monotonicity of the quotient of two Maclaurin series. For parts (c) and (d) let us consider the functions $f, g, h : (0,\infty) \to \mathbb{R}$, defined by $f_\nu(x) = xI_\nu(x)/\sinh x$, $g_\nu(x) = x^{1/2} I_\nu(x)/\sinh x$ and $h_\nu(x) = xI'_\nu(x)/I_\nu(x)$. Owing to Gronwall [6, Eq. 17] it is known that the function $\nu \mapsto h_\nu(x)$ is increasing on $[0,\infty)$ for all $x > 0$ fixed. Consequently for all $x > 0$ we have

$$xf'_\nu(x)/f_\nu(x) = h_\nu(x) + 1 - x\coth x \geqslant h_0(x) + 1 - x\coth x > 0, \quad \nu \geqslant 0,$$
$$xg'_\nu(x)/g_\nu(x) = h_\nu(x) + 1/2 - x\coth x \geqslant h_{1/2}(x) + 1/2 - x\coth x = 0, \quad \nu \geqslant 1/2.$$

The inequality $h_0(x) + 1 - x\coth x > 0$ was proved in [2], while the relation $h_{1/2}(x) + 1/2 - x\coth x = 0$ follows from $I_{1/2}(x) = \sqrt{2/(\pi x)}\sinh x$.

Now, we are going to prove the sharpness of parts (b), (d), (e), (f) and (g). For this consider the functions $q, r, s, u, v : (0,\infty) \to \mathbb{R}$, defined by $q_\nu(x) = x^{\alpha_\nu} I_\nu(x) e^{-x}$, $r_\nu(x) = x^{\beta_\nu} I_\nu(x)/\sinh x$, $s_\nu(x) = x^{\gamma_\nu} I_\nu(x)/\cosh x$, $u_\nu(x) = x^{\delta_\nu} I_\nu(x)/\sinh x$ and $v_\nu(x) = x^{\epsilon_\nu} I_\nu(x)/\sinh x$. Clearly we have

$$xq'_\nu(x)/q_\nu(x) = h_\nu(x) - x + \alpha_\nu.$$

Thus the problem of finding the smallest value of α_ν for which the function q_ν is increasing on $(0,\infty)$ reduces to the problem of finding the minimum of the function $x \mapsto h_\nu(x) - x$. On the other hand owing to Gronwall (p. 275 [6]) for $\nu > -1$ fixed and x large the following asymptotic formula is valid

$$h_\nu(x) = x - \frac{1}{2} + \frac{4\nu^2 - 1}{8x} + \mathcal{O}\left(\frac{1}{x^2}\right), \tag{1.3}$$

which leads to [6, Eq. 5]

$$h_\nu(x) - x > \lim_{x\to\infty}[h_\nu(x) - x] = -1/2,$$

where $\nu \geqslant 1/2$ and $x > 0$. This shows that if $\nu \geqslant 1/2$, then the smallest constant α_ν for which the function q_ν is increasing on $(0,\infty)$ is indeed 1/2. With this the proof of part (b) is complete.

Now, let us focus on parts (d) and (g). Clearly we have

$$xr'_\nu(x)/r_\nu(x) = \varphi_\nu(x) + \beta_\nu - 1$$

and

$$xv'_\nu(x)/v_\nu(x) = \varphi_\nu(x) + \epsilon_\nu - 1,$$

where $\varphi_\nu : (0,\infty) \to \mathbb{R}$ is defined by

$$\varphi_\nu(x) = xf'_\nu(x)/f_\nu(x) = h_\nu(x) + 1 - x\coth x.$$

Thus the problem of finding the smallest value of β_ν (ϵ_ν respectively) for which the function r_ν (v_ν respectively) is increasing on $(0,\infty)$ reduces to the problem of finding the minimum of the function φ_ν. By using the well-known recurrence relation $I'_\nu(x)/I_\nu(x) = \nu/x + I_{\nu+1}(x)/I_\nu(x)$, the expression $\varphi_\nu(x)$ can be rewritten as

$$\varphi_\nu(x) = \nu + xI_{\nu+1}(x)/I_\nu(x) + 1 - x\coth x, \tag{1.4}$$

which leads to

$$\lim_{x\to 0}\varphi_\nu(x) = \nu$$

for all $\nu > -1$ fixed. On the other hand it is known that [3, Eq. 2.17]

$$I_{\nu+1}(x)/I_\nu(x) > \coth x - 1/x,$$

where $-1 < \nu < 1/2$ and $x > 0$. Thus in fact we have that

$$\varphi_\nu(x) > \lim_{x\to 0}\varphi_\nu(x) = \nu$$

for all $x > 0$ and $-1 < \nu < 1/2$. Observe that if $\nu = 1/2$, then $\varphi_\nu(x) = 1/2$ for all $x > 0$, and hence in the above inequality we have equality. These show that if $-1 < \nu \leqslant 1/2$, then the smallest constant ϵ_ν for which the function v_ν is increasing on $(0,\infty)$ is indeed $1 - \nu$. On the other hand using (1.3) we obtain that for $\nu > -1$ fixed and x large the following asymptotic formula is valid

$$\varphi_\nu(x) = \frac{1}{2} + x(1 - \coth x) + \frac{4\nu^2 - 1}{8x} + \mathcal{O}\left(\frac{1}{x^2}\right).$$

Now, since $\lim_{x\to\infty} x(1 - \coth x) = 0$, from the above asymptotic formula we deduce that

$$\lim_{x\to\infty}\varphi_\nu(x) = 1/2$$

for all $\nu > -1$ fixed. Recall that due to Gronwall [6, Eq. 17] it is known that the function $\nu \mapsto h_\nu(x)$ is increasing on $[0,\infty)$ for all $x > 0$ fixed. This implies that the function $\nu \mapsto \varphi_\nu(x)$ is also increasing on $[0,\infty)$ for all $x > 0$ fixed. Thus for all $x > 0$ and $\nu > 1/2$ we have

$$\varphi_\nu(x) > \lim_{x\to\infty}\varphi_\nu(x) = \varphi_{1/2}(x) = 1/2,$$

which in turn implies that if $\nu \geqslant 1/2$, then the smallest constant β_ν for which the function r_ν is increasing on $(0,\infty)$ is indeed 1/2. This completes the proof of sharpness of parts (d) and (g). Now let us focus on part (e). It is easy to verify that

$$xs'_\nu(x)/s_\nu(x) = h_\nu(x) - x\tanh x + \gamma_\nu = \nu + xI_{\nu+1}(x)/I_\nu(x) - x\tanh x + \gamma_\nu.$$

On the other hand [3, Eq. 2.14]

$$I_{\nu+1}(x)/I_\nu(x) < \tanh x$$

for all $x > 0$ and $\nu > -1/2$. Denoting

$$\psi_\nu(x) = \nu + xI_{\nu+1}(x)/I_\nu(x) - x\tanh x,$$

this leads to

$$\psi_\nu(x) < \lim_{x\to 0}\psi_\nu(x) = \nu$$

for all $x > 0$ and $\nu > -1/2$. Observe that if $\nu = -1/2$, then $\psi_\nu(x) = -1/2$ for all $x > 0$, and hence in the above inequality we have equality. These show that if $\nu \geqslant -1/2$, then the largest constant γ_ν for which the function s_ν is decreasing on $(0,\infty)$ is indeed $-\nu$.

Finally, since

$$xu'_\nu(x)/u_\nu(x) = \varphi_\nu(x) + \delta_\nu - 1,$$

and [3, Eq. 2.16]

$$I_{\nu+1}(x)/I_\nu(x) < \coth x - 1/x,$$

where $\nu > 1/2$ and $x > 0$, from which we have

$$\varphi_\nu(x) < \lim_{x\to 0}\varphi_\nu(x) = \nu$$

for all $x > 0$ and $\nu > 1/2$, we deduce that the largest value of δ_ν for which the function u_ν is decreasing on $(0,\infty)$ for all $\nu \geqslant 1/2$ is indeed $\delta_\nu = 1 - \nu$. This completes the proof of this lemma. □

2. Lower and upper bounds for the generalized Marcum Q-function

In this section we are going to establish some new tight lower and upper bounds for the generalized Marcum Q-function by using the results of Lemma 1. In the followings $\mathrm{erfc} : \mathbb{R} \to (0,2)$ stands for the complementary error function, which is defined by

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\mathrm{d}t.$$

Moreover, for all $\lambda > 0$ and $x \in \mathbb{R}$ let

$$\Gamma(\lambda) = \int_0^\infty t^{\lambda-1}e^{-t}\mathrm{d}t \quad \text{and} \quad \Gamma(\lambda,x) = \int_x^\infty t^{\lambda-1}e^{-t}\mathrm{d}t$$

denote, as usual, the Euler gamma and the upper incomplete gamma functions. Notice that for all $x \in \mathbb{R}$

$$\Gamma\left(\frac{1}{2},x^2\right) = \sqrt{\pi}\,\mathrm{erfc}(x).$$

Finally, as usual, for $k \in \{0,1,2,\ldots,n\}$ we denote by

$$C_n^k = \frac{n!}{(n-k)!k!}$$

the binomial coefficient and we use the familiar notations

$$(2k)!! = 2\cdot 4\cdots(2k) \quad \text{and} \quad (2k-1)!! = 1\cdot 3\cdots(2k-1).$$

2.1. First case: $b \geqslant a > 0$

Our first main result of this subsection reads as follows.

Theorem 1. *If $\nu \geqslant 1$ and $b \geqslant a > 0$, then the following inequalities hold*

$$Q_\nu(a,b) \geqslant \frac{bI_{\nu-1}(ab)}{a^{\nu-1}e^{ab}} \int_{b-a}^{\infty} (u+a)^{\nu-1} e^{-u^2/2} \mathrm{du}, \tag{2.1}$$

$$Q_\nu(a,b) \geqslant \frac{bI_{\nu-1}(ab)}{2a^{\nu-1}\sinh(ab)} \left[\int_{b-a}^{\infty} (u+a)^{\nu-1} e^{-u^2/2} \mathrm{du} - \int_{b+a}^{\infty} (u-a)^{\nu-1} e^{-u^2/2} \mathrm{du}\right], \tag{2.2}$$

$$Q_\nu(a,b) \leqslant \frac{I_{\nu-1}(ab)}{2(ab)^{\nu-1}\cosh(ab)} \left[\int_{b-a}^{\infty} (u+a)^{2\nu-1} e^{-u^2/2} \mathrm{du} + \int_{b+a}^{\infty} (u-a)^{2\nu-1} e^{-u^2/2} \mathrm{du}\right]. \tag{2.3}$$

Moreover, the inequality (2.1) holds true for all $\nu > 0$, while the inequality (2.3) for all $\nu \geqslant 1/2$.

Proof. Since from part (a) of Lemma 1 the function $x \mapsto xI_\nu(x)e^{-x}$ is strictly increasing on $(0,\infty)$, it follows that for all $t \geqslant b$ and $\nu > -1$ we have

$$I_\nu(t) \geqslant \frac{e^t b}{e^b t} I_\nu(b). \tag{2.4}$$

Changing in (2.4) t with at, b with ab and ν with $\nu - 1$, from (1.1) we obtain

$$Q_\nu(a,b) \geqslant \frac{1}{a^{\nu-1}} \int_b^{\infty} t^\nu e^{-\frac{t^2+a^2}{2}} \frac{e^{at}(ab)}{e^{ab}(at)} I_{\nu-1}(ab)\mathrm{dt} = \frac{bI_{\nu-1}(ab)}{a^{\nu-1}e^{ab}} \int_b^{\infty} t^{\nu-1} e^{-(t-a)^2/2} \mathrm{dt} = \frac{bI_{\nu-1}(ab)}{a^{\nu-1}e^{ab}} \int_{b-a}^{\infty} (u+a)^{\nu-1} e^{-u^2/2} \mathrm{du}.$$

Analogously, from part (c) of Lemma 1 the function $x \mapsto xI_\nu(x)/\sinh x$ is strictly increasing on $(0,\infty)$ for all $\nu \geqslant 0$ and hence for all $t \geqslant b$ and $\nu \geqslant 0$ we have

$$I_\nu(t) \geqslant \frac{b \sinh t}{t \sinh b} I_\nu(b). \tag{2.5}$$

Changing in (2.5) t with at, b with ab and ν with $\nu - 1$, from (1.1) we obtain

$$\begin{aligned} Q_\nu(a,b) &\geqslant \frac{1}{a^{\nu-1}} \int_b^{\infty} t^\nu e^{-\frac{t^2+a^2}{2}} \frac{(ab)\sinh(at)}{(at)\sinh(ab)} I_{\nu-1}(ab)\mathrm{dt} = \frac{bI_{\nu-1}(ab)}{2a^{\nu-1}\sinh(ab)} \int_b^{\infty} t^{\nu-1}[e^{-(t-a)^2/2} - e^{-(t+a)^2/2}]\mathrm{dt} \\ &= \frac{bI_{\nu-1}(ab)}{2a^{\nu-1}\sinh(ab)} \left[\int_{b-a}^{\infty} (u+a)^{\nu-1} e^{-u^2/2} \mathrm{du} - \int_{b+a}^{\infty} (u-a)^{\nu-1} e^{-u^2/2} \mathrm{du}\right]. \end{aligned}$$

Similarly, from part (e) of Lemma 1 the function $x \mapsto x^{-\nu}I_\nu(x)/\cosh x$ is decreasing on $(0,\infty)$ for all $\nu \geqslant -1/2$, and thus for all $t \geqslant b$ and $\nu \geqslant -1/2$ we have

$$I_\nu(t) \leqslant \frac{t^\nu \cosh t}{b^\nu \cosh b} I_\nu(b). \tag{2.6}$$

Now, changing in (2.5) t with at, b with ab and ν with $\nu - 1$, and using again (1.1) we get that

$$\begin{aligned} Q_\nu(a,b) &\leqslant \frac{1}{a^{\nu-1}} \int_b^{\infty} t^\nu e^{-\frac{t^2+a^2}{2}} \frac{(at)^{\nu-1}}{(ab)^{\nu-1}} \frac{\cosh(at)}{\cosh(ab)} I_{\nu-1}(ab)\mathrm{dt} = \frac{I_{\nu-1}(ab)}{2(ab)^{\nu-1}\cosh(ab)} \int_b^{\infty} t^{2\nu-1}[e^{-(t-a)^2/2} + e^{-(t+a)^2/2}]\mathrm{dt} \\ &= \frac{I_{\nu-1}(ab)}{2(ab)^{\nu-1}\cosh(ab)} \left[\int_{b-a}^{\infty} (u+a)^{2\nu-1} e^{-u^2/2} \mathrm{du} + \int_{b+a}^{\infty} (u-a)^{2\nu-1} e^{-u^2/2} \mathrm{du}\right]. \end{aligned}$$

With this the proof is complete. □

Now, by using the well-known Newton binomial theorem, from Theorem 1 we immediately obtain the following result. We note that the coefficients $A_m(\alpha)$, which appear in the following result, were given explicitly in [4] and as we can see below they are related to the upper incomplete gamma function.

Corollary 1. *If $n \in \{1,2,3,\ldots\}$ and $b \geqslant a > 0$, then the following inequalities hold*

$$Q_n(a,b) \geqslant \frac{bI_{n-1}(ab)}{a^{n-1}e^{ab}} \sum_{j=0}^{n-1} C_{n-1}^j a^j A_{n-j-1}(b-a), \tag{2.7}$$

$$Q_n(a,b) \geqslant \frac{bI_{n-1}(ab)}{2a^{n-1}\sinh(ab)} \sum_{j=0}^{n-1} C_{n-1}^j a^j [A_{n-j-1}(b-a) - (-1)^j A_{n-j-1}(b+a)], \tag{2.8}$$

$$Q_n(a,b) \leqslant \frac{I_{n-1}(ab)}{2(ab)^{n-1}\cosh(ab)} \sum_{j=0}^{2n-1} C_{2n-1}^j a^j [A_{2n-j-1}(b-a) + (-1)^j A_{2n-j-1}(b+a)], \tag{2.9}$$

where for all $m \in \{0,1,2,\ldots\}$ the coefficients occurring in the above inequalities are defined by

$$A_m(\alpha) = \int_\alpha^\infty u^m e^{-u^2/2}\,\mathrm{d}u = 2^{\frac{m-1}{2}}\Gamma\left(\frac{m+1}{2},\frac{\alpha^2}{2}\right),$$

which can be rewritten as follows:

$$A_0(\alpha) = \sqrt{\frac{\pi}{2}}\,\mathrm{erfc}\left(\frac{\alpha}{\sqrt{2}}\right),\ A_1(\alpha) = e^{-\alpha^2/2}$$

and for all $k \in \{1,2,3,\ldots\}$

$$A_{2k}(\alpha) = e^{-\alpha^2/2}\sum_{i=1}^{k}\alpha^{2i-1}\frac{(2k-1)!!}{(2i-1)!!} + (2k-1)!!A_0(\alpha),$$

$$A_{2k+1}(\alpha) = e^{-\alpha^2/2}\sum_{i=1}^{k}\alpha^{2i}\frac{(2k)!!}{(2i)!!} + (2k)!!A_1(\alpha).$$

We note that by using parts (b), (d) and (f) of Lemma 1 we may obtain sharper lower and upper bounds than those presented in Theorem 1 and Corollary 1.

Theorem 2. *If $\nu \geqslant 3/2$ and $b \geqslant a > 0$, then the following inequalities hold true*

$$Q_\nu(a,b) \geqslant \frac{\sqrt{b}I_{\nu-1}(ab)}{a^{\nu-1}e^{ab}}\int_{b-a}^{\infty}(u+a)^{\nu-1/2}e^{-u^2/2}\mathrm{d}u, \tag{2.10}$$

$$Q_\nu(a,b) \geqslant \frac{\sqrt{b}I_{\nu-1}(ab)}{2a^{\nu-1}\sinh(ab)}\left[\int_{b-a}^{\infty}(u+a)^{\nu-1/2}e^{-u^2/2}\mathrm{d}u - \int_{b+a}^{\infty}(u-a)^{\nu-1/2}e^{-u^2/2}\mathrm{d}u\right], \tag{2.11}$$

$$Q_\nu(a,b) \leqslant \frac{bI_{\nu-1}(ab)}{2(ab)^{\nu-1}\sinh(ab)}\left[\int_{b-a}^{\infty}(u+a)^{2\nu-2}e^{-u^2/2}\mathrm{d}u - \int_{b+a}^{\infty}(u-a)^{2\nu-2}e^{-u^2/2}\mathrm{d}u\right]. \tag{2.12}$$

Since the proofs of the above results go along the lines introduced in the proof of Theorem 1, they are not included in this paper. Observe that (2.12) in particular yields for all $n \in \{2,3,4,\ldots\}$ and $b \geqslant a > 0$

$$Q_n(a,b) \leqslant \frac{bI_{n-1}(ab)}{2(ab)^{n-1}\sinh(ab)}\sum_{j=0}^{2n-2}C_{2n-2}^{j}a^j[A_{2n-j-2}(b-a) - (-1)^jA_{2n-j-2}(b+a)], \tag{2.13}$$

which is tighter than (2.9) in the case when $n \in \{2,3,4,\ldots\}$.

Finally, we are going to present another generalization of Wang's lower bound in (3.33). Taking into account the result stated in part (g) of Lemma 1, we obtain that (2.12) is reversed:

Theorem 3. *If $0 < \nu \leqslant 3/2$ and $b \geqslant a > 0$, then*

$$Q_\nu(a,b) \geqslant \frac{bI_{\nu-1}(ab)}{2(ab)^{\nu-1}\sinh(ab)}\left[\int_{b-a}^{\infty}(u+a)^{2\nu-2}e^{-u^2/2}\mathrm{d}u - \int_{b+a}^{\infty}(u-a)^{2\nu-2}e^{-u^2/2}\mathrm{d}u\right]. \tag{2.14}$$

2.2. Second case: $a > b > 0$

Our first main result of this subsection reads as follows.

Theorem 4. *If $\nu \geqslant 1$ and $a > b > 0$, then the following inequalities hold*

$$Q_\nu(a,b) \geqslant 1 - \frac{bI_{\nu-1}(ab)}{2a^{\nu-1}\sinh(ab)}\left[\int_{-a}^{b-a}(u+a)^{\nu-1}e^{-u^2/2}\mathrm{d}u - \int_{a}^{b+a}(u-a)^{\nu-1}e^{-u^2/2}\mathrm{d}u\right], \tag{2.15}$$

$$Q_\nu(a,b) \leqslant 1 - \frac{I_{\nu-1}(ab)}{2(ab)^{\nu-1}\cosh(ab)}\left[\int_{-a}^{b-a}(u+a)^{2\nu-1}e^{-u^2/2}\mathrm{d}u + \int_{a}^{b+a}(u-a)^{2\nu-1}e^{-u^2/2}\mathrm{d}u\right]. \tag{2.16}$$

Moreover, the inequality (2.16) holds true for all $\nu \geqslant 1/2$.

Proof. In order to prove (2.15), by using part (c) of Lemma 1, we conclude that for all $0 < t \leqslant b$ and $\nu \geqslant 0$ the inequality (2.5) is reversed, i.e., we have

$$I_\nu(t) \leqslant \frac{b}{t}\frac{\sinh t}{\sinh b}I_\nu(b). \tag{2.17}$$

Changing in (2.17) t with at, b with ab and ν with $\nu-1$, in view of (1.2) we have

$$Q_\nu(a,b) \geqslant 1-\frac{1}{a^{\nu-1}}\int_0^b t^\nu e^{-\frac{t^2+a^2}{2}}\frac{ab}{at}\frac{\sinh(at)}{\sinh(ab)}I_{\nu-1}(ab)\mathrm{d}t = 1-\frac{bI_{\nu-1}(ab)}{2a^{\nu-1}\sinh(ab)}\int_0^b t^{\nu-1}[e^{-(t-a)^2/2}-e^{-(t+a)^2/2}]\mathrm{d}t$$
$$= 1-\frac{bI_{\nu-1}(ab)}{2a^{\nu-1}\sinh(ab)}\left[\int_{-a}^{b-a}(u+a)^{\nu-1}e^{-u^2/2}\mathrm{d}u-\int_a^{b+a}(u-a)^{\nu-1}e^{-u^2/2}\mathrm{d}u\right].$$

Analogously, by using again the fact that the function $x\mapsto x^{-\nu}I_\nu(x)/\cosh x$ is decreasing on $(0,\infty)$, it follows that for all $0\leqslant t\leqslant b$ and $\nu\geqslant -1/2$ the inequality (2.6) is reversed, i.e., we have

$$I_\nu(t) \geqslant \frac{t^\nu\cosh t}{b^\nu\cosh b}I_\nu(b). \tag{2.18}$$

Changing in (2.18) t with at, b with ab and ν with $\nu-1$ in view of (1.2) we get that

$$Q_\nu(a,b) \leqslant 1-\frac{1}{a^{\nu-1}}\int_0^b t^\nu e^{-\frac{t^2+a^2}{2}}\frac{(at)^{\nu-1}\cosh(at)}{(ab)^{\nu-1}\cosh(ab)}I_{\nu-1}(ab)\mathrm{d}t = 1-\frac{I_{\nu-1}(ab)}{2(ab)^{\nu-1}\cosh(ab)}\int_0^b t^{2\nu-1}\left[e^{-(t-a)^2/2}+e^{-(t+a)^2/2}\right]\mathrm{d}t$$
$$= 1-\frac{I_{\nu-1}(ab)}{2(ab)^{\nu-1}\cosh(ab)}\left[\int_{-a}^{b-a}(u+a)^{2\nu-1}e^{-u^2/2}\,\mathrm{d}u+\int_a^{b+a}(u-a)^{2\nu-1}e^{-u^2/2}\mathrm{d}u\right]. \qquad \square$$

Similarly, as in the previous subsection, by using the well-known Newton binomial theorem, from Theorem 4 we immediately obtain the following result. We note that the coefficients $B_m(\alpha)$, which appear in the following result, were given explicitly in [4].

Corollary 2. *If $n\in\{1,2,3,\ldots\}$ and $a>b>0$, then the following inequalities hold*

$$Q_n(a,b) \geqslant 1-\frac{bI_{n-1}(ab)}{2a^{n-1}\sinh(ab)}\sum_{j=0}^{n-1}C_{n-1}^j a^j\left[B_{n-j-1}(a)-(-1)^jB_{n-j-1}(-a)\right], \tag{2.19}$$

$$Q_n(a,b) \leqslant 1-\frac{I_{n-1}(ab)}{2(ab)^{n-1}\cosh(ab)}\sum_{j=0}^{2n-1}C_{2n-1}^j a^j\left[B_{2n-j-1}(a)+(-1)^jB_{2n-j-1}(-a)\right]. \tag{2.20}$$

The coefficients occurring in the above lower and upper bounds are defined for all $m\in\{0,1,2,\ldots\}$ as

$$B_m(\alpha) = \int_{-\alpha}^{b-\alpha}u^m e^{-u^2/2}du,$$

which for all $k\in\{1,2,3,\ldots\}$ can be rewritten as

$$B_{2k}(\alpha) = -\sum_{i=1}^{k}\left[e^{-\alpha^2/2}\alpha^{2i-1}+e^{-(b-\alpha)^2/2}(b-\alpha)^{2i-1}\right]\frac{(2k-1)!!}{(2i-1)!!}+(2k-1)!!B_0(\alpha),$$
$$B_{2k+1}(\alpha) = \sum_{i=1}^{k}\left[e^{-\alpha^2/2}\alpha^{2i}-e^{-(b-\alpha)^2/2}(b-\alpha)^{2i}\right]\frac{(2k)!!}{(2i)!!}+(2k)!!B_1(\alpha),$$

where

$$B_0(\alpha) = \sqrt{\frac{\pi}{2}}\left[erfc\left(-\frac{\alpha}{\sqrt{2}}\right)-erfc\left(\frac{b-\alpha}{\sqrt{2}}\right)\right] \quad \text{and} \quad B_1(\alpha) = e^{-\alpha^2/2}-e^{-(b-\alpha)^2/2}.$$

We note that, as in the previous subsection, by using parts (d) and (f) of Lemma 1 we may obtain sharper lower and upper bounds than those presented in Theorem 4 and Corollary 2 (see Theorem 5 above). Since the proofs of the following results go along the lines introduced in the proof of Theorem 4, they are not included in this paper.

Theorem 5. *If $\nu\geqslant 3/2$ and $a>b>0$, then the following inequalities hold*

$$Q_\nu(a,b) \geqslant 1-\frac{\sqrt{b}I_{\nu-1}(ab)}{2a^{\nu-1}\sinh(ab)}\left[\int_{-a}^{b-a}(u+a)^{\nu-1/2}e^{-u^2/2}\mathrm{d}u-\int_a^{b+a}(u-a)^{\nu-1/2}e^{-u^2/2}\mathrm{d}u\right], \tag{2.21}$$

$$Q_\nu(a,b) \leqslant 1-\frac{bI_{\nu-1}(ab)}{2(ab)^{\nu-1}\sinh(ab)}\left[\int_{-a}^{b-a}(u+a)^{2\nu-2}e^{-u^2/2}\mathrm{d}u-\int_a^{b+a}(u-a)^{2\nu-2}e^{-u^2/2}\mathrm{d}u\right]. \tag{2.22}$$

Observe that (2.22) in particular yields

$$Q_n(a,b) \leqslant 1-\frac{bI_{n-1}(ab)}{2(ab)^{n-1}\sinh(ab)}\sum_{j=0}^{2n-2}C_{2n-2}^j a^j[B_{2n-j-2}(a)-(-1)^jB_{2n-j-2}(-a)], \tag{2.23}$$

which for $n \in \{2,3,4,\ldots\}$ is more stringent than (2.20). Moreover, by using part (g) of Lemma 1, we obtain that (2.22) is reversed:

Theorem 6. *If* $0 < \nu \leqslant 3/2$ *and* $a > b > 0$, *then*

$$Q_\nu(a,b) \geqslant 1 - \frac{bI_{\nu-1}(ab)}{2(ab)^{\nu-1}\sinh(ab)}\left[\int_{-a}^{b-a}(u+a)^{2\nu-2}e^{-u^2/2}\mathrm{d}u - \int_a^{b+a}(u-a)^{2\nu-2}e^{-u^2/2}\mathrm{d}u\right]. \tag{2.24}$$

3. Sharpness of the bounds and comparison with other existing bounds

In this section our aim is to discuss the sharpness of the bounds deduced in the previous section and to compare them with the known bounds from [2,4].

3.1. First case: $b \geqslant a > 0$

Recall that recently Baricz and Sun (Theorem 1 [4]), in order to generalize the results of Corazza and Ferrari [5], proved that if $\nu \geqslant 1$ and $b \geqslant a > 0$, then the following inequalities hold

$$\sqrt{\frac{\pi}{2}}\frac{b^\nu I_{\nu-1}(ab)}{a^{\nu-1}e^{ab}}\,\mathrm{erfc}\left(\frac{b-a}{\sqrt{2}}\right) \leqslant Q_\nu(a,b) \leqslant \frac{I_{\nu-1}(ab)}{(ab)^{\nu-1}e^{ab}}\int_{b-a}^{\infty}(u+a)^{2\nu-1}e^{-u^2/2}\mathrm{d}u. \tag{3.25}$$

Moreover, the right-hand side of (3.25) holds true for all $\nu \geqslant 1/2$. Consequently, if $n \in \{1,2,3,\ldots\}$ and $b \geqslant a > 0$, then the inequalities (Corollary 1 [4])

$$\frac{b^n I_{n-1}(ab)}{a^{n-1}e^{ab}}A_0(b-a) \leqslant Q_n(a,b) \leqslant \frac{I_{n-1}(ab)}{(ab)^{n-1}e^{ab}}\sum_{j=0}^{2n-1}C_{2n-1}^j a^j A_{2n-j-1}(b-a) \tag{3.26}$$

hold. In the proof of the left-hand side of (3.25) we used that $x \mapsto x^{\nu+1}I_\nu(x)e^{-x}$ is increasing on $(0,\infty)$ for all $\nu \geqslant 0$, and consequently it follows that for all $t \geqslant b$ and $\nu \geqslant 0$ we have

$$I_\nu(t) \geqslant \frac{e^t b^{\nu+1}}{e^b t^{\nu+1}}I_\nu(b). \tag{3.27}$$

Since for all $\nu \geqslant 0$ the lower bound in (2.4) is more stringent than the lower bound in (3.27), we conclude that the lower bound in (2.1) is more tighter than the lower bound in (3.25). Consequently (2.7) improves the left-hand side of (3.26). Moreover, it is important to note here that (2.1) holds for all $\nu > 0$, while (3.25) just for $\nu \geqslant 1$.

Since the function $x \mapsto e^{-x}\sinh x$ is strictly increasing on $(0,\infty)$ we conclude that the lower bound in (2.5) is tighter than the lower bound in (2.4). This in turn implies that for all $\nu \geqslant 1$ the lower bound in (2.2) is more stringent that the lower bound in (2.1), and consequently (2.8) improves (2.7). It is also worth mentioning here that, in order to generalize the result of Wang [15, Eq. 11], recently, Baricz (Theorem 1 [2]) proved that if $\nu \geqslant 1$ and $b \geqslant a > 0$, then the following inequality holds

$$Q_\nu(a,b) \geqslant \sqrt{\frac{\pi}{2}}\frac{b^\nu I_{\nu-1}(ab)}{2a^{\nu-1}\sinh(ab)}\left[\mathrm{erfc}\left(\frac{b-a}{\sqrt{2}}\right) - \mathrm{erfc}\left(\frac{b+a}{\sqrt{2}}\right)\right]. \tag{3.28}$$

In particular, this yields (Corollary 1 [2])

$$Q_n(a,b) \geqslant \sqrt{\frac{\pi}{2}}\frac{b^n I_{n-1}(ab)}{2a^{n-1}\sinh(ab)}\left[\mathrm{erfc}\left(\frac{b-a}{\sqrt{2}}\right) - \mathrm{erfc}\left(\frac{b+a}{\sqrt{2}}\right)\right], \tag{3.29}$$

where $n \in \{1,2,3,\ldots\}$ and $b \geqslant a > 0$. In [2, Remark 3] it was also shown that (3.28) improves the left-hand side of (3.25). The key tool in the proof of (3.28) it was the fact that for all $t \geqslant b$ and $\nu \geqslant 0$ we have

$$I_\nu(t) \geqslant \frac{b^{\nu+1}}{t^{\nu+1}}\frac{\sinh t}{\sinh b}I_\nu(b). \tag{3.30}$$

Clearly, for $\nu \geqslant 0$ the lower bound in (2.5) is more tighter than the lower bound in (3.30), and thus the lower bound in (2.2) is more stringent than the lower bound in (3.28). Consequently, the lower bound in (2.8) is sharper than the lower bound in (3.29).

In the proof of the right-hand side of (3.25) we used the fact that the function $x \mapsto x^{-\nu}e^{-x}I_\nu(x)$ is strictly decreasing on $(0,\infty)$ for all $\nu \geqslant -1/2$ and thus for all $t \geqslant b$ and $\nu \geqslant -1/2$ we have

$$I_\nu(t) \leqslant \frac{e^t t^\nu}{e^b b^\nu}I_\nu(b). \tag{3.31}$$

Since the function $x \mapsto e^{-x}\cosh x$ is strictly decreasing on $(0,\infty)$, it follows that for $\nu \geqslant -1/2$ the upper bound in (2.6) is more stringent than the upper bound in (3.31), which in turn implies that the upper bound in (2.3) is more tighter than the upper bound in (3.25). Moreover, this reveals that inequality (2.9) improves the right-hand side of (3.26). Finally, observe that from part (e) of Lemma 1 we have that actually the exponent ν in (2.6) is the least possible. This in turn implies that in (2.3) the power $2\nu-1$ cannot be changed by any smaller constant, and hence (2.3) in this sense is sharp.

It is important to note here that for $n = 1$ the lower bounds in (2.7) and (3.26) reduces to

$$Q_1(a,b) \geqslant \frac{bI_0(ab)}{e^{ab}}A_0(b-a) = \sqrt{\frac{\pi}{2}}\frac{bI_0(ab)}{e^{ab}}\mathrm{erfc}\left(\frac{b-a}{\sqrt{2}}\right), \tag{3.32}$$

which has been deduced by Corazza and Ferrari [5, Eq. 9]. Thus (2.7) provides actually a better generalization of the above result than (3.25). Similarly, if we choose $n = 1$ in (2.8) and (3.29), then we reobtain the result of Wang [15, Eq. 11]

$$Q_1(a,b) \geqslant \frac{bI_0(ab)}{2\sinh(ab)}[A_0(b-a) - A_0(b+a)] = \sqrt{\frac{\pi}{2}}\frac{bI_0(ab)}{2\sinh(ab)}\left[\mathrm{erfc}\left(\frac{b-a}{\sqrt{2}}\right) - \mathrm{erfc}\left(\frac{b+a}{\sqrt{2}}\right)\right]. \tag{3.33}$$

From the above discussion it is clear that (2.8) provides a better generalization of Wang's result than (3.29). Moreover, it is also clear that Wang's lower bound in (3.33) is more tighter than the Corazza and Ferrari's lower bound in (3.32).

Observe that for $n = 1$ the inequality (2.9) reduces to

$$\begin{aligned} Q_1(a,b) &\leqslant \frac{I_0(ab)}{2\cosh(ab)}\sum_{j=0}^{1} C_1^j a^j [A_{1-j}(b-a) + (-1)^j A_{1-j}(b+a)] \\ &= \frac{I_0(ab)}{2\cosh(ab)}\left\{e^{-(b-a)^2/2} + e^{-(b+a)^2/2} + a\sqrt{\frac{\pi}{2}}\left[\mathrm{erfc}\left(\frac{b-a}{\sqrt{2}}\right) - \mathrm{erfc}\left(\frac{b+a}{\sqrt{2}}\right)\right]\right\}. \end{aligned} \tag{3.34}$$

On the other hand, the right-hand side of (3.26) in particular yields a known result of Corazza and Ferrari [5, Eq. 7], i.e.,

$$Q_1(a,b) \leqslant \frac{I_0(ab)}{e^{ab}}\sum_{j=0}^{1} C_1^j a^j A_{1-j}(b-a) = \frac{I_0(ab)}{e^{ab}}\left[e^{-(b-a)^2/2} + a\sqrt{\frac{\pi}{2}}\,\mathrm{erfc}\left(\frac{b-a}{\sqrt{2}}\right)\right]. \tag{3.35}$$

Since (2.9) improves the right-hand side of (3.26), it is clear that our new upper bound (3.34) is more stringent than the upper bound in (3.35). We note that recently the upper bound in (3.35) has been improved also by Wang [15, Eq. 8], and further by Baricz [2, Remark 4]. However, due to their different nature these upper bounds are not directly comparable with our new upper bound in (3.34).

It is important to note here that very recently Sun et al. [13] used Wang's lower bound (3.33) and our new upper bound (3.34) in order to deduce a new approximation for the bit error probability of differential quaternary phase shift keying with Gray coding over an additive white Gaussian noise channel.

Observe also that for $\nu \geqslant 3/2$ inequality (2.11) is sharper than (2.10), which is better than (2.1). Moreover, for $\nu \geqslant 3/2$ the lower bound in (2.11) is more stringent than the lower bound in (2.2), which is more tighter than the lower bound in (2.1). Analogously, since the function $x \mapsto x^{-1}\tanh x$ is strictly decreasing on $(0,\infty)$, we conclude that for $\nu \geqslant 3/2$ the upper bound in (2.12) is tighter than the upper bound in (2.3). It is worth mentioning here that the sharpness of parts (b), (d) and (f) of Lemma 1 guarantees that in (2.10) and (2.11) the exponent 1/2 is the least possible, i.e., the power $\nu - 1/2$ cannot be changed by any larger constant; and similarly, in (2.12) the exponent $2\nu - 2$ cannot be changed by any smaller constant. Thus inequalities 2.10, 2.11 and 2.12 are sharp in this sense.

Finally, we note that it is easy to see that for $1 \leqslant \nu \leqslant 3/2$ the upper bound in (2.14) is more stringent than the upper bound in (2.2), and in particular when $\nu = 1$ it reduces to (3.33). This is actually the third generalization of Wang's result (3.33), and although is tighter than (2.2), is not very useful, since holds just for $0 < \nu \leqslant 3/2$. All the same, part (g) of Lemma 1 guarantees that in the upper bound in (2.14) the exponent $2\nu - 2$ is the least possible and together with (2.11) this yields the best possible extension of Wang's result (3.33).

3.2. Second case: $a > b > 0$

Recently, in order to extend the result of Wang [15, Eq. 27], Baricz (Theorem 2 [2]) deduced the following lower bound

$$Q_\nu(a,b) \geqslant 1 - \sqrt{\frac{\pi}{2}}\frac{b^\nu I_{\nu-1}(ab)}{2a^{\nu-1}\sinh(ab)}\left[\mathrm{erfc}\left(\frac{a-b}{\sqrt{2}}\right) + \mathrm{erfc}\left(\frac{a+b}{\sqrt{2}}\right) - 2\mathrm{erfc}\left(\frac{a}{\sqrt{2}}\right)\right], \tag{3.36}$$

where $\nu \geqslant 1$ and $a > b > 0$. Observe that for $n \in \{1,2,3,\ldots\}$ this reduces to (Corollary 2 [2])

$$Q_n(a,b) \geqslant 1 - \sqrt{\frac{\pi}{2}}\frac{b^n I_{n-1}(ab)}{2a^{n-1}\sinh(ab)}\left[\mathrm{erfc}\left(\frac{a-b}{\sqrt{2}}\right) + \mathrm{erfc}\left(\frac{a+b}{\sqrt{2}}\right) - 2\,\mathrm{erfc}\left(\frac{a}{\sqrt{2}}\right)\right]. \tag{3.37}$$

The key tool in the proof of (3.36) it was the fact that the function $x \mapsto x^{\nu+1}I_\nu(x)/\sinh x$ is strictly increasing on $(0,\infty)$ for all $\nu \geqslant 0$, i.e., for $0 < t \leqslant b$ and $\nu \geqslant 0$ the inequality (3.30) is reversed. It is easy to verify that the upper bound in (2.17) is more tighter than the upper bound in the reversed form of (3.30). This in turn implies that the lower bound in (2.15) is more stringent than the lower bound in (3.36). In particular, inequality (2.19) improves (3.37).

Recall that in order to generalize the result of Corazza and Ferrari [5, Eq. 12], the authors (Theorem 2 [4]) proved that for all $\nu \geqslant 1/2$ and $a > b > 0$ the following inequality holds

$$Q_\nu(a,b) \leqslant 1 - \frac{I_{\nu-1}(ab)}{(ab)^{\nu-1}e^{ab}} \int_{-a}^{b-a} (u+a)^{2\nu-1} e^{-u^2/2} \mathrm{d}u, \tag{3.38}$$

which in particular reduces to (Corollary 2 [4])

$$Q_n(a,b) \leqslant 1 - \frac{I_{n-1}(ab)}{(ab)^{n-1}e^{ab}} \sum_{j=0}^{2n-1} C_{2n-1}^j a^j B_{2n-j-1}(a), \tag{3.39}$$

where $n \in \{1,2,3,\ldots\}$. In order to deduce (3.38), the authors used that for all $\nu \geqslant -1/2$ and $0 \leqslant t \leqslant b$ the inequality (3.31) is reversed. Since the function $x \mapsto e^{-x}\cosh x$ is strictly decreasing on $(0,\infty)$, it follows that for $\nu \geqslant -1/2$ the lower bound in (2.18) is tighter than the lower bound in the reversed form of (3.31). Consequently, we deduce that for all $\nu \geqslant 1/2$ the upper bound in (2.16) is sharper than the upper bound in (3.38), and in particular, (2.20) improves (3.39). On the other hand, part (e) of Lemma 1 guarantees that the exponent ν in (2.18) is the least possible, which in turn implies that the power $2\nu - 1$ in (2.16) cannot be changed by any larger constant.

It is worth mentioning here that when $n = 1$ the inequalities (2.19) and (3.37) reduce to

$$Q_1(a,b) \geqslant 1 - \frac{bI_0(ab)}{2\sinh(ab)}[B_0(a) - B_0(-a)] = 1 - \frac{bI_0(ab)}{2\sinh(ab)}\sqrt{\frac{\pi}{2}}\left[\operatorname{erfc}\left(\frac{a-b}{\sqrt{2}}\right) + \operatorname{erfc}\left(\frac{a+b}{\sqrt{2}}\right) - 2\operatorname{erfc}\left(\frac{a}{\sqrt{2}}\right)\right], \tag{3.40}$$

where $a > b > 0$. This result was deduced recently by Wang [15, Eq. 27]. Now, since (2.19) improves (3.37), and the lower bound in (2.15) is more stringent than the lower bound in (3.36), we conclude that (2.15) is better a extension of (3.40) than (3.36).

Now, if we choose $n = 1$ in (2.20), then for all $a > b > 0$ we get

$$\begin{aligned} Q_1(a,b) &\leqslant 1 - \frac{I_0(ab)}{2\cosh(ab)} \sum_{j=0}^{1} C_1^j a^j [B_{1-j}(a) + (-1)^j B_{1-j}(-a)] \\ &= 1 - \frac{I_0(ab)}{2\cosh(ab)}\left\{2e^{-a^2/2} - e^{-(b-a)^2/2} - e^{-(b+a)^2/2} + a\sqrt{\frac{\pi}{2}}\left[\operatorname{erfc}\left(\frac{a-b}{\sqrt{2}}\right) + \operatorname{erfc}\left(\frac{a+b}{\sqrt{2}}\right) - 2\operatorname{erfc}\left(\frac{a}{\sqrt{2}}\right)\right]\right\}. \end{aligned} \tag{3.41}$$

On the other hand from (3.39) we obtain

$$Q_1(a,b) \leqslant 1 - \frac{I_0(ab)}{e^{ab}} \sum_{j=0}^{1} C_1^j a^j B_{1-j}(a) = 1 - \frac{I_0(ab)}{e^{ab}}\left\{e^{-a^2/2} - e^{-(b-a)^2/2} + a\sqrt{\frac{\pi}{2}}\left[\operatorname{erfc}\left(-\frac{a}{\sqrt{2}}\right) - \operatorname{erfc}\left(\frac{b-a}{\sqrt{2}}\right)\right]\right\}, \tag{3.42}$$

which was proved recently by Corazza and Ferrari [5, Eq. 12]. Since (2.20) improves (3.39) it is clear that our new upper bound in (3.41) is more tighter than the upper bound in (3.42). It is worth mentioning that recently the upper bound in (3.42) has been improved by Wang [15, Eq. 17], and further by Baricz (Remark 6 [2]). However, due to their different nature these upper bounds are not directly comparable with our new upper bound in (3.41).

We note that inequalities (2.21) and (2.22) complete the inequalities (2.11) and (2.12). Moreover, observe that for $\nu \geqslant 3/2$ inequality (2.21) improves (2.15). Analogously, since the function $x \mapsto x^{-1}\tanh x$ is strictly decreasing on $(0,\infty)$, we conclude that for $\nu \geqslant 3/2$ the upper bound in (2.22) is tighter than the upper bound in (2.16). Furthermore, the sharpness in parts (d) and (f) of Lemma 1 guarantees that in (2.21) the exponent $\nu - 1/2$ cannot be replaced by any larger constant, while in (2.22) the exponent $2\nu - 2$ cannot be changed by any smaller constant. These show that our results are sharp. Thus, for $\nu \geqslant 3/2$ the inequality (2.21) provides the best possible extension of (3.40).

Finally, we note that it is easy to verify that the lower bound in (2.24) for $1 \leqslant \nu \leqslant 3/2$ is tighter than the lower bound in (2.15). Moreover, it can be easily verified that for $\nu = 1$ inequality (2.24) reduces to (3.40). This is actually the third generalization of Wang's result (3.40), and although is tighter than (2.15), is not very useful, since holds just for $0 < \nu \leqslant 3/2$. All the same, part (g) of Lemma 1 guarantees that in the upper bound in (2.24) the exponent $2\nu - 2$ is the greatest possible and together with (2.21) this yields the best possible extension of Wang's result (3.40).

4. Numerical results

In this section, our aim is to present some numerical results in order to compare our proposed bounds for the integer order generalized Marcum Q-function with other known bounds in the literature. Since our bounds are valid for either $b \geqslant a$ or $b < a$, we show the comparisons in two cases.

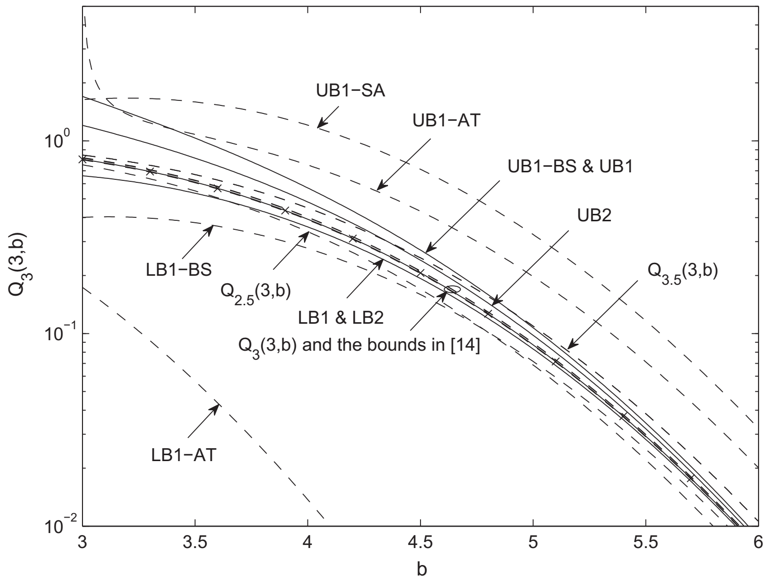

Fig. 1. Numerical results for $Q_\nu(a,b)$ of integer order ν and its upper and lower bounds versus b for the case $b > a = 3$ and $\nu = 3$. 'x: exact, Dashed lines: previous bounds. The lines in the ring: exact value and the bounds of [14]. Solid lines: our proposed bounds, i.e., LB1, LB2, UB1 and UB2.

4.1. First case: $b \geqslant a > 0$

For this case, we have derived two lower bounds for $Q_\nu(a,b)$ with integer order ν, i.e., (2.7) and (2.8). We denote them as LB1 and LB2, respectively. We have also obtained two upper bounds, i.e., (2.9) for integer order $\nu \geqslant 1$ and (2.13) for integer order $\nu \geqslant 2$, which are denoted by UB1 and UB2, respectively. The previous lower bounds for $Q_\nu(a,b)$ of integer order ν include LB1-AT in [1, the first line in Eq. 18][2], $Q_{\nu-0.5}(a,b)$ in [8, Eq. 11, Eq. 14], LB1-BS in [4, Eq. 4, Eq. 8] and $Q_{\nu-LB1}(a,b)$ in [14, Eq. 52]. Similarly, the previous upper bounds include UB1-SA in [11, Eq. 8], UB1-AT in [1, Eq. 17], $Q_{\nu+0.5}(a,b)$ in [8, Eq. 11, Eq. 14], UB1-BS in [4, Eq. 8], $Q_{\nu-UB1}(a,b)$ in [14, Eq. 55] and $Q_{\nu-UB2}(a,b)$ in [14, Eq. 56].

The numerical results for the case $a = 3$, $\nu = 3$ and $b \in (3,6)$ are illuminated in Fig. 1. It is known that the log-concavity based bounds are very tight, even in terms of relative errors [14]. In the case of Fig. 1, the log-concavity based bounds $Q_{\nu-LB1}(a,b)$, $Q_{\nu-UB1}(a,b)$ and $Q_{\nu-UB2}(a,b)$ are much tighter than our proposed bounds. The numerical results of the bounds of $Q_3(3,b)$ for $b \in [6,24]$ are illustrated in Table 1. In this table we show the exact value of $Q_3(3,b)$ and the bounds LB2, UB2, $Q_{3-LB1}(3,b)$ and $Q_{3-UB1}(3,b)$ together with the relative error of each considered bound with respect to the Marcum function $Q_3(3,b)$, expressed as $100 \times (\text{bound} - Q_3(3,b))/Q_3(3,b)$ and indicated as $\varepsilon\%$. We can find that the tightness of our bounds improves faster than that of the log-concavity based bounds. Moreover, if the value of b is enough large (e.g. $b \geqslant 17$), our proposed bounds are tighter than the log-concavity based bounds[3].

If we compare our novel bounds with the existing bounds proposed earlier than [14], one can find in Fig. 1 that our bounds are tighter than these existing bounds for large values of b, and looser than some of the existing bounds (e.g. $Q_{\nu-0.5}(a,b)$ and $Q_{\nu+0.5}(a,b)$) for small values of b (i.e., close to a). These hold true also for larger values of a and ν.

The analytical comparisons in the previous section are supported by our numerical results. We can find that UB2 is tighter than UB1 and UB1-BS, LB1 and LB2 are tighter than LB1-BS, respectively. However, the difference between LB1 and LB2 and the difference between UB1 and UB1-BS are not significant. If we choose a smaller value of a, e.g. $a = 1$, it is simpler to find that LB2 and UB1 are tighter than LB1 and UB1-BS, respectively.

In [4], we proved that the relative errors of the bounds LB1-BS and UB1-BS converge to zero as b tends to infinity. According the previous section, our novel bounds are tighter than the bounds LB1-BS and UB1-BS. Therefore, the relative errors of our bounds also converge to zero as b tends to infinity.

4.2. Second case: $a > b > 0$

For this case we deduced a lower bound for integer order generalized Marcum Q-function, i.e., (2.19), which we denote as LB3. We have also obtained two upper bounds, i.e., (2.20) for integer order $\nu \geqslant 1$ and (2.23) for integer order $\nu \geqslant 2$. We denote them as UB3 and UB4, respectively. The previous lower bounds include LB2-AT in [1, Eq. 20], LB3-AT in [1, Eq. 21],

[2] There is a mistake in the formula of $Q_\nu(0,b)$ given in [1]. For a correct version, the readers are referred to [14].

[3] Note that $Q_{\nu-UB1}(a,b)$ is tighter than $Q_{\nu-UB2}(a,b)$ and $Q_{\nu+0.5}(a,b)$, $Q_{\nu-LB1}(a,b)$ is tighter than $Q_{\nu-0.5}(a,b)$ [14].

Table 1
Relative error ε% of the bounds of $Q_3(3,b)$.

b	$Q_3(3,b)$	LB2	LB2 ε%	UB2	UB2 ε%	$Q_{3\text{-}LB1}(3,b)$	$Q_{3\text{-}LB1}(3,b)$ ε%	$Q_{3\text{-}UB1}(3,b)$	$Q_{3\text{-}UB1}(3,b)$ε%
6	0.0078	0.0057	−2.95	0.0083	7.44	0.0077	−0.64	0.0079	1.93
7	2.625e−4	2.564e−4	−1.95	2.744e−4	4.96	2.600e−4	−0.57	2.660e−4	1.72
8	3.265e−6	3.220e−6	−1.38	3.380e−6	3.53	3.248e−6	−0.51	3.315e−6	1.53
9	1.500e−8	1.484e−8	−1.02	1.540e−8	2.65	1.493e−8	−0.46	1.520e−8	1.38
10	2.524e−11	2.504e−11	−0.79	2.576e−11	2.06	2.514e−11	−0.41	2.556e−11	1.25
11	1.555e−14	1.546e−14	−0.62	1.581e−14	1.64	1.549e−14	−0.38	1.573e−14	1.14
12	3.506e−18	3.488e−18	−0.50	3.553e−18	1.34	3.494e−18	−0.35	3.543e−18	1.05
13	2.892e−22	2.880e−22	−0.42	2.925e−22	1.12	2.883e−22	−0.32	2.920e−22	0.97
14	8.733e−27	8.702e−27	−0.35	8.816e−27	0.95	8.707e−27	−0.30	8.811e−27	0.90
15	9.655e−32	9.626e−32	−0.30	9.733e−32	0.81	9.628e−32	−0.28	9.736e−32	0.84
16	3.910e−37	3.900e−37	−0.26	3.937e−37	0.70	3.900e−37	−0.26	3.941e−37	0.79
17	5.801e−43	5.788e−43	−0.22	5.837e−43	0.62	5.787e−43	−0.25	5.844e−43	0.74
18	3.155e−49	3.149e−49	−0.20	3.172e−49	0.54	3.148e−49	−0.23	3.177e−49	0.70
19	6.290e−56	6.279e−56	−0.17	6.321e−56	0.48	6.277e−56	−0.22	6.332e−56	0.66
20	4.600e−63	4.592e−63	−0.16	4.620e−63	0.43	4.590e−63	−0.21	4.629e−63	0.63
21	1.234e−70	1.232e−70	−0.14	1.239e−70	0.39	1.231e−70	−0.20	1.241e−70	0.60
22	1.214e−78	1.212e−78	−0.13	1.218e−87	0.35	1.212e−78	−0.19	1.221e−78	0.57
23	4.384e−87	4.379e−87	−0.11	4.400e−78	0.32	4.376e−87	−0.18	4.408e−87	0.55
24	5.811e−96	5.805e−96	−0.10	5.828e−96	0.29	5.801e−96	−0.17	5.842e−96	0.52

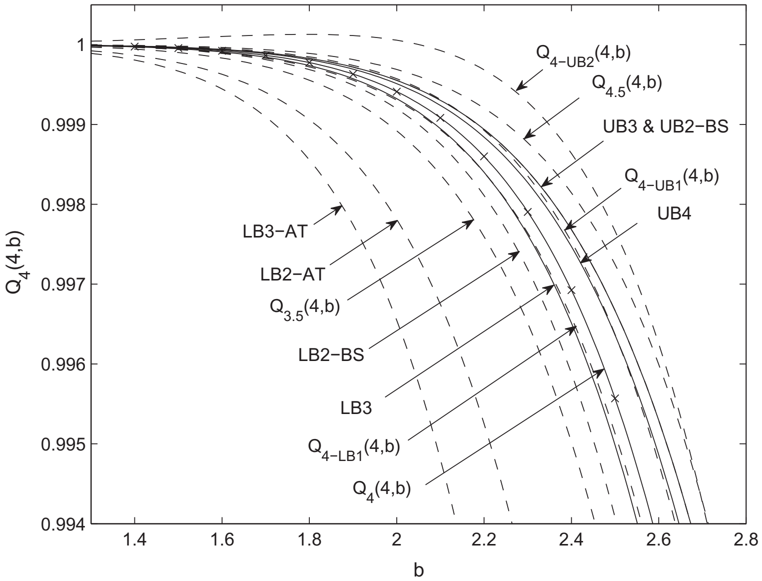

Fig. 2. Numerical results for $Q_v(a,b)$ of integer order v and its upper and lower bounds versus b for the case $b < a = 4$ and $v = 4$. x: exact, Dashed lines: previous bounds. Solid lines: our proposed bounds, i.e., LB3, UB3 and UB4.

$Q_{v-0.5}(a,b)$ in [8, Eq. 11, Eq. 14], LB2-BS in [4, Eq. (18)] and $Q_{v-LB1}(a,b)$ in [14, Eq. 52]. The upper bounds previously introduced in the literature include $Q_{v+0.5}(a,b)$ in [8, Eq. 11, Eq. 14], UB2-BS in [4, Eq. 19], $Q_{v-UB1}(a,b)$ in [14, Eq. 55] and $Q_{v-UB2}(a,b)$ in [14, Eq. 56].

The numerical results for the case $b < a = 4$ and $v = 4$ are illuminated in Fig. 2. One can find that our new bounds are tighter than the existing bounds for small values of b, when the values of $Q_v(a,b)$ are quite close to 1. For larger values of b (i.e., close to a), our new bounds are looser than the log-concavity based bounds given in [14][4]. These results hold true also for larger values of a and v.

The analytical comparisons in the previous section for the case $b < a$ are also supported by our numerical results. We can find that UB4 is tighter than UB3 and UB2-BS, and LB3 is tighter than LB2-BS. However, the difference between UB3 and UB2-BS is not significant. If we choose a smaller a, e.g. $a = 1$, it is easy to see that UB3 is tighter UB2-BS.

[4] More numerical results for larger values of b (i.e., close to a) are shown in Fig. 8 of [14].

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