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AN INTEGRAL EQUATION METHOD FOR THE NUMERICAL SOLUTION OF LAPLACE'S EQUATION WITHOUT GREEN'S FUNCTION

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The common way to establish an integral equation for the solution of Laplace's equation uses the Green's function of the given equation. It will be shown in this paper that an integral equation can also be constructed by using a particular solution of the Laplace equation as the Kernel of the integral equation.

1. Introduction

If one wishes to solve Laplace's equation, one generally uses a numerical equivalent of the operator ∇^2. However, it is even possible to transform the given equation into an equivalent integral equation. The integral equation is then solved numerically. For the solution of the Laplace equation in a two dimensional area S, the corresponding integral equation only involves the boundary C of S. This means that a two dimensional problem is replaced by the numerical solution of a one-dimensional integral equation. This feature reduces the complexity of the problem, saves computation time and storage locations if the problem is solved numerically.

Integral equations for potential problems have been used in various applications [1–6]. The same techniqye has also been applied to other problems such as the diffusion equation [7, 8], drift–diffusion problems [10,11], electro–magnetic scattering [12,13], the calculation of eigenvalues [14] and semiconductor problems [15].

Firstly, the construction of an integral equation for a potential problem is briefly reviewed. Let $\phi(\bar{r})$ $[\nabla^2\phi = 0]$ be the potential in an area S with boundary C, on which the potential $\phi = f(\bar{r})$ is a known function. In order to construct an integral equation, one uses the Green's function of the Laplace equation:

$$G(\bar{r}\,|\bar{r}\,') = \frac{1}{2\pi}\ln|\bar{r} - \bar{r}\,'| \,. \tag{1}$$

The unknown potential $\phi(\bar{r})$ is then written as:

$$\phi(\bar{r}) = \oint_C \rho(\bar{r}\,')\, G(\bar{r}\,|\bar{r}\,')\, \mathrm{d}C' \,, \tag{2}$$

where $\rho(\bar{r})$ is an unknown source function defined along the boundary C. Imposing the boundary condition $\phi = f(\bar{r})$ on the proposed solution (2) yields:

$$\oint_C \rho(\bar{r}\,')\, G(\bar{r}\,|\bar{r}\,')\, \mathrm{d}C' = f(\bar{r}) \,, \qquad \bar{r} \in \mathrm{C} \,. \tag{3}$$

(3) is an integral equation for the unknown function ρ. Once the integral equation has been solved, the potential ϕ can be calculated in every point by performing the integration (2). It is evident that the integral equation (3) only involves the boundary C, proving that the dimension of the problem has been reduced. It should be noted here that integral equations can still be established for more complicated boundary conditions than $\phi = f$ [16].

It has been proved that it is always possible to write a given solution ϕ as an integral like eq. (2). One can also argue that the Green's function is a particular solution of Laplace's equation $[\nabla^2 G = \delta(\boldsymbol{r} - \boldsymbol{r}')]$ and hence a superposition of solutions such as the integral (2) still represents a solution of Laplace's equation. Obviously, only the boundary conditions are then to be considered. However, one may wonder why the Green's function is used in eqs. (2) and (3). If eq. (2) should be replaced by:

$$\phi(\boldsymbol{r}) = \oint_C \rho(\boldsymbol{r}')\, H(\boldsymbol{r}\,|\boldsymbol{r}')\, \mathrm{d}C' \,, \tag{4}$$

$H(r|r')$ being a particular solution of the Laplace equation, the relation (4) will still satisfy Laplace's equation. The construction of an integral equation is readily performed by imposing the boundary conditions on the relation (4). In this paper several types of kernels $H(r|r')$ will be considered in order to determine their ability for the numerical solution of an integral equation. If the kernels are polynomials, they are called kernels of the first kind. Other functions $H(r|r')$ will be denoted further on as kernels of the second kind.

In order to solve the integral equation numerically, the boundary C of S is divided into m intervals ΔC_i ($i = 1, ..., m$). Assuming ρ be a constant ρ_i in ΔC_i, the relation (4) can be replaced by:

$$\phi(r) = \sum_{j=1}^{m} \rho_j \int_{\Delta C_j} H(r|r')\, dC' . \tag{5}$$

Imposing the boundary condition $\phi = f$, gives rise to:

$$\sum_{j=1}^{m} \rho_j \int_{\Delta C_j} H(r_i|r')\, dC' = f(r_i) , \quad i = 1, ..., m , \tag{6}$$

where r_i can be taken as the centre point of the interval ΔC_i. The integral equation is now replaced by the numerical solution of a linear algebraic set (6) with m unknowns $\rho_1, ..., \rho_m$. Once this has been performed, the ϕ-value in an arbitrary point r can be evaluated by eq. (5).

Obviously, one can imagine many other numerical approximations for eq. (4), by using higher order approximating functions for ρ e.g. This will not be done here as we are primarily interested in the application of new Kernels $H(r|r')$ instead of the Green's function $G(r|r')$. Moreover, it has been found experimentally that higher order approximations do not always improve the accuracy when solving the integral equation numerically [17].

2. Kernels of the first kind

We define kernels of the first kind as polynomials in x, y, x' and y'. As we are interested in two-dimensional problems, a polynomial satisfying Laplace's equation can be easily found by mentioning that the real and imaginary part of an analytic function is always harmonic. By taking an arbitrary linear combination of $\mathrm{Re}[(z - z')^n]$ and $\mathrm{Im}[(z - z')^n]$, [$z = x + jy$, $z' = x' + jy'$] one gets a suitable kernel $H(r|r')$.

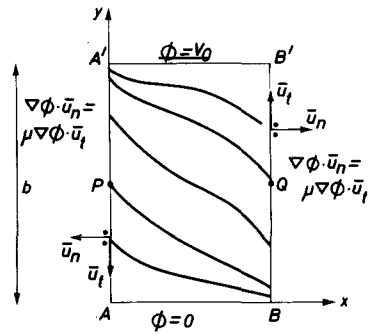

Fig. 1. Rectangular geometry used to establish the integral equation method.

In order to check the accuracy of the method, the potential problem shown on fig. 1, will be solved numerically. The boundary conditions are:

$$\phi = V_0 , \qquad r \in A'B' , \tag{7}$$

$$\phi = 0 , \qquad r \in AB , \tag{8}$$

$$\nabla\phi \cdot \boldsymbol{u}_n = \mu\nabla\phi \cdot \boldsymbol{u}_t , \qquad r \in AA', BB' . \tag{9}$$

The boundary condition (9) describes the field problem in a Hall generator [16]. A sketch of the equipotential lines for $\mu = 1$ is also shown on fig. 1. For $\mu = 0$, the solution of the problem is easily found to be:

$$\phi = V_0 y/b . \tag{10}$$

For $\mu = 1$, the so-called Hall voltage V_H (= potential difference between P and Q) is known accurately ($V_H = 0.5226$ for $V_0 = 1$). This value can also be used to check the accuracy of the results.

By using kernels of the first kind, H being a polynomial of degree n, very accurate results were obtained for $\mu = 0$, in spite of the ill-conditioned matrix [18]. For $\mu \neq 0$, the matrix was still ill-conditioned for $2n < m$ (m being the number of unknowns), but the results became meaningless. For $2n \geqslant m$ the conditioning of the matrix was good but only the results obtained with $m = 2n$ were acceptable.

The results obtained with kernels of the first kind can be easily explained by considering a parti-

cular example. Let $H(r|r')$ be given by:

$$H(r|r') = \mathrm{Re}[(z - z')^2] = (x - x')^2 - (y - y')^2. \quad (11)$$

The potential $\phi(r)$ is then given by:

$$\phi(r) = \oint_C \rho(r')\, H(r|r')\, \mathrm{d}C' = (x^2 - y^2) \oint_C \rho(r')\, \mathrm{d}C'$$

$$- 2x \oint_C x'\, \rho(r')\, \mathrm{d}C' + 2y \oint_C y' \rho(r')\, \mathrm{d}C'$$

$$+ \oint_C (x'^2 - y'^2)\, \rho(r')\, \mathrm{d}C'$$

$$= (x^2 - y^2) A_0 - 2xA_1 + 2yA_2 + A_3 . \quad (12)$$

This proves that there can be only four linear independent unknowns A_0, A_1, A_2 and A_3, regardless of the dimension m of the algebraic set. This explains why ill conditioned matrices were obtained. For $m \leqslant 2n$, the number of equations becomes smaller than the number of independent unknowns, so that the algebraic set is no longer ill-conditioned. The reason why in-spite of the ill-conditioning, very accurate results are found for $\mu = 0$, is that the analytic solution (10) is incorporated in the proposed solution (12). If $\mu \neq 0$, the analytic solution cannot be presented exactly by a polynomial as (12). One obtains only acceptable results for $2n = m$, as the number of independent unknowns fits the dimension of the algebraic set.

3. Kernels of the second kind

Kernels of the second kind are generally defined as linear combinations of $\mathrm{Re}[(z - z')^\alpha]$ and $\mathrm{Im}[(z - z')^\alpha]$, α being a non-integer real number. Kernels of the first kind ($\alpha = n$ = integer) show the disadvantage that the number of divisions m of the boundary is directly related to n. With a given Kernel, it is not possible to increase m in order to improve the accuracy. Secondly, the method is fundamentally a polynomial approximation for the unknown potential which is fitted to the boundary conditions in m points along C. It should then rather be classified as a parametric method [19]. Therefore, the applicability of Kernels of the second kind will be outlined now. It will be found that this method gives results comparable to those obtained with the classical method based on the Green's function as the kernel of the integral equation.

In order to establish the integral equation, one has to evaluate the potential ϕ_p caused by a constant source ρ_i in ΔC_i:

$$\phi_p = \rho_i \int_{\Delta C_i} \mathrm{Re}(z - z')^\alpha \, \mathrm{d}C' . \quad (13)$$

By putting ΔC_i along the real axis, the integral (13) can be easily evaluated by complex integrations:

$$\phi_p = \rho_i \, \mathrm{Re}\Big[\int_{-|\Delta C_i|/2}^{+|\Delta C_i|/2} (z - z')^\alpha \, \mathrm{d}z'\Big]$$

$$= \frac{\rho_i}{\alpha + 1} \mathrm{Re}[(z + |\Delta C_i|/2)^{\alpha+1}$$

$$-(z - |\Delta C_i|/2)^{\alpha+1}] . \quad (14)$$

The same formula holds if $\mathrm{Im}[(z - z')^\alpha]$ is used as the kernel. Even the components of $\nabla\phi$ are found in a similar way. This means that the coefficients of the algebraic set (6) are easily evaluated by complex integrations.

In order to define the kernel of the second kind unambiguously, one has to draw a cut-line in the complex plane. In order to avoid meaningless results, this cut-line may not go through the area S. For the function $(z - z')^\alpha$, z' being always located on the boundary C, the cut-line has been chosen along the externally directed normal (fig. 2).

It has also been found experimentally that the numerical results could be improved by taking a linear combination of $\mathrm{Re}[(z - z')^\alpha]$ and $\mathrm{Im}[(z - z')^\alpha]$ where ζ is a complex number chosen in such a way

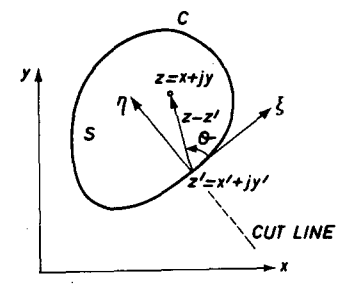

Fig. 2. Angular dependence of the kernel.

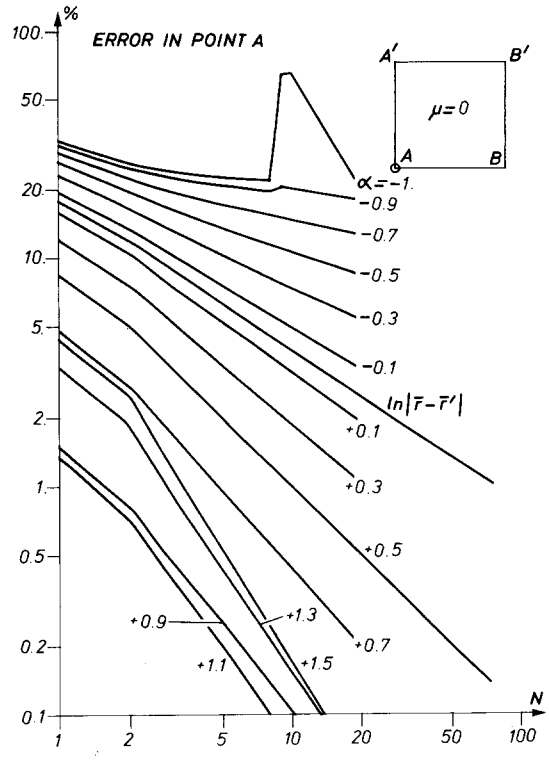

Fig. 3. Error in point A as a function of N.

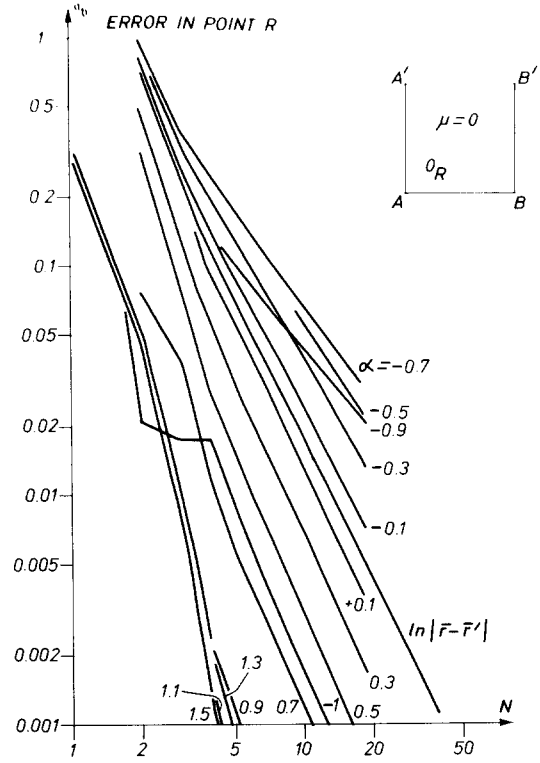

Fig. 4. Error in point R as a function of N.

that the angular dependence on θ (fig. 2). of the kernel be a minimum in the half plane $\eta \geqslant 0$.

The problem has been solved numerically for a square shaped geometry (fig. 1). The discretisation procedure was performed as outlined in the introduction. Fig. 3 shows the absolute value of the error on the potential ϕ at the point A as a function of N for $V_0 = 1$ and $\mu = 0$. N is the number of divisions per side. Different values of α are considered and the error values were normalised to the maximum attainable potential $V_0 = 1$. Note that the point A does not coincide with one of the r_i points mentioned in section 1, where the boundary condition $\phi = 0$ (or V_0) is fulfilled exactly. Fig. 4 shows similar results for a point R located inside S.

Finally, fig. 5 represented the error on the so-called Hall-voltage, which is the potential difference between the points P and Q (fig. 1). These results are obtained for $\mu = 1$. The exact Hall voltage is then $V_H = 0.5226$.

From the results shown in figs. 4–6 several conclusions can be drawn. By comparing fig. 4 and fig. 5, the error in a point on the boundary C is larger than in a point located inside S. This is a common fact for integral equation methods as the discretisation process takes place on the boundary C. The numerical noise introduced by this will be smoothed out for points at a sufficient distance from C. By comparing fig. 4 and fig. 6 one observes that the parameter μ has little influence on the accuracy in contrast to the results obtained with kernels of the first kind. This can be easily understood as the exact solution (10) for $\mu = 0$ is not incorporated in the proposed solution (4) if $H(\mathbf{r}|\mathbf{r}')$ is a kernel of the second kind. As a general rule it can be stated that the optimum values of α are situated in the interval $(-1, +2)$. However, α may not approach to an integer number as the kernel becomes then of the first kind. The value $\alpha = 0.5$ has proved to give good results in all situations. In order to compare with the classical method, the results obtained with the Green's function integral equation technique are also drawn on figs. 4–6. The drops of the curves as functions of N are very regular except for very low values of N. The same

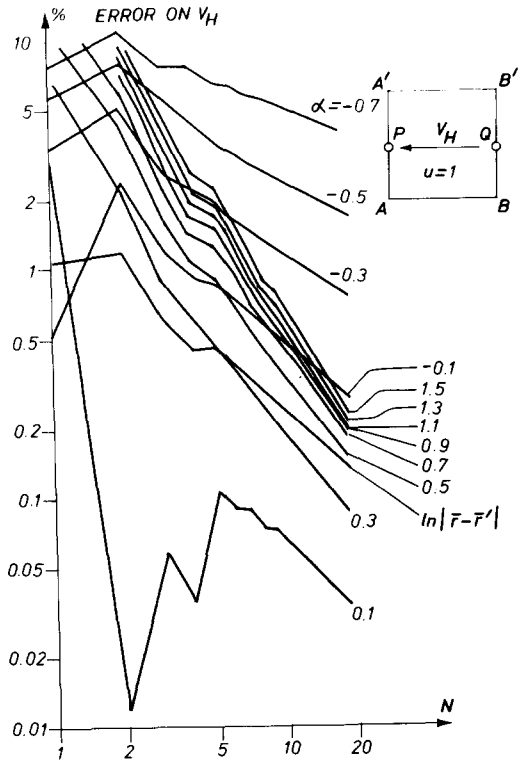

Fig. 5. Error on Hall voltage as a function of N.

remark holds for the Green's function integral equation. The behaviour of the drop shows a sharp contrast between kernels of the second and the first kind, where very irregular curves were found.

4. Relation with the Green's function

By investigating the results obtained with $\alpha = \pm 0.1$ and the Green's function, one is intended to think that the particular case $\alpha = 0$ is nothing else than the Green's function. Nevertheless, the kernel $(z - z')^\alpha$ with $\alpha = 0$ has no significance. However, by performing the limit $\alpha \to 0$ carefully, the Green's function is obtained indeed:

$$\lim_{\alpha \to 0} \frac{(z - z')^\alpha - 1}{\alpha} = \lim_{\alpha \to 0} \frac{1 + \alpha \ln(z - z') + \ldots - 1}{\alpha}$$

$$= \ln(z - z').$$

This proves that the Green's function can be considered as the limiting kernel for $\alpha \to 0$.

5. Conclusion

An integral equation method has been presented to solve Laplace's equation numerically. Instead of the Green's function, a particular solution was used as the kernel of the integral equation. As there is an infinity of particular solutions of Laplace's equation, only functions which can be written as a linear combination of the real and imaginary part of $(z - z')^\alpha$ have been considered. If α differs from an integer value, very good results were obtained. The method may then be compared to the classical Green's function technique.

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