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Continuous Numerical Solutions and Error Bounds for Time Dependent Systems of Partial Differential Equations: Mixed Problems

L. JÓDAR AND E. PONSODA
Department of Applied Mathematics, Polytechnical University of Valencia
P. O. Box 22.012, Valencia, Spain

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Abstract—The aim of this paper is to construct continuous numerical solutions with a prefixed accuracy in a bounded domain $\Omega(t_0, t_1) = [0,p] \times [t_0, t_1]$, for mixed problems of the type $u_t(x,t) - D(t)u_{xx}(x,t) = 0$, $0 < x < p$, $t > 0$, subject to $u(0,t) = u(p,t) = 0$ and $u(x,0) = F(x)$. Here, $u(x,t)$ and $F(x)$ are r-component vectors and $D(t)$ is a $\mathbb{C}^{r\times r}$ valued two-times continuously differentiable function, so that $D(t_1)D(t_2) = D(t_2)D(t_1)$ for $t_2 \ge t_1 > 0$ and there exists a positive number δ such that every eigenvalue z of $(D(t) + D^H(t))/2$ with $t > 0$ is bigger than δ.

Keywords—Partial differential equation, Numerical solution, Error bound, Multistep method, Logarithmic norm.

1. INTRODUCTION

Coupled partial differential equations appear in many different problems, such as for magnetohydrodynamic flows [1], in the study of temperature distribution within a composite heat conductor [2], mechanics [3,4], diffusion problems [5,6], nerve conduction problems [7,8], biochemistry [9], armament models [10], etc. Discrete numerical methods for solving coupled partial differential equations are widely studied in the literature [11–13], however, the analytic solution of a system of partial differential equations may satisfy an important physical property and the numerical solution may not. This motivates the search for the analytic solution or the analytic-numerical solution of the problem.

In this paper, we consider coupled time dependent partial differential problems of the type

$$u_t(x,t) - D(t)u_{xx}(x,t) = 0, \qquad 0 < x < p, \qquad t > 0, \tag{1.1}$$

$$u(0,t) = u(p,t) = 0, \qquad t > 0, \tag{1.2}$$

$$u(x,0) = F(x), \qquad 0 \le x \le p, \tag{1.3}$$

where $u = (u_1, u_2, \dots, u_r)^\top, F(x)$ are vectors in $\mathbb{C}^r$ and $D(t)$ is an invertible $\mathbb{C}^{r\times r}$ valued two-times continuously differentiable function satisfying the conditions:

$$\text{There exist a positive number } \delta > 0 \text{ such that}$$
$$\text{every eigenvalue } z \text{ of the matrix } \frac{D(t) + D^H(t)}{2} \text{ verifies } z \ge \delta, \tag{1.4}$$

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Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX where $D^H(t)$ denotes the conjugate transpose of $D(t)$, and

$$D(t_1)D(t_2) = D(t_2)D(t_1) \qquad \text{for } t_1 \geq t_2 \geq 0. \tag{1.5}$$

Sufficient conditions on $D(t)$ in order to verify the property (1.5) may be found in [14,15]. Concrete situations where problem (1.1)–(1.3) arises can be found in [16–18]. For the case where $D(t)$ is a constant matrix, problem (1.1)–(1.3) has been treated in [19].

This paper is organized as follows. In Section 2, we construct an exact series solution of problem (1.1)–(1.3) under the hypotheses (1.4) and (1.5). Section 3 deals with the construction of a continuous numerical solution of problem (1.1)–(1.3) in a bounded domain $\Omega(t_0, t_1) = [0,p] \times [t_0, t_1]$ with $t_0 > 0$, with a prefixed accuracy. This fact is achieved in two steps. First, the infinite series solution is truncated in an appropriate way. Second, we approximate the general term of the truncated finite series using recent results given in [20].

Throughout this paper, the set of all eigenvalues of a matrix D in $\mathbb{C}^{r\times r}$ is denoted by $\sigma(D)$. We denote by $\|D\|$ the 2-norm of D, [21, p. 56]:

$$\|D\| = \sup_{x\neq 0} \frac{\|Dx\|_2}{\|x\|_2},$$

where for a vector $y \in \mathbb{C}^r$, $\|y\|_2 = \left(y^\top y\right)^{1/2}$ is the usual Euclidean norm of y. The norm $\|D\|$ coincides with the square root of the maximum of the set $\left\{|z|;\ z \in \sigma(D^H D)\right\}$, see [22, p. 41]. In accordance with [23; 24, p. 110; 25, p. 59], the logarithmic norm $\mu(D)$ of the matrix D is defined by

$$\mu(D) = \lim_{h\to 0,\, h>0} \frac{\|I + hD\| - 1}{h},$$

and the following properties hold

$$|\mu(D)| \leq \|D\|, \tag{1.6}$$

$$\mu(\alpha D) = \alpha\mu(D), \qquad \text{for all } \alpha \geq 0, \tag{1.7}$$

$$\mu(D) \geq \operatorname{Re}(Z), \qquad \text{for every eigenvalue } z \text{ of } D, \tag{1.8}$$

$$\mu(D) = \max\left\{z;\ z \in \sigma\left(\frac{D + D^H}{2}\right)\right\}. \tag{1.9}$$

2. AN EXACT SERIES SOLUTION

Let us consider the vector initial value problem

$$T_n'(t) + \left(\frac{n\pi}{p}\right)^2 D(t)T_n(t) = 0, \qquad T_n(0) = c_n, \quad t > 0, \tag{2.1}$$

where c_n is the n^{th} sine-Fourier coefficient of $F(x)$:

$$c_n = \frac{2}{p}\int_0^p F(x)\sin\left(\frac{n\pi x}{p}\right)dx, \qquad n \geq 1.$$

Under the hypothesis (1.5), one gets

$$D(t)\int_0^t D(s)\,ds = \int_0^t D(s)\,ds\ D(t), \qquad t > 0,$$

and from [15], the unique solution of (2.1) is given by

$$T_n(t) = \exp\left[-\left(\frac{n\pi}{p}\right)^2 \int_0^t D(s)\,ds\right] c_n, \qquad t \geq 0, \quad n \geq 1.$$

Let us consider now the sequence of functions $\{u_n(x,t)\}_{n\geq 1}$ defined by

$$u_n(x,t) = \exp\left[-\left(\frac{n\pi}{p}\right)^2 \int_0^t D(s)\,ds\right] c_n \sin\left(\frac{n\pi x}{p}\right), \tag{2.2}$$

and note that

$$u_n(x,0) = u_n(x,p) = 0, \qquad t > 0,$$

and for $0 < x < p$, $t > 0$,

$$\frac{\partial u_n(x,t)}{\partial t} - D(t)\frac{\partial^2 u_n(x,t)}{\partial x^2} = \left[T_n'(t) + \left(\frac{n\pi}{p}\right)^2 T_n(t)\right] \sin\left(\frac{n\pi x}{p}\right) = 0. \tag{2.3}$$

This suggests a candidate series solution of problem (1.1)–(1.3) of the form

$$\begin{aligned} u(x,t) &= \sum_{n\geq 1} u_n(x,t) = \sum_{n\geq 1} T_n(t)\sin\left(\frac{n\pi x}{p}\right) \\ &= \sum_{n\geq 1} \exp\left[-\left(\frac{n\pi}{p}\right)^2 \int_0^t D(s)\,ds\right] c_n \sin\left(\frac{n\pi x}{p}\right). \end{aligned} \tag{2.4}$$

Assuming for a moment that $u(x,t)$ is well defined, note that $u(x,0) = \sum_{n\geq 1} c_n \sin((n\pi x)/p)$ and hence, the initial value condition (1.3) is satisfied if $F(x)$ satisfies any of the sufficient conditions for the convergence to $F(x)$ of its sine-Fourier series (see [26, p. 57] for instance).

To prove that $u(x,t)$ defined by (2.4) is a rigorous solution of problem (1.1)–(1.3), note that from [24, p. 111] and the property (1.7) of the logarithmic norm, it follows that

$$\|T_n(t)\| \leq \|c_n\| \exp\left[\left(\frac{n\pi}{p}\right)^2 \int_0^t \mu(-D(s))\,ds\right], \qquad t \geq 0. \tag{2.5}$$

Now we use a local argument. Let $t_1 > t_0 > 0$, and let us consider the domain

$$\Omega(t_0,t_1) = [0,p] \times [t_0,t_1]. \tag{2.6}$$

From Bendixon's theorem (see [27, p. 395]), one gets

$$\operatorname{Re} z \geq \min\left\{w;\, w \in \sigma\left(\frac{D(t) + D^H(t)}{2}\right)\right\}, \qquad z \in \sigma(D(t)), \tag{2.7}$$

and from property (1.8) of the logarithmic norm, we have

$$\mu(D(t)) \geq \operatorname{Re} z, \qquad z \in \sigma(D(t)). \tag{2.8}$$

Let us introduce the scalar function $d(t_0,t_1)$, defined by

$$d(t_0,t_1) = \min_{t_0\leq u\leq t_1} \left\{w,\, w \in \sigma\left(\frac{D(u) + D^H(u)}{2}\right)\right\}, \tag{2.9}$$

and note that from hypothesis (1.4), one gets $d(t_0,t_1) \geq \delta > 0$ and from (2.7)–(2.9), it follows that

$$\mu(D(t)) \geq d(t_0,t_1) \geq \delta, \qquad t \in [t_0,t_1]. \tag{2.10}$$

From (2.5) and (2.10), we can write

$$\|T_n(t)\| \leq \|c_n\| \exp\left[-t\, d(t_0,t_1)\left(\frac{n\pi}{p}\right)^2\right] \leq \|c_n\| \exp\left[-t\left(\frac{n\pi}{p}\right)^2 \delta\right]. \tag{2.11}$$

Since, from the Riemann-Lebesgue lemma, the sequence $\{c_n\}_{n\geq 1}$ is bounded:

$$\|c_n\| \leq M, \qquad n \geq 1 \tag{2.12}$$

from (2.11), one concludes the absolute convergence of the series (2.4) in the domain $\Omega(t_0, t_1)$. From (2.1), it follows that $T_n'(t) = -\left(\frac{n\pi}{p}\right)^2 D(t)T_n(t)$. Thus, the series obtained by termwise partial differentiation with respect to the variable t in (2.4) takes the form

$$\sum_{n\geq 1} \frac{\partial u_n(x,t)}{\partial t} = \sum_{n\geq 1} T_n'(t) \sin\left(\frac{n\pi x}{p}\right) = -D(t) \sum_{n\geq 1} \left(\frac{n\pi}{p}\right)^2 T_n(t) \sin\left(\frac{n\pi x}{p}\right),$$

and from (2.11) and (2.12) for $(x,t) \in \Omega(t_0, t_1)$, one gets

$$\begin{aligned} \sum_{n\geq 1} \left\| \frac{\partial u_n(x,t)}{\partial t} \right\| &\leq \gamma(t_0,t_1) \sum_{n\geq 1} \left(\frac{n\pi}{p}\right)^2 \|c_n\| \exp\left[-t\left(\frac{n\pi}{p}\right)^2 \delta\right] \\ &\leq \gamma(t_0,t_1)\, M \sum_{n\geq 1} \left(\frac{n\pi}{p}\right)^2 \exp\left[-t_0\left(\frac{n\pi}{p}\right)^2 \delta\right], \end{aligned} \tag{2.13}$$

where

$$\gamma(t_0, t_1) = \max\left\{\|D(t)\|,\ t_0 \leq t \leq t_1\right\}.$$

Taking two-times termwise partial differentiation with respect to the variable x in (2.4), one gets

$$\sum_{n\geq 1} \frac{\partial^2 u_n(x,t)}{\partial x^2} = -\sum_{n\geq 1} \left(\frac{n\pi}{p}\right)^2 T_n(t) \sin\left(\frac{n\pi x}{p}\right) c_n,$$

and from (2.11) for $(x,t) \in \Omega(t_0, t_1)$, it follows that

$$\sum_{n\geq 1} \left\| \frac{\partial^2 u_n(x,t)}{\partial x^2} \right\| \leq M \sum_{n\geq 1} \left(\frac{n\pi}{p}\right)^2 \exp\left[-t_0\left(\frac{n\pi}{p}\right)^2 \delta\right]. \tag{2.14}$$

From the derivation theorem for functional series, [28, Theorem 9.14], and from (2.11),(2.13) and (2.14), it follows that $u(x,t)$ defined by (2.4) is a continuous, two-times partial differentiable with respect to x, and once with respect t, satisfying (1.1),(1.2). From the previous comments and [26, Corollary 1, p. 57], the following result has been established.

THEOREM 2.1. *Let us consider the problem* (1.1)–(1.3), *where* $D(t)$ *is a* $\mathbb{C}^{r\times r}$ *valued continuous function in* $t \geq 0$ *satisfying the conditions* (1.4)–(1.5), *and let* $F(x)$ *be a continuous function in [0,p] such that* $F(0) = F(p) = 0$, *and each of its components* f_j *for* $1 \leq j \leq r$ *satisfies one of the conditions:*

(i) $f_j(x)$ *is locally of bounded variation at any point;*
(ii) $f_j(x)$ *admits one-sided derivatives* $\left(f_j'\right)_R(x)$ *and* $\left(f_j'\right)_L(x)$ *at any point* $x \in [0,p]$.

Then, u(x,t) defined by (2.7) *is a solution of problem* (1.1)–(1.3).

EXAMPLE 1. Let us consider the problem (1.1)–(1.3) in $\mathbb{C}^{2\times 2}$ where

$$D(t) = \begin{bmatrix} 1 & e^t \\ -e^t & 1 \end{bmatrix}, \qquad t > 0,$$

and $F(x)$ any function in $\mathbb{C}^2$ satisfying the hypotheses of Theorem 2.1 in $[0,p]$.

It is straightforward to show that $D(t)$ satisfies the property (1.5), is invertible, and

$$\|D(t)\| = \sqrt{1+e^{2t}}, \qquad t > 0,$$
$$\frac{D(t)+D^H(t)}{2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad t > 0.$$

Thus, the condition (1.4) is satisfied with $\delta = 1$ and $d(t_0, t_1)$ defined by (2.9) is equal to one, i.e., $d(t_0, t_1) = 1$ for every $t_1 \geq t_0 > 0$.

REMARK 1. The condition (1.4) can be replaced by the less restrictive condition obtained with $\delta = 0$. The result of Theorem 2.1 remains true if we assume that $F(x)$ is four-times continuously differentiable function with $F^{(i)}(0) = F^{(i)}(p) = 0$ for $0 \leq i \leq 2$. In this case, the sine-Fourier coefficients $\{c_n\}_{n\geq 1}$ verify $\|c_n\| \leq Z/n^4$, where $Z = 1/p \int_0^p \|F^{(4)}(x)\| \, dx$, [29, p. 93]. Under these alternative conditions apart from the continuity of $D(t)$ in $[0, \infty[$, the series $u(x,t)$ defined by (2.4) is a rigorous solution of problem (1.1)–(1.3).

3. CONTINUOUS NUMERICAL SOLUTIONS AND ERROR BOUNDS

The solution $u(x,t)$ provided by Theorem 2.1 presents two drawbacks from a computational point of view. First, the exact solution $u(x,t)$ is represented by an infinite series, and second, it involves the computation of matrix exponentials of integrals which is not an easy task [30].

In this section, we try to overcome these difficulties. More precisely, we are interested in addressing the following problem. Given t_0 and t_1 with $0 < t_0 < t_1$ and $\epsilon > 0$, how can an approximate solution of problem (1.1)–(1.3) be constructed, with an error smaller than ϵ uniformly for $(x,t) \in \Omega(t_0, t_1) = [0,p] \times [t_0, t_1]$.

For the sake of clarity in the presentation of the following, we summarize briefly some results recently given in Section 2 of [20]. Let $S(t)$ be a two-times continuously differentiable $\mathbb{C}^{r\times r}$ valued function and let us consider the initial value problem

$$Y'(t) = S(t)Y(t); \qquad Y(0) = C, \quad 0 \leq t \leq b, \tag{3.1}$$

and let $\{Y_m\}_{m=0}^N$ the finite sequence defined by $t_m = mh$, $h > 0$,

$$Y_m = \prod_{j=0}^{m-1} \left\{ I - \frac{h}{2} S(t_{m-j}) \right\}^{-1} \left\{ I + \frac{h}{2} S(t_{m-j-1}) \right\} C, \qquad 1 \leq m \leq N, \qquad Nh = b,$$

where $h < 2/k_0$ and

$$k_i = \max \left\{ \|S^{(i)}\|; \ 0 \leq t \leq b \right\}, \qquad 0 \leq i \leq 2.$$

From [20, Theorem 2], the discretization error $e_m = Y(t_m) - Y_m$ satisfies

$$\|e_m\| \leq \|C\| \frac{h^2 b}{6} \exp\left[3k_0 b\right] \{k_0^3 + 3k_1 k_0 + k_2\}, \qquad 1 \leq m \leq N, \qquad Nh = b,$$

and if we define

$$S_Y(t) = \frac{1}{h} \left\{ (t_{m+1} - t) Y_m + (t - t_m Y_{m+1}) \right\}, \qquad \text{for } t_m \leq t < t_{m+1}, \qquad 0 \leq m \leq N-1,$$

then the difference between the theoretical solution $Y(t)$ of (3.1) and $S_Y(t)$ satisfies

$$\|Y(t) - S_Y(t)\| \leq h^2 \gamma, \qquad 0 \leq t \leq b, \tag{3.2}$$
$$\gamma = \exp\left[bk_0\right] \|C\| \left\{ \frac{r}{8}(k_1 + k_0^2) + \frac{b}{6} \exp[2bk_0](k_0^3 + 3k_1 k_0 + k_2) \right\}. \tag{3.3}$$

Let M be defined by (2.12), then from (2.10)–(2.12), one gets

$$\sum_{n>n_0} \|u_n(x,t)\| \le M \sum_{n>n_0} \exp\left[-t_0\left(\frac{n\pi}{p}\right)^2 d(t_0,t_1)\right].$$

Let n_0 be the first positive integer, so that

$$\sum_{n>n_0} \exp\left[-t_0\left(\frac{n\pi}{p}\right)^2 d(t_0,t_1)\right] < \frac{\epsilon}{2M}. \tag{3.4}$$

Note that from (3.4), one gets that

$$S_{n_0}(x,t) = \sum_{n=1}^{n_0} u_n(x,t) = \sum_{n=1}^{n_0} \exp\left[-\left(\frac{n\pi}{p}\right)^2 \int_0^t D(s)\,ds\right] c_n \sin\left(\frac{n\pi x}{p}\right)$$

is an approximate solution of problem (1.1)–(1.3), whose error with respect to the exact solution $u(x,t)$ given by Theorem 2.1, satisfies

$$\|u(x,t) - S_{n_0}(x,t)\| < \frac{\epsilon}{2}, \qquad \text{uniformly for } (x,t) \in \Omega(t_0,t_1). \tag{3.5}$$

Now, we consider n_0 initial value problems defined by

$$\begin{aligned} &T_n(t) + \left(\frac{n\pi}{p}\right)^2 D(t)T_n(t) = 0, \qquad 0 \le t \le t_1, \\ &T_n(0) = c_n = \frac{2}{p}\int_0^p F(x)\sin\left(\frac{n\pi x}{p}\right)dx. \end{aligned} \tag{3.6}$$

Let us denote by

$$k_{n,i} = \left(\frac{n\pi}{p}\right)^2 \max\left\{\|D^{(i)}(t)\|;\ 0 \le t \le t_1\right\}, \qquad 0 \le i \le 2, \quad 1 \le n \le n_0,$$

let N_n be a positive integer and $h_n > 0$ such that $N_n h_n = t_1$ and

$$h_n < \frac{2}{k_{0,n}} = \left(\frac{p}{\pi}\right)^2 \frac{1}{n^2 \max\{\|D(t)\|; 0 \le t \le t_1\}}, \qquad 1 \le n \le n_0, \tag{3.7}$$

and let us consider the sequences $\{Y_{m,n}\}_{\substack{0\le m\le N_n\\ 1\le n\le n_0}}$ defined by

$$Y_{0,n} = c_n, \tag{3.8}$$

$$\begin{gathered} Y_{m,n} = \prod_{j=0}^{m-1}\left\{I + \frac{h_n}{2}\left(\frac{n\pi}{p}\right)^2 D(t_{m-j,n})\right\}^{-1}\left\{I - \frac{h_n}{2}\left(\frac{n\pi}{p}\right)^2 D(t_{m-j-1,n})\right\}, \\ 1 \le n \le n_0, \qquad 1 \le m \le N_n, \qquad N_n h_n = t_1, \end{gathered} \tag{3.9}$$

and $\{t_{i,n}\}_{i=0}^{N_n}$ is a partition of the interval $[0,t_1]$ with $t_{i+1,n} - t_{i,n} = h_n$.

Note that from the properties (1.6) and (1.9) of the logarithmic norm and from the definition (2.9), the condition (3.7) can be written in the form

$$h_n < \left(\frac{p}{n\pi}\right)^2 \frac{1}{d(0,t_1)}, \qquad 1 \le n \le n_0, \tag{3.10}$$

where

$$d(0,t_1) = \min_{0\le u\le t_1} \left\{ w;\ w \in \sigma\left(\frac{D(u)+D^H(u)}{2}\right)\right\}. \tag{3.11}$$

Now let us introduce the continuous B-spline function $S_{T_n}(t)$ which interpolates $\{Y_{m,n}\}$ in $[0,t_1]$ defined by

$$\begin{gathered} edS_{T_n(t)} = \frac{1}{h_n}\left\{(t_{m+1,n}-t)Y_{m,n} + (t-t_{m,n})Y_{m+1,n}\right\}, \\ t_{m,n} \le t < t_{m+1,n}, \qquad 0 \le m \le N_n - 1, \qquad 1 \le n \le n_0. \end{gathered} \tag{3.12}$$

From (3.2),(3.3), the difference $T_n(t) - S_{T_n}(t)$ between the exact solution $T_n(t)$ of problem (3.6) and the aproximation $S_{T_n}(t)$ satisfies

$$\|T_n(t) - S_{T_n}(t)\| \le h_n^2\gamma_n, \qquad t\in[t_0,t_1], \qquad 1\le n\le n_0, \tag{3.13}$$

where

$$\begin{aligned} \gamma_n = \exp\left[\left(\frac{n\pi}{p}\right)^2 t_1 d_0\right] \|c_n\| \Bigg\{ &\frac{r}{8}\left[\left(\frac{n\pi}{p}\right)^2 d_1 + \left(\frac{n\pi}{p}\right)^4 d_0^2\right] \\ &+ \frac{t_1}{6}\exp\left[2\left(\frac{n\pi}{p}\right)^2 t_1 d_0\right]\left[\left(\frac{n\pi}{p}\right)^6 d_0^3 + 3\left(\frac{n\pi}{p}\right)^4 d_1 d_0 + \left(\frac{n\pi}{p}\right)^2 d_2\right]\Bigg\}, \end{aligned} \tag{3.14}$$

and

$$d_i = \max\left\{\|D^{(i)}\|;\ 0\le t\le t_1\right\}, \qquad 0\le i\le 2. \tag{3.15}$$

Taking h_n small enough so that

$$0 < h_n < \frac{\epsilon}{2\,n_0\sqrt{\gamma_n}}, \qquad 1\le n\le n_0, \tag{3.16}$$

then from (3.13) and (3.16), it follows that

$$\|T_n(t) - S_{T_n}(t)\| \le \frac{\epsilon}{2n_0}, \qquad 0\le t\le t_1, \qquad 1\le n\le n_0. \tag{3.17}$$

Let us consider the approximate solution

$$U_{n_0}(x,t) = \sum_{n=1}^{n_0} S_{T_n}(t)\sin\left(\frac{n\pi x}{p}\right), \qquad (x,t)\in\Omega(t_0,t_1). \tag{3.18}$$

If h_n satifies the condition

$$h_n < \min\left\{\left(\frac{p}{n\pi}\right)^2 \frac{1}{d(0,t_1)}, \frac{\epsilon}{2\,n_0\sqrt{\gamma_n}}\right\}, \qquad 1\le n\le n_0, \tag{3.19}$$

then conditions (3.10) and (3.16) hold, and from (3.4),(3.17) and (3.18), it follows that

$$\begin{aligned} \|U_{n_0}(x,t) - S_{n_0}(x,t)\| &\le \sum_{n=1}^{n_0} \|T_n(t) - S_{T_n}(t)\| \left|\sin\left(\frac{n\pi x}{p}\right)\right| \\ &< n_0\left(\frac{\epsilon}{2n_0}\right) = \frac{\epsilon}{2}, \qquad (x,t)\in\Omega(t_0,t_1). \end{aligned} \tag{3.20}$$

From (3.5) and (3.20), one gets

$$\|u(x,t) - U_{n_0}(x,t)\| < \epsilon, \qquad (x,t)\in\Omega(t_0,t_1).$$

Summarizing, the following result has been established.

THEOREM 3.1. *Consider the previous notation and the hypotheses of Theorem 2.1. Let* $\epsilon > 0$ *and* $t_1 > t_0 > 0$. *If* n_0 *is the first positive integer satisfying* (3.4) *and* $S_{T_n}(t)$ *is defined by* (3.12) *where* h_n *and* N_n *are chosen, so that* $N_n h_n = t_1$ *and satisfying* (3.19) *for* $1 \le n \le n_0$, *then* $U_{n_0}(x,t)$ *defined by* (3.18) *is an approximate solution of problem* (1.1)–(1.3), *whose error with respect to the exact solution* $u(x,t)$ *given by Theorem 2.1 is smaller than* ϵ *in* $\Omega(t_0,t_1) = [0,p] \times [t_0,t_1]$.

Theorems 2.1 and 3.1 contain a procedure for providing an approximate solution of problem (1.1)–(1.3) under the hypotheses (1.4) and (1.5), and the two-times continuously differentiability of $D(t)$. As indicated by Theorem 3.1, arbitrarily good (uniformly) approximations are always available. The process itself may be summarized as follows, for given $D(t), F(x), t_0, t_1$ and $\epsilon > 0$:

STEP 1.

- Find constants M and $d(t_0,t_1)$ defined by (2.12) and (2.9), respectively.
- Determine the first positive integer n_0 such that (3.4) holds.

STEP 2.

- Compute constants $d(0,t_1)$ and d_i for $0 \le i \le 2$ defined by (3.11) and (3.15), respectively.
- Compute γ_n defined by (3.14) for $1 \le n \le n_0$.
- Determine h_n satisfying (3.19) for $1 \le n \le n_0$.
- Compute $\{Y_{m,n}\}$ defined by (3.8),(3.9) for $1 \le n \le n_0$, $0 \le m \le n_0$, and $S_{T_n}(t)$ for $t_0 \le t \le t_1$ defined by (3.12).
- Compute $U_{n_0}(x,t)$ given by (3.18).

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