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On the range of non-vanishing p-torsion cohomology for $GL_n(\mathbb{F}_p)$ ☆

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Abstract

The range of non-vanishing of $H^*(GL_n(\mathbb{F}_p), \mathbb{F}_p)$ is not know in general. In this paper we construct a cohomology class in $H^*(GL_n(\mathbb{F}_p), \mathbb{F}_p)$ of very low degree, namely $2p-2$, and we prove that it is nonzero if $p \geqslant n$.

1. Introduction

The group $GL_n(\mathbb{F}_p)$ is a very important group, extensively used in number theory and automorphic forms. A conjecture of Ash (see [3], also [4]) relates Hecke eigenclasses of $H^*(GL_n(\mathbb{Z}), \mathbb{F}_p)$ and $H^*(GL_n(\mathbb{F}_p), \mathbb{F}_p)$ (or, in general, $H^*(\Gamma, V)$ for some subgroup of finite index Γ of $GL_n(\mathbb{Z})$ and some finite-dimensional $\mathbb{F}_p$ vector space V) with continuous semisimple representations of the absolute Galois group $G_{\mathbb{Q}}$ into $GL_n(\mathbb{F}_p)$.

In general, we do not know what is the range where the $\mathbb{F}_p$ cohomology of $GL_n(\mathbb{F}_p)$ is non-vanishing. We have some vanishing results, like that of Maazen [9], stating that for $p > 2$:

$$H^k\big(GL_n(\mathbb{F}_p), \mathbb{F}_p\big) = 0 \quad \text{for } k < n.$$

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Quillen [11] proved that the cohomology groups stabilize to zero, i.e.,

$$H^*\big(GL_\infty(\mathbb{F}_p), \mathbb{F}_p\big) = 0$$

in positive dimensions.

A natural question that arises is the following: What is the smallest m such that $H^m(GL_n(\mathbb{F}_p), \mathbb{F}_p) \neq 0$?

In this paper we give a very low upper bound for this m. Namely, we will prove that $m \leqslant 2p-2$ under the mild assumption $p \geqslant n$ (i.e., almost all p). For that, we will construct a class in $H^*(GL_n(\mathbb{F}_p), \mathbb{F}_p)$ of degree $2p-2$ and we will prove that it is nonzero if $p \geqslant n$.

Our class proves that if $p \geqslant n$, then $H^{2p-2}(GL_n(\mathbb{F}_p), \mathbb{F}_p) \neq 0$. We suspect that our class is the Bockstein of a class from $H^{2p-3}(GL_n(\mathbb{F}_p), \mathbb{F}_p)$ and we conjecture that $2p-3$ is the smallest degree where the cohomology is nonzero.

The only classes defined for general $H^*(GL_n(\mathbb{F}_p), \mathbb{F}_p)$ that we know of have been found by Milgram and Priddy in [10]. These classes are detected on certain maximal p-tori of block form. Our class is not one of those since our class is zero when restricted to all maximal p-tori of block form. Also, our class is not even in the ring generated by the Milgram and Priddy classes, since it has smaller degree than any of them.

In Section 3 we will compute the Hecke algebra $\mathcal{H}(GL_n(\mathbb{F}_p)//U_n)$, by giving its generators and finding some relations between them, that we will need later.

In Section 4 we will construct the new class as an element of $H^*(U_n, \mathbb{F}_p)$ and we will prove that it is $GL_n(\mathbb{F}_p)$-invariant using the Hecke algebra we computed in Section 3. Here U_n is a p-Sylow subgroup of $GL_n(\mathbb{F}_p)$ and it consists of all upper triangular matrices with 1 on the diagonal.

2. Notations

Let $G = GL_n(\mathbb{F}_p)$, B be the subgroup of $GL_n(\mathbb{F}_p)$ consisting of upper triangular matrices, $U = U_n$ be the subgroup of $GL_n(\mathbb{F}_p)$ consisting of upper triangular matrices with 1 on the diagonal, $T = T_n$ be the subgroup of $GL_n(\mathbb{F}_p)$ consisting of diagonal matrices (the torus), and W be the subgroup of $GL_n(\mathbb{F}_p)$ consisting of matrices obtained by permuting the rows of the identity matrix corresponding to each permutation of S_n.

3. The Hecke algebra $\mathcal{H}(GL_n(\mathbb{F}_p)//U_n)$ over $\mathbb{Z}$

In this section, we will compute the $\mathbb{Z}$-Hecke algebra $\mathcal{H}(G//B)$ and $\mathcal{H}(G//U)$, where $G = GL_n(\mathbb{F}_p)$, while B and $U = U_n(\mathbb{F}_p)$ are as above. We have the Bruhat decomposition:

$$B\backslash G/B = \coprod_{w \in W} BwB,$$

where W was defined above.

Proposition 3.1. *With the above notations, $\mathcal{H}(G//B)$ is generated by the double cosets $Bs_iB = (s_i)$ where $s_i \in W$ corresponds to the transposition $(i, i+1)$. The relations between the double cosets (s_i) in $\mathcal{H}(G//B)$ are the following*:

$$(s_i)(s_j) = (s_j)(s_i), \quad \text{if } |i-j| > 1, \qquad (s_i)(s_{i+1})(s_i) = (s_{i+1})(s_i)(s_{i+1}),$$
$$(s_i)(s_i) = p \cdot (1) + (p-1)(s_i).$$

Proof. See [7, p. 3]. □

We now turn to $\mathcal{H}(G//U)$. As in [7], for $w \in S_n$ define

$$l(w) = \min\{k\colon\ w = s_{i_1} \dots s_{i_k}\}.$$

Let $d(w) = \deg BwB$ (regarded as a B-double coset). Recall that $\deg BwB$ is defined as the number d of left cosets Bw_i such that

$$BwB = \coprod_{1 \leqslant i \leqslant d} Bw_i.$$

It is also equal to $[B : B \cap w^{-1}Bw]$.

We have $d(w) = p^{l(w)}$ since it is enough to check this on s_i, because $d(\cdot)$ is multiplicative on minimal products of s_i and $l(\cdot)$ is additive on minimal products of s_i. Since U is normal in B, we have $B = \coprod_{t \in T} Ut$ where $T = T_n$ is the subgroup of $GL_n(\mathbb{F}_p)$ consisting of diagonal matrices. Also observe that W normalizes T. We then have

$$\coprod_{t \in T} UtwU = BwB = BwU = \coprod_{i=\overline{1\dots d(w)}} Bwu_i = \coprod_{i=\overline{1\dots d(w)},\, t \in T} Utwu_i, \tag{1}$$

where wu_i is a system of single B-coset representatives for BwB with $u_i \in U$. Using the Bruhat decomposition, we get from here that

$$U \backslash G / U = \coprod_{w \in W,\, t \in T} UtwU. \tag{2}$$

Since

$$UtwU \supset \coprod_{i=1}^{d(w)} Utwu_i \quad \text{for each } t \in T$$

and when we take the union for all $t \in T$, we get equality (see (1)), we actually have

$$UtwU = \coprod_{i=1}^{d(w)} Utwu_i \quad \text{for each } t \in T.$$

Let us denote the double coset UxU by (x). We obtain therefore that $\deg(tw) = d(w) = \deg(w)$, in $\mathcal{H}(G//U)$.

Proposition 3.2. *With the above notations, $\mathcal{H}(G//U)$ is generated by the double cosets (s_i) and (t) with $t \in T$. The relations between these generators in $\mathcal{H}(G//U)$ are the following*:

$$(ts_i) = (t)(s_i), \qquad (s_i t) = (s_i)(t), \qquad (tt') = (t)(t'),$$
$$(s_i)(s_j) = (s_j)(s_i), \quad \textit{if } |i-j| > 1,$$
$$(s_i)(s_{i+1})(s_i) = (s_{i+1})(s_i)(s_{i+1}),$$
$$(s_i)(s_i) = p(1) + \sum_{kl=-1} \bigl(\operatorname{diag}(1,\ldots,1,k,l,1,\ldots,1)s_i\bigr),$$

where k is at position i in $\operatorname{diag}(1,\ldots,1,k,l,1,\ldots,1)$.

Remark 3.1. We do not need to prove that these are *all* the relations between the generators. We will only use later that the generators satisfy *these* relations.

Proof. We saw above (in Eq. (2)) that $\mathcal{H}(G//U)$ is generated by the double cosets (tw) with $t \in T$, $w \in W$. Let now $t, t' \in T$ and $w, w' \in W$ be such that $l(w) + l(w') = l(ww')$.

Since $(tw) \cdot (t'w')$ as a set contains $(twt'w')$ and

$$\deg(tw)\deg(t'w') = \deg(w)\deg(w') = \deg(ww') = \deg(twt'w')$$

(because we know that $\deg(ww') = \deg(t_1ww')$ and $twt'w'$ can be written as t_1ww'), we get that

$$(tw)(t'w') = (twt'w'). \tag{3}$$

From here, by giving appropriate values to t, t', w, w', we get that

$$(t)(w) = (tw), \quad (w)(t) = (wt), \quad \text{and} \quad (tt') = (t)(t').$$

Also from here, since for $|i-j| > 1$ we have $l(s_i) + l(s_j) = l(s_is_j)$, we get

$$(s_i)(s_j) = (s_is_j) = (s_js_i) = (s_j)(s_i).$$

If $w \in W$, write $w = s_{i_1}\ldots s_{i_k}$, a minimal decomposition in product of transpositions. Then $l(w) = l(s_{i_1}) + l(s_{i_2}) + \cdots + l(s_{i_k})$ and from (3) we get

$$(w) = (s_{i_1})\ldots(s_{i_k}).$$

The permutations of positions $i, i+1, i+2$ form a group isomorphic to S_3. There are three transpositions there. Two of them are s_i and s_{i+1}. The third is $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$. Since this is a minimal decomposition of this transposition (because it cannot be a product of 2 transpositions and it is not an elementary transposition s_j), we get that

$$(s_i)(s_{i+1})(s_i) = (s_i s_{i+1} s_i) = (s_{i+1} s_i s_{i+1}) = (s_{i+1})(s_i)(s_{i+1}).$$

We now want to prove the relation for $(s_i)(s_i)$. We will prove that

$$U s_i U s_i U = U 1 U \cup \coprod_{kl=-1} U \operatorname{diag}(1, \ldots, 1, k, l, 1, \ldots, 1) s_i U, \tag{4}$$

where k is at position i. Because

$$s_i = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & I_{n-i-1} \end{pmatrix} \quad \text{with } s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we get that

$$U s_i U s_i U = \begin{pmatrix} U_{i-1} & * & * \\ 0 & U_2 s U_2 s U_2 & * \\ 0 & 0 & U_{n-i-1} \end{pmatrix},$$

so we see that without loss of generality we may assume $U = U_2$. In this case an element of U has the form $A = \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ and thus a nontrivial element of sUs is of the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1/a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1/a \\ a & 0 \end{pmatrix} \begin{pmatrix} 1 & 1/a \\ 0 & 1 \end{pmatrix}.$$

This implies that

$$UsUsU = U \cup \coprod_{a \neq 0} U \begin{pmatrix} 0 & -1/a \\ a & 0 \end{pmatrix} U = U \cup \coprod_{kl=-1} U \operatorname{diag}(k, l) s U.$$

We thus obtained the relation (4). From here we get that

$$(s_i)^2 = m(1) + \sum_{kl=-1} m_i \bigl(\operatorname{diag}(1, \ldots, 1, k, l, 1, \ldots, 1) s_i\bigr)$$

for some integers $m, m_i > 0$. Now since for any $t \in T$, $\deg(ts_i) = p$, $\deg(1) = 1$, and $\deg(s_i)^2 = p^2$, we have no other choice than $m = p$, $m_i = 1$ so we get the following relation:

$$(s_i)^2 = p(1) + \sum_{kl=-1} \bigl(\operatorname{diag}(1, \ldots, 1, k, l, 1, \ldots, 1) s_i\bigr). \qquad \square$$

4. The new class

As we saw in the previous section, the Hecke algebra $\mathcal{H}(GL_n(\mathbb{F}_p)//U_n)$ is generated by the double cosets of the diagonal matrices and the double cosets of the s_i, where s_i is the matrix corresponding to the transposition $(i, i+1)$.

Given a finite group G and a p-Sylow subgroup H, we know from [5, p. 84] that res_H^G is a monomorphism between $H^*(G, \mathbb{F}_p)$ and $H^*(H, \mathbb{F}_p)$. We want to give a necessary and sufficient condition in terms of Hecke operators for a class in $H^*(H, \mathbb{F}_p)$ to be in $H^*(G, \mathbb{F}_p)$.

We first recall the definition of the Hecke operators.

From [3], recall that a Hecke pair (Γ, S) consists of a subgroup Γ of $GL_n(\mathbb{Z})$ containing $\Gamma(N)$ for some N, and a semigroup S of $GL_n(\mathbb{Q})$ such that $\Gamma \subset S$. $\Gamma(N)$ is the group of matrices in $SL_n(\mathbb{Z})$ congruent to the identity mod N.

As in [3], given a Hecke pair (Γ, S) and a left S-module M, we define an action of the Hecke algebra $\mathcal{H}(S//\Gamma)$ on $H^*(\Gamma, M)$. We first define the action of $\Gamma s \Gamma$ for $s \in S$ as the Hecke operator T_s defined below:

$$T_s(\beta) = \operatorname{tr}_{\Gamma \cap s\Gamma s^{-1} \to \Gamma} \operatorname{res}_{\Gamma \cap s\Gamma s^{-1}} s^*(\beta) \quad \text{for any } \beta \in H^*(\Gamma, M).$$

We extend this action to the entire Hecke algebra $\mathcal{H}(S//\Gamma)$ by linearity. It is proved in [12] that $H^*(\Gamma, M)$ has a structure of a right $\mathcal{H}(S//\Gamma)$-module via the Hecke operator action described above.

The following lemma is [5, Example 2, p. 85].

Lemma 4.1. *Let G be a finite group and H be a p-Sylow subgroup. A cohomology class $\beta \in H^*(H, \mathbb{F}_p)$ is in $H^*(G, \mathbb{F}_p)$ if and only if the action of all the Hecke operators on β is* punctual, *i.e., $T_x(\beta) = \deg(x)\beta$ for all $x \in \mathcal{H}(G//H)$.*

Proof. If $\beta \in H^*(H, \mathbb{F}_p)$ is the restriction of a class in $H^*(G, \mathbb{F}_p)$ by [5, Theorem 10.3, p. 84], β is G-invariant, i.e., $\operatorname{res}^H_{H \cap gHg^{-1}} \beta = \operatorname{res}^{gHg^{-1}}_{H \cap gHg^{-1}} g^*\beta$ for any $g \in G$. But then

$$\begin{aligned} T_g(\beta) &= \operatorname{tr}_{H \cap gHg^{-1} \to H} \operatorname{res}^{gHg^{-1}}_{H \cap gHg^{-1}} g^*\beta = \operatorname{tr}_{H \cap gHg^{-1} \to H} \operatorname{res}^H_{H \cap gHg^{-1}} \beta \\ &= \big(H : H \cap gHg^{-1}\big)\beta = \deg T_g \beta. \end{aligned}$$

By linearity, we get that the action of all the Hecke operators is punctual.

We now prove the other implication. Suppose that all the Hecke operators act punctually on β. Let $w = \operatorname{tr}_{H \to G} \beta$. Let S be a system of representatives for the $H - H$ double cosets of G. Then

$$\begin{aligned} \operatorname{res}_H w = \operatorname{res}_H \operatorname{tr}_{H \to G} \beta &= \sum_{s \in S} \operatorname{tr}_{H \cap sHs^{-1} \to H} \operatorname{res}^{sHs^{-1}}_{H \cap sHs^{-1}} s^*\beta = \sum_{s \in S} T_s(\beta) \\ &= \sum_{s \in S} (\deg T_s)\beta = \sum_{s \in S} \big(H : H \cap sHs^{-1}\big)\beta = (G : H)\beta. \end{aligned}$$

The last equality holds because $(H : H \cap sHs^{-1})$ is exactly the number of simple right cosets that compose HsH. So by taking the union of all double cosets HsH and decomposing each into simple cosets, we get all the simple cosets of G/H.

Since $(G : H)$ is prime to p, we have that

$$\beta = \mathrm{res}_H \frac{1}{(G:H)} w. \qquad \square$$

Lemma 4.2. *A class $\beta \in H^*(U_n, \mathbb{F}_p)$ is in $H^*(GL_n(\mathbb{F}_p), \mathbb{F}_p)$ if and only if*:

- *$T_t(\beta) = \beta$ for any $t \in T_n$, and*
- *$T_{s_i}(\beta) = 0$ for $1 \leqslant i \leqslant n-1$.*

Proof. By applying the previous lemma, $\beta \in H^*(U_n, \mathbb{F}_p)$ is in $H^*(GL_n(\mathbb{F}_p), \mathbb{F}_p)$ if and only if all the Hecke operators act punctually on β.

Because the Hecke action is compatible with the multiplication in the Hecke algebra, it is enough to check that the elements of T_n (the subgroup of diagonal matrices) and the s_i act punctually on our class β. This is because these elements generate the Hecke algebra.

This ends our proof since the degree of the torus elements is 1 (the double coset is also a single coset since T_n normalizes U_n) and the degree of the s_i is p. $\square$

Definition 4.1. Let $\beta_i : U_n \to \mathbb{F}_p$ be defined by $\beta_i((a_{k,l})) = a_{i,i+1}$. Then we have $\beta_i \in \mathrm{Hom}(U_n, \mathbb{F}_p) = H^1(U_n, \mathbb{F}_p)$.

Define $\alpha_i = \delta(\beta_i)$ where $\delta : H^*(U_n, \mathbb{F}_p) \to H^*(U_n, \mathbb{F}_p)$ is the Bockstein operator.

Recall that the Bockstein operator $\delta : H^n(G, \mathbb{Z}/p\mathbb{Z}) \to H^{n+1}(G, \mathbb{Z}/p\mathbb{Z})$ is the connecting homomorphism in the long exact sequence arising from the exact sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0.$$

Let also $H_i = \ker(\beta_i)$.

Proposition 4.3. *Let $t = \mathrm{diag}(t_1, \ldots, t_n) \in T_n$. Then*

$$T_t(\alpha_i) = \frac{t_{i+1}}{t_i} \alpha_i.$$

Proof. Since $tU_nt^{-1} = U_n$, we have that

$$T_t(\alpha_i) = \mathrm{tr}_{U_n \to U_n} t^*(\alpha_i) = t^*(\alpha_i) = \frac{t_{i+1}}{t_i} \alpha_i. \qquad \square$$

For $U_2 \cong \mathbb{Z}/p$ we see that $H^{\mathrm{ev}}(U_2)$ (even cohomology) is a polynomial ring in one indeterminate generated by the element $\alpha \in H^2(U_2)$ corresponding to the canonical morphism $U_2 \to \mathbb{F}_p$. From the above proposition, we see that α^k is invariant under the action of T_2 if and only if $(p-1)|k$. It is easy to see that $T_{s_1} \equiv 0$, so $\alpha^{k(p-1)} \in H^*(GL_2(\mathbb{F}_p))$. Let $\chi_2 = \alpha^{p-1}$.

In general, embed U_k into U_n for $k < n$ as follows:

$$U_k \to U_n, \qquad A \to \begin{pmatrix} A & 0 \\ 0 & I_{n-k} \end{pmatrix}.$$

We also have a map in the other direction:

$$U_n \to U_k, \qquad \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \to A.$$

Because the composition of the above two maps is the identity $U_k \to U_k$, in cohomology the second map induces an injection $H^*(U_k) \hookrightarrow H^*(U_n)$.

For U_3, let $\chi_3 = \chi_2 + T_{s_2}(\chi_2)$. Here we regard χ_2 as an element of $H^*(U_3)$ via the embedding $H^*(U_2) \hookrightarrow H^*(U_3)$ defined above. It is easy to see that

$$U_3 \cap s_2 U_3 s_2^{-1} = \left\{ A \in U_3\colon\ A = \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

and let us denote this subgroup by H. Then we can write

$$\chi_3 = \alpha^{p-1} + \operatorname{tr}_{H \to U_3} s_2^*\big(\alpha^{p-1}\big).$$

Observe that $s_2^*(\alpha) = \gamma$ where $\gamma \in H^2(H, \mathbb{F}_p)$ comes from the morphism

$$\gamma : H \to \mathbb{F}_p, \qquad \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to b$$

via the Bockstein, thus we get that

$$\chi_3 = \alpha^{p-1} + \operatorname{tr}_{H \to U_3} \gamma^{p-1}.$$

Let us now define $\chi_3' = \beta^{p-1} + T_{s_1}(\beta^{p-1}) = \beta^{p-1} + \operatorname{tr}_{H_p \to U_3} \gamma_1^{p-1}$, where $\beta \in H^2(U_3)$ respectively $\gamma_1 \in H^2(H_p)$ come from the morphisms

$$\beta : U_3 \to \mathbb{F}_p, \qquad \begin{pmatrix} 1 & * & * \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \to b, \qquad \gamma_1 : H_p \to \mathbb{F}_p, \qquad \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \to c.$$

Proposition 4.4. *With the above notations we have*:

$$\chi_3 = \chi_3'.$$

Proof. First we have that χ_3 and χ_3' actually come from $H^{2(p-1)}(U_3,\mathbb{Z})$ via reduction mod p. This is easy to see, since we can define similar elements χ_3 and χ_3' in $H^{2(p-1)}(U_3,\mathbb{Z})$ and the transfer map $\mathrm{tr}_{H\to U_3}$ commutes with reduction mod p.

It is enough to prove that $\chi_3=\chi_3'$ in $H^*(U_3,\mathbb{Z})$, since then their images in $H^*(U_3,\mathbb{F}_p)$ will be equal. In this proof from now on, we will be working with $\mathbb{Z}$ coefficients.

Now we will prove that the restriction of χ_3 and χ_3' to all the subgroups A_i defined below is the same mod p (i.e., their difference is a multiple of p).

We define the subgroups $A_i\leqslant GL_3(\mathbb{F}_p)$:

$$A_i=\left\{\begin{pmatrix}1&k&*\\0&1&ik\\0&0&1\end{pmatrix}: k\in\mathbb{F}_p\right\},\quad \text{for } i=0,1,\dots,p-1 \qquad\text{and}\qquad A_p=\begin{pmatrix}1&0&*\\0&1&*\\0&0&1\end{pmatrix}.$$

We first compute the restriction of χ_3 to all A_i. Since the subgroup H from the definition of χ_3 is actually A_0, we have that $HA_i=U_3$ for $i=1,2,\dots,p$ (since H is of index p in U_3 and HA_i is a subgroup strictly larger than H). Thus by the double coset formula [6, Theorem 4.2.6, p. 41], we have

$$\mathrm{res}_{A_i}\,\mathrm{tr}_{H\to U_3}\,\gamma^{p-1}=\mathrm{tr}_{H\cap A_i\to A_i}\,\mathrm{res}_{H\cap A_i}\,\gamma^{p-1}=0\mod p,\quad \text{for } i=1,2,\dots,p,$$

since it is known [2, Corollary 5.9, p. 72]) that the transfer map from a proper subgroup to an elementary abelian group is zero when we are working with $\mathbb{F}_p$ coefficients, and the transfer map commutes with reduction mod p. So the image in $H^*(U_3,\mathbb{F}_p)$ of $\mathrm{res}_{A_i}\,\mathrm{tr}_{H\to U_3}\,\gamma^{p-1}$ is 0, so $\mathrm{res}_{A_i}\,\mathrm{tr}_{H\to U_3}\,\gamma^{p-1}=0$ mod p in $H^*(U_3,\mathbb{Z})$. We thus have that

$$\mathrm{res}_{A_i}\,\chi_3=\mathrm{res}_{A_i}\,\alpha^{p-1}\mod p,\quad \text{for } i=1,2,\dots,p.$$

Let $\alpha_i\in H^2(A_i)$ be defined by the morphism $\alpha_i: A_i\to\mathbb{Q}/\mathbb{Z}$ given by

$$\alpha_i\begin{pmatrix}1&k&*\\0&1&ik\\0&0&1\end{pmatrix}\to k/p.$$

Then $\mathrm{res}_{A_i}\,\alpha=\alpha_i$ if $i<p$ and $\mathrm{res}_{A_p}\,\alpha=0$ so we can rewrite the above equation as follows:

$$\mathrm{res}_{A_i}\,\chi_3=\alpha_i^{p-1}\quad \text{for } i=1,2,\dots,p-1 \qquad\text{and}\qquad \mathrm{res}_{A_p}\,\chi_3=0,$$

everything being mod p. Now for $H=A_0$ the matrices

$$C_i=\begin{pmatrix}1&0&0\\0&1&i\\0&0&1\end{pmatrix}\quad \text{with } i=0,1,\dots,p-1$$

are a complete system of double (and single) H coset representatives so we have

$$\operatorname{res}_H \chi_3 = \alpha_0^{p-1} + \operatorname{res}_H \operatorname{tr}_{H\to U_3} \gamma^{p-1} = \alpha_0^{p-1} + \sum_{i=0}^{p-1} \operatorname{res}_H C_i^*(\gamma)^{p-1}$$

$$= \alpha_0^{p-1} + \sum_{i=0}^{p-1} (\operatorname{res}_H \gamma + i\alpha_0)^{p-1} = \alpha_0^{p-1} + (p-1)\alpha_0^{p-1} = 0,$$

also mod p. Here we used the binomial formula for each $(\operatorname{res}_H \gamma + i\alpha_0)^{p-1}$ and we kept into account that $\sum_{i=0}^{p-1} i^k = 0 \bmod p$ for $1 \leqslant k < p-1$ and $\sum_{i=0}^{p-1} i^{p-1} = p-1 \bmod p$. In conclusion, we have that $\operatorname{res}_{A_0} \chi_3 = \operatorname{res}_{A_p} \chi_3 = 0 \bmod p$ and $\operatorname{res}_{A_i} \chi_3 = \alpha_i^{p-1} \bmod p$ for $i = 1, 2, \dots, p-1$.

Similarly to what we did above, we check that $\operatorname{res}_{A_i} \operatorname{tr}_{A_p\to U_3} \gamma_1^{p-1} = 0 \bmod p$ for $i = 0, 1, \dots, p-1$ and $\operatorname{res}_{A_p} \operatorname{tr}_{A_p\to U_3} \gamma_1^{p-1} = -\operatorname{res}_{A_p} \beta^{p-1} \bmod p$. We also see that $\operatorname{res}_{A_i} \beta = i\alpha_i$ for $i = 0, 1, \dots, p-1$ so $\operatorname{res}_{A_0} \beta^{p-1} = 0$ and $\operatorname{res}_{A_i} \beta^{p-1} = \alpha_i^{p-1}$ for $i = 1, \dots, p-1$.

Putting these all together, we get that $\operatorname{res}_{A_0} \chi_3' = \operatorname{res}_{A_p} \chi_3' = 0 \bmod p$ and $\operatorname{res}_{A_i} \chi_3' = \alpha_i^{p-1} \bmod p$ for $i = 1, 2, \dots, p-1$. This implies that $\operatorname{res}_{A_i} \chi_3 = \operatorname{res}_{A_i} \chi_3' \bmod p$ for $i = 0, 1, \dots, p$, i.e., χ_3 and χ_3' have the same restriction mod p on all A_i.

We can obtain $H^{2(p-1)}(U_3, \mathbb{Z})$ from:

Theorem 4.5 [8, Theorem 6.26, p. 523]. *The cohomology ring of*

$$G = \big(A, B : A^p = B^p = [A, B]^p = \big[A, [A, B]\big] = \big[B, [A, B]\big] = 1\big),$$

for p odd, is as follows: $H^*(G, \mathbb{Z}) = \mathbb{Z}[\alpha, \beta, \mu, \nu, \zeta, c_1, \dots, c_{p-2}]$, $\deg\alpha = \deg\beta = 2$, $\deg\mu = \deg\nu = 3$, $\deg\zeta = 2p$, $\deg c_i = 2i + 2$, *with relations*

(0) $p\alpha = p\beta = p\mu = p\nu = pc_i = p^2\zeta = 0$,
(1) $\alpha\mu = \beta\nu$,
(2) $\alpha^p\mu = \beta^p\nu$,
(3) $\mu^2 = \nu^2 = 0$,
(4) $c_ic_j = \alpha c_i = \beta c_i = \mu c_i = \nu c_i = 0$, $1 \leqslant i, j < p-2$,
(5) $c_ic_{p-2} = 0$, $1 \leqslant i < p-2$, $c_{p-2}^2 = \alpha^{p-1}\beta^{p-1}$,
(6) $\alpha c_{p-2} = \alpha\beta^{p-1}$, $\beta c_{p-2} = \beta\alpha^{p-1}$,
(7) $\mu\alpha^{p-1} = \mu c_{p-2}$, $\nu\beta^{p-1} = \nu c_{p-2}$,
(8) $\alpha\beta^p = \beta\alpha^p$.

If $p > 3$, then $c_2 = d\mu\nu$ for some $d \in \mathbb{Z}_p^*$. If $p = 3$, then $p\zeta = e\mu\nu$, some $e \in \mathbb{Z}_p^*$. a, λ act as follows:

(i) $\alpha^a = \beta$, $\mu^a = -\nu$, $c_i^a = \epsilon_i c_i$, $\epsilon_i = \pm 1$, $\epsilon_{p-2} = 1$, $\epsilon_2 = -1$ *if* $p > 3$.
(ii) $\alpha^\lambda = \alpha$, $\beta^\lambda = \beta + \alpha$, $\nu^\lambda = \mu + \nu$, $c_i^\lambda = c_i$, $1 \leqslant i < p-2$, $[B, [A, B]]c_{p-2}^\lambda = c_{p-2} + (\beta + \alpha)^{p-1} - \beta^{p-1}$, $\zeta^\lambda = \zeta$. *Here* $a, \lambda : G \to G$ *are*: $a : A \to B, B \to A$, $\lambda : B \to B, A \to AB$. *If* $H = \langle B, C\rangle$ (*where* $C = [A, B] = B^{-1}A^{-1}BA$,

$\gamma \in H^2(H, \mathbb{Z})$ corresponding to $C \to 1/p, B \to 0$) we may take $c_i = \operatorname{Cor} \gamma^{i+1}$, $i < i < p-2$, $c_{p-2} = \operatorname{Cor} \gamma^{p-1} + \beta^{p-1}$, and $\zeta = \mathcal{N}(\gamma)$.

From here we see that $H^{2(p-1)}(U_3, \mathbb{Z})$ is generated by $\alpha^i \beta^{p-1-i}$ ($i = 0, 1, \ldots, p-1$) and $c_{p-2} = \chi_3'$ (χ_3' was defined just before Proposition 4.4). These are all the generators for $H^{2(p-1)}(U_3, \mathbb{Z})$ because the other potential generators are zero. We can get other potential generators by multiplying a c_i for $i < p-2$ with one of $\alpha, \beta, \mu, \nu, c_j$ ($j < p-2$), but this product is zero. We could also get other potential generators for $p > 3$ by multiplying $\mu\nu$ with something, but $\mu\nu = c_2/d$, $d \in \mathbb{F}_p^*$ so we have already taken this potential generator into consideration.

Because of this we can write

$$\chi_3 - \chi_3' = f(\alpha, \beta) + a\chi_3',$$

where $f(X, Y) \in \mathbb{F}_p[X, Y]$ (since $p\alpha = p\beta = 0$) is a homogeneous polynomial of degree $p-1$ and $a \in \mathbb{F}_p$ (since $p\chi_3' = 0$). Restricting to all A_i, we get

$$f(X, 0) = f(0, X) = 0, \qquad f(X, iX) + aX^{p-1} = 0 \quad \text{for } i = 1, 2, \ldots, p-1$$

because $A_i \simeq \mathbb{F}_p^2$.

From here, by considering the homogeneous polynomial $g(X, Y) = f(X, Y) + aX^{p-1}$, we get that $g(X, iX) = 0$ for $i = 1, \ldots, p-1$ and $g(0, X) = 0$. By making the change of variable $X \leftarrow iX$ for $i \neq 0$, we get that $g(iX, X) = 0$ for $i = 0, \ldots, p-1$ so the polynomial $h(X) = g(X, 1)$ has the property $h(i) = 0$ for $i = 0, \ldots, p-1$, but it is of degree $p-1$ so it must be identically 0. So $g(X, Y) \equiv 0$ and $f(X, Y) = -aX^{p-1}$ and from $f(X, 0) = 0$ we get that $a = 0$ so $f(X, Y) \equiv 0$. This implies that $\chi_3 - \chi_3' = 0$. □

Proposition 4.6. $\chi_3 \in H^*(GL_3(\mathbb{F}_p), \mathbb{F}_p)$.

Proof. Because of Lemma 4.2, we just have to check that $T_t(\chi_3) = \chi_3$ for all $t \in T_3$ and $T_{s_i}(\chi_3) = 0$.

We have, for $t = \operatorname{diag}(t_1, t_2, t_3)$:

$$\begin{aligned} T_t(\chi_3) &= T_t(\alpha^{p-1}) + T_t(T_{s_2}\alpha^{p-1}) = (t_2/t_1)^{p-1}\alpha^{p-1} + T_{s_2 t'}(\alpha^{p-1}) \\ &= \alpha^{p-1} + T_{s_2}T_{t'}(\alpha^{p-1}) = \alpha^{p-1} + T_{s_2}(\alpha^{p-1}) = \chi_3, \end{aligned}$$

since we saw that $(s_i)(t) = (s_i t) = (t' s_i) = (t')(s_i)$ for some $t' \in T_3$.

For T_{s_1} we have

$$\begin{aligned} T_{s_1}(\chi_3) &= T_{s_1}(\beta^{p-1}) + T_{(s_1)(s_1)}(\beta^{p-1}) = T_{s_1}(\beta^{p-1}) + T_{p(1)+\sum_{i=1}^{p-1}(t_i s_1)}(\beta^{p-1}) \\ &= T_{s_1}(\beta^{p-1}) + \sum_{i=1}^{p-1} T_{(t_i s_1)}(\beta^{p-1}) = pT_{s_1}(\beta^{p-1}) = 0. \end{aligned}$$

The fact that $T_{s_2}(\chi_3) = 0$ is done similarly, but using the other definition of χ_3, namely $\chi_3 = \alpha^{p-1} + T_{s_2}(\alpha^{p-1})$. $\square$

Definition 4.2. Define iteratively $\chi_n = \chi_{n-1} + T_{s_{n-1}}(\chi_{n-1}) \in H^*(U_n, \mathbb{F}_p)$, where χ_2 and χ_3 have already been defined. Here we used the embedding of U_{n-1} in U_n that has been described earlier.

Definition 4.3. Define $H_k \leqslant U_n,\ k = 1, \ldots, n-1$ to be the subgroups

$$H_k = \{A \in U_n\colon\ A = (a_{ij})_{i,j},\ a_{k,k+1} = 0\}.$$

Remark 4.1. It is easy to check that $H_i = U_n \cap s_i U_n s_i^{-1}$.

Before we go to our main theorem, we will need the following functoriality property.

Lemma 4.7. *Let G be a finite group and H a normal subgroup of G. Let G' be another subgroup of G such that there exists a split exact sequence*

$$1 \to K \to G \xrightarrow{\pi} G' \to 1$$

for some subgroup K of G. Let $H' = H \cap G'$. If $K \subset H$, then the map $G'/H' \hookrightarrow G/H$ induced by the inclusion is an isomorphism and there exists an induced split exact sequence

$$1 \to K \to H \to H' \to 1.$$

Also $\mathrm{tr}_{H \to G}\, x = \mathrm{tr}_{H' \to G'}\, x$ *for any* $x \in H^*(H') \hookrightarrow H^*(H)$.

Proof. From the split exact sequence we have that $G'K = G$ since any element of G can be written as a product $\pi(x) \in G'$ and an element of K, namely $(\pi(x))^{-1}x$. Then $G'H = G$ since $K \subset H$. From one of the isomorphism theorems for groups, we have that $G'/H \cap G' \simeq G'H/H$ so we get that $G'/H' \simeq G/H$, the map being that induced by the inclusion.

Now if $x \in H$, then $(\pi(x))^{-1}x \in K \subset H$, so $\pi(x) \in H$. But $\pi(x) \in G'$ so $\pi(x) \in H'$. Reciprocally, any element $y \in H'$ is in G' so $\pi(y) = y$; therefore $\pi|_H : H \to H'$ is surjective. Restricting now the given exact sequence to H, we get a split exact sequence

$$1 \to K \to H \to H' \to 1.$$

To prove now the equality of the transfer maps, we can suppose, by dimension shifting, that $x \in H^0(H')$. Then we can find a system S of representatives for $G'/H' \simeq G/H$. Thus S will also be a system of representatives for G/H. Then

$$\mathrm{tr}_{H' \to G'}\, x = \sum_{s \in G'/H'} s^* x = \sum_{s \in S} s^* x \in H^*(G') \subset H^*(G),$$

so $\mathrm{tr}_{H' \to G'}\, x = \sum_{s \in S} s^* x = \sum_{s \in G/H} s^* x = \mathrm{tr}_{H \to G}\, x \in H^*(G)$. $\square$

Theorem 4.8. $\chi_n \in H^*(GL_n(\mathbb{F}_p), \mathbb{F}_p)$.

Proof. We first prove that

$$T_t(\chi_n) = \chi_n \quad \text{for all } t \in T_n.$$

We do that by proving that $T_t(\chi_k) = \chi_k$ in U_n, for $k = 2, \dots, n$. We proceed by induction on k.

Case $k = 2$ is trivial: $T_t(\chi_2) = T_t(\alpha^{p-1}) = (t_2/t_1)^{p-1}\alpha^{p-1} = \alpha^{p-1}$.

Suppose case k is proved; let us prove it for $k+1$:

$$T_t(\chi_{k+1}) = T_t\big(\chi_k + T_{s_k}(\chi_k)\big) = \chi_k + T_{s_k}T_{t'}(\chi_k) = \chi_k + T_{s_k}(\chi_k) = \chi_{k+1},$$

where $t' \in T$ is such that $s_k t = t' s_k$.

We are left to prove that

$$T_{s_i}(\chi_n) = 0 \quad \text{for } i = 1, 2, \dots, n-1.$$

We proceed by induction on n. We already saw that for $n = 2$ and $n = 3$ the theorem is true, so the above relation is verified.

Suppose now that the above relation is true for n and $n-1$ and let us prove it for $n+1$, $n \geqslant 3$. We have

$$T_{s_i}(\chi_{n+1}) = T_{s_i}(\chi_n) + T_{s_i}T_{s_n}(\chi_n).$$

If $i < n-1$, we have $(s_n)(s_i) = (s_i)(s_n)$ so

$$T_{s_i}(\chi_{n+1}) = T_{s_i}(\chi_n) + T_{s_n}T_{s_i}(\chi_n) = 0 + 0 = 0,$$

because Lemma 4.7 says that $T_{s_i}x$, $x \in H^*(U_{n-1})$ is the same when regarded in U_{n-1} and in U_n. The induction hypothesis implies that $T_{s_i}(\chi_n) = 0$.

For $i = n-1$ we have

$$\begin{aligned} T_{s_{n-1}}(\chi_{n+1}) &= T_{s_{n-1}}(\chi_n) + T_{s_{n-1}}T_{s_n}(\chi_n) = 0 + T_{s_{n-1}}T_{s_n}\big(\chi_{n-1} + T_{s_{n-1}}(\chi_{n-1})\big) \\ &= T_{s_{n-1}}T_{s_n}(\chi_{n-1}) + T_{s_{n-1}}T_{s_n}T_{s_{n-1}}(\chi_{n-1}) \\ &= T_{s_{n-1}}T_{s_n}(\chi_{n-1}) + T_{s_n}T_{s_{n-1}}T_{s_n}(\chi_{n-1}) = 0 + 0 = 0. \end{aligned}$$

We used here

$$T_{s_n}(\chi_{n-1}) = \mathrm{tr}_{H_n \to G}\,\mathrm{res}_{H_n}\big(s_n^*\chi_{n-1}\big) = \mathrm{tr}_{H_n \to G}(\mathrm{res}_{H_n}\,\chi_{n-1}) = p\chi_{n-1} = 0$$

and the relation $(s_{n-1})(s_n)(s_{n-1}) = (s_n)(s_{n-1})(s_n)$.

For $i = n$ we have

$$T_{s_n}(\chi_{n+1}) = T_{s_n}(\chi_n) + T_{s_n s_n}(\chi_n) = T_{s_n}(\chi_n) + \sum_{i=1}^{p-1} T_{t_i s_n}(\chi_n)$$
$$= T_{s_n}(\chi_n) + \sum_{i=1}^{p-1} T_{s_n}(\chi_n) = pT_{s_n}(\chi_n) = 0,$$

since we saw that $(s_i)^2 = p(1) + \sum_{j=1}^{p-1}(t_j)(s_i)$ where t_j are some elements of the torus T_{n+1} and we already saw that the elements of T_{n+1} act trivially on χ_n. $\square$

Now that we proved that this class is invariant to the whole Hecke algebra, we ask ourselves: is this class non-zero? This class is of degree $2(p-1)$ and it is known that $H^k(GL_n(\mathbb{F}_p), \mathbb{F}_p) = 0$ for $k < n$ by a theorem of Maazen (see [10]).

So if $2(p-1) < n$, our class will be zero. But we can prove the following theorem.

Theorem 4.9. *If $p \geqslant n$, then $\chi_n \neq 0$.*

Proof. Let

$$U = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in M_n(\mathbb{F}_p).$$

Then the subgroup $E = \langle I_n + U \rangle \leqslant U_n$ is elementary abelian, because $I_n + U$ has order p. Actually $(I_n + U)^p = I_n^p + U^p = I_n$ since $U^p = 0$ ($U^n = 0$ and $p \geqslant n$).

We have $EH_i = U_n$ for all $i = 1, \ldots, n-1$ since H_i is a subgroup of index p in U_n and $E \not\subset H_i$. Because of this, the $E - H_i$ double coset decomposition of U_n has only one coset and we have

$$\begin{aligned}\operatorname{res}_E \chi_n &= \operatorname{res}_E \chi_{n-1} + \operatorname{res}_E \operatorname{tr}_{H_{n-1}\to U_n} \operatorname{res}_{H_{n-1}}\bigl(s_{n-1}^*(\chi_{n-1})\bigr) \\ &= \operatorname{res}_E \chi_{n-1} + \operatorname{tr}_{0\to E} \operatorname{res}_0\bigl(s_{n-1}^*(\chi_{n-1})\bigr) = \operatorname{res}_E \chi_{n-1} + 0 = \operatorname{res}_E \chi_{n-1}.\end{aligned}$$

We can repeat the computation and we successively get that

$$\operatorname{res}_E \chi_n = \operatorname{res}_E \chi_{n-1} = \cdots = \operatorname{res}_E \chi_3 = \operatorname{res}_E \chi_2 = \operatorname{res}_E \alpha^{p-1} = \alpha_E^{p-1} \neq 0,$$

where $\alpha_E \in H^2(E)$ is the generator of the polynomial part of $H^*(E)$. $\square$

Remark 4.2. Observe that for $n = 2, 3$, the class we defined is an important generator of $H^*(GL_n(\mathbb{F}_p), \mathbb{F}_p)$: the class χ_2 is α^{p-1}, a generator of $H^*(GL_2(\mathbb{F}_p), \mathbb{F}_p)$. Note that the cohomology $H^*(GL_2(\mathbb{F}_p), \mathbb{F}_p)$ has only two generators, one being α^{p-1} while the other is nilpotent of degree $2p-3$ (see [1]).

The class χ_3 is the image of the generator

$$b_{p-2} \in H^*\big(GL_3(\mathbb{F}_p), \mathbb{Z}\big)_{(p)}$$

of $H^*(GL_3(\mathbb{F}_p), \mathbb{Z})_{(p)}$(from [13]) via the reduction mod p map.

Remark 4.3. The only classes defined for general $H^*(GL_n(\mathbb{F}_p), \mathbb{F}_p)$ that we know of have been found by Milgram and Priddy in [10]. These classes are detected on certain maximal p-tori of block form. Our class is not one of those since our class is zero when restricted to all maximal p-tori of block form:

Proposition 4.10. *If E is an elementary abelian subgroup* (p*-torus*) *of* $GL_n(\mathbb{F}_p)$ *of block form*:

$$E = \begin{pmatrix} I_k & * \\ 0 & I_{n-k} \end{pmatrix} \quad \textit{for some } k$$

and $n > 2$, *then* $\operatorname{res}_E \chi_n = 0$.

Proof. We do this by induction on n.

For $n = 3$ this has been done already in the proof of Proposition 4.4, since there are only two maximal p-tori of block form in U_3, namely H_0 and H_p so E must be one of them.

Suppose now that we proved that $\operatorname{res}_E \chi_n = 0$ for all p tori E of block form of U_n, and let's prove that $\operatorname{res}_E \chi_{n+1} = 0$. We have

$$\operatorname{res}_E \chi_{n+1} = \operatorname{res}_E \chi_n + \operatorname{res}_E \operatorname{tr}_{H_n \to U_{n+1}} s_n^* \chi_n.$$

But actually $\chi_n \in H^*(U_n)$ where the embedding of U_n in U_{n+1} has been defined earlier in this chapter. We have the commutative diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & E \cap U_n \arrow[d] \\ U_{n+1} \arrow[r] & U_n, \end{tikzcd}$$

where the horizontal maps are obtained by truncating a $(n+1) \times (n+1)$ matrix to the $n \times n$ matrix from the upper left-hand corner. From here we get a commutative diagram in cohomology:

$$\begin{tikzcd} H^*(U_n) \arrow[r, hook] \arrow[d, "\operatorname{res}"] & H^*(U_{n+1}) \arrow[d, "\operatorname{res}"] \\ H^*(E \cap U_n) \arrow[r, hook] & H^*(E), \end{tikzcd}$$

so we get that $\operatorname{res}_E \chi_n = \operatorname{res}_{E \cap U_n} \chi_n$. Since $E \cap U_n$ is a p-torus of block form in U_n, we get by the induction hypothesis that $\operatorname{res}_{E \cap U_n} \chi_n = 0$ so $\operatorname{res}_E \chi_n = 0$.

To compute $\mathrm{res}_E\, \mathrm{tr}_{H_n \to U_{n+1}}\, s_n^* \chi_n$ we have two cases.

The first case is $E \not\subset H_n$. Then $EH_n = U_{n+1}$, so by the double coset formula

$$\mathrm{res}_E\, \mathrm{tr}_{H_n \to U_{n+1}}\, s_n^* \chi_n = \mathrm{tr}_{E\cap H_n \to E}\, \mathrm{res}_{E\cap H_n}\, s_n^* \chi_n = 0,$$

since the transfer map $\mathrm{tr}_{E' \to E}$ is identically zero if E' is a proper subgroup of the elementary abelian subgroup E. From here we get $\mathrm{res}_E\, \chi_{n+1} = 0 + 0 = 0$.

The second case is $E \subset H_n$. Then the matrices

$$t_i = \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & 1 & i \\ 0 & 0 & 1 \end{pmatrix}, \quad i = 0, \dots, p-1$$

form a system of representatives for the $E - H_n$ double cosets of U_{n+1}. By the double coset formula

$$\mathrm{res}_E\, \mathrm{tr}_{H_n \to U_{n+1}}\, s_n^* \chi_n = \sum_{i=0}^{p-1} \mathrm{res}_E\, t_i^* s_n^* \chi_n = \sum_{i=0}^{p-1} t_i^* s_n^*\, \mathrm{res}_E\, \chi_n = 0,$$

since t_i and s_n normalize E. Thus $\mathrm{res}_E\, \chi_{n+1} = 0 + 0 = 0$. □

Looking again at the classes defined by Milgram and Priddy, we see that the only classes that they defined explicitly for $p > 2$ and $n > 2$ are of degree bigger than $2p - 2$. So our class is not even in the ring generated by these classes.

It is likely that our class is the Bockstein of a class in $H^{2p-3}(GL_n(\mathbb{F}_p), \mathbb{F}_p)$.

The question is now: can there be non-zero classes in $H^*(GL_n(\mathbb{F}_p), \mathbb{F}_p)$ of degree less than $2p - 3$?

For $n = 2$ from [1] we get that the smallest degree of a nonzero class is $2p - 3$. From this only known example, we make the following conjecture.

Conjecture 4.11. *If $n \geqslant 2$ and $p \geqslant 3$, then*

$$H^k\big(GL_n(\mathbb{F}_p), \mathbb{F}_p\big) = 0 \quad \text{for } k < 2p - 3.$$

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