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Stabilization of hybrid neutral stochastic differential delay equations by delay feedback control

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ABSTRACT

This paper is concerned with the problem of exponential mean-square stabilization of hybrid neutral stochastic differential delay equations with Markovian switching by delay feedback control. A delay feedback controller is designed in the drift part so that the controlled system is mean-square exponentially stable. We discussed two types of structure controls; that is, state feedback and output injection. The stabilization criteria are derived in terms of linear matrix inequalities.

1. Introduction

A stochastic differential equation (SDE) with Markovian switching is known as a hybrid system which can be described by a set of SDEs with transitions between models determined by a Markovian chain in a finite mode set. As an important class of hybrid systems, SDEs with Markovian switching are usually used to model many practical systems, for example, electric power systems, the control system of a solar thermal central receiver, financial systems, etc. (see e.g. [1–5]). One of the important hot topics in the study of hybrid SDEs is the analysis of stability. A great number of significant results on this topic have been reported in the literature; see, for instance, [4–23] and the references therein.

There is now an intensive literature in the area of stabilization of SDEs. The problem of mean square exponential stabilization by state feedback controllers for a class of SDEs with Markovian switching was investigated in [24], while the problem of stabilization of hybrid stochastic differential delay equations (SDDEs) with Markovian switching by non-delay feedback controllers was addressed in [25]. The stabilization problem of hybrid SDDEs with Markovian switching by delay feedback controllers was addressed in [26]. Furthermore, the almost surely exponential stabilization problem of hybrid SDDEs by stochastic feedback controllers was investigated in [27]. Even though the stabilization of stochastic control systems has been widely studied [20,24–29], relatively little is known about the stabilization of hybrid neutral stochastic differential delay equations with Markovian switching. The purpose of this paper is to discuss the exponential mean-square stabilization of hybrid neutral stochastic differential delay equations with Markovian switching by delay feedback controllers.

Notation: Throughout this paper, for real symmetric matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semi-definite (respectively, positive definite). $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, and the notation $|\cdot|$ refers to the Euclidean vector norm. The notation M^T represents the transpose of the matrix M. If M is a matrix, its operator norm is denoted by $\|M\| = \sup\{|Mx| : |x| = 1\}$. If M is a symmetric matrix, $\lambda_{\max}(M)$ and $\lambda_{\min}(M)$ represent its largest and smallest eigenvalue, respectively. The symmetric terms in a symmetric matrix are denoted by $*$. $0_{m\times n}$ denotes zero matrix with $m \times n$ dimensions. $a \vee b$ denotes the maximum of a and b, while $a \wedge b$ denotes the minimum of a and b. $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ denotes a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, where Ω is a sample space. $\mathcal{F}$ is the σ-algebra of subset of the sample space and $\mathcal{P}$ is the probability measure on $\mathcal{F}$. Let $\tau > 0$ and $C([-\tau, 0]; \mathbb{R}^n)$ denote the family of all continuous $\mathbb{R}^n$-valued function on $[-\tau, 0]$. Denote by $C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ the family of all $\mathcal{F}_0$-measurable bounded $C([-\tau, 0]; \mathbb{R}^n)$-valued random variables $\xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\}$. $\mathbb{E}(\cdot)$ is the expectation operator with respect to some probability measure $\mathcal{P}$. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

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2. Problem formulation

In this paper, we consider the exponential mean-square stabilization problem of hybrid neutral stochastic differential delay equation (NSDDE) by delay feedback controllers. Given an unstable hybrid neutral stochastic differential delay equation

$$\begin{aligned}\mathrm{d}[x(t)-D(x(t-\delta),r(t))] &= f(x(t),x(t-\tau),t,r(t))\mathrm{d}t\\ &\quad + g(x(t),x(t-\tau),t,r(t))\mathrm{d}\omega(t),\end{aligned} \tag{1}$$

where $\{r(t)\}$ is a continuous time Markovian process with right continuous trajectories taking values in a finite set $\mathcal{S} = \{1,2,\ldots,N\}$ with transition probabilities given by

$$\Pr\{r(t+\Delta t)=j \mid r(t)=i\} = \begin{cases}\pi_{ij}\Delta t + o(\Delta t), & i \neq j\\ 1+\pi_{ii}\Delta t + o(\Delta t), & i=j,\end{cases}$$

in which $\Delta t > 0$, $\lim_{\Delta t\to 0}(o(\Delta t)/\Delta t) = 0$, and $\pi_{ij} \geqslant 0$, for $j \neq i$, is the transition rate from mode i at time t to mode j at time $t+\Delta t$ and $\pi_{ii} = -\sum_{j=1,\, j\neq i}^{N} \pi_{ij}$. $\omega(t)$ is a scalar Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$, which is independent from the Markov chain $\{r(t),\ t \geq 0\}$ and satisfies $\mathbb{E}\{\mathrm{d}\omega(t)\} = 0$, $\mathbb{E}\{\mathrm{d}\omega^2(t)\} = \mathrm{d}t$. The purpose of this paper is to design a delay feedback control $u(x(t),x(t-\tau),r(t))$ in the drift part, based on the current and past state and mode, so that the controlled system

$$\begin{aligned}&\mathrm{d}[x(t)-D(x(t-\delta),r(t))]\\ &\quad = [f(x(t),x(t-\tau),t,r(t)) + u(x(t),x(t-\tau),r(t))]\mathrm{d}t\\ &\qquad + g(x(t),x(t-\tau),t,r(t))\mathrm{d}\omega(t),\end{aligned} \tag{2}$$

is exponentially stable in mean square. For the sake of simplicity, we only consider the underlying unstable hybrid system

$$\begin{aligned}\mathrm{d}[x(t)-D(x(t-\tau),r(t))] &= f(x(t),x(t-\tau),t,r(t))\mathrm{d}t\\ &\quad + g(x(t),x(t-\tau),t,r(t))\mathrm{d}\omega(t).\end{aligned} \tag{3}$$

Assume that the initial data are given by $r(0) = r_0$ and

$$\{x(s): -\tau \leq s \leq 0\} = \xi \in C^b_{\mathcal{F}_0}([-\tau,0];\ \mathbb{R}^n). \tag{4}$$

Denote by $x(t,\xi)$ the solution of Eq. (3) with initial conditions (4) and $r(0) = r_0$. Then by the theory of hybrid NSDDEs (see e.g. [8]), Eq. (3) on $t \geq 0$ is exponentially stable in mean square, if

$$\limsup_{t\to\infty} \frac{1}{t}\log(\mathbb{E}|x(t;\xi)|^2) < 0$$

for any $\xi \in C^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$. Set $\eta(t) = x(t) - D(x(t-\tau),r(t))$, and design a linear delay feedback controller

$$\begin{aligned}&u(x(t-\tau),r(t)) = F(r(t))G(r(t))\eta(t-\tau)\\ &\quad = F(r(t))G(r(t))\big(x(t-\tau) - D(x(t-2\tau),r(t))\big),\end{aligned} \tag{5}$$

where $F(r(t)) : \mathcal{S} \to \mathbb{R}^{n\times l}$, $G(r(t)) : \mathcal{S} \to \mathbb{R}^{l\times n}$, and one of them is given while the other needs to be designed. These two cases are usually known as (see e.g. [26,27]):

- State feedback: design $F(\cdot)$ when $G(\cdot)$ is given.
- Output injection: design $G(\cdot)$ when $F(\cdot)$ is given.

According to Eq. (3), the controlled system becomes

$$\begin{aligned}&\mathrm{d}[x(t)-D(x(t-\tau),r(t))]\\ &\quad = [f(x(t),x(t-\tau),t,r(t)) + u(x(t-\tau),r(t))]\mathrm{d}t\\ &\qquad + g(x(t),x(t-\tau),t,r(t))\mathrm{d}\omega(t)\\ &\quad = [f(x(t),x(t-\tau),t,r(t)) + F(r(t))G(r(t))\\ &\qquad \times \big(x(t-\tau) - D(x(t-2\tau),r(t))\big)]\mathrm{d}t\\ &\qquad + g(x(t),x(t-\tau),t,r(t))\mathrm{d}\omega(t).\end{aligned} \tag{6}$$

The controlled system (6) is a neutral stochastic differential delay equation with Markovian switching, where some initial data are required to be known. In general, we always impose the initial condition $\{x(s) : -2\tau \leq s \leq 0\} = \xi \in C^b_{\mathcal{F}_0}([-2\tau,0];\mathbb{R}^n)$ and fix the initial state r_0 arbitrarily for Markov chain $r(t)$ but let the initial data ξ vary in $C^b_{\mathcal{F}_0}([-2\tau,0];\mathbb{R}^n)$ for the NSDDE (6) (see e.g. [3,15,26]). In this paper, we shall regard the controlled system (6) as an NSDDE on $t \geq 2\tau$ with initial data $\{x(s) : 0 \leq s \leq 2\tau\}$, then by the theory of hybrid NSDDEs (see e.g. [15,26]), we have $\mathbb{E}(|x(t)|^2) < \infty$, for both $0 \leq t \leq 2\tau$ and $t \geq 2\tau$. These can be interpreted as follows: let Eq. (3) evolve from time 0 to 2τ and observe the whole segment $\{x(t) : 0 \leq t \leq 2\tau\}$ with fixed initial state r_0 arbitrarily for Markov chain and $\xi \in C^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$. As time evolves, we design the delay feedback control $u(x(t-\tau),r(t)) = F(r(t))G(r(t))\big(x(t-\tau) - D(x(t-2\tau),r(t))\big)$ from the moment of 2τ based on the past observation $\{x(t) : 0 \leq t \leq 2\tau\}$. Accordingly we shall regard the controlled system (6) as an NSDDE on $t \geq 2\tau$ with initial data $\{x(t) : 0 \leq t \leq 2\tau\}$ (see e.g. [26]). The solution of Eq. (6) with initial data $\xi \in C^b_{\mathcal{F}_0}([0,2\tau];\mathbb{R}^n)$ is denoted by $x(t,\xi)$, then we have the following definition:

Definition 1. For any initial data $\xi \in C^b_{\mathcal{F}_0}([0,2\tau];\mathbb{R}^n)$ and fixed initial state r_0 arbitrarily for Markov chain $r(t)$, Eq. (6) is said to be exponentially stable in mean square if

$$\limsup_{t\to\infty} \frac{1}{t}\log(\mathbb{E}|x(t;\xi)|^2) < 0.$$

Remark 1. Usually, we design the feedback controller $u(x(t),r(t))$ which only depends on the current state $x(t)$. However, there exists time delay τ between the time when the system state is observed and the time when the feedback control reaches the system in practice. Naturally, a delay feedback control $U(x(t-\tau),r(t))$ which depends on the past states $x(t-\tau)$ should be considered. In [26], the authors studied the stabilization of non-neutral stochastic differential delay equations with Markovian switching by delay feedback control $u(x(t-\tau)) = F(r(t)G(r(t)))x(t-\tau)$. However, $\eta(t) = x(t) - D(x(t-\tau),r(t))$ is regarded as a whole state in this paper; it is thus more natural to design the delay feedback control $u(x(t-\tau),r(t)) = F(r(t))G(r(t))\eta(t-\tau)$.

Remark 2. In this paper, the linear delay feedback control is with the structure of the form $u(x(t-\tau),r(t)) = F(r(t))G(r(t))\eta(t-\tau)$. When $G(r(t))$ is given, the feedback control is a state feedback. In the case when $F(r(t))$ is given, it is a special form of output feedback.

3. Stabilization of linear hybrid NSDDE

In this section, we are given an n-dimensional unstable linear hybrid NSDDE

$$\begin{aligned}\mathrm{d}[x(t)-D(r(t))x(t-\tau)] &= [A(r(t))x(t) + B(r(t))x(t-\tau)]\mathrm{d}t\\ &\quad + [C(r(t))x(t) + H(r(t))x(t-\tau)]\mathrm{d}\omega(t)\end{aligned} \tag{7}$$

on $t \geq \tau$. For notational simplicity, in the sequel, a matrix $D(r(t))$ will be denoted by D_i for each possible $r(t) = i,\ i \in \mathcal{S}$; for example, $A(r(t)) = A_i,\ B(r(t)) = B_i,\ C(r(t)) = C_i,\ H(r(t)) = H_i$, and so on. Here, we assume $\|D_i\| < 1$ for all $i \in \mathcal{S}$. We design a delay feedback control $u(x(t-\tau),r(t))$ in the drift part so that the controlled system

$$\begin{aligned}&\mathrm{d}[x(t)-D(r(t))x(t-\tau)]\\ &\quad = [A(r(t))x(t) + B(r(t))x(t-\tau) + u(x(t-\tau),r(t))]\mathrm{d}t\\ &\qquad + [C(r(t))x(t) + H(r(t))x(t-\tau)]\mathrm{d}\omega(t),\end{aligned} \tag{8}$$

is exponentially stable in mean square. The given NSDDE (7) is linear, accordingly we shall consider a linear delay feedback controller

$$\begin{aligned}u(x(t-\tau), r(t)) &= F(r(t))G(r(t))\eta(t-\tau)\\ &= F(r(t))G(r(t))\big(x(t-\tau) - D(r(t))x(t-2\tau)\big).\end{aligned}$$

Then, Eq. (8) can be re-written as

$$\begin{aligned}\mathrm{d}[x(t) - D(r(t))x(t-\tau)] = {} & [A(r(t))x(t) + B(r(t))x(t-\tau)\\ & + F(r(t))G(r(t))\eta(t-\tau)]\mathrm{d}t\\ & + [C(r(t))x(t) + H(r(t))x(t-\tau)]\mathrm{d}\omega(t). \end{aligned} \tag{9}$$

Our purpose is to design $F(\cdot)$ when $G(\cdot)$ is given or design $G(\cdot)$ when $F(\cdot)$ is given so that $\mathbb{E}|x(t;\xi)|^2$ will tend to zero exponentially. Next, we will consider the delay feedback control in above two cases in Sections 3.1 and 3.2, respectively.

3.1. State feedback: Design F_i when G_i is given

Let $\hat{x}_t := \{x(t+s), -3\tau \le s \le 0\}$ for $t \ge 3\tau$. Then, it can be observed that $\{\hat{x}_t, r(t), t \ge 3\tau\}$ is a Markov process with initial state $(\{x(s) : 0 \le s \le 3\tau\}, r_0)$ [8]. Denote by $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S}; \mathbb{R}_+)$ the family of real-valued functions $V(x,t,i)$ on $\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S}$ which are continuously twice differentiable in x and once in t. If $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S}; \mathbb{R}_+)$, define an operator $\mathcal{L}V$ from $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S}$ to $\mathbb{R}$ by (see e.g. [13])

$$\begin{aligned}\mathcal{L}V(x,y,t,i) = {} & V_t(x - D(y,i), t, i) + V_x(x - D(y,i), t, i)\\ & \times f(x,y,i,t) + \frac{1}{2}\mathrm{trace}[g^T(x,y,t,i)\\ & \times V_{xx}(x - D(y,i), t, i)g(x,y,t,i)]\\ & + \sum_{j\in\mathcal{S}} \pi_{ij} V(x - D(y,i), t, j),\end{aligned} \tag{10}$$

where

$$V_t(x,t,i) = \frac{\partial V(x,t,i)}{\partial t},$$

$$V_x(x,t,i) = \Big(\frac{\partial V(x,t,i)}{\partial x_1}, \ldots, \frac{\partial V(x,t,i)}{\partial x_n}\Big),$$

$$V_{xx}(x,t,i) = \Big(\frac{\partial^2 V(x,t,i)}{\partial x_k \partial x_l}\Big)_{n\times n}.$$

Furthermore, if $V(\hat{x}_t, t, r(t))$ is defined on $C^b_{\mathcal{F}_0}([-3\tau, 0]; \mathbb{R}^n) \times \mathbb{R}_+ \times \mathcal{S}$, that is, $V(\hat{x}_t, t, r(t)) : C^b_{\mathcal{F}_0}([-3\tau, 0]; \mathbb{R}^n) \times \mathbb{R}_+ \times \mathcal{S} \to \mathbb{R}_+$, then the operator $\mathcal{L}V$ associated with Eq. (3) is defined in the following way:

$$\begin{aligned}\mathcal{L}V(\hat{x}_t, t, i) = {} & V_t\big(x(t) - D(x(t-\tau), i), t, i\big)\\ & + V_x\big(x(t) - D(x(t-\tau), i), t, i\big)\\ & \times f(x(t), x(t-\tau), t, i)\\ & + \frac{1}{2}\mathrm{trace}[g^T(x(t), x(t-\tau), t, i)\\ & \times V_{xx}\big(x(t) - D(x(t-\tau), i), t, i\big)\\ & \times g(x(t), x(t-\tau), t, i)]\\ & + \sum_{j\in\mathcal{S}} \pi_{ij} V\big(x(t) - D(x(t-\tau), i), t, j\big),\end{aligned}$$

for each $r(t) = i \in \mathcal{S}$ (see e.g. [15,30]). Now, we choose a stochastic Lyapunov–Krasovskii functional candidate as

$$\begin{aligned}V(\hat{x}_t, t, r(t)) = {} & \eta(t)^T P(r(t))\eta(t) + \int_{t-\tau}^{t} x(s)^T Q(r(s))x(s)\mathrm{d}s\\ & + \int_{-\tau}^{0}\int_{t+\theta}^{t} x(s)^T R x(s)\mathrm{d}s\mathrm{d}\theta\\ & + \int_{-\tau}^{0}\int_{t+\theta}^{t} x(s-\tau)^T S x(s-\tau)\mathrm{d}s\mathrm{d}\theta\\ & + \int_{-\tau}^{0}\int_{t+\theta}^{t} x(s-2\tau)^T W x(s-2\tau)\mathrm{d}s\mathrm{d}\theta,\end{aligned} \tag{11}$$

for $t \ge 3\tau$. Here $P_i > 0,\ Q_i > 0,\ R > 0,\ S > 0,\ W > 0$.

Remark 3. The double integrals item $\int_{-\tau}^{0}\int_{t+\theta}^{t} x(s-2\tau)^T W x(s-2\tau)\mathrm{d}s\mathrm{d}\theta$ is introduced into the Lyapunov–Krasovskii functional (11); naturally we shall regard the controlled system (9) as an NSDDE on $t > 3\tau$ with initial data $\{x(t) : 0 \le t \le 3\tau\}$ and fixed initial state r_0 arbitrarily for Markov chain $r(t)$.

Before we provide our main result, a useful lemma is presented in the following.

Lemma 3.1. *If there are scalars $\lambda_1 > \lambda_2 \ge 0$ and $\lambda_3 > 0,\ \lambda_4 > 0$, such that*

$$\begin{aligned}\mathbb{E}(\mathcal{L}V(\hat{x}_t, t, r(t))) \le {} & -\lambda_1\mathbb{E}|x(t)|^2 + \lambda_2\mathbb{E}|x(t-\tau)|^2\\ & + \lambda_3\mathbb{E}|x(t-2\tau)|^2 - \lambda_4\mathbb{E}\int_{t-3\tau}^{t}|x(s)|^2\mathrm{d}s,\end{aligned} \tag{12}$$

for all $t \ge 3\tau$, then, we have

$$\limsup_{t\to\infty} \frac{1}{t}\log(\mathbb{E}|x(t;\xi)|^2) \le -\gamma,$$

where $\gamma > 0$ satisfies

$$\gamma \le \frac{\lambda_4}{\max\big\{\big(\lambda_{\max\, i\in\mathcal{S}}(Q_i) + \tau\lambda_{\max}(R)\big), \big(\tau\lambda_{\max}(S)\big), \big(\tau\lambda_{\max}(W)\big)\big\}}, \tag{13}$$

$$2\gamma\, \lambda_{\max\, i\in\mathcal{S}}(P_i) + (\lambda_2 + 2\gamma\, \lambda_{\max\, i\in\mathcal{S}}(P_i))e^{\gamma\tau} + \lambda_3 e^{2\gamma\tau} \le \lambda_1, \tag{14}$$

$$\gamma < \frac{1}{2\tau}\log\frac{1}{\lambda_{\max\, i\in\mathcal{S}}(D_i)}. \tag{15}$$

Proof. From (11), it can be seen that

$$\begin{aligned}\mathbb{E}V(\hat{x}_t, t, r(t)) \le {} & \lambda_{\max\, i\in\mathcal{S}}(P_i)\mathbb{E}|\eta(t)|^2 + \lambda_{\max\, i\in\mathcal{S}}(Q_i)\int_{t-\tau}^{t}\mathbb{E}|x(s)|^2\mathrm{d}s\\ & + \tau\lambda_{\max}(R)\int_{t-\tau}^{t}\mathbb{E}|x(s)|^2\mathrm{d}s + \tau\lambda_{\max}(S)\\ & \times \int_{t-\tau}^{t}\mathbb{E}|x(s-\tau)|^2\mathrm{d}s + \tau\lambda_{\max}(W)\int_{t-\tau}^{t}\mathbb{E}|x(s-2\tau)|^2\mathrm{d}s\\ = {} & \lambda_{\max\, i\in\mathcal{S}}(P_i)\mathbb{E}|\eta(t)|^2 + \lambda_{\max\, i\in\mathcal{S}}(Q_i)\\ & \times \int_{t-\tau}^{t}\mathbb{E}|x(s)|^2\mathrm{d}s + \tau\lambda_{\max}(R)\int_{t-\tau}^{t}\mathbb{E}|x(s)|^2\mathrm{d}s\\ & + \tau\lambda_{\max}(S)\int_{t-2\tau}^{t-\tau}\mathbb{E}|x(s)|^2\mathrm{d}s + \tau\lambda_{\max}(W)\int_{t-3\tau}^{t-2\tau}\mathbb{E}|x(s)|^2\mathrm{d}s\\ \le {} & 2\,\lambda_{\max\, i\in\mathcal{S}}(P_i)(\mathbb{E}|x(t)|^2 + \mathbb{E}|x(t-\tau)|^2)\\ & + \Big[\lambda_{\max\, i\in\mathcal{S}}(Q_i) + \tau\lambda_{\max}(R)\Big]\int_{t-\tau}^{t}\mathbb{E}|x(s)|^2\mathrm{d}s\\ & + \tau\lambda_{\max}(S)\int_{t-2\tau}^{t-\tau}\mathbb{E}|x(s)|^2\mathrm{d}s\\ & + \tau\lambda_{\max}(W)\int_{t-3\tau}^{t-2\tau}\mathbb{E}|x(s)|^2\mathrm{d}s.\end{aligned} \tag{16}$$

Denote $\mu_1 = 2\lambda_{\max i\in\mathcal{S}}(P_i)$, $\mu_2 = \max\big\{\big(\lambda_{\max i\in\mathcal{S}}(Q_i) + \tau\lambda_{\max}(R)\big), \big(\tau\lambda_{\max}(S)\big), \big(\tau\lambda_{\max}(W)\big)\big\}$, then

$$\mathbb{E}V(\hat{x}_t, t, r(t)) \le \mu_1(\mathbb{E}|x(t)|^2 + \mathbb{E}|x(t-\tau)|^2) + \mu_2\int_{t-3\tau}^{t}\mathbb{E}|x(s)|^2\mathrm{d}s. \tag{17}$$

By Itô differential formula, we have

$$e^{\gamma t}\mathbb{E}V(\hat{x}_t, t, r(t)) = C + \int_{3\tau}^{t} e^{\gamma s}\big[\gamma\mathbb{E}V(\hat{x}_s, s, r(s)) + \mathbb{E}\big(\mathcal{L}V(\hat{x}_s, s, r(s))\big)\big]\mathrm{d}s,$$

where $C = e^{3\gamma\tau}\mathbb{E}V(\hat{x}_{3\tau}, 3\tau, r(3\tau))$. Then, it follows from (12) to (17) that

$$\begin{aligned}
&e^{\gamma t}\mathbb{E}V(\hat{x}_t, t, r(t))\\
&\le C + \int_{3\tau}^{t} e^{\gamma s}\Big[(-\lambda_1 + \gamma\mu_1)\mathbb{E}|x(s)|^2 + (\lambda_2 + \gamma\mu_1)\mathbb{E}|x(s-\tau)|^2\\
&\quad + \lambda_3\mathbb{E}|x(s-2\tau)|^2 + (-\lambda_4 + \gamma\mu_2)\int_{s-3\tau}^{s}\mathbb{E}|x(u)|^2\mathrm{d}u\Big]\mathrm{d}s\\
&\le C + (-\lambda_1 + \gamma\mu_1)\int_{3\tau}^{t} e^{\gamma s}\mathbb{E}|x(s)|^2\mathrm{d}s + (\lambda_2 + \gamma\mu_1)\\
&\quad\times\int_{2\tau}^{t-\tau} e^{\gamma(s+\tau)}\mathbb{E}|x(s)|^2\mathrm{d}s + \lambda_3\int_{\tau}^{t-2\tau} e^{\gamma(s+2\tau)}\mathbb{E}|x(s)|^2\mathrm{d}s\\
&\le C + (-\lambda_1 + \gamma\mu_1)\int_{3\tau}^{t} e^{\gamma s}\mathbb{E}|x(s)|^2\mathrm{d}s + (\lambda_2 + \gamma\mu_1)\\
&\quad\times\int_{2\tau}^{3\tau} e^{\gamma(s+\tau)}\mathbb{E}|x(s)|^2\mathrm{d}s + (\lambda_2 + \gamma\mu_1)\\
&\quad\times\int_{3\tau}^{t} e^{\gamma(s+\tau)}\mathbb{E}|x(s)|^2\mathrm{d}s + \lambda_3\int_{\tau}^{3\tau} e^{\gamma(s+2\tau)}\mathbb{E}|x(s)|^2\mathrm{d}s\\
&\quad + \lambda_3\int_{3\tau}^{t} e^{\gamma(s+2\tau)}\mathbb{E}|x(s)|^2\mathrm{d}s\\
&\le C + (-\lambda_1 + \gamma\mu_1 + (\lambda_2 + \gamma\mu_1)e^{\gamma\tau} + \lambda_3 e^{2\gamma\tau})\\
&\quad\times\int_{3\tau}^{t} e^{\gamma s}\mathbb{E}|x(s)|^2\mathrm{d}s + \big[((\lambda_2 + \gamma\mu_1)e^{\gamma\tau}) \vee (\lambda_3 e^{2\gamma\tau})\big]\\
&\quad\times\int_{\tau}^{3\tau} e^{\gamma(s+\tau)}\mathbb{E}|x(s)|^2\mathrm{d}s\\
&\le C + \big[((\lambda_2 + \gamma\mu_1)e^{\gamma\tau}) \vee (\lambda_3 e^{2\gamma\tau})\big]\int_{\tau}^{3\tau} e^{\gamma s}\mathbb{E}|x(s)|^2\mathrm{d}s.
\end{aligned}$$

Note

$$\begin{aligned}
&\mathbb{E}V(\hat{x}_{3\tau}, 3\tau, r(3\tau))\\
&\le \mu_1\mathbb{E}|x(3\tau)|^2 + \mu_1\mathbb{E}|x(2\tau)|^2 + \mu_2\int_0^{3\tau}\mathbb{E}|x(s)|^2\mathrm{d}s\\
&\le (2\mu_1 + 3\mu_2\tau)\mathbb{E}\|\xi\|^2,
\end{aligned}$$

then, we have

$$e^{\gamma t}\mathbb{E}V(\hat{x}_t, t, r(t)) \le C_r\mathbb{E}\|\xi\|^2,$$

where $C_r = (2\mu_1 + 3\mu_2\tau)e^{3\gamma\tau} + \frac{1}{\gamma}\big[((\lambda_2 + \gamma\mu_1)e^{\gamma\tau}) \vee (\lambda_3 e^{2\gamma\tau})\big](e^{3\gamma\tau} - e^{\gamma\tau})$.

On the other hand,

$$e^{\gamma t}\mathbb{E}V(\hat{x}_t, t, r(t)) \ge e^{\gamma t}\lambda_{\min\, i\in\mathcal{S}}(P_i)\mathbb{E}|\eta(t)|^2,$$

thus, we obtain

$$\mathbb{E}|x(t) - D(x(t-\tau), r(t))|^2 \le \frac{C_r}{\lambda_{\min\, i\in\mathcal{S}}(P_i)}\mathbb{E}\|\xi\|^2 e^{-rt}.$$

Denote $k = \lambda_{\max i\in\mathcal{S}}(D_i) \in (0, 1)$, then by the inequality (15) and the Lemma 4.6 of [8], we have

$$\mathbb{E}|x(t;\xi)|^2 \le \Big[\frac{\sqrt{k}e^{3\gamma\tau}}{1-\sqrt{k}} + \frac{C_r}{(1-\sqrt{k})(1-k)\lambda_{\min\, i\in\mathcal{S}}(P_i)}\Big]\mathbb{E}\|\xi\|^2,$$

which implies

$$\limsup_{t\to\infty}\frac{1}{t}\log(\mathbb{E}|x(t;\xi)|^2) \le -\gamma.$$

This completes the proof. □

Now, we have the following result.

Theorem 3.1. *Assume that for chosen positive-definite $n\times n$ matrices R, S, W, M and a scalar $\alpha > 0$, the following LMIs*

$$\begin{bmatrix} \overline{\Omega}_{i1} & \Omega_{i2} & Y_iG_i\\ * & \Omega_{i3} & 0\\ * & * & -M \end{bmatrix} < 0, \quad i\in\mathcal{S}, \tag{18}$$

$$\sum_{j\ne i}\pi_{ij}Q_j < R, \quad i\in\mathcal{S}, \tag{19}$$

have solutions $\bar{\tau} > 0$ and P_i, Q_i, Y_i with $P_i > 0$, $Q_i > 0$ and $Y_i \in \mathbb{R}^{n\times l}$, where

$$\begin{aligned}
\overline{\Omega}_{i1} &= P_iA_i + A_i^TP_i + Y_iG_i + (Y_iG_i)^T + Q_i + \bar{\tau}R\\
&\quad + \bar{\tau}S + \bar{\tau}W + C_i^TP_iC_i + \sum_{j\in\mathcal{S}}\pi_{ij}P_j,
\end{aligned}$$

$$\Omega_{i2} = P_iB_i - A_i^TP_iD_i - Y_iG_iD_i + C_i^TP_iH_i - \sum_{j\in\mathcal{S}}\pi_{ij}P_jD_i,$$

$$\begin{aligned}
\Omega_{i3} &= -Q_i + H_i^TP_iH_i + \alpha D_i^TD_i - 2D_i^TP_iB_i\\
&\quad - 2D_i^TY_iG_i + D_i^T\sum_{j\in\mathcal{S}}\pi_{ij}P_jD_i,
\end{aligned}$$

and

$$\bar{\lambda} = -\lambda_{\max\, i\in\mathcal{S}}(\overline{\Omega}_i), \tag{20}$$

$$\overline{\Omega}_i = \begin{bmatrix} \overline{\Omega}_{i1} + (Y_iG_i)M^{-1}(Y_iG_i)^T & \Omega_{i2}\\ * & \Omega_{i3} \end{bmatrix}. \tag{21}$$

Choose three positive numbers β_1, β_2, β_3, such that

$$\begin{aligned}
&\lambda_{\max\, i\in\mathcal{S}}(|\pi_{ii}|Q_i) > \lambda_{\max}(M)\beta_1,\\
&\lambda_{\max}(S) > \lambda_{\max}(M)\beta_2, \qquad \lambda_{\max}(W) > \lambda_{\max}(M)\beta_3,
\end{aligned} \tag{22}$$

$$\bar{\lambda} + \bar{\tau}\lambda_{\min}(S) + \bar{\tau}\lambda_{\max}(W) > -\bar{\lambda} + \bar{\tau}\lambda_{\max}(S) \ge 0. \tag{23}$$

Let $\tau^* \in (0, \bar{\tau}]$ be the largest number which obeys

$$\max_{j\in\mathcal{S}}(6\tau^*\|A_i\|^2 + 4\|C_i\|^2) \le \beta_1, \tag{24}$$

$$\max_{j\in\mathcal{S}}(6\tau^*\|B_i\|^2 + 6\tau^*\|P_i^{-1}Y_iG_i\|^2 + 4\|H_i\|^2) \le \beta_2, \tag{25}$$

$$\max_{j\in\mathcal{S}}(6\tau^*\|P_i^{-1}Y_iG_iD_i\|^2) \le \beta_3. \tag{26}$$

Then, if $\tau \le \tau^*$, by setting

$$F_i = P_i^{-1}Y_i, \quad i\in\mathcal{S}, \tag{27}$$

the controlled system (9) is exponentially stable in mean square.

Proof. Let $\mathcal{L}$ be the weak infinitesimal generator of the random process $\{\hat{x}_t, r(t),\ t \geq 3\tau\}$. Then, by Itô differential formula, for each $r(t) = i \in \mathcal{S}$, it can be verified that

$$\begin{aligned}\mathcal{L}V(\hat{x}_t, t, i) = {} & 2\eta(t)^T P_i[A_i x(t) + B_i x(t-\tau) \\ & + F_i G_i \eta(t-\tau)] + x(t)^T Q_i x(t) \\ & - x(t-\tau)^T Q_i x(t-\tau) + \tau x(t)^T R x(t) \\ & + \tau x(t-\tau)^T S x(t-\tau) - \int_{t-\tau}^{t} x(s)^T R x(s) \mathrm{d}s \\ & - \int_{t-\tau}^{t} x(s-\tau)^T R x(s-\tau)\mathrm{d}s \\ & + \tau x(t-2\tau)^T W x(t-2\tau) \\ & - \int_{t-\tau}^{t} x(s-2\tau)^T W x(s-2\tau)\mathrm{d}s \\ & + [C_i x(t) + H_i x(t-\tau)]^T \\ & \times P_i[C_i x(t) + H_i x(t-\tau)] + \eta(t)^T \sum_{j\in\mathcal{S}} \pi_{ij} P_j \eta(t) \\ & + \int_{t-\tau}^{t} x(s)^T \sum_{j\in\mathcal{S}} \pi_{ij} Q_j x(s)\mathrm{d}s. \end{aligned} \tag{28}$$

Note

$$\begin{aligned} & 2x(t)^T P_i F_i G_i \eta(t-\tau) \\ & = 2x(t)^T P_i F_i G_i \eta(t) - 2x(t)^T P_i F_i G_i(\eta(t) - \eta(t-\tau)) \\ & \leq 2x(t)^T P_i F_i G_i \eta(t) + x(t)^T (P_i F_i G_i) M^{-1} \\ & \quad \times (P_i F_i G_i)^T x(t) + (\eta(t) - \eta(t-\tau))^T \\ & \quad \times M(\eta(t) - \eta(t-\tau)), \end{aligned} \tag{29}$$

and

$$\begin{aligned} & -2x(t-\tau)^T D_i^T P_i F_i G_i D_i x(t-2\tau) \leq \alpha x(t-\tau)^T D_i^T D_i x(t-\tau) \\ & \quad + \alpha^{-1} |(P_i F_i G_i D_i) x(t-2\tau)|^2. \end{aligned} \tag{30}$$

We define

$$\xi(t) = [x(t)^T \quad x(t-\tau)^T]^T,$$

then it follows from (18), and (27)–(30) that

$$\begin{aligned}\mathcal{L}V(\hat{x}_t, t, i) \leq {} & \xi(t)^T \Omega_i \xi(t) - \tau x(t)^T S x(t) \\ & - \tau x(t)^T W x(t) + \tau x(t-\tau)^T S x(t-\tau) \\ & + \tau x(t-2\tau)^T W x(t-2\tau) \\ & + \alpha^{-1} |P_i F_i G_i D_i x(t-2\tau)|^2 \\ & - |\pi_{ii}| \int_{t-\tau}^{t} x(s)^T Q_i x(s)\mathrm{d}s \\ & - \int_{t-\tau}^{t} x(s-\tau)^T S x(s-\tau)\mathrm{d}s \\ & - \int_{t-\tau}^{t} x(s-2\tau)^T W x(s-2\tau)\mathrm{d}s \\ & + (\eta(t) - \eta(t-\tau))^T M(\eta(t) - \eta(t-\tau)), \end{aligned} \tag{31}$$

where

$$\Omega_i = \begin{bmatrix} \Omega_{i1} + (Y_i G_i) M^{-1} (Y_i G_i)^T & \Omega_{i2} \\ * & \Omega_{i3} \end{bmatrix}, \tag{32}$$

$$\begin{aligned}\Omega_{i1} = {} & P_i A_i + A_i^T P_i + Y_i G_i + (Y_i G_i)^T + Q_i \\ & + \tau R + \tau S + \tau W + C_i^T P_i C_i + \sum_{j\in\mathcal{S}} \pi_{ij} P_j. \end{aligned}$$

On the other hand,

$$\begin{aligned}\eta(t) - \eta(t-\tau) = {} & \int_{t-\tau}^{t} [A(r(s))x(s) + B(r(s))x(s-\tau) \\ & + F(r(s))G(r(s))\eta(s-\tau)]\mathrm{d}s \\ & + \int_{t-\tau}^{t} [C(r(s))x(s) + H(r(s))x(s-\tau)]\mathrm{d}\omega(s). \end{aligned}$$

Thus, we have

$$\begin{aligned} & \mathbb{E}|\eta(t) - \eta(t-\tau)|^2 \leq 2\tau \mathbb{E} \int_{t-\tau}^{t} |A(r(s))x(s) + B(r(s))x(s-\tau) \\ & \quad + F(r(s))G(r(s))\eta(s-\tau)|^2 \mathrm{d}s \\ & \quad + 2\mathbb{E} \int_{t-\tau}^{t} |C(r(s))x(s) + H(r(s))x(s-\tau)|^2 \mathrm{d}s \\ & \leq \max_{i\in\mathcal{S}} (6\tau \|A_i\|^2 + 4\|C_i\|^2) \mathbb{E} \int_{t-\tau}^{t} |x(s)|^2 \mathrm{d}s \\ & \quad + \max_{i\in\mathcal{S}} (6\tau \|B_i\|^2 + 6\tau \|P_i^{-1} Y_i G_i\|^2 \\ & \quad + 4\|H_i\|^2) \mathbb{E} \int_{t-\tau}^{t} |x(s-\tau)|^2 \mathrm{d}s \\ & \quad + \max_{i\in\mathcal{S}} (6\tau \|P_i^{-1} Y_i G_i D_i\|^2) \mathbb{E} \int_{t-\tau}^{t} |x(s-2\tau)|^2 \mathrm{d}s. \end{aligned} \tag{33}$$

Recalling (24)–(26), we obtain

$$\begin{aligned}\mathbb{E}|\eta(t) - \eta(t-\tau)|^2 \leq {} & \beta_1 \mathbb{E} \int_{t-\tau}^{t} |x(s)|^2 \mathrm{d}s \\ & + \beta_2 \mathbb{E} \int_{t-\tau}^{t} |x(s-\tau)|^2 \mathrm{d}s \\ & + \beta_3 \mathbb{E} \int_{t-\tau}^{t} |x(s-2\tau)|^2 \mathrm{d}s. \end{aligned} \tag{34}$$

Applying the Schur complement to (18) gives $\overline{\Omega}_i < 0$. As $\tau \leq \bar{\tau}$, it can be obtained that $\Omega_i < 0$. Set $\lambda = -\lambda_{\max_{i\in\mathcal{S}}}(\Omega_i)$, then we have $\lambda > 0$ and

$$\begin{aligned}\mathbb{E}(\mathcal{L}V(\hat{x}_t, t, i)) \leq {} & -(\lambda + \tau\lambda_{\min}(S) + \tau\lambda_{\min}(W))\mathbb{E}|x(t)|^2 \\ & + (-\lambda + \tau\lambda_{\max}(S))\mathbb{E}|x(t-\tau)|^2 \\ & + \left[\left(\alpha^{-1} | \lambda_{\max_{i\in\mathcal{S}}} \|Y_i G_i D_i\|^2\right) \vee \left(\tau\lambda_{\min}(W)\right)\right] \\ & \times \mathbb{E}|x(t-2\tau)|^2 - (|\pi_{ii}| \lambda_{\max_{i\in\mathcal{S}}}(Q_i) \\ & - \lambda_{\max}(M)\beta_1)\mathbb{E} \int_{t-\tau}^{t} |x(s)|^2 \mathrm{d}s - (\lambda_{\max}(S) \\ & - \lambda_{\max}(M)\beta_2)\mathbb{E} \int_{t-\tau}^{t} |x(s-\tau)|^2 \mathrm{d}s \\ & - (\lambda_{\max}(W) - \lambda_{\max}(M)\beta_3) \\ & \times \mathbb{E} \int_{t-\tau}^{t} |x(s-2\tau)|^2 \mathrm{d}s \\ \leq {} & -\lambda_1 \mathbb{E}|x(t)|^2 + \lambda_2 \mathbb{E}|x(t-\tau)|^2 \\ & + \lambda_3 \mathbb{E}|x(t-2\tau)|^2 - \lambda_4 \int_{t-3\tau}^{t} \mathbb{E}|x(s)|^2 \mathrm{d}s, \end{aligned} \tag{35}$$

where

$$\lambda_1 = \lambda + \tau\lambda_{\min}(S) + \tau\lambda_{\min}(W), \qquad \lambda_2 = -\lambda + \tau\lambda_{\max}(S),$$

$$\lambda_3 = \left[\left(\alpha^{-1} | \lambda_{\max_{i\in\mathcal{S}}} \|Y_i G_i D_i\|^2\right) \vee \left(\tau\lambda_{\min}(W)\right)\right],$$

$$\lambda_4 = \max_{i\in\mathcal{S}}\Big\{\lambda_{\max}(|\pi_{ii}|Q_i) - \lambda_{\max}(M)\beta_1, \lambda_{\max}(S) - \lambda_{\max}(M)\beta_2, \lambda_{\max}(W) - \lambda_{\max}(M)\beta_3\Big\}.$$

From (22) to (23), we observe that $\lambda_1 > \lambda_2 \geq 0$, $\lambda_3 > 0$ and $\lambda_4 > 0$. Then, by Lemma 3.1, it is easy to show that the controlled system (9) is exponentially stable in mean square. This completes the proof. □

In addition, for the convenience of discussing the output injection, the inequality (28) in Theorem 3.1 will be further processed in the following corollary.

Corollary 3.1. *Assume that for chosen positive-definite $n\times n$ matrices R, S, W, M, and a scalar $\alpha > 0$, the following LMIs*

$$\begin{bmatrix} \overline{\Lambda}_{i1} & P_iB_i & Y_iG_i & Y_iG_i & 0 \\ * & \Lambda_{i2} & 0 & 0 & (Y_iG_i)^T \\ * & * & -M & 0 & 0 \\ * & * & * & -Pi & 0 \\ * & * & * & * & -Pi \end{bmatrix} < 0, \quad i \in \mathcal{S} \tag{36}$$

$$\sum_{j\neq i}\pi_{ij}Q_j < R, \quad i \in \mathcal{S} \tag{37}$$

have solutions $\bar{\tau} > 0$ and P_i, Q_i, Y_i with $P_i > 0$, $Q_i > 0$ and $Y_i \in \mathbb{R}^{n\times l}$, where

$$\overline{\Lambda}_{i1} = P_iA_i + A_i^TP_i + Y_iG_i + (Y_iG_i)^T + Q_i + \bar{\tau}R + \bar{\tau}S + \bar{\tau}W + A_i^TP_iA_i + 2C_i^TP_iC_i + 2\sum_{j\neq i}\pi_{ij}P_j,$$

$$\Lambda_{i2} = -Q_i + 4D_i^TP_iD_i + \alpha D_i^TD_i + B_i^TP_iB_i + 2H_i^TP_iH_i + 2D_i^T\sum_{j\neq i}\pi_{ij}P_jD_i$$

and

$$\lambda = -\lambda_{\max_{i\in\mathcal{S}}}(\overline{\Lambda}_i), \tag{38}$$

$$\overline{\Lambda}_i = \begin{bmatrix} \overline{\Lambda}_{i1} + (Y_iG_i)M^{-1}(Y_iG_i)^T + (Y_iG_i)P_i^{-1}(Y_iG_i)^T & P_iB_i \\ * & \Lambda_{i2} + (Y_iG_i)^TP_i^{-1}(Y_iG_i) \end{bmatrix}. \tag{39}$$

Choose three positive numbers $\beta_1, \beta_2, \beta_3$, such that

$$\begin{aligned} &\lambda_{\max_{i\in\mathcal{S}}}(|\pi_{ii}|Q_i) > \lambda_{\max}(M)\beta_1, \\ &\lambda_{\max}(S) > \lambda_{\max}(M)\beta_2, \qquad \lambda_{\max}(W) > \lambda_{\max}(M)\beta_3, \end{aligned} \tag{40}$$

$$\bar{\lambda} + \bar{\tau}\lambda_{\min}(S) + \bar{\tau}\lambda_{\max}(W) > -\bar{\lambda} + \bar{\tau}\lambda_{\max}(S) \geq 0. \tag{41}$$

Let $\tau^* \in (0, \bar{\tau}]$ be the largest number which obeys

$$\max_{j\in\mathcal{S}}(6\tau^*\|A_i\|^2 + 4\|C_i\|^2) \leq \beta_1, \tag{42}$$

$$\max_{j\in\mathcal{S}}(6\tau^*\|B_i\|^2 + 6\tau^*\|P_i^{-1}Y_iG_i\|^2 + 4\|H_i\|^2) \leq \beta_2, \tag{43}$$

$$\max_{j\in\mathcal{S}}(6\tau^*\|P_i^{-1}Y_iG_iD_i\|^2) \leq \beta_3. \tag{44}$$

Then, if $\tau \leq \tau^*$, by setting

$$F_i = P_i^{-1}Y_i, \quad i \in \mathcal{S}, \tag{45}$$

the controlled system (9) is exponentially stable in mean square.

Proof. Note

$$\begin{aligned} -2x(t)^TD_i^TP_iA_ix(t-\tau) &= -2x(t)^TA_i^TP_iD_ix(t-\tau) \\ &\leq x(t)^TA_i^TP_iA_ix(t) + x(t-\tau)^TD_i^TP_iD_ix(t-\tau), \\ 2x(t)^TC_i^TP_iH_ix(t-\tau) &\leq x(t)^TC_i^TP_iC_ix(t) + x(t-\tau)^TH_i^TP_iH_ix(t-\tau), \\ -2x(t)^T\sum_{j\in\mathcal{S}}\pi_{ij}P_jD_ix(t-\tau) &\leq x(t)^T\sum_{j\in\mathcal{S}}|\pi_{ij}|P_jx(t) + x(t-\tau)^TD_i^T \times \sum_{j\in\mathcal{S}}|\pi_{ij}|P_jD_ix(t-\tau), \\ -2x(t)^TP_iF_iG_iD_ix(t-\tau) &\leq x(t)^T(P_iF_iG_i)P_i^{-1}(P_iF_iG_i)^Tx(t) + x(t-\tau)^TD_i^TP_iD_ix(t-\tau), \\ -2x(t-\tau)^TD_i^TP_iB_ix(t-\tau) &\leq x(t-\tau)^TD_i^TP_iD_ix(t-\tau) + x(t-\tau)^TB_i^TP_iB_ix(t-\tau), \\ -2x(t-\tau)^TD_i^TP_iF_iG_ix(t-\tau) &\leq x(t-\tau)^TD_i^TP_iD_ix(t-\tau) + x(t-\tau)^T(P_iF_iG_i)^TP_i^{-1} \times (P_iF_iG_i)x(t-\tau). \end{aligned}$$

Then, following a similar line to the proof of Theorem 3.1, we have the desired result. □

Remark 4. The stability analysis of Theorem 3.1 and Corollary 3.1 is based on the Lyapunov functional defined by (11). It is worth pointing out that the design of delay feedback control can also handle the cases of hybrid non-neutral stochastic differential equations and hybrid non-neutral stochastic differential delay equations with Markovian switching.

3.2. Output injection: Design G_i when F_i is given

Now, we discuss the second case that the F_i is given but G_i needs to be designed. we present the following theorem.

Theorem 3.2. *Assume that for chosen positive-definite $n\times n$ matrices R, S, W, M, and positive scalars $\alpha, \eta_{1i}, \eta_{2i}, \eta_{3i}$, $i \in \mathcal{S}$, the following LMIs*

$$\begin{bmatrix} \overline{\Psi}_i & \Phi_{i1} & \Phi_{i2} & \Phi_{i3} & \Phi_{i4} & \Phi_{i5} & \Phi_{i6} \\ * & -Z_i & 0 & 0 & 0 & 0 & 0 \\ * & * & -Z_i & 0 & 0 & 0 & 0 \\ * & * & * & -Z_i & 0 & 0 & 0 \\ * & * & * & * & -Z_i & 0 & 0 \\ * & * & * & * & 0 & -Z_i & 0 \\ * & * & * & * & * & * & -V_i \end{bmatrix} < 0, \quad i \in \mathcal{S} \tag{46}$$

$$\sum_{j\neq i}\pi_{ij}Q_j < R, \quad i \in \mathcal{S} \tag{47}$$

$$-2X_i + (1+\eta_{1i})I < 0, \quad i \in \mathcal{S}, \tag{48}$$

$$-2X_i + (1+\eta_{2i})I < 0, \quad i \in \mathcal{S}, \tag{49}$$

$$-2I + (1+\eta_{3i})Q_i < 0, \quad i \in \mathcal{S}, \tag{50}$$

have solutions $\bar{\tau} > 0$ and X_i, Q_i, Y_i with $X_i > 0$, $Q_i > 0$, and $Y_i \in \mathbb{R}^{l\times n}$, where Eqs. (51) and (52) are given in Box I. Choose three positive scalars $\beta_1, \beta_2, \beta_3$, such that

$$\begin{aligned} &\lambda_{\max_{i\in\mathcal{S}}}(|\pi_{ii}|Q_i) > \lambda_{\max}(M)\beta_1, \\ &\lambda_{\max}(S) > \lambda_{\max}(M)\beta_2, \qquad \lambda_{\max}(W) > \lambda_{\max}(M)\beta_3, \end{aligned} \tag{53}$$

$$\bar{\lambda} + \bar{\tau}\lambda_{\min}(S) + \bar{\tau}\lambda_{\max}(W) > -\bar{\lambda} + \bar{\tau}\lambda_{\max}(S) \geq 0. \tag{54}$$

Let $\tau^* \in (0, \bar{\tau}]$ be the largest number which obeys

$$\max_{j\in\mathcal{S}}(6\tau^*\|A_i\|^2 + 4\|C_i\|^2) \leq \beta_1, \tag{55}$$

$$\max_{j\in\mathcal{S}}(6\tau^*\|B_i\|^2 + 6\tau^*\|F_iY_iX_i^{-1}\|^2 + 4\|H_i\|^2) \leq \beta_2, \tag{56}$$

$$\max_{j\in\mathcal{S}}(6\tau^*\|F_iY_iX_i^{-1}D_i\|^2) \leq \beta_3. \tag{57}$$

$$\overline{\Psi}_i = \begin{bmatrix} \Psi_{i1} & B_iX_i & F_iY_i & F_iY_i & 0 & X_i\bar{R} & X_i\bar{S} & X_i\bar{W} & X_i \\ * & -\eta_{1i}Q_i & 0 & 0 & (F_iY_i)^T & 0 & 0 & 0 & 0 \\ * & * & -\eta_{2i}M & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -X_i & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -X_i & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\bar{\tau}I & 0 & 0 & 0 \\ * & * & * & * & * & * & -\bar{\tau}I & 0 & 0 \\ * & * & * & * & * & * & * & -\bar{\tau}I & 0 \\ * & * & * & * & * & * & * & * & -\eta_{3i}Q_i \end{bmatrix}, \tag{51}$$

$$\Psi_{i1} = A_iX_i + X_iA_i^T + F_iY_i + (F_iY_i)^T, \qquad Z_i = \mathrm{diag}(X_1, X_2, \ldots, X_N),$$

$$V_i = \mathrm{diag}(X_1, \ldots, X_{i-1}, I, X_{i-1}, \ldots, X_N), \qquad \Phi_{ik} = \begin{bmatrix}\hat{\Phi}_{ik}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}^T, \quad k = 1, 2,$$

$$\Phi_{il} = \begin{bmatrix}0 & \hat{\Phi}_{il}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}^T, \quad l = 3, 4, 5, 6, \qquad \bar{R}^2 = R, \qquad \bar{S}^2 = S, \qquad \bar{W}^2 = W,$$

$$\hat{\Phi}_{i1} = \left(\sqrt{\pi_{i1}}X_i, \ldots, \sqrt{\pi_{i(i-1)}}X_i,\ X_iA_i^T,\ \sqrt{\pi_{i(i+1)}}X_i, \ldots, \sqrt{\pi_{iN}}X_i\right),$$

$$\hat{\Phi}_{i2} = \left(\sqrt{\pi_{i1}}X_i, \ldots, \sqrt{\pi_{i(i-1)}}X_i,\ \sqrt{2}X_iC_i^T,\ \sqrt{\pi_{i(i+1)}}X_i, \ldots, \sqrt{\pi_{iN}}X_i\right),$$

$$\hat{\Phi}_{i3} = \left(\sqrt{0.5\pi_{i1}}X_iD_i^T, \ldots, \sqrt{0.5\pi_{i(i-1)}}X_iD_i^T,\ 2X_iD_i^T,\ \sqrt{0.5\pi_{i(i+1)}}X_iD_i^T, \ldots, \sqrt{0.5\pi_{iN}}X_iD_i^T\right),$$

$$\hat{\Phi}_{i4} = \left(\sqrt{0.5\pi_{i1}}X_iD_i^T, \ldots, \sqrt{0.5\pi_{i(i-1)}}X_iD_i^T,\ X_iB_i^T,\ \sqrt{0.5\pi_{i(i+1)}}X_iD_i^T, \ldots, \sqrt{0.5\pi_{iN}}X_iD_i^T\right),$$

$$\hat{\Phi}_{i5} = \left(\sqrt{0.5\pi_{i1}}X_iD_i^T, \ldots, \sqrt{0.5\pi_{i(i-1)}}X_iD_i^T,\ \sqrt{2}X_iH_i^T,\ \sqrt{0.5\pi_{i(i+1)}}X_iD_i^T, \ldots, \sqrt{0.5\pi_{iN}}X_iD_i^T\right),$$

$$\hat{\Phi}_{i6} = \left(\sqrt{0.5\pi_{i1}}X_iD_i^T, \ldots, \sqrt{0.5\pi_{i(i-1)}}X_iD_i^T,\ \sqrt{\alpha}X_iD_i^T,\ \sqrt{0.5\pi_{i(i+1)}}X_iD_i^T, \ldots, \sqrt{0.5\pi_{iN}}X_iD_i^T\right),$$

and

$$\overline{\lambda} = -\lambda_{\max_{i\in\mathcal{S}}}(\overline{\Theta}_i), \tag{52}$$

$$\overline{\Theta}_i = \begin{bmatrix}\overline{\Theta}_{1i} & X_i^{-1}B_i \\ * & \overline{\Theta}_{2i}\end{bmatrix},$$

$$\begin{aligned}\overline{\Theta}_{1i} \ &= X_i^{-1}A_i + A_i^TX_i^{-1} + X_i^{-1}F_iY_iX_i^{-1} + X_i^{-1}(F_iY_i)^TX_i^{-1} + Q_i + \bar{\tau}R + \bar{\tau}S + \bar{\tau}W + 2C_i^TX_i^{-1}C_i + A_i^TX_i^{-1}A_i \\ &\quad + 2\sum_{j\neq i}\pi_{ij}X_j^{-1} + (X_i^{-1}F_iY_iX_i^{-1})M^{-1}(X_i^{-1}F_iY_iX_i^{-1})^T + (X_i^{-1}F_iY_iX_i^{-1})X_i(X_i^{-1}F_iY_iX_i^{-1})^T,\end{aligned}$$

$$\overline{\Theta}_{2i} = -Q_i + 3D_i^TX_i^{-1}D_i + B_i^TX_j^{-1}B_i + 2H_i^TX_i^{-1}H_i + 2D_i^T\sum_{j\neq i}\pi_{ij}X_j^{-1}D_i + (X_i^{-1}F_iY_iX_i^{-1})^TX_i(X_i^{-1}F_iY_iX_i^{-1}).$$

Box I.

Then, if $\tau \le \tau^*$, *by setting*

$$G_i = Y_iX_i^{-1}, \quad i \in \mathcal{S}, \tag{58}$$

the controlled system (9) *is exponentially stable in mean square.*

Proof. Let V be the same as defined by (11) and $P_i = X_i^{-1}$. Here, we only need to prove

$$\Lambda_i < 0, \quad i \in \mathcal{S}, \tag{59}$$

where

$$\Lambda_i = \begin{bmatrix}\Lambda_{i1} + (P_iF_iG_i)M^{-1}(P_iF_iG_i)^T + (P_iF_iG_i)P_i^{-1}(P_iF_iG_i)^T & P_iB_i \\ * & \Lambda_{i2} + (P_iF_iG_i)^TP_i^{-1}(P_iF_iG_i)\end{bmatrix},$$

$$\begin{aligned}\Lambda_{i1} \ &= P_iA_i + A_i^TP_i + P_iF_iG_i + (P_iF_iG_i)^T + Q_i + \tau R + \tau S \\ &\quad + \tau W + A_i^TP_iA_i + 2C_i^TP_iC_i + 2\sum_{j\neq i}\pi_{ij}P_j.\end{aligned}$$

By the Schur complement, (59) is equivalent to Eq. (60) which is given in Box II.

Pre- and post-multiplying (60) by X_i, we have Eq. (61) which is given in Box III.

Note that, for given scalars $\eta_{1i} > 0,\ \eta_{2i} > 0,\ \eta_{3i} > 0,\ i \in \mathcal{S}$,

$$0 \le (X_i - \eta_{1i}I)Q_i(X_i - \eta_{1i}I) = X_iQ_iX_i - \eta_{1i}X_iQ_i - \eta_{1i}Q_iX_i + \eta_{1i}^2Q_i,$$

$$0 \le (X_i - \eta_{2i}I)M(X_i - \eta_{2i}I) = X_iMX_i - \eta_{2i}X_iM - \eta_{2i}MX_i + \eta_{2i}^2M,$$

$$0 \le (I - \eta_{3i}Q_i)Q_i^{-1}(I - \eta_{3i}Q_i) = Q_i^{-1} - 2\eta_{3i}I + \eta_{3i}^2Q_i.$$

Then, from (48)–(50), we get

$$\begin{aligned}-X_iQ_iX_i \ &\le \ \frac{\eta_{1i}}{2}(-2X_i + (1+\eta_{1i})I)Q_i \\ &\quad + \frac{\eta_{1i}}{2}Q_i(-2X_i + (1+\eta_{1i})I) - \eta_{1i}Q_i < -\eta_{1i}Q_i,\end{aligned} \tag{62}$$

$$\begin{aligned}-X_iMX_i \ &\le \ \frac{\eta_{2i}}{2}(-2X_i + (1+\eta_{2i})I)M \\ &\quad + \frac{\eta_{2i}}{2}M(-2X_i + (1+\eta_{2i})I) - \eta_{2i}M < -\eta_{2i}M,\end{aligned} \tag{63}$$

$$\begin{aligned}-Q_i^{-1} \ &\le \ \eta_{3i}(-2I + \eta_{3i}Q_i) \\ &= \ \eta_{3i}(-2I + (1+\eta_{3i})Q_i) - \eta_{3i}Q_i < -\eta_{3i}Q_i.\end{aligned} \tag{64}$$

On the other hand, $\tau \le \tau^* \le \bar{\tau}^{-1}$ implies $-\tau^{-1} \le -\bar{\tau}$. Thus, (61) is equivalent to Eq. (65) which is given in Box IV.

By the Schur complements, (46) is equivalent to (65). This completes the proof. $\square$

$$\begin{bmatrix} \widetilde{\Lambda}_{i1} & P_iB_i & P_iF_iG_i & P_iF_iG_i & 0 & \bar{R}P_i & \bar{S}P_i & \bar{W}P_i \\ * & \Lambda_{i2} & 0 & 0 & (P_iF_iG_i)^T & 0 & 0 & 0 \\ * & * & -M & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -P_i & 0 & 0 & 0 & 0 \\ * & * & * & * & -P_i & 0 & 0 & 0 \\ * & * & * & * & * & -\tau^{-1}P_i^2 & 0 & 0 \\ * & * & * & * & * & * & -\tau^{-1}P_i^2 & 0 \\ * & * & * & * & * & * & * & -\tau^{-1}P_i^2 \end{bmatrix} < 0, \quad i \in \mathcal{S}, \tag{60}$$

where

$$\widetilde{\Lambda}_{i1} = P_iA_i + A_i^TP_i + P_iF_iG_i + (P_iF_iG_i)^T + Q_i + A_i^TP_iA_i + 2C_i^TP_iC_i + 2\sum_{j\neq i}\pi_{ij}P_j,$$

$$\bar{R}^2 = R, \qquad \bar{S}^2 = S, \qquad \bar{W}^2 = W.$$

Box II.

$$\begin{bmatrix} X_i\widetilde{\Lambda}_{i1}X_i & B_iX_i & F_iY_i & F_iY_i & 0 & X_i\bar{R} & X_i\bar{S} & X_i\bar{W} & X_i \\ * & X_i\Lambda_{i2}X_i & 0 & 0 & (F_iY_i)^T & 0 & 0 & 0 & 0 \\ * & * & -X_iMX_i & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -X_i & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -X_i & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\tau^{-1}I & 0 & 0 & 0 \\ * & * & * & * & * & * & -\tau^{-1}I & 0 & 0 \\ * & * & * & * & * & * & * & -\tau^{-1}I & 0 \\ * & * & * & * & * & * & * & * & -Q_i^{-1} \end{bmatrix} < 0, \quad i \in \mathcal{S}. \tag{61}$$

Box III.

$$\begin{bmatrix} \widetilde{\Psi}_{i1} & B_iX_i & F_iY_i & F_iY_i & 0 & X_i\bar{R} & X_i\bar{S} & X_i\bar{W} & X_i \\ * & \widetilde{\Psi}_{i2} & 0 & 0 & (F_iY_i)^T & 0 & 0 & 0 & 0 \\ * & * & -\eta_{2i}M & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -X_i & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -X_i & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\bar{\tau}I & 0 & 0 & 0 \\ * & * & * & * & * & * & -\bar{\tau}I & 0 & 0 \\ * & * & * & * & * & * & * & -\bar{\tau}I & 0 \\ * & * & * & * & * & * & * & * & -\eta_{3i}Q_i \end{bmatrix} < 0, \quad i \in \mathcal{S}, \tag{65}$$

where

$$\widetilde{\Psi}_{i1} = \Psi_{i1} + X_iA_i^TX_i^{-1}A_iX_i + 2X_iC_i^TX_i^{-1}C_iX_i + 2X_i\sum_{j\neq i}\pi_{ij}X_j^{-1}X_i$$

$$\widetilde{\Psi}_{i2} = -\eta_{1i}Q_i + 4X_iD_i^TX_i^{-1}D_iX_i + \alpha X_iD_i^TD_iX_i + X_iB_i^TX_i^{-1}B_iX_i + 2X_iH_i^TX_i^{-1}H_iX_i + 2X_iD_i^T\sum_{j\neq i}\pi_{ij}X_j^{-1}D_iX_i.$$

Box IV.

4. Stabilization of nonlinear hybrid NSDDE

In this section, we will discuss the more general nonlinear stabilization problem. Consider an n-dimensional unstable nonlinear hybrid NSDDE

$$\begin{aligned} \mathrm{d}[x(t) - D(x(t-\tau), r(t))] &= f(x(t), x(t-\tau), t, r(t))\mathrm{d}t \\ &\quad + g(x(t), x(t-\tau), t, r(t))\mathrm{d}\omega(t) \end{aligned} \tag{66}$$

on $t \geq \tau$, where f, g are both mappings from $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S}$ to $\mathbb{R}^n$. Throughout this paper, assume that both f and g satisfy the local Lipschitz condition and obey the linear growth condition. In the following, we are required to design a delay feedback controller $u(x(t-\tau), r(t))$ so that the underlying controlled system

$$\begin{aligned} &\mathrm{d}[x(t) - D(x(t-\tau), r(t))] \\ &\quad = [f(x(t), x(t-\tau), t, r(t)) + u(x(t-\tau), r(t))]\mathrm{d}t \\ &\qquad + g(x(t), x(t-\tau), t, r(t))\mathrm{d}\omega(t) \end{aligned} \tag{67}$$

will be exponentially stable in mean square. Denote $\eta(t) = x(t) - D(x(t-\tau), r(t))$. Here, we use a linear delay feedback controller; that is,

$$u(x(t-\tau), r(t)) = F(r(t))G(r(t))\eta(t-\tau). \tag{68}$$

As a result, the controlled system (67) is replaced by

$$\begin{aligned} &\mathrm{d}[x(t) - D(x(t-\tau), r(t))] \\ &\quad = [f(x(t), x(t-\tau), t, r(t)) + F(r(t))G(r(t))\eta(t-\tau)]\mathrm{d}t \\ &\qquad + g(x(t), x(t-\tau), t, r(t))\mathrm{d}\omega(t). \end{aligned} \tag{69}$$

The Lyapunov functional defined by (11) is also employed here, and we still regard the controlled system (69) as an NSDDE on $t > 3\tau$ with initial data $\{x(t) : 0 \leq t \leq 3\tau\}$ and fixed initial state r_0 arbitrarily for Markov chain $r(t)$. In the next sub-section, we will use the linear control $F(r(t))G(r(t))\eta(t-\tau)$ to stabilize the nonlinear system (66). For the nonlinear terms $2\eta(t)^TP(r(t))f(x(t),$

$x(t-\tau), t, r(t))$, $g(x(t), x(t-\tau), t, r(t))^T P(r(t)) g(x(t), x(t-\tau), t, r(t))$ and $D(x, r(t))^T M D(x, r(t))$ will appear in $\mathcal{L}V(\hat{x}_t, t, i)$ with $\mathcal{L}V$ being defined by (10), it is natural to impose some conditions on the nonlinear coefficients f and g. Moreover, we still need to estimate $\mathbb{E}|\eta(t)-\eta(t-\tau)|^2$. For these purposes, we impose the following hypotheses:

Assumption 4.1. Assume there are symmetric matrices $P_i > 0$, $U_i > 0$ and $V_i > 0$, such that

$$\begin{aligned} &2\eta(t)^T P_i f(x(t), x(t-\tau), t, r(t)) \\ &\quad + g(x(t), x(t-\tau), t, r(t))^T P_i g(x(t), x(t-\tau), t, r(t)) \\ &\leq x(t)^T U_i x(t) + x(t-\tau)^T V_i x(t-\tau), \end{aligned} \tag{70}$$

for all $(x(t), x(t-\tau), t, r(t)) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S}$;

Assumption 4.2. Assume

$$D(x, i)^T M D(x, i) \leq k_i^2 x^T M x, \quad \forall x \in \mathbb{R}^n \tag{71}$$

for all symmetric matrix M, where $k_i = \|D(x, i)\| \in (0, 1)$, $i \in \mathcal{S}$.

Assumption 4.3. Assume there are four positive constants δ_1, δ_2, δ_3, and δ_4, such that

$$|f(x(t), x(t-\tau), t, i)|^2 \leq \delta_1 |x(t)|^2 + \delta_2 |x(t-\tau)|^2, \tag{72}$$

$$|g(x(t), x(t-\tau), t, i)|^2 \leq \delta_3 |x(t)|^2 + \delta_4 |x(t-\tau)|^2 \tag{73}$$

for all $(x(t), x(t-\tau), t, r(t)) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S}$.

4.1. State feedback

Here, we consider the matrix G_i is given but F_i needs to be designed.

Theorem 4.1. *Suppose Assumptions 4.1–4.3 hold. Assume that for chosen positive-definite $n \times n$ matrices R, S, W, M, the following LMIs*

$$\begin{bmatrix} \overline{\Gamma}_{1i} & 0 & Y_i G_i & Y_i G_i & 0 \\ * & \Gamma_{2i} & 0 & 0 & k_i (Y_i G_i)^T \\ * & * & -M & 0 & 0 \\ * & * & * & -P_i & 0 \\ * & * & * & * & -P_i \end{bmatrix} < 0, \quad i \in \mathcal{S}, \tag{74}$$

$$\sum_{j \neq i} \pi_{ij} Q_j < R, \quad i \in \mathcal{S}, \tag{75}$$

have solutions $\bar{\tau} > 0$ and P_i, Q_i, Y_i with $P_i > 0$, $Q_i > 0$ and $Y_i \in \mathbb{R}^{n \times l}$, where

$$\bar{\lambda} = -\lambda_{\max_{i \in \mathcal{S}}}(\overline{\Gamma}_i), \tag{76}$$

$$\overline{\Gamma}_i = \begin{bmatrix} \overline{\Gamma}_{1i} + (Y_i G_i) M^{-1} (Y_i G_i)^T + (Y_i G_i) P_i^{-1} (Y_i G_i)^T & 0 \\ * & \Gamma_{2i} + (Y_i G_i)^T P_i^{-1} (Y_i G_i) \end{bmatrix}, \tag{77}$$

$$\overline{\Gamma}_{1i} = U_i + Y_i G_i + (Y_i G_i)^T + Q_i + \bar{\tau} R + \bar{\tau} S + \bar{\tau} W + 2 \sum_{j \neq i} \pi_{ij} P_j$$

$$\Gamma_{2i} = V_i - Q_i + k_i^2 P_i + 2k_i^2 \sum_{j \neq i} \pi_{ij} P_j.$$

Choose three positive numbers β_1, β_2, β_3, such that

$$\begin{aligned} &\lambda_{\max_{i \in \mathcal{S}}}(|\pi_{ii}| Q_i) > \lambda_{\max}(M) \beta_1, \\ &\lambda_{\max}(S) > \lambda_{\max}(M) \beta_2, \qquad \lambda_{\max}(W) > \lambda_{\max}(M) \beta_3, \end{aligned} \tag{78}$$

$$\bar{\lambda} + \bar{\tau} \lambda_{\min}(S) + \bar{\tau} \lambda_{\max}(W) > -\bar{\lambda} + \bar{\tau} \lambda_{\max}(S) \geq 0. \tag{79}$$

Let $\tau^* \in (0, \bar{\tau}]$ be the largest number which obeys

$$8\tau^* \delta_1 + 4\delta_3 \leq \beta_1, \tag{80}$$

$$8\tau^* \delta_2 + 8\tau^* \max_{i \in \mathcal{S}} \|P_i^{-1} Y_i G_i\|^2 + 4\delta_4 \leq \beta_2, \tag{81}$$

$$8\tau^* \max_{i \in \mathcal{S}} (k_i^2 \|P_i^{-1} Y_i G_i\|^2) \leq \beta_3. \tag{82}$$

Then, if $\tau \leq \tau^*$, by setting

$$F_i = P_i^{-1} Y_i, \quad i \in \mathcal{S}, \tag{83}$$

the controlled system (69) is exponentially stable in mean square.

Proof. We derive from (11) that

$$\begin{aligned} \mathcal{L}V(\hat{x}_t, t, i) = \; & 2\eta(t)^T P_i [f(x(t), x(t-\tau), t, i) \\ & + F_i G_i \eta(t-\tau)] + x(t)^T Q_i x(t) \\ & - x(t-\tau)^T Q_i x(t-\tau) + \tau x(t)^T \\ & \times R x(t) + \tau x(t-\tau)^T S x(t-\tau) \\ & + \tau x(t-2\tau)^T W x(t-2\tau) \\ & - \int_{t-\tau}^{t} x(s)^T R x(s) \mathrm{d}s - \int_{t-\tau}^{t} x(s-\tau)^T \\ & \times R x(s-\tau) \mathrm{d}s - \int_{t-\tau}^{t} x(s-2\tau)^T W x(s-2\tau) \mathrm{d}s \\ & + g(x(t), x(t-\tau), t, i)^T P_i g(x(t), x(t-\tau), t, i) \\ & + \eta(t)^T \sum_{j \in \mathcal{S}} \pi_{ij} P_j \eta(t) \\ & + \int_{t-\tau}^{t} x(s)^T \sum_{j \in \mathcal{S}} \pi_{ij} Q_j x(s) \mathrm{d}s. \end{aligned} \tag{84}$$

Note

$$\begin{aligned} 2\eta(t)^T P_i F_i G_i \eta(t-\tau) = \; & 2\eta(t)^T P_i F_i G_i \eta(t) \\ & - 2\eta(t)^T P_i F_i G_i [\eta(t) - \eta(t-\tau)], \end{aligned} \tag{85}$$

and

$$\begin{aligned} & -2x(t)^T P_i F_i G_i [\eta(t) - \eta(t-\tau)] \\ & \quad \leq x(t)^T (P_i F_i G_i) M^{-1} (P_i F_i G_i)^T x(t) \\ & \qquad + [\eta(t) - \eta(t-\tau)]^T M [\eta(t) - \eta(t-\tau)], \end{aligned} \tag{86}$$

$$\begin{aligned} & -2D(x(t-\tau), i)^T P_i F_i G_i [\eta(t) - \eta(t-\tau)] \\ & \quad \leq [\eta(t) - \eta(t-\tau)]^T M [\eta(t) - \eta(t-\tau)] \\ & \qquad + D(x(t-\tau), i)^T (P_i F_i G_i) M^{-1} (P_i F_i G_i)^T D(x(t-\tau), i). \end{aligned} \tag{87}$$

On the other hand,

$$\begin{aligned} & -2x(t)^T P_i F_i G_i D(x(t-\tau), i) \\ & \quad \leq x(t)^T (P_i F_i G_i) P_i^{-1} (P_i F_i G_i)^T x(t) \\ & \qquad + D(x(t-\tau), i)^T P_i D(x(t-\tau), i), \end{aligned} \tag{88}$$

$$\begin{aligned} & -2D(x(t-\tau), i)^T P_i F_i G_i D(x(t-\tau), i) \\ & \quad \leq D(x(t-\tau), i)^T P_i D(x(t-\tau), i) \\ & \qquad + D(x(t-\tau), i)^T (P_i F_i G_i)^T P_i^{-1} (P_i F_i G_i) D(x(t-\tau), i), \end{aligned} \tag{89}$$

and

$$\begin{aligned} & \eta(t)^T \sum_{j \in \mathcal{S}} \pi_{ij} P_j \eta(t) \\ & \quad = x(t)^T \sum_{j \in \mathcal{S}} \pi_{ij} P_j x(t) - 2x(t)^T \sum_{j \in \mathcal{S}} \pi_{ij} P_j D(x(t-\tau), i) \\ & \qquad + D(x(t-\tau), i)^T \sum_{j \in \mathcal{S}} \pi_{ij} P_j D(x(t-\tau), i) \end{aligned}$$

$$\begin{aligned}&\le 2x(t)^T\sum_{j\neq i}\pi_{ij}P_jx(t)\\&\quad+2D(x(t-\tau),i)^T\sum_{j\neq i}\pi_{ij}P_jD(x(t-\tau),i).\end{aligned}\tag{90}$$

Then, by Assumptions 4.1–4.2 and (75), we have

$$\begin{aligned}\mathcal{L}V(\hat{x}_t,t,i)\le{}&\xi(t)^T\Gamma_i\xi(t)-\tau x(t)^TSx(t)\\&-\tau x(t)^TWx(t)+\tau x(t-\tau)^TSx(t-\tau)\\&+\tau x(t-2\tau)^TWx(t-2\tau)\\&-|\pi_{ii}|\int_{t-\tau}^t x(s)^TQ_ix(s)\mathrm{d}s\\&-\int_{t-\tau}^t x(s-\tau)^TSx(s-\tau)\mathrm{d}s\\&-\int_{t-2\tau}^t x(s-\tau)^TWx(s-2\tau)\mathrm{d}s\\&+2[\eta(t)-\eta(t-\tau)]^TM[\eta(t)-\eta(t-\tau)],\end{aligned}\tag{91}$$

where

$$\Gamma_i=\begin{bmatrix}\Gamma_{1i}+(P_iF_iG_i)M^{-1}(P_iF_iG_i)^T+(P_iF_iG_i)P_i^{-1}(P_iF_iG_i)^T & 0\\ * & \Gamma_{2i}+(P_iF_iG_i)^TP_i^{-1}(P_iF_iG_i)\end{bmatrix},\tag{92}$$

$$\Gamma_{1i}=U_i+P_iF_iG_i+(P_iF_iG_i)^T+Q_i+\tau R+\tau S+\tau W+2\sum_{j\neq i}\pi_{ij}P_j.$$

Observe

$$\begin{aligned}\eta(t)-\eta(t-\tau)={}&\int_{t-\tau}^t[f(x(s),x(s-\tau),t,r(s))\\&+F(r(s))G(r(s))\eta(s-\tau)]\mathrm{d}s\\&+\int_{t-\tau}^t g(x(s),x(s-\tau),s,r(s))\mathrm{d}\omega(s).\end{aligned}$$

Thus, by Assumption 4.3, we have

$$\begin{aligned}&\mathbb{E}|\eta(t)-\eta(t-\tau)|^2\\&\le 2\tau\mathbb{E}\int_{t-\tau}^t|f(x(s),x(s-\tau),t,r(s))\\&\quad+F(r(s))G(r(s))\eta(s-\tau)|^2\mathrm{d}s\\&\quad+2\mathbb{E}\int_{t-\tau}^t|g(x(s),x(s-\tau),s,r(s))|^2\mathrm{d}s\\&\le(4\tau\delta_1+2\delta_3)\mathbb{E}\int_{t-\tau}^t|x(s)|^2\mathrm{d}s\\&\quad+(4\tau\delta_2+4\tau\max_{i\in\mathcal{S}}\|P_i^{-1}Y_iG_i\|^2+2\delta_4)\mathbb{E}\int_{t-\tau}^t|x(s-\tau)|^2\mathrm{d}s\\&\quad+\max_{j\in\mathcal{S}}(4\tau k_i^2\|P_i^{-1}Y_iG_i\|^2)\mathbb{E}\int_{t-\tau}^t|x(s-2\tau)|^2\mathrm{d}s.\end{aligned}\tag{93}$$

Recalling (80)–(82), we obtain

$$\begin{aligned}&2\mathbb{E}|\eta(t)-\eta(t-\tau)|^2\\&\le\beta_1\mathbb{E}\int_{t-\tau}^t|x(s)|^2\mathrm{d}s+\beta_2\mathbb{E}\int_{t-\tau}^t|x(s-\tau)|^2\mathrm{d}s\\&\quad+\beta_3\mathbb{E}\int_{t-\tau}^t|x(s-2\tau)|^2\mathrm{d}s.\end{aligned}\tag{94}$$

By a similar method, it is easy to see that

$$\begin{aligned}\mathbb{E}(\mathcal{L}V(\hat{x}_t,t,i))\le{}&-\lambda_1\mathbb{E}|x(t)|^2+\lambda_2\mathbb{E}|x(t-\tau)|^2\\&+\lambda_3\mathbb{E}|x(t-2\tau)|^2-\lambda_4\int_{t-3\tau}^t\mathbb{E}|x(s)|^2\mathrm{d}s,\end{aligned}\tag{95}$$

where

$$\lambda_1=\lambda+\tau\lambda_{\min}(S)+\tau\lambda_{\min}(W),$$
$$\lambda_2=-\lambda+\tau\lambda_{\max}(S),\qquad\lambda_3=\tau\lambda_{\min}(W),$$
$$\lambda=-\lambda_{\max_{i\in\mathcal{S}}}(\Gamma_i),$$
$$\begin{aligned}\lambda_4={}&\max\Big\{\lambda_{\max_{i\in\mathcal{S}}}(|\pi_{ii}|Q_i)-\lambda_{\max}(M)\beta_1,\ \lambda_{\max}(S)-\lambda_{\max}(M)\beta_2,\\&\lambda_{\max}(W)-\lambda_{\max}(M)\beta_3\Big\},\end{aligned}$$

with $\lambda_1>\lambda_2\ge 0$, $\lambda_3>0$ and $\lambda_4>0$. Then, by Lemma 3.1, the controlled system (69) is exponentially stable in mean square. The proof is completed. □

4.2. Output injection

Now, let us discuss the case that F_i is given while G_i needs to be designed. The proof of this theorem is omitted here as it is similar to that of Theorem 3.2.

Theorem 4.2. *Let Assumptions 4.1–4.3 hold. Assume that for chosen positive-definite $n\times n$ matrices R, S, W, M, and positive numbers $\eta_{1i}\,\eta_{2i}$, η_{3i}, η_{4i}, η_{5i}, $i\in\mathcal{S}$, the following LMIs*

$$\begin{bmatrix}\overline{\Sigma}_i & \Theta_{i1} & \Theta_{i2}\\ * & -Z_i & 0\\ * & * & -Z_i\end{bmatrix}<0,\quad i\in\mathcal{S},\tag{96}$$

$$\sum_{j\neq i}\pi_{ij}Q_j<R,\quad i\in\mathcal{S},\tag{97}$$

$$-2X_i+(1+\eta_{1i})I<0,\quad i\in\mathcal{S},\tag{98}$$
$$-2X_i+(1+\eta_{2i})I<0,\quad i\in\mathcal{S},\tag{99}$$
$$-2I+(1+\eta_{3i})Q_i<0,\quad i\in\mathcal{S},\tag{100}$$
$$-2I+(1+\eta_{4i})U_i<0,\quad i\in\mathcal{S},\tag{101}$$
$$-2I+(1+\eta_{5i})V_i<0,\quad i\in\mathcal{S},\tag{102}$$

have solutions $\bar{\tau}>0$ and X_i, Q_i, Y_i with $X_i>0$, $Q_i>0$ and $Y_i\in\mathbb{R}^{l\times n}$, where $\overline{\Sigma}_i$ is given in Box V

$$\Sigma_{i1}=F_iY_i+(F_iY_i)^T,\qquad Z_i=\mathrm{diag}(X_1,X_2,\dots,X_N),$$
$$\bar{R}^2=R,\qquad\bar{S}^2=S,\qquad\bar{W}^2=W,$$

$$\Theta_{i1}=\begin{bmatrix}\hat{\Theta}_{i1}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}^T,$$

$$\Theta_{i2}=\begin{bmatrix}0 & \hat{\Theta}_{i2}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}^T,$$

$$\hat{\Theta}_{i1}=\left(\sqrt{2\pi_{i1}}X_i,\dots,\sqrt{2\pi_{i(i-1)}}X_i,\ 0,\ \sqrt{2\pi_{i(i+1)}}X_i,\dots,\sqrt{2\pi_{iN}}X_i\right),$$

$$\hat{\Theta}_{i2}=\left(\sqrt{2\pi_{i1}}X_i,\dots,\sqrt{2\pi_{i(i-1)}}X_i,\ k_iX_i,\ \sqrt{2\pi_{i(i+1)}}X_i,\dots,\sqrt{2\pi_{iN}}X_i\right),$$

and

$$\bar{\lambda}=-\lambda_{\max_{i\in\mathcal{S}}}(\overline{\Xi}_i),\tag{103}$$

$$\overline{\Xi}_i=\begin{bmatrix}\overline{\Xi}_{i1} & 0\\ * & \overline{\Xi}_{i2}\end{bmatrix},$$

$$\begin{aligned}\overline{\Xi}_{i1}={}&U_i+X_i^{-1}F_iY_iX_i^{-1}+X_i^{-1}(F_iY_i)^TX_i^{-1}\\&+X_iQ_iX_i+\bar{\tau}R+\bar{\tau}S+\bar{\tau}W+2\sum_{j\neq i}\pi_{ij}X_j^{-1}\\&+(X_i^{-1}F_iY_iX_i^{-1})M^{-1}(X_i^{-1}F_iY_iX_i^{-1})^T\\&+(X_i^{-1}F_iY_iX_i^{-1})X_i(X_i^{-1}F_iY_iX_i^{-1})^T,\end{aligned}$$

$$\begin{aligned}\overline{\Xi}_{i2}={}&V_i-Q_i+k_i^2X_i^{-1}+(X_i^{-1}F_iY_iX_i^{-1})^T\\&\times X_i(X_i^{-1}F_iY_iX_i^{-1})+2k_i^2\sum_{j\neq i}\pi_{ij}X_j^{-1}.\end{aligned}$$

$$\overline{\Sigma}_i = \begin{bmatrix} \Sigma_{i1} & 0 & F_iY_i & F_iY_i & 0 & X_i\bar{R} & X_i\bar{S} & X_i\bar{W} & X_i & X_i & 0 \\ * & -\eta_{1i}Q_i & 0 & 0 & k_i(F_iY_i)^T & 0 & 0 & 0 & 0 & 0 & X_i \\ * & * & -\eta_{2i}M & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -X_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -X_i & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\bar{\tau}I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\bar{\tau}I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -\bar{\tau}I & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & -\eta_{3i}Q_i & 0 & 0 \\ * & * & * & * & * & * & * & * & * & -\eta_{4i}U_i & 0 \\ * & * & * & * & * & * & * & * & * & * & -\eta_{5i}V_i \end{bmatrix},$$

Box V.

Choose three positive numbers $\beta_1, \beta_2, \beta_3$, such that

$$\lambda_{\max_{i\in\mathcal{S}}}(|\pi_{ii}|Q_i) > \lambda_{\max}(M)\beta_1, \qquad \lambda_{\max}(S) > \lambda_{\max}(M)\beta_2,$$

$$\lambda_{\max}(W) > \lambda_{\max}(M)\beta_3, \tag{104}$$

$$\overline{\lambda} + \bar{\tau}\lambda_{\min}(S) + \bar{\tau}\lambda_{\max}(W) > -\overline{\lambda} + \bar{\tau}\lambda_{\max}(S) \geq 0. \tag{105}$$

Let $\tau^* \in (0, \bar{\tau}]$ be the largest number which obeys

$$8\tau^*\delta_1 + 4\delta_3 \leq \beta_1, \tag{106}$$

$$8\tau^*\delta_2 + 8\tau^* \max_{i\in\mathcal{S}} \|P_i^{-1}Y_iG_i\|^2 + 4\delta_4 \leq \beta_2, \tag{107}$$

$$8\tau^* \max_{i\in\mathcal{S}}(k_i^2\|P_i^{-1}Y_iG_i\|^2) \leq \beta_3. \tag{108}$$

Then, if $\tau \leq \tau^*$, by setting

$$G_i = Y_iX_i^{-1}, \quad i \in \mathcal{S}, \tag{109}$$

the controlled system (69) is exponentially stable in mean square.

5. Example

In this section, we will provide two examples to illustrate the effectiveness of the proposed method.

Example 1. Consider a 2-dimensional hybrid NSDDE (7) with the system matrices given below:

$$A_1 = \begin{bmatrix} -2 & -1.6 \\ 1 & -3.4 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & -2 \\ 2.5 & -1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \qquad D_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \qquad H_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}.$$

The mode switching is governed by the transition rate matrix

$$\Pi := \begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix}.$$

The computer simulation (Figs. 1 and 2) shows this hybrid NSDDE (7) is not exponentially stable. Choose $\alpha = 0.001$, $\beta_1 = 6$, $\beta_2 = 20$, $\beta_3 = 3.9$, and

$$R = \begin{bmatrix} 61 & 0 \\ 0 & 61 \end{bmatrix}, \qquad S = \begin{bmatrix} 101 & 0 \\ 0 & 101 \end{bmatrix}, \qquad W = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix},$$

$$M = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}, \qquad G_1 = \begin{bmatrix} 1 & 0.5 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} 0.2 & 1 \end{bmatrix}.$$

Then by Theorem 3.1, we can get the maximum time delay $\bar{\tau} = 0.0019$ and $\overline{\lambda} = 0.0838$,

$$P_1 = \begin{bmatrix} 1.6367 - 0.856 \\ -0.85621.2002 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 3.1288 - 1.2792 \\ -1.27921.7886 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 6.0991 - 1.3560 \\ -1.35603.1333 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 7.6601 - 0.7533 \\ -0.75333.9147 \end{bmatrix}, \qquad Y_1 = \begin{bmatrix} -2.4172 \\ -1.2405 \end{bmatrix},$$

$$Y_2 = \begin{bmatrix} -0.1021 \\ -4.3173 \end{bmatrix}$$

and

$$\lambda_{\max_{i\in\mathcal{S}}}(|\pi_{ii}|Q_i) = 31.2237 > \lambda_{\max}(M)\beta_1 = 30,$$

$$\lambda_{\max}(S) = 101 > \lambda_{\max}(M)\beta_2 = 100,$$

$$\lambda_{\max}(W) = 20 > \lambda_{\max}(M)\beta_3 = 19.5,$$

$$\lambda_1 = [\overline{\lambda} + \bar{\tau}\lambda_{\min}(S) + \bar{\tau}\lambda_{\max}(W)] = 0.3137$$
$$> \lambda_2 = -\overline{\lambda} + \bar{\tau}\lambda_{\max}(S) = 0.1081.$$

Let $\tau = 0.001 < 0.0019$, which obeys (24)–(26), that is,

$$\max_{j\in\mathcal{S}}(6\tau\|A_i\|^2 + 4\|C_i\|^2) = 1.0847 \leq \beta_1 = 6,$$

$$\max_{j\in\mathcal{S}}(6\tau\|B_i\|^2 + 6\tau\|P_i^{-1}Y_iG_i\|^2 + 4\|H_i\|^2) = 1.2408 \leq \beta_2 = 20,$$

$$\max_{j\in\mathcal{S}}(6\tau\|P_i^{-1}Y_iG_iD_i\|^2) = 0.0064 \leq \beta_3 = 3.9.$$

Then, by setting

$$F_1 = \begin{bmatrix} -3.2185 \\ -3.3296 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} -1.4408 \\ -3.4442 \end{bmatrix},$$

the controlled NSDDE (9) is exponentially stable in mean square. Fig. 3 shows the jump mode $r(t)$, and Fig. 4 shows the controlled NSDDE (9) is exponentially stable in mean square.

The technique of delay feedback control in this paper can also be used to handle the case of hybrid unstable stochastic differential equations with Markovian switching. The following example illustrates the effectiveness and advantages of our results.

Example 2. Consider the controlled hybrid NSSDE (7) with the system matrices given below:

$$A_1 = \begin{bmatrix} 1 & -1 \\ 1 & -5 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -5 & -1 \\ 1 & 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix},$$

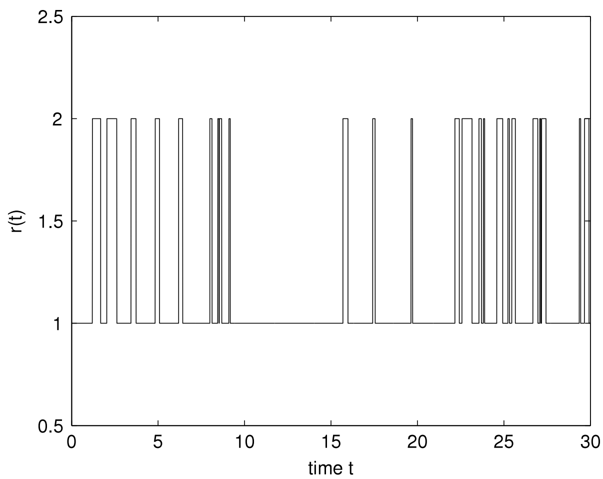

Fig. 1. Jump mode $r(t)$.

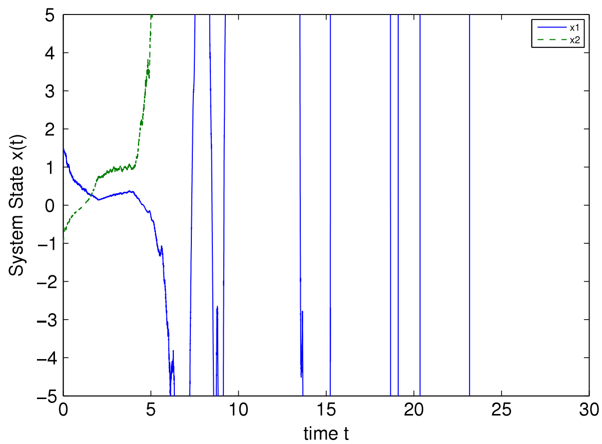

Fig. 2. System state $x(t)$.

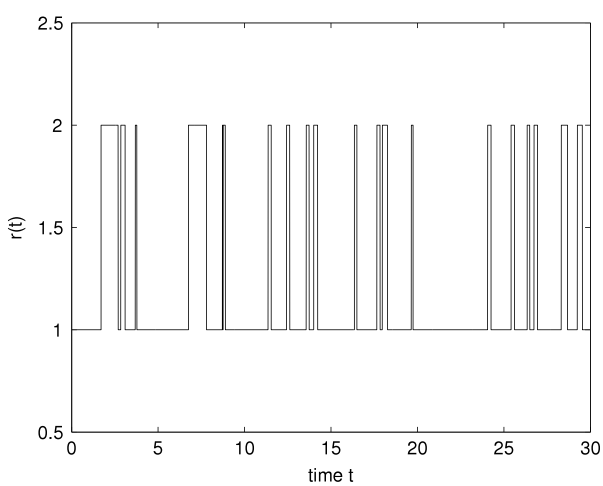

Fig. 3. Jump mode $r(t)$.

$$B_1 = B_2 = D_1 = D_2 = H_1 = H_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Suppose the transition rate matrix is given by

$$\Pi := \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

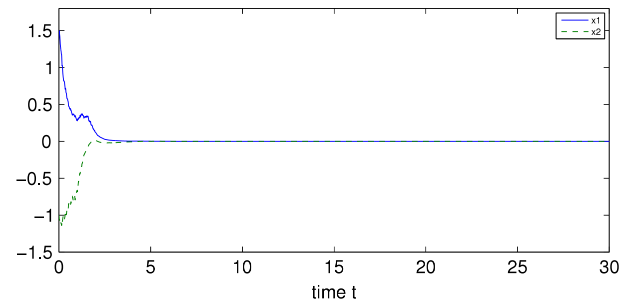

Fig. 4. The controlled system state $x(t)$.

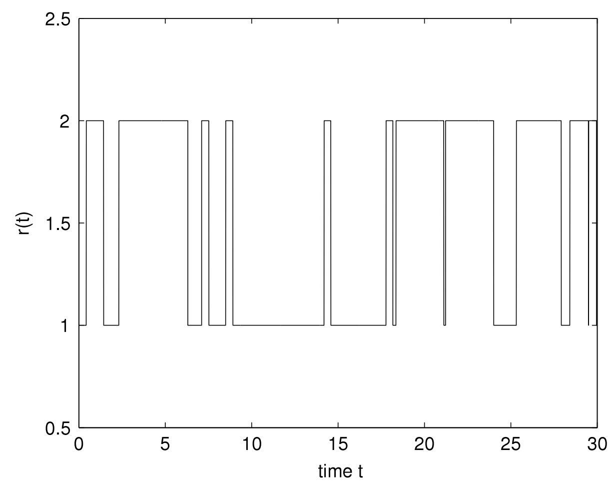

Fig. 5. Jump mode $r(t)$.

Then the controlled NSDDE becomes the controlled SDDE (2.3) in the Example 1 of [26]. Choose $\alpha = 0.001,\ \beta_1 = 6,\ \beta_2 = 10,\ \beta_3 = 1$,

$$R = \begin{bmatrix} 61 & 0 \\ 0 & 61 \end{bmatrix}, \qquad S = \begin{bmatrix} 101 & 0 \\ 0 & 101 \end{bmatrix},$$

$$W = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad M = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}.$$

By Theorem 3.2, we can get the maximum time delay $\bar{\tau} = 0.0178$ and $\bar{\lambda} = 0.3089$,

$$P_1 = \begin{bmatrix} 0.4789 & -0.7760 \\ -0.7760 & 3.1204 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 3.1204 & 0.7760 \\ 0.7760 & 0.4789 \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} 2.3080 & -4.6048 \\ -4.6048 & 23.9438 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 23.9438 & 4.6048 \\ 4.6048 & 2.3080 \end{bmatrix}, \qquad Y_1 = \begin{bmatrix} -9.9820 \\ -0.0017 \end{bmatrix},$$

$$Y_2 = \begin{bmatrix} 0.0017 \\ -9.9820 \end{bmatrix}.$$

Let $\tau = 0.01$, which obeys (24)–(26), by setting

$$F_1 = \begin{bmatrix} -34.9145 \\ -8.6830 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} 8.6830 \\ -34.9145 \end{bmatrix},$$

then the controlled SDDE (2.3) of [26] is exponentially stable in mean square. Fig. 5 shows the jump mode $r(t)$, and Fig. 6 shows the controlled SDDE (2.3) of [26] is exponentially stable in mean square.

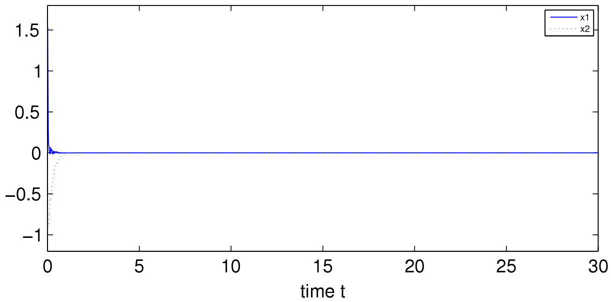

Fig. 6. The controlled system state $x(t)$.

6. Conclusions

In this paper, the problem of stabilization of hybrid neutral stochastic differential delay equations with Markovian switching by delay feedback controls has been considered. The stabilization criteria are derived in terms of linear matrix inequalities. These make the design of delay feedback controls can be more easy in practice. The technique of delay feedback control in this paper is also applicable to the cases of hybrid stochastic differential equations and hybrid stochastic differential delay equations. The idea can be developed into stochastic stabilization by stochastic feedback control.

Acknowledgments

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References

[1] O.L.V. Costa, E.O. Assumpcao Filhoa, E.K. Boukasb, R.P. Marquesa, Constrained quadratic state feedback control of discrete-time Markovian jump linear systems, Automatica 35 (1999) 617–626.
[2] M.K. Ghosh, A. Arapostathis, S.I. Marcus, Ergodic control of switching diffusions, SIAM J. Control Optim. 35 (1997) 1952–1988.
[3] X. Mao, Stochastic Differential Equations and their Applications, second ed., Horwood Publishing, Chichester, 2007.
[4] G. Yin, R. Liu, R. Zhang, Recursive algorithms for stock liquidation: a stochastic optimization approach, SIAM J. Optim. 13 (2002) 240–263.
[5] G. Yin, R. Zhang, Continuous-Time Markov Chains and Applications: A Singular Perturbation Approach, Springer-Verlag, New York, 1998.
[6] W. Chen, L. Wang, Delay-dependent stability for neutral-type neural networks with time-varying delays and Markovian jumping parameters, Neurocomputing 120 (2013) 569–576.
[7] L. Huang, X. Mao, Delay-dependent exponential stability of neutral stochastic delay systems, IEEE Trans. Automat. Control 54 (2009) 147–152.
[8] V. Kolmanovskii, N. Koroleva, T. Maizenberg, X. Mao, A. Matasov, Neutral stochastic differential delay equations with Markovian switching, Stoch. Anal. Appl. 21 (2003) 819–847.
[9] Q. Luo, X. Mao, Y. Shen, New criteria on exponential stability of neutral stochastic differential equations, Systems Control Lett. 55 (2006) 826–834.
[10] W. Li, H. Su, K. Wang, Global stability analysis for stochastic coupled systems on networks, Automatica 47 (2011) 215–220.
[11] Marija Milošvić, Almost sure exponential stability of solutions to highly nonlinear neutral stochastic differential equations with time-dependent delay and the Euler–Maruyama approximation, Math. Comput. Modelling 57 (2013) 887–899.
[12] X. Mao, Exponential stability of stochastic delay interval systems with Markovian switching, IEEE Trans. Automat. Control 47 (2002) 1604–1612.
[13] X. Mao, Y. Shen, C. Yuan, Almost surely asymptotic stability of neutral stochastic differential delay equations with Markovian switching, Stochastic Process. Appl. 118 (2008) 1385–1406.
[14] X. Mao, Y. Shen, A. Gray, Almost sure exponential stability of backward Euler–Maruyama discretizations for hybrid stochastic differential equations, J. Comput. Appl. Math. 235 (2011) 1213–1226.
[15] X. Mao, C. Yuan, Stochastic Differential Equations with Markovian Switching, Imperial College Press, 2006.
[16] S. Xu, T. Chen, J. Lam, Robust H_∞ filtering for uncertain Markovian jump systems with mode-dependent time delays, IEEE Trans. Automat. Control 48 (2003) 900–908.
[17] S. Xu, J. Lam, D.W.C. Ho, Delay-dependent asymptotic stability of neural networks with time-varying delays, Internat. J. Bifur. Chaos 18 (2008) 245–250.
[18] S. Xu, J. Lam, C. Yang, Robust H_∞ control for uncertain linear neutral delay system, Optimal Control Appl. Methods 23 (2002) 113–123.
[19] D. Yue, Q. Han, Delay-dependent exponential stability of stochastic systems with time-varying delay, nonlinearity, and Markovian switching, IEEE Trans. Automat. Control 50 (2005) 217–222.
[20] C. Yuan, X. Mao, Robust stability and controllability of stochastic differential delay equations with Markovian switching, Automatica 40 (2004) 343–354.
[21] C. Yuan, J. Zou, X. Mao, Stability in distribution of stochastic differential delay equations with Markovian switching, Systems Control Lett. 5 (2003) 195–207.
[22] B. Zhang, S. Xu, G. Zong, Y. Zou, Delay-dependent exponential stability for uncertain stochastic Hopfield neural networks with time-varying delays, IEEE Trans. Circuits Syst. I. Regul. Pap. 56 (2009) 1241–1247.
[23] Y. Kao, C. Wang, Global stability analysis for stochastic coupled reaction–diffusion systems on networks, Nonlinear Anal. RWA 14 (2013) 1457–1465.
[24] C. Yuan, J. Lygeros, Stabilization of a class of stochastic differential equations with Markovian switching, Systems Control Lett. 54 (2005) 819–833.
[25] X. Mao, G. Yin, C. Yuan, Stabilization and destabilization of hybrid systems of stochastic differential equations, Automatica 43 (2007) 264–273.
[26] X. Mao, J. Lam, L. Huang, Stabilisation of hybrid stochastic differential equations by delay feedback control, Systems Control Lett. 57 (2008) 927–953.
[27] F. Deng, Q. Luo, X. Mao, Stochastic stabilization of hybrid differential equations, Automatica 48 (2012) 2321–2328.
[28] W. Chen, W. Zheng, Y. Shen, Delay-dependent stochastic stability and H_∞-control of uncertain neutral stochastic systems with time delay, IEEE Trans. Automat. Control 54 (2009) 1660–1667.
[29] B. Zhang, S. Xu, Y. Zou, Output feedback stabilization for delayed large-scale stochastic systems with Markovian jumping parameters, Asian J. Control 11 (2009) 457–460.
[30] L. Shaikhet, Lyapunov Functionals and Stability of Stochastic Functional Differential Equations, Springer, 2013.