ELSEVIER

Topology and its Applications 87 (1998) 105–115

TOPOLOGY AND ITS APPLICATIONS

On the discrete Conley index in the invariant subspace ☆

Andrzej Szymczak [a,*], Klaudiusz Wójcik [b], Piotr Zgliczyński [b]

[a] Georgia Institute of Technology, School of Mathematics, Atlanta, GA 30332-0160, USA
[b] Jagiellonian University, Institute of Mathematics, Reymonta 4, 30-059 Kraków, Poland

Received 15 November 1996; revised 28 April 1997

Abstract

We present theorems concerning the relations between the discrete homotopy Conley index in the affine invariant subspace and the index calculated in the entire space or in the half space. Our basic tool is the notion of a representable index pair, i.e., an index pair composed of hypercubes. As an application we prove an index theorem for homeomorphisms $f:\mathbb{R}^{n-1}\times[0,+\infty)\to\mathbb{R}^{n-1}\times[0,+\infty)$ of compact attraction. © 1998 Elsevier Science B.V.

Keywords: Discrete Conley index; Fixed point index; Permanence

AMS classification: 58G10; 54H20

0. Introduction

In this paper we study continuous maps $f:\mathbb{R}^{n-1}\times[0,+\infty)\to\mathbb{R}^{n-1}\times[0,+\infty)$ such that $E=\mathbb{R}^{n-1}\times\{0\}$ is invariant under f. Let $S\subset E$ be an isolated invariant set for such an f. Then S is also an isolated invariant set with respect to $f|_E$. We investigate the relationship between the Conley indices of S with respect to f and $f|_E$. We show that if the unstable set of S is contained in E then these indices are the same and if the stable set of S is contained in E then the index with respect to f is trivial. In mathematical ecology a map f is said to be permanent (or uniformly persistent) when E is a repeller (see [6,7]). Our Theorem 2 implies that if the index of S with respect to f is nontrivial then there exists an $x\in\mathbb{R}^n\times(0,\infty)$ such that $f^n(x)$ tends to E. In particular if such a set S exists then f is not permanent. In the context of flows the results in this direction were obtained in [3,20] and [21]. It is well known [2] that if $x\in E$ is an isolated fixed

☆ Research supported by Polish Scientific Grant no 0449/P3/94/06 and by the KBN grant 2 P03A 040 10.

* Corresponding author. E-mail: andrzej@math.gatech.edu.

PII S0166-8641(97)00164-8

point of f then the fixed point index of x with respect to f and the fixed point index with respect to $f|_E$ can be arbitrary integers, with the values unrelated to each other even for C^∞ maps. Using Mrozek's result [11] we obtain the relationship between the fixed point indices of S with respect to f and $f|_E$ under the assumption that S is of attracting or repelling type. As an application we prove a fixed point index theorem for homeomorphisms of compact attraction.

1. Main results

Let (X, ρ) be a metric space, $f: X \to X$ a continuous map and $K \subset X$. We define the unstable and stable sets of K by

$$W^u(K) = \big\{x \in X\colon \text{ there exists } \sigma: \mathbb{Z}^- \to X \text{ such that } \sigma(0) = x,\ f\big(\sigma(i-1)\big) = \sigma(i) \text{ for } i \in \mathbb{Z}^- \text{ and } \lim_{i \to -\infty} \rho\big(\sigma(i), K\big) = 0\big\}.$$

$$W^s(K) = \big\{x \in X\colon \lim_{i \to \infty} \rho\big(f^i(x), K\big) = 0\big\}.$$

For an isolated invariant set S with respect to f the discrete Conley index of S will be denoted by $h(S, f, X)$ (the next section explains what we mean by the Conley index). Fix $n \in \mathbb{N}$ and let $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times [0, \infty)$ and $E = \mathbb{R}^{n-1} \times \{0\}$.

In Section 4 we shall prove the following two theorems

Theorem 1. *Suppose that $f: X \to X$ ($X = \mathbb{R}^n_+$ or $\mathbb{R}^n$) is a continuous map such that $f(E) \subset E$ and $S \subset E$ is an isolated invariant set for f such that $W^u(S) \subset E$. Then $h(S, f, X) = h(S, f|_E, E)$.*

Theorem 2. *Suppose that $f: \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is a continuous map such that $f(E) \subset E$ and $S \subset E$ is an isolated invariant set for f satisfying $W^s(S) \subset E$. Then $h(S, f, \mathbb{R}^n_+) = 0$.*

2. Preliminaries

$\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}^-$ and $\mathbb{N}$ will denote the sets of real, integer, nonnegative integer, nonpositive integer and natural numbers, respectively. We shall consider sequences indexed by either $\mathbb{N}$ or $\mathbb{Z}$. Sequences of the latter type will be referred to as double-sided sequences.

For a pair $P = (P_1, P_2)$ of compact sets, by P_1/P_2 we denote the pointed space resulting from P_1 when the points of P_2 are identified to a single distinguished point, denoted by $[P_2]$. $\mathcal{H}top$ will stand for the homotopy category of pointed topological spaces. For a basepoint preserving map g its homotopy class will also be denoted by g. This ambiguity should not cause misunderstanding.

Let X denote a fixed locally compact metric space and f be a continuous map of X into itself. For a given the set $N \subset X$, by $\operatorname{Inv}^+_f N$, $\operatorname{Inv}^-_f N$ and $\operatorname{Inv}_f N$ we shall denote the positively invariant, negatively invariant and invariant part of N relative to f, respectively, defined by (see [15, Section 3])

$$\begin{aligned}
\operatorname{Inv}_f^+ N &= \{x \in X:\ f^i(x) \in N \text{ for all } i \in \mathbb{Z}^+\},\\
\operatorname{Inv}_f^- N &= \{x \in X:\ \text{there exists } \sigma:\mathbb{Z}^- \to N \text{ such that}\\
&\qquad \sigma(0) = x \text{ and } f(\sigma(i-1)) = \sigma(i) \text{ for } i \in \mathbb{Z}^-\},\\
\operatorname{Inv}_f N &= \operatorname{Inv}_f^+ N \cap \operatorname{Inv}_f^- N.
\end{aligned}$$

A set S is said to be *invariant* if $f(S) = S$. This is easily seen to be equivalent to $S = \operatorname{Inv}_f S$. S is called *an isolated invariant set* if it admits a compact neighborhood N such that $S = \operatorname{Inv}_f N$. Such a neighborhood N is then called *an isolating neighborhood of* S.

Now, let us recall the definition of the *category of objects equipped with a morphism* (see [18] for details) over a given category $\mathcal{K}$, denoted by $\mathcal{K}_m$. Put

$$Ob(\mathcal{K}_m) = \{(X,\alpha):\ X \in Ob(\mathcal{K}) \text{ and } \alpha \in Mor_{\mathcal{K}}(X,X)\}$$

and

$$Mor_{\mathcal{K}_m}((X,\alpha),(X',\alpha')) = M((X,\alpha),(X',\alpha'))/\equiv$$

where

$$M((X,\alpha),(X',\alpha')) = \{\beta \in Mor_{\mathcal{K}}(X,X'):\ \beta\circ\alpha = \alpha'\circ\beta\} \times \mathbb{Z}^+$$

and $\equiv$ is the equivalence relation in the above set defined by

$$(\beta,d) \equiv (\overline{\beta},\overline{d}) \Leftrightarrow \exists_{k\in\mathbb{Z}^+}\beta\circ\alpha^{\overline{d}+k} = \overline{\beta}\circ\alpha^{d+k}.$$

The morphism represented by $(\beta,d) \in M((X,\alpha),(X',\alpha'))$ will be denoted by $[\beta,d]$. The composition of morphisms in $\mathcal{K}_m$ is defined by

$$[\beta',d']\circ[\beta,d] = [\beta'\circ\beta, d'+d].$$

In the sequel we shall use the notation $[X,\alpha]$ for the class of all objects in $\mathcal{K}_m$ isomorphic to an object (X,α).

Any morphism in $\mathcal{H}top$ being the homotopy class of a constant map will be denoted by 0. The trivial isomorphism class in $\mathcal{H}top_m$ will be denoted in the same way, i.e., we put $0 = [C,0]$ where C is any pointed space. This class is independent of the choice of C. This notation is ambiguous, but it will always be clear from context what is meant by 0. We note that $[X,\alpha] = 0$ if and only if $\alpha^m = 0$ for some $m \in \mathbb{Z}^+$.

In order to assign an index to each isolated invariant set, we need the notion of an index pair.

Definition 1. A compact pair $P = (P_1,P_2)$ of subsets of N will be called *an index pair* for an isolated invariant set S (with respect to f) iff the following three conditions are satisfied.

(i) $S = \operatorname{Inv}_f \operatorname{cl}(P_1 \setminus P_2) \subset \operatorname{int}(P_1 \setminus P_2)$,
(ii) $f(P_1 \setminus P_2) \subset P_1$,
(iii) $f(P_2) \cap P_1 \subset P_2$.

For an index pair $P = (P_1, P_2)$, f induces the continuous map $f_P : P_1/P_2 \to P_1/P_2$ which will be called *the index map* (cf. [10,16,18]). The (homotopy) *Conley index* of S, denoted by $h(S, f, X)$ is defined as the class of all objects in the $\mathcal{H}top_m$ category isomorphic to $(P_1/P_2, f_P)$. This class can be shown to be independent of the choice of an index pair P for S.

In the sequel we shall use the following simple

Proposition 3. *Let (Y, g) be an object in $\mathcal{H}top_m$, Y' a subspace of Y containing the distinguished point and $g' : Y \to Y$ a continuous map. Assume that $g(Y') \subset Y'$, $g'(Y) \subset Y'$ and, for some $m \in \mathbb{Z}^+$, g^m is homotopic to g' relative to Y'. Then, the objects (Y, g) and $(Y', \bar{g})$ are isomorphic in $\mathcal{H}top_m$, where $\bar{g} : Y' \to Y'$ denotes the restriction of g to Y'.*

Proof. Consider the following two morphisms in $\mathcal{H}top_m$:

$$[i, 0] : (Y', \bar{g}) \to (Y, g),$$
$$[\bar{g}' \circ g^m, 2m] : (Y, g) \to (Y', \bar{g}),$$

where $i : Y' \to Y$ is the inclusion map and $\bar{g}' : Y \to Y'$ is the restriction of g'. The latter morphism is well-defined since, by our assumptions on g and g',

$$\begin{aligned}(\bar{g}' \circ g^m) \circ g &= \bar{g}' \circ g \circ g^m \sim \bar{g}' \circ g \circ g' = \bar{g}^m \circ \bar{g} \circ \bar{g}' \\ &= \bar{g} \circ \bar{g}^m \circ \bar{g}' = \bar{g} \circ \bar{g}' \circ g' \sim \bar{g} \circ (\bar{g}' \circ g^m).\end{aligned}$$

The two morphisms turn out to be reciprocal isomorphisms, since

$$i \circ \bar{g}' \circ g^m = g' \circ g^m \sim g^{2m} \quad \text{and} \quad \bar{g}' \circ g^m \circ i = \bar{g}^{2m}. \qquad \Box$$

3. Existence of representable index pairs

Let n be a fixed natural number. For an $\varepsilon > 0$ we define a family Ω_ε of subsets of $\mathbb{R}^n$ by

$$\Omega_\varepsilon = \left\{ \prod_{i=1}^{n} \left[k_i\varepsilon, (k_i + 1)\varepsilon\right]:\ k_i \in \mathbb{Z} \text{ for } i = 1, 2, \ldots, n \right\}.$$

In what follows we shall mainly be concerned with subsets of $\mathbb{R}^n$ being unions of elements of Ω_ε. Such sets will be called *ε-representable* (or just *representable* if ε is of no importance). Similarly, a pair of sets will be called *representable* if and only if both of its components are ε-representable for some positive ε. For an ε-representable set R we put

$$R_\varepsilon = \{K \in \Omega_\varepsilon:\ K \subset R\}.$$

Note that each ε-representable set R is ε/M-representable for any $M \in \mathbb{N}$, so that the set $R_{\varepsilon/M}$ is also well-defined. For a set $A \subset \Omega_\varepsilon$ by $|A|$ we denote the union of all elements of A. The aim of this section is to prove the following

Lemma 4. *Let f be a continuous map of a representable subset R of $\mathbb{R}^n$ into itself. Any isolated invariant set for f admits a representable index pair.*

Although a proof is essentially contained in [19], we give it in a detailed way for the sake of completeness. An alternative proof can be based on [8] using the method of [17, Lemma 5.1].

Proof. In what follows, by int, bd and cl we denote the interior, boundary and closure relative to R. Assume R is ε-representable. For any $M \in \mathbb{N}$ let the function $\mathcal{F}_M : R_{\varepsilon/M} \to 2^{R_{\varepsilon/M}}$ be defined by

$$\mathcal{F}_M(K) = \{L \in R_{\varepsilon/M}\colon\ L \cap f(K) \neq \emptyset\}$$

for all $K \in R_{\varepsilon/M}$. For a set $D \subset R_{\varepsilon/M}$ we put

$$\operatorname{Inv}_{\mathcal{F}_M} D = \{K \in \Omega_{\varepsilon/M}\colon \text{ there exists a double-sided sequence } \{K_i\} \subset D \text{ such that } K_0 = K \text{ and } K_{i+1} \in \mathcal{F}_M(K_i) \text{ for all } i \in \mathbb{Z}\}.$$

Let N be an isolating neighborhood for an invariant set S. Without the loss of generality we can assume that N is ε/M-representable for some $M \in \mathbb{N}$. Let us begin with showing that, for some $k \in \mathbb{N}$,

$$|\operatorname{Inv}_{\mathcal{F}_{kM}} N_{\varepsilon/(kM)}| \cup f\big(|\operatorname{Inv}_{\mathcal{F}_{kM}} N_{\varepsilon/(kM)}|\big) \subset \operatorname{int} N. \tag{1}$$

We show (1) by contradiction. Assume it does not hold for any k. Then, for any $k \in \mathbb{N}$, there exists a double-sided sequence $\{K_i^k\} \subset N_{\varepsilon/(kM)}$ such that

$$\big(K_0^k \cup f(K_0^k)\big) \cap \operatorname{cl}(R \setminus N) \neq \emptyset \tag{2}$$

and

$$K_{i+1}^k \in \mathcal{F}_{kM}(K_i^k). \tag{3}$$

For each $i \in \mathbb{Z}$ and $k \in \mathbb{N}$ let x_i^k be any point of K_i^k. Using the compactness of N and the diagonal method one can obtain a strictly increasing sequence of natural numbers $\{k_j\}$ such that for each $i \in \mathbb{Z}$ the limit

$$x_i^* = \lim_{j \to \infty} x_i^{k_j}$$

exists. Clearly, (2) implies that either $x_0^* \in \operatorname{cl}(R \setminus N)$ or $f(x_0^*) \in \operatorname{cl}(R \setminus N)$. Moreover, by (3), $x_{i+1}^* = f(x_i^*)$. We conclude that either x_0^* or x_1^* belongs to $\operatorname{Inv}_f N \cap \operatorname{cl}(R \setminus N)$, which results in a contradiction since N was assumed to be an isolating neighborhood and $x_i^* \in N$ for each i. Hence (1) is proved.

Let $\delta = \varepsilon/(kM)$. At this point we are able to give a formula for a δ-representable index pair for S. Let:

$$A = \operatorname{Inv}_{\mathcal{F}_{kM}} N_\delta,$$
$$B = \{K \in R_\delta\colon\ K \in \mathcal{F}_{kM}(L) \text{ for some } L \in A\}.$$

We will finish the proof by showing that the pair

$$(P_1, P_2) = (|B|, |B \setminus A|)$$

is an index pair for S. We note that, by (1), $P_1 \subset N$.

To begin with, notice that $\operatorname{cl}(P_1 \setminus P_2) = |A| \subset N$, so that $\operatorname{Inv}_f \operatorname{cl}(P_1 \setminus P_2) \subset S$. On the other hand, for any $x \in S$ there exists a double-sided sequence $\{x_i\} \subset N$ such that $x_0 = x$ and $f(x_i) = x_{i+1}$ for all $i \in \mathbb{Z}$. Let K_i be any element of N_δ containing x_i. Then, $K_{i+1} \in \mathcal{F}_{kM}(K_i)$ for all i. Therefore, $K_0 \in A$. To sum up, any $K \in N_\delta$ containing x belongs to A. Therefore, $x \in \operatorname{int}|A|$, so that $S \subset \operatorname{int}|A|$ and the condition (i) in the definition of the index pair follows.

In order to show (ii), notice that for any $x \in P_1 \setminus P_2 \subset |A|$ there is a $K \in A$ such that $x \in K$. By the definition of B and $\mathcal{F}_{kM}$,

$$f(x) \in \operatorname{int}|\mathcal{F}_{kM}(K)| \subset |B| = P_1$$

and (ii) follows.

It remains to prove (iii). Take any $K \in B \setminus A$. We begin with showing that $\mathcal{F}_{kM}(K) \cap A = \emptyset$. Since $K \in B$, there is a $K' \in A$ with $K \in \mathcal{F}_{kM}(K')$. Assume there is a $K'' \in \mathcal{F}_{kM}(K) \cap A$. The definition of A implies the existence of double-sided sequences $\{K_i'\}$ and $\{K_i''\}$ of elements of N_δ satisfying the following conditions: $K_0' = K'$, $K_0'' = K''$, $K_{i+1}' \in \mathcal{F}_{kM}(K_i')$ and $K_{i+1}'' \in \mathcal{F}_{kM}(K_i'')$ for all $i \in \mathbb{Z}$. Define the double-sided sequence $\{K_i\} \subset N_\delta$ by

$$K_i = \begin{cases} K_{i+1}' & \text{for } i \in \mathbb{Z} \setminus \mathbb{Z}^+, \\ K_{i-1}'' & \text{for } i \in \mathbb{N}, \\ K & \text{for } i = 0. \end{cases}$$

Clearly, $K_{i+1} \in \mathcal{F}_{kM}(K_i)$ for all $i \in \mathbb{Z}$. Hence $K_0 = K \in \operatorname{Inv}_{\mathcal{F}_{kM}} N_\delta = A$, which is a contradiction. Therefore, the following condition, even stronger that (iii), holds

$$\begin{aligned} f(P_2) \cap \operatorname{cl}(P_1 \setminus P_2) &\subset |A| \cap f(|B \setminus A|) \\ &\subset \bigcup_{K \in B \setminus A} |A| \cap \operatorname{int}|\mathcal{F}_{kM}(K)| = \emptyset. \qquad \square \end{aligned} \tag{4}$$

4. Proofs of the main theorems

Proof of Theorem 1. Let $f\colon X \to X$ be a continuous map such that $f(E) \subset E$, where $X = \mathbb{R}^n_+$ or $\mathbb{R}^n$. Let S be an isolated invariant set for f such that $W^u(S) \subset E$. By Lemma 4 there exists an ε-representable index pair $P = (P_1, P_2)$ for S for some $\varepsilon > 0$. Let $Y = P_1/P_2$, $g = f_P$,

$$Y_0 = \{[x] \in Y\colon\ x \in (\mathbb{R}^{n-1} \times [-\varepsilon/2, \varepsilon/2]) \cap P_1\} \cup \{[P_2]\},$$
$$Y' = \{[x] \in Y\colon\ x \in E \cap P_1\} \cup \{[P_2]\}.$$

Note that the distinguished point is added in an artificially-looking way in order to deal properly with the case of P_2 being disjoint with E. Notice that by ε-representability of P, Y' is a strong deformation retract of Y_0 with a deformation $r : Y_0 \times [0,1] \to Y$ given by

$$r([x_1, x_2, \dots, x_n], t) = [x_1, x_2, \dots, x_{n-1}, (1-t)x_n].$$

Since $W^u(S) \subset E$, there is an $m \in \mathbb{N}$ such that $g^m(Y) \subset Y_0$. To see this, for $k \in \mathbb{N}$ define

$$F_k = \{x \in X\colon \text{ there is a } y \in X \text{ such that } f^i(y) \in \mathrm{cl}(P_1 \setminus P_2) \\ \text{for } i = 0, 1, \dots, k \text{ and } f^k(y) = x\}.$$

Clearly,

$$\bigcap_{k \in \mathbb{N}} F_k = \mathrm{Inv}_f^- \, \mathrm{cl}(P_1 \setminus P_2) \subset W^u(S) \subset E.$$

Since $\{F_k\}$ is a decreasing sequence of compact sets, there is an $m \in \mathbb{N}$ such that $F_m \subset \mathbb{R}^{n-1} \times [-\varepsilon/2, \varepsilon/2]$, which results in $g^m(Y) \subset Y_0$. Now, one can apply Proposition 3 with $g' = r_1 \circ g^m$, where $r_1 : Y_0 \to Y$ is defined by $r_1(y) = r(y, 1)$, to show that (Y, g) and $(Y', \bar{g})$ ($\bar{g}$ is the restriction of g) are isomorphic in $\mathcal{H}top_m$. Hence,

$$h(S, f, X) = [Y, g] = [Y', \bar{g}] = h(S, f|_E, E). \qquad \square$$

Before we prove Theorem 2 let us make one remark. At first look Theorem 2 appears to be essentially obvious. Why should not one just check that an arbitrarily small perturbation of the form $f_\varepsilon(x) = f(x) + \varepsilon e_n$, where e_n is the normal vector of the boundary, destroys the isolated invariant set S and use the continuation invariance of the Conley index? However, there is an error in this reasoning: we cannot be sure that the invariant set for the perturbed map is indeed an empty set.

Proof of Theorem 2. Let $f : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be a continuous map leaving E invariant and S be an isolated invariant set of f such that $W^s(S) \subset E$. Let $P = (P_1, P_2)$ be an ε-representable index pair for S. Put $Y = P_1/P_2$, $g = f_P$, $y_0 = [P_2] \in Y$ and

$$Y_* = \{[x] \in Y\colon x \in (\mathbb{R}^{n-1} \times [\varepsilon/2, \infty)) \cap P_1\} \cup \{[P_2]\}.$$

One easily sees that Y_* is mapped by g^m into the distinguished point for some $m \in \mathbb{N}$ (if it was not, then for each m there would exist an $x_m \in \mathbb{R}^{n-1} \times [\varepsilon/2, \infty)$ with $f^i(x_m) \in P_1 \setminus P_2$ for $i = 0, 1, \dots, m$. Clearly, any accumulation point of the $\{x_m\}$ sequence is an element of $\mathrm{Inv}_f^+ \, \mathrm{cl}(P_1 \setminus P_2) \subset W^s(S)$ and, at the same time, is not contained in E. This is a contradiction.) Notice that Y_* is a strong deformation retract of Y. Indeed, a deformation $r : Y \times [0,1] \to Y$ can be defined by

$$r([x_1, x_2, \dots, x_n], t) = [x_1, x_2, \dots, x_{n-1}, \max(x_n, t\varepsilon/2)].$$

Let $\bar{r} = r(\cdot, 1)$. Since $g^m \circ \bar{r}(Y) = g^m(Y_*) = \{y_0\}$,

$$g^{m+1} = g^m \circ g \sim g^m \circ \bar{r} \circ g = 0.$$

Thus, $h(S, f, X) = [Y, g] = 0. \quad \square$

5. An application

Let us recall the relationship between the Conley index and the fixed point index first. In what follows, by H^* we shall denote a cohomology functor with coefficients in the field of rational numbers. An isomorphism class of objects in $\mathcal{H}top_m$ is said to be of *finite type* if and only if it for any of its representatives (or equivalently, for one of its representatives) (Y, g) there exists a $k \in \mathbb{N}$ such that the graded vector subspace $\operatorname{Im} H^*(g)^k$ of $H^*(Y)$ is of finite type (cf. [17, Section 4]). For such a class $\mathcal{I}$ we define its *Lefschetz number*, denoted by $\Lambda(\mathcal{I})$ by

$$\Lambda(\mathcal{I}) = \Lambda(g_0^*) = \sum_{q=0}^{\infty} (-1)^q \operatorname{tr} g_0^q,$$

where by g_0^* we denote the graded endomorphism of $\operatorname{Im} H^*(g)^k$ being the restriction of $H^*(g)$. It can be shown that the right-hand side does not depend on the choice of (Y, g) and k. The following theorem, which is a very special case of [11, Theorem 4], can easily be obtained as a corollary of Lemma 4.

Theorem 5. *If f is a continuous map of a representable set $R \subset \mathbb{R}^n$ into itself and S is an isolated invariant set for f then $h(S, f, R)$ is of finite type and*

$$\operatorname{ind}(f, S) = \Lambda\big(h(S, f, R)\big),$$

where by $\operatorname{ind}(f, S)$ *we denote the fixed point index of f on the interior of any isolating neighborhood of S.*

Proof (*Sketch*). Take a representable index pair $P = (P_1, P_2)$ for S, constructed in Lemma 4. Since $h(S, f, R) = [P_1/P_2, f_P]$ and $H^*(P_1/P_2)$ is of finite type, the index is of finite type.

Consider the semidynamical system $\overline{f}_P$ on P_1/P_2 treated as a topological space, not a pointed space ($\overline{f}_P$ is induced by the index map f_P). Let $\pi : P_1 \to P_1/P_2$ be the projection map, $\pi(x) = [x]$. Clearly, $\pi(S)$ is an isolated invariant set for $\overline{f}_P$ and, since the dynamics of f around S and the dynamics of $\overline{f}_P$ around $\pi(S)$ are topologically equivalent (π is a local conjugacy on a small neighborhood of S),

$$\operatorname{ind}\big(\overline{f}_P, \pi(S)\big) = \operatorname{ind}(f, S).$$

Moreover, (4) implies that the distinguished point $[P_2]$ has a neighborhood which is mapped by $\overline{f}_P$ into $[P_2]$. Therefore, it is an isolated fixed point of index 1. We conclude that

$$\begin{aligned}\operatorname{ind}(f, S) &= \operatorname{ind}\big(\overline{f}_P, \pi(S)\big) = \operatorname{ind}(\overline{f}_P, P_1/P_2) - \operatorname{ind}\big(\overline{f}_P, [P_2]\big)\\ &= \Lambda\big(H^*(\overline{f}_P)\big) - 1 = \Lambda\big(H^*(f_P)\big) = \Lambda\big(h(S, f, R)\big). \qquad \square\end{aligned}$$

For the rest of this section, fix a continuous map f of $\mathbb{R}^n_+$ into itself such that $f(E) \subset E$. An isolated invariant set $S \subset E$ will be called *of attracting type* if $W^u(S) \subset E$ and *of* *repelling type* if $W^s(S) \subset E$. As an immediate consequence of Theorems 1, 2 and 5 we obtain the following

Corollary 6. *If S is of attracting type then* $\operatorname{ind}(f, S) = \operatorname{ind}(f|_E, S)$. *If it is of repelling type then* $\operatorname{ind}(f, S) = 0$.

Recall that the map f is *of compact attraction* if and only if there exists a compact set A such that for every $x \in \mathbb{R}^n_+$ the ω-limit set of x intersects A. For such an f, the set $\operatorname{Fix}(f)$ of all its fixed points can be shown to be compact and of index 1 (the second statement is a special case of the normalization property of the fixed point index proven in [5]). Also, $\operatorname{Fix}(f) \cap E$, being the set of all fixed points of $f|_E$, is compact and of index 1 with respect to $f|_E$.

Now we are ready to prove the main result of this section.

Theorem 7. *Assume that f is of compact attraction and*

$$\operatorname{Fix}(f) \subset A \cup R \cup K,$$

where A, R, K are pairwise disjoint isolated invariant sets, $A \subset E$ is of attracting type, $R \subset E$ is of repelling type and $K \cap E = \emptyset$. Then

$$\operatorname{ind}(f, K) + \operatorname{ind}(f|_E, A) = 1$$

and

$$\operatorname{ind}(f, K) = \operatorname{ind}(f|_E, R).$$

Proof. Additivity of the fixed point index yields:

$$1 = \operatorname{ind}(f, K) + \operatorname{ind}(f, A) + \operatorname{ind}(f, R)$$

and

$$1 = \operatorname{ind}(f|_E, A) + \operatorname{ind}(f|_E, R).$$

By Corollary 6

$$\operatorname{ind}(f|_E, A) = \operatorname{ind}(f, A)$$

and

$$\operatorname{ind}(f, R) = 0.$$

Obviously, Theorem 7 follows from these identities. □

Remark 8. The same problem for flows on $\mathbb{R}^n_+$ was first investigated by A. Capietto and B.M. Garay [3]. Their approach works only for flows induced by a vector field and some special isolated invariant sets called saturated. By application of the time duality of the Conley index [14], results of [20] extend work Capietto and Garay to the case of any continuous flow and attracting and repelling type sets. Actually, Corollary 15 and Theorem 16 in [20] is a special case of our Theorem 7 and Corollary 6, by results of [12].

Theorem 9. *Let f be as above. Let S be an isolated invariant set (not necessarily contained in E). Suppose that $h(S, f, \mathbb{R}^n_+) \neq 0$. Then either there exists $x \in \mathbb{R}^n_+ \setminus E$ such that $\omega(x) \cap E \neq \emptyset$ or there exists a compact invariant set S_1 which is contained in $\mathbb{R}^n_+ \setminus E$.*

Proof. Suppose that

$$\forall x \in \mathbb{R}^n_+ \setminus E \quad \omega(x) \cap E = \emptyset. \tag{5}$$

We have to show that there exists compact invariant set S_1 such that $S_1 \cap E = \emptyset$. Suppose that $S \subset E$. From (5) follows that $W^s(S) \subset E$. From this and Theorem 2 we obtain $h(S, f, \mathbb{R}^n_+) = 0$, which is in contradiction with our assumptions. So we proved that S is not contained in E.

Let us take $x \in S \setminus E$. Obviously $S_1 := \omega(x)$ is nonempty compact invariant set. From (5) follows that $S_1 \cap E = \emptyset$. This finishes the proof. □

References

[1] C. Bowszyc, Fixed point theorem for the pairs of spaces, Bull. Acad. Polon. Sci. 16 (11) (1968) 845–850.
[2] R.F. Brown, R.E. Greene and H. Schirmer, Fixed points of maps extensions, Lecture Notes in Math. 1411 (Springer, Berlin) 24–45.
[3] A. Capietto, B.M. Garay, Saturated invariant sets and boundary behaviour of differential systems, J. Math. Anal. Appl. 176 (1993) 166–181.
[4] A. Dold, Lectures on Algebraic Topology (Springer, Berlin, 1972).
[5] C.C. Fenske and H.O. Peitgen, On fixed points of zero index in asymptotic fixed point theory, Pacific J. Math. 66 (2) (1976) 391–410.
[6] J. Hofbauer and K. Sigmund, The Theory of Evolution and Dynamical Systems (Cambridge Univ. Press, Cambridge, 1988).
[7] J. Hofbauer and J.W.H. So, Uniform persistence and repellers for maps, Proc. Amer. Math. Soc. 107 (1989) 1137–1142.
[8] T. Kaczyński and M. Mrozek, Conley index for discrete multivalued dynamical systems, Topology Appl. 65 (1995) 83–96.
[9] M. Mrozek, Leray functor and cohomological Conley for discrete dynamical systems, Trans. Amer. Math. Soc. 318 (1990) 149–178.
[10] M. Mrozek, Shape index and other indices of Conley type for local maps on locally compact Hausdorff spaces, Fund. Math. 145 (1994) 15–37.
[11] M. Mrozek, Index pairs and the fixed point index for semidynamical systems with discrete time, Fund. Math. 133 (1989) 179–194.
[12] M. Mrozek, The Conley index on compact ANR's is of finite type, Results in Math. 18 (1990) 306–313.
[13] M. Mrozek, The Morse equation in Conley's index theory for homeomorphisms, Topology Appl. 38 (1991) 45–60.
[14] M. Mrozek and R. Srzednicki, Time duality of the Conley index, Results in Math. 24 (1993) 161–167.
[15] M. Mrozek and K. Rybakowski, A cohomological Conley index for maps on metric spaces, J. Differential Equations 90 (1991) 143–171.

[16] J.W. Robbin and D. Salamon, Dynamical systems, shape theory and the Conley index, Ergodic Theory Dynamical Systems 8* (1988) 375–393.
[17] A. Szymczak, The Conley index and symbolic dynamics, Topology 35 (1996) 287–299.
[18] A. Szymczak, The Conley index for discrete semidynamical systems, Topology Appl. 66 (1995) 215–240.
[19] A. Szymczak, A combinatorial procedure for finding isolating neighborhoods and index pairs, Proc. Royal Soc. Edinburgh, to appear.
[20] K. Wójcik, An attraction result and the index theorem for continuous flow on $R^n \times [0, +\infty)$, Ann. Polon. Math., to appear.
[21] K. Wójcik, Conley index and permanence in dynamical systems, to appear.