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Exact spin–spin correlation functions of Bethe lattice Ising and BEG models in external fields

N.Sh. Izmailian [1], Chin-Kun Hu *

Institute of Physics, Academia Sinica, Nankang, Taipei 11529, Taiwan

Abstract

We develop a transfer matrix method to compute exactly the spin–spin correlation functions for Bethe lattice Ising model and Blume–Emery–Griffiths (BEG) model in the external magnetic field h and for any temperature T. The correlation length $\xi(T,h)$ obtained from the spin–spin correlation function shows interesting scaling and divergent behavior as $h \to 0$ and T approaches the critical temperature T_c. © 1998 Elsevier Science B.V. All rights reserved

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1. Introduction

In critical phenomena, the correlation function contains important information about a phase transition system [1] and is often studied by theoretical calculations [2–4] and experimental measurements [5–7]. It is widely believed that the singular behavior of physical quantities at the critical temperature T_c of a second-order phase transition is related to the divergence of the correlation length ξ at T_c [1]. However, the nontrivial spin–spin correlation functions are exactly known only in a few models, including the two-dimensional Ising model in zero magnetic field and at any temperatures [3], the planar Ising model in a magnetic field and at T_c [4]. A long-standing problem of statistical mechanics is the exact solution of the spin–spin correlation function for a phase transition model in an external magnetic field and at any temperature T. In this paper, we develop a transfer matrix method to compute exactly the spin-spin correlation functions $\langle s_0 s_n \rangle$ for Bethe lattice Ising model [8] and Blume–Emery–Griffiths (BEG) model [9] in the external magnetic field h and for any temperature T. The correlation length $\xi(T,h)$ obtained from the spin–spin correlation function shows interesting scaling

* Corresponding author.
[1] On leave from Yerevan Physics Institute, Yerevan, Armenia.

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and divergent behavior as $h \to 0$ and $t \to 0$, where $t = (T - T_c)/T_c$. Our results also solve a long-standing puzzle in the critical phenomena of the Bethe lattice Ising-type model.

The Cayley tree [10–14] and the Bethe lattice [15–21] have been widely used in solid state and statistical physics. The Bethe lattices have attracted particular interest because they usually reflect essential features of systems, even when conventional mean-field theories fail [20]. The difference between the Cayley tree and the Bethe lattice has been discussed by Baxter [8]. In the Cayley tree, the surface plays a very important role because the sites on the surface is a finite fraction of the total sites even in the thermodynamic limit. As a consequence, the spin models on the Cayley tree exhibits quite unusual type of phase transition [10–12] without long-range order [12–14]; the calculated correlation functions do not show singular behavior [12–14].

To overcome this problem one usually considers only properties of sites deep in the interior of the Cayley tree. The union of such equivalent sites can be regarded as forming the Bethe lattice [8]. Thus, the Bethe lattice is assumed to have translation symmetry like any regular lattices. It has been shown that the magnetization M and the magnetic susceptibility χ of the central spin s_0 of the Ising model on the Bethe lattice with coordination number q have singular behaviors near T_c and $h = 0$, where $J_c = 1/T_c = \frac{1}{2}\ln q/(q-2)$, with the critical exponents $\beta = 1/2$, $\delta = 3$, and $\gamma = 1$ [8]. However, there is no previous calculation which shows that ξ of the Bethe lattice Ising model or other Ising-type models has singular behavior.

In this paper, we demonstrate the crucial role of Bethe lattice dimensionality in determining critical behavior of the correlation length and show clearly that correlation length ξ diverges at the critical point of M and χ for the Bethe lattice Ising model with critical exponent $\nu = 1$, which is different from the mean-field critical exponent $\nu = 1/2$ considered by Tsallis and Magalhães in a recent review paper [22], but is consistent with the critical exponent of the localization length associated with density–density correlator in the Bethe lattice Anderson model obtained by a supersymmetry method [17,18]. Our result gives independent support to the idea that mean-field approximation and Bethe lattice approach are not equivalent in principle [20]. We also find that for small $h > 0$ and $t \to 0$, $\xi \sim h^{-2/3}$. In this paper we also obtain exact spin–spin correlation function for the BEG model [9] and find that at the tricritical point, $\xi \sim t^{-1/2}$ with tricritical exponent $\nu_t = 1/2$.

2. Spin-1/2 Ising model

Now we first consider the following Ising Hamiltonian on the Bethe lattice G:

$$-\beta H = J \sum_{\langle ij \rangle} s_i s_j + h \sum_i s_i \,, \tag{1}$$

where $\beta = (k_B T)^{-1}$ and $s_i = +1$ or -1. The first sum runs over all nearest-neighbor pairs of G and the second over all sites of G. The calculation on a Bethe lattice is done recursively [8]. If the Bethe lattice is "cut" in central point with a spin s_0, it will disintegrate into q identical branches and the partition function of the model $Z_N = \sum_s \exp(-\beta H)$ will take the form

$$Z_N = \sum_{s_0} e^{hs_0} g_N^q(s_0) = e^h g_N^q(+) + e^{-h} g_N^q(-)\,, \tag{2}$$

where $g_N(s_0)$ is in fact the partition function of one branch and N is number of generations. Cutting each branch in the site s_1 (nearest to the central site) we obtain

$$g_N(s_0) = \sum_{s_1} \exp(hs_1 + Js_0s_1) g_{N-1}^{q-1}(s_1)\,. \tag{3}$$

For the magnetization ($m = \langle s_0 \rangle$) of the spin in the central site we obtain

$$m = \frac{1}{Z_N} \sum_{s_0} s_0 \exp(hs_0) g_N^q(s_0) = \frac{\exp(2h) - x_N^q}{\exp(2h) + x_N^q}\,, \tag{4}$$

where $x_N = g_N(-)/g_N(+)$.

It is now convenient, for calculating the spin-spin correlation function, to write down the expression for partition function in the following form:

$$Z_N = \sum_{s_0 s_1 \ldots s_n} \exp\left(J \sum_{i=0}^{n-1} s_i s_{i+1} + h \sum_{i=0}^{n} s_i \right) \times g_N^{q-1}(s_0) g_{N-1}^{q-2}(s_1) \ldots g_{N-n+1}^{q-2}(s_{n-1}) g_{N-n}^{q-1}(s_n)\,, \tag{5}$$

where n denotes the number of steps from the central point 0. Summing over $s_n, s_{n-1}, \ldots, s_1$, we obtain Eq. (2) again. The two-spin correlation function $\Gamma(n) = \langle s_0 s_n \rangle$, can be written as

$$\Gamma(n) = \frac{\sum s_0 s_n \exp\left(J \sum_{i=0}^{n-1} s_i s_{i+1} + h \sum_{i=0}^{n} s_i \right) x_N^{q-1}(s_0) x_{N-1}^{q-2}(s_1) \ldots x_{N-n}^{q-1}(s_n)}{\sum \exp\left(J \sum_{i=0}^{n-1} s_i s_{i+1} + h \sum_{i=0}^{n} s_i \right) x_N^{q-1}(s_0) x_{N-1}^{q-2}(s_1) \ldots x_{N-n}^{q-1}(s_n)}\,, \tag{6}$$

where $x_{N-k}(s) = g_{N-k}(s)/g_{N-k}(+)$ ($x_{N-k}(+) = 1$ and $x_{N-k}(-) = x_{N-k}$).

We are interested in the case when the series of solution of recursion relations given by Eq. (3) converges to a stable point as $N \to \infty$. In this case $\lim x_{N-n}(-) = x$ for all finite n and the recursion equations becomes

$$x^{q-1} \exp(-2h) = \frac{x \exp(2J) - 1}{\exp(2J) - x}\,. \tag{7}$$

Now consider 2×2 matrices $\mathbf{V}$ and $\mathbf{S}$

$$\mathbf{V} = \begin{pmatrix} V_{++} & V_{+-} \\ V_{-+} & V_{--} \end{pmatrix}, \qquad \mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The elements of V are given by

$$V_{ss'} = \exp\left(Jss' + h\frac{s+s'}{2}\right)\left[x(s)x(s')\right]^{(q-2)/2}, \tag{8}$$

where s and s' independently take values ± 1. Now also consider the vector $\boldsymbol{R}$ and transposed vector $\boldsymbol{R}^T = (r_1, r_2)$, which have elements $r_1^2 = e^h$ and $r_2^2 = e^{-h}x^q$.

Since $\mathbf{V}$ is real-symmetric ($V_{+-} = V_{-+} = V$), it can be diagonalized by a 2×2 matrix $\mathbf{P}$ with the matrix elements $p_{ss'}$ which satisfy

$$\frac{p_{++}}{p_{-+}} = \frac{\lambda_1 - V_{--}}{V} \quad \text{and} \quad \frac{p_{--}}{p_{+-}} = \frac{\lambda_2 - V_{++}}{V}, \tag{9}$$

where λ_1 and λ_2 are the eigenvalues of $\mathbf{V}$ and are

$$\lambda_1 = 2\sinh(2J)\frac{\exp(J+h)}{(\exp(2J)-x)},$$
$$\lambda_2 = (2\cosh(2J) - x - x^{-1})\frac{\exp(J+h)}{(\exp(2J)-x)}. \tag{10}$$

The spin–spin correlation function $\Gamma(n)$ can be written as

$$\Gamma(n) = \frac{\mathbf{R}^T\mathbf{S}\mathbf{V}^n\mathbf{S}\mathbf{R}}{\mathbf{R}^T\mathbf{V}^n\mathbf{R}} = \frac{\mathbf{R}^T\mathbf{S}\mathbf{P}\begin{pmatrix}\lambda_1^n & 0\\ 0 & \lambda_2^n\end{pmatrix}\mathbf{P}^{-1}\mathbf{S}\mathbf{R}}{\mathbf{R}^T\mathbf{P}\begin{pmatrix}\lambda_1^n & 0\\ 0 & \lambda_2^n\end{pmatrix}\mathbf{P}^{-1}\mathbf{R}}. \tag{11}$$

After some algebraic manipulation, we can finally obtain

$$\Gamma(n) = m^2 + (1-m^2)\lambda^n, \tag{12}$$

where

$$\lambda = \frac{\lambda_2}{\lambda_1} = \frac{2\cosh(2J) - x - x^{-1}}{2\sinh(2J)}, \tag{13}$$

$$m = \langle s_0\rangle = \frac{x^{-1} - x}{x + x^{-1} - 2\exp(-2J)}. \tag{14}$$

Let us now consider the general behavior of $\Gamma(n)$ in the critical region. First consider the case $h=0$ and $T=T_c$. From Eq. (12), we obtain $\Gamma(n) = (q-1)^{-n}$. The dimension of the Bethe lattice is defined by $d_n = \ln C_n/\ln n$ which tends to infinity with $n\to\infty$ for $q>2$ and equals to 1 for $q=2$, where $C_n = (q(q-1)^n - 2)/(q-2)$ is the total number of sites. We should note that for $q=2$ the Bethe lattice becomes the ordinary one-dimensional chain. In the limit of large n, d_n for all $q>2$ becomes

$$d = \frac{n}{\ln n}\ln(q-1).$$

Thus we can write, for large n,

$$\Gamma(n) = (q-1)^{-n} = n^{-d}. \tag{15}$$

Near the critical point, setting as usual $t=(T-T_c)/T_c$, we write $\Gamma(n)$ as

$$\Gamma(n)=\frac{\exp(-n/\xi)}{n^d}\,, \tag{16}$$

where the correlation length ξ is given by

$$\xi=\left[\ln\frac{1}{(q-1)\lambda}\right]^{-1}=\left[\ln\left(\frac{1}{q-1}\coth\frac{J_c}{1+t}\right)\right]^{-1}\sim\frac{q-1}{q(q-2)J_c}t^{-1}\,. \tag{17}$$

Thus, we find that the correlation length ξ increases as the critical point is approached according to $\xi\sim t^{-\nu}$, with critical exponent $\nu=1$. While the Bethe lattice Ising model exhibits in general mean-field like phase transition with "classical" exponents, the critical behavior of the correlation length near the transition point coincides with the correlation length behavior in one-dimensional chain with critical exponent $\nu=1$, which differs from the "classical value" $\nu=1/2$ [1]. Similar behavior has also been observed in the localization length associated with density–density correlator in the Bethe lattice Anderson model obtained by a supersymmetry method [17,18].

If h and t are both sufficiently small and not zero in the critical region, then based on Eqs. (7), (13) and (17) the general behavior of the correlation length ξ should be described by a scaling function F,

$$\xi=t^{-1}F(ht^{-3/2})\,, \tag{18}$$

where $F(x)=\left(f_1+f_2x^{2/3}\right)^{-1}$ with

$$f_1=\frac{q(q-2)}{2(q-1)}\ln\frac{q}{q-2}\quad\text{and}\quad f_2^3=9\,\frac{q(q-2)}{(q-1)^2}\,.$$

For a small $h>0$ and $t\to 0$, Eq. (18) implies

$$\xi=f_2^{-1}h^{-2/3}\,, \tag{19}$$

i.e. the critical exponent is 2/3.

The bulk susceptibility per lattice site χ or the linear response against the field is derived from Eq. (4):

$$\begin{aligned}\chi=\frac{\partial m}{\partial h}&=\chi_0\left[1-\frac{(q-1)(2\cosh(2J)-x-x^{-1})}{2\sinh(2J)}\right]^{-1}\\&=\frac{\chi_0}{1-(q-1)\lambda}\,,\end{aligned} \tag{20}$$

where χ_0 is nonsingular part of the magnetic susceptibility and is given by

$$\begin{aligned}\chi_0&=2e^{-2J}\frac{(2\cosh(2J)-x-x^{-1})(2e^{2J}-x-x^{-1})}{\sinh(2J)(x+x^{-1}-2e^{-2J})^2}\\&=(1-m^2)(1+\lambda)\,.\end{aligned} \tag{21}$$

By means of the fluctuation relation $\chi=\sum(\Gamma(n)-m^2)$ we recover Eq. (20) from Eq. (12). From Eqs. (17) and (19) we can easily establish the relation between χ and ξ in the critical region: $\chi\sim\xi$.

3. Spin-1 Ising model

The transfer-matrix method can be extended to calculate spin–spin correlation function for other Ising-type spin models on the Bethe and Bethe-like lattices. Let us consider, for example, a spin-1 Ising model, which is known as the Blume–Emery–Griffiths (BEG) model [9] and has played an important role in the development of the theory of tricritical phenomena [23]. The BEG model on the Bethe lattice has been studied in Refs. [15,19].

The Hamiltonian of the spin-1 Ising model on the Bethe lattice is given by

$$-\beta H = J\sum_{\langle ij\rangle} s_i s_j + K\sum_{\langle ij\rangle} s_i^2 s_j^2 - \Delta\sum_i s_i^2 + h\sum_i s_i\,, \tag{22}$$

where $s_i = +1, 0, -1$ is the spin variable at site i. The first term describes the ferromagnetic coupling (J) between s_i and s_j and the second term describes the biquadratic coupling (K). Both interactions are restricted to the q nearest-neighbor pairs of spins. The third term describes the single-ion anisotropy (Δ) and the last term represents the effects of an external magnetic field (h).

We can extend the above transfer matrix method for the Ising model to the BEG model with a 3×3 transfer matrix $\mathbf{V}$. The eigenvalues of $\mathbf{V}$ are

$$\lambda_1 = \frac{b}{b-u} \quad \text{and} \quad \frac{\lambda_{2,3}}{\lambda_1} = C \pm \sqrt{C^2 - D}\,. \tag{23}$$

Here

$$C = -\frac{u}{2b} + \frac{(ab+a+b)(u^2+u-av^2)}{2ab[(u+1)^2 - v^2]}$$

and

$$D = \frac{(b-u)(u^2 - a^2v^2)}{ab[(u+1)^2 - v^2]}\,,$$

where $b = \exp(K)\cosh(J) - 1$, $a = b/(\exp(K)\sinh(J))$ and u, v are the solution of following equations:

$$\exp(2h) = \frac{u-av}{u+av}\left(\frac{u+1+v}{u+1-v}\right)^{q-1}, \tag{24}$$

$$\exp(2\Delta) = \frac{4(b-u)^2}{u^2 - a^2v^2}\left[(u+1)^2 - v^2\right]^{q-1}. \tag{25}$$

The correlation function $\Gamma(n)$ is given by

$$\Gamma(n) = m_1^2 + \frac{(m_1^2 - m_2)\lambda_3 + (m_3 - m_1^2)\lambda_1}{\lambda_2 - \lambda_3}\left(\frac{\lambda_2}{\lambda_1}\right)^n + \frac{(m_1^2 - m_2)\lambda_2 + (m_3 - m_1^2)\lambda_1}{\lambda_3 - \lambda_2}\left(\frac{\lambda_3}{\lambda_1}\right)^n, \tag{26}$$

with

$$m_1 = \langle s_0 \rangle = -v\frac{u(a+1)+a}{b+u^2+av^2}\,, \tag{27}$$

$$m_2 = \langle s_0^2 \rangle = \frac{u(u+1)+av^2}{b+u^2+av^2}\,, \tag{28}$$

$$m_3 = \langle s_0 s_1 \rangle = \frac{bu^2+a^3(b+1)v^2}{ab(b+u^2+av^2)}\,. \tag{29}$$

The global phase diagram of the Bethe lattice BEG model has been studied in detail in the Refs. [15,19]. The λ-line of the phase transition in the $(J,\,K,\,\Delta)$ space is given by following conditions:

$$h=0 \quad \text{and} \quad \exp(\Delta_\lambda) = \frac{2(b-u_c)}{u_c}(u_c+1)^{q-1}\,, \tag{30}$$

where $u_c = a/(q-1-a)$. In terms of the T, Δ_λ/J and K/J, Eq. (30) of the λ-line implies a relation $T = T_c(\Delta_\lambda/J, K/J)$ which locates the critical temperature as a function of Δ_λ/J and K/J.

The critical line starts at $\Delta_\lambda \to -\infty$, $T_c/J = \frac{1}{2}\ln q/(q-2)$, which corresponds the critical temperature of the spin-1/2 Ising model. We note that for $\Delta \to -\infty$, the state $s_i = 0$ are suppressed, and the Hamiltonian Eq. (22) reduces to the spin-1/2 Ising model with interaction J and external magnetic field h. For certain values of K/J the system possesses a tricritical point at which the phase transition changes from the second order to the first order. A tricritical point satisfies the following equation [19]:

$$\frac{u_c+1}{b-u_c} = q-2+\frac{q-3}{2qu_c}\,. \tag{31}$$

In critical region $\Gamma(n)$ has an asymptotic decay of the form

$$\Gamma(n) \sim \frac{\exp(-n/\xi)}{n^d}\,, \tag{32}$$

where ξ is correlation length and is given by

$$\xi = \left[\ln\frac{1}{(q-1)(\lambda_2/\lambda_1)}\right]^{-1}.$$

Using Taylor expansion for ξ near the λ-line, we find that ξ increases as the critical point is approached according to $\xi \sim t^{-1}$, with critical exponent $v=1$, everywhere on the λ-line except the tricritical point, where $\xi \sim t^{-1/2}$ with tricritical exponent $v_t = 1/2$.

4. Summary

Gujrati [20] has showed that in many cases the behavior of spin models on Bethe or Bethe-like lattices are qualitatively correct even when conventional mean-field theories fail. By a proper choice of these lattices, it is possible to satisfy frustrations, gauge symmetries, etc., which are usually lost in conventional mean-field calculations, because of lack of correlations. The correlations are present on the Bethe-like lattices, and in this paper we have given the exact expression for such correlations. It should be noted that we can obtain the proper singular behavior of ξ because we have used the proper thermodynamic limit ($N \to \infty$) to obtain the recursion equations and the correlation function, i.e. Eqs. (7), (8) and (12) for the Bethe lattice Ising model and Eqs. (24)–(26) for the Bethe lattice BEG model and we have demonstrated the crucial role of Bethe lattice dimensionality in determining critical behavior of the correlation length.

In conclusion, it must be remarked that the transfer-matrix methods discussed in this paper can be extended easily to obtain correlation functions with singular correlation length for other spin models, e.g. Potts model, multilayer Ising model, general spin model, etc., on the Bethe and Bethe-like structures [24]. In particular, our results for the spin model on the Bethe lattice can be extended easily to Husimi lattices, because they can be related with each other in terms of the star–triangle transformation [8].

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