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From quantum trajectories to classical orbits

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Abstract

The evolution of open quantum systems can be "unraveled" into individual "trajectories" in a variety of ways. In the mesoscopic regime, quantum jump (QJ) trajectories approach a diffusive limit similar to quantum state diffusion (QSD). In the classical limit, both unravelings show the rise of classical orbits for both regular and chaotic systems. © 1997 Published by Elsevier Science B.V.

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Quantum mechanics is nonlocal. Classical mechanics is local. Though it is widely believed that quantum mechanics is the fundamental theory, attempts to describe classical phenomena by quantum equations are fraught with difficulties. Not only do the calculations become extremely cumbersome, but they are conceptually more difficult.

There is no particular consensus on what it means to cross the "quantum → classical" border. Many criteria have been suggested: rapid decay of macroscopic superpositions, localization to coherent states in phase space, decoherence, near-determinism, positivity of the Wigner distribution, and nonviolation of the Bell inequalities [1–5]. In fact, it is rarely necessary to choose: as a rule, all of these are satisfied for macroscopic systems. Both modern experiments and the emerging field of nanotechnology, however, increasingly challenge this divide. As we probe the mesoscopic region, where both quantum and classical effects are important, it behooves us to have a better idea of what "classical" means. This is not only of theoretical interest: a better understanding should make it possible to produce more efficient numerical models of mesoscopic systems.

In this Letter, we show how the use of quantum trajectories, both continuous and discontinuous, illuminates the scales where neither a purely quantum nor a purely classical description is practical. These models not only simplify computation (one of their major original motivations), but describe when and how quantum nonlocality disappears in the classical limit. We illustrate our results by plotting regular and chaotic quantum trajectories, and show the limit where they recover classical regular and chaotic orbits.

The Schrödinger equation is the basic dynamical law of nonrelativistic physics. However, strictly speaking, it applies only to the entire Universe as the only

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truly closed system. All other systems are open. It is well known that the environment is crucial for the emergence of classical features in a quantum system. The best-known example is the case of a measurement, leading to sharp values of some physical quantity. More generically, environments induce decoherence, which is closely related to the rise of classical properties [1–3]. Consequently, we shall concentrate on open quantum systems in the Markovian limit.

Markovian open quantum systems are usually described by a master equation:

$$\dot{\rho} = -\mathrm{i}[H,\rho] + \sum_m (L_m \rho L_m^\dagger - \tfrac{1}{2}\{L_m^\dagger L_m, \rho\}), \quad (1)$$

where ρ is the density matrix for the system, H its Hamiltonian, and the linear operators L_m describe the effects of the environment. However, Eq. (1) is not entirely satisfactory for our purpose. It describes only mean values, computed over both quantum and classical probabilities. Hence, density matrices by themselves do not tell us which features can be described classically and which require a quantum description [6].

Let us illustrates this distinction for the mean values of the position operator q and its square q^2. These values give little idea of the actual degree of localization of the particle. A large value of the spread $[\mathrm{Tr}(q^2\rho) - \mathrm{Tr}(q\rho)^2]^{1/2}$ could correspond either to delocalized particles (resulting from, e.g., the spreading of a single wavepacket) or to localized particles whose position is classically uncertain (resulting from, e.g., Brownian motion). The evolution of localized particles can be efficiently computed with classical or semiclassical models, while delocalized particles necessarily require a less efficient but more complete quantum description.

In the latter case, generally, there is nothing which corresponds to classical phase space trajectories. This is one of the chief difficulties in characterizing quantum chaos in a way similar to that used in classical dynamics.

By *unraveling* the evolution of the density operator, one obtains, as we shall see, classical mixtures (i.e., classical probabilities) of quantum pure states (i.e., quantum probabilities). This allows one to distinguish quantum from classical and, at the same time, provides a powerful tool for practical computations.

In such an unraveling, one describes the system in terms of a normalized pure state $|\psi(t)\rangle$ which follows a stochastic "trajectory" in Hilbert space. By averaging the pure state projector $|\psi\rangle\langle\psi|$ over all possible trajectories with appropriate weights, one reproduces the density operator $\rho = M(|\psi\rangle\langle\psi|)$. This is analogous classically to replacing the Fokker–Planck equation for probability densities with a stochastic Langevin equation for single trajectories.

Unfortunately, unlike the case of classical Brownian motion, the unraveling of the master equation (1) is not unique. Thus, there is ambiguity in how one separates classical and quantum uncertainties, related to the ambiguity in identifying density matrices with quantum ensembles. In this paper we consider two well known unravelings of (1).

In quantum state diffusion (QSD), the (Itô) stochastic evolution equation for the pure state $|\psi(t)\rangle$ reads

$$\begin{aligned} |\mathrm{d}\psi(t)\rangle &= -\mathrm{i}H|\psi(t)\rangle\,\mathrm{d}t \\ &\quad - \tfrac{1}{2}\sum_j (L_j^\dagger L_j - 2\langle L_j^\dagger\rangle_\psi L_j + |\langle L_j\rangle_\psi|^2)|\psi(t)\rangle\,\mathrm{d}t \\ &\quad + \sum_j (L_j - \langle L_j\rangle_\psi)|\psi(t)\rangle\,\mathrm{d}\xi_j, \end{aligned} \quad (2)$$

where the "noises" $\mathrm{d}\xi_j$ are complex-valued Wiener processes of zero mean $M(\mathrm{d}\xi_j) = 0$ and correlations $M(\mathrm{d}\xi_j\,\mathrm{d}\xi_k) = 0$, $M(\mathrm{d}\xi_j^*\,\mathrm{d}\xi_k) = \delta_{jk}\,\mathrm{d}t$. This equation describes a continuous nondifferentiable evolution similar to the familiar diffusive paths of a classical Brownian particle, but in Hilbert space instead of real space. QSD is the only continuous unraveling which satisfies the same symmetry properties as the master equation itself [7].

Our second example is the quantum jumps (QJ) unraveling, which is closely related to photon counting. However, it can be defined for any Lindblad master equation [8,9]. The stochastic increment for the wave-function is

$$\begin{aligned} |\mathrm{d}\psi(t)\rangle &= -\mathrm{i}H|\psi(t)\rangle\,\mathrm{d}t \\ &\quad - \tfrac{1}{2}\sum_j (L_j^\dagger L_j - \langle L_j^\dagger L_j\rangle_\psi)|\psi(t)\rangle\,\mathrm{d}t \\ &\quad + \sum_j \left(\frac{L_j|\psi(t)\rangle}{\sqrt{\langle L_j^\dagger L_j\rangle_\psi}} - |\psi(t)\rangle\right)\mathrm{d}N_j. \end{aligned} \quad (3)$$

The discrete Poissonian noises dN_j assume the values 0 or 1. Most of the time $dN_j = 0$ and the evolution is continuous and differentiable. However, whenever $dN_j = 1$ there is a "jump" to the state $L_j|\psi(t)\rangle/\sqrt{\langle L_j^\dagger L_j\rangle_t}$. The dN_j processes have mean values $M_{|\psi\rangle}(dN_j) = \langle L_j^\dagger L_j\rangle_\psi dt$ and correlations $dN_j\,dt = 0$ and $dNj\,dN_k = \delta_{jk}\,dN_j$. This means, essentially, that jumps occur randomly with an average rate $\langle L_j^\dagger L_j\rangle$.

It can be shown that in the case of continuous measurements, the QJ and QSD equations describe the state of the system conditioned on a given measurement record, photon counting for QJ [8,9] and balanced heterodyne detection for QSD [10]. However, both can be defined abstractly simply as unravelings of the master equation, and we will use them in this sense.

Let us illustrate these two unravelings for a simple example: the damped harmonic oscillator at finite temperature. $H = \omega a^\dagger a$, $L_1 = \sqrt{\bar{n}\gamma}a^\dagger$ and $L_2 = \sqrt{(\bar{n}+1)\gamma}a$ where $\bar{n}$ is the thermal equilibrium mean photon number, $\bar{n} = \langle a^\dagger a\rangle_\rho$, and γ is the inverse relaxation time. For QSD one can show that any initial states tends to a coherent state: $|\psi(t)\rangle \to |\alpha_t\rangle$, where $a|\alpha_t\rangle = \alpha_t|\alpha_t\rangle$ and $\alpha_t = (\langle q\rangle_\psi + i\langle p\rangle_\psi)/\sqrt{2}$. Furthermore, the evolution of α_t is governed by a classical equation,

$$d\alpha_t = -i\omega\alpha_t\,dt - \tfrac{1}{2}\gamma\alpha_t\,dt + \sqrt{\bar{n}\gamma}\,d\xi_t. \tag{4}$$

Hence, for this example at least, the QSD equation fully describes how the environment localizes the quantum state down to a minimum Gaussian wavepacket, and how this wavepacket follows a classical trajectory.

In Refs. [7,11,12] it was argued that this is quite general, and not peculiar to the harmonic oscillator. In Ref. [13] the QSD equation was applied to the kicked anharmonic oscillator (KAOS) system; in this case localization takes place despite competition with the delocalizing nonlinear Hamiltonian. In general, when the system remains small (in the sense that it does not explore much of the phase space in units of $\hbar$), the QSD trajectory presents no definite structure. However, when the parameters are such that the system explores a larger portion of phase space, the quantum trajectories exhibit a clear structure which approaches the classical strange attractor.

Similar behavior has been demonstrated for other chaotic systems, including the weak link capacitor and quantum kicked rotor [4], and the forced damped Duffing oscillator [14]. As we approach the classical limit, the structure of the strange attractor begins to appear.

The case of QJ is quite different. No result analogous to that for QSD holds in the harmonic oscillator case. However, in Ref. [5] the KAOS system was studied using QJ, and again the classical strange attractor appeared whenever the system explored enough of phase space.

This result was very puzzling at the time. We can now show that this behavior is also generic for QJ. In particular, we shall see that whenever the system is far from the origin of phase space (i.e., the harmonic oscillator ground state) its dynamics become very similar to QSD.

Consider a system in a state $|\psi(t)\rangle$ with a Hamiltonian H and environment operators $L_1 = \sqrt{\gamma_1}a$ and $L_2 = \sqrt{\gamma_2}a^\dagger$. When the oscillator is well localized relative to its distance from the origin, so that $|\alpha_t| \gg \Delta\alpha_t$, then the rate of jumps $\langle L^\dagger L\rangle_t \approx |\alpha_t|^2$ is large, while the size of the jumps are small: $\langle\psi(t)|(a^\dagger - \alpha_t^*)(a - \alpha_t)|\psi(t)\rangle = \Delta\alpha_t$, and similarly for $a^\dagger$. (This corresponds to approaching the classical limit.) Hence, when the energy of the system is large relative to $\hbar$, the "jumpy" evolution approaches closer and closer to a diffusion process like QSD, describing localized wavepackets following classical trajectories. In order to study this limit, let us define

$$|\phi(t)\rangle = D(-\alpha_t)|\psi(t)\rangle, \tag{5}$$

where $\alpha_t = \langle a\rangle_{\psi(t)}$ and $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ is the displacement operator. $|\phi(t)\rangle$ is the state $|\psi(t)\rangle$ displaced such that the mean position and momentum vanish: $\langle\phi(t)|a|\phi(t)\rangle = 0$. Note that $|\alpha_t|$ measures the phase space "distance" of the state $|\psi(t)\rangle$ from the origin $|0\rangle$. Let $\Delta\alpha_t$ measure the width of the state ψ_t,

$$\Delta\alpha_t^2 \equiv \langle a^\dagger a\rangle_\psi - \langle a^\dagger\rangle_\psi\langle a\rangle_\psi = \langle\phi(t)|a^\dagger a|\phi(t)\rangle. \tag{6}$$

We can now develop the QJ equation (3) to first order in $\Delta\alpha/|\alpha|$. To remove irrelevant phases, we use the one-dimensional projector $P_t \equiv |\psi(t)\rangle\langle\psi(t)|$. From Eq. (3) one obtains

$$dP_t = -i[H, P_t]\,dt + \sum_j \left(-\tfrac{1}{2}\{L_j^\dagger L_j, P_t\} + \langle L_j^\dagger L_j\rangle_\psi P_t\right) dt + \left(\frac{L_j P_t L_j^\dagger}{\langle L_j^\dagger L_j\rangle_\psi} - P_t\right) dN_j. \quad (7)$$

Using $a|\psi\rangle = \alpha|\psi\rangle + D(\alpha)a|\phi\rangle$ and $aP_ta^\dagger = |\alpha_t|^2 P_t + \alpha_t^*(a-\alpha_t)P_t + \alpha_t P_t(a^\dagger - \alpha_t^*) + O^2(\Delta\alpha/|\alpha|)$, one deduces

$$\frac{aP_ta^\dagger}{\langle a^\dagger a\rangle_t} = P_t + \frac{a-\alpha_t}{\alpha_t}P_t + P_t\frac{a^\dagger - \alpha_t^*}{\alpha_t^*} + O\left(\frac{\Delta\alpha^2}{|\alpha|^2}\right), \quad (8)$$

and similarly for $a^\dagger P_t a$. (These are our only environment operators.) Inserting this into (7) yields

$$dP_t \approx -i[H, P_t]\,dt + \sum_j (L_j P_t L_j^\dagger - \tfrac{1}{2}\{L_j^\dagger L_j, P_t\})\,dt + (L_j - \langle L_j\rangle_\psi)P_t\frac{\langle L_j^\dagger\rangle}{|\langle L_j\rangle|}\,dW_j + P_t(L_j^\dagger - \langle L_j^\dagger\rangle_\psi)\frac{\langle L_j\rangle}{|\langle L_j\rangle|}\,dW_j, \quad (9)$$

where we have made the approximation

$$\frac{\langle L_j^\dagger\rangle}{\sqrt{\langle L_j^\dagger L_j\rangle}}\left(\frac{dN_j}{\sqrt{\langle L_j^\dagger L_j\rangle}} - \sqrt{\langle \hat{L}_j^\dagger L_j\rangle}\,dt\right) \approx \frac{\langle L_j^\dagger\rangle}{|\langle L_j\rangle|}\,dW_j, \quad (10)$$

with the dW_j independent standard real Wiener processes ($dW_i\,dW_j = \delta_{ij}\,dt$). Hence,

$$|d\psi(t)\rangle \approx -iH|\psi(t)\rangle\,dt - \tfrac{1}{2}\sum_j (L_j^\dagger L_j - 2\langle L_j^\dagger\rangle_\psi L_j + |\langle L_j\rangle_\psi|^2)|\psi(t)\rangle\,dt + \sum_j (L_j - \langle L_j\rangle_\psi)|\psi(t)\rangle\frac{\langle L_j^\dagger\rangle}{|\langle L_j\rangle|}\,dW_j. \quad (11)$$

Note that Eq. (11) preserves the norm of $|\psi(t)\rangle$ and recovers (1) in the mean: $\rho(t) = M(|\psi(t)\rangle\langle\psi(t)|)$ at all times. It is almost identical to the QSD equation (2), except that the noise is a real Wiener process multiplied by a phase which depends on the phase space position of the oscillator. The solutions of Eqs. (2) and (11) are very similar, as we shall see.

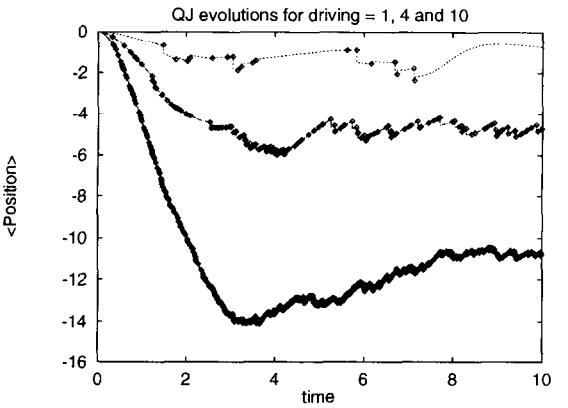

Fig. 1. The harmonic oscillator with $\omega = 1$ at finite temperature, the equilibrium displaced by a constant force β; curves are plotted for the values $\beta = 1, 4, 10$ and average thermal excitation $\bar{n} = 0.2$. Each jump is marked; as the oscillator is displaced further from the origin, the jumps become frequent and small in effect, illustrating the transition from a "jumpy" trajectory to a diffusion process.

We can see this by going to the classical limit and plotting the trajectories for the QJ equation. We have done this for an extension of the thermal model presented in (4) above. Note how in this case the jumps occur very frequently as we move away from the origin of phase space (see Fig. 1). This also works for the chaotic cases already studied with QSD; in the QJ case as well, we can see the emergence of classical structure (see Fig. 2).

A similar derivation can be done for any choice of environment operators L which are linear in a and $a^\dagger$ (e.g., x and p).

We have seen how the use of stochastic evolution equations for pure states enables us to draw useful pictures of elementary quantum phenomena. These same equations provide powerful tools for practical numerical computations; in particular, one can exploit the existence of localized solutions to reduce the numerical difficulty of solving the equations, using the technique of *moving bases* [15–18] to make computer simulations more efficient. Quantum unravelings have long suggested the possibility of separating quantum and classical uncertainties for open systems; however, the ambiguity in the choice of unraveling has prevented

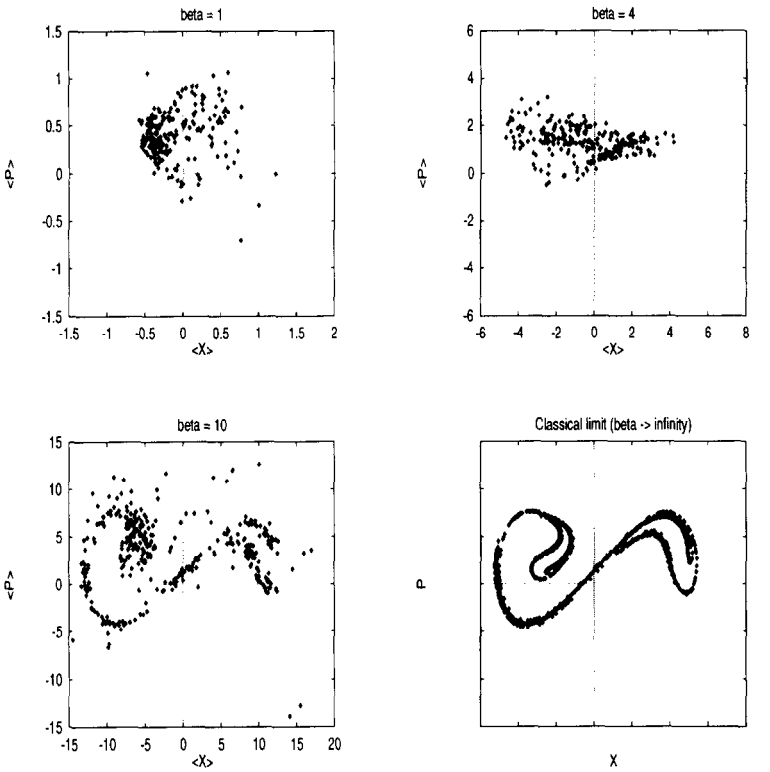

Fig. 2. The Poincaré section of the forced damped Duffing oscillator $\ddot{x} = x - x^3/\beta^2 - 2\Gamma\dot{x} + g\beta\cos(\omega t)$ in the chaotic regime ($\Gamma = 0.125, g = 0.3, \omega = 1$). The value β gives the scale of the system, with $\beta \to \infty$ the classical limit; this system classically exhibits dissipative chaos, and has already been investigated in this limit using QSD (see Ref. [14]). We keep $\hbar = 1$ constant. As the scale increases for $\beta = 1, 4, 10$, the structure of the strange attractor clearly emerges, showing the diffusive limit of quantum jumps. This is similar to the behavior observed for QSD. The classical result is included for comparison.

any conclusions from being drawn as to the exact meaning of this separation.

We now see that, as one approaches the classical limit, it is possible to make this separation in a similar way for *different* unravelings. This suggests that, in a sense, the details do not matter: there is a single classical limit towards which a broad class of different "quantum trajectory" techniques tend. In treating mesoscopic physical systems, this should make it possible to determine unambiguously which characteristics may be given classical and which quantum descriptions. This, in turn, may contribute greatly to the numerical solution of practical problems.

It has been shown that both QSD and QJ correspond to different sets of decoherent histories [19–21], as developed by Griffiths, Omnès, and Gell-Mann and Hartle [3]. This result thus argues that as one goes to the classical limit, these two sets of histories become very similar; and therefore, in the language of Gell-Mann and Hartle, that there may be a unique "quasiclassical realm". In both cases, the dynamics automatically picks out the correct "pointer basis" for the system.

We have some numerical evidence that yet another unraveling, the orthogonal jumps of Diósi, also reproduces classical trajectories in the classical limit. This ortho-jump unraveling corresponds to a set of exactly decoherent histories as well [22].

In the case of heterodyne measurement, it has already been shown that diffusive behavior can arise from the QJ equation [10], and in fact reproduces the QSD equation exactly. In this Letter, however, we have seen that diffusive behavior arises even in the normal photon-counting form of the QJ equation.

We should also stress the limitations of this approach. In the first place, it depends on the "sys-

tem/environment split", which will always include a certain ambiguity. Second, we have limited ourselves to the Markovian approximation. This is mainly because little is known about the more general case. We expect that similar considerations apply to non-Markovian systems.

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References

[1] E. Joos and H.D. Zeh, Z. Phys. B, 59 (1985) 223.
[2] W.H. Zurek, Phys. Today 36 (October 1991).
[3] R. Griffiths, J. Stat. Phys. 36 (1984) 219;
R. Omnès, Rev. Mod. Phys. 64 (1992) 339;
H.F. Dowker and J.J. Halliwell, Phys. Rev. D 46 (1992) 1580;
M. Gell-Mann and J.B. Hartle, Phys. Rev. D 47 (1993) 3345.
[4] T.P. Spiller, J.F. Ralph, T.D. Clark, R.J. Prance and H. Prance, J. Low Temp. Phys. 101 (1995) 1037.
[5] M. Rigo and N. Gisin, Quantum Semiclass. Opt. 8 (1996) 255.
[6] M. Rigo, G. Alber, F. Mota-Furtado and P.F. O'Mahony, submitted to Phys. Rev. A.
[7] N. Gisin and I.C. Percival, J. Phys. A 25 (1992) 5677; 26 (1993) 2233, 2245.
[8] J. Dalibard, Y. Castin, and K. Mølmer, Phys. Rev. Lett. 68 (1992) 580; J. Opt. Soc. Am. 10 (1993) 524.
[9] H.J. Carmichael, Lecture notes in physics m18. An open systems approach to quantum optics (Springer, Berlin, 1993).
[10] H.M. Wiseman and G.J. Milburn, Phys. Rev. A 47 (1993) 642;
P.L. Knight and B. Garraway, in: Quantum dynamics of simple systems, eds. G.L. Oppo et al., 44th Scottish University Summer School in Physics (1996) 199.
[11] I.C. Percival, J. Phys. A 27 (1994) 1003.
[12] J.J. Halliwell and A. Zoupas, Phys. Rev. D 52 (1995) 7294.
[13] T.P. Spiller and J.F. Ralph, Phys. Lett. A 194 (1994) 235.
[14] T.A. Brun, J. Phys. A 29 (1995) 2077.
[15] R. Schack, T. Brun and I.C. Percival, J. Phys. A 28 (1995) 5401.
[16] T. Steimle, G. Alber and I.C. Percival, J. Phys. A 28 (1995) L491.
[17] M. Holland, S. Marksteiner, P. Marte and P. Zoller, Phys. Rev. Lett. 76 (1996) 3683.
[18] R. Schack and T. Brun, quant-ph/9608004, to appear in Comp. Phys. Comm.
[19] L. Diósi, N. Gisin, J.J. Halliwell and I.C. Percival, Phys. Rev. Lett. 21 (1995) 203.
[20] T.A. Brun, quant-ph/9606025, to appear in Phys. Rev. Lett.
[21] Ting Yu, Imperial College Preprint TP/95-96/44, gr-qc/9605071.
[22] L. Diósi, Phys. Lett. A 114 (1986) 451, 185 (1994) 5;
J.P. Paz and W. Zurek, Phys. Rev. D 48 (1993) 2728.