Modeling Retail Trade Areas Using Higher-Order, Multiplicatively Weighted Voronoi Diagrams

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Voronoi (Thiessen) trade area models are geometrically based procedures for generating theoretical trade areas using store characteristics and assumptions about consumer behavior. Existing models assume that customers patronize only one facility resulting in mutually exclusive trade areas. We introduce two new Voronoi models which include the assumption that a customer selects from the k (k = 1, 2, 3, ...) nearest most attractive facilities. When the customer is indifferent between the k facilities, the trade areas are given by the order-k, multiplicatively weighted Voronoi diagram. If the customer shows a preference for nearer facilities, the trade areas are given by the ordered, order-k, multiplicatively weighted Voronoi diagram. Both models produce overlapping trade areas. The ordered, order-k, multiplicatively weighted Voronoi diagram model also enables us to examine the effects of different preference levels on the sales estimates for a facility. The new models are demonstrated using data relating to a chain of supermarkets in the twin cities of Kitchener-Waterloo in Ontario, Canada.

INTRODUCTION

A variety of techniques has been used to delimit retail trade areas. This paper focuses on one technique, Voronoi (Thiessen) diagram models, a geometrically based procedure for delimiting theoretical trade areas for a set of similar (including competing) facilities in space (Jones and Simmons, 1993, p. 348). The general form of Voronoi models is that they combine information on store locations and attributes with assumptions about consumer behavior to generate the trade areas. They are most useful in those situations where detailed consumer patronage data is either unavailable or deemed too costly or time consuming to acquire. Another positive feature, not found in some other models, is that they can be used in either a descriptive or a predictive way. For example, when applied to the outlets of a

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single retail chain, Voronoi diagrams provide a visual representation of the chain's locational strategy, including its evolution over time. Voronoi diagrams can also be used to identify potential sites for new facilities, as well as indicating the impact of these and other changes on the existing set of facilities (Ghosh and McLafferty, 1987, p.67). Unlike other models, such as the those based on the multiplicative competitive interaction (MCI) model (Nakanishi and Cooper, 1974) or the multinomial logit (MNL) model (Arnold et al., 1980), the Voronoi models do not require complex statistical calibration procedures. In sum, when used in appropriate situations, Voronoi models provide reasonable approximations of real trade areas, quickly and inexpensively, and without the requirement of extensive retail expertise on the part of the analyst (Jones and Simmons, 1993, p. 372). For example, all the trade area maps in this paper were defined in under an hour using a personal computer.

So far, two kinds of Voronoi diagrams have been used to model retail trade areas. The simpler of these, the ordinary Voronoi diagram (OVD), considers only the locations of the facilities and assumes that customers patronize the nearest facility (e.g., Ghosh and McLafferty, 1987, p.65; Jones and Mock, 1984; Jones and Simmons, 1993, p. 348; West, 1981; West and Von Hohenbalken, 1984). The other type, the multiplicatively weighted Voronoi diagram (MWVD), considers both locational and non-locational attributes of facilities (represented as an aggregate measure of attractiveness) and assumes that customers select stores on the basis of a trade-off between distance and attractiveness (e.g., Boots, 1980; O'Kelly and Miller, 1989; Von Hohenbalken and West, 1984). Both types of Voronoi model assign customers to only one facility so that the resulting trade areas are spatial monopolies.

The main contribution of this paper is to extend the existing Voronoi models by incorporating an additional assumption that customers patronize more than one facility. We do this by combining the MWVD with another kind of Voronoi diagram known as higher-order Voronoi diagrams. The latter contains the assumption that customers patronize the k ($k = 2$, 3, 4, ...) nearest facilities. When customers are indifferent between the k nearest facilities, the trade areas are given by the order-k, multiplicatively weighted Voronoi diagram (OKMWVD). If customers express a preference for nearer facilities, the trade areas are given by the ordered, order-k, multiplicatively weighted Voronoi diagram (OOKMWVD). One feature of the new models is that resulting trade areas are no longer mutually exclusive. Since such an outcome is more consistent with real trade areas, we contend that the models we introduce represent improvements over the existing ones.

We begin by reviewing the OVD and MWVD models in Section 2. Next, in Section 3, we describe the order-k Voronoi diagram (OKVD) and the ordered, order-k Voronoi diagram (OOKVD). In Section 4 we combine these two Voronoi diagrams with the MWVD to create the two new trade area models, the OKMWVD and the OOKMWVD. We illustrate the use of these two new models in Section 5 using data relating to a chain of supermarkets in the twin cities of Kitchener-Waterloo in Ontario, Canada and by showing how these diagrams may be used to estimate sales volumes. Finally, Section 6 provides a summary together with suggestions for future work.

Note that numerous algorithms have been published for all types of Voronoi diagrams used in this paper (for summaries see Okabe et al., 1992, pp. 261-267; Edelsbrunner, 1993). In addition, software is available from several sources. One listing of such software can be found on the Web at **http://www.geom.umn.edu/locate/cglib**.

EXISTING VORONOI DIAGRAM MODELS

Here we define formally the ordinary Voronoi diagram (OVD) and the multiplicatively weighted Voronoi diagram (MWVD). Our definitions follow those of Okabe et al. (1992) where additional information regarding the OVD and MWVD, especially their properties, may be found. Since these definitions make extensive use of set theory notation, readers unfamiliar with such notation should consult the Appendix.

Ordinary Voronoi Diagram

Consider a set of n facilities ($2 \le n \le \infty$), $P = \{p_1, p_2, \ldots, p_n\}$ in a two-dimensional space. Let the location of facility p_j be indicated by $\mathbf{x}_j = (x_{j1}, x_{j2})$. Let each facility be spatially distinct, i.e., $\mathbf{x}_j \neq \mathbf{x}_l$ for $j \neq l, j, l \in I_n = \{1, 2, \ldots, n\}$. Let p be an arbitrary location in the plane indicated by $\mathbf{x} = (x_1, x_2)$. Then the euclidean distance between p and p_j is $d(p, p_j) = \|\mathbf{x} - \mathbf{x}_j\| = [(x_1 - x_{j1})^2 + (x_2 - x_{j2})^2]^{1/2}$.
We call the region

$$V(p_j) = \{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_j\| \le \|\mathbf{x} - \mathbf{x}_l\| \text{for } j \neq l, l \in I_n\}$$

the ordinary Voronoi polygon (OVP) of facility p_j and the set given by $\vartheta(P) = \{V(p_1), \ldots, V(p_n)\}$ the ordinary Voronoi diagram (OVD) of P. Clearly, $V(p_j)$ contains all locations which are closer to facility p_j than to any other facility. If customers are arbitrarily located, trade areas generated by assuming that customers use the nearest facility are equivalent to OVPs. The applicability of trade areas defined by the OVD is confined to those situations where undifferentiated products are sold from facilities that are similar in all respects and which are located in an homogeneous space without barriers. Such situations are most likely to occur for facilities offering low order goods at neighbourhood and community levels. However, in many other circumstances, other factors besides accessibility influence a customer's choice of facility (Beaumont, 1987, p. 29; Craig et al., 1984) so that customers may shop at other than the closest facility. Some of these factors may be incorporated by extending the OVD.

Multiplicatively Weighted Voronoi Diagram

Consider a set of n facilities as in the previous section but in addition attach a weight w_j ($w_j > 0$) to facility p_j. We use this weight to define a weighted distance $d_w(p, p_j)$from an arbitrary location p to p_j. There are many functional forms of weighted distance (see Okabe et al., 1992, pp. 126–141) but the one of interest here is referred to as the multiplicatively weighted distance and has the form

$$d_w(p, p_j) = (1/w_j)\|\mathbf{x} - \mathbf{x}_j\|, \; w_j > 0. \qquad (1)$$

Then the multiplicatively weighted Voronoi polygon (MWVP) $V_w(p_j)$ of facility p_j is given by $V_w(p_j) = \{\mathbf{x} | (1/w_j)\|\mathbf{x}-\mathbf{x}_j\| \leq (1/w_l)\|\mathbf{x}-\mathbf{x}_l\|, j \neq l\}$ and the set $\vartheta_w(P) = \{V_w(p_1), ..., V_w(p_n)\}$ is called the multiplicatively weighted Voronoi diagram (MWVD) of P.

To interpret this model we note that several other trade area models, including those based on the Reilly gravitational, Huff, and spatial interaction formulations, assume that a customer's evaluation of a facility depends on both the facility's location and attractiveness. Often (e.g., Ghosh and McLafferty, 1987, p. 90) this evaluation is expressed in terms of a utility U_{ij} of facility j for customer i, where U_{ij} has the general form

$$U_{ij} = A_j^{\alpha} d_{ij}^{-\beta} \ (\alpha, \beta \geq 0) \tag{2}$$

where A_j is a measure of the attractiveness of facility j and d_{ij} is a function of the distance from customer i to facility j. Comparison of equations (1) and (2) reveals that they have identical forms and further, if customer i is located at p and $\alpha = \beta = 1$,

$$d_w(p, p_j) = 1/U_{ij}.$$

Thus, the MWVP of facility p_j is equivalent to the trade area which is demarcated by assigning to p_j all customers for which that facility maximizes U_{ij}. Note, however, that in the MWVD the parameters α and β are provided by the analyst while in the spatial interaction models they are estimated from observed consumer behavior.

Unlike the situation in the OVD, in the MWVD, customer behavior is modelled as a trade-off between distance and attractiveness so that not all customers are assumed to patronize the nearest facility. This property makes the MWVD more attractive for empirical applications. Unfortunately, the MWVD still shares with the OVD the unrealistic feature of assigning customers to only one facility. However, this limitation can be overcome by combining the MWVD with two variants of another type of Voronoi diagram. This so-called higher-order Voronoi diagram is described in the next section.

HIGHER-ORDER VORONOI DIAGRAMS

Order-*k* Voronoi Diagram

As with the two Voronoi diagrams described in the previous section, we begin with a set of facilities $P = \{p_1, ..., p_n\}$ but now, instead of dealing with individual facilities, we consider subsets of k facilities selected from P. To illustrate this, let us first consider the situation where $k = 2$, that is we are considering pairs of facilities.

Let $A^{(2)}(P) = \{P_1^{(2)}, \ldots, P_i^{(2)}, \ldots, P_l^{(2)}\}$, where $P_i^{(2)} = \{p_{i1}, p_{i2}\}, p_{i1}, p_{i2} \in P$ and $l = {}_nC_2$, be all the possible subsets of P which consist of two facilities. As in the previous section, let p represent an arbitrary location in the plane and $d(p, p_{ij})$ the euclidean distance from p to p_{ij}. We then define the order-2 Voronoi polygon of $P_i^{(2)}$ as

$$V(P_i^{(2)}) = \{p \mid d(p, p_{i1}) \le d(p, p_j) \text{ and } d(p, p_{i2}) < d(p, p_j) \text{ for } p_j \in P \backslash P_i^{(2)}\}. \quad (3)$$

Thus, $V(P_i^{(2)})$ consists of all locations for which either p_{i1} or p_{i2} is the first or second nearest facility. The assignment rule used to define $V(P_i^{(2)})$ in equation (3) is equivalent to one in which the longer of the distances from p_{i1} and p_{i2} is shorter than or equal to the shortest distance among the distances from p to the remaining facilities of P (i.e., the facilities in $P \backslash \{p_{i1}, p_{i2}\}$). Thus, $V(P_i^{(2)})$ may also be written as

$$V(P_i^{(2)}) = \{p \mid \max_{P_h} \{d(p, p_h) \mid p_h \in P_i^{(2)}\} \le \min_{P_j} \{d(p, p_j) \mid p_j \in P \backslash P_i^{(2)}\}\}. \quad (4)$$

It is straightforward to extend the above arguments to the general case of subsets of k facilities. Let $A^{(k)}(P)$ be the set of all possible subsets consisting of k facilities selected from the set of facilities P, i.e.,

$$A^{(k)}(P) = \{P_1^{(k)}, \ldots, P_i^{(k)}, \ldots, P_l^{(k)}\}, \quad (5)$$

where $P_i^{(k)} = \{p_{i1}, \ldots, p_{ik}\}, p_{ij} \in P$, and $l = {}_nC_k$. Then, analogously to equation (4), we define

$$V(P_i^{(k)}) = \{p \mid \max_{P_h} \{d(p, p_h) \mid p_h \in P_i^{(k)}\} \le \min_{P_j} \{d(p, p_j) \mid p_j \in P \backslash P_i^{(k)}\}\}$$

and call this the order-k Voronoi polygon associated with $P_i^{(k)}$. We call the set $\vartheta(A^{(k)}(P)) = \vartheta^{(k)} = \{V(P_1^{(k)}), \ldots, V(P_l^{(k)})\}$, the order-$k$ Voronoi diagram (OKVD) generated by P. Figure 1 shows an example of an order-2 Voronoi diagram. The pair of numbers in each polygon in this figure represents the two closest facilities for all locations in the polygon. For example, (2, 4) indicates that the two closest facilities are numbers 2 and 4.

Ordered, Order-*k* Voronoi Diagram

In the OKVD we are not concerned with which of the k facilities in $P_i^{(k)}$ is the nearest, second nearest, ..., or kth nearest. However, in the ordered, order-k Voronoi diagram (OOKVD), this order is considered explicitly. Again, we begin by examining the situation where $k = 2$, i.e., for a pair of facilities. Formally, let $A^{<2>}(P)$ be the set of all ordered pairs of facilities obtained from the set of facilities $P = \{p_1, \ldots, p_n\}$, i.e., $A^{<2>}(P) = \{P_1^{<2>}, \ldots, P_i^{<2>}, \ldots, P_l^{<2>}\}$ where $P_i^{<2>} = (p_{i1}, p_{i2}), p_{i1}, p_{i2} \in P$, and $l = n(n-1)$.
As above, let p be an arbitrary location in the plane and $d(p, p_{ij})$ the euclidean distance from p to p_{ij}. For a set $P_i^{<2>}$ in $A^{<2>}(P)$, we define $V(P_i^{<2>}) = \{p \mid d(p, p_{i1}) \le d(p, p_{i2}) \le d(p, p_j), p_j \in P \backslash \{p_{i1}, p_{i2}\}\}$.

We call the set $V(P_i^{<2>})$ the ordered, order-2 Voronoi polygon associated with $P_i^{<2>}$. Thus, $V(P_i^{<2>})$ consists of all locations for which p_{i1} and p_{i2} are the first and second nearest facilities, respectively.

These definitions can be extended to the general case of k facilities. Let $A^{<k>}(P)$ be the set of all ordered k-tuples of facilities obtained from $P = \{p_1, \ldots, p_n\}$, i.e.,

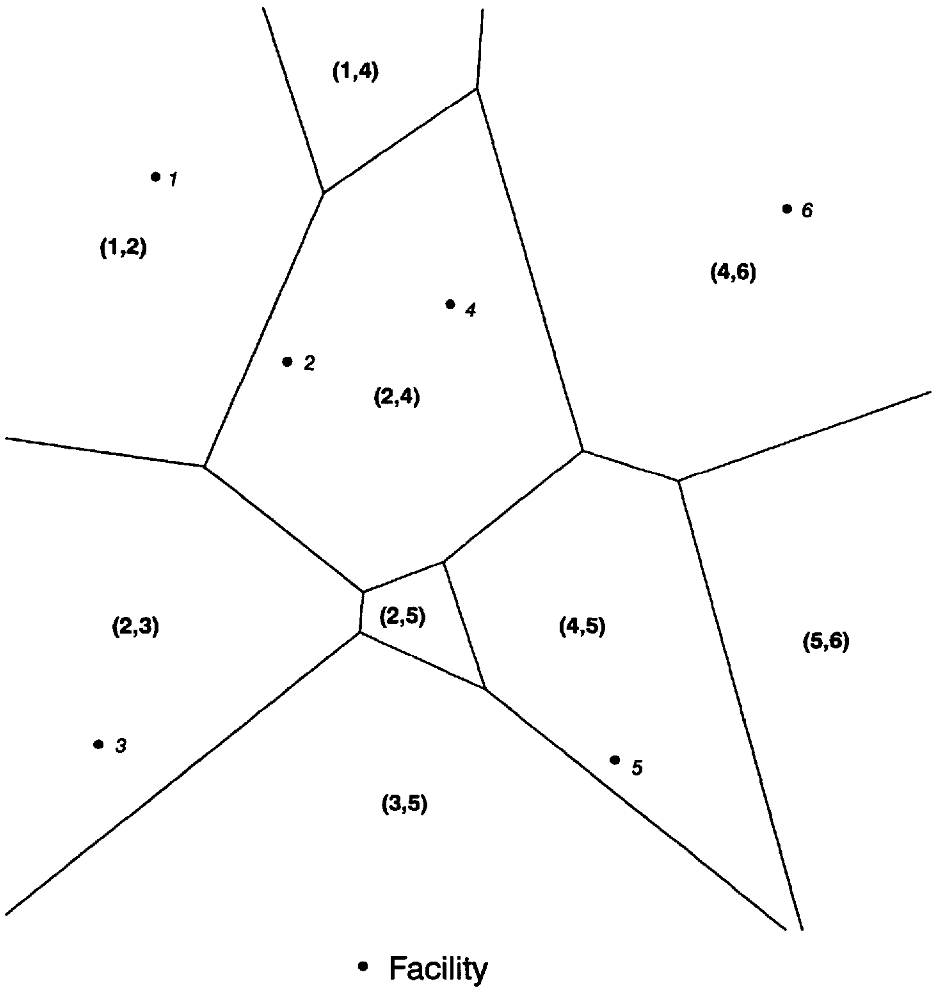

Figure 1. *An Order-2 Voronoi diagram (The numbers in parentheses in each region are the two nearest facilities for all locations within the region.)*

$$A^{\langle k\rangle}(P) = \{P_1^{\langle k\rangle}, \ldots, P_i^{\langle k\rangle}, \ldots, P_l^{\langle k\rangle}\} \tag{6}$$

where $P_i^{\langle k\rangle} = (p_{i1}, \ldots, p_{ik}), p_{ij} \in P, j \in I_k$, and $l = n(n-1) \ldots (n-k+1)$. For a set $P_i^{\langle k\rangle}$ in $A^{\langle k\rangle}$ we define

$$V(P_i^{\langle k\rangle}) = \{p \mid d(p, p_{i1}) \leq d(p, p_{i2}) \leq \ldots \leq d(p, p_{ik}) \leq d(p, p_j), p_j \in P \backslash \{p_{i1}, \ldots, p_{ik}\}\}.$$

We call $V(P_i^{\langle k\rangle})$ the ordered, order-k Voronoi polygon associated with $P_i^{\langle k\rangle}$ and the set

$$\vartheta(A^{<k>}(P)) = \vartheta^{<k>} = \{V(P_1{}^{<k>}), \ldots, V(P_l{}^{<k>})\}$$

the ordered, order-k Voronoi diagram (OOKVD) of P. The ordered, order-2 Voronoi diagram corresponding to the order-2 Voronoi diagram of Figure 1 is shown in Figure 2. This time the pair of numbers in parentheses in each polygon represents the first and second nearest facilities for all locations in the polygon. For example, (2, 4) indicates that facilities 2 and 4 are the first and second nearest facilities, respectively.

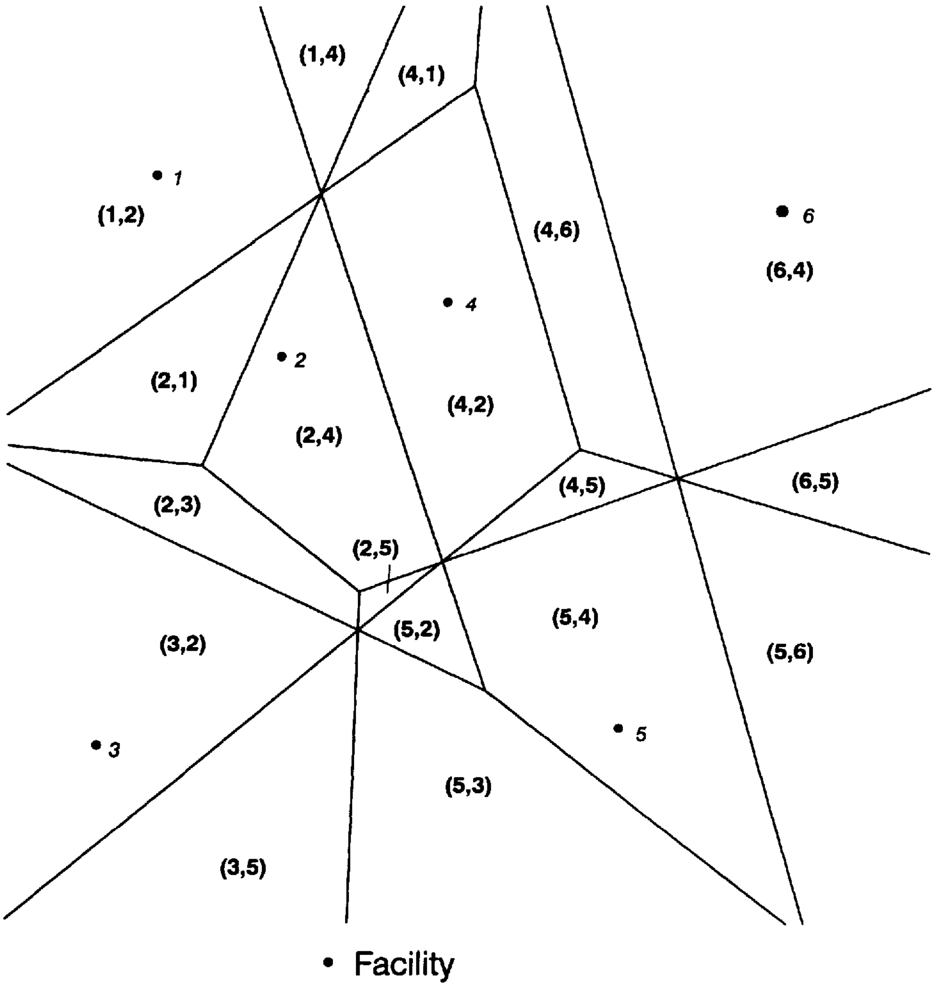

Figure 2. An Ordered, Order-2 Voronoi diagram (The numbers in parentheses in each region are the first and second nearest facilities, respectively, for all locations within the region

HIGHER-ORDER, MULTIPLICATIVELY WEIGHTED VORONOI DIAGRAMS

In this section we combine the two higher-order diagrams described in the previous sections with the MWVD of an earlier mentioned section. We begin by defining the new diagrams and then provide an interpretation of them in the context of retail trade area models.

Order-*k* Multiplicatively Weighted Voronoi Diagram

Once more, let $P = \{p_1, ..., p_n\}$ be the set of facilities and w_i ($w_i > 0$) be a weight attached to facility p_i. Let $A^{(k)}(P)$ be defined as in equation (5) and $d_w(p, p_{ij})$ be the weighted euclidean distance from an arbitrary location p to p_{ij} as defined by equation (1). Then, $V_w(P_i^{(k)})$, the order-k, multiplicatively weighted Voronoi polygon associated with $P_i^{(k)}$ is given by

$$V_w(P_i^{(k)}) = \{p | \max_{P_h} \{d_w(p, p_h) | p_h \in P_i^{(k)}\} \le \min_{P_j} \{d_w(p, p_j) | p_j \in P \backslash P_i^{(k)}\}\}$$

and the set

$$\vartheta_w(A^{(k)}(P)) = \vartheta_w^{(k)} = \{V_w(P_1^{(k)}), ..., V_w(P_l^{(k)})\}$$

is the order-k, multiplicatively weighted Voronoi diagram (OKMVD) of P.

Ordered, Order-*k* Multiplicatively Weighted Voronoi Diagram

Define P, w_i, p and $d_w(p, p_{ij})$ as in the previous section and define $A^{<k>}(P)$ as in equation (6). The ordered, order-k multiplicatively weighted Voronoi polygon of $P_i^{<k>}$ is given by

$$V_w(P_i^{<k>}) = \{p | \ d_w(p, p_{i1}) \le d_w(p, p_{i2}) \le \ldots \le d_w(p, p_{ik}) \le d_w(p, p_j), p_j \in P \backslash \{p_{i1}\}, \ldots, p_{ik}\}\},$$

while the set $\vartheta_w(A^{<k>}(P)) = \vartheta_w^{<k>} = \{V_w(P_1^{<k>}), \ldots, V_w(P_l^{<k>})\}$ constitutes the ordered, order-k multiplicatively weighted Voronoi diagram (OOKMWVD) of P.

Higher-order, Multiplicatively Weighted Voronoi Diagrams as Retail Trade Area Models

Here we interpret the higher-order, multiplicatively weighted Voronoi diagrams described in the two previous sections in terms of retail trade area models. For simplicity, we limit our interpretation to the case where $k = 2$. Then, the OKMWVD and the OOKMWVD are equivalent to retail trade area models incorporating the following assumptions:

1. a number of facilities, n, of the same type (e.g., supermarkets), are located in a finite, planar region, S (e.g., a city);
2. customers patronize one or more of the facilities;
3. an individual facility, j, is assigned a weight, w_j ($w_j > 0$), on the basis of its attractiveness to customers in terms of one or more attributes (e.g., price, size, parking, age, etc.);
4. the utility U_{ij}, of facility j for customer i is an inverse function of the distance travelled by i to reach j, d_{ij}, and a direct function of the attractiveness of j, w_j (see equation (2));
5. customers limit their purchases to the two facilities with the highest utilities;

6a. the customer is indifferent between the two facilities (i.e., the customer is equally likely to select either of the facilities)
or

6b. the customer displays a preference for the facility with the higher utility (i.e., the customer is more likely to select the facility with the higher utility).

Assumption 6a means that the appropriate trade areas are given by the OKMWVD, while assumption 6b is consistent with trade areas given by the OOKMWVD. Note that, since individual polygons (trade areas) defined by the OKMWVD and the OOKMWVD are associated with two or more facilities, the trade areas of individual centres necessarily overlap. For example, the cross-hatched area in Figure 3 (labelled (4, 5)) is the region common to the trade areas of facilities 4 and 5. In the case of an order-2 model, the trade area of an individual store will include all locations for which the facility represents either the highest or the second highest utility. To see this explicitly, let

$V_w^{(2)}(p_i, p_j)$ be the order-2 Voronoi polygon associated with the pair of facilities p_i, p_j.

Then, the trade area of facility p_i is given by $\bigcup_j V_w^{(2)}(p_i, p_j) j \neq i$. For example, in Figure 3 the trade area of facility 5 consists of the polygons labelled (2, 5) (3, 5), (4, 5) and (5, 8). Thus, unlike the existing Voronoi models, these new Voronoi models do not define trade areas which are spatial monopolies.

ILLUSTRATIONS OF THE NEW MODELS

In this section we demonstrate how the new models may be used to define trade areas and how, in turn, these may be used to estimate sales volumes. Although we use real data for this purpose, the reader should recognize that this section is not intended as a retail case study. In particular, to facilitate the illustration of the technique and to focus attention on its distinctive features, we simplify some procedures (e.g., the generation of weights) and assumptions (e.g., those relating to the spatial distribution of customers and income) which are not specific to the model. None of these simplifications are a requirement of the models and the analyst is free to adopt more sophisticated procedures and assumptions, if desired, in an actual case study.

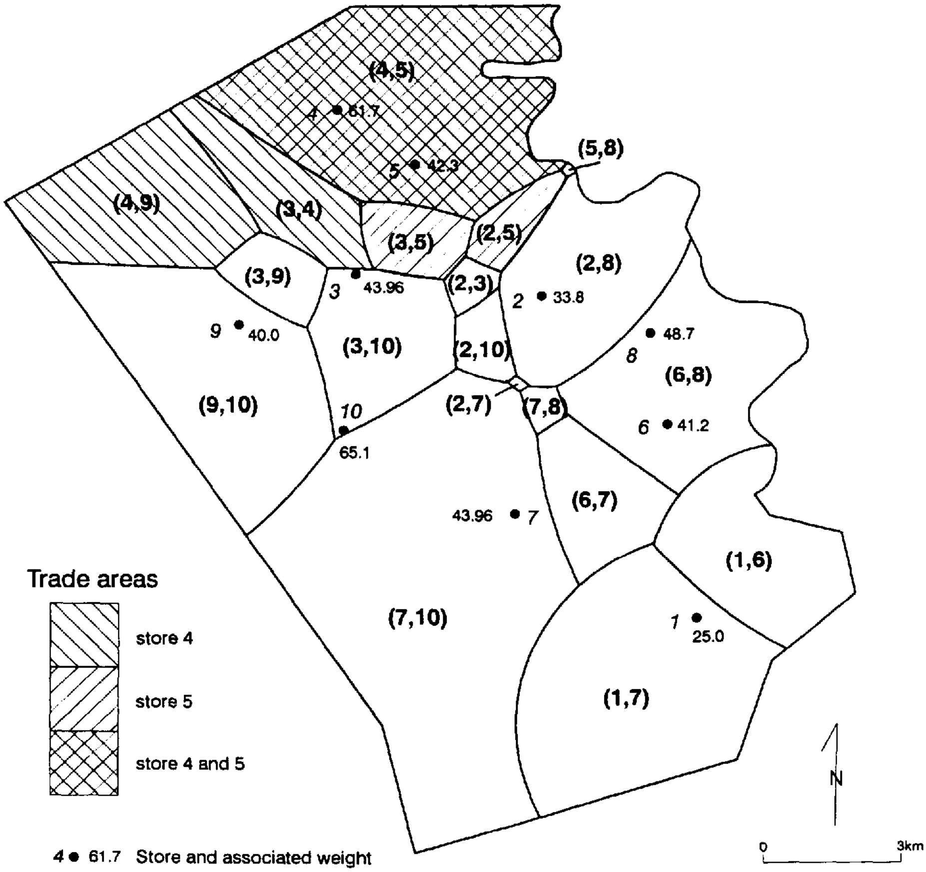

Figure 3. O2MWVD for Zehrs' Supermarkets, Kitchener-Waterloo (The numbers in parentheses in each region are the two most attractive supermarkets for all locations within the region)

The data used relates to supermarkets owned by the Zehrs chain which were operational at the end of 1995 in the twin cities of Kitchener-Waterloo (K-W), Ontario, Canada (population approximately 270,000 in 1995). Although other supermarket chains and an independent operated in K-W at this time, Zehrs dominated the local market, accounting for 10 of a total of 22 supermarkets. Further, a 1995 NADbank survey reported that 83 per cent of residents of the Kitchener Census Metropolitan Area (which includes K-W) interviewed, shopped at Zehrs at least once in the previous month (*Kitchener-Waterloo Record*, 11 November 1995, pp. B7-B8). In contrast, the second place chain, Dutch Boy (with 6 stores), was visited by only 41 percent of respondents. The locations of the Zehrs supermarkets are shown in Figures 3-6.

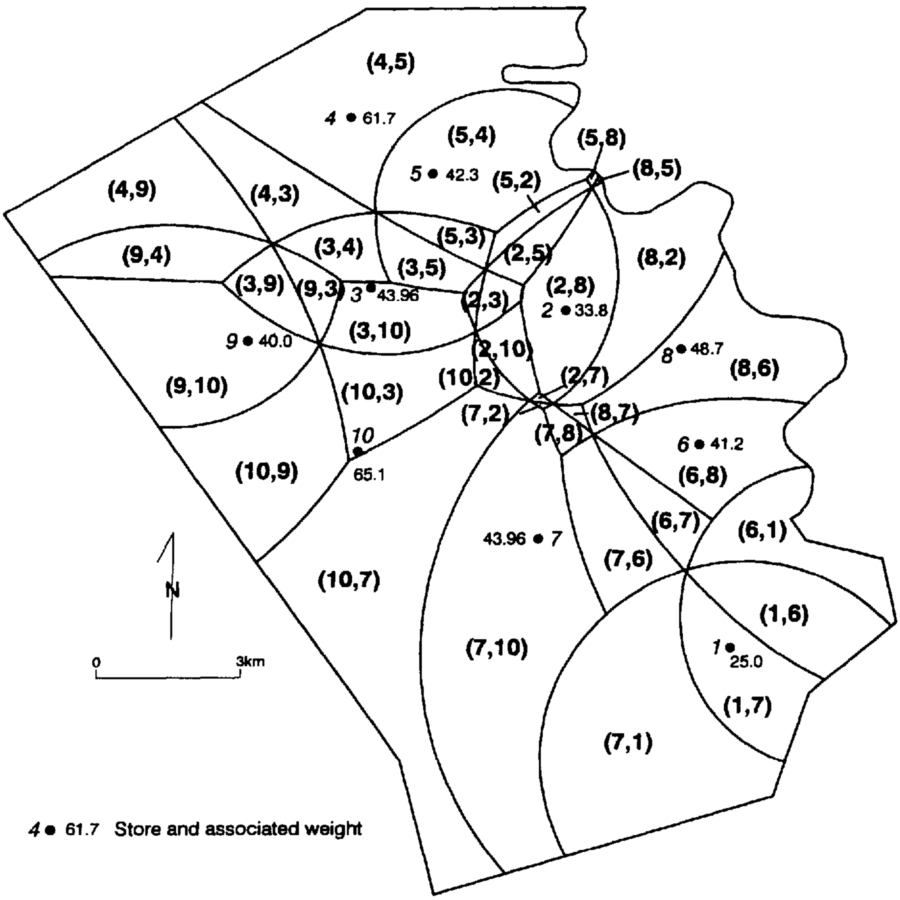

Figure 4. OO2MWVD for Zehr's Supermarkets, Kitchener-Waterloo (The numbers in parentheses in each region are the first and second most attractive supermarkets, respectively, for all locations within the region.)

Trade Areas

Implementation of our models requires that we weight each of the 10 Zehrs supermarkets. Here we use a simple procedure which incorporates only the easily measured store characteristics described in Table 1. Note that these variables do not include any relating to prices since we are considering only those supermarkets associated with a single chain. Next, the value of each variable for each supermarket was standardized by expressing it as a proportion of the sum of the values of that variable for all supermarkets. Then a principal components analysis (PCA) was performed on the standardized variables. The PCA resulted in three components being extracted (see Table 2), which together accounted for 90.1 percent of the total variance. The variable which loaded strongest on each component,

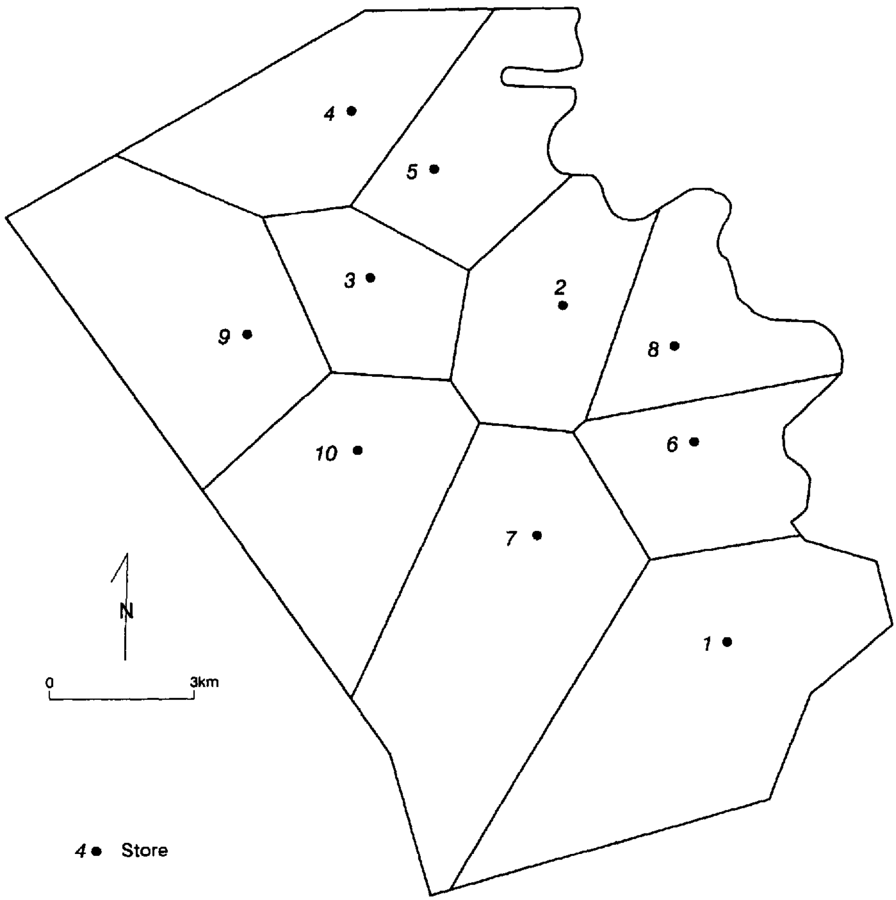

Figure 5. OVD for Zehrs' Supermarkets, Kitchener-Waterloo

SIZE (component 1), CTRSIZE (component 2) and ANCIL (component 3), was selected as representative of that component. Then the values of each of these variables were multiplied by the proportion of the total variance accounted for by its corresponding factor. Finally, the adjusted values of the three variables for each supermarket were summed to generate the supermarket's weight. The resulting weights can be seen in Figures 3, 4 and 6.

Figures 3 and 4 show the OKMWVD and OOKMVWD, respectively, for the Zehrs supermarkets for $k = 2$ (hereafter called O2MWVD and OO2MWVD, respectively). Order-2 was selected because the NADbank survey cited above indicated that, on average, a customer visited 1.66 different supermarkets in a one month period. Note that in Figures 3 and 4, the bracketed pairs of numbers represent the two most attractive facilities for customers located within that area and that only those trade area boundaries within K-W are shown. The trade areas in Figures 3 and 4 can be compared with those which would result if either of the existing Voronoi diagram models is used. Figure 5 shows the trade areas

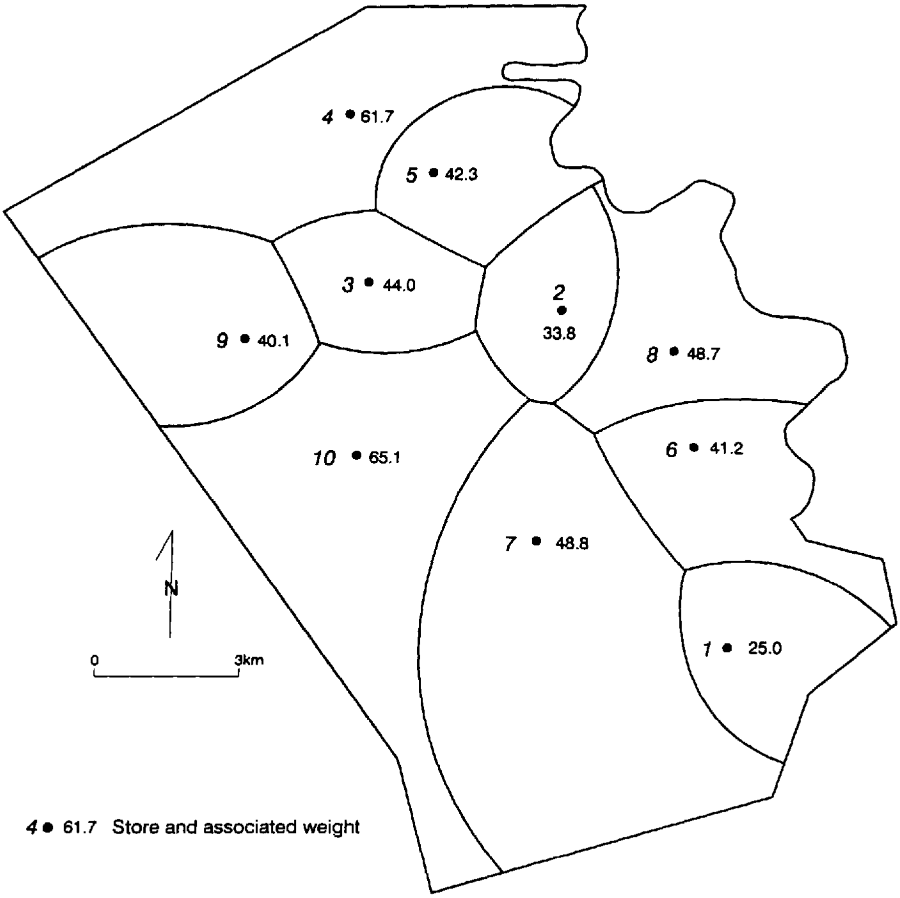

Figure 6. MWVD for Zehrs' Supermarkets, Kitchener-Waterloo

defined by the OVD. Recall that spatial proximity is the only consideration involved in demarcating these areas. This results in non-overlapping trade areas, which, in this case, are also more uniform in size than those of the O2MWVD and OO2MWVD. This is because the OVD under emphasizes the influence of more attractive stores and vice versa. While this feature is not a limitation of the MWVD, which considers the relative attractiveness of the stores, the MWVD still produces mutually exclusive trade areas as shown in Figure 6. We feel that comparisons such as these between the O2MWVD and the OO2MWVD on the one hand and the OVD and MWVD on the other, clearly demonstrate the improvements offered by the former models.

Sales Estimates

Any of the trade area configurations shown in Figures 3-6 can be used to generate sales estimates for individual supermarkets. Normally, at this stage, the analyst would obtain

TABLE 1

Variables Measuring Supermarket Attractiveness

Variable	*Description*
SIZE	Size of the store in square feet.
CTRSIZE	Size of the centre (i.e., mall or plaza) in which the store is located in square feet. If the store is stand alone, then SIZE = CTRSIZE.
AGE	Number of years since store opened.
TYPE	Number of stores located in the centre.
ANCHOR	Number of other anchor stores located in the centre.
HOURS	Total number of hours store is open during a one week period.
ANCIL	Number of ancillary features located in store (i.e., bank machine, debit card, post office).
INSTORE	Number of features related specifically to grocery shopping that store offers (i.e., bulk food, bakery, pharmacist, fish counter, wine shop, flower shop, small appliances, clothing, hardware, jewellery, furniture, restaurant, hair stylist).

maps showing the distribution of demographic and other customer information for the study area, over which the delimited trade areas could be overlain. In this illustration, in order to focus on our procedure, we assume a uniform distribution of sales potential (i.e., a uniform distribution of customers and disposable income). Thus, for both the OVD and the MWVD, in which trade areas are non-overlapping, a supermarket is seen to capture all of the sales potential in its trade area so that the volume of sales received is directly proportional to the size of the trade area.

In the O2MWVD customers are considered indifferent with respect to the two most attractive stores and patronize each with a probability of ½. Thus, the sales volume of store i is proportional to $\frac{1}{2}\sum_{\substack{j \\ j \neq i}} \left|V_w^{(2)}(p_i, p_j)\right|$, where $\left|V_w^{(2)}(p_i, p_j)\right|$ is the area of $V_w^{(2)}(p_i, p_j)$. That is to say, store i captures one-half of the sales potential from each region for which it is one of the two most attractive stores. In the OO2MWVD, while customers patronize two

TABLE 2

Rotated Component Loadings

Variable	*Component 1*	*Component 2*	*Component 3*
SIZE	0.92640*	0.06329	0.08212
CTRSIZE	−0.09227	0.96878*	−0.20270
AGE	−0.89798	0.19232	0.14972
TYPE	−0.14778	0.91347	−0.28782
ANCHOR	−0.04712	0.91746	0.24463
HOURS	0.84897	−0.23672	0.32841
ANCIL	0.12465	−0.10914	0.95082*
INSTORE	0.90035	−0.04623	0.12521
Variance Explained (%)	48.8	28.7	12.6

Note: * Variables which best summarize their respective components stores, they display a preference for the more attractive one, say, p_i, from the pair p_i, p_j. If P_{ij} ($1/2 < P_{ij} < 1$) is the probability of selecting p_i, $(1 - P_{ij})$ is the probability of selecting p_j. Then, the sales volume of store i is proportional to

$$\sum_{\substack{j \\ j \neq i}} P_{ij} \left| V_w^{<2>}(p_i, p_j) \right| + \sum_{\substack{j \\ j \neq i}} (1 - P_{ji}) \left| V_w^{<2>}(p_j, p_i) \right|, \tag{7}$$

where $|V_w{}^{<2>}(p_i, p_j)|$ is the area of $V_w{}^{<2>}(p_i, p_j)$. Thus, store i is seen as capturing a portion P_{ij} of the sales potential of the region for which it is the most attractive store and store j is the second most attractive store, and a proportion $(1 - P_{ji})$ of the sales potential of the region for which it is the second most attractive store and store j is the most attractive store. Note that as P_{ij} tends to 1, the sales estimates for any store in the OO2MWVD approach those of the MWVD. The probabilities P_{ij} and $(1 - P_{ij})$ can be considered equivalent to the proportions of market penetration by the two supermarkets in the region in question. The value of P_{ij} can be determined either endogenously (in relation to store characteristics) or exogenously. For example, in the former instance, for two stores F_1 and F_2 with associated weights w_1 and w_2 respectively, if F_1 is the more attractive store, we might set the probability of a customer patronizing F_1 as

$$P_{12} = 0.5 + 0.5\,[w_1/(w_1 + w_2)].$$

This format insures that the more attractive store will be assigned a probability between 0.5 and 1. However, if the probabilities are assigned exogenously, by adopting different values of P_{ij}, the analyst can examine the sensitivity of sales to different levels of preference for the more attractive store.

As an illustration of the potential of such an analysis, we examine three supermarkets which differ in terms of their attractiveness, supermarkets 1 (low), 7 (moderate) and 10 (high). The sales estimates derived using the methods described above for the different Voronoi models are summarized in Table 3. To simplify the presentation of the analysis for the OO2MWVD, we assume that the probability of a customer selecting the more attractive store from the pair of most attractive stores is the same for all stores. Thus, the terms P_{ij} and $(1 - P_{ji})$ in equation (7) are replaced by constant values P and $(1 - P)$, respectively. The OO2MWVD is then examined for three different values of P.

From Table 3 we see that, for supermarkets 1 and 10, the MWVD and the O2MWVD produce the lowest and highest sales volumes, respectively, with the OO2MWVD models occupying intermediate positions. The opposite is true for supermarket 7. These differences in the relative fortunes of the stores arise as a result of the particular spatial configuration of stores and their associated weights. The results indicate that, of the five scenarios represented by the Voronoi models, the one in which supermarkets 1 and 10 attract all of the customers for which they are the most attractive store and no one else, produces their lowest sales. Higher sales occur when these stores capture an increasing proportion of customers

TABLE 3

Sales Estimates for Selected Supermarkets for Different Voronoi Models

STORE	*WEIGHT*	*Sales Estimates*				
		MWVD	*O2MWVD*	*OO2MWVD (0.9./0.1)*	*OO2MWVD (0.75/0.25)*	*OO2MWVD (0.6/0.4)*
1	25.01	10.942	16.066	12.018	13.536	15.054
7	48.78	43.038	35.608	41.624	39.368	37.112
10	65.05	33.630	35.461	33.908	34.490	35.073

Notes: MWVD Multiplicatively weighted Voronoi diagram
O2MWVD order-2, multiplicatively weighted Voronoi diagram
OO2MWVD order-2, multiplicatively weighted Voronoi diagram. Numbers in Parentheses represent the probabilities of the store being selected by those customers for which it is the first and second most attractive store, respectively

for which they are the second most attractive store, even though this entails a corresponding reduction in their share of customers for which they are the most attractive store. Indeed, their highest sales are achieved in that situation where customers are indifferent between the two most attractive stores (i.e., when these stores capture equal proportions of those customers for which they are the first or second most attractive store). In this situation, supermarket 1would receive a 47 percent increase in sales relative to the MWVD scenario. Supermarket 7 on the other hand achieves highest sales in that situation in which it captures all of those customers for which it is the most attractive store. Sales estimates do not increase in those scenarios in which it attracts an increasing share of those customers for which it is the second most attractive store.

As expressed in equation (7), we recognize that the value of P_{ij} need not be the same for all stores, and might even be different for different competitors of the same store. In such circumstances, the number of scenarios to be considered will be much larger than those in the above example. Nevertheless, we feel this example is sufficient to indicate that results such as those in Table 3 have obvious implications for stores in terms of marketing strategy. For example, they can indicate where advertising and sales promotional efforts might best be aimed. However, if a store decides to increase its market share by adjusting features which influence its attractiveness, the weight attached to that store would change and so too would all the Voronoi diagrams described in this paper, with the exception of the ordinary Voronoi diagram (which ignores store attractiveness). Fortunately, in such circumstances it is not necessary to construct anew the entire diagram. Because the Voronoi diagram is a local geometric structure, it is only necessary to re-construct those edges of the diagram which are associated with the store with changed weight.

SUMMARY AND CONCLUSIONS

We have presented two new trade area models, the OKMWVD and the OOKMWVD, which extend existing types of Voronoi diagram models by incorporating assumptions about consumer choice behavior. These new models are more realistic in that they generate trade areas which are both overlapping and probabilistic rather than deterministic spatial monopolies. In addition, we demonstrated how the OOKMWVD can be used to examine the sensitivity of a store's potential sales to changes in customers' relative preferences for the stores they consider the most attractive. As illustrated in our supermarket example, such information could have important ramifications for the marketing strategies adopted by stores. Although this illustration considered supermarkets belonging to only one chain, competing chains can be readily accommodated by simply adding their stores to the spatial pattern. While adding supermarkets can lead to changes in the weights of stores already included in the pattern, if the weights are derived in a relative fashion (as in the illustration), it will always produce changes in one or more trade areas. When the set of supermarkets consists of more than one chain, we have the additional possibility of considering aspects of spatial competition between chains such as the extent to which their trade areas overlap (Von Hohenbalken and West, 1984).

Although our research suggests that the new models provide a more effective method of trade area definition, several areas remain to be explored. First, trade area demarcation is sensitive to how distance is measured. While euclidean distance was used in this study, other metrics, such as the Manhattan, may be more appropriate in different circumstances. We may also wish to consider the effects of physical and psychological barriers on consumer movement. Second, we used a value of $k = 2$ (customers patronize two stores) since we were dealing with a low order function. Other values of k might be more appropriate for other kinds of functions. Indeed, it would be interesting to examine how the relative size of a store's trade area responds to changes in k. Again, in empirical situations, such knowledge could be helpful in developing an appropriate marketing strategy. Finally, although we used a multiplicative weighting scheme, other weighting schemes are possible including additive and compound forms (Okabe et al., 1992). Adoption of such schemes would permit the incorporation of additional characteristics of both consumer and store behavior into the models.

APPENDIX A: SET THEORY NOTATION

A set P consisting of n elements, $p_1, ..., p_n$, is written as $P = \{p_1, ..., p_n\}$.

$I_n = \{1, 2, ..., n\}$ is the set of natural numbers from 1 to n, where the suffix n indicates the maximum integer in the set.

A set of elements x satisfying some conditions is written $\{x | \text{conditions}\}$.

If an element x belongs to a set P we denote it by $x \in P$.

The complement of set B with respect to set A, denoted by $A \backslash B$, is the set of all elements of A that do not belong to B.

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