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Estimation of $P(Z < Y)$ for correlated stochastic time series models

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Abstract

Let Z and Y represent two time series that are not necessarily independent, and Z_{n+L}, Y_{m+k} denote their values respectively at future times $n+L$ and $m+k$, where $n+L=m+k$. Autoregressive (AR), Moving Average (MA), and Autoregressive Moving Average (ARMA) models are employed both under stationary and non-stationary conditions to estimate $P(Z_{n+L} < Y_{m+k}|\underline{z},\underline{y})$, where $\underline{z}=z_1,\ldots,z_n$, $\underline{y}=y_1,\ldots,y_m$, represent samples of size n and m from the time series Z and Y, respectively. Simulation studies are used to assess the accuracy of the estimation. © 1999 Elsevier Science Inc. All rights reserved.

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1. Introduction

Let $\underline{z}=z_1,\ldots,z_n$ and $\underline{y}=y_1,\ldots,y_m$ represent observed values of two correlated time series, Z and Y, respectively. Z_{n+L} and Y_{m+k} denote values of Z and Y at the future times $n+L$ and $m+k$, respectively, $L,k=1,2,3,\ldots$ Often in practice, statisticians, quality control engineers or other applied practitioners of statistics face the problem of estimating $Q=P(Z_{n+L} < Y_{m+k}|\underline{z},\underline{y})$, where $n+L=m+k$. Some applications in which estimation of Q can be of the interest are:

1. Reliability analysis of stress-strength relationships in which a component with a random stress (Z) is associated with a random strength (Y). The reliability of the component is measured by the probability that the strength exceeds the stress.

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2. In situations where Z and Y represent lifetimes of two devices and the estimation of the probability that one of the devices will fail before the other is needed.
3. Z and Y represent prices of two stocks. Prices change according to stochastic time series models, and at a future time, $n+L=m+k$, $P(Z_{n+L} < Y_{m+k}|\underline{z}, \underline{y})$ is to be estimated.

Guttman et al. (1988) found confidence bounds for Q for the case in which explanatory variables are associated with Z and Y, and that their distributions are assumed to be normal. Aminzadeh (1997) derived confidence bounds for Q for the exponential distribution by using a regression model. In the normal case some results for Q can be found in Reiser and Guttman (1986, 1987). Aminzadeh (1991) obtained approximate confidence bounds for Q in a 1-way ANOVA random model and Chao (1982) compared several estimators of Q for the exponential case based on Maximum Likelihood Estimates (MLE), in the article explanatory variables are not considered to be part of the inference.

In this article it is assumed that values of the random variables Z and Y change over time. Thus, the assumption of independence for Z_i $(i=1,2,\ldots)$ and similarly for Y_i $(i=1,2,\ldots)$ which is made in Guttman et al. (1988), Aminzadeh (1991) and Aminzadeh (1997), is not valid in this article. Autoregressive (AR), Moving Average (MA), and Autoregressive Moving Average (ARMA) models will be used to represent behavior of the time series Z and Y. In practice there might be some correlation between the time series Z and Y, therefore the assumption of independence seems to be very limited. In this article it is assumed that white noise processes associated with Z and Y have the bivariate normal distribution. The stationary condition is used for the models in Sections 2–4, and estimation of Q when time series are non-stationary is discussed in Section 5. The main assumption here is that time series analysis have been performed on the series Z and Y, appropriate models are fitted, and estimation of Q is of the interest. Although for the purpose of this article it is not necessary to assume that behavior of Z and Y series are represented by the same model, however for the simplicity of presentation of the results, this assumption is made.

2. Autoregressive (AR) models

Let the series Z and Y be represented by autoregressive models of order p

$$Z_t - \mu = \sum_{i=1}^{p} \phi_i (Z_{t-i} - \mu) + a_t, \tag{1}$$

$$Y_t - \xi = \sum_{i=1}^{p} \theta_i (Y_{t-i} - \xi) + b_t,$$

respectively, where, $\phi_i, \theta_i (i = 1, 2, \dots, p)$ are the autoregressive parameters, and μ, ξ are mean of the series Z and Y, respectively. a_t and b_t are white-noise processes with $E(a_t) = E(b_t) = 0$, $\mathrm{Cov}(a_t, a_{t\pm s}) = \mathrm{Cov}(b_t, b_{t\pm r}) = 0$, $r, s \neq 0$, $\mathrm{Var}(a_t) = \sigma_z^2$, $\mathrm{Var}(b_t) = \sigma_y^2$. Since the assumption of independence is very limited in practice, we assume that the joint probability distribution of a_t and b_t is a bivariate normal distribution, so that, $\mathrm{Cov}(a_{t\pm s}, b_{t\pm r}) = \rho\sigma_z\sigma_y, \forall\ r$ and s.

Let $Z_n(L) = E(Z_{n+L}|\underline{z})$ and $Y_m(k) = E(Y_{m+k}|\underline{y})$. Using Eq. (1) we get

$$Z_n(L) = \mu + \sum_{i=1}^{p} \phi_i (Z_n(L-i) - \mu). \tag{2}$$

Now, let $e_n(L) = Z_{n+L} - Z_n(L)$ represent the difference between actual and expected value of Z at the time $n + L$. It can be shown that

$$e_n(L) = \sum_{v=0}^{L-1} R_v a_{n+L-v},$$

where, $R_0 = 1, R_1 = \phi_1, \dots, R_v = \sum_{i=1}^{p} \phi_i R_{v-i}$. Using the assumptions for the white-noise process, a_t, the variance of $e_n(L)$ is given by

$$\mathrm{Var}(e_n(L)) = \mathrm{Var}(Z_{n+L}) = \sigma_z^2 \sum_{v=0}^{L-1} R_v^2.$$

Similarly, letting $\epsilon_m(k) = Y_{m+k} - Y_m(k)$, the following results can be obtained for the time series Y,

$$Y_m(L) = \xi + \sum_{i=1}^{p} \theta_i (Y_m(k-i) - \xi), \qquad \epsilon_m(k) = \sum_{w=0}^{k-1} \delta_w b_{m+k-w},$$

$$\mathrm{Var}(Y_{m+k}) = \sigma_y^2 \sum_{w=0}^{k-1} \delta_w^2,$$

$\delta_0 = 1, \delta_1 = \theta_1, \delta_w = \sum_{i=1}^{p} \theta_i \delta_{w-i}$. Since $e_n(L)$ and $\epsilon_m(k)$ are both linear combination of normal variables, their marginal distributions are normal. Therefore Z_{n+L}, Y_{m+k} are normally distributed as well. Hence,

$$Q = P(Z_{n+L} < Y_{m+k}|\underline{z}, \underline{y}) = \Phi(\delta),$$

$$\begin{aligned} \delta &= \frac{Y_m(k) - Z_n(L)}{\sqrt{\mathrm{Var}(e_n(L)) + \mathrm{Var}(\epsilon_m(k)) - 2\mathrm{Cov}(e_n(L), \epsilon_m(k))}}, \\ &= \frac{(\xi - \mu) + \sum_{i=1}^{p} \theta_i (Y_m(k-i) - \xi) - \sum_{i=1}^{p} \phi_i (Z_n(L-i) - \mu)}{\sqrt{\sigma_z^2 \sum_{v=0}^{L-1} R_v^2 + \sigma_y^2 \sum_{w=0}^{K-1} \delta_w^2 - 2\rho\sigma_z\sigma_y \sum_{v=0}^{L-1} \sum_{w=0}^{k-1} R_v \delta_w}}. \end{aligned} \tag{3}$$

$\Phi(\cdot)$ is the cdf of the standard normal distribution. For variety of models that will be considered in this article, a similar procedure to this section will be used to derive closed form formulas for δ. Estimation of Q requires estimates of the parameters involved in δ. SAS, MINITAB or other statistical packages provide estimates for parameters of a specified time series model. For AR(1) model that will be used in simulation studies (Section 6), it can be shown that

$$\delta = \frac{(\xi - \mu) + \theta^k (y_m - \xi) - \phi^L (z_n - \mu)}{\sqrt{\frac{\sigma_z^2(1-\phi^{2L})}{1-\phi^2} + \frac{\sigma_y^2(1-\theta^{2k})}{1-\theta^2} - 2\rho\sigma_z\sigma_y \sum_{v=1}^{L} \sum_{w=1}^{k} \phi^{L-v}\theta^{k-w}}}, \tag{4}$$

where ϕ and θ are AR parameters of AR(1) models for the Z and Y series, respectively.

3. Moving average (MA) models

The moving average models of order q for the series Z and Y are defined as

$$Z_t - \mu = a_t - \sum_{j=1}^{q} \gamma_j a_{t-j}, \qquad Y_t - \xi = b_t - \sum_{j=1}^{q} \varphi_j b_{t-j},$$

where, γ_j, φ_j $(j = 1, 2, \ldots, q)$ are MA parameters.

For these models the following can be shown:

$$Z_n(L) = \mu, \qquad e_n(L) = \sum_{j=0}^{q} \gamma_j a_{n+L-j}, \quad \gamma_o = -1,$$

$$\mathrm{Var}(Z_{n+L}) = \sigma_z^2 \left(1 + \sum_{j=1}^{q} \gamma_j^2 \right),$$

$$Y_m(k) = \xi, \qquad \epsilon_m(k) = \sum_{j=0}^{q} \varphi_j b_{m+k-j}, \quad \varphi_o = -1,$$

$$\mathrm{Var}(Y_{m+k}) = \sigma_y^2 \left(1 + \sum_{j=1}^{q} \varphi_j^2 \right),$$

and as the result,

$$\delta = \frac{\xi - \mu}{\sqrt{\sigma_z^2(1 + \sum_{j=1}^{q} \gamma_j^2) + \sigma_y^2(1 + \sum_{j=1}^{q} \varphi_j^2) - 2\rho\sigma_z\sigma_y \sum_{j=0}^{q} \sum_{j=0}^{q} \gamma_j \varphi_j}}.$$

4. Autoregressive moving average (ARMA) models

Autoregressive moving average models which are used more often in practice will be considered in this section. ARMA(p,q) models are given by

$$Z_t - \mu = \sum_{i=1}^{p} \phi_i (Z_{t-i} - \mu) - \sum_{j}^{q} \gamma_j a_{t-j} + a_t,$$

$$Y_t - \xi = \sum_{i=1}^{p} \theta_i (Y_{t-i} - \xi) - \sum_{j}^{q} \varphi_j b_{t-j} + b_t.$$

It can be shown that

$$Z_n(L) = \mu + \sum_{i=1}^{p} \phi_i (Z_n(L-i) - \mu), \tag{5}$$

$$e_n(L) = \mu + \sum_{i=1}^{p} \phi_i (Z_{n+L-i} - Z_n(L-i)) - \sum_{j=1}^{q} \gamma_j a_{n+L-j} + a_{n+L},$$

and as a result

$$\mathrm{Var}(Z_{n+L}) = \mathrm{Var}(e_n(L)) = \sigma_z^2 \left(\sum_{v=0}^{L-1} R_v^2 + \sum_{j=1}^{q} \gamma_j^2 \right),$$

where, R_v and δ_w are defined in Section 2. Similarly for the series Y, we get

$$Y_m(k) = \xi + \sum_{i=1}^{p} \theta_i (Y_m(k-i) - \xi),$$

$$\epsilon_m(k) = \xi + \sum_{i=1}^{p} \theta_i (Y_{m+k-i} - Y_m(k-i)) - \sum_{j}^{q} \varphi_j b_{m+k-j} + b_{m+k},$$

$$\mathrm{Var}(Y_{m+k}) = \mathrm{Var}(\epsilon_m(k)) = \sigma_y^2 \left(\sum_{w=0}^{k-1} \delta_w^2 + \sum_{j=1}^{q} \varphi_j^2 \right).$$

δ for ARMA(p,q) takes the form,

$$\delta = \frac{(\xi - \mu) + \sum_{i=1}^{p} \theta_i (Y_m(k-i) - \xi) - \sum_{i=1}^{p} \phi_i (Z_n(L-i) - \mu)}{\sqrt{\mathrm{Var}(e_n(L)) + \mathrm{Var}(\epsilon_m(k)) - 2\rho\sigma_z\sigma_y \mathrm{Cov}(e_n(L), \epsilon_m(k))}},$$

where, $\mathrm{Cov}((e_n(L), \epsilon_m(k))$ can be written using the representations for $e_n(L)$, and $\epsilon_m(k)$, similar to the previous sections.

In Section 6, ARMA(1,1) will be used for simulation studies. For the ARMA(1,1) models it can be shown that

$$z_n(L) = \mu + \phi^L (z_n - \mu,\ y_m(k) = \xi + \theta^k (y_m - \xi), \tag{6}$$

$$\mathrm{Var}(e_n(L)) = \sigma_z^2 \left[1 + \gamma^2 \phi^{2(L-1)} + (\gamma - \phi)^2 \sum_{v=1}^{L-1} \phi^{2(L-v-1)} \right], \tag{7}$$

$$\mathrm{Var}(\epsilon_m(k)) = \sigma_y^2 \left[1 + \varphi^2 \theta^{2(k-1)} + (\varphi - \theta)^2 \sum_{w=1}^{k-1} \theta^{2(k-w-1)} \right],$$

$$\begin{aligned}\mathrm{Cov}(e_n(L),\epsilon_m(k)) = \rho\sigma_z\sigma_y\Bigg[&1-(\varphi-\theta)\sum_{w=1}^{k-1}\theta^{(k-w-1)}-\varphi\theta^{k-1}\\ &-(\gamma-\phi)\sum_{v=1}^{L-1}\phi^{(L-v-1)}\\ &+(\varphi-\theta)(\gamma-\phi)\sum_{v=1}^{L-1}\sum_{w=1}^{k-1}\theta^{(k-w-1)}\phi^{(L-v-1)}\\ &+(\gamma-\phi)(\varphi\theta)^{k-1}\sum_{v=1}^{L-1}\phi^{(L-v-1)}-\gamma\phi^{L-1}\\ &+\gamma\phi^{L-1}(\varphi-\theta)\sum_{w=1}^{k-1}\theta^{(k-w-1)}+\ \gamma\varphi\phi^{L-1}\theta^{k-1}\Bigg].\end{aligned}$$

5. Non-stationary ARMA models

Often the condition of the stationary that we have assumed in Sections 2–4 is not valid for a time series. In practice the difference transformations on a non-stationary series are used to make the series satisfy the stationary conditions. The degree of difference usually is small in practice. Note that differences of a stationary process is still a stationary process. Therefore in this section only differences of degree one and two will be considered. ARMA(p,d,q) denotes an autoregressive moving average time series of order (p,q) that can be made stationary after applying the difference transformation of degree d. The difference transformation of degree d for a non-stationary series Z is defined by $X_t=(1-B)^dZ_t$, where B is a backward-shift operator. For example, $B^dZ_t=Z_{t-d}$.

5.1. ARMA(1,1,1)

Let Z be a non-stationary time series and its first-degree difference series, X_t, defined by $X_t=(1-B)Z_t=Z_t-Z_{t-1}$, can be modeled by ARMA(1,1). Also let Y be a non-stationary time series, and its first-degree difference is defined by $T_t=Y_t-Y_{t-1}$, which is also modeled by ARMA(1,1). Without loss of generality assume that the means of the series X and T are zero. We have

$$X_t=\phi X_{t-1}-\gamma a_{t-1}+a_t,$$

$$T_t=\theta T_{t-1}-\varphi b_{t-1}+b_t.$$

Since $X_t=Z_t-Z_{t-1}$, then $Z_t=X_t+Z_{t-1}$. Now applying this relation iteratively, we get

$$Z_{n+L} = \sum_{v=1}^{L} X_{n+v} + z_n. \tag{8}$$

Note that $\sum_{v=1}^{L} X_{n+v}$ in Eq. (8) is a random variable. Since X is modeled by ARMA(1,1), and its mean is assumed to be zero, using Eq. (6) we get

$$Z_n(L) = E(Z_{n+L}|\underline{z}) = z_n + \sum_{v=1}^{L} \phi^v X_n = z_n + (z_n - z_{n-1}) \sum_{v=1}^{L} \phi^v.$$

$$e_n(v) = X_{n+v} - X_n(v) = a_{n+v} - (\gamma - \phi) \sum_{s=1}^{v-1} \phi^{(v-s-1)} a_{n+s} - \gamma \phi^{(v-1)} a_n,$$

hence,

$$\mathrm{Var}(Z_{n+L}) = \sigma_z^2 \left(\sum_{v=1}^{L} \left[1 + (\gamma - \phi)^2 \sum_{s=1}^{v-1} \phi^{2(v-s-1)} + \gamma^2 \phi^{2(v-1)} \right] \right).$$

Similarly, for the series Y we have:

$$Y_m(k) = y_m + \sum_{w=1}^{k} \theta^w T_m = y_m + (y_m - y_{m-1}) \sum_{w=1}^{k} \theta^w,$$

$$\epsilon_m(w) = b_{m+w} - (\varphi - \theta) \sum_{d=1}^{w-1} \theta^{(w-d-1)} b_{m+d} - \varphi \theta^{(w-1)} b_m,$$

$$\mathrm{Var}(\epsilon_m(k)) = \sigma_y^2 \left(\sum_{w=1}^{k} \left[1 + (\varphi - \theta)^2 \sum_{d=1}^{w-1} \theta^{2(w-d-1)} + \varphi^2 \theta^{2(w-1)} \right] \right).$$

Now,

$$\delta = \frac{y_m + (y_m - y_{m-1}) \sum_{w=1}^{k} \theta^w - z_n - (z_n - z_{n-1}) \sum_{v=1}^{L} \phi^v}{\sqrt{\mathrm{Var}(e_n(L)) + \mathrm{Var}(\epsilon_m(k)) - 2\mathrm{Cov}(e_n(L), \epsilon_m(k))}},$$

where,

$$\begin{aligned} \mathrm{Cov}(e_n(L), \epsilon_m(k)) = \rho \sigma_z \sigma_y \sum_{v=1}^{L} \sum_{w=1}^{k} \Bigg[& 1 - (\varphi - \theta) \sum_{d=1}^{w-1} \theta^{(w-d-1)} \\ & - \varphi \theta^{w-1} - (\gamma - \phi) \sum_{s=1}^{v-1} \phi^{(v-s-1)} \\ & + (\varphi - \theta)(\gamma - \phi) \sum_{s=1}^{v-1} \sum_{d=1}^{w-1} \theta^{(w-d-1)} \phi^{(v-s-1)} \\ & + (\gamma - \phi)(\varphi \theta)^{v-1} \sum_{s=1}^{v-1} \phi^{(v-s-1)} - \gamma \phi^{v-1} \\ & + \gamma \phi^{v-1} (\varphi - \theta) \sum_{d=1}^{w-1} \theta^{(w-d-1)} + \gamma \varphi \phi^{v-1} \theta^{w-1} \Bigg]. \end{aligned}$$

5.2. ARIMA(1,2,1)

Suppose Z and Y are non-stationary time series and their second-degree difference series,

$$X_t = (1-B)^2 Z_t = Z_t - 2Z_{t-1} + Z_{t-2}, \tag{9}$$

$$T_t = (1-B)^2 Y_t = Y_t - 2Y_{t-1} + Y_{t-2},$$

are stationary ARMA(1,1) processes as defined in Section 4. From Eq. (9), we get $Z_t = X_t + 2Z_{t-1} - Z_{t-2}$, which implies

$$Z_{n+L} = (L+1)z_n - Lz_{n-1} + \sum_{v=1}^{L}(L-v+1)X_{n+v}.$$

Since $X_{n+1}, X_{n+2}, \ldots$ are all random variables and $z_n, z_{n-1}, \ldots$ are all observed, using Eq. (6)

$$Z_n(L) = \sum_{v=1}^{L}(L-v+1)(\phi^v X_n) + (L+1)z_n - Lz_{n-1}.$$

Now, to find variance of Z_{n+L}, a general formula for $e_n(L)$ has to be found. Noting that by the assumptions, mean of X is zero, it can be shown that

$$e_n(1) = a_{n+1} - \gamma a_n,$$

$$e_n(L) = a_{n+L} + \sum_{v=1}^{L-1}\left(\sum_{s=1}^{v+1} s\phi^{v+1-s} - \gamma\sum_{s=1}^{v} s\phi^{v-s}\right)a_{n+L-v} - \gamma\sum_{v=1}^{L} v\phi^{L-v}a_n, \quad L > 1$$

and as the result

$$\mathrm{Var}(Z_{n+L}) = \sigma_z^2\left[1 + \left(\sum_{v=1}^{L-1}\left[\sum_{s=1}^{v+1} s\phi^{(v+1-s)} - \gamma\sum_{s=1}^{v} s\phi^{(v-s)}\right]^2\right) + \gamma^2\left(\sum_{v=1}^{L} v\phi^{L-v}\right)^2\right].$$

Similarly for the time series Y we have

$$Y_m(k) = \sum_{w=1}^{k}(k-w+1)(\theta^w T_m) + (k+1)y_m - ky_{m-1},$$

$$\epsilon_m(k) = b_{m+k} + \sum_{w=1}^{k-1}\left(\sum_{d=1}^{w+1} d\theta^{w+1-d} - \varphi\sum_{d=1}^{w} d\theta^{w-d}\right)b_{m+k-w} - \varphi\sum_{w=1}^{k} w\theta^{k-w}b_m,$$

$$\mathrm{Var}(Y_{m+k}) = \sigma_y^2\left[1 + \left(\sum_{w=1}^{k-1}\left[\sum_{d=1}^{w+1} d\theta^{(w+1-d)} - \varphi\sum_{d=1}^{w} d\theta^{(w-d)}\right]^2\right)\right.$$
$$\left. + \varphi^2\left(\sum_{w=1}^{k} w\theta^{k-w}\right)^2\right].$$

For the ARMA(1,2,1) model, the $\mathrm{Cov}(e_n(L), \epsilon_m(k))$ is given by

$$\begin{aligned}
\mathrm{Cov}(e_n(L), \epsilon_m(k)) = \rho\sigma_z\sigma_y\Bigg[1 &+ \sum_{w=1}^{k-1}\left(\sum_{d=1}^{w+1} d\theta^{w+1-d} - \varphi\sum_{d=1}^{w} d\theta^{w-d}\right)\\
&- \varphi\sum_{w=1}^{k} w\theta^{k-w} + \sum_{v=1}^{L-1}\left(\sum_{s=1}^{v+1} s\phi^{v+1-s} - \gamma\sum_{s=1}^{v} s\phi^{v-s}\right)\\
&+ \sum_{v=1}^{L-1}\sum_{w=1}^{k-1}\left(\sum_{s=1}^{v+1} s\phi^{v+1-s} - \gamma\sum_{s=1}^{v} s\phi^{v-s}\right)\\
&\times\left(\sum_{d=1}^{w+1} d\theta^{w+1-d} - \varphi\sum_{d=1}^{w} d\theta^{w-d}\right)\\
&- \varphi\sum_{v=1}^{L-1}\sum_{w=1}^{k}\left(\sum_{s=1}^{v+1} s\phi^{v+1-s} - \gamma\sum_{s=1}^{v} s\phi^{v-s}\right)\left(w\theta^{k-w}\right)\\
&- \gamma\sum_{v=1}^{L} v\phi^{L-v} - \gamma\sum_{v=1}^{L}\sum_{w=1}^{k-1}(v\phi^{L-v})\left(\sum_{d=1}^{w+1} d\theta^{w+1-d} - \varphi\sum_{d=1}^{w} d\theta^{w-d}\right)\\
&+ \gamma\varphi\sum_{v=1}^{L}\sum_{w=1}^{k} vw\phi^{L-v}\theta^{k-w}\Bigg].
\end{aligned}$$

In Sections 5.1 and 5.2 first and second degree difference transformation are considered. A higher degree difference transformations, or a different order for ARMA models can be employed to estimate Q. The main job is to write $Z_n(L), Y_m(k), \mathrm{Var}(e_n(L)), \mathrm{Var}(\epsilon_m(k))$, and as the result, δ in closed forms in such a way that estimated values of parameters obtained from fitting time series models can be used to get an estimate for Q.

6. Simulation

To assess the validity of the method of estimation for Q, AR(1) and ARMA(1,1) models are employed in simulation studies. For selected values of parameters, samples of size $n = 100$ for the series Z and $m = 80$ for the series Y based on the AR(1) model are generated 100 times.

Since by the assumption, the Z and Y series are correlated, simulation programs are designed accordingly to generate a_t and b_t white noise processes which have the bivariate normal distribution $N(0, 0, \sigma_z, \sigma_y, \rho)$. Morrison (Morrison, 1976, pp. 89–90) gives equations of ellipses that represent the relationship of bivariate normal variables. To generate the white noise processes a_t, b_t, first two independent normal variables W_1 and W_2 with the distributions $N(0, \sqrt{1+\rho})$ and $N(0, \sqrt{1-\rho})$, respectively, are generated. a_t, b_t are found using the equations $a_t = (\sigma_z/\sqrt{2})(w_1 + w_2)$, $b_t = (\sigma_y/\sqrt{2})(w_1 - w_2)$.

For each of 100 runs, δ values using (4), and as the result hundred true values of $Q = P(Z_{n+L} < Y_{m+k})$ (each based on simulated data) are found. Also using each of 100 pairs of samples, AR(1) models are fitted for Z and Y series. Having estimated values of the parameters for each model, the corresponding one hundred $\hat{Q}$ values are obtained. A similar simulation studies are employed for ARMA(1,1) models based on the results given in section 4. Tables 1 and 2 list simulated values of Mean Square Error, $\mathrm{MSE} = \zeta = E(\hat{Q} - Q)^2$. Examination of Tables 1 and 2 reveals that: (1) As the difference between mean of the time series, μ and ξ increases, MSE decreases, as expected. (2) As the

Table 1
Simulated values of ζ for AR(1) Models with $n = 100$, $m = 80$

$(\phi\ \theta\ \mu\ \xi\ \sigma_z\ \sigma_y)$ (L, k, ρ):ζ		
(0.8, −0.6, 2, 2.2, 1.2, 1.5)	(0.3, −0.7, 5, 4, 0.8, 1.2)	(0.5, −0.1, 10, 2, 4, 1)
(20, 40, 0.05):$\zeta = 0.0056$	(20, 40, 0.05):$\zeta = 0.00157$	(20, 40, 0.05):$\zeta = 0.00054$
(05, 25, 0.05):$\zeta = 0.0041$	(05, 25, 0.05):$\zeta = 0.00145$	(05, 25, 0.05):$\zeta = 0.00052$
(20, 40, 0.20):$\zeta = 0.0116$	(20, 40, 0.20):$\zeta = 0.00162$	(20, 40, 0.20):$\zeta = 0.00067$
(05, 25, 0.20):$\zeta = 0.0042$	(05, 25, 0.20):$\zeta = 0.00159$	(05, 25, 0.20):$\zeta = 0.00066$
(20, 40, 0.40):$\zeta = 0.0146$	(20, 40, 0.40):$\zeta = 0.00174$	(20, 40, 0.40):$\zeta = 0.00074$
(05, 25, 0.40):$\zeta = 0.0059$	(05, 25, 0.40):$\zeta = 0.00169$	(05, 25, 0.40):$\zeta = 0.00072$

Table 2
Simulated values of ζ for ARMA(1,1) Models with $n = 100$, $m = 80$

$(\phi\ \gamma\ \theta\ \varphi)$ $(\mu\ \xi\ \sigma_z\ \sigma_y)$ (L, k, ρ):ζ		
(0.8, 0.3, −0.6, 0.5)	(0.3, 0.9, −0.7, −0.6)	(0.5, 0.4, −0.1, 0.5)
(2, 2.2, 1.2, 1.5)	(5, 4, 0.8, 1.2)	(10, 2, 4, 1)
(20, 40, 0.05):$\zeta = 0.0064$	(20, 40, 0.05):$\zeta = 0.00134$	(20, 40, 0.05):$\zeta = 0.00034$
(05, 25, 0.05):$\zeta = 0.0059$	(05, 25, 0.05):$\zeta = 0.00123$	(05, 25, 0.05):$\zeta = 0.00030$
(20, 40, 0.20):$\zeta = 0.0125$	(20, 40, 0.20):$\zeta = 0.00142$	(20, 40, 0.20):$\zeta = 0.00059$
(05, 25, 0.20):$\zeta = 0.0056$	(05, 25, 0.20):$\zeta = 0.00139$	(05, 25, 0.20):$\zeta = 0.00051$
(20, 40, 0.40):$\zeta = 0.0137$	(20, 40, 0.40):$\zeta = 0.00156$	(20, 40, 0.40):$\zeta = 0.00063$
(05, 25, 0.40):$\zeta = 0.0069$	(05, 25, 0.40):$\zeta = 0.00146$	(05, 25, 0.40):$\zeta = 0.00058$

correlation, ρ increases, MSE increases. This means for highly (positively) correlated time series the estimation error for Q is somewhat greater. Tables suggest that this is particularly the case when the difference between μ and ξ is small, as the difference between μ and ξ gets larger, higher correlation does not increase MSE as much. (3) As values of L and k decrease, MSE decreases as well, meaning that error of estimation for Q for shorter periods is smaller. Since in practice estimation of Q for negative correlation is not of interest, in the simulations studies negative values of ρ are not used.

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