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Characterization of graphs with equal domination and covering number

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Abstract

Let G be a simple graph of order $n(G)$. A vertex set D of G is dominating if every vertex not in D is adjacent to some vertex in D, and D is a covering if every edge of G has at least one end in D. The domination number $\gamma(G)$ is the minimum order of a dominating set, and the covering number $\beta(G)$ is the minimum order of a covering set in G. In 1981, Laskar and Walikar raised the question of characterizing those connected graphs for which $\gamma(G)=\beta(G)$. It is the purpose of this paper to give a complete solution of this problem. This solution shows that the recognition problem, whether a connected graph G has the property $\gamma(G)=\beta(G)$, is solvable in polynomial time. As an application of our main results we determine all connected extremal graphs in the well-known inequality $\gamma(G)\leqslant\lfloor n(G)/2\rfloor$ of Ore (1962), which extends considerable a result of Payan and Xuong from 1982. With a completely different method, independently around the same time, Cockayne, Haynes and Hedetniemi also characterized the connected graphs G with $\gamma(G)=\lfloor n(G)/2\rfloor$. © 1998 Elsevier Science B.V. All rights reserved

1. Terminology

We consider finite, undirected, and simple graphs G with the vertex set $V(G)$ and the edge set $E(G)$. For $A\subseteq V(G)$ let $G[A]$ be the subgraph induced by A. A subgraph H of G with $V(H)=V(G)$ is called a factor of G. $N(x)=N(x,G)$ denotes the set of vertices adjacent to the vertex x and $N[x]=N[x,G]=N(x)\cup\{x\}$. More generally, we define $N(X)=N(X,G)=\bigcup_{x\in X}N(x)$ and $N[X]=N[X,G]=N(X)\cup X$ for a subset X of $V(G)$. The vertex v is an end vertex if $d(v,G)=1$, and an isolated vertex if $d(v,G)=0$, where $d(x)=d(x,G)=|N(x)|$ is the degree of $x\in V(G)$. Let $\Omega=\Omega(G)$ be the set of end vertices and let $I=I(G)$ be the set of isolated vertices of G. We denote by $\delta=\delta(G)$ the minimum degree and by $n=n(G)=|V(G)|$ the order of G. We write C_n for a cycle of length n and K_n for the complete graph of order n. A star is a

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complete bipartite graph $K_{1,m}$ with $m \geqslant 2$, and the unique vertex v of this star of degree m is called the center.

A set $D \subseteq V(G)$ is a dominating set of G if $N[D,G]=V(G)$, and D is a covering of G if every edge of G has at least one end in D. The domination number $\gamma(G)$, and the covering number $\beta(G)$ of G is the cardinality of a smallest dominating set, and a smallest covering of G, respectively. A set of pairwise non-adjacent vertices of G is an independent set of G. The cardinality of a maximum independent set is called the independence number $\alpha(G)$ of the graph G. Let T be a non-trivial tree. If we substitute each edge in T by two parallel edges and then subdivide each edge we call this resulting graph a C_4-cactus. Let G be a graph and $\mathscr{F}=\{H_x \mid x \in V(G) \text{ and } H_x \neq \emptyset\}$ be a family of graphs disjoint from each other and from G, indexed by the vertices of G. The corona $G \circ \mathscr{F}$ of the graph G and the family $\mathscr{F}$ is the disjoint union of G and the graphs H_x, $x \in V(G)$, with additional edges joining each vertex v of G to all vertices of H_v. If all graphs of the family $\mathscr{F}$ are isomorphic to one and the same graph H, then we shall write $G \circ H$ instead of $G \circ \mathscr{F}$.

2. Preliminary results

The first three propositions are easy to prove and well-known.

Proposition 2.1. *If G is a graph, then $\gamma(G) \leqslant \alpha(G)$, and if $I(G)=\emptyset$, then $\gamma(G) \leqslant \beta(G)$.*

Proposition 2.2 (Gallai [3]). *If G is a graph, then $\alpha(G)+\beta(G)=n(G)$.*

Proposition 2.3 (Ore [6]). *If G is a graph with $I(G)=\emptyset$, then $\gamma(G) \leqslant \lfloor n(G)/2 \rfloor$.*

Proposition 2.4 (Volkmann [9]). *Let G be graph with $I(G)=\emptyset$ and $\gamma(G)=\beta(G)$. If H is a factor of G without isolated vertices, then $\gamma(H)=\beta(H)=\gamma(G)=\beta(G)$, and $\gamma(H')=\beta(H')$ for each component H' of H.*

In the proofs of our main results Proposition 2.4 will be frequently applied. The following observation, which Teschner and the second author have made in 1995, is suitable for separating the problem into two parts. For reason of completeness, we shall give here the short proof.

Proposition 2.5 (Teschner, Volkmann [10, p. 221]). *If G is a connected graph with $\gamma(G)=\beta(G)$, then $\delta(G) \leqslant 2$.*

Proof. Suppose to the contrary that $\delta(G) \geqslant 3$. If T is a minimum covering of G, then T is also a dominating set, and because of $\gamma(G)=\beta(G)=|T|$, even a minimum dominating set of G. Since $V(G)-T$ is an independent set, it follows $N(x,G) \subseteq T$ for all $x \in V(G)-T$. If we choose three vertices $u, v, w \in N(a, G)$ for a vertex $a \in V(G)-T$, then we observe that $(T \cup \{a\})-\{u, v\}$ is also a dominating set of G. This contradiction yields the desired result. □

In 1981, Laskar and Walikar [5] posed the problem of characterizing those graphs for which the domination number is equal to the covering number. Until now, however, only a few classes of such graphs have been characterized. For example, Payan and Xuong [7] in 1982, and independently Fink, Jacobson, Kinch and Roberts [2] in 1985 have shown that $G = C_4$ or $G = H \circ K_1$ for an arbitrary connected graph H, are all graphs of even order with $\gamma(G) = \beta(G) = n(G)/2$. Furthermore, Stracke (see [9]) characterized in 1990 all trees G, and Volkmann [9] in 1994 all chordal and unicyclic graphs G for which $\gamma(G) = \beta(G)$. It is the purpose of this paper to give a complete solution of the 15 years old problem of Laskar and Walikar. Because of Proposition 2.5, it is sufficient to investigate graphs with minimum degree two or one. As an interesting application of our results, we determine the connected graphs G with $\gamma(G) = \lfloor n(G)/2 \rfloor$, i.e., we obtain all extremal graphs in the well-known inequality of Ore [6] from Proposition 2.3. With a completely different method, independently around the same time, Cockayne, Haynes and Hedetniemi [1] also characterized the graphs G with $\gamma(G) = \lfloor n(G)/2 \rfloor$.

3. Graphs with minimum degree two

Lemma 3.1. *Let G be a connected graph with $\gamma(G) = \beta(G)$ and $\delta(G) = 2$. Then G is a bipartite graph such that the smaller partite set is a minimum dominating set as well as a minimum covering of G.*

Proof. Let D be a minimum covering of G. Then $V(G)-D$ is an independent set, and by the hypothesis D is also a minimum dominating set. Assume that there exists an edge uv in the induced subgraph $G[D]$. From $\delta(G) = 2$ it follows that $D-u$ is a dominating set of G, a contradiction. Therefore, G is a bipartite graph with the partition sets $D, V(G)-D$ such that $|D| \leqslant |V(G)-D|$ and D is a minimum dominating set and a minimum covering of G. □

Lemma 3.2. *Let G be a connected, bipartite graph with $\gamma(G) = \beta(G)$, $\delta(G) = 2$ and the partite sets A, B. If $|A| \leqslant |B|$, then for each pair $x, y \in A$ of distance 2, there exist at least two common neighbours of x and y of degree 2.*

Proof. Let $x, y \in A$ be vertices of distance two and $b \in B$ a common neighbour. Because of $\delta(G) = 2$ and since $(A - \{x, y\}) \cup \{b\}$ is not a dominating set of G, there exists a vertex $u \neq b$ in B with $N(u, G) = \{x, y\}$. If $d(b, G) \geqslant 3$, then $(A - \{x, y\}) \cup \{u\}$ is also not a dominating set of G, and hence there exists a further vertex $v \neq u$ in B with $N(v, G) = \{x, y\}$. □

Now we are able to present a characterization of all connected graphs G with $\delta(G)=2$ and $\gamma(G)=\beta(G)$.

Theorem 3.3. *Let G be a connected graph with $\delta(G)=2$. Then $\gamma(G)=\beta(G)$ if and only if G is bipartite such that for every pair x and y of distance 2 of the smaller partite set, there exist at least two common neighbours of x and y of degree 2.*

Proof. First assume that $\gamma(G)=\beta(G)$. Lemma 3.1 yields that G is a bipartite graph. If A, B are the partite sets of G with $|A|\leqslant|B|$, then the desired result follows from Lemma 3.2.

Conversely, let G be a bipartite graph with the partite sets A, B and assume without loss of generality that $|A|\leqslant|B|$. Furthermore, for every pair $x, y\in A$ of distance 2 there exist at least two common neighbours of degree 2. Obviously, A is a covering of G. Suppose that A is not a minimum dominating set of G. Now we choose a minimum dominating set D of G such that $D\cap A=A'$ is of maximum cardinality. With $A''=A-A'$ and $B''=D\cap B$, we immediately deduce that $1\leqslant|B''|<|A''|$. In addition, the maximality of $|A'|$ yields $|N(b,G)\cap A''|\geqslant 2$ for all $b\in B''$. Let a_1, a_2 be two different vertices in $N(b,G)\cap A''$ for a vertex $b\in B''$. Since a_1, a_2 are vertices of distance two, there exist two common neighbours b_1 and b_2 in B of degree 2. Hence, the vertices b_1 and b_2 are in B'', but now $(D-\{b_1,b_2\})\cup\{a_1,a_2\}$ is also a minimum dominating set of G, a contradiction to the maximality of A'. Consequently, A is a minimum dominating set, and according to Proposition 2.1, also a minimum covering, which completes the proof. $\square$

Theorem 3.4. *Let G be a connected graph with $\gamma(G)=\beta(G)$ and $\delta(G)=2$. Then G is a bipartite graph and if $G\neq C_4$, then the partite set A of G with $2\leqslant|A|<|V(G)-A|$ is the unique minimum covering of G.*

Proof. According to Lemma 3.1, G is a bipartite graph. In addition, if A, B are the partite sets with $2\leqslant|A|\leqslant|B|$, then A is a minimum covering of G. Suppose that there exists a further minimum covering D with $D\cap B\neq\emptyset$. We choose a vertex $a_1\in A-D$. Since D is a covering, it follows that $N(a_1,G)\subseteq D$. Let $b\in N(a_1,G)$ and $a_2\neq a_1$ a further neighbour of b. By Lemma 3.2 there exist two common neighbours b_1, b_2 of a_1 and a_2 of degree two. Clearly, $b_1, b_2\in D$, but $a_2\notin D$. If $G\neq C_4$, it follows that $d(a_1,G)\geqslant 3$ or $d(a_2,G)\geqslant 3$. Without loss of generality we may assume that $u\neq b_1, b_2$ is a neighbour of a_1. If $N(u,G)=\{a_1,a_2\}$, then $(D-\{u,b_1,b_2\})\cup\{a_1,a_2\}$ is also a covering set, a contradiction. If there is a neighbour $a_3\neq a_1, a_2$ of u, then we deduce from Lemma 3.2 that there exist two common neighbours b_3, b_4 of a_1 and a_3 of degree two. But now $(D-\{b_1,b_2,b_3,b_4\})\cup\{a_1,a_2,a_3\}$ is a covering of G. This contradiction finishes the proof. $\square$

Theorem 3.5. *Let G be a connected graph with $\gamma(G)=\beta(G)$, $\delta(G)=2$ and let D be a minimum covering of G. Then there exists a connected induced C_4-cactus subgraph $H=H(D)$ containing D with $d(x,G)=d(x,H)=2$ for all vertices $x \in (V(G)-D)\cap V(H)$.*

Proof. Since the statement of Theorem 3.5 is valid for $G=C_4$, we now assume that $G \neq C_4$. According to Theorem 3.4, G is a bipartite graph, and the partite set A of G with $2 \leqslant |A| < |V(G)-A|$ is the unique minimum covering of G. Let $b \in B = V(G)-A$ and $\{a_1,a_2\} \subseteq N(b,G)$. Following Lemma 3.2, there exist two common neighbours w,z of a_1,a_2 of degree 2 in G. Hence, the cycle $a_1wa_2za_1$ is a connected induced C_4-cactus subgraph $H'=H(a_1,a_2)$ containing $A'=\{a_1,a_2\}$ with $d(x,G)=d(x,H')=2$ for all vertices $x \in B \cap V(H')$. Now let $A' \subseteq A$ be of maximum cardinality with those properties, and suppose that $A' \neq A$. Since G is connected and bipartite, there exists a vertex $v \in B$ and two vertices $u_1,u_2 \in N(v,G)$ such that $u_1 \in A'$ and $u_2 \in A-A'$. Applying again Lemma 3.2, there are two common neighbours of u_1,u_2 of degree 2 in G. Consequently, we can extend $H(A')$ to a connected and induced C_4-cactus subgraph $H(A' \cup \{u_2\})$ with the desired properties, a contradiction to the choice of A'. □

Corollary 3.6. *Let G be a connected C_4-cactus with the partite sets A and B. If $|A| \leqslant |B|$, then $|A|=\gamma(G)=\beta(G)$ and $|B|=2|A|-2$.*

Proof. From Theorem 3.3 we deduce that $|A|=\gamma(G)=\beta(G)$. If p is the number of C_4-cycles of G, then by induction on p it is easy to see that $|A|=p+1$ and $|B|=2p$, which yields $|B|=2|A|-2$. □

The next result follows from Theorems 3.4, 3.5, and Corollary 3.6.

Corollary 3.7. *Let G be a connected graph with $\gamma(G)=\beta(G)$ and $\delta(G)=2$. Then G is a bipartite graph. If A and B are the partite sets with $|A| \leqslant |B|$, then $|B| \geqslant 2|A|-2$.*

4. Graphs with minimum degree one

Theorem 4.1. *Let G be a connected graph with $\delta(G)=1$. Then $\gamma(G)=\beta(G)$ if and only if*

(i) *The subgraph $G^* = G - N[\Omega(G),G]$ is bipartite, and the components of G^* are isolated vertices, bipartite graphs $G_1,\dots,G_l$ with $\gamma(G_i)=\beta(G_i)$ and $\delta(G_i)=2$ for $i=1,\dots,l$, or bipartite graphs $L_1,\dots,L_m$ with $\gamma(L_j)=\beta(L_j)$ and $\delta(L_j)=1$ for $j=1,\dots,m$.*

(ii) *For $i=1,\dots,l$, the graph G_i', induced by the vertices of G_i which are not adjacent to a vertex of $N(\Omega(G),G)$, is connected with $\delta(G_i')=2$ and $\gamma(G_i')=\beta(G_i')=\beta(G_i)$.*

(iii) *The graphs L_j are either stars $S_1,\dots,S_k$ with at least three vertices, where the center of S_i is not adjacent to a vertex of $N(\Omega(G),G)$ for $i=1,\dots,k$, or non-stars $H_1,\dots,H_r$ such that $H_j'=H_j-\Omega(H_j)$ is a connected graph with $\delta(H_j')=2$ and $\gamma(H_j')=\beta(H_j')=\beta(H_j)$ for $j=1,\dots,r$. Furthermore, for $j=1,\dots,r$, the graph H_j'', induced by the vertices of H_j' which are not adjacent to a vertex of $N(\Omega(G),G)$ is connected with $\delta(H_j'')=2$ and $\gamma(H_j'')=\beta(H_j'')=\beta(H_j')$.*

Proof. In the first step of the proof we shall show that (i)–(iii) are valid if $\gamma(G)=\beta(G)$. Without loss of generality we consider a minimum covering D with $N(\Omega(G),G)\subseteq D$.

(i) If $G'=G-E(G[N(\Omega(G),G)])$, then we shall show that G' is a bipartite graph with the partite sets D and $V(G)-D$ such that $\gamma(G')=\beta(G')=|D|$. Since G' is a factor of G without isolated vertices, Proposition 2.4 yields $\gamma(G')=\beta(G')=|D|$. Clearly, $V(G')-D$ is an independent set. Suppose that there exists an edge uv in the induced subgraph $G'[D]$. The construction of G' implies $d(x,G')\geqslant 2$ for every $x\in N(u,G')$ or for every $x\in N(v,G')$. Let $d(x,G')\geqslant 2$ for every $x\in N(u,G')$. But now $D-\{u\}$ is also a dominating set of G', a contradiction. Hence, G' is bipartite with the partite sets D and $V(G)-D$.

Consequently, G^* is a bipartite graph, and for each component H of G^* we have $\delta(H)=0$, $\delta(H)=1$, or $\delta(H)\geqslant 2$. Let $G_1,\dots,G_l$ and $L_1,\dots,L_m$ be the components with $\delta(G_i)\geqslant 2$ and $\delta(L_j)=1$, respectively. Now the factor

$$G'[N[\Omega(G),G]\cup I(G^*)]\cup G_1\cup\cdots\cup G_l\cup L_1\cup\cdots\cup L_m$$

of G' contains no isolated vertex, and so we conclude with $A_i=D\cap V(G_i)$ that $\gamma(G_i)=\beta(G_i)=|A_i|$ for $i=1,\dots,l$ and $\gamma(L_j)=\beta(L_j)$ for $j=1,\dots,m$. As a consequence of Proposition 2.5, we see that $\delta(G_i)=2$ for $i=1,\dots,l$.

(ii) The set A_i is a minimum covering of G_i for $i\in\{1,\dots,l\}$. Since D is a partite set of the bipartite graph G', containing the subsets A_i and $N(\Omega(G),G)$, there is no vertex of A_i adjacent to a vertex of $N(\Omega(G),G)$. Now let $G_i'\subseteq G_i$ be the graph induced by the vertices which are not adjacent to a vertex of $N(\Omega(G),G)$. Then $d(x,G_i')\geqslant 2$ for each $x\in V(G_i')-A_i$. First, we show that G_i' is connected. Suppose to the contrary that there exist two components J_1 and J_2 of G_i' with $A_i^{(1)}=A_i\cap V(J_1)$ and $A_i^{(2)}=A_i\cap V(J_2)$. Since G_i is connected, there exists a vertex $v\in V(G_i)-A_i$ in the neighbourhood of $N(\Omega(G),G)$ and two vertices $a_1,a_2\in N(v,G_i)$ such that $a_1\in A_i^{(1)}$ and $a_2\in A_i^{(2)}$. Now it is easy to see that $D-\{a_1,a_2\}\cup\{v\}$ is a dominating set of G, a contradiction. Using analogous arguments we obtain $d(x,G_i')\geqslant 2$ for every $x\in A_i$. Note that $A_i=A_i\cap V(G_i')$. This means that G_i' is a connected graph with $\delta(G_i')\geqslant 2$. Since the inequality $\gamma(G_i')<\beta(G_i')$ immediately yields a contradiction to $\gamma(G)=\beta(G)$, we also deduce that $\gamma(G_i')=\beta(G_i')$, and hence Proposition 2.5 implies $\delta(G_i')=2$ for $i=1,\dots,l$. The inequality $\gamma(G_i')<\gamma(G_i)$ also contradicts the fact that $\gamma(G)=\beta(G)$, and therefore the proof of (ii) is complete.

(iii) Now we consider the components $L_1,\dots,L_m$. The set $T_i=D\cap V(L_i)$ is a minimum covering and dominating set of L_i for $i=1,\dots,m$. Then $\Omega(L_i)\subseteq N(N(\Omega(G),$

$G),G)$, $N(\Omega(L_i),L_i)\subseteq T_i$, and $L_i'=L_i-\Omega(L_i)$ is connected for $i=1,\dots,m$. If $\delta(L_i')=0$, then L_i is a star, and it is a simple matter to show that $L_i\neq K_2$, and that there exists no edge in G from the center of L_i to $N(\Omega(G),G)$ for $i=1,\dots,m$. If $\delta(L_i')\geqslant 2$, then we can immediately deduce from Propositions 2.4 and 2.5 $\gamma(L_i')=\beta(L_i')$ and $\delta(L_i')=2$.

Next we shall show that $\delta(L_i')=1$ is not possible. Suppose to the contrary that there exists an $i\in\{1,\dots,m\}$ with $\delta(L_i')=1$. If $L_i'=K_2$, then it is not difficult to obtain a contradiction to $\gamma(G)=\beta(G)$. Hence, we assume that $L_i'\neq K_2$. Now the factor

$$G'[N[\Omega(G),G]\cup I(G^*)\cup\Omega(L_i)]\cup\left(\bigcup_{i=1}^{l}G_i\right)\cup L_1\cup\cdots\cup L_{i-1}\cup L_i'\cup L_{i+1}\cup\cdots\cup L_m$$

of G' contains no isolated vertex, and so we conclude that $\gamma(L_i')=\beta(L_i')$. Furthermore, $\Omega(L_i')$ is independent with $\Omega(L_i')\subseteq D\cap V(L_i')=T_i'$. Therefore, $N(\Omega(L_i'),L_i')\subseteq V(G)-D$ is also independent. For $x\in V(L_i')-\Omega(L_i')$, we have $d(x,L_i')\geqslant 2$. Let $v\in N(\Omega(L_i'),L_i')$, $U=\{u_1,\dots,u_p\}=\Omega(L_i')\cap N(v,L_i')$ and $w\in N(v,L_i')-\Omega(L_i')$. Note that $|U\cup\{w\}|\geqslant 2$ and therefore, if $p\geqslant 2$, the vertex set $N(v,L_i')-\Omega(L_i')$ could be empty. In addition, we have $N(N(\Omega(L_i'),L_i'),L_i')\subseteq T_i'$, in particular $w\in T_i'$. Now the factor

$$G'[N[\Omega(G),G]\cup I(G^*)\cup\Omega(L_i)\cup U\cup\{v,w\}]$$
$$\cup\left(\bigcup_{i=1}^{l}G_i\right)\cup\left(\bigcup_{j\neq i}L_j\right)\cup(L_i'-(U\cup\{v,w\}))$$

has no isolated vertices, especially it contains the component J with

$$J=G'[N[\Omega(G),G)]\cup I(G^*)\cup\Omega(L_i)\cup U\cup\{v,w\}].$$

But then $\gamma(J)=|N(\Omega(G),G)|+|\{v\}|<|N(\Omega(G),G)|+|U\cup\{w\}|=\beta(J)$, a contradiction to Proposition 2.4.

Therefore, the components $H_1,\dots,H_r$ of G^* of minimum degree one which are not stars have the properties that $H_j'=H_j-\Omega(H_j)$ is connected with $\delta(H_j')\geqslant 2$ for $j=1,\dots,r$. Since the inequality $\gamma(H_j')<\beta(H_j')$ immediately yields a contradiction to $\gamma(G)=\beta(G)$, we also deduce that $\gamma(H_j')=\beta(H_j')$, and hence Proposition 2.5 implies $\delta(H_j')=2$ for $j=1,\dots,r$. The inequality $\gamma(H_j')<\gamma(H_j)$ also contradicts the fact that $\gamma(G)=\beta(G)$, and hence $\gamma(H_j')=\beta(H_j')=\beta(H_j)=\gamma(H_j)$ for $j=1,\dots,r$.

The proof that H_j'' is connected with $\delta(H_j'')=2$ and $\gamma(H_j'')=\beta(H_j'')=\beta(H_j')$ for $j=1,\dots,r$ is similar to (ii) and is therefore omitted.

In the second step of the proof we shall show that $\gamma(G)=\beta(G)$ if G fulfills the conditions (i)–(iii). First we choose the factor

$$F=G\left[N[\Omega(G),G]\cup I(G^*)\cup\left(\bigcup_{i=1}^{k}S_i\right)\cup\left(\bigcup_{i=1}^{r}\Omega(H_i)\right)\right]\cup\left(\bigcup_{i=1}^{l}G_i\right)\cup\left(\bigcup_{i=1}^{r}H_i'\right)$$

of G that contains no isolated vertices. Now let $A_1,\dots,A_l$ be a minimum covering of $G_1,\dots,G_l$, and $s_1,\dots,s_k$ be the centers of the stars $S_1,\dots,S_k$, respectively. Without loss of generality we choose minimum coverings $B_1,\dots,B_r$ of $H_1,\dots,H_r$, respectively such that $B_i \cap \Omega(H_i) = \emptyset$ for $i = 1, \ldots, r$. Then we see by Lemma 3.1 and the condition $\gamma(H_j') = \beta(H_j') = \beta(H_j)$ that

$$D = N(\Omega(G), G) \cup \{s_1, \ldots, s_k\} \cup A_1 \cup \cdots \cup A_l \cup B_1 \cup \cdots \cup B_r$$

is a minimum covering and minimum dominating set of the factor F. Furthermore, D is also a covering of G, and hence $|D| = \gamma(F) = \beta(F) = \beta(G)$. Without loss of generality we now choose a minimum dominating set D_γ of G with $N(\Omega(G), G) \cup \{s_1, \ldots, s_k\} \subseteq D_\gamma$. Suppose $|D_\gamma| < |D|$. We shall investigate two possible cases.

Case 1. There exists an $i \in \{1, \ldots, l\}$ with $|D_\gamma \cap V(G_i)| < |D \cap V(G_i)| = \beta(G_i) = |A_i|$. Now we choose D_γ such that $|D_\gamma \cap A_i|$ is of maximum cardinality. From $\gamma(G_i') = \beta(G_i)$ it follows that $|D_\gamma \cap A_i| \leqslant |A_i| - 2$, and that there exists a vertex $u \in D_\gamma \cap (V(G_i) - A_i)$. From the maximum cardinality of $|D_\gamma \cap A_i|$ we deduce that there exist at least two neighbours a_1, a_2 of u in $A_i - D_\gamma$. By condition (i), Theorems 3.4 and 3.3, there exist two common neighbours b_1, b_2 of a_1 and a_2 of degree 2 in G_i. Hence, $b_1, b_2 \in D_\gamma$. But now $(D_\gamma - \{b_1, b_2\}) \cup \{a_1, a_2\}$ is also a minimum dominating set of G, a contradiction to the choice of D_γ.

Case 2. There exists an $i \in \{1, \ldots, r\}$ with

$$|D_\gamma \cap V(H_i)| < |D \cap V(H_i)| = \beta(H_i) = |B_i|.$$

Without loss of generality we choose D_γ such that $D_\gamma \cap \Omega(H_i) = \emptyset$. Because of $D \cap \Omega(H_i) = \emptyset$, it follows $|D_\gamma \cap V(H_i')| < |D \cap V(H_i')| = \beta(H_i') = \beta(H_i) = |B_i|$. Hence, we can apply similiar arguments as in Case 1, and the proof is complete. □

From the proof of Theorem 4.1 we can immediately deduce the following result.

Corollary 4.2. *We have* $2\gamma(G_i) < n(G_i)$ *for* $i = 1, \ldots, l$ *and* $2\gamma(L_j) < n(L_j)$ *for* $j = 1, \ldots, m$ *in Theorem* 4.1.

Remark 4.3. Because of Proposition 2.5, Lemma 3.1, Theorems 3.3 and 4.1, the recognition problem, whether a connected graph G has the property $\gamma(G) = \beta(G)$, is solvable in polynomial time.

5. Graphs with $\gamma(G) = \lfloor n(G)/2 \rfloor$

As an application of our main results we shall characterize all connected graphs G with $\gamma(G) = \lfloor n(G)/2 \rfloor$.

Lemma 5.1. *Let G be a graph without isolated vertices and D be a minimum dominating set of G. If $D' \subseteq D$ such that $G' = G - N[D', G]$ contains no isolated vertices, then*

$$2|D'| \leqslant |N[D', G]| \leqslant 2|D'| + n(G) - 2\gamma(G).$$

Proof. Let $D'' = D - D'$. Since D is a minimum dominating set, D' is contained in the neighbourhood of $N[D', G] - D'$. Therefore, $(N[D', G] - D') \cup D''$ is also a dominating set of G, which yields the inequality $2|D'| \leqslant |N[D', G]|$. Now suppose that there exists a set $D' \subseteq D$ with $|N[D', G]| > 2|D'| + n(G) - 2\gamma(G)$. According to Proposition 2.3 and the estimation $|D''| \leqslant \gamma(G')$, we obtain the contradiction

$$\begin{aligned} n(G) &= 2|D'| + 2|D''| - 2\gamma(G) + n(G) < |N[D', G]| + 2\gamma(G') \\ &\leqslant |N[D', G]| + n(G') = n(G). \qquad \square \end{aligned}$$

If G is a connected graph with $\gamma(G) = \lfloor n(G)/2 \rfloor$, then $2\gamma(G) = n(G)$ or $2\gamma(G) = n(G) - 1$. The Propositions 2.1 and 2.2 imply $\gamma(G) = \alpha(G) = \beta(G)$ in the case $2\gamma(G) = n(G)$, and $\alpha(G) = \gamma(G)$ or $\beta(G) = \gamma(G)$ in the case $2\gamma(G) = n(G) - 1$. For this reason we shall distinguish the two cases $\gamma(G) = \beta(G)$ and $\gamma(G) = \alpha(G)$ in our characterization of all connected graphs with $\gamma(G) = \lfloor n(G)/2 \rfloor$.

Theorem 5.2. *Let G be a connected graph with $\gamma(G) = \lfloor n(G)/2 \rfloor$ and $\gamma(G) = \beta(G)$.*

- *If $n(G)$ is even, then $G = C_4$ or $G = H \circ K_1$ for an arbitrary connected graph H.*
- *If $n(G)$ is odd and $\delta(G) = 2$, then G is a C_4-cactus, consisting of two cycles or G is a C_4 together with a further vertex which is adjacent to two non adjacent vertices of the C_4, i.e. the complete bipartite graph $K_{2,3}$.*
- *If $n(G)$ is odd and $\delta(G) = 1$, then the following five cases are possible:*
  (1) $|N(\Omega(G), G)| = |\Omega(G)| - 1$ *and* $G - N[\Omega(G), G] = \emptyset$.
  (2) $|N(\Omega(G), G)| = |\Omega(G)|$ *and* $G - N[\Omega(G), G]$ *is an isolated vertex.*
  (3) $|N(\Omega(G), G)| = |\Omega(G)|$ *and* $G - N[\Omega(G), G]$ *is a star of order three.*
  (4) $|N(\Omega(G), G)| = |\Omega(G)|$ *and* $G - N[\Omega(G), G]$ *is a bipartite graph* G_1 *with* $n(G_1) = 5$, $\gamma(G_1) = \beta(G_1) = \delta(G_1) = 2$, *and the graph* G_1', *induced by the vertices of* G_1 *which are not adjacent to a vertex of* $N(\Omega(G), G)$, *is a* C_4.
  (5) $|N(\Omega(G), G)| = |\Omega(G)|$ *and* $G - N[\Omega(G), G]$ *is a bipartite graph* H_1 *with one end vertex* u, *which is also a cut vertex of* G, *and* $H_1' = H_1 - u = C_4$.

Proof. First let $n(G)$ even. If $\delta(G) = 2$, then Theorem 3.4 yields $G = C_4$. If $\delta(G) = 1$, and D is a minimum dominating set of G, then we assume without loss of generality $N(\Omega(G), G) \subseteq D$. With $D' = N(\Omega(G), G)$ in Lemma 5.1, we deduce from $|\Omega(G)| \geqslant |N(\Omega(G), G)|$ that $|N(\Omega(G), G)| = |\Omega(G)|$ and $N[N(\Omega(G), G), G] - N[\Omega(G), G] = \emptyset$. Furthermore, since G is connected, we obtain $G = H \circ K_1$ for an arbitrary connected graph H.

Now we discuss the case $n(G) = 2p + 1$ and $\gamma(G) = \beta(G) = p$. If $\delta(G) = 2$, then Lemma 3.1 implies that G is a bipartite graph. If A and B are the partite sets with $|A| \leqslant |B|$, then we deduce from Corollary 3.7 that $2 \leqslant |A| \leqslant 3$. If $|A| = 3$, then G is a C_4-cactus, consisting of two cycles, and if $|A| = 2$, then G is the complete bipartite graph $K_{2,3}$.

In the remaining case $\delta(G) = 1$ let D be a minimum dominating set of G. Without loss of generality we may assume that $N(\Omega(G), G) \subseteq D$. With $D' = N(\Omega(G), G)$ in

Lemma 5.1, we deduce that $|N(\Omega(G),G)| \leqslant |\Omega(G)| \leqslant |N(\Omega(G),G)|+1$. If $|N(\Omega(G),G)| = |\Omega(G)|-1$, then we see as above that $G-N[\Omega(G),G]=\emptyset$. If $|N(\Omega(G),G)|=|\Omega(G)|$, then, together with Corollary 3.7, Theorem 4.1, and Corollary 4.2, it is straightforward to verify the cases (2)–(5). □

The case $n(G)$ even in Theorem 5.2 is done by Payan and Xuong [7] and Fink, Jacobson, Kinch and Roberts [2].

Theorem 5.3. *Let G be a connected graph with $\gamma(G)=\lfloor n(G)/2 \rfloor$, $n(G)=2p+1$, and $\gamma(G)=\alpha(G)=p$. Then the following cases are possible.*

(1) *$G=C_3$ or G consists of a cycle C_3 and a graph $H \circ K_1$ (H not necessarily connected) and arbitrary additional edges between H and one or two vertices of the cycle C_3 such that G is connected.*

(2a) *$G=C_5$ or G is a cycle $C_5=v_1v_2v_3v_4v_5v_1$ with one or two chords of the form v_1v_3 and v_2v_5.*

(2b) *G consists of a cycle $C_5=v_1v_2v_3v_4v_5v_1$ and a graph $H \circ K_1$ (H not necessarily connected) and arbitrary additional edges between H and v_1 such that G is connected. Furthermore, one or two chords of the form v_1v_4 and v_2v_5 are also admissible.*

(2c) *G consists of a cycle $C_5=v_1v_2v_3v_4v_5v_1$ and a graph $H \circ K_1$ (H not necessarily connected) and arbitrary additional edges between H and v_1 and v_3 such that G is connected. Furthermore, the chord v_1v_3 is also admissible.*

(3) *$G=C_7$ or G is a cycle $C_7=v_1v_2v_3v_4v_5v_6v_7v_1$ with one, two, or three chords of the form v_1v_4, v_1v_5, and v_2v_5.*

Proof. First, we show that $|N(X,G)| \geqslant |X|$ for all subsets X of $V(G)$. If not, then it is not very difficult to see that there exists an independent set S with $|S|>|N(S,G)|$ and hence there is a maximal independent set I containing S. Furthermore, I and $(I-S)\cup N(S,G)$ are dominating sets of G. If $\gamma(G)=\alpha(G)$, these two sets are minimum dominating sets, a contradiction.

Hence, by a theorem of Tutte [8] (see also [10, p. 136]), G has a factor F, whose components are either 1-regular or cycles of odd length. Our hypotheses and Proposition 2.1 imply

$$p=\gamma(G)\leqslant\gamma(F)\leqslant\alpha(F)\leqslant p,$$

which yields $p=\gamma(F)=\alpha(F)$. Now it is easy to see that F contains exactly one odd cycle C_3, C_5, or C_7. Furthermore, the graph J, induced by the vertices of the 1-regular components of F, has even order with $2\gamma(J)=2\alpha(J)=2\beta(J)=n(J)$. According to Theorem 5.2, the components of J are cycles of length 4 or graphs of the form $L\circ K_1$. It is straightforward to verify that C_4 is not a component of J. Consequently, $J=H\circ K_1$ (H not necessarily connected). In addition, if there is an edge between the odd cycle and an end vertex of a component of J, then this component is necessarily a complete graph K_2. Finally, a simple case analysis yields the desired result. □

A different proof and further applications of the Theorems 5.1 and 5.2 can be found in the interesting paper of Cockayne, Haynes and Hedetniemi [1]. Very recently we noticed in the Mathematical Reviews that Hartnell and Rall [4] also examined the class of graphs in which some minimum dominating set covers all the edges. Laskar wrote about [4] in MR 96c:05146: "The authors attempt to give a structeral characterization of this class".

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