Unmediated communication in games with complete and incomplete information [☆]

Dino Gerardi

Department of Economics, Yale University, 28 Hillhouse Avenue, New Haven, CT 06511, USA

Abstract

We study the effects of adding unmediated communication to static, finite games of complete and incomplete information. We characterize $S^U(G)$, the set of outcomes of a game G, that are induced by sequential equilibria of cheap talk extensions. A cheap talk extension of G is an extensive-form game in which players communicate before playing G. A reliable mediator is not available and players exchange private or public messages that do not affect directly their payoffs. We first show that if G is a game of complete information with five or more players and rational parameters, then $S^U(G)$ coincides with the set of correlated equilibria of G. Next, we demonstrate that if G is a game of incomplete information with at least five players, rational parameters and full support (i.e., all profiles of types have positive probability), then $S^U(G)$ is equal to the set of communication equilibria of G.
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1. Introduction

Communication allows players to share information with one another and to coordinate their actions. By doing this, in many games players can attain outcomes that are not feasible otherwise. In this paper, we study the effects of adding communication to static, finite games of complete and incomplete information.

A system of communication specifies the rules according to which players communicate. Adding one to a game defines a new, extended game. In a game with communication players exchange messages before choosing their actions. The notions of correlated equilibrium [2], communication equilibrium and the revelation principle (see Myerson [21] and Forges [11]) characterize the set of outcomes that players can achieve in games with communication. In particular, this set of outcomes coincides with the set of correlated equilibria if the game is one of complete information, and with the set of communication equilibria if the game is one of incomplete information. This is a very powerful and useful result. Although there are infinitely many ways in which players can exchange messages, it is very easy to characterize what they can achieve with communication. In fact, correlated and communication equilibria are simply defined by a number of linear inequalities. Therefore, the set of correlated or communication equilibria has a simple and tractable mathematical structure (it is a convex polyhedron). Consider, for example, a game of complete information, and suppose that we are interested in finding the highest payoff that a given player can achieve with communication. In principle, there are infinitely many games with communication to consider. However, given the characterization above, it is enough to find the best correlated equilibrium for that player. This simply involves maximizing a linear function over a convex polyhedron.

Since correlated and communication equilibria are important and commonly used solution concepts, it is worthwhile to examine carefully the conditions under which they can be applied. Myerson [21] and Forges [11] show that the set of correlated or communication equilibria coincides with the set of outcomes that players can achieve with communication under two critical assumptions. First, not only do players exchange messages with one another, but they can also communicate with a reliable and impartial mediator (*mediated* communication). Secondly, the *Nash equilibrium* concept is used to analyze the games with communication.

In many situations, however, the assumption that there is a reliable mediator is too strong, and players are restricted to exchange messages among themselves. In a bargaining setting, for example, the buyer and the seller usually undergo a number of rounds of direct conversations before reaching a final agreement, and no third party mediates their positions or filters their messages. Similarly, political leaders often communicate directly with their advisors. It is true that the mediator could be considered a machine that is programmed to perform a certain number of operations. Still, one has to assume that an impartial individual, different from the players, is available to program the machine. We cannot rely on the players, who would take advantage of the opportunity, and would program the machine in their best interest. When a mediator is not available, and players can only communicate with one another, we say that communication is direct or *unmediated*.

As far as the solution concept is concerned, Nash equilibrium is a legitimate candidate. However, a game with communication is an extensive-form game, and Nash equilibrium does not prevent players from using non credible threats and behaving irrationally in events that are reached with probability zero. In a game with communication, it is therefore appropriate to use a stronger solution concept, such

as *sequential equilibrium* [15], which requires players to behave optimally on and off the equilibrium path.

The goal of this paper is to characterize the outcomes that players can achieve when an impartial mediator is not available and we adopt a solution concept stronger than Nash equilibrium. We further address the related question of whether it is possible to use correlated and communication equilibria when the assumptions made by Myerson [21] and Forges [11] are not satisfied.

Depending on the presence of a mediator and on the adopted solution concept, the existing literature on games with communication can be classified into four categories: (i) mediated communication, Nash equilibrium; (ii) mediated communication, sequential equilibrium; (iii) unmediated communication, Nash equilibrium; (iv) unmediated communication, sequential equilibrium. To illustrate our results in more details, and to describe how our findings contribute to the existing body of work, we now discuss these categories.

We have already mentioned that in case (i) the set of correlated or communication equilibria describes the outcomes that players can achieve with communication. To implement a correlated or communication equilibrium, a mediator usually performs lotteries and sends players private messages. However, Lehrer and Sorin [18] show that correlated and communication equilibria can be implemented in communication systems in which the mediator behaves deterministically and sends only public messages (see also Lehrer [17]).

What happens when communication is mediated and the solution concept is stronger than Nash equilibrium? For static games of complete information, the characterization of the outcomes attainable with communication in terms of correlated equilibria trivially extends to the case in which the sequential equilibrium concept is used.[1] A similar result can be derived for games of incomplete information with full support (i.e., games in which all profiles of players' types have positive probability). In this case, the set of outcomes that players can achieve with mediated communication and the sequential equilibrium concept coincides with the set of communication equilibria. This result does not hold in games without full support. In Section 3, we show that there exist communication equilibria of games without full support that cannot be implemented. We also demonstrate the failure of the revelation principle in games that do not have full support.

There is an extensive literature on games with unmediated communication. Aumann and Hart [3] use the Nash equilibrium concept and characterize the set of outcomes implementable for two-player games in which only one player has private information. Amitai [1] generalizes Aumann and Hart's [3] results to two-player games in which both sides have private information. Urbano and Vila [24,25] demonstrate that if the two players have bounded rationality and can solve only

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[1] However, Dhillon and Mertens [10] show that the outcomes achievable with communication cannot be described by correlated equilibria when the perfect equilibrium concept is used to analyze the extended games with communication. They demonstrate the failure of the revelation principle in this context, and provide a characterization of the outcomes achievable with mediated communication only in two-player games.

problems of limited computational complexity, then it is possible to implement all correlated equilibria and all communication equilibria without mediation. Results have also been obtained for specific games with two players, like double auctions [20], the "battle of the sexes" [5], and games with an informed expert and an uninformed decision maker (see Crawford and Sobel [9] and Krishna and Morgan [16], among others).

For games with four or more players, Barany [6] and Forges [12] provide a complete characterization of the effects of unmediated communication when the solution concept is Nash equilibrium. Barany [6] considers static games of complete information and shows that any rational correlated equilibrium (i.e., any correlated equilibrium with rational components) can be implemented. Moreover, if a game also has rational parameters, then the set of outcomes attainable with unmediated communication coincides with the set of correlated equilibria (see Forges [12]). Games of incomplete information are considered by Forges [12]. She demonstrates that if a game has four or more players and rational parameters, then the set of communication equilibria completely characterizes the outcomes achievable with unmediated communication.[2]

To implement a correlated equilibrium, Barany [6] constructs a scheme of unmediated communication such that an action profile is chosen according to the correlated equilibrium distribution, and each player learns only her own action. Each message is sent by two players to a third one, and public verification of the past record is possible. To prevent unilateral deviations in the communication phase, Barany [6] assumes that a player who receives two different messages stops the communication process and reports that cheating has occurred. Then messages are verified, and the deviator is identified and punished by the opponents who play a "minmax" strategy against her in the original game. If a receiver deviates and reports that cheating has occurred when it has not, then she is punished at her "minmax" level. Note that there are two situations in which Barany's [6] equilibrium fails to satisfy the sequential rationality criterion [7,8]. First, the strategy profile that minmaxes a deviator is not necessarily a Nash equilibrium of the original game. Second, a receiver who observes a deviation may not have an incentive to report that cheating has occurred. These problems extend to Forges [12], since she uses the communication scheme proposed by Barany [6] to prove her result.

Ben-Porath [8] uses the solution concept of sequential equilibrium. He provides sufficient conditions for a communication equilibrium to be implemented with unmediated communication in games with three or more players. Specifically, Ben-Porath [8] shows that a communication equilibrium can be implemented provided that the game admits a Bayesian–Nash equilibrium in which the payoff of every type of each player is lower than the communication equilibrium payoff. The basic idea is

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[2] Forges [12] proposes a mechanism of communication in which players exchange messages at the ex-ante stage (before they learn their types) and at the interim stage (after each player learns her own type). However, Gerardi [13] shows that in games with four or more players and rational parameters, it is possible to implement all communication equilibria even if communication takes place only at the interim stage.

that if a player deviates during the communication phase, then she is punished by her opponents, who play the Bayesian–Nash equilibrium. Using this approach one cannot implement, for example, a communication equilibrium in which the payoff of a player is lower than all Bayesian–Nash equilibrium payoffs: no punishment is available to prevent the player from deviating in the communication phase.

We propose a different system of communication that avoids this problem. We provide a complete characterization of the set of outcomes that can be implemented with unmediated communication in games with five or more players and rational parameters. Our solution concept in the extended games with communication is sequential equilibrium. We show that in games of complete information an outcome can be implemented with direct communication if and only if it is a correlated equilibrium. For games with incomplete information and full support, we demonstrate that the set of outcomes achievable with unmediated communication coincides with the set of communication equilibria.

Our results provide support for the use of correlated and communication equilibria to describe the effects of communication. If a game has at least five players, rational parameters and full support, one can use the simple structure of the set of correlated or communication equilibria even if an impartial mediator is not available and players are sequentially rational.

The rest of the paper is organized as follows. In Section 2, we analyze games with complete information and in Section 3 we study games with incomplete information. Section 4 contains the proof of the main theorem and Section 5 concludes.

2. Games with complete information

In this section, we restrict attention to games of complete information and characterize the set of outcomes that players can achieve with unmediated communication.

Let $\Gamma = \langle P_1, \ldots, P_1, S_1, \ldots, S_I, u_1, \ldots, u_I \rangle$ be a finite normal-form game, where $P_1, \ldots, P_I$ are the players, S_i is the set of actions available to player P_i, $S = \prod_{i=1}^{I} S_i$ is the set of action profiles, and $u_i : S \to \mathbb{R}$ is the payoff function of player P_i. We let $S_{-i} = \prod_{j \neq i} S_j$ denote the set of profiles of actions of players different from P_i. The set of probability distributions over S is denoted by $\Delta(S)$.

We extend Γ by allowing players to communicate before they choose their actions. A *cheap talk extension* of Γ is an extensive-form game that consists of a communication phase and an action phase. During the communication phase, players exchange "cheap" messages, i.e., messages that do not affect *directly* their payoffs. Then, in the action phase, the original game Γ is played.

To describe our results on unmediated communication, it is convenient first to recapitulate the more general case of mediated communication. We consider cheap talk extensions in which players exchange messages among themselves and communicate with an impartial mediator (see, for example, Myerson [23]).

A strategy profile in a cheap talk extension of Γ induces an outcome in Γ, i.e., a probability distribution over S. Let $N(\Gamma)$ denote the set of outcomes in Γ induced by Nash equilibria of cheap talk extensions of Γ (where mediated communication is allowed), and $S(\Gamma)$ denote the set of outcomes induced by sequential equilibria. In other words, a probability distribution $r \in \Delta(S)$ is in $N(\Gamma)$ (respectively, $S(\Gamma)$) if and only if there exists a cheap talk extension of Γ and a Nash equilibrium (respectively, sequential equilibrium) of that extension that induces r.

To characterize the sets $N(\Gamma)$ and $S(\Gamma)$, we need to introduce the notion of correlated equilibrium. A probability distribution $r \in \Delta(S)$ is a correlated equilibrium of Γ if and only if:

$$\sum_{s \in S} r(s)(u_i(s) - u_i(s_{-i}, \delta_i(s_i))) \geqslant 0, \quad i = 1, \ldots, I, \quad \forall \delta_i : S_i \to S_i. \tag{1}$$

We let $C(\Gamma)$ denote the set of correlated equilibria of Γ. Being defined by finitely many linear inequalities, $C(\Gamma)$ is a convex polyhedron.

In the analysis below, we shall devote special attention to the class of correlated equilibria with rational components. We say that a correlated equilibrium r is *rational* if for every action profile in S, the probability $r(s)$ is a rational number.

Any correlated equilibrium is in the set $S(\Gamma)$ (and therefore in $N(\Gamma)$). Let r be a correlated equilibrium and consider the following cheap talk extension of Γ. The mediator randomly selects an action profile in S according to r and informs each player P_i only of the ith component of the chosen action profile. Then the players simultaneously choose their actions. Inequality (1) guarantees the existence of a sequential equilibrium in which each player follows the mediator's recommendation. Clearly, the induced outcome in Γ is the correlated equilibrium r.

The revelation principle for normal-form games states that any probability distribution in $N(\Gamma)$ is a correlated equilibrium of Γ [11,21]. Thus, it follows that both $N(\Gamma)$ and $S(\Gamma)$ coincide with $C(\Gamma)$, the set of correlated equilibria of Γ.

We now consider unmediated communication. We assume that a reliable mediator is not available and consider cheap talk extensions defined as follows: We start by introducing *plain* cheap talk extensions. Given a normal-form game Γ, a plain cheap talk extension of Γ consists of a finite number of steps of communication, at the end of which Γ is played. For each step, the plain cheap talk extension specifies the senders, that is, the players who are allowed to send a message. Further, for each sender, it specifies the receivers and the set of messages. Plain cheap talk extensions allow for simultaneous messages (if the sets of senders contain more than one player) and sequential messages (if the sets of senders are singleton). Moreover, we can have private messages, public messages, and intermediate situations in which a sender sends a message to a subset of her opponents.

We are ready to define a cheap talk extension with unmediated communication. In the first step, a number of players (possibly zero) send *public* messages. To each vector of messages the cheap talk extension associates a plain cheap talk extension.

In a plain cheap talk extension, the identity of senders and receivers and the set of messages of each step are predetermined and do not depend on previous messages. In contrast, in a cheap talk extension, the public messages sent in the first step may

affect the set of messages and the identity of senders and receivers in subsequent steps. Of course, if the number of players sending public messages in the first step is zero, then the cheap talk extension is plain. In other words, plain cheap talk extensions are a special case of cheap talk extensions with unmediated communication.

Let $S^{U}(\Gamma)$ denote the set of outcomes of Γ induced by a sequential equilibrium of some cheap talk extension with unmediated communication. Clearly, since mediated communication is more general than unmediated communication, $S^{U}(\Gamma)$ is included in $S(\Gamma)$ (and, therefore, in $N(\Gamma)$ and $C(\Gamma)$).

Our first result shows that, if there are at least five players, the set $S^{U}(\Gamma)$ contains all rational correlated equilibria of Γ (i.e., all correlated equilibria with rational components).

When a correlated equilibrium r belongs to $S^{U}(\Gamma)$, we say that r can be implemented.

Theorem 1. *Let Γ be a finite normal-form game with five or more players, and let r be a rational correlated equilibrium of Γ. Then $r \in S^{U}(\Gamma)$.*

The proof of Theorem 1 is in Section 4, where we construct $\bar{\Gamma}(r)$, a finite plain cheap talk extension of Γ with unmediated communication, and a sequential equilibrium $\Psi(r)$ of $\bar{\Gamma}(r)$ that induces the distribution r on S.

To implement a rational correlated equilibrium r, we propose a scheme of communication such that, if all players follow it, an action profile in S is chosen according to the distribution r. Moreover, when the communication phase ends, player P_i, $i = 1, \ldots, I$, learns *only* which action in S_i she has to play. Therefore, each player has an incentive to play the action that she learns, provided that her opponents behave likewise.

To give an intuition of our proof and explain how players learn their actions, let us consider the following simple example. There are three players, P_1, P_2 and P_3, and three action profiles, s, s', and s''. Given an action profile, say s, let $pr_3(s)$ denote the action of P_3 in s. We assume $pr_3(s) = pr_3(s')$, and $pr_3(s) \neq pr_3(s'')$. The action profiles s and s' specify the same action for P_3, while s'' specifies a different action. Suppose we want the players to select an action profile at random, according to the uniform distribution. Moreover, P_3 has to learn her action in the chosen profile, while P_1 and P_2 do not have to learn anything.[3] This can be accomplished in the following way. P_1 selects α and σ, two permutations on $\{s, s', s''\}$, at random, according to the uniform distribution, and independently of each other. P_2 selects x, an element of $\{s, s', s''\}$, at random, according to the uniform distribution. The chosen profile is $\sigma(x)$. Clearly, $\sigma(x)$ has a uniform distribution and is unknown to all players. We require P_1 to send the permutation $\alpha\sigma$ to P_2. Since α and σ are independent of each other and uniformly distributed, P_2 does not learn anything about σ (the conditional distribution of σ,

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[3] To keep the example as simple as possible, we require that only P_3 learns her action. Obviously, to implement a correlated equilibrium, each player has to learn her action in the chosen profile (see Section 4 for details).

given $\alpha\sigma$, is uniform). Then P_2 sends the element $\alpha\sigma(x)$ to P_3. Finally, P_1 sends $pr_3\circ\alpha^{-1}$ (a mapping from $\{s, s', s''\}$ to P_3's set of actions) to P_3. At this point P_3 computes $pr_3\circ\alpha^{-1}(\alpha\sigma(x)) = pr_3(\sigma(x))$ and learns her action in the chosen profile.

This system of communication, although simple, presents a serious problem: P_1 and P_2 might have an incentive to deviate from it. If, for example, P_2 sends P_3 a message different from $\alpha\sigma(x)$, P_3 will not learn the correct action. Clearly, in a game P_2 might prefer to induce P_3 not to learn the correct action.

We construct a more complex communication scheme that solves this problem and gives all players an incentive to follow it. We require that a message is sent by three different players to a fourth one. In equilibrium, all senders report the correct message. Moreover, we show that it is sequentially rational for a receiver to follow the message sent by the *majority* of the senders. This implies that a sender does not have profitable deviations during the communication stage. Even if she deviates, the receiver will receive the correct message from the other two senders.

Our communication scheme differs from the systems of communication proposed by Barany [6] and Ben-Porath [7,8] in other important aspects. In fact, the idea of using the majority rule to prevent unilateral deviations in the conversation phase cannot be applied to their communication schemes if there are only five players. Similarly to the example above, in Barany [6] and Ben-Porath [7,8], a combination of two random variables determines the action profile in S that players play once communication is over (in our example the two variables are σ and x). Clearly, it is crucial that a player does not know both random variables, otherwise she learns her opponents' actions. However, this requirement cannot be satisfied if there are five players and every message is sent by three different players. In this case, at least one player must know both random variables. In our communication scheme, the action profile chosen depends on four random variables. In this way, we are able to construct a communication scheme in which every message is sent by three players and no player learns her opponents' actions.

This also explains our need for at least five players. A message goes from three players to a fourth one. This already requires four players. But if there are only four players all messages are public, and all players can learn the chosen action profile. At least another player is needed to generate private messages. Thus, five is the smallest number of players needed to use our communication scheme.

Finally, in contrast to Barany [6] and Ben-Porath [7], our system of communication does not require public verification of past messages. In the game that we present is Section 4, players can exchange oral messages.

Theorem 1 considers only rational correlated equilibria. An obvious question is whether players can implement correlated equilibria that involve probabilities which are irrational numbers. We now show that a correlated equilibrium with irrational components can be implemented provided that it can be expressed as a convex combination of rational correlated equilibria. Let $r_1, \ldots, r_K$ be K rational correlated equilibria of a game, Γ, with five or more players. Consider the correlated equilibrium $r = \sum_{k=1}^{K} \rho_k r_k$, where $\rho_1, \ldots, \rho_K$ are positive numbers (rational or irrational) such that $\sum_{k=1}^{K} \rho_k = 1$. Theorem 1 guarantees that for every rational

correlated equilibrium r_k, $k = 1, \ldots, K$, there exists a plain cheap talk extension $\bar{\Gamma}(r_k)$, and a sequential equilibrium $\Psi(r_k)$ of $\bar{\Gamma}(r_k)$ that induces the distribution r_k on S.

To implement r, we let players perform a "jointly controlled lottery"[4] in which every correlated equilibrium r_k is selected with probability ρ_k. Specifically, consider the following cheap talk extension $\tilde{\Gamma}(r)$. Players P_1 and P_2 simultaneously announce (to all players) two positive numbers in the unit interval. Let ω^i denote the number announced by player P_i, $i = 1, 2$. Let $\omega = \chi(\omega^1, \omega^2)$, where

$$\chi(\omega^1, \omega^2) := \begin{cases} \omega^1 + \omega^2, & \text{if } \omega^1 + \omega^2 \leqslant 1, \\ \omega^1 + \omega^2 - 1, & \text{if } \omega^1 + \omega^2 > 1, \end{cases} \tag{2}$$

and define ρ_0 to be equal to zero. If $\omega \in (\sum_{j=0}^{k-1} \rho_j, \sum_{j=0}^{k} \rho_j]$, for some $k = 1, \ldots, K$, then the game $\bar{\Gamma}(r_k)$ is played.[5]

Although Kreps and Wilson [15] define sequential equilibria only for finite games, it is easy to extend their definition to $\tilde{\Gamma}(r)$. Notice that after P_1 and P_2 announce ω^1 and ω^2, respectively, a proper subgame is induced. We require that the equilibrium strategies and beliefs of $\tilde{\Gamma}(r)$, when restricted to a subgame $\bar{\Gamma}(r_k)$, constitute a sequential equilibrium of $\bar{\Gamma}(r_k)$. Moreover, player P_i, $i = 1, 2$, chooses the number ω^i to maximize her expected payoff (given her strategy in the rest of the game and her opponents' strategies).

Consider the following assessment. In the first step, both P_1 and P_2 randomly select a number in the interval $(0, 1]$, according to the uniform distribution. If the cheap talk extension $\bar{\Gamma}(r_k)$ is selected, players play the sequential equilibrium $\Psi(r_k)$. It is easy to check that this assessment constitutes a sequential equilibrium of the game $\tilde{\Gamma}(r)$. The only thing to note here is that, since ω^2 is independent of ω^1 and uniformly distributed, ω is also independent of ω^1 and uniformly distributed. This implies that P_1 is indifferent between all possible announcements in the first step (clearly, the same argument can be applied to P_2). Therefore, we have proved that the correlated equilibrium r can be implemented.

For an important class of games, we are able to provide a complete characterization of the set of outcomes that can be implemented. We say that a normal-form game Γ is rational if all its parameters are rational numbers, i.e., if for every $i = 1, \ldots, I$, and for every strategy profile s in S, the payoff $u_i(s)$ is a rational number. When Γ is rational, any correlated equilibrium can be expressed as a convex combination of rational correlated equilibria.[6] Thus, any correlated equilibrium is in the set $S^U(\Gamma)$. Since it is always the case that $S^U(\Gamma)$ is contained in $C(\Gamma)$, we conclude that the two sets are identical. The next corollary summarizes our results.

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[4] A jointly controlled lottery [4] is a communication scheme that allows two or more players to randomly select an outcome. The scheme is immune against unilateral deviations in the sense that no player can, by her own decision, influence the probability distribution.

[5] Notice that the cheap talk extension $\tilde{\Gamma}(r)$ is not plain.

[6] If Γ is rational, the vertices of $C(\Gamma)$ are rational correlated equilibria.

Corollary 1. *Let Γ be a finite normal-form game with five or more players. Let r be a convex combination of rational correlated equilibria of Γ. Then $r \in S^{U}(\Gamma)$. If Γ is rational, $S^{U}(\Gamma) = C(\Gamma)$.*

For any two-player game Γ, it is easy to show that $S^{U}(\Gamma)$ is the convex hull of the Nash equilibrium outcomes of Γ. Intuitively, when there are only two players, all messages are public, and therefore, only lotteries over Nash equilibria can be implemented (see Aumann and Hart [3]). Thus, in the class of games with rational parameters, a complete characterization of the set $S^{U}(\Gamma)$ is unavailable only when Γ has three or four players. Obviously, our communication scheme cannot be applied if there are less than five players, but this does not rule out the possibility that all correlated equilibria can be implemented with some other communication scheme. We do not have any example of a correlated equilibrium in a game with three or four players that cannot be implemented. Barany [6] provides the example of a rational correlated equilibrium in a three-player game that cannot be implemented (in Nash equilibrium) if the message spaces are finite. However, Forges [12] shows that the correlated equilibrium considered by Barany [6] is induced by a Nash equilibrium in a cheap talk extension where a continuum of messages is used. For a game Γ with three or four players, a partial characterization of the set $S^{U}(\Gamma)$ is provided by Ben-Porath [8], who gives sufficient conditions for a rational correlated equilibrium to be implemented.

We conclude this section with a comment on our definition of unmediated communication. The cheap talk extensions defined in this section are not the most general form of unmediated communication. One could think, for example, of situations in which a sender can choose the receivers of her message. However, Corollary 1 shows that restricting attention to our cheap talk extensions is without loss of generality in games with five or more players and rational parameters. For these games, allowing for more general forms of unmediated communication does not expand the set of outcomes that players can implement.

3. Games with incomplete information

We now consider games of incomplete information and show how unmediated communication allows players to expand the set of equilibrium outcomes.

Let $G = \langle P_1, \ldots, P_I, T_1, \ldots, T_I, S_1, \ldots, S_I, p, u_1, \ldots, u_I \rangle$ be a finite Bayesian game. Players are $P_1, \ldots, P_I$, and S_i denotes the set of actions of P_i. As in Section 2, S denotes the set of action profiles, S_{-i} is the set of profiles of actions of players different from P_i, and $\Delta(S)$ denotes the set of probability distributions over S. In addition, T_i is the set of types of P_i, and $T = \prod_{i=1}^{I} T_i$ is the set of type profiles. We let $T_{-i} = \prod_{j \neq i} T_j$ denote the set of profiles of types of players different from P_i. The payoffs of P_i are described by $u_i : T \times S \rightarrow \mathbb{R}$. Finally, p is a probability distribution

over T.[7] We say that game G has *full support* if every profile of types occurs with positive probability, i.e., if $p(t) > 0$ for every t in T. Games in which players' types are independent constitute an obvious example of games with full support.

We proceed as in Section 2 and extend G by introducing pre-play communication. Although we are interested in unmediated communication, it is convenient to start our analysis by considering first the case of mediated communication.

A strategy profile in a cheap talk extension of G induces an outcome in G, i.e., a mapping from T to $\Delta(S)$. As in the previous section, we let $N(G)$ denote the set of outcomes in G induced by a Bayesian–Nash equilibrium of some cheap talk extension of G (where mediated communication is allowed). Similarly, $S(G)$ denotes the set of outcomes in G induced by a sequential equilibrium of some cheap talk extension of G.

The set $N(G)$ can be easily characterized in terms of communication equilibria. A function $q : T \to \Delta(S)$ is a communication equilibrium if and only if:

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\begin{aligned}
&\sum_{t_{-i} \in T_{-i}} \sum_{s \in S} p(t_{-i}|t_i) q(s|(t_{-i}, t_i)) u_i((t_{-i}, t_i), s) \\
&\geqslant \sum_{t_{-i} \in T_{-i}} \sum_{s \in S} p(t_{-i}|t_i) q(s|(t_{-i}, t_i')) u_i((t_{-i}, t_i), (s_{-i}, \delta_i(s_i))), \\
&\quad i = 1, \dots, I, \quad \forall (t_i, t_i') \in T_i^2 \quad \forall \delta_i : S_i \to S_i.
\end{aligned}
\tag{3}
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We let $CE(G)$ denote the set of communication equilibria of G. $CE(G)$ is defined by finitely many linear inequalities, and therefore is a convex polyhedron. We also say that a communication equilibrium q is rational if, for every action profile s in S and every type profile t in T, the probability $q(s|t)$ is a rational number.

To see that a communication equilibrium q belongs to the set $N(G)$, consider the following cheap talk extension $G^{\mathrm{D}}(q)$, usually called the *canonical* game. First, each player P_i sends the mediator a message in T_i. The mediator, after receiving a vector of messages t, randomly selects an action profile in S according to the probability distribution $q(\cdot|t)$ and informs each player P_i only of the ith component of the chosen profile. Finally, all players simultaneously choose their actions. It follows from inequality (3) that there exists a Bayesian–Nash equilibrium of $G^{\mathrm{D}}(q)$ in which every player reports her type truthfully to the mediator and follows the mediator's recommendation. The notion of communication equilibrium is a generalization of the notion of correlated equilibrium for games with incomplete information. In a correlated equilibrium, a mediator allows players to coordinate their actions. In a communication equilibrium, the mediator has two roles: she helps players coordinate their actions and exchange their private information. The two notions of equilibria coincide in games with complete information.

On the other hand, it follows from the revelation principle that any outcome in $N(G)$ is a communication equilibrium (see Myerson [21] and Forges [11]). Therefore, for any finite Bayesian game G, the sets $N(G)$ and $CE(G)$ coincide.

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[7] For notational simplicity, we assume that beliefs in G are consistent.

The characterization of the set $S(G)$ is, in general, less immediate. Clearly, for any game G, $S(G)$ is included in $CE(G)$. Further, the set $S(G)$ is non-empty (sequential equilibria exist in any extension of G) and convex (the mediator conducts a lottery among different extensions and announces publicly the outcome of the lottery). To investigate whether any communication equilibrium of G is in $S(G)$, we first need some definitions.

Given a communication equilibrium q of G, consider the canonical game $G^D(q)$. We say that q is *regular* if there is a sequential equilibrium of $G^D(q)$ in which each player reports her type truthfully to the mediator and obeys the mediator's recommendation *after* being honest (a player may disobey the mediator's recommendation if she did not report her type truthfully). Clearly, any regular communication equilibrium of G belongs to $S(G)$.

When G has full support, it is easy to show that any communication equilibrium is regular. When all type profiles have positive probability, a player never learns that some other player lied to the mediator. This fact and inequality (3) imply that for a player who reports her type sincerely, it is sequentially rational to obey the mediator's suggestion. Moreover, inequality (3) also guarantees that no player has an incentive to lie to the mediator. Thus, if G has full support, $S(G)$ is equal to $CE(G)$.

This result does not hold in games without full support. As we show in the next example, it is possible to construct a game G', without full support, such that the set $S(G')$ is strictly included in the set of communication equilibria $CE(G')$.

Example 1. *A Bayesian game G' with $CE(G') \nsubseteq S(G')$.*

G' is a two-player game. The set of types of players P_1 and P_2 are $T_1 = \{t_1^1, t_1^2\}$ and $T_2 = \{t_2^1, t_2^2\}$, respectively. P_1 has to choose an action from the set $S_1 = \{s^1, s^2, s^3\}$, while P_2 does not choose an action. The probability distribution over the set of type profiles is:

$$p(t_1^1, t_2^1) = p(t_1^1, t_2^2) = p(t_1^2, t_2^1) = \tfrac{1}{3}, \quad p(t_1^2, t_2^2) = 0.$$

Payoffs are described in Table 1, where, for each combination of type profile and action, we report the corresponding vector of payoffs (the first entry denotes P_1's payoff).

First, let us consider the game G' without communication. In any Bayesian–Nash equilibrium, type t_1^2 plays action s^2, and type t_1^1 chooses either s^1, or s^3, or a randomization between the two actions.

We now provide a complete characterization of the set $S(G')$. Consider a cheap talk extension of G'. Since type t_1^2 of player P_1 knows that player P_2 has type t_2^1, sequential rationality implies that in every information set in which t_1^2 has to choose an action from S_1, she will play s^2.

Further, type t_1^1 of P_1 never chooses action s^2. In fact, independent of P_2's type, action s^2 is dominated by actions s^1 and s^3. We now show that if q belongs to $S(G')$, then $q(t_1^1, t_2^1) = q(t_1^1, t_2^2)$. Both types of P_2 prefer action s^3 to any other action when

Table 1
Payoffs of the game G'

	t_2^1	t_2^2
t_1^1	s^1 $(1,0)$	s^1 $(0,0)$
	s^2 $(-1,0)$	s^2 $(-1,0)$
	s^3 $(0,2)$	s^3 $(1,1)$
t_1^2	s^1 $(-1,-1)$	s^1 $(0,0)$
	s^2 $(1,1)$	s^2 $(0,0)$
	s^3 $(0,1)$	s^3 $(0,0)$

P_1 has type t_1^1. Suppose, by contradiction, that there exists a sequential equilibrium of a cheap talk extension that induces an outcome q, with $q(s^3|t_1^1,t_2^1) > q(s^3|t_1^1,t_2^2)$. Then type t_2^2 would have an incentive to deviate and mimic the behavior of type t_2^1 to increase the probability of action s^3. Similarly, suppose that a sequential equilibrium induces an outcome q such that $q(s^3|t_1^1,t_2^1) < q(s^3|t_1^1,t_2^2)$. Type t_2^1 knows that, independent of her strategy, type t_1^2 of P_1 will play action s^2. Therefore, t_2^1 has an incentive to mimic the behavior of t_2^2: if P_1 has type t_1^1, the probability of action s^3 will increase.

We conclude that $S(G')$ coincides with the set of equilibrium outcomes of G'. If the players are sequentially rational, communication cannot expand the set of equilibrium outcomes of G'. Given this, it is very easy to show that $S(G')$ is strictly included in the set of communication equilibria of G'. A communication equilibrium that does not belong to $S(G')$ is, for example, q' defined by

$$q'(t_1^1,t_2^1) = q'(t_1^2,t_2^2) = s^1, \quad q'(t_1^1,t_2^2) = s^3, \quad q'(t_1^2,t_2^1) = s^2,$$

where we adopt the convention of writing, for example, $q'(t_1^1,t_2^1) = s^1$ to denote $q'(s^1|t_1^1,t_2^1) = 1$.

Although not all communication equilibria of G' belong to $S(G')$, the revelation principle is still valid in the above example. Any element of $S(G')$ is a regular communication equilibrium and can be implemented with the canonical game. This result could suggest that for a Bayesian game G, $S(G)$ coincides with the set of regular communication equilibria. However, it turns out that this conjecture is incorrect. As the next example demonstrates, when a game G does not have full support, $S(G)$ may contain communication equilibria that are not regular.

Example 2. *A communication equilibrium that cannot be implemented with the canonical game.*

Consider the following three-person game G''. The set of types of players P_1 and P_2 are $T_1 = \{t_1^1, t_1^2\}$, and $T_2 = \{t_2^1, t_2^2\}$, respectively. P_3 does not have private information and is the only player to choose an action, from the set $S_3 =$

Table 2
Payoffs of the game G''

	t_2^1		t_2^2	
t_1^1	s^1	$(-1,-1,1)$	s^1	$(1,0,0)$
	s^2	$(1,1,0)$	s^2	$(1,1,1)$
	s^3	$(0,0,-1)$	s^3	$(-3,0,1)$
	s^4	$(0,0,-1)$	s^4	$(1,0,-1)$
t_1^2	s^1	$(0,1,0)$	s^1	$(0,0,0)$
	s^2	$(1,1,1)$	s^2	$(0,0,0)$
	s^3	$(0,1,-1)$	s^3	$(0,0,0)$
	s^4	$(0,-3,1)$	s^4	$(0,0,0)$

$\{s^1, s^2, s^3, s^4\}$. The probability distribution over the set of profiles of types is given by

$$p(t_1^1, t_2^1) = p(t_1^1, t_2^2) = p(t_1^2, t_2^1) = \tfrac{1}{3}, \quad p(t_1^2, t_2^2) = 0.$$

Finally, Table 2 describes the vector of payoffs for each pair of type profile and action (the first entry refers to P_1, the second one to P_2).

The communication equilibrium of G'' that maximizes P_3's expected payoff is unique and equal to q'', where

$$q''(t_1^1, t_2^1) = s^1, \quad q''(t_1^1, t_2^2) = q''(t_1^2, t_2^1) = s^2, \quad q''(s^3|t_1^2, t_2^2) = q''(s^4|t_1^2, t_2^2) = \tfrac{1}{2}.$$

We now show that the communication equilibrium q'' is not regular. Consider the canonical game $G''^{\mathrm{D}}(q'')$. P_3 has four information sets, one for each recommendation that the mediator can send. After receiving a recommendation, P_3 computes her beliefs over the set $\{(t_1^1, t_2^1), (t_1^1, t_2^2), (t_1^2, t_2^1)\}$. In any consistent assessment of $G''^{\mathrm{D}}(q'')$, P_3's beliefs after recommendation s^3 must coincide with her beliefs after s^4.[8] Notice that it is sequentially rational for P_3 to choose action s^3 only if she assigns probability one to the profile of types (t_1^1, t_2^2). On the other hand, it is optimal for P_3 to play s^4 only if she assigns probability one to (t_1^2, t_2^1). But since P_3 has the same beliefs after s^3 and s^4, it cannot be sequentially rational to obey both recommendations. We conclude that q'' cannot be implemented with the canonical game $G''^{\mathrm{D}}(q'')$.

However, q'' does belong to the set $S(G'')$. Consider the following cheap talk extension $\widehat{G''}(q'')$. Players P_1 and P_2 simultaneously report their messages to the mediator. Then the mediator recommends an action to P_3. Finally, P_3 chooses an action from S_3. The set of messages of P_i, $i = 1, 2$, is $\hat{T}_i = \{t_i^1, t_i^2, t_i^3, t_i^4\}$, where t_i^3 and

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[8] The only thing that P_3 learns after receiving recommendations s^3 or s^4 is that the mediator received message t_1^2 from P_1, and t_2^2 from P_2. The mediator performs the lottery between s^3 and s^4 after P_1 and P_2 send their messages. Therefore, to prove the consistency of an assessment of $G''^{\mathrm{D}}(q'')$, we cannot choose trembles for P_1 or P_2 that depend on the outcome of the mediator's lottery. This, in turn, implies that P_3's beliefs after s^3 cannot differ from her beliefs after s^4.

Table 3
Mediator's recommendation in $\widehat{G''}(q'')$

	t_2^1	t_2^2	t_2^3	t_2^4
t_1^1	s^1	s^2	s^1	s^1
t_1^2	s^2	$\frac{1}{2}s^3, \frac{1}{2}s^4$	s^2	s^2
t_1^3	s^1	s^2	s^3	s^1
t_1^4	s^1	s^2	s^1	s^4

t_i^4 are two arbitrary messages. In Table 3, we report the mediator's recommendation to P_3 for each vector of reports of P_1 and P_2.

The mediator adopts a deterministic behavior unless P_1 and P_2 report messages t_1^2 and t_2^2, respectively. In this case the mediator randomizes, with equal probability, between recommendations s^3 and s^4.

It is easy to show that the cheap talk extension $\widehat{G''}(q'')$ admits a sequential equilibrium in which P_1 and P_2 "reveal their types truthfully" (i.e., t_i^j sends message $t_i^j, i = 1, 2, j = 1, 2$) and P_3 obeys the mediator's recommendation (clearly, this equilibrium induces q''). Intuitively, by introducing the new messages t_i^3 and t_i^4, $i = 1, 2$, we allow player P_3 to have *different* beliefs after recommendations s^3 and s^4 (recommendation s^3 can be induced by reports (t_1^3, t_2^3), and s^4 by reports (t_1^4, t_2^4)). In this way, both obedience to s^3 and obedience to s^4 can be sequentially rational (see Gerardi [13] for details).

A complete characterization of the set $S(G)$ when G is a game without full support is still an open question and beyond the scope of this paper. In Gerardi [13], we provide the solution for a special class of games. Specifically, we characterize $S(G)$ when G is a game in which one player is uninformed and has to choose an action, while all other players have private information but do not choose an action.

We now assume that an impartial mediator is not available and turn to unmediated communication. A cheap talk extension of G is defined as follows. First, Nature selects a type profile t according to the probability distribution p and each player learns her own type. Then, in the communication phase, players exchange "cheap" messages as described in Section 2. Finally, in the action phase, players simultaneously choose their actions.

We denote by $S^U(G)$ the set of outcomes in a Bayesian game G induced by a sequential equilibrium of some cheap talk extension with unmediated communication. Clearly, $S^U(G)$ is included in $CE(G)$.

We say that a communication equilibrium can be implemented if it belongs to $S^U(G)$. We are ready to state our first result for games of incomplete information.

Theorem 2. *Let G be a finite Bayesian game with five or more players, and let q be a rational and regular communication equilibrium of G. Then $q \in S^U(G)$.*

We omit the proof of Theorem 2 (we refer the interested reader to Gerardi [14]) and present an informal description of it. Our proof consists of two steps. In the first step, we use a result due to Forges [12]. Consider a communication equilibrium (not necessarily rational) q of a finite Bayesian game G with at least *four* players. Forges [12] constructs a cheap talk extension $G^{F}(q)$ in which, first, the players receive messages from a mediator, then exchange public and private messages, and finally choose their actions. Notice that in $G^{F}(q)$ the players do not send messages to the mediator, i.e., the mediator is a correlation device. Forges [12] shows that q is the outcome induced by a Bayesian–Nash equilibrium of $G^{F}(q)$. We demonstrate that if q is regular, then the equilibrium in $G^{F}(q)$ that implements q can be made sequential. However, communication in $G^{F}(q)$ is still mediated, since the mediator has to send messages to the players at the beginning of $G^{F}(q)$. In the second step of our proof, we use the fact that the communication equilibrium is rational and that there are at least five players. Under these assumptions, we show that the players, after learning their types, can use the communication scheme presented in Section 4 to generate the mediator's messages. In other words, we construct a finite plain cheap talk extension $\bar{G}(q)$ which starts with the communication scheme described in Section 4. At the end of it, the players exchange public and private messages as in $G^{F}(q)$ and finally choose their actions. We conclude our proof by showing that $\bar{G}(q)$ admits a sequential equilibrium $\Phi(q)$ that induces q.

Theorem 2 does not pertain to communication equilibria that are not regular or that have some irrational components. However, these equilibria can be implemented, provided that they can be expressed as a convex combination of rational and regular communication equilibria. Notice that in games without full support, convex combinations of regular communication equilibria are not necessarily regular. It follows that the set of regular communication equilibria of a game G (without full support) may be strictly included in the set $S^{U}(G)$.

As in Section 2, in order to implement a convex combination of rational and regular communication equilibria, we require the players to conduct a jointly controlled lottery among different cheap talk extensions. Specifically, consider a game G with at least five players. Let q be a convex combination of K regular and rational communication equilibria $q_1, \ldots, q_K$, with weights $\rho_1, \ldots, \rho_K$, respectively. It follows from Theorem 2 that for every q_k $(k = 1, \ldots, K)$ there exists a plain cheap talk extension $\bar{G}(q_k)$ and a sequential equilibrium $\Phi(q_k)$ of $\bar{G}(q_k)$ that induces q_k. We construct the following cheap talk extension $\tilde{G}(q)$. At the beginning of $\tilde{G}(q)$, players P_1 and P_2 simultaneously announce to all players two positive numbers in the unit interval (P_i announces ω^i, $i = 1, 2$). Let $\omega = \chi(\omega^1, \omega^2)$, where $\chi(\omega^1, \omega^2)$ is defined in Eq. (2) and let ρ_0 be equal to zero. If $\omega \in (\sum_{j=0}^{k-1} \rho_j, \sum_{j=0}^{k} \rho_j)$, for some $k = 1, \ldots, K$, then the cheap talk extension $\bar{G}(q_k)$ is played.

The cheap talk extension $\tilde{G}(q)$ is not a finite game, since the first two players can announce any number in the interval (0,1]. Moreover, after P_1 and P_2 report ω^1 and ω^2, respectively, a proper subgame is not induced, since the players have private information about their types. In general, consistent beliefs have not been defined in

games with infinite strategy sets. Some authors have extended the notion of sequential equilibrium only to specific classes of infinite games.[9] However, for our purposes, it is enough to consider assessments of $\tilde{G}(q)$ in which all types of P_i, $i = 1, 2$, select ω^i at random, according to the uniform distribution. We call these assessments *simple*. For a simple assessment, it is easy to define consistent beliefs. In this case, observing ω^1 and ω^2 does not provide any information about the types of P_1 and P_2. A simple assessment is a sequential equilibrium if the strategies and the beliefs, when restricted to a game $\bar{G}(q_k)$, form a sequential equilibrium of $\bar{G}(q_k)$. Moreover, for every type of P_i, $i = 1, 2$, selecting ω^i at random according to the uniform distribution is optimal among *all* behavioral strategies.

Consider the simple assessment where the players play the sequential equilibrium $\Phi(q_k)$ if game $\bar{G}(q_k)$ is selected, i.e., if $\omega \in (\sum_{j=0}^{k-1} \rho_j, \sum_{j=0}^{k} \rho_j)$. The analysis at the end of Section 2 shows that for every type of P_i ($i = 1, 2$), it is optimal to choose ω^i randomly according to the uniform distribution. Therefore, we conclude that q can be implemented.

For some games, it is possible to provide a precise characterization of the set of communication equilibria that can be implemented. We say that a game has rational parameters if for every $i = 1, \dots, I$, every action profile s in S and every profile of types t in T, the payoffs $u_i(t, s)$ and the probability $p(t)$ are rational numbers. If G has rational parameters, the vertices of $CE(G)$ are rational communication equilibria. Moreover, the vertices of $CE(G)$ are regular communication equilibria if G has full support (remember that in this case all communication equilibria are regular). Therefore, when G has at least five players, rational parameters, and full support, every communication equilibrium can be implemented. We can summarize our findings as follows.

Corollary 2. *Let G be a finite Bayesian game with five or more players. Let q be a convex combination of rational and regular communication equilibria of G. Then $q \in S^U(G)$. If G has full support and rational parameters, $S^U(G) = CE(G)$.*

It is an open question whether it is possible to implement communication equilibria that cannot be expressed as convex combinations of rational and regular communication equilibria. As Corollary 2 suggests, a complete characterization of the set of outcomes that can be implemented with unmediated communication is not available for games without full support or with irrational parameters or with less than five players. As has already been stated, for a game G of incomplete information with at least three players, a partial characterization of $S^U(G)$ is provided by Ben-Porath [8].

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[9] Manelli [19], for example, considers signaling games.

4. Proof of Theorem 1

In this section, we prove that when a normal-form game Γ has five or more players, any rational correlated equilibrium can be implemented. Given a rational correlated equilibrium r of Γ, we construct a finite plain cheap talk extension $\bar{\Gamma}(r)$ and a sequential equilibrium $\Psi(r)$ of $\bar{\Gamma}(r)$ that induces the distribution r on S.

We first illustrate how the players can generate the probability distribution r. Since r is rational, there exists a positive integer $\tilde{m}$ (greater than one) and, for every s in S, a non-negative integer $\tilde{m}_s$ such that $r(s) = \frac{\tilde{m}_s}{\tilde{m}+1}$. Define $X = \{1, \ldots, \tilde{m}, \tilde{m}+1\}$, and let $\{X_s\}_{s\in S}$ be a partition of X such that $|X_s| = \tilde{m}_s$ for every s in S. For $i = 1, \ldots, I$, let the projection $pr_i : S \to S_i$ be defined by $pr_i(s) = s_i$ if $s = (s_1, \ldots, s_i, \ldots, s_I)$. We extend each projection pr_i to X as follows:

$$pr_i(x) = s_i \quad \text{if} \quad x \in X_s \quad \text{and} \quad pr_i(s) = s_i.$$

It is easy to verify that if an element $y \in X$ is randomly selected according to the uniform distribution and if every player P_i chooses the action $pr_i(y)$, then every action profile s in S is chosen with probability $r(s)$.

Let $\Lambda(X)$ denote the set of permutations on X and let $m + 1$ denote the cardinality of $\Lambda(X)$. The sets $X, \Lambda(X)$, and the projections $pr_1, \ldots, pr_I$ are common knowledge among the players.

It is useful to divide the game $\bar{\Gamma}(r)$ into several steps. For each step, we illustrate the game and present equilibrium behavioral strategies (hereafter simply called equilibrium strategies) and equilibrium beliefs. Then, we prove that the assessment $\Psi(r)$ which we propose is sequentially rational. Finally, we outline the proof that $\Psi(r)$ is consistent (see Gerardi [14] for details).

Step 0. Random choices.

In this step, which consists of several substeps, players "jointly select" random permutations on X and a random element of X. We will explain later how the joint selections are made. We first list the random choices that players make. $P_1, P_2, P_3, P_4, P_6, \ldots, P_I$ jointly select permutation σ.[10] $P_2, P_3, P_4, P_5, P_6, \ldots, P_I$ choose two permutations, τ and β_{41}. P_2, P_3, P_4, P_5 choose permutation β_{31}. $P_1, P_3, P_4, P_5, P_6, \ldots, P_I$ select two permutations, φ and β_{42}. P_1, P_3, P_4, P_5 choose permutation β_{32}. P_1, P_2, P_5 select an element $x \in X$. Finally, for $i = 1, \ldots, 4$, players in the set $\{P_1, P_2, P_3, P_4\} \setminus \{P_i\}$ choose permutation α_i.

We let $\mathfrak{I}_i$ denote the set of random choices known to P_i at the end of Step 0. It is convenient to summarize Step 0 in Table 4, where for every player P_i, we list the elements of the set $\mathfrak{I}_i$.

There is a separate substep for each random choice. Every choice is made according to the uniform distribution over the underlying probability space and every choice is made independently of all others.

We now describe how the players jointly select a random permutation, or an element of X. Consider, for example, the substep in which $P_1, P_2, P_3, P_4, P_6, \ldots, P_I$

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[10] The proof we present is valid both for the cases $I = 5$ and $I > 5$. Clearly, any reference to player P_k, with $k > 5$, is relevant only if there are more than five players.

Table 4
Random choices of Step 0

Player	Random choices										
P_1	σ		φ	x		α_2	α_3	α_4			β_{32}	β_{42}
P_2	σ	τ		x	α_1		α_3	α_4	β_{31}	β_{41}		
P_3	σ	τ	φ		α_1	α_2		α_4	β_{31}	β_{41}	β_{32}	β_{42}
P_4	σ	τ	φ		α_1	α_2	α_3		β_{31}	β_{41}	β_{32}	β_{42}
P_5		τ	φ	x					β_{31}	β_{41}	β_{32}	β_{42}
P_6	σ	τ	φ							β_{41}		β_{42}
$\vdots$	$\vdots$	$\vdots$	$\vdots$							$\vdots$		$\vdots$
P_I	σ	τ	φ							β_{41}		β_{42}

have to select the random permutation σ. The two players with the lowest indices (P_1 and P_2 in this case) make two announcements simultaneously. Specifically, P_i, $i = 1, 2$, announces a permutation $\sigma^i \in \Lambda(X)$ to players in the set $\{P_1, P_2, P_3, P_4, P_6, \ldots, P_I\} \backslash \{P_i\}$. The chosen permutation will be $\sigma = \sigma^1 \sigma^2$ (notice that σ is common knowledge among $P_1, P_2, P_3, P_4, P_6, \ldots, P_I$). In equilibrium, P_i selects a permutation σ^i at random, according to the uniform distribution on $\Lambda(X)$. This implies that the random permutation σ is uniformly distributed.

A similar procedure is used to make the remaining random choices.[11] In every substep, the two players who have to make an announcement choose their messages randomly, according to the uniform distribution on the underlying probability space, independently of the messages that they have already sent or received.

Equilibrium beliefs are very simple. At the end of Step 0, a player either knows the realization of a given random variable or believes that the random variable is uniformly distributed.

Sketch of $\bar{\Gamma}(r)$. To provide the reader with a better understanding of our construction, let us outline the rest of the game $\bar{\Gamma}(r)$ before presenting the next steps. The state $y := \sigma\tau\varphi(x)$ determines the action profile of the original game Γ that players choose in the action phase. In equilibrium, y has a uniform distribution on X since σ, τ, φ and x are uniformly distributed. Notice that no player knows the realization of y at the end of Step 0. This is crucial, since a player who is informed about y knows the actions of the game Γ that her opponents play when communication is over. However, Step i, $i = 1, \ldots, 4$, is carefully designed to make P_i learn $\alpha_i(y)$ and nothing more (P_5 learns $\alpha_2(y)$ in Step 2, and for $k > 5$, P_k learns $\alpha_4(y)$ in Step 4). The permutations $\alpha_1, \ldots, \alpha_4$ prevent a player from learning the state y. In Step 5 P_i, $i = 1, \ldots, 4$, learns the function $pr_i\alpha_i^{-1}$ (P_5 learns $pr_5\alpha_2^{-1}$, and for $k > 5$, P_k learns $pr_k\alpha_4^{-1}$). So player P_i learns her action $pr_i(y)$. Since the action profile $(pr_1(y), \ldots, pr_I(y))$ is chosen according to the correlated equilibrium distribution r, P_i

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[11] To select the random element x, P_1 and P_2 simultaneously announce $x^1 \in X$ and $x^2 \in X$, respectively. The chosen element is $x = x^1 + x^2$, where $+$ is mod $(\tilde{m} + 1)$.

has exactly the same information as in the case in which she receives recommendation $pr_i(y)$ from a mediator who implements r. In the sixth and last step, every player P_i chooses action $pr_i(y)$. Notice that $pr_i(y)$ is optimal for P_i given her information and her opponents' actions.

After Step 0, the game $\bar{\Gamma}(r)$ proceeds as follows.

Step 1. P_2, P_3 and P_4 send P_1 the permutation $\alpha_1\sigma\tau$.

Player P_i, $i = 2, 3, 4$, sends P_1 a permutation on X. The three senders report their messages simultaneously. Let a be a permutation on X. The meaning of message a in Step 1 is "The realization of the random permutation $\alpha_1\sigma\tau$ is a". To simplify the presentation we say that in Step 1, P_2, P_3 and P_4 send P_1 the permutation $\alpha_1\sigma\tau$ (we adopt this terminology to describe the next steps).

The equilibrium strategies prescribe that each sender reports the realization of $\alpha_1\sigma\tau$ to P_1. For example, if the realizations of the random permutations α_1, σ, and τ are $\widehat{\alpha_1}, \hat{\sigma}$, and $\hat{\tau}$, respectively, in equilibrium P_2, P_3 and P_4 send P_1 the message $\widehat{\alpha_1}\hat{\sigma}\hat{\tau}$.

Before presenting equilibrium beliefs, let us clarify our terminology. As we will see, in Steps 1–5 of the game $\bar{\Gamma}(r)$, each message is sent by three players to a fourth one. We will often consider the message sent either by the majority of the senders, if a majority exists, or by the first sender in the description of the step (P_2 in Step 1), if the senders report three different messages. For expositional reasons, we refer to this message simply as the message sent by the majority of the senders (we shall use the upper bar to denote it). Moreover, we occasionally refer to the message sent by the majority as the message that the receiver "learns".

We are now ready to describe equilibrium beliefs. At the end of Step 1, P_1 assigns probability one to the event that the realization of $\alpha_1\sigma\tau$ coincides with $\overline{\alpha_1\sigma\tau}$, the message sent by the majority of the senders. Further, P_1 continues to believe that $\tau, \alpha_1, \beta_{31}$, and β_{32} are uniformly distributed. It is important to notice that P_1 does not change her beliefs about τ and α_1. Intuitively, knowing the realization of $\alpha_1\sigma\tau$ does not make any realization of τ more likely than the others. Every realization of τ is made compatible with the realization of $\alpha_1\sigma\tau$ by one (and only one) realization of α_1. Since α_1 and τ are independent and uniformly distributed, the marginal distribution of τ, given $\alpha_1\sigma\tau$, is uniform.

Notice that P_1 knows (from Step 0) the permutation φ and the element x. On the equilibrium path, she receives the realization of the permutation $\alpha_1\sigma\tau$ (from all senders) and therefore learns $\alpha_1\sigma\tau\varphi(x) = \alpha_1(y)$.

We provide a brief description of Steps 2–4 since they are similar to Step 1. These steps consist of several substeps in which three players send a random variable to a fourth one. As in Step 1, in every information set of Steps 2–4, the equilibrium strategies prescribe that the senders report the realization of the corresponding random variable. Moreover, independently of the triple of messages received, every receiver of Steps 2–4 has the following beliefs: (i) all her senders knew the realization of the random variable they had to report (see below); (ii) the realization of that random variable is the message sent by the majority of her senders; (iii) all the random variables that she did not select in Step 0 are uniformly distributed (this implies that at the end of Step 4 each players believes that y is uniformly distributed).

Step 2.

In this step, P_5 and P_2 learn $\alpha_2(y)$. Since P_2 does not know the permutation φ, she receives the permutation $\alpha_2\sigma\tau\varphi$. This message should be sent by three senders. However, only two players, P_3 and P_4, know the realizations of σ, τ and φ. We divide Step 2 into two substeps. In Substep 2.1, P_5 (who knows τ and φ) receives the permutation $\alpha_2\sigma$ from P_1, P_3 and P_4. In Substep 2.2, P_3, P_4 and P_5 send the permutation $\alpha_2\sigma\tau\varphi$ to P_2. Notice that both P_2 and P_5 knows the realization of x. Thus, in equilibrium, they can compute $\alpha_2\sigma\tau\varphi(x) = \alpha_2(y)$.

Substep 2.2 is the first step in which a player, P_5, does not learn in Step 0 all the components of the random variable that she has to send (in fact, P_5 receives $\alpha_2\sigma$ in Step 2.1). It is perhaps worthwhile to specify in detail P_5's equilibrium strategy and P_2's beliefs in this step. As mentioned above, in Substep 2.1 P_5 assigns probability one to the event that the realization of $\alpha_2\sigma$ is $\overline{\alpha_2\sigma}$, the message sent by the majority of her senders. Therefore, in equilibrium, P_5 sends the permutation $\overline{\alpha_2\sigma}\tau\varphi$ to P_2. Let us now consider P_2. First, she believes that the realization of $\alpha_2\sigma\tau\varphi$ is $\overline{\alpha_2\sigma\tau\varphi}$. Second, P_2 assigns probability one to the event that in Substep 2.1 P_5 learned the realization of $\alpha_2\sigma$, i.e., that P_5 received the realization of $\alpha_2\sigma$ from the majority of the senders. Third, P_2 believes that $\varphi, \alpha_2, \beta_{32}$, and β_{42} (the random variables that she did not select in Step 0) are uniformly distributed.

Step 3.

This step is designed to make P_3 learn the random element $\alpha_3(y)$. Since no player knows y, we need some preliminary steps in which the senders of the message to P_3 learn the necessary information. Specifically, in Substep 3.1, P_5 receives the permutation $\alpha_3\sigma$ from P_1, P_2 and P_4. In Substep 3.2, P_2, P_4 and P_5 send player P_1 the permutation $\beta_{31}\alpha_3\sigma\tau$. Then, in Substep 3.3, P_2 receives the permutation $\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi$ from P_4, P_1 and P_5. Finally, in Substep 3.4, P_1, P_2 and P_5 send the element $\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi(x)$ to P_3. Notice that P_3 knows both β_{32} and β_{31}. Therefore, she can compute $\beta_{31}^{-1}\beta_{32}^{-1}(\beta_{32}\beta_{31}\alpha_3\sigma\tau\varphi(x)) = \alpha_3(y)$.

Step 4.

The purpose of Step 4 is to let P_4 (and $P_6, ..., P_I$, if $I > 5$) learn the realization of the random element $\alpha_4(y)$.

In Substep 4.1, P_1, P_2 and P_3 send P_5 the permutation $\alpha_4\sigma$. In Substep 4.2, P_1 receives the permutation $\beta_{41}\alpha_4\sigma\tau$ from P_2, P_3 and P_5. Then, in Substep 4.3, P_3, P_1 and P_5 report the permutation $\beta_{42}\beta_{41}\alpha_4\sigma\tau\varphi$ to P_2. In Substep 4.4, P_1, P_2 and P_5 send P_4 the element $\beta_{42}\beta_{41}\alpha_4\sigma\tau\varphi(x)$. P_4 knows the permutations β_{41} and β_{42}, and so she learns $\alpha_4(y)$. As usual, $\overline{\beta_{42}\beta_{41}\alpha_4\sigma\tau\varphi(x)}$ denotes the message sent by the majority of the senders to P_4 in Substep 4.4.

Step 4 ends here if there are exactly five players, otherwise it continues as follows. Consider $k = 6, ..., I$. In Substep 4.$(k-1)$, P_1, P_2 and P_5 send P_k the element $\beta_{42}\beta_{41}\alpha_4\sigma\tau\varphi(x)$.[12] The message sent by the majority of the senders to P_k is denoted by $\overline{\beta_{42}\beta_{41}\alpha_4\sigma\tau\varphi(x)}^k$.

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[12] Then P_k can use the permutations β_{41} and β_{42} to compute $\alpha_4(y)$.

Step 5:

At this point of the game, every player knows an element of the set X. However, this element alone does not reveal any information about the state y. In this step every player P_i, $i = 1, \ldots, I$, learns which action in the set S_i corresponds to y. We divide Step 5 into the following substeps.

Substep 5.1. P_2, P_3 and P_4 send P_1 the function $pr_1\alpha_1^{-1}$.

In this step, P_2, P_3 and P_4 simultaneously send three messages to P_1. The (finite) set of feasible messages contains any mapping from X to S_1 that can be generated by applying the function pr_1 to some permutation on X (i.e., a message is feasible if and only if it can be expressed as $pr_1\varsigma$, where ς is a permutation on X).

The equilibrium strategies prescribe that each sender sends P_1 the function $pr_1\alpha_1^{-1}$. Note that all senders know the realization of the random permutation α_1, and the projection pr_1 is common knowledge among the players.

As usual, according to equilibrium beliefs, the message sent by the majority (which we denote by $\overline{pr_1\alpha_1^{-1}}$) coincides with the realization of the random function $pr_1\alpha_1^{-1}$.

Note that not all permutations on X are compatible[13] with the message $\overline{pr_1\alpha_1^{-1}}$. Thus, at the end of this step, P_1 assigns probability zero to some realizations of α_1. This, combined with the fact that P_1 knows the realization of $\alpha_1\sigma\tau$ and σ, implies that some realizations of the random permutation τ have zero probability. Therefore, the conditional distribution of the state y cannot be uniform over X anymore. We will come back to this point when we show that the assessment $\Psi(r)$ is sequentially rational.

The rest of Step 5 is similar to Substep 5.1. Specifically, consider $i = 2, 3, 4$. In Substep 5.i, players in the set $\{P_1, P_2, P_3, P_4\}\backslash\{P_i\}$ send P_i the function $pr_i\alpha_i^{-1}$. In Substep 5.5, P_1, P_3 and P_4 send P_5 the function $pr_5\alpha_2^{-1}$. If there are more than five players, P_1, P_2 and P_3 send P_k the function $pr_k\alpha_4^{-1}$ in Substep 5.k (where $k = 6, \ldots, I$). Sets of feasible messages, equilibrium strategies, and equilibrium beliefs are similar to those described in Substep 5.1.

Step 6. The game Γ is played.

In Step 6, all players simultaneously choose an action and then the game $\bar{\Gamma}(r)$ ends. P_i's set of (pure) strategies in Step 6 is S_i, her set of actions in the original game.

Equilibrium strategies are formally described in Table 5. Roughly speaking, to choose an action, a player applies the projection function learned in Step 5 to the element of X computed before Step 5. Consider, for example, P_1. In Step 1, she receives the permutation $\overline{\alpha_1\sigma\tau}$ from the majority of the senders. P_1 knows (from Step 0) the realizations of φ and x and computes the element $\overline{\alpha_1\sigma\tau}\varphi(x)$. In Substep 5.1, the majority of the senders send P_1 the function $\overline{pr_1\alpha_1^{-1}}$. Her equilibrium strategy in Step 6 is to choose the action $\overline{pr_1\alpha_1^{-1}}\,\overline{\alpha_1\sigma\tau}\varphi(x)$. The other players adopt similar strategies (see Table 5). Notice that the equilibrium strategy of a player in Step 6 does not depend on the messages that she sends in Steps 1–5.

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[13] A permutation ς on X is compatible with message $\overline{pr_1\alpha_1^{-1}}$ if $pr_1\varsigma^{-1} = \overline{pr_1\alpha_1^{-1}}$.

Table 5
Equilibrium strategies in Step 6

Player	Equilibrium strategies
P_1	$\overline{pr_1\alpha_1^{-1}}\overline{\alpha_1\sigma\tau}\varphi(x)$
P_2	$\overline{pr_2\alpha_2^{-1}}\overline{\alpha_2\sigma\tau}\varphi(x)$
P_3	$\overline{pr_3\alpha_3^{-1}}\beta_{31}^{-1}\beta_{32}^{-1}\overline{\beta_{32}\beta_{31}\alpha_3\sigma\tau}\varphi(x)$
P_4	$\overline{pr_4\alpha_4^{-1}}\beta_{41}^{-1}\beta_{42}^{-1}\overline{\beta_{42}\beta_{41}\alpha_4\sigma\tau}\varphi(x)$
P_5	$\overline{pr_5\alpha_2^{-1}}\overline{\alpha_2\sigma}\tau\varphi(x)$
P_6	$\overline{pr_6\alpha_4^{-1}}\beta_{41}^{-1}\beta_{42}^{-1}\overline{\beta_{42}\beta_{41}\alpha_4\sigma\tau}\varphi(x)^6$
$\vdots$	
P_I	$\overline{pr_I\alpha_4^{-1}}\beta_{41}^{-1}\beta_{42}^{-1}\overline{\beta_{42}\beta_{41}\alpha_4\sigma\tau}\varphi(x)^I$

Sequential rationality. We now demonstrate that the assessment $\Psi(r)$ is sequentially rational. In particular, we restrict attention to player P_1 and show that, given her beliefs, she does not have profitable deviations. Our proof can be easily applied to any other player. Given the so-called "one-shot-deviation principle", we only need to verify that deviations in a single information set are not profitable for P_1.

Consider an information set in Step 6. P_1 knows the realization of $\alpha_1\sigma\tau$ and the realizations of the random variables in the set $\tilde{\mathfrak{I}}_1$, where

$$\tilde{\mathfrak{I}}_1 = \{\sigma, \varphi, x, \alpha_2, \alpha_3, \alpha_4, \beta_{32}, \beta_{42}, \beta_{31}, \alpha_3\sigma\tau, \beta_{41}\alpha_4\sigma\tau, pr_1\alpha_1^{-1}\}.$$

Clearly, this implies that P_1 knows the realization of $pr_1(y)$, which we denote by s_1. Notice that y is independent of $\tilde{\mathfrak{I}}_1$, since the random permutation τ is independent of $\tilde{\mathfrak{I}}_1$.

In any information set of Step 6, P_1 assigns probability one to the event that every player P_k, $k = 2, \ldots, I$, chooses the action $pr_k(y)$. We now compute the probability that P_1's opponents play the action profile $(s_2, \ldots, s_I)$ given her information. We have:

$$\Pr(pr_2(y) = s_2, \ldots, pr_I(y) = s_I | pr_1(y) = s_1, \tilde{\mathfrak{I}}_1, \alpha_1\sigma\tau)$$

$$= \frac{\Pr(pr_1(y) = s_1, pr_2(y) = s_2, \ldots, pr_I(y) = s_I, \tilde{\mathfrak{I}}_1, \alpha_1\sigma\tau)}{\Pr(pr_1(y) = s_1, \tilde{\mathfrak{I}}_1, \alpha_1\sigma\tau)}$$

$$= \frac{\Pr(pr_1(y) = s_1, pr_2(y) = s_2, \ldots, pr_I(y) = s_I, \tilde{\mathfrak{I}}_1)}{\Pr(pr_1(y) = s_1, \tilde{\mathfrak{I}}_1)}$$

$$= \frac{\Pr(pr_1(y) = s_1, pr_2(y) = s_2, \ldots, pr_I(y) = s_I)}{\Pr(pr_1(y) = s_1)} = r(s_2, \ldots, s_I | s_1), \quad (4)$$

where r is the correlated equilibrium distribution that we want to implement. The third equality follows from y being independent of $\tilde{\mathfrak{I}}_1$ and the last equality comes from the fact that y is uniformly distributed over X. To explain the second equality,

we need to introduce additional notation. Let $\hat{\varphi}$, $\hat{x}$, and $\widehat{pr_1\alpha_1^{-1}}$ denote the realizations of φ, x, and $pr_1\alpha_1^{-1}$, respectively. We say that the permutation $\widehat{\alpha_1\sigma\tau}$ on X is compatible with $\tilde{\mathfrak{I}}_1$ and s_1 if $\widehat{pr_1\alpha_1^{-1}}\widehat{\alpha_1\sigma\tau}\hat{\varphi}(\hat{x}) = s_1$. We let Q denote the number of realizations of $\alpha_1\sigma\tau$ that are compatible with $\tilde{\mathfrak{I}}_1$ and s_1. Then it is easy to show that for any compatible permutation $\widehat{\alpha_1\sigma\tau}$, the following holds:

$$\Pr(pr_1(y) = s_1, pr_2(y) = s_2, \ldots, pr_I(y) = s_I, \tilde{\mathfrak{I}}_1, \alpha_1\sigma\tau = \widehat{\alpha_1\sigma\tau})$$

$$= \frac{1}{Q}\Pr(pr_1(y) = s_1, pr_2(y) = s_2, \ldots, pr_I(y) = s_I, \tilde{\mathfrak{I}}_1),$$

and

$$\Pr(pr_1(y) = s_1, \tilde{\mathfrak{I}}_1, \alpha_1\sigma\tau = \widehat{\alpha_1\sigma\tau}) = \frac{1}{Q}\Pr(pr_1(y) = s_1, \tilde{\mathfrak{I}}_1).$$

The two equalities above show that all realizations of $\alpha_1\sigma\tau$ compatible with $\tilde{\mathfrak{I}}_1$ and s_1 are equally likely.

According to Eq. (4), P_1 has the same information as in the case where she receives recommendation s_1 from a reliable mediator who implements the correlated equilibrium r. Thus, the action s_1 maximizes P_1's expected payoff. In other words, the equilibrium strategy of Step 6 is optimal for player P_1.

Consider now any P_1's information set in Steps 1–5. Remember that P_1 assigns probability one to the event that all senders know the realization of the permutation that they report. Since every receiver follows the message sent by the majority of the senders and since P_1's action in Step 6 does not depend on her messages in Steps 1–5, we conclude that a deviation is not profitable.

Finally, we examine Step 0. Consider, for example, the substep in which P_1 and P_2 send two messages (σ^1 and σ^2, respectively) to determine the permutation $\sigma = \sigma^1\sigma^2$. Since σ^2 is uniformly distributed, the permutation σ is independent of σ^1 and uniformly distributed. Any strategy is optimal in this substep for P_1, including the uniform randomization over the set $\Lambda(X)$. Clearly, a similar argument can be used to show that P_1 does not have profitable deviations in any other substep of Step 0. We conclude that the assessment $\Psi(r)$ is sequentially rational.

Consistency. A detailed proof of the consistency of $\Psi(r)$ is in Gerardi [14] where we construct a sequence of completely mixed strategies that converges to the equilibrium strategies. We then compute beliefs along the sequence and show that, in the limit, they coincide with equilibrium beliefs. In what follows, we only outline our proof and provide an intuition for our result.

As far as consistency in Step 0 is concerned, we use the equilibrium strategies to construct a (constant) sequence of completely mixed strategies (in Step 0 all strategies are played with positive probability). This implies that, along the sequence, each random variable chosen in Step 0 has a uniform distribution on the underlying probability space. The assessment $\Psi(r)$ trivially satisfies consistency in Step 0.[14]

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[14] Remember that at the end of Step 0, a player either knows the realization of a random variable or believes that the random variable is uniformly distributed.

For each Step j, $j = 1, 2.1, \ldots, 5.I$, we construct $\{\varepsilon_{j,n}\}_{n=1}^{\infty}$, a sequence of positive numbers in the unit interval, converging to zero. We assume that in every information set of Step j, the first sender follows her equilibrium strategy with probability $1 - \varepsilon_{j,n}^3$. All other deviations are equally likely. We refer to these completely mixed strategies by saying that the first sender of Step j trembles from her equilibrium strategy with probability $\varepsilon_{j,n}^3$. We also assume that the other two senders of Step j tremble from their equilibrium strategies with probability $\varepsilon_{j,n}^2$. Notice that for each Step j, $j = 1, 2.1, \ldots, 5.I$, our sequence of mixed strategies satisfies the following properties: (i) the first sender is less likely to deviate than any of the other two senders; (ii) the probability that any two senders deviate converges to zero faster than the probability that only the other sender deviates.

Consider Step 1. The senders, P_2, P_3 and P_4, know the realization of $\alpha_1 \sigma \tau$ from Step 0. Properties (i) and (ii) above guarantee that in the limit (i.e., as the probability of deviating converges to zero) the majority of the senders report the true message. This proves the consistency of the assessment $\Psi(r)$ in Step 1. In exactly the same way, we show that $\Psi(r)$ satisfies consistency in Substeps 2.1, 3.1, 4.1 and 5.1, ..., 5.I.

To prove the consistency of $\Psi(r)$ in Substep 2.2, we assume that for every $n = 1, 2, \ldots, \varepsilon_{2.1,n} = \varepsilon_{2.2,n}$ (that is, trembles in Substep 2.1 have the same probability as trembles in Substep 2.2).[15] To understand the role that this assumption plays, let us suppose that in Substep 2.2 P_2 receives the same message from P_3 and P_4, and a different one from P_5. There are three possible reasons why P_5 sent a different message: (i) P_5 received the true realization of $\alpha_2 \sigma$ in Substep 2.1, but she deviated in Substep 2.2. (ii) P_5 received a wrong message in Substep 2.1, but followed her equilibrium strategy in Substep 2.2. (iii) P_5 received a wrong message in Substep 2.1, and then deviated. Note that only one deviation (by P_5) is needed for case (i). In case (ii), we need at least two deviations in Substep 2.1. Finally, at least three deviations are required for case (iii). Since we assume that the probabilities of deviations in Substeps 2.1 and 2.2 are the same, in the limit we assign probability one to case (i), which requires the fewest deviations.[16]

The analysis of Step 3 is slightly complicated by the fact that a player can be a receiver after being a sender. We require that the trembles of the different substeps converge to zero at different speeds. To see why this assumption is needed, let us consider P_1. Suppose that the realization of $\alpha_3 \sigma$ is $\widehat{\alpha_3}\hat{\sigma}$, and that P_1 deviates in Substep 3.1 sending P_5 message $\widehat{\alpha_3}'\hat{\sigma}$ (where $\widehat{\alpha_3}'$ is different from $\widehat{\alpha_3}$). Further, suppose that in Substep 3.2, P_1 receives the same message $\widehat{\beta_{31}\alpha_3}'\hat{\sigma}\hat{\tau}$ from P_2 and P_4

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[15] In Substep 2.2, the receiver P_2 believes that the realization of $\alpha_2 \sigma \tau \varphi$ is $\overline{\alpha_2 \sigma \tau \varphi}$ and that P_5 learned the realization of $\alpha_2 \sigma$ in Substep 2.1.

[16] Another interesting case arises when P_2 receives the same message from P_4 and P_5 and a different one from P_3. We can justify this triple of messages with a single deviation by P_3 in Substep 2.2. A deviation by P_4 requires at least another deviation. It is true that P_3 trembles with probability $\varepsilon_{2.2,n}^3$ in Substep 2.2, while P_4 trembles with probability $\varepsilon_{2.2,n}^2$, but any other deviation occurs at least with probability $\varepsilon_{2.2,n}^2$ or $\varepsilon_{2.1,n}^2$. Since $\varepsilon_{2.2,n}$ is equal to $\varepsilon_{2.1,n}$, we conclude that (in the limit) with probability one P_3 deviated in Substep 2.2 and P_5 learned the truth in Substep 2.1.

and the different message $\widehat{\beta_{31}}\widehat{\alpha_3}'\hat{\sigma}\hat{\tau}$ from P_5 (where $\widehat{\beta_{31}}$ and $\hat{\tau}$ are two permutations on X). The following two situations are both compatible with this triple of messages. In the first scenario, P_5 learned the correct permutation $\widehat{\alpha_3}\hat{\sigma}$, but then deviated in Substep 3.2. The probability of this tremble is of the same order as $\varepsilon_{3.2,n}^2$. In the second scenario, P_5 learned the permutation $\widehat{\alpha_3}'\hat{\sigma}$ and then followed her equilibrium strategy. The probability of this event is an infinitesimal of the same order as $\varepsilon_{3.1,n}^2$. In Substep 3.2, P_1 can assign probability zero to the last event only if we assume that $\lim_{n \to \infty} \frac{\varepsilon_{3.1,n}}{\varepsilon_{3.2,n}} = 0$, i.e., only if the trembles are much less likely in Substep 3.1 than in Substep 3.2.

The argument above also implies the necessity of assuming that $\lim_{n \to \infty} \frac{\varepsilon_{3.2,n}}{\varepsilon_{3.3,n}} = 0$. In this way, we can show that P_2's beliefs in Substep 3.3 are consistent for the case in which P_2 follows her equilibrium strategy in Substep 3.1, but deviates in Substep 3.2. The case in which P_2 deviates both in Substep 3.1 and in Substep 3.2 requires a further assumption: $\lim_{n \to \infty} \frac{\varepsilon_{3.1,n}}{\varepsilon_{3.3,n}^2} = 0$. To give an intuition for this requirement, suppose that the realizations of σ, τ, α_3, and β_{31} are $\hat{\sigma}$, $\hat{\tau}$, $\widehat{\alpha_3}$, and $\widehat{\beta_{31}}$, respectively. Suppose also that P_2 sends message $\widehat{\alpha_3}'\hat{\sigma}$ in Substep 3.1 and message $\widehat{\alpha_3}''\hat{\sigma}$ in Substep 3.2 (where $\widehat{\alpha_3}$, $\widehat{\alpha_3}'$ and $\widehat{\alpha_3}''$ are three different permutations). Suppose that P_5 learns the permutation $\widehat{\alpha_3}'\hat{\sigma}$ sent by P_2 (the probability of this event is an infinitesimal of the same order as $\varepsilon_{3.1,n}^2$). Then, the probability that in Substep 3.2 P_1 learns the permutation $\widehat{\beta_{31}}\widehat{\alpha_3}''\hat{\sigma}\hat{\tau}$ (sent by P_2) converges to one. If in Substep 3.3 every sender follows her equilibrium strategy, P_2 receives three different messages. Now consider the event in which both P_1 and P_5 receive the true permutation. Then the probability that three different messages are sent in Substep 3.3 is an infinitesimal of the same order as $\varepsilon_{3.3,n}^4$. To assign probability zero to the first event (in the limit), we need to assume that $\lim_{n \to \infty} \frac{\varepsilon_{3.1,n}}{\varepsilon_{3.3,n}^2} = 0$.

For Substep 3.4, we assume that $\varepsilon_{3.4,n} = \varepsilon_{3.3,n}$, for every n. This assumption is sufficient to guarantee the consistency of $\Psi(r)$ in Substep 3.4. To provide some intuition, suppose that P_3 receives the same message from two senders, but a different one from the third. Consider the two following scenarios. It is possible that all senders know the truth[17] and only one of them deviated in Substep 3.4. Alternatively, it is possible that some of the senders did not learn the truth. However, this implies that at least two players deviated in Substeps 3.1–3.3. Since deviations in Substep 3.4 are at least as likely as deviations in earlier steps, we conclude that, in the limit, the latter event has probability zero.[18]

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[17] The probability of this event converges to one as n goes to infinity.

[18] The case where P_3 receives three different messages can be analyzed in a similar way. If all senders know the truth, the probability of receiving three different messages is an infinitesimal of the same order as $\varepsilon_{3.4,n}^4$. If P_1 and/or P_5 did not learn the truth, then at least two players deviated in Substeps 3.1–3.2. Since $\varepsilon_{3.4,n}$ converges to zero faster than $\varepsilon_{3.1,n}$ and $\varepsilon_{3.2,n}$, we assign probability one to the event that both P_1 and P_5 know the truth. Given this, the probability that P_2 does not learn the truth in Substep 3.3 is an infinitesimal of the same order as $\varepsilon_{3.3,n}^4$. It is true that $\varepsilon_{3.3,n}$ is equal to $\varepsilon_{3.4,n}$, but note that P_3 cannot receive

Finally, the proof of the consistency of $\Psi(r)$ in Step 4 is identical to the proof for Step 3 (we assume $\varepsilon_{4.j,n} = \varepsilon_{3.j,n}$ for $j = 1, \ldots, 4$ and $\varepsilon_{4.j,n} = \varepsilon_{3.4,n}$ for $j = 5, \ldots, I-1$).

5. Conclusion

In this paper we characterize the outcomes of static games that sequentially rational players can implement with direct communication. We show that for a large class of games there is no difference between mediated and unmediated communication, in the sense that both forms of communication allow players to implement the same outcomes. Specifically, if a game has five or more players, rational parameters, and full support, then the set of outcomes that can be implemented with unmediated communication coincides with the set of communication equilibria.

We use sequential equilibrium to analyze cheap talk extensions of a static game. However, our results provide a complete characterization of the effects of unmediated communication even for the case in which a solution concept weaker than sequential equilibrium, but stronger than Nash equilibrium (such as subgame-perfect equilibrium or perfect Bayesian equilibrium) is considered. In fact, by using a solution concept weaker than sequential equilibrium, one can always implement all the outcomes than we implement in this paper. If the solution concept is stronger than Nash equilibrium, only communication equilibria can be implemented. Thus, if a game satisfies our assumptions (five or more players, rational parameters and full support), the set of communication equilibria coincides with the set of outcomes that are induced by perfect Bayesian or subgame-perfect equilibria of cheap talk extensions with unmediated communication.

Future research is needed to obtain a complete characterization of the effects of unmediated communication in games that do not satisfy our assumptions. Moreover, we restrict attention to static games. For extensive-form games, two important articles deal with mediated communication. Forges [11] extends the notion of correlated and communication equilibria to multistage games. Myerson [22] introduces a sequential rationality criterion in the context of multistage games with mediated communication. An interesting extension of this paper would be the analysis of unmediated communication in dynamic games.

Acknowledgments

(footnote continued)

three different messages if both P_1 and P_5 follow their equilibrium strategies in Substep 3.4. At least one other deviation is needed. We conclude that all senders know the truth and two of them deviated in the last substep.

References

[1] M. Amitai, Cheap-talk with incomplete information on both sides, Discussion Paper No. 90, Center for Rationality, The Hebrew University of Jerusalem, 1996.

[2] R.J. Aumann, Subjectivity and correlation in randomized strategies, J. Math. Econom. 1 (1974) 67–96.

[3] R.J. Aumann, S. Hart, Long cheap talk, Econometrica, forthcoming.

[4] R.J. Aumann, M. Maschler, R.E. Stearns, Repeated games of incomplete information: an approach to the non-zero sum case, in: R.J. Aumann, M. Maschler (Eds.), Repeated Games with Incomplete Information, The MIT Press, Cambridge, 1995 (Chapter V).

[5] J.S. Banks, R.L. Calvert, A battle-of-the-sexes game with incomplete information, Games Econom. Behav. 4 (1992) 347–372.

[6] I. Barany, Fair distribution protocols or how the players replace fortune, Math. Oper. Res. 17 (1992) 327–340.

[7] E. Ben-Porath, Correlation without mediation: expanding the set of equilibrium outcomes by 'cheap' pre-play procedures, J. Econom. Theory 80 (1998) 108–122.

[8] E. Ben-Porath, Cheap talk in games with incomplete information, J. Econom. Theory 108 (2003) 45–71.

[9] V. Crawford, J. Sobel, Strategic information transmission, Econometrica 50 (1982) 1431–1451.

[10] A. Dhillon, J.F. Mertens, Perfect correlated equilibria, J. Econom. Theory 68 (1996) 279–302.

[11] F. Forges, An approach to communication equilibria, Econometrica 54 (1986) 1375–1385.

[12] F. Forges, Universal mechanisms, Econometrica 58 (1990) 1341–1364.

[13] D. Gerardi, Strategic decision-making and information transmission, Ph.D. Thesis, Northwestern University, June 2001.

[14] D. Gerardi, Unmediated communication in games with complete and incomplete information, Cowles Foundation Discussion Paper No. 1371, 2002.

[15] D.M. Kreps, R. Wilson, Sequential equilibria, Econometrica 50 (1982) 863–894.

[16] V. Krishna, J. Morgan, The art of conversation, Mimeo, Penn State University and University of California at Berkeley, 2002.

[17] E. Lehrer, Mediated talk, Int. J. Game Theory 25 (1996) 177–188.

[18] E. Lehrer, S. Sorin, One-shot public mediated talk, Games Econom. Behav. 20 (1997) 131–148.

[19] A.M. Manelli, Cheap talk and sequential equilibria in signaling games, Econometrica 64 (1996) 917–942.

[20] S.A. Matthews, A. Postlewaite, Pre-play communication in two-person sealed-bid double auction, J. Econom. Theory 48 (1989) 238–263.

[21] R.B. Myerson, Optimal coordination mechanisms in generalized principal-agent problems, J. Math. Econom. 10 (1982) 67–81.

[22] R.B. Myerson, Multistage games with communication, Econometrica 54 (1986) 323–358.

[23] R.B. Myerson, Game Theory: Analysis of Conflict, Harvard University Press, Cambridge, 1991.

[24] A. Urbano, J. Vila, Computational complexity and communication: coordination in two-player games, Econometrica 70 (2002) 1893–1927.

[25] A. Urbano, J. Vila, Computationally restricted unmediated talk under incomplete information, Mimeo, The Valencian Institute of Economic Research, 2002.