Linear Control Laws for Singular Linear Systems

by MANOEL SOBRAL, JR.

Instituto Tecnológico de Aeronáutica
São Jose dos Campos, São Paulo, Brazil

ABSTRACT: *For a class of quadratic performance indices the optimal control law is a combination of maximum effort (bang-bang) and singular. The singular control law is linear and it is optimal in a hyperplane in the n-dimensional state-space. For practical purposes it is desirable to restrict the class of admissible control laws to be linear. This investigation presents a method of finding a linear control law which is optimal in the sense that it is the singular control law in the singular surface and the best possible linear law elsewhere. Classical calculus of variations and the more sophisticated maximum principle of Pontryagin are the mathematical tools used; the former provides a simple and straight forward method for obtaining the singular solutions while the latter is used to extend the linear singular law to the entire state space.*

Introduction

In any practical optimization problem the control signal is constrained in some way. In mathematical terminology, the set U of admissible control signals $u(t)$ is defined to be a certain set of functions of the real variable t. In most problems U is defined as the set of all piecewise continuous functions $u(t)$ such that the absolute value $|u(t)|$ is uniformly bounded by a fixed number M. Another way of constraining the control signal is to include it in the performance index. This is done in general by adding a term to the performance index which is the integral of some non-negative definite form in u—for instance λu^2. For obvious physical reasons the performance index is then said to include the "cost" of control. Consider now the following optimization problem. Given a linear plant characterized by the equation (in state space notation)

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u}, \tag{1}$$

where $\mathbf{x}$ is a $n \times 1$ state vector, $\mathbf{A}$, $\mathbf{B}$ are $n \times n$ matrices and $\mathbf{u}$ is the $n \times 1$ control vector, it is desired to find $\mathbf{u}(t)$ which minimizes the performance index[1]

$$S(\mathbf{u}) = \int_0^T \mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)dt, \tag{2}$$

[1] Throughout this paper, the superscript T denotes the transpose of a matrix.

where $\mathbf{Q}$ and $\mathbf{R}$ are non-negative definite matrices of constants. It is a well known result of optimal control theory (**1**, **2**) that the optimal control law $\mathbf{u}(t)$ is a linear combination of the state variable of the system, i.e.,

$$\mathbf{u}(t) = \mathbf{C}(t)\mathbf{x}(t). \tag{3}$$

Consider now the singular problem, i.e., $\mathbf{R} \equiv \mathbf{0}$. If the performance index above does not include the cost of control, it has been shown (**3**, **4**), that $\mathbf{u}(t)$ is linear when $\mathbf{x}(t)$ lies in a certain hyperplane Γ in the n-dimensional state space. However, outside of Γ the control is made by impulses (assuming no bounds in $|\mathbf{u}|$). If $\mathbf{u}$ is constrained in magnitude, the familiar bang-bang control is optimal outside of Γ_r (portion of Γ where the linear control law satisfies the constraint on the magnitude of $\mathbf{u}$). Therefore, in the case $\mathbf{R} \equiv \mathbf{0}$, the optimal control is not linear everywhere but only in Γ_r. For practical reasons we may desire to have linear control everywhere. Therefore, the following question is posed: which (if any) of the control laws given by $\mathbf{u} = \mathbf{Cx}$ is the best approximation for the bang-bang control (i.e., the control which minimizes the performance index) outside of Γ_r? In the following paragraph a concise formulation of the problem is presented; only single input-single output linear time invariant plants are considered.

1. Formulation of the Problem

Consider a linear, time invariant plant described by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t), \tag{4}$$

where $\mathbf{A}$ is an $n \times n$ matrix of constants, $\mathbf{x}$ is the $n \times 1$ state vector, $\mathbf{b}$ is a $n \times 1$ constant matrix and $u(t)$ is the control signal. Assuming complete controllability for system Eq. 4, there is no loss of generality (**5**) in taking $\mathbf{A}$, $\mathbf{b}$, $\mathbf{x}$ to have the forms

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \vdots & & & \\ -a_0 & -a_1 & -a_2 & & -a_{n-1} \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \tag{5}$$

$$\mathbf{x} = \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ \overset{(n-1)}{x} \end{bmatrix}.$$

Consider now the functional

$$S(u) = \int_0^\infty [\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t)]dt, \tag{6}$$

where $\mathbf{Q}$ is non-negative definite, $n \times n$ matrix of constants. The set U of admissible controls is defined as being the set of all signals $u(t)$ such that

$$|u(t)| \leq k_A \tag{7}$$

and

$$u(t) = \mathbf{c}^T\mathbf{x}(t), \tag{8}$$

where $\mathbf{x}(t)$ is the solution of system Eq. 4 and $\mathbf{c}$ is a $n \times 1$ constant vector. Since Eq. 7 is to be valid at all times, it is obvious (in view of Eq. 8) that the initial conditions must be bounded. Therefore let the set of admissible initial conditions be X, X being the set of all $\mathbf{x}_o$ such that

$$\|\mathbf{x}_o\| \leq k_\beta, \tag{9}$$

where $\|\mathbf{x}_o\|$ = euclidean norm of $\mathbf{x}_o$. The set U is then the set of all controls given by Eq. 8, which satisfy Eq. 7 at all times and for any initial condition satisfying Eq. 9. The optimization problem is to find u in U which minimizes the functional Eq. 6 subject to the restrictions

$$\mathbf{x}(o) = \mathbf{x}_o \tag{10a}$$

and

$$\mathbf{x}(\infty) = \mathbf{o}. \tag{10b}$$

Moreover, u must be the singular optimal control on the singular surface. Denoting the integrand in Eq. 6 as $F(x, \dot{x}, \cdots \overset{(n-j)}{x})$, two cases will be treated separately: (a) $J = 1$ and (b) $j > 1$. Case (a) is called the case of singularity of first order and is treated in the next section. Case (b) is called the case of singularity of order j and will be treated in the succeeding section.

Systems of Singularity of First Order

2. Singular Linear Control Law

In this section performance indices, in which the order of the highest derivative of $x(t)$ in the integrand is $n - 1$, are considered. The singular trajectories are obtained in a simple and straight-forward manner (**6**); then conditions which the desired linear control law must satisfy in order to be the optimal control law on the singular surface are derived. The integrand in the performance index Eq. 6 is written as

$$F(x, \dot{x}, \ddot{x}, \cdots \overset{(n-1)}{x}) = \sum_{i,j=1}^{n} q_{i,j} \overset{(i-1)}{x} \overset{(j-1)}{x}. \tag{11}$$

From the calculus of variations a fundamental necessary condition for an extremum is that the integrand Eq. 11 satisfies the Euler Eq. 7

$$\sum_{k=o}^{n-1} (-1)^k \frac{d^k}{dt^k}\left(\frac{\partial_F}{\partial \overset{(k)}{x}}\right) = 0. \tag{12}$$

From (11)

$$\frac{\partial_F}{\partial \overset{(k)}{x}} = 2 \sum_{i=1}^{n} q_{k+1,i} \overset{(i-1)}{x} \tag{13}$$

therefore the Euler equation is

$$2 \sum_{k=o}^{n-1} \sum_{i=o}^{n-1} (-1)^k q_{k+1,i+1} \overset{(i+k)}{x} = 0. \tag{14}$$

The above equation is even in the derivatives of x, i.e., it can be written as

$$2 \sum_{p=o}^{n-1} K_p \overset{(2p)}{x} = 0, \tag{15}$$

where

$$K_p = \sum_{\substack{k,l \\ k+l=2p}} q_{k+1,l+1}; \qquad p = 0, 1, \cdots, n-1. \tag{16}$$

Taking the characteristic equation associated with Eq. 15 we obtain

$$P(s^2) = 2 \sum_{p=o}^{n-1} K_p s^{2p} = 0 \tag{17}$$

$P(s^2)$ is even in s; therefore it can be factored into a unique product of a Hurwitz and an anti-Hurwitz polynomial; since the final state of the system is the stable origin (as given by Eq. 10), the unbounded solutions of Eq. 15 must be discarded. Hence, if we write

$$P(s^2) = H(s)H(-s), \tag{18}$$

the differential equation which the optimal trajectory must satisfy is

$$\sum_{i=o}^{n-1} h_i \overset{(i)}{x} = 0, \qquad h_{n-1} = 1. \tag{19}$$

The trajectories given by Eq. 19 are the so-called singular solutions (**3**, **4**). Since Eq. 19 is an equation of order $n - 1$ and the system equation is of order n, the Euler equation gives the optimal trajectories only for some initial conditions. These initial conditions form a hyperplane Γ_1 in the n-dimensional state space. Obviously Γ_1 is the set of all $\mathbf{x}_o$ such that

$$\sum_{i=o}^{n-1} h_i \overset{(i)}{x}_o = 0. \tag{20}$$

In the following we derive conditions that the desired linear control law $u(t) = \mathbf{c}^T\mathbf{x}(t)$ must satisfy in order that Eq. 19 be satisfied by $\mathbf{x}(t)$ when

$\mathbf{x}_o$ is in Γ_1. To do so the system Eq. 4 is written

$$\sum_{i=o}^{n-1} (a_i - c_i) \overset{(i)}{x} + \overset{(n)}{x} = 0. \tag{21}$$

The characteristic equation associated with Eq. 21 is

$$H_T(s) = \sum_{i=o}^{n-1} (a_i - c_i)s^i + s^n = 0. \tag{22}$$

If the solution of Eq. 21 is to be identical to the solution of Eq. 19 when $\mathbf{x}_o$ is in Γ_1, the system Eq. 21 must contain all modes of Eq. 19. Hence, we must equate

$$H_T(s) = Z(s)H(s), \tag{23}$$

where $Z(s) = (s + z_o)$. Equating coefficients of the same power in s, we obtain the relationships

$$z_o = \frac{a_o - c_o}{h_o} \tag{24}$$

and

$$c_i = a_i - z_o h_i - h_{i-1}, \qquad i = 1, \cdots, n-1. \tag{25}$$

Equation 25 expresses every component of $\mathbf{c}$ (but the first) as a function of c_o. Hence by imposing the condition that the linear control law $u = \mathbf{c}^T\mathbf{x}$ be the optimal singular control on the singular surface we have reduced the n-variable problem to a single variable one. Note also that if Eqs. 25 are satisfied, the control $u = \mathbf{c}^T\mathbf{x}$ is independent of c_o when $\mathbf{x}$ is on the singular surface Γ_1.[2] In view of Eqs. 23 and 24, the necessary and sufficient condition for assuring a stable system is simply that

$$a_o - c_o > 0. \tag{26}$$

Note that under the stated end point condition (Eq. 10b), Eq. 26 is a condition for the existence of a solution.

3. Extended Linear Control Law

In order to extend the linear control to the entire state space, we must find a vector $\mathbf{c}^*$ which has the following properties: (a) Its components c_i^*, $i = 1, \cdots, n-1$ must satisfy Eq. 22; (b) the control $u^*(t) = (\mathbf{c}^*)^T x(t)$ must be admissible, i.e., u^* must be in U; (c) u^* must minimize the performance index Eq. 6, i.e.,

$$\min_{u \epsilon U} S(u) = S(u^*).$$

If Pontryagin's formulation is used,[3] the problem is to find $\mathbf{c}$ such that the Hamiltonian (8)

$$M(\mathbf{p}, \mathbf{x}, \mathbf{c}) = p^T\dot{\mathbf{x}} - \mathbf{x}^T\mathbf{Q}\mathbf{x} \tag{27}$$

[2] Some characteristics of the linear control law are discussed in (9).

[3] After completion of this work certain difficulties in applying Pontryagin's Max. Principle to this problem have come to the attention of the author. This is now being studied and as soon as the investigation is completed a note will be sent to this journal.

is maximized (in $\mathbf{c}$). The maximization is subject to the constraint

$$u^2 = \mathbf{x}^T\mathbf{c}\mathbf{c}^T\mathbf{x} \leq k_A{}^2, \tag{28}$$

for every $\mathbf{x}_o$ such that Eq. 9 holds. Moreover, Eqs. 25 must be satisfied. The auxiliary variable $\mathbf{p}$ in Eq. 27 is defined by its components

$$(\dot{p})_i = -\frac{\partial_M}{\partial \overset{(i-1)}{x}}, \qquad i = 1, \cdots, n. \tag{29}$$

Substitution of Eq. 4 into Eq. 27 gives

$$M(\mathbf{p}, \mathbf{x}, \mathbf{c}) = p_n\mathbf{c}^T x + \mathbf{p}^T\mathbf{A}\mathbf{x} - \mathbf{x}^T\mathbf{Q}\mathbf{x}. \tag{30}$$

Note that if we set $\frac{\partial_M}{\partial_{c_i}} = 0$, the condition $p_n(t) = 0$ (singular solution) is obtained. Since the only term in Eq. 30 dependent on $\mathbf{c}$ is

$$N(\mathbf{p}, \mathbf{x}, \mathbf{c}) = p_n\mathbf{c}^T\mathbf{x}, \tag{31}$$

we have to maximize $N(\mathbf{p}, \mathbf{x}, \mathbf{c})$ restricted to Eq. 28. Furthermore, since Eq. 25 is to be satisfied, the only variable in $\mathbf{c}$ is c_o. Using Lagrange multipliers, we set

$$\frac{\partial}{\partial c_o}[N(\mathbf{p}, \mathbf{x}, \mathbf{c}) + \rho\mathbf{x}^T\mathbf{c}\mathbf{c}^T\mathbf{x}] = 0. \tag{32}$$

Performing the differentiation indicated in Eq. 32, we obtain

$$p_n \frac{\partial_{\mathbf{c}}{}^T}{\partial c_o}\mathbf{x} + \rho\mathbf{x}^T\left[\frac{\partial \mathbf{c}}{\partial c_o}\mathbf{c}^T + \mathbf{c}\frac{\partial_{\mathbf{c}}{}^T}{\partial c_o}\right]\mathbf{x} = 0, \tag{33}$$

or

$$p_n \frac{\partial_{\mathbf{c}}{}^T}{\partial c_o}\mathbf{x} + 2\rho\mathbf{x}^T\mathbf{c}\frac{\partial_{\mathbf{c}}{}^T}{\partial c_o}\mathbf{x} = 0. \tag{34}$$

The function $p_n(t)$ can be shown (**9**) to be given by

$$p_n(t) = \frac{\sqrt{q_{11}q_{nn}}}{c_o - a_o}\sum_{i=o}^{n-1} h_i \overset{(i)}{x}(t). \tag{35}$$

Note that p_n is the n^{th} component of $\mathbf{p}$; $\mathbf{p}$ is the solution of

$$\dot{\mathbf{p}} = -\mathbf{F}^T\mathbf{p} + 2\mathbf{Q}\mathbf{x}, \tag{36}$$

where

$$\mathbf{F} = \mathbf{A} + \mathbf{b}\mathbf{c}^T. \tag{37}$$

Let

$$\mathbf{h} = \begin{bmatrix} h_o \\ \vdots \\ h_{n-1} \end{bmatrix},$$

we have

$$p_n(t) = \frac{\sqrt{q_{11}q_{nn}}}{c_o - a_o}\mathbf{x}^T\mathbf{h}. \tag{38}$$

Substituting Eq. 38 into Eq. 34, we obtain

$$\mathbf{x}^T \left[\frac{\sqrt{q_{11}q_{nn}}}{c_o - a_o} \mathbf{h} + 2\rho \mathbf{c} \right] \frac{\partial \mathbf{c}^T}{\partial c_o} \mathbf{x} = 0. \tag{39}$$

Let

$$\mathbf{D} = \left[\frac{\sqrt{q_{11}q_{nn}}}{c_o - a_o} \mathbf{h} + 2\rho \mathbf{c} \right] \left[\frac{\partial \mathbf{c}^T}{\partial c_o} \right], \tag{40}$$

it is easily seen that $\mathbf{D}$ (a dyad) is of rank at most one and its only (possibly) non-zero eigenvalue is

$$\lambda(\mathbf{D}) = \left[\frac{\partial \mathbf{c}^T}{\partial c_o} \right] \left[\frac{\sqrt{q_{11}q_{nn}}}{c_o - a_o} \mathbf{h} + 2\rho \mathbf{c} \right]. \tag{41}$$

In view of Eq. 39, it is necessary that all the eigenvalues of $\mathbf{D}$ be zero. Performing the differentiation indicated in Eq. 41 and setting $\lambda(\mathbf{D}) = 0$ we obtain

$$\frac{\sqrt{q_{11}q_{nn}}}{2\rho} \mathbf{h}^T\mathbf{h} = (a_o - c_o)\mathbf{h}^T\mathbf{c}. \tag{42}$$

Equation 42 relates the optimal value of c_o to the Lagrange multiplier ρ. The remaining problem is to find ρ in such a way that the constraint in $u^2(t)$ be satisfied whenever $\mathbf{x}_o$ is in Γ_1. A numerical iterative procedure is unavoidable in general. However, the following considerations simplify considerably the search for the optimal ρ (or equivalently, c_o). If Eq. 25 is substituted into Eq. 31, we obtain

$$N(\mathbf{p}, \mathbf{x}, \mathbf{c}) = p_n \sum_{i=o}^{n-1} (a_i - h_{i-1}) \overset{(i)}{x} - p_n z_o \mathbf{h}^T\mathbf{x}. \tag{43}$$

The only term in $N(\mathbf{p}, \mathbf{x}, \mathbf{c})$ which depends on c_o is $p_n z_o \mathbf{h}^T\mathbf{x}$. Since by Eq. 38 and Eq. 26 p_n has the sign opposite to the sign of $\mathbf{h}^T\mathbf{x}$, in order to maximize $N(\mathbf{p}, \mathbf{x}, \mathbf{c})$ (in $\mathbf{c}$) we must maximize z_o. That is, to make c_o as small as the constraint in $|u(t)|$ allows. This result confirms the intuitive notion that the free pole at $-\dfrac{a_o - c_o}{h_o}$ should be placed as far as possible from the origin. Note that Eq. 42 also carries this information since a loose constraint in $|u|$ should result in a "small' value for ρ and hence a "large" value for $(a_o - c_o)$. On the other hand, since (note that $\mathbf{cc}^T$ is of rank one)

$$\max_{\|\mathbf{x}_o\| \leq k_B} u^2(0) = \max_{\|\mathbf{x}_o\| \leq k_B} \mathbf{x}_o^T \mathbf{c}\mathbf{c}^T \mathbf{x}_o = \mathbf{c}^T\mathbf{c} k_B^2, \tag{44}$$

a necessary condition for $\mathbf{c}$ to satisfy the constraint in u for all $\mathbf{x}_o$ such that $\|\mathbf{x}_o\| \leq k_B$ is

$$c^T c \leq \left(\frac{k_A}{k_B} \right)^2. \tag{45}$$

Condition Eq. 45 is by no means sufficient in general. However, if $\mathbf{x}^T\mathbf{x}$ is a Lyapunov function for system Eq. 4, Eq. 35 is also sufficient (because in this case $\|\mathbf{x}(t)\|^2 \leq \|\mathbf{x}_o\|^2$).

4. *Illustrative Example*

Given the system equation (Fig. 1)

$$\ddot{x} = u \tag{46}$$

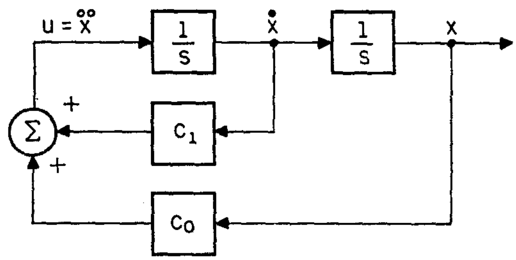

Fig. 1. System for Example 4.

and the performance index

$$\int_o^\infty (x^2 + \dot{x}^2)dt, \tag{47}$$

find a linear control law

$$u = c_o x + c_1 \dot{x}, \tag{48}$$

such that u minimizes the performance index Eq. 47 subject to the following restrictions

$$\begin{aligned} \mathbf{x}(o) &= \mathbf{x}_o, & \|x_o\| &\leq 1, \\ \mathbf{x}(\infty) &= o, & |u(t)|^2 &\leq 5. \end{aligned} \tag{49}$$

From the performance index, the Euler equations are

$$\ddot{x} - x = 0; \tag{50}$$

the stable part is found to be

$$\dot{x} + x = 0. \tag{51}$$

Therefore,

$$h = \begin{bmatrix} 1 \\ \overline{1} \end{bmatrix}. \tag{52}$$

From Eqs. 24 and 25

$$z_o = - c_o \tag{53}$$

$$c_1 = c_o - 1. \tag{54}$$

Equation 45 gives

$$c_o^2 + (c_o - 1)^2 \leq 5. \tag{55}$$

Then c_o must satisfy the inequalities

$$-1 \leq c_o \leq 2. \tag{56}$$

But $c_o < o$ for stability; hence, the optimal c_o lies in the interval

$$-1 \leq c_o < o. \tag{57}$$

On the other hand Eq. 42 gives

$$\frac{1}{\rho} = c_o(-2c_o + 1), \tag{58}$$

and, therefore, should lie in the interval

$$-\infty < \rho \leq -\tfrac{1}{3}. \tag{59}$$

Since the optimal c_o should be the smallest possible, the numerical procedure starts upon our letting $c_o = -1$ (or alternatively, $\rho = -\frac{1}{3}$). In this particular case since the function $V(t) = \mathbf{x}^T(t)\mathbf{x}(t)$ is non-increasing,

$$\frac{dV}{dt} = -4\dot{x}^2(t), \tag{60}$$

the optimal value of c_o is -1. The singular surface Γ_1 for this example is shown in Fig. 2. Also shown in this figure is the set X of admissible initial condition $\mathbf{x}_o{}^T = \frac{1}{5}(-1, -2)$; for this particular initial condition $u^2(o) = 5$.

Systems of Singularity of Order j

5. Singular Linear Control Law

In this section performance indices n which the integrand is

$$F(x, \dot{x}, \cdots, \overset{n-j}{x}) = \sum_{i,k=1}^{n-j+1} q_{i,k} \overset{(i-1)}{x} \overset{(k-1)}{x} \tag{61}$$

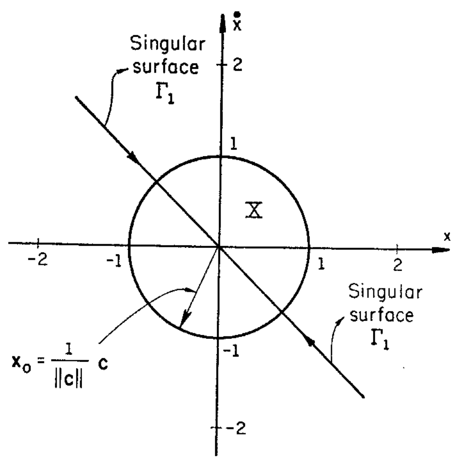

FIG. 2. Sets X, Γ_1 and initial condition $\mathbf{x}_o = \frac{k_A}{\|\mathbf{c}\|}\mathbf{c}$ for Example 4.

are considered. As before, singular trajectories are obtained; then conditions which the linear control must satisfy in order to be the optimal law on the singular surface, are derived. First, in view of Eq. 61 the Euler equation is of order $2(n - j)$. Proceeding as in Section 2, we obtain the stable Euler equation

$$\sum_{i=o}^{n-j} h_i \overset{(i)}{x}(t) = 0. \tag{62}$$

The singular surface is then the hyperplane Γ_j, defined as the set of all $\mathbf{x}_o$ such that

$$\sum_{i=o}^{n-j} h_i \overset{(i+k)}{x_o} = 0, \qquad k = 0, 1, \cdots, j - 1. \tag{63}$$

The conditions on $\mathbf{c}$ are obtained by equating the polynomial $H_T(s)$ (given by Eq. 22 to the product of the two following polynomials

$$H_T(s) = Z(s)H_j(s), \tag{64}$$

where

$$H_j(s) = \sum_{i=o}^{n-j} h_i s^i, \qquad h_{n-j} = 1. \tag{65}$$

In view of Eq. 65, $Z(s)$ is a polynomial of order j in s. Equating terms of the same power in Eq. 64, we obtain

$$a_i - c_i = \sum_{k=o}^{j} h_{i-k} z_k; \qquad i = 0, 1, \cdots, n - 1. \tag{66}$$

If we solve the first j equations of Eq. 66 for the z_k's as functions of $c_o, c_1, \cdots, c_{j-1}$, the remaining $(n - j)c_i$'s will then be given by

$$c_i = a_i - \sum_{k=o}^{j} h_{i-k} z_k(c_o, c_1, \cdots, c_{j-1}), \qquad i = j, \cdots, n - 1. \tag{67}$$

Therefore, analogous to the special case $j = 1$, the restriction that the linear control law be optimal on Γ_j reduces the n-variable problem to a j-variable problem. Again, stability is assured if the polynomial $Z(s)$ is Hurwitz.

6. Extended Linear Control Law

Using Pontryagin's formulation, we arrive (as in the second section) at the problem of maximizing in $\mathbf{c}$ the function $N(\mathbf{p}, \mathbf{x}, \mathbf{c})$ (given by Eq. 31), subject to the constraint relation Eq. 28. As before, $\mathbf{c}$ must also satisfy the singularity conditions, i.e., Eq. 67. Using the Lagrange multiplier technique, we set

$$\frac{\partial}{\partial c_m}\left[N(\mathbf{p}, \mathbf{x}, \mathbf{c}) + \mathbf{x}^T\mathbf{c}\mathbf{c}^T\mathbf{x}\right] = 0; \qquad m = o, \cdots, j - 1. \tag{68}$$

Performing the differentiation in Eq. 68, we arrive at

$$p_n \frac{\partial \mathbf{c}^T}{\partial c_m} \mathbf{x} + 2\mathbf{x}^T\mathbf{c} \frac{\partial \mathbf{c}^T}{\partial c_m} \mathbf{x} = 0, \qquad m = o, 1, \cdots, j - 1. \tag{69}$$

As it has been shown (**9**), $p_n(t)$ is given by

$$p_n(t) = \sum_{i=o}^{j-1} P_i \frac{d^i}{dt^i} \sum_{l=o}^{n-j} h_l \overset{(l)}{x}, \tag{70}$$

where the coefficients P_i are functions of $c_o, c_1, \cdots, c_{j-1}$. Let $\mathbf{h}^m$ represent an $n \times 1$ vector such that: (a) its first $m - 1$ components are zero; (b) for $i \leq m + n - j$, the $(m + i)^{\text{th}}$ component is h_i and (c) for $i > m + n - j$, the $(m + i)^{\text{th}}$ component is zero. With this notation we can write

$$p_n(t) = \sum_{i=o}^{j-1} P_i \mathbf{x}^T \mathbf{h}^i. \tag{71}$$

Substituting Eq. 71 into Eq. 69, yields

$$\mathbf{x}^T \left[\sum_{i=o}^{j-1} P_i \mathbf{h}^i + 2\rho \mathbf{c} \frac{\partial \mathbf{c}^T}{\partial c_m} \right] \mathbf{x} = o, \qquad m = o, \cdots, j - 1. \tag{72}$$

Let

$$\mathbf{D}_m = \sum_{i=o}^{j-1} P_i \mathbf{h}^i + 2\rho \mathbf{c} \frac{\partial \mathbf{c}^T}{\partial c_m}, \qquad m = o, \cdots, j - 1. \tag{73}$$

As before $\mathbf{D}_m$ is recognized to be a matrix of rank one, the only (possibly) non-zero eigenvalue being

$$\lambda(\mathbf{D}_m) = \frac{\partial \mathbf{c}^T}{\partial c_m} \sum_{i=o}^{j-1} P_i \mathbf{h}^i + 2\rho \mathbf{c}. \tag{74}$$

A necessary condition for Eq. 72 to be true is

$$\lambda(\mathbf{D}_m) = 0; \qquad m = o, 1, \cdots, j - 1. \tag{75}$$

The j Eqs. 75 are then solved for the c_i's, $i = o, \cdots, j - 1$, as functions of ρ. Again the choice of the Lagrange multiplier ρ is made through an iterative procedure. Analogous to the case $j = 1$, Eq. 45 gives a necessary condition that the c_i's must satisfy.

7. Illustrative Example

Given the system equation

$$\overset{oo}{x} = u, \tag{76}$$

and the performance index

$$\int_o^\infty x^2(t) dt, \tag{77}$$

find a linear control law $u(t)$ such that it minimizes the performance index Eq. 77 subject to restrictions

$$\mathbf{x}(o) = \mathbf{x}_o, \qquad \|\mathbf{x}_o\| \leq 1 \tag{78}$$

and

$$x(\infty) = \mathbf{0} \tag{79}$$

and

$$u^2(t) \leq 2. \tag{80}$$

The Euler equation degenerates to $x = o$. Hence

$$\mathbf{h}^o = \begin{bmatrix} 1 \\ \hline 0 \end{bmatrix}, \qquad \mathbf{h}^1 = \begin{bmatrix} 0 \\ \hline 1 \end{bmatrix}. \tag{81}$$

From Eq. 67 we obtain

$$z_o = -c_o \tag{82}$$

and

$$z_1 = -c_1. \tag{83}$$

The coefficients P_i are found to be

$$P_o = -\frac{1}{c_o} \tag{84}$$

and

$$P_1 = -\frac{1}{c_o c_1}. \tag{85}$$

Therefore

$$\frac{\partial_{\mathbf{c}}{}^T}{\partial c_o} = [1, 0] \tag{86}$$

and

$$\frac{\partial_{\mathbf{c}}{}^T}{\partial c_o} = [0, 1]. \tag{87}$$

Equation 75 yields

$$2\rho c_o + \frac{1}{c_o} = 0 \tag{88a}$$

and

$$2\rho c_1 - \frac{1}{c_o c_1} = 0. \tag{88b}$$

Hence

$$c_1{}^2 = -c_o. \tag{89}$$

The system equation is then

$$\overset{oo}{x} + -c_o\dot{x} - c_o x = 0; \tag{90}$$

that is a damping ratio of 0.5. The value of c_o is restricted by Eq. 45 to lie in the interval

$$-1 \leq c_o \leq 2. \tag{91}$$

For stability, $c_o < 0$; hence the optimal c_o lies in the interval

$$-1 \leq c_o < 0. \tag{92}$$

It can be verified that the solution

$$c_o = -1 \tag{93}$$

minimizes the performance index; moreover since $V(t) = \mathbf{x}^T\mathbf{x}$ is non-decreasing,

$$\frac{dV}{dt} = -2\dot{x}^2(t), \tag{94}$$

it satisfies the constraint in $u^2(t)$.

Conclusions

An optimization problem with a new kind of constraint has been presented. Besides the usual constraint in magnitude, the control signal is restricted to the class of linear controls. Because the performance index (a non-negative definite quadratic form) does not include the "cost of control," singular solutions are present. The linear control law obtained, besides being optimal on the singular surface, is optimal outside of it in the sense that no other linear law gives a smaller value for the given performance index while satisfying the magnitude constraint for all admissible initial conditions. The computational procedure involved in a given problem is straight-forward and easily executed. The extension of these results to multi-imput systems is deemed possible as a simple theoretical (although perhaps computationally arduous) task.

Acknowledgments

The author gratefully acknowledges the guidance of Professor J. B. Cruz, Jr., and thanks Professor R. A. Rohrer for his invaluable criticism and suggestions. Discussions with other members of the Systems Theory Group of the Coordinated Science Laboratory have also proven helpful.

* * *

This work is based upon a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Electrical Engineering, University of Illinois, Urbana, Ill.

Bibliography

(1) R. E. Kalman, "The Theory of Optimal Control and the Calculus of Variations," Tech. Rept. 61–3, *RIAS*, 1961.

(2) G. L. Colina, and P. Dorato, "Application of Pontryagin's Maximum Principle: Linear Control Systems," Res. Rep. No. 10–15–62, Poly. Inst. Brooklyn, Microwave Research Institute, June, 1962.

(3) W. M. Wonham and C. D. Johnson, "Optimal Bang-Bang Control With Quadratic Performance Index," Proc. Fourth Jt. Auto. Cont. Conf., pp. 101–112, 1963.

(4) C. D. Johnson, and J. E. Gibson, "Singular Solutions in Problems of Optimal Control," *IEEE Trans. Auto. Cont.*, Vol. Ac-8, pp. 4–15, Jan., 1963.
(5) W. M. Wonham, and C. D. Johnson, *op. cit.*, p. 101.
(6) R. A. Rohrer and M. Sobral, Jr., "The Euler-Poisson Equation and Optimal Linear Control," Tech. Rep. No. R-176, Coordinated Science Lab., Univ. of Ill., Urbana, Aug., 1963.
(7) L. E. Elsgolc, "Calculus of Variations," Reading, Mass., Addison-Wesley Pub. Co., Inc., p. 44, 1962.
(8) L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, "The Mathematical Theory of Optimal Processes," New York, Interscience Pub., pp. 17–21, 1962.
(9) M. Sobral, Jr., "Linear Control Laws for Singular Linear Systems," Rep. R-186, Coordinated Science Laboratory, Univ. of Ill., Urbana, Feb., 1964.