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Discussion on: "On the Generation of Random Stable Polynomials"

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This is an interesting and well written survey paper on the random generation of stable polynomials. Different methods of random generation in the spaces of polynomial coefficients, polynomial roots and Levinson-Durbin (LD) parameters are compared. For random generation of stable polynomials different stability conditions have been used. Most of the generation algorithms are based on necessary and sufficient conditions (direct root generation, Hit-and-Run algorithm, discrete Hermite-Bieler method, LD parametrization). However, the uniform Levinson-Durbin (ULD) parametrization plays the central role as the most simple, accurate and numerically stable method for generation of both discrete (Schur) and continuous (Hurwitz) polynomials.

There are several very interesting open problems mentioned in this paper and a number of appealing directions can be considered for further investigations. For example, it is very interesting to explore the root distribution issue of generated stable polynomials.

This discussion focuses on two issues.

First, the introduction of a new name (LD parameters) for the parameters obtained via Schur-Cohn stability test is a bit questionable. The Schur-Cohn stability test and Levinson recursion are well known. However, the parameters obtained by this recursive procedure have already too many different names in different research fields:

- reflection coefficients and Schur-Szego coefficients in mathematics,
- PARCOR (partial correlation) coefficients in estimation theory,
- reflection coefficients and k-parameters in signal processing,
- reflection coefficients in time series analysis and in transmission line theory,
- reflection coefficients and canonical parameters in control,
- Fam-Medich (FM) parameters in randomized methods.

"Reflection coefficients" seems to be the most popular name for these parameters, having also a clear meaning in the transmission line theory. However, in mathematics and control this name is rather meaningless. The authors refer to these parameters as Levinson-Durbin (LD) parameters, since this parametrization first appeared in their works. But the works of Schur and Cohn appeared many decades earlier than the works of Durbin and Levinson. So, maybe "Schur-Cohn parameters" or "Schur-Levinson parameters" are better candidates?

Second, uniform distribution for LD parameters leads to the uniform distribution of the polynomial coefficients. Unfortunately, all the roots of ULD-polynomials cluster close to the stability boundary for high order polynomials. The question is how to modify the proposed methods so that the root location will be better?

One possibility to overcome this drawback is to combine the Hit-and-Run algorithm with reflection coefficients (or LD parameters) approach, introduced in [1]. We call this algorithm *Hit-and-Run along reflection segments* (HRRS) because it is based on the next stable segment lemma.

Lemma. [1] *Through an arbitrary Schur stable point in the polynomial coefficient space $q \in R^n$ with reflection coefficients (LD parameters) $|t_k| < 1$, $k = 1, \ldots, n$ one can put n stable line segments*

$$Q^k(-1, 1) = conv\{q|t_k \in (-1, 1)\}$$

where $conv\{q|t_k \in (-1, 1)\}$ denotes the convex hull obtained by varying a single reflection coefficient (LD parameter) t_k in the interval $(-1, 1)$.

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The idea of this algorithm is following:

1. Choose an initial point $q^0 \in R^n$ with reflection coefficients (LD parameters) $|t_k| < 1, k = 1, \ldots, n$.
2. Find randomly a level number $k \in \{1, \ldots, n\}$.
3. Determine the direction of the segment by choosing $t_k(j+1) \in (t_k(j), -sign(t_k(j)))$ where j is the sample number $j = 1, \ldots, N$.
4. Find randomly a point q^{j+1} on the segment $Q^k(t_k(j), t^{max})$ where $t^{max} \in (t_k(j), -sign(t_k(j)))$.
5. Return to step 2.

The preliminary results obtained by this algorithm are promising. HRRS algorithm has some advantages over HR algorithm:

- The "bad mixing" is avoided because of the condition of step 3.
- It is very simple and numerically stable algorithm because no "boundary oracle" is needed and all the computations are performed in the reflection coefficient (LD parameter) space.

In principle, the properties of the HRRS algorithm are similar to those of LD algorithm:

- If $t^{max} = -sign(t_k(j))$, the results are similar to ULD applications.
- If $|t^{max}| < |t_k(0)|$, the results similar to truncated ULD are obtained.
- If t^{max} is not fixed but decreasing on every step, $t^{max}(j+1) = ct^{max}(j)$, $|c| < 1$, then the roots of generated polynomials can be moved farther from the stability boundary even for high order polynomials.

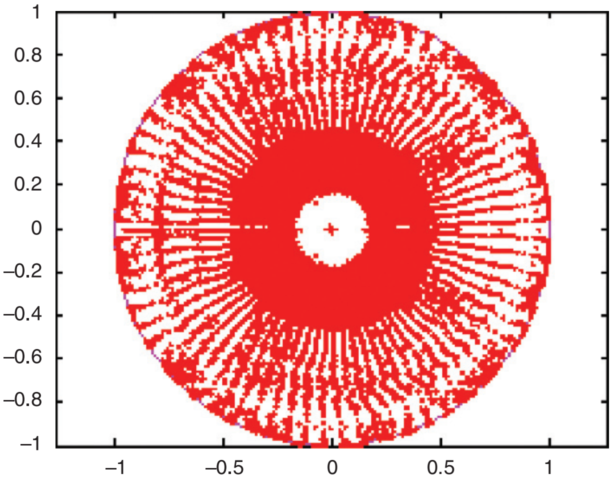

Fig. 1. The roots of sampled polynomials of degree 80.

In Fig. 1 the roots of polynomials generated by *Hit-and-Run along reflection segments* algorithm are depicted for $n = 80$, $N = 10000$, $t^{max}(0) = 0.999$, $c = 0.95$ and $t_k(q^0) = 0.001$, $k = 1, \ldots, 80$.

References

1. Nurges Ü. New stability conditions via reflection coefficients of polynomials. *IEEE TAC*. 2005; 50(9): 1354–1360.

Final Comments by the Authors P. Shcherbakov and F. Dabbene

First of all, we would like to thank Prof. Nurges for his interest towards our work and his nice words in the introduction of his discussion note. We briefly address the two main discussion points raised in his note.

About the naming "LD coefficients," we thoroughly agree with Prof. Nurges that it is always difficult to come up with the correct name, and that a possibility was to go for the most popular one, as he suggested. We however preferred to follow a historical perspective: our choice was motivated by the fact that these coefficients were referred to as LD-coefficient in the original paper (ref. [7] in the bibliography list in our paper), where uniform random sampling of ARMA models was first proposed.

Regarding the new method suggested by Prof. Nurges, based on the combination of HR and RC, we indeed found it very interesting and worth analyzing. On the other hand, our first intuition is that no rigorous assertions can be made about the distribution of the coefficients and/or roots of the polynomials obtained in such a way. Also, looking at Fig. 1, the root distribution seems to be somewhat "regular" and highly non-uniform.

Finally, we would like to briefly comment on the issue of uniform distribution of the polynomial coefficients and the consequent clustering of roots close to the stability boundary. As we point out in Remark 1 of our paper, this behavior is not always negative. Contrary, it could be desirable in many practical application were the generated transfer function represents the dynamics of an uncertainty affecting the system.