ELSEVIER

Composites Science and Technology 62 (2002) 719–727

COMPOSITES SCIENCE AND TECHNOLOGY

www.elsevier.com/locate/compscitech

Investigation of non-linear free vibrations of fully clamped symmetrically laminated carbon-fibre-reinforced PEEK (AS4/APC2) rectangular composite panels

B. Harras[a], R. Benamar[b,*], R.G. White[c]

[a]Laboratoire de Modélisation Numérique et Analyse des Structures, LAMNAS FST de Fès-Saiss, Route d'Immouzer, BP 2202 Fès, Morocco
[b]Laboratoire d'Etudes et de Recherches en Simulation, Instrumentation et Mesure, LERSIM, E.G.T, Ecole Mohammadia d'Ingénieurs, Université Mohammed V, BP 765 Agdal, Rabat, Morocco
[c]Department of Aeronautics and Astronautics, University of Southampton, Highfield, Southampton S017 1BJ, UK

Received 15 March 2001; received in revised form 3 July 2001; accepted 26 July 2001

Abstract

In an earlier publication, the optimum parameters for ultrasonic welding of carbon-fibre/PEEK (AS4/APC-2) composites have been determined; the physico-chemical properties and fracture surface morphology of modes I and II have been described. The objective of this paper is further to extend our understanding of the dynamic behaviour of the AS4/APC-2 by the theoretical modelling of its geometrically non-linear dynamic behaviour at large vibration amplitudes. In previous work, a theoretical model based on Hamilton's principle and spectral analysis has been used to study the non-linear free vibration of CFRP (carbon-fibre-reinforced plastic) and hybrid composite panels. The present paper investigates the first non-linear mode shape of symmetrically laminated AS4/APC2 composite panels. The large vibration amplitudes problem, reduced to a set of non-linear algebraic equations, is solved numerically. Results are summarised for various plate aspect ratios and vibration amplitudes, giving the dependence of the first non-linear mode shape and associated frequencies on the amplitudes of vibration, and showing higher bending stresses near to the clamps at large deflections, compared with those predicted by linear theory. © 2002 Published by Elsevier Science Ltd.

Keywords: B. Non-linear behaviour; C. Laminate theory; Resonance frequencies

1. Introduction

The use of engineering thermoplastics in both structural and non-structural applications is rapidly increasing [1]. There is currently a great deal of interest in the use of thermoplastic polymers as matrices in fibre-reinforced composites for high-performance applications such as those encountered in the aerospace industry [1]. Thermoplastics offer a number of potential advantages over the commonly used thermosets. These include greater toughness, ease of processing, a better high-temperature performance, a better resistance to solvents and other chemicals, a lower moisture absorption, a better flame resistance, an infinite shelf life, and the possibility of recycling [1,2].

Among thermoplastic composites, a particular technical challenge and design opportunity is presented by the high performance continuous fibre composites based on resins such as PEEK [poly(ether-ether-ketone)]. This material has significantly better properties than currently available thermoset composites, particularly with respect to fracture toughness, damage tolerance and resistance to hot, and wet environments [3].

As described in [4], PEEK is the foremost thermoplastic matrix that may replace epoxies in many aerospace composites. The outstanding property of PEEK is its high fracture toughness, which is 50–100 times higher than epoxies. Another important advantage of PEEK is its low water absorption, which is less than 0.5% at 23 °C compared to 4–5% for conventional aerospace epoxies. Being semi-crystalline, it does not dissolve in common solvents. However, it may absorb some of these solvents, most notably methylene chloride [4]. The amount of solvent decreases with increasing crystallinity [4].

* Corresponding author.
E-mail address: rbenamar@emi.ac.ma (R. Benamar).

0266-3538/02/$ - see front matter © 2002 Published by Elsevier Science Ltd.
PII: S0266-3538(01)00133-6

Although the incentive for the development of advanced thermoplastic composites came from the aerospace industry, they are potentially attractive for many other demanding applications.

In an earlier publication of [5], the method for determination of the optimum parameters for ultrasonic welding of carbon-fibre/PEEK composites was described. In [6], we have described the physicochemical properties and fracture surface morphology of modes I and II specimens obtained in the earlier work. The objective of the work reported here was to further extend our understanding of the dynamic behaviour of the carbon/PEEK at large amplitude, via the theoretical modelling of its geometrically non-linear dynamic behaviour.

In a series of papers by Benamar and co-workers [7–16], a theoretical model based on Hamilton's principle and spectral analysis has been developed to study the non-linear free and steady state periodic forced vibration of beams, and the non-linear free vibration of homogeneous and composite plates and shells. The model has been reduced to the solution of a set of non-linear algebraic equations, which has been performed numerically using appropriate algorithms in order to obtain a set of non-linear mode shapes for the structure considered in each case, with the corresponding amplitude dependent non-linear frequencies. The model presents many advantages; it is not subject to the practical limitation of weak non-linearity in its formulation, as was the case for some models for non-linear vibration based on the perturbation procedure developed in [17]; its formulation is quite simple compared to some finite element approaches; periodic solutions can be obtained directly with any desired accuracy as solutions of the set of non-linear algebraic equations; once the most significant contributing functions are known, engineering applications can be made easily using data tables or rapid computer programs using only a small number of appropriate functions; and once the contributions of the functions are calculated, the resulting strains and stresses can be obtained directly, using the analytical expressions for the derivatives of the basic functions, which allows information to be obtained about the non-linear frequencies and mode shapes, and also leads to interesting indications on the dangerous zones of stress concentration.

This model is based on analytical and numerical procedures and the assumptions are the separation of the time and space variables and harmonic dependence in time. More recently, this model was applied to analyse the geometrically non-linear free dynamic response of rectangular CFRP (carbon fibre reinforced plastic) laminated plates [18]. The results obtained were in a good qualitative agreement with experimental measurements and with numerical published results, based on the HFEM (Hierarchical Finite Element Method) [19,20]. In [21], the theoretical model has been applied to study the linear and non-linear dynamic response of the hybrid composite panel made of new aircraft structural material called Glare 3.

The aim of this investigation was to characterise the linear and non-linear dynamic response of AS4/APC2 unidirectional composite panels. The general formulation of the model for non-linear vibration of symmetrically laminated plates at large vibration amplitudes is presented, and more details may be found in the above mentioned references. Periodic displacement was assumed since the motion of plates vibrating freely with amplitude displacement of their thickness is generally periodic [22]. Fully clamped boundaries have been considered here because they are adequate to model many real panel-type situations, such as aircraft wing panels [23] and are the easiest to achieve in practice, compared with the simply supported boundaries, for experimental measurements. The natural frequencies are determined. The first non-linear mode shape is examined. The effect of non-linearity on the non-linear resonant frequencies and the non-linear fundamental mode shape and associated bending stress patterns at large vibration amplitudes is investigated. Comparison of the change in natural frequency at large vibration amplitudes between AS4/APC2, CFRP and isotropic plates is presented. The second and higher non-linear mode shapes, and the analysis of the fibre orientation effect on the extent of the non-linearity will be presented later.

2. Formulation

A thin rectangular plate of a length a in the x-direction, a width b in the y-direction, and a thickness h in the z-direction is considered. the origin of the Cartesian co-ordinate system is located in one corner of the mid-plane with the z axis being perpendicular to this plane as shown in Fig. 1. For such a plate, the deflection in the x and y directions are represented here by clamped-clamped beam functions. These functions, which satisfy

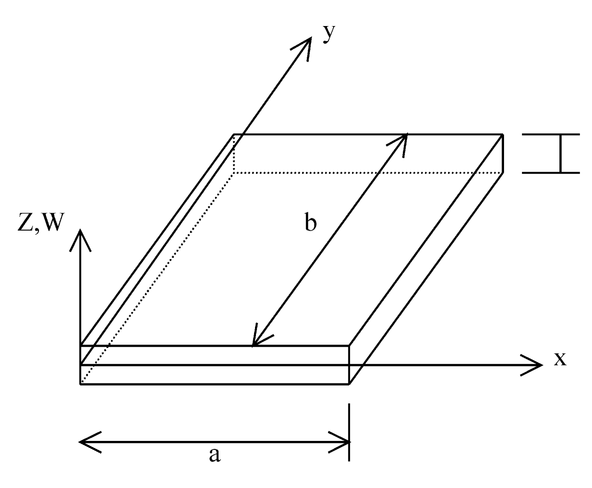

Fig. 1. Plate notation.

all the fully clamped theoretical boundary conditions, i.e. zero displacement and zero slope along the four plate edges, have been used and shown to be appropriate in previous studies of the vibration of fully clamped rectangular isotropic and laminated plates [7, 8, 18].

Using the Kirchhoff hypothesis of classical thin plates, one can express the total strains as:

$$\varepsilon_j = \overline{\varepsilon_j} + \kappa_j z \quad j = 1, 2, 6 \tag{1}$$

where subscripts 1 and 2 refer to in-plane normal effects in the directions x and y, and subscript 6 refers to in-plane shear action.

Considering Von Kármán type geometric non-linearity, one can write

$$\begin{aligned} \overline{\varepsilon_1} &= \frac{\partial U}{\partial x} + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2 \quad \overline{\varepsilon_2} = \frac{\partial V}{\partial y} + \frac{1}{2}\left(\frac{\partial W}{\partial y}\right)^2 \\ \overline{\varepsilon_1} &= \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} + \frac{\partial W}{\partial x}\frac{\partial W}{\partial y} \end{aligned} \tag{2a–c}$$

Assuming the Kirchhoff hypothesis, one can express the curvatures changes as:

$$\kappa_1 = -\frac{\partial^2 W}{\partial x^2} \quad \kappa_2 = -\frac{\partial^2 W}{\partial y^2} \quad \kappa_6 = -2\frac{\partial^2 W}{\partial x \partial y} \tag{3a–c}$$

where U, V and W are the mid-point displacements along the x, y and z directions respectively.

For a thin plate with an arbitrary number of anisotropic layers of arbitrary arrangement and thickness, the constitutive relations are:

$$\left\{\frac{N_i}{M_i}\right\} = \left(\frac{A_{ij}}{B_{ij}} \middle| \frac{B_{ij}}{D_{ij}}\right)\left\{\frac{\overline{\varepsilon_j}}{\kappa_j}\right\} i, j = 1, 2, 6 \tag{4}$$

where the stress and moment resultants are defined as:

$$(N_i, M_i) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (1, z)\sigma_i \mathrm{d}z \quad i = 1, 2, 6 \tag{5}$$

The symmetric matrices A_{ij}, B_{ij}, D_{ij} are defined as

$$\left(A_{ij}, B_{ij}, D_{ij}\right) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(1, z, z^2\right) Q_{ij} \mathrm{d}z \tag{6}$$

and Q_{ij} are the reduced stiffness coefficients which can be related to the more familiar engineering elastic moduli as [24]:

$$\{\sigma_i\} = \left[Q_{ij}\right]\{\overline{\varepsilon_i}\} \tag{7}$$

in which σ_i (i = 1, 2, 6) are the in-plane stress components.

For symmetric laminations with the fibre orientations $\theta = 0°$ [4]:

$$B_{ij} = 0, A_{16} = A_{26} = D_{16} = D_{26} = 0 \tag{8}$$

The expression for the bending strain energy V_b, axial strain energy V_a and kinetic energy T are given by [25].

$$\begin{aligned} V_b = \frac{1}{2}\int_S \Bigg\{ & D_{11}\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + 2D_{12}\frac{\partial^2 W}{\partial y^2}\frac{\partial^2 W}{\partial x^2} + D_{22}\left(\frac{\partial^2 W}{\partial y^2}\right)^2 \\ & + 4D_{16}\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial xy} + 4D_{26}\frac{\partial^2 W}{\partial y^2}\frac{\partial^2 W}{\partial xy} + 4D_{66}\left(\frac{\partial^2 W}{\partial xy}\right)^2 \Bigg\} \mathrm{d}S \end{aligned} \tag{9}$$

$$\begin{aligned} V_a = \frac{1}{2}\int_S \Bigg\{ & \frac{A_{11}}{4}\left(\frac{\partial W}{\partial x}\right)^4 + \frac{A_{22}}{4}\left(\frac{\partial W}{\partial y}\right)^4 + \left[\frac{A_{12}}{2} + A_{66}\right] \\ & \left(\frac{\partial W}{\partial y}\right)^2\left(\frac{\partial W}{\partial x}\right)^2 + A_{16}\left(\frac{\partial W}{\partial x}\right)^3\frac{\partial W}{\partial y} + A_{26}\left(\frac{\partial W}{\partial y}\right)^3\frac{\partial W}{\partial x} \Bigg\} \mathrm{d}S \end{aligned} \tag{10}$$

and

$$T = \frac{1}{2}\rho h \int_S \left(\frac{\partial W}{\partial t}\right)^2 \mathrm{d}S \tag{11}$$

Where S is the plate surface [0,a]×[0,b] and dS is the elementary surface dx dy.

In the above expressions, the assumption of neglecting the in-plane displacements U and V in the energy expressions has been made as for the fully clamped rectangular isotropic plates analysis considered in [7,11]. The range of validity of this assumption has been extensively discussed in light of the experimental and numerical results obtained for the frequency amplitude dependence and the bending stress estimates obtained at large vibrations amplitudes. The results obtained via this assumption were compared with the previous ones based on various methods such as the finite element method, the method based on Berger's approximation, the ultraspherical polynomial method and the elliptic function method. It was found that the percentage error in the non-linear frequency estimates based on this assumption, for amplitudes up to 1.5 times thickness, does not exceed 1.3%. Also, in the experimental investigation of the non-linear behaviour of fully clamped rectangular plates at large vibration amplitudes presented in reference [13], it was found that the rate of increase in bending stresses estimates, obtained from measured data, was in very good agreement with that obtained from the theory, in which the assumption of zero in-plane displacements was made. In the above references concerned with the isotropic case, the conclusion was that this assumption, which allows a great simplification in the modelling and a great reduction in the computation time when calculating the non-linear mode shapes, the associated non-linear frequencies, and the non-linear bending stress patterns, is valid for a reasonable range of vibrations amplitudes and plates aspect ratios. In a more recent work concerned with symmetrically laminated plates [18], the validity of this assumption has also been examined, via comparison of the non-linear frequency estimates it leads to, with those obtained by the HFEM, for two symmetrically laminated plates, with different lay-up and thicknesses. The ratio ω^*_{nl}/ω^*_l calculated from the present model is very close to the results produced by the HFEM with the in-plane and out-of-plane shape functions $p_i = p_o = 5$ respectively. The error does not exceed 1.01% in the non-linear frequency ratio ω^*_{nl}/ω^*_l estimates for amplitudes up to 1 times the plate thickness.

The good agreement found has shown that the assumption of neglecting U and V may be used in the geometrically non-linear analysis of laminated plates with a reasonable accuracy.

As mentioned above, panel vibrations of the order of one time the thickness, which is the range of interest, are generally periodic [20,22], so the transverse displacements are assumed in the form:

$$W = a_i w_i(x, y)\sin\omega t \tag{12}$$

in which the usual summation convention for the repeated indexes i is used. i is summed over the range 1 to n, with n being the number of basic functions considered.

Discretization of the strain and kinetic energy expressions can be carried out leading to:

$$\begin{aligned} V_b &= \frac{1}{2} a_i a_j k_{ij} \sin^2\omega t \\ V_a &= \frac{1}{2} a_i a_j a_k a_l b_{ijkl} \sin^4\omega t \\ T &= \frac{1}{2} a_i a_j \omega^2 m_{ij} \cos^2\omega t \end{aligned} \tag{13a–c}$$

in which m_{ij}, k_{ij} and b_{ijkl} are the mass tensor, the rigidity tensor and the geometrical non-linearity tensor respectively and the summation convention for repeated indices is used. The expressions for these tensors are:

$$\begin{aligned} k_{ij} &= \frac{ah^5E}{b^3} k^*_{ij} \quad b_{ijkl} = \frac{ah^5E}{b^3} b^*_{ijkl} \\ m_{ij} &= \rho h^3 ab m^*_{ij} \end{aligned} \tag{14a–c}$$

where the non-dimensional tensors m^*_{ij}, k^*_{ij} and b^*_{ijkl} are given in terms of integrals of the non-dimensional basic functions w^*_i's, defined in [18], and their first and second partial derivatives, by:

$$m^*_{ij} = \int_{S^*} w^*_i w^*_j \mathrm{d}x^* \mathrm{d}y^* \tag{15}$$

$$\begin{aligned} k^*_{ij} = \int_{S^*} \Bigg\{ & D^*_{11}\alpha^4 \left[\frac{\partial^2 w^*_i}{\partial x^{*2}} \frac{\partial^2 w^*_j}{\partial x^{*2}} \right] + D^*_{22} \left[\frac{\partial^2 w^*_i}{\partial y^{*2}} \frac{\partial^2 w^*_j}{\partial y^{*2}} \right] \\ & + D^*_{12}\alpha^2 \left[\frac{\partial^2 w^*_i}{\partial x^{*2}} \frac{\partial^2 w^*_j}{\partial y^{*2}} + \frac{\partial^2 w^*_i}{\partial y^{*2}} \frac{\partial^2 w^*_j}{\partial x^{*2}} \right] \\ & + 2D^*_{16}\alpha^3 \left[\frac{\partial^2 w^*_i}{\partial x^{*2}} \frac{\partial^2 w^*_j}{\partial x^* y^*} + \frac{\partial^2 w^*_i}{\partial x^* y^*} \frac{\partial^2 w^*_j}{\partial x^{*2}} \right] \\ & + 2D^*_{26}\alpha \left[\frac{\partial^2 w^*_i}{\partial y^{*2}} \frac{\partial^2 w^*_j}{\partial x^* y^*} + \frac{\partial^2 w^*_i}{\partial x^* y^*} \frac{\partial^2 w^*_j}{\partial y^{*2}} \right] \\ & + 4D^*_{66}\alpha^4 \left[\frac{\partial^2 w^*_i}{\partial x^* y^*} \frac{\partial^2 w^*_j}{\partial x^* y^*} \right] \Bigg\} \mathrm{d}S^* \end{aligned} \tag{16}$$

In the third, fourth and fifth terms between brackets in the above expression, pairs of terms obtained by interchanging i and j were used instead of one term. This was done as in [8,18,26] in order to make the k^*_{ij} tensor symmetric, as required by the theory [8,18,26], which does not change the sum $\frac{1}{2} a_i a_j k^*_{ij}$ corresponding to the bending strain energy V_a.

$$\begin{aligned} b^*_{ijkl} = \int_{S^*} \Bigg\{ & \frac{A^*_{11}}{4}\alpha^4 \left[\frac{\partial w^*_i}{\partial x^*} \frac{\partial w^*_j}{\partial x^*} \frac{\partial w^*_k}{\partial x^*} \frac{\partial w^*_l}{\partial x^*} \right] \\ & + \frac{A^*_{22}}{4} \left[\frac{\partial w^*_i}{\partial y^*} \frac{\partial w^*_j}{\partial y^*} \frac{\partial w^*_k}{\partial y^*} \frac{\partial w^*_l}{\partial y^*} \right] + \left[\frac{A^*_{12}}{4} + \frac{A^*_{66}}{2} \right] \\ & \alpha^2 \left\{ \left[\frac{\partial w^*_i}{\partial y^*} \frac{\partial w^*_j}{\partial y^*} \frac{\partial w^*_k}{\partial x^*} \frac{\partial w^*_l}{\partial x^*} \right] + \left[\frac{\partial w^*_i}{\partial x^*} \frac{\partial w^*_j}{\partial x^*} \frac{\partial w^*_k}{\partial y^*} \frac{\partial w^*_l}{\partial y^*} \right] \right\} \\ & + \frac{A^*_{16}}{4}\alpha^3 \left\{ \left[\frac{\partial w^*_i}{\partial x^*} \frac{\partial w^*_j}{\partial x^*} \frac{\partial w^*_k}{\partial x^*} \frac{\partial w^*_l}{\partial y^*} \right] + \left[\frac{\partial w^*_i}{\partial x^*} \frac{\partial w^*_j}{\partial x^*} \frac{\partial w^*_k}{\partial y^*} \frac{\partial w^*_l}{\partial x^*} \right] \right\} \\ & + \frac{A^*_{26}}{2}\alpha \left\{ \left[\frac{\partial w^*_i}{\partial x^*} \frac{\partial w^*_j}{\partial y^*} \frac{\partial w^*_k}{\partial y^*} \frac{\partial w^*_l}{\partial y^*} \right] \right. \\ & \left. + \left[\frac{\partial w^*_i}{\partial y^*} \frac{\partial w^*_j}{\partial x^*} \frac{\partial w^*_k}{\partial y^*} \frac{\partial w^*_l}{\partial y^*} \right] \right\} \Bigg\} \mathrm{d}S^* \end{aligned} \tag{17}$$

As mentioned above for k^*_{ij}, the last three terms have been written twice, with interchanging appropriate indices in each case, in order to obtain a b^*_{ijkl} tensor fulfilling the symmetry requirements assumed in the derivation procedure: $b^*_{ijkl} = b^*_{klij}$ and $b^*_{ijkl} = b^*_{jikl}$, which does not change the sum $\frac{1}{2} a_i a_j a_k a_l b^*_{ijkl}$ giving the axial strain energy V_a. An illustration of this procedure has been given in [18].

In Eqs. (16) and (17) $A^*_{ij} = (A_{ij}/hE)$ and $D^*_{ij} = (D_{ij}/h^3E)$, in which H is the plate thickness and E is a reference Young's modulus, whose numerical value was taken as 7 10^{10} N/m, which is a typical value for aluminium alloys.

Neglecting energy dissipation, the equation of motion derived from Hamilton's principle is:

$$\delta\int_0^{2\pi}(V-T)\mathrm{d}t=0 \tag{18}$$

where $V=V_\mathrm{a}+V_\mathrm{b}$

Insertion of Eq. (13a–c) in (18), and derivation with respect to the unknown constants a_i, leads to the following set of non-linear algebraic Eq. (19), as shown in [18]:

$$3a_ia_ja_kb^*_{ijkl}+2a_ik^*_{ir}-2a_i\omega^{*^2}m^*_{ir}=0 \quad i=1,\ldots\ldots,n \tag{19}$$

which has to be solved numerically.

To complete the formulation, we adopt the procedure developed in [18,26] to obtain the first non-linear mode. As no dissipation is considered here, a supplementary equation can be obtained by applying the principle of conservation of energy, which can be written as:

$$V_{\max}=T_{\max} \tag{20a}$$

This leads to the following equation:

$$\omega^{*^2}=\frac{a_ia_jk^*_{ij}+a_ia_ja_ka_lb^*_{ijkl}}{a_ia_jm^*_{ij}} \tag{20b}$$

This expression for ω^{*^2} is substituted in Eq. (19) to obtain a system of n non-linear algebraic equations leading to the contribution coefficients a_i, $i=1$ to n.

ω and ω^* are the non-linear frequency and non-dimensional non-linear frequency parameters related by:

$$\omega^2=\frac{h^2E}{\rho b^4}\omega^{*^2} \tag{21}$$

To obtain the first non-linear mode shape of the plate considered, the contribution of the first basic function is first fixed and the other basic functions contributions are calculated via the numerical solutions of the remaining $(n-1)$ non-linear algebraic equations.

The stress in the kth layer can be expressed in terms of the laminate middle surface strains and curvatures as:

$$\{\sigma_k\}=\left[\overline{Q}\right]_k.\{\varepsilon\} \tag{22}$$

in which $\{\sigma\}^T_k=\left[\,\sigma_x \quad \sigma_y \quad \tau_{xy}\,\right]$ and terms of matrix $\left[\overline{Q}\right]$ can be obtained by the relationships given in [24].

The maximum bending strains ε_{xb} and ε_{yb} obtained for $z=h/2$ are given by

$$\varepsilon_{xb}=(h/2)\left(\partial^2W/\partial x^2\right) \quad \varepsilon_{yb}=(h/2)\left(\partial^2W/\partial y^2\right) \tag{23a and b}$$

For the plate considered here, and for $z=h/2$ which corresponds to a layer for which $\theta=0^\circ$, we have $Q_{16}=Q_{26}=0$. Using the stress-strain relationship (22) the stresses at $z=h/2$ can be obtained as:

$$\begin{aligned}\sigma_{xb}&=\frac{hQ_{11}}{2}\frac{\partial^2w}{\partial x^2}+\frac{hQ_{12}}{2}\frac{\partial^2w}{\partial y^2}\\ \sigma_{yb}&=\frac{hQ_{12}}{2}\frac{\partial^2w}{\partial x^2}+\frac{hQ_{22}}{2}\frac{\partial^2w}{\partial y^2}\end{aligned} \tag{24a and b}$$

In terms of the non-dimensional parameters defined in the previous work [18,26], non-dimensional bending stresses σ^*_{xb} and σ^*_{yb} can be defined by:

$$\begin{aligned}\sigma^*_{xb}&=\alpha^2\beta\left(\frac{\partial^2w^*}{\partial x^{*2}}\right)+\left(\frac{\partial^2w^*}{\partial y^{*2}}\right)\\ \sigma^*_{yb}&=\alpha^2\left(\frac{\partial^2w^*}{\partial x^{*2}}\right)+\gamma\left(\frac{\partial^2w^*}{\partial y^{*2}}\right)\end{aligned} \tag{25a and b}$$

where $\alpha=\frac{b}{a}$, $\beta=\frac{Q_{11}}{Q_{12}}$ and $\gamma=\frac{Q_{22}}{Q_{12}}$ The relationships between the dimensional and non-dimensional stresses are:

$$\sigma=\frac{Q_{12}h^2}{2b^2}\sigma^* \tag{26}$$

which is valid for both dimensional and non-dimensional pairs of stresses defined by Eqs. (24a and b)–(25a and b).

3. Applications to fully clamped rectangular laminated AS4/APC2 composite panels

3.1. Introduction

Two types of AS4/APC2 rectangular plates are analysed. Their geometrical and material properties are defined in Table 1. The properties have been taken from references [26,27]. It can be seen that the material properties of the plates are very close. The calculation was made for the two plates using the present model in order to estimate how slight differences in the material properties would affect the non-linear dynamic parameters. The plates 1 and 2 are symmetrcically laminated plates. Their bending and twisting stiffnesses are uncoupled. Also, as in symmetric laminates, the in-plane and bending stiffnesses are uncoupled [28].

3.2. General presentation of numerical results

Regarding the geometrically non-linear vibrations analysis of laminated plates, there are not many numerical or experimental results which can be used for comparison. The validity and accuracy of this model results has been discussed largely in previous publications [7,11,13,18] as mentioned in Section 2.

The plates' natural frequencies are shown in Table 2. The difference between the results for the two plates is very small. Table 3 shows the comparison of the non-linear frequency ratio of the first mode shape of the two laminated plates AS4/APC2, with those obtained for some laminated composites plates in the literature [19,25]. The lay-up and material properties of the plates used for comparison are listed in Table 4. It is noted that the values of the frequency ratio for different plates varies slightly, which is due to the differences in the thickness, the lay up, and the material properties.

The non-linear dynamic behaviour of symmetrically laminated plates depends on many factors of an individual plate. It is very difficult to say for an arbitrary laminated plate how these factors such as thickness, lay-

Table 1
Geometric and material properties of the plates

	Plate 1	Plate 2
Number of layers	16	16
Orientation of principal axes	$(0°)_{16}$	$(0°)_{16}$
a (mm)	450	450
b (mm)	300	300
h (mm)	2	2
E_x (GPa)	131	134
E_y (GPa)	8.7	8.9
G_{xy} (GPa)	5	5.1
Crystallinity	0.58	0.66
ν_{xy}	0.28	0.28
ρ(kg/m^3)	1540	1540

Table 2
Linear natural frequencies (rad/s) of plates 1 and 2, $\alpha = 2/3$

Natural frequency (rad/s)	Plate 1	Plate 2
ω_{l1}	58,010	58,390
ω_{l2}	96,265	96,894
ω_{l3}	139,016	139,929
ω_{l4}	164,947	166,034
ω_{l5}	165,643	166,714
ω_{l6}	221,086	222,510
ω_{l7}	260,756	262,488
ω_{l8}	264,970	266,712
ω_{l9}	285,962	287,819

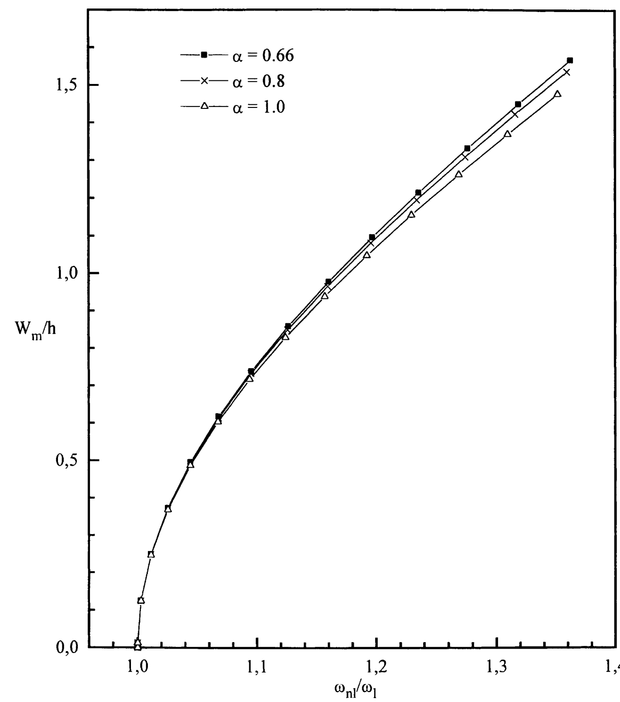

Fig. 2. Plate 1. Amplitudes calculated at point $(x^*, y^*) = (0.5, 0.5)$ for $\alpha = 2/3$, 0.8 and 1.

Table 3
Comparison of the non-linear frequency ratio ω_{nl}/ω_l (1st mode) of the two laminated AS4/APC-2 Plates with those of some laminated plates [19,25], $\alpha = 2/3$

Plate h (mm)	Plate 3	Plate 1	Plate 2	Plate 4	Plate 5
	1.0	2	2	2.144	2.72
W_m/h					
0.2	1.0058	1.0083	1.0079	1.0094	1.0067
0.4	1.0232	1.0305	1.0302	1.0370	1.0267
0.6	1.0516	1.0652	1.0648	1.0812	1.0589
0.8	1.0903	1.1123	1.1119	1.1396	1.1023
1.0	1.1382	1.1684	1.1679	1.2100	1.1554
1.2	1.1941	1.2312	1.2308	1.2903	1.2168
1.4	–	1.3015	1.3009	1.3786	1.2852
1.5	–	1.3392	1.3386	1.4254	1.3217

Table 4
Lay-up and material properties of the plates [18,19] used for comparison in Table 3

Plate	Orientation of principal axes	E_x (GPa)	E_y (GPa)	G_{xy} (GPa)	ν_{xy}	ρ(kg/m^3)
3	$(90°,-45°,45°,0°)_{sym,}$	120.5	9.63	3.58	0.32	1540
4	$(45°,-45°,0°,0°,45°,-45°,90°,90°)_{sym,}$	131	13.03	6.41	0.38	1630
5	$(45°,-45°,0°,-45°,45°,-45°,0°,45°)_{sym,}$	173	7.2	3.76	0.29	1540

up, material properties, etc., affect its non-linear dynamic properties unless calculations are made in each case. The dependence of the non-linear frequency on the amplitude of vibration is plotted in Figs. 2 and 3, which present the backbone curves of plates 1 and 2 obtained

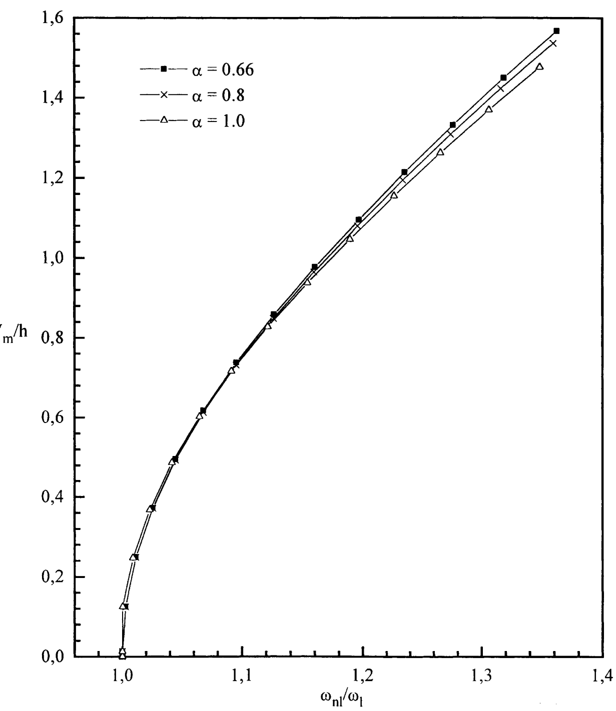

Fig. 3. Plate 2. Amplitudes calculated at point $(x^*,y^*)=(0.5, 0.5)$ for $\alpha=2/3$, 0.8 and 1.

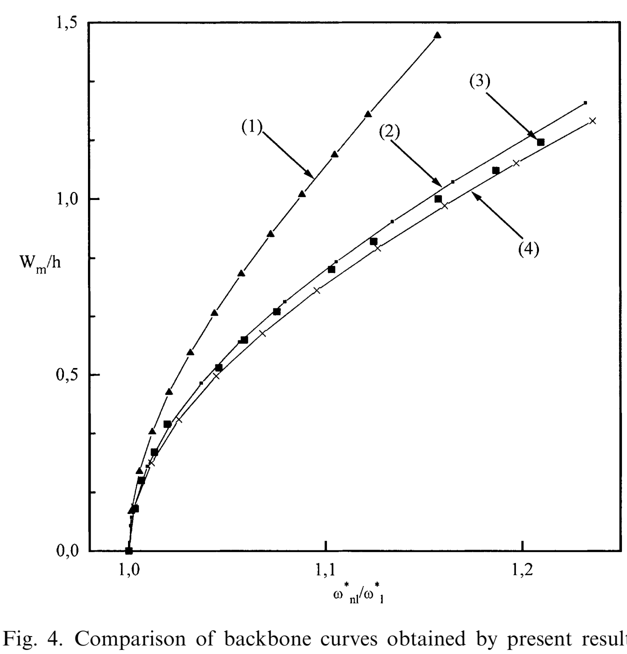

Fig. 4. Comparison of backbone curves obtained by present results for AS4/APC-2 panel and those published in [8,18,20]. $(x^*,y^*)=(0.5, 0.5)$ and $\alpha=0.6$ (2) Results obtained from the present model for CFRP plate 5 [18] (3) Values taken from [20] for CFRP plate 5, read from graph. (4) AS4/APC-2 panel. (1) Isotropic plates [8].

at $(x^*, y^*)=(0.5, 0.5)$ for $\alpha=b\ a=2/3$, 0.8 and 1. W_m represents the maximum amplitude of vibration displacement attained during the cycle at the point (x^*, y^*). The dependence of the non-linear frequency on the amplitude of vibration is plotted in Fig. 4, for the first mode of fully clamped AS4/APC2, CFRP plate 5 and isotropic plates having an aspect ratio $\alpha=0.6$, with the ratio of non-dimensional non-linear frequency to the linear frequency ω^*_{nl}/ω^*_l against the maximum of non-dimensional transverse displacement W_m/h. The curves show that the first non-linear mode shape of fully clamped AS4/APC2 panel exhibits a higher change in frequency with amplitude compared with that obtained for an isotropic plate of the same thickness. Slight differnce in change of the frequency between AS4/APC2 panel and CFRP plate 5 may be noticed. Also, the ratio ω^*_{nl}/ω^*_l calculated from the present model for the CFRP plate 5 is very close to the results produced by the HFEM [20]. All curves show the amplitude dependence of the first non-linear mode shape. It is noted that the non-linearity increases with the aspect ratio α. This

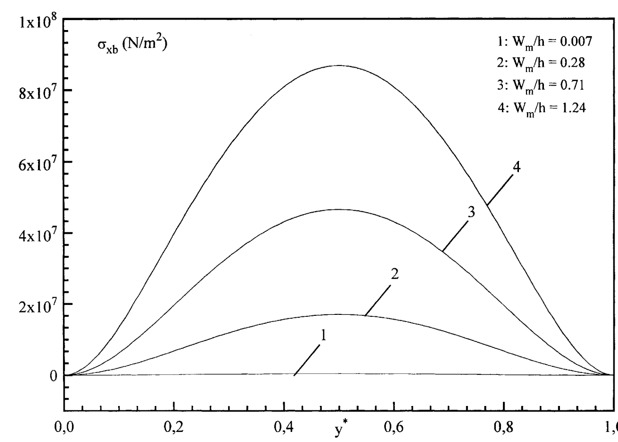

Fig. 5. Plate 1. The bending stress distribution σ_{xb} for $\alpha=0.66$ (along $x^*=0.025$) of the 1st mode at $W_m/h=0.007$, 0.28, 0.71 and 1.24.

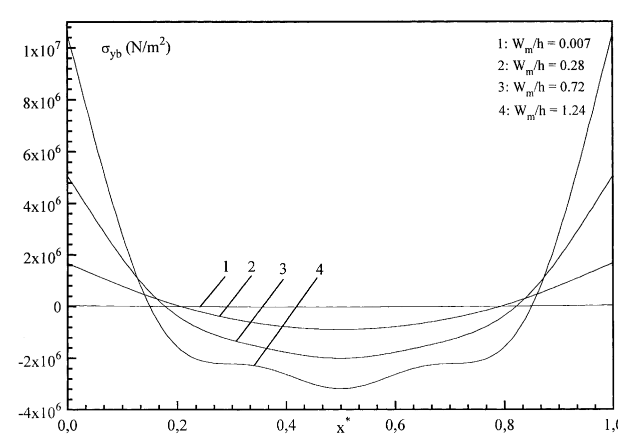

Fig. 6. Plate 1. The bending stress distribution σ_{yb} for $\alpha=0.66$ (along $y^*=0.25$) of the 1st mode at $W_m/h=0.007$, 0.28, 0.72 and 1.24.

result is in qualitative agreement with that obtained in a previous works [8,9,11–13,18,20,21].

The bending stress distribution associated with the rectangular plate 1 first non-linear mode is plotted in Fig. 4 for $x^* = 0.025$ and $\alpha = 2/3$. It can be seen in this figure, corresponding to a region which is very close to the clamps, that the bending can exhibit a quite unusual distribution, with a positive bending stress along the whole section parallel to the y direction, i.e. the symmetric direction of the first mode. The bending stress distribution, associated with the first non-linear mode, is plotted for various plate aspect ratios and various sections parallel to directions of symmetry of the mode in Figs. 5–9. All curves show the amplitude dependence of the stress distribution, and a high increase of the bending stress near to the clamps, compared with the rate of increase in predicted linear theory.

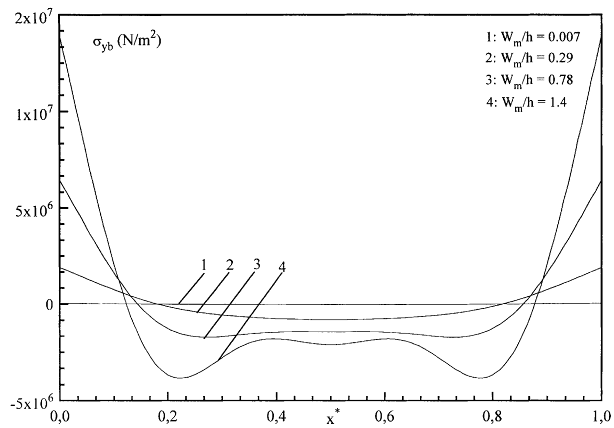

Fig. 7. Plate 1. The bending stress distribution σ_{yb} for $\alpha = 1$ (along $y^* = 0.25$) of the 1st mode at $W_m/h = 0.007$, 0.29, 0.78 and 1.4.

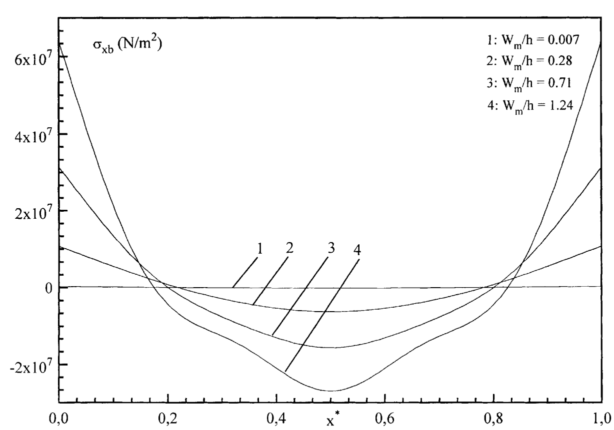

Fig. 8. Plate 1. The bending stress distribution σ_{xb} for $\alpha = 0.66$ (along $y^* = 0.25$) of the 1st mode at $W_m/h = 0.007$, 0.28, 0.71 and 1.24.

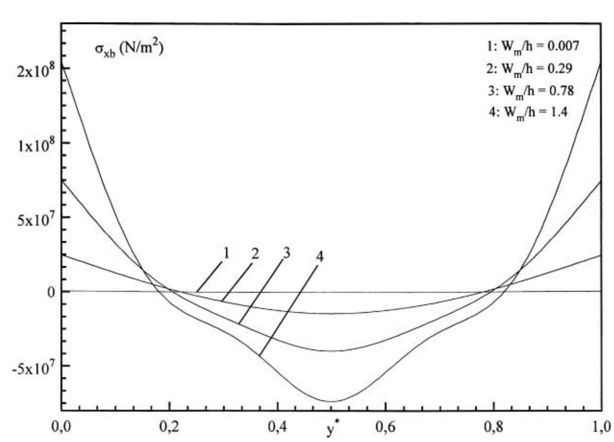

Fig. 9. Plate 1. The bending stress distribution σ_{xb} for $\alpha = 1$ (along $x^* = 0.025$) of the 1st mode at $W_m/h = 0.007$, 0.29, 0.78 and 1.4.

4. Conclusion

The natural frequencies of two fully clamped rectangular symmetrically laminated plates AS4/APC2 with different material properties has been obtained numerically.

The free response of the fully clamped rectangular laminated plates AS4/APC2 at large amplitudes of vibration has been investigated using the non-linear model developed in [7,8] and used in [18,21]. This model has been successfully used in the present work in order to further extend our understanding of the dynamic behaviour of the AS4/APC2 at large vibration amplitudes. The amplitude dependence of the first non-linear mode shape of fully clamped rectangular plates, for various aspect ratios and vibration amplitudes, has been obtained via the iterative solution of a set of non-linear algebraic equations, involving a fourth order tensor due to the geometrical non-linearity. The amplitude-dependent first non-linear mode shape was expressed as a series of plate functions obtained as products of clamped–clamped x and y beam functions.

Considering the results obtained, numerical data corresponding to various values of the plate aspect ratio and maximum of displacement up to 1.5 the plate thickness are plotted. The slope of the backbone curves varies with the aspect ratio. When the aspect ratio increases the non-linearity slightly increases. As a consequence of the effect of the non-linearity, all curves have shown the amplitude dependence of the stress distribution, and a high increase of the bending stress near to the clamps, compared with the rate of increase obtained in the linear theory.

It appears from the present work that the non-linear model developed in references [7,8] allows the estimation of the amplitude dependent first non-linear mode shape, the associated non-linear frequency, and qualitative estimation of the bending stresses associated with the first non-linear mode. Further investigations are needed in order to allow higher order non-linear modes to be estimated and to find out how the estimated non-linear modes can be simply used in the non-linear forced response analysis. Also, the effects of such a non-linearity on the fatigue life of highly excited plate-type structures, working in severe environment, has to be investigated. There is considerable relevance to the acoustic fatigue problem in aircraft structures.

References

[1] Cole KC. A review of recent developments in joining high-performance thermoplastic composites. IMI91RT-31200-1155-G, CNRC33030, 1991.

[2] Carlsson LA editors. Thermoplastic composite materials. Composites materials series vol. 7. Amsterdam: Elsevier, 1991.

[3] Taylor NS, Jones SB. Feasibility of welding thermoplastic composite materials. Bonding and repair of composites TA418.9.C6B711, 14 July 1989.

[4] Mallick PK. Fibre-reinforced composites: materials, manufacturing and design. 2nd ed (rev and expanded). Marcel Dekker, 1993, p. 152.

[5] Harras B, Cole KC, Vu-Khanh T. Optimisation of the ultrasonic welding of Peek-carbon composites. J Reinf Plast Compo 1996; 15:174–82.

[6] Harras B, Cole KC, Vu-Khanh T. Characterisation of the ultrasonically welded Peek-carbon composites. In: Proceedings of 29th International SAMPE Technical Conference, 1997. p. 475–85.

[7] Benamar R, Bennouna MMK, White RG. The effects of large vibration amplitudes on the fundamental mode shape of thin elastic structures, part I: simply supported and clamped-clamped beams. Journal of Sound and Vibration 1991;149:179–95.

[8] Benamar R, Bennouna MMK, White RG. The effects of large vibration amplitudes on the fundamental mode shape of thin elastic structures, part II: fully clamped rectangular isotropic plates. Journal of Sound and Vibration 1993;164:295–316.

[9] Benamar R, Bennouna MMK, White RG. The effects of large vibration amplitudes on the fundamental mode shape of a fully clamped, symmetrically laminated rectangular plate. In: Proceeding of the Fourth International Conference on Recent Advances in Structural Dynamics, Southampton, 1990.

[10] Moussaoui F, Benamar R, White RG. The effects of large vibration amplitudes on the mode shapes and natural frequencies of thin elastic shells, Part I: coupled transverse-circumferential mode shapes of isotropic circular shells of infinite length. Journal of Sound and Vibration 2000;232:917–43.

[11] El Kadiri M, Benamar R, White RG. The non-linear free vibration of fully clamped rectangular plates: second non-linear mode for various plate aspect ratios. Journal of Sound and Vibration 1999;228:333–58.

[12] Benamar R, Bennouna MMK, White RG. Spatial distribution of the harmonic distortion induced by large vibration amplitudes of fully clamped beams and rectangular plates. In: Proceedings of the 8th IMAC, 1990. p. II-1352–58.

[13] Benamar R, Bennouna MMK, White RG. The effects of large vibration amplitudes on the mode shapes and natural frequencies of thin elastic structures, part III: fully clamped rectangular isotropic plates-measurements of the mode shape amplitude dependence and the spatial distribution of harmonic distortion. Journal of Sound and Vibration 1994;175:377–424.

[14] Azrar L, Benamar R, White RG. A semi-analytical approach to the non-linear dynamic response problem S–S and C–C beams at large vibration amplitudes Part I: general theory and application to the single mode approach to free and forced vibration analysis. Journal of Sound and Vibration 1999;224:183–207.

[15] Beidouri Z, Benamar R. The effects of large vibration amplitudes on the mode shapes and natural frequencies of C–C–C–SS rectangular isotropic plates. to be submitted to Journal of Sound and Vibration.

[16] Haterbouch M, Benamar R, White RG. The effects of large vibration amplitudes on the mode shapes and natural frequencies of clamped circular isotropic plates. To be submitted to Journal of Sound and Vibration.

[17] Azrar L, Benamar R, Potier-ferry M. An Asymptotic-numerical method for large-amplitude free vibrations of thin elastic plates. Journal of Sound and Vibration 1999;220:695–727.

[18] Harras B, Benamar R, White RG. Geometrically non-linear free vibration of fully clamped symmetrically laminated rectangular composite plates. Journal of Sound and Vibration in press.

[19] Han W, Petyt M. Geometrically nonlinear vibration analysis of thin, rectangular plates using the hierarchical finite element method-II: 1st mode of laminated plates and higher modes of isotropic and laminated plates. Computers and structures 1997;63:309–18.

[20] Ribeiro P, Petyt M. Multi-modal geometrical non-linear free vibration of fully clamped composite laminated plates. Journal of Sound and Vibration 1999;225:127–52.

[21] Harras B, Benamar R, White RG. Linear and non-linear dynamic characterisation of the Glare composite panel. Accepted for oral presentation in ICCM-13, 2001.

[22] Ribeiro R. Geometrical nonlinear vibration of beams and plates by the hierarchical finite element method. PhD thesis, University of Southampton, UK, 1998.

[23] Han W. The analysis of isotropic and laminated rectangular plates including geometrical non-linearity using the p-version finite element method. PhD thesis, Institute of Sound and Vibration Research, UK, 1993.

[24] Jones RM, (editor). Mechanics of composite materials. International Student 1975 p. 51.

[25] Ashton JE, Whitney JM. Theory of laminated plates. Technomic Publication, 1970.

[26] Benamar R, Bennouna MMK, White RG. The effects of large vibration amplitudes on the fundamental mode shape of a fully clamped, symmetrically laminated rectangular plate. In: Proceeding of the Fourth International Conference on Recent Advances in Structural Dynamics, Southampton, 1990. p. 749–60.

[27] Tsai SW. Composites Design, 4th ed., Think composites, 1988.

[28] China CY. Nonlinear analysis of plates. New York: McGraw-Hill, 1980.