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A BSDE approach to Nash equilibrium payoffs for stochastic differential games with nonlinear cost functionals

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Abstract

In this paper, we study Nash equilibrium payoffs for two-player nonzero-sum stochastic differential games via the theory of backward stochastic differential equations. We obtain an existence theorem and a characterization theorem of Nash equilibrium payoffs for two-player nonzero-sum stochastic differential games with nonlinear cost functionals defined with the help of doubly controlled backward stochastic differential equations. Our results extend former ones by Buckdahn et al. (2004) [3] and are based on a backward stochastic differential equation approach.

Keywords: Nash equilibrium; Stochastic differential games; Backward stochastic differential equations; Stochastic backward semigroups

1. Introduction

Since the pioneering work of Isaacs [11], differential games and stochastic differential games have been investigated by many authors. Fleming and Souganidis [7] were the first to study zero-sum stochastic differential games and obtained that the lower and the upper value functions of

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such games satisfy the dynamic programming principle and coincide under the Isaacs condition. Recently, based on the ideas of Fleming and Souganidis [7], Buckdahn et al. [3] studied Nash equilibrium payoffs for two-player nonzero-sum stochastic differential games, while Buckdahn and Li [4] generalized at one hand the results of Fleming and Souganidis [7] for stochastic differential games and simplified the approach considerably by using backward stochastic differential equations. We refer the reader to Fleming and Souganidis [7] and Friedman [8] for a description of earlier results. In the present paper, we bring ideas of the both papers [3,4] together, in order to study Nash equilibrium payoffs for two-player nonzero-sum stochastic differential games with nonlinear cost functionals.

As concerns deterministic differential games, since the work of Kononenko [13] in the framework of positional strategies and Tolwinski et al. [19] in the framework of Friedman strategies, it is well known that deterministic nonzero-sum differential games admit Nash equilibrium payoffs. Recently, Buckdahn et al. [3] generalized the above result to two-player nonzero-sum stochastic differential games and obtained an existence and a characterization for two-player nonzero-sum stochastic differential games. On the other hand, since the works of Case [5] and Friedman [8], Nash equilibrium payoffs should be the solution of Hamilton–Jacobi equations. Based on these ideas, Bensoussan and Frehse [1] and Mannucci [15] generalized the above result to stochastic differential games using the existence of smooth enough solutions for a system of parabolic partial differential equations, while Hamadène et al. [10], Hamadène [9] and Lepeltier et al. [14] used a saddle point argument in the framework of backward stochastic differential equations. But both methods rely heavily on the assumption of the non degeneracy of diffusion coefficients.

In this paper, we investigate Nash equilibrium payoffs for two-player nonzero-sum stochastic differential games. The generalization of the earlier result by Buckdahn et al. [3] concerns the following aspects. First, our cost functionals are defined by controlled backward stochastic differential equations, and the admissible control processes depend on events occurring before the beginning of the stochastic differential game. Thus, our cost functionals are not necessarily deterministic. Second, since our cost functionals are nonlinear, we cannot apply the methods used in [3]. We make use of the notion of stochastic backward semigroups introduced by Peng [17], and the theory of backward stochastic differential equations. Finally, each player has his own backward stochastic differential equation, controlled also by the adversary player, which defines his own cost functional.

Beyond the theoretical interest of this paper, the result of the paper is also applicable in finance and economics. For instance, we can consider an application of our theoretical result to a problem arising in financial markets. Let the financial market consist of a risk-free asset and risky stocks and consider two investors (players) in this financial market. Both investors try to maximize their payoff functionals, which are, in general, different. To maximize them, they have to use investment strategies with delays. Indeed, although both investors react immediately to the financial market, the financial market is not so quick in reacting to the moves of both investors. The above described problem leads to a two-player nonzero-sum stochastic differential game. We can use our theoretical result to get an existence theorem and a characterization theorem of Nash equilibrium payoffs for this game.

Our paper is organized as follows. In Section 2, we introduce some notations and preliminaries concerning backward stochastic differential equations, which we will need in what follows. In Section 3, we give the main results of this paper and their proofs, i.e., an existence theorem and a characterization theorem of Nash equilibrium payoffs for two-player nonzero-sum stochastic differential games as well as their proofs.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the classical Wiener space, i.e., for the given terminal time $T > 0$, we consider $\Omega = C_0([0, T]; \mathbb{R}^d)$ as the space of continuous functions $h : [0, T] \to \mathbb{R}^d$ such that $h(0) = 0$, endowed with the supremum norm, and let $\mathbb{P}$ be the Wiener measure on the Borel σ-field $\mathcal{B}(\Omega)$ over Ω, with respect to which the coordinate process $B_t(\omega) = \omega_t, \omega \in \Omega, t \in [0, T]$, is a d-dimensional standard Brownian motion. We denote by $\mathcal{N}_{\mathbb{P}}$ the collection of all $\mathbb{P}$-null sets in Ω and define the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$, which is generated by the coordinate process B and completed by all $\mathbb{P}$-null sets:

$$\mathcal{F}_t = \sigma\{B_s, s \leq t\} \vee \mathcal{N}_{\mathbb{P}}, \quad t \in [0, T],$$

where $\mathcal{N}_{\mathbb{P}}$ is the set of all $\mathbb{P}$-null sets.

Let us introduce the following spaces, which will be needed in what follows.

- $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^n) = \Big\{\xi \mid \xi : \Omega \to \mathbb{R}^n$ is an $\mathcal{F}_T$ -measurable random variable such that $\mathbb{E}[|\xi|^2] < +\infty\Big\}$,
- $\mathcal{H}^2(0, T; \mathbb{R}^d) = \Big\{\varphi \mid \varphi : \Omega \times [0, T] \to \mathbb{R}^d$ is an $\{\mathcal{F}_t\}$-adapted process such that $\mathbb{E}\int_0^T |\varphi_t|^2 \mathrm{d}t < +\infty\Big\}$,
- $S^2(0, T; \mathbb{R}) = \Big\{\varphi \mid \varphi : \Omega \times [0, T] \to \mathbb{R}$ is an $\{\mathcal{F}_t\}$-adapted continuous process such that $\mathbb{E}[\sup_{0\leq t\leq T} |\varphi_t|^2] < +\infty\Big\}$.

We consider the BSDE with data (f, ξ):

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}B_s, \quad 0 \leq t \leq T. \tag{2.1}$$

Here $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is such that, for any $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, $f(\cdot, y, z)$ is progressively measurable. We make the following assumptions:

(H1) (Lipschitz condition): There exists a positive constant L such that for all $(t, y_i, z_i) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d,\ i = 1, 2,$

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq L(|y_1 - y_2| + |z_1 - z_2|).$$

(H2) $f(\cdot, 0, 0) \in \mathcal{H}^2(0, T; \mathbb{R})$.

The following existence and uniqueness theorem was established by Pardoux and Peng [16].

Lemma 2.1. *Let the assumptions* (H1) *and* (H2) *hold. Then, for all* $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, *BSDE* (2.1) *has a unique solution*

$$(Y, Z) \in S^2(0, T; \mathbb{R}) \times \mathcal{H}^2(0, T; \mathbb{R}^d).$$

We recall the well-known comparison theorem for solutions of BSDEs, which has been established by El Karoui et al. [6] and Peng [17].

Lemma 2.2. *Let $\xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, and f^1 and f^2 satisfy* (H1) *and* (H2). *We denote by (Y^1, Z^1) and (Y^2, Z^2) the solutions of BSDEs with data (f^1, ξ^1) and (f^2, ξ^2), respectively, and we suppose that*
(i) $\xi^1 \le \xi^2$, $\mathbb{P}$-*a.s.*,
(ii) $f^1(t, Y_t^2, Z_t^2) \le f^2(t, Y_t^2, Z_t^2)$, $\mathrm{d}t\mathrm{d}\mathbb{P}$-*a.e.*
Then, we have $Y_t^1 \le Y_t^2$, a.s., for all $t \in [0, T]$. Moreover, if $\mathbb{P}(\xi^1 < \xi^2) > 0$, then $\mathbb{P}(Y_t^1 < Y_t^2) > 0$, $t \in [0, T]$, and in particular, $Y_0^1 < Y_0^2$.

By virtue of the notations introduced in the above lemma, for some $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, we put

$$f^1(s, y, z) = f(s, y, z) + \varphi_1(s), \qquad f^2(s, y, z) = f(s, y, z) + \varphi_2(s).$$

Then we have the following lemma. For the proof the readers can refer to El Karoui et al. [6], and Peng [17].

Lemma 2.3. *Suppose that $\xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, f satisfies* (H1) *and* (H2) *and $\varphi_1, \varphi_2 \in \mathcal{H}^2(0, T; \mathbb{R})$. We denote by (Y^1, Z^1) and (Y^2, Z^2) the solution of BSDEs (2.1) with data (f^1, ξ^1) and (f^2, ξ^2), respectively. Then we have the following estimate:*

$$\begin{aligned}
&|Y_t^1 - Y_t^2|^2 + \frac{1}{2}\mathbb{E}\left\{\int_t^T \mathrm{e}^{\beta(t-s)}[|Y_s^1 - Y_s^2|^2 + |Z_s^1 - Z_s^2|^2]\mathrm{d}s \mid \mathcal{F}_t\right\} \\
&\quad \le \mathbb{E}\left\{\mathrm{e}^{\beta(T-t)}|\xi^1 - \xi^2|^2 \mid \mathcal{F}_t\right\} + \mathbb{E}\left\{\int_t^T \mathrm{e}^{\beta(t-s)}|\varphi_1(s) - \varphi_2(s)|^2\mathrm{d}s \mid \mathcal{F}_t\right\},
\end{aligned}$$

where $\beta = 16(1 + L^2)$ and L is the Lipschitz constant in (H1).

3. Nash equilibrium payoffs for nonzero-sum stochastic differential games

The objective of this section is to investigate Nash equilibrium payoffs for two-player nonzero-sum stochastic differential games with nonlinear cost functionals. An existence theorem (Theorem 3.20) and a characterization theorem (Theorem 3.16) of Nash equilibrium payoffs for two-player nonzero-sum stochastic differential games are the main results of this section.

Let U and V be two compact metric spaces. Here U is considered as the control state space of the first player, and V as that of the second one. The associated sets of admissible controls will be denoted by $\mathcal{U}$ and $\mathcal{V}$, respectively. The set $\mathcal{U}$ is formed by all U-valued $\mathbb{F}$-progressively measurable processes, and $\mathcal{V}$ is the set of all V-valued $\mathbb{F}$-progressively measurable processes.

For given admissible controls $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, we consider the following control system

$$\begin{cases} \mathrm{d}X_s^{t,\zeta;u,v} = b(s, X_s^{t,\zeta;u,v}, u_s, v_s)\mathrm{d}s + \sigma(s, X_s^{t,\zeta;u,v}, u_s, v_s)\mathrm{d}B_s, \quad s \in [t, T], \\ X_t^{t,\zeta;u,v} = \zeta, \end{cases} \tag{3.1}$$

where $t \in [0, T]$ is regarded as the initial time, and $\zeta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$ as the initial state. The mappings

$$b : [0, T] \times \mathbb{R}^n \times U \times V \to \mathbb{R}^n,$$
$$\sigma : [0, T] \times \mathbb{R}^n \times U \times V \to \mathbb{R}^{n \times d}$$

are supposed to satisfy the following conditions:

(H3.1) For all $x \in \mathbb{R}^n$, $b(\cdot, x, \cdot, \cdot)$ and $\sigma(\cdot, x, \cdot, \cdot)$ are continuous in (t, u, v).
(H3.2) There exists a positive constant L such that, for all $t \in [0, T]$, $x, x' \in \mathbb{R}^n$, $u \in U$, $v \in V$,

$$|b(t, x, u, v) - b(t, x', u, v)| + |\sigma(t, x, u, v) - \sigma(t, x', u, v)| \leq L|x - x'|.$$

It is obvious that, under the above conditions, for any $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, the control system (3.1) has a unique strong solution $\{X_s^{t,\zeta;u,v},\ 0 \leq t \leq s \leq T\}$, and we also have the following estimates.

Lemma 3.1. *For all $p \geq 2$, there exists a positive constant C_p such that, for all $t \in [0, T]$, $\zeta, \zeta' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$, $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$,*

$$\mathbb{E}\left\{\sup_{t \leq s \leq T} |X_s^{t,\zeta;u,v}|^p | \mathcal{F}_t\right\} \leq C_p(1 + |\zeta|^p), \quad \mathbb{P}\text{-a.s.},$$
$$\mathbb{E}\left\{\sup_{t \leq s \leq T} |X_s^{t,\zeta;u,v} - X_s^{t,\zeta';u,v}|^p | \mathcal{F}_t\right\} \leq C_p|\zeta - \zeta'|^p, \quad \mathbb{P}\text{-a.s.}$$

Here the constant C_p only depends on p, the Lipschitz constant and the linear growth of b and σ.

For given admissible controls $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, we consider the following BSDE:

$$Y_s^{t,\zeta;u,v} = \Phi(X_T^{t,\zeta;u,v}) + \int_s^T f(r, X_r^{t,\zeta;u,v}, Y_r^{t,\zeta;u,v}, Z_r^{t,\zeta;u,v}, u_r, v_r)\mathrm{d}r - \int_s^T Z_r^{t,\zeta;u,v}\mathrm{d}B_r, \quad t \leq s \leq T, \tag{3.2}$$

where $X^{t,\zeta;u,v}$ is introduced in Eq. (3.1) and

$$\Phi = \Phi(x) : \mathbb{R}^n \to \mathbb{R},$$
$$f = f(t, x, y, z, u, v) : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times U \times V \to \mathbb{R}$$

satisfy the following conditions:

(H3.3) For all $(x, y, z) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d$, $f(\cdot, x, y, z, \cdot, \cdot)$ is continuous in (t, u, v).
(H3.4) There exists a positive constant L such that, for all $t \in [0, T]$, $x, x' \in \mathbb{R}^n$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$, $u \in U$ and $v \in V$,

$$|f(t, x, y, z, u, v) - f(t, x', y', z', u, v)| + |\Phi(x) - \Phi(x')| \leq L(|x - x'| + |y - y'| + |z - z'|).$$

It is by now standard that under the above assumptions Eq. (3.2) admits a unique solution $(Y^{t,\zeta;u,v}, Z^{t,\zeta;u,v}) \in S^2(0, T; \mathbb{R}) \times \mathcal{H}^2(0, T; \mathbb{R}^d)$. Moreover, in [4] it was shown that the following holds:

Proposition 3.2. *There exists a positive constant C such that, for all $t \in [0, T]$, $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, $\zeta, \zeta' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$,*

$$|Y_t^{t,\zeta;u,v}| \leq C(1 + |\zeta|), \quad \mathbb{P}\text{-a.s.},$$
$$|Y_t^{t,\zeta;u,v} - Y_t^{t,\zeta';u,v}| \leq C|\zeta - \zeta'|, \quad \mathbb{P}\text{-a.s.}$$

We now introduce subspaces of admissible controls and give the definition of admissible strategies.

Definition 3.3. The space $\mathcal{U}_{t,T}$ (resp. $\mathcal{V}_{t,T}$) of admissible controls for Player I (resp., II) on the interval $[t,T]$ is defined as the space of all processes $\{u_r\}_{r\in[t,T]}$ (resp., $\{v_r\}_{r\in[t,T]}$), which are $\mathbb{F}$-progressively measurable and take values in U (resp., V).

Definition 3.4. A nonanticipating strategy with delay (NAD strategy) for Player I is a measurable mapping $\alpha : \mathcal{V}_{t,T} \to \mathcal{U}_{t,T}$, which satisfies the following properties:

(1) α is a nonanticipative strategy, i.e., for every $\mathbb{F}$-stopping time $\tau : \Omega \to [t,T]$, and for $v_1, v_2 \in \mathcal{V}_{t,T}$ with $v_1 = v_2$ on $[\![t,\tau]\!]$, it holds $\alpha(v_1) = \alpha(v_2)$ on $[\![t,\tau]\!]$. (Recall that $[\![t,\tau]\!] = \{(s,\omega) \in [t,T]\times\Omega, t \le s \le \tau(\omega)\}$.)
(2) α is a nonanticipative strategy with delay, i.e., for all $v \in \mathcal{V}_{t,T}$, there exists an increasing sequence of stopping times $\{S_n(v)\}_{n\ge1}$ with
 (i) $t = S_0(v) \le S_1(v) \le \cdots \le S_n(v) \le \cdots \le T$,
 (ii) $S_n(v) < S_{n+1}(v)$ on $\{S_n(v) < T\}, n \ge 0$,
 (iii) $\mathbb{P}(\bigcup_{n\ge1}\{S_n(v) = T\}) = 1$,
 such that, for all $n \ge 1$, $\Lambda \in \mathcal{F}_t$ and $v, v' \in \mathcal{V}_{t,T}$, we have: if $v = v'$ on $[\![t, S_{n-1}(v)]\!] \bigcap(\Lambda \times [t,T])$, then
 (iv) $S_l(v) = S_l(v')$, on Λ, $\mathbb{P}$-a.s., $1 \le l \le n$,
 (v) $\alpha(v) = \alpha(v')$, on $[\![t, S_n(v)]\!] \bigcap(\Lambda \times [t,T])$.

The set of all NAD strategies for Player I for games over the time interval $[t,T]$ is denoted by $\mathcal{A}_{t,T}$. The set of all NAD strategies $\beta : \mathcal{U}_{t,T} \to \mathcal{V}_{t,T}$ for Player II for games over the time interval $[t,T]$ is defined symmetrically and denoted by $\mathcal{B}_{t,T}$.

We have the following lemma, which is useful in what follows.

Lemma 3.5. *Let $\alpha \in \mathcal{A}_{t,T}$ and $\beta \in \mathcal{B}_{t,T}$. Then there exists a unique couple of admissible control processes $(u,v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ such that*

$$\alpha(v) = u, \qquad \beta(u) = v.$$

Such a result can be found already in [3]. However, since our definition of NAD strategies differs, we shall provide its proof.

For given control processes $u(\cdot) \in \mathcal{U}_{t,T}$ and $v(\cdot) \in \mathcal{V}_{t,T}$, we introduce now the associated cost functional

$$J(t,x;u,v) := Y_s^{t,x;u,v}|_{s=t}, \quad (t,x) \in [0,T]\times\mathbb{R}^n.$$

(Recall that $Y^{t,x;u,v}$ is defined by BSDE (3.2) with $\zeta = x \in \mathbb{R}^n$.) We define the lower and the upper value functions W and U, resp., of the game: for all $(t,x) \in [0,T]\times\mathbb{R}^n$, we put

$$W(t,x) := \operatorname*{esssup}_{\alpha\in\mathcal{A}_{t,T}} \operatorname*{essinf}_{\beta\in\mathcal{B}_{t,T}} J(t,x;\alpha,\beta),$$

and

$$U(t,x) := \operatorname*{essinf}_{\beta\in\mathcal{B}_{t,T}} \operatorname*{esssup}_{\alpha\in\mathcal{A}_{t,T}} J(t,x;\alpha,\beta).$$

Here we use Lemma 3.5 to identify $(X^{t,x;\alpha,\beta}, Y^{t,x;\alpha,\beta}, Z^{t,x;\alpha,\beta}) = (X^{t,x;u,v}, Y^{t,x;u,v}, Z^{t,x;u,v})$, and, in particular, $J(t,x;\alpha,\beta) = J(t,x;u,v)$, where $(u,v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ is the couple of controls associated with $(\alpha,\beta) \in \mathcal{A}_{t,T} \times \mathcal{B}_{t,T}$ by the relation $(\alpha(v), \beta(u)) = (u, v)$.

Remark 3.6. For the convenience of the reader we recall the notion of the essential infimum and the essential supremum of families of random variables (see, e.g., [12] for more details). Given a family of $\mathcal{F}$-measurable real valued random variables ξ_α ($\alpha \in I$), an $\mathcal{F}$-measurable random variable ξ is said to be $\operatorname*{essinf}_{\alpha\in I} \xi_\alpha$, if

(i) $\xi \le \xi_\alpha$, $\mathbb{P}$-a.s., for all $\alpha \in I$;
(ii) if for any random variable η such that $\eta \le \xi_\alpha$, $\mathbb{P}$-a.s., for all $\alpha \in I$, it holds that $\eta \le \xi$, $\mathbb{P}$-a.s.

We introduce the notion of $\operatorname*{esssup}_{\alpha\in I} \xi_\alpha$ by the following relation:

$$\operatorname*{esssup}_{\alpha\in I} \xi_\alpha = -\operatorname*{essinf}_{\alpha\in I}(-\xi_\alpha).$$

Remark 3.7. Lemma 3.5 guarantees that for NAD strategies $\alpha \in \mathcal{A}_{t,T}$ and $\beta \in \mathcal{B}_{t,T}$ there exists a unique associated couple $(u,v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ of admissible controls such that $\alpha(v) = u$, $\beta(u) = v$. For general nonanticipative strategies we can, in general, not get such a couple of controls. Let us give an example: we suppose that $U = V$ and $\varphi, \psi : U \to U$ are measurable functions such that $\psi \circ \varphi$ does not have a fixed point. We define

$$\begin{aligned} &\alpha(v)_s = \varphi(v_s), \quad s \in [t,T],\ v \in \mathcal{V}_{t,T}, \\ &\beta(u)_s = \psi(u_s), \quad s \in [t,T],\ u \in \mathcal{U}_{t,T}. \end{aligned}$$

Then α and β are nonanticipative strategies for Player I and II, respectively. But there is no couple $(u,v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ such that $\alpha(v) = u$, $\beta(u) = v$. Indeed, if there existed such couple of controls $(u,v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$, we would have, for $s \in [t,T]$,

$$\begin{aligned} &u_s = \alpha(v)_s = \varphi(v_s), \\ &v_s = \beta(u)_s = \psi(u_s) = \psi \circ \varphi(v_s). \end{aligned}$$

But this means that v_s is a fixed point of $\psi \circ \varphi$, which contradicts the assumptions of the absence of fixed points.

Let us now give the proof of Lemma 3.5.

Proof. We give the proof in two steps.

Step 1: For $(u,v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ we denote by $\{S_n(v)\}_{n\ge1}$ (resp., $\{T_n(u)\}_{n\ge1}$) the sequence of stopping times associated with $\alpha \in \mathcal{A}_{t,T}$ (resp., $\beta \in \mathcal{B}_{t,T}$) by Definition 3.4. Then, for arbitrarily given $(u,v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ we define the optional set

$$\Gamma := \bigcup_{n\ge1}\Big([S_n(v)] \cup [T_n(u)]\Big),$$

where $[S_n(v)]$ (resp., $[T_n(u)]$) denotes the graph of $S_n(v)$ (resp., $T_n(u)$). Then, for $\omega \in \Omega$, we have

$$\Gamma(\omega) = \Big\{S_n(v)(\omega), T_l(u)(\omega), n, l \ge 1,\ \text{s.t. } S_n(v)(\omega) < T, T_l(u)(\omega) < T\Big\} \bigcup \{T\},$$

and we observe that $\Gamma(\omega)$ is a finite set.

We denote by D_Γ the first hitting time of Γ, and we define a sequence of $\{\mathcal{F}_r\}$-stopping times as follows:

$$\tau_0 = t,$$
$$\tau_1(u,v) = D_\Gamma (= S_1(v) \wedge T_1(u)),$$
$$\tau_2(u,v) = D_{\Gamma\setminus[\tau_1(u,v)]} \wedge T,$$
$$\vdots$$
$$\tau_n(u,v) = D_{\Gamma\setminus\cup_{i=1}^{n-1}[\tau_i(u,v)]} \wedge T, \quad n \geq 1.$$

Recall that $a \wedge b = \min\{a, b\},\ a, b \in \mathbb{R}$.

We notice that $\tau_1(u, v)$ is independent of (u, v), and for $n \geq 2$, $\tau_n(u, v)$ depends only on $(u, v)|_{[\![t,\tau_{n-1}(u,v)]\!]}$. Indeed, this is a direct consequence of point (2) in Definition 3.4 and the definition of $\{\tau_n(u, v)\}_{n\geq1}$.

From the definition of $\{\tau_n\}_{n\geq1}$ it follows that, for all $(u, v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$,

(i) $t = \tau_0 \leq \tau_1(u, v) \leq \cdots \leq \tau_n(u, v) \leq \cdots \leq T$,
(ii) $\tau_n(u, v) < \tau_{n+1}(u, v)$, on $\{\tau_n(u, v) < T\}, n \geq 0$. Moreover, since $\Gamma(\omega)$ is a finite set, $\mathbb{P}(d\omega)$-a.s., $\mathbb{P}(\bigcup_{n\geq1}\{\tau_n(u, v) = T\}) = 1$.
(iii) For $n \geq 1$ and all $(u, v), (u', v') \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$, it holds: if $(u, v) = (u', v')$ on $[\![t, \tau_{n-1}(u, v)]\!]$, then $\tau_l(u, v) = \tau_l(u', v')$, $1 \leq l \leq n$, and $\alpha(v) = \alpha(v')$ and $\beta(u) = \beta(u')$, on $[\![t, \tau_n(u, v)]\!]$.

Step 2: For $\alpha \in \mathcal{A}_{t,T}$ and $\beta \in \mathcal{B}_{t,T}$, we let $\{\tau_n\}_{n\geq1}$ be constructed as above. Since neither τ_1 depends on the controls nor (α, β) restricted to $[\![t, \tau_1]\!]$ does, the process

$$(u^0, v^0) := (\alpha(v_0), \beta(u_0)), \quad \text{for arbitrary } (u_0, v_0) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T},$$

is such that $\alpha(v^0) = u^0$ and $\beta(u^0) = v^0$, on $[\![t, \tau_1]\!]$.

Taking into account that τ_2 only depends on the controls restricted to $[\![t, \tau_1]\!]$, and $(\alpha(v^0), \beta(u^0))|_{[\![t,\tau_2(u^0,v^0)]\!]}$ only depends on the controls (u^0, v^0) restricted to $[\![t, \tau_1]\!]$, we can define

$$(u^1, v^1) := (\alpha(v^0), \beta(u^0)),$$

and since we have $(u^1, v^1) = (u^0, v^0)$ on $[\![t, \tau_1]\!]$, it follows that $(u^1, v^1) = (\alpha(v^1), \beta(u^1))$, on $[\![t, \tau_2(u^1, v^1)]\!]$. Repeating the above argument we put

$$(u^n, v^n) := (\alpha(v^{n-1}), \beta(u^{n-1})) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}.$$

Then, since due to $(n-1)$th iteration step $(\alpha(v^{n-1}), \beta(u^{n-1})) = (u^{n-1}, v^{n-1})$, on $[\![t, \tau_n(u^{n-1}, v^{n-1})]\!]$, we also have $(u^{n-1}, v^{n-1}) = (u^n, v^n)$, on $[\![t, \tau_n(u^{n-1}, v^{n-1})]\!]$, and, thus, also $\tau_l(u^n, v^n) = \tau_l(u^{n-1}, v^{n-1})$, $0 \leq l \leq n+1$. Hence,

$$\tau_n(u^{n-1}, v^{n-1}) = \tau_n(u^n, v^n) \leq \tau_{n+1}(u^n, v^n) = \tau_{n+1}(u^{n+1}, v^{n+1}), \quad n \geq 1,$$

from which we deduce the existence of the limit of stopping times

$$\tau := \lim_{n\to\infty} \tau_n(u^n, v^n) \leq T.$$

For arbitrarily given $(u_0, v_0) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ we define

$$(u, v) := \sum_{n\geq0} (u^n, v^n) 1_{[\![\tau_{n-1}(u^{n-1},v^{n-1}),\tau_n(u^n,v^n)[\![} + (u_0, v_0) 1_{[\![\tau,T]\!]}.$$

Obviously, $(u, v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$, and since $(u, v) = (u^n, v^n)$ on $[\![t, \tau_n(u^n, v^n)]\!]$, we have $\tau_l(u, v) = \tau_l(u^n, v^n), 0 \le l \le n+1, n \ge 0$ (see the above property (iii)). But this allows to conclude from (ii) that

$$\mathbb{P}\left(\bigcup_{n\ge 1}\{\tau_n(u^n, v^n) = T\}\right) = \mathbb{P}\left(\bigcup_{n\ge 1}\{\tau_n(u, v) = T\}\right) = 1.$$

Consequently, since the above defined process $(u, v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ has the property that

$$\begin{aligned}(\alpha(v), \beta(u)) &= (\alpha(v^n), \beta(u^n)) \quad \text{on } [\![t, \tau_n(u, v)]\!] \text{ (nonanticipativity of } (\alpha, \beta))\\ &= (u^n, v^n) = (u, v) \quad \text{on } [\![t, \tau_n(u, v)]\!],\end{aligned}$$

we have $(\alpha(v), \beta(u)) = (u, v)$ on $[t, T] \times \Omega$, $\mathrm{d}s\mathrm{d}\mathbb{P}$-a.e. The proof is complete. □

The following lemmas were established in [2] under a slightly different definition of NAD strategies. However, their validity in our new framework can be checked easily.

Lemma 3.8. *Under assumptions* (H3.1)–(H3.4)*, for all* $(t, x) \in [0, T] \times \mathbb{R}^n$*, the value functions* $W(t, x)$ *and* $U(t, x)$ *are deterministic functions.*

Lemma 3.9. *There exists a positive constant* C *such that, for all* $t, t' \in [0, T]$ *and* $x, x' \in \mathbb{R}^n$*, we have*

(i) $W(t, x)$ *is* $\frac{1}{2}$*-Hölder continuous in* t*:*

$$|W(t, x) - W(t', x)| \le C(1 + |x|)|t - t'|^{\frac{1}{2}};$$

(ii) $|W(t, x) - W(t, x')| \le C|x - x'|$.

The same properties hold true for the function U.

Remark 3.10. From the above lemma it follows, in particular, that the functions W and U are of at most linear growth, i.e., there exists a positive constant C such that, for all $t \in [0, T]$ and $x \in \mathbb{R}^n$, $|W(t, x)| \le C(1 + |x|)$.

We now recall the notion of stochastic backward semigroups, which was introduced by Peng [17] and translated by Buckdahn and Li [4] into the framework of stochastic differential games. For a given initial condition (t, x), a positive number $\delta \le T - t$, for admissible control processes $u(\cdot) \in \mathcal{U}_{t,t+\delta}$ and $v(\cdot) \in \mathcal{V}_{t,t+\delta}$, and a real-valued random variable $\eta \in L^2(\Omega, \mathcal{F}_{t+\delta}, \mathbb{P}; \mathbb{R})$, we define

$$G^{t,x;u,v}_{t,t+\delta}[\eta] := \overline{Y}^{t,x;u,v}_t,$$

where $(\overline{Y}^{t,x;u,v}_s, \overline{Z}^{t,x;u,v}_s)_{t\le s\le t+\delta}$ is the unique solution of the following BSDE with time horizon $t + \delta$:

$$\begin{aligned}\overline{Y}^{t,x;u,v}_s &= \eta + \int_s^{t+\delta} f(r, X^{t,x;u,v}_r, \overline{Y}^{t,x;u,v}_r, \overline{Z}^{t,x;u,v}_r, u_r, v_r)\mathrm{d}r\\ &\quad - \int_s^{t+\delta} \overline{Z}^{t,x;u,v}_r \mathrm{d}B_r, \quad t \le s \le t + \delta,\end{aligned}$$

and $X^{t,x;u,v}$ is the unique solution of Eq. (3.1) with $\zeta = x \in \mathbb{R}^n$.

We observe that for the solution $Y^{t,x;u,v}$ of BSDE (3.2) with $\zeta = x \in \mathbb{R}^n$ we have

$$J(t,x;u,v) = Y_t^{t,x;u,v} = G_{t,T}^{t,x;u,v}[\Phi(X_T^{t,x;u,v})] = G_{t,t+\delta}^{t,x;u,v}[Y_{t+\delta}^{t,x;u,v}]$$
$$= G_{t,t+\delta}^{t,x;u,v}[J(t+\delta, X_{t+\delta}^{t,x;u,v}; u, v)].$$

Remark 3.11. For the special case that f is independent of (y,z) we have

$$G_{s,T}^{t,x;u,v}[\Phi(X_T^{t,x;u,v})] = G_{s,t+\delta}^{t,x;u,v}[Y_{t+\delta}^{t,x;u,v}]$$
$$= \mathbb{E}\left[Y_{t+\delta}^{t,x;u,v} + \int_s^{t+\delta} f(r, X_r^{t,x;u,v}, u_r, v_r)\mathrm{d}r \mid \mathcal{F}_s\right], \quad s \in [t, t+\delta].$$

In particular,

$$G_{t,T}^{t,x;u,v}[\Phi(X_T^{t,x;u,v})] = \mathbb{E}\left[\Phi(X_T^{t,x;u,v}) + \int_t^T f(r, X_r^{t,x;u,v}, u_r, v_r)\mathrm{d}r \mid \mathcal{F}_t\right].$$

For more details on stochastic backward semigroups, the reader is referred to Peng [17] and Buckdahn and Li [4]. Let us also recall the following dynamic programming principle for the value functions of stochastic differential games. Its proof can be found in [2].

Proposition 3.12. *Under assumptions* (H3.1)–(H3.4) *the following dynamic programming principle holds: for all* $0 < \delta \leq T - t,\ x \in \mathbb{R}^n$,

$$W(t,x) = \operatorname*{esssup}_{\alpha \in \mathcal{A}_{t,t+\delta}} \operatorname*{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}} G_{t,t+\delta}^{t,x;\alpha,\beta}[W(t+\delta, X_{t+\delta}^{t,x;\alpha,\beta})],$$

and

$$U(t,x) = \operatorname*{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}} \operatorname*{esssup}_{\alpha \in \mathcal{A}_{t,t+\delta}} G_{t,t+\delta}^{t,x;\alpha,\beta}[U(t+\delta, X_{t+\delta}^{t,x;\alpha,\beta})].$$

After having recalled some basics on two-player zero-sum stochastic differential games, let us introduce the framework of two-player nonzero-sum stochastic differential games where each of the both players has his own terminal as well as running cost functionals Φ_j and f_j, respectively, $j = 1, 2$. More precisely, for arbitrarily given admissible controls $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, we consider the following BSDEs, $j = 1, 2$,

$$ {}^jY_s^{t,\zeta;u,v} = \Phi_j(X_T^{t,\zeta;u,v}) + \int_s^T f_j(r, X_r^{t,\zeta;u,v}, {}^jY_r^{t,\zeta;u,v}, {}^jZ_r^{t,\zeta;u,v}, u_r, v_r)\mathrm{d}r$$
$$ - \int_s^T {}^jZ_r^{t,\zeta;u,v}\mathrm{d}B_r, \quad t \leq s \leq T,$$

where $X^{t,\zeta;u,v}$ is introduced by Eq. (3.1) and

$$\Phi_j = \Phi_j(x) : \mathbb{R}^n \to \mathbb{R},$$
$$f_j = f_j(t,x,y,z,u,v) : [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times U \times V \to \mathbb{R},$$

are assumed to satisfy the conditions (H3.3) and (H3.4). In addition, in order to simplify the arguments, we also suppose that

(H3.5) b, σ, Φ_j and $f_j,\ j = 1, 2$, are bounded.

The associated stochastic backward semigroups are denoted by ${}^jG^{t,x;u,v}_{t,s}$, $t \le s \le T$, $j = 1, 2$, and for the associated cost functionals $J_j(t, x; u, v) = {}^jY^{t,x;u,v}_t$, we have

$$J_j(t,x;u,v) = {}^jG^{t,x;u,v}_{t,T}[\Phi_j(X^{t,x;u,v}_T)] = {}^jG^{t,x;u,v}_{t,t+\delta}[{}^jY^{t,x;u,v}_{t+\delta}]$$
$$= {}^jG^{t,x;u,v}_{t,t+\delta}[J_j(t+\delta, X^{t,x;u,v}_{t+\delta}, u, v)],$$

$(t, x) \in [0, T] \times \mathbb{R}^n$, $(u, v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$, $0 \le \delta \le T - t$, $j = 1, 2$.

For what follows, we assume that the Isaacs condition holds in the following sense: for all $(t, x, y, p) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ and $A \in \mathbb{S}^n$ (recall that $\mathbb{S}^n$ denotes the set of $n \times n$ symmetric matrices) and for $j = 1, 2$, we have

$$\sup_{u\in U}\inf_{v\in V}\left\{\frac{1}{2}\mathrm{tr}(\sigma\sigma^T(t,x,u,v)A) + \langle p, b(t,x,u,v)\rangle + f_j(t,x,y,p^T\sigma(t,x,u,v),u,v)\right\}$$
$$= \inf_{v\in V}\sup_{u\in U}\left\{\frac{1}{2}\mathrm{tr}(\sigma\sigma^T(t,x,u,v)A) + \langle p, b(t,x,u,v)\rangle\right.$$
$$\left. + f_j(t,x,y,p^T\sigma(t,x,u,v),u,v)\right\}. \tag{3.3}$$

Under Isaacs condition (3.3) we have, similar to [2],

$$W_1(t,x) = \operatorname*{esssup}_{\alpha\in\mathcal{A}_{t,T}}\operatorname*{essinf}_{\beta\in\mathcal{B}_{t,T}} J_1(t,x;\alpha,\beta) = \operatorname*{essinf}_{\beta\in\mathcal{B}_{t,T}}\operatorname*{esssup}_{\alpha\in\mathcal{A}_{t,T}} J_1(t,x;\alpha,\beta),$$

and

$$W_2(t,x) = \operatorname*{essinf}_{\alpha\in\mathcal{A}_{t,T}}\operatorname*{esssup}_{\beta\in\mathcal{B}_{t,T}} J_2(t,x;\alpha,\beta) = \operatorname*{esssup}_{\beta\in\mathcal{B}_{t,T}}\operatorname*{essinf}_{\alpha\in\mathcal{A}_{t,T}} J_2(t,x;\alpha,\beta), \tag{3.4}$$

$(t, x) \in [0, T] \times \mathbb{R}^n$.

Finally, we complete the preparation with the definition of the Nash equilibrium payoff of stochastic differential games, which is similar to the definition introduced by Buckdahn et al. [3].

Definition 3.13. A couple $(e_1, e_2) \in \mathbb{R}^2$ is called a Nash equilibrium payoff at the point (t, x) if for any $\varepsilon > 0$, there exists $(\alpha_\varepsilon, \beta_\varepsilon) \in \mathcal{A}_{t,T} \times \mathcal{B}_{t,T}$ such that, for all $(\alpha, \beta) \in \mathcal{A}_{t,T} \times \mathcal{B}_{t,T}$,

$$\begin{aligned} &J_1(t,x;\alpha_\varepsilon,\beta_\varepsilon) \ge J_1(t,x;\alpha,\beta_\varepsilon) - \varepsilon, \quad \mathbb{P}\text{-a.s.},\\ &J_2(t,x;\alpha_\varepsilon,\beta_\varepsilon) \ge J_2(t,x;\alpha_\varepsilon,\beta) - \varepsilon, \quad \mathbb{P}\text{-a.s.}\end{aligned} \tag{3.5}$$

and

$$|\mathbb{E}[J_j(t,x;\alpha_\varepsilon,\beta_\varepsilon)] - e_j| \le \varepsilon, \quad j = 1, 2.$$

Remark 3.14. We attract the reader's attention to the fact that $J_j(t, x; \alpha, \beta)$, $j = 1, 2$, are random variables. In our existence result (Theorem 3.20) we will construct cost functionals $J_j(t, x; \alpha_\varepsilon, \beta_\varepsilon)$, $\varepsilon > 0$, $j = 1, 2$, which are deterministic.

By virtue of Lemma 3.5 we can easily get the following lemma.

Lemma 3.15. *For any $\varepsilon > 0$ and $(\alpha_\varepsilon, \beta_\varepsilon) \in \mathcal{A}_{t,T} \times \mathcal{B}_{t,T}$, (3.5) holds if and only if, for all $(u, v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$,*

$$\begin{aligned} &J_1(t, x; \alpha_\varepsilon, \beta_\varepsilon) \geq J_1(t, x; u, \beta_\varepsilon(u)) - \varepsilon, \quad \mathbb{P}\text{-a.s.}, \\ &J_2(t, x; \alpha_\varepsilon, \beta_\varepsilon) \geq J_2(t, x; \alpha_\varepsilon(v), v) - \varepsilon, \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{3.6}$$

We now give the characterization theorem of Nash equilibrium payoffs for nonzero-sum stochastic differential games.

Theorem 3.16. *Let $(t, x) \in [0, T] \times \mathbb{R}^n$. Under Isaacs condition (3.3), $(e_1, e_2) \in \mathbb{R}^2$ is a Nash equilibrium payoff at point (t, x) if and only if for all $\varepsilon > 0$, there exist $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ such that for all $s \in [t, T]$ and $j = 1, 2$,*

$$\mathbb{P}\left({}^jY_s^{t,x;u^\varepsilon,v^\varepsilon} \geq W_j(s, X_s^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon \mid \mathcal{F}_t \right) \geq 1 - \varepsilon, \quad \mathbb{P}\text{-a.s.}, \tag{3.7}$$

and

$$|\mathbb{E}[J_j(t, x; u^\varepsilon, v^\varepsilon)] - e_j| \leq \varepsilon. \tag{3.8}$$

Remark 3.17. The above theorem generalizes the results of [3,18] from the case of classical cost functionals without running costs to nonlinear cost functionals which running costs f_j, $j = 1, 2$, depend on (y, z). Moreover, in our framework the controls can depend on events occurring before time t.

We prepare the proof of this theorem by the following two lemmas.

Lemma 3.18. *Let $(t, x) \in [0, T] \times \mathbb{R}^n$ and $u \in \mathcal{U}_{t,T}$ be arbitrarily fixed. Then,*

(i) *for all $\delta \in [0, T - t]$ and $\varepsilon > 0$, there exists an NAD strategy $\alpha \in \mathcal{A}_{t,T}$ such that, for all $v \in \mathcal{V}_{t,T}$,*

$$\begin{aligned} &\alpha(v) = u, \quad \text{on } [t, t + \delta], \\ &{}^2Y_{t+\delta}^{t,x;\alpha(v),v} \leq W_2(t + \delta, X_{t+\delta}^{t,x;\alpha(v),v}) + \varepsilon, \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

(ii) *for all $\delta \in [0, T - t]$ and $\varepsilon > 0$, there exists an NAD strategy $\alpha \in \mathcal{A}_{t,T}$ such that, for all $v \in \mathcal{V}_{t,T}$,*

$$\begin{aligned} &\alpha(v) = u, \quad \text{on } [t, t + \delta], \\ &{}^1Y_{t+\delta}^{t,x;\alpha(v),v} \geq W_1(t + \delta, X_{t+\delta}^{t,x;\alpha(v),v}) - \varepsilon, \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

Proof. We only give the proof of (i). Indeed, (ii) can be proved by a symmetric argument. We begin with observing that putting $\beta^{v'}(u') = v'$, for all $u' \in \mathcal{U}_{t+\delta,T}$, defines for every $v' \in \mathcal{V}_{t+\delta,T}$ an element $\beta^{v'} \in \mathcal{B}_{t+\delta,T}$ and allows to regard $\mathcal{V}_{t+\delta,T}$ as a subset of $\mathcal{B}_{t+\delta,T}$. Consequently, from (3.4), for any $y \in \mathbb{R}^n$, we have

$$\begin{aligned} W_2(t + \delta, y) &= \operatorname*{essinf}_{\alpha \in \mathcal{A}_{t+\delta,T}} \operatorname*{esssup}_{\beta \in \mathcal{B}_{t+\delta,T}} J_2(t + \delta, y; \alpha, \beta) \\ &\geq \operatorname*{essinf}_{\alpha \in \mathcal{A}_{t+\delta,T}} \operatorname*{esssup}_{v \in \mathcal{V}_{t+\delta,T}} J_2(t + \delta, y; \alpha(v), v), \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

Therefore, for $\varepsilon_0 > 0$, there exists $\alpha_y \in \mathcal{A}_{t+\delta,T}$ such that

$$W_2(t+\delta, y) \geq \underset{v \in \mathcal{V}_{t+\delta,T}}{\text{esssup}}\, J_2(t+\delta, y; \alpha_y(v), v) - \varepsilon_0, \quad \mathbb{P}\text{-a.s.} \tag{3.9}$$

(The existence of $\alpha_y \in \mathcal{A}_{t+\delta,T}$ can be shown with the techniques used in the proof of Lemma 3.8 in [4].)

Let $\{O_i\}_{i\geq 1} \subset \mathcal{B}(\mathbb{R}^n)$ be a partition of $\mathbb{R}^n$ such that $\sum_{i\geq 1} O_i = \mathbb{R}^n$, $O_i \neq \emptyset$, and $\text{diam}(O_i) \leq \varepsilon_0,\ i \geq 1$. Let $y_i \in O_i,\ i \geq 1$. We put, for $v \in \mathcal{V}_{t,T}$,

$$\alpha(v)_s = \begin{cases} u_s, & s \in [t, t+\delta], \\ \sum_{i\geq 1} 1_{\{X_{t+\delta}^{t,x;u,v} \in O_i\}} \alpha_{y_i}(v|_{[t+\delta,T]})_s, & s \in (t+\delta, T]. \end{cases} \tag{3.10}$$

The such introduced mapping $\alpha : \mathcal{V}_{t,T} \to \mathcal{U}_{t,T}$ is an NAD strategy. Indeed,

(i) *The mapping α is nonanticipative.*

Proof. For every $(\mathcal{F}_r)$-stopping time $\tau : \Omega \to [t, T]$, and for $v_1, v_2 \in \mathcal{V}_{t,T}$ with $v_1 = v_2$ on $[\![t, \tau]\!]$, we decompose v_1, v_2 into $v_1^1, v_2^1 \in \mathcal{V}_{t,t+\delta}$ and $v_1^2, v_2^2 \in \mathcal{V}_{t+\delta,T}$ such that $v_i^1 = v_i$ on $[t, t+\delta]$, and $v_i^2 = v_i$ on $[t+\delta, T]$, $i = 1, 2$. In order to abbreviate, we will write for this: $v_1 = v_1^1 \oplus v_1^2$ and $v_2 = v_2^1 \oplus v_2^2$. Then we have $v_1^1 = v_2^1$ on $[\![t, \tau \wedge (t+\delta)]\!]$ and $v_1^2 = v_2^2$ on $[\![\tau \wedge (t+\delta), \tau]\!]$. It is obvious that $\alpha(v_1) = u = \alpha(v_2)$ on $[t, t+\delta]$ and, hence, also on $[\![t, \tau \wedge (t+\delta)]\!]$. Since $v_1^1 = v_2^1$ on $[\![t, \tau \wedge (t+\delta)]\!]$, we have $X_{t+\delta}^{t,x;\alpha(v_1^1),v_1^1} = X_{t+\delta}^{t,x;\alpha(v_2^1),v_2^1}$ on $\{\tau > t+\delta\}$, $\mathbb{P}-a.s.$ Therefore, because of the nonanticipativity of $\alpha_{y_i},\ i \geq 1$,

$$\alpha(v_1) = \sum_{i\geq 1} 1_{\{X_{t+\delta}^{t,x;\alpha(v_1^1),v_1^1} \in O_i\}} \alpha_{y_i}(v_1^2) = \sum_{i\geq 1} 1_{\{X_{t+\delta}^{t,x;\alpha(v_2^1),v_2^1} \in O_i\}} \alpha_{y_i}(v_2^2) = \alpha(v_2)$$

on $[\![\tau \wedge (t+\delta), \tau]\!]$. $\square$

(ii) *The mapping α is a nonanticipative strategy with delay.*

Proof. For $v = v^1 \oplus v^2 \in \mathcal{V}_{t,t+\delta} \times \mathcal{V}_{t+\delta,T}$, we have

$$\alpha(v) = u \oplus \sum_{i\geq 1} 1_{\{X_{t+\delta}^{t,x;u,v^1} \in O_i\}} \alpha_{y_i}(v^2).$$

Let $\{S_n^i(v^2)\}_{n\geq 1}$ be the sequence of the stopping times associated with $\alpha_{y_i} \in \mathcal{A}_{t+\delta,T}$ in the sense of Definition 3.4. Then, putting $S_0 = t$, $S_1 = t+\delta$,

$$S_{n+1}(v) = \sum_{i\geq 1} 1_{\{X_{t+\delta}^{t,x;u,v^1} \in O_i\}} S_n^i(v^2), \quad n \geq 1.$$

We have that $\{S_n(v)\}_{n\geq 1}$ satisfies the condition (2) in Definition 3.4. Thus, α is a nonanticipative strategy with delay. $\square$

From Lemma 3.9, (3.9) and (3.10) it follows that, for $v \in \mathcal{V}_{t,T}$,

$$\begin{aligned} W_2(t+\delta, X_{t+\delta}^{t,x;\alpha(v),v}) &\geq \sum_{i\geq 1} 1_{\{X_{t+\delta}^{t,x;\alpha(v),v} \in O_i\}} W_2(t+\delta, y_i) - C\varepsilon_0 \\ &\geq \sum_{i\geq 1} 1_{\{X_{t+\delta}^{t,x;\alpha(v),v} \in O_i\}} J_2(t+\delta, y_i; \alpha_{y_i}(v|_{[t+\delta,T]}), v) - C\varepsilon_0 \\ &= \sum_{i\geq 1} 1_{\{X_{t+\delta}^{t,x;\alpha(v),v} \in O_i\}} J_2(t+\delta, y_i; \alpha(v), v) - C\varepsilon_0. \end{aligned}$$

Thus, from Proposition 3.2,

$$W_2(t+\delta, X_{t+\delta}^{t,x;\alpha(v),v}) \geq \sum_{i\geq 1} 1_{\{X_{t+\delta}^{t,x;\alpha(v),v}\in O_i\}} J_2(t+\delta, X_{t+\delta}^{t,x;\alpha(v),v};\alpha(v),v) - C\varepsilon_0$$
$$= J_2(t+\delta, X_{t+\delta}^{t,x;\alpha(v),v};\alpha(v),v) - C\varepsilon_0.$$

Here C is a constant which can vary from line to line, but which is independent of $v \in \mathcal{V}_{t,T}$.

Putting $\varepsilon_0 = \varepsilon C^{-1}$ in the latter estimate, we obtain

$$W_2(t+\delta, X_{t+\delta}^{t,x;\alpha(v),v}) \geq J_2(t+\delta, X_{t+\delta}^{t,x;\alpha(v),v};\alpha(v),v) - \varepsilon, \quad v \in \mathcal{V}_{t,T}.$$

The proof is complete. □

The proof of Theorem 3.16 necessitates the following lemma.

Lemma 3.19. *There exists a positive constant C such that, for all $(u,v),(u',v') \in \mathcal{U}_{t,T}\times\mathcal{V}_{t,T}$, and for all $\mathcal{F}_r$-stopping times $S:\Omega\to[t,T]$ with $X_S^{t,x;u,v} = X_S^{t,x;u',v'}$, $\mathbb{P}$-a.s., it holds, for all real $\tau\in[t,T]$,*

$$\mathbb{E}\left[\sup_{0\leq s\leq\tau} |X_{(S+s)\wedge T}^{t,x;u,v} - X_{(S+s)\wedge T}^{t,x;u',v'}|^2 |\mathcal{F}_t\right] \leq C\tau, \quad \mathbb{P}\text{-a.s.}$$

This lemma is the result of a straight forward estimate using the fact that b and σ are bounded.

Let us give now the proof of Theorem 3.16.

Proof of Theorem 3.16 (*Sufficiency of* (3.7) *and* (3.8))**.**

Let $\varepsilon > 0$ be arbitrarily fixed. For $\varepsilon_0 > 0$ being specified later we suppose that $(u^{\varepsilon_0}, v^{\varepsilon_0}) \in \mathcal{U}_{t,T}\times\mathcal{V}_{t,T}$ satisfies (3.7) and (3.8), i.e., for all $s\in[t,T]$ and $j=1,2$,

$$\mathbb{P}\left({}^{j}Y_s^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}} \geq W_j(s, X_s^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}) - \varepsilon_0 \mid \mathcal{F}_t\right) \geq 1-\varepsilon_0, \quad \mathbb{P}\text{-a.s.}, \tag{3.11}$$

and

$$|\mathbb{E}[J_j(t,x;u^{\varepsilon_0},v^{\varepsilon_0})] - e_j| \leq \varepsilon_0. \tag{3.12}$$

Let us fix some partition: $t = t_0 \leq t_1 \leq \cdots \leq t_m = T$ of $[t,T]$ and $\tau = \sup_i |t_i - t_{i+1}|$. We apply Lemma 3.18 to u^{ε_0} and $t+\delta = t_1,\ldots,t_m$, successively. Then, for $\varepsilon_1 > 0$ (ε_1 depends on ε and is specified later) there exist NAD strategies $\alpha_i \in \mathcal{A}_{t,T}$, $i=1,\ldots,m$, such that, for all $v\in\mathcal{V}_{t,T}$,

$$\alpha_i(v) = u^{\varepsilon_0}, \quad \text{on } [t,t_i],$$
$${}^{2}Y_{t_i}^{t,x;\alpha_i(v),v} \leq W_2(t_i, X_{t_i}^{t,x;\alpha_i(v),v}) + \varepsilon_1, \quad \mathbb{P}\text{-a.s.} \tag{3.13}$$

For all $v\in\mathcal{V}_{t,T}$, we define

$$S^v = \inf\left\{s\geq t \mid \lambda(\{r\in[t,s] : v_r \neq v_r^{\varepsilon_0}\}) > 0\right\},$$

$$t^v = \inf\left\{t_i \geq S^v \mid i=1,\ldots,m\right\} \wedge T,$$

where λ denotes the Lebesgue measure on the real line $\mathbb{R}$. Obviously, S^v and t^v are stopping times, and we have $S^v \leq t^v \leq S^v + \tau$.

Let

$$\alpha_\varepsilon(v) = \begin{cases} u^{\varepsilon 0}, & \text{on } [\![t, t^v]\!], \\ \alpha_i(v), & \text{on } (t_i, T] \times \{t^v = t_i\},\ 1 \le i \le m. \end{cases}$$

It is easy to check that α_ε is an NAD strategy. From (3.13) it follows that

$$\begin{aligned} {}^2Y_{t^v}^{t,x;\alpha_\varepsilon(v),v} &= \sum_{i=1}^m {}^2Y_{t_i}^{t,x;\alpha_\varepsilon(v),v} 1_{\{t^v = t_i\}} \\ &\le \sum_{i=1}^m W_2(t_i, X_{t_i}^{t,x;\alpha_\varepsilon(v),v}) 1_{\{t^v = t_i\}} + \varepsilon_1 \\ &= W_2(t^v, X_{t^v}^{t,x;\alpha_\varepsilon(v),v}) + \varepsilon_1, \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{3.14}$$

In what follows we will show that, for all $\varepsilon > 0$ and $v \in \mathcal{V}_{t,T}$,

$$J_2(t, x; \alpha_\varepsilon(v), v) \le J_2(t, x; u^{\varepsilon 0}, v^{\varepsilon 0}) + \varepsilon, \qquad \alpha_\varepsilon(v^{\varepsilon 0}) = u^{\varepsilon 0}. \tag{3.15}$$

This relation as well as the symmetric one for J_1 will lead to the sufficiency of (3.7) and (3.8). For the proof of (3.15), we note that by (3.14), Lemmas 2.2 and 2.3 there exists a positive constant C such that

$$\begin{aligned} J_2(t, x, \alpha_\varepsilon(v), v) &= {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[{}^2Y_{t^v}^{t,x,\alpha_\varepsilon(v),v}] \\ &\le {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[W_2(t^v, X_{t^v}^{t,x;\alpha_\varepsilon(v),v}) + \varepsilon_1] \\ &\le {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[W_2(t^v, X_{t^v}^{t,x;\alpha_\varepsilon(v),v})] + C\varepsilon_1. \end{aligned} \tag{3.16}$$

Thanks to Lemmas 3.9 and 3.19 as well as the definitions of t^v and α_ε we have

$$\begin{aligned} &\mathbb{E}\left[|W_2(t^v, X_{t^v}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}) - W_2(t^v, X_{t^v}^{t,x;\alpha_\varepsilon(v),v})|^2 | \mathcal{F}_t\right] \\ &\le C\mathbb{E}\left[|X_{t^v}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}} - X_{t^v}^{t,x;\alpha_\varepsilon(v),v}|^2 | \mathcal{F}_t\right] \\ &\le C\tau, \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

Thus, from Lemma 2.3 it follows that

$$\begin{aligned} &|{}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[W_2(t^v, X_{t^v}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}})] - {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[W_2(t^v, X_{t^v}^{t,x;\alpha_\varepsilon(v),v})]| \\ &\le C\mathbb{E}[|W_2(t^v, X_{t^v}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}) - W_2(t^v, X_{t^v}^{t,x;\alpha_\varepsilon(v),v})|^2 | \mathcal{F}_t]^{\frac{1}{2}} \\ &\le C\tau^{\frac{1}{2}}, \end{aligned}$$

and the above inequality and (3.16) yield

$$\begin{aligned} J_2(t, x, \alpha_\varepsilon(v), v) &\le |{}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[W_2(t^v, X_{t^v}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}})] \\ &\qquad - {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[W_2(t^v, X_{t^v}^{t,x;\alpha_\varepsilon(v),v})]| \\ &\quad + {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[W_2(t^v, X_{t^v}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}})] + C\varepsilon_1 \\ &\le {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[W_2(t^v, X_{t^v}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}})] + C\varepsilon_1 + C\tau^{\frac{1}{2}}. \end{aligned}$$

For $s \in [t, T]$, we put

$$\Omega_s = \left\{ {}^2Y_s^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}} \ge W_2(s, X_s^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}) - \varepsilon_0 \right\}.$$

By the inequality $a \leq b + |a - b|$, $a, b \in \mathbb{R}$, we have

$$
\begin{aligned}
J_2(t, x; \alpha_\varepsilon(v), v) &\leq {}^2G^{t,x;\alpha_\varepsilon(v),v}_{t,t^v}[W_2(t^v, X^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}_{t^v})] + C\varepsilon_1 + C\tau^{\frac{1}{2}} \\
&= {}^2G^{t,x;\alpha_\varepsilon(v),v}_{t,t^v}\left[\sum_{i=1}^m W_2(t_i, X^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}_{t_i}) 1_{\{t^v = t_i\}}\right] + C\varepsilon_1 + C\tau^{\frac{1}{2}} \\
&\leq {}^2G^{t,x;\alpha_\varepsilon(v),v}_{t,t^v}\left[\sum_{i=1}^m W_2(t_i, X^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}_{t_i}) 1_{\{t^v = t_i\}} 1_{\Omega_{t_i}}\right] + C\varepsilon_1 + C\tau^{\frac{1}{2}} \\
&\quad + \left| {}^2G^{t,x;\alpha_\varepsilon(v),v}_{t,t^v}\left[\sum_{i=1}^m W_2(t_i, X^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}_{t_i}) 1_{\{t^v = t_i\}}\right] \right. \\
&\quad \left. - {}^2G^{t,x;\alpha_\varepsilon(v),v}_{t,t^v}\left[\sum_{i=1}^m W_2(t_i, X^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}_{t_i}) 1_{\{t^v = t_i\}} 1_{\Omega_{t_i}}\right] \right|.
\end{aligned}
\tag{3.17}
$$

Using Lemma 2.3 again as well as the boundedness of W_2, we see that

$$
\begin{aligned}
&\left| {}^2G^{t,x;\alpha_\varepsilon(v),v}_{t,t^v}\left[\sum_{i=1}^m W_2(t_i, X^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}_{t_i}) 1_{\{t^v = t_i\}}\right] \right. \\
&\quad \left. - {}^2G^{t,x;\alpha_\varepsilon(v),v}_{t,t^v}\left[\sum_{i=1}^m W_2(t_i, X^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}_{t_i}) 1_{\{t^v = t_i\}} 1_{\Omega_{t_i}}\right] \right| \\
&\leq C\mathbb{E}\left[\sum_{i=1}^m |W_2(t_i, X^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}_{t_i})|^2 1_{\{t^v = t_i\}} 1_{\Omega^c_{t_i}} |\mathcal{F}_t\right]^{\frac{1}{2}} \\
&\leq C\sum_{i=1}^m \mathbb{P}(\Omega^c_{t_i}|\mathcal{F}_t)^{\frac{1}{2}} \leq Cm\varepsilon_0^{\frac{1}{2}},
\end{aligned}
\tag{3.18}
$$

where we have used (3.11) for the latter estimate. Observing that

$$
{}^2Y^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}_{t_i} \geq W_2(t_i, X^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}_{t_i}) - \varepsilon_0, \quad \text{on } \Omega_{t_i},
$$

we deduce from Lemmas 2.2 and 2.3 that

$$
\begin{aligned}
&{}^2G^{t,x;\alpha_\varepsilon(v),v}_{t,t^v}\left[\sum_{i=1}^m W_2(t_i, X^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}_{t_i}) 1_{\{t^v = t_i\}} 1_{\Omega_{t_i}}\right] \\
&\leq {}^2G^{t,x;\alpha_\varepsilon(v),v}_{t,t^v}\left[\sum_{i=1}^m ({}^2Y^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}_{t_i} + \varepsilon_0) 1_{\{t^v = t_i\}} 1_{\Omega_{t_i}}\right] \\
&\leq {}^2G^{t,x;\alpha_\varepsilon(v),v}_{t,t^v}\left[\sum_{i=1}^m {}^2Y^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}_{t_i} 1_{\{t^v = t_i\}} 1_{\Omega_{t_i}} + \varepsilon_0\right] \\
&\leq {}^2G^{t,x;\alpha_\varepsilon(v),v}_{t,t^v}\left[\sum_{i=1}^m {}^2Y^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}_{t_i} 1_{\{t^v = t_i\}} 1_{\Omega_{t_i}}\right] + C\varepsilon_0.
\end{aligned}
$$

Hence, taking into account ${}^2Y_{t^v}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}} = \sum_{i=1}^m {}^2Y_{t_i}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}} 1_{\{t^v=t_i\}}$ and that, in analogy to (3.18)

$$\left|{}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}\left[\sum_{i=1}^m {}^2Y_{t_i}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}} 1_{\{t^v=t_i\}}1_{\Omega_{t_i}}\right] - {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}\left[\sum_{i=1}^m {}^2Y_{t_i}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}} 1_{\{t^v=t_i\}}\right]\right| \le Cm\varepsilon_0^{\frac{1}{2}},$$

we see that

$$\begin{aligned}
&{}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}\left[\sum_{i=1}^m W_2(t_i, X_{t_i}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}})1_{\{t^v=t_i\}}1_{\Omega_{t_i}}\right]\\
&\le {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[{}^2Y_{t^v}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}] + C\varepsilon_0 + Cm\varepsilon_0^{\frac{1}{2}}\\
&\le |{}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[{}^2Y_{t^v}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}] - {}^2G_{t,t^v}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}[{}^2Y_{t^v}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}]|\\
&\quad + {}^2G_{t,t^v}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}[{}^2Y_{t^v}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}] + C\varepsilon_0 + Cm\varepsilon_0^{\frac{1}{2}}\\
&= |{}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[{}^2Y_{t^v}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}] - {}^2G_{t,t^v}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}[{}^2Y_{t^v}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}]|\\
&\quad + J_2(t,x;u^{\varepsilon 0},v^{\varepsilon 0}) + C\varepsilon_0 + Cm\varepsilon_0^{\frac{1}{2}}.
\end{aligned}$$

In the frame of the proof of (3.15), we also need the following estimate

$$|{}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[{}^2Y_{t^v}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}] - {}^2G_{t,t^v}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}[{}^2Y_{t^v}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}]| \le C\tau^{\frac{1}{2}}.$$

In order to verify this relation we let, for all $s \in [t, t^v]$,

$$y_s = {}^2G_{s,t^v}^{t,x;\alpha_\varepsilon(v),v}[{}^2Y_{t^v}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}],$$

and we consider the BSDE solved by $y = (y_s)$

$$y_s = {}^2Y_{t^v}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}} + \int_s^{t^v} f_2(r, X_r^{t,x;\alpha_\varepsilon(v),v}, y_r, z_r, \alpha_\varepsilon(v_r), v_r)\mathrm{d}r - \int_s^{t^v} z_r \mathrm{d}B_r,$$

as well as

$$\begin{aligned}
{}^2Y_s^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}} &= {}^2Y_{t^v}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}\\
&\quad + \int_s^{t^v} f_2(r, X_r^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}, {}^2Y_r^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}, {}^2Z_r^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}, u_r^{\varepsilon 0}, v_r^{\varepsilon 0})\mathrm{d}r\\
&\quad - \int_s^{t^v} {}^2Z_r^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}\mathrm{d}B_r, \quad s \in [t, t^v].
\end{aligned}$$

We notice that $\alpha_\varepsilon(v) = u^{\varepsilon 0}$, on $[\![t, t^v]\!]$, $v = v^{\varepsilon 0}$, on $[\![t, S^v]\!]$. (Of course, these equalities, in particular, the latter one, are understood as $\mathrm{d}s\mathrm{d}\mathbb{P}$-a.e.) Thanks to Lemma 2.3 we have

$$\begin{aligned}
&|{}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[{}^2Y_{t^v}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}] - {}^2G_{t,t^v}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}[{}^2Y_{t^v}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}]|^2\\
&\le C\mathbb{E}\left[\int_t^{S^v} |f_2(r, X_r^{t,x;\alpha_\varepsilon(v),v}, y_r, z_r, \alpha_\varepsilon(v)_r, v_r)\right.\\
&\quad \left. - f_2(r, X_r^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}, y_r, z_r, u_r^{\varepsilon 0}, v_r^{\varepsilon 0})|^2\mathrm{d}r|\mathcal{F}_t\right]
\end{aligned}$$

$$+C\mathbb{E}\left[\int_{S^v}^{t^v}|f_2(r,X_r^{t,x;\alpha_\varepsilon(v),v},y_r,z_r,\alpha_\varepsilon(v)_r,v_r)\right.$$
$$\left.-\ f_2(r,X_r^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}},y_r,z_r,u_r^{\varepsilon_0},v_r^{\varepsilon_0})|^2\mathrm{d}r|\mathcal{F}_t\right]$$
$$=C\mathbb{E}\left[\int_{S^v}^{t^v}|f_2(r,X_r^{t,x;\alpha_\varepsilon(v),v},y_r,z_r,\alpha_\varepsilon(v)_r,v_r)\right.$$
$$\left.-\ f_2(r,X_r^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}},y_r,z_r,u_r^{\varepsilon_0},v_r^{\varepsilon_0})|^2\mathrm{d}r|\mathcal{F}_t\right]$$
$$\le C\mathbb{E}\left[\int_{S^v}^{t^v}1_{\{v_r\neq v_r^{\varepsilon_0}\}}\mathrm{d}r|\mathcal{F}_t\right]\le C\mathbb{E}[t^v-S^v|\mathcal{F}_t]\le C\tau.$$

Therefore, we have

$$^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}\left[\sum_{i=1}^m W_2(t_i,X_{t_i}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}})1_{\{t^v=t_i\}}1_{\Omega_{t_i}}\right]$$
$$\le C\tau^{\frac{1}{2}}+J_2(t,x;u^{\varepsilon_0},v^{\varepsilon_0})+C\varepsilon_0+Cm\varepsilon_0^{\frac{1}{2}},$$

and (3.17), (3.18) as well as the above inequality yield

$$J_2(t,x;\alpha_\varepsilon(v),v)\le J_2(t,x;u^{\varepsilon_0},v^{\varepsilon_0})+C\varepsilon_0+Cm\varepsilon_0^{\frac{1}{2}}+C\varepsilon_1+C\tau^{\frac{1}{2}}.$$

We can choose $\tau>0$, $\varepsilon_0>0$, and $\varepsilon_1>0$ such that $C\varepsilon_0+Cm\varepsilon_0^{\frac{1}{2}}+C\varepsilon_1+C\tau^{\frac{1}{2}}\le\varepsilon$ and $\varepsilon_0<\varepsilon$. Thus,

$$J_2(t,x;\alpha_\varepsilon(v),v)\le J_2(t,x;u^{\varepsilon_0},v^{\varepsilon_0})+\varepsilon,\qquad v\in\mathcal{V}_{t,T}.$$

By a symmetric argument we can construct $\beta_\varepsilon\in\mathcal{B}_{t,T}$ such that, for all $u\in\mathcal{U}_{t,T}$,

$$J_1(t,x;u,\beta_\varepsilon(u))\le J_1(t,x;u^{\varepsilon_0},v^{\varepsilon_0})+\varepsilon,\qquad \beta_\varepsilon(u^{\varepsilon_0})=v^{\varepsilon_0}.\tag{3.19}$$

Finally, by virtue of (3.15), (3.19), (3.12) and Lemma 3.15 we can see that $(\alpha_\varepsilon,\beta_\varepsilon)$ satisfies Definition 3.13. Therefore, (e_1,e_2) is a Nash equilibrium payoff. □

Proof of Theorem 3.16 (*Necessity of* (3.7) *and* (3.8))**.**

We assume that $(e_1,e_2)\in\mathbb{R}^2$ is a Nash equilibrium payoff at the point (t,x). Then, for all sufficiently small $\varepsilon>0$, there exists $(\alpha_\varepsilon,\beta_\varepsilon)\in\mathcal{A}_{t,T}\times\mathcal{B}_{t,T}$ such that, for all $(\alpha,\beta)\in\mathcal{A}_{t,T}\times\mathcal{B}_{t,T}$

$$\begin{aligned}&J_1(t,x;\alpha_\varepsilon,\beta_\varepsilon)\ge J_1(t,x;\alpha,\beta_\varepsilon)-\varepsilon^4,\quad \mathbb{P}\text{-a.s.},\\&J_2(t,x;\alpha_\varepsilon,\beta_\varepsilon)\ge J_2(t,x;\alpha_\varepsilon,\beta)-\varepsilon^4,\quad \mathbb{P}\text{-a.s.},\end{aligned}\tag{3.20}$$

and

$$|\mathbb{E}[J_j(t,x;\alpha_\varepsilon,\beta_\varepsilon)]-e_j|\le\varepsilon^4,\quad j=1,2.\tag{3.21}$$

Moreover, from Lemma 3.5 we know that there exists a unique couple $(u^\varepsilon,v^\varepsilon)$ such that

$$\alpha_\varepsilon(v^\varepsilon)=u^\varepsilon,\qquad \beta_\varepsilon(u^\varepsilon)=v^\varepsilon.$$

Let us argue by contradiction. For this we observe that (3.21) means that (3.8) holds. Assuming that (3.7) does not hold true, we have, for all $\varepsilon' > 0$, the existence of some $\varepsilon \in (0, \varepsilon')$ (for which we use the notations introduced above) and $\delta \in [0, T - t]$ such that, for some $j \in \{1, 2\}$, say for $j = 1$,

$$\mathbb{P}\left(\mathbb{P}({}^1Y_{t+\delta}^{t,x;u^\varepsilon,v^\varepsilon} < W_1(t+\delta, X_{t+\delta}^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon \mid \mathcal{F}_t) > \varepsilon\right) > 0. \tag{3.22}$$

Put

$$A = \left\{ {}^1Y_{t+\delta}^{t,x;u^\varepsilon,v^\varepsilon} < W_1(t+\delta, X_{t+\delta}^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon \right\} \in \mathcal{F}_{t+\delta}. \tag{3.23}$$

By applying Lemma 3.18 to u^ε and $t + \delta$ we see that, there exists an NAD strategy $\widetilde{\alpha} \in \mathcal{A}_{t,T}$, such that, for all $v \in \mathcal{V}_{t,T}$,

$$\begin{aligned} &\widetilde{\alpha}(v) = u^\varepsilon, \quad \text{on } [t, t+\delta], \\ &{}^1Y_{t+\delta}^{t,x;\widetilde{\alpha}(v),v} \geq W_1(t+\delta, X_{t+\delta}^{t,x;\widetilde{\alpha}(v),v}) - \frac{\varepsilon}{2}, \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{3.24}$$

By virtue of Lemma 3.5 there exists a unique couple $(u, v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ such that

$$\widetilde{\alpha}(v) = u, \qquad \beta_\varepsilon(u) = v.$$

We observe that this, in particular, means that $u = u^\varepsilon$ on $[t, t + \delta]$. Let us define now a control $\widetilde{u} \in \mathcal{U}_{t,T}$ as follows:

$$\widetilde{u} = \begin{cases} u^\varepsilon, & \text{on } ([t, t+\delta) \times \Omega) \cup ([t+\delta, T] \times A^c), \\ u, & \text{on } [t+\delta, T] \times A. \end{cases}$$

Since $\beta_\varepsilon \in \mathcal{B}_{t,T}$ is nonanticipative it follows that $\beta_\varepsilon(\widetilde{u}) = \beta_\varepsilon(u^\varepsilon) = v^\varepsilon$ on $[t, t + \delta]$, and for all $s \in [t + \delta, T]$,

$$\beta_\varepsilon(\widetilde{u})_s = \begin{cases} \beta_\varepsilon(u)_s = v_s, & \text{on } A, \\ \beta_\varepsilon(u^\varepsilon)_s = v_s^\varepsilon, & \text{on } A^c. \end{cases}$$

Then we have

$$\begin{aligned} &X^{t,x;\widetilde{u},\beta_\varepsilon(\widetilde{u})} = X^{t,x;u^\varepsilon,v^\varepsilon}, \quad \text{on } [t, t+\delta], \\ &X^{t,x;\widetilde{u},\beta_\varepsilon(\widetilde{u})} = \begin{cases} X^{t,x;\widetilde{\alpha}(v),v}, & \text{on } [t+\delta, T] \times A, \\ X^{t,x;u^\varepsilon,v^\varepsilon}, & \text{on } [t+\delta, T] \times A^c, \end{cases} \end{aligned}$$

and standard arguments show that also

$${}^1Y_{t+\delta}^{t,x;\widetilde{u},\beta_\varepsilon(\widetilde{u})} = \begin{cases} {}^1Y_{t+\delta}^{t,x;\widetilde{\alpha}(v),v}, & \text{on } A, \\ {}^1Y_{t+\delta}^{t,x;u^\varepsilon,v^\varepsilon}, & \text{on } A^c. \end{cases}$$

Therefore,

$$\begin{aligned} J_1(t, x; \widetilde{u}, \beta_\varepsilon(\widetilde{u})) &= {}^1Y_t^{t,x;\widetilde{u},\beta_\varepsilon(\widetilde{u})} = {}^1G_{t,t+\delta}^{t,x;\widetilde{u},\beta_\varepsilon(\widetilde{u})}[{}^1Y_{t+\delta}^{t,x;\widetilde{u},\beta_\varepsilon(\widetilde{u})}] \\ &= {}^1G_{t,t+\delta}^{t,x;\widetilde{u},\beta_\varepsilon(\widetilde{u})}[{}^1Y_{t+\delta}^{t,x;\widetilde{u},\beta_\varepsilon(\widetilde{u})} 1_A + {}^1Y_{t+\delta}^{t,x;\widetilde{u},\beta_\varepsilon(\widetilde{u})} 1_{A^c}] \\ &= {}^1G_{t,t+\delta}^{t,x;\widetilde{u},\beta_\varepsilon(\widetilde{u})}[{}^1Y_{t+\delta}^{t,x;\widetilde{\alpha}(v),v} 1_A + {}^1Y_{t+\delta}^{t,x;u^\varepsilon,v^\varepsilon} 1_{A^c}]. \end{aligned}$$

Thanks to Lemma 2.2 and (3.24) we have

$$\begin{aligned}J_1(t,x;\widetilde{u},\beta_\varepsilon(\widetilde{u})) &= {}^1G^{t,x;\widetilde{u},\beta_\varepsilon(\widetilde{u})}_{t,t+\delta}[{}^1Y^{t,x;\widetilde{\alpha}(v),v}_{t+\delta}1_A + {}^1Y^{t,x;u^\varepsilon,v^\varepsilon}_{t+\delta}1_{A^c}]\\ &\geq {}^1G^{t,x;\widetilde{u},\beta_\varepsilon(\widetilde{u})}_{t,t+\delta}\left[\left(W_1(t+\delta,X^{t,x;\widetilde{\alpha}(v),v}_{t+\delta}) - \frac{\varepsilon}{2}\right)1_A + {}^1Y^{t,x;u^\varepsilon,v^\varepsilon}_{t+\delta}1_{A^c}\right]\\ &= {}^1G^{t,x;\widetilde{u},\beta_\varepsilon(\widetilde{u})}_{t,t+\delta}\left[W_1(t+\delta,X^{t,x;u^\varepsilon,v^\varepsilon}_{t+\delta})1_A + {}^1Y^{t,x;u^\varepsilon,v^\varepsilon}_{t+\delta}1_{A^c} - \frac{\varepsilon}{2}1_A\right].\end{aligned}$$

Hence, from (3.23) it follows that

$$\begin{aligned}J_1(t,x;\widetilde{u},\beta_\varepsilon(\widetilde{u})) &\geq {}^1G^{t,x;\widetilde{u},\beta_\varepsilon(\widetilde{u})}_{t,t+\delta}\left[W_1(t+\delta,X^{t,x;u^\varepsilon,v^\varepsilon}_{t+\delta})1_A + {}^1Y^{t,x;u^\varepsilon,v^\varepsilon}_{t+\delta}1_{A^c} - \frac{\varepsilon}{2}1_A\right]\\ &\geq {}^1G^{t,x;\widetilde{u},\beta_\varepsilon(\widetilde{u})}_{t,t+\delta}\left[({}^1Y^{t,x;u^\varepsilon,v^\varepsilon}_{t+\delta}+\varepsilon)1_A + {}^1Y^{t,x;u^\varepsilon,v^\varepsilon}_{t+\delta}1_{A^c} - \frac{\varepsilon}{2}1_A\right]\\ &= {}^1G^{t,x;\widetilde{u},\beta_\varepsilon(\widetilde{u})}_{t,t+\delta}\left[{}^1Y^{t,x;u^\varepsilon,v^\varepsilon}_{t+\delta} + \frac{\varepsilon}{2}1_A\right]\\ &= {}^1G^{t,x;u^\varepsilon,v^\varepsilon}_{t,t+\delta}\left[{}^1Y^{t,x;u^\varepsilon,v^\varepsilon}_{t+\delta} + \frac{\varepsilon}{2}1_A\right].\end{aligned} \tag{3.25}$$

Let

$$y_s = {}^1G^{t,x;u^\varepsilon,v^\varepsilon}_{s,t+\delta}\left[{}^1Y^{t,x;u^\varepsilon,v^\varepsilon}_{t+\delta} + \frac{\varepsilon}{2}1_A\right], \quad s\in[t,t+\delta].$$

This process is the solution of the following BSDE:

$$\begin{aligned}y_s &= {}^1Y^{t,x;u^\varepsilon,v^\varepsilon}_{t+\delta} + \frac{\varepsilon}{2}1_A + \int_s^{t+\delta} f_1(r,X^{t,x;u^\varepsilon,v^\varepsilon}_r, y_r, z_r, u^\varepsilon_r, v^\varepsilon_r)\mathrm{d}r\\ &\quad - \int_s^{t+\delta} z_r\mathrm{d}B_r, \quad s\in[t,t+\delta],\end{aligned}$$

which we compare with

$$\begin{aligned}{}^1Y^{t,x;u^\varepsilon,v^\varepsilon}_s &= {}^1Y^{t,x;u^\varepsilon,v^\varepsilon}_{t+\delta} + \int_s^{t+\delta} f_1(r,X^{t,x;u^\varepsilon,v^\varepsilon}_r, {}^1Y^{t,x;u^\varepsilon,v^\varepsilon}_r, {}^1Z^{t,x;u^\varepsilon,v^\varepsilon}_r, u^\varepsilon_r, v^\varepsilon_r)\mathrm{d}r\\ &\quad - \int_s^{t+\delta} {}^1Z^{t,x;u^\varepsilon,v^\varepsilon}_r\mathrm{d}B_r, \quad s\in[t,t+\delta].\end{aligned}$$

Putting

$$\overline{y}_s = y_s - {}^1Y^{t,x;u^\varepsilon,v^\varepsilon}_s, \qquad \overline{z}_s = z_s - {}^1Z^{t,x;u^\varepsilon,v^\varepsilon}_s, \quad s\in[t,t+\delta],$$

we have

$$\begin{aligned}\overline{y}_s &= \int_s^{t+\delta}[f_1(r,X^{t,x;u^\varepsilon,v^\varepsilon}_r, y_r, z_r, u^\varepsilon_r, v^\varepsilon_r)\\ &\quad - f_1(r,X^{t,x;u^\varepsilon,v^\varepsilon}_r, {}^1Y^{t,x;u^\varepsilon,v^\varepsilon}_r, {}^1Z^{t,x;u^\varepsilon,v^\varepsilon}_r, u^\varepsilon_r, v^\varepsilon_r)]\mathrm{d}r\\ &\quad + \frac{\varepsilon}{2}1_A - \int_s^{t+\delta}\overline{z}_r\mathrm{d}B_r, \quad s\in[t,t+\delta].\end{aligned} \tag{3.26}$$

For notational simplification let us assume that the Brownian motion B is one dimensional, and we introduce, for $r\in[t,t+\delta]$,

$$\begin{aligned}a_r &= 1_{\{\overline{y}_r\neq 0\}}(\overline{y}_r)^{-1}\Big(f_1(r,X^{t,x;u^\varepsilon,v^\varepsilon}_r, y_r, z_r, u^\varepsilon_r, v^\varepsilon_r)\\ &\quad - f_1(r,X^{t,x;u^\varepsilon,v^\varepsilon}_r, {}^1Y^{t,x;u^\varepsilon,v^\varepsilon}_r, z_r, u^\varepsilon_r, v^\varepsilon_r)\Big),\end{aligned}$$

$$b_r = 1_{\{\bar{z}_r \neq 0\}}(\bar{z}_r)^{-1}\Big(f_1(r, X_r^{t,x;u^\varepsilon,v^\varepsilon}, {}^1Y_r^{t,x;u^\varepsilon,v^\varepsilon}, z_r, u_r^\varepsilon, v_r^\varepsilon) \\ - f_1(r, X_r^{t,x;u^\varepsilon,v^\varepsilon}, {}^1Y_r^{t,x;u^\varepsilon,v^\varepsilon}, {}^1Z_r^{t,x;u^\varepsilon,v^\varepsilon}, u_r^\varepsilon, v_r^\varepsilon)\Big).$$

Then, from the Lipschitz property of f_1 we see that $|a_r| \le L$, $|b_r| \le L$, $r \in [t, t+\delta]$, and BSDE (3.26) takes the following form:

$$\bar{y}_s = \frac{\varepsilon}{2}1_A + \int_s^{t+\delta} [a_r\bar{y}_r + b_r\bar{z}_r]\mathrm{d}r - \int_s^{t+\delta} \bar{z}_r\mathrm{d}B_r, \quad s \in [t, t+\delta].$$

By putting

$$Q_s = \exp\left(\int_t^s a_r\mathrm{d}r + \int_t^s b_r\mathrm{d}B_r - \frac{1}{2}\int_t^s |b_r|^2\mathrm{d}r\right), \quad s \in [t, t+\delta],$$

applying Itô's formula to $\bar{y}_s Q_s$, and then taking the conditional expectation, we deduce that

$$\bar{y}_t = \frac{\varepsilon}{2}\mathbb{E}[1_A Q_{t+\delta}|\mathcal{F}_t].$$

By the Schwarz inequality we have

$$\mathbb{P}(A|\mathcal{F}_t)^2 = (\mathbb{E}[1_A|\mathcal{F}_t])^2 \le \mathbb{E}[1_A Q_{t+\delta}|\mathcal{F}_t]\mathbb{E}[Q_{t+\delta}^{-1}|\mathcal{F}_t].$$

We observe that

$$\begin{aligned}\mathbb{E}[Q_{t+\delta}^{-1}|\mathcal{F}_t] &= \mathbb{E}\left[\exp\left(-\int_t^{t+\delta} a_r\mathrm{d}r - \int_t^{t+\delta} b_r\mathrm{d}B_r + \frac{1}{2}\int_t^{t+\delta} |b_r|^2\mathrm{d}r\right) \mid \mathcal{F}_t\right] \\ &\le \exp(L\delta + L^2\delta)\mathbb{E}\left[\exp\left(-\int_t^{t+\delta} b_r\mathrm{d}B_r - \frac{1}{2}\int_t^{t+\delta} |b_r|^2\mathrm{d}r\right) \mid \mathcal{F}_t\right] \\ &= \exp(L\delta + L^2\delta).\end{aligned}$$

Let

$$\Delta = \left\{\mathbb{P}({}^1Y_{t+\delta}^{t,x;u^\varepsilon,v^\varepsilon} < W_1(t+\delta, X_{t+\delta}^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon \mid \mathcal{F}_t) > \varepsilon\right\}\Big(=\{\mathbb{P}(A|\mathcal{F}_t) > \varepsilon\}\Big).$$

Then,

$$\begin{aligned}\bar{y}_t &= \frac{\varepsilon}{2}\mathbb{E}[1_A Q_{t+\delta}|\mathcal{F}_t] \\ &\ge \frac{\exp(-L\delta - L^2\delta)\varepsilon}{2}(\mathbb{E}[1_A|\mathcal{F}_t])^2 = \frac{\exp(-L\delta - L^2\delta)\varepsilon}{2}(\mathbb{P}(A|\mathcal{F}_t))^2 \\ &> \frac{\varepsilon^3}{2}C_0 1_\Delta,\end{aligned} \tag{3.27}$$

for $C_0 = \exp(-L\delta - L^2\delta)$, where we use (3.22) in the last inequality. Combining (3.27) with

$$\bar{y}_t = y_t - {}^1Y_t^{t,x;u^\varepsilon,v^\varepsilon} = {}^1G_{t,t+\delta}^{t,x;u^\varepsilon,v^\varepsilon}\left[{}^1Y_{t+\delta}^{t,x;u^\varepsilon,v^\varepsilon} + \frac{\varepsilon}{2}1_A\right] - {}^1G_{t,t+\delta}^{t,x;u^\varepsilon,v^\varepsilon}[{}^1Y_{t+\delta}^{t,x;u^\varepsilon,v^\varepsilon}],$$

we have

$${}^1G_{t,t+\delta}^{t,x;u^\varepsilon,v^\varepsilon}\left[{}^1Y_{t+\delta}^{t,x;u^\varepsilon,v^\varepsilon} + \frac{\varepsilon}{2}1_A\right] > {}^1G_{t,t+\delta}^{t,x;u^\varepsilon,v^\varepsilon}[{}^1Y_{t+\delta}^{t,x;u^\varepsilon,v^\varepsilon}] + \frac{\varepsilon^3}{2}C_0 1_\Delta,$$

and (3.25) then yields

$$J_1(t,x;\widetilde{u},\beta_\varepsilon(\widetilde{u})) > J_1(t,x;\alpha_\varepsilon,\beta_\varepsilon) + \frac{\varepsilon^3}{2}C_0 1_\Delta.$$

We can choose $\varepsilon' \in (0,1)$ sufficiently small such that $\frac{\varepsilon'^3}{2}C_0 > \varepsilon'^4$. (Recall that $\varepsilon' > 0$ has been introduced at the beginning of the proof, assuming that (3.7) does not hold true.) Then this relation is also satisfied by $\varepsilon \in (0,\varepsilon')$: $\frac{\varepsilon^3}{2}C_0 > \varepsilon^4$. Since $\mathbb{P}(\Delta) > 0$, the above inequality contradicts with (3.20) for $\alpha(\cdot) = \widetilde{u}$. The proof is complete. □

We now give the existence theorem of a Nash equilibrium payoff.

Theorem 3.20. *Let Isaacs condition (3.3) hold. Then for all $(t,x) \in [0,T] \times \mathbb{R}^n$, there exists a Nash equilibrium payoff at (t,x).*

Let us admit the following proposition for the moment. We shall give its proof after.

Proposition 3.21. *Under the assumptions of Theorem 3.16, for all $\varepsilon > 0$, there exists $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ independent of $\mathcal{F}_t$ such that, for all $t \le s_1 \le s_2 \le T$, $j = 1,2$,*

$$\mathbb{P}\Big(W_j(s_1, X_{s_1}^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon \le {}^jG_{s_1,s_2}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(s_2, X_{s_2}^{t,x;u^\varepsilon,v^\varepsilon})]\Big) > 1-\varepsilon.$$

Let us begin with the proof of Theorem 3.20.

Proof. By Theorem 3.16 we only have to prove that, for all $\varepsilon > 0$, there exists $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ which satisfies (3.7) and (3.8) for $s \in [t,T]$, $j = 1,2$.

For $\varepsilon > 0$, we consider $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ given by Proposition 3.21, i.e., in particular, $(u^\varepsilon, v^\varepsilon)$ is independent of $\mathcal{F}_t$, and we put $s_1 = s, s_2 = T$ in Proposition 3.21. This yields (3.7). We also observe that the fact that $(u^\varepsilon, v^\varepsilon)$ is independent of $\mathcal{F}_t$ implies that $J_j(t,x;u^\varepsilon,v^\varepsilon)$, $j = 1,2$, are deterministic and $\Big\{(J_1(t,x;u^\varepsilon,v^\varepsilon), J_2(t,x;u^\varepsilon,v^\varepsilon)), \varepsilon > 0\Big\}$ is a bounded sequence. Consequently, we can choose an accumulation point of this sequence, as $\varepsilon \to 0$. Let us denote this point by (e_1,e_2). Obviously, this combined with (3.7) allows to conclude from Theorem 3.16 that (e_1,e_2) is a Nash equilibrium payoff at (t,x). We also refer to the fact that since $(u^\varepsilon, v^\varepsilon)$ is independent of $\mathcal{F}_t$, the conditional probability $\mathbb{P}(\cdot|\mathcal{F}_t)$ of the event $\Big\{W_j(s_1, X_{s_1}^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon \le {}^jG_{s_1,s_2}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(s_2, X_{s_2}^{t,x;u^\varepsilon,v^\varepsilon})]\Big\}$ coincides with its probability. Indeed, also $\Big\{W_j(s_1, X_{s_1}^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon \le {}^jG_{s_1,s_2}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(s_2, X_{s_2}^{t,x;u^\varepsilon,v^\varepsilon})]\Big\}$ is independent of $\mathcal{F}_t$. The proof is complete. □

Before we present the proof of the above proposition, we give the following lemmas, which will be needed.

Lemma 3.22. *For all $\varepsilon > 0$, and all $\delta \in [0, T-t]$ and $x \in \mathbb{R}^n$, there exists $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ independent of $\mathcal{F}_t$, such that*

$$W_1(t,x) - \varepsilon \le {}^1G_{t,t+\delta}^{t,x;u^\varepsilon,v^\varepsilon}[W_1(t+\delta, X_{t+\delta}^{t,x;u^\varepsilon,v^\varepsilon})], \quad \mathbb{P}\text{-a.s.},$$

and

$$W_2(t,x) - \varepsilon \le {}^2G_{t,t+\delta}^{t,x;u^\varepsilon,v^\varepsilon}[W_2(t+\delta, X_{t+\delta}^{t,x;u^\varepsilon,v^\varepsilon})], \quad \mathbb{P}\text{-a.s.}$$

Proof. Let $\mathbb{F}^t = (\mathcal{F}_s^t)_{s\in[t,T]}$ denote the filtration generated by $(B_s - B_t)_{s\in[t,T]}$ and augmented by the $\mathbb{P}$-null sets. By $\mathcal{U}_{s,T}^t$ (resp., $\mathcal{V}_{s,T}^t$) we denote the set of $\mathbb{F}^t$-adapted processes $\{u_r\}_{r\in[s,T]}$ (resp., $\{v_r\}_{r\in[s,T]}$) taking their values in U (resp., V). Moreover, let $\mathcal{A}_{s,T}^t$ (resp., $\mathcal{B}_{s,T}^t$) denote the set of NAD strategies which map from $\mathcal{V}_{s,T}^t$ into $\mathcal{U}_{s,T}^t$ (resp., $\mathcal{U}_{s,T}^t$ into $\mathcal{V}_{s,T}^t$). With this setting we replace the framework of SDEs driven by a Brownian motion $B = (B_s)_{s\in[0,T]}$ by that of SDEs driven by a Brownian motion $(B_s - B_t)_{s\in[t,T]}$. We also translate the above arguments from the framework of SDEs to the associated BSDEs. Then, proceeding in the same manner as above, but now in our new framework, we have the Isaacs condition, for $j = 1, 2,\ s \in [t, T]$,

$$\sup_{u\in U}\inf_{v\in V}\left\{\frac{1}{2}\mathrm{tr}(\sigma\sigma^T(s,x,u,v)A) + \langle p, b(s,x,u,v)\rangle + f_j(s,x,y,p^T\sigma(s,x,u,v),u,v)\right\}$$
$$= \inf_{v\in V}\sup_{u\in U}\left\{\frac{1}{2}\mathrm{tr}(\sigma\sigma^T(s,x,u,v)A) + \langle p, b(s,x,u,v)\rangle + f_j(s,x,y,p^T\sigma(s,x,u,v),u,v)\right\}$$

for the associated value functionals

$$\widetilde{W}_1(s,x) = \operatorname*{esssup}_{\alpha\in\mathcal{A}_{s,T}^t}\operatorname*{essinf}_{\beta\in\mathcal{B}_{s,T}^t} J_1(s,x;\alpha,\beta) = \operatorname*{essinf}_{\beta\in\mathcal{B}_{s,T}^t}\operatorname*{esssup}_{\alpha\in\mathcal{A}_{s,T}^t} J_1(s,x;\alpha,\beta),$$

and

$$\widetilde{W}_2(s,x) = \operatorname*{essinf}_{\alpha\in\mathcal{A}_{s,T}^t}\operatorname*{esssup}_{\beta\in\mathcal{B}_{s,T}^t} J_2(s,x;\alpha,\beta) = \operatorname*{esssup}_{\beta\in\mathcal{B}_{s,T}^t}\operatorname*{essinf}_{\alpha\in\mathcal{A}_{s,T}^t} J_2(s,x;\alpha,\beta),$$

$(s,x) \in [t,T]\times\mathbb{R}^n$.

For $j = 1, 2$, from [2] we know that W_j restricted to $[t,T]\times\mathbb{R}^n$ and $\widetilde{W}_j$ are inside the class of continuous functions with at most polynomial growth and the unique viscosity solutions of the same Hamilton–Jacobi–Bellman–Isaacs equation. Consequently, they coincide

$$\widetilde{W}_j(s,x) = W_j(s,x), \quad (s,x)\in[t,T]\times\mathbb{R}^n,\ j = 1,2.$$

From the dynamic programming principle for $\widetilde{W}_j$ and by observing that $\mathcal{V}_{t,T}^t \subset \mathcal{B}_{t,T}^t$ we have

$$W_1(t,x) = \widetilde{W}_1(t,x) = \operatorname*{esssup}_{\alpha\in\mathcal{A}_{t,T}^t}\operatorname*{essinf}_{\beta\in\mathcal{B}_{t,T}^t}{}^1G_{t,t+\delta}^{t,x;\alpha,\beta}[W_1(t+\delta, X_{t+\delta}^{t,x;\alpha,\beta})]$$
$$\le \operatorname*{esssup}_{\alpha\in\mathcal{A}_{t,T}^t}\operatorname*{essinf}_{v\in\mathcal{V}_{t,T}^t}{}^1G_{t,t+\delta}^{t,x;\alpha(v),v}[W_1(t+\delta, X_{t+\delta}^{t,x;\alpha(v),v})].$$

Consequently, for $\varepsilon > 0$ and $\delta > 0$, there exists $\alpha_\varepsilon \in \mathcal{A}_{t,T}^t$ such that, for all $v \in \mathcal{V}_{t,T}^t$,

$$W_1(t,x) - \varepsilon \le {}^1G_{t,t+\delta}^{t,x;\alpha_\varepsilon(v),v}[W_1(t+\delta, X_{t+\delta}^{t,x;\alpha_\varepsilon(v),v})], \quad \mathbb{P}\text{-a.s.}$$

The symmetric argument allows to show that the existence of $\beta_\varepsilon \in \mathcal{B}_{t,T}^t$ such that, for all $u \in \mathcal{U}_{t,T}^t$,

$$W_2(t,x) - \varepsilon \le {}^2G_{t,t+\delta}^{t,x;u,\beta_\varepsilon(u)}[W_2(t+\delta, X_{t+\delta}^{t,x;u,\beta_\varepsilon(u)})], \quad \mathbb{P}\text{-a.s.}$$

In the same way as shown in Lemma 3.5, we get the existence of $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}^t_{t,T} \times \mathcal{V}^t_{t,T}$ such that

$$\alpha_\varepsilon(v^\varepsilon) = u^\varepsilon, \qquad \beta_\varepsilon(u^\varepsilon) = v^\varepsilon.$$

Therefore, we have

$$W_1(t,x) - \varepsilon \leq {}^1G^{t,x;u^\varepsilon,v^\varepsilon}_{t,t+\delta}[W_1(t+\delta, X^{t,x;u^\varepsilon,v^\varepsilon}_{t+\delta})],$$

and

$$W_2(t,x) - \varepsilon \leq {}^2G^{t,x;u^\varepsilon,v^\varepsilon}_{t,t+\delta}[W_2(t+\delta, X^{t,x;u^\varepsilon,v^\varepsilon}_{t+\delta})].$$

The proof is complete. □

We also need the following lemma.

Lemma 3.23. *Let $n \geq 1$ and let us fix some partition $t = t_0 < t_1 < \cdots < t_n = T$ of the interval $[t, T]$. Then, for all $\varepsilon > 0$, there exists $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ independent of $\mathcal{F}_t$, such that, for all $i = 0, \ldots, n-1$,*

$$W_1(t_i, X^{t,x;u^\varepsilon,v^\varepsilon}_{t_i}) - \varepsilon \leq {}^1G^{t,x;u^\varepsilon,v^\varepsilon}_{t_i,t_{i+1}}[W_1(t_{i+1}, X^{t,x;u^\varepsilon,v^\varepsilon}_{t_{i+1}})], \quad \mathbb{P}\text{-a.s.},$$

and

$$W_2(t_i, X^{t,x;u^\varepsilon,v^\varepsilon}_{t_i}) - \varepsilon \leq {}^2G^{t,x;u^\varepsilon,v^\varepsilon}_{t_i,t_{i+1}}[W_2(t_{i+1}, X^{t,x;u^\varepsilon,v^\varepsilon}_{t_{i+1}})], \quad \mathbb{P}\text{-a.s.}$$

Proof. We shall give the proof by induction. By the above lemma, it is obvious for $i = 0$. We now assume that $(u^\varepsilon, v^\varepsilon)$ independent of $\mathcal{F}_t$, is constructed on the interval $[t, t_i)$ and we shall define it on $[t_i, t_{i+1})$. From the above lemma it follows that, for all $y \in \mathbb{R}^n$, there exists $(u^y, v^y) \in \mathcal{U}_{t_i,T} \times \mathcal{V}_{t_i,T}$ independent of $\mathcal{F}_t$, such that,

$$W_j(t_i, y) - \frac{\varepsilon}{2} \leq {}^jG^{t_i,y;u^y,v^y}_{t_i,t_{i+1}}[W_j(t_{i+1}, X^{t_i,y;u^y,v^y}_{t_{i+1}})], \quad \mathbb{P}\text{-a.s}, \ j = 1, 2. \tag{3.28}$$

Let us fix arbitrarily $j = 1, 2$. Moreover, for $y, z \in \mathbb{R}^n$ and $s \in [t_i, t_{i+1}]$, we put

$$y^1_s = {}^jG^{t_i,y;u^y,v^y}_{s,t_{i+1}}[W_j(t_{i+1}, X^{t_i,y;u^y,v^y}_{t_{i+1}})], \quad \text{and} \quad y^2_s = {}^jG^{t_i,z;u^y,v^y}_{s,t_{i+1}}[W_j(t_{i+1}, X^{t_i,z;u^y,v^y}_{t_{i+1}})],$$

and we consider the BSDEs:

$$y^1_s = W_j(t_{i+1}, X^{t_i,y;u^y,v^y}_{t_{i+1}}) + \int_s^{t_{i+1}} f_j(r, X^{t_i,y;u^y,v^y}_r, y^1_r, z^1_r, u^y_r, v^y_r)\mathrm{d}r - \int_s^{t_{i+1}} z^1_r \mathrm{d}B_r,$$
$$s \in [t_i, t_{i+1}],$$

and

$$y^2_s = W_j(t_{i+1}, X^{t_i,z;u^y,v^y}_{t_{i+1}}) + \int_s^{t_{i+1}} f_j(r, X^{t_i,z;u^y,v^y}_r, y^2_r, z^2_r, u^y_r, v^y_r)\mathrm{d}r - \int_s^{t_{i+1}} z^2_r \mathrm{d}B_r,$$
$$s \in [t_i, t_{i+1}].$$

By virtue of Lemmas 2.3, 3.1 and 3.9 we have

$$\begin{aligned}&|{}^jG^{t_i,y;u^y,v^y}_{t_i,t_{i+1}}[W_j(t_{i+1}, X^{t_i,y;u^y,v^y}_{t_{i+1}})] - {}^jG^{t_i,z;u^y,v^y}_{t_i,t_{i+1}}[W_j(t_{i+1}, X^{t_i,z;u^y,v^y}_{t_{i+1}})]|^2\\ &\quad \leq C\mathbb{E}[|W_j(t_{i+1}, X^{t_i,y;u^y,v^y}_{t_{i+1}}) - W_j(t_{i+1}, X^{t_i,z;u^y,v^y}_{t_{i+1}})|^2|\mathcal{F}_{t_i}]\end{aligned}$$

$$
\begin{aligned}
&+C\mathbb{E}\left[\int_{t_i}^{t_{i+1}} \Big| f_j(r, X_r^{t_i,y;u^y,v^y}, y_r^1, z_r^1, u_r^y, v_r^y) \right.\\
&\left. -\ f_j(r, X_r^{t_i,z;u^y,v^y}, y_r^1, z_r^1, u_r^y, v_r^y)\Big|^2 \mathrm{d}r \Big| \mathcal{F}_{t_i}\right]\\
&\le C\mathbb{E}\left[\left|X_{t_{i+1}}^{t_i,y;u^y,v^y} - X_{t_{i+1}}^{t_i,z;u^y,v^y}\right|^2 \Big| \mathcal{F}_{t_i}\right] + C\mathbb{E}\left[\int_{t_i}^{t_{i+1}} \left|X_r^{t_i,y;u^y,v^y} - X_r^{t_i,z;u^y,v^y}\right|^2 \mathrm{d}r \Big| \mathcal{F}_{t_i}\right]\\
&\le C|y-z|^2.
\end{aligned}
$$

Therefore, by the above inequality, Lemma 3.9 and (3.28)

$$
\begin{aligned}
W_j(t_i, z) - \varepsilon &\le W_j(t_i, y) - \varepsilon + C|y-z|\\
&\le {}^jG_{t_i,t_{i+1}}^{t_i,y;u^y,v^y}[W_j(t_{i+1}, X_{t_{i+1}}^{t_i,y;u^y,v^y})] - \frac{\varepsilon}{2} + C|y-z|\\
&\le {}^jG_{t_i,t_{i+1}}^{t_i,z;u^y,v^y}[W_j(t_{i+1}, X_{t_{i+1}}^{t_i,z;u^y,v^y})] - \frac{\varepsilon}{2} + C|y-z|\\
&\le {}^jG_{t_i,t_{i+1}}^{t_i,z;u^y,v^y}[W_j(t_{i+1}, X_{t_{i+1}}^{t_i,z;u^y,v^y})], \quad \mathbb{P}\text{-a.s.},
\end{aligned}
$$

for $C|y-z| \le \frac{\varepsilon}{2}$.

Let $\{O_i\}_{i\ge 1} \subset \mathcal{B}(\mathbb{R}^n)$ be a partition of $\mathbb{R}^n$ with $\mathrm{diam}(O_i) < \frac{\varepsilon}{2C}$ and let $y_l \in O_l$. Then, for $z \in O_l$,

$$
W_j(t_i, z) - \varepsilon \le {}^jG_{t_i,t_{i+1}}^{t_i,z;u^{y_l},v^{y_l}}[W_j(t_{i+1}, X_{t_{i+1}}^{t_i,z;u^{y_l},v^{y_l}})], \quad \mathbb{P}\text{-a.s.}, \tag{3.29}
$$

and we define

$$
u^\varepsilon = \sum_{l\ge 1} 1_{O_l}(X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon}) u^{y_l}, \qquad v^\varepsilon = \sum_{l\ge 1} 1_{O_l}(X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon}) v^{y_l}.
$$

Therefore, we have

$$
\begin{aligned}
&{}^jG_{t_i,t_{i+1}}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_{i+1}, X_{t_{i+1}}^{t,x;u^\varepsilon,v^\varepsilon})]\\
&= {}^jG_{t_i,t_{i+1}}^{t_i,X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon};u^\varepsilon,v^\varepsilon}\left[\sum_{l\ge 1} W_j(t_{i+1}, X_{t_{i+1}}^{t_i,X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon};u^\varepsilon,v^\varepsilon}) 1_{O_l}(X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon})\right]\\
&= {}^jG_{t_i,t_{i+1}}^{t_i,X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon};u^\varepsilon,v^\varepsilon}\left[\sum_{l\ge 1} W_j(t_{i+1}, X_{t_{i+1}}^{t_i,X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon};u^{y_l},v^{y_l}}) 1_{O_l}(X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon})\right]\\
&= \sum_{l\ge 1} {}^jG_{t_i,t_{i+1}}^{t_i,X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon};u^{y_l},v^{y_l}}[W_j(t_{i+1}, X_{t_{i+1}}^{t_i,X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon};u^{y_l},v^{y_l}})] 1_{O_l}(X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon}).
\end{aligned}
$$

The latter relation follows from the uniqueness of solutions of BSDEs. From (3.29) it follows that

$$
\begin{aligned}
{}^jG_{t_i,t_{i+1}}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_{i+1}, X_{t_{i+1}}^{t,x;u^\varepsilon,v^\varepsilon})] &\ge \sum_{l\ge 1}[W_j(t_i, X_{t_i}^{t,x;u^{y_l},v^{y_l}}) - \varepsilon] 1_{O_l}(X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon})\\
&= \sum_{l\ge 1} W_j(t_i, X_{t_i}^{t,x;u^{y_l},v^{y_l}}) 1_{O_l}(X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon\\
&= W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon.
\end{aligned}
$$

The proof is complete. $\square$

Finally, we give the proof of Proposition 3.21.

Proof. Let $t = t_0 < t_1 < \cdots < t_n = T$ be a partition of $[t, T]$, and $\tau = \sup_i(t_{i+1} - t_i)$. From Lemma 3.9 it follows that, for all $j = 1, 2$, $0 \le k \le n$, $s \in [t_k, t_{k+1})$ and $(u, v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$,

$$\begin{aligned}
&\mathbb{E}[|W_j(t_k, X_{t_k}^{t,x;u,v}) - W_j(s, X_s^{t,x;u,v})|^2] \\
&\quad \le 2\mathbb{E}[|W_j(t_k, X_{t_k}^{t,x;u,v}) - W_j(s, X_{t_k}^{t,x;u,v})|^2] \\
&\qquad + 2\mathbb{E}[|W_j(s, X_{t_k}^{t,x;u,v}) - W_j(s, X_s^{t,x;u,v})|^2] \\
&\quad \le C|s - t_k|(1 + \mathbb{E}[|X_{t_k}^{t,x;u,v}|^2]) + C\mathbb{E}[|X_{t_k}^{t,x;u,v} - X_s^{t,x;u,v}|^2] \\
&\quad \le C\tau.
\end{aligned} \tag{3.30}$$

Here and after C is a constant which may be different from line to line.

By virtue of Lemma 3.23 we let $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ be defined as in Lemma 3.23 for $\varepsilon = \varepsilon_0$, where $\varepsilon_0 > 0$ will be specified later. Then, we have for all i, $0 \le i \le n$,

$$W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon_0 \le {}^j G_{t_i,t_{i+1}}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_{i+1}, X_{t_{i+1}}^{t,x;u^\varepsilon,v^\varepsilon})], \quad \mathbb{P}\text{-a.s.}$$

For $t \le s_1 \le s_2 \le T$, we suppose, without loss of generality, that $t_{i-1} \le s_1 \le t_i$ and $t_k \le s_2 \le t_{k+1}$, for some $1 \le i < k \le n-1$. Therefore, by Lemmas 2.2 and 2.3 we have

$$\begin{aligned}
{}^j G_{t_i,t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_{k+1}, X_{t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon})] &= {}^j G_{t_i,t_k}^{t,x;u^\varepsilon,v^\varepsilon}[{}^j G_{t_k,t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_{k+1}, X_{t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon})]] \\
&\ge {}^j G_{t_i,t_k}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_k, X_{t_k}^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon_0] \\
&\ge {}^j G_{t_i,t_k}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_k, X_{t_k}^{t,x;u^\varepsilon,v^\varepsilon})] - C\varepsilon_0 \\
&\ge \cdots \ge {}^j G_{t_i,t_{i+1}}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_{i+1}, X_{t_{i+1}}^{t,x;u^\varepsilon,v^\varepsilon})] - C(k-i)\varepsilon_0 \\
&\ge W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon}) - C(k-i+1)\varepsilon_0
\end{aligned}$$

and the above inequality yields

$$\begin{aligned}
{}^j G_{s_1,t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_{k+1}, X_{t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon})] &= {}^j G_{s_1,t_i}^{t,x;u^\varepsilon,v^\varepsilon}[{}^j G_{t_i,t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_{k+1}, X_{t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon})]] \\
&\ge {}^j G_{s_1,t_i}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon}) - C(k-i+1)\varepsilon_0] \\
&\ge {}^j G_{s_1,t_i}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon})] - C(k-i+1)\varepsilon_0 \\
&\ge {}^j G_{s_1,t_i}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon})] - \frac{\varepsilon}{2},
\end{aligned}$$

where we put $\varepsilon_0 = \frac{\varepsilon}{2Cn}$. Let us put

$$\begin{aligned}
I_1 &= {}^j G_{s_1,t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_{k+1}, X_{t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon})] - {}^j G_{s_1,t_i}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon})] + \frac{\varepsilon}{2} \ge 0, \\
I_2 &= {}^j G_{s_1,s_2}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(s_2, X_{s_2}^{t,x;u^\varepsilon,v^\varepsilon})] - W_j(s_1, X_{s_1}^{t,x;u^\varepsilon,v^\varepsilon}) + \frac{\varepsilon}{2}.
\end{aligned} \tag{3.31}$$

We assert that

$$\mathbb{E}[|I_1 - I_2|^2] \le C\tau.$$

Indeed, setting

$$y_s = {}^j G_{s,t_i}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon})], \quad s \in [s_1, t_i],$$

we have the associated BSDEs:

$$y_s = W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon}) + \int_s^{t_i} f_j(r, X_r^{t,x;u^\varepsilon,v^\varepsilon}, y_r, z_r, u_r^\varepsilon, v_r^\varepsilon)\mathrm{d}r - \int_s^{t_i} z_r \mathrm{d}B_r,$$
$$s \in [s_1, t_i].$$

On the other hand, putting

$$y_s' = W_j(s_1, X_{s_1}^{t,x;u^\varepsilon,v^\varepsilon}), \quad s \in [s_1, t_i],$$

we have by Lemma 2.3

$$\begin{aligned}
&|{}^jG_{s_1,t_i}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon})] - W_j(s_1, X_{s_1}^{t,x;u^\varepsilon,v^\varepsilon})|^2 \\
&\quad \le C\mathbb{E}[|W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon}) - W_j(s_1, X_{s_1}^{t,x;u^\varepsilon,v^\varepsilon})|^2|\mathcal{F}_{s_1}] \\
&\qquad + C\mathbb{E}\left[\int_{s_1}^{t_i} |f_j(r, X_r^{t,x;u^\varepsilon,v^\varepsilon}, y_r, z_r, u_r^\varepsilon, v_r^\varepsilon)|^2 \mathrm{d}r|\mathcal{F}_{s_1}\right].
\end{aligned}$$

Therefore, from the boundedness of f_j and the independence of $\mathcal{F}_t$ of $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$,

$$\begin{aligned}
&\mathbb{E}[|{}^jG_{s_1,t_i}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon})] - W_j(s_1, X_{s_1}^{t,x;u^\varepsilon,v^\varepsilon})|^2|\mathcal{F}_t] \\
&\quad \le C\mathbb{E}[|W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon}) - W_j(s_1, X_{s_1}^{t,x;u^\varepsilon,v^\varepsilon})|^2|\mathcal{F}_t] + C(t_i - s_1) \\
&\quad = C\mathbb{E}[|W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon}) - W_j(s_1, X_{s_1}^{t,x;u^\varepsilon,v^\varepsilon})|^2] + C(t_i - s_1).
\end{aligned}$$

From (3.30) it follows that

$$\mathbb{E}[|{}^jG_{s_1,t_i}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon})] - W_j(s_1, X_{s_1}^{t,x;u^\varepsilon,v^\varepsilon})|^2] \le C\tau. \tag{3.32}$$

By a similar argument we have

$$\mathbb{E}[|{}^jG_{s_2,t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_{k+1}, X_{t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon})] - W_j(s_2, X_{s_2}^{t,x;u^\varepsilon,v^\varepsilon})|^2] \le C\tau. \tag{3.33}$$

For $s \in [s_1, s_2]$ we let

$$y_s^1 = {}^jG_{s,t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_{k+1}, X_{t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon})] = {}^jG_{s,s_2}^{t,x;u^\varepsilon,v^\varepsilon}[{}^jG_{s_2,t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_{k+1}, X_{t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon})]],$$

and

$$y_s^2 = {}^jG_{s,s_2}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(s_2, X_{s_2}^{t,x;u^\varepsilon,v^\varepsilon})],$$

and we consider the associated BSDEs:

$$\begin{aligned}
y_s^1 &= {}^jG_{s_2,t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_{k+1}, X_{t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon})] + \int_s^{s_2} f_j(r, X_r^{t,x;u^\varepsilon,v^\varepsilon}, y_r^1, z_r^1, u_r^\varepsilon, v_r^\varepsilon)\mathrm{d}r \\
&\quad - \int_s^{s_2} z_r^1 \mathrm{d}B_r, \quad s \in [s_1, s_2],
\end{aligned}$$

and

$$y_s^2 = W_j(s_2, X_{s_2}^{t,x;u^\varepsilon,v^\varepsilon}) + \int_s^{s_2} f_j(r, X_r^{t,x;u^\varepsilon,v^\varepsilon}, y_r^2, z_r^2, u_r^\varepsilon, v_r^\varepsilon)\mathrm{d}r - \int_s^{s_2} z_r^2 \mathrm{d}B_r,$$
$$s \in [s_1, s_2].$$

By virtue of Lemma 2.3 we have

$$\begin{aligned}&|{}^{j}G^{t,x;u^\varepsilon,v^\varepsilon}_{s_1,t_{k+1}}[W_j(t_{k+1},X^{t,x;u^\varepsilon,v^\varepsilon}_{t_{k+1}})]-{}^{j}G^{t,x;u^\varepsilon,v^\varepsilon}_{s_1,s_2}[W_j(s_2,X^{t,x;u^\varepsilon,v^\varepsilon}_{s_2})]|^2\\&\quad\le C\mathbb{E}[|{}^{j}G^{t,x;u^\varepsilon,v^\varepsilon}_{s_2,t_{k+1}}[W_j(t_{k+1},X^{t,x;u^\varepsilon,v^\varepsilon}_{t_{k+1}})]-W_j(s_2,X^{t,x;u^\varepsilon,v^\varepsilon}_{s_2})|^2|\mathcal{F}_{s_1}].\end{aligned}$$

Consequently, from (3.33) it follows that

$$\mathbb{E}[|{}^{j}G^{t,x;u^\varepsilon,v^\varepsilon}_{s_1,t_{k+1}}[W_j(t_{k+1},X^{t,x;u^\varepsilon,v^\varepsilon}_{t_{k+1}})]-{}^{j}G^{t,x;u^\varepsilon,v^\varepsilon}_{s_1,s_2}[W_j(s_2,X^{t,x;u^\varepsilon,v^\varepsilon}_{s_2})]|^2]\le C\tau.$$

By the above inequality and (3.32) we get

$$\mathbb{E}[|I_1-I_2|^2]\le C\tau.$$

Consequently,

$$\mathbb{P}\left(I_2\le-\frac{\varepsilon}{2}\right)\le\mathbb{P}\left(|I_1-I_2|\ge\frac{\varepsilon}{2}\right)\le\frac{4\mathbb{E}[|I_1-I_2|^2]}{\varepsilon^2}\le\frac{4C\tau}{\varepsilon^2}\le\varepsilon,$$

where we choose $\tau\le\frac{\varepsilon^3}{4C}$, and from (3.31) it follows that

$$\mathbb{P}\Big(W_j(s_1,X^{t,x;u^\varepsilon,v^\varepsilon}_{s_1})-\varepsilon\le{}^{j}G^{t,x;u^\varepsilon,v^\varepsilon}_{s_1,s_2}[W_j(s_2,X^{t,x;u^\varepsilon,v^\varepsilon}_{s_2})]\Big)\ge1-\varepsilon.$$

The proof is complete. $\square$

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