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# On Pappus' configuration in non-commutative projective geometry

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## Abstract

In non-commutative projective geometry there exist Pappus' configurations whose diagonal points are not collinear. In this paper, we consider two lines  $r, s$  in a projective plane and the points  $A, B$  on  $r$ ,  $A', B', C'$  on  $s$ , and we investigate which points  $X$  on  $r$  lead to collinear diagonal points in the corresponding Pappus' configuration. A geometric interpretation of this result is given, showing that these are exactly all the fixed points of a projectivity of the line  $r$ . Finally, we show that the system of fixed points of a large class of projectivities of the line  $r$  may be considered as the set of points  $X$  on  $r$  such that the diagonal points of a suitable Pappus' configuration defined by  $X$  and other points, are collinear. © 2001 Elsevier Science Inc. All rights reserved.

**Keywords:** Pappus' configuration; Diagonal points; Eigenvalue; Eigenvector; Perspectivity; Projectivity

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## 1. Historical notes

Pappus' theorem extends in geometry from third century A.D. to the present. Its history begins with the famous work *Mathematical Collection* of the Hellenistic mathematician Pappus of Alexandria within the tradition of Greek geometry. Pappus' work is about Euclidean geometry, so he had to distinguish the case of parallel lines (Proposition 138 of *Collection*) from the case of incident lines (Proposition 139).

Later, this theorem will play a central role in modern projective geometry. In 1640, Blaise Pascal, in his work *Essays pour les coniques*, obtained a result about a hexagon inscribed in a conic that generalizes Pappus' theorem. Pascal's result is proved using projective methods, in particular, using Desargues' idea of points at

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infinity. Further, Pappus' theorem plays a decisive role in the foundation of Euclidean and projective geometry. In his book of 1899, the famous *Grundlagen der Geometrie*, David Hilbert gave an axiomatization of Euclidean geometry based on modern set theory. Following von Staudt's ideas Hilbert showed that it is possible to define an algebraic system whose properties depend on the axioms chosen for the geometry. In particular, he proved that Desargues' theorem holds if and only if the algebraic system is a division ring and that Pappus' theorem is true if and only if the division ring is a commutative field. With the aim of showing the independence of Pappus' theorem from the other axioms of geometry he was able to construct a geometrical structure based on a new non-commutative division ring whose elements are formal Laurent series.

Hilbert's spirit in the foundation of geometry fascinated Veblen's and Young's minds, and in 1910 they gave an axiomatic treatment of projective geometry in the book *Projective geometry*. In the introduction of this book they write:

The starting point of any strictly logical treatment of geometry (and indeed of any branch of mathematics) must then be a set of undefined elements and relations, and a set of unproved propositions involving them; and from these all other propositions (theorems) are to be derived by the methods of formal logic.

They accepted Hilbert's point of view, refusing the self-evidence of the unproved propositions that they prefer to call *assumptions* rather than axioms or postulates. They introduced the following assumption in the foundation of projective geometry: *If a line intersects two sides of a triangle, then the line intersects the third side*, nowadays known as *Veblen–Young axiom*. Veblen and Young also proved that the commutativity of the coordinate system corresponding to a projective space is equivalent to the so-called *Fundamental theorem of projective geometry*, which says: *A projectivity between one-dimensional primitive forms is uniquely determined when three pairs of homologous elements are given* [5, p. 95], and they noted that this is equivalent to the theorem of Pappus.

## 2. Introduction

Let  $\pi = PG(2, K)$  be a (left) projective plane over a division ring  $K$ . Let  $r, s$  be distinct lines of  $\pi$ ,  $A, B$  points on  $r$  and  $A', B', C'$  points on  $s$  such that  $(A', B', A, B)$  is a projective frame of  $\pi$ , that is,  $A' = \langle a' \rangle$ ,  $B' = \langle b' \rangle$ ,  $A = \langle a \rangle$ ,  $B = \langle b \rangle$ , with  $b = a' + b' + a$ . Any point  $C'$  on  $s$  distinct from  $A'$  and  $B'$  has, with respect to the basis  $(a', b', a)$ , projective coordinates  $(y_1, y_2, 0)$ , with  $y_1 \neq 0$  and  $y_2 \neq 0$ , that is,  $C' = y_1 a' + y_2 b'$ . Further, any point  $X$  on  $r$  has projective coordinates  $(x_1, x_1, x_2)$  (see Fig. 1).

Consider now in  $\pi$  the perspectivities

$$\omega_1 : r \rightarrow A \vee C' \quad \text{with centre } A',$$

$$\omega_2 : A \vee C' \rightarrow B \vee C' \quad \text{with centre } L = (A \vee B') \cap (A' \vee B),$$

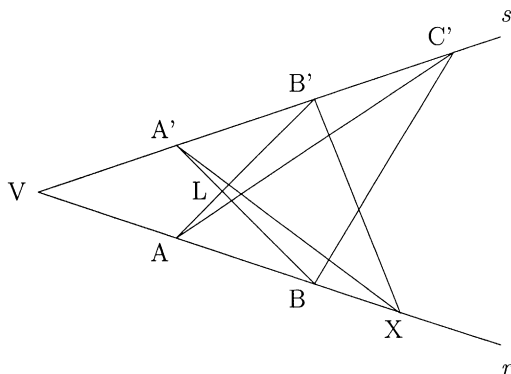


Fig. 1.

$\omega_3 : B \vee C' \rightarrow r$  with centre  $B'$ ,

and the projectivity of the line  $r$

$$\omega_{(A,B;A',B',C')} = \omega_3 \circ \omega_2 \circ \omega_1. \quad (1)$$

It is clear that the fixed points of  $\omega_{(A,B;A',B',C')}$  are exactly all the points  $P$  on  $r$  such that the diagonal points of the Pappus' configuration defined by  $A, B, P$  and  $A', B', C'$  are collinear. Therefore, it suffices to investigate the system of fixed points of the projectivity  $\omega_{(A,B;A',B',C')}$ . Let  $\omega$  be a projectivity of a projective space over a (left)  $K$ -vector space  $V$ , induced by a linear automorphism  $f$  of  $V$ . A point  $P = \langle u \rangle$  is fixed by  $\omega$  if and only if  $f(u) = \lambda u$  for some  $\lambda \neq 0$ .

From this fact we see that in order to find the system of fixed points of  $\omega$  we have to study the system of eigenvalues and eigenvectors of the linear transformation  $f$ . First we note that if  $\lambda$  is an eigenvalue of  $f$ , then every element conjugate to  $\lambda$  is also an eigenvalue of  $f$ , so the set of all eigenvalues of  $f$  is union of conjugacy classes. Next we can obtain an explicit description of the eigenset  $I_\lambda = \{u \in V : f(u) = \lambda u\}$  as shown in the following:

**Proposition 2.1.** *Let  $\lambda$  be an eigenvalue of a linear transformation  $f$ . If  $\{e_1, \dots, e_k\}$  is a maximal linearly independent subset of the eigenset  $I_\lambda$ , we have*

$$I_\lambda = \{a_1 e_1 + \dots + a_k e_k : a_i \in C(\lambda)\},$$

where  $C(\lambda)$  is the centralizer of  $\lambda$  in the ground field  $K$ .

**Proof.** Let  $x \in I_\lambda \setminus \{e_1, \dots, e_k\}$ . Then the set  $\{e_1, \dots, e_k, x\}$  is linearly dependent and  $x$  belongs to the subspace spanned by  $\{e_1, \dots, e_k\}$ . So we have

$$x = a_1 e_1 + \dots + a_k e_k,$$

where  $a_1, \dots, a_k$  are elements of  $K$ . By applying  $f$  we obtain

$$\lambda a_1 e_1 + \dots + \lambda a_k e_k = \lambda x = f(x) = a_1 \lambda e_1 + \dots + a_k \lambda e_k,$$

and from this we deduce  $\lambda a_i = a_i \lambda$  for all  $i$ .

Suppose now  $x = a_1 e_1 + \dots + a_k e_k$  with  $a_i \in C(\lambda)$ . Then we have

$$f(x) = a_1 \lambda e_1 + \dots + a_k \lambda e_k = \lambda(a_1 e_1 + \dots + a_k e_k) = \lambda x,$$

and so  $x$  belongs to  $I_\lambda$ .  $\square$

The eigenset  $I_\lambda$  is not, in general, a subspace of  $V$ . We can now consider the subspace spanned by  $I_\lambda$  and show the connection which exists among subspaces spanned by eigenset corresponding to conjugate eigenvalues.

**Proposition 2.2.** *Let  $\lambda_1$  and  $\lambda_2$  be eigenvalues of a linear transformation  $f$ . Then we have*

$$\lambda_1 \text{ and } \lambda_2 \text{ are conjugate if and only if } \langle I_{\lambda_1} \rangle = \langle I_{\lambda_2} \rangle.$$

**Proof.** Suppose  $\lambda_2 = \beta \lambda_1 \beta^{-1}$  with  $\beta \neq 0$  and let  $\{e_1, \dots, e_t\}$  be a maximal linearly independent subset of  $I_{\lambda_1}$ . Then  $\{\beta e_1, \dots, \beta e_t\}$  is a maximal linearly independent subset of  $I_{\lambda_2}$  and we have

$$\langle I_{\lambda_1} \rangle = \langle e_1, \dots, e_t \rangle = \langle \beta e_1, \dots, \beta e_t \rangle = \langle I_{\lambda_2} \rangle.$$

Suppose now  $\langle I_{\lambda_1} \rangle = \langle I_{\lambda_2} \rangle$ . Let  $\{e_1, \dots, e_t\}$  be a maximal linearly independent subset of  $I_{\lambda_1}$ . This set is a basis of the subspace  $\langle I_{\lambda_1} \rangle = \langle I_{\lambda_2} \rangle$ . So, if  $u$  is a non-zero vector of  $I_{\lambda_2}$ , we have  $u = a_1 e_1 + \dots + a_t e_t$  with  $a_i \in K$ . By applying  $f$  we obtain

$$\lambda_2 a_1 e_1 + \dots + \lambda_2 a_t e_t = \lambda_2 u = f(u) = a_1 \lambda_1 e_1 + \dots + a_t \lambda_1 e_t,$$

and so  $\lambda_2 a_i = a_i \lambda_1$  for all  $i$ . Observe that it is not possible to have  $a_1 = \dots = a_t = 0$ , otherwise we would have  $u = 0$ . We can then suppose  $a_1 \neq 0$ , so we obtain  $\lambda_2 = a_1 \lambda_1 a_1^{-1}$ .  $\square$

**Lemma 2.3.** *Let  $e_1, \dots, e_t$  be a minimal collection of linearly dependent eigenvectors corresponding to the eigenvalues  $\lambda_1, \dots, \lambda_t$  (respectively) of a linear transformation  $f$ . Then the  $\lambda_i$  are all conjugate of each other.*

**Proof.** Consider a dependency (unique upto left scalar multiplication)  $a_1 e_1 + \dots + a_t e_t = 0$ . Applying  $f$  we obtain  $a_1 \lambda_1 e_1 + \dots + a_t \lambda_t e_t = 0$ . So there is a constant  $c$  such that  $a_i \lambda_i = c a_i$  for all  $i$ . This means that all  $\lambda_i$  are conjugate to  $c$ , so they are all conjugate of each other.  $\square$

Next result is about subspaces spanned by eigensets corresponding to non-conjugate eigenvalues.

**Proposition 2.4.** *Let  $\lambda_1, \dots, \lambda_n$  be eigenvalues of a linear transformation  $f$  such that any two of them are not conjugate. Then each one of the subspaces  $\langle I_{\lambda_1} \rangle, \dots, \langle I_{\lambda_n} \rangle$  intersects the span of the others only in the zero vector.*

**Proof.** If  $B_j = \{e_{j1}, \dots, e_{jk_j}\}$  is a maximal linearly independent subset of  $I_{\lambda_j}$  for each  $j \in \{1, \dots, n\}$ , then  $B_j$  is a basis of  $\langle I_{\lambda_j} \rangle$ . Suppose, on the contrary, that there is a non-zero vector  $x$  such that

$$x \in \langle I_{\lambda_i} \rangle \cap \left( \sum_{j \neq i} \langle I_{\lambda_j} \rangle \right).$$

Then we have

$$x = a_{i1}e_{i1} + \dots + a_{ik_i}e_{ik_i} = \sum_{j \neq i} \sum_{k=1}^{k_j} a_{jk}e_{jk}.$$

The values  $a_{i1}, \dots, a_{ik_i}$  cannot be all zero, so the set  $B = B_1 \cup \dots \cup B_n$  is linearly dependent. Let now  $S$  be a minimal linearly dependent subset of  $B$ .  $S$  must contain two eigenvectors  $e_s$  and  $e_t$  corresponding to different eigenvalues  $\lambda_s$  and  $\lambda_t$  (otherwise  $S$  would be independent). Then from the previous lemma  $\lambda_s$  and  $\lambda_t$  are conjugate, a contradiction.  $\square$

Let  $\{C_1, \dots, C_k\}$  be the set of distinct conjugacy classes of eigenvalues of  $f$ , and let  $\lambda_i$  be a representative of  $C_i$  for every  $i = 1, \dots, k$ . Then the system of fixed points of  $\omega$  is given by

$$\bigcup_{i=1}^k \{\langle x \rangle : x \in I_{\lambda_i} \setminus \{0\}\}.$$

Let now  $\mathcal{B}_i = \{e_{i1}, \dots, e_{ik_i}\}$  be a maximal linearly independent subset of  $I_{\lambda_i}$ . Then we have

$$I_{\lambda_i} = \{a_{i1}e_{i1} + \dots + a_{ik_i}e_{ik_i} : a_{ij} \in C(\lambda_i)\}.$$

Hence, the fixed points of  $\omega$  contained in the projective subspace  $\langle I_{\lambda_i} \rangle$  are, with respect to the basis  $\mathcal{B}_i$ , all the points coordinatized by the division subring  $C(\lambda_i)$ , that is, they form a *subgeometry* of  $\langle I_{\lambda_i} \rangle$ .

We can summarize these results as follows:

**Proposition 2.5.** *Let  $\omega$  be a projectivity of a projective space of finite-dimension  $n$  over a division ring  $K$  induced by a linear automorphism  $f$ . If  $C_1 = [\lambda_1], \dots, C_t = [\lambda_t]$  are all the conjugacy classes of the eigenvalues of  $f$  ( $n \geq t$ ) and  $I_{\lambda_1}, \dots, I_{\lambda_t}$  are the eigensets corresponding to  $\lambda_1, \dots, \lambda_t$ , then the system of fixed points of  $\omega$  is the union of all the subgeometries formed by the points of  $\langle I_{\lambda_i} \rangle$  coordinatized by the centralizer of  $\lambda_i$  in  $K$ .*

### 3. Fixed points and Pappus' configuration

If the projective space is a line, then for a projectivity  $\omega$  the following cases can occur:

- (i)  $\text{Fix}(\omega) = \emptyset$ .
- (ii)  $\text{Fix}(\omega)$  is a one-point set.
- (iii)  $\text{Fix}(\omega)$  is a two-points set.
- (iv)  $\text{Fix}(\omega)$  is projective subline.

Since  $V, A, B$  ( $V = r \cap s$ ) are fixed points of  $\omega_{(A,B;A',B',C')}$ , it follows that the system of fixed points of this projectivity is a projective subline of  $r$ .

**Proposition 3.1.** *With the notations of Section 2 the system of fixed points of  $\omega_{(A,B;A',B',C')}$  is the projective subline of the points of  $r$  coordinatized by the centralizer in  $K$  of  $y_2^{-1}y_1$ .*

**Proof.** Let  $X = \langle x_1a' + x_1b' + x_2a \rangle$  be any point of  $r$ . Then we have

$$\begin{aligned}\omega_1(X) &= \langle x_1y_2^{-1}y_1a' + x_1b' + x_2a \rangle, \\ (\omega_2 \circ \omega_1)(X) &= \langle x_1y_2^{-1}y_1a' + (x_1 - x_2 + x_2y_2^{-1}y_1)b' + x_2y_2^{-1}y_1a \rangle, \\ \omega_{(A,B;A',B',C')}(X) &= \langle x_1y_2^{-1}y_1a' + x_1y_2^{-1}y_1b' + x_2y_2^{-1}y_1a \rangle.\end{aligned}$$

Let  $V_2$  be the two-dimensional vector space corresponding to the line  $r$ . Then the automorphism  $f$  of  $V_2$  defined by

$$f : x_1a' + x_1b' + x_2a \in V_2 \rightarrow x_1y_2^{-1}y_1a' + x_1y_2^{-1}y_1b' + x_2y_2^{-1}y_1a \in V_2$$

induces the projectivity  $\omega_{(A,B;A',B',C')}$ . Observe that vectors  $a, b$  of  $V_2$  are a maximal linearly independent subset of eigenvectors of  $f$  corresponding to the eigenvalue  $\lambda = y_2^{-1}y_1$ . Then the corresponding eigenset  $I_\lambda$  of  $f$  is

$$I_\lambda = \{\alpha a + \beta b : \alpha, \beta \in C(\lambda)\},$$

where  $C(\lambda)$  is the centralizer of  $\lambda$  in  $K$ . From this fact it follows that the system of fixed points of  $\omega_{(A,B;A',B',C')}$  is

$$\text{Fix}(\omega_{(A,B;A',B',C')}) = \{\langle \alpha a + \beta b \rangle : (\alpha, \beta) \neq (0, 0), \alpha, \beta \in C(\lambda)\},$$

that is, it coincides with the set of all the points on  $r$  coordinatized by the centralizer of  $\lambda = y_2^{-1}y_1$ .  $\square$

**Corollary 3.2.** *The set of all points  $X$  on  $r$ , such that the diagonal points of the Pappus' configuration defined by  $A', B', C'$  and  $A, B, X$  are collinear, is a projective subline of  $r$  coordinatized by a suitable centralizer in  $K$ .*

We will prove now that the system of fixed points of a projectivity of a line  $r$  into itself, with at least three fixed points, is exactly the set of all points  $X$  on  $r$  such that

the diagonal points of a suitable Pappus' configuration defined by  $X$  and other fixed points are collinear.

**Proposition 3.3.** *Let  $r$  be a line of  $\pi$  and  $\varphi$  a projectivity of  $r$  with at least three fixed points. Then for any two fixed points  $A, B$  of  $\varphi$ , there exist a line  $s$  distinct from  $r$ , not containing either  $A$  or  $B$  and three points  $A', B', C'$  on  $s$  satisfying the following property: The set of points  $X$  on  $r$ , such that diagonal points of the Pappus' configuration defined by  $A, B, X$  and  $A', B', C'$  are collinear, is exactly the set of fixed points of  $\varphi$ .*

**Proof.** Let  $f$  be a linear automorphism that induces  $\varphi$ , and let  $A = \langle a \rangle$ ,  $B = \langle b^* \rangle$  two fixed points of  $\varphi$ . Then we have

$$f(a) = \lambda a, \quad f(b^*) = \mu b^*,$$

$\varphi$  has a subline of fixed points, and so  $f$  has only one conjugacy class of eigenvalues. Hence,  $\lambda$  and  $\mu$  are conjugate, that is,  $\lambda = \beta\mu\beta^{-1}$  ( $\beta \neq 0$ ). Consider now the vector  $b = \beta b^*$ . Then we have

$$f(b) = \beta\mu b^* = (\beta\mu\beta^{-1})(\beta b^*) = \lambda b.$$

Let  $B = \langle b \rangle$  and choose projective coordinates of  $B$  in such a way that they determine an eigenvector of  $f$  corresponding to the eigenvalue  $\lambda$ . Then the vectors  $a, b$  are a maximal linearly independent subset of eigenvectors of  $f$  corresponding to the eigenvalue  $\lambda$ , and the system of fixed points of  $\varphi$  is

$$\text{Fix}(\varphi) = \{ \langle \alpha a + \beta b \rangle : (\alpha, \beta) \neq (0, 0), \alpha, \beta \in C(\lambda) \}.$$

Consider now a vector  $a'$  outside the subspace  $\langle a, b \rangle$  and the vector  $b' = b - a - a'$ . From this choice it follows that  $(A' = \langle a' \rangle, B' = \langle b' \rangle, A = \langle a \rangle, B = \langle b \rangle)$  is a projective frame of  $\pi$ . Consider the point  $C' = \langle \lambda a' + b' \rangle$  on the line  $s$  through  $A'$  and  $B'$ . Then by Proposition 3.1 we have

$$\text{Fix}(\omega_{(A, B; A', B', C')}) = \{ \langle \alpha a + \beta b \rangle : (\alpha, \beta) \neq (0, 0), \alpha, \beta \in C(\lambda) \},$$

from which follows  $\text{Fix}(\varphi) = \text{Fix}(\omega_{(A, B; A', B', C')})$  as we wanted.  $\square$

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