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A noncontractible cell-like compactum whose suspension is contractible

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ABSTRACT

We prove that: (1) Every compact metrizable space is weakly homotopy equivalent to a cell-like compactum; and (2) There exists a noncontractible cell-like compactum whose suspension is contractible (this gives an affirmative answer to the Bestvina-Edwards problem).

1. INTRODUCTION

Several years ago, Bestvina and Edwards [3; Problem 677] originated the following interesting question: *Does there exist a noncontractible cell-like compactum whose suspension is contractible?* In our previous paper [2], we constructed a noncontractible cell-like compactum whose *reduced* suspension is a contractible ANR. However, its unreduced suspension turned out to be noncontractible, so the question above remained open. In the present paper we finally provide a solution – by proving that the answer to the Bestvina-Edwards problem is affirmative.

2. PRELIMINARIES

Let T be the following well-known subspace of the plane – the topologist's sine curve:

$$T = \{(a,b) \in \mathbb{R}^2 \mid 0 < a \leq 1,\ b = \sin\frac{1}{a} \text{ or } a = 0,\ -1 \leq b \leq 1\}.$$

Let $t_0 = (0,-1)$ and $t_1 = (1, \sin 1)$. For a space S we shall denote the cone over it by $C(S)$. The points of $C(S)$ are parametrized by symbols $[s, \tau]$ ($s \in S$, $\tau \in [0, 1]$), where every point $[s, 0]$ is indentified with $s \in S$ and every point $[s, 1]$ is indentified with the vertex of the cone, for all $s \in S$.

Lemma 2.1. *For every compact metrizable space X and every point $x_0 \in X$, the compactum*

$$\tilde{X} = (X \times T) \cup C((\{x_0\} \times T) \cup (X \times \{t_0\}))$$

is cell-like.

Proof. Consider T as the intersection $\bigcap_{i=1}^{\infty} T_i$ of a decreasing sequence of contractible finite polyhedra $\{T_i\}_{i \in \mathbb{N}}$. Then clearly

$$\tilde{X} = \bigcap_{i=1}^{\infty} ((X \times T_i) \cup C((\{x_0\} \times T_i) \cup (X \times \{t_0\}))).$$

Since the spaces $(X \times T_i) \cup C((\{x_0\} \times T_i) \cup (X \times \{t_0\}))$ are obviously contractible, $\tilde{X}$ is cell-like. $\square$

Lemma 2.2. *Let X and $\tilde{X}$ be as in Lemma 2.1. Then for every locally connected compact metrizable space Y and every mapping $\varphi : Y \to \tilde{X}$ there exist a subspace $\tilde{X}_\varphi \subset \tilde{X}$, containing $\varphi(Y) \cup (X \times \{t_1\})$ and a retraction $r_\varphi : \tilde{X}_\varphi \to X \times \{t_1\}$ which is a homotopy equivalence (where (T, t_1) is as explained above).*

Proof. Let $Z = X \vee [0, \frac{1}{2})$ be the bouquet of the space X and the half-open interval $[0, \frac{1}{2})$ relative to the base points $x_0 \in X$ and $0 \in [0, \frac{1}{2})$. We can consider $Z \times T$ as a subset of $\tilde{X}$. Suppose that for every $n \in \mathbb{N}$, there exist a point $y_n \in Y$ such that

$$\varphi(y_n) = (z_n, (a_n, b_n)) \in Z \times T \subset \tilde{X} \text{ and } 0 < a_n < \frac{1}{n}.$$

Since Y is metrizable and compact, there exists a subsequence $\{y_{n_i}\}_{i \in \mathbb{N}} \subset Y$, converging to some point $y^* \in Y$. Consider the open set $\varphi^{-1}(\tilde{X} \backslash \{\vartheta\})$, where ϑ is the vertex of the cone

$$C((\{x_0\} \times T) \cup (X \times \{t_0\})).$$

Since Y is locally connected, there exists an open neighborhood U_{y^*} in $\varphi^{-1}(\tilde{X} \backslash \{\vartheta\})$, which is path-connected. The image $\varphi(U_{y^*})$ must then also be path-connected.

However, $\varphi(y^*)$ and $\varphi(y_{n_i})$ cannot be connected by a path in $\tilde{X} \backslash \{\vartheta\}$. So, there exists an index n_0 such that the image $\varphi(Y)$ lies in

$$\tilde{X}_\varphi = \tilde{X} \backslash (Z \times \{(a, b) \in T \mid 0 < a < \frac{1}{n_0}\}).$$

It is now evident that the space $X \times \{t_1\}$ is a strong deformation retract of $\tilde{X}_\varphi$, i.e. there exist a retraction

$$r_\varphi : \tilde{X}_\varphi \to X \times \{t_1\}$$

which is a homotopy equivalence. $\square$

3. MAIN RESULTS

Theorem 3.1. *Every compact metrizable space is weakly homotopy equivalent to a cell-like compactum.*

Proof. According to Lemma 2.1, it suffices to prove that the inclusion $i : X \to \tilde{X}$ defined by $i(x) = (x, t_1) \in \tilde{X}$, for every $x \in X$, is a weak homotopy equivalence. That is, for any finite polyhedron P, the natural mapping of homotopy classes $\mathcal{P} : [P, X] \to [P, \tilde{X}]$ is bijective.

Let $f : P \to X$ be any mapping such that the composition $i \circ f : P \to \tilde{X}$ is a homotopically trivial mapping, i.e. $i \circ f$ can be extended to the cone over P,

$$F : C(P) \to \tilde{X}.$$

Since $C(P)$ is locally connected, it follows by Lemma 2.2 that the following diagram is homotopically commutative:

$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d] & X \arrow[d, hook, "i"] & \\
C(P) \arrow[r, "F"] \arrow[ur, dashed] \arrow[dr, "\tilde{F}"'] & \tilde{X} & X \times \{t_1\} \arrow[l, hook'] \arrow[ul, "p"'] \\
& \tilde{X}_F \arrow[u, hook, "i_F"] \arrow[ur, "r_F"'] &
\end{tikzcd}$$

where p is a projection and r_F is a strong deformation retraction. Therefore f can be extended to the mapping

$$p \circ r_F \circ \tilde{F} : C(P) \to X$$

and thus $\mathcal{P}$ is a monomorphism.

Let now $g : P \to \tilde{X}$ be any mapping. By Lemma 2.2, we obtain the following homotopically commutative diagram:

$$\begin{tikzcd}
& X \arrow[d] & \\
P \arrow[r, "g"] \arrow[ur, dashed] \arrow[dr, "\tilde{g}"'] & \tilde{X} \arrow[r, phantom, "\supset"] & X \times \{t_1\} \arrow[ul, "p"'] \\
& \tilde{X}_g \arrow[u, "i_g"] \arrow[ur, "r_g"'] &
\end{tikzcd}$$

So g is homotopic to $i \circ p \circ r_g \circ \tilde{g}$ and $\mathcal{P}$ is a bijective mapping. $\square$

Theorem 3.2. *Let X be an arbitrary noncontractible acyclic compact ANR. Then $\tilde{X}$ (defined in Lemma 2.1) is a cell-like noncontractible compactum and its suspension $\Sigma\tilde{X}$ is contractible.*

Proof. Since X is a noncontractible acyclic ANR, its fundamental group $\pi_1(X)$ is nontrivial, therefore by Theorem 3.1, $\pi_1(\tilde{X})$ is nontrivial and $\tilde{X}$ is a cell-like noncontractible compactum.

Consider the following subspaces of $\tilde{X}$:

$$A_1 = X \times T, \quad A_2 = C(X \times \{t_0\}),$$
$$A = A_1 \cup A_2, \text{ and } B = C(\{x_0\} \times T).$$

The suspension $\Sigma\tilde{X} = \Sigma(A \cup B)$ is a natural union of the suspensions ΣA and ΣB. Since ΣX is an AR, the segment $\Sigma\{x_0\} \subset \Sigma X$ is a strong deformation retract of ΣX. Let

$$h : (\Sigma X) \times I \to \Sigma X$$

be a homotopy from the identity mapping to the retraction r onto $\Sigma\{x_0\}$.

For every point $t \in T$, we have a natural embedding $\Sigma i_t : \Sigma X \to \Sigma A_1$, induced by the map $i_t : X \to A_1$, defined by $i_t(x) = (x, t)$. Let $H_{11} : (\Sigma A_1) \times I \to \Sigma A_1$ be a homotopy which is uniquely determined by h, such that for every t, the following diagram is commutative:

$$\begin{array}{ccc} (\Sigma X) \times I & \xrightarrow{h} & \Sigma X \\ \Sigma i_t \times \mathrm{id} \downarrow & & \downarrow \Sigma i_t \\ (\Sigma A_1) \times I & \xrightarrow{H_{11}} & \Sigma A_1 \end{array}$$

Since ΣA_2 is an AR, the mapping

$$\Sigma((X \times \{t_0\}) \cup C(\{x_0, t_0\})) \times I \to \Sigma A_2,$$

which is defined as the restriction of H_{11} onto $(\Sigma(X \times \{t_0\})) \times I$, and as the projection

$$(\Sigma C((x_0, t_0))) \times I \to \Sigma C((x_0, t_0))$$

onto the second summand, can be extended to

$$H_{12} : \Sigma A_2 \times I \to \Sigma A_2.$$

The space ΣA is considered as a sum of ΣA_1 and ΣA_2. Homotopies H_{11} and H_{12} coincide on $(\Sigma(A_1 \cap A_2)) \times I$, therefore we have the homotopy

$$H_1 : (\Sigma A) \times I \to \Sigma A.$$

Define $H_2 : (\Sigma B) \times I \to \Sigma B$ as a natural contraction to a point (note that B is a contractible space and so is ΣB).

Now define a homotopy $H : (\Sigma(A \cup B)) \times I \to \Sigma(A \cup B)$ by the following formula:

$$H(z,t) = \begin{cases} H_1(z,2t) & \text{if } z \in A \text{ and } t \in [0,\frac{1}{2}], \\ \{z\} & \text{if } z \in B \text{ and } t \in [0,\frac{1}{2}], \\ H_2(H_1(z,1),2t-1) & \text{if } z \in A \text{ and } t \in [\frac{1}{2},1], \\ H_2(z,2t-1) & \text{if } z \in B \text{ and } t \in [\frac{1}{2},1]. \end{cases}$$

The mapping H is then a homotopy between the identity and the constant mapping, so $\Sigma\tilde{X}$ is indeed a contractible space. □

4. EPILOGUE

There exist 2-dimensional noncontractible acyclic ANR compacta. For instance, let P be a CW complex, constructed according to the following presentation (cf. [1]):

$$\langle a, b \mid b^{-2}aba, b^{-3}a^5 \rangle.$$

Then by Theorems 3.1 and 3.2, $\tilde{P}$ is a 3-dimensional noncontractible cell-like compactum whose suspension is contractible.

Question 4.1. Do there exist 1- or 2-dimensional noncontractible cell-like compacta whose suspensions are contractible?

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