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Composite beams with weak shear connection

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Abstract

In the past, the theories of composite beams with weak shear connection only dealt with the problem of bending on a symmetry plane. In this paper a beam model that also takes into account the other state of stress due to torsion and bending on a plane parallel to the connection is presented, in order to analyze the consequence of a flexible connection in these usual states of deformation. In kinematics it is assumed that each component of the cross-section is rigid in its own plane and only a relative slip can occur at the interface while the contact is preserved. The system of balance equations and relevant boundary conditions is established by the virtual work principle. Such a model is used for analyzing the torsion problem of a closed box section and some numerical results referring to a rectangular box section are presented. © 2001 Published by Elsevier Science Ltd.

Keywords: Composite beams; Interface slip; Weak shear connection; Torsion

1. Introduction

Composite beams are widely used in the field of construction and they are creating growing interest in different engineering sectors. There are many ways to obtain a connection between components made by different materials. If the layers are connected by means of strong adhesives, the connection can be considered as "rigid" in the sense that no displacement discontinuities occur at the interface between components, so that the beam can be modeled as a non-homogeneous beam and the analysis can be based on usual models (see Reddy (1993)). In many other cases involving engineering applications, like steel–concrete beams or a layered wood system connected with nails, the connection is "weak" in the sense that displacement discontinuities occur at the interface. In particular, the connection is often weak in shear, permitting only the relative slip but preserving the contact.

Early studies on beams with weak shear connection consider the linear problem and are due to Newmark (Newmark et al., 1951). Numerous recent papers extend his approach, also consider dynamics (see e.g. Girhammer and Pan (1993)), the case of viscoelastic materials (see e.g. Tarantino and Dezi (1992) and Bradford and Gilbert (1992)) and the non-linear behavior of materials (see e.g. Oehlers and Sved (1995) and

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Nomenclature

$\boldsymbol{a}$	unitary vector on the cross-section plane
A	cross-section area (weighed by E)
A	differential operators of the displacement formulation
b	bi-moment due to external forces
B	bi-moment due to beam stress
$\boldsymbol{c}$	vector of displacement unknowns
C	first inertial moment of the cross-section area (weighed by E)
$\boldsymbol{C}$	vector of assigned displacements for kinematic boundary conditions
D	total stiffness of the connection on the cross-section
D, D^*	compatibility and equilibrium differential operators
$\boldsymbol{d}$	slip at the interface
$\boldsymbol{e}$	unitary vector in the ζ-direction
E	normal Young's modulus
$\boldsymbol{E}$	strain tensor
f	shear stress at the connection
$\boldsymbol{g}$	vector of resultants of external forces
G	shear Young's modulus
$\boldsymbol{G}$	vector of assigned stress resultants for static boundary conditions
$\boldsymbol{h}$	contact forces
H	first inertial moment of the connection stiffness
I	second inertial moment of the cross-section area (weighed by E)
J	contribution of the beam to the DSV torsional stiffness
k	connection stiffness
K	second inertial moment of the connection stiffness
K	constitutive operator relating $\boldsymbol{c}$ to $\boldsymbol{R}$
L	length of the beam
n	resultant of external forces in the axial direction
$\boldsymbol{n}$	outward normal
N	resultant of beam stress in the axial direction
N	differential operator furnishing the generalized displacements
m	resultant moment of external forces
M	resultant moment of beam stress
$\boldsymbol{p}$	forces per unit volume
q	resultant of external forces in transverse directions
$\boldsymbol{R}$	vector of resultants on the cross-section
S	cross-section
$\boldsymbol{S}$	stress tensor
$\boldsymbol{r}$	position of the material points on the cross-section plane
$\boldsymbol{s}$	displacement vector
s	curvilinear co-ordinate of Σ
t	thickness of thin-walled sections
T	stress flux of closed thin-walled sections
v	displacement component in the y-direction
V	volume occupied by the beam
w	displacement component in the ζ-direction
x	co-ordinate on the cross-section plane
y	co-ordinate on the cross-section plane
$\boldsymbol{\gamma}$	shear strain vector
Γ	boundary of the cross-section
Λ	interface line in the cross section
ν	component of resultant of connection stress
μ	component of resultant moment of connection stress
β	resultant of axial connection stress weighed by warping
ε	axial strain
φ	twisting rotation angle
$\boldsymbol{\vartheta}$	rigid rotation of the cross-section
$\boldsymbol{\rho}$	position vector of mean line points
σ	normal stress
Σ	mean line in thin-walled sections
$\boldsymbol{\tau}$	shear stress
ζ	axial co-ordinate
ω	shape function describing warping
Ω	area closed by Σ

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a, b	indexes denoting a component of the cross-section
T	indexes denoting the twisting center

Nguyen et al. (1998)). An extensive list of references on this topic in not in the aim of this work. These studies deal with the plane problem of bending where a symmetric beam undergoes loads lying on the symmetry plane.

The aim of the research developed by the author consists in defining a model for the analysis of three-dimensional problems, where not only bending on the symmetry plane is considered, but also transverse bending and torsion. The torsion problem has a notable relevance in the design of composite box section used in bridge construction, where the external loads induce an important and non-uniform torque moment on the beam and the consequence of such an internal action requires investigation. In particular it is necessary to know the consequence of the connection deformability on the stress distribution and on the structure stiffness. Furthermore the coupling between twisting and transverse displacements (if any) should be analyzed in order to evaluate the transverse state of stress on the connection. The results presented can however be applied to other situations involving beam with weak shear connection.

The problem is approached by assuming a kinematical model where the two components of the cross-section are rigid in their own plane while the axial displacements are obtained as a sum of the bending displacement and a warping function furnishing a state of stress balanced in uniform torsion. Only displacements parallel to the connection plane can be discontinuous while the contact is preserved. This provides a constraint between the component displacements and requires that the mean displacement in the direction which is orthogonal to the connection and rotation around an axis parallel to the beam axis are the same for the two parts. The warping function can be determined by solving two differential problems defined in the two portions of the cross-section and by introducing a jump condition ensuring the continuity of the shear stress at the interface.

The duality relations between resultants of the active stress and descriptors of strain, the eleven equilibrium equations and the relevant boundary conditions are obtained from the virtual work principle. The problem is subsequently approached by assuming the displacements as unknown and a system of six differential equations is finally obtained. In the particular case of symmetric cross-section the system can be split into two separate systems, the former describes problems involving axial stretch and bending on the plane orthogonal to the connection plane (as in the Newmark's theory) and the latter describes problems involving torsion and bending on the plane parallel to the connection plane. No further decompositions are possible and under generic loads twisting and transverse bending of the two cross-section components are coupled.

Finally, a simple but meaningful application is developed. This shows some particular aspects regarding the effects of connection flexibility on torsion and examines a beam with a thin-walled rectangular box section where the weak connection is located at the top of the webs (a classical solution for steel–concrete composite bridge). When the connection stiffness varies the center of twisting moves and the unit warping function changes its shape. Under non-uniform torsion the torque moment can be decomposed into three parts, the first two are related to the derivatives of the twisting angle as in De Saint Venant and Vlasov theories while the third component arises as a consequence of the discontinuities at the interface and is related to the first and third derivatives of the transverse slip between the two parts of the cross-section.

2. Kinematic assumptions

2.1. Displacement field

In the reference configuration the composite beam occupies the cylindrical region $V = S \times [0, L]$ generated by translating its cross-section S with regular boundary Γ along a rectilinear axis, orthogonal to the cross-section. The cross-section is subdivided into two parts "a" and "b" by a rectilinear line Λ describing the position of a continuous connection ($V^{\mathrm{a}} \cup V^{\mathrm{b}} = V$, $S^{\mathrm{a}} \cup S^{\mathrm{b}} = S$, $\Gamma^{\mathrm{a}} \cup \Gamma^{\mathrm{b}} = \Gamma$).

The unit vector $\boldsymbol{e}$ is oriented in the direction of the axis and the two orthogonal unit vectors $\boldsymbol{a}_x$, $\boldsymbol{a}_y$ are parallel to the cross-section plane. The unit vector $\boldsymbol{a}_x$ is parallel to the connection line Λ.

The generic beam point at the reference configuration is located at

$$\boldsymbol{r} + \zeta\boldsymbol{e} = x\boldsymbol{a}_x + y\boldsymbol{a}_y + \zeta\boldsymbol{e}, \quad (\boldsymbol{r}, \zeta) \in V = S \times [0, L] \tag{1}$$

while the connection is located at $y = y_c$ and its points are defined in the set $\Lambda \times [0, L]$.

The starting point consists of assuming that each component of the cross-section is rigid in its own plane. This is substantially verified in thin-walled beams where intermediate diaphragms are usually disposed. As a consequence, the following representation can be adopted for the admissible motion:

$$\boldsymbol{s}(\boldsymbol{r}, \zeta) = \boldsymbol{s}_0^{\mathrm{a}}(\zeta) + \boldsymbol{\vartheta}^{\mathrm{a}}(\zeta) \times \boldsymbol{r} - \boldsymbol{P}_\zeta\boldsymbol{\vartheta}^{\mathrm{a}}(\zeta) \times \boldsymbol{r}_{\mathrm{T}} + \psi(\boldsymbol{r}, \zeta)\boldsymbol{e} \quad \text{on } V^{\mathrm{a}} \tag{2a}$$

$$\boldsymbol{s}(\boldsymbol{r}, \zeta) = \boldsymbol{s}_0^{\mathrm{b}}(\zeta) + \boldsymbol{\vartheta}^{\mathrm{b}}(\zeta) \times \boldsymbol{r} - \boldsymbol{P}_\zeta\boldsymbol{\vartheta}^{\mathrm{b}}(\zeta) \times \boldsymbol{r}_{\mathrm{T}} + \psi(\boldsymbol{r}, \zeta)\boldsymbol{e} \quad \text{on } V^{\mathrm{b}} \tag{2b}$$

The two functions $\boldsymbol{s}_0^{\mathrm{a}}(\zeta) = u_0^{\mathrm{a}}(\zeta)\boldsymbol{a}_x + v_0^{\mathrm{a}}(\zeta)\boldsymbol{a}_y + w_0^{\mathrm{a}}(\zeta)\boldsymbol{e}$ and $\boldsymbol{s}_0^{\mathrm{b}}(\zeta) = u_0^{\mathrm{b}}(\zeta)\boldsymbol{a}_x + v_0^{\mathrm{b}}(\zeta)\boldsymbol{a}_y + w_0^{\mathrm{b}}(\zeta)\boldsymbol{e}$ describe rigid translations of the cross-section at ζ, the first two components of $\boldsymbol{\vartheta}^{\mathrm{a}}$ and $\boldsymbol{\vartheta}^{\mathrm{b}}$ describe its rigid rotations around an axis lying on the cross-section plane and crossing the origin (bending) while the third components describe the rotation around an axis parallel to the beam axis and intersecting the cross-section at $\boldsymbol{r}_{\mathrm{T}}$ (twisting, $\boldsymbol{P}_\zeta = \boldsymbol{e} \otimes \boldsymbol{e}$ is the projector along the axis), finally the two functions ψ^{a} and ψ^{b} describe the axial displacements which are not produced by rigid motions (warpings).

A discontinuity of displacement field can occur at the connection plane. In the theory presented it is assumed that the connection, denoted as shear connection, does not permit a displacement jump in the direction orthogonal to the connection plane and only discontinuities parallel to the x–ζ plane may occur, so that the jump condition can be expressed as follows (see Fig. 1):

$$[\boldsymbol{s}]_{y=y_c} \cdot \boldsymbol{a}_y = 0 \tag{3}$$

This provides a constraint for the previous displacement field and implies the following conditions:

$$\boldsymbol{P}_\zeta\boldsymbol{\vartheta}^{\mathrm{a}} = \boldsymbol{P}_\zeta\boldsymbol{\vartheta}^{\mathrm{b}} = \varphi(\zeta)\boldsymbol{e} \tag{4}$$

$$\boldsymbol{s}_0^{\mathrm{a}} \cdot \boldsymbol{a}_y = \boldsymbol{s}_0^{\mathrm{b}} \cdot \boldsymbol{a}_y = v_0(\zeta)\boldsymbol{a}_y \tag{5}$$

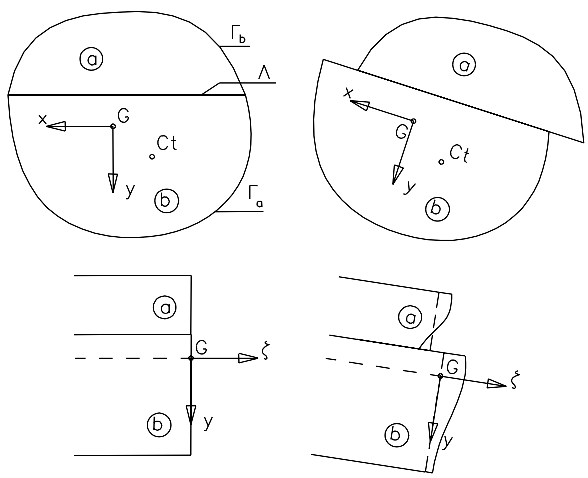

Fig. 1. Kinematic model.

In other words the twisting rotation φ and the translation in the y-direction v_0 must be the same for the two components while axial translations, transversal translations along the x-direction, bending rotations may differ from each other and the warping is discontinuous at the connection.

In order to obtain a beam theory, in the sense that the displacement field is expressed on the basis of a finite number of function of ζ, it is necessary to reduce the set of admissible displacements in the ζ-direction. In the sequel it will be assumed that the mean rotation around an axis lying on the section plane is proportional to the first derivative of the axis translation (Kirchhoff bending) and the further axial displacement is proportional to the first derivative of the rotation around $\boldsymbol{r}_T$ and can be described by means of a shape function $\omega(\boldsymbol{r})$, usually denoted as the unit warping function (Vlasov torsion):

$$\boldsymbol{\vartheta}^a \cdot \boldsymbol{a}_x = \boldsymbol{\vartheta}^b \cdot \boldsymbol{a}_x = -v_0' \tag{6}$$

$$\boldsymbol{\vartheta}^a \cdot \boldsymbol{a}_y = u_0^{a\prime} \tag{7}$$

$$\boldsymbol{\vartheta}^b \cdot \boldsymbol{a}_y = u_0^{b\prime} \tag{8}$$

$$\psi = \varphi'\omega(\boldsymbol{r}) \tag{9}$$

where apexes denote the derivative with respect to ζ. As a consequence, the displacement field assumes the following form:

$$\boldsymbol{s}^a = u_0^a \boldsymbol{a}_x + v_0 \boldsymbol{a}_y + \big(-xu_0^{a\prime} - v_0' y + w_0^a + \varphi'\omega\big)\boldsymbol{e} + \varphi \boldsymbol{e} \times (\boldsymbol{r} - \boldsymbol{r}_T) \tag{10a}$$

$$\boldsymbol{s}^b = u_0^b \boldsymbol{a}_x + v_0 \boldsymbol{a}_y + (-xu_0^{b\prime} - v_0' y + w_0^b + \varphi'\omega)\boldsymbol{e} + \varphi \boldsymbol{e} \times (\boldsymbol{r} - \boldsymbol{r}_T) \tag{10b}$$

Such a kinematic model should be sufficient to describe the rule of the connection flexibility in transverse bending and torsion. The functions describing the displacement field can be collected in the vector $\boldsymbol{c}(\zeta) = \left[v_0, w_0^a, w_0^b, u_0^a, u_0^b, \varphi\right]$. A more complex model can be however assumed by introducing further shape functions in order to obtain a more refined description of behavior in shear (see e.g. Laudiero and Savoia (1990), Dall'Asta and Leoni (1997)).

2.2. Strain field

The expression of the strain tensor field can be obtained directly from the displacements. It contains only the terms describing the shear strain involving the ζ-axis and axial strain along the same axis, so that it can be posed in the form

$$\boldsymbol{E} = \tfrac{1}{2}\boldsymbol{\gamma} \otimes \boldsymbol{\gamma} + \varepsilon \boldsymbol{e} \otimes \boldsymbol{e} \tag{11}$$

where the vector $\boldsymbol{\gamma}$ and the strain ε have the expressions (∇_S is the gradient of the functions defined on the section):

$$\boldsymbol{\gamma} = \varphi'(\nabla_S \omega + \boldsymbol{e} \times (\boldsymbol{r} - \boldsymbol{r}_T)) \tag{12}$$

$$\varepsilon = -xu_0^{b\prime\prime} - yv_0'' + w_0^{b\prime} + \varphi''\omega \quad \text{on } V^a \tag{13a}$$

$$\varepsilon = -xu_0^{a\prime\prime} - yv_0'' + w_0^{a\prime} + \varphi''\omega \quad \text{on } V^b \tag{13b}$$

The description of strain is completed by the slip occurring at the connection and furnished by the following expression

$$\boldsymbol{d}(x;\zeta) = \left[\left(\boldsymbol{I} - \boldsymbol{P}_y \right) \boldsymbol{s} \right]_{y=y_c} = \left(u_0^{\mathrm{b}} - u_0^{\mathrm{a}} \right) \boldsymbol{a}_x + \left(w_0^{\mathrm{b}} - w_0^{\mathrm{a}} - x \left(u_0^{\mathrm{b}\prime} - u_0^{\mathrm{a}\prime} \right) + \varphi' [\omega]_{y=y_c} \right) \boldsymbol{e}, \quad (x,\zeta) \in \Lambda \times [0,L] \tag{14}$$

3. Constitutive relations

The material forming the beam is transversally isotropic but it is not necessarily homogeneous on the section. The shear and axial stresses are related to strain by the following relation

$$\boldsymbol{\tau} = G(\boldsymbol{r})\boldsymbol{\gamma} \tag{15}$$

$$\sigma = E(\boldsymbol{r})\varepsilon \tag{16}$$

where E is the elastic modulus and G the shear modulus.

The tensor of active stress has the same structure of the strain tensor and can be written in the form

$$\boldsymbol{S} = \boldsymbol{\tau} \otimes \boldsymbol{\tau} + \sigma \boldsymbol{e} \otimes \boldsymbol{e} \tag{17}$$

It is assumed here that the connection stiffness is the same in the ζ-direction and x-direction, it can vary along the connection line, so that the connection constitutive law can be expressed as follows:

$$\boldsymbol{f} = k(x)\boldsymbol{d} \tag{18}$$

where $\boldsymbol{f}$ is the shear stress transmitted between the two components.

The problem will be analyzed by assuming linear elastic constitutive laws for the beam materials and the connection. In the following session the warping function will be determined and it will be shown that it depends on the constitutive parameters, so that it is not a trivial matter the extension of this formulation to the case of non-linear or viscoelastic behavior, even if this remains an interesting problem in structural mechanics.

4. Warping function

The shape function $\omega(\boldsymbol{r})$ describing the cross-section warping is obtained with the same criterion adopted in the Vlasov theory of torsion: under a displacement field consisting of uniform twisting, i.e. $\boldsymbol{s} = \varphi \boldsymbol{e} \times (\boldsymbol{r} - \boldsymbol{r}_{\mathrm{T}}) + \varphi' \omega \boldsymbol{e}$ with $\varphi'' = 0$, the function ω must furnish a stress field which is locally balanced when body forces on V and contact forces on the mantle $\Gamma \times (0, L)$ are not present. Only shear stresses exist and they must satisfy the equilibrium conditions on the interior points of the section, on the boundary and at the interface between the two components (div_S is the divergence of the functions defined on the section, $\boldsymbol{n}$ is the outward normal defined on Γ)

$$\mathrm{div}_S \, \boldsymbol{\tau} = 0 \quad \text{on } S^{\mathrm{a}} \text{ and on } S^{\mathrm{b}} \tag{19a}$$

$$\boldsymbol{\tau} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma^{\mathrm{a}} \text{ and on } \Gamma^{\mathrm{b}} \tag{19b}$$

$$\lfloor \boldsymbol{\tau} \cdot \boldsymbol{a}_y \rfloor_{y=y_c} = 0 \quad \text{on } \Lambda \tag{19c}$$

$$\boldsymbol{\tau} \cdot \boldsymbol{a}_y = \boldsymbol{f} \quad \text{on } \Lambda \tag{19d}$$

As a consequence, the warping function must be a solution of the following differential problem (Δ_S is the Laplace operator of functions defined on the section and the commas denote the partial derivatives)

$$\Delta_S(G\omega) = 0 \quad \text{on } S^{\mathrm{a}} \text{ and on } S^{\mathrm{b}} \tag{20a}$$

$$(\nabla_S(\omega) + \boldsymbol{e} \times (\boldsymbol{r} - \boldsymbol{r}_\mathrm{T})) \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma^\mathrm{a} \text{ and on } \Gamma^\mathrm{b} \tag{20b}$$

$$\lfloor G\omega_{,y} \rfloor_{y=y_\mathrm{c}} + [G]_{y=y_\mathrm{c}}(y - y_\mathrm{T}) = 0 \quad \text{on } \Lambda \tag{20c}$$

$$G\omega_{,y} + G(y - y_\mathrm{T}) = k[\omega]_{y=y_\mathrm{c}} \quad \text{on } \Lambda \tag{20d}$$

The warping function does not depend only on the geometrical characteristics of the cross-section but finding it also involves constitutive entities. In the very usual case in which each part of the section is homogeneous, i.e. $\nabla_S G^\mathrm{a} = \nabla_S G^\mathrm{b} = \boldsymbol{0}$, and the connection is homogeneous, i.e. $k_{,x} = 0$, the warping function depends on the two ratios G^a/k and G^b/k between the shear stiffness of the two components and the stiffness of the connection.

The solution is not unique and it is defined once the mean value and the centre of rotation $\boldsymbol{r}_\mathrm{T}$ are chosen. For convenience they will be chosen in such a way that the normal stresses deriving from non-uniform twisting does not produce axial resultant and bending moments.

5. Balance conditions

A balance condition for the composite beams can be obtained on the basis of the virtual work principle; this permits identifying the dynamic entities which are duals of the kinematic entities previously introduced for describing the displacement field of the beam. Such dynamic entities are resultants of the stresses which makes work (active stress). In a second step the local equilibrium conditions are obtained by integrating by parts (Euler problem).

The beam is supposed to be subjected to external actions consisting of force per unit volume $\boldsymbol{p}(\boldsymbol{r}, \zeta)$ acting on the interior points and contact forces $\boldsymbol{h}(\boldsymbol{r}, \zeta)$ acting on the boundary. The virtual work for a virtual field of displacements $\hat{\boldsymbol{s}}$ and related strain $\hat{\varepsilon}$, $\hat{\boldsymbol{\gamma}}$ and slip $\hat{\boldsymbol{d}}$, has the following expression:

$$\mathsf{L} = \int_0^L \int_S \sigma\hat{\varepsilon} + \boldsymbol{\tau} \cdot \hat{\boldsymbol{\gamma}} \, \mathrm{d}S \, \mathrm{d}\zeta + \int_0^L \int_\Lambda \boldsymbol{f} \cdot \hat{\boldsymbol{d}} \, \mathrm{d}\Lambda \, \mathrm{d}\zeta - \int_V \boldsymbol{p} \cdot \hat{\boldsymbol{s}} \, \mathrm{d}V - \int_{\partial V} \boldsymbol{h} \cdot \hat{\boldsymbol{s}} \, \mathrm{d}\partial V \tag{21}$$

It is assumed that contact forces act on the mantle only and there is no work related to stress and displacements of the end sections. The presence of kinematic constraints or prescribed forces at the beam extremities affects the boundary conditions only and the relevant results are reported in Appendix A. Once the kinematic model is introduced, the expression can be posed in the following form, involving only quantities defined on the cross-section:

$$\mathsf{L} = \int_0^L \boldsymbol{R} \cdot \mathsf{D}\hat{\boldsymbol{c}} \, \mathrm{d}\zeta - \int_0^L \boldsymbol{q} \cdot \mathsf{N}\hat{\boldsymbol{c}} \, \mathrm{d}\zeta \tag{22}$$

The first term concerning the internal work shows that particular duality relations can be evidenced between the component of the vector $\boldsymbol{R}(\zeta) = [\boldsymbol{R}_\mathrm{b}, \boldsymbol{R}_\mathrm{c}]$ with $\boldsymbol{R}_\mathrm{b}(\zeta) = [M_x, N^\mathrm{a}, N^\mathrm{b}, M_y^\mathrm{a}, M_y^\mathrm{b}, M_\zeta, B]$ and $\boldsymbol{R}_\mathrm{c}(\zeta) = \left[v_\zeta v_x, \mu_y, \beta\right]$, which contains the resultants of active stresses, and the kinematic entities representing generalized strain derived from $\boldsymbol{c}(\zeta)$ by means of the differential operator D.

The expression of stress resultants are reported in the sequel (the apex "a" or "b" in the $\boldsymbol{R}$-components refers to integration carried out on the relevant portion of section):

$$N(\zeta) = \int_S \sigma \, \mathrm{d}S \tag{23a}$$

$$M_x(\zeta) = \int_S y\sigma \, \mathrm{d}S \tag{23b}$$

$$M_y(\zeta) = \int_S -x\sigma \, \mathrm{d}S \tag{23c}$$

$$M_\zeta(\zeta) = \int_S |(\boldsymbol{r} - \boldsymbol{r}_\mathrm{T}) \times \tau| \, \mathrm{d}S \tag{23d}$$

$$B(\zeta) = \int_S \omega\sigma \, \mathrm{d}S \tag{23e}$$

$$v_x(\zeta) = \int_\Lambda f_x \, \mathrm{d}\Lambda \tag{23f}$$

$$v_\zeta(\zeta) = \int_\Lambda f_\zeta \, \mathrm{d}\Lambda \tag{23g}$$

$$\mu_y(\zeta) = \int_\Lambda -x f_\zeta \, \mathrm{d}\Lambda \tag{23h}$$

$$\beta(\zeta) = \int_\Lambda f_\zeta [\omega]_{y=y_\mathrm{c}} \, \mathrm{d}\Lambda \tag{23i}$$

The seven components of $\boldsymbol{R}_\mathrm{b}$ are the usual beam resultants of stress of the Vlasov theory. The difference consists in the resultant of axial stress and in the bending moments around the y-axis, which are split into two parts related to the components "a" and "b" as consequences of their different mean axial strains and curvatures. The four components of $\boldsymbol{R}_\mathrm{c}$ concern the resultants of the connection stresses. The first two v_ζ and v_x are simply the resultants in the ζ-direction and x-direction, the third describes the resultant moment around the y-axis while the last is similar to the bi-moment and weighs the stress by means of the jump in the warping function.

The differential operator D is defined as follows

$$\mathsf{D}_\mathrm{c} = \begin{bmatrix} \mathsf{D}_\mathrm{b} \\ \mathsf{D}_\mathrm{c} \end{bmatrix} \boldsymbol{c} \tag{24}$$

with

$$\mathsf{D}_\mathrm{b} = \begin{bmatrix} -\partial^2 & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \partial & \vdots & \vdots & \vdots & \cdots \\ \cdots & \vdots & \partial & \vdots & \vdots & \cdots \\ \cdots & \vdots & \vdots & \partial^2 & \vdots & \cdots \\ \cdots & \vdots & \vdots & \vdots & \partial^2 & \cdots \\ \cdots & \vdots & \vdots & \vdots & \vdots & \partial \\ \cdots & \cdots & \cdots & \cdots & \cdots & \partial^2 \end{bmatrix}, \qquad \mathsf{D}_\mathrm{c} = \begin{bmatrix} \cdots & -1 & 1 & \cdots & \cdots & \cdots \\ \cdots & \vdots & \vdots & -1 & 1 & \cdots \\ \cdots & \vdots & \vdots & -\partial & \partial & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \partial \end{bmatrix} \tag{25a, b}$$

where D_b and D_c respectively furnish the generalized strain working for the beam and connection resultants (∂ denotes the derivative operator with respect to ζ).

The external work consists of a term describing the contribution in the interior of the beam, obtained by means of the resultants collected in $\boldsymbol{q} = [q_y, m_x, n^{\mathrm{a}}, n^{\mathrm{b}}, q_x^{\mathrm{a}}, m_y^{\mathrm{a}}, q_x^{\mathrm{b}}, m_y^{\mathrm{b}}, m_\zeta, b]$ and the generalized displacements $\mathsf{N}\boldsymbol{c}$, where N maps $\boldsymbol{c}$ into $\mathsf{N}\boldsymbol{c} = [v_0, -v_0', w_0^{\mathrm{a}}, w_0^{\mathrm{b}}, u_0^{\mathrm{a}}, u_0^{\mathrm{a}\prime}, u_0^{\mathrm{b}}, u_0^{\mathrm{b}\prime}, \varphi, \varphi']$. The components of $\boldsymbol{q}$ are reported in the sequel (the apex "a" or "b" in the $\boldsymbol{q}$-components refers to the integration carried out on the relevant portion of section):

$$n(\zeta) = \int_S p_\zeta \,\mathrm{d}S + \int_\Gamma h_\zeta \,\mathrm{d}\Gamma \tag{26a}$$

$$q_x(\zeta) = \int_S p_x \,\mathrm{d}S + \int_\Gamma h_x \,\mathrm{d}\Gamma \tag{26b}$$

$$q_y(\zeta) = \int_S p_y \,\mathrm{d}S + \int_\Gamma h_y \,\mathrm{d}\Gamma \tag{26c}$$

$$m_x(\zeta) = \int_S y p_\zeta \,\mathrm{d}S + \int_\Gamma y h_\zeta \,\mathrm{d}\Gamma \tag{26d}$$

$$m_y(\zeta) = \int_S -x p_\zeta \,\mathrm{d}S + \int_\Gamma -x h_\zeta \,\mathrm{d}\Gamma \tag{26e}$$

$$m_\zeta(\zeta) = \int_S |(\boldsymbol{r} - \boldsymbol{r}_{\mathrm{T}}) \times \boldsymbol{p}| \,\mathrm{d}S + \int_\Gamma |(\boldsymbol{r} - \boldsymbol{r}_{\mathrm{T}}) \times \boldsymbol{h}| \,\mathrm{d}\Gamma \tag{26f}$$

$$b(\zeta) = \int_S \omega p_\zeta \,\mathrm{d}S + \int_\Gamma \omega h_\zeta \,\mathrm{d}\Gamma \tag{26g}$$

By integrating by parts it is possible to derive the local formulation of the equilibrium problem consisting of differential equations in the interior of the domain and boundary conditions at the ends. The differential system assumes the following form

$$\mathsf{D}^*\boldsymbol{R} = [\,\mathsf{D}_{\mathrm{b}}^* \quad \mathsf{D}_{\mathrm{c}}^*\,]\boldsymbol{R} = \boldsymbol{g} \tag{27}$$

with

$$\mathsf{D}_{\mathrm{b}}^* = \begin{bmatrix} -\partial^2 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & -\partial & \vdots & \vdots & \vdots & \vdots & \cdots \\ \cdots & \vdots & -\partial & \vdots & \vdots & \vdots & \cdots \\ \cdots & \vdots & \vdots & \partial^2 & \vdots & \vdots & \cdots \\ \cdots & \vdots & \vdots & \vdots & \partial^2 & \vdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & -\partial & \partial^2 \end{bmatrix}, \qquad \mathsf{D}_{\mathrm{c}}^* = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ -1 & \vdots & \vdots & \cdots \\ 1 & \vdots & \vdots & \cdots \\ \cdots & -1 & \partial & \cdots \\ \cdots & 1 & -\partial & \cdots \\ \cdots & \cdots & \cdots & -\partial \end{bmatrix} \tag{28a, b}$$

where D^* is the formal adjoint of D, and the vector $\boldsymbol{g} = [q_y + m_x', n^{\mathrm{a}}, n^{\mathrm{b}}, q_x^{\mathrm{a}} - m_y^{\mathrm{a}\prime}, q_x^{\mathrm{b}} - m_y^{\mathrm{b}\prime}, m_\zeta - b']$ collects the contribution of the external actions. The relevant boundary conditions are reported in the appendix.

The problem is ruled by six equilibrium conditions expressing the equilibrium in the y-direction of the whole beam, as a consequence of an equal translation in the y-direction, the equilibrium in the axial direction and x-direction for each component separately, as a consequence of the jump in the displacements in such directions, and a last equilibrium condition ensuring the balance with respect to moments around the beam axis of the whole cross-section, as a consequence of an equal twisting rotation. The interaction between the axial forces on the two components and the connection force is already known from the analysis of the symmetric beams under bending (Newmark) while the last three equations describe the interaction arising in the transverse direction (x-direction) and in the twisting.

In particular, the two components bend differently in the transverse direction and the flexible connection affects the transverse equilibrium by means of two terms, the former is proportional to the transverse slip providing two opposite transverse forces between the components and the latter is proportional to the moment around the y-axis furnished by the connection force and affects the equilibrium by way of the derivative. Finally, three terms contribute to the rotational equilibrium around the centre of twist. The torque moment M_ζ describes the effect of the active shear stress whose distribution is ruled by the warping function as in the De Saint Venant theory. The other two terms concern the effect of the reactive shear stress which arises for balancing the normal stresses and the axial connection forces. The term related to β is a characteristic term of the beam with shear connection while the contribution related to B also exists in the Vlasov theory. Differently from that case, the normal stress s is now produced both by non-uniform warping and by the independent bending of the two components on the x–z plane.

6. Displacement formulation

The problem can be approached by assuming the displacements as unknown and obtaining the expression of the stress resultants by introducing the constitutive behavior of the materials. Once the following quantities, containing the inertial properties of the cross-section and constitutive parameters, are defined:

$$A = \int_S E \, \mathrm{d}S \tag{29a}$$

$$C_x = \int_S E y \, \mathrm{d}S \quad C_y = \int_S E x \, \mathrm{d}S \quad C_\omega = \int_S E \omega \, \mathrm{d}S \tag{29b, c, d}$$

$$I_{xx} = \int_S E y^2 \, \mathrm{d}S \quad I_{yy} = \int_S E x^2 \, \mathrm{d}S \quad I_{xy} = \int_S E x y \, \mathrm{d}S \tag{29e, f, g}$$

$$I_{x\omega} = \int_S E x \omega \, \mathrm{d}S \quad I_{y\omega} = \int_S E y \omega \, \mathrm{d}S \quad I_{\omega\omega} = \int_S E \omega^2 \, \mathrm{d}S \tag{29h, i, j}$$

$$J = \int_S G \boldsymbol{r} \times (\nabla\omega - \boldsymbol{e} \times \boldsymbol{r}) \, \mathrm{d}S \tag{29k}$$

$$D = \int_{x_1}^{x_2} k \, \mathrm{d}x \tag{29l}$$

$$H_x = \int_{x_1}^{x_2} k x \, \mathrm{d}x \quad H_\omega = \int_{x_1}^{x_2} k [\omega]_{y=y_c} \, \mathrm{d}x \tag{29m, n}$$

$$K_{xx} = \int_{x_1}^{x_2} k x^2 \, \mathrm{d}x \quad K_{x\omega} = \int_{x_1}^{x_2} k x [\omega]_{y=y_c} \, \mathrm{d}x \quad K_{\omega\omega} = \int_{x_1}^{x_2} k [\omega]_{y=y_c}^2 \, \mathrm{d}x \tag{29o, p, q}$$

it becomes possible to build up a symmetric operator K furnishing the vector of stress resultants from the quantities $\mathsf{D}\boldsymbol{c}$ defining the system strain, i.e.

$$\boldsymbol{R} = \mathsf{K}\mathsf{D}\boldsymbol{c} \tag{30}$$

the expression of the component of K are reported in Appendix A. They are evaluated by assuming that the origin of the reference system is posed at the centroid of the section, in the sense that

$$C_x = 0, \quad C_y = 0 \tag{31a, b}$$

It should however be observed that this does not imply that first order inertial moments vanish for the single components ($C_x^{\mathrm{a}} \neq 0, C_y^{\mathrm{a}} \neq 0, C_x^{\mathrm{b}} \neq 0, C_y^{\mathrm{b}} \neq 0$). The warping function ω and the position of $\boldsymbol{r}_{\mathrm{T}}$ are chosen such that

$$C_\omega = 0 \tag{32a}$$

$$I_{x\omega} = 0, \quad I_{y\omega} = 0 \tag{32b, c}$$

Even in this case the integrals evaluated on the single section component can differ from zero. Differently from the case of a classic beam theory (see Gjelsvik (1981)), only a little simplification derives from such an usual position and only one stiffness term disappears in K (the term $I_{y\omega}$).

The equilibrium conditions can be obtained as previously and lead to a differential problem in the unknown $\boldsymbol{c} = [\boldsymbol{c}^{\mathrm{F}}, \boldsymbol{c}^{\mathrm{T}}]$ consisting of the vector $\boldsymbol{c}^{\mathrm{F}} = \left[v_0, w_0^{\mathrm{a}}, w_0^{\mathrm{b}}\right]$ grouping the displacements describing axial stretch and bending in the plane orthogonal to the connection and the vector $\boldsymbol{c}^{\mathrm{T}} = \left[u_0^{\mathrm{a}}, u_0^{\mathrm{b}}, \varphi\right]$ grouping the displacements describing twisting and bending in the plane parallel to the connection. The differential equations assume the following form:

$$\mathsf{A}\boldsymbol{c} = \begin{bmatrix} \mathsf{A}^{\mathrm{FF}} & \mathsf{A}^{\mathrm{FT}} \\ \mathsf{A}^{\mathrm{TF}} & \mathsf{A}^{\mathrm{TT}} \end{bmatrix} \begin{bmatrix} \boldsymbol{c}^{\mathrm{F}} \\ \boldsymbol{c}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{g}^{\mathrm{F}} \\ \boldsymbol{g}^{\mathrm{T}} \end{bmatrix} = \boldsymbol{g} \tag{33}$$

where $\boldsymbol{g} = \lfloor \boldsymbol{g}^{\mathrm{F}}, \boldsymbol{g}^{\mathrm{T}} \rfloor$ is the previously defined vector involving the resultant of the external forces and now divided into the two parts $\boldsymbol{g}^{\mathrm{F}} = [q_y + m_x', n^{\mathrm{a}}, n^{\mathrm{b}}]$ and $\boldsymbol{g}^{\mathrm{T}} = [q_x^{\mathrm{a}} - m_y^{\mathrm{a}\prime}, q_x^{\mathrm{b}} - m_y^{\mathrm{b}\prime}, m_\zeta - b']$. The differential operator $\mathsf{A} = \mathsf{D}^*\mathsf{K}\mathsf{D}$ is self-adjoint and can be split in the following four components:

$$\mathsf{A}^{\mathrm{FF}} = \begin{bmatrix} \partial^2 I_{yy}\partial^2, & -\partial^2 C_y^{\mathrm{a}}\partial, & -\partial^2 C_y^{\mathrm{b}}\partial \\ \partial C_y^{\mathrm{a}}\partial^2, & -\partial A^{\mathrm{a}}\partial + D, & -D \\ \partial C_y^{\mathrm{b}}\partial^2, & -D & -\partial A^{\mathrm{b}}\partial + D \end{bmatrix} \tag{34a}$$

$$\mathsf{A}^{\mathrm{FT}} = \begin{bmatrix} \partial^2 I_{xy}^{\mathrm{a}}\partial^2, & -\partial^2 C_x^{\mathrm{a}}\partial + \partial H_x, & -\partial H_x \\ \partial^2 I_{xy}^{\mathrm{b}}\partial^2, & -\partial H_x, & -\partial^2 C_x^{\mathrm{b}}\partial + \partial H_x \\ -\partial^2 I_{y\omega}\partial^2, & \partial^2 C_\omega^{\mathrm{a}}\partial + \partial H_\omega & \partial^2 C_\omega^{\mathrm{b}}\partial - \partial H_\omega \end{bmatrix} \tag{34b}$$

$$\mathsf{A}^{\mathrm{TF}} = \left(\mathsf{A}^{\mathrm{FT}}\right)^* = \begin{bmatrix} \partial^2 I_{xy}^{\mathrm{a}}\partial^2, & \partial^2 I_{xy}^{\mathrm{b}}\partial^2, & -\partial^2 I_{y\omega}\partial^2 \\ \partial C_x^{\mathrm{a}}\partial^2 - H_x\partial, & H_x\partial, & -\partial C_\omega^{\mathrm{a}}\partial^2 - H_\omega\partial \\ \partial H_x, & \partial C_x^{\mathrm{b}}\partial^2 - H_x\partial & -\partial C_\omega^{\mathrm{b}}\partial^2 + H_\omega\partial \end{bmatrix} \tag{34c}$$

$$\mathsf{A}^{\mathrm{TT}} = \begin{bmatrix} \partial^2 I_{xx}^{\mathrm{a}}\partial^2 - \partial K_{xx}\partial + L, & \partial K_{xx}\partial - L, & -\partial^2 I_{x\omega}^{\mathrm{a}}\partial^2 - \partial K_{x\omega}\partial \\ \partial K_{xx}\partial - L, & \partial^2 I_{xx}^{\mathrm{b}}\partial^2 - \partial K_{xx}\partial + L, & -\partial^2 I_{x\omega}^{\mathrm{b}}\partial^2 + \partial K_{x\omega}\partial \\ -\partial^2 I_{x\omega}^{\mathrm{a}}\partial^2 - \partial K_{x\omega}\partial, & -\partial^2 I_{x\omega}^{\mathrm{b}}\partial^2 + \partial K_{x\omega}\partial & \partial^2 I_{\omega\omega}\partial^2 - \partial(J + K_{\omega\omega})\partial \end{bmatrix} \tag{34d}$$

The differential problem also requires the definition of 10 boundary conditions at each end, they can be of natural or static type and are reported in Appendix A.

The problem solution in its variational format can be sought for $\boldsymbol{c} \in H^2 \times H^1 \times H^1 \times H^2 \times H^2 \times H^2$ (H^n is the Hilbert space of functions whose n-derivative is square-integrable on $[0, L]$). If K is positive definite and rigid motions are avoided, a unique solution exists. It is interesting to observe that a different regularity is required for the stiffness term concerning the beam, grouped in K_{b} in Appendix A, and the stiffness term of the connection, grouped in K_c in Appendix A. The former must lie in L^∞ while less regularity is required for the latter which may lie in H^{-1}, so that pointwise connections may also be considered.

In numerous cases of technical interest, the cross-section is symmetric with respect to the y–ζ plane. This provides considerable simplification in the expression of the balance conditions and permits some interesting observations.

The symmetry implies that the reference axes are principal axes and the following stiffness terms vanish

$$C_y^a = C_y^b = 0 \tag{35a}$$

$$I_{xy} = I_{xy}^a = I_{xy}^b = 0 \tag{35b}$$

$$I_{y\omega} = I_{y\omega}^a = I_{y\omega}^b = 0 \tag{35c}$$

$$H_x = H_\omega = 0 \tag{35d}$$

As a consequence, the operators A^{FT}, A^{TF}, and the related boundary conditions, vanish and the problem is split into two independent parts, the former, related to $\boldsymbol{c}^T$, describes the bending and stretch in the symmetry plane and the latter, related to $\boldsymbol{c}^F$, describes bending in the transverse directions and twisting. No further splitting is usually possible, so that stretch in the axial direction and bending in y–ζ plane are always coupled, as already known from the Newmark's theory, and, in general, transverse bending imply twisting rotations.

7. Application: rectangular box section

The case of bending is already known and analyzed in numerous previous papers. In the following the torsion of a beam with rectangular closed section, described in Fig. 2, will be analyzed. The example refers to a steel box section with constant thickness and the connection is located between the upper deck and the lower "C" section. The upper deck stiffness is equivalent to the stiffness of a concrete deck with a thickness of 200 mm and the results can be considered valid even for such a usual type of composite cross-section. The problem is sufficiently simple to permit obtaining some results in closed form and showing some aspects of interest in applications.

The first problem consists in evaluating the warping function. In this regard the usual assumptions for thin-walled closed section can be introduced and ω can be derived from the shear flux, avoiding the solution of Eq. (20). It is convenient to describe the positions of the cross-section points by means of the expression

$$\boldsymbol{r} = \boldsymbol{\rho}(s) + n\boldsymbol{a}_n \tag{36}$$

where $\boldsymbol{\rho}$ is such that $|\boldsymbol{\rho}_{,s}| = 1$ and furnishes the positions of the points of the mean line Σ, $\boldsymbol{a}_s = \boldsymbol{\rho}_{,s}$ is the unit tangent vector, $\boldsymbol{a}_n = \boldsymbol{e} \times \boldsymbol{a}_s$ is the unit normal vector, so that the co-ordinates (s, n) are defined in the set $\Sigma \times (t(s)/2, -t(s)/2)$, where $t(s)$ represents the thickness at s. It is assumed that the tangential stress is a

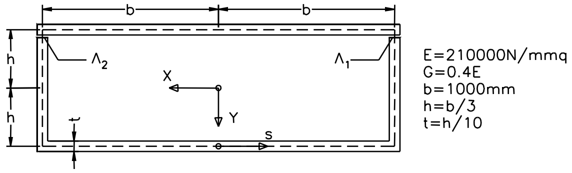

Fig. 2. Rectangular box section.

constant in the thickness and is oriented like the tangent vector. i.e. $\boldsymbol{\tau} = \tilde{\tau}(s)\boldsymbol{a}_s(\tilde{(\cdot)}$ denotes the value of $(\cdot)$ at the mean curve). From the equilibrium it follows that the stress flux $T = \tilde{\tau}(s)\boldsymbol{t}(s)$ is a constant. The connection line Λ reduces to two segments Λ_α ($\alpha = 1, 2$) intersecting the curve Σ at s_α. The connection stiffness is a constant on Λ_α. The resultants of the connection forces and stiffness on Λ_α will be denoted by $v_{\zeta\alpha}(\zeta) = \tilde{f}_{\zeta\alpha}(\zeta)\Lambda_\alpha$, $v_{x\alpha}(\zeta) = \tilde{f}_{x\alpha}(\zeta)\Lambda_\alpha$, $D_\alpha(\zeta) = \tilde{k}_\alpha(\zeta)\Lambda_\alpha$.

From the equilibrium of the cross-section it is easy to prove that the stress flux on the two section parts and the connection forces have the expressions

$$T = v_{\zeta\alpha} = \frac{M_\zeta}{2\Omega} \tag{37}$$

where Ω is the area of the surface closed by the curve Σ (nothing changes with respect to Bredt's theory). The stiffness of the cross-section can be deduced from the energy balance and furnishes the relation

$$J + K_{\omega\omega} = \frac{4\Omega^2}{\left(\int_\Sigma \frac{1}{Gt}\,\mathrm{d}s + \sum_\alpha \frac{1}{D_\alpha}\right)} \tag{38}$$

In the specific case of rectangular box sections the following expression have been obtained for the torsional stiffness ($D_1 = D_2 = D/2$)

$$J + K_{\omega\omega} = \frac{64b^2h^2}{4\frac{b+h}{Gt} + 2\frac{1}{D_1}} \tag{39}$$

and Fig. 3 reports the trend of the torsional stiffness versus the connection stiffness. The torsional stiffness is reported by means of the ratio between expression (41) and the torsional stiffness of a closed section with rigid connection J_∞. A very large field of values is spanned by the stiffness for usual values of D_α/G, that in the example may vary from 0.005 to 0.05.

At this point the warping function at the mean line $\tilde{\omega}$ can be evaluated by requiring that the shear strain obtained from compatibility Eq. (12) be equal to the shear strain deduced from the stress flux. A similar

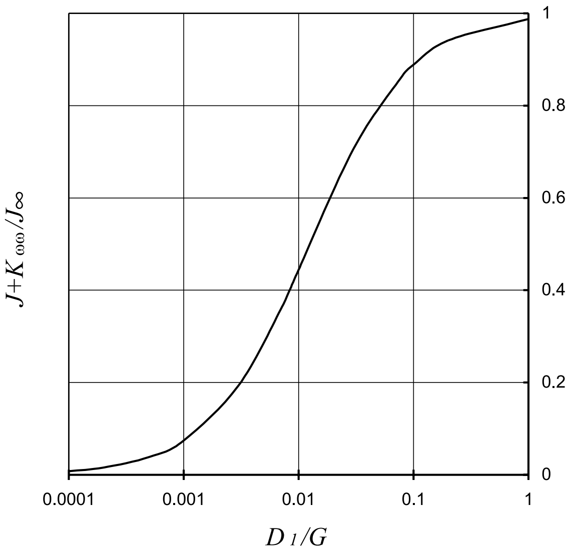

Fig. 3. Torsional stiffness versus connection flexibility.

relation can be required between the displacement jump and the connection force at the connection. This leads to the following conditions:

$$\tilde{\omega}(s) - \tilde{\omega}(s_0) = \frac{1}{2\Omega} \int_{s_0}^{s} \left(\frac{J + K_{\omega\omega} - \mathbf{e} \times (\boldsymbol{r} - \boldsymbol{r}_\mathrm{T}) \cdot \boldsymbol{a}_s}{Gt} \right) \mathrm{d}s \quad \text{on } \Sigma_\mathrm{a} \text{ and on } \Sigma_\mathrm{b} \tag{40}$$

$$[\tilde{\omega}]_{s=s_\alpha} = \frac{v_{\zeta\alpha}}{D_\alpha} \tag{41}$$

It should be observed that the rotation around the twisting axis provides stress and strain on the cross-section components only, whose total strain is the sum of $\tilde{u}_{s,\zeta}$, due to rotation, and $\tilde{w}_{,s}$, due to warping, while the force and the slip at the interface are controlled by the warping only, by means of the jump $[\tilde{w}]_{s=s_\alpha}$.

The warping function is piecewise rectilinear and has a jump at the connection. Its analytical expression is cumbersome and is not reported here; however, Fig. 4 shows its diagram on a half section. The curves refer to a different value of the connection stiffness ($D_\alpha/G = 1, 0.1, 0.01, 0.001$).

In the example the concentrated variation $[\tilde{w}]_{s=s_\alpha}$ of the warping at the interface has an opposite sign with respect to the derivative $\tilde{w}_{,s}$ in its neighborhood because the component of the flux related to the only rotation (not affecting the slip at interface) is larger than the total flux of Eq. (37), required from the equilibrium. A different situation occurs when b is smaller than h.

When the connection stiffness decreases the jump increases and the shape of the warping function notably differs from that of a section with a rigid connection. It does not tend to a null function for a connection stiffness approaching zero, as occurs for the open section, because the two components are not free to rotate around their center of torsion but they are constrained to rotate around the global center of torsion, which has the expression

$$y_\mathrm{T} = \frac{3DGth^2}{(b + 3h)(Db + Dh + Gt)} \tag{42}$$

The position of the rotation center of the cross-section versus the connection stiffness is described in Fig. 5a. It is located at the geometrical centre of the box section when $D = \infty$ and it shifts by increasing its distance from the position of the connection.

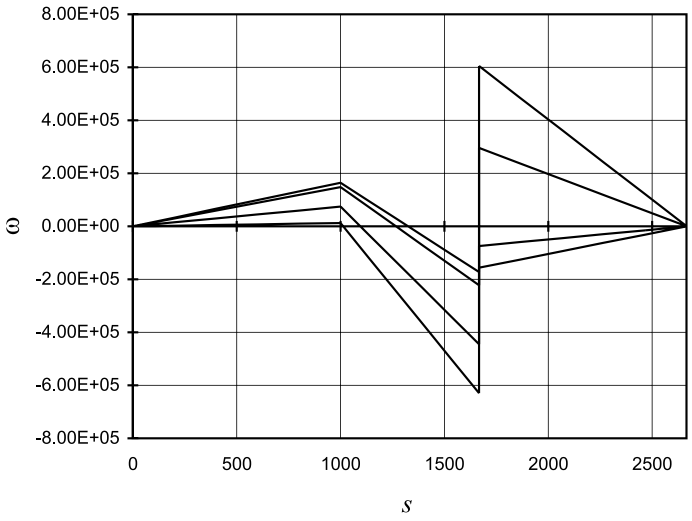

Fig. 4. Warping function on half a section.

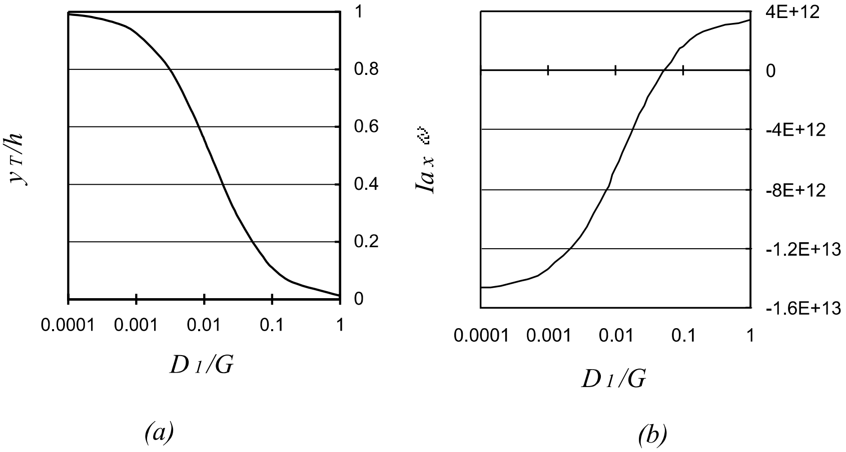

Fig. 5. (a) Rotation center position versus connection stiffness, (b) $I^{\mathrm{a}}_{x\omega}$ versus connection stiffness.

As a consequence of the discontinuity of the warping function and the translation of the rotation centre, warping inverts its trend on the upper component (component "a") for sufficiently weak connections. This means that the constitutive term $I^{\mathrm{a}}_{x\omega}$, which weighs the first moment of the warping function with respect to the symmetry axis and controls the coupling between rotation and the transverse displacements in the non-uniform torsion, changes its sign. The term $I^{\mathrm{a}}_{x\omega}$ has the following expression and Fig. 5b reports its value versus the connection stiffness.

$$I^{\mathrm{a}}_{x\omega} = -\frac{2b^3ht(bGt + 9hGt - Db^2 - 2hDb + 3Dh^2)}{3(b + 3h)(Db + Dh + Gt)} \tag{43}$$

Once the warping function has been determined, all the inertial quantities can be evaluated and the problem of non-uniform torsion can be approached. In order to analyze this particular problem, it is convenient to make a change of the variables describing the problem. The three equations describing the torsion problem can be rewritten by substituting the two transversal displacements with the following functions

$$u_{\mathrm{m}}(\zeta) = \frac{u_0^{\mathrm{a}} + u_0^{\mathrm{b}}}{2} \tag{44}$$

$$\delta(\zeta) = u_0^{\mathrm{a}} - u_0^{\mathrm{b}} \tag{45}$$

describing the mean transverse displacement (x-direction) and the transverse slip between the two components. In the case of simple torsion ($q_x^{\mathrm{a}} = q_x^{\mathrm{b}} = 0$) and constant cross-section the field equations reduce to a system of two differential equations involving δ and φ only

$$\frac{I^{\mathrm{a}}_{xx}I^{\mathrm{b}}_{xx}}{I_{xx}}\delta'''' - K_{xx}\delta'' + D\delta + I^{\mathrm{a}}_{x\omega}\varphi'''' + K_{x\omega}\varphi'' = 0 \tag{46a}$$

$$+I^{\mathrm{a}}_{x\omega}\delta'''' + K_{x\omega}\delta'' + I_{\omega\omega}\varphi'''' - (J + K_{\omega\omega})\varphi'' = m_\zeta \tag{46b}$$

while the mean displacement can be derived from the relation $u_{\mathrm{m}}'''' = -\delta''''\left(I^{\mathrm{b}}_{xx} - I^{\mathrm{a}}_{xx}\right)/2I_{xx}$.

Some numerical results have been evaluated for a beam ($L/h = 20$) with end constraints preventing twisting rotations ($\varphi = 0$) and relative transversal displacements ($\delta = 0$). This is usual in steel–concrete composite bridge where rigid diaphragms connecting both the components are located at the beam extremities while intermediate diaphragms only ensure that the sections of each component remain rigid in its

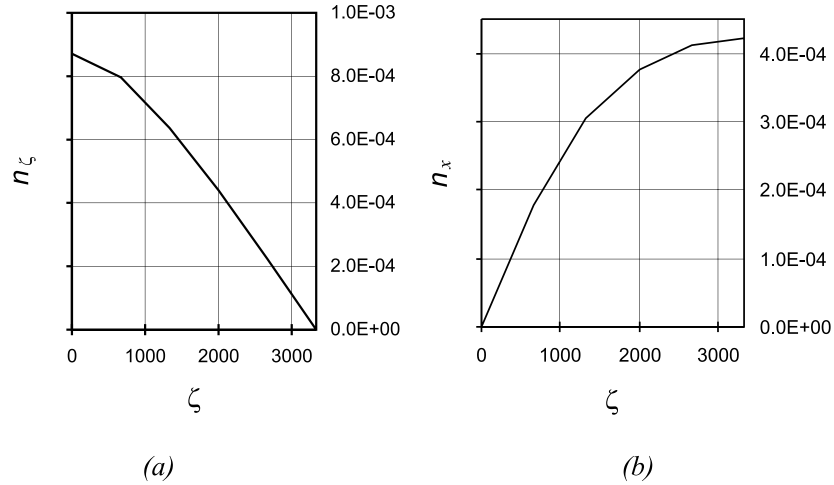

Fig. 6. (a) Axial connection force on half a beam, (b) transverse connection force on half a beam.

own plane. An approximated solution has been obtained by expressing $\varphi(\zeta)$ and $\delta(\zeta)$ by means of a summation of sinusoidal terms. This does not furnish accurate results near the ends because all the even derivatives are constrained to be null, but equally permits obtaining qualitative information.

Under non-uniform torsion the total section deforms in its own plane and a transverse slip occurs, so that connection forces arises both in the axial and transverse direction, as described in Fig. 6a and b for $D_\alpha/G = 0.01$ and $m_\zeta = 1$. The entity of these forces is of interest in the design of the connection stiffness and strength. The axial force influence attains its maximum value near the support, where the connection force deriving from vertical load also attains the maximum value, while the transverse force reaches maximum value at midspan where the connection force induced by vertical load is usually small.

The total torsion moment on the beam can be decomposed into three contributions:

$$M_{\mathrm{DSV}} = (J + K_{\omega\omega})\varphi' \tag{47}$$

which is proportional to the first derivative of the twisting angle, as in the De Saint Venant theory,

$$M_\omega = -I_{\omega\omega}\varphi''' \tag{48}$$

produced by the non-uniform warping, as in the Vlasov theory, and

$$M_\delta = -I^{\mathrm{a}}_{x\omega}\delta''' - K_{x\omega}\delta' \tag{49}$$

which is a characteristic term of the problem of beams with weak shear connection and derives from the transversal relative slip δ. The diagrams of the total torsion moment and its contributions on half a beam is reported in Fig. 7 for $D_\alpha/G = 0.01$ and $m_\zeta = 1$.

In beams with a stiff connection non-uniform twisting induces an opposite bending of the upper and lower flange and the relevant shear furnishes a contribution to the torque moment. In this case, the connection flexibility permits a relative transverse displacement of the two components which reduces the total bending of the upper and lower part, so that the moment M_ω due to the sole twisting is penalized by the moment M_δ due to the relative slip which has an opposite sign. The importance of the three terms strongly vary with the connection stiffness and in Fig. 8 the ratios between the three contributions and the total torsion moment at the supports are reported. In the case examined the reduction of the De Saint Venant moment becomes notable when the connection stiffness is lower than $0.01G$ and the two contribution due to non-uniform twisting and relative slip can become larger than the total moment. Their effects, in term of

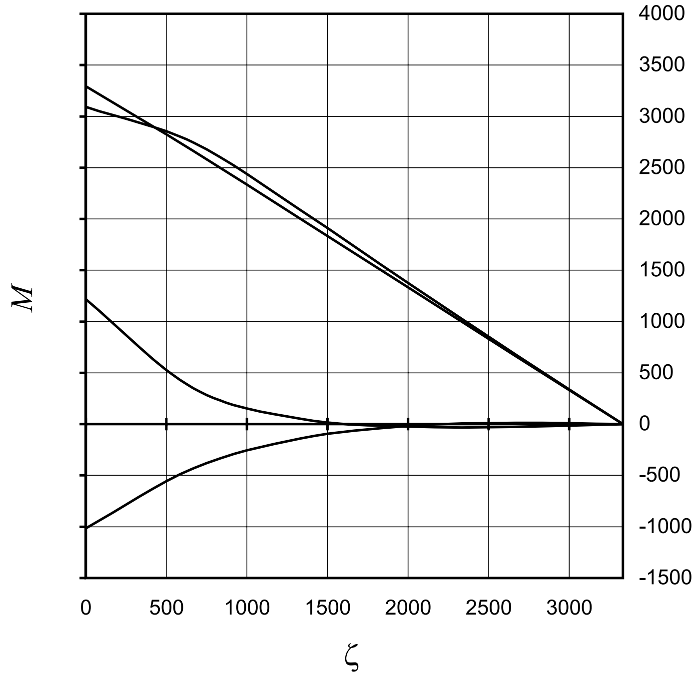

Fig. 7. Torsion moment components and total torsion moment on half a beam.

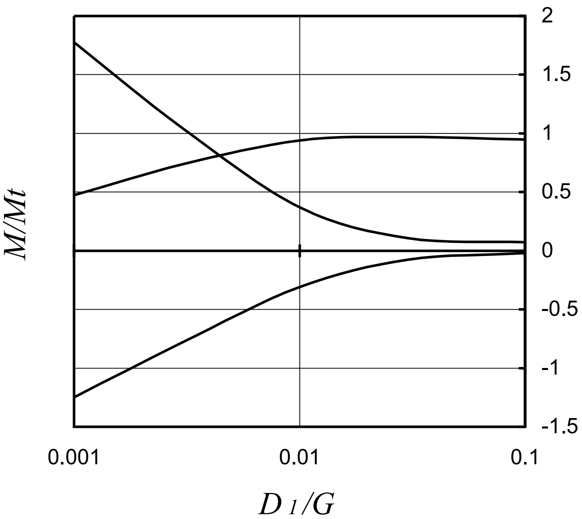

Fig. 8. Torsion moment components versus connection stiffness at supports.

stress, may be not very large because they can have an opposite sign. In Fig. 9 the total axial stress and the two contribution due to non-uniform twisting and transverse bending are reported. In this case the normal stress induced by d is small even if M_δ is almost as large as M_ω. It should however be remembered that M_δ consists of two contributions and the normal stress is related to $I^{a}_{x\omega}\delta'''$ only (transverse relative bending).

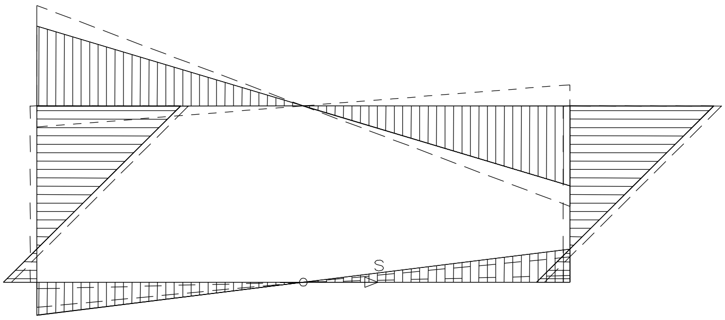

Fig. 9. Axial stress at midspan, dashed line refers to the contribution due to warping and dash-dot line to the contribution due to transverse bending. Diagram external to the section means traction.

8. Conclusions

The problem of composite beams with weak shear connection is of considerable importance in engineering and in particular in composite bridges that are creating growing interest in civil construction.

The existing theories of composite beams deal with the bending problem on the symmetry plane and are inadequate to furnish information about other situations, like torsion and transverse bending.

The paper furnishes a beam model for the general case of a beam under three-dimensional loads. It is based on the rigidity of the two section components in their own plane and a discontinuity of tangential displacements is permitted at the connection plane while the contact is preserved.

The differential problem furnishing the discontinuous warping function is presented. The balance conditions are derived from the virtual works principle and a displacement formulation is finally obtained.

Some new aspects of interest are underlined: the coupling arising between transverse bending and twisting, the influence of connection deformability on the solution of the uniform torsion problem and decomposition of the torque moment into three components in the non-uniform torsion problem.

Some specific results on rectangular box sections are reported.

Appendix A

A.1. Boundary conditions

It is assumed that the displacements at the boundary are free and the contact force $\boldsymbol{h}$ is known (static or Neumann boundary conditions). The virtual work done by the contact force at each base can be posed in the form

$$\boldsymbol{G}_L \cdot \mathsf{N}_L\hat{\boldsymbol{c}} + \boldsymbol{G}_0 \cdot \mathsf{N}_0\hat{\boldsymbol{c}} \tag{A.1}$$

where the vector $\boldsymbol{G}$ has the expression $\boldsymbol{G} = [\overline{V}_y, -\overline{M}_x, \overline{N}^{\mathrm{a}}, \overline{N}^{\mathrm{b}}, \overline{V}_x^{\mathrm{a}}, \overline{M}_y^{\mathrm{a}}, \overline{V}_x^{\mathrm{b}}, \overline{M}_y^{\mathrm{b}}, \overline{M}_\zeta, \overline{B}]$ and the pedex 0 or L refers to the considered end section on which it is evaluated. The quantities are the resultant of the contact force and the bar denotes that they are evaluated with respect to the known contact force and their expression is the same as the previously defined resultants of stress, once σ, τ_x, τ_y is replaced by h_ζ, h_x, h_y. The two shear resultants, not previously defined, have the expressions

$$\overline{V}_{x0} = \int_S -h_x \,\mathrm{d}S \quad \overline{V}_{y0} = \int_S -h_y \,\mathrm{d}S \tag{A.2a, b}$$

The operators N_0 and N_L are the traces of N for $\zeta = 0$ and $\zeta = L$ respectively. By integrating by parts it is possible to obtain the field differential equations discussed previously and the following boundary conditions:

$$\mathsf{F}_{0R}\boldsymbol{R} + \mathsf{F}_{0g}\boldsymbol{g} + \boldsymbol{G}_0 = 0, \quad \mathsf{F}_{LR}\boldsymbol{R} + \mathsf{F}_{Lg}\boldsymbol{g} - \boldsymbol{G}_L = 0 \tag{A.3a, b}$$

where F_{0R} and F_{LR} are the traces of the operator F_R which maps $\boldsymbol{R}$ into $\mathsf{F}_R\boldsymbol{R} = [M_x', -M_x, N^{\mathrm{a}}, N^{\mathrm{b}}, -M_y^{\mathrm{a}\prime} - \mu_y, -M_y^{\mathrm{a}}, -M_y^{\mathrm{b}\prime} + \mu_y, -M_y^{\mathrm{b}}, -B' + M_\zeta + \beta, B]$ and F_{0g} and F_{Lg} are the traces of the operator F_g which maps $\boldsymbol{g}$ into $\mathsf{F}_g\boldsymbol{g} = [-m_x, 0, 0, 0, m_y^{\mathrm{a}}, 0, 0, 0, -b, 0]$.

In the opposite case in which the displacements at the end sections are completely assigned (kinematic or Dirichelet boundary conditions) the work done by the stress as the expression

$$\mathsf{F}_{LR}\boldsymbol{R}\cdot\boldsymbol{C}_L - \mathsf{F}_{0R}\boldsymbol{R}\cdot\boldsymbol{C}_0 = 0 \tag{A.4}$$

where the known vector $\boldsymbol{C}$ has the expression $\mathbf{C} = \left[\overline{v}_0, \overline{v}_0', \overline{w}_0^{\mathrm{a}}, \overline{w}_0^{\mathrm{a}}, \overline{u}_0^{\mathrm{a}}, \overline{u}_0^{\mathrm{a}\prime}, \overline{u}_0^{\mathrm{b}}, \overline{u}_0^{\mathrm{b}\prime}, \overline{\varphi}, \overline{\varphi}'\right]$ and the pedex 0 or L refers to the considered end section on which it is evaluated.

Integration by parts leads to the following kinematic conditions

$$\mathsf{N}_0\boldsymbol{c} - \boldsymbol{C}_0 = 0, \quad \mathsf{N}_L\boldsymbol{c} - \boldsymbol{C}_L = 0 \tag{A.5a, b}$$

The boundary condition can obviously be of mixed type.

A.2. Operator K

The operator K can be decomposed in the two operators K_b and K_c furnishing the beam and connection stress resultants from the beam and connection generalized strain in the following manner

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{R}_\mathrm{b} \\ \boldsymbol{R}_\mathrm{c} \end{bmatrix} = \begin{bmatrix} \mathsf{K}_\mathrm{b} & 0 \\ 0 & \mathsf{K}_\mathrm{c} \end{bmatrix} \mathsf{D}\boldsymbol{c} = \mathsf{K}\mathsf{D}\boldsymbol{c} \tag{A.6}$$

where

$$\mathsf{K}_\mathrm{b} = \begin{bmatrix} I_{yy} & C_y^{\mathrm{a}} & C_y^{\mathrm{b}} & -I_{xy}^{\mathrm{a}} & -I_{xy}^{\mathrm{b}} & \cdots & \cdots \\ C_y^{\mathrm{a}} & A^{\mathrm{a}} & \vdots & -C_x^{\mathrm{a}} & \vdots & \vdots & C_\omega^{\mathrm{a}} \\ C_y^{\mathrm{b}} & \vdots & A^{\mathrm{b}} & \vdots & -C_x^{\mathrm{b}} & \vdots & C_\omega^{\mathrm{b}} \\ -I_{xy}^{\mathrm{a}} & -C_x^{\mathrm{a}} & \vdots & I_{xx}^{\mathrm{a}} & \vdots & \vdots & -I_{x\omega}^{\mathrm{a}} \\ -I_{xy}^{\mathrm{b}} & \vdots & -C_x^{\mathrm{b}} & \vdots & I_{xx}^{\mathrm{b}} & \vdots & -I_{x\omega}^{\mathrm{b}} \\ \cdots & \vdots & \vdots & \vdots & \vdots & J & \cdots \\ \cdots & C_\omega^{\mathrm{a}} & C_\omega^{\mathrm{b}} & -I_{x\omega}^{\mathrm{a}} & -I_{x\omega}^{\mathrm{b}} & \cdots & I_{\omega\omega} \end{bmatrix} \tag{A.7a}$$

$$\mathsf{K}_\mathrm{c} = \begin{bmatrix} D & \cdots & -H_x & H_\omega \\ \cdots & D & \vdots & \cdots \\ -H_x & \vdots & K_{xx} & -K_{x\omega} \\ H_\omega & \cdots & -K_{x\omega} & K_{\omega\omega} \end{bmatrix} \tag{A.7b}$$

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