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Separating subdivision of bicycle wheel inequalities over cut polytopes

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Abstract

In this paper, we give a separation algorithm for a class of valid inequalities for cut polytopes and report the upper bound of the maximum cut problem obtained by separating over this class of inequalities in some computational experiments. © 1998 Elsevier Science B.V. All rights reserved.

Keywords: Maximum cut; Separation algorithm

1. Introduction

Let $G=(V,E)$ be a simple connected graph. Given $S\subseteq V$, $\delta(S)$, the *cut induced by* S, is the set of edges with exactly one end in S. (Note that we allow $S=\emptyset$ and V.) The *maximum cut problem* is to find a cut of the largest size. Given $A\subseteq E$, z^A, the incidence vector of A is defined as follows: $z_e^A=1$ if and only if $e\in A$. The *cut polytope*, P_G^C, for G is the convex hull of incidence vectors of all the cuts of G. The *separation problem* for a class of valid inequalities, $\mathscr{C}$, is the following: Given z^*, either conclude that z^* satisfies every inequality in $\mathscr{C}$ or find an inequality in $\mathscr{C}$ that z^* violates. Barahona and Mahjoub [1] introduced a class of inequalities known as the bicycle wheel inequalities and Gerards [6] solved the corresponding separation problem. Barahona and Mahjoub [1] also gave an operation, known as the subdivision of edges, to enlarge this class of inequalities. Our objective is to extend the algorithm in Gerards [6] to include these inequalities with the same computational complexity; this is given in Section 3. However, the running time of this algorithm in its most general form is $O(n^5)$. It will be impractical to employ such an algorithm in an cutting plane implementation. So we use instead an $O(n^3)$ heuristic when testing the effectiveness of these cutting planes for random graphs; the results from these computational experiments are presented in Section 4.

2. Preliminaries

The most basic types of valid inequalities for cut polytopes are $z_e\geqslant 0$ and $z_e\leqslant 1$ where $e\in E$; these are, respectively, known as the *lower and upper bound inequalities*. Let C be the edge-set of a cycle in G. Then $\sum(z_e : e\in F)-\sum(z_e : e\in C\backslash F)\leqslant |F|-1$, where F

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is subset of C of odd size, is a valid inequality, called a *cycle inequality*, for P_G^C. In fact, a cycle inequality is facet-inducing if and only if the cycle is "chordless". This class of inequalities was introduced in Barahona and Mahjoub [1]. They also gave a separation algorithm for the class consisting of lower bound, upper bound and cycle inequalities. The following result is the crux of their algorithm.

Proposition 2.1. *Let $G=(V,E)$ be a graph. Let w_1 and w_2 be two non-negative weight functions on the edges. Then the following problem for every v_1 and v_2 in V can be solved in $O(|V|^3)$ time: $\min\{\sum(w_1(e) : e \in F) + \sum(w_2(e) : e \in P \backslash F) : P$ is the edge-set of a walk from v_1 to v_2 and $F \subseteq P$ is odd.$\}$.*

This proposition generalizes the well-known algorithm for finding a minimum weight odd cycle in a graph in $O(|V|^3)$ time. A graph is called a *bicycle wheel of size* p if it consists of a cycle of length p and two vertices that are adjacent to each other and to every vertex in the cycle. Let W be the edge-set of a bicycle wheel of size $2k+1$ where $k \geqslant 1$. Then the inequality $\sum(z_e : e \in W) \leqslant 2(2k+1)$ is valid for P_G^C where G is a graph having this bicycle wheel as a subgraph; it is a *bicycle wheel inequality*. For example, $\sum(z_e : e \in W) \leqslant 10$ where W is the set of edges of the graph in Fig. 1a. Throughout the discussion, we will assume the vertices on the cycle are $v_1, v_2, \ldots, v_{2k+1}$ and the two universal vertices are v_{0_1}, v_{0_2} in a bicycle wheel. We may also refer to $\{v_{0_1}, v_{0_2}\}$ as the *hub*. Barahona and Mahjoub [1] also showed that bicycle wheel inequalities are facet-inducing. Gerards [6] gave a separation algorithm for the class consisting of lower bound, upper bound, cycle and bicycle wheel inequalities.

3. Bicycle inequalities with subdivisions

Given $G=(V,E)$ and $z^* \in \Re^{|E|}$, the separation algorithm given in Gerards [6] can be stated as follows: Assume that there is no violated cycle inequality. For every pair of adjacent vertices u and v, construct a graph $G'=(V',E')$ where $V_{u,v} = \{w \in V : (u,w),(v,w) \in E\}$ and $E_{u,v} = \{(w,w') \in E : w,w' \in V_{u,v}\}$ and the weight for $(w,w') \in E_{u,v}$ is $2 - z^*_{(w,w')} - 1/2(z^*_{(u,w)} + z^*_{(v,w)} + z^*_{(u,w')} + z^*_{(v,w')})$. Then there is a violated bicycle wheel inequality if and only if the minimum weight odd cycle in this auxiliary graph has weight less than $z^*_{(u,v)}$. So the running time of this algorithm is $O(mn^3)$ where $n=|V|$ and $m=|E|$. Our goal is to find a separation algorithm for a larger class of inequalities without increasing the complexity of this algorithm. The class of bicycle wheel inequalities can indeed be enlarged by applying the following result of Barahona and Mahjoub [1] to every edge of a given bicycle wheel W.

Theorem 3.1. *Let $G=(V_G,E_G)$ be a graph and $I : a^{\mathrm{T}}z \leqslant c$ be a valid inequality for P_G^C. Let (i,j) be an edge with $a_{(i,j)} \neq 0$. Let $G'=(V',E')$ be a graph obtained from G in the following way: Vertices $i_1, i_2, \ldots, i_k$ are added; the edge-set $P = \{(i,i_1),(i_1,i_2),\ldots,(i_k,j)\}$ is added; and (i,j) is removed. Let P^+ and P^- define a partition of P with $|P^+|$ odd if $a_{(i,j)}>0$, and $|P^-|$ even if $a_{(i,j)}<0$. Let $a' \in \Re^{E'}$ be defined as follows: $a'_e = a_e$ if $e \in E_G \cap E'$, $a'_e = a_{(i,j)}$ if $e \in P^+$ and $a'_e = -a_{(i,j)}$ if $e \in P^-$. Then $I' : a'^{\mathrm{T}}z \leqslant c + (|P^+|-1)a_{(i,j)}$ if $a_{(i,j)}>0$ and $I' : a'^{\mathrm{T}}z \leqslant c - |P^-|a_{(i,j)}$ if $a_{(i,j)}<0$ is valid for $P_{G'}^C$. Moreover, if I is facet-inducing for P_G^C, then I' is facet-inducing for $P_{G'}^C$.*

Let P_e be the path that replaces the edge e. Then the inequality becomes

$$\sum_{e\in W}\sum_{f\in P_e^+} z_f - \sum_{e\in W}\sum_{f\in P_e^-} z_f \leqslant 2(2k+1) + \sum_{e\in W}(|P_e^+|-1).$$

(Note that each P_e^+ is odd.) Fig. 1b is obtained from Fig. 1a by such replacements. For example, if we set $P^+_{(3,4)} = \{(4,13)\}$, $P^+_{(1,4)} = \{(4,12)\}$, $P^+_{(1,2)} = \{2,10\}$ and $P^+_{(4,5)} = \{(4,8),(8,9),(9,5)\}$, then the corresponding inequality can be simplified to $\sum(z_e : e \in W\backslash F) - \sum(z_e : e \in F) \leqslant 12$ where W is the set of edges of the graph in Fig. 1b and $F = \{(3,13),(1,11),(11,12),(1,10)\}$. Since a bicycle wheel inequality is facet-inducing, such a new inequality is also facet-inducing for P_G^C if its support graph is an induced subgraph of G. In general, it is still a valid inequality since I is a valid inequality for P_G^C implies I is valid

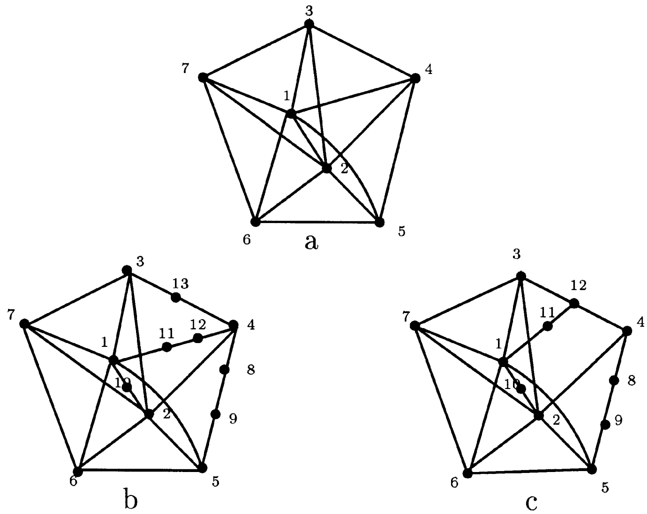

Fig. 1. Examples.

for $P^C_{G_1}$ if $G \subseteq G_1$. If we let $w_f = 1 - z_f$, then the inequality can be rewritten as

$$\sum_{e\in W}\left(\sum_{f\in P_e^+} w_f + \sum_{f\in P_e^-} z_f\right) \geqslant 2k+2. \tag{1}$$

This indeed enlarged the class substantially. From Eq. (1), one is tempted to say every P_e has minimum weight of the type in Proposition 2.1 in a most violated inequality of the form (1). However this will create a problem because these minimum weight paths P_e's may intersect and may not even be paths. We can overcome this problem by enlarging the class further to include inequalities by allowing the replacement paths to be walks and allowing them to intersect each other. The validity of such inequalities follows from the following simple observation.

Proposition 3.2. *Let I be a valid inequality for P^C_G. Suppose G_1 is obtained from G through a sequence of identifications of vertices that produces no loops. Let G' be the graph obtained from G_1 by deleting the multiple edges of it. Then I' is valid for $P^C_{G'}$ where I' is obtained from I by adding up the coefficients of the multiple edges* (*before their deletion*).

For example, the graph in Fig. 1c is obtained from Fig. 1b by identifying 12 and 13. However $P_{(3,4)}$ and $P_{(1,4)}$ intersect each other. To be precise, $P_{(3,4)} = \{(3,12),(4,12)\}$ and $P_{(1,4)} = \{(1,11),(11,12),(4,12)\}$. If we set $P^+_{(3,4)} = \{(4,12)\}$, $P^+_{(1,4)} = \{(4,12)\}$, $P^+_{(1,2)} = \{2,10\}$ and $P^+_{(4,5)} = \{(4,8),(8,9),(9,5)\}$, then the corresponding inequality can be simplified to $\sum(z_e : e \in W\backslash(F \cup \{(4,12)\})) + 2z_{(4,12)} - \sum(z_e : e \in F) \leqslant 12$ where W is the set of edges of the graph in Fig. 1c and $F = \{(1,11),(11,12),(1,10),(3,12)\}$. The inequalities in this enlarged class are the *generalized bicycle wheel inequalities*. Such a generalized inequality still has the form in (1). (Of course, P_e^+ and P_e^- are now multisets rather than sets.)

So given a generalized bicycle wheel inequality, it must come from a bicycle wheel inequality. Let W be the underlying bicycle wheel corresponding to this bicycle wheel inequality. Let C_i^1 be the cycle consisting of the edges (v_{0_1}, v_i), (v_{0_1}, v_{i+1}) and (v_i, v_{i+1}), and C_i^2 be the cycle consisting of the edges (v_{0_2}, v_i), (v_{0_2}, v_{i+1}) and (v_i, v_{i+1}) in the original bicycle wheel. (We take $2k+2$ to be 1.) Then Eq. (1) can be rewritten as

$$\sum_{i=1}^{2k+1} \frac{1}{2}\left(\left(\sum_{e\in C_i^1}\left(\sum_{f\in P_e^+} w_f + \sum_{f\in P_e^-} z_f\right) - 1\right)\right.$$
$$\left.+\left(\sum_{e\in C_i^2}\left(\sum_{f\in P_e^+} w_f + \sum_{f\in P_e^-} z_f\right) - 1\right)\right) \geqslant g,$$

where

$$g = 1 - \left(\sum_{f \in P^{+}_{(v_{0_1}, v_{0_2})}} w_f + \sum_{f \in P^{-}_{(v_{0_1}, v_{0_2})}} z_f \right).$$

So given z^*, if we want to find a generalized bicycle inequality that is most violated by z^*, it must be of the form

$$\sum_{i=1}^{2k+1} \frac{1}{2} \left(\left(\sum_{e \in C_i^1} (\alpha(e)) - 1 \right) + \left(\sum_{e \in C_i^2} (\alpha(e)) - 1 \right) \right) \geqslant 1 - (\alpha(v_{0_1}, v_{0_2})), \tag{2}$$

where $\alpha(e)$ solves $\min\{\sum(w^*(f): f \in F) + \sum(z^*(f): f \in P \backslash F): P$ is a walk from u to v and $F \subseteq P$ is odd.$\}$ if $e = (u, v)$. (w^* is w with z replaced by z^*). These $\alpha(e)$ can be found by using Proposition 2.1.

Given z^* and an underlying graph $G = (V, E)$, we assume it satisfies all lower bound, upper bound and cycle inequalities. The observations in the above paragraph lead to the following algorithm to find a most violated generalized bicycle wheel inequality with $\{v_{0_1}, v_{0_2}\}$ as the hub: Construct an auxiliary graph $H = (V_H, E_H)$ with H being a complete graph and $V_H = V \backslash \{v_{0_1}, v_{0_2}\}$ and find a minimum weight odd cycle of H where the weight

$$w_H(u, v) = \alpha(u, v) + \tfrac{1}{2}(\alpha(v_{0_1}, u) + \alpha(v_{0_1}, v) + \alpha(v_{0_2}, u) + \alpha(v_{0_2}, v)) - 1.$$

(Note that this corresponds to the term inside the outer summation in Eq. (2).) We need to check that $w_H(u, v) \geqslant 0$. (We note that in Gerards' separation algorithm for bicycle wheel inequalities, $\alpha(u, v)$ is replaced by $1 - z^*(u, v)$ in its auxiliary graph.) To see this, we observe that

$$w_H(u, v) = \tfrac{1}{2}(\alpha(u, v) + \alpha(v_{0_1}, u) + \alpha(v_{0_1}, v) - 1) + \tfrac{1}{2}(\alpha(u, v) + \alpha(v_{0_2}, u) + \alpha(v_{0_2}, v) - 1).$$

It is not difficult to check that the closed walks consisting of a solution of each of $\alpha(u, v)$, $\alpha(v_{0_1}, u)$ and $\alpha(v_{0_1}, v)$ ($\alpha(u, v)$, $\alpha(v_{0_2}, u)$ and $\alpha(v_{0_2}, v)$) has an odd number of w_f^*'s. Since z^* satisfies all the cycle inequalities, $w_H(u, v) \geqslant 0$. Let C be an odd cycle of minimum weight in H. Then there is a violated generalized bicycle wheel inequality with $\{v_{0_1}, v_{0_2}\}$ as the hub of its support graph if and only if $\sum(w_H(f): f \in C) < 1 - \alpha(v_{0_1}, v_{0_2})$. Hence we have the following result.

Theorem 3.3. *The separation algorithm for the class consisting of lower bound, upper bound, cycle and generalized bicycle wheel inequalities can be solved in polynomial time.*

The running time of this algorithm is $O(n^2)$ (where n is the number of vertices) times the running time of finding a minimum weight odd cycle in the auxiliary graph, given $O(n^5)$. So if the graph is dense, this algorithm is of the same complexity of the algorithm in Gerards [6]. If we consider a *slightly* smaller class, that is, we only consider the $\{v_{0_1}, v_{0_2}\}$ as the hub if (v_{0_1}, v_{0_2}) is an edge. (Note that we still include those configurations that has a path from v_{0_1} to v_{0_2}.) Then the complexity is $O(mn^3)$ (where m is the number of edges) as before. From a more practical point of view, it seems that our algorithm requires extra calculations, namely, shortest path calculations to find the weight of the edges of the auxiliary graph; however, these values are actually readily available. A typical implementation will run this separation subroutine when there is no violated cycle inequality; so these shortest path calculations are performed in the separation subroutine for the cycle inequalities.

Cutting plane algorithms usually make rather fast progress initially but eventually require a lot of steps, each of which gives a very small improvement. One can expect this to happen when one uses the cycle inequalities. Therefore one may expect that the requirement of the separation algorithm, that there be no violated cycle inequalities, would be a big disadvantage. However, it is possible to use the algorithm even if such violations exist. Suppose we set up the auxiliary graph H for the fixed hub $\{v_{0_1}, v_{0_2}\}$. To ensure $w_H(u, v) \geqslant 0$ for all $(u, v) \in H$, we only require certain cycle inequalities not to be violated by the current point z^*. (For instance, the cycle inequalities corresponding to the cycles that use either v_{0_1} or v_{0_2} will do.) So it is possible for one to find violated generalized bicycle wheel inequalities when not all cycle inequalities are satisfied by the current point. Moreover, even if some of the edge-weights for H are negative, it is still possible to proceed with the algorithm by replacing those negative weights by a weight of 0. (Of course, in this case, there is no guarantee that we can find a violated generalized bicycle wheel inequality even if one exists.) In fact, one can identify a violated cycle inequality if $w_H(u,v)$ is negative since we know that $w_H(u,v)$ can be written as a "slack" of two cycle inequalities.

4. Computational experiments

A popular method to solve a maximum cut problem is to use a cutting plane method, for example, Saigal [8] uses lower bound, upper bound and cycle inequalities with various heuristics and tricks to solve problems where the underlying graphs are grid graphs with a universal vertex. As we have mentioned earlier, the complexity of our separation algorithm is n^2 (where n is the number of vertices) times the running time of finding a minimum weight odd cycle in the auxiliary graph. This $O(n^5)$ running time is too much for a separation subroutine. So we consider a heuristic and look at the effect of adding such a heuristic separation subroutine of generalized bicycle wheel inequalities in a generic cutting plane algorithm. By generic, we mean we employ no tricks to speed up the implementation, for example, we made no attempt to find violated generalized bicycle wheel inequalities before all the violated cycle inequalities are found. Moreover, we also take a very conservative approach when dropping non-active constraints from the current LP, we only drop a constraint if it is non-active for the last twenty iterations; generalized bicycle wheel inequalities are not dropped.

One way to cut down the running time of our algorithm is to separate a subset of the inequalities by not running through all the possible choices of the hub. Another idea is to make the auxiliary search graph smaller. We employ both of these ideas in our heuristic. The heuristic is the following: Randomly pick seven sets of vertices of size r (r is fixed). For each of the seven sets, say the elements are $v_1, v_2, \ldots, v_r$, use $\{v_1, v_2\}$ as the hub and construct the auxiliary graph H using only $v_3, \ldots, v_r$. Since a good upper bound is essential in any cutting plane algorithm, it is interesting to compare the upper bound obtained by using lower bound, upper bound and cycle inequalities, and the upper bound obtained by further including the generalized bicycle wheel inequalities. We compare these two upper bounds against a heuristic lower bound for graphs that are randomly generated. Our heuristic lower bound is computed by repeatedly choosing cuts in the graph at random to search for a local maximum. (Our local maximum is a cut $\delta(S)$ that is at least as large as cuts that can be obtained from S by adding one or two vertices to it or by deleting one or two vertices from it.)

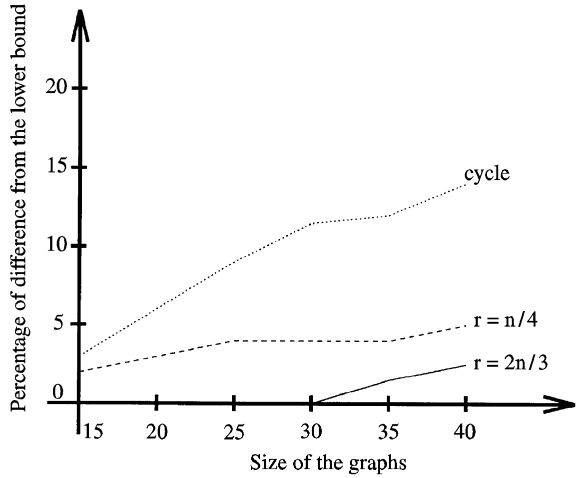

Fig. 2. Density is 0.7.

Our LP solver is CPLEX. We tested graphs generated randomly of size 15, 20, 25, 30, 35 and 40 with density 0.7, 0.8 and 0.9. (We also tested graphs with density less than or equal to 0.6 but the effect is small. This is not surprising since these graphs are generated randomly and they probably do not have K_5 as minors; Barahona and Mahjoub [1] showed that for such graphs, lower bound, upper bound and cycle inequalities are enough to describe their cut polytopes.) The test results are plotted in Figs. 2, 3 and 4 for density equal to 0.7, 0.8 and 0.9, respectively. The dotted curve represent the percentage that the bound, obtained from separating over the lower bound, upper bound and cycle inequalities, comes within our lower bound. The dashed (solid) curve represent the percentage that the bound, obtained from separating over the lower bound, upper bound, cycle and generalized bicycle wheel inequalities with heuristic and $r = n/4$ ($r = 2n/3$), comes within our lower bound. For density equal to 0.9, $r = 2n/3$ makes no significant gain over $r = n/4$. One may wonder whether the inclusion

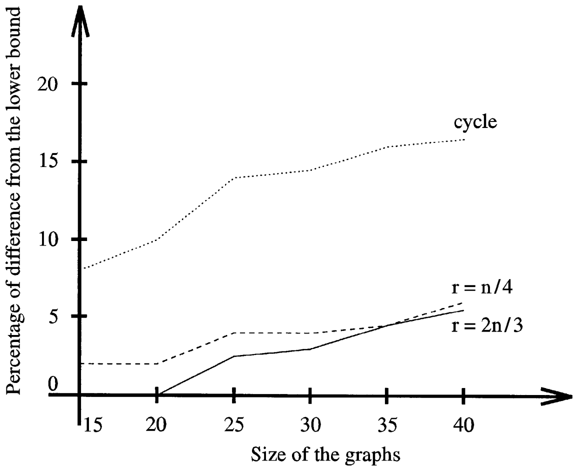

Fig. 3. Density is 0.8.

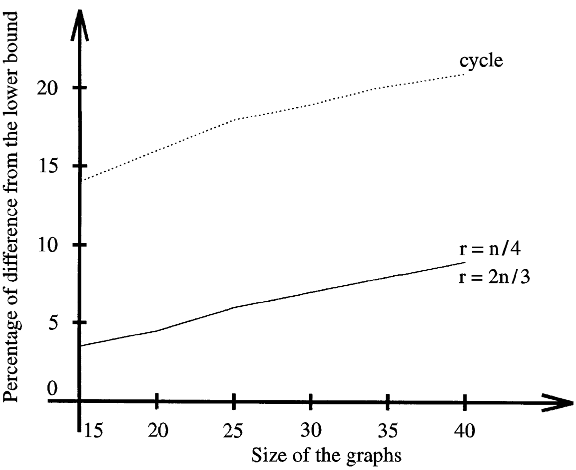

Fig. 4. Density is 0.9.

of these generalized bicycle wheel inequalities in addition to the bicycle wheel inequalities has any impact in a cutting plane method. From our experiments, these generalized bicycle wheel inequalities have a significant impact on decreasing the upper bound in the sense that many violated generalized bicycle wheel inequalities are found by the separation subroutine. Furthermore, they seem to be most effective when applied to those tested graphs with density equal to 0.7.

5. Remarks

It is known that any valid inequality for stable set polytope of G can be transformed to a valid inequality for cut polytope of G' (G with a universal vertex). (See DeSimone [5] and Padberg [7].) By using known valid inequalities for stable set polytopes whose support graph is a "wheel" with a hub of size p, one can obtain valid inequalities for cut polytopes whose support graph is a "wheel" with a hub of size $p+1$. These classes of inequalities can be separated using a similar technique. (See Cheng [3].) Although the inequalities arising from "wheel" with hub of size two are not the generalized bicycle wheel inequalities, they can be obtained from them through a series of transformations from Barahona and Mahjoub [1].

The idea of allowing paths to be walks and allowing them to intersect when finding a separation algorithm first appears in Cheriyan et al. [2] in their study of the maximum 2-satisfiability problem. They used "cycle" and "wheel" inequalities in a cutting plane algorithm to solve instances of maximum 2-satisfiability problem as well as instances from other problems, including the maximum cut problem, by transforming them into a maximum 2-satisfiability problem. (A 2-satisfiability problem can be expressed as a bidirectional graph. The underlying graphs of the "cycle" and "wheel" inequalities are "cycle" and "wheel" with a "hub" of size one.) These "cycle" and "wheel" inequalities are related to the cycle and generalized bicycle wheel inequalities for cut polytopes. Although, the "wheel" inequalities are related to the generalized bicycle wheel inequalities, the bound does not change when solving a maximum cut problem in this setting. (See Cunningham, Tunçel and Wang [4] for a theoretical treatment.)

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