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On Some Kripke Complete and Kripke Incomplete Intermediate Predicate Logics

Abstract. The Kripke-completeness and incompleteness of some intermediate predicate logics is established. In particular, we obtain a Kripke-incomplete logic $(\mathbf{H}_* + A + D + K)$ where $\mathbf{H}_*$ is the intuitionistic predicate calculus, A is a disjunction-free propositional formula, $D = \forall x(P(x) \vee Q) \supset \forall x P(x) \vee Q$, $K = \neg\neg\forall x(P(x) \vee \neg P(x))$ (the negative answer to a question of T. Shimura).

Kripke semantics is a powerful, useful and convenient tool for dealing with propositional logics (such as intermediate, modal etc.). On the other hand, in the predicate case many simple and natural intermediate logics happen to be incomplete, and examples of completeness are rather seldom. Probably the first general completeness result for intermediate predicate logics with the axiom of constant domain was proved by T. Shimura [6]. Namely, he studied Kripke-completeness for predicate logics of the form (i) $(\mathbf{H}_* + D + \Gamma)$ and (ii) $(\mathbf{H}_* + D + K + \Gamma)$, with $\mathbf{H}_*$ being the intuitionistic predicate logic, Γ being some set of propositional formulas, D and K being the following axioms:

$$D: \qquad \forall x(P(x) \vee Q) \supset \forall x P(x) \vee Q,$$

$$K: \qquad \neg\neg\forall x(P(x) \vee \neg P(x)),$$

where P is a unary predicate letter and Q is a propositional letter. Namely, he proved the Kripke-completeness of the logics (i) and (ii) for every set Γ of "subframe" formulas, i.e. canonical formulas of the form $X(M)$ (or, equivalently, for every set Γ of implicational formulas). Also he proved the Kripke-completeness of the logics (ii) for sets Γ of "cofinal subframe" formulas, i.e. canonical formulas of the form $X(M, \bot)$ (in other words, for sets Γ of disjunction-free propositional formulas) satisfying some additional technical condition C (see [6], Theorem 3.11). It is known [3, 5] that the logic $(\mathbf{H}_* + N_5 + D)$ is Kripke-incomplete where N_5 is the following cofinal subframe formula

$$N_5: \qquad \neg\neg q \vee \neg q.$$

On the other hand, Shimura in [6] asked whether the condition C in his Theorem 3.11 may be omitted, that is whether every logic (ii) for a set Γ of

Presented by **Hiroakira Ono**; *Received* April 25, 1995; *Revised* December 13, 1995

Studia Logica **61**: 281–292, 1998.

disjunction-free propositional formulas is Kripke-complete. Here we give a negative answer to this question. Namely, we consider two cofinal subframe formulas:

$$N_7: \qquad \neg\neg q \vee (\neg\neg q \supset q),$$
$$\overline{N}: \qquad \neg\neg p \supset (q \supset p) \vee (\neg q \supset p).$$

Note that in terms of [6, par. 3], $(\mathbf{H}_* + N_7) = (\mathbf{H}_* + X''(M_2, \perp))$ and $(\mathbf{H}_*+\overline{N}) = (\mathbf{H}_*+X''(M_3, \perp))$ (canonical formulas of some Kripke frames). It is easily seen that the condition C holds for N_7 but not for N. Hence Kripke-completeness of $(\mathbf{H}_* + N_7 + K + D)$ follows from [6] (see Corollary 3.13(2)). On the other hand, Shimura asked (Question 3.14 in [6]) whether $(\mathbf{H}_* + \overline{N} + K + D)$ is Kripke-complete.

In this paper we show that the predicate logics $(\mathbf{H}_* + \overline{N} + K + D)$, $(\mathbf{H}_* + \overline{N} + D)$, $(\mathbf{H}_* + N_7 + D)$, $(\mathbf{H}_* + \overline{N})$, $(\mathbf{H}_* + N_7)$ are Kripke-incomplete, and the logics $(\mathbf{H}_* + \overline{N} + K)$, $(\mathbf{H}_* + N_7 + K)$ are Kripke-complete. Cf. this situation to the case of the axiom N_5: the logics $(\mathbf{H}_* + N_5)$, $(\mathbf{H}_* + N_5 + K)$, $(\mathbf{H}_* + N_5 + K + D)$ are Kripke-complete [1], and the logic $(\mathbf{H}_* + N_5 + D)$ is Kripke-incomplete [3, 5]. Note that $(\mathbf{H} + \overline{N}) \subset (\mathbf{H} + N_7) \subset (\mathbf{H} + N_5)$ since $N_5 \supset N_7$ and $(\neg\neg(p \wedge q) \vee (\neg\neg(p \wedge q) \supset (p \wedge q))) \supset \overline{N}$ are theorems of the intuitionistic propositional calculus $\mathbf{H}$. Note also that formulas N_5, N_7, $\overline{N}$ coincide with F_3, F_4, F_{16} from [2] respectively, and N_5, N_7 are formulas from Nishimura's sequence.

The following diagram (Fig. 1) shows the resulting picture of Kripke-completeness and incompleteness for the predicate logics mentioned above ("$\oplus$" means "complete" and "$\ominus$" means "incomplete").

Due to the incompleteness of $(\mathbf{H}_* + \overline{N} + K + D)$ we obtain a Kripke-incomplete logic $(\mathbf{H}_*+A+K+D)$ for a disjunction-free propositional formula A; thus the question 3.14 in [6] is answered negatively.

1. A *predicate Kripke frame* is a pair (M, U) in which M is a propositional frame (i.e. a partially ordered set with the least element 0_M) and U is a domain mapping on M satisfying the standard condition $\forall_{u,v\in M}(u \leq v \Rightarrow \emptyset \neq U(u) \subseteq U(v))$. (M, U) is called a *Kripke frame with constant domain* if U is a constant mapping: $\forall_{u\in M}\ U(u) = U(0_M)$. The formula D is valid in (M, U) iff (M, U) is a frame with constant domain. M is called a *K-frame* if $\forall_{u\in M}\exists_{v\in \overline{M}}\ u \leq v$ (here $\overline{M}$ denotes the set of all maximal elements of M). The formula K is valid in every (M, U) over a K-frame M.

Elements u, v of M are called *compatible* ($u\,C\,v$) if $\exists_{w\in M}(u \leq w \wedge v \leq w)$. M is an *N_5-frame* if $\forall_{u,v\in M}\ u\,C\,v$. M is an *N_7-frame* if $\forall_{u\in M\setminus \overline{M}}\forall_{v\in M}\ u\,C\,v$. M is an $\overline{N}$-frame if $\forall_{u,v\in M\setminus\overline{M}}\ u\,C\,v$. Obviously, $M \vDash N_5$ iff M is an

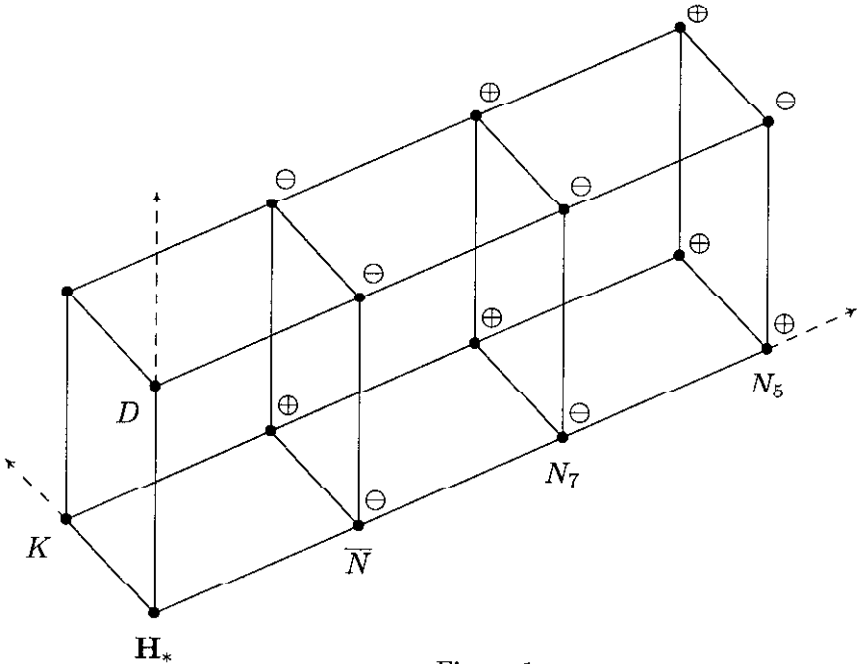


Figure 1.

N_5-frame; and analogously for N_7 and $\overline{N}$. M is a *KN_7-frame* if $\overline{M} \neq \emptyset$ and $\forall_{u \in M \setminus \overline{M}} \forall_{v \in \overline{M}}\, u \leq v$.

LEMMA 1. *M is an N_7-frame iff M is an N_5-frame or a KN_7-frame.*

PROOF. If a K-frame M validates N_7 then M is a KN_7-frame. Now let M be not a K-frame and $M \models N_7$. Fix $w_0 \in M$ such that $\forall_{w \geq w_0}\, w \notin \overline{M}$. Then $\forall_{w \geq w_0} \forall_{u \in M}\, u\, C\, w$. Then for any $u, v \in M$ there exist $w_1, w_2 \in M$ such that $w_0 \leq w_1 \leq w_2$ and $u \leq w_1$, $v \leq w_2$. Thus M is an N_5-frame. ∎

Let $\overline{N}^{(n)} = \forall \boldsymbol{x}(q \wedge \neg\neg P_1(\boldsymbol{x}) \supset P_1(\boldsymbol{x})) \vee \forall \boldsymbol{x}(\neg q \wedge \neg\neg P_2(\boldsymbol{x}) \supset P_2(\boldsymbol{x}))$, where q is a propositional letter, P_1 and P_2 are n-ary predicate letters, $\boldsymbol{x} = (x_1, \ldots, x_n)$, $n \in \omega$.

LEMMA 2. (1) $(\mathbf{H} + N_7) \vdash (\neg\neg q \supset \neg\neg p) \vee (\neg\neg p \supset q)$.
(2) $(\mathbf{H} + \overline{N}) \vdash (q \wedge \neg\neg p_1 \supset p_1) \vee (\neg q \wedge \neg\neg p_2 \supset p_2)$.
(3) $(\mathbf{H}_* + \overline{N} + K) \vdash \overline{N}^{(n)}$ *for every* $n \in \omega$.

PROOF. For (1), note that $\neg\neg(p \wedge q) \vee (\neg\neg(p \wedge q) \supset (p \wedge q))$ is an instance of N_7; for (2), analogously apply $\neg\neg C \supset (q \supset C) \vee (\neg q \supset C)$ where $C = (\neg\neg p_1 \supset p_1) \wedge (\neg\neg p_2 \supset p_2)$; and (3) follows from (2) since $(\mathbf{H}_* + K) \vdash \neg\neg\forall \boldsymbol{x}(\neg\neg P_i(\boldsymbol{x}) \supset P_i(\boldsymbol{x}))$ for $i = 1, 2$. ∎

2. THEOREM 1. *The predicate logics* $(\mathbf{H}_* + N_7 + K)$ *and* $(\mathbf{H}_* + \overline{N} + K)$ *are Kripke-complete.*

PROOF for $\mathscr{L} = (\mathbf{H}_* + N_7 + K)$. We use the technique from [1] to construct a Kripke model for an arbitrary $\mathscr{L}$-saturated set of sentences Γ (in a language $\mathcal{L}$). Consider the KN_7-frame $M = M_0 \cup \overline{M}$ in which M_0 is the subordination frame, i.e. the set of all finite sequences of natural numbers, with the partial order $u \leq v \iff \exists_{w \in M_0}\, u * w = v$ ($*$ means the concatenation), and with the least element $0_M = \langle\ \rangle$; $\overline{M}$ is the denumerable set of maximal elements of M, and $\forall_{u \in M_0} \forall_{v \in \overline{M}}\, u < v$. Fix a system of disjoint denumerable sets of constant symbols $\{C_u : u \in M_0\}$ and a system of languages $\{\mathcal{L}_u : u \subset M\}$ such that $\mathcal{L}_{\langle\ \rangle} = \mathcal{L}$, $\mathcal{L}_{u*\langle k\rangle} = (\mathcal{L}_u + C_{u*\langle k\rangle})$, $\mathcal{L}_v = \overline{\mathcal{L}} = (\mathcal{L} + \bigcup_{u \in M_0} C_u)$ for $v \in \overline{M}$ (note that the constants from $C_{\langle\ \rangle}$ are not used in $\mathcal{L}_u$ for $u \in M$). Also we fix a system $\{\Gamma_u : u \in M_0\}$ of $\mathscr{L}$-saturated sets of sentences in the languages $\mathcal{L}_u$ such that:

(i) $\Gamma_{\langle\ \rangle} = \Gamma$ and $\forall_{u,v \in M_0}(u \leq v \Rightarrow \Gamma_u \subseteq \Gamma_v)$;

(ii) if $\neg\neg(B_1 \supset B_2) \in \Gamma_u$, $(B_1 \supset B_2) \notin \Gamma_u$ then $B_1 \in \Gamma_{u*\langle k\rangle}$, $B_2 \notin \Gamma_{u*\langle k\rangle}$ for some $k \in \omega$;

(iii) if $\neg\neg\forall x B(x) \in \Gamma_u$, $\forall x B(x) \notin \Gamma_u$ then $B(c) \notin \Gamma_{u*\langle k\rangle}$ for some $k \in \omega$ and $c \in C_{u*\langle k\rangle}$.

(We assume that $\mathcal{L}$ contains no constants from C_u and that each of $\Gamma_{u*\langle k\rangle}$ is obtained either by (ii) or by (iii).)

LEMMA 3. (1) $\neg\neg A \in \Gamma_{u*\langle k\rangle} \Longrightarrow \neg\neg A \in \Gamma_u$ *for every sentence* A *of the language* $\mathcal{L}_u$.

(2) *The set of sentences* $\overline{\Gamma} = \bigcup_{u \in M_0} \Gamma_u$ *is* $\mathscr{L}$*-consistent.*

PROOF. (1) By the construction of $\Gamma_{u*\langle k\rangle}$, we have a sentence B such that $\neg\neg B \in \Gamma_u$, $B \notin \Gamma_{u*\langle k\rangle}$. Now if $\neg\neg A \in \Gamma_{u*\langle k\rangle} \setminus \Gamma_u$ then $\neg\neg A \supset B \notin \Gamma_u$, and thus Γ_u is $\mathscr{L}$-inconsistent since $\mathscr{L} \vdash (\neg\neg B \supset \neg\neg A) \vee (\neg\neg A \supset B)$, see Lemma 2(1).

(2) We repeat word by word the proof of Lemma 5.3 from [1]. The crucial point is:

$$B_i(\boldsymbol{a}_i, \boldsymbol{b}_i) \subset \Gamma_{v_i*\langle k_i\rangle} \Rightarrow \exists x_i B(x_i, \boldsymbol{b}_i) \in \Gamma_{v_i*\langle k_i\rangle} \Rightarrow$$
$$\Rightarrow \neg\neg\exists x_i B(x_i, \boldsymbol{b}_i) \in \Gamma_{v_i*\langle k_i\rangle} \Rightarrow \neg\neg\exists x_i B(x_i, \boldsymbol{b}_i) \in \Gamma_{v_i},$$

where $\boldsymbol{a}_i \subseteq C_{v_i*\langle k_i\rangle}$, $\boldsymbol{b}_i \subseteq \mathcal{L}_{v_i}$. ∎

Now, due to axiom K, the $\mathscr{L}$-consistency of $\overline{\Gamma}$ implies its $\mathbf{C}_*$-consistency, where $\mathbf{C}_*$ is the classical predicate logic ("Glivenko's theorem", cf. Proposition 3.1 from [1]). Thus for every sentence A of the language $\mathcal{L}_u (u \in M_0)$

the following holds: if $\neg\neg A \notin \Gamma_u$ then $\neg\neg A \notin \overline{\Gamma}$ and we obtain a $\mathbf{C}_*$-saturated extension (in the language $\overline{\mathcal{L}}$) of $\mathbf{C}_*$-consistent set $\overline{\Gamma} \cup \{\neg A\}$. Let $\{\Gamma_v : v \subset \overline{M}\}$ be the set of such $\mathbf{C}_*$-saturated extensions (for all $u \in M_0$ and $\neg\neg A \notin \Gamma_u$). Finally we construct the predicate Kripke frame (M, U), in which $U(u)$ is the set of constants of the language $\mathcal{L}_u$ (for $u \in M$), and the valuation $\vDash$ on (M, U) such that: $u \vDash A$ iff $A \in \Gamma_u$ for every $u \in M$ and every sentence A of the language $\mathcal{L}_u$ (cf. Lemma 4.3 from [1]).

PROOF for $\mathscr{L} = (\mathbf{II}_* + \overline{N} + K)$. Consider the $\overline{N}$-frame $M = M_0 \cup \overline{M}$ where M_0 is the subordination frame, $\overline{M} = \{w_{v,i} : v \in M_0, i \in \omega\} \cup \{w_{v,v'} : v, v' \in M_0, \neg(v\, C\, v' \text{ in } M_0)\}$ and $u < w_{v,i} \iff u \leq v$ in M_0, $u < w_{v,v'} \iff u \leq v$ or $u \leq v'$ in M_0. Fix $\{C_u : u \in M_0\}$, $\{\mathcal{L}_u : u \in M\}$, $\{\Gamma_u : u \in M_0\}$ as above. Let $\{\Gamma_{w_{vi}} : i \in \omega\}$ be the set of $\mathbf{C}_*$-saturated extensions (in the language $\overline{\mathcal{L}}$) of $\Gamma_v \cup \{\neg A\}$ for $\neg\neg A \notin \Gamma_v$, $v \in M_0$, and $\Gamma_{w_{vv'}}$ be $\mathbf{C}_*$-saturated extension of $\Gamma_v \cup \Gamma_{v'}$ — this set is $\mathbf{C}_*$-consistent due to the axiom K and the following lemma.

LEMMA 4. *The set $\Gamma_v \cup \Gamma_{v'}$ is $\mathscr{L}$-consistent for every $v, v' \in M_0 \setminus \{\langle\, \rangle\}$.*

PROOF. Suppose the contrary. Then there exist $A_1(\boldsymbol{b}) \in \Gamma_v$, $A_2(\boldsymbol{c}) \in \Gamma_{v'}$ such that $\mathscr{L} \vdash \neg(A_1(\boldsymbol{b}) \wedge A_2(\boldsymbol{c}))$ (here $\boldsymbol{b}$ and $\boldsymbol{c}$ are disjoint lists of constants from $\mathcal{L}_v \setminus \mathcal{L}_u$ and $\mathcal{L}_{v'} \setminus \mathcal{L}_u$ where $u \leq v$, $u \leq v'$, $\mathcal{L}_v \cap \mathcal{L}_{v'} = \mathcal{L}_u$). Then $\mathscr{L} \vdash \forall \boldsymbol{z}(A_2(\boldsymbol{z}) \supset \neg\exists \boldsymbol{y} A_1(\boldsymbol{y}))$, thus $A \in \Gamma_v$, $\neg A \in \Gamma_{v'}$, where $A = \exists \boldsymbol{y} A_1(\boldsymbol{y})$ is the sentence of the language $\mathcal{L}_u$. Now by the construction of Γ_v and $\Gamma_{v'}$ we have sentences $B_1(\boldsymbol{a})$, $B_2(\boldsymbol{a})$ such that $\neg\neg B_1(\boldsymbol{a}) \in \Gamma_v$, $B_1(\boldsymbol{a}) \notin \Gamma_v$, $\neg\neg B_2(\boldsymbol{a}) \in \Gamma_{v'}$, $B_2(\boldsymbol{a}) \notin \Gamma_{v'}$ (here $\boldsymbol{a}$ is the list of all constants occurring in B_1 or B_2 from $(\mathcal{L}_v \cup \mathcal{L}_{v'}) \setminus \mathcal{L}_u$. Therefore $\forall \boldsymbol{x}(A \wedge \neg\neg B_1(\boldsymbol{x}) \supset B_1(\boldsymbol{x})) \notin \Gamma_u$ and $\forall \boldsymbol{x}(\neg A \wedge \neg\neg B_2(\boldsymbol{x}) \supset B_2(\boldsymbol{x})) \notin \Gamma_u$, in contradiction to Lemma 2(3). ■

3. THEOREM 2. *Every predicate logic $\mathscr{L}$ such that $(\mathbf{H}_* + \overline{N}) \subseteq \mathscr{L} \subseteq (\mathbf{H}_* + N_7)$ is Kripke incomplete.*

PROOF. The formulas $\overline{N}^{(n)}$ (for all $n \in \omega$) are valid in every predicate Kripke frame validating $\overline{N}$. And now we use the semantics of Kripke bundles [5] to show that

$$(\mathbf{H}_* + N_7) \nvdash \overline{N}^{(1)}.$$

A *Kripke bundle* is a triple $\mathscr{F} = (M, D, \pi)$, in which M and D are preordered sets and π is a p-morphism of D onto M. We set $D_u = \pi^{-1}(u)$ for $u \in M$, $D_0 = M$, $D_1 = D$ and $D_n = \bigcup_{u \in M}(D_u)^n$ for $n \in \omega$, $n \geqslant 2$. We define also the preordering $\leq_n$ on D_n:

$$(a_1, \ldots, a_n) \leq_n (b_1, \ldots, b_n) \text{ iff } \bigwedge_{i \leqslant n}(a_i \leq b_i \text{ in } D) \wedge \bigwedge_{i \neq j}(a_i = a_j \Rightarrow b_i = b_j).$$

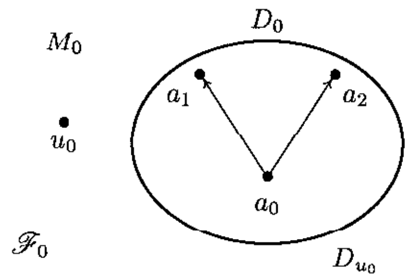
A valuation on $\mathscr{F}$: $u \vDash A(a_1, \dots, a_n)$ (for $u \in M$, $a_1, \dots, a_n \in D_u$) satisfies the following conditions:

(i) $u \vDash B_1 \wedge B_2$ iff $u \vDash B_1$ and $u \vDash B_2$,
$u \vDash B_1 \vee B_2$ iff $u \vDash B_1$ or $u \vDash B_2$,
$u \vDash B_1 \supset B_2\,(a_1, \dots, a_n)$ iff $\forall_{v \geq u} \forall_{b_1,\dots,b_n \subset D_v}$
$(a_1, \dots, a_n) \leq_n (b_1, \dots, b_n) \wedge v \vDash B_1(b_1, \dots, b_n) \Rightarrow v \vDash B_2(b_1, \dots, b_n)$,
$u \nvDash \bot$,
$u \vdash \forall x B(a_1, \dots, a_n, x)$ iff $\forall_{v \geq u} \forall_{b_1,\dots,b_n,c \in D_v}$
$(a_1, \dots, a_n) \leq_n (b_1, \dots, b_n) \Rightarrow v \vDash B(b_1, \dots, b_n, c)$,
$u \vDash \exists x B(a_1, \dots, a_n, x)$ iff $\exists_{c \in D_u}\, u \vDash B(b_1, \dots, b_n, c)$;

(ii) (monotonicity condition):
$(a_1, \dots, a_n) \leq_n (b_1, \dots, b_n) \wedge u \vDash A\,(a_1, \dots, a_n) \Rightarrow v \vDash A\,(b_1, \dots, b_n)$
for every formula A (or, equivalently, for every atomic A).

The predicate logic of a Kripke bundle $\mathscr{F}$ is the set of all formulas which are strongly valid in $\mathscr{F}$ (a formula A is called strongly valid if every substitution instance A' of A is true under every valuation $\vDash$ on $\mathscr{F}$, that is $u \vDash A'(a_1, \dots, a_n)$ for every $u \in M, a_1, \dots, a_n \in D_u$). This set contains all theorems of $\mathbf{H}_*$ and is closed under *modus ponens*, generalization and substitution (see [5]).

Figure 2.



Let us consider the Kripke bundle $\mathscr{F}_0 = (M_0, D_0, \pi_0)$ where $M_0 = \{u_0\}$, $D_0 = \{a_0, a_1, a_2\}$, $a_0 < a_i$ for $i \neq 0$ and $\pi_0(a_i) = u_0$ for $i \leqslant 2$ (Fig. 2). ■

LEMMA 5. (1) *The formula* $\overline{N}^{(1)}$ *is not strongly valid in* $\mathscr{F}_0$.
(2) *The formula* N_7 *is strongly valid in* $\mathscr{F}_0$.

PROOF. (1) $u_0 \nvDash \forall x(Q(a_0) \wedge \neg\neg P_1(a_0, x) \supset P_1(a_0, x))$ and $u_0 \nvDash \forall x(\neg Q(a_0) \wedge \neg\neg P_2(a_0, x) \supset P_2(a_0, x))$ if $u_0 \vDash Q(a_1)$, $u_0 \vDash Q(a_2)$ and $u_0 \nvDash P_i(a_i, a_0)$, $u_0 \vDash P_i(a_i, a_j)$ for all $i, j \in \{1, 2\}$.

(2) Suppose that $u_0 \nvDash \neg\neg A \vee (\neg\neg A \supset A)\,(a_0, a_1, a_2)$. Then $u_0 \vDash \neg A(a_i, a_1, a_2)$ for some $i \leqslant 2$ and $u_0 \vDash \neg\neg A(a_0, a_1, a_2)$, $u_0 \nvDash A(a_0, a_1, a_2)$ (since $u_0 \vDash (\neg\neg A \supset A)(a_j, a_1, a_2)$ for $j \neq 0$), which contradicts to (a_0, a_1, a_2) $\leq_3 (a_i, a_1, a_2)$. ■

REMARK 1. We can demonstrate the Kripke incompleteness of $(\mathbf{H}_* + N_7)$ using the formula $N_5 \vee K$ (see Lemma 1). Also we can take a rather natural formula $K' = \neg\neg\forall x(\neg\neg P(x) \vee \neg P(x))$ instead of $\overline{N}^{(1)}$ in the proof of Theorem 2. Note that $(\mathbf{H}_* + N_7 + (N_5 \vee K)) \vdash \overline{N}^{(n)}$ and $(\mathbf{H}_* + (N_5 \vee K)) \vdash K'$ but $(\mathbf{H}_* + K') \nvdash N_5 \vee K$; consider the Kripke frame with the denumerable constant domain (M, U), $M = \{u_0, v_0\} \cup \{w_i : i \in \omega\}$, $u_0 < v_0$, $u_0 < w_0 < w_1 < w_2 < \ldots$ (Fig. 3). Note that the logic $(\mathbf{H}_* + N_7 + (N_5 \vee K))$ is Kripke incomplete too, since $(\mathbf{H}_* + N_7 + (N_5 \vee K)) \nvdash N_5 \vee \forall y \neg\neg\forall x(P(x,y) \vee \neg P(x,y))$; consider the Kripke bundle $\mathscr{F} = (M, D, \pi)$, $M = \{u_0, v_0, v_1\}$, $u_0 < v_i$, $D = \{a_0, b_0, b_1, c_0, c_1\}$, $a_0 < b_i$, $c_i \leq c_j$, $\pi(a_0) = u_0$, $\pi(b_i) = v_i$, $\pi(c_i) = v_1$ for $i, j \leqslant 1$ (Fig. 4).

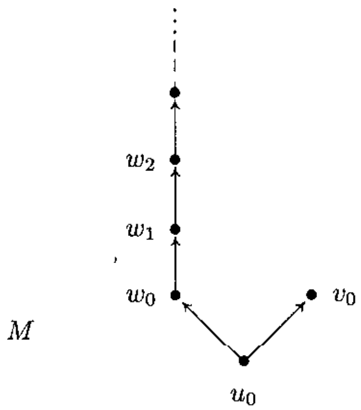


Figure 3.

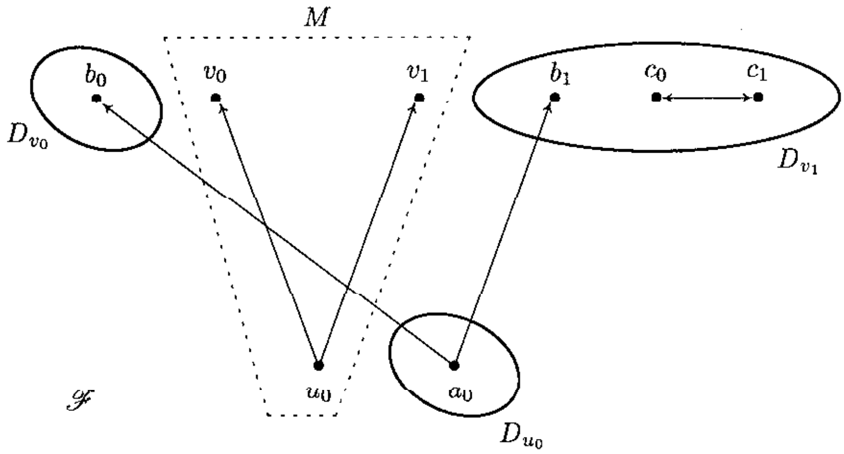


Figure 4.

S. Ghilardi (Theorem 5.4 from [4]) proved that moreover $(\mathbf{H}_*+P_2\wedge W_2)\nvdash K'$ (where P_2 and W_2 are propositional axioms of height 2 and width 2), by using a C-set quite similar to the Kripke bundle $\mathscr{F}_0$. One can easily see that the bundle $\mathscr{F}_0$ also fits for the proof (recall that C-sets are generalizations of Kripke bundles). On the other hand, in the proof of Theorem 5.6 from [4] C-sets are necessary, and Kripke bundles are not sufficient (see [8]).

T. Shimura pointed out that S. Ghilardi's Theorem 5.4 from [4] may be easily strengthened. Namely, let $\mathscr{L}$ be a Kripke complete predicate logic, $\mathscr{L}\subseteq(\mathbf{H}_*+P_2\wedge W_2)$, and M be an $\mathscr{L}$-frame refuting K'.

S. Ghilardi proved that M contains a subset N isomorphic to the infinite binary tree T. In other words, there exists a subreduction from M to T and thus the implicational fragment of $\mathscr{L}$ is intuitionistic (i.e. it coincides with the corresponding fragment of $\mathbf{H}_*$). But in fact every two incomparable points of N are incompatible in M. Now one can easily prove that there exists a cofinal subreduction from M to an arbitrary finite tree (for the terminology see [6]). Therefore we have

PROPOSITION 1. *Let $\mathscr{L}$ be a Kripke complete predicate logic, $\mathscr{L}\subseteq(\mathbf{H}_*+P_2\wedge W_2)$. Then the disjunction-free propositional fragment of $\mathscr{L}$ is intuitionistic.*

It is clear that our Theorem 2 is an immediate consequence of this strengthened form of Theorem 5.4 from [4].

REMARK 2. The Kripke completion of $(\mathbf{H}_*+N_7)$ is $(\mathbf{H}_*+N_5)\cap(\mathbf{H}_*+N_7+K)$. We can show that in fact this logic is not finitely axiomatizable (by applying a method from [7]).

REMARK 3. Obviously, $(\mathbf{H}_*+\overline{N}+D)\vdash\overline{N}^{(n)}$ for all $n\in\omega$. Also $(\mathbf{H}_*+N_7+D)\vdash(N_5\vee K)$. Namely, $\neg\neg(q\wedge(P(x)\vee\neg P(x)))\vee(\neg\neg(q\wedge(P(x)\vee\neg P(x)))\supset(q\wedge(P(x)\vee\neg P(x))))$ and $\neg\neg(\neg q\wedge(P(x)\vee\neg P(x)))\vee(\neg\neg(\neg q\wedge(P(x)\vee\neg P(x)))\supset(\neg q\wedge(P(x)\vee\neg P(x))))$ imply $\neg\neg q\vee\neg q\vee(\neg\neg q\vee\neg q\supset P(x)\vee\neg P(x))$, thus (using D) we obtain $\neg\neg q\vee\neg q\vee(\neg\neg q\vee\neg q\supset\forall x(P(x)\vee\neg P(x)))$, and now $N_5\vee K$ follows obviously. Therefore $(\mathbf{H}_*+N_7+D)=(\mathbf{H}_*+N_5+D)\cap(\mathbf{H}_*+N_7+K+D)$, and the formula $N_5\vee K$ (or K') does not allow us to prove Kripke incompleteness of the logic $(\mathbf{H}_*+N_7+D)$ (and similarly for $(\mathbf{H}_*+\overline{N}+D)$ and $\overline{N}^{(n)}$). So we establish incompleteness of these logics by slightly modifying the incompleteness proof for $(\mathbf{H}_*+N_5+D)$ [3, 5].

4. THEOREM 3. *Every predicate logic $\mathscr{L}$ such that $(\mathbf{H}_*+\overline{N}+D)\subseteq\mathscr{L}\subseteq(\mathbf{H}_*+N_5+D)$ or $(\mathbf{H}_*+\overline{N}+D)\subseteq\mathscr{L}\subseteq(\mathbf{H}_*+\overline{N}+K+D)$ is Kripke incomplete.*

PROOF. Let us consider the formula (cf. [3, 5]):

$$\Phi: \quad \neg\forall x P(x) \ \wedge\ \forall x((q_1 \supset q_2 \vee P(x)) \vee (q_2 \supset q_1 \vee P(x))) \ \supset \\ \supset\ (q_1 \wedge \neg\neg q_2 \supset q_2) \vee (q_2 \wedge \neg\neg q_1 \supset q_1)$$

where P is a unary predicate letter, q_1 and q_2 are propositional letters.

LEMMA 6. *The formula Φ is valid in every predicate Kripke frame with constant domain (M, U) validating $\overline{N}$.*

PROOF. Suppose that $u \models \neg\forall x P(x)$, $u \models \forall x((q_1 \supset q_2 \vee P(x)) \vee (q_2 \supset q_1 \vee P(x)))$ and $u \not\models (q_1 \wedge \neg\neg q_2 \supset q_2)$, $u \not\models (q_2 \wedge \neg\neg q_1 \supset q_1)$ for some $u \in M$. There exist $v_1, v_2 \in M$ such that $u \leq v_1$, $u \leq v_2$, $v_1 \models q_1$, $v1 \models \neg\neg q_2$, $v_1 \not\models q_2$, $v_2 \models q_2$, $v_2 \models \neg\neg q_1$, $v_2 \not\models q_1$. Then $v_1, v_2 \notin \overline{M}$, and thus $v_1 C v_2$ (in $\overline{N}$-frame M). Fix $w \in M$ and $a \in U(0_M)$ such that $v_1 \leq w$, $v_2 \leq w$, $w \not\models P(a)$ (since $u \models \neg\forall x P(x)$). Then $u \not\models (q_1 \supset q_2 \vee P(a)) \vee (q_2 \supset q_1 \vee P(a))$. This is a contradiction. ■

Let us consider the Kripke bundles $\mathscr{F}' = (M', D', \pi')$ and $\mathscr{F}'' = (M'', D'', \pi'')$ such that $M' = M'' = \{u_0, v_1, v_2, w_0\}$, $u_0 < v_i < w_0$ $(i = 1, 2)$, $D'_{u_0} = D''_{u_0} = \{b'_n, c'_n : n > 0\}$, $D'_{v_1} = D''_{v_1} = \{b\} \cup \{c_n : n > 0\}$, $D'_{v_2} = D''_{v_2} = \{c\} \cup \{b_n : n > 0\}$, $D''_{w_0} - \{d_0, d_1\}$, $D''_{w_0} = \{d_n : n \geqslant 0\}, \ldots < b'_2 < b'_1 < b, \ldots < c'_2 < c'_1 < c, \ldots < c_2 < c_1 < b < d_0, \ldots < b_2 < b_1 < c < d_0, b'_n < b_n < d_n, c'_n < c_n < d_n$ for $n > 0$ (items for $d_n, n > 1$ are omitted in $\mathscr{F}'$), and $\ldots < d_2 < d_1 < d_0$ only in $\mathscr{F}''$ (see Fig. 5 and 6).

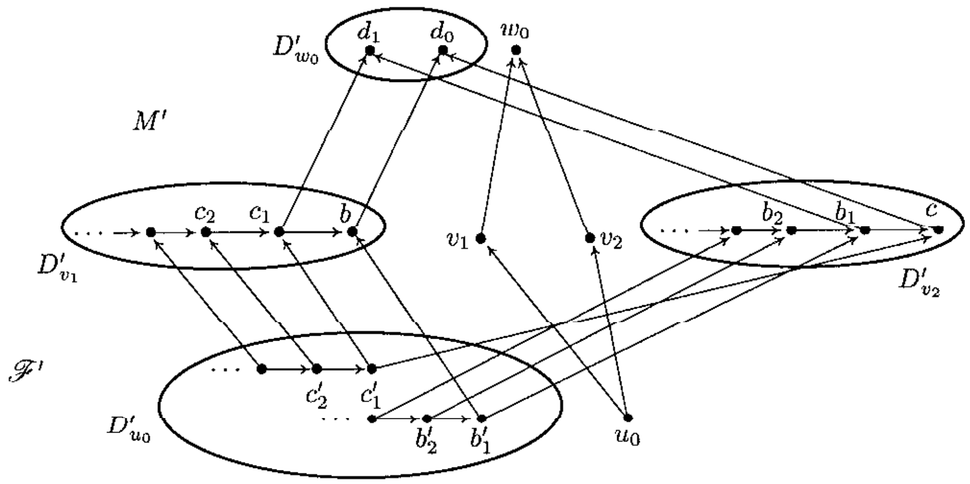


Figure 5.

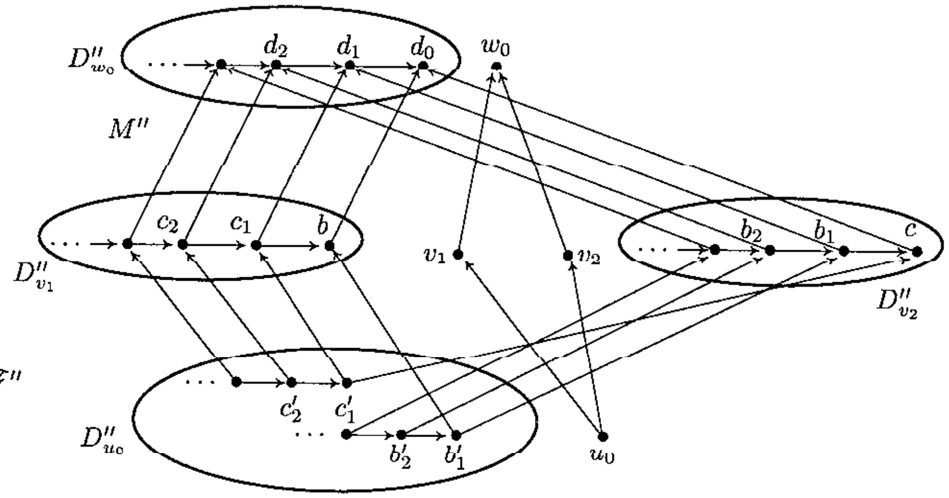


Figure 6.

V. Shchtman used a similar bundle to prove the Kripke incompleteness of $(\mathbf{H}_* + N_5 + D)$ [5].

LEMMA 7. (1) *The formula Φ is not strongly valid in $\mathscr{F}'$ and in $\mathscr{F}''$.*

(2) *The formula N_5 is strongly valid in $\mathscr{F}''$.*

(3) *The formulas K and $\overline{N}$ are strongly valid in $\mathscr{F}'$.*

PROOF. (1) Consider the following valuation: $u \models q_i \Leftrightarrow u \geq v_i$, $u \models P(a) \Leftrightarrow (u = v_1 \wedge a = b) \vee (u = v_2 \wedge a = c) \vee (u = w_0 \wedge a = d_0)$. Then $u_0 \not\models \Phi$, since $w_0 \not\models P(d_1)$, $u_0 \models (q_1 \supset q_2 \vee P(b'_n))$, $u_0 \models (q_2 \supset q_1 \vee P(c'_n))$, $v_1 \not\models (q_1 \wedge \neg\neg q_2 \supset q_2)$, $v_2 \not\models (q_2 \wedge \neg\neg q_1 \supset q_1)$.

(2) Suppose that $u_0 \not\models (\neg\neg A \vee \neg A)(b'_1, c'_1, \dots, b'_k, c'_k)$. Then we have $u_i \geq u_0$ and $a'_{n_i}, a''_{n_i} \in D''_{u_i}$ such that $b'_n \leq a'_{n_i}, c'_n \leq a''_{n_i}$ for $n \leqslant k$, $i = 1, 2$ and $u_1 \models A(a'_{n_1}, a''_{n_1} : n \leqslant k)$, $u_2 \models \neg A(a'_{n_2}, a''_{n_2} : n \leqslant k)$, which contradicts to $(a'_{n_i}, a''_{n_i} : n \leqslant k) \leq_{2k} (d_0, \dots, d_0)$ for $i = 1, 2$ (since do is the "greatest" individual in $\mathscr{F}''$).

(3) Suppose that $u_0 \not\models \neg\neg\forall x(A(x, \boldsymbol{a}) \vee \neg A(x, \boldsymbol{a}))$, $\boldsymbol{a} = (b'_n, c'_n : n \leqslant k)$. Then $u \models \neg\forall x(A(x, \boldsymbol{a}') \vee \neg A(x, \boldsymbol{a}'))$ for some $u \in M'$, $\boldsymbol{a}' \in (D'_u)^{2k}$ and $w_0 \models \neg\forall x(A(x, \boldsymbol{d}) \vee \neg A(x, \boldsymbol{d}))$ for some $\boldsymbol{d} \in (D'_{w_0})^{2k}$. This is a contradiction.

Now suppose that $u_0 \not\models ((\neg\neg A \supset (B \supset A) \vee (\neg B \supset A))(\boldsymbol{a})$. Then $u' \models \neg\neg A(\boldsymbol{a}') \wedge B(\boldsymbol{a}')$, $u' \not\models A(\boldsymbol{a}')$, $u'' \models \neg\neg A(\boldsymbol{a}'') \wedge \neg B(\boldsymbol{a}'')$, $u'' \not\models A(\boldsymbol{a}'')$ for some $u', u'' \in M'$, $\boldsymbol{a}' \in (D'_{u'})^{2k}$, $\boldsymbol{a}'' \in (D'_{u''})^{2k}$. Therefore $u' \neq w_0$, $u'' \neq w_0$ and $\boldsymbol{a}' \leq_{2k} \boldsymbol{d}$, $\boldsymbol{a}'' \leq_{2k} \boldsymbol{d}$ where $\boldsymbol{d} = (d_0, \dots, d_0)$. Thus $w_0 \models B(\boldsymbol{d})$ and $w_0 \models \neg B(\boldsymbol{d})$. This is a contradiction. ∎

A Kripke bundle $\mathscr{F} = (M, D, \pi)$ is called a *D-bundle* iff $\forall_{u \leq v} \forall_{b \in D_v} [\{a \in D_u : a \leq b\}$ is infinite or $(u = v, \forall_{d \in D_u \setminus \{b\}}\ b \not\leq d)]$.

LEMMA 8. (1) *$\mathscr{F}'$ and $\mathscr{F}''$ are D-bundles.*

(2) *The formula D is strongly valid in every D-bundle $\mathscr{F} = (M, D, \pi)$.*

PROOF. (1) is obvious.

(2) Suppose that $u \vDash \forall x (A(c_1, \dots, c_n, x) \vee B(c_1, \dots, c_n))$, $u \nvDash \forall x A(c_1, \dots, c_n, x)$, $u \nvDash D(c_1, \dots, c_n)$. Then there exist $v \geq u$ and $d_1, \dots, d_n, b \in D_v$ such that $(c_1, \dots, c_n) \leq_n (d_1, \dots, d_n)$ and $v \nvDash A(d_1, \dots, d_n, b)$. Choose $a \in D_u$ such that $a \leq b$ and $a \notin \{c_1, \dots, c_n\}$) (or $a = b$ if $u = v$, $\forall_{d \in D_u \setminus \{b\}}\ b \not\leq d$). Then $(c_1, \dots, c_n, a) \leq_{n+1} (d_1, \dots, d_n, b)$ and $u \nvDash A(c_1, \dots, c_n, a)$, which contradicts to $u \vDash A(c_1, \dots, c_n, a) \vee B(c_1, \dots, c_n)$. ∎

Thus, $(\mathbf{H}_* + N_5 + D) \nvdash \Phi$ and $(\mathbf{H}_* + \overline{N} + K + D) \nvdash \Phi$.

COROLLARY 1. *The predicate logics $(\mathbf{H}_* + N_7 + D)$, $(\mathbf{H}_* + \overline{N} + D)$ and $(\mathbf{H}_* + \overline{N} + K + D)$ are Kripke incomplete.*

COROLLARY 2. *There exists a disjunction-free propositional formula A such that predicates logics $(\mathbf{H}_* + A + D)$ and $(\mathbf{H}_* + A + K + D)$ are Kripke incomplete.*

Acknowledgment. The author would like to express his gratitude to anonymous referees for very useful comments and suggestions.

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