Drawdown Distribution Due to Well Fields in Coupled Leaky Aquifers 2. Finite Aquifer System

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Solutions for the drawdown distribution due to the operation of well fields in coupled leaky aquifers of infinite areal extent are available. This paper is concerned with the drawdown distribution caused by the pumping of well fields in coupled leaky aquifers of finite areal extent. A transform to solve the flow problem is obtained by the modification of the zero-order Hankel transform. Two cases of the problem are analyzed. Case 1 concerns the upper aquifer when it is confined or unconfined and the drawdown in it is very small compared with its saturated thickness. Case 2 concerns the upper unconfined aquifer when the drawdown in it can be significant in relation to its saturated thickness. Approximate solutions and solutions for special cases are obtained in addition to the exact solutions. The solutions are in the form of series that can readily be evaluated.

Drawdown distribution due to pumpage from several wells tapping the same aquifer can often be calculated more conveniently by assuming that the pumpage from the well field is uniformly distributed over the area. A brief summary of some studies related to drawdown distribution due to well fields is given by *Saleem and Jacob* [1973], who also present the mathematical solutions for the drawdown distribution caused by the well fields in coupled leaky aquifers of infinite areal extent. The purpose of this paper is to develop analytical solutions for the drawdown distribution due to well fields in coupled leaky aquifers of finite areal extent. Pumping from one or both of the aquifers can be negative or zero, or one aquifer can be pumped while the other is being replenished.

Figure 1 is a diagrammatic representation of a coupled leaky aquifer system of finite areal extent. The aquifers of the system are assumed to be individually homogeneous, isotropic, and uniform in thickness, and the angle of tilt is small. The hydraulic conductivity and storativity of aquifers and the leakage coefficient of the aquitard remain invariant with time and in space. The storativity of the aquitard is negligible, and the contrasts between the hydraulic conductivities of the two aquifers and the conductivity of the aquitard are so great that the flow is essentially vertical in the aquitard and horizontal in the aquifers. The well fields are operated at constant rates, and the distribution of pumpage from the two aquifers is uniform over the two concentric circular well field areas of radii R_1 and R_2.

Two cases of the finite aquifer system problem are analyzed:

1. The drawdown in the upper aquifer is small in relation to its saturated thickness. In this case the two aquifers are effectively confined.

2. The drawdown in the upper unconfined aquifer is significant in relation to its saturated thickness.

CASE 1

The problem is to determine the drawdown distribution within and outside the radii of well fields in the coupled leaky aquifer system, as caused by the pumping from one or both of the well fields. It is assumed here, in addition to the assumptions previously made, that the drawdown in the unconfined aquifer is small in relation to its saturated thickness.

The flow system can be approximated by the following boundary value problem [e.g., *Hantush*, 1967; *Saleem and Jacob*, 1973]:

$$\frac{\partial^2 s_1}{\partial r^2} + \frac{1}{r}\frac{\partial s_1}{\partial r} - \frac{1}{B_1{}^2}(s_1 - s_2) + \frac{Q_1}{T_1} f_1(r) = \frac{1}{\nu_1}\frac{\partial s_1}{\partial t} \tag{1}$$

$$\frac{\partial^2 s_2}{\partial r^2} + \frac{1}{r}\frac{\partial s_2}{\partial r} + \frac{1}{B_2{}^2}(s_1 - s_2) + \frac{Q_2}{T_2} g(r) = \frac{1}{\nu_2}\frac{\partial s_2}{\partial t} \tag{2}$$

$$s_1(r, 0) = s_2(r, 0) = 0 \tag{3}$$

$$\partial s_1(r, t)/\partial r = \partial s_2(r, t)/\partial r = 0 \qquad r = r_e \tag{4}$$

$$\partial s_1(0, t)/\partial r = \partial s_2(0, t)/\partial r = 0 \tag{5}$$

in which

$$s_1 = (h_{i1} - h_1) \qquad s_2 = (h_{i2} - h_2) \tag{6}$$

$$\nu_n = T_n/S_n \qquad B_n{}^2 = T_n/(K'/b') \qquad n = 1, 2 \tag{7}$$

$$\begin{aligned} f_1(r) &= 1 \qquad 0 < r < R_1 \\ f_1(r) &= 0 \qquad r > R_1 \end{aligned} \tag{8}$$

$$\begin{aligned} g(r) &= 1 \qquad 0 < r < R_2 \\ g(r) &= 0 \qquad r > R_2 \end{aligned} \tag{9}$$

[1] Deceased.

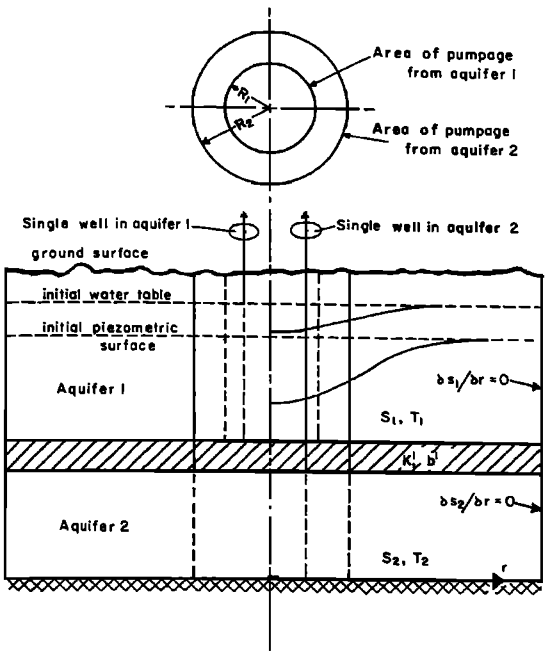

Fig. 1. Diagrammatic representation of well fields in coupled leaky aquifers of finite areal extent; R_1 and R_2 are radii of well fields in aquifers 1 and 2, respectively.

where subscripts 1 and 2 correspond to aquifers 1 and 2, respectively; T, S, ν, and h are transmissivity, storativity, hydraulic diffusivity, and hydraulic head, respectively; h_{i1} and h_{i2} are the initial heads; s_1 and s_2 are the drawdowns due to the operation of the well fields in the system; K'/b' is the leakage coefficient or 'leakance'; Q_1 and Q_2 denote constant rates of withdrawal per unit area; R_1 and R_2 are radii of well fields; r is the radial distance from the center of the well fields to any point in the flow field; and r_e is the radius of the aquifer system.

Solution

In order to solve the flow problem the zero-order Hankel transform of a function $f(r)$ is modified and defined as

$$H_0[f(r)] = V(\alpha_n) = \frac{2^{1/2}}{r_e}\int_0^{r_e} rf(r)\,\frac{J_0(\alpha_n r/r_e)}{J_0(\alpha_n)}\,dr \qquad (10)$$

where α_n are positive roots of $J_0'(\alpha) = 0$. By expanding a function in a Fourier-Bessel series it can be shown that the inversion formula for the transformation represented by (10) is

$$f(r) = \frac{2^{1/2}}{r_e}\sum_{m=1}^{\infty}\frac{V(\alpha_n)J_0(\alpha_n r/r_e)}{J_0(\alpha_n)} \qquad (11)$$

where $V(\alpha_n)$ is the transform of the function $f(r)$ defined by (10) and J_0 is the zero-order Bessel function of the first kind.

Applying (10) to $\nabla^2\bar{s}$ and integrating by parts, one obtains

$$H_0[\nabla^2\bar{s}] = H_0\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\bar{s}}{dr}\right)\right] = \left[\left(r\frac{d\bar{s}}{dr}\cdot\frac{J_0(\alpha_n r/r_e)}{J_0(\alpha_n)}\right)_{r=0}^{r=r_e} - \int_0^{r_e} r\frac{d\bar{s}}{dr}\cdot\frac{\alpha_n}{r_e}\,\frac{J_0'(\alpha_n r/r_e)}{J_0(\alpha_n)}\,dr\right](2^{1/2}/r_e) \qquad (12)$$

The first term on the right-hand side vanishes if $(d\bar{s}/dr)_{r=r_e} = 0$ or

$$H_0\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\bar{s}}{dr}\right)\right] = \left[-\left(\bar{s}r\,\frac{\alpha_n}{r_e}\,\frac{J_0'(\alpha_n r/r_e)}{J_0(\alpha_n)}\right)_{r=0}^{r=r_e} + \frac{\alpha_n}{r_e}\int_0^{r_e}\frac{\bar{s}}{J_0(\alpha_n)}\left(J_0'(\alpha_n r/r_e) + \frac{\alpha_n}{r_e}J_0''(\alpha_n r/r_e)\right)dr\right](2^{1/2}/r_e) \qquad (13)$$

Again the first term on the right side vanishes because $J_1(\alpha_n) = 0$, and by means of the fact that $J_0(\alpha_n r/r_e)$ satisfies the differential equation

$$J_0''(\alpha_n r/r_e) + \frac{r_e}{\alpha_n r}J_0'(\alpha_n r/r_e) + J_0(\alpha_n r/r_e) = 0 \qquad (14)$$

(13) becomes

$$H_0[(1/r)\,d/dr\,(r\,d\bar{s}/dr)] = -(\alpha_n^2/r_e^2)\bar{F}(\alpha_n) \qquad (15)$$

in which $\bar{F}(\alpha_n)$ is the integral transform, defined by equation 10, of $\bar{s}(r)$.

Successive application of the Laplace transform with respect to t and the modified Hankel transform, defined by (10), with respect to r to the flow problem defined by (1)–(9) yields, after simplification,

$$\bar{F}_1(\alpha_n, p) = Q_1R_1J_1(\alpha_nR_1/r_e)\,\frac{2^{1/2}\nu_1(p + c_2\nu_2)}{T_1\alpha_nJ_0(\alpha_n)p[(p + c)^2 - \mu^2]} + \frac{Q_22^{1/2}R_2J_1(\alpha_nR_2/r_e)\nu_1\nu_2}{T_2\alpha_nJ_0(\alpha_n)B_1^2p[(p + c)^2 - \mu^2]} \qquad (16)$$

$$\bar{F}_2(\alpha_n, p) = Q_1R_1J_1(\alpha_nR_1/r_e)\cdot\frac{2^{1/2}\nu_1\nu_2}{B_2^2\alpha_nT_1J_0(\alpha_n)p[(p + c)^2 - \mu^2]} + \frac{2^{1/2}Q_2R_2\nu_2J_1(\alpha_nR_2/r_e)(p + c_1\nu_1)}{T_2\alpha_nJ_0(\alpha_n)p[(p + c)^2 - \mu^2]} \qquad (17)$$

where $\bar{F}_1(\alpha_n, p)$ and $\bar{F}_2(\alpha_n, p)$ are the modified Hankel transforms, defined by (10), of the Laplace transforms of the variables $s_1(r, t)$ and $s_2(r, t)$, respectively; α_n and p are the parameters of respective transformations; and

$$c_1 = \alpha_n^2/r_e^2 + 1/B_1^2 \qquad c_2 = \alpha_n^2/r_e^2 + 1/B_2^2 \qquad (18)$$

$$c = 0.5(c_1\nu_1 + c_2\nu_2) \qquad (19)$$

$$\mu^2 = 0.25[\alpha_n^2(\nu_1 - \nu_2)/r_e^2 + \nu_1/B_1^2 - \nu_2/B_2^2]^2 + \nu_1\nu_2/(B_1^2B_2^2) \qquad (20)$$

Application of the inverse Laplace transform [*Erdelyi*, 1954] and the inverse modified Hankel transform, defined by (11), to (16) and (17) yields

$$s_1(r, t) = \frac{2}{r_e}\sum_{n=1}^{\infty}Q_1R_1J_1(\alpha_nR_1/r_e)\,\frac{c_2\nu_1\nu_2}{T_1\alpha_n(c^2 - \mu^2)}\cdot\left[1 - e^{-ct}\left(\cosh\mu t + \frac{cc_2\nu_2 - c^2 + \mu^2}{c_2\nu_2\mu}\sinh\mu t\right)\right]\cdot\frac{J_0(\alpha_nr/r_e)}{J_0^2(\alpha_n)} + \frac{2}{r_e}\sum_{n=1}^{\infty}\frac{Q_2R_2\nu_1\nu_2J_1(\alpha_nR_2/r_e)}{T_2B_1^2\alpha_n(c^2 - \mu^2)}\cdot\left[1 - e^{-ct}\left(\cosh\mu t + \frac{c}{\mu}\sinh\mu t\right)\right]\frac{J_0(\alpha_nr/r_e)}{J_0^2(\alpha_n)} \qquad (21)$$

$$s_2(r, t) = \frac{2}{r_e}\sum_{n=1}^{\infty} Q_1R_1J_1(\alpha_nR_1/r_e)\frac{\nu_1\nu_2}{B_2^2T_1\alpha_n(c^2-\mu^2)} \cdot\left[1 - e^{-ct}\left(\cosh \mu t + \frac{c}{\mu}\sinh \mu t\right)\right]\frac{J_0(\alpha_n r/r_e)}{J_0^2(\alpha_n)} + \frac{2}{r_e}\sum_{n=1}^{\infty}\frac{Q_2R_2\nu_1\nu_2c_1J_1(\alpha_nR_2/r_e)}{T_2\alpha_n(c^2-\mu^2)}\left[1 - e^{-ct}\left(\cosh \mu t + \frac{cc_1\nu_1 - c^2 + \mu^2}{c_1\nu_1\mu}\sinh \mu t\right)\right]\frac{J_0(\alpha_n r/r_e)}{J_0^2(\alpha_n)} \qquad (22)$$

Approximate solutions. The drawdowns s_1 and s_2 in the two aquifers at large and small values of time can be determined from $\bar{F}_1$ and $\bar{F}_2$ at small and large values of the Laplace transform parameter p. Equations 16 and 17 are rewritten as

$$\bar{F}_1(\alpha, p) = \frac{2^{1/2}Q_1R_1J_1(\alpha_nR_1/r_e)(p + \alpha_n^2\nu_2/r_e^2 + \nu_2/B_2^2)}{\alpha_n\nu_2T_1J_0(\alpha_n)pM_1M_2} + \frac{2^{1/2}Q_2R_2J_1(\alpha_nR_2/r_e)}{\alpha_nT_2B_1^2J_0(\alpha_n)pM_1M_2} \qquad (23)$$

$$\bar{F}_2(\alpha, p) = \frac{2^{1/2}Q_2R_2J_1(\alpha_nR_2/r_e)(p + \alpha_n^2\nu_1/r_e^2 + \nu_1/B_1^2)}{\alpha_n\nu_1T_2J_0(\alpha_n)pM_1M_2} + \frac{2^{1/2}Q_1R_1J_1(\alpha_nR_1/r_e)}{\alpha_nT_1B_2^2J_0(\alpha_n)pM_1M_2} \qquad (24)$$

where

$$M_1 = [\alpha_n^2/r_e^2 + 0.5(p/\nu_1 + p/\nu_2 + 1/B_1^2 + 1/B_2^2)] + 0.5[(p/\nu_1 - p/\nu_2 + 1/B_1^2 - 1/B_2^2)^2 + 4/(B_1^2B_2^2)]^{1/2} \qquad (25)$$

time are derived by comparing the quantity $(p/\nu_1 - p/\nu_2)$ with $4/B_1^2B_2^2$ under the radical sign in (25) and (26) [*Saleem and Jacob*, 1973; *Hantush*, 1967].

Solution for relatively large times: If $(p/\nu_1 - p/\nu_2)^2 < 0.20/(B_1B_2)^2$, i.e., if the time t is of the order of $t > 3.3(|\nu_2 - \nu_1|)B_1B_2/(\nu_1\nu_2)$, the radicals in (25) and (26) can be approximated by $(1/B_1^2 + 1/B_2^2)$, and (23) and (24) are written as

$$\bar{F}_1(\alpha, p) = \frac{2^{1/2}Q_1\nu_\nu^2R_1J_1(\alpha_nR_1/r_e)(p + \alpha_n^2\nu_2/r_e^2 + \nu_2/B_2^2)}{\alpha_nT_1J_0(\alpha_n)\nu_2p(p + \alpha_n^2\nu_\nu/r_e^2)(p + \alpha_n^2\nu_\nu/r_e^2 + \nu_\nu/B^2)} + \frac{2^{1/2}Q_2\nu_\nu^2R_2J_1(\alpha_nR_2/r_e)}{\alpha_nT_2J_0(\alpha_n)B_1^2p(p + \alpha_n^2\nu_\nu/r_e^2)(p + \alpha_n^2\nu_\nu/r_e^2 + \nu_\nu/B^2)} \qquad (27)$$

$$\bar{F}_2(\alpha, p) = \frac{2^{1/2}Q_2R_2\nu_\nu^2J_1(\alpha_nR_2/r_e)(p + \alpha_n^2\nu_1/r_e^2 + \nu_1/B_1^2)}{\alpha_nT_2J_0(\alpha_n)\nu_1p(p + \alpha_n^2\nu_\nu/r_e^2)(p + \alpha_n^2\nu_\nu/r_e^2 + \nu_\nu/B^2)} + \frac{2^{1/2}Q_1\nu_\nu^2R_1J_1(\alpha_nR_1/r_e)}{\alpha_nT_1B_2^2J_0(\alpha_n)p(p + \alpha_n^2\nu_\nu/r_e^2)(p + \alpha_n^2\nu_\nu/r_e^2 + \nu_\nu/B^2)} \qquad (28)$$

where

$$\nu_\nu = 2\nu_1\nu_2/(\nu_1 + \nu_2) \qquad (29)$$

$$1/B^2 = 1/B_1^2 + 1/B_2^2 \qquad (30)$$

Application of the inverse Laplace transform and the inverse transform, defined by (11), to (27) and (28) yields

$$s_1(r, t) = \frac{2Q_1R_1}{r_e\nu_2T_1}\sum_{n=1}^{\infty}\frac{J_1(\alpha_nR_1/r_e)J_0(\alpha_nr/r_e)}{\alpha_nJ_0^2(\alpha_n)}\cdot\{\nu_2(\alpha_n^2/r_e^2 + 1/B_2^2)/[(\alpha_n^2/r_e^2)(\alpha_n^2/r_e^2 + 1/B^2)] - [(\nu_2 - \nu_\nu)\alpha_n^2/r_e^2 + \nu_2/B_2^2]\cdot\exp(-\alpha_n^2\nu_\nu t/r_e^2)/(\alpha_n^2/r_e^2B^2) + [(\nu_2 - \nu_\nu)\alpha_n^2/r_e^2 + \nu_2/B_2^2 - \nu_\nu/B^2]\cdot\exp[-(\alpha_n^2/r_e^2 + 1/B^2)\nu_\nu t]/[(\alpha_n^2/r_e^2 + 1/B^2)/B^2]\} + \frac{2Q_2R_2}{r_eT_2B_1^2}\sum_{n=0}^{\infty}\frac{J_1(\alpha_nR_2/r_e)J_0(\alpha_nr/r_e)}{\alpha_nJ_0^2(\alpha_n)}\{1/[(\alpha_n^2/r_e^2 + 1/B^2)\alpha_n^2/r_e^2] + \exp[-(\alpha_n^2/r_e^2 + 1/B^2)\nu_\nu t]/[(\alpha_n^2/r_e^2 + 1/B^2)/B^2] - \exp(-\alpha_n^2\nu_\nu t/r_e^2)/(\alpha_n^2/r_e^2B^2)\} \qquad (31)$$

$$s_2(r, t) = \frac{2Q_2R_2}{r_e\nu_1T_2}\sum_{n=1}^{\infty}\frac{J_1(\alpha_nR_2/r_e)J_0(\alpha_nr/r_e)}{\alpha_nJ_0^2(\alpha_n)}\cdot\{\nu_1(\alpha_n^2/r_e^2 + 1/B_1^2)/[(\alpha_n^2/r_e^2)(\alpha_n^2/r_e^2 + 1/B^2)] - [(\nu_1 - \nu_\nu)\alpha_n^2/r_e^2 + \nu_1/B_1^2]\cdot\exp(-\alpha_n^2\nu_\nu t/r_e^2)/(\alpha_n^2/r_e^2B^2) + [(\nu_1 - \nu_\nu)\alpha_n^2/r_e^2 + \nu_1/B_1^2 - \nu_\nu/B^2]\cdot\exp[-(\alpha_n^2/r_e^2 + 1/B^2)\nu_\nu t]/[(\alpha_n^2/r_e^2 + 1/B^2)/B^2]\} + \frac{2Q_1R_1}{r_eT_1B_2^2}\sum_{n=1}^{\infty}\frac{J_1(\alpha_nR_1/r_e)J_0(\alpha_nr/r_e)}{\alpha_nJ_0^2(\alpha_n)}\{1/[(\alpha_n^2/r_e^2 + 1/B^2)\alpha_n^2/r_e^2] - \exp(-\alpha_n^2\nu_\nu t/r_e^2)/(\alpha_n^2/r_e^2B^2) + \exp[-(\alpha_n^2/r_e^2 + 1/B^2)\nu_\nu t]/[(\alpha_n^2/r_e^2 + 1/B^2)/B^2]\} \qquad (32)$$

$$M_2 = [\alpha_n^2/r_e^2 + 0.5(p/\nu_1 + p/\nu_2 + 1/B_1^2 + 1/B_2^2)] - 0.5[(p/\nu_1 - p/\nu_2 + 1/B_1^2 - 1/B_2^2)^2 + 4/(B_1^2B_2^2)]^{1/2} \qquad (26)$$

The criteria for solutions at large and small values of

Solution for short times: If $0.5(2/B_1B_2)^2 < 0.05(p/\nu_1 - p/\nu_2)^2$, i.e., if time t is of the order of $t \le 0.22(|\nu_2 - \nu_1|)B_1B_2/\nu_1\nu_2$. the radicals in (25) and (26) can be approximated by $(p/\nu_1 - p/\nu_2 + 1/B_1^2 - 1/B_2^2)$. Thus (23) and (24) become

$$\bar{F}_1(\alpha, p) = \frac{2^{1/2}Q_1R_1J_1(\alpha_nR_1/r_e)\nu_1}{\alpha_nT_1J_0(\alpha_n)p(p + \alpha_n^2\nu_1/r_e^2 + \nu_1/B_1^2)} + \frac{2^{1/2}Q_2R_2J_1(\alpha_nR_2/r_e)\nu_1\nu_2}{\alpha_nT_2B_1^2J_0(\alpha_n)p(p + \alpha_n^2\nu_1/r_e^2 + \nu_1/B_1^2)(p + \alpha_n^2\nu_2/r_e^2 + \nu_2/B_2^2)} \quad (33)$$

$$\bar{F}_2(\alpha, p) = \frac{2^{1/2}Q_2R_2J_1(\alpha_nR_2/r_e)\nu_2}{\alpha_nT_2J_0(\alpha_n)p(p + \alpha_n^2\nu_2/r_e^2 + \nu_2/B_2^2)} + \frac{2^{1/2}Q_1R_1J_1(\alpha_nR_1/r_e)\nu_1\nu_2}{\alpha_nB_2^2T_1J_0(\alpha_n)p(p + \alpha_n^2\nu_1/r_e^2 + \nu_1/B_1^2)(p + \alpha_n^2\nu_2/r_e^2 + \nu_2/B_2^2)} \quad (34)$$

which after inversion yield

$$\begin{aligned} s_1(r, t) = {} & \frac{2Q_1R_1}{r_eT_1}\sum_{n=1}^{\infty}\frac{J_1(\alpha_nR_1/r_e)J_0(\alpha_nr/r_e)}{\alpha_nJ_0^2(\alpha_n)(\alpha_n^2/r_e^2 + 1/B_1^2)}\{1 - \exp[-(\alpha_n^2/r_e^2 + 1/B_1^2)\nu_1t]\} \\ & + \frac{2Q_2R_2}{r_eT_2B_1^2}\sum_{n=1}^{\infty}\frac{J_1(\alpha_nR_2/r_e)J_0(\alpha_nr/r_e)}{\alpha_nJ_0^2(\alpha_n)}\{1/[(\alpha_n^2/r_e^2 + 1/B_1^2)(\alpha_n^2/r_e^2 + 1/B_2^2)] \\ & + \nu_2\cdot\exp[-(\alpha_n^2/r_e^2 + 1/B_1^2)\nu_1t]/[(\alpha_n^2\nu_1/r_e^2 - \alpha_n^2\nu_2/r_e^2 + \nu_1/B_1^2 - \nu_2/B_2^2)(\alpha_n^2/r_e^2 + 1/B_1^2)] \\ & - \nu_1\cdot\exp[-(\alpha_n^2/r_e^2 + 1/B_2^2)\nu_2t]/[(\alpha_n^2\nu_1/r_e^2 - \alpha_n^2\nu_2/r_e^2 + \nu_1/B_1^2 - \nu_2/B_2^2)(\alpha_n^2/r_e^2 + 1/B_2^2)]\} \end{aligned} \quad (35)$$

$$\begin{aligned} s_2(r, t) = {} & \frac{2Q_2R_2}{r_eT_2}\sum_{n=0}^{\infty}\frac{J_1(\alpha_nR_2/r_e)J_0(\alpha_nr/r_e)}{\alpha_nJ_0^2(\alpha_n)(\alpha_n^2/r_e^2 + 1/B_2^2)}\{1 - \exp[-(\alpha_n^2/r_e^2 + 1/B_2^2)\nu_2t]\} \\ & + \frac{2Q_1R_1}{r_eT_1B_2^2}\sum_{n=1}^{\infty}\frac{J_1(\alpha_nR_1/r_e)J_0(\alpha_nr/r_e)}{\alpha_nJ_0^2(\alpha_n)}\{1/[(\alpha_n^2/r_e^2 + 1/B_1^2)(\alpha_n^2/r_e^2 + 1/B_2^2)] \\ & + \nu_2\cdot\exp[-(\alpha_n^2/r_e^2 + 1/B_1^2)\nu_1t]/[(\alpha_n^2\nu_1/r_e^2 - \alpha_n^2\nu_2/r_e^2 + \nu_1/B_1^2 - \nu_2/B_2^2)(\alpha_n^2/r_e^2 + 1/B_1^2)] \\ & - \nu_1\cdot\exp[-(\alpha_n^2/r_e^2 + 1/B_2^2)\nu_2t]/[(\alpha_n^2\nu_1/r_e^2 - \alpha_n^2\nu_2/r_e^2 + \nu_1/B_1^2 - \nu_2/B_2^2)(\alpha_n^2/r_e^2 + 1/B_2^2)]\} \end{aligned} \quad (36)$$

CASE 2

The flow problem of case 2 is essentially the same as the problem of case 1 except that the drawdown in the upper unconfined aquifer can be significant in relation to its depth of saturation. All other assumptions applicable to the problem of case 1 are applicable here. The flow in the system is governed by (1)–(8) except that (1) and (2) are modified [e.g., *Saleem and Jacob*, 1973; *Hantush*, 1962]:

$$\nabla^2h_1^2 - (2K'/b'K_1)(h_1 - h_2) = (2\theta/K_1)\,\partial h_1/\partial t \quad (37)$$

where

$$\nabla^2h_1^2 = \partial^2h_1^2/\partial r^2 + (1/r)\,\partial h_1^2/\partial r \quad (38)$$

θ and K_1 are the specific yield and the hydraulic conductivity of the unconfined aquifer, respectively; and h_1 and h_2 are hydraulic heads in the unconfined and confined aquifers, respectively. The other symbols are as defined earlier. Equation 37 is nonlinear in h_1 and therefore is difficult to solve. By replacing $(h_{i1} + h_1)/2$ with a weighted mean depth of the flow profile, D, and by multiplying the right-hand side with h_1/D the following approximate differential equations are obtained [*Saleem and Jacob*, 1973] for the case when the upper aquifer is unconfined:

$$\frac{\partial^2d_1}{\partial r^2} + \frac{1}{r}\frac{\partial d_1}{\partial r} - \frac{1}{\beta_1^2}\left(\frac{d_1}{2D} - d_2\right) + \frac{2Q_1}{K_1}f_1(r) = \frac{1}{\eta_1}\frac{\partial d_1}{\partial t} \quad (39)$$

$$\frac{\partial^2d_2}{\partial r^2} + \frac{1}{r}\frac{\partial d_2}{\partial r} + \frac{1}{B_2^2}\left(\frac{d_1}{2D} - d_2\right) + \frac{Q_2}{T_2}g(r) = \frac{1}{\nu_2}\frac{\partial d_2}{\partial t} \quad (40)$$

in which

$$d_1 = (h_{i1}^2 - h_1^2) \qquad d_2 = (h_{i2} - h_2) \quad (41)$$

$$\beta_1^2 = K_1/2(K'/b') \qquad \eta_1 = K_1D/\theta \quad (42)$$

Solution

The flow problem is represented by (39) and (40) and (3)–(5). The solution of the problem is obtained by applying the Laplace transform and the modified Hankel transform (equation 10) to the equations, and after simplification one obtains

$$\bar{G}_1(\alpha_n, p) = \frac{2(2)^{1/2}Q_1\eta_1R_1J_1(\alpha_nR_1/r_e)(p + c_2\nu_2)}{K_1\alpha_nJ_0(\alpha_n)p[(p + l)^2 - \beta^2]} + \frac{2^{1/2}Q_2\eta_1\nu_2R_2J_1(\alpha_nR_2/r_e)}{T_2\beta_1^2\alpha_nJ_0(\alpha_n)p[(p + l)^2 - \beta^2]} \quad (43)$$

$$\bar{G}_2(\alpha_n, p) = \frac{2^{1/2}Q_2\nu_2R_2J_1(\alpha_nR_2/r_e)(p + c_1\eta_1)}{T_2\alpha_nJ_0(\alpha_n)p[(p + l)^2 - \beta^2]} + \frac{2^{1/2}Q_1\eta_1\nu_2R_1J_1(\alpha_nR_1/r_e)}{K_1DB_2^2\alpha_nJ_0(\alpha_n)p[(p + l)^2 - \beta^2]} \quad (44)$$

where $\bar{G}_1(\alpha_n, p)$ and $\bar{G}_2(\alpha_n, p)$ are the modified Hankel transforms, defined by (10), of the Laplace transforms of the variables $d_1(r, t)$ and $d_2(r, t)$, respectively; and

$$l = 0.5(c_1\eta_1 + c_2\nu_2) \quad (45a)$$

$$\beta^2 = 0.25[\alpha_n^2(\eta_1 - \nu_2)/r_e^2 + \eta_1/(2D\beta_1^2) - \nu_2/B_2^2]^2 + \eta_1\nu_2/(2D\beta_1^2B_2^2) \quad (45b)$$

Equations 43 and 44 on inversion ultimately yield

$$d_1(r,\ t) = \frac{4Q_1R_1\eta_1\nu_2}{r_eK_1}\sum_{n=1}^{\infty}\frac{c_2J_1(\alpha_nR_1/r_e)J_0(\alpha_nr/r_e)}{(l^2-\beta^2)\alpha_nJ_0^{\,2}(\alpha_n)}\cdot\left[1-\exp(-lt)\left(\cosh(\beta t)+\frac{lc_2\nu_2-l^2+\beta^2}{c_2\nu_2\beta}\sinh(\beta t)\right)\right]$$
$$+\frac{2Q_2R_2\eta_1\nu_2}{r_eT_2\beta_1^{\,2}}\sum_{n=1}^{\infty}\frac{J_1(\alpha_nR_1/r_e)J_0(\alpha_nr/r_e)}{(l^2-\beta^2)\alpha_nJ_0^{\,2}(\alpha_n)}\cdot\{1-\exp(-lt)[\cosh(\beta t)+(l/\beta)\sinh(\beta t)]\} \tag{46}$$

$$d_2(r,\ t) = \frac{2Q_1R_1\eta_1\nu_2}{r_eK_1DB_2^2}\sum_{n=1}^{\infty}\frac{J_1(\alpha_nR_1/r_e)J_0(\alpha_nr/r_e)}{(l^2-\beta^2)\alpha_nJ_0^{\,2}(\alpha_n)}\cdot\{1-\exp(-lt)[\cosh(\beta t)+(l/\beta)\sinh(\beta t)]\}$$
$$+\frac{2Q_2R_2\eta_1\nu_2}{r_eT_2}\sum_{n=1}^{\infty}\frac{c_1J_1(\alpha_nR_1/r_e)J_0(\alpha_nr/r_e)}{(l^2-\beta^2)\alpha_nJ_0^{\,2}(\alpha_n)}\cdot\left[1-\exp(-lt)\left(\cosh(\beta t)+\frac{lc_1\eta_1+\beta^2-l^2}{c_1\eta_1\beta}\sinh(\beta t)\right)\right] \tag{47}$$

Approximate solutions. The drawdown distributions in the two aquifers at large and small values of time are determined exactly as they were in case 1. Equations 43 and 44 are set in the form

$$\bar{G}_1(\alpha_n,\ p) = \frac{2(2)^{1/2}Q_1R_1J_1(\alpha_nR_1/r_e)[p+\alpha_n^{\,2}\nu_2/r_e^{\,2}+\nu_2/B_2^{\,2}]}{K_1\nu_2\alpha_nJ_0(\alpha_n)pN_1N_2}+\frac{2^{1/2}Q_2R_2J_1(\alpha_nR_2/r_e)}{T_2\beta_1^{\,2}\alpha_nJ_0(\alpha_n)pN_1N_2} \tag{48}$$

$$\bar{G}_2(\alpha_n,\ p) = \frac{2^{1/2}Q_2R_2J_1(\alpha_nR_2/r_e)[p+\alpha_n^{\,2}\eta_1/r_e^{\,2}+\eta_1/(2D\beta_1^{\,2})]}{T_2\eta_1\alpha_nJ_0(\alpha_n)pN_1N_2}+\frac{2^{1/2}Q_1R_1J_0(\alpha_nR_1/r_e)}{K_1DB_2^{\,2}\alpha_nJ_0(\alpha_n)pN_1N_2} \tag{49}$$

where

$$N_1 = [\alpha_n^{\,2}/r_e^{\,2}+0.5(p/\eta_1+p/\nu_2+1/\hat{B}_1^{\,2}+1/B_2^{\,2})]$$
$$+0.5[(p/\eta_1-p/\nu_2+1/\hat{B}_1^{\,2}$$
$$-1/B_2^{\,2})^2+4/(\hat{B}_1^{\,2}B_2^{\,2})]^{1/2} \tag{50}$$

$$N_2 = [\alpha_n^{\,2}/r_e^{\,2}+0.5(p/\eta_1+p/\nu_2+1/\hat{B}_1^{\,2}+1/B_2^{\,2})]$$
$$-0.5[(p/\eta_1-p/\nu_2+1/\hat{B}_1^{\,2}$$
$$-1/B_2^{\,2})^2+4/(\hat{B}_1^{\,2}B_2^{\,2})]^{1/2} \tag{51}$$

$$\hat{B}_1^{\,2} = 2D\beta_1^{\,2} \tag{52}$$

The criteria for solutions at large and small values of time are derived by comparing $(p/\eta_1 - p/\nu_2)$ with $4/(\hat{B}_1^2B_2^2)$ under radicals in (50) and (51).

Solution for large times: If $(p/\eta_1 - p/\nu)^2 < 0.2/(\hat{B}_1^2B_2^2)$ or $t > 3.3(|\nu_2-\eta_1|)B_2\hat{B}_1/\eta_1\nu_2$, the radicals in (50) and (51) can be approximated by $[1/(\hat{B}_1^2+1/B_2^2)]$, and (48) and (49) are modified accordingly and after application of the inverse transforms ultimately yield

$$d_1(r,\ t) = \sum_{n=1}^{\infty}\frac{4Q_1R_1J_1(\alpha_nR_1/r_e)J_0(\alpha_nr/r_e)}{r_eK_1\nu_2\alpha_nJ_0^{\,2}(\alpha_n)}\cdot\{(\alpha_n^{\,2}/r_e^{\,2}+1/B_2^{\,2})\nu_2/[(\alpha_n^{\,2}/r_e^{\,2}+1/B^2)\alpha_n^{\,2}/r_e^{\,2}]$$
$$-[(\nu_2-\eta)\alpha_n^{\,2}/r_e^{\,2}+\nu_2/B_2^{\,2}]\cdot\exp(-\alpha_n^{\,2}\eta t/r_e^{\,2})/[\alpha_n^{\,2}/(r_e^{\,2}B^2)]$$
$$+[(\nu_2-\eta)\alpha_n^{\,2}/r_e^{\,2}+\nu_2/B_2^{\,2}-\eta/B^2]\cdot\exp[-(\alpha_n^{\,2}/r_e^{\,2}+1/B^2)\eta t]/[(\alpha_n^{\,2}/r_e^{\,2}+1/B^2)/B^2]\}$$
$$+\sum_{n=1}^{\infty}\frac{2Q_2R_2J_1(\alpha_nR_2/r_e)J_0(\alpha_nr/r_e)}{r_eT_2\beta_1^{\,2}\alpha_nJ_0^{\,2}(\alpha_n)}\{1/[(\alpha_n^{\,2}/r_e^{\,2}+1/B_2^{\,2})\alpha_n^{\,2}/r_e^{\,2}]$$
$$+\exp[-(\alpha_n^{\,2}/r_e^{\,2}+1/B^2)\eta t]/[(\alpha_n^{\,2}/r_e^{\,2}+1/B^2)/B^2]-\exp(-\alpha_n^{\,2}\eta t/r_e^{\,2})/(\alpha_n^{\,2}/r_e^{\,2}B^2)\} \tag{53}$$

$$d_2(r,\ t) = \sum_{n=1}^{\infty}\frac{2Q_2R_2J_1(\alpha_nR_2/r_e)J_0(\alpha_nr/r_e)}{r_eT_2\eta_1\alpha_nJ_0^{\,2}(\alpha_n)}\cdot\{\eta_1[\alpha_n^{\,2}/r_e^{\,2}+1/(2D\beta_1^{\,2})]/[(\alpha_n^{\,2}/r_e^{\,2}+1/B^2)\alpha_n^{\,2}/r_e^{\,2}]$$
$$-[(\eta_1-\eta)\alpha_n^{\,2}/r_e^{\,2}+\eta_1/\hat{B}_1^{\,2}]\cdot\exp[-(\alpha_n^{\,2}/r_e^{\,2})\eta t]/[\alpha^2/(r_e^{\,2}B^2)]$$
$$+[(\eta_1-\eta)\alpha_n^{\,2}/r_e^{\,2}+\eta_1/\hat{B}_1^{\,2}-\eta/B^2]\cdot\exp[-(\alpha_n^{\,2}/r_e^{\,2}+1/B^2)\eta t]/[(\alpha_n^{\,2}/r_e^{\,2}+1/B^2)/B^2]\}$$
$$+\sum_{n=1}^{\infty}\frac{2Q_1R_1J_1(\alpha_nR_1/r_e)J_0(\alpha_nr/r_e)}{r_eK_1DB_2^{\,2}\alpha_nJ_0^{\,2}(\alpha_n)}\{1/[(\alpha_n^{\,2}/r_e^{\,2}+1/B^2)\alpha_n^{\,2}/r_e^{\,2}]$$
$$+\exp[-(\alpha_n^{\,2}/r_e^{\,2}+1/B^2)\eta t]/[(\alpha_n^{\,2}/r_e^{\,2}+1/B^2)/B^2]-\exp(-\alpha_n^{\,2}\eta t/r_e^{\,2})/[\alpha_n^{\,2}/(r_e^{\,2}B^2)]\} \tag{54}$$

where

$$\eta = 2\eta_1\nu_2/(\eta_1+\nu_2) \tag{55}$$

Solution for short times: If $0.5[4/(2\hat{B}_1^2B_2^2)] < 0.05(p/\eta_1 - p/\nu_2)^2$, i.e., if time t is of the order of $t \le 0.22(|\nu_2 - \eta_1|)\hat{B}_1B_2/(\eta_1\nu_2)$, the radicals in (50) and (51) can be approximated by $(p/\eta_1 - p/\nu_2 + 1/\hat{B}_1^2 - 1/B_2^2)$, and (48) and (49) are modified accordingly. Application of the inverse transforms to the resulting equations ultimately yields

$$d_1(r, t) = \sum_{n=1}^{\infty} \frac{4Q_1R_1J_1(\alpha_nR_1/r_e)J_0(\alpha_nr/r_e)}{r_eK_1\alpha_nJ_0^2(\alpha_n)[\alpha_n^2/r_e^2 + 1/\hat{B}_1^2]} \{1 - \exp[-(\alpha_n^2/r_e^2 + 1/\hat{B}_1^2)\eta_1t]\}$$
$$+ \sum_{n=1}^{\infty} \frac{2Q_2R_2J_1(\alpha_nR_2/r_e)J_0(\alpha_nr/r_e)}{r_eT_2\beta_1^2\alpha_nJ_0^2(\alpha_n)} \{1/[(\alpha_n^2/r_e^2 + 1/\hat{B}_1^2)(\alpha_n^2/r_e^2 + 1/B_2^2)]$$
$$+ \nu_2 \exp[-(\alpha_n^2/r_e^2 + 1/\hat{B}_1^2)\eta_1t]/[(\alpha_n^2\eta_1/r_e^2 - \alpha_n^2\nu_2/r_e^2 + \eta_1/\hat{B}_1^2 - \nu_2/B_2^2)(\alpha_n^2/r_e^2 + 1/\hat{B}_1^2)]$$
$$- \eta_1 \exp[-(\alpha_n^2/r_e^2 + 1/B_2^2)\nu_2t]/[(\alpha_n^2\eta_1/r_e^2 - \alpha_n^2\nu_2/r_e^2 + \eta_1/\hat{B}_1^2 - \nu_2/B_2^2)(\alpha_n^2/r_e^2 + 1/B_2^2)]\} \quad (56)$$

$$d_2(r, t) = \sum_{n=1}^{\infty} \frac{2Q_2R_2J_1(\alpha_nR_2/r_e)J_0(\alpha_nr/r_e)}{r_eT_2\alpha_nJ_0^2(\alpha_n)(\alpha_n/r_e^2 + 1/B_2^2)} \{1 - \exp[-(\alpha_n^2/r_e^2 + 1/B_2^2)\nu_2t]\}$$
$$+ \sum_{n=1}^{\infty} \frac{2Q_1R_1J_1(\alpha_nR_1/r_e)J_0(\alpha_nr/r_e)}{r_eK_1DB_2^2\alpha_nJ_0^2(\alpha_n)} \{1/[(\alpha_n^2/r_e^2 + 1/B_2^2)(\alpha_n^2/r_e^2 + 1/\hat{B}_1^2)]$$
$$+ \nu_2 \exp[-(\alpha_n^2/r_e^2 + 1/\hat{B}_1^2)\eta_1t]/[(\alpha_n^2\eta_1/r_e^2 - \alpha_n^2\nu_2/r_e^2 + \eta_1/\hat{B}_1^2 - \nu_2/B_2^2)(\alpha_n^2/r_e^2 + 1/\hat{B}_1^2)]$$
$$- \eta_1 \exp[-(\alpha_n^2/r_e^2 + 1/B_2^2)\nu_2t]/[(\alpha_n^2\eta_1/r_e^2 - \alpha_n^2\nu_2/r_e^2 + \eta_1/\hat{B}_1^2 - \nu_2/B_2^2)(\alpha_n^2/r_e^2 + 1/B_2^2)]\} \quad (57)$$

Special Solutions

The solutions derived in the previous sections for the two cases can be simplified to obtain solutions for special cases.

One well field operating. If only one of the well fields is operating, the drawdown distributions in the two aquifers for cases 1 and 2 are obtained by letting the discharge from the nonpumping well field go to zero in the respective solutions. It should be pointed out that the drawdown in the nonpumping aquifer is not zero. There will be some induced drawdown due to pumpage from the other aquifer, because the two aquifers are coupled through the aquitard.

Well field in single aquifers. The drawdown distribution due to the operation of a well field in single isolated aquifers can be obtained by letting the leakage coefficient of the aquitard and the pumpage from the other aquifer (Q_2) approach zero and by making all hydraulic diffusivities equal to that of the isolated aquifer. Equation 21, after the above substitutions, yields

$$s(r, t) = \frac{2QRr_e}{T} \sum_{n=1}^{\infty} [1 - \exp(-\alpha_n^2\nu t/r_e^2)] \cdot \frac{J_1(\alpha_nR/r_e)J_0(\alpha_nr/r_e)}{\alpha_n^3J_0^2(\alpha_n)} \quad (58)$$

Equation 58 describes the drawdown distribution s due to the operation of a well field of radius R in a confined aquifer of transmissivity T.

A similar expression can be derived for the case when the isolated aquifer is an unconfined aquifer from (46) (case 2):

$$d_1(r, t) = \frac{4QRr_e}{K} \sum_{n=1}^{\infty} [1 - \exp(-\alpha_n^2\nu t/r_e^2)] \cdot \frac{J_1(\alpha_nR/r_e)J_0(\alpha_nr/r_e)}{\alpha_n^3J_0^2(\alpha_n)} \quad (59)$$

Equation 59 predicts the drawdown d in a single unconfined aquifer of radius r_e due to pumpage from it.

Concluding Remarks

Solutions for the problem of drawdown distribution due to well fields in coupled leaky aquifers of finite areal extent are obtained. Solutions are in the form of infinite series of Bessel functions and hyperbolic and exponential functions. These functions are extensively tabulated [*Dwight*, 1961; *Abramowitz and Stegun*, 1964]. *Gray and Mathews* [1966] also give tables of Bessel functions and the first 50 roots of $J_0(x) = 0$ and of $J_1(x) = 0$.

The drawdown distributions in coupled leaky aquifers derived in this paper are based on the assumption that the storativity of the aquitard is negligible. Several investigators have pointed out that the storativity of the aquitard can be significant [e.g., *Hantush*, 1960; *Domenico and Mifflin*, 1965; *Neuman and Witherspoon*, 1969; *Saleem*, 1973]. *Neuman and Witherspoon* [1969] discussed the limitations of assuming no storage in the aquitard. *Bredehoeft and Pinder* [1970] pointed out that storage in the aquitard is important only during an initial period when dimensionless time, expressed in terms of the aquitard parameters, is approximately 0.1. The solutions derived in the previous sections are not applicable during this initial period.

The solutions for the drawdown distribution due to well fields in coupled leaky aquifers are needed for their management [*Saleem and Jacob*, 1971]. The solutions can also be used for the prediction of rise in hydraulic head due to artificial recharge in finite coupled leaky aquifers by changing the sign of terms representing flow rates. The method of superposition [*Carslaw and Jaeger*, 1959] can be used in conjunction with solutions developed in this paper to obtain solutions for more complicated cases involving recovery. Steady state solutions can be obtained by letting time approach infinity in the solutions for the two cases.

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Notation

- b' thickness of aquitard.
- c parameter defined by (19).
- c_1, c_2 parameters defined by (18).
- d_1, d_2 drawdowns in aquifers 1 and 2, respectively, defined by (41).
- $f_1(r)$ parameter defined by (8).
- $g(r)$ parameter defined by (9).

h_1, h_2 hydraulic heads in aquifers 1 and 2, respectively.
h_{i1}, h_{i2} initial values of h_1, h_2, respectively.
p Laplace transform parameter.
r radial distance from the center of well fields to any point in the surrounding area.
r_e radius of the aquifer system.
s_1, s_2 drawdowns in aquifers 1 and 2, respectively, defined by (6).
B parameter defined by (30).
B_1, B_2 leakage factors for the two aquifers, equal to $(T_n/K'/b')^{0.5}$, $n = 1, 2$.
$\hat{B}_1$ leakage factor for unconfined aquifer, defined by (52).
D weighted mean depth of the flow profile for aquifer 1.
$\bar{F}_1, \bar{F}_2$ parameters defined by (16) and (17), respectively.
$\bar{G}_1, \bar{G}_2$ parameters defined by (43) and (44), respectively.
H_0 zero-order Hankel transform of a function modified here and defined by (10).
$J_0(x)$ zero-order Bessel function of the first kind.
$J_0'(x)$, $J_0''(x)$ first and second derivatives of $J_0(x)$, respectively.
$J_1(x)$ first-order Bessel function of the first kind.
K_1 hydraulic conductivity of aquifer 1.
K' hydraulic conductivity of aquitard.
M_1, M_2 parameters defined by (25) and (26).
N_1, N_2 parameters defined by (50) and (51).
Q_1, Q_2 volumetric flow rates per unit area of well fields in aquifers 1 and 2, respectively.
R_1, R_2 radii of well fields in aquifers 1 and 2, respectively.
S_1, S_2 storativities of aquifers 1 and 2, respectively.
T_1, T_2 transmissivities of aquifers 1 and 2, respectively.
α_n positive roots of $J_1(x) = 0$.
β parameter defined by (45*b*).
β_1 parameter defined by (42).
η parameter defined by (55).
η_1 hydraulic diffusivity of the unconfined aquifer.
θ specific yield of the unconfined aquifer.
μ parameter defined by (20).
ν_1, ν_2 hydraulic diffusivities of aquifers 1 and 2, respectively.
ν_ν parameter defined by (29).
l parameter defined by (45*a*).

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