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Generalized Elliptic Genus and Cobordism Class of Nonspin Real Grassmannians

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Abstract. We prove that Witten's generalized elliptic genus is rigid under certain group actions and constant (identically zero) on the real Grassmannians $\mathbb{G}r_4(\mathbb{R}^{2m+5})$, $m \geq 1$, for appropriate vector bundles.

We also prove that all the Pontryagin numbers of these Grassmannians are zero, i.e. they have cobordism class zero, which implies the vanishing of all genera.

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1. Introduction

We shall be considering the index of the following operator suggested by Witten [19, 20] for a compact, oriented, smooth $2n$-dimensional (spin) manifold M,

$$\partial\!\!\!/ \otimes \Delta(V) \otimes \bigotimes_{i=1}^{\infty} \bigwedge\nolimits_{q^i} V \otimes \bigotimes_{j=1}^{\infty} S_{q^j} T, \tag{1}$$

where $T = TM_c$ is the complexified tangent bundle,

$$S_a T = \sum_{j=0}^{\infty} a^j S^j T, \quad \bigwedge\nolimits_a V = \sum_{j=0}^{\infty} a^j \bigwedge\nolimits^j V,$$

and $S^j T$, $\bigwedge^j V$ denote the jth symmetric and exterior tensor powers of T and V, respectively. Here, V is an oriented real (spin) vector bundle of even rank endowed with a connection and $w_2(V) = w_2(M)$; $\Delta(V)$ is the corresponding (possibly only locally defined) spin bundle, and $\partial\!\!\!/ \otimes \Delta(V)$ is a globally defined twisted Dirac

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operator on M. The index of (1) defines a generalized elliptic genus with modular properties [1, 10].

$$\tau_q(M, V/T) = \sum_{i=0}^{\infty} \mathrm{ind}(\partial\!\!\!/ \otimes \Delta(V) \otimes R_i) \cdot q^i,$$

where each R_i is the coefficient of q^i in the tensor product

$$\bigotimes_{i=1}^{\infty} \bigwedge\nolimits_{q^i} V \otimes \bigotimes_{j=1}^{\infty} S_{q^j} T.$$

The first few terms of this elliptic genus are

$$\begin{aligned}
R_0 &= 1,\\
R_1 &= V \oplus T,\\
R_2 &= \bigwedge\nolimits^2 V \oplus V \oplus S^2T \oplus T \oplus V \otimes T,\\
R_3 &= \bigwedge\nolimits^3 V \oplus V \oplus S^3T \oplus T \oplus 2V\\
&\quad \otimes T \oplus V \otimes S^2T \oplus \bigwedge\nolimits^2 V \otimes T \oplus \bigwedge\nolimits^2 V \oplus S^2T,
\end{aligned}$$

so that the first term is $\mathrm{ind}(\partial\!\!\!/ \otimes \Delta(V))$.

By setting $V = T$ one recovers the original elliptic genus $\tau_q(M)$ whose coefficients are indices of twisted signature operators well defined on any compact, oriented, smooth $2n$-dimensional manifold M. Particular attention has been paid to the elliptic genus on spin manifolds with circle actions [1, 12, 13, 18–20] for which Witten conjectured rigidity theorems under such actions, proved by Taubes [18], Bott and Taubes [1]. Roughly speaking, this means that if M admits a circle action so that the equivariant signatures $\tau(M, R_i)_g = \mathrm{ind}(\partial\!\!\!/ \otimes \Delta(T) \otimes R_i)_g$ and equivariant elliptic genus $\tau_q(M)_g$ are defined for $g \in S^1$, then

$$\tau_q(M)_g = \tau_q(M),$$

i.e. they do not depend on g. The statement of rigidity for the generalized elliptic genus is analogous, and was conjectured by Witten [20] and proved by Bott and Taubes [1]. In [2, 4, 5], however, we have considered nonspin manifolds with finite second homotopy group on which the rigidity theorem for the original elliptic genus $\tau_q(M) = \tau_q(M, T/T)$ still holds. We shall explore the rigidity properties of $\tau_q(M, V/T)$ on general π_2-finite manifolds in a future paper.

Hirzebruch and Slodowy proved in [7] that on many spin homogeneous spaces the original elliptic genus is constant and equal to the signature of the space. In this note, we consider the real Grassmannians $\mathbb{G}r_4(\mathbb{R}^{2m+5})$, $m \geq 1$, which have finite π_2 and are not spin. We show that the elliptic genus (1) is rigid under certain S^1 actions and vanishes completely on these Grassmannians for appropriate vector bundles V, therefore extending the results of [7] to the generalized elliptic genus on nonspin π_2-finite manifolds. Finally, we compute all the Pontryagin numbers of these Grassmannians via further index calculations, thus determining their cobordism class. This proves the vanishing of all genera, including the one above, but not the rigidity result.

The paper is organized as follows. In Section 2 we recall the twistor construction on quaternion-Kähler manifolds and the Bott–Borel–Weil Theorem. In Section 3 we review the nonrigidity of the elliptic genus on the complex projective plane, we state and prove the rigidity of the generalized elliptic genus on the real Grassmannians, and determine the cobordism class of the Grassmannians.

2. Preliminaries

Since the Grassmannians under consideration are quaternion-Kähler manifolds, we review in this section some facts of quaternion-Kähler geometry that will be used. For further details, see [17].

2.1. QUATERNION-KÄHLER MANIFOLDS AND TWISTOR SPACES

An oriented, connected, irreducible $4n$-dimensional Riemannian manifold M is called a quaternion-Kähler manifold, $n \geq 2$, if its linear holonomy is contained in the group $\mathrm{Sp}(n)\mathrm{Sp}(1)$. If $n = 1$, one must require the manifold to be Einstein and self-dual, since $\mathrm{Sp}(1)\mathrm{Sp}(1) = \mathrm{SO}(4)$. Due to the reduction of the structure bundle, the complexified tangent bundle of M can be seen as a product

$$TM_c = E \otimes H,$$

where E and H are locally defined bundles corresponding to the standard complex representations $\mathbb{C}^{2n}$ and $\mathbb{C}^2$ of $\mathrm{Sp}(n)$ and $\mathrm{Sp}(1)$ respectively. The real Grassmannians we are considering belong to a family of quaternion-Kähler symmetric spaces with positive scalar curvature. There is one of the form G/K for each compact simple Lie group G, where K is the centralizer of a three dimensional subalgebra of $\mathfrak{g} = \mathrm{Lie}(G)$ corresponding to a highest root [21]. They are called Wolf spaces and include the quaternionic projective spaces $\mathbb{HP}^n$, the complex Grassmannians $\mathbb{G}r_2(\mathbb{C}^{n+2})$, the real Grassmannians $\mathbb{G}r_4(\mathbb{R}^{n+4})$, and five exceptional symmetric spaces.

Although quaternion-Kähler manifolds are not spin in general, there are well-defined Dirac operators coupled to vector bundles, such as $\partial\!\!\!/ \otimes \bigwedge^p E \otimes S^q H$ with $p+q-n$ even. This gives rise to a theory of quaternionic spinors which has proved to be very successful in studying the geometry and topology of quaternion-Kähler manifolds [17].

The twistor space Z of a quaternion-Kähler manifold M is the projectivization of H, $Z = \mathbb{P}(H)$, which is globally defined and fibers over $\pi\colon Z \longrightarrow M$ with fibre $\mathbb{CP}^1$. In fact, Z admits a natural complex structure. When the quaternion-Kähler manifold has nonzero scalar curvature, the twistor space admits a complex contact structure given as follows. The Levi-Civita connection on M determines a horizontal holomorphic distribution $\mathcal{D}$ giving the following short exact sequence

$$0 \longrightarrow \mathcal{D} \longrightarrow T^{1,0}Z \longrightarrow L \longrightarrow 0,$$

where L is the quotient line bundle. In the case of positive scalar curvature, the line bundle L is positive, making Z a Fano manifold that can be embedded into a complex projective space by some power of L. Furthermore,

$$\mathcal{D} \cong \pi^* E \otimes L^{1/2} \qquad \text{and} \qquad \pi^* H \cong L^{1/2} \oplus L^{-1/2}.$$

A remarkable fact [14, 16] is that the cohomology of the Dirac operators on M coupled to $S^q H$ is isomorphic to that of the basic Dolbeaut complex on Z coupled to the line bundle $L^{(q-n)/2}$, so that

$$\mathrm{ind}(\partial\!\!\!/ \otimes S^q H) = \chi(Z, \mathcal{O}(L^{(q-n)/2})),$$

where $q - n$ is even. These complexes can be tensored with quaternionic bundles, such as E or its powers to give further identities. For instance [9],

$$\mathrm{ind}(\partial\!\!\!/ \otimes \textstyle\bigwedge^p E \otimes S^q H) = \chi(Z, \mathcal{O}(\bigwedge^p(\pi^* E) \otimes L^{(q-n)/2})),$$

with $p + q - n$ even, or equivalently

$$\mathrm{ind}(\partial\!\!\!/ \otimes \textstyle\bigwedge^p E \otimes S^q H) = \chi(Z, \mathcal{O}(\bigwedge^p \mathcal{D}^* \otimes L^k)),$$

with $k = (p + q - n)/2$. This is the content of the twistor transform. In particular, the twistor fibration gives $\pi^*(4c_2(H)) = -c_1(L)^2$ on the twistor space, a fact that will be used later.

Notice that the twistor space of a Wolf space is then a homogeneous space where the isotropy group is the centralizer of a one-dimensional torus. This fact enables us to use the Bott–Borel–Weil Theorem (see below) in our index calculations in Section 3.2.

2.2. THE GRASSMANNIAN

Let

$$M = \mathbb{G}r_4(\mathbb{R}^{2m+5}) = \frac{\mathrm{SO}(2m+5)}{\mathrm{SO}(2m+1) \times \mathrm{SO}(4)}.$$

Since the isotropy group is $\mathrm{SO}(2m+1) \times \mathrm{SO}(4)$, its complexified tangent bundle factors as follows

$$TM_c = W \otimes (U \otimes H),$$

where W and $U \otimes H$ are homogeneous vector bundles. W corresponds to the (complex) standard representations of $\mathrm{SO}(2m+1)$, and $U \otimes H$ corresponds to the product of two copies of the standard representations of the two copies of $\mathrm{SU}(2)$

since $\mathrm{SO}(4) = \mathrm{SU}(2) \times_{\mathbb{Z}_2} \mathrm{SU}(2)$. In fact, U and H constitute the half spinor representations in 4 dimensions. The twistor space takes the form

$$Z = \frac{\mathrm{SO}(2m+5)}{\mathrm{SO}(2m+1) \times U(2)}$$

with fibre $\mathbb{CP}^1$ over the Grassmannian. It is a Kähler manifold of complex dimension $4m+3$ and has a positive complex line bundle associated to the determinant of the standard representation of $U(2)$. In this case, the twistor transform gives

$$\widehat{A}(M, P(W) \otimes S^i U \otimes S^j H) = \chi(Z, \pi^*(P(W) \otimes S^i U) \otimes L^{(j-(2m+1))/2}), \quad (2)$$

where $P(W)$ is a tensor bundle associated to W. We will use the following shorthand notation to denote the twistor transform of an index

$$P(W) \otimes S^i U \otimes S^j H \rightsquigarrow P(W) \otimes S^i U \otimes L^{(j-(2m+1))/2}.$$

Although the indices in (2) will represent indices of elliptic operators only if $i + j + 2m + 1$ is even, the corresponding Euler characteristic for the right-hand side can be computed for any values of i and j by means of the Bott–Borel–Weil Theorem and the Weyl dimension formula. Furthermore, since we are in a homogeneous set-up, the Bott–Borel–Weil Theorem identifies the relevant cohomology spaces as representations of $\mathrm{SO}(2m+5)$.

2.3. BOTT–BOREL–WEIL THEOREM

Here we recall the Bott–Borel–Weil Theorem as quoted in [8, theorem 5], which identifies the spaces $H^{0,k}(Y, V)$ for certain complex homogeneous spaces Y and homogeneous vector bundles V on them. Let

G = a compact connected Lie group,
T = torus in G,
$L = Z_G(T)$ the centralizer of the torus T,
T extended to a maximal torus $\tilde{T}$ in L,
$\triangle = \{\text{roots of } (\mathfrak{g}, \tilde{\mathfrak{t}})\}$,
$\triangle(\mathfrak{l}) = \{\text{roots of } (\mathfrak{l}, \tilde{\mathfrak{t}})\} \subset \triangle$,
$\triangle^+$ = positive system of roots for $\triangle$ chosen with $\triangle(\mathfrak{l})$ generated by simple roots,
$\delta = \frac{1}{2} \sum_{\alpha \in \triangle^+} \alpha$,
W = Weyl group.

The space $Y = G/L$ is a complex manifold.

THEOREM 2.1 (Bott–Borel–Weil Theorem)). *Let $V(\lambda)$ be an irreducible representation of L with highest weight λ.*

(a) *If* $\langle \lambda + \delta, \alpha \rangle = 0$ *for some* $\alpha \in \Delta$, *then* $H^{0,k}(G/L, V(\lambda)) = 0$ *for all* k.
(b) *If* $\langle \lambda + \delta, \alpha \rangle \neq 0$ *for all* $\alpha \in \Delta$, *define*

$$q = \sharp\{\alpha \in \Delta^+ \mid \langle \lambda + \delta, \alpha \rangle < 0\}.$$

Choose $w \in W$ *so that* $w(\lambda + \delta)$ *is dominant and set* $\mu = w(\lambda + \delta) - \delta$. *Then*

$$H^{0,k}(G/L, V(\lambda)) = \begin{cases} 0 & \text{if } k \neq q, \\ F(\mu) & \text{if } k = q, \end{cases} \tag{3}$$

where $F(\mu)$ *is a finite dimensional irreducible representation of G with highest weight* μ.

The dimension of the representation $F(\mu)$ is given by the Weyl dimension formula

$$\dim F(\mu) = \prod_{\alpha \in \Delta^+} \frac{\langle \alpha, \mu + \delta \rangle}{\langle \alpha, \delta \rangle}.$$

3. Rigidity

We begin this section by examining the nonrigidity of the original elliptic genus on the complex projective plane. After that we proceed to the case of the real Grassmannians.

3.1. COMPLEX PROJECTIVE PLANE

Consider the original elliptic genus on an oriented, compact, connected $2n$-dimensional manifold M

$$\tau_q(M) = \tau_q(M, T/T) = \sum_{i=0}^{\infty} \mathrm{ind}(d_s \otimes R_i) \cdot q^i, \tag{4}$$

where $d_s = \partial\!\!\!/ \otimes \Delta(T)$ is the signature operator and

$$\begin{aligned} R_0 &= 1 \\ R_1 &= 2T \\ R_2 &= 2(T^{\otimes 2} + T) \\ R_3 &= 2T^{\otimes 3} + \bigwedge^3 T + S^3 T + 4T^{\otimes 2} + 2T, \end{aligned}$$

so that the constant term in $\tau_q(M)$ is the signature of M, $\tau(M)$.

Let $X = \mathbb{CP}^2$ considered as a quaternion-Kähler manifold. Since X is not spin and $\pi_2(X) = \mathbb{Z}$, the elliptic genus is not rigid on X. For instance,

$$\tau_q(X) = 1 + 32q + \text{higher-order terms},$$

while the equivariant elliptic genus

$$\tau_q(X)_g = 1 \quad \text{(independent of } q),$$

where g is a projective involution of X with fixed point set a projective line and a point [7]. The first two terms of the elliptic genus on X are $\tau(X) = 1$ and $2\tau(X, T) = 32$, while the first two terms of the equivariant one are $\tau(X)_g = 1$ and $2\tau(X, T)_g = 0$.

The indices $\tau(X) = 1$ and $\tau(X, T) = 16$ are, in fact, the (virtual) dimensions of certain vector spaces, and $\tau(X)_g = 1$ and $\tau(X, T)_g = 0$ are the traces of g on them. Since X is a homogeneous space, we can actually compute these vector spaces as representations of SU(3). We have that

$$TX_c = E \otimes H,$$

where this time E corresponds to a self-adjoint rank 2 representation of $U(1)$ and H to the standard representation of SU(2). The spin representation corresponding to T is then $\Delta(T) = E \oplus H$. The signature operator $d_s = \partial\!\!\!/ \otimes (E \oplus H)$ and $d_s \otimes T = \partial\!\!\!/ \otimes (E \oplus H) \otimes (E \otimes H)$. By twistor transform

$$E \oplus H \rightsquigarrow E \otimes L^{-1} \oplus 1,$$

where 1 denotes a rank 1 trivial bundle, and

$$(E \oplus H) \otimes (E \otimes H) \rightsquigarrow 1 \oplus S^2E \oplus E \otimes L^{1/2} \oplus E \otimes L^{-1/2}.$$

Let $F(\lambda_1, \lambda_2, \lambda_3)$ denote the complex irreducible representation of SU(3) with dominant weight $(\lambda_1, \lambda_2, \lambda_3)$, where the coordinates are such that $F(1, 0, 0) = \mathbb{C}^3$ and $F(1, 1, 0) = \mathfrak{su}(3)$ are the standard and adjoint representations of SU(3) respectively. By Wolf's description, the root giving rise to the one $U(1)$ factor of the isotropy of the twistor space which produces L is $(1, 0, -1)$ in the Cartan subalgebra of $\mathfrak{su}(3)$. The embedded weight of E is $(1, 0, 1)$. By using the Bott–Borel–Weil theorem, the SU(3)-representation corresponding to $\tau(X)$ is the one-dimensional trivial representation $V(0, 0, 0)$, so that the trace of g on it is 1. The SU(3) representation corresponding to $\tau(X, T) = 16$ is

$$2\, F(0, 0, 0) \oplus F(1, 0, 1) \oplus F(1, 0, 0,) \oplus F(0, 0, -1),$$

where $\dim F(0, 0, 0) = 1$, $\dim F(1, 0, 1) = 8$, $\dim F(1, 0, 0,) = 3$, $\dim F (0, 0, -1) = 3$. The trace of g on these spaces is

$$2 + 0 + (-1) + (-1) = 0,$$

by the Weyl character formula, so that $\tau(X, T)_g = 0$.

Remark. In the case of the real Grassmannians, all the relevant (virtual) vector spaces turn out to be trivial vector spaces, so that there is no nontrivial vector space on which to take a trace of a group element, which renders the rigidity result as well as the vanishing.

3.2. THE RIGIDITY ON THE GRASSMANNIANS

We will use the notation established in Section 2.2.

THEOREM 3.1. *Let $V = U \otimes H$. The spin representation of V is $\Delta(V) = U \oplus H$ and the twisted Dirac operator $\partial\!\!\!/ \otimes (U \oplus H)$ is well defined. Then the generalized elliptic genus $\tau_q(\mathbb{G}r_4(\mathbb{R}^{2m+5}), V/T)$ is rigid under the action of* $\mathrm{SO}(2m+5)$ *and vanishes identically.*

Remark. Notice that we have dropped the condition $p_1(V) = p_1(T)$.

Proof. The spin bundle $\Delta(U \otimes H) = U \oplus H$ is only locally defined. However, its tensor product with itself

$$\begin{aligned}\Delta(U \otimes H) \otimes \Delta(U \otimes H) &= (U \oplus H) \otimes (U \oplus H)\\ &= 2 \oplus S^2U \oplus S^2H \oplus U \otimes H\end{aligned}$$

is globally defined, and its sections are operated on by $\partial\!\!\!/ \otimes \Delta(U \otimes H)$.

Let $\{e_i,\ i = 1, \ldots, m+2\}$ denote the canonical basis of $\mathbb{R}^{m+2}$. Let

$$\mathfrak{H} = \mathrm{span}(\{\alpha_j = e_j - e_{j+1},\ j = 1, \ldots, m+1\} \cup \{\alpha_{m+2} = e_{m+2}\})$$

be the Cartan subalgebra of $\mathfrak{so}(2m+5)$ generated by the simple roots α_j. Thus, half the sum of the positive roots is $\delta = (1/2)(2m+3, 2m+1, \ldots, 1)$.

On the twistor space

$$Z = \frac{\mathrm{SO}(2m+5)}{\mathrm{SO}(2m+1) \times U(2)},$$

the fundamental representation Q of $U(2)$ has dominant weight $(1, 0)$ so that $L = \det(Q)$ has weight $(1, 1)$, which will be embedded in $\mathfrak{H}$ as $(1, 1, 0, \ldots, 0)$. The representation U of $\mathrm{SU}(2)$ corresponds to $U = Q \otimes L^{1/2}$ and has weight $(1, 0) + (-1/2, -1/2) = (1/2, -1/2)$, embedded as $(1/2, -1/2, 0, \ldots, 0)$. The standard representation W of $\mathrm{SO}(2m+1)$ has dominant weight $(1, 0, \ldots, 0) \in \mathbb{R}^m$ which will be embedded as e_3.

First, let us review a few of the bundles R_i

$$\begin{aligned}R_0 &= 1,\\ R_1 &= [U \otimes H] \oplus [W \otimes U \otimes H],\\ R_2 &= [S^2U \oplus S^2H] \oplus [U \otimes H] \oplus [W \otimes U \otimes H]\\ &\quad \oplus [W \otimes (S^2U \otimes S^2H \oplus S^2U \oplus S^2H \oplus 1)],\\ &\quad \oplus [S^2W \otimes S^2U \otimes S^2H \oplus \textstyle\bigwedge^2 W \otimes S^2U \oplus \bigwedge^2 W \otimes S^2H],\end{aligned}$$

Now, before applying twistor transform we have to take the product with $\Delta(U \otimes H) = U \oplus H$,

$$\begin{aligned}
\Delta(V) \otimes R_0 &= U \oplus H, \\
\Delta(V) \otimes R_1 &= (1 \oplus W) \otimes (S^2U \otimes H \oplus H \oplus U \otimes S^2H \oplus U), \\
\Delta(V) \otimes R_2 &= (1 \oplus W) \otimes (S^2U \otimes H \oplus H \oplus U \otimes S^2H \oplus U) \\
&\quad \oplus W \otimes (U \oplus H) \oplus (1 \oplus W \oplus {\textstyle\bigwedge}^2 W) \\
&\quad \otimes (S^3U \oplus U \oplus S^2U \otimes H \oplus U \otimes S^2H \oplus S^2H \oplus H) \\
&\quad \oplus (1 \oplus W) \otimes (S^3U \otimes S^2H \oplus U \otimes S^2H \oplus S^2U \\
&\quad \otimes S^3H \oplus S^2U \otimes H).
\end{aligned}$$

For example, the twistor transform gives

$$\begin{aligned}
&U \rightsquigarrow U \otimes L^{(-(2m+1))/2}, \\
&H \rightsquigarrow L^{-m}, \\
&S^2U \otimes H \rightsquigarrow S^2U \otimes L^{-m}, \\
&U \otimes S^2H \rightsquigarrow U \otimes L^{(1-2m)/2}, \\
&W \otimes S^2U \otimes H \rightsquigarrow W \otimes S^2U \otimes L^{-m}, \\
&W \otimes U \otimes S^2H \rightsquigarrow W \otimes U \otimes L^{(1-2m)/2},
\end{aligned}$$

etc. Since each one of these bundles correspond to irreducible representations of the isotropy group, by the Bott–Borel–Weil Theorem we only need to use their highest weights in the following calculations. The highest weights of each of the representations embedded into the Cartan subalgebra of $\mathfrak{so}(2m+5)$ are

$$\begin{aligned}
&U \otimes L^{(-(2m+1))/2} \mapsto (-m, -1-m, 0, \dots, 0), \\
&L^{-m} \mapsto (-m, -m, 0, \dots, 0), \\
&S^2U \otimes L^{-m} \mapsto (1-m, -1-m, 0, \dots, 0), \\
&U \otimes L^{(1-2m)/2} \mapsto (1-m, -m, 0, \dots, 0), \\
&W \otimes S^2U \otimes L^{-m} \mapsto (1-m, -1-m, 1, 0, \dots, 0), \\
&W \otimes U \otimes L^{(1-2m)/2} \mapsto (1-m, -m, 1, 0, \dots, 0),
\end{aligned}$$

Add $\delta = (1/2)(2m+3, 2m+1, \dots, 1)$ to each one of these weights:

$$\begin{aligned}
&(-m, -1-m, 0, 0, \dots, 0) + \delta = \tfrac{1}{2}(3, -1, 2m-1, 2m-3, \dots 1), \\
&(-m, -m, 0, 0, \dots, 0) + \delta = \tfrac{1}{2}(3, 1, 2m-1, 2m-3, \dots 1),
\end{aligned}$$

$$(1-m, -1-m, 0, 0, \ldots, 0) + \delta = \tfrac{1}{2}(5, -1, 2m-1, 2m-3, \ldots 1),$$

$$(1-m, -m, 0, 0, \ldots, 0) + \delta = \tfrac{1}{2}(5, 1, 2m-1, 2m-3, \ldots 1),$$

$$(1-m, -1-m, 1, 0, \ldots, 0) + \delta = \tfrac{1}{2}(5, -1, 2m+1, 2m-3, \ldots 1),$$

$$(1-m, -m, 1, 0, \ldots, 0) + \delta = \tfrac{1}{2}(5, 1, 2m+1, 2m-3, \ldots 1),$$

which already show a pattern. In general, since the decompositions of the tangent space $TM_c = W \otimes U \otimes H$, the bundle $V = U \otimes H$ and the spinor bundle $\Delta(U \otimes H) = U \oplus H$ are symmetrical in U and H, every R_i that contains a summand with a factor $S^iU \otimes S^jH$ will contain a symmetrical one containing $S^jU \otimes S^iH$. Hence, the corresponding weights plus δ will be

$$\gamma_{i,j} = \tfrac{1}{2}(i+j+2, j-i, * \ldots, *)$$

and

$$\gamma_{j,i} = \tfrac{1}{2}(j+i+2, i-j, * \ldots, *),$$

respectively. Therefore, the two resulting vectors are related by a reflection in the Weyl group so that the two values of q computed by the Bott–Borel–Weil theorem for each one of them have different parity, and give the same $\mathrm{SO}(2m+5)$-representation but with different sign. This means that there is no vector space on which to take the trace of a transformation. This implies the rigidity of this elliptic genus as well as its vanishing. □

From the main argument we get the following corollary:

COROLLARY 3.1. *Let V be a vector bundle which contains symmetric powers of U and H is a symmetrical manner. The twisted Dirac operator $\partial\!\!\!/ \otimes \Delta(V)$ is well defined. Then the elliptic genus $\tau_q(\mathbb{G}r_4(\mathbb{R}^{2m+5}), V/T)$ is rigid under the action of* $\mathrm{SO}(2m+5)$ *and vanishes identically.*

3.3. COBORDISM CLASS

As a final observation, let us point out that the vanishing of these genera provide enough linear equations to determine all the Pontryagin numbers of the Grassmannian, and therefore its cobordism class [11].

Let $M = \mathbb{G}r_4(\mathbb{R}^{2m+5})$. As mentioned before, the tangent bundle factors

$$TM_c = W \otimes V,$$

with $V = U \otimes H$. The sum of the bundles W and V is trivial

$$W \oplus V = 2m+5,$$

which implies

$$\begin{aligned} TM_c &= (2m+5-V)\otimes V, \\ &= (2m+5)U\otimes H - (S^2U\otimes S^2H \oplus S^2U \oplus S^2H + 1), \end{aligned}$$

which shows that all the Pontryagin classes are given in terms of the classes of S^2U and S^2H, i.e. $c_2(U)$ and $c_2(H)$. Thus, all the Pontryagin numbers are linear combinations of the characteristic numbers

$$\langle c_2(H)^{2m+1-j}c_2(U)^j, [M]\rangle.$$

By using the same approach for calculating indices of coupled Dirac operators as in 3.2, we can evaluate all the characteristic numbers involving $c_2(U)$ and $c_2(H)$.

PROPOSITION 3.1. *Evaluation on the fundamental class* $[M]$ *yields*

$$\begin{aligned} &\langle 4^{2m+1}c_2(H)^{2m+1-j}c_2(U)^j, [M]\rangle \\ &\quad = (-1)^j 2\binom{2m+1}{j}\binom{4m+4}{2m+1}\binom{4m+4}{2j+1}^{-1}. \end{aligned}$$

Proof. By the calculations above, the polynomial

$$\begin{aligned} f_j(k) &= \operatorname{ind}(\not{\partial}\otimes S^{2m+1+2k}H\otimes S^{2j}U) \\ &= \frac{(2k+2j+2m+3)(2k+2m+2)(2k-2j+2m+1)(2j+1)}{(2m+3)(2m+2)(2m+1)^2} \\ &\quad \times \binom{k+j+2m+1}{2m}\binom{k-j+2m}{2m}. \end{aligned}$$

Since U corresponds to the standard representation of $\mathrm{Sp}(1) = \mathrm{SU}(2)$, the tensor powers of the virtual representation $S^2U - 3$, where 3 denotes a trivial representation of dimension 3, satisfy

$$(S^2H-3)^{\otimes j} = \sum_{i=0}^{j}\binom{2m+1}{i}S^{2(j-i)}H,$$

which is proved by induction and using the Clebsch–Gordan formula. Therefore, the polynomial in k for each j,

$$\begin{aligned} h_j(k) &= \widehat{A}(M, S^{2m+1+2k}H\otimes(S^2U-3)^{\otimes j}) \\ &= \sum_{i=0}^{j}\binom{2m+1}{i}f_{j-i}(k) \\ &= (-1)^j\frac{(2j+1)!(2m-2j-1)!(k+m+1)}{j!(2m-j+1)!(2m+3)(2m+2)}\binom{k+2m+1}{2m+1} \\ &\quad \times (4m^2+8km+4k^2+8k+2m(4-j)+3(1-j))\binom{k+2m-j}{2m-2j-1}. \end{aligned}$$

$h_j(k)$ has degree $4m + 3 - 2j$ and the coefficient of $k^{4m+3-2j}$ is, on the one hand,

$$\frac{2}{(4m+3-2j)!}\langle 4^{2m+1}c_2(H)^{2m+1-j}c_2(U)^j, [M]\rangle$$

since the lowest dimensional component of $\mathrm{ch}((S^2U - 3)^{\otimes j})$ is $c_2(U)^j$. On the other hand, from the expression for $h_j(k)$, such a coefficient is given as follows

$$(-1)^j \frac{4(2j+1)!}{j!(2m-j+1)!(2m+3)!},$$

which proves the proposition. □

Notice that, while the Pontryagin numbers are symmetric linear combinations of the pairings computed above, the pairings themselves are antisymmetric. For instance,

$$\langle 4^{2m+1}c_2(H)^{2m+1}, [M]\rangle = -\langle 4^{2m+1}c_2(U)^{2m+1}, [M]\rangle.$$

COROLLARY 3.2. *The oriented cobordism class of the real Grassmannian* $\mathbb{G}r_4(\mathbb{R}^{2m+5})$, $m \geq 1$, *is equal to zero.*

In particular, this proves that other genera vanish completely. For instance, there is another index-type expression [10]

$$\tilde{\tau}_q(M, V/T) = \sum_{i=0}^{\infty} \widehat{A}(M, R_i') \cdot q^i, \tag{5}$$

where the R_i' come from the following product

$$\sum_{i=0}^{\infty} R_i' \cdot q^i = \bigotimes_{i=1}^{\infty} \bigwedge\nolimits_{-q^{2i-1}} V \otimes \bigotimes_{j=1}^{\infty} S_{q^{2j}} T,$$

and

$$\widehat{A}(M, R_i') = \langle \widehat{A}(M)\mathrm{ch}(R_i'), [M]\rangle$$

is a twisted $\widehat{A}$-genus. In our context, these $\widehat{A}$-genera are not indices of well-defined twisted Dirac operators.

COROLLARY 3.3. *Let V be a vector bundle given in terms of symmetric powers of U and H is a symmetrical manner. Then the elliptic genus* $\tilde{\tau}_q(\mathbb{G}r_4(\mathbb{R}^{2m+5}), V/T)$ *vanishes identically. Furthermore, if* $V = 0$ *one obtains the so-called Witten genus* [6] *which vanishes identically on* $\mathbb{G}r_4(\mathbb{R}^{2m+5})$.

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