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Anytime clausal reasoning *

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Given any incomplete clausal propositional reasoner satisfying certain properties, we extend it to a family of increasingly-complete, sound, and tractable reasoners. Our technique for generating these reasoners is based on restricting the length of the clauses used in chaining (Modus Ponens). Such a family of reasoners constitutes an anytime reasoner, since each propositional theory has a complete reasoner in the family. We provide an alternative characterization, based on a fixed-point construction, of the reasoners in our anytime families. This fixed-point characterization is then used to define a transformation of propositional theories into logically equivalent theories for which the base reasoner is complete; such theories are called "vivid". Developing appropriate notions of vividness and techniques for compiling theories into vivid theories has already generated considerable interest in the KR community. We illustrate our approach by developing an anytime family based on Boolean constraint propagation.

1. Introduction

Given a logical knowledge base represented by a clausal propositional theory, it is important in many AI systems to determine formulae that are logically entailed. Since this reasoning problem is intractable[1] [7], many practical AI systems use sound, incomplete, and tractable reasoners [8]. One such reasoner, boolean constraint propagation (BCP) [27,28], is widely used for incomplete linear-time reasoning with clausal propositional theories. Since BCP is quite weak, for example, it even cannot detect the inconsistency in the theory $\{(P \vee Q), (P \vee \neg Q), (\neg P \vee Q), (\neg P \vee \neg Q)\}$, is not suitable for several important applications that require more reasoning capabilities [19].

We extend BCP to a family $\vdash_0^{\mathrm{BCP}}, \vdash_1^{\mathrm{BCP}}, \ldots$ of reasoners such that each $\vdash_i^{\mathrm{BCP}}$ is tractable, each $\vdash_{i+1}^{\mathrm{BCP}}$ is at least as complete as $\vdash_i^{\mathrm{BCP}}$, and each theory has a $\vdash_i^{\mathrm{BCP}}$ complete for reasoning with it. These reasoners could be used for providing a quick "first cut" to a problem, which can be later improved. Given any reasoning task, one could start with $\vdash_0^{\mathrm{BCP}}$, and successively proceed to the next reasoner if more resources are available. Such a family is called an *anytime family* of reasoners, a notion inspired by [3], where anytime reasoners are complete reasoners that provide partial answers

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[1] For the purposes of this paper, a problem is tractable iff the corresponding decision problem is known to be in PTIME [22].

even if stopped prematurely; the completeness of the answer improves with the time used in computing the answer.

We specify BCP using algebraic semantics (cf. [17]), that is, by a set of equalities between formulas and between theories, which generates an equivalence relation $=_{\text{BCP}}$ on the set of theories. A formula ψ can be *inferred using* $=_{\text{BCP}}$ from a theory Γ, denoted by $\Gamma \vdash_{\text{BCP}} \psi$, iff the theory $\Gamma \cup \{\neg\psi\} =_{\text{BCP}} \{\mathbf{f}\}$, where the formula **f** denotes *false*. Using algebraic semantics allows us to generalize our approach to any *admissible* relation, a notion defined by a list of desirable properties.

The reasoner $\vdash_{\text{BCP}}$ does not allow chaining over inferred clauses. For example, given the theory

$$\Gamma_0 = \{(P \vee Q), (P \vee \neg Q), (\neg P \vee S \vee T), (\neg P \vee S \vee \neg T)\},$$

it turns out that while $\Gamma_0 \vdash_{\text{BCP}} (P)$ and $\Gamma_0 \cup \{(P)\} \vdash_{\text{BCP}} (S)$, it is **not** the case that $\Gamma_0 \vdash_{\text{BCP}} (S)$. We show that adding arbitrary chaining to $\vdash_{\text{BCP}}$ leads to a complete but intractable reasoner! The reasoners in the anytime family are obtained by restricting the size of the clauses over which chaining is allowed. For the above theory, we obtain that $\Gamma_0 \vdash_0^{\text{BCP}} (P)$, $\Gamma_0 \vdash_1^{\text{BCP}} (P)$, $\Gamma_0 \nvdash_0^{\text{BCP}} (S)$ and $\Gamma_0 \vdash_1^{\text{BCP}} (S)$.

We present an alternative characterization of the reasoners in the anytime families using lattice-theoretic fixed-points. Although the reasoners are defined by allowing chaining over clauses, the fixed-point construction uses only a restricted set of clauses, called basic clauses. For any admissible relation $=_R$, any theory Γ, and any number k, we define an operator $T_{R,\Gamma,k}$ on sets of basic clauses. After defining Viv(R, Γ, k) to be the theory obtained by adding the fixed-point of this operator to Γ, we show that the set of clauses inferable from Viv(R, Γ, k) using $\vdash_R$ is exactly the set of clauses inferable from Γ using $\vdash_k^R$, the kth reasoner in the anytime family based on $\vdash_R$. We also present an O$(m^{2k} f(n + m^k))$ time algorithm for computing Viv(R, Γ, k), where f is the time complexity of $\vdash_R$, n is the size of Γ, and m is the number of distinct atoms in Γ.

Our fixed-point construction can be used for knowledge compilation [16,26,31], by transforming arbitrary clausal theories to logically equivalent theories for which $\vdash_R$ is complete for inferring clauses. Since $\vdash_R$ is usually an efficient reasoner, such transformed theories are said to be *vivid*, with respect to $\vdash_R$. The term “vivid” is inspired by [24], where vivid theories are ones where an answer can be “read off” quickly.[2] For a knowledge base that is queried frequently, compiling it into a vivid theory could significantly improve the efficiency of answering the queries. In fact, even approximate compilation, where the transformed theory is not logically equivalent to the original theory, if of considerable interest, as illustrated in [5,31]. We show that

[2] A strict interpretation of “reading off” quickly the answers of clausal queries would require a vivid knowledge base to explicitly contain all the prime implicants [30]: a clause is then entailed iff the knowledge base contains a subclause of the clause. The problem with this approach is that vivifying even some Horn theories, which are already vivid with respect to $\vdash_{\text{BCP}}$, leads to an exponential blow-up in their sizes. Note, however, that any knowledge base that is vivid using the strict interpretation is also vivid with respect to $\vdash_{\text{BCP}}$ using our definition.

for each theory Γ there is a k for which Viv(R, Γ, k) is vivid with respect to $\vdash_R$. In particular, we show that Horn, positive, negative, and satisfiable 2-CNF theories are vivid with respect to $\vdash_{\mathrm{BCP}}$, and that Viv(BCP, Γ, 1) for any unsatisfiable 2-CNF theory Γ is also vivid with respect to $\vdash_{\mathrm{BCP}}$.

The rest of the paper is organized as follows. In section 2, we present some basic terminology and definitions. In section 3, we define reasoners based on admissible relation on theories. In section 4, we extend any admissible reasoner to an anytime family of reasoners. In section 5, we present the fixed-point construction, which is related to the anytime family in section 6. In section 7, we compare our anytime family with other incomplete reasoners in the literature, before concluding in section 8.

2. Preliminaries

We restrict our attention to clausal formulas of propositional logic [29]. The *atoms* are taken from the denumerable set $\{P, Q, \ldots\}$. A *literal* is either an atom (called *positive literal*) or a negated atom (called *negative literal*), for example, P and $\neg P$. A *clause* is a finite disjunction of zero or more literals, enclosed within parentheses, for example, $(P \vee \neg Q)$. A *theory* is a finite set of clauses. The notions of subclauses, satisfiability, entailment ($\models$), and logical equivalence ($\equiv$) are defined as usual.

We normally use variables p, q, etc. to denote atoms; variables α, β, etc. to denote literals; variables ψ, φ, etc. to denote clauses; and variables Γ, Δ, etc. to denote theories.

The *size*, $|\psi|$, of a clause $\psi = (\alpha_1 \vee \cdots \vee \alpha_n)$ is n, that is, the number of literals in the clause. The *empty clause*, ( ), containing no literals is the only clause of size 0; it is denoted by **f**. A clause of size 1 is called a *unit clause*, and of size 2 is called a *binary clause*. A *Horn clause* is a clause containing at most one positive literal, a *positive clause* is a clause containing no negative literal, and a *negative clause* is a clause containing no positive literal. A k*-CNF theory* is a theory in which each clause has at most k literals.

The *Herbrand base*, HB(ψ), of any clause ψ is the set of atoms appearing in literals of ψ. The Herbrand base, HB(Γ), of any theory Γ, is the set of atoms appearing in Γ:

$$\mathrm{HB}(\Gamma) = \bigcup \bigl\{ \mathrm{HB}(\psi) \mid \psi \in \Gamma \bigr\}.$$

The *complement* of a literal toggles the negation connective: $\sim p = \neg p$ and $\sim \neg p = p$ for any atom p. The *complement*, $\sim \psi$, of a clause $(\alpha_1 \vee \cdots \vee \alpha_n)$ is defined to be the theory $\{(\sim \alpha_1), \ldots, (\sim \alpha_n)\}$.[3]

[3] We have defined the complement of a clause to be a theory, because only clauses are allowed as formulas. This non-standard notion of complement is justified because $\{\neg\psi\} \equiv \sim(\psi)$ for any clause ψ in propositional logic. Since we enclose clauses within parentheses, there should be no confusion between complements of literals and complements of unit clauses.

3. Reasoners based on admissible relations

In this section, we investigate reasoners based on admissible relations on theories. These reasoners, the weakest of which is BCP, will form the basis of the anytime families presented in the later sections. In section 3.1, we use refutation based on equivalence relations on theories for defining reasoners. In section 3.2, we define a special class of equivalence relations, called admissible relations. In section 3.3, we present some properties of reasoners defined using admissible relations.

3.1. Equality-based reasoners

Suppose $=_R$ is any equivalence relation on theories. Relation $=_R$ partitions the set of all theories into equivalence classes. Intuitively, all theories in the same equivalence class are supposed to have the same logical content. Since $\{\mathbf{f}\}$ is the prototypical inconsistent theory, it follows that a theory should be deemed inconsistent, based on $=_R$, iff the theory is in the equivalence class of $\{\mathbf{f}\}$. This allows us to develop a reasoner based on $=_R$ using refutation: a clause can be inferred from a theory iff the theory augmented by the complement of the clause is in the equivalence class of $\{\mathbf{f}\}$. In particular, the *consequence relation* $\vdash_R$ based on any equivalence relation $=_R$ on theories is defined as follows:

Definition 1. For any equivalence relation $=_R$ on theories, any theory Γ, and any clause ψ:

$$\Gamma \vdash_R \psi \quad \text{iff} \quad \Gamma \cup \sim \psi =_R \{\mathbf{f}\}.$$

Consequence relations are usually called *reasoners*. Since the equivalence closure of any relation is an equivalence relation, any relation on theories can be used to specify a reasoner in this manner. For example, consider the relation $\doteq$ given by the schemas B1 and B2 of figure 1. The equivalence closure of the relation $\doteq$ is defined to be the relation $=_{\mathrm{BCP}}$, and the reasoner based on $=_{\mathrm{BCP}}$ is denoted by $\vdash_{\mathrm{BCP}}$.

Example 1. Consider the theory $\Gamma_0 = \{(P \vee Q), (P \vee \neg Q), (\neg P \vee S \vee T), (\neg P \vee S \vee \neg T)\}$ and the clauses $\psi_0 = (P)$ and $\psi_1 = (S)$. Since $\Gamma_0 \cup \sim \psi_0 =_{\mathrm{BCP}} \{\mathbf{f}\}$, it follows that $\Gamma_0 \vdash_{\mathrm{BCP}} (P)$. Since $\Gamma_0 \cup \sim \psi_1 \neq_{\mathrm{BCP}} \{\mathbf{f}\}$, it follows that $\Gamma_0 \nvdash_{\mathrm{BCP}} (S)$. Note that $\Gamma_0 \models (P)$ and $\Gamma_0 \models (S)$. Thus, $\vdash_{\mathrm{BCP}}$ is *not* complete.

$$\text{B1.} \quad \{\mathbf{f}\} \cup \Gamma \doteq \{\mathbf{f}\},$$

$$\text{B2.} \quad \big\{(\alpha), (\sim \alpha \vee \alpha_1 \vee \cdots \vee \alpha_n)\big\} \cup \Gamma \doteq \big\{(\alpha), (\alpha_1 \vee \cdots \vee \alpha_n)\big\} \cup \Gamma.$$

Figure 1. Equivalence relation for BCP, where Γ is any theory, α's are any literals, and n is any natural number.

Unit resolution is a well-known refinement of resolution, where at least one resolvent in each resolution step is required to be a unit clause. It is well known [28] that BCP is a variant of unit resolution, in the sense that BCP can obtain **f** from a theory iff the theory has a unit refutation, that is, **f** can be obtained from the theory using unit resolution steps. It follows that for any theory Γ, $\Gamma =_{\mathrm{BCP}} \{\mathbf{f}\}$ iff Γ has a unit refutation. [27] presents a linear time algorithm for BCP, which essentially extends a linear-time Horn satisfiability algorithm [18]. Further improvement in the efficiency of this algorithm has been made recently [33]. Relation $=_{\mathrm{BCP}}$ and the reasoner $\vdash_{\mathrm{BCP}}$ are used for examples throughout this paper.

3.2. *Admissible relations on theories*

We now motivate several properties of equivalence relations on theories such that the reasoners based on them are suitable for reasoning with logical knowledge bases. We then prove that $=_{\mathrm{BCP}}$ is the weakest relation that satisfies these properties.

Soundness. An equivalence relation $=_R$ on theories is *sound* iff for any theories Γ and Δ, if $\Gamma =_R \Delta$ then $\Gamma \equiv \Delta$. Unsound equivalence relations force logically distinct theories into the same equivalence class, undesirably bestowing them identical logical content.

Modularity. An equivalence relation $=_R$ on theories is *modular* iff for any theories Γ, Γ' and Δ, if $\Gamma =_R \Gamma'$ then $\Gamma \cup \Delta =_R \Gamma' \cup \Delta$. Modular relations allow parts of theories to be replaced by equivalent theories (with respect to $=_R$). Since theories that represent knowledge bases are usually built incrementally, modularity allows replacing the knowledge base at any stage by an equivalent knowledge base (with respect to $=_R$).

Independence. An equivalence relation $=_R$ on theories is *independent* iff for any theories Γ and Δ, if HB(Γ) and HB(Δ) are disjoint then $\Gamma \cup \Delta =_R \{\mathbf{f}\}$ iff either $\Gamma =_R \{\mathbf{f}\}$ or $\Delta =_R \{\mathbf{f}\}$. Reasoners based on independent equivalence relations allow easy composition of disjoint knowledge bases, possibly developed by different people.

Irredundancy. An equivalence relation $=_R$ is *irredundant* iff for any theory Γ and any clause ψ, if ψ contains a pair of complementary literals or some clause in Γ is a subclause of ψ, and $\Gamma \cup \{\psi\} =_R \{\mathbf{f}\}$, then $\Gamma =_R \{\mathbf{f}\}$. Adding super clauses or trivially true clauses to a knowledge base does not change the conclusions made by reasoners based on an irredundant equivalence relations.

Simplification. An equivalence relation $=_R$ is *simplifying* iff $\{(\alpha), (\sim \alpha \vee \alpha_1 \vee \cdots \vee \alpha_n)\} =_R \{(\alpha), (\alpha_1 \vee \cdots \vee \alpha_n)\}$, for any number n, and any literals α's. Reasoners based on simplifying equivalence relations are able to simulate at least unit resolution.

Tractability. An equivalence relation $=_R$ is *tractable* iff there is an polynomial time algorithm for determining whether $\Gamma =_R \{\mathbf{f}\}$, given any theory Γ as input. Reasoners based on tractable equivalence relations are also tractable.

Definition 2. A relation on theories is said to be *admissible* iff it is an equivalence relation that is sound, modular, independent, irredundant, simplifying, and tractable.

Theorem 1 shows that the relation $=_{\mathrm{BCP}}$ on theories is the weakest admissible relation, that is, $=_{\mathrm{BCP}} \subseteq =_R$ for any admissible relation $=_R$.

Theorem 1. Relation $=_{\mathrm{BCP}}$ is admissible. For any admissible relation $=_R$ on theories and any theories Γ and Δ, if $\Gamma =_{\mathrm{BCP}} \Delta$ then $\Gamma =_R \Delta$.

Proof. We first show that $=_{\mathrm{BCP}}$ satisfies all the requirements of an admissible relation. By definition, $=_{\mathrm{BCP}}$ is an equivalence relation. $=_{\mathrm{BCP}}$ is sound, because lhs(B1) $\equiv$ rhs(B1), lhs(B2) $\equiv$ rhs(B2), and $\equiv$ is also an equivalence relation.[4] It follows directly from B2 that $=_{\mathrm{BCP}}$ is simplifying. Tractability of $=_{\mathrm{BCP}}$ follows directly from the linear-time algorithm for BCP [27].

For modularity, note that lhs(B1) $\cup\, \Delta =_{\mathrm{BCP}}$ rhs(B1) $\cup\, \Delta$ and lhs(B2) $\cup\, \Delta =_{\mathrm{BCP}}$ rhs(B2) $\cup\, \Delta$, for any theory Δ. The claim can then be proved by straight-forward induction on the number of transitive rules used in the derivation of $\Gamma =_{\mathrm{BCP}} \Gamma'$: suppose $\Gamma =_{\mathrm{BCP}} \Gamma'' =_{\mathrm{BCP}} \Gamma'$, then it follows that $\Gamma \cup \Delta =_{\mathrm{BCP}} \Gamma'' \cup \Delta =_{\mathrm{BCP}} \Gamma' \cup \Delta$.

Independence is also proved by induction on the derivation of $\Gamma \cup \Delta =_{\mathrm{BCP}} \{\mathbf{f}\}$. In the base case, it follows from B1 and B2 that $\Gamma \cup \Delta \doteq \{\mathbf{f}\}$ iff either $\Gamma \doteq \{\mathbf{f}\}$ or $\Delta \doteq \{\mathbf{f}\}$. The only non-trivial inductive cases are obtained by using reflexive and by using transitive rules. In the reflexive case, $\Gamma \cup \Delta = \{\mathbf{f}\}$ iff at least one of Γ or Δ is $\{\mathbf{f}\}$ and the other is $\emptyset$. The transitive case follows directly from the observation that for any theory Ω, $\Gamma \cup \Delta \doteq \Omega$ iff there are theories Γ' and Δ' such that $\Omega = \Gamma' \cup \Delta'$, $\Gamma \doteq \Gamma'$ and $\Delta \doteq \Delta'$.

For irredundancy, if ψ has a pair of complementary literals then it is clear that ψ cannot contribute to **f** using BCP. Now, suppose $\Gamma \cup \{\psi\} =_{\mathrm{BCP}} \{\mathbf{f}\}$ and $\Gamma = \Gamma' \cup \{\mu\}$, where μ is a subclause of ψ. Consider the derivation

$$\Gamma \cup \{\psi\} = \Gamma_1 \doteq \Gamma_2 \doteq \cdots \doteq \Gamma_n = \{\mathbf{f}\}.$$

It is easy to show, by contradiction on the earliest violation, that for each n there is a Γ'_n such that $\Gamma =_{\mathrm{BCP}} \Gamma'_n$ and $\Gamma_n = \Gamma'_n \cup \{\psi_n\}$, where ψ_n is a super-clause of some clause in Γ'_n. It then follows that $\Gamma =_{\mathrm{BCP}} \{\mathbf{f}\}$.

To prove the second claim of the theorem, all we need to show is that lhs(B1) $=_R$ rhs(B1) and lhs(B2) $=_R$ rhs(B2), for any admissible relation $=_R$. The former follows from independence and reflexivity ($\{\mathbf{f}\} =_R \{\mathbf{f}\}$), while the latter follows from the simplification and modularity of $=_R$. □

It is easy to obtain admissible relations other than $=_{\mathrm{BCP}}$. For example, consider k-resolution [4] where the result and at least one parent of each resolution step is restricted to be a clause of size at most k. For any $k \geqslant 0$, consider the relation $=_k$ defined as: For any theories Γ and Δ, $\Gamma =_k \Delta$ iff

[4] lhs and rhs denote the left-hand side and right-hand side, respectively.

(1) both Γ and Δ have k-refutations, or

(2) neither Γ nor Δ has a k-refutation.

Note that $=_1$ is identical to $=_{\mathrm{BCP}}$, because unit resolution is 1-resolution. It follows, using ideas similar to that in the proof of theorem 1, that $=_k$ is admissible for any k. It is also easy to verify that all $=_k$'s are distinct.

3.3. *Properties of admissible reasoners*

A reasoner $\vdash_R$ defined using an admissible relation $=_R$ is called an *admissible reasoner*. Theorem 2 shows some properties of admissible reasoners:

Theorem 2. For any admissible relation $=_R$, any theories Γ and Δ, and any clause ψ:

1. (*Invariance under* $=_R$) If $\Gamma =_R \Delta$ and $\Gamma \vdash_R \psi$ then $\Delta \vdash_R \psi$.
2. (*Soundness*) If $\Gamma \vdash_R \psi$ then $\Gamma \vDash \psi$.
3. (*Relation to* $\vdash_{\mathrm{BCP}}$) If $\Gamma \vdash_{\mathrm{BCP}} \psi$ then $\Gamma \vdash_R \psi$.
4. (*Monotonicity*) If $\Gamma \subseteq \Delta$ and $\Gamma \vdash_R \psi$ then $\Delta \vdash_R \psi$.
5. (*Partial completeness*) If $\mathrm{HB}(\Gamma) \subseteq \mathrm{HB}(\psi)$ and $\Gamma \vDash \psi$ then $\Gamma \vdash_R \psi$.

Proof. Invariance under $=_R$ follows directly from the definition of $\vdash_R$. Soundness follows directly from the soundness of $=_R$. Relation to $\vdash_{\mathrm{BCP}}$ follows directly from theorem 1. Monotonicity follows directly from the modularity and independence of $=_R$. For proving partial completeness, all we need to show is that $\Gamma' =_R \{\mathbf{f}\}$, where $\Gamma' = \Gamma \cup \sim \psi$. There are three cases.

Case I. Suppose there is an atom p such that both p and $\neg p$ are literals in ψ. Since $=_R$ is simplifying, $\{(p), (\neg p)\} =_R \{(p), \mathbf{f}\}$. Since $\{(p), (\neg p)\} \subseteq \sim \psi$, and $=_R$ is modular and independent, $\Gamma' =_R \{\mathbf{f}\}$.

Case II. Suppose there is a subclause φ of ψ in Γ. Since $=_R$ is simplifying, $\{\varphi\} \cup \sim \psi =_R \{\mathbf{f}\} \cup \sim \psi$. Since $=_R$ is modular and independent, $\Gamma' =_R \{\mathbf{f}\}$.

Otherwise. Let Γ'' be the theory obtained by removing all literals of ψ from the clauses in Γ. Since $=_R$ is simplifying, $\Gamma' =_R \Gamma'' \cup \sim \psi$. Since $\mathrm{HB}(\Gamma) \subseteq \mathrm{HB}(\psi)$, each clause in Γ'' has a subclause in $\sim \psi$, that is, $\Gamma'' \cup \sim \psi \equiv \sim \psi$. Since $=_R$ is sound, $\Gamma' \equiv \sim \psi$. Since ψ does not contain any pair of complementary literals (case I), $\sim \psi$ is satisfiable, that is, $\Gamma \nvDash \psi$. $\square$

Although $\vdash_{\mathrm{BCP}}$ is incomplete in general, partial completeness of theorem 2 presents a case when it is complete, that is, when the clause mentions every atom in the theory. We now show that $\vdash_{\mathrm{BCP}}$ is complete for inferring arbitrary clauses from some restricted theories:

1. *Horn theories.* Consider any Horn theory Γ and any clause ψ. Since $\sim \psi$ is a set of literals, $\Gamma \cup \sim \psi$ is equivalent (with respect to $=_R$) to some Horn theory. Since BCP is known to be complete for determining satisfiability of Horn theories [28], $\Gamma \models \psi$ iff $\Gamma \cup \sim \psi =_{\mathrm{BCP}} \{\mathbf{f}\}$.
2. *Positive theories.* Consider any positive theory Γ and any clause ψ. Since $\sim \psi$ is a set of literals and $=_{\mathrm{BCP}}$ is simplifying, there are theories Γ' and Γ'' over disjoint atoms, such that $\Gamma \cup \sim \psi =_{\mathrm{BCP}} \Gamma' \cup \Gamma''$, Γ' has only unit clauses, and Γ'' has only positive clauses. $\Gamma' \cup \Gamma''$ is satisfiable unless Γ' contains both (p) and $(\neg p)$ for some atom p, or Γ'' contains $\mathbf{f}$; $\Gamma' \cup \Gamma'' =_{\mathrm{BCP}} \{\mathbf{f}\}$ in both cases. Therefore $\vdash_{\mathrm{BCP}}$ is complete for inferring clauses from positive theories.
3. *Negative theories.* The argument is identical to that for positive theories.
4. *Satisfiable 2-CNF theories.* The argument is identical to that for positive theories, except that Γ'' has only binary clauses, and possibly $\mathbf{f}$.

Thus, the tractable but incomplete reasoner $\vdash_{\mathrm{BCP}}$ is complete for inferring clauses from Horn, positive, negative, and satisfiable 2-CNF theories. These theories are called vivid with respect to $\vdash_{\mathrm{BCP}}$. In general, a theory is said to be vivid with respect to a reasoner if the reasoner is complete for inferring clauses from the theory:

Definition 3. A theory Γ is called *vivid* with respect to a reasoner $\vdash$ iff for any clause ψ, if $\Gamma \models \psi$ then $\Gamma \vdash \psi$.

Since any admissible reasoner $\vdash_R$ is at least as complete as $\vdash_{\mathrm{BCP}}$, it follows that Horn, positive, negative, and satisfiable 2-CNF theories are vivid with respect to $\vdash_R$.

4. Anytime families of reasoners

In this section, we show how to extend any admissible reasoner to an anytime family of reasoners. In section 4.1, we extend any admissible reasoner to a complete reasoner. In section 4.2, we restrict the complete reasoners to obtain the anytime families, and present some properties of the family.

4.1. Complete reasoners

For any admissible $=_R$, the reasoner $\vdash_R$ is sound but may be incomplete. A common source of incompleteness in $\vdash_R$ is its inability to use previously inferred clauses for inferring new clauses. For the theory Γ_0 of example 1, both $\Gamma_0 \vdash_{\mathrm{BCP}} (P)$ and $\Gamma_0 \cup \{(P)\} \vdash_{\mathrm{BCP}} (S)$, but $\Gamma_0 \nvdash_{\mathrm{BCP}} (S)$. In other words, while clause (P) can be inferred from Γ_0 and (S) can be inferred if (P) is added to Γ_0, (S) cannot be inferred from Γ_0 itself. Thus, $\vdash_{\mathrm{BCP}}$ is unable to use the previously inferred clause (P) to infer the new clause (S).

The reasoner $\vdash_R$ can be extended by adding an inference rule that provides this capability of chaining on clauses.

Definition 4. The reasoner $\vdash^R$ based on any equivalence relation $=_R$ on theories is defined using the following two inference rules:

$$\text{C1.}\quad \frac{\Gamma \vdash_R \varphi}{\Gamma \vdash^R \varphi},$$

$$\text{C2.}\quad \frac{\Gamma \vdash^R \psi;\ \ \Gamma, \psi \vdash^R \varphi}{\Gamma \vdash^R \varphi},$$

where Γ is any theory, and ψ and φ are any clauses.

It follows from rule C1 that $\vdash^R$ is at least as complete as $\vdash_R$. It is rule C2 that provides the capability of using previously inferred clauses to infer new clauses. Note that Γ, ψ in rule C2 denotes the theory $\Gamma \cup \{\psi\}$, following the usual convention in presenting inference rules. As usual, inferring a clause from a theory using $\vdash^R$ requires a proper derivation, that is, a finite sequence of steps, each of the form either $\Gamma \vdash_R \psi$ or $\Gamma \vdash^R \psi$, such that all steps of the latter form must be obtained from earlier steps using one of the two inference rules. It is trivial to verify that $\Gamma_0 \vdash^{\mathrm{BCP}} (S)$ in the above example. Since any admissible reasoner $\vdash_R$ and the rules C1 and C2 are sound, the reasoner $\vdash^R$ is also sound. In section 6, we will show that $\vdash^R$ is also complete for any admissible relation $=_R$.

4.2. Restricted reasoners

Since $\vdash^R$, for any admissible relation $=_R$, is sound and complete, it is also intractable, and thus, not very useful for our purposes. However, it is possible to restrict $\vdash^R$ for obtaining reasoners that are more complete than $\vdash_R$, but are still tractable. For instance, if ψ in rule C2 is restricted to be a unit clause, then the restricted $\vdash^R$ is tractable; we will later show that this reasoner is also complete for inferring clauses from 2-CNF theories, which $\vdash_{\mathrm{BCP}}$ is not.

Thus, restricting the size of ψ in rule C2 of $\vdash^R$ seems to be a reasonable approach for obtaining tractable consequence relations. The following inference system defines a family $\vdash^R_k$ of reasoners, where k is any natural number, and $=_R$ is any equivalence relation:

Definition 5. For any equivalence relation $=_R$ on theories and any natural number k, the reasoner $\vdash^R_k$ is defined using the following two inference rules:

$$\text{A1.}\quad \frac{\Gamma \vdash_R \varphi}{\Gamma \vdash^R_k \varphi},$$

$$\text{A2.}\quad \frac{\Gamma \vdash^R_k \psi;\ \ \Gamma, \psi \vdash^R_k \varphi}{\Gamma \vdash^R_k \varphi}\quad \text{for } |\psi| \leqslant k,$$

where Γ is any theory, and ψ and φ are any clauses.

Rule A1 is identical to rule C1 of $\vdash^R$. Rule A2 adds a restriction on rule C2 such that chaining is allowed only over clauses of sizes at most k.

For the theory Γ_0 of example 1, it is easy to verify that $\Gamma_0 \vdash_0^{\text{BCP}} (P)$, $\Gamma_0 \vdash_1^{\text{BCP}} (P)$, $\Gamma_0 \nvdash_0^{\text{BCP}} (S)$, and $\Gamma_0 \vdash_1^{\text{BCP}} (S)$. The proof for $\Gamma_0 \vdash_1^{\text{BCP}} (S)$ is as follows:

(1) From $\Gamma_0 \cup \sim (P) =_{\text{BCP}} \{\mathbf{f}\}$, we obtain that $\Gamma_0 \vdash_{\text{BCP}} (P)$.

(2) From (1) and rule A1, we obtain that $\Gamma_0 \vdash_1^{\text{BCP}} (P)$.

(3) From $\Gamma_0 \cup \{(P)\} \cup \sim (S) =_{\text{BCP}} \{\mathbf{f}\}$, we obtain that $\Gamma_0 \cup \{(P)\} \vdash_{\text{BCP}} (S)$.

(4) From (3) and rule A1, we obtain that $\Gamma_0 \cup \{(P)\} \vdash_1^{\text{BCP}} (S)$.

(5) From (2) and (4) and rule A2, we obtain that $\Gamma_0 \vdash_1^{\text{BCP}} (S)$.

Note that step (5) is not allowed when $k = 0$, because the size of clause (P) is 1. More examples of $\vdash_k^{\text{BCP}}$ will be given in section 6.

We now present two useful properties of the reasoners $\vdash_k^R$.

Soundness. For any admissible relation $=_R$ and any number k, the reasoner $\vdash_k^R$ is *sound*, that is, for any theory Γ and any clause ψ, if $\Gamma \vdash_k^R \psi$ then $\Gamma \models \psi$. This directly follows from the soundness of $\vdash_R$ and of rules A1 and A2.

Monotonicity. The reasoning power of $\vdash_k^R$ *grows monotonically* with $=_R$ and k, that is, for any admissible relations $=_R$ and $=_{R'}$, any numbers k and k', any theory Γ and any clause ψ, if $k \leqslant k'$, $=_R \subseteq =_{R'}$, and $\Gamma \vdash_k^R \psi$ then $\Gamma \vdash_{k'}^{R'} \psi$. This follows from the observation that any derivation of $\Gamma \vdash_k^R \psi$ can be transformed into a derivation of $\Gamma \vdash_{k'}^{R'} \psi$, just be replacing each k by k' and R by R'.

Several other properties will be given in section 6. Because of these properties, the family $\vdash_k^R$ (for $k = 0, 1, \ldots$) of reasoners for any admissible relation $=_R$ is called an anytime family of propositional reasoners.

5. Constructing vivid theories

In this section, we present a method for constructing vivid theories. In section 5.1, we use $\vdash_R$ to define an operator $T_{R,\Gamma,k}$ on theories, and show that this operator always has a least fixed-point, denoted by $\text{lfp}(T_{R,\Gamma,k})$ and called the kth fixed-point of Γ with respect to $=_R$. We define $\text{Viv}(R, \Gamma, k)$ to be the theory $\Gamma \cup \text{lfp}(T_{R,\Gamma,k})$. In section 5.2, we show some properties of this fixed-point and that for each Γ there is a k for which $\text{Viv}(R, \Gamma, k)$ is vivid with respect to $\vdash_R$. In section 5.3, we present an algorithm for computing the fixed-points. In section 6, we will relate Viv to the anytime family of reasoners based on $=_R$.

5.1. Fixed-point construction

It is not necessary to consider all possible clauses in defining the operator T_*. For any (finite) theory Γ, we restrict our attention to only those clauses that are built from the atoms in Γ such that all literals in a clause have distinct atoms; these clauses are called *basic clauses*. A basic clause with at most k literals is called a k-clause. The *extended Herbrand base*, $E(\Gamma)$, of a theory Γ is the set of all basic clauses, and the *k-extended Herbrand base*, $E(\Gamma, k)$, is the set of all k-clauses. We normally use variables μ, π, etc. to denote basic clauses.

The operator $T_{R,\Gamma,k}$ on any set S of k-clauses produces the set of k-clauses that can be inferred from $\Gamma \cup S$ using $\vdash_R$:

Definition 6. For any theory Γ and any $k \in \mathcal{N}$, the function $T_{R,\Gamma,k} : 2^{E(\Gamma,k)} \to 2^{E(\Gamma,k)}$ is defined as

$$T_{R,\Gamma,k}(S) = \{\mu \in E(\Gamma, k) \mid \Gamma \cup S \vdash_R \mu\},$$

where S is any subset of $E(\Gamma, k)$.

Since the powerset of any set is a complete lattice with respect to the subset relation, $(2^{E(\Gamma,k)}, \subseteq)$ is also a complete lattice. Since $E(\Gamma, k)$ is always finite, this complete lattice is also finite. The next lemma shows that the operator $T_{R,\Gamma,k}$ on this lattice is monotonic in its arguments and parameters:

Lemma 3. For any theories Γ and Δ, any $k, p \in \mathcal{N}$, and any subset M of $E(\Gamma, k)$ and S of $E(\Delta, p)$, if $\Gamma \subseteq \Delta$, $k \leqslant p$, and $M \subseteq S$ then $T_{R,\Gamma,k}(M) \subseteq T_{R,\Delta,p}(S)$.

Proof. Since $\Gamma \subseteq \Delta$ and $k \leqslant p$, it follows that $E(\Gamma, k) \subseteq E(\Delta, p)$. Now consider any basic clause μ:

$$\begin{aligned}
\mu \in T_{R,\Gamma,k}(M) &\Rightarrow \mu \in E(\Gamma, k) \text{ and } \Gamma \cup M \vdash_R \mu \quad \text{(definition)} \\
&\Rightarrow \mu \in E(\Delta, p) \text{ and } \Delta \cup S \vdash_R \mu \\
&\qquad \big(E(\Gamma, k) \subseteq E(\Delta, p),\ \Gamma \subseteq \Delta,\ M \subseteq S,\ \text{theorem 2}\big) \\
&\Rightarrow \mu \in T_{R,\Delta,p}(S) \quad \text{(definition)}.
\end{aligned}$$

Thus, $T_{R,\Gamma,k}(M) \subseteq T_{R,\Delta,p}(S)$. □

Since $T_{R,\Gamma,k}$ is a monotonic operator over a finite lattice, it has a least fixed-point [32], which can also be characterized using the ordinal powers of $T_{R,\Gamma,k}$, defined in the usual manner (cf. [25]): For any theory Γ and any $k \in \mathcal{N}$:

$$\begin{aligned}
&T_{R,\Gamma,k}\uparrow 0 = \emptyset, \\
&T_{R,\Gamma,k}\uparrow n = T_{R,\Gamma,k}\big(T_{R,\Gamma,k}\uparrow(n-1)\big) \quad (\text{if } n \in \mathcal{N}), \\
&T_{R,\Gamma,k}\uparrow \omega = \bigcup \{T_{R,\Gamma,k}\uparrow n \mid n \in \mathcal{N}\}.
\end{aligned}$$

Intuitively $T_{R,\Gamma,k}\uparrow 1$ is the set of all k-clauses that can be inferred from Γ alone using $\vdash_R$, $T_{R,\Gamma,k}\uparrow 2$ is the set of all k-clauses that can be inferred from Γ and the clauses in $T_{R,\Gamma,k}\uparrow 1$ using $\vdash_R$, and so on. The following corollary of lemma 3 follows from [32]:

Corollary 4. For any admissible relation $=_R$, any theory Γ, and any number k, the least fixed-point lfp($T_{R,\Gamma,k}$) of $T_{R,\Gamma,k}$ is given by $T_{R,\Gamma,k}{\uparrow}\omega$.

We refer to lfp($T_{R,\Gamma,k}$) as the kth fixed-point of Γ with respect to $=_R$; k is said to be the *index* of this fixed-point. The least fixed-point is used to define a function Viv from the set of theories and natural numbers to the set of theories:

Definition 7. For any theory Γ and any number k, Viv(R, Γ, k) is defined to be the theory $\Gamma \cup$ lfp($T_{R,\Gamma,k}$).

Note that Viv(R, Γ, k) augments the theory Γ, rather than replacing it, by the theory lfp($T_{R,\Gamma,k}$), since this allows more clauses to be inferred from it using $\vdash_R$.

Example 2. Consider $\Gamma = \{(P \vee Q), (\neg P \vee Q), (P \vee \neg Q)\}$:

$$
\begin{aligned}
&T_{\mathrm{BCP},\Gamma,0}{\uparrow}n = \emptyset && \text{(for all } n \geqslant 0\text{)},\\
&T_{\mathrm{BCP},\Gamma,1}{\uparrow}n = \big\{(P), (Q)\big\} && \text{(for all } n \geqslant 1\text{)},\\
&T_{\mathrm{BCP},\Gamma,k}{\uparrow}n = \big\{(P), (Q)\big\} \cup \Gamma && \text{(for all } n \geqslant 1 \text{ and all } k \geqslant 2\text{)}.
\end{aligned}
$$

Now consider the theory $\Delta = \Gamma \cup \{(\neg P \vee \neg Q)\}$:

$$
\begin{aligned}
&T_{\mathrm{BCP},\Delta,0}{\uparrow}n = \emptyset \quad \text{(for all } n \geqslant 0\text{)},\\
&T_{\mathrm{BCP},\Delta,1}{\uparrow}1 = \big\{(P), (\neg P), (Q), (\neg Q)\big\},\\
&T_{\mathrm{BCP},\Delta,1}{\uparrow}2 = \{\mathbf{f}\} \cup T_{\mathrm{BCP},\Delta,1}{\uparrow}1 = E(\Delta, 1).
\end{aligned}
$$

The least fixed-points are given by

$$
\begin{aligned}
&\mathrm{lfp}(T_{\mathrm{BCP},\Gamma,0}) = \mathrm{lfp}(T_{\mathrm{BCP},\Delta,0}) = \emptyset,\\
&\mathrm{lfp}(T_{\mathrm{BCP},\Gamma,1}) = \big\{(P), (Q)\big\}, && \mathrm{lfp}(T_{\mathrm{BCP},\Delta,1}) = E(\Delta, 1),\\
&\mathrm{lfp}(T_{\mathrm{BCP},\Gamma,2}) = \Gamma \cup \big\{(P), (Q)\big\}, && \mathrm{lfp}(T_{\mathrm{BCP},\Delta,2}) = E(\Delta).
\end{aligned}
$$

Note that **f** is a basic clause in lfp($T_{\mathrm{BCP},\Delta,1}$) but not in lfp($T_{\mathrm{BCP},\Delta,0}$), although it is in $E(\Delta, 0)$. This is possible, intuitively, since obtaining **f** from Δ using $\vdash_{\mathrm{BCP}}$ requires that at least one of the basic clauses in the set $\{P, Q, \neg P, \neg Q\}$ be added to Δ; this happens in lfp($T_{\mathrm{BCP},\Delta,1}$) but not in lfp($T_{\mathrm{BCP},\Delta,0}$). In general, for a theory Γ higher values of k may lead to more clauses of sizes smaller than k to be in lfp($T_{R,\Gamma,k}$) due to such "feedback" effects.

5.2. *Properties of the fixed-points*

Theorem 5 shows some properties of the fixed-point lfp($T_{R,\Gamma,k}$): monotonicity, soundness, tractability, and eventual completeness:

Theorem 5. For any admissible relation $=_R$, any theories Γ, Γ' and Δ, any numbers k and p, and any basic clause $\mu \in E(\Gamma)$:

1. (*Monotonicity*) If $E(\Gamma, k) \subseteq E(\Delta, p)$ and $\Gamma =_R \Gamma' \subseteq \Delta$ then $\mathrm{lfp}(T_{R,\Gamma,k}) \subseteq \mathrm{lfp}(T_{R,\Delta,p})$.
2. (*Soundness*) If $\mu \in \mathrm{lfp}(T_{R,\Gamma,k})$ then $\Gamma \vDash \mu$.
3. (*Eventual completeness*) If $m \geqslant |\mathrm{HB}(\Gamma)|$ and $\Gamma \vDash \mu$ then $\mu \in \mathrm{lfp}(T_{R,\Gamma,m})$.
4. (*Tractability*) For a fixed k, $\mathrm{lfp}(T_{R,\Gamma,k})$ can be computed in $\mathrm{O}(m^{2k} f(n + m^k))$ time, where f is the time complexity of $\vdash_R$, n is the size of Γ, and m is the size of $\mathrm{HB}(\Gamma)$.

Proof. For monotonicity, all we need to show is that for all n, $T_{R,\Gamma,k}{\uparrow}n \subseteq T_{R,\Delta,p}{\uparrow}n$. We show this by induction on n.

The base case, when $n = 0$, is trivial since $T_{R,\Gamma,k}{\uparrow}0 = \emptyset$. For the inductive case, when $n > 0$, the hypothesis is that $T_{R,\Gamma,k}{\uparrow}(n-1) \subseteq T_{R,\Delta,p}{\uparrow}(n-1)$. For any basic clause μ:

$$
\begin{aligned}
\mu \in T_{R,\Gamma,k}{\uparrow}n &\Rightarrow \mu \in T_{R,\Gamma,k}(T_{R,\Gamma,k}{\uparrow}(n-1)) && \text{(definition)}\\
&\Rightarrow \Gamma \cup T_{R,\Gamma,k}{\uparrow}(n-1) \vdash_R \mu && \text{(definition)}\\
&\Rightarrow \Delta \cup T_{R,\Gamma,k}{\uparrow}(n-1) \vdash_R \mu && \text{(modularity and theorem 2)}\\
&\Rightarrow \Delta \cup T_{R,\Delta,p}{\uparrow}(n-1) \vdash_R \mu && \text{(modularity and inductive hypothesis)}\\
&\Rightarrow \mu \in T_{R,\Delta,p}(T_{R,\Delta,p}{\uparrow}(n-1)) && \text{(definition and } E(\Gamma,k) \subseteq E(\Delta,p))\\
&\Rightarrow \mu \in T_{R,\Delta,p}{\uparrow}n && \text{(definition).}
\end{aligned}
$$

Thus, $T_{R,\Gamma,k}{\uparrow}n \subseteq T_{R,\Delta,p}{\uparrow}n$. Note the dependence on Γ' in going from Γ to Δ in the above sequence.

Using a similar induction, soundness follows directly from the soundness of $\vdash_R$.

Eventual completeness is proved by contradiction. All we need to prove is that the theorem holds for $m = |\mathrm{HB}(\Gamma)|$; other cases would then follow directly from theorem 5. Assume now that the claim is false for $m = |\mathrm{HB}(\Gamma)|$, i.e., there is some theory Γ, some basic clause $\mu \in E(\Gamma)$ such that $\Gamma \vDash \mu$ but $\mu \notin \mathrm{lfp}(T_{R,\Gamma,m})$. For this fixed Γ, let μ be a maximal basic clause for which the theorem does not hold. Since μ is a basic clause, size of μ is at most m.

Case 1. Size of μ is m, i.e, $\mathrm{HB}(\mu) = \mathrm{HB}(\Gamma) \supseteq \mathrm{HB}(\Gamma)$. Since $\Gamma \vDash \mu$, we obtain from theorem 2 that $\Gamma \vdash_R \mu$, i.e., $\mu \in \mathrm{lfp}(T_{R,\Gamma,m})$.

Case 2. Size of μ is less than m, i.e., there is an atom $p \in \mathrm{HB}(\Gamma) - \mathrm{HB}(\mu)$. Thus, both $\mu \vee (p)$ and $\mu \vee (\neg p)$ are in $E(\Gamma, m)$. Since $\Gamma \vDash \mu$, it follows that $\Gamma \vDash \mu \vee (p)$. Since μ is a maximal clause that violates the theorem, $\mu \vee (p) \in \mathrm{lfp}(T_{R,\Gamma,m})$. Similarly, $\mu \vee (\neg p) \in \mathrm{lfp}(T_{R,\Gamma,m})$. Since $\{\mu \vee (p), \mu \vee (\neg p)\} \vdash_R \mu$, we obtain from theorem 2 that $\mathrm{lfp}(T_{R,\Gamma,m}) \vdash_R \mu$. Since this is a fixed-point, it follows that $\mu \in \mathrm{lfp}(T_{R,\Gamma,m})$.

Since we arrive at a contradiction in all cases, eventual completeness is proved.

Tractability will be proved in section 5.3. $\square$

For monotonicity, note that a sufficient condition for ensuring $E(\Gamma, k) \subseteq E(\Delta, p)$ is that HB(Γ) $\subseteq$ HB(Δ) and $k \leqslant p$, and that a sufficient condition for HB(Γ) $\subseteq$ HB(Δ) is that $\Gamma \subseteq \Delta$. The following example shows that monotonicity may be violated if HB(Γ) $\not\subseteq$ HB(Δ). Consider $\Gamma = \{P, \neg P, Q\}$ and $\Delta = \{P, \neg P\}$. Since $\Gamma =_{\text{BCP}} \{\mathbf{f}\} =_{\text{BCP}} \Delta$, we have $\Gamma =_{\text{BCP}} \Delta$. However, HB($\Gamma$) $= \{P, Q\} \not\subseteq \{P\} =$ HB(Δ). Also,

$$\text{lfp}(T_{\text{BCP},\Gamma,1}) = \{(P), (\neg P), (Q), (\neg Q), \mathbf{f}\} \not\subseteq \{(P), (\neg P), \mathbf{f}\} = \text{lfp}(T_{\text{BCP},\Delta,1}).$$

This idea can be used to create similar examples where the two theories are satisfiable.

It follows from soundness that the clauses added to a theory Γ for obtaining Viv(R, Γ, k) for any number k are logically entailed by Γ. Thus, any theory Γ is logically equivalent to the theory Viv(R, Γ, k) for any number k.

It follows from eventual completeness that if $k \geqslant |\text{HB}(\Gamma)|$ then $\text{lfp}(T_{R,\Gamma,k}) = \text{lfp}(T_{R,\Gamma,|\text{HB}(\Gamma)|})$, since $E(\Gamma, k) = E(\Gamma) = E(\Gamma, |\text{HB}(\Gamma)|)$. Thus, the sequence of least fixed-points for increasing k's converge by the time $k = |\text{HB}(\Gamma)|$. Since we are dealing with only finite theories, this value is also finite. Thus, we have the following corollary of theorem 5:

Corollary 6. For any admissible relation $=_R$ and any theory Γ, $\Gamma \equiv$ Viv(R, Γ, m) and Viv(R, Γ, m) is vivid with respect to $\vdash_R$ for any $m \geqslant |\text{HB}(\Gamma)|$.

In section 4, we showed that satisfiable 2-CNF theories are vivid for $\vdash_{\text{BCP}}$. The theory Δ of example 2 shows that some unsatisfiable 2-CNF theories are not vivid for $\vdash_{\text{BCP}}$. However, lemma 7 shows a weaker result for those theories:

Lemma 7. For any unsatisfiable 2-CNF theory Γ, Viv(BCP, Γ, 1) is vivid for $\vdash_{\text{BCP}}$.

Proof. We prove, by constructing a model, that if Γ is a 2-CNF theory and $\mathbf{f} \notin$ Viv(BCP, Γ, 1), then Γ is satisfiable. Consider the following procedure:

1. $\Delta :=$ Viv(BCP, Γ, 1);
2. for each p in HB(Γ) do
3. if $\Delta \vdash_{\text{BCP}} p$ then
4. $\quad\Delta := \Delta \cup \{p\}$
5. else $\Delta := \Delta \cup \{\neg p\}$.

All we need to show is that $\Delta \neq_{\text{BCP}} \{\mathbf{f}\}$ is an invariant maintained by the "for loop". We prove this by induction on the number of iterations of the loop; let Δ_n be the value of Δ after the nth iteration. The invariant holds before entering the loop, when $n = 0$, because $\Delta_0 =$ Viv(BCP, Γ, 1) $\nvdash_{\text{BCP}} \mathbf{f}$. Since $=_{\text{BCP}}$ is simplifying, for any Δ_i there are theories Γ'_i and Γ''_i over disjoint atoms, such that $\Delta_i =_{\text{BCP}} \Gamma'_i \cup \Gamma''_i$, Γ'_i has only unit clauses (or $\mathbf{f}$), Γ''_i has only binary clauses, and $\Gamma''_i \subseteq \Delta_0$. After the ith iteration, in which suppose literal α is added to Δ, if $\Delta_i \vdash_{\text{BCP}} \mathbf{f}$ then it follows from independence of $=_{\text{BCP}}$ that $\Gamma''_i \cup \{(\alpha)\} \vdash_{\text{BCP}} \mathbf{f}$. Since $\vdash_{\text{BCP}}$ is monotonic, we have

Algorithm Compute-R-lfp(Γ, k):

1. compute HB(Γ) and $E(\Gamma, k)$;
2. $\Delta := \Gamma$;
3. repeat
4. for each $\mu \in E(\Gamma, k) - \Delta$ do
5. if $\Delta \vdash_R \mu$
6. then $\Delta := \Delta \cup \{\mu\}$;
7. until no more changes in Δ;
8. return Δ

End (Compute-R-lfp).

Figure 2. Algorithm for computing Viv(R, Γ, k).

$\Delta_0 \cup \{(\alpha)\} \vdash_{\mathrm{BCP}} \mathbf{f}$. From the fix-point construction, we obtain that $(\alpha) \in \Delta_0$, that is, $\Delta_0 \vdash_{\mathrm{BCP}} \mathbf{f}$, a contradiction. □

5.3. *Algorithm for vivification*

A straight-forward way for computing the least fixed-point lfp($T_{R,\Gamma,k}$) for any theory Γ and any number k is given in the algorithm Compute-R-lfp of figure 2. After computing the Herbrand base and the k-extended Herbrand base, the fixed-point is built incrementally: starting with an empty set, keep adding to it the basic clauses in $E(\Gamma, k)$ that can be inferred from this set and Γ using $\vdash_R$. For any input theory Γ and any number k, it is easy to verify that Compute-R-lfp returns lfp($T_{R,\Gamma,k}$).

To compute the worst-case cost for a fixed $=_R$ and k, consider a theory Γ of size (sum of the sizes of the clauses) n and m distinct atoms. The number of distinct basic clauses is O(m^k). Since each iteration of the repeat loop adds at least one new clause to Δ, there are at most O(m^k) iterations. Since each clause may have to be tried in each iteration, at most O(m^{2k}) tests of $\vdash_R$ are needed. Since the maximum size of lfp($T_{R,\Gamma,k}$) is also O(m^k), the total cost of all these tests is O($m^{2k} f(n + m^k)$) time, where f is the cost of $\vdash_R$.

6. Vivification for anytime reasoning

In this section, we present the main result of this paper that relates Viv to the anytime family of reasoners. We first prove some intermediate results.

Although rule A2 explicitly allows using only one previously inferred clause, lemma 8 shows that chaining can be performed over more than one clause:

Lemma 8. For any clauses $\psi_0, \psi_1, \ldots$ such that $|\psi_i| \leqslant k$ for each i, if $\Gamma \vdash_k^R \psi_n$; $\Gamma, \psi_n \vdash_k^R \psi_{n-1}$; $\ldots$; $\Gamma, \psi_n, \ldots, \psi_1 \vdash_k^R \psi_0$ then $\Gamma \vdash_k^R \psi_0$.

Proof. It follows from the assumptions that for any $i \in 0 \ldots n$, $\Gamma, \psi_n, \ldots, \psi_{i+1} \vdash_k^R \psi_0$. The claim follows from $i = n$. □

We now show that inferring basic clauses using the extended Herbrand base of a theory is the only interesting case; inferring any other clause is either trivial or is equivalent to inferring some basic clause. For this, we need a definition:

Definition 8. For any theory Γ and any clause ψ, the clause ψ_Γ is defined as follows:

1. If ψ contains complementary literals, then $\psi_\Gamma = \mathbf{t}$;[5]
2. Otherwise, ψ_Γ is the clause obtained from ψ by removing all literals p and $\neg p$ such that $p \notin \mathrm{HB}(\Gamma)$.

It is well known that $\Gamma \models \psi$ iff $\Gamma \models \psi_\Gamma$. Lemma 9 shows that this extends to our reasoners as well:

Lemma 9. For any admissible relation $=_R$, any theory Γ, any clause ψ, and any number k: $\Gamma \vdash_k^R \psi$ iff either $\psi_\Gamma = \mathbf{t}$ or $\Gamma \vdash_k^R \psi_\Gamma$.

Proof. If ψ contains complementary literals, then the claim follows from the simplification property of $=_R$, otherwise the claim follows from the independence of $=_R$. □

Now we can prove the main result of this paper:

Theorem 10. For any admissible relation $=_R$, any theory Γ, any clause ψ, and any number k: $\mathrm{Viv}(R, \Gamma, k) \vdash_R \psi$ iff $\Gamma \vdash_k^R \psi$.

Proof. It follows from lemma 9 that it is sufficient to prove the claim when ψ is a basic clause in $E(\Gamma)$. Recall that $\mathrm{Viv}(R, \Gamma, k) = \Gamma \cup \mathrm{lfp}(T_{R,\Gamma,k})$.

Only if. Suppose $\mathrm{Viv}(R, \Gamma, k) \vdash_R \psi$. From monotonicity of $\vdash_k^R$, we have $\Gamma \cup \mathrm{lfp}(T_{R,\Gamma,k}) \vdash_k^R \psi$. For any $\mu \in \mathrm{lfp}(T_{R,\Gamma,k})$, it follows from the definition and finiteness of the fixed-point and lemma 8 that $\Gamma \vdash_k^R \mu$.

If. Suppose $\Gamma \vdash_k^R \psi$. We show that $\mathrm{Viv}(R, \Gamma, k) \vdash_R \psi$ by induction on the length of the derivation for $\Gamma \vdash_k^R \psi$. In the base case, where $\Gamma \vdash_R \psi$, it follows from theorem 2 that $\mathrm{Viv}(R, \Gamma, k) \vdash_R \psi$.

For the inductive case, there is a clause φ such that $|\varphi| \leqslant k$, $\Gamma \vdash_k^R \varphi$, and $\Gamma, \varphi \vdash_k^R \psi$. Using the inductive assumption, we have $\mathrm{Viv}(R, \Gamma, k) \vdash_R \varphi$ and $\mathrm{Viv}(R, \Gamma \cup \{\varphi\}, k) \vdash_R \psi$. There are two mutually-exclusive and exhaustive cases:

1. φ has a pair of complementary literals: it follows from the irredundancy of $=_R$ that $\mathrm{Viv}(R, \Gamma, k) \vdash_R \psi$.

[5] The symbol $\mathbf{t}$ can be thought of as the negation of clause $\mathbf{f}$, that is, $\mathbf{t} = \neg\mathbf{f}$. Although $\mathbf{t}$ is technically not a clause, it is logically equivalent to the clause $(p \vee \neg p)$ for any atom p.

2. Otherwise: it follows from lemma 9 that Viv$(R, \Gamma, k) \vdash_R \varphi_\Gamma$. Thus, $\varphi_\Gamma \in$ Viv(R, Γ, k), since φ_Γ is a basic clause in $E(\Gamma)$. Since φ_Γ is a subclause of φ, it then follows from irredundancy of $=_R$ and Viv$(R, \Gamma \cup \{\varphi\}, k) \vdash_R \psi$ that Viv$(R, \Gamma, k) \vdash_R \psi$.

Thus, Viv$(R, \Gamma, k) \vdash_R \psi$ in all cases. □

It then follows from theorem 5 that for any number k and any admissible relation $=_R$, the reasoner $\vdash_k^R$ is *tractable*. In particular, the algorithm that computes Viv(R, Γ, k) and then determines whether Viv$(R, \Gamma, k) \vdash_R \psi$ takes O$(m^{2k} f(n+m^k+l))$ time, for fixed $=_R$ and k, where f is the time complexity of $\vdash_R$, n is the size of Γ, l is the size of ψ, and m is the size of HB(Γ).

It also follows from theorem 10 that $\vdash_k^R$ is complete for Γ iff $\vdash_R$ is complete for Viv(R, Γ, k). Moreover, a theory is vivid with respect to $\vdash_R$ iff $\vdash_R$ for it is identical to $\vdash_k^R$ for each k. Using theorem 5 and lemma 9, we obtain the following corollary of theorem 10:

Corollary 11. For any admissible relation $=_R$, any theory Γ, any $k \geqslant |\text{HB}(\Gamma)|$, and any clause ψ: $\Gamma \vDash \psi$ iff $\Gamma \vdash_k^R \psi$.

Note that k does not depend on ψ. Thus, for any admissible relation $=_R$ and any theory Γ, the family $\vdash_k^R$ $(k = 0, 1, \ldots)$ of reasoners is *eventually complete*, that is, there is a number k such $\vdash_k^R$ is complete for inferring clauses from Γ.

Since it follows from the tractability of $\vdash_k^R$ that it is *not* complete, eventual completeness is a much weaker requirement than completeness. In particular, eventual completeness does not require that the *same* $\vdash_k^R$ be complete for inferring clauses from each theory; different k's are allowed for different theories!

Since $\vdash^R$ allows chaining over all clauses, we obtain the next corollary of theorem 10, which shows that $\vdash^R$ is complete, for any admissible relation $=_R$:

Corollary 12. For any admissible relation $=_R$, any theory Γ, and any clause ψ: $\Gamma \vDash \psi$ iff $\Gamma \vdash^R \psi$.

Consider the algorithm R-infer given below:

Algorithm R-infer(Γ, ψ):

1. compute HB(Γ);
2. for $k = 0$ to $|\text{HB}(\Gamma)|$ do
3. if $\Gamma \vdash_k^R \psi$ then return ("yes")
4. else print ("no until level", k);
5. return ("no");

End (R-infer).

Algorithm R-infer(Γ, ψ) returns "yes" iff $\Gamma \models \psi$, and "no" otherwise. If its execution is stopped at any time, we know that $\Gamma \not\vdash_k^R \psi$, where k is the last value for which the statement "no until level k" was printed. Thus, R-infer is an anytime algorithm, since it, even if stopped prematurely, provides meaningful partial answer that improves as more time is spent.

7. Comparison with earlier approaches

We compare the anytime family $\vdash_k^{\mathrm{BCP}}$ of reasoners with some other tractable reasoners presented in the literature. Although work on knowledge compilation (cf. [16, 26]) has a similar flavor of adding clauses to theories for obtaining completeness, there has been no attempt yet to obtain anytime families. Although some of the work reported below allow non-clausal formulas, we focus only on clausal formulas.

7.1. Relevance logic and RP-entailment

Belnap [2] presented a 4-valued model-theory for PC, called relevance logic, whose entailment relation, say $\models_B$, is strictly weaker than $\models$, the entailment relation for classical 2-valued model theory. Intuitively, relevance logic allows equivalences based on the properties of logical operators such as commutativity, associativity, distributivity, De Morgan's laws and double negation [1]; for example, $\{\psi \vee \neg\neg\mu\} \models_B \mu \vee \psi$, for any clause ψ and μ. It also allows inferring clauses from their subclauses; for example, $\{(P \vee Q)\} \models_B (P \vee Q \vee R)$. However, relevance logic blocks chaining; for example, $\{(P), (\neg P \vee Q)\} \not\models_B (Q)$.

Levesque [23] presented a logic of implicit and explicit beliefs, where explicit beliefs are obtained using the $\models_B$ entailment, and proved that $\Gamma \models_B \psi$ can be determined in $\mathrm{O}(|\Gamma||\psi|)$ time; the entailment holds iff each clause in ψ is a superclause of some clause in Γ.

Frisch [20] presented a 3-valued model-theory for PC, whose entailment relation, $\models_{RP}$, is strictly stronger than $\models_B$ but strictly weaker than $\models$. He proved that $\Gamma \models_{RP} \psi$ can also be determined in $\mathrm{O}(|\Gamma||\psi|)$ time. He also argued that it is the strongest propositional logic that is sound but allows no chaining, and proved that $\Gamma \models_{RP} \psi$ iff $\Gamma \cup \{p \vee \neg p \mid p \in \mathrm{HB}(\Gamma \cup \{\psi\})\} \models_B \psi$. For example, $\models_{RP} (P \vee \neg P)$ but $\not\models_B (P \vee \neg P)$.

It follows from the RP-decision theorem for facts [20] and the semantics of conjunction that $\Gamma \models_{RP} \psi$ iff either each clause in ψ is a superclause of some clause in Γ or ψ has complimentary literals. In either of these cases, $\Gamma \vdash_{\mathrm{BCP}} \psi$, because of irredundancy of $=_{\mathrm{BCP}}$. Thus, $\vdash_{\mathrm{BCP}}$ is at least as strong as $\models_{RP}$. Since $\{(P), (\neg P \vee Q)\} \not\models_{RP} (Q)$ and $\{(P), (\neg P \vee Q)\} \vdash_{\mathrm{BCP}} (Q)$, it then follows that $\vdash_{\mathrm{BCP}}$ is strictly stronger than both $\models_B$ and $\models_{RP}$.

7.2. Approximate entailment

Cadoli and Schaerf [6] parameterized $\models_{RP}$ by sets of lemmas: their entailment relation $\models_S^3$ is defined using a 3-valued model theory which restricts each atom in the set S to the traditional 2 values.[6] Intuitively, the logic allows chaining on the atoms in the set S; for example, if $P \in S$ then $\{(P), (\neg P \vee Q)\} \models^3_S (Q)$. They show that the entailment $\Gamma \models^3_S \psi$ can also be determined in $O(|\Gamma||\psi|2^{|S|})$ time. For the general case, $\models^3_S$ is intractable.

For any set S of atoms, if $P \notin S$ then $\{(P), (\neg P \vee Q)\} \not\models^3_S Q$. Thus, $\vdash_{\text{BCP}}$ is not weaker than any entailment $\models^3_S$, except when S contains all atoms in the language. For any number k, let S be the set $\{P_1, \ldots, P_{k+2}\}$ and let Γ be the theory containing all $(k+2)$-clauses built from the atoms in S. It follows that $\Gamma \models^3_S \mathbf{f}$ and $\Gamma \not\vdash^{\text{BCP}}_k \mathbf{f}$. Thus, for each number k there is a set S of size $k+2$ such that $\models^3_S$ is not weaker than $\vdash^{\text{BCP}}_k$. Thus, the two families of entailments are incomparable.

7.3. Bounded resolution

Gallo and Scutellà [21] built a hierarchy, $\Gamma = \Gamma_0, \Gamma_1, \ldots$, of classes of theories in PC, such that for each Γ_k, the satisfiability problem is solvable in $O(n^{k+1})$ time, where n is the size of the theory. Büning [4] defined k-resolution, a restriction on resolution that at least one parent must have at most k literals, and showed that k-resolution is refutation complete for Γ_{k-1}, but refutation-incomplete for Γ_k.

k-resolution can be used to define a family of tractable entailment relations: $\Gamma \vdash^k_B \psi$ iff $\Gamma \cup \{\neg\psi\}$ has a refutation using k-resolution. Although the exact relation between $\vdash^{\text{BCP}}_k$ and $\vdash^k_B$ is still open, the following example shows that $\vdash^{\text{BCP}}_2$ is sometimes stronger than $\vdash^2_B$.

Consider the theory Γ containing the following clauses:

$$\begin{array}{ll}
(\neg P \vee \neg Q \vee S), & (\neg T \vee \neg U \vee \neg P \vee Q \vee V), \\
(\neg P \vee \neg Q \vee \neg S), & (\neg T \vee \neg U \vee \neg P \vee Q \vee \neg V), \\
(\neg T \vee U \vee P), & (\neg T \vee U \vee \neg P \vee Q \vee W), \\
(\neg T \vee \neg U \vee P), & (\neg T \vee U \vee \neg P \vee Q \vee \neg W).
\end{array}$$

Since there is no clause in Γ with 2 literals, it follows that $(\neg T)$ can not be obtained from Γ using 2-resolution. Now consider the least fixed-point $\text{lfp}(T_{\text{BCP},\Gamma,2})$: Since $\Gamma \cup \{(P), (Q)\} =_R \{\mathbf{f}\}$, the clause $(\neg P \vee \neg Q)$ is in the fixed-point. Since $\Gamma \cup \{(\neg P \vee \neg Q), (T), (U)\} =_R \{\mathbf{f}\}$, the clause $(\neg T \vee \neg Q)$ is also in the fixed-point. Since $\Gamma \cup \{(\neg P \vee \neg Q), (T), (\neg U)\} =_R \{\mathbf{f}\}$, the clause $(\neg T \vee Q)$ is also in the fixed-point. Thus, $(\neg T)$ is also in the fixed-point.

Now, consider the theory Γ' obtained from Γ by switching T and $\neg T$, and by replacing all other atoms by pairwise-distinct new atoms. Using the same argument given above, we obtain that (T) can not be obtained from Γ' using 2-resolution and that (T) is in the fixed-point $\text{lfp}(T_{\text{BCP},\Gamma',2})$. It then follows that $\Gamma \cup \Gamma' \vdash^{\text{BCP}}_2 \mathbf{f}$, but $\Gamma \cup \Gamma' \not\vdash^B_2 \mathbf{f}$.

[6] [6] also defines a family of unsound but complete entailment relations, using a similar idea.

7.4. Access-Limited Logics (ALL)

Crawford and Kuipers [9,11] present ALL, a logic that attempts to formalize the access limitations that are inherent in a network-structured knowledge base. ALL allows retrieving only those assertions that are reachable by following an available access path. It is shown that if the access paths are bounded then reasoning is tractable. This system exhibits Socratic completeness in the sense that all facts that are logical consequence of the knowledge base can be deduced after a sequence of preliminary queries. They define a family $\vdash^k_{\text{ALL}}$ of entailment relation such that only k nesting of preliminary queries are allowed for inference in $\vdash^k_{\text{ALL}}$. Although the exact relation between $\vdash^{\text{BCP}}_k$ and $\vdash^k_{\text{ALL}}$ is still open, the following example [10] shows that $\vdash^{\text{BCP}}_2$ is sometimes stronger than $\vdash^2_{\text{ALL}}$.

Consider the theory Γ containing the following clauses:

$$\begin{array}{ll}
(\neg P \vee Q \vee U), & (\neg P \vee \neg Q \vee S \vee W),\\
(\neg P \vee Q \vee \neg U), & (\neg P \vee \neg Q \vee S \vee \neg W),\\
(P \vee R \vee V), & (P \vee \neg T \vee S \vee X),\\
(P \vee R \vee \neg V), & (P \vee \neg T \vee S \vee \neg X).
\end{array}$$

It can be verified by an exhaustive case analysis that (S) can't be inferred from Γ using $\vdash^2_{\text{ALL}}$. However, (S) is in the fixed-point lfp($T_{\text{BCP},\Gamma,2}$), since the following clauses are also in the fixed-point: $(\neg P \vee Q)$, $(P \vee R)$, $(\neg P \vee S)$ and $(P \vee S)$.

8. Conclusions

We presented a technique for developing anytime families of reasoners based on admissible equivalence relations on propositional clausal theories. Using a fixed-point construction, we provided an alternative characterization of the anytime family. The fixed-point was also used for compiling theories into equivalent vivid theories that allow efficient inferencing. We showed that the reasoners in the weakest anytime family, which is based on BCP, is incomparable to previously-known incomplete reasoners.

Although we have used the size of clauses as the bound for restricting chaining, there are other possibilities. For instance, we can restrict chaining over clauses in some explicitly provided set of clauses, that is explicitly provided: for any such set S of clauses, the condition in rule A2 is changed to $\psi \in S$. By this variation, we can define almost an infinite number of different reasoners!

We have implemented the reasoners $\vdash^{\text{BCP}}_k$ $(k = 0, \ldots, n)$ in the anytime family based on BCP. Our preliminary experiments suggest that the fixed-point become prohibitively large even for $k = 4$ for most 3-CNF theories with 100 distinct atoms and 1000 clauses. Since most of the clauses in the fixed-point are redundant, our current work involves developing efficient and practical compilation techniques based on this approach. We are also exploring the use of this approach for developing fast algorithms for unsatisfiability testing.

Although this paper focussed on clausal theories, our approach can be generalized to non-clausal theories. For this, we first extend BCP to an efficient non-clausal reasoner [12] and then use the vivification algorithm presented in [13] to develop the anytime families [14]. The notion of admissible relation for non-clausal case is much more complicated. We are also working on developing a model-theoretic semantics for the anytime reasoners.

References

[1] A.R. Anderson and N.D. Belnap, *Entailment, the Logic of Relevance and Neccessity* (Princeton University Press, 1975).

[2] N.D. Belnap, A useful four-valued logic, in: *Modern Uses of Multiple-Valued Logics*, eds. G. Epstein and J.M. Dunn (Reidel, 1977).

[3] M. Boddy and T. Dean, Solving time dependent planning problems, Technical report, Department of Computer Science, Brown University (1988).

[4] H.K. Buning, On generalized Horn formulas and k-resolution, Theoretical Computer Science 116 (1993) 405–413.

[5] M. Cadoli, Panel on knowledge compilation and approximations: terminology, questions, and references, in: *Fourth International Symposium on Artificial Intelligence and Mathematics (AI/MATH-96)*, Florida (1996) pp. 183–186.

[6] M. Cadoli and M. Schaerf, Approximation in concept description languages, in: *Principles of Knowledge Representation and Reasoning: Proceedings of the Third International Conference (KR'92)*, eds. B. Nebel, C. Rich and W. Swartout (Morgan Kaufmann, Cambridge, MA, 1992) pp. 330–341.

[7] S.A. Cook, The complexity of theorem proving procedures, in: *Proceedings Third Annual ACM Symposium on the Theory of Computing* (1971) pp. 151–158.

[8] J. Crawford, ed., *Proceedings of the AAAI Workshop on Tractable Reasoning* (American Association for Artificial Intelligence, San Jose, CA, 1992).

[9] J. Crawford and B. Kuipers, Towards a theory of access-limited logic for knowledge representation, in: *Proceedings First International Conference on Principles of Knowledge Representation and Reasoning (KR'89)* (1989) pp. 67–78.

[10] J.M. Crawford, Personal communication (1994).

[11] J.M. Crawford and B.J. Kuipers, Negation and proof by contradiction in access-limited logic, in: *Proceedings Ninth National Conference on Artificial Intelligence (AAAI-91)* (1991) pp. 897–903.

[12] M. Dalal, Efficient propositional constraint propagation, in: *Proceedings Tenth National Conference on Artificial Intelligence (AAAI-92)* (American Association for Artificial Intelligence, San Jose, CA, 1992) pp. 409–414.

[13] M. Dalal, Tractable deduction in knowledge representation systems, in: *Principles of Knowledge Representation and Reasoning: Proceedings of the Third International Conference (KR'92)*, eds. B. Nebel, C. Rich and W. Swartout (Morgan Kaufmann, Cambridge, MA, 1992) pp. 393–402.

[14] M. Dalal, Tractable reasoning in knowledge representation systems, Technical Report CUCS-017-95, Department of Computer Science, Columbia University, New York, NY (July 1995). file://ftp.cs.columbia.edu/reports/reports-1995/cucs-017-95.ps.gz.

[15] M. Dalal, Anytime families of tractable propositional reasoners, in: *Fourth International Symposium on Artificial Intelligence and Mathematics (AI/MATH-96)*, Florida (1996) pp. 42–45.

[16] A. del Val, An analysis of approximate knowledge compilation, in: *Proceedings Fourteenth International Joint Conference on Artificial Intelligence (IJCAI-95)* (1995) pp. 830–836.

[17] R. Dionne, E. Mays and F.J. Oles, The equivalence of model-theoretic and structural subsumption in description logics, in: *Proceedings Thirteenth International Joint Conference on Artificial*

Intelligence (IJCAI-93) (1993) pp. 710–716.

[18] W.F. Dowling and J.H. Gallier, Linear-time algorithms for testing the satisfiability of propositional Horn formulae, Journal of Logic Programming 1(3) (1984) 267–284.

[19] J. Doyle and R. Patil, Two theses of knowledge representation: language restrictions, taxanomic classification, and the utility of representation services, Artificial Intelligence 48(3) (1991) 261–297.

[20] A.M. Frisch, Inference without chaining, in: *Proceedings Tenth International Joint Conference on Artificial Intelligence (IJCAI-87)* (1987) pp. 515–519.

[21] G. Gallo and M.G. Scutellà, Polynomially solvable satisfiability problems, Information Processing Letters 29 (1988) 221–227.

[22] M. Garey and D. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness* (Freeman, New York, 1979).

[23] H.J. Levesque, A logic of implicit and explicit belief, in: *Proceedings National Conference on Artificial Intelligence (AAAI-84)* (1984) pp. 198–202.

[24] H.J. Levesque, Making believers out of computers, Artificial Intelligence 30 (1986) 81–108.

[25] J.W. Lloyd, *Foundations of Logic Programming* (Springer, Berlin, 2 ed., 1987).

[26] P. Marquis, Knowledge compilation using theory prime implicates, in: *Proceedings Fourteenth International Joint Conference on Artificial Intelligence (IJCAI-95)* (1995) pp. 837–843.

[27] D. McAllester, An outlook on truth maintenance, Memo 551, MIT AI Lab (August 1980).

[28] D. McAllester, Truth maintenance, in: *Proceedings Eighth National Conference on Artificial Intelligence (AAAI-90)* (1990) pp. 1109–1116.

[29] E. Mendelson, *Introduction to Mathematical Logic* (Van Nostrand, Princeton, NJ, 1964).

[30] R. Reiter and J. de Kleer, Foundations of assumption-based truth maintenance systems: Preliminary report, in: *Proceedings Sixth National Conference on Artificial Intelligence (AAAI-87)* (1987) pp. 183–188.

[31] B. Selman and H. Kautz, Knowledge compilation using Horn approximations, in: *Proceedings Ninth National Conference on Artificial Intelligence (AAAI-91)* (1991) pp. 904–909.

[32] A. Tarski, A lattice-theoretical fixpoint theorem and its applications, Pacific J. Math. 5 (1955) 285–309.

[33] H. Zhang and M.E. Stickel, An efficient algorithm for unit propagation, in: *Fourth International Symposium on Artificial Intelligence and Mathematics (AI/MATH-96)*, Florida (1996) pp. 166–169.