Journal of Mathematical Sciences, Vol. 113, No. 1, 2003

PROPERTIES OF ENDOMORPHISM RINGS OF ABELIAN GROUPS. II

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Introduction

This work is the second half of a review on endomorphism rings of Abelian groups; it contains Parts 5, 6, and 7 of this review (Secs. 24–44). Parts 1–4 of this review are being published in another issue of the same journal [272]. Most of the results are presented with proofs.

In Part 5, isomorphism theorems and realization theorems for endomorphism rings of Abelian groups are considered. In Part 6, hereditary endomorphism rings and related topics are studied. In Part 7, completely transitive Abelian groups and their endomorphism rings are studied.

We accept the Zermelo–Fraenkel axiomatic set ZFC of set theory (including the choice axiom and the Zorn lemma). The terms "class" and "set" are used in the ordinary set-theoretical sense. We note that $\mathbf{N}$, $\mathbf{Z}$, and $\mathbf{Q}$ denote the set of all positive integers, the group or ring of all integers, and the group or field of all rational numbers, respectively. Further, p is always a prime integer, F_p is the field consisting of p elements, and $Z(n)$ is a cyclic group of order n. The end of the proof is denoted by $\square$.

The ring of all endomorphisms of an Abelian group A is denoted by $E(A)$ or R; the center of the ring $E(A)$ is usually denoted by C.

Part 5

ISOMORPHISM OR REALIZATION THEOREMS FOR ENDOMORPHISM RINGS

By an isomorphism theorem for endomorphism rings is usually meant a theorem that states that two groups (probably in a given class) with isomorphic endomorphism rings are isomorphic. Therefore, the classical formulation of an isomorphism theorem states that $G \cong H$ for two groups G and H with $E(G) \cong E(H)$. At present, there are many variants of this formulation. A weak assertion is possible, where the isomorphism of groups is not required. More precisely, we can seek isomorphism conditions of endomorphism rings of given groups G and H. On the other hand, a strong assertion is possible, where a given isomorphism of endomorphism rings $\psi : E(G) \to E(H)$ is induced by some isomorphism of groups $\varphi : G \to H$ (this means that $\psi(\eta) = \varphi\eta\varphi^{-1}$ for all $\eta \in E(G)$). This type of isomorphism theorem answers the related problem: find out all groups in a fixed class such that all automorphisms of endomorphism rings of the groups are inner. Finally, there is a particularly strong type of isomorphism theorem, where one group is fixed and the other group runs over a quite large class of groups. Continuous isomorphisms of finitely topologized endomorphism rings are very useful.

In the general formulation, a realization theorem for endomorphism rings states that rings from this class can be represented up to an isomorphism as endomorphism rings of groups in a given class of groups. In such realization theorems, the starting point is some class of rings. In other realization theorems, the starting point is some class of groups, and it is more natural to call them characterization theorems. Such theorems give a ring-theoretical description of endomorphism rings of groups in this class of groups. Among realization theorems, there are theorems which can be called theorems on split realization. In these theorems, the endomorphism ring is a direct sum of a subring isomorphic to a given ring and some special ideal of endomorphisms. Therefore, theorems on split realization give examples of a partial realization of

Translated from Itogi Nauki i Tekhniki, Seriya Sovremennaya Matematika i Ee Prilozheniya. Tematicheskie Obzory. Vol. 85, Algebra-21, 2003.

rings by endomorphism rings of groups. This superficially weaker realization is effectively applied to the reduction of some group-theoretical problems to solvable problems on some classes of rings.

An important application of realization theorems is the search for various pathological direct decompositions of groups. The technology of such an application is constructing a ring with some properties. Then this ring is realized either totally or partially as the endomorphism ring of some group. At the last stage, it is proved that the group has the required decomposition.

Isomorphism theorems and realization theorems are interrelated. For a given class of groups, an isomorphism theorem implies that the structure of endomorphism rings is no more simple than the structure of groups. Consequently, if the structure of such groups is not known, then a ring-theoretical description of their endomorphism rings does not guarantee any new information on groups. This is confirmed by the Baer–Kaplansky isomorphism theorem for torsion groups (Theorem 24.1) and the characterization theorem of Liebert for endomorphism rings of separable p-groups (Fuchs [165, Theorem 109.1]).

On the other hand, realization theorems often provide one with many nonisomorphic groups with given endomorphism ring. In such examples, the realization theorem acts as a nonisomorphism theorem.

In the first four sections of this part, we consider problems related to isomorphism theorems. We prove the classical Baer–Kaplansky theorem for torsion groups. For completely transitive torsion-free groups, we have a quite satisfactory isomorphism theorem for topological endomorphism rings. Then we generalize the Baer–Kaplansky theorem to arbitrary groups and study the effect of the divisible part of a group on the probability of fulfillment of the isomorphism theorem for the group. As in the case of torsion-free groups, it is essential to consider just topological isomorphisms of endomorphism rings. Finally, the isomorphism problem for endomorphism rings is studied for mixed groups of torsion-free rank 1. It is not surprising that the problem has a negative solution even in the case of such mixed groups. We recall that for two groups G and H, we say that a ring isomorphism

$$\psi : E(G) \to E(H)$$

is induced by a group isomorphism $\varphi : G \to H$ if $\psi(\eta) = \varphi\eta\varphi^{-1}$ for each $\eta \in E(G)$.

The last three sections are devoted to realization theorems. Two of these sections contain the following well-known remarkable results of Corner: the theorem on split realization for endomorphism rings of separable p-groups and the realization theorem for endomorphism rings of countable torsion-free groups. In the last section of this part, we present one theorem on split realization for endomorphism rings of mixed groups.

24. The Baer–Kaplansky Theorem

Torsion groups have many idempotent endomorphisms. The proof of the following perfect theorem uses the well-known methods of using of idempotent endomorphisms.

Theorem 24.1 (Baer [49], Kaplansky [234]). *If A and C are two torsion groups with isomorphic endomorphism rings, then every isomorphism $E(A) \to E(C)$ is induced by some isomorphism $A \to C$.*

Proof. It is sufficient to consider p-groups. Indeed, $A = \sum^{\oplus} A_p$ and $C = \sum^{\oplus} C_p$, where A_p and C_p are the p-components of the groups A and C, respectively. Then $E(A) = \Pi E(A_p)$ and $E(C) = \Pi E(C_p)$. Since

$$E(A_p) = \bigcap_{(n,p)=1} nE(A) \qquad \text{and} \qquad E(C_p) = \bigcap_{(n,p)=1} nE(C),$$

every ring isomorphism $E(A) \to E(C)$ maps from $E(A_p)$ into $E(C_p)$. Therefore, it is sufficient to assume that A and C are p-groups. Further, we deal with some fixed ring isomorphism $\psi : E(A) \to E(C)$. For $\eta \in E(A)$, we merely write $\psi(\eta) = \eta^*$.

First, let the group A be cocyclic (i.e., A is a cyclic or quasi-cyclic p-group). Then the group C is indecomposable; therefore, C is cocyclic. Examples 3.2 and 3.4 show that $A \cong C$ in this case.

The remaining part of the proof is decomposed into three cases.

1. The group A is bounded. Therefore, it is a direct sum of cyclic p-groups. Let g be one of the generating elements of a cyclic direct summand of the group A of maximal order p^k. If $\varepsilon : A \to \langle g \rangle$ is a projection, then ε is an idempotent of the ring $E(A)$ and ε^* is an idempotent of the ring $E(C)$. Therefore, $\varepsilon^* C$ is a direct summand of the group C. By property (d) from Sec. 3, ψ induces the ring isomorphism $E(\langle g \rangle) \to E(\varepsilon^* C)$. By the previous remark, $\varepsilon^* C$ is a cyclic group $\langle h \rangle$ of order p^k. Now we can construct the required isomorphism $\varphi : A \to C$. For any element $a \in A$, we choose an endomorphism $\eta \in E(A)$ such that $a = \eta g$ (the η exists by the assumption on the order of the element g). We define $\varphi : A \to C$ such that $\varphi a = \eta^* h$. This definition is correct (i.e., it does not depend on the choice of the endomorphism η), because if $a = \eta_1 g$ and $\eta_1 \in E(A)$, then $(\eta - \eta_1)g = 0$ and $(\eta - \eta_1)\varepsilon = 0$, whence $((\eta - \eta_1)\varepsilon)^* = (\eta^* - \eta_1^*)\varepsilon^* = 0$ and $(\eta^* - \eta_1^*)h = 0$. We take one more element $b \in A$ and choose $\zeta \in E(A)$ such that $b = \zeta g$. Then $a + b = (\eta + \zeta)g$ and $\varphi(a + b) = (\eta + \zeta)^* h = \eta^* h + \zeta^* h = \varphi a + \varphi b$ (i.e., φ preserves the addition). If $\varphi a = \eta^* h = 0$, then $(\eta\varepsilon)^* = \eta^* \varepsilon^* = 0$. Therefore, $\eta\varepsilon = 0$, $a = \eta\varepsilon g = 0$, and $\ker \varphi = 0$. Further, $p^k A = 0$ implies $p^k E(A) = 0$. Therefore, $p^k E(C) = 0$ and $p^k C = 0$. Therefore, C is a bounded group and $\langle h \rangle$ is its direct summand of maximal order p^k. For every element $c \in C$, there exists $\theta \in E(C)$ such that $c = \theta h$. Let $\theta = \eta^*$ for some $\eta \in E(A)$. Then $c = \theta h = \eta^* h = \varphi a$, where $a = \eta g$. We have obtained that φ is a bijection (i.e., φ is an isomorphism). Finally, if $\xi \in E(A)$, then we write the element $c = \varphi a$ in the form $c = \eta^* h$ for some $\eta \in E(A)$. Then $\xi^* c = \xi^* \eta^* h = \varphi(\xi \eta g) = \varphi \xi a = (\varphi \xi \varphi^{-1})c$ (i.e., $\psi(\xi) = \xi^* = \varphi \xi \varphi^{-1}$). Therefore φ induces ψ.

2. The group A has the form $A = B \oplus D$, where B is a bounded group and D is a nonzero divisible group. Let $\langle g \rangle$ be a cyclic direct summand of maximal order p^k in the group B, E be a quasi-cyclic direct summand of the group D, and $d_1, \ldots, d_n, \ldots$ be a generator system of the group E such that $pd_1 = 0$ and $pd_{n+1} = d_n$ for $n \geq 1$. Let $\varepsilon : A \to \langle g \rangle$ and $\pi : A \to E$ be projections. As in item 1, $\varepsilon^* C$ is a cyclic direct summand of the group C. Let $\varepsilon^* C = \langle h \rangle$, and let $e_1, \ldots, e_n, \ldots$ be a generator system of the group $\pi^* C$ such that $pe_1 = 0$ and $pe_{n+1} = e_n$ for $n \geq 1$. We take an element $a \in A$. Then $a = a_1 + a_2$ for some $a_1 \in B$ and $a_2 \in D$). We take an endomorphism $\eta \in E(A)$ such that $\eta g = a_1$ and $\eta d_n = a_2$ for some n. We set

$$\varphi a = \eta^*(h + e_n).$$

It is sufficient to prove that φa does not depend on the choice η and n. We take $\eta_1 \in E(A)$ such that $\eta_1 g = a_1$ and $\eta_1 d_m = a_2$ (we can assume that $m \geq n$). Then $(\eta - \eta_1)\varepsilon = 0$ and $(p^{m-n}\eta - \eta_1)d_m = 0$. Then the endomorphism $(p^{m-n}\eta - \eta_1)\pi$ annihilates $E[p^m]$. Example 3.4 shows that this means that this endomorphism is divided by p^m. The endomorphism $((p^{m-n}\eta - \eta_1)\pi)^*$ is divided by p^m. Therefore, it annihilates the element e_m. We have $\eta^* h = \eta_1^* h$ and $\eta^* e_n = p^{m-n}\eta^* e_m = \eta_1^* e_m$. Therefore, $\eta^*(h + e_n) = \eta_1^*(h + e_m)$. As in item 1, we verify the property that the mapping φ preserves the addition, is bijective, and induces ψ.

3. We consider the case where the group A has an unbounded basis subgroup. By the well-known property of basis subgroups, there exist decompositions

$$A = \langle a_1 \rangle \oplus \cdots \oplus \langle a_k \rangle \oplus A_k \quad (k \in \mathbf{N})$$

such that $A_k = \langle a_{k+1} \rangle \oplus A_{k+1}$ and $0(a_k) = p^{n_k}$, where $1 \leq n_1 < \cdots < n_k < \cdots$. Let ε_k be the natural projection $A \to \langle a_k \rangle$. For subscripts j and k ($j \neq k$), let ξ_{jk} be the endomorphism of the group A such that $\xi_{jk}(a_k) = a_j$ for $j < k$, $\xi_{jk}(a_k) = p^{n_j - n_k} a_j$ for $j > k$, and $\xi_{jk}(\mathrm{Ker}(\varepsilon_k)) = 0$. In this case,

(1) ε_k are pairwise orthogonal idempotents;

(2) $\xi_{jk}\varepsilon_k = \xi_{jk} = \varepsilon_j \xi_{jk}$ for all $j \neq k$;

(3) $\xi_{kj}\xi_{jk} = p^{|n_j - n_k|}\varepsilon_k$ for all $j \neq k$;

(4) $\xi_{ij}\xi_{jk} = \xi_{ik}$ if $i < j < k$ or $i > j > k$.

The endomorphisms ε_k^* and ξ_{jk}^* of the group C satisfy conditions (1)–(4). By item (d) from Sec. 3, every subgroup $\varepsilon_k^* C$ is a cyclic direct summand of the group C and the order of $\varepsilon_k^* C$ is equal to the order of $\varepsilon_k A$. By (2), $\xi_{k,k+1}^*$ maps from $\varepsilon_{k+1}^* C$ into $\varepsilon_k^* C$. Let $\varepsilon_k^* C = \langle c_k \rangle$. We prove that it is possible to choose generators c_k such that $\xi_{k,k+1}^* c_{k+1} = c_k$ for all k. Indeed, if $c_1, \ldots, c_k$ are chosen and the element c'_{k+1}

generates the subgroup $\varepsilon^*_{k+1}C$, then $\xi^*_{k,k+1}c'_{k+1} = tc_k$ for some integer t. Consequently, (3) implies that $\xi^*_{k+1,k}tc_k = p^{n_{k+1}-n_k}c'_{k+1}$. Taking into account the orders of elements, we obtain $(p,t) = 1$. We take an element $c_{k+1} = sc'_{k+1}$, where $st \equiv 1(\operatorname{mod} p^{n_{k+1}})$. Then $\xi^*_{k,k+1}c_{k+1} = c_k$. By property (4), $\xi^*_{jk}c_k = c_j$ for all $j < k$.

For any element $a \in A$, we choose $\eta \in E(A)$ such that $\eta a_k = a$ for some k. Let $\varphi : a \to \eta^* c_k$. The mapping φ is well defined (if $\eta_1 a_j = a$ and $j \geq k$, then $(\eta\xi_{kj} - \eta_1)\varepsilon_j = 0$, whence $(\eta^*\xi^*_{kj} - \eta^*_1)\varepsilon^*_j = 0$, which means $\eta^* c_k = \eta^*_1 c_j$). It can be verified that φ is an isomorphism inducing ψ. $\square$

The above theorem has an interesting consequence. Assume that $A = C$. Let $\psi : E(A) \to E(A)$ be an automorphism of the ring $E(A)$. By Theorem 24.1, there exists an automorphism φ of the group A such that $\psi(\eta) = \varphi\eta\varphi^{-1}$ for every $\eta \in E(A)$. Since automorphisms of the group A coincide with invertible elements of the ring $E(A)$, the last relation means that ψ is an inner automorphism of the ring $E(A)$.

Corollary 24.2 (Baer [49])**.** *For a torsion group, every automorphism of its endomorphism ring is inner.*

The method used in the proof of Theorem 24.1 arises in [234], where the author generalized the corresponding Baer theorem [49] on endomorphism rings of bounded groups. We call this method the Kaplansky method. It has the following content. For a torsion group, indecomposable idempotents of its endomorphism ring correspond to direct summands of a group that is isomorphic to $Z(p^k)$ or $Z(p^\infty)$. For constructing an isomorphism from a group A onto a group C, Kaplansky used the transference of properties of such summands by endomorphisms to obtain required elements of the group C.

Exercise 1. If V and W are two vector spaces over the same division ring, then every isomorphism of the rings of linear transformations of the spaces V and W is induced by some isomorphism $V \to W$.

Exercise 2 (Wolfson [449])**.** Let A and C be two p-adic torsion-free modules. Then every isomorphism of endomorphism rings of modules A and C is induced by some isomorphism $A \to C$.

Exercise 3. Prove that it is impossible to extend Theorem 24.1 and Exercise 2 to arbitrary p-adic modules. For this purpose, compare the groups $Z(p^\infty)$ and I_p.

Exercise 4. Give an example of two nonisomorphic torsion-free groups of rank 1 with isomorphic endomorphism rings. Let A and C be two torsion-free groups of rank 1 and idempotent type (an idempotent type is defined in the next section). Prove that if $E(A) \cong E(C)$ then $A \cong C$.

25. Continuous Isomorphisms of Endomorphism Rings of Torsion-Free Groups

For arbitrary torsion-free groups, any analog of the Baer–Kaplansky theorem does not hold. It is sufficient to consider groups of rank 1. Torsion-free groups can have very few endomorphisms. In the general case, the endomorphism ring weakly affects the original group. From general considerations, theorems on isomorphisms of endomorphism rings of torsion-free groups are more expectable in the case of groups with many endomorphisms or groups from quite narrow classes of groups. This is confirmed in this section, where we study isomorphisms of endomorphism rings of homogeneous completely transitive torsion-free groups. In addition, we consider bidirectionally continuous (with respect to finite topologies) isomorphisms of endomorphism rings. Such isomorphisms are called topological. A topological isomorphism $\psi : E(A) \to E(C)$ of endomorphism rings is a ring isomorphism ψ such that ψ and ψ^{-1} are continuous with respect to finite topologies of the rings $E(A)$ and $E(C)$. The continuity of isomorphisms of endomorphism rings is necessary in the sense that every group isomorphism $A \to C$ induces a topological isomorphism of endomorphism rings $E(A) \to E(C)$. Indeed, let $\varphi : A \to C$ be an isomorphism, and let $\psi : E(A) \to E(C)$ be the induced isomorphism (i.e., $\psi(\eta) = \varphi\eta\varphi^{-1}$ for all $\eta \in E(A)$). We take any neighborhood of zero U_X of the ring $E(C)$, where X is some subset of the group C and $U_X = \{\beta \in E(C) \,|\, \beta X = 0\}$ (see the beginning of Sec. 14). We recall that a basis of neighborhoods of zero of the ring $E(C)$ in the finite topology consists of the family of left ideals U_X for all subsets X. Then $\psi^{-1}U_X = U_{\varphi^{-1}X}$, where $U_{\varphi^{-1}X} = \{\alpha \in E(A) \,|\, \alpha(\varphi^{-1}X) = 0\}$ is a neighborhood of zero of the ring

$E(A)$. Conversely, ψ maps from neighborhoods of zero of the ring $E(A)$ into neighborhoods of zero of the ring $E(C)$. Therefore ψ is a topological isomorphism. In connection with the Baer–Kaplansky theorem, we note that we can assert without the use of this theorem that every isomorphism $E(A) \to E(C)$ for reduced torsion groups A and C is continuous. This directly follows from Proposition 14.2(1).

All used notions related to torsion-free groups are defined in Sec. 2. In addition, a type is called an *idempotent type* if it contains a characteristic consisting of symbols 0 and ∞. The type of a torsion-free group A of rank 1 is an idempotent type if and only if A is isomorphic to the additive group of some subring of the field of rational numbers $\mathbf{Q}$ (also see Example 3.3 in Sec. 3). For an element a of a torsion-free group G, the p-height and the characteristic are denoted by $h_p(a)$ and $\chi(a)$, respectively. If $M \subseteq G$, then $\langle M \rangle_*$ is a pure subgroup of the group G generated by the subset M. The type of a homogeneous group G is denoted by $t(G)$ (in particular, this touches on a group G of rank 1). At the end of the section, we give a short summary of other results on isomorphisms of endomorphism rings of torsion-free groups.

It is convenient to reformulate Definition 22.3 of a completely transitive torsion-free group in terms of characteristics. A torsion-free group G is said to be *completely transitive* if for any two its nonzero elements a and b such that $\chi(a) \leq \chi(b)$, there exists an endomorphism $\alpha \in E(G)$ such that $\alpha a = b$. Instead of elements, we can use pure subgroups of rank 1 in this definition.

Lemma 25.1. *A torsion-free group G is completely transitive if and only if for each of its pure subgroups A of rank 1, each homomorphism $A \to G$ is induced by some endomorphism of the group G.*

Proof. First, we assume that the group G is completely transitive. Let $\alpha : A \to G$ be a nonzero homomorphism, where A is a pure subgroup of rank 1 of the group G. We take a nonzero element $a \in A$. Since $\chi(a) \leq \chi(\alpha a)$, there exists $\beta \in E(G)$ such that $\beta a = \alpha a$. Every nonzero homomorphism from the group of rank 1 into a torsion-free group is a monomorphism. Therefore, $\beta_{|A} = \alpha$ (i.e., α is extended to an endomorphism β of the group G). For the proof of the converse assertion, take two nonzero elements a and b of G with $\chi(a) \leq \chi(b)$. We set $A = \langle a \rangle_*$ and $B = \langle b \rangle_*$. The groups A and B have rank 1. Therefore, there exists a homomorphism $\alpha : A \to B$ that maps from a into b. This α is induced by some endomorphism of the group G. Therefore, the group G is completely transitive. $\square$

For homogeneous groups, the above lemma can be sharpened.

Lemma 25.2. (1) *A homogeneous torsion-free group G is completely transitive if and only if for any two pure subgroups A and B in G of rank 1, there exists an endomorphism α of the group G such that $\alpha A = B$.*

(2) *A homogeneous completely transitive torsion-free group G is irreducible. In addition, if G has an idempotent type, then G is an endocyclic group and each of its pure subgroups contains a generating element of the $E(G)$-module G.*

Proof. (1) *Necessity.* Since G is a homogeneous group, we can choose two nonzero elements $a \in A$ and $b \in B$ such that $\chi(a) = \chi(b)$. If $\alpha \in E(G)$ and $\alpha a = b$, then it is clear that $\alpha A = B$.

Sufficiency. Let $0 \neq a,\ b \in G$, and $\chi(a) \leq \chi(b)$. We set $A = \langle a \rangle_*$ and $B = \langle b \rangle_*$. Then $\beta A = B$ for some $\beta \in E(G)$. Since $\beta a, b \in B$ and the group B has rank 1, we have $n(\beta a) = mb$, where n and m are nonzero integers and $(n, m) = 1$. The restriction $\beta_{|A}$ is an isomorphism from A onto B. Therefore, $\chi(\beta a) = \chi(a)$, whence $\chi(\beta a) \leq \chi(b)$. Considering the relations $n(\beta a) = mb$ and $(n, m) = 1$, we obtain $mG = G$. Consequently, the group G has the endomorphism $\dfrac{n}{m}1$. In this case, $\alpha a = b$, where $\alpha = \dfrac{n}{m}\beta$.

(2) Assume that H is a nonzero pfi-subgroup of the group G (i.e., H is a pure fully characteristic subgroup in G). Let $a \in H$ and $b \in G$ be some nonzero elements. Since the group G is homogeneous, we can choose a positive integer n such that $\chi(a) \leq \chi(nb)$. Then $\alpha a = nb$, where $\alpha \in E(G)$. Therefore, $nb \in H$, since the subgroup H is fully characteristic. Consequently, $b \in H$, since H is pure. Therefore, $H = G$. The group G does not have proper pfi-subgroups; therefore, G is irreducible.

If the type of G is an idempotent type, then every nonzero pure subgroup of G contains an element $a \neq 0$ with characteristic $\chi(a)$ consisting of symbols 0 and ∞. Then $\chi(a) \leq \chi(b)$ for every $b \in G$. Therefore, $b \in E(G)a$ and $E(G)a = G$ (i.e., the element a generates the $E(G)$-module G). □

The next proposition reduces the study of homogeneous completely transitive torsion-free groups to the study of homogeneous completely transitive torsion-free groups of idempotent type. Using this property and Lemma 25.2, we can apply Theorem 8.5 on irreducible endofinite groups to the study of homogeneous completely transitive groups. We note that if G and A are two torsion-free groups and A has rank 1, then every element of $G \otimes A$ has the form $g \otimes a$ for some elements $g \in G$ and $a \in A$.

Proposition 25.3. *A homogeneous torsion-free group G is completely transitive if and only if $G \cong F \otimes A$, where F is a homogeneous completely transitive torsion-free group of idempotent type, A is a torsion-free group of rank 1, and $pF = F$ for every prime integer p such that $pA = A$. In this case, the mapping $\alpha \to \alpha \otimes 1$ ($\alpha \in E(F)$) defines a ring isomorphism $E(F) \to E(G)$.*

Proof. Let G be a homogeneous completely transitive group. We choose a nonzero element $a \in G$. Let $A = \langle a \rangle_*$. We prove that $E(G)a \cong \mathrm{Hom}(A, G)$. If $\alpha \in E(G)$, then we associate the homomorphism $\alpha_{|A} \in \mathrm{Hom}(A, G)$ with the element $\alpha a \in E(G)a$. If $\beta \in \mathrm{Hom}(A, G)$, then the homomorphism β is extended to some endomorphism α of the group G by Lemma 25.1. In addition, $\alpha_{|A} = \beta$. Consequently, the correspondence $\alpha a \to \alpha_{|A}$ is an isomorphism from $E(G)a$ onto $\mathrm{Hom}(A, G)$.

Let $F = E(G)a$. We prove that F is a homogeneous completely transitive group of idempotent type. We take two arbitrary pure subgroups X' and Y' of rank 1 of the group F. Let X and Y be the pure subgroups of the group G generated by X' and Y', respectively. Then $X \cap F = X'$ and $Y \cap F = Y'$. We choose $\alpha \in E(G)$ such that $\alpha X = Y$ (Lemma 25.2). It is clear that $\alpha' X' = Y'$, where $\alpha' = \alpha_{|F}$. We note that F is a fully characteristic subgroup of the group G; therefore, $\alpha' \in E(F)$. We have obtained that $X' \cong Y'$; therefore, F is a homogeneous group. By Lemma 25.2, it is completely transitive. By construction, a is a generating element of the $E(F)$-module F. Since endomorphisms do not decrease characteristics of elements, $\chi(a) \leq \chi(x)$ for all $x \in F$ (the characteristics are taken in F). The last assertion implies that $\chi(a)$ consists of 0 and ∞. This means that the element a has an idempotent type; therefore, the group F has an idempotent type.

Further, we use the canonical homomorphism $\theta_G : \mathrm{Hom}(A, G) \otimes A \to G$ such that $\theta_G(\varphi \otimes b) = \varphi b$ for elements $\varphi \in \mathrm{Hom}(A, G)$ and $b \in A$. Since $\mathrm{im}\, \theta_G = \sum\limits_{\varphi : A \to G} \mathrm{im}\, \varphi$, it is clear that θ_G is an epimorphism. If $\varphi \otimes b \neq 0$, then $\varphi \neq 0$ and $b \neq 0$. Therefore, $\varphi b \neq 0$, since the group A has rank 1 and nonzero homomorphisms $A \to G$ are monomorphisms. We have obtained that θ_G is an isomorphism. Therefore, $G \cong F \otimes A$, where the groups F and A satisfy the conditions of the proposition.

Conversely, assume that F and A satisfy the conditions of the proposition. We prove that the group $G = F \otimes A$ is completely transitive. First, if $f \in F$ and $a \in A$, then $\chi(f \otimes a) = \chi(f) + \chi(a)$. We obtain that G is a homogeneous group. We take two arbitrary pure subgroups X and Y of rank 1 of the group G and choose nonzero elements $x \otimes a \in X$ and $y \otimes b \in Y$ ($x, y \in F$ and $a, b \in A$) such that $\chi(x \otimes a) = \chi(y \otimes b)$. Further, we choose nonzero integers n and m such that $na = mb$. Then $nm(x \otimes a) = mx \otimes na$ and $nm(y \otimes b) = ny \otimes mb$. Since $\chi(nm(x \otimes a)) = \chi(nm(y \otimes b))$, we have $\chi(mx) = \chi(ny)$. Let $\alpha mx = ny$, where $\alpha \in E(F)$. Then $\alpha \otimes 1 \in E(G)$ and $(\alpha \otimes 1)(mx \otimes a) = ny \otimes a$. Therefore, $(\alpha \otimes 1)X = Y$, and the group G is completely transitive by Lemma 25.2.

Finally, we prove the existence of the required ring isomorphism $E(F) \cong E(G)$. If $x \otimes a = y \otimes a$ for some elements $x, y \in F$ and $a \in A$, then $x = y$, since we have the natural isomorphism $F \otimes \langle a \rangle \cong F$. Therefore, the considered mapping of rings is a monomorphism. Let $\gamma \in E(G)$. We prove that γ is induced by some $\alpha \in E(F)$ (i.e., $\gamma = \alpha \otimes 1$). Let $x \in F$, $a \in A$, and $\gamma(x \otimes a) = z \otimes b$, where $z \in F$ and $b \in A$. The elements a and b have equivalent characteristics. Therefore, there exists only a finite set of prime integers p such that $h_p(a) > h_p(b)$ and $h_p(a) < \infty$ in this case. Let p be one of these integers. Then

$h_p(a) = h_p(b) + k$. We have

$$h_p(x \otimes a) = h_p(x) + h_p(a), \quad h_p(z \otimes b) = h_p(z) + h_p(b),$$

and $h_p(x \otimes a) \le h_p(z \otimes b)$. Therefore, $h_p(z) = h_p(x) + l$, where $l \ge k$ (l is a positive integer or ∞). Therefore, $z = p^k z'$, $z' \in F$, and $z \otimes b = z' \otimes p^k b$. Here $h_p(a) = h_p(p^k b)$; in addition, $h_q(b) = h_q(p^k b)$ for all prime integers $q \ne p$. We find elements $z_1 \in F$ and $b_1 \in A$ such that $z \otimes b = z_1 \otimes b_1$ and $\chi(a) \le \chi(b_1)$. The last inequality implies that $na = mb_1$ for some positive integers n and m such that $(n, m) = 1$ and $mA = A$. By assumption, $mF = F$. Therefore, $z_1 = mz_2$ for some $z_2 \in F$. Now we have $z_1 \otimes b_1 = mz_2 \otimes b_1 = z_2 \otimes mb_1 = nz_2 \otimes a$. Let $y = nz_2$. We prove that for every element $x \otimes a \in F \otimes A$, the image $\gamma(x \otimes a)$ is equal to $y \otimes a$ for a unique $y \in F$. The uniqueness follows from the existence of the isomorphism $F \otimes \langle a \rangle \cong F$. The correspondence $\alpha : x \to y$ ($x \in F$) is an endomorphism of the group F and $\gamma = \alpha \otimes 1$. $\square$

We now present the main result of this section.

Theorem 25.4. *If G and H are two homogeneous completely transitive torsion-free groups with topologically isomorphic endomorphism rings, then $G \otimes B \cong H \otimes A$, where B and A are torsion-free groups of rank 1 and of types $t(H)$ and $t(G)$, respectively. More precisely, assume that the groups $G \cong F_1 \otimes A$ and $H \cong F_2 \otimes B$ are written as in Proposition 25.3, where F_1 and F_2 are homogeneous completely transitive groups of idempotent types. Then $E(F_1)$ and $E(F_2)$ are topologically isomorphic rings and every topological ring isomorphism $\psi : E(F_1) \to E(F_2)$ is induced by some group isomorphism $\varphi : F_1 \to F_2$.*

Proof. If the second assertion is proved, then $F_1 \cong F_2$ and

$$G \otimes B \cong (F_1 \otimes A) \otimes B \cong (F_2 \otimes B) \otimes A \cong H \otimes A.$$

We prove the second assertion. The mappings $\alpha \to \alpha \otimes 1$ and $\beta \to \beta \otimes 1$ ($\alpha \in E(F_1)$ and $\beta \in E(F_2)$) are ring isomorphisms $E(F_1) \to E(G)$ and $E(F_2) \to E(H)$, respectively (Proposition 25.3). It is clear that these isomorphisms are topological. Consequently, $E(F_1)$ and $E(F_2)$ also are topologically isomorphic rings.

Let $\psi : E(F_1) \to E(F_2)$ be a topological ring isomorphism. We set $V_i = F_i \otimes \mathbf{Q}$ and $S_i = E(F_i) \otimes \mathbf{Q}$ ($i = 1, 2$). Then V_i is a faithful irreducible S_i-module by Lemma 25.2 and Proposition 5.8. Further, we set $D_i = \operatorname{End}_{S_i} V_i$ and $L_i = \operatorname{End}_{D_i} V_i$ ($i = 1, 2$). By the Schur lemma, D_i is a division ring. By the Chevalley–Jacobson density theorem on irreducible modules, the ring S_i is dense in the finite topology of the ring L_i (see Sec. 1). As before, the ring $E(F_i)$ is identified with its image under the canonical embedding $E(F_i) \to S_i$, $i = 1, 2$ (see Sec. 5). Then the finite topology of the ring $E(F_i)$ coincides with the topology induced by the finite topology of the ring L_i. Therefore, $\psi \otimes 1_{\mathbf{Q}}$ is a topological isomorphism of the rings S_1 and S_2; it is also denoted by ψ. Since S_i is dense in the complete ring L_i ($i = 1, 2$), we have that ψ is uniquely extended to an isomorphism of the rings L_1 and L_2; it is also denoted by ψ. We write η^* instead of $\psi(\eta)$.

We fix some generating element g of the $E(F_1)$-module F_1 (we use Lemma 25.2). Let W be the subspace of the D_1-space V_1 generated by the element g, and let $\pi : V_1 \to W$ be a projection. Then $\pi \in L_1$ and $\pi^2 = \pi$. Therefore, $(\pi^*)^2 = \pi^*$ and $\pi^* : V_2 \to \pi^* V_2$ is a projection. In addition,

$$D_1 \cong \operatorname{End}_{D_1} W \cong \pi L_1 \pi \cong \pi^* L_2 \pi^* \cong \operatorname{End}_{D_2} \pi^* V_2.$$

Therefore, $\dim_{D_2} \pi^* V_2 = 1$ (regarding the isomorphism $\operatorname{End}_{D_1} W \cong \pi L_1 \pi$ and the similar isomorphism for the ring L_2, see property (b) from Sec. 3). In $\pi^* V_2 \cap F_2$, we choose some generating element h of the $E(F_2)$-module F_2.

We define $\varphi : F_1 \to F_2$ as follows. For any element $a \in F_1$, we take $\eta \in E(F_1)$ such that $a = \eta g$. We set $\varphi a = \eta^* h$ and prove that φ is a mapping (i.e., φ does not depend of the choice of η). If $a = \eta_1 g$ with $\eta_1 \in E(F_1)$, then $(\eta - \eta_1)g = 0$. Therefore, $(\eta - \eta_1)\pi = 0$, since $\dim_{D_1} W = 1$. Therefore,

$$(\eta^* - \eta_1^*)\pi^* = 0 \quad \text{and} \quad (\eta^* - \eta_1^*)h = (\eta^* - \eta_1^*)\pi^* h = 0;$$

therefore, the definition of φ is correct.

It is easy to verify that φ is a group homomorphism. If $a = \eta g \neq 0$, then $\eta\pi \neq 0$ and $\eta^*\pi^* \neq 0$. Therefore, $\eta^* h = \eta^*\pi^* h \neq 0$, since $\dim_{D_2} \pi^* V_2 = 1$. Thus, φ is a monomorphism. For any element $c \in F_2$, we have $c = \theta h$ for some $\theta \in E(F_2)$. We set $\eta = \psi^{-1}(\theta)$ and $a = \eta g$. Then $\varphi a = \eta^* h = \theta h = c$ (i.e., φ is an isomorphism).

We prove that φ induces ψ. Let $\xi \in E(F_1)$, and let c be any element of F_2. Then $c = \theta h$ for some $\theta \in E(F_2)$. We set $a = \varphi^{-1} c$. We have $a = \eta g$, where $\eta \in E(F_1)$. Then $c = \eta^* h$. Now we have

$$\xi^* c = \xi^* \eta^* h = (\xi\eta)^* h = \varphi((\xi\eta)g) = \varphi\xi(\eta g) = \varphi\xi(\varphi^{-1}c) = (\varphi\xi\varphi^{-1})c.$$

Therefore, $\psi(\xi) = \xi^* = \varphi\xi\varphi^{-1}$ for every $\xi \in E(F_1)$. Therefore, φ induces ψ. □

It is easy to verify that the proof of Theorem 25.4 implicitly uses the Kaplansky method. We consider several consequences of the above theorem.

Corollary 25.5. *If G and H are two homogeneous completely transitive groups of idempotent or equal types, then every topological ring isomorphism $E(G) \to E(H)$ is induced by some group isomorphism $G \to H$. A similar assertion holds if G and H are homogeneous completely transitive torsion-free groups and the group G is almost divisible.*

Proof. In the case of idempotent types, we can assume that $G = F_1$ and $H = F_2$ in Theorem 25.4. If $t(G) = t(H)$, then $G \cong F_1 \otimes A$ and $H \cong F_2 \otimes A$ (see the notation of Theorem 25.4). In these isomorphisms, we identify the left-hand sides with the right-hand sides. Let $\psi : E(G) \to E(H)$ be a topological isomorphism. If $\alpha \in E(F_1)$, then by Proposition 25.3, the image $\psi(\alpha \otimes 1)$ is equal to $\beta \otimes 1$ for a unique $\beta \in E(F_2)$. The correspondence $\alpha \to \beta$ gives a topological isomorphism $\psi' : E(F_1) \to E(F_2)$. Assume that ψ' is induced by an isomorphism $\varphi : F_1 \to F_2$. For every $\alpha \in E(F_1)$, we have

$$\psi(\alpha \otimes 1) = \psi'(\alpha) \otimes 1 = (\varphi\alpha\varphi^{-1}) \otimes 1 = (\varphi \otimes 1)(\alpha \otimes 1)(\varphi \otimes 1)^{-1}.$$

Consequently, the isomorphism $\varphi \otimes 1$ between G and H induces ψ.

For an almost divisible group G, the isomorphism $E(G) \cong E(H)$ implies $t(G) = t(H)$, and we obtain the case we already considered. □

Corollary 25.5 can be reformulated.

Corollary 25.6. *Two homogeneous completely transitive torsion-free groups are isomorphic if and only if their types are equal and the endomorphism rings are topologically isomorphic.*

On the other hand, we can determine when endomorphism rings of groups of arbitrary types are topologically isomorphic.

Corollary 25.7. *Let G and H be two homogeneous completely transitive torsion-free groups, and let F_1 and F_2 be the corresponding groups from Theorem 25.4. The rings $E(G)$ and $E(H)$ are topologically isomorphic if and only if $F_1 \cong F_2$.*

Let a be a nonzero element of the group G. It follows from the proof of Proposition 25.3 that $F_1 = E(G)a$. Since the group G is completely transitive, it is also clear that $E(G)a = \{g \in G \,|\, \chi(g) \geq \chi(a)\}$. In addition, $F_1 \cong \mathrm{Hom}(A, G)$, where A is a group of rank 1 of type $t(G)$; therefore, the group F_1 is uniquely determined by the group G. A similar assertion holds for the group F_2.

As in the case of torsion groups, the application of Corollary 25.5 to a homogeneous completely transitive group leads to the following property (see the remark before Corollary 24.2).

Corollary 25.8. *For a homogeneous completely transitive torsion-free group G, every topological automorphism of the endomorphism ring of G is inner.*

We apply Theorem 25.4 to a homogeneous separable torsion-free group. Such a group is completely transitive by Example (a) in Sec. 22. Let G be a separable torsion-free group. By Proposition 14.2(2), the set of left annihilators of all primitive idempotents of the ring $E(G)$ is a subbasis of neighborhoods of zero of the finite topology on the ring $E(G)$. Repeating the argument used for torsion groups, we obtain that every isomorphism of endomorphism rings of two separable torsion-free groups is continuous with respect to the finite topologies. Therefore, Theorem 25.4 implies the following result.

Corollary 25.9 (Hauptfleisch [203])**.** *If G and H are two homogeneous separable torsion-free groups and $E(G) \cong E(H)$, then $G \otimes B \cong H \otimes A$, where B and A are groups of rank 1 of types $t(H)$ and $t(G)$, respectively.*

In the following definition, we formalize statements presented at the beginning of this part on one of the possible generalizations of an ordinary statement of the isomorphism problem. Let $\mathcal{X}$ be a class of groups. A group $G \in \mathcal{X}$ is said to be determined by its endomorphism ring (resp., topological endomorphism ring) in the class $\mathcal{X}$ if $G \cong H$ for every group $H \in \mathcal{X}$ such that $E(G)$ and $E(H)$ are isomorphic (resp., topologically isomorphic) rings. Therefore, a torsion group is determined by its endomorphism ring in the class of all torsion groups (Theorem 24.1), and a homogeneous almost divisible completely transitive torsion-free group is determined by its topological endomorphism ring in the class of all homogeneous completely transitive torsion-free groups (Corollary 25.5).

There is no description of torsion-free groups determined by their endomorphism rings in the class of all torsion-free groups. Such groups exist (e.g., a divisible group; see also Exercises 6 and 7). The works of Sebeldin [394–398] are devoted to the problem of determination of torsion-free groups by endomorphism rings in various special classes of groups. In [394], he proved that a completely decomposable torsion-free group G is determined by its endomorphism ring in the class of all such groups if and only if every direct summand of rank 1 of the group G is almost divisible. Sebeldin [395,396] obtained a similar result on the determination of a vector group in the class of all vector groups.

Bazzoni and Metelli [52] have proved that a separable torsion-free group G is determined by its endomorphism ring in the class of all such groups if and only if every direct summand of rank 1 of the group G is almost divisible. This generalizes the above Sebeldin theorem on endomorphism rings of completely decomposable groups (see also Exercise 3). In addition, they proved that if a separable torsion-free group G has at least one direct summand of rank 1 that is not almost divisible, then there exists at least $2^{\aleph_0}$ pairwise nonisomorphic separable torsion-free groups H with $E(G) \cong E(H)$. In their work, Bazzoni and Metelli have obtained practically complete solutions of basic problems on endomorphism rings of separable torsion-free groups. For arbitrary such groups G and H, they have obtained necessary and sufficient conditions for the isomorphism $E(G) \cong E(H)$. In addition, they present a ring-theoretic characterization of endomorphism rings of separable torsion-free groups.

Taking into account the proof of Theorem 25.4, solve Exercises 1 and 2 generalizing this theorem.

Exercise 1. Let G and H be two irreducible endocyclic torsion-free groups, and let every pure C-submodule in G contain some generating element of the $E(G)$-module G (C is the center of the ring $E(G)$). Prove that every topological isomorphism $E(G) \to E(H)$ is induced by some isomorphism $G \to H$.

Exercise 2. If G and H are two irreducible endofinite torsion-free groups whose endomorphism rings are topologically isomorphic, then G and H are quasi-isomorphic groups.

Exercise 3 (Bazzoni–Metelli [52])**.** Let G and H be two separable torsion-free groups such that $E(G) \cong E(H)$.

(a) If G is a homogeneous group, then H is a homogeneous group.

(b) If G is a completely decomposable group, then H is a completely decomposable group whose rank is equal to the rank of G;

(c) $|G| = |H|$.

Exercise 4 (Sebeldin [398])**.** Prove that a torsion-free group is determined by its endomorphism ring in the class of all groups if and only if it is determined by its endomorphism ring in the class of all torsion-free groups and it is not a reduced algebraically compact group.

Exercise 5 (Sebeldin [398]). Let G be a torsion-free group, and let v be a characteristic such that the type defined by v is the greatest lower bound of the set of types of all elements of the group G. We set $vG = \{g \in G \,|\, \chi(g) \geq v\}$. Prove that $E(G) \cong E(vG)$.

Exercise 6 (Sebeldin [398]). Let G be a torsion-free group containing a pure subgroup of rank 1 that is not almost divisible. Using Exercise 5, prove that G is not determined by its endomorphism ring in the class of all torsion-free groups.

Exercise 7. Prove that a reduced algebraically compact torsion-free group is determined by its endomorphism ring in the class of all torsion-free groups if and only if it has a finite number of p-adic components (i.e., it is almost divisible).

The next exercise generalizes Corollary 25.9.

Exercise 8 (Webb [443]). If the set of types of all direct summands of rank 1 of a separable torsion-free group G contains the least type, then G is said to be *pointwise* in this type. Prove that if G and H are pointwise separable torsion-free groups with isomorphic endomorphism rings, then: (1) $G \otimes B \cong H \otimes A$ for some groups B and A of rank 1; (2) if G and H are pointwise in idempotent types, then $G \cong H$.

In the following exercises, we use the subgroups vG defined in Exercise 5 to obtain a satisfactory description of fully characteristic subgroups of homogeneous completely transitive torsion-free groups.

Exercise 9. A homogeneous torsion-free group G is completely transitive if and only if each of its nonzero fully characteristic subgroups is equal to vG for some characteristic v. In this case, the lattices of fully characteristic subgroups of the group G and the torsion-free group of rank 1 of type $t(G)$ are isomorphic.

Exercise 10. Using the lattice of all characteristics, describe the lattice of all fully characteristic subgroups of a homogeneous completely transitive torsion-free group.

Exercise 11. Prove that fully characteristic subgroups of the torsion-free group G coincide with subgroups of the form nG $(n \geq 0)$ if and only if G is a homogeneous completely transitive group of idempotent type.

Exercise 12. For a homogeneous completely transitive torsion-free group G, prove that the group $\sum\limits_{\mathfrak{M}}^{\oplus} G$ is completely transitive for every cardinal number $\mathfrak{M}$.

26. Endomorphism Rings of Groups with Large Divisible Subgroups

We now pass to the study of the isomorphism problem of endomorphism rings of arbitrary groups. As in the preceding section, we primarily consider topological isomorphisms that reflect the structure of the group far more completely. The examples of the groups $Z(p^\infty)$ and I_p show that the use of ordinary isomorphisms of endomorphism rings does not give the possibility of distinguishing torsion groups from torsion-free groups in the general case. First, we present one generalization of the Baer–Kaplansky theorem. Then we prove that the group with a sufficiently large divisible subgroup is determined by its topological endomorphism ring in the class of all groups. In this section, we follow the work of May [306].

We pass to the theme of this section. First, we have the following useful property. Let A be a group, and let ε be an idempotent of the ring $E(A)$. It is easy to verify that the canonical isomorphism $E(\varepsilon A) \cong \varepsilon E(A)\varepsilon$ indicated in item (b) of Sec. 3 is topological provided that the finite topology is defined on $E(\varepsilon A)$ and the topology induced by the finite topology of the ring $E(A)$ is defined on $\varepsilon E(A)\varepsilon$.

There exists a topological version of the Baer–Kaplansky theorem for arbitrary groups. For a group G, the torsion part of G (i.e., the largest torsion subgroup of G) is denoted by $T(G)$.

Theorem 26.1 (May [306]). (1) *Let G and H be two groups, and let $\psi : E(G) \to E(H)$ be a topological isomorphism. Then there exists an isomorphism $\varphi : T(G) \to T(H)$ such that $\psi(\eta)$ and $\varphi\eta\varphi^{-1}$ coincide on $T(H)$ for every $\eta \in E(G)$.*

(2) *Let T be a torsion group, and let H be a group. Then every topological isomorphism $E(T) \to E(H)$ is induced by some isomorphism $T \to H$.*

Proof. (1) We can construct the required isomorphism φ by using the Kaplansky method. It is possible to restrict ourself to the torsion parts of the groups G and H. We only need to specify the following circumstance. If ε is an idempotent of the ring $E(G)$ and $\varepsilon G \cong Z(p^k)$ ($k \in \mathbf{N}$), then it is clear that $\psi(\varepsilon)H \cong Z(p^k)$. Let $\varepsilon G \cong Z(p^\infty)$. Then $E(\varepsilon G) \cong Q_p^*$ by Example 3.4, whence $E(\psi(\varepsilon)H) \cong Q_p^*$. Therefore, either $\psi(\varepsilon)H \cong Z(p^\infty)$ or $\psi(\varepsilon)H \cong I_p$ (Example 3.5). By the properties mentioned before the theorem, the rings $E(\varepsilon G)$ and $E(\psi(\varepsilon)H)$ are topologically isomorphic. Since the finite topologies on $E(Z(p^\infty))$ and $E(I_p)$ are different (the first topology is p-adic and the second topology is discrete, see Exercises 3 and 4 from Sec. 14), we have $\psi(\varepsilon)H \cong Z(p^\infty)$. Therefore, there is no any obstacle to the use of the Kaplansky method.

(2) Let $\psi : E(T) \to E(H)$ be a topological isomorphism, and let y be an element of the group H. Since ψ is continuous, there exist elements $x_1, \dots, x_n \in T$ such that $\psi(\alpha)y = 0$ for any $\alpha \in E(T)$ such that $\alpha x_i = 0$ for all $i = 1, \dots, n$. However, there exists a number $m > 0$ such that $mx_i = 0$ for all i. Therefore, $\psi(m1_T)y = m\psi(1_T)y = m1_H y = my = 0$. Therefore, H is a torsion group and the use of (1) completes the proof. □

In the remaining main part of the section, we show that if a group has a nontrivial divisible subgroup, then this has a substantial effect on topological isomorphisms of endomorphism rings. First, we need to prove two lemmas.

Lemma 26.2 (May [306]). *Let D be a torsion-free divisible group, and let D^* be a group. Then every isomorphism $E(D) \cong E(D^*)$ is induced by some isomorphism $D \cong D^*$.*

Proof. Let $\phi : E(D) \to E(D^*)$ be an isomorphism. For convenience, $\phi(\alpha)$ is denoted by α^* ($\alpha \in E(D)$). Assume that ε is an idempotent of the ring $E(D)$. We assert that $\varepsilon D \cong Z(p^\infty)$ if and only if $\varepsilon^* D^* \cong Z(p^\infty)$ and that D^* is a divisible group. By assumption, the group D contains a direct summand that is isomorphic to $\mathbf{Q}$. Let π be the projection of the group D on this summand. Then $E(\pi D) \cong \mathbf{Q}$. Therefore, $E(\pi^* D^*) \cong \mathbf{Q}$. It is clear that $\pi^* D^* \cong \mathbf{Q}$. Thus, the group D^* contains a direct summand that is isomorphic to $\mathbf{Q}$. Now assume that $\varepsilon D \cong Z(p^\infty)$. Then $E(\varepsilon D) \cong Q_p^*$ and $E(\varepsilon^* D^*) \cong Q_p^*$. Therefore, either $\varepsilon^* D^* \cong Z(p^\infty)$ or $\varepsilon^* D^* \cong I_p$. However, the second isomorphism is impossible, since $\operatorname{Hom}(\mathbf{Q}, Z(p^\infty)) \neq 0$ and $\operatorname{Hom}(\mathbf{Q}, I_p) = 0$ (consider item (b) from Sec. 3). Thus, $\varepsilon^* D^* \cong Z(p^\infty)$. If we assume that D^* is not a divisible group, then it contains a nonzero summand without summands isomorphic to $\mathbf{Q}$ or $Z(p^\infty)$ for some p. Therefore, the group D also has such a summand; this contradicts the divisibility of D.

We prepare for using the Kaplansky method. Let P be the set of all prime integers relevant to D (we recall that P is the set of all p such that the p-component of the group D is nonzero). We choose idempotents $\pi, \varepsilon_p \in E(D)$ ($p \in P$) such that

$$\pi D \cong \pi^* D^* \cong \mathbf{Q} \qquad \text{and} \qquad \varepsilon_p D \cong \varepsilon_p^* D^* \cong Z(p^\infty)$$

and fix a nonzero element $x \in \pi D$. In the case $P = \varnothing$, we take every nonzero element $x' \in \pi^* D^*$. Since D and D^* are vector spaces over $\mathbf{Q}$, the Kaplansky method is applicable. Consequently, we can assume that the set P is not empty. Let

$$\zeta : \pi D \longrightarrow \sum_{p \in P}^{\oplus} \varepsilon_p D$$

be a fixed epimorphism whose kernel is generated by elements x and $p^{-\infty}x$ for all $p \notin P$, where $p^{-\infty}x$ is a root of the equation $py = x$ in the group πD. Since ζ maps from the group πD into the torsion part of the group D, a similar assertion holds for ζ^*. Consequently, we can choose an element $x' \neq 0$ in the kernel of the homomorphism ζ^*.

There exists $\zeta_1 \in \operatorname{Hom}(\pi D, D)$ such that the homomorphism $\zeta_1^* \in \operatorname{Hom}(\pi^* D^*, D^*)$ has the kernel $\langle x', p^{-\infty}x', p \notin P \rangle$. The image of this homomorphism ζ_1 is a torsion group, since ζ_1^* has a similar property;

consequently, $m\zeta(x) = 0$ for some $m > 0$. Therefore, $m\zeta_1 = \gamma\zeta$ for some $\gamma \in E(D)$, whence $m\zeta_1^* = \gamma^*\zeta^*$. Therefore, the p-height of the element x' in $\ker \zeta^*$ is equal to zero for almost all integers $p \in P$ and is finite for remaining $p \in P$. Replacing x' by some multiple of it, we can assume that $\langle x', p^{-\infty}x', p \notin P\rangle$ is the kernel of the homomorphism ζ^*.

We now define $\varphi : D \to D^*$. If $d \in D$ and β is an endomorphism of D such that $\beta x = d$, then we set $\varphi d = \beta^* x'$. If $\beta_1 x = d$ for some $\beta_1 \in E(D)$, then $(\beta - \beta_1)\pi = \gamma\zeta$ for some γ. Consequently, $(\beta^* - \beta_1^*)\pi^* = \gamma^*\zeta^*$. This implies that $\beta^* x' = \beta_1^* x'$ and φ is well defined. By symmetry, it is easily verified that φ has an inverse homomorphism and $\alpha^*\varphi = \varphi\alpha$ for every $\alpha \in E(D)$. Then $\phi(\alpha) = \varphi\alpha\varphi^{-1}$ and φ induces ϕ. □

Lemma 26.3 (May [306]). *Let G be a group, and let D be the divisible part of G. Assume that D contains isomorphic copies of the groups $\mathbf{Q}$ and $Z(p^\infty)$ for every p and G can be isomorphically embedded in D. Then every isomorphism $E(G) \to E(G^*)$ is induced by some isomorphism $G \to G^*$.*

Proof. Let ϕ be a fixed isomorphism $E(G) \to E(G^*)$. We denote $\phi(\alpha)$ by α^* for $\alpha \in E(G)$. We choose an idempotent $\varepsilon \in E(G)$ with $\varepsilon G = D$ and we set $D^* = \varepsilon^* G^*$. Lemma 26.2 gives the isomorphism $\psi : D \to D^*$ such that $\psi\beta = \beta^*\psi$ for every $\beta \in E(D)$. By assumption, we can fix an embedding $\delta : G \to D$. We assert that $\delta^* : G^* \to D^*$ is an embedding. Assume that $0 \neq g' \in G^*$. Since D^* contains a copy of the group $Z(p^\infty)$ for every p, there exists $\beta \in \operatorname{Hom}(G, D)$ such that $\beta^*(g') \neq 0$. However, $\beta = \gamma\delta$ for some $\gamma \in E(D)$; therefore, $0 \neq \beta^*(g') = \gamma^*\delta^*(g')$ implies $\delta^*(g') \neq 0$. Therefore, δ^* is an injective mapping.

We prove that $\psi(\delta G) = \delta^* G^*$. First, assume that there exists $g \in G$ with $\psi(\delta g) \notin \delta^* G^*$. As above, we have

$$\beta^*(\psi(\delta g)) \neq 0 \qquad \text{and} \qquad \beta^*(\delta^* G^*) = 0$$

for some $\beta \in E(D)$. Then $\beta\delta = 0$, whence $0 = \psi\beta\delta(g) = \beta^*\psi\delta(g) \neq 0$; this is a contradiction. Now assume that there exists $g' \in G^*$ such that $\delta^*(g') \notin \psi(\delta G)$. We choose $\beta \in E(D)$ such that $\beta^*(\psi(\delta G)) = 0$ and $\beta^*(\delta^* g') \neq 0$. Then $\psi\beta\delta(G) = 0$, whence $\beta\delta(G) = 0$. Therefore, $\beta^*\delta^*(G^*) = 0$; this is a contradiction.

Let $\varphi : G \to G^*$ be a uniquely defined isomorphism such that $\delta^*\varphi = \psi\delta$ on the group G. To complete the proof, it is sufficient to prove that $\varphi\alpha = \alpha^*\varphi$ for every $\alpha \in E(G)$. First, let $\beta \in \operatorname{Hom}(G, D)$. Then $\beta = \gamma\delta$ for some $\gamma \in E(D)$. Consequently,

$$\psi\beta = \psi\gamma\delta = \gamma^*\psi\delta = \gamma^*\delta^*\varphi = \beta^*\varphi.$$

If $\alpha \in E(G)$, then $\delta\alpha \in \operatorname{Hom}(G, D)$. Therefore,

$$\delta^*\varphi\alpha = \psi\delta\alpha = (\delta\alpha)^*\psi = \delta^*\alpha^*\psi.$$

However, δ^* is an injective map. Therefore, $\varphi\alpha = \alpha^*\varphi$. □

Our intuition suggests that theorems on isomorphisms of endomorphism rings are quite possible if there are many endomorphisms that can be used for distinguishing between elements of the group.

Theorem 26.4 (May [306]). *Let G be a group containing isomorphic copies of the groups $\mathbf{Q}$ and $Z(p^\infty)$ for every p. For every group H, each topological isomorphism $E(G) \to E(H)$ is induced by some isomorphism $G \to H$.*

Proof. Let $\phi : E(G) \to E(H)$ be some topological isomorphism. We set

$$\tilde{G} = \sum_\lambda{}^{\oplus} G \qquad \text{and} \qquad \tilde{H} = \sum_\lambda{}^{\oplus} H,$$

where $\lambda = |G|$. Then the ring $E(\tilde{G})$ (resp. $E(\tilde{H})$) is naturally isomorphic to the ring of all column-convergent $\lambda \times \lambda$-matrices over $E(G)$ (resp., over $E(H)$) (Theorem 14.3). It follows from the definition of column-convergent matrices and the conditions for ϕ that ϕ induces the isomorphism $\tilde{\phi} : E(\tilde{G}) \to E(\tilde{H})$. Since the group $\tilde{G}$ can be embedded in its divisible part, Lemma 26.3 gives the isomorphism $\tilde{\varphi} : \tilde{G} \to \tilde{H}$

inducing $\tilde{\phi}$. Let G_0 and H_0 denote some summands of the groups $\tilde{G}$ and $\tilde{H}$ corresponding to a certain subscript, and let $\varepsilon : \tilde{G} \to G_0$ and $\omega : \tilde{H} \to H_0$ be projections. Then $\tilde{\phi}(\varepsilon) = \omega$ implies $\tilde{\varphi}\varepsilon = \omega\tilde{\varphi}$. Therefore, $\tilde{\varphi}G_0 \subseteq H_0$. By symmetry considerations, $\tilde{\varphi}^{-1}H_0 \subseteq G_0$. Let φ be the restriction $\tilde{\varphi}$ to G_0. Then φ induces ϕ. □

We present examples showing that the theorem is not true if the group $\mathbf{Q}$ or one of the groups $Z(p^\infty)$ is absent in G. First, we extend the notion of the finite topology to arbitrary homomorphism groups. Let A and B be two groups. For a finite subset $X \subseteq A$, we set $U_X = \{\varphi \in \mathrm{Hom}(A,B) \,|\, \varphi X = 0\}$. Here, U_X is a submodule of the left $E(B)$-module $\mathrm{Hom}(A,B)$, and $\mathrm{Hom}(A,B)$ is turned into a topological $E(B)$-module if its basis of neighborhoods of zero consists of submodules U_X for all finite subsets X and the finite topology is considered on the ring $E(B)$. The topological $E(B)$-module $\mathrm{Hom}(A,B)$ is a topological group such that the mapping

$$E(B) \times \mathrm{Hom}(A,B) \longrightarrow \mathrm{Hom}(A,B),\ (\alpha,\varphi) \longrightarrow \alpha\varphi$$

is continuous with respect to finite topologies on $E(B)$ and $\mathrm{Hom}(A,B)$. The following lemma is directly verified.

Lemma 26.5. *Let*

$$A = \sum_{i=1}^{m}{}^{\oplus} A_i \qquad \text{and} \qquad B = \sum_{j=1}^{n}{}^{\oplus} B_j$$

be groups. We naturally identify $\mathrm{Hom}(A,B)$ *with* $\prod_{i,j\geq 1} \mathrm{Hom}(A_i,B_j)$*. We take the finite topology on every group* $\mathrm{Hom}(A_i,B_j)$*. Then the finite topology on* $\mathrm{Hom}(A,B)$ *is the product topology on the product of these groups. Conversely, if the finite topology is defined on* $\mathrm{Hom}(A,B)$*, then the subspace topology on every group* $\mathrm{Hom}(A_i,B_j)$ *coincides with the finite topology.*

We recall that a basis of neighborhoods of zero of the product topology on the group $\prod_{i,j\geq 1} \mathrm{Hom}(A_i,B_j)$ consists of products $\prod_{i,j\geq 1} V_{ij}$, where V_{ij} runs over a basis of neighborhoods of zero of the group $\mathrm{Hom}(A_i,B_j)$. Details of the product topology are contained in §13 of Fuchs' work [164].

We need two torsion-free groups F and F' with a quite specific structure. We follow the construction of these groups presented in Example 5 in §88 and Theorem 88.4 of Fuchs' work [165]. Using primarily the notation in this construction, we fix a prime integer p, set $r = 2$, and construct the groups F and F' as some subgroups of the $\mathbf{Q}$-space $\mathbf{Q}a_1 \oplus \mathbf{Q}a_2$ with basis $\{a_1,a_2\}$. We choose two p-adic integers π and π' that are algebraically independent over $\mathbf{Q}$. We describe the construction of the group F starting from the number π; the construction of the group F' is similar to the construction of the group F (only primes are added to all letters, except for a_1, a_2, and p). Let

$$\pi = \sum_{i\geq 0} s_i p^i, \quad 0 \leq s_i < p, \qquad \text{and let} \quad \pi_n = \sum_{i=0}^{n-1} s_i p^i \quad \text{for } n \geq 0.$$

We denote by x_n a unique solution of the equation $p^n x_n = a_1 + \pi_n a_2$. We note that $x_0 = a_1$ and $px_{n+1} = x_n + s_n a_2$, since the group is torsion-free. We define F and F' by $F = \langle a_1, a_2, x_i,\ i \geq 1\rangle$ and $F' = \langle a_1, a_2, x_i',\ i \geq 1\rangle$. In Fuchs' work [165], it is shown that $E(F) = \mathbf{Z} = E(F')$ (this is easily verified; in fact, $\mathrm{Hom}(F,F') = 0 = \mathrm{Hom}(F',F)$). Before constructing examples, we need one more lemma.

Lemma 26.6 (May [306])**.** *Let D be a divisible group that is either a p-group or a group without p-component. Then there exists a topological isomorphism of left $E(D)$-modules* $\mathrm{Hom}(F,D) \cong \mathrm{Hom}(F',D)$.

Proof. We denote $B = \langle a_1, a_2\rangle$. First, assume that the group D is a group without p-component (i.e., D does not have elements of order p^n for all positive integers n). Since F/B is a p-group, $\mathrm{Hom}(F/B,D) = 0$, and it is possible to identify $\mathrm{Hom}(F,D)$ with $\mathrm{Hom}(B,D)$. The same is applicable to F'; therefore, the left $E(D)$-modules $\mathrm{Hom}(F, D)$ and $\mathrm{Hom}(F', D)$ are equal. To prove that this identification is topological, we take an element $f \in F$. Then $p^k f \in B$ for some k. If homomorphisms $\alpha \in \mathrm{Hom}(F, D)$ and $\alpha' \in \mathrm{Hom}(F', D)$ correspond to one another under the identification, then $p^k \alpha(f) = \alpha'(p^k f)$. Consequently, $\alpha(f) = 0$ if and only if $\alpha'(p^k f) = 0$, and the correspondence is topological.

Now assume that D is a p-group. Since the kernel of every homomorphism $F \to D$ contains the subgroup $p^k B$ for some k, we have

$$\mathrm{Hom}(F, D) = \bigcup_{k \geq 0} \mathrm{Hom}(F/p^k B, D),$$

where we have carried out natural identifications. A similar assertion holds for F'. To obtain the required correspondence between $\mathrm{Hom}(F, D)$ and $\mathrm{Hom}(F', D)$, it is sufficient to find a family of isomorphisms $\varphi_k : F/p^k B \to F'/p^k B$ commuting with the canonical mappings

$$\theta_k : F/p^{k+1}B \to F/p^k B \qquad \text{and} \qquad \theta'_k : F'/p^{k+1}B \to F'/p^k B$$

in the sense that $\varphi_k \theta_k = \theta'_k \varphi_{k+1}$ for all $k \geq 0$. Indeed, if we have such isomorphisms, then there is a family of induced isomorphisms

$$\varphi_k^* : \mathrm{Hom}(F'/p^k B, D) \longrightarrow \mathrm{Hom}(F/p^k B, D),$$

and

$$\theta_k^* \varphi_k^* = (\varphi_k \theta_k)^* = (\theta'_k \varphi_{k+1})^* = \varphi_{k+1}^* (\theta'_k)^*.$$

The relation $\theta_k^* \varphi_k^* = \varphi_{k+1}^* (\theta'_k)^*$ means that φ_{k+1}^* coincides with φ_k^* on $\mathrm{Hom}(F'/p^k B, D)$. Consequently, there exists an isomorphism $\mathrm{Hom}(F', D) \to \mathrm{Hom}(F, D)$ coinciding with φ_k^* on $\mathrm{Hom}(F'/p^k B, D)$ for all $k \geq 0$; this isomorphism is the required isomorphism (in fact, we deal here with direct spectra and induced mappings of direct limits).

We define new elements

$$x_{nk} = x_n + \sum_{i=0}^{k-1} s_{n+i} p^i a_2 \quad (k \geq 0),$$

and, similarly, x'_{nk}. We note that $x_{n0} = x_n$, $x_{0k} = a_1 + \pi_k a_2$, and $p x_{n+1,k} = x_{nk} + p^k s_{n+k} a_2$. Then $F = \langle a_2, x_{nk},\, n \geq 0 \rangle$. A similar relation holds for F'. Now we can define the mappings $\varphi_k : F/p^k B \to F'/p^k B$ by

$$\varphi_k(a_2 + p^k B) = a_2 + p^k B \qquad \text{and} \qquad \varphi_k(x_{nk} + p^k B) = x'_{nk} + p^k B \quad (n \geq 0).$$

These mappings are well-defined isomorphisms, since it is obvious that they have inverse homomorphisms. For the proof of the permutability of φ_k and φ_{k+1} with the canonical mappings θ_k and θ'_k, we note that $x_{n,k+1} = x_{nk} + p^k s_{n+k} a_2$. Therefore,

$$x_{n,k+1} + p^k B = x_{nk} + p^k B,$$

and the permutability is obvious.

Finally, we prove that the obtained correspondence between $\mathrm{Hom}(F, D)$ and $\mathrm{Hom}(F', D)$ is topological. Assume that $\alpha \in \mathrm{Hom}(F, D)$ corresponds to $\alpha' \in \mathrm{Hom}(F', D)$. Let $f \in F$. Then $f \in \langle a_2, x_n \rangle$ for some n. Assume that $\alpha'(a_2) = \alpha'(x'_n) = 0$. We choose k such that $\alpha(p^k B) = \alpha'(p^k B) = 0$; therefore, $\alpha' \varphi_k = \alpha$. It is clear that $\alpha(a_2) = \alpha'(a_2) = 0$ and

$$\alpha(x_n) = \alpha(x_{nk}) = \alpha' \varphi_k(x_{nk} + p^k B) = \alpha'(x'_{nk} + p^k B) = \alpha'(x'_n) = 0.$$

Therefore, $\alpha(f) = 0$. Using symmetry considerations, we complete the proof of the lemma. $\square$

We construct examples announced previously.

Example 26.7 (May [306]). (1) We set

$$D = \mathbf{Q} \oplus \sum_{q \neq p}^{\oplus} Z(q^\infty), \qquad G = F \oplus D \qquad \text{and} \qquad H = F' \oplus D.$$

Then $E(F) = E(F') = \mathbf{Z}$ and $F \not\cong F'$; therefore, $G \not\cong H$. The ring $E(G)$ is isomorphic to the matrix ring

$$\begin{pmatrix} \mathbf{Z} & 0 \\ \mathrm{Hom}(F, D) & E(D) \end{pmatrix}$$

and $E(H)$ is isomorphic to a similar matrix ring (only F is replaced by F'). Let $\phi : \mathrm{Hom}(F, D) \to \mathrm{Hom}(F', D)$ be one of the topological isomorphisms of left $E(D)$-modules whose existence is proved in Lemma 26.6. Using Lemma 26.5, it is directly verified that the correspondence

$$\begin{pmatrix} z & 0 \\ \alpha & \rho \end{pmatrix} \longrightarrow \begin{pmatrix} z & 0 \\ \phi(\alpha) & \rho \end{pmatrix}, \qquad z \in \mathbf{Z}, \quad \alpha \in \mathrm{Hom}(F, D), \quad \rho \in E(D)$$

defines a topological ring isomorphism $E(G) \cong E(H)$. Therefore, Theorem 26.4 is not true if one of the groups $Z(p^\infty)$ is absent.

(2) We set

$$D = \sum_{q \neq p}^{\oplus} Z(q^\infty), \qquad G = F \oplus Z(p^\infty) \oplus D, \qquad \text{and} \qquad H = F' \oplus Z(p^\infty) \oplus D.$$

As above, $G \not\cong H$. The ring $E(G)$ is isomorphic to the corresponding matrix ring considered as the following row:

$$E(G) \cong \mathbf{Z} \oplus \mathrm{Hom}(F, Z(p^\infty)) \oplus \mathrm{Hom}(F, D) \oplus E(Z(p^\infty) \oplus D).$$

A similar assertion holds for $E(H)$ (we only replace F by F'). Lemmas 26.5 and 26.6 imply that $E(G)$ and $E(H)$ are topologically isomorphic. Therefore, Theorem 26.4 is not true if the group $\mathbf{Q}$ is omitted.

Exercise 1. In connection with the existence of the topological isomorphism $E(G) \to E(H)$ from Example 26.7, prove the following more general property. Assume that A, A', and B are some groups such that $\mathrm{Hom}(B, A) = 0 = \mathrm{Hom}(B, A')$, and there are a topological ring isomorphism $\psi : E(A) \to E(A')$ and a topological isomorphism of left $E(B)$-modules $\phi : \mathrm{Hom}(A, B) \to \mathrm{Hom}(A', B)$ such that $\phi(\eta\alpha) = \phi(\eta)\psi(\alpha)$ for all $\eta \in \mathrm{Hom}(A, B)$ and $\alpha \in E(A)$. We set $G = A \oplus B$ and $G' = A' \oplus B$. Then the rings $E(G)$ and $E(G')$ are topologically isomorphic. We note that ϕ is called a semilinear isomorphism from the right $E(A)$-module $\mathrm{Hom}(A, B)$ onto the right $E(A')$-module $\mathrm{Hom}(A', B)$.

Exercise 2. If F and F' are the groups from Lemma 26.6, then prove that $E(F) \cong E(F') \cong \mathbf{Z}$ and $\mathrm{Hom}(F, F') = \mathrm{Hom}(F', F) = 0$.

Exercise 3. A reduced torsion-free group that is a p-adic module is determined by its endomorphism ring in the class of all torsion-free groups.

Exercise 4. Prove that for completely decomposable torsion-free groups A and B of finite rank, the following conditions are equivalent:

(a) $r(\mathrm{Hom}(A, C)) = r(\mathrm{Hom}(B, C))$ for every completely decomposable torsion-free group C of finite rank;
(b) $r(\mathrm{Hom}(C, A)) = r(\mathrm{Hom}(C, B))$ for every group C from (a);
(c) $A \cong B$.

Exercise 5. Let A be a reduced torsion-free group of finite rank, B be a reduced torsion-free group, and $r_p(\mathrm{Hom}(C, A)) = r_p(\mathrm{Hom}(C, B))$ for every p and each torsion-free group C such that $r_p(C) < \infty$ for all p. Prove that B has finite rank.

Exercise 6. Let A be a torsion-free group of finite rank, and let B be a group such that $\mathrm{Hom}(B, \mathbf{Z}) = 0$. If $r_p(A) = r_p(B)$ and $r_p(\mathrm{Hom}(A, C)) = r_p(\mathrm{Hom}(B, C))$ for all p and torsion-free groups C of finite rank, then B is a torsion-free group of finite rank.

Exercise 7. Let A and B be two torsion-free groups of finite rank. If $\operatorname{Hom}(A, C) \sim \operatorname{Hom}(B, C)$ for every torsion-free group C of finite rank, then prove that $A \sim B$. The converse assertion is also true.

27. Isomorphisms of Endomorphism Rings of Mixed Groups of Torsion-Free Rank 1

Theorem 26.4 contains a quite specific condition that guarantees the existence of an isomorphism theorem for a group. When searching for other possible isomorphism theorems, it is natural to consider mixed groups after torsion groups and torsion-free groups. A mixed group contains nonzero elements of finite order and elements of infinite order. Since isomorphism theorems exist quite rarely for torsion-free groups, the same is true for mixed groups. Indeed, two central questions have a negative answer even in the case of mixed groups of torsion-free rank 1 with a divisible factor torsion-free group. We mean the following questions. Is it true that an isomorphism $E(G) \cong E(H)$ implies an isomorphism $G \cong H$? Is every automorphism of the ring $E(G)$ inner? All the results proved in the section are taken from the work of May and Toubassi [312].

At the end of the section, we briefly list basic properties concerning the isomorphism problem of endomorphism rings of mixed Q_p-modules and Q_p^*-modules. The structure of these modules is simpler than the structure of mixed groups. Under certain restrictions, May obtained several isomorphism theorems for these modules. The use of topological isomorphisms of endomorphism rings is a particularity of the theorems.

As before, the torsion part of a group G is denoted by $T(G)$. If $M \subseteq G$, then $\langle M \rangle$ is the subgroup generated in G by the subset M. If G is a mixed group, then $G/T(G)$ is a torsion-free group called the *torsion-free part of the group G.* Its rank is called the *torsion-free rank of the mixed group G.* Therefore, if G is a mixed group of torsion-free rank 1, then $G/T(G)$ is isomorphic to some subgroup of the group $\mathbf{Q}$. We recall that $h_p(a)$ (or $h_p^A(a)$) denotes the p-height of an element a with respect to some group A. For an element a of the group A, we need a more general notion of the generalized p-height of a; it is denoted by $h_p^*(a)$. The generalized p-height arises in connection with subgroups $p^\sigma A$, where σ is an ordinal number. The subgroups $p^\sigma A$ are defined as follows: $p^0 A = A$, $p^{\sigma+1} A = p(p^\sigma A)$, and $p^\sigma A = \bigcap\limits_{\rho<\sigma} p^\rho A$ if σ is a limit ordinal number. Let τ be the least ordinal number with $p^{\tau+1} A = p^\tau A$. It is clear that $p^\tau A$ is a maximal p-divisible subgroup of the group A. If an element a is not contained in $p^\tau A$, then there exists a unique ordinal number σ such that $a \in p^\sigma A \setminus p^{\sigma+1} A$; we set $h_p^*(a) = \sigma$. In the case $a \in p^\tau A$, we set $h_p^*(a) = \infty$ and assume that ∞ exceeds every ordinal number. Obviously, this notion is more delicate than the notion of p-height; it allows one to distinguish elements of infinite height. The ascending sequence $(h_p^*(a), h_p^*(pa), \dots, h_p^*(p^n a), \dots)$ of ordinal numbers and symbols ∞ is called the *p-indicator of the element a.*

We consider some special notation. Let G be a mixed group of torsion-free rank 1, x be one of its elements of infinite order, and p be a prime integer. We define

$$I(G) = \{\alpha \in E(G) \mid \alpha G \subseteq T(G)\}, \qquad I_p(G) = \{\alpha \in I(G) \mid \alpha G \subseteq G_p\}$$

and

$$I_p(G, x) = \{\alpha \in I_p(G) \mid \alpha x = 0\}.$$

Here, $I(G)$ and $I_p(G)$ are ideals and $I_p(G, x)$ is a left ideal of the ring $E(G)$. We denote by $P(G)$ the set of all prime integers p such that the p-indicator of the element x contains the symbol ∞. We note that this definition does not depend on the choice of a concrete element x of infinite order. We formulate our principal result.

Theorem 27.1 (May and Toubassi [312]). *Let G and H be two mixed groups of torsion-free rank 1, and let $x \in G$ and $y \in H$ be some elements of infinite order. If $E(G)$ and $E(H)$ are isomorphic rings, then $P(G) = P(H)$. In this case, there is a one-to-one correspondence between isomorphisms $E(G) \to E(H)$ and families of ring isomorphisms $\varphi_p : I_p(G) \to I_p(H)$ such that $\varphi_p(I_p(G, x)) = I_p(H, y)$ for almost all p.*

This theorem follow directly from three propositions that we now prove. We prepare for the first proposition. We assume that the groups G and H satisfy the conditions of the theorem.

Let $\phi : E(G) \to E(H)$ be a ring isomorphism. The Kaplansky method used in the proof of Theorem 26.1 is also applicable in this case. One should account forthe fact that the groups G and H do not have direct summands that are isomorphic to one of the groups of p-adic integers I_p. Therefore, there exists an isomorphism $\varphi : T(G) \to T(H)$ such that $\phi(\alpha)_{|T(H)} = \varphi\alpha\varphi^{-1}$ for every $\alpha \in E(G)$. Since the proof of assertion (1) of Theorem 26.1 uses only mappings on torsion subgroups, the isomorphism φ exists if we have some isomorphism $\phi : I(G) \to I(H)$ and if $\alpha \in I(G)$. We use the isomorphism φ to identify $T(G)$ with $T(H)$ in this section throughout. We denote the common torsion subgroup of the groups G and H by T. If ϕ is some isomorphism of one of the types described above, then the image $\phi(\alpha)$ is usually denoted by α^*. Therefore, the mappings α and α^* act on the group T.

We need one simple lemma. For a positive integer m, we set $T_m = \sum\limits_{p|m}^{\oplus} T_p$ and $\operatorname{supp}(m) = \{p \,|\, m \vdots p\}$.

Lemma 27.2. *Let m be an integer such that $\operatorname{supp}(m) \subseteq P(G)$. Then $G = F_m \oplus T_m$, where F_m is some m-divisible subgroup of the group G (the m-divisibility means that $mF_m = F_m$).*

Proof. Let A be the maximal m-divisible subgroup of the group G. It is clear that $A \cap T_m$ is the maximal divisible subgroup of the group T_m. We denote it by D_m. We choose a group F_m such that $A = F_m \oplus D_m$. To complete the proof, it remains to prove that $G = A + T_m$. To do this, it is sufficient to prove that $A + T_m$ contains every element $x \in G$ of infinite order. There exists a positive integer n such that $h_p(nx) = \infty$ for every $p \in \operatorname{supp}(m)$ and $\operatorname{supp}(n) \subseteq \operatorname{supp}(m)$. Therefore, $nx \in A$, whence $nx = na$ for some $a \in A$. Therefore, $x - a \in T_m$ and $x \in A + T_m$. □

Proposition 27.3. *Let $\phi : E(G) \to E(H)$ be an isomorphism. Then $\phi(I(G)) = I(H)$ and $P(G) = P(H)$.*

Proof. By symmetry, it is sufficient to prove that $\phi(I(G)) \subseteq I(H)$ and $P(G) \subseteq P(H)$. First, assume that H has a subgroup isomorphic to the group of rational numbers $\mathbf{Q}$. In this case, H splits as a mixed group. Consequently, there exists an idempotent $\varepsilon \in E(G)$ such that $\varepsilon^* H = T$. Thus, ε is an idempotent of the ring $E(G)$ acting as the identity mapping on T. Since $\varepsilon \neq 1$, we obtain $\varepsilon G = T$. If $\alpha \in I(G)$, then $\alpha = \varepsilon\alpha$; therefore, $\alpha^* = \varepsilon^*\alpha^* \in I(H)$. Therefore, we can assume that every divisible subgroup of the group H is a torsion group. We fix some elements $x \in G$ and $y \in H$ of infinite order. We set $I(G, x) = \{\alpha \in I(G) \,|\, \alpha x = 0\}$. The group $I(G, x)$ is naturally isomorphic to the group $\operatorname{Hom}(G/\langle x\rangle, T(G))$ (we note that $I(G) = \operatorname{Hom}(G, T(G))$. Since $G/\langle x\rangle$ is a torsion group, the groups $\operatorname{Hom}(G/\langle x\rangle, T(G))$ and $I(G, x))$ are algebraically compact (Fuchs [164, Theorem 46.1]). We define the homomorphism

$$\psi : I(G, x) \to H \quad \text{as} \quad \psi(\beta) = \beta^*(y) \quad \text{for} \quad \beta \in I(G, x)$$

and set $A = \psi(I(G, x))$. Then the torsion-free rank of the group A/A^1 does not exceed 1 and the group is algebraically compact as a homomorphic image of the algebraically compact group $I(G, x)$, where $A^1 = \bigcap\limits_{n \in \mathbf{N}} nA$ is the first Ulm subgroup of the group A. Taking into account the structure of algebraically compact groups (see Sec. 2 and Fuchs [164, § 40]), we obtain that A/A^1 is a bounded group. It is easy to see that the group A is a direct sum of a divisible group and a bounded group. Since every divisible subgroup of the group H is a torsion group, $A \subseteq T$. Let $\alpha \in I(G)$. Then $n\alpha \in I(G, x)$ for some positive integer n, whence $n\alpha^*(y) \in T$. Therefore $\alpha^* \in I(H)$. We have proved that $\phi(I(G)) \subseteq I(H)$.

Now assume that $p \in P(G)$. Then $G = F_p \oplus T_p$ by Lemma 27.2, and the multiplication by p is an automorphism of the group F_p. Let ε be the projection G on F_p. Then $\varepsilon^* H$ is a summand of the group H, where the multiplication by p is an automorphism. Since $1 - \varepsilon \in I(G)$, we obtain that $1 - \varepsilon^* \in I(H)$ by the above argument. Consequently, $\varepsilon^* H$ contains elements of infinite order, and we obtain that $p \in P(H)$. □

Lemma 27.4. *A prime integer p belongs to $P(G)$ if and only if there exists an endomorphism $\alpha \in E(G)$ inducing the multiplication by the rational number $1/p$ on the factor group G/T.*

Proof. First, assume that such α exists. Let an element $x \in G$ have an infinite order. Then $\alpha(px) = x + t$ for some $t \in T$. If $n = 0(t)$, then $\alpha(pnx) = nx$; therefore $h_p(nx) = \infty$. Therefore, $p \in P(G)$. Conversely, assume that $p \in P(G)$. Lemma 27.2 implies that $G = F_p \oplus T_p$. Since the multiplication by p is an automorphism of the group G_p, there exists $\alpha \in E(G)$ inducing the multiplication by $1/p$ on G/T. $\square$

Proposition 27.5. *Assume that $\phi : I(G) \to I(H)$ is a ring isomorphism and $P(G) = P(H)$. Then ϕ can be uniquely extended to a ring isomorphism $E(G) \to E(H)$.*

Proof. First, we show that if ϕ is extended to the ring $E(G)$, then this extension is unique. Assume that ϕ_1 and ϕ_2 are two such extensions. Since $(\phi_1 - \phi_2)I(G) = 0$, we have that $\phi_1 - \phi_2$ induces some homomorphism

$$E(G)/I(G) \longrightarrow E(H)/I(H).$$

However, $E(G)/I(G)$ and $E(H)/I(H)$ can be considered as subgroups in $E(G/T)$ and $E(H/T)$, respectively; the last groups are torsion-free groups of rank 1. Therefore, $\phi_1 - \phi_2 = 0 \iff (\phi_1 - \phi_2)(\alpha) = 0$ for some $\alpha \notin I(G)$. Since $(\phi_1 - \phi_2)(1) = 0$, we obtain $\phi_1 = \phi_2$.

We prove that ϕ can be extended. To define our mapping, we need some preparations. For every positive integer m with $\operatorname{supp}(m) \subseteq P(G)$, we have a decomposition $G = F_m \oplus T_m$ by Lemma 27.2. We denote by ε_m the natural projection onto T_m. Since ε_m^* is also an idempotent in $I(H)$, it gives a similar decomposition of the group H with $\varepsilon_m^*(H) = T_m$. For any integers n and m as above, we have the mapping $\rho(n, m) \in E(G)$ inducing multiplication by the fraction n/m on G/T. The multiplication by m is invertible on F_m. Therefore, we set $\rho(n, m) = (n/m)(1 - \varepsilon_m)$. Since $P(G) = P(H)$, we can similarly define $\rho'(n, m) \in E(H)$: $\rho'(n, m) = (n/m)(1 - \varepsilon_m^*)$. We construct $\Theta : E(G) \to E(H)$. Let $\alpha \in E(G)$. Assume that α induces multiplication by the rational number n/m on the group G/T. By Lemma 27.4, we can assume that $\operatorname{supp}(m) \subseteq P(G)$. Since $\alpha - \rho(n, m)$ induces the zero mapping on G/T, we have $\alpha = \rho(n, m) + \beta$ for some $\beta \in I(G)$. We set $\Theta(\alpha) = \rho'(n, m) + \beta^*$. We prove that Θ is a ring homomorphism; in this case, it is an isomorphism, since symmetry considerations imply that Θ has an inverse homomorphism.

We begin with the proof of the property that our mapping is well defined. For this purpose, we assume that $\alpha = \rho(n_1, m_1) + \beta_1 = \rho(n_2, m_2) + \beta_2$. We denote

$$\rho_i = \rho(n_i, m_i), \qquad \rho_i' = \rho'(n_i, m_i) \quad (i = 1, 2).$$

We have to prove that $(\rho_2 - \rho_1)^* = \rho_2' - \rho_1'$. Since ρ_1 and ρ_2 induce the same mapping on G/T, we have $n_1/m_1 = n_2/m_2$. Let $n_1/m_1 = n/m$, where $(n, m) = 1$. We need the following assertion (*): if γ is an endomorphism of H such that $m\gamma = 0$ and $\gamma\varepsilon_m^* = 0$, then $\gamma = 0$. This follows from the property that $\gamma\varepsilon_m^* = 0$ implies that γH is an m-divisible group. Let $\gamma = (\rho_2 - \rho_1)^* - (\rho_2' - \rho_1')$. Then

$$m\gamma = (m\rho_2 - m\rho_1)^* - (m\rho_2' - m\rho_1') = (n\varepsilon_{m_2} - n\varepsilon_{m_1})^* - (n\varepsilon_{m_2}^* - n\varepsilon_{m_1}^*) = 0.$$

Taking into account that $\varepsilon_m = \varepsilon_{m_i}\varepsilon_m$ $(i = 1, 2)$, we have

$$\gamma\varepsilon_m^* = (\rho_2\varepsilon_m - \rho_1\varepsilon_m)^* - (\rho_2'\varepsilon_m^* - \rho_1'\varepsilon_m^*)$$

$$= \left(\frac{n_2}{m_2}(1 - \varepsilon_{m_2})\varepsilon_{m_2}\varepsilon_m - \frac{n_1}{m_1}(1 - \varepsilon_{m_1})\varepsilon_{m_1}\varepsilon_m\right)^* - \left(\frac{n_2}{m_2}(1 - \varepsilon_{m_2}^*)\varepsilon_{m_2}^*\varepsilon_m^* - \frac{n_1}{m_1}(1 - \varepsilon_{m_1}^*)\varepsilon_{m_1}^*\varepsilon_m^*\right) = 0.$$

Therefore, $(\rho_2 - \rho_1)^* = \rho_2' - \rho_1'$.

Let $\alpha_1, \alpha_2 \in E(G)$. We choose integers n_1, n_2, and m such that α_i induces n_i/m on G/T $(i = 1, 2)$. Then $\alpha_i = \rho(n_i, m) + \beta_i$ for some $\beta_i \in I(G)$. The mapping Θ is additive, since

$$\Theta(\alpha_1 + \alpha_2) = \Theta\left(\frac{n_1 + n_2}{m}(1 - \varepsilon_m) + (\beta_1 + \beta_2)\right) = \frac{n_1 + n_2}{m}(1 - \varepsilon_m^*) + (\beta_1 + \beta_2)^* = \Theta(\alpha_1) + \Theta(\alpha_2).$$

We prove that Θ is multiplicative. We set $\rho_i = \rho(n_i, m)$ and $\rho'_i = \rho'(n_i, m)$. First,

$$\Theta(\rho_1\rho_2) = \Theta\left(\frac{n_1}{m}(1-\varepsilon_m)\frac{n_2}{m}(1-\varepsilon_m)\right) = \Theta\left(\frac{n_1n_2}{m^2}(1-\varepsilon_m)\right) = \frac{n_1n_2}{m^2}(1-\varepsilon_m^*) = \rho'_1\rho'_2.$$

Taking into account this relation, it remains to prove that $(\beta_1\rho_2)^* = \beta_1^*\rho'_2$ and $(\rho_1\beta_2)^* = \rho'_1\beta_2^*$. We set $\gamma = (\beta_1\rho_2)^* - \beta_1^*\rho'_2$. Now

$$m\gamma = (\beta_1 n_2(1-\varepsilon_m))^* \; - \; \beta_1^* n_2(1-\varepsilon_m^*) = 0$$

and

$$\gamma\varepsilon_m^* = (\beta_1\rho_2\varepsilon_m)^* \; - \; \beta_1^*\rho'_2\varepsilon_m^* = 0 - 0 = 0.$$

Therefore, $(\beta_1\rho_2)^* = \beta_1^*\rho'_2$ by assertion (*). Similarly, $(\rho_1\beta_2)^* = \rho'_1\beta_2^*$, and the proof is completed. □

To examine the local action of the isomorphism ϕ, we need the following lemma.

Lemma 27.6. *Let $\phi : I(G) \to I(H)$ be an isomorphism, $\alpha \in I(G)$, and let π_p be the projection of the group T on T_p. Then $\phi(\pi_p\alpha) = \pi_p\phi(\alpha)$.*

Proof. Assume that $\beta \in I(H)$ and $\beta(T_q) = 0$ for some q. We prove the following assertion. There exists $\varepsilon \in I(G)$ such that $\varepsilon_{|T} = \varepsilon\pi_q$, $\phi(\varepsilon)_{|T} = \phi(\varepsilon)\pi_q$, and $\pi_q\beta = \phi(\varepsilon)\beta$. We denote

$$T_{-q} = \sum_{p\neq q}^{\oplus} T_p.$$

First, assume that G/T_{-q} splits as a mixed group. Then there exists a mapping $\varepsilon : G \to T$ such that $\varepsilon_{|T} = \pi_q$. Therefore, $\phi(\varepsilon)_{|T} = \pi_q$, and the assertion is easily verified. Now we assume that G/T_{-q} does not split. In this case, the factor group G/T is q-divisible. It follows from $\pi_q\beta(T) = 0$ that $\pi_q\beta(H)$ is a divisible subgroup of the group T_q. Let $\varepsilon : G \to T$ be the projection on this subgroup. Then $\phi(\varepsilon)$ acts as the identity mapping on it; therefore, $\pi_q\beta = \phi(\varepsilon)\pi_q\beta$. Now the assertion is verified in this case.

To prove the lemma, we set $\beta = \phi(\pi_p\alpha) - \pi_p\phi(\alpha)$. Here, $\beta \in I(H)$ and $\beta(T_q) = 0$ for every prime integer q. It is sufficient to prove that $\pi_q\beta = 0$ for every q. We apply the above assertion with $q \neq p$. We have

$$\pi_q\beta = \phi(\varepsilon)\beta = \phi(\varepsilon\pi_p\alpha) - \phi(\varepsilon)\pi_p\phi(\alpha) = \phi(\varepsilon\pi_q\pi_p\alpha) - \phi(\varepsilon)\pi_q\pi_p\phi(\alpha) = 0.$$

Applying the assertion with $q = p$, we obtain

$$\pi_q\beta = \phi(\varepsilon)\beta = \phi(\varepsilon\pi_p\alpha) - \phi(\varepsilon)\pi_p\phi(\alpha) = \phi(\varepsilon\alpha) - \phi(\varepsilon)\phi(\alpha) = 0.$$

□

Proposition 27.7. *Let $x \in G$ and $y \in H$ be two elements of infinite order. Every isomorphism $\phi : I(G) \to I(H)$ induces the isomorphism $\varphi_p : I_p(G) \to I_p(H)$ for every p such that $\varphi_p(I_p(G,x)) = I_p(H,y)$ for almost all p. This correspondence is a bijection between the isomorphisms ϕ and families of isomorphisms $\{\varphi_p \,|\, p$ is a prime integer$\}$ satisfying the above conditions.*

Proof. We take some isomorphism ϕ. If $\alpha \in I_p(G)$, then $\phi(\alpha) = \phi(\pi_p\alpha) = \pi_p\phi(\alpha) \in I_p(H)$. Therefore, the restriction ϕ to $I_p(G)$ is the mapping $I_p(G) \to I_p(H)$, which is denoted by φ_p. We prove the condition on the family $\{\varphi_p\}$. Let P be some set of prime integers. Assume that $\alpha_p \in I_p(G,x)$ $(p \in P)$ satisfy the relations $\varphi_p(\alpha_p) \notin I_p(H,y)$ for every $p \in P$. Since $G/\langle x\rangle$ is a torsion group, there exists $\alpha \in I(G,x)$ such that $\pi_p\alpha = \alpha_p$ for all $p \in P$. Therefore,

$$\pi_p\phi(\alpha)(y) = \phi(\pi_p\alpha)(y) = \varphi_p(\alpha_p)(y) \neq 0$$

for every $p \in P$. We obtain that the set P is finite. Consequently,

$$\varphi_p(I_p(G,x)) \subseteq I_p(H,y)$$

for almost all p. By symmetry, we obtain the first assertion of the proposition.

Let $\{\varphi_p\}$ be a family of isomorphisms such that $\varphi_p(I_p(G,x)) = I_p(H,y)$ for almost every p. We define the mappings

$$g : I(G) \to \prod_p I_p(G) \qquad \text{and} \qquad h : I(H) \to \prod_p I_p(H),$$

where $g(\alpha) = (\pi_p\alpha)$ and $h(\beta) = (\pi_p\beta)$. We note that g and h are ring monomorphisms. We also define the ring isomorphism

$$\prod \varphi_p : \prod_p I_p(G) \longrightarrow \prod_p I_p(H)$$

by "gluing" all the isomorphisms φ_p. If $\alpha \in I(G)$, then $\pi_p\alpha(x) = 0$ for almost all p. Since $H/\langle my\rangle$ is a torsion group, there exists a $\beta \in I(H)$ such that $\pi_p\beta = \varphi_p(\pi_p\alpha)$ for every p. Therefore

$$(\Pi\varphi_p)g(\alpha) = (\varphi_p(\pi_p\alpha)) = h(\beta).$$

Consequently, $(\Pi\varphi_p)g(I(G)) \subseteq h(I(H))$. By symmetry, $(\Pi\varphi_p)g(I(G)) = h(I(H))$. Therefore, we have a ring isomorphism $\phi : I(G) \to I(H)$ such that $(\Pi\varphi_p)g = h\phi$. Consequently, if $\alpha \in I_p(G)$, then

$$\varphi_p(\alpha) = \varphi_p(\pi_p\alpha) = \pi_p\phi(\alpha) = \phi(\pi_p\alpha) = \phi(\alpha).$$

This means that ϕ induces the family $\{\varphi_p\}$ given previously. It remains to prove that the correspondence is injective. If an isomorphism ϕ' also induces the family $\{\varphi_p\}$, then $\phi'(\pi_p\alpha) = \phi(\pi_p\alpha)$ for every $\alpha \in I(G)$. Therefore, $\pi_p\phi'(\alpha) = \pi_p\phi(\alpha)$ for all p and α. Therefore, $\phi = \phi'$. $\square$

Before constructing the examples mentioned at the beginning of the section, we define the notion of the height matrix $H(a)$ of an element a of a group A. The height matrix combines the information obtained by using the p-indicator (see the beginning of the section) of an element for all prime numbers p.

Let $p_1, \dots, p_n, \dots$ be the sequence of all prime numbers considered in ascending order. We associate with the element a of the group A the following $\omega \times \omega$-matrix:

$$\begin{pmatrix} h^*_{p_1}(a) & h^*_{p_1}(p_1a) & \dots & h^*_{p_1}(p_1^k a) & \dots \\ \dots & \dots & \dots & \dots & \dots \\ h^*_{p_n}(a) & h^*_{p_n}(p_na) & \dots & h^*_{p_n}(p_n^k a) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} = (\sigma_{nk})$$

whose elements are ordinal numbers or the symbol ∞. The element σ_{nk} at the nth row and kth column of the matrix $H(a)$ coincides with the generalized p_n-height of the element $p_n^k a$, and the nth row is the p_n-indicator of the element a defined in this section. The above matrix is called the height matrix $H(a)$ of the element $a \in A$ in the group A. The following remark follows directly from the definition: the matrix $H(p_na)$ is obtained from the matrix $H(a)$ by replacing the nth row $\sigma_{n0}, \dots, \sigma_{nk}, \dots$ of the matrix $H(a)$ by the row $\sigma_{n1}, \dots, \sigma_{n,k+1}, \dots$. We take two elements a and b of infinite order in the group A. Assume that $sa = tb$ for some nonzero integers s and t. It follows from the remark that the nth rows of the matrices $H(a) = (\sigma_{nk})$ and $H(b) = (\rho_{nk})$ can differ from one another only in the case of $p_n \mid st$, and for this p_n there are nonzero integers l and m such that the following condition holds: $\sigma_{n,k+l} = \rho_{n,k+m}$ for all k. Now two $\omega \times \omega$-matrices (σ_{nk}) and (ρ_{nk}) are said to be *equivalent* if the nth rows of both matrices coincide for almost all n, and for all remaining n there exist integers $l, m \geq 0$ (depending on n) such that the above condition holds. If the torsion-free rank of the group A is equal to 1, then any two elements $a, b \in A$ of infinite order depend on one another; therefore, their height matrices $H(a)$ and $H(b)$ are equivalent.

Example 27.8. There exist nonisomorphic mixed groups G and H of torsion-free rank 1 such that $E(G) \cong E(H)$ and $G/T(G) \cong H/T(H) \cong \mathbf{Q}$.

Proof. As G, we take the group generated by an element x and the family of elements $\{x_{pi} \mid p$ is a prime integer, $i \geq 1\}$ with the relations $p^i x_{pi} = x$ for every p and i. The group H is generated by an element y

and the family of elements $\{y_p, y_{pi} \mid p \text{ is a prime integer, } i \geq 1\}$ with the relations $py_p = y$ and $p^i y_{pi} = y_p$ for every p and i. We set

$$t_{pi} = x_{pi} - px_{p,i+1} \qquad \text{and} \qquad s_{pi} = y_{pi} - py_{p,i+1}.$$

Then t_{pi} and s_{pi} are torsion elements of order p^i and

$$T(G)_p = \sum_i{}^{\oplus} \langle t_{pi} \rangle, \qquad T(H)_p = \sum_i{}^{\oplus} \langle s_{pi} \rangle.$$

It is easy to verify that

$$h_p^*(x) = h_p^*(y) = \omega \qquad \text{and} \qquad p^\omega(T(G)_p) = 0,\ p^\omega(T(H)_p) = 0$$

for every p. In addition, $G/T(G) \cong H/T(H) \cong \mathbf{Q}$.

Let p be a prime integer. For endomorphisms $\alpha \in I_p(G)$ and $\beta \in I_p(H)$, the consideration of heights implies that $\alpha(x) = 0$ and $\beta(y_p) = 0$. Then $\beta(y_q) = 0$ for every $q \neq p$. We denote

$$\bar{G} = G/\langle x \rangle \quad \text{and} \quad \bar{H} = H/\langle y_q \mid q \text{ is a prime integer} \rangle.$$

Further, let $\overline{T(G)}_p$ and $\overline{T(H)}_p$ be the images of the subgroups $T(G)_p$ and $T(H)_p$ in the groups $\bar{G}$ and $\bar{H}$ with respect to the canonical homomorphisms $G \to \bar{G}$ and $H \to \bar{H}$, respectively. We also note that $I_p(G, x) = I_p(G)$ and $I_p(H, y) = I_p(H)$ by the first statement of this paragraph. Then we have the natural ring isomorphisms

$$I_p(G) \cong \mathrm{Hom}(\bar{G}, \overline{T(G)}_p) \qquad \text{and} \qquad I_p(H) = \mathrm{Hom}(\bar{H}, \overline{T(H)}_p).$$

It is clear that there exists an isomorphism $\bar{G} \to \bar{H}$ such that $\bar{x}_{qi} \to \bar{y}_{qi}$ (q is a prime integer and $i \geq 1$). This isomorphism maps from $\overline{T(G)}_p$ onto $\overline{T(H)}_p$; therefore, it induces the ring isomorphism $\varphi_p : I_p(G) \to I_p(H)$ for every p. By Theorem 27.1, the family $\{\varphi_p\}$ induces the ring isomorphism $E(G) \to E(H)$ (we note that $P(G) = P(H) = \varnothing$). It remains to mention that the groups G and H are not isomorphic, since the height matrices $H(x)$ and $H(y)$ of the elements x and y are not equivalent. ☐

Example 27.9. There exists a mixed group G of torsion-free rank 1 whose endomorphism ring has outer automorphisms (i.e., $E(G)$ has automorphisms that are not inner).

Proof. Let G be the group generated by an element x and a family of elements $\{z_p \mid p \text{ is a prime integer}$ $(p > 3)\}$ satisfying the relations $px = p^2 z_p$ for every $p > 3$. We set $t_p = x - pz_p$. The element t_p has order p, and

$$T(G) = \sum_{p>3}{}^{\oplus} \langle t_p \rangle.$$

For every $p > 3$, we define $\Theta_p : G \to G$ by using the relations $\Theta_p(x) = x + t_p$ and $\Theta_p(z_q) = z_q$ ($q > 3$). It is easy to prove that Θ_p is an automorphism of the group G with $\Theta_p(t_p)2t_p$. Further, we define the ring isomorphism

$$\varphi_p : I_p(G) \to I_p(G) \qquad \text{as} \quad \varphi_p(\alpha) = \Theta_p \alpha \Theta_p^{-1}.$$

We note that if $\alpha \in I_p(G)$, then $\alpha(pz_p) = 0$. Therefore, $\alpha(\Theta_p(x)) = \alpha(x + t_p) = 2\alpha(x)$. We have

$$I_p(G, x) = I_p(G, \Theta_p(x)) = \varphi_p(I_p(G, x))$$

for every $p > 3$. Theorem 27.1 implies that the family $\{\varphi_p\}$ induces some automorphism of the ring $E(G)$. This automorphism is denoted by $*$. We note that if $\alpha \in I_p(G)$, then $\alpha^* = \Theta_p \alpha \Theta_p^{-1}$.

Now assume that there exists an automorphism $\Theta : G \to G$ inducing the automorphism $*$ of the ring $E(G)$ (i.e., $\alpha^* = \Theta^{-1} \alpha \Theta$ for all $\alpha \in E(G)$). Since the group G has torsion-free rank 1, there exist two nonzero integers m and n with $m\Theta(x) = nx$. The height matrices of the elements $\Theta(x)$ and x are equal; therefore, $m = \pm n$. Therefore, we have $\Theta(x) = \pm x + s$ for some $s \in T(G)$. We fix some prime integer p

that does not divide the order of the element s. As above, we have $\Theta(z_p) = \pm z_p + s'$ for some $s' \in T(G)$. We assert that $\Theta(t_p) = \pm t_p$. We have

$$\Theta(t_p) = \Theta(x - pz_p) = \pm t_p + kz_p + s - ps',$$

where k is an integer. Since $\Theta(t_p) \in T(G)_p$, we have $k = 0$. Since p does not divide the order of the element $s - ps'$, we have $\Theta(t_p) = \pm t_p$. We now define $\alpha \in I_p(G)$ by $\alpha(x) = 0$, $\alpha(z_q) = 0$ $(q > 3,\ q \neq p)$, and $\alpha(z_p) = t_p$. We note that $\alpha(t_q) = 0$ for all $q > 3$. Since

$$\alpha^* \Theta_p = \Theta_p \alpha \qquad \text{and} \qquad \Theta\alpha^* = \alpha\Theta,$$

we have $\Theta\Theta_p\alpha = \alpha\Theta\Theta_p$. Applying both these endomorphisms to z_p, we obtain

$$\Theta\Theta_p\alpha(z_p) = \pm 2t_p \qquad \text{and} \qquad \alpha\Theta\Theta_p(z_p) = \pm t_p.$$

However, the relation $\pm 2t_p = \pm t_p$ is impossible, since $p > 3$; this is a contradiction. $\square$

The presented examples show that isomorphism theorems do not hold for mixed groups of torsion-free rank 1. In this case, we can enlarge the problem and seek isomorphism conditions of endomorphism rings of two given mixed groups G and H of torsion-free rank 1. Toubassi and May [421] solved this problem for the group G with simply presented torsion subgroup $T(G)$. For such a group G, they also obtained a necessary condition which is satisfied provided every automorphism of the ring $E(G)$ is inner.

Our intuition suggests that the existence of isomorphism theorems is more expectable in the case of Q_p-modules. In fact, a Q_p-module G is an Abelian group such that $nG = G$ for all integers n with $(n, p) = 1$. Q_p-homomorphisms coincide with $\mathbf{Z}$-homomorphisms. May and Toubassi [313] proved the following theorem about such modules. Let M and N be two mixed Q_p-modules of torsion-free rank 1, and let the torsion submodule $T(M)$ be simply presented. Then every isomorphism of endomorphism rings $E(M)$ and $E(N)$ is induced by some isomorphism of modules M and N. Unfortunately, this theorem cannot be extended to mixed modules of other torsion-free ranks, even if we assume that the modules are reduced and their torsion-free factor modules are divisible. In the work of Göbel and May [180], counterexamples for all finite torsion-free ranks $r \geq 2$ were found. They proved that for every fixed $r \geq 2$, there exists a continuum of pairwise nonisomorphic reduced Q_p-modules G_i with isomorphic endomorphism rings. In addition,

$$G_i/T(G_i) \cong \sum_r{}^{\oplus}\mathbf{Q} \qquad \text{and} \qquad T(G_i) \cong \sum_{k \geq 1}{}^{\oplus} Z(p^k)$$

for all i (see also Sec. 30).

It is well known that modules over the completion Q_p^* of the ring Q_p with respect to the p-adic topology (i.e., p-adic modules) have a simpler structure than Q_p-modules. Seeking isomorphism theorems for mixed modules, it is natural to consider p-adic modules. We note that homomorphisms of reduced p-adic modules coincide with ordinary group homomorphisms (see Exercise 1 of Sec. 6 for this topic).

May [307] considered the isomorphism problem for endomorphism rings of p-adic modules. He proved the following theorem. Let G and H be two reduced torsion-free p-adic modules of finite rank. Assume that the module $T(G)$ is totally projective. Then every isomorphism $E(G) \cong E(H)$ is induced by some isomorphism $G \cong H$. Simple examples show that the rank restriction cannot be completely removed. May [307] constructed an unexpected example. It turns out that there exist a separable p-group T and an infinite family (G_i) of pairwise nonisomorphic reduced p-adic modules of torsion-free rank 1 such that $T(G_i) \cong T$ and $E(G_i) \cong E(G_j)$ for all i and j. This example is a serious obstacle for seeking isomorphism theorems. Therefore, we study topological isomorphisms of endomorphism rings. We deal with the Baer–Kaplansky theorem (Theorem 24.1) and the Wolfson theorem (Exercise 2 of Sec. 24). There exists one common property of these theorems: the property that isomorphisms of endomorphism rings are actually topological isomorphisms. In both cases, the finite topology arises in the endomorphism ring itself. For reduced torsion groups, this property is mentioned at the beginning of Sec. 25; standard properties of these modules imply that the same considerations are applicable to torsion-free modules. There is a mixed module such that the finite topology of its endomorphism ring cannot be defined in terms of the endomorphism ring without consideration the original group. Therefore, it is natural that the finite topology on endomorphism rings of mixed p-adic modules is useful for isomorphism theorems.

Exercise 1. Let G be a mixed group, and let $\{x_i \mid i \in I\}$ be a system of its elements such that $\{x_i + T(G) \mid i \in I\}$ is a maximal linearly independent system of elements of the group $G/T(G)$. We set $I(G, x_i) = \{\alpha \in E(G) \mid \alpha G \subseteq T(G)$ and $\alpha x_i = 0$ for all $i \in I\}$. Prove that $I(G, x_i)$ is a left ideal of the ring $E(G)$ and

$$I(G, x_i) = \mathrm{Hom}\Big(G/\sum_{i\in I}\langle x_i\rangle,\, T(G)\Big);$$

therefore, $I(G, x_i)$ is an algebraically compact group.

Exercise 2. In the situation of Exercise 1, we set

$$I_p(G, x_i) = \{\alpha \in I(G, x_i) \mid \alpha G \subseteq T(G)_p\}.$$

Prove that

$$I(G, x_i) = \prod_p I_p(G, x_i).$$

Exercise 3. Let G be a p-adic module containing a direct summand that is isomorphic to Q_p^*. Prove that for every p-adic module H, each topological isomorphism $E(G) \to E(H)$ is induced by some isomorphism $G \to H$. If G and H are two reduced modules, then the assertion holds for every isomorphism $E(G) \cong E(H)$.

Exercise 4. Find nonisomorphic groups whose endomorphism rings are isomorphic regular rings.

28. The Corner Theorem on Split Realization

In the works of Liebert [289] and Pierce [351], a complete characterization of endomorphism rings of separable p-groups is given. The result of Liebert is also presented in §109 of Fuchs' work [165]. These theorems are very interesting, but it is difficult to apply them to group problems. This is clear, since all properties of p-groups are exactly reflected in their endomorphism rings by the Baer–Kaplansky theorem. In this section, we prove the theorem of Corner on a partial realization for endomorphism rings of some separable p-groups. This realization theorem reduces some group-theoretical problems to problems for one acceptable class of rings.

In this section, we use different notions related to p-groups and the p-adic topology which are contained in Sec. 20. We assume that groups, rings, and modules have the p-adic topology. The closure, density, completeness, and completion are considered in the p-adic topology. The completion is called the *p-adic completion*. A topological space with a countable dense subspace is said to be *separable*. At the end of the section, it will be convenient to consider formal infinite series of elements of the group (similar series in finitely topologized endomorphism rings are considered in Sec. 14). We say that a series $\sum_{i\geq 1} a_i$ of elements of a group G converges in the p-adic topology to an element $a \in G$ if the sequence of partial sums $a_1, a_1 + a_2, \dots$ has the limit a. We recall that p-groups without elements of infinite height are said to be *separable* (Sec. 20). In the same section, a periodically complete p-group is defined as the torsion part $T(\hat{B})$ of the p-adic completion $\hat{B}$ of the direct sum B of cyclic p-groups. This group is denoted by $\bar{B}$. The subgroup B is a basis subgroup for $\bar{B}$, and $\bar{B}$ is the closure of B. The closure of a pure subgroup $D \subseteq \bar{B}$ in the group $\bar{B}$ is denoted by $\bar{D}$. Since the group $\hat{B}$ is complete, it is clear that $\bar{D}$ is a direct summand of the group $\bar{B}$.

We now define those endomorphisms that are used in realizations of endomorphism rings of some p-groups. An endomorphism ω of a p-group G is said to be *small* if for every positive integer e, there exists a positive integer n such that $\omega(p^nG)[p^e] = 0$ (see [350]). It is clear that small endomorphisms of the p-group G form an ideal of the ring $E(G)$; this ideal is denoted by $E_s(G)$. Pierce [350] proved that the additive group of the factor ring $E(G)/E_s(G)$ is a complete p-adic module (we do not use this property). The Corner theorem on the split realization is a partial conversion of this result. It states that if the additive group of some ring A is the completion in the p-adic topology of a free p-adic module of rank $\leq\aleph_0$, then there exists a separable p-group G such that $E(G) \cong A \oplus E_s(G)$ (group direct sum). In this case, the ring $E(G)$ is called the *split extension* of the ring A with respect to the ideal $E_s(G)$.

In this section, p denotes some fixed prime integer. The letters e, k, m, and n denote positive integers. The main result will follow from one assertion; it is convenient to formulate this assertion now.

Proposition 28.1. *Let $\bar{B}$ be a periodically complete p-group with an unbounded basis subgroup B of cardinality $\leq 2^{\aleph_0}$, and let ϕ be a separable closed subring of the ring $E(\bar{B})$ that does not move B and satisfies the following condition*:

(C) *if* $\varphi \in \phi$ *and* $\varphi(p^n\bar{B})[p] = 0$ *for some* n, *then* $\varphi \in p\phi$.

Then there exists a pure subgroup G of the group $\bar{B}$ such that G contains B and $E(G) = \phi \oplus E_s(G)$.

The assumptions of Proposition 28.1 hold good up to Theorem 28.11. We give the following remark. Let G be a pure subgroup of the group $\bar{B}$ containing B. Since $\bar{B}$ is the torsion part of a complete group and the subgroup G is dense in $\bar{B}$, every homomorphism $G \to \bar{B}$ is uniquely extended to an endomorphism of the group $\bar{B}$. We identify such a homomorphism with the indicated extension. In Proposition 28.1, the ring $E(G)$ is identified with the subring of all endomorphisms of the group $\bar{B}$ that do not move G. Under this assumption,

$$E_s(G) = E_s(\bar{B}) \cap E(G).$$

For a given set F of endomorphisms of the group $\bar{B}$ and the subgroup $H \subseteq \bar{B}$, we denote by F_H the set of all restrictions $\varphi_{|H}$ $(\varphi \in F)$.

We consider several simple lemmas.

Lemma 28.2. *The set $\phi_{\bar{B}[p^e]}$ is countable for every e.*

Proof. Since ϕ is separable, it contains a countable dense subset F. We consider any $\varphi \in \phi$. Since F is dense, there exists $\psi \in F$ such that $\varphi - \psi \in p^eE(\bar{B})$. It is clear that $\varphi - \psi$ annihilates $\bar{B}[p^e]$; therefore,

$$\varphi_{|\,\bar{B}[p^e]} = \psi_{|\,\bar{B}[p^e]}.$$

Therefore,

$$\phi_{\bar{B}[p^e]} = F_{\bar{B}[p^e]},$$

and $\phi_{\bar{B}[p^e]}$ is countable. □

Lemma 28.3. *Every infinite subset X of $\bar{B}$ is contained in the closure $\bar{D}$ of some direct summand D of the group B with $|D| \leq |X|$.*

Proof. Every element of the set X is the limit of a Cauchy sequence of some elements in B. We gather all elements of such sequences for all elements of X in one set. Since B is a direct sum of cyclic groups, this infinite set is contained in some direct summand D of the group B of the same cardinality. Then $X \subseteq \bar{D}$, $|D| \leq |X|$, and $\bar{D}$ is a direct summand of the group $\bar{B}$. □

Lemma 28.4. *Let D be a direct summand of the group B such that*

(D_1) *if* $\varphi \in \phi$ *and* $\varphi(p^n\bar{D})[p] = 0$ *for some* n, *then* $\varphi \in p\phi$.

Then for all e, we have

(D_e) *if* $\varphi \in \phi$ *and* $\varphi(p^n\bar{D})[p^e] = 0$ *for some* n, *then* $\varphi \in p^e\phi$.

The proof is obtained by induction on the integer e. Assume that (D_{e-1}) holds for some $e > 1$. We consider $\varphi \in \phi$ such that $\varphi(p^n\bar{D})[p^e] = 0$. It is clear that $\varphi(p^n\bar{D})[p^{e-1}] = 0$, whence $\varphi = p^{e-1}\psi$ for some $\psi \in \phi$ by (D_{e-1}). Then

$$\psi(p^{n+e-1}\bar{D})[p] \subseteq \varphi(p^n\bar{D})[p^e] = 0.$$

Therefore, $\psi \in p\phi$ and $\varphi \in p^e\phi$.

We define some other notation and definitions. Let $\mathcal{D}$ be the set of all unbounded countable direct summands D of the group B satisfying conditions (D_1) and (D_e) of Lemma 28.4. For every e and each $D \in \mathcal{D}$, we denote by $\Lambda_e(D)$ the set of all homomorphisms $\lambda : \bar{D} \to \bar{B}$ such that the following two conditions hold:

(L_e) $(\lambda - \varphi)(p^n\bar{D})[p^e] \neq 0$ for all n and each $\varphi \in \phi_{\bar{D}}$;
(L'_e) $\lambda(p^m\bar{D})[p^{e-1}] = 0$ for some m.

By T, we denote the set of all ordered triples (e, D, λ) with a positive integer e, $D \in \mathcal{D}$, and $\lambda \in \Lambda_e(D)$.

Lemma 28.5. *Every countable subset from $\bar{B}$ is contained in the closure $\bar{D}$ for an appropriate $D \in \mathcal{D}$.*

Proof. By Lemma 28.2, all elements of the set $(\phi \setminus p\phi)_{\bar{B}[p]}$ can be numbered as $\varphi_1, \varphi_2, \ldots$. Property (C) from Proposition 28.1 implies that for any two k and n, there exists an element $x_{kn} \in (p^n\bar{B})[p]$ with $\varphi_k x_{kn} \neq 0$. We take a countable subset X of $\bar{B}$. Lemma 28.3 states that there is a countable direct summand D of the group B such that $\bar{D}$ contains X and all elements x_{kn}. Since $\bar{D}$ is a direct summand of the group $\bar{B}$, we have $x_{kn} \in (p^n\bar{D})[p]$ for all k and n. By the choice of elements x_{kn}, the summand D is not bounded and satisfies (D_1). Therefore $D \in \mathcal{D}$. □

Lemma 28.6. *The following relations hold*:

(a) $|\Lambda_e(D)| = 2^{\aleph_0}$ *for all e and all $D \in \mathcal{D}$*;
(b) $|T| = 2^{\aleph_0}$.

Proof. Since $|\bar{B}| = 2^{\aleph_0}$ and every homomorphism $\bar{D} \to \bar{B}$ is determined by its restriction to the countable subgroup D, we have $|\Lambda_e(D)| = 2^{\aleph_0}$. To obtain the converse inequality, we consider D as a countable direct sum of unbounded subgroups D_k $(k \in \mathbf{N})$. For every subset $I \subseteq \mathbf{N}$, let π_I be a homomorphism $\bar{D} \to \bar{B}$ that acts as the identity mapping on D_k for $k \in I$ and as the zero mapping on D_k for $k \notin I$. The cardinality of the set of all such homomorphisms π_I is continuum. All the mappings $p^{e-1}\pi_I$ annihilate $\bar{D}[p^{e-1}]$, where any two of them are distinct on $(p^n\bar{D})[p^e]$ for every n. Removing the countable subset consisting of those $p^{e-1}\pi_I$ which coincide with any element of the countable set $\phi_{\bar{B}[p^e]}$ on every subgroup $(p^n\bar{D})[p^e]$, we obtain $2^{\aleph_0}$ different elements from $\Lambda_e(D)$. This implies (a), whence we immediately obtain (b). □

A subgroup G of the group $\bar{B}$ is ϕ-invariant if $\varphi G \subseteq G$ for every $\varphi \in \phi$. The following lemma is very important for the subsequent results.

Lemma 28.7. *Let G be a pure ϕ-invariant subgroup of the group $\bar{B}$ such that G contains B and for every triple $(e, D, \lambda) \in T$, there exists an element $x \in G \cap \bar{D}$ with $\lambda x \notin G$. Then $E(G) = \phi \oplus E_s(G)$.*

Proof. We consider an endomorphism $\varphi \in \phi \cap E_s(\bar{B})$ and any positive integer e. Since φ is small, $\varphi(p^n\bar{B})[p^e] = 0$ for some n. Therefore, $\varphi \in p^e\phi$ by (C) and Lemma 28.4 (with $D = B$). Therefore, φ annihilates $\bar{B}[p^e]$ and $\varphi = 0$, since e is arbitrary. We have proved that the sum $\phi + E_s(\bar{B})$ is a direct sum.

We now prove that $E(G) \subseteq \phi \oplus E_s(\bar{B})$. Let $\omega \in E(G)$. First, we verify that for every e,

$$(\omega - \varphi)(p^n\bar{B})[p^e] = 0 \quad \text{for some } n \text{ and some } \varphi \in \phi.$$

Assume the contrary. Let e be the least positive integer for which the assertion is not true. Then there exist m and $\psi \in \phi$ such that $\lambda(p^m\bar{B})[p^{e-1}] = 0$, where $\lambda = \omega - \psi$. By Lemma 28.2, there exists a countable subset $F \subseteq \phi$ such that $F_{\bar{B}[p^e]} = \phi_{\bar{B}[p^e]}$, and for every n and each $\varphi \in F$, we can choose $y_{\varphi,n} \in (p^n\bar{B})[p^e]$ such that $(\lambda - \varphi)y_{\varphi,n} \neq 0$. By Lemma 28.5, there exists $D \in \mathcal{D}$ whose closure $\bar{D}$ contains all elements $y_{\varphi,n}$. Then we obtain $\lambda_{|\bar{D}} \in \Lambda_e(D)$. By assumption, there exists $x \in G \cap \bar{D}$ with $\lambda x \notin G$. However, $\lambda x = \omega x - \psi x \in G$; this is a contradiction. We can assert that $\omega - \varphi$ is a small endomorphism,

$$\omega - \varphi \in E_s(\bar{B}), \quad \omega \in \phi \oplus E_s(B), \qquad \text{and} \qquad E(G) \subseteq \phi \oplus E_s(\bar{B}).$$

Since the group G is ϕ-invariant, $\phi \subseteq E(G)$. Applying the modular law, we obtain that

$$E(G) = E(G) \cap (\phi \oplus E_s(\bar{B})) = \phi \oplus (E(G) \cap E_s(\bar{B})) = \phi \oplus E_s(G).$$

□

We also need three auxiliary lemmas.

Lemma 28.8. *For given e and $D \in \mathcal{D}$, let F be the set of homomorphisms $\bar{D} \to \bar{B}$ such that $F_{\bar{D}[p^e]}$ is countable and $\varphi(p^n\bar{D})[p^e] \neq 0$ for every $\varphi \in F$ and every n. Then for every m, there exists a set S of cardinality $2^{\aleph_0}$ of nonzero elements of $(p^m\bar{D})[p^e]$ such that $\varphi(x-y) \neq 0$ for different $x, y \in S \cup \{0\}$ and every $\varphi \in F$.*

Proof. Without loss of generality, we can assume that the set F is countable. We number elements of the set F as $\varphi_1, \varphi_2, \ldots$. Let $\psi_1, \psi_2, \ldots$ be any infinite sequence taken from this numbering. We denote by Δ the set of all functions $N \to \{0,1\}$.

It follows from the assumption that for every $\varphi \in F$, there exists an element $x \in \bar{D}[p^e]$ such that $\varphi x \neq 0$ and the height $h(x)$ exceeds the largest integer in a preseribed finite set of positive integers. By easy induction that uses this remark, there exists a sequence $x_1, x_2, \ldots$ of elements in $(p^m\bar{D})[p^e]$ such that for every k, we have:

(a) $\psi_k x_k \neq 0$;
(b) $h(\sum_{i<k} \psi_j(\delta_i - \varepsilon_i))x_i < h(x_k)$ in all situations, where $1 \leq j \leq k$ and $\delta, \varepsilon \in \Delta$ satisfy the relation $\sum_{i<k} \psi_j(\delta_i - \varepsilon_i)x_i \neq 0$;
(c) $h(x_1) < h(x_2) < \cdots$.

It follows from (c) that for every $\delta \in \Delta$, the series $\sum_i \delta_i x_i$ converges in the p-adic topology of the group $\bar{B}$ to some element of $(p^m\bar{D})[p^e]$. We consider elements $\delta, \varepsilon \in \Delta$. Assume that $\delta_j \neq \varepsilon_j$ for some positive integer j; for example, $\delta_j = 1$ and $\varepsilon_j = 0$. Now (a) and (b) with $k = j$ imply that

$$\sum_{i \leq j} \psi_j(\delta_i - \varepsilon_i)x_i \neq 0.$$

Since homomorphisms ψ_j cannot decrease heights, (c) and (b) (with $k > j$) imply that

$$h\Big(\sum_{i \leq j} \psi_j(\delta_i - \varepsilon_i)x_i\Big) < h\Big(\sum_{i > j} \psi_j(\delta_i - \varepsilon_i)x_i\Big).$$

Consequently,

$$\sum_i \psi_j(\delta_i - \varepsilon_i)x_i \neq 0 \qquad \text{provided} \quad \delta, \varepsilon \in \Delta \quad \text{and} \quad \delta_j \neq \varepsilon_j. \tag{$*$}$$

For every k, let $x_{k_1}, x_{k_2}, \ldots$ be a subsequence of elements x_j corresponding the infinite set of integers j with $\psi_j = \varphi_k$. For every $\delta \in \Delta$, we assume that

$$x_\delta = \sum_i \delta_i x_{k_i}.$$

Then $x_0 = 0$ and all elements x_δ are contained in $(p^m\bar{D})[p^e]$. Since every homomorphism $\bar{D} \to \bar{B}$ is continuous in the p-adic topology, $(*)$ directly implies that $\varphi(x_\delta - x_\varepsilon) \neq 0$ for different $\delta, \varepsilon \in \Delta$ and every $\varphi \in F$. Since $|\Delta| = 2^{\aleph_0}$, we complete the proof of the lemma. □

We say that an element $x \in \bar{B}[p^e]$ is e-strong if for every $\varphi \in \phi$, the relation $\varphi x = 0$ implies $\varphi \in p^e\phi$.

Lemma 28.9. *Let $(e, D, \lambda) \in T$, and let H be a subgroup of $\bar{B}$ with $|H| < 2^{\aleph_0}$. Then there exists an e-strong element $x \in \bar{D}[p^e]$ such that $\lambda x \neq 0$ and the direct sum $(B + H) \oplus \phi x \oplus \langle \lambda x \rangle$ exists in $\bar{B}$.*

Proof. By (L'_e), we can choose m with $\lambda(p^m\bar{D})[p^{e-1}] = 0$. It follows from (D_e), (L'_e), and Lemma 28.2 that we can apply Lemma 28.8 to the set

$$F = (\phi \setminus p^e\phi)_{\bar{D}} \cup (\lambda + \phi_{\bar{D}}).$$

Let S be the corresponding set of cardinality $2^{\aleph_0}$ of nonzero elements from $(p^m\bar{D})[p^e]$. We note that $px \in (p^m\bar{D})[p^{e-1}]$ for every $x \in S$. By our choice of the integer m, we obtain

$$\lambda(px) = 0 \qquad \text{for all} \quad x \in S. \tag{$**$}$$

Now we prove that for different $x, y \in S \cup \{0\}$, if $\varphi \in \phi_{\bar{D}}$ and an integer s satisfies

$$\varphi x + s(\lambda x) = \varphi y + s(\lambda y),$$

then $\varphi x = s(\lambda x) = \varphi y = s(\lambda y) = 0$. If $p \nmid s$, then we can choose an integer t such that $st \equiv 1(\bmod p)$. Then $(\lambda + t\varphi)(x - y) = 0$, which is impossible, since $\lambda + t\varphi \in \lambda + \phi_{\bar{D}}$. Therefore, $p|s$. By $(**)$, this implies $s(\lambda x) = s(\lambda y) = 0$. Therefore, $\varphi(x - y) = 0$, whence $\varphi \notin F$. Therefore, $\varphi \in p^e\phi_{\bar{D}}$; this implies $\varphi x = \varphi y = 0$, since $x, y \in \bar{D}[p^e]$.

Taking $y = 0$, we have that the sum $\phi x + \langle \lambda x \rangle$ is a direct sum for every $x \in S$. Now assume that $(B + H) \cap (\phi x \oplus \langle \lambda x \rangle) \neq 0$ for every $x \in S$. It is clear that $|H + \phi D + \lambda D| < 2^{\aleph_0}$; we also note that ϕD is countable (this follows from Lemma 28.2). Consequently, Lemma 28.3 states that there exists a direct summand D_0 of the group B such that $|D_0| < 2^{\aleph_0}$ and $H + \phi D + \lambda D \subseteq \bar{D}_0$. Since $S \subseteq \bar{D}$, we have $\phi x \oplus \langle \lambda x \rangle \subseteq \bar{D}_0$ for every $x \in S$. In addition, since $B \cap \bar{D}_0 = D_0$ and $H \subseteq \bar{D}_0$, the modular law implies $(B+H) \cap \bar{D}_0 = \bar{D}_0 + H$. Consequently, our assumption implies that $(D_0 + H) \cap (\phi x \oplus \langle \lambda x \rangle) \neq 0$ for every $x \in S$. This means that for every $x \in S$, there exist $\varphi \in \phi$, an integer s, and $h \in D_0 + H$ such that

$$\varphi x + s(\lambda x) = h \neq 0.$$

However, $|S| = 2^{\aleph_0}$ and there exist at least $2^{\aleph_0}$ acceptable triples (φ, s, h). On the other hand, $|D_0 + H| < 2^{\aleph_0}$; therefore, we can guarantee the existence of two different elements $x, y \in S$ providing the same triple (φ, s, h). For this triple, we have $\varphi x + s(\lambda x) = \varphi y + s(\lambda y) = h \neq 0$. This contradicts the property proved in the preceding paragraph. This contradiction implies the lemma. □

Lemma 28.10. *Let H be a ϕ-invariant pure subgroup of the group $\bar{B}$ containing B, and let x_i $(i \in I)$ be a family of elements from $\bar{B}$ such that*

(a) *every x_i is e_i-strong for some e_i;*
(b) *in $\bar{B}$, there exists a direct sum $H' = H \oplus \sum\limits_{i \in I}^{\oplus} \phi x_i$.*

Then there exists a ϕ-invariant pure subgroup G in $\bar{B}$ such that $G[p] \subseteq H' \subseteq G$.

Proof. Since $H \supseteq B$, the factor group $\bar{B}/H$ is divisible. For every $i \in I$, there exists a sequence $x_i^0 = x_i, x_i^1, x_i^2, \dots$ of elements of $\bar{B}$ such that $x_i^{n-1} - px_i^n \in H$ for all n. Then $\varphi x_i^{n-1} - \varphi p x_i^n \in \varphi H \subseteq H$ for all $i \in I$, $\varphi \in \phi$, and n; consequently, the subgroups

$$H_r = H + \sum_{i \in I} \phi x_i^r \qquad (r = 0, 1, 2, \dots)$$

form an ascending sequence in $\bar{B}$. We set $G = \bigcup\limits_{r \geq 0} H_r$. It is clear that G contains H' (H' coincides with H_0) and G is invariant with respect to ϕ. By construction, every element of H_{n-1}/H is divided by p in H_n/H. Therefore G/H is a divisible group. Since the subgroup H is pure in $\bar{B}$, we have that G is pure in $\bar{B}$.

To complete the proof, we take any element $y \in G[p]$ and assume that r is the least integer such that $y \in H_r$. We need to prove that $r = 0$ ($G[p] \subseteq H'$ holds in this case). Assume that $r \geq 1$. We have

$$y = h + \sum_{i \in I} \varphi_i x_i^r,$$

where $h \in H$ and φ_i $(i \in I)$ are some elements of ϕ that are equal to zero for almost all i. Then

$$\left[p^r h + \sum_{i \in I} \varphi_i (p^r x_i^r - x_i)\right] + \sum_{i \in I} \varphi_i x_i = p^r y = 0,$$

where the expression in square brackets is an element of H. Therefore, $\varphi_i x_i = 0$ $(i \in I)$ by (c). By (a), we obtain $\varphi_i \in p^{e_i}\phi \subseteq p\phi$ $(i \in I)$ and

$$y = h + \sum_{i \in I} \varphi_i x_i^r \in H + \sum_{i \in I} \phi p x_i^r = H_{r-1},$$

i.e., $y \in H_{r-1}$. This contradicts the minimality of the integer r. □

Proof of Proposition 28.1. By Lemma 28.6(b), we can index elements of the set T using ordinal numbers α as $(e_\alpha, D_\alpha, \lambda_\alpha)$ $(\alpha < \Omega)$, where Ω is the least ordinal number of cardinality $2^{\aleph_0}$. We inductively construct elements x_α $(\alpha < \Omega)$ such that the following two conditions hold for all $\beta \leq \Omega$:

(I_β) for all $\alpha < \beta$, x_α is an e_α-strong element in $\bar{D}_\alpha$ and $\lambda_\alpha x_\alpha \neq 0$;
(II_β) the direct sum $B \oplus \sum_{\alpha<\beta}^{\oplus} (\phi x_\alpha \oplus \langle \lambda_\alpha x_\alpha \rangle)$ exists in $\bar{B}$.

Assume that $0 < \gamma \leq \Omega$ and we have constructed the initial segment of the sequence of elements x_α such that conditions (I_β) and (II_β) hold for all $\beta < \gamma$. If γ is a limit ordinal number, then elements x_α were constructed for all $\alpha < \gamma$. It is easy to prove that (I_γ) and (II_γ) hold. Now we assume that γ is a nonlimit ordinal number. Let $\gamma = \beta + 1$ for some ordinal number β. By assumption, the elements x_α $(\alpha < \beta)$ were constructed and they satisfy (I_β) and (II_β). Since all subgroups $\phi x_\alpha \oplus \langle \lambda_\alpha x_\alpha \rangle$ $(\alpha < \beta)$ are countable by Lemma 28.2, the minimality of Ω implies that their direct sum H has cardinality $<2^{\aleph_0}$. Consequently, the application of Lemma 28.9 to this subgroup H and the triple $(e_\beta, D_\beta, \lambda_\beta) \in T$ implies that x_β satisfies (I_γ) and (II_γ). The use of transfinite induction completes the construction of the elements x_α $(\alpha < \Omega)$.

Now we number elements x_α $(\alpha < \Omega)$ as x_t by using elements $t \in T$. For every $t \in T$, we write $t = (e_t, D_t, \lambda_t)$. It follows from (I_Ω) and (II_Ω) that the following conditions hold:

(I) for every $t \in T$, the element x_t is an e_t-strict element in $\bar{D}_t$ and $\lambda_t x_t \neq 0$;
(II) the direct sum $B \oplus \sum_{t \in T}^{\oplus} (\phi x_t \oplus \langle \lambda x_t \rangle)$ exists in $\bar{B}$.

Therefore, it is clear that the ϕ-invariant pure subgroup B of the group $\bar{B}$ and the family of elements x_t $(t \in T)$ satisfy the conditions of Lemma 28.10. Therefore, there exists a ϕ-invariant pure subgroup G of the group $\bar{B}$ such that

$$G[p] \subseteq B \oplus \sum_{t \in T}^{\oplus} \phi x_t \subseteq G.$$

Taking into account (II), we obtain that all subgroups $\langle \lambda_t x_t \rangle$ $(t \in T)$ have zero intersection with the subgroup $G[p]$. Using the standard induction by the order, we obtain that they have zero intersection with the group G. Therefore $\lambda_t x_t \notin G$ $(t \in T)$. We have constructed a ϕ-invariant pure subgroup G of the group $\bar{B}$ containing B such that $G \cap \bar{D}$ contains some element x $(= x_t)$ such that $\lambda x \notin G$. By Lemma 28.7, $E(G) = \phi \oplus E_s(G)$.

□

We have completed the preparatory stage and can formulate the following result.

Theorem 28.11 (Corner [96]). *Let A be a ring, and let the additive group of A be the completion of a free p-adic module of at most countable rank. Then there exists a separable p-group G such that $E(G) = A \oplus E_s(G)$.*

Proof. The group of p-adic integers I_p is the completion in the p-adic topology of the group of integers $\mathbf{Z}$. Therefore, the additive group of the ring A is the completion of a free Abelian group of at most countable rank. Consequently, the ring A is complete and separable in the p-adic topology, and the group A/p^kA is equal to a direct sum of at most $\aleph_0$ cyclic groups of order p^k for every positive integer k. The factor groups A/p^kA are naturally considered as left A-modules. If k, n, and e are any positive integers such that $k \geq n+e$, then it is easy to prove that

$$(p^n(A/p^kA))[p^e] = p^{k-e}A/p^kA.$$

Therefore, if $a \in A$ and $k \geq n+e$, then

$$a(p^n(A/p^kA))[p^e] = 0 \quad \Longleftrightarrow \quad a \in p^eA. \qquad (*)$$

We form a direct sum:

$$B = \sum_{k\in\mathbf{N}}^{\oplus} A/p^kA.$$

We consider the group B and its torsion completion $\bar{B}$ (i.e., the torsion part of the completion of the group B) as natural left A-modules. For every element $a \in A$, let $\varphi_a : \bar{B} \to \bar{B}$ be an endomorphism of the left multiplication by the element a of the group $\bar{B}$. The mapping $a \to \varphi_a$ $(a \in A)$ is a ring homomorphism $A \to E(\bar{B})$; its image is denoted by ϕ. It is clear that the subgroup B is invariant with respect to ϕ. It follows from $(*)$ that

$$\varphi_a(p^n\bar{B})[p^e] = 0 \quad \Longleftrightarrow \quad a \in p^eA$$

for an element $a \in A$ and any n and e. Taking $e=1$, we obtain that ϕ satisfies condition (C) of Proposition 28.1. Further, if $\varphi_a = 0$ for some $a \in A$, then $(*)$ implies that $a \in p^eA$ for all e. Therefore, $a = 0$ and the ring homomorphism $A \to E(\bar{B})$ $(a \to \varphi_a)$ is an embedding. Therefore, A is topologically isomorphic to the subring ϕ. Since A is separable and complete, the subring ϕ is separable and closed in $E(\bar{B})$. Now we obtain that all conditions of Proposition 28.1 hold, and the assertion of the theorem follows from this proposition. □

The following two propositions unveil the character of possible applications of the proved theorem.

A p-group is said to be *essentially indecomposable* if at least one summand in each of its nontrivial decompositions into a direct sum of two summands is bounded.

Proposition 28.12. *There exists an essentially indecomposable separable p-group.*

Proof. By Theorem 28.11, there exists a separable p-group G such that

$$E(G) = Q_p^* \oplus E_s(G).$$

We consider any direct decomposition $G = G_1 \oplus G_2$; let π_1 and π_2 be the corresponding projections. Their images $\bar{\pi}_1$ and $\bar{\pi}_2$ in the factor ring $E(G)/E_s(G)$ are also idempotents whose sum is equal to the identity element. Since $E(G)/E_s(G) \cong Q_p^*$, either $\bar{\pi}_1 = 0$ or $\bar{\pi}_2 = 0$. For definiteness, let $\bar{\pi}_1 = 0$. Then $\pi_1 \in E_s(G)$. This implies that for some positive integer n, $(p^nG_1)[p] = \pi_1(p^nG)[p] = 0$. Therefore, $p^nG_1 = 0$, which is what was required. □

The second proposition is related to the following well-known notions referring to direct decompositions. Two direct decompositions of the group A, $A = \sum_{i\in I}^{\oplus} B_i$ and $A = \sum_{j\in J}^{\oplus} C_j$, are said to be *isomorphic* if there exists a bijection $f : I \to J$ such that $B_i \cong C_{f(i)}$ $(i \in I)$. If $A = \sum_i^{\oplus} B_i$, where every group B_i is a direct sum and $B_i = \sum_j^{\oplus} B_{ij}$, then the decomposition $A = \sum_i^{\oplus}\sum_j^{\oplus} B_{ij}$ is called an *extension* or a *refinement* of the first decomposition. A group A satisfies the isomorphism property of extensions of direct decompositions if any two direct decompositions of the group A have isomorphic extensions.

Proposition 28.13. *There exists a separable p-group that does not satisfy the isomorphism property of extensions of direct decompositions.*

Proof. Let S be the semigroup with 1 with generators a and b and defining relations $a^2 = a$ and $b^2 = b$. Let A be the semigroup ring of the semigroup S over the ring $\mathbf{Z}$, $A = \mathbf{Z}S$. Every element from S is uniquely represented as a word consisting of symbols a and b that do not contain syllables a^2 or b^2. All such words can be written as the following sequence:

$$1, \quad a, \quad b, \quad ba; \quad \ldots \quad ; \quad (ab)^n, \quad (ab)^n a, \quad b(ab)^n, \quad b(ab)^n a; \quad (ab)^{n+1}, \quad \ldots .$$

The additive group of the ring A is free. Indeed, different products of symbols a and b form a basis. By Theorem 28.11, there exists a separable p-group G with $E(G) = \hat{A} \oplus E_s(G)$, where $\hat{A}$ is the p-adic completion of the ring A. Since elements a and b are idempotents of the ring $E(G)$, we have two direct decompositions:

$$G = aG \oplus (1-a)G = bG \oplus (1-b)G. \tag{$*$}$$

All idempotents a, $1-a$, b, and $1-b$ are not contained in $E_s(G)$; therefore, the argument used in the proof of Proposition 28.12 shows that all summands in decompositions $(*)$ are not bounded. Consequently, for the proof of the property that decompositions $(*)$ do not have isomorphic extensions, it is sufficient to prove that every unbounded direct summand of the group aG or $(1-a)G$ is not isomorphic to any direct summand of the group bG or $(1-b)G$.

Let $a^* = a$ or $1-a$, and let $b^* = b$ or $1-b$. In order to obtain a contradiction, we assume that some unbounded direct summand U of the group a^*G is isomorphic to some direct summand V of the group b^*G. Therefore, there exist two idempotents $u, v \in E(G)$ such that $U = uG$, $V = vG$, $u = a^*ua^*$, and $v = b^*vb^*$. By Proposition 3.10, there exist elements $x, y \in E(G)$ such that $x = a^*xb^*$, $y = b^*ya^*$, $xy = u$, and $yx = v$; in particular, $u = u^2 = a^*xb^*ya^*$. Since the summand $U = uG$ is not bounded, $u \notin E_s(G)$. Consequently, we have

$$\bar{u} = \bar{u}^2 = a^*\bar{x}b^*\bar{y}a^* \neq 0, \tag{$**$}$$

where $\bar{x}, \bar{y} \in \hat{A}$ and the overbar is used to denote the canonical homomorphism $E(G) \to \hat{A}$ with the kernel $E_s(G)$. We prove that relations $(**)$ are impossible. Considering the construction of the ring A, it is easy to verify that the ring $\hat{A}$ admits an automorphism of order 2 that maps from the elements a and b into a^* and b^*, respectively. Since we focus our attention on $(**)$, we can assume that $a^* = a$ and $b^* = b$; therefore, $\bar{u} = a\bar{x}b\bar{y}a$. It is convenient to represent elements of the ring $\hat{A}$ as formal finite series of elements from A, i.e., elements of S and their integral multiples. We have considered above the elements of the semigroup S in the form of a certain sequence. Writing the elements $\bar{x}$ and $\bar{y}$ in such a manner, we obtain an expression for $\bar{u}$ in the following form:

$$\bar{u} = \sum_{i \geq 1} \xi_i (ab)^i a,$$

where (ξ_i) is some sequence of elements of Q_p^* that converges to zero. It is clear that $(ab)^i a \cdot (ab)^j a = (ab)^{i+j} a$. Since $\bar{u}^2 = \bar{u}$, an easy calculation shows that all ξ_i are equal to zero. Therefore, $\bar{u} = 0$; this contradicts $(**)$. □

Exercises 1–8 are results from the work of Pierce [350]. In these exercises, G and H are p-groups. A homomorphism $\omega : G \to H$ is said to be *small* if for every e, there exists an integer n such that $\omega(p^n G)[p^e] = 0$.

Exercise 1. Prove that all small homomorphisms from the group G into a group H form the subgroup $\mathrm{Hom}_s(G, H)$ of the group $\mathrm{Hom}(G, H)$, and $\mathrm{Hom}(G, H)/\mathrm{Hom}_s(G, H)$ is a torsion-free group.

If the order of an element a of a group G is equal to p^n, then the integer n is called the *exponent of the element* a; it is denoted by $e(a)$.

Exercise 2. A homomorphism $\omega : G \to H$ is small if and only if the following condition holds: for every k, there exists an integer n such that $e(a) \geq n$ implies that $e(\omega a) \leq e(a) - k$ for every $a \in G$.

Exercise 3. Let $B = \sum_{i \in I}^{\oplus} \langle b_i \rangle$ be a basis subgroup of a group G, and let $c_i \in H$ $(i \in I)$ be elements such that $e(c_i) \leq e(b_i)$ and for every $k \geq 0$ there exists an integer n such that $e(c_i) \leq e(b_i) - k$ provided $e(c_i) \geq n$. Then there exists a uniquely defined small homomorphism $\omega : G \to H$ such that $\omega(b_i) = c_i$ $(i \in I)$.

In Exercises 4 and 5, Exercise 3 is used.

Exercise 4. Let E be a pure subgroup of the group G. Every small homomorphism from the group E into a group H can be extended to a small homomorphism from the group G into the group H.

Exercise 5. Every small homomorphism from the group $p^n G$ into the group $p^n H$ can be extended to a small homomorphism from the group G into the group H.

Exercise 6. Prove that: (a) $E_s(G) = E(G) \iff G$ is a bounded group; (b) $E_s(G) = 0 \iff G$ is a divisible group.

Exercise 7. Let G_1 and G_2 be two p-groups, and let $\varphi : G_1 \to G_2$ be a homomorphism. Then

(a) $\mathrm{Hom}_s(G_1 \oplus G_2, H) \cong \mathrm{Hom}_s(G_1, H) \oplus \mathrm{Hom}_s(G_2, H)$;

(b) there exists the induced mapping

$$\varphi^* : \mathrm{Hom}_s(G_2, H) \to \mathrm{Hom}_s(G_1, H).$$

Formulate and prove assertions dual to (a) and (b).

If R is a ring (maybe without identity element), then the Jacobson radical $J(R)$ of R is the largest left ideal consisting of left-quasi-regular elements (see the introduction to Part 7).

Exercise 8. Let $H_s(G) = \{\alpha \in E_s(G) \,|\, x \in G[p],\, h(x) < \infty \implies h(x) \leq h(\alpha x)\}$, $K_s(G) = \{\alpha \in E_s(G) \,|\, \alpha G[p] = 0\}$, and $L_s(G) = \{\alpha \in E_s(G) \,|\, \alpha G \subseteq pG$. Prove that $H_s(G)$, $K_s(G)$ and $L_s(G)$ are ideals of the ring $E_s(G)$ and $J(E_s(G)) = K_s(G) + L_s(G) = H_s(G)$.

Exercise 9 (Corner [96])**.** Let q be a positive integer. Then there exists a separable p-group G such that for any two positive integers m and n,

$$G^m \cong G^n \iff m \equiv n (\mod q).$$

29. The Corner Theorem on Countable Torsion-Free Rings

This section is devoted to the proof of another famous theorem of Corner: a realization theorem for countable reduced torsion-free rings. We recall that group terms applied to a ring are related to the additive group of the ring. Therefore, terms such as a torsion-free ring of finite rank, a reduced ring, or a pure subring have an exact meaning. As in the preceding section, we substantially use the p-adic topology and the $\mathbf{Z}$-adic topology on the group or the ring (the definition of these topologies is given in Sec. 2). It is easy to verify that for any ring A, the addition and the multiplication are continuous in both topologies, since nA are always ideals of the ring A. Therefore, A is a topological ring. If a group A is divided by every prime integer $\neq p$, then the p-adic topology coincides with the $\mathbf{Z}$-adic topology on A.

We present basic properties on the $\mathbf{Z}$-adic completion and the p-adic completion of groups and rings. The completion with respect to any linear topology is explicitly studied in §13 of Fuchs' work [164]. Sections 39 and 40 of this book contain information on the structure of groups that are complete in the $\mathbf{Z}$-adic topology. Cauchy sequences and convergent sequences were defined in Sec. 20. A group is said to be *complete in a given topology* if it is a Hausdorff group and every Cauchy sequence in the group A has a limit in A. With the use of Cauchy sequences or inverse limits, we can form the $\mathbf{Z}$-adic and p-adic completion of the group A (i.e., the completion with respect to the $\mathbf{Z}$-adic topology and the p-adic topology, respectively). By $\hat{A}$ and $\hat{A}_p$, we denote the $\mathbf{Z}$-adic completion and the p-adic completion, respectively. If the group A is Hausdorff in the $\mathbf{Z}$-adic topology, then it can be embedded as a dense pure subgroup in $\hat{A}$; in so doing, the completion topology on the group $\hat{A}$ coincides with the $\mathbf{Z}$-adic topology. A torsion-free group is a Hausdorff group if and only if the group is reduced.

If B is a pure subgroup of the group A, then $B \cap nA = nB$ for all n. Consequently, the induced topology on the group B coincides with the $\mathbf{Z}$-adic topology. Therefore, we obtain the following property. If B is a dense pure subgroup of the group $\hat{A}$ containing A, then $\hat{B} = \hat{A}$. It is easy to verify that the subgroup B of the group A is dense in A if and only if A/B is a divisible group. Indeed, the factor group A/B is divisible if and only if $nA + B = A$ for every nonzero integer n. The last condition is equivalent to the density of the subgroup B in A.

If a ring A is Hausdorff in the $\mathbf{Z}$-adic topology, we can similarly consider the completion $\hat{A}$, which is a topological ring. The structure of complete groups and rings is well known. Completions of the ring of integers $\mathbf{Z}$ play a particular role here. We have

$$\hat{Z}_p = Q_p^* \qquad \text{and} \qquad \hat{\mathbf{Z}} = \prod_p Q_p^*.$$

If a Hausdorff group A is considered as a $\mathbf{Z}$-module, then it is a topological $\mathbf{Z}$-module. By continuity, the canonical bilinear mapping $\mathbf{Z} \times A \to A$, which is defined by $(k, a) \to ka$ for all $k \in \mathbf{Z}$ and $a \in A$, is extended to a bilinear mapping $\hat{\mathbf{Z}} \times \hat{A} \to \hat{A}$; in so doing, all module axioms hold. Therefore, the completion $\hat{A}$ is a topological $\hat{\mathbf{Z}}$-module and the p-adic completion $\hat{A}_p$ is a $\hat{Z}_p$-module. Similarly, the completion $\hat{A}$ of the Hausdorff ring A is a topological $\hat{\mathbf{Z}}$-algebra. Thus, the $\mathbf{Z}$-adic completion of the group (the ring) A has the form

$$\hat{A} = \prod_p A_p,$$

where $\hat{A}_p$ is the p-adic completion of the group (the ring) A. We note that $\hat{A}_p$ coincides with the p-adic completion of the reduced part of the tensor product $A \otimes Q_p$. As is indicated in Sec. 2, the group is complete in the $\mathbf{Z}$-adic topology if and only if the group is reduced and algebraically compact.

We will also use the well-known possibility of extending continuous mappings in complete groups. It is essential that all group homomorphisms are continuous in the $\mathbf{Z}$-adic topology. We summarize all of the preceding in the following proposition.

Proposition 29.1. *A Hausdorff group A is contained as a dense pure subgroup in its $\mathbf{Z}$-adic completion $\hat{A}$, which is torsion-free provided the group A is torsion-free. The Hausdorff ring A is a dense pure subring in its $\mathbf{Z}$-adic completion $\hat{A}$. In addition, the $\mathbf{Z}$-adic completion is a $\hat{\mathbf{Z}}$-module, and every homomorphism of Hausdorff groups $\alpha : A \to B$ is uniquely extended to a homomorphism $\hat{\alpha} : \hat{A} \to \hat{B}$ of $\hat{\mathbf{Z}}$-modules.*

Theorem 29.2 (Corner [93])**.** *Every countable reduced torsion-free ring R with identity element is isomorphic to the endomorphism ring $E(A)$ of some countable reduced torsion-free group A.*

Proof. First, we assume that the ring R is an algebra over the ring Q_p of rational numbers whose denominators are coprime to p. This is equivalent to the assumption that $qR = R$ for all prime numbers $q \neq p$. For the subsequent presentation, it is convenient to denote such a ring R by R_p. Since the ring R_p is reduced, the p-adic topology in this ring is a Hausdorff topology. We form the completion $\hat{R}_p$ of the ring R_p in the p-adic topology. According to Proposition 29.1, R_p is a Q_p^*-algebra containing R_p as a pure dense subring. Since the ring R is countable, R contains a finite or countable set $\{\xi_n\}$ of elements that is a maximal independent set over Q_p^*. Consequently, for every $\alpha \in R_p$, there exists a relation

$$p^n \alpha = \pi_1 \xi_1 + \cdots + \pi_k \xi_k,$$

where n is a positive integer and $\pi_i \in Q_p^*$. The elements π_i are uniquely defined up to factors of the form p^m. Therefore, we can consider the pure subring S_p of the ring Q_p^* generated by the subring Q_p and elements π_i taken for all $\alpha \in R_p$. It is clear that the ring S_p is countable.

Now assume that $\gamma_1\alpha_1+\dots+\gamma_m\alpha_m=0$ for some $\alpha_1,\dots,\alpha_m\in R_p$, where the elements $\gamma_1,\dots,\gamma_m\in Q_p^*$ are linearly independent over S_p. Choosing a sufficiently large integer n, we have

$$p^n\alpha_j=\sum_i \pi_{ij}\xi_i,$$

where $\pi_{ij}\in S_p$. Therefore,

$$\sum_{i,j}\gamma_j\pi_{ji}\xi_i=0 \qquad \text{and} \qquad \sum_j \gamma_j\pi_{ji}=0,$$

since the elements ξ_i are independent. Consequently, all the elements α_j are equal to zero.

For every $\alpha\in R_p$, we choose p-adic integers ρ_α and σ_α such that the set $\{\rho_\alpha,\sigma_\alpha\,|\,\alpha\in R_p\}$ is algebraically independent over S_p. The algebraic independence means that there is no finite set of numbers ρ_α and σ_α that are roots of every polynomial with coefficients in S_p. This is possible, since S_p is countable and the cardinality of Q_p^* is continuum. For these ρ_α and σ_α, we set

$$\varepsilon_\alpha=\rho_\alpha 1+\sigma_\alpha\alpha\in\hat{R}_p.$$

Let A denote the pure subgroup

$$\langle R_p, R_p\varepsilon_\alpha\,|\,\alpha\in R_p\rangle_*$$

of the group $\hat{R}_p$. It is clear that A is a countable reduced torsion-free group.

We prove that the endomorphism ring $E(A)$ of the constructed group A is naturally isomorphic to the ring R. It is clear that the group is a left R_p-module, and different elements of R_p have different actions on A, since $1\in A$. Therefore, R_p can be considered as a subring of the ring $E(A)$ if every $\alpha\in R_p$ is identified with the endomorphism of the left multiplication of the group A by the element α.

It is much more difficult to prove that $R_p=E(A)$. To do this, we take any endomorphism $\eta\in E(A)$. Since $A\supseteq R_p$, we have that A is a dense pure subgroup in $\hat{R}_p$. Therefore, the endomorphism η is extended to a unique Q_p^*-endomorphism $\hat{\eta}$ of the group $\hat{R}_p$. Using this endomorphism $\hat{\eta}$, we obtain

$$\eta\varepsilon_\alpha=\hat{\eta}(\rho_\alpha 1+\sigma_\alpha\alpha)=\rho_\alpha(\hat{\eta}1)+\sigma_\alpha(\hat{\eta}\alpha)=\rho_\alpha(\eta 1)+\sigma_\alpha(\eta\alpha) \tag{$*$}$$

for $\alpha\in R_p$. It follows from the definition of the group A that for some positive integers k and n, we have the following relations:

$$p^k(\eta\varepsilon_\alpha)=\beta_0+\sum_{i=1}^n\beta_i\varepsilon_{\alpha_i}, \qquad p^k(\eta 1)=\gamma_0+\sum_{i=1}^n\gamma_i\varepsilon_{\alpha_i}, \qquad \text{and} \qquad p^k(\eta\alpha)=\delta_0+\sum_{i=1}^n\delta_i\varepsilon_{\alpha_i},$$

where $\beta,\gamma,\delta\in R_p$. For simplicity, we assume that $\alpha=\alpha_1$. Substituting in $(*)$, we obtain that

$$\beta_0+\sum_{i=1}^n\beta_i(\rho_{\alpha_i}1+\sigma_{\alpha_i}\alpha_i)=\rho_\alpha\left[\gamma_0+\sum_{i=1}^n\gamma_i(\rho_{\alpha_i}1+\sigma_{\alpha_i}\alpha_i)\right]+\sigma_\alpha\left[\sigma_0+\sum_{i=1}^n\sigma_i(\rho_{\alpha_i}1+\sigma_{\alpha_i}\alpha_i)\right].$$

Since the p-adic numbers ρ_α and σ_α are algebraically independent over S_p, these numbers and their products are linearly independent over S_p. Using the above property of combinations of elements from R_p with linearly S_p-independent p-adic coefficients and comparing coefficients in the left-hand and right-hand sides of the last relation, we obtain $\beta_1=\gamma_0$ and $\beta_1\alpha=\delta_0$; all other elements β,γ, and δ are equal to zero. Therefore, $p^k(\eta 1)=\gamma_0$ and $p^k(\eta\alpha)=\gamma_0\alpha$. Setting $\eta 1=\gamma$, we obtain the relation $\eta\alpha=\gamma\alpha$ for every $\alpha\in R_p$. Therefore, η acts on R_p as the left multiplication by γ. A similar assertion holds for $\hat{\eta}$ and $\eta=\hat{\eta}_{|A}$. Therefore, $E(A)=R$ provided R is a Q_p-algebra.

We now pass to the case where R is any ring that satisfies the conditions of the theorem. Since R is a reduced ring, it is a Hausdorff ring and its $\mathbf{Z}$-adic completion $\hat{R}$ contains R as a dense pure subring. We have a representation

$$\hat{R}=\prod_p\hat{R}_p,$$

where $\hat{R}_p$ is the p-adic completion of the reduced part R_p of the tensor product $R \otimes Q_p$. We note that for an element $\alpha \in R$,

$$\alpha = (\alpha_p) \in \prod_p R_p, \qquad \text{where} \quad \alpha_p \in R_p$$

(i.e., the canonical embedding $R \to \hat{R}$ turns R into a subring of the ring $\prod\limits_p R_p$).

For every $\alpha \in R$, we choose ρ_α and σ_α as follows: $\rho_\alpha = (\rho_{\alpha_p})$ and $\sigma_\alpha = (\sigma_{\alpha_p})$, where ρ_{α_p} and σ_{α_p} are p-adic integers that are algebraically independent over S_p (S_p is similarly constructed from R_p). If $R_p = 0$ for some p, then we take $\rho_{\alpha_p} = \sigma_{\alpha_p} = 0$. We define the elements

$$\varepsilon_\alpha = \rho_\alpha 1 + \sigma_\alpha \alpha \in \hat{R}$$

and the group A by

$$A = \langle R, R\varepsilon_\alpha \,|\, \alpha \in R\rangle_*.$$

This is a countable reduced dense subgroup in $\hat{A}$. It is clear that R can be embedded in $E(A)$ as a subring. For $\eta \in E(A)$, we consider a unique extension to some endomorphism $\hat{\eta} \in E(\hat{R})$. Since the subgroups $\hat{R}_p$ are fully characteristic in $\hat{R}$, $\hat{\eta}$ does not move every such subgroup. The considered local case shows that $\hat{\eta}$ acts on $\hat{R}_p$ as the left multiplication by the R_p-component of the element $\eta 1 = \gamma \in R$. Therefore, η coincides with the left multiplication by γ on the ring $\hat{R}$. Therefore, η coincides with the left multiplication by γ on the group A. $\square$

At present, it is known that for the ring R, by Theorem 29.2, there exists $\mathfrak{M}$ of groups A_i such that $E(A_i) \cong R$ and $\mathrm{Hom}(A_i, A_j) = 0$ for all $i \neq j$, where $\mathfrak{M}$ is any previously given cardinal number (see Dugas–Göbel [121], Corner–Göbel [99]). A system of such groups A_i is called an *R-rigid system.*

The theorem does not directly imply that A is a group of finite rank if R is a ring of finite rank. However, this is always possible.

Theorem 29.3 (Corner [93]). *Every reduced torsion-free ring R of finite rank n with identity element is isomorphic to the endomorphism ring of a reduced torsion-free group A of rank not exceeding $2n$.*

The proof is a modification of the proof of Theorem 29.2. Let $\alpha_1, \dots, \alpha_n$ be a maximal linearly independent system of elements of the group R^+, and let $\alpha_1 = 1$. We choose elements $\rho_i = (\rho_{ip})$ $(i = 1, \dots, n)$ such that the p-adic integers $\rho_{1p}, \dots, \rho_{np}$ are algebraically independent over S_p for any p. We take the element $\varepsilon = \rho_1\alpha_1 + \cdots + \rho_n\alpha_n$. It is clear that ε is contained in the $\mathbf{Z}$-adic completion $\hat{R}$ of the ring R. We now set

$$A = \langle R,\, R\varepsilon\rangle_*.$$

It is clear that A is a reduced torsion-free group of rank at most $2n$. As in the preceding theorem, R is embedded as a subring in $E(A)$. Thus, $R \subseteq E(A)$. To prove the inverse inclusion, we take any endomorphism $\eta \in E(A)$ and extend it to an endomorphism $\hat{\eta}$ of the group $\hat{R}$. This is possible, since A is a pure dense subgroup of the group $\hat{R}$. Then

$$\eta\varepsilon = \sum_{i=1}^{n} \rho_i \eta \alpha_i.$$

It follows from the definition of the group A that for some positive integer m and elements $\beta_i, \gamma_i \in R$, we have the relations

$$m(\eta\varepsilon) = \beta_0 \qquad \text{and} \qquad m(\eta\alpha_i) = \beta_i + \gamma_i\varepsilon \quad (i = 1, \dots, n).$$

Substituting them in the relation for $\eta\varepsilon$, we obtain

$$\beta_0 + \gamma_0\Big(\sum_i \rho_i\alpha_i\Big) = \sum_i \rho_i\Big(\beta_i + \gamma_i \sum_j \rho_j\alpha_j\Big).$$

The assertion on combinations of elements of S_p obtained in the proof of Theorem 29.2 implies

$$\beta_0 = 0, \gamma_0\alpha_i = \beta_i \quad (i = 1, \ldots, n), \qquad \gamma_i\alpha_j + \gamma_j\alpha_i = 0 \quad (i, j = 1, \ldots, n).$$

For $i = j = 1$, the last relation implies $\gamma_1 = 0$. For $j = 1$, we obtain $\gamma_i = 0$ $(i = 1, \ldots, n)$. Therefore, $m(\eta\varepsilon) = \gamma_0\varepsilon$ and $m(\eta\alpha_i) = \beta_i = \gamma_0\alpha_i$. Setting $\eta 1 = \gamma \in R$, we obtain $m(\eta 1) = \gamma_0$ (recall that $\alpha_1 = 1$) and $m(\eta\alpha_i) = \gamma_0\alpha_i = m(\gamma\alpha_i)$. Therefore, $\eta\alpha_i = \gamma\alpha_i$ $(i = 1, \ldots, n)$. Therefore, $\eta\alpha = \gamma\alpha$ for all $\alpha \in R$. This shows that the endomorphism η coincides with the left multiplication by γ and $E(A) = R$.

□

It is easy to verify that the group A constructed in Theorem 29.3 has rank $2n$. However, this upper bound for rank cannot be decreased (this will soon be verified). The following lemma is related to Exercise 1.

Lemma 29.4. *Let G be a group such that $E(G)$ is a reduced torsion-free ring of cardinality $<2^{\aleph_0}$. Then G is a reduced torsion-free group.*

Proof. It is known that for every direct summand of the group G, the ring $E(G)$ contains a subring that is isomorphic to the endomorphism ring of this summand (property (b) of Sec. 3). Consequently, the group G cannot contain direct summands that are isomorphic to $\mathbf{Q}$, since $E(\mathbf{Q}) \cong \mathbf{Q}$ and $\mathbf{Q}$ is a divisible group; it does not contain direct summands that are isomorphic to some quasi-cyclic group $Z(p^\infty)$, since $E(Z(p^\infty)) \cong Q_p^*$ and $|Q_p^*| = 2^{\aleph_0}$. Finally, G cannot contain direct summands of the form $Z(p^k)$, since $E(Z(p^k)) \cong Z_{p^k}$ and Z_{p^k} is a torsion ring. The group G is reduced, since it does not contain groups $\mathbf{Q}$ and $Z(p^\infty)$. In addition, G is torsion-free, since it does not contain the groups $Z(p^k)$. □

Proposition 29.5 (Corner [93]). *For every positive integer $n \geq 2$, there exists a torsion-free ring of rank n that is not isomorphic to the endomorphism ring of any group of rank $<2n$.*

Proof. Let us have $n \geq 2$, and let p be a prime integer. There exists an irreducible polynomial $f(x)$ of degree n over the ring of integers $\mathbf{Z}$ such that the equation $f(x) = 0$ has no solution in the ring of $s \times s$-matrices over Q_p^* for $s = 1, 2, \ldots, n-1$. For example, we can take $f(x) = x^n - p$. The well-known Eisenstein criterion guarantees the irreducibility of $f(x)$ over $\mathbf{Z}$. Assume that for some nonzero $s < n$, there exists an $s \times s$-matrix M over Q_p^* such that $f(M) = 0$. Then $M^n = pE_s$ and $(\det M)^n = p^s$, where E_s is the identity $s \times s$-matrix and $\det M$ is the determinant of the matrix M. Since $s \geq 1$ and p is a prime element of the ring Q_p^*, we have $p \mid \det M$. Therefore, $p^n \mid p^s$; this is impossible, since $n > s$. Let $Q(\omega)$ be an extension of the field $\mathbf{Q}$ by using some root ω of the polynomial $f(x)$, and let $Q_p[\omega]$ be the corresponding subring of the field $Q(\omega)$, which is denoted by A. It is clear that A is a reduced torsion-free ring of rank n. In addition, A is a domain, and some nonzero integral multiple of every element from $Q(\omega)$ is contained in $Q_p[\omega]$.

Assume that there is a reduced torsion-free group G of rank $<2n$ with $E(G) \cong A$. We identify $E(G)$ with A. We prove that G is a torsion-free module over the ring A. Let $\alpha g = 0$, where $g \in G$ and $0 \neq \alpha \in A$. Then A contains some integral multiple of the inverse element for α in the field $Q(\omega)$; therefore, there exists an element $\beta \in A$ such that $\beta\alpha = t1$ for some positive integer t. Therefore, $tg = \beta(\alpha g) = 0$, whence $g = 0$. Let r be the rank of the group G as an A-module (r is the dimension of the $Q(\omega)$-space $Q(\omega) \otimes_A G$). It is directly verified that the rank of the group G is equal to rn. By assumption, G is a group of rank $<2n$; therefore, $r = 1$ and G is a group of rank n.

The group G is a Q_p-module; therefore, it is divided by every prime integer $\neq p$. The same is also true for every pure subgroup of it. Let $\mathbf{Z}e_1 \oplus \cdots \oplus \mathbf{Z}e_s$ be a p-basis subgroup of the group G, and let $H = Q_pe_1 \oplus \cdots \oplus Q_pe_s$ be the free Q_p-module generated by this subgroup (H is a pure subgroup in G called a basis submodule of the Q_p-module G, and $e_1, \ldots, e_s$ is a free basis of the Q_p-module H). Since the factor group G/H is divisible, the pure subgroup H is dense in G (see the beginning of the section). Therefore, $\hat{G} = \hat{H} = \hat{Q}_pe_1 \oplus \cdots \oplus \hat{Q}_pe_s$. We now extend the endomorphism ω of the group G to a

Q_p^*-endomorphism $\hat{\omega}$ of the group $\hat{G}$. Since extensions of endomorphisms of the group G to $\hat{G}$ are unique and $f(\omega) = 0$, we have $f(\hat{\omega}) = 0$. Taking the matrix representation of the endomorphism $\hat{\omega}$ with respect to the basis $e_1, \dots, e_s$ (see Theorem 3.11), we obtain that the equation $f(x) = 0$ is solvable in the ring of $s \times s$-matrices over Q_p^*. Therefore, $s = n$ by our choice of the polynomial $f(x)$, and $G = Q_p e_1 \oplus \cdots \oplus Q_p e_n$. This implies that $E(G)$ is isomorphic to the ring of all $n \times n$-matrices over Q_p that have rank n^2 and cannot be isomorphic to the ring A.

We have proved that the ring A is not isomorphic to the endomorphism ring of any reduced torsion-free group of rank $\leq 2n$. The use of Lemma 29.4 completes the proof. $\square$

Some quite general situations are known in which a torsion-free ring of rank n can be realized as the endomorphism ring of some torsion-free group of rank n. Zassenhaus [451] proved that if R is a ring such that R^+ is a free group of rank n, then $R \cong E(A)$ for some torsion-free group A of rank n. Butler [74] proved that if R is a torsion-free ring of rank n such that $R \otimes Q_p$ is a free Q_p-module for every p with $pR \neq R$, then $R \cong E(A)$ for some torsion-free group A of rank n.

Now we consider two simple general applications of Theorem 29.3 that are important for the theory.

Corollary 29.6 (Corner [93]). *Every rational algebra of finite dimension n is isomorphic to the quasi-endomorphism algebra of some torsion-free group of rank $2n$.*

Proof. Let A be a $\mathbf{Q}$-algebra of dimension n. If $v_1, \dots, v_n$ is a basis of the algebra A, then coordinates of all products $v_i v_j$ $(i, j = 1, \dots, n)$ in the basis are called *structural constants* of the algebra A with respect to the basis $v_1, \dots, v_n$. Replacing these basis elements by some integral multiples, we can obtain the situation where all structural constants with respect to the new basis are integers. In addition, the identity element of the algebra A can be included in this basis. Thus, the algebra A has a basis $e_1, \dots, e_n$ $(e_1 = 1)$ such that the free subgroup generated in A by the elements $e_1, \dots, e_n$ is closed with respect to multiplication, and, therefore, forms a subring A' of the ring A. By Theorem 29.3, there exists a torsion-free group G of rank $2n$ whose endomorphism ring is isomorphic to A'. By construction, $E(G) \otimes \mathbf{Q} \cong A' \otimes \mathbf{Q} = A$. The quasi-endomorphism algebra of the group G is isomorphic to the algebra A. $\square$

Corollary 29.7 (Corner [93]). *A (not necessarily commutative) group U of finite order n is isomorphic to the automorphism group of some Abelian torsion-free group G if and only if U is isomorphic to the group of invertible elements of some torsion-free ring A. In this case, U can be realized as the automorphism group of some torsion-free group of rank $\leq 2n$.*

Proof. If U is the automorphism group of some torsion-free group G, then U is the group of invertible elements of the ring $E(G)$, which is a torsion-free ring (see Exercise 1). Conversely, if a finite group U is the group of invertible elements of some torsion-free ring A, then U is also the group of invertible elements of the subring A' generated by U. Elements of the ring A' are linear combinations of elements of U with integral coefficients. Therefore, the additive group of the ring A' is a free group of rank not exceeding the order of the group U. By Theorem 29.3, there exists a torsion-free group G of rank $\leq 2n$ whose endomorphism ring is isomorphic to A'; therefore, the automorphism group of G is isomorphic to U. $\square$

There exist many applications of Theorems 29.2 and 29.3 to the proof of the existence of countable torsion-free groups with some properties that can be expressed by endomorphisms. Among these applications, we mention examples of different pathological direct decompositions of torsion-free groups. These applications are well known; therefore, we do not present them (e.g., see the works of Corner [93, 94] and §95 of Fuchs' work [165]. Some applications are presented in the exercises. We present a rough scheme for proving the existence of torsion-free groups with certain properties. First, we construct a suitable countable reduced torsion-free ring R. Theorem 29.2 implies the existence of a countable torsion-free group G with endomorphism ring $E(G) \cong R$. Then we verify that the group G has the required properties. For example, the surprising result of Corner on the existence of a countable group without nonzero indecomposable direct summands contains a ring R such that every nonzero idempotent e of it is the sum of two nonzero idempotents e_1 and e_2 with $e_1e_2 = e_2e_1 = 0$. It follows from the standard correspondence between direct decompositions of the group and decompositions of idempotent endomorphisms (Proposition 3.9) that the group G is superdecomposable. In this theorem and in some other theorems, the semigroup ring $\mathbf{Z}S$ of some countable semigroup S over the ring $\mathbf{Z}$ is taken as the ring R.

Exercise 1. (a) Prove that the endomorphism ring of a reduced torsion-free group is a reduced torsion-free ring.

(b) What can we say about a group whose endomorphism ring is a reduced torsion-free ring?

Exercise 2 (Corner [93])**.** Prove that the rings $Q_p^* \times Q_p^*$, $\mathbf{Q} \times \mathbf{Q}$, and $F_p \times F_p$ are not endomorphism rings of any group. Therefore, we cannot remove from Theorem 29.2 each of the following three conditions on the ring R: (1) R is countable, (2) R is reduced, and (3) R is torsion-free.

Exercise 3. If S is a countable semigroup with identity element, then the semigroup ring $\mathbf{Z}S$ of the semigroup S over the ring $\mathbf{Z}$ is a countable reduced torsion-free ring; therefore, $\mathbf{Z}S$ can be represented as the endomorphism ring.

Exercise 4. The matrix rings

$$\begin{pmatrix} \mathbf{Z} & \mathbf{Q} \\ 0 & \mathbf{Q} \end{pmatrix}, \qquad \begin{pmatrix} F_p & 0 \\ F_p & \mathbf{Z} \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} Q_p^* & 0 \\ \mathcal{A}_p & \mathbf{Q} \end{pmatrix}$$

($\mathcal{A}_p$ is a field of p-adic numbers) are not isomorphic to the endomorphism ring of any group. However, each of these rings is anti-isomorphic to the endomorphism ring of some group. Two rings R and S are said to be anti-isomorphic if there exists an additive isomorphism $\omega : R \to S$ such that $\omega(xy) = \omega(y)\omega(x)$ for all $x, y \in R$.

Exercise 5. Prove that the matrix ring

$$\begin{pmatrix} Z_{p^m} & Z_{p^m} \\ Z_{p^m} & Z_{p^n} \end{pmatrix} \quad (m \leq n)$$

is isomorphic and anti-isomorphic to the endomorphism ring of some group.

Exercise 6 (Gewirtzman [174])**.** (a) Let G and H be free or divisible torsion-free groups. If $E(G)$ and $E(H)$ are anti-isomorphic rings, then $r(G) = r(H) < \infty$.

(b) If G is a free group of finite rank or a divisible torsion-free group of finite rank, then the ring $E(G)$ has an anti-automorphism.

Exercise 7 (Corner [93])**.** There exists a countable group without indecomposable direct summands.

Exercise 8 (Corner [93])**.** Let q be a positive integer. There exists a countable torsion-free group A such that for positive integers m and n,

$$A^m \cong A^n \iff m \equiv n \pmod{q}.$$

Exercise 9 (Corner [93])**.** There exist countable torsion-free groups A, B, and C such that $A \cong B \oplus C$, $B \cong A \oplus C$, and $A \not\cong B$.

30. The Realization Problem for Endomorphism Rings of Mixed Groups

There exist several variants of realization theorems for endomorphism rings of mixed groups. All these assertions are very similar to Theorem 28.11 of Corner on split realization for endomorphism rings of separable p-groups. For a given torsion-free ring A, a mixed group G with a certain torsion subgroup $T(G)$ such that $G/T(G)$ is a divisible group and $E(G) = A \oplus \mathrm{Hom}(G, T(G))$ (a group direct sum) is constructed. Similar realizations are found in the works of Dugas [117] and Dugas–Göbel [127]. Corner and Göbel [99] have obtained a realization theorem for endomorphism rings of mixed groups.

We have chosen for presentation one specific realization theorem on endomorphism rings of mixed groups proved by Franzen and Goldsmith [161], where close relations with the "torsion-free case" are obtained. Franzen and Goldsmith constructed some functor from the category of reduced torsion-free groups into the category of reduced mixed groups with several good properties. The use of this functor allows one to directly transfer results from the category of torsion-free groups to the category of mixed groups. Using other methods, Göbel and May [180] have also constructed a similar functor.

Instead of the ordinary category of groups, it is more convenient to use the so-called Walker category denoted by Walk. In the category Walk, objects are groups and the set of morphisms from a group G into a group H is

$$\operatorname{Hom}_W(G,H) = \operatorname{Hom}(G,H)/\operatorname{Hom}_t(G,H),$$

where $\operatorname{Hom}_t(G,H) = \operatorname{Hom}(G,T(H))$. In particular, $E_t(G) = \operatorname{Hom}_t(G,G)$, and $E_W(G)$ is the endomorphism ring of the group G in the category Walk. Therefore, the above realization $E(G) = A \oplus \operatorname{Hom}(G,T(G))$ for the ring A can be rewritten in the form $E(G) = A \oplus E_t(G)$, and we can say that the ring A is isomorphic to the endomorphism ring $E_W(G)$ of the mixed group G in the category Walk.

Theorem 28.11 of Corner can be transformed to a result on representations of some rings by endomorphism rings of objects in an acceptable category. Let $\mathcal{P}$ be the category with p-groups as objects and the set of morphisms

$$\operatorname{Hom}_{\mathcal{P}}(G,H) = \operatorname{Hom}(G,H)/\operatorname{Hom}_s(G,H),$$

where $\operatorname{Hom}_s(G,H)$ is the subgroup of small homomorphisms (see Sec. 28, Exercise 1). Then Theorem 28.11 can be reformulated as an assertion about an isomorphism of the ring A and the endomorphism ring $E_{\mathcal{P}}(G)$ of a p-group G in the category $\mathcal{P}$.

The category notions used below are defined in Sec. 1. Sec. 10 of Fuchs' work [164] contains information on pull-backs.

A torsion group G is said to be *separable* if each of its finite subsets can be embedded in a direct summand of the group G, which is a direct sum of cyclic groups. The group G is separable if and only if every p-component of G has no elements of infinite p-height (see the remark before Proposition 20.4).

Assume that T is a separable p-group that is not a periodically complete p-group. Let $\hat{T}$ be the torsion part of the completion of the group T in the p-adic topology. Then $\hat{T}$ is a periodically complete p-group and T is a pure dense subgroup in $\hat{T}$. In addition, $\hat{T}/T$ is a divisible p-group, since T is dense (see Secs. 2 and 29). A divisible p-group is a direct sum of copies of the group $Z(p^\infty)$. Now it is clear that there exists a subgroup $T' \subseteq \hat{T}$ such that T is a pure subgroup in T' and $T'/T \cong Z(p^\infty)$.

If T is a torsion separable group, then considering that $\mathbf{Q}/\mathbf{Z} \cong \sum\limits_p^{\oplus} Z(p^\infty)$ and T is a direct sum of separable p-groups, we obtain the following property. There exists a separable torsion group T' such that T is a pure subgroup in T' and $T'/T \cong \mathbf{Q}/\mathbf{Z}$. In other words, we have a pure exact sequence of groups

$$0 \to T \to T' \overset{\alpha}{\to} \mathbf{Q}/\mathbf{Z} \to 0. \qquad (*)$$

We fix this sequence up to the end of the section.

We take any group X. It is known that the sequence $(*)$ induces another purely exact sequence (Fuchs [164, Theorem 60.4])

$$0 \to T \otimes X \to T' \otimes X \overset{\alpha_X}{\to} \mathbf{Q}/\mathbf{Z} \otimes X \to 0.$$

We denote this sequence by $(*_X)$.

The canonical homomorphism $\mathbf{Q} \to \mathbf{Q}/\mathbf{Z}$ induces the epimorphism $\eta_X : \mathbf{Q} \otimes X \to \mathbf{Q}/\mathbf{Z} \otimes X$. Therefore, we have the diagram

$$\begin{array}{ccc} & & \mathbf{Q} \otimes X \\ & & \big\downarrow{\scriptstyle \eta_X} \\ T' \otimes X & \xrightarrow{\alpha_X} & \mathbf{Q}/\mathbf{Z} \otimes X \end{array}.$$

In such a situation, we can construct a pull-back, i.e., a commutative diagram

$$\begin{array}{ccc} H(X) & \xrightarrow{\pi_X} & \mathbf{Q}\otimes X \\ \Big\downarrow{\scriptstyle\sigma_X} & & \Big\downarrow{\scriptstyle\eta_X} \\ T'\otimes X & \xrightarrow{\alpha_X} & \mathbf{Q}/\mathbf{Z}\otimes X \end{array},$$

where the group $H(X)$ is uniquely defined up to isomorphism. As $H(X)$, we can take the subgroup of the direct sum $(\mathbf{Q}\otimes X)\oplus(T'\otimes X)$ consisting of all pairs (a,b), where $\eta_X(a)=\alpha_X(b)$; let $\pi_X:(a,b)\to a$ and $\sigma_X:(a,b)\to b$. Since η_X is an epimorphism, σ_X is also an epimorphism. The obtained pull-back can be extended to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T\otimes X & \longrightarrow & H(X) & \xrightarrow{\pi_X} & \mathbf{Q}\otimes X & \longrightarrow & 0 \\ & & \Big\| & & \Big\downarrow{\scriptstyle\sigma_X} & & \Big\downarrow{\scriptstyle\eta_X} & & \\ 0 & \longrightarrow & T\otimes X & \longrightarrow & T'\otimes X & \xrightarrow{\alpha_X} & \mathbf{Q}/\mathbf{Z}\otimes X & \longrightarrow & 0 \end{array},$$

where the mapping $T\otimes X\to H(X)$ is defined by $b\to(0,b)$ for $b\in T\otimes X$. It follows from the construction of a pull-back that the kernel $\ker\sigma_X$ is isomorphically mapped by π_X onto $\ker\eta_X$. In turn, the kernel $\ker\eta_X$ is canonically isomorphic to the factor group $X/T(X)$. Indeed, the exact sequence

$$\mathbf{Z}\otimes X\xrightarrow{\varkappa_X}\mathbf{Q}\otimes X\xrightarrow{\eta_X}\mathbf{Q}/\mathbf{Z}\otimes X\longrightarrow 0$$

($\varkappa_X$ is induced by the embedding $\varkappa:\mathbf{Z}\to\mathbf{Q}$) implies that $\ker\eta_X=\operatorname{im}\varkappa_X\cong(\mathbf{Z}\otimes X)/\ker\varkappa_X$. Further, $\ker\varkappa_X=T(\mathbf{Z}\otimes X)$ and $\mathbf{Z}\otimes X$ and X are canonically isomorphic groups. Therefore, $\ker\eta_X\cong X/T(X)$ and $\ker\sigma_X\cong X/T(X)$.

Continuing the consideration of the diagram, we note that $T\otimes X$ is a torsion group and $\mathbf{Q}\otimes X$ is a torsion-free group. Therefore, $T\otimes X$ is the torsion part of the group $H(X)$ and $\mathbf{Q}\otimes X$ is its torsion-free part (i.e., the factor group with respect to the torsion subgroup). Therefore, the torsion-free ranks of $H(X)$ and X are equal (it is clear that the rank of the torsion-free group $\mathbf{Q}\otimes X$ is equal to the torsion-free rank of the group X). We note that if X is a reduced torsion-free group, then $H(X)$ is also a reduced group. Therefore, $H(X)$ does not split (i.e., $H(X)$ is not a direct sum a torsion group and of a torsion-free group).

For a group G, we denote by $U(G)$ the first Ulm subgroup $\bigcap_{n\geq 1} nG$ of the group G. The following lemma holds.

Lemma 30.1. $\ker\sigma_X=U(H(X))$.

Proof. We first show that if T'_p is the p-component of the group T' for some p and B_p is a p-basis subgroup of the group X, then $T'_p\otimes X\cong T'_p\otimes B_p$. We take a p-pure exact sequence $0\to B_p\to X\to X/B_p\to 0$ (the p-pure exactness means that B_p is a p-pure subgroup of the group X). We have the induced sequence

$$0\to T'_p\otimes B_p\to T'_p\otimes X\to T'_p\otimes X/B_p\to 0,$$

which is also p-pure exact. Since T'_p is a p-group and X/B_p is a p-divisible group by the definition of a p-basis subgroup, $T'_p\otimes X/B_p=0$, whence the required isomorphism follows.

Thus, we have

$$T'\otimes X\cong\sum_p{}^{\oplus}(T'_p\otimes B_p),$$

where p runs over some set of prime numbers. Since T'_p is a separable p-group, $U(T'_p\otimes B_p)=0$ for every p. Indeed, B_p is a direct sum of cyclic groups $\langle b_i\rangle$; therefore, $T'_p\otimes B_p$ is a direct sum of the groups $T'_p\otimes\langle b_i\rangle$. If $\langle b_i\rangle\cong\mathbf{Z}$ or $\langle b_i\rangle\cong Z(p^n)$, then $T'_p\otimes\langle b_i\rangle\cong T'_p$ or $T'_p\otimes\langle b_i\rangle\cong T'_p/p^nT'_p$, respectively. Now it is clear that $U(T'_p\otimes B_p)=0$; therefore, $U(T'\otimes X)=0$. However, $\sigma_XU(H(X))\subseteq U(T'\otimes X)=0$, whence $U(H(X))\subseteq\ker\sigma_X$.

Conversely, let z be an element from $\ker \sigma_X$, and let n be any nonzero integer. Then there exists an element $y \in H(X)$ with $z - ny = x \in T(H(X))$ by the divisibility of the factor group $H(X)/T(H(X))$. Then

$$x = \sigma_X(x) = \sigma_X(z) - \sigma_X(ny) = -\sigma_X(ny) \in n(T' \otimes X) \cap (T' \otimes X)$$

(recall that $(H(X)) = T \otimes X$). Since the sequence $(*_X)$ is pure exact, we obtain

$$n(T' \otimes X) \cap (T \otimes X) = n(T \otimes X).$$

Therefore, $x \in n(T \otimes X)$ and $z = ny + x \in nH(X)$. Therefore,

$$z \in \bigcap_{n \geq 1} nH(X) = U(h(X)) \qquad \text{and} \qquad \ker \sigma_X \subseteq U(H(X)).$$

$\square$

The construction of the group $H(X)$ is functorial in the sense indicated below. We naturally obtain a covariant functor from the category of groups into the category of mixed groups if we associate every group X with $H(X)$ and for any homomorphism $f : X \to Y$ define a homomorphism $H(f) : H(X) \to H(Y)$ according to the following rule. Let

$$g_1 = 1 \otimes f : \mathbf{Q} \otimes X \to \mathbf{Q} \otimes Y \qquad \text{and} \qquad g_2 = 1 \otimes f : T' \otimes X \to T' \otimes Y$$

be induced homomorphisms. We have two pull-backs

$$\begin{array}{ccc} H(X) & \xrightarrow{\pi_X} & \mathbf{Q} \otimes X \\ \big\downarrow{\scriptstyle \sigma_X} & & \big\downarrow{\scriptstyle \eta_X} \\ T' \otimes X & \xrightarrow{\alpha_X} & \mathbf{Q}/\mathbf{Z} \otimes X \end{array} \qquad \text{and} \qquad \begin{array}{ccc} H(Y) & \xrightarrow{\pi_Y} & \mathbf{Q} \otimes Y \\ \big\downarrow{\scriptstyle \sigma_Y} & & \big\downarrow{\scriptstyle \eta_Y} \\ T' \otimes Y & \xrightarrow{\alpha_Y} & \mathbf{Q}/\mathbf{Z} \otimes Y \end{array} .$$

For a pair $(a, b) \in H(X)$, we set $H(f)(a, b) = (g_1(a), g_2(b))$. Let us prove that $H(f)(a, b) \in H(Y)$. By the definition of the group $H(Y)$, we need to prove that $\eta_Y(g_1(a)) = \alpha_Y(g_2(b))$. This is proved by a direct calculation by using the relation $\eta_X(a) = \alpha_X(b)$. Therefore, $H(f)$ is a homomorphism, and we obtain the required functor.

We also define two functors U and F. For the group X, we set

$$U(X) = \bigcap_{n \geq 1} nX$$

and for $f : X \to Y$, we set

$$U(f) = f_{|U(X)}.$$

Then U is a covariant functor from the category of groups into itself, which is an example of a subfunctor of the identity functor. Further, F is the so-called functor of the transfer to factor groups with respect to the torsion part (i.e., $F(X) = X/T(X)$, and $F(f)$ is the homomorphism $F(X) \to F(Y)$ induced by f on factor groups with respect to torsion parts).

Proposition 30.2. *The functors UH and F are naturally equivalent.*

Proof. For every group X, we construct some isomorphism $\phi_X : UH(X) \to F(X)$. By Lemma 30.1, $UH(X) = \ker \sigma_X$. It was noted previously that there is an isomorphism $\ker \sigma_X \xrightarrow{\pi_X} \ker \eta_X$ and the subgroup $\ker \eta_X$ is canonically isomorphic to $X/T(X)$. Let

$$\omega_X : \ker \eta_X \to X/T(X) = F(X)$$

be the last isomorphism. We set $\phi_X = \omega_X \pi'_X$, where π'_X is the restriction π_X to $\ker \sigma_X$. It is verified that $\phi_Y UH(f) = F(f)\phi_X$ for any homomorphism $f : X \to Y$. This means that the isomorphisms ϕ_X define a natural functor equivalence $\phi : UH \to F$. $\square$

We denote by $\mathcal{F}$ the category of reduced torsion-free groups.

Theorem 30.3 (Franzen–Goldsmith [161], Göbel–May [180]). *Let T be a separable torsion group, and let T' be a pure extension of the group T by using $\mathbf{Q}/\mathbf{Z}$ such that T' is a separable group. Then there exists a complete embedding $\bar{H} : \mathcal{F} \to$ Walk such that for every group $X \in \mathcal{F}$, the following assertions hold*:

(1) *$\bar{H}(X)$ is a reduced nonsplit group of the same torsion-free rank as that of X*;
(2) *$T(\bar{H}(X)) \cong T \otimes X$*;
(3) *$\bar{H}(X)/T(\bar{H}(X))$ is a divisible group*;
(4) *$U\bar{H}(X) = X$ and $\bar{H}(X)/U\bar{H}(X) \cong T' \otimes X$.*

Proof. A complete embedding $\bar{H} : \mathcal{F} \to$ Walk is a covariant functor such that for any two objects $X, Y \in \mathcal{F}$, the mapping

$$\mathrm{Hom}(X, Y) \to \mathrm{Hom}_W(\bar{H}(X), \bar{H}(Y)), \qquad f \to \bar{H}(f)$$

is an isomorphism and $\bar{H}$ is injective on objects.

For a group $X \in \mathcal{F}$, we set

$$\bar{H}(X) = H(X) \qquad \text{and} \qquad \bar{H}(f) = H(f) + \mathrm{Hom}_t(H(X), H(Y))$$

for a homomorphism $f : X \to Y$. Then (1)–(3) are proved. By Proposition 30.2, the functors UH and F are naturally equivalent and F is the identity functor on $\mathcal{F}$. Consequently, we can identify $U\bar{H}(X)$ with X. Since the mapping σ_X in the above pull-back is an epimorphism, $H(X)/\ker \sigma_X \cong T' \otimes X$. However, $\ker \sigma_X = UH(X)$ by Lemma 30.1. Therefore, $\bar{H}(X)/U\bar{H}(X) \cong T' \otimes X$.

It remains to prove that $\bar{H}$ effects a complete embedding. Let

$$h : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(H(X), H(Y))$$

be the homomorphism induced by H (i.e., $h(f) = H(f)$ for $f : X \to Y$), and let

$$u : \mathrm{Hom}(H(X), H(Y)) \to \mathrm{Hom}(X, Y)$$

be the homomorphism induced by U (i.e., $u(g) = U(g)$ for $g : H(X) \to H(Y)$) (recall that $UH(X) = X$ and $UH(Y) = Y$). Then uh is the identity mapping on $\mathrm{Hom}(X, Y)$. Therefore, h is a monomorphism and u is an epimorphism. Further, we prove that $\ker u = \mathrm{Hom}_t(H(X), H(Y))$. If $g \in \ker u$, then $g = 0$ on $UH(X)$ and $\ker g \supseteq UH(X)$. However, $H(X)/UH(X)$ is a torsion group by (4). Therefore, $\mathrm{im}\, g \cong H(X)/\ker g$ is a torsion group as a homomorphic image of the group $H(X)/UH(X)$. Therefore, $\mathrm{im}\, g \subseteq T(H(Y))$ and $g \in \mathrm{Hom}_t(H(X), H(Y))$. On the other hand, if $g \in \mathrm{Hom}_t(H(X), H(Y))$ (i.e., $\mathrm{im}\, g$ is a torsion group), then $g(UH(X)) = 0$, since

$$g(UH(X)) \subseteq UH(Y) \cap T(H(Y)) = 0.$$

Therefore, $g \in \ker u$ and $\ker u = \mathrm{Hom}_t(H(X), H(Y))$. Therefore, the mapping u induces the isomorphism

$$\bar{u} : \mathrm{Hom}_W(\bar{H}(X), \bar{H}(Y)) = \mathrm{Hom}(H(X), H(Y))/\mathrm{Hom}_t(H(X), H(Y)) \to \mathrm{Hom}(X, Y)$$

such that if

$$\bar{h} : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(H(X), H(Y)/\mathrm{Hom}_t(H(X), H(Y))$$

is the composition of h with the canonical homomorphism on the factor group, then $\bar{h}$ and $\bar{u}$ are mutually inverse isomorphisms. This implies that the mapping

$$\mathrm{Hom}(X, Y) \to \mathrm{Hom}_W(\bar{H}(X), \bar{H}(Y)), \quad f \to \bar{H}(f)$$

is an isomorphism and $\bar{H}$ is a complete embedding. □

Corollary 30.4. *In the situation of Theorem* 30.3*, the ring $E(H(X))$ is a split extension of the ring $E(X)$ by using the ideal $E_t(H(X))$, i.e.,*

$$E(H(X)) = E(X) \oplus E_t(H(X)).$$

Proof. By the proof of the theorem, there exist ring homomorphisms

$$E(X) \xrightarrow{h} E(H(X)) \xrightarrow{u} E(X)$$

such that $uh = 1_{E(X)}$ and $\ker u = E_t(H(X))$. $\square$

Corollary 30.5. *Let A be a countable reduced torsion-free ring. Then there exists a countable reduced mixed group G such that $G/T(G)$ is a divisible group and $E_W(G) \cong A$, i.e., $E(G) \cong A \oplus E_t(G)$. If A has finite rank n, then the torsion-free rank of the group G is equal to $2n$.*

Proof. By Theorem 29.2, there exists a countable reduced group X with $E(X) \cong A$. The group $H(X)$ plays the role of G. It remains to use Theorem 30.3 and Corollary 30.4 (in so doing, we choose an acceptable torsion group in Theorem 30.3; for example, we can take an unbounded countable direct sum of cyclic groups). If the rank of the ring A is equal to n, then by Theorem 29.3 and its proof, the group X can be chosen so that it has rank $2n$. $\square$

Corollary 30.5 admits a considerable generalization. The remark after the proof of Theorem 29.2 shows that for a countable reduced torsion-free ring A, there exist $\mathfrak{M}$ ($\mathfrak{M}$ is any cardinal number) groups X_i such that $E(X_i) \cong A$ and $\mathrm{Hom}(X_i, X_j) = 0$ for all $i \neq j$. Therefore, we can assert that there exist $\mathfrak{M}$ reduced mixed groups G_i such that $G_i/T(G_i)$ is a divisible group, $E_W(G_i) \cong A$, and $\mathrm{Hom}_W(G_i, G_j) = 0$ for all $i \neq j$.

Using the functor $\bar{H}$ from Theorem 30.3, we can transfer different properties of torsion-free groups X formulated in terms of their endomorphism rings to the corresponding mixed groups $H(X)$. We consider two results on such a transfer; two other results are presented in the exercises.

A set $\{X_i\}_{i\in I}$ of nonzero groups is called a **Z**-*rigid system* if $E(X_i) \cong \mathbf{Z}$ and $\mathrm{Hom}(X_i, X_j) = 0$ for all $i \neq j$. It is known that for every infinite cardinal number $\mathfrak{M}$, there exists a **Z**-rigid system consisting of $2^{\mathfrak{M}}$ reduced torsion-free groups of cardinality $\mathfrak{M}$ (Fuchs [165, Theorem 89.2], Shelah [407]). Using the above remark about the transference of properties, we obtain two corollaries.

Corollary 30.6. *For every infinite cardinal number $\mathfrak{M}$, there exist $2^{\mathfrak{M}}$ mixed groups G_i of torsion-free rank $\mathfrak{M}$ such that $G_i/T(G_i)$ is a divisible group, $E_W(G_i) \cong \mathbf{Z}$, and $\mathrm{Hom}_W(G_i, G_j) = 0$ for all $i \neq j$.*

Corollary 30.7. *Every group G_i from the previous corollary is essentially indecomposable in the following sense. In any direct decomposition $G_i = E_1 \oplus E_2$ of this group, one of the summands E_1 or E_2 is a torsion group* (*see the definition of an essentially indecomposable p-group before Proposition* 28.12).

Proof. We have $E(G_i) = \mathbf{Z} \oplus E_t(G_i)$. The remaining part of the proof is similar to the proof of Proposition 28.12. $\square$

The embedding of the category of reduced torsion-free groups into the category of reduced mixed groups, which was found by Göbel and May [180] and which is denoted by G, has the following additional important feature. If X_i ($i \in I$) are reduced torsion-free groups such that they are pure dense subgroups in the same group complete in the **Z**-adic topology and $E(X_i) = E(X_j)$ for all $i, j \in I$, then $E(G(X_i)) = E(G(X_j))$ for all $i, j \in I$. In constructing torsion-free groups with given endomorphism ring (see Dugas and Göbel [121], Corner and Göbel [99]), we usually obtain groups X_i with equal (under identification) endomorphism rings. Using this functor G in the situation after Corollary 30.5, we can obtain $\mathfrak{M}$ groups G_i with isomorphic endomorphism rings.

Exercise 1 (Franzen–Goldsmith [161])**.** There is another method for constructing the functor H from Theorem 30.3. We take a mixed group $M = H(Z)$ of torsion-free rank 1. Prove that the functors H and $M \otimes -$ are naturally equivalent. Here we have $(M \otimes -)X = M \otimes X$ for a reduced torsion-free group X and $(M \otimes -)f = 1 \otimes f : M \otimes X \to M \otimes Y$ for a homomorphism $f : X \to Y$ of reduced torsion-free groups X and Y.

Using Exercises 8 and 9 from Sec. 29, prove Exercises 2 and 3.

Exercise 2. Let q be a positive integer. There exists a countable mixed group G such that for positive integers m and n,

$$G^m \cong G^n \quad \Longleftrightarrow \quad m \equiv n (\operatorname{mod} q).$$

Exercise 3. There exist mixed groups G, E, and F such that

$$G \cong E \oplus F, \qquad E \cong G \oplus F, \qquad \text{and} \qquad G \ncong E.$$

Exercise 4. We fix a prime integer p. Prove that there exists a mixed group G of torsion-free rank continuum such that $T(G) \cong \sum\limits_{i \geq 1}^{\oplus} Z(p^i)$, $G/T(G)$ is a divisible group, $U(G) \cong I_p$ and

$$E(G) = Q_p^* \oplus E_t(G).$$

In addition, such a group G is essentially indecomposable.

Exercise 5. Describe Artinian E-rings and self-injective E-rings.

Remarks. Brief reviews of problems related to isomorphisms of endomorphism rings are given in Sec. 25 for torsion-free groups and in Sec. 27 for mixed groups, Q_p-modules, and Q_p^*-modules. Here, we note that there are studies devoted to isomorphisms of endomorphism groups and endomorphism semigroups. The additive group and the multiplicative semigroup of the endomorphism ring are called the *endomorphism group* and the *endomorphism semigroup,* respectively. Isomorphisms of endomorphism groups were considered in the works of Pierce [350], Sebeldin [395–397], and others. The works of Sebeldin [399] and Puusemp [363] are devoted to isomorphisms of endomorphism semigroups. In [363], it is shown that it is sufficient to require the isomorphism of endomorphism semigroups of torsion groups in the Baer–Kaplansky theorem.

There exists a large bibliography on isomorphisms of endomorphism semigroups or rings of modules. Many publications are partially devoted to endomorphism rings or semigroups of modules close to free modules or generators (see, for example, the works [315–317, 319] of Mikhalev and the reviews of Mikhalev [318] and Markov–Mikhalev–Skornyakov–Tuganbaev [304]). The classical formulation of the isomorphism problem of endomorphism rings for modules admits many variations. For example, we can consider modules over different rings. In this case, it is required that all isomorphisms of endomorphism rings be induced by semilinear isomorphisms of the modules. A semilinear isomorphism ${}_RA \to {}_SB$ of left modules ${}_RA$ and ${}_SB$ over the rings R and S, respectively, consists of a ring isomorphism $\omega : R \to S$ and an additive isomorphism $\varphi : A \to B$ such that $\varphi(ra) = \omega(r)\varphi(a)$ for any $r \in R$ and $a \in A$. For example, every isomorphism of rings of linear transformations of two vector spaces over arbitrary division rings is induced by some semilinear isomorphism of these spaces (Baer [50]). In the context of category theory, Walker posed the following general question. When does the isomorphism of endomorphism rings $\operatorname{End}_R A \cong \operatorname{End}_S B$ imply the category equivalence $F : R - \mathrm{mod} \to S - \mathrm{mod}$ and the isomorphism $B \cong FA$ of S-modules? We give a stronger formulation. When does there exist a category equivalence $F : R-\mathrm{mod} \to S-\mathrm{mod}$ such that $FA = B$ and F induces a given ring isomorphism $\varphi : \operatorname{End}_R A \to \operatorname{End}_S B$ (see Theorem 16.20 of Walker in Faith's work [141])?

We return to realization theorems. Although the endomorphism ring of a separable p-group G is not completely described in Theorem 28.11 of Corner, the known part (i.e., the ring A) allows us to obtain various properties of direct decompositions of a p-group G. Theorem 29.2 describes some of the endomorphism rings of countable reduced torsion-free groups. In 1965, Corner [95] characterized the class of all such endomorphism rings as topological rings. He proved that a topological ring A is topologically isomorphic to the endomorphism ring (with the finite topology) of some countable reduced torsion-free group if and only if A is a complete Hausdorff ring having a basis of neighborhoods of zero consisting of a descending sequence of left ideals $\{N_k\}$, $k = 1, 2, \ldots$ such that A/N_k is a countable reduced torsion-free group for every k.

In the first half of the eighties, the study of realization theorems was the main part of the theory of endomorphism rings. The work of Corner [93] is the starting point of many studies on the realization problem for endomorphism rings of Abelian groups and p-adic modules. Dugas and Göbel [120–124] proved Theorems 29.2 and 28.11 of Corner and the above theorem without the countability assumption. Unfortunately, the proofs of these powerful results are beyond the scope of our work. Therefore, we consider only the classical part of the considered field of the theory of endomorphism rings: the theorems of Corner. We refer the interested reader to the indicated works of Dugas and Göbel and to the review of Göbel [176] containing the history of the question, methods, and a large bibliography. In the work of Corner and Göbel [99], a unified method for attacking realization theorems is developed, which unifies "the torsion case," "the torsion-free case," and "the mixed case." Similar to theorems of Corner, theorems of Dugas and Göbel and other realization theorems obtained in the eighties have important applications to many group-theoretical questions. Among these questions, there are Kaplansky test problems and their different generalizations, pathological direct decompositions, the existence of large indecomposable groups with special properties, and the existence of "very" decomposable (i.e., superdecomposable) groups and their generalizations. As a result, the existence of pathological decompositions for Abelian groups seems to be a norm. They demonstrate the following remark of Kaplansky on Abelian torsion-free groups "... anything that can conceivably happen actually does happen" (Kaplansky [234]). There exist applications of another type. For example, to the problem of constructing rigid and semirigid classes of groups with different properties, and to some problems related to torsion classes of groups. These applications are briefly reflected in the review of Göbel [176], which also contains the corresponding bibliography.

We also note that works of Dugas and Göbel [126], Rychkov [384], and Dugas, Irwin, and Khabbaz [132] contain theorems on the split realization for endomorphism rings of different classes of separable torsion-free groups. The work of Dugas, Göbel, and Goldsmith [129] contains a theorem on split realization for endomorphism rings of modules over a complete discrete valuation ring.

Problem 21 (Fuchs)**.** Obtain a general characterization of endomorphism rings of p-groups. Which of them are endomorphism rings of totally projective p-groups?

The works of Pierce [351] and Liebert [289] contain a characterization of endomorphism rings of separable p-groups.

Problem 22. Is every automorphism of the endomorphism ring of a homogeneous completely transitive torsion-free group inner or, equivalently, topological (see Theorem 25.4)?

Problem 23 (May)**.** Is Theorem 26.4 true if isomorphisms of endomorphism rings are topological?

Problem 24. Find conditions under which two vector groups have isomorphic endomorphism rings.

For separable torsion-free groups, such conditions are obtained by Bazzoni and Metelli [52].

Problem 25. (a) Which groups are determined by their endomorphism rings (topological endomorphism rings) in the class of all groups?

(b) Which torsion-free groups are determined by their endomorphism rings (topological endomorphism rings) in the class of all torsion-free groups?

The notion of a group determined by its endomorphism ring in the given class of groups is defined at the end of Sec. 25. Divisible torsion-free groups and finite groups are determined by their endomorphism rings in the class of all groups. There are many groups determined by topological endomorphism rings in the class of all groups. The class of such groups contains all torsion groups (Theorem 26.1) and the groups from Theorem 26.4. Concerning item (b) of the problem, see Exercises 6 and 7 in Sec. 25.

Problem 26 (May)**.** Let G and H be two mixed groups of torsion-free rank 1.

(a) When is $E(G)$ isomorphic to $E(H)$?

(b) When is every automorphism of the ring $E(G)$ inner?

The necessary condition for (a) is contained in Theorem 27.1. The work of Toubassi and May [421] gives a solution of the problem for a group G whose torsion subgroup is simply presented.

Problem 27 (Kaplansky–May)**.** Let G and H be two mixed p-adic modules.

(a) Find conditions under which every isomorphism (topological isomorphism) $E(G) \cong E(H)$ is induced by some isomorphism $G \cong H$.

(b) When is each automorphism (topological automorphism) of the ring $E(G)$ inner?

Interesting results related to Problem 27 are obtained in the works of May [306, 307].

Problem 28. Characterize endomorphism rings of homogeneous completely transitive torsion-free groups.

Problem 29. (a) For different classes of groups (e.g., torsion groups, separable groups, vector groups, torsion-free groups of finite rank, or mixed groups of torsion-free rank 1), find conditions under which two groups from this class have anti-isomorphic endomorphism rings.

(b) Describe groups whose endomorphism rings have anti-automorphisms. In particular, consider groups from classes listed in (a).

Anti-isomorphisms of endomorphism rings are studied in the paper of Gewirtzman [174] (see Exercises 4–6 of Sec. 29).

Problem 30. Study isomorphisms of endomorphism semigroups of groups from different classes of groups. In particular, do this for endomorphism semigroups of separable groups, vector groups, and mixed groups of torsion-free rank 1.

The papers of Sebeldin [399] and Puusemp [363] are devoted to isomorphisms of endomorphism semigroups of groups.

Part 6

HEREDITARY ENDOMORPHISM RINGS AND RELATED TOPICS

The title of this, most largest, part is not completely exact. In addition to groups with hereditary endomorphism rings, faithful groups are studied, endoflat groups reappear, and equivalences of some categories of Abelian groups and categories of modules over endomorphism rings are found. Although these studies are essentially used in the sections devoted to hereditary endomorphism rings, they are of great importance in their own right. Various applications of the main results are presented.

The study of hereditary endomorphism rings is useful for many reasons. We will see that some classical theorems of Baer on homogeneous completely decomposable torsion-free groups remain true if groups with hereditary endomorphism rings play the role of groups of rank 1. This is most pronounced for torsion-free groups of finite rank. Moreover, if analogs of these theorems hold for a group, then the endomorphism ring of the group is hereditary under some assumptions. Nevertheless, the class of groups with hereditary endomorphism rings is large. It contains different classes of groups. Many torsion-free groups of finite rank described in the last two decades have hereditary endomorphism rings. Classical theorems on homogeneous completely decomposable groups include the following assertions (see Fuchs [165, §§ 86, 87]).

(1) Let G be a completely decomposable homogeneous group of type τ. If the subgroup B of the group G is a homogeneous group of type τ, then B is a completely decomposable group.

(2) If K is a pure subgroup of the torsion-free group B such that B/K is a completely decomposable homogeneous group of type τ and all elements from $B \setminus K$ have type τ, then K is a direct summand of the group B.

(3) A homogeneous torsion-free group G is separable if and only if each of its pure subgroups of finite rank is a direct summand of G.

Assertion (1) is called *"the Baer–Kolettis theorem,"* and assertion (2) is called *"the Baer lemma."*

In the last three sections of this part, we study the naturally arising problem of the description of groups with hereditary endomorphism rings.

We often use categories that are either complete subcategories of the category of Abelian groups or complete subcategories of the category of right modules over the endomorphism ring of some group.

31. Self-Small Groups

A group A is said to be *self-small* if the image of every homomorphism $\varphi : A \to \sum\limits_{i\in I}^{\oplus} A_i$, where all the groups $A_i \cong A$ and I is an arbitrary set of subscripts, is contained in a finite sum of some groups A_i. Therefore, the smallness is some finiteness condition. Finitely generated groups are examples of self-small groups. We present different criteria of self-smallness proved by Arnold and Murley [36]. A large part of them are related to endomorphism rings.

If A is a group and $\{A_i\}_{i\in I}$ is a set of groups isomorphic to A, then it is convenient to assume that all the groups A_i in the direct sum $\sum\limits_{i\in I}^{\oplus} A_i$ are equal to A.

We begin with the following simple remarks.

(a) In the definition of a self-small group, it is possible to consider only countable direct sums of copies of the group A.

Indeed, assume that the image of every homomorphism

$$A \to \sum_{i=1}^{\infty}{}^{\oplus} A_i \quad (A_i = A)$$

is contained in a finite sum of some groups A_i. Let

$$\varphi : A \to \sum_{i\in I}{}^{\oplus} A_i$$

be a homomorphism, where $A_i \cong A$ $(i \in I)$ and I is an arbitrary set of subscripts. Assume that the image φA is not contained in a finite sum of any summands A_i. In this case, there exists a countable set of the groups A_i (e.g., $A_1, A_2, \dots$) such that $\pi_j\varphi \neq 0$ for all $j = 1, 2, \dots$, where $\pi_j : \sum\limits_{i\in I}^{\oplus} A_i \to A_j$ are natural projections. Then the homomorphism

$$\pi\varphi : A \to \sum_{i=1}^{\infty}{}^{\oplus} A_i, \qquad \text{where} \quad \pi : \sum_{i\in I}{}^{\oplus} A_i \to \sum_{i=1}^{\infty}{}^{\oplus} A_i$$

is a projection, has the property that the image $(\pi\varphi)A$ is not contained in a finite sum of any summands A_i.

(b) A direct summand of a self-small group is a self-small group.

(c) A self-small group cannot be an infinite direct sum of nonzero summands.

For the proof of (c), assume that a self-small group A is equal to the sum $\sum\limits_{i=1}^{\infty\,\oplus} B_i$, where all $B_i \neq 0$. We denote by $\varkappa_i : B_i \to A$ the embedding $(i \in I)$. We consider the homomorphism $\varphi : A \to \sum\limits_{i=1}^{\infty\,\oplus} A_i$ $(A_i = A)$ that maps from B_i into A_i and coincides with $\varkappa_i$ on B_i for every $i \geq 1$. However, the existence of such a homomorphism contradicts the self-smallness of the group A.

Let A be a group. For any nonempty subset X of the group A, let

$$X^* = \{\alpha \in E(A) \mid \alpha X = 0\}.$$

We set

$$Y^* = \{\alpha \in A \mid \alpha a = 0 \qquad \text{for all} \quad \alpha \in Y\}$$

for any nonempty subset Y of the ring $E(A)$. Then X^* is a left ideal of the ring $E(A)$ and Y^* is a subgroup of the group A; in so doing,

$$X \subseteq X^{**} \quad \text{and} \quad X^* = X^{***}, \qquad Y \subseteq Y^{**} \quad \text{and} \quad Y^* = Y^{***}.$$

A left ideal I of the ring $E(A)$ is called an *annihilator left ideal* if $I = X^*$ for some subset $X \subseteq A$. A subgroup B of the group A is called a *kernel* subgroup if $B = Y^*$ for some subset $Y \subseteq E(A)$. The proof of the following lemma follows directly from definitions and the relations $X^* = X^{***}$ and $Y^* = Y^{***}$.

Lemma 31.1. (a) *A left ideal I of the ring $E(A)$ is an annihilator left ideal if and only if $I = I^{**}$. A subgroup B of the group A is a kernel subgroup if and only if $B = B^{**}$.*

(b) *The correspondences*

$$B \to B^*, \qquad I \to I^*$$

are mutually inverse and are order-preserving correspondences between all kernel subgroups of the group A and all annihilator left ideals of the ring $E(A)$.

In terms of kernel subgroups and annihilator ideals, we formulate one general result, which will be used in subsequent studies.

Proposition 31.2. *For a group A, the following conditions are equivalent*:

(a) *A is not a self-small group*;

(b) *there exists a chain $B_1 \subseteq \ldots \subseteq B_n \subseteq \ldots$ of proper subgroups of the group A such that $A = \bigcup\limits_{n=1}^{\infty} B_n$ and $B_n^* \neq 0$ for all n*;

(c) *there exists a chain $B_1 \subseteq \ldots \subseteq B_n \subseteq \ldots$ of proper kernel subgroups of the group A such that $A = \bigcup\limits_{n=1}^{\infty} B_n$*;

(d) *there exists a chain $I_1 \supseteq \ldots \supseteq I_n \supseteq \ldots$ of nonzero annihilator left ideals of the ring $E(A)$ such that $A = \bigcup\limits_{n=1}^{\infty} I_n^*$. In this case, $\bigcap\limits_{n=1}^{\infty} I_n = 0$.*

Proof. (a) $\Longrightarrow$ (b). We form a direct sum $\sum\limits_{i=1}^{\infty}{}^{\oplus} A_i$ with $A_i = A$ for all i. Let $\pi_j : \sum\limits_{i=1}^{\infty}{}^{\oplus} A_i \to A_j$ be the projection ($j \geq 1$). Since A is not a self-small group, there exists a homomorphism $\varphi : A \to \sum\limits_{i=1}^{\infty}{}^{\oplus} A_i$ such that $\pi_i \varphi \neq 0$ for all i. For every $n \geq 1$, we set

$$B_n = \{x \in A \mid \pi_i \varphi(x) = 0 \qquad \text{for all} \quad i > n\}.$$

Then $B_1 \subseteq \ldots \subseteq B_n \subseteq \ldots$ is a chain of proper subgroups of the group A and $A = \bigcup\limits_{n=1}^{\infty} B_n$. It is also clear that $B_n^* \neq 0$ for all n.

(b) $\Longrightarrow$ (c). In (b), we replace every subgroup B_n by B_n^{**}. Doing so, we obtain an ascending chain of kernel subgroups with $A \bigcup\limits_{n=1}^{\infty} B_n^{**}$ (since $B_n \subseteq B_n^{**}$). If we assume that $B_n^{**} = A$, then $B_n^* = B_n^{***} = A^* = 0$; this is impossible by (b). Consequently, all subgroups B_n^{**} are proper.

The implication (c) $\Longrightarrow$ (d) follows from Lemma 31.1. Using (c), we set $I_n = B_n^*$. Then $I_n^* = B_n$ and $\bigcap\limits_{n=1}^{\infty} I_n = 0$.

(d) $\Longrightarrow$ (a). First, we show that there exists an infinite subset $S \subseteq E(A)$ such that the set $S \setminus (X^* \cap S)$ is finite for all finite subsets X from A. For every n such that $I_n \neq I_{n+1}$, we choose some $\alpha_n \in I_n \setminus I_{n+1}$. For every n such that $I_n = I_{n+1}$, we set $\alpha_n = 0$. Since $\bigcap\limits_{n=1}^{\infty} I_n = 0$ and $I_n \neq 0$, we obtain that $S = \{\alpha_n \mid n \in \mathbf{N}\}$ is an infinite subset in $E(A)$. If X is a finite subset of the group A, then $X \subseteq I_n^*$ for some n, since $A = \bigcup\limits_{n=1}^{\infty} I_n^*$. Consequently, $I_m = I_m^{**} \subseteq X^*$ for all $m \geq n$. Therefore,

$$S \setminus (X^* \cap S) \subseteq \{\alpha_1, \ldots, \alpha_{n-1}\}$$

and $S \setminus (X^* \cap S)$ is a finite set.

For every $\alpha \in S$, we denote by A_α a copy of the group A. We form a direct sum $\sum\limits_{\alpha \in S}^{\oplus} A_\alpha$ and define the homomorphism

$$\varphi : A \to \sum_{\alpha \in S}^{\oplus} A_\alpha, \qquad \varphi(a) = \sum_{\alpha \in S} \alpha(a),$$

where $\alpha(a) \in A_\alpha$ $(\alpha \in S)$. Such φ exists, since $S \setminus (\{a\}^* \cap S)$ is a finite set for every $a \in A$; therefore, for a given α, almost all αa are equal to 0. Since S is an infinite set,

$$\varphi A \not\subseteq \sum_{\alpha \in S'}^{\oplus} A_\alpha$$

for every finite subset $S' \subseteq S$. Therefore, A is not a self-small group. $\square$

Corollary 31.3. (a) *Let $\{B_i\}_{i \in I}$ be a family of self-small groups, and let $A = \prod\limits_{i \in I} B_i$. Assume that* $\mathrm{Hom}(B_i, B_j) = 0$ *for all $i \neq j$ and $\alpha = 0$ for every $\alpha \in E(A)$ such that $\alpha\big(\sum\limits_{i \in I}^{\oplus} B_i\big) = 0$. Then A is a self-small group.*

(b) *Let $\{B_i\}_{i \in I}$ be a family of self-small reduced torsion-free groups such that for every prime integer p and any different subscripts i and j, the relation $pB_i \neq B_i$ implies $pB_j = B_j$. Then $\prod\limits_{i \in I} B_i$ is a self-small group.*

Proof. (a) Assume that A is not a self-small group. By Proposition 31.2(c), we can choose a chain $A_1 \subseteq \ldots \subseteq A_n \subseteq \ldots$ of proper kernel subgroups of the group A with $A = \bigcup\limits_{n=1}^{\infty} A_n$. We prove that for every $i \in I$, there exists the least integer $n(i)$ such that $B_i \subseteq A_{n(i)}$. Assume the contrary. Then $B_i \cap A_1 \subseteq \ldots \subseteq B_i \cap A_n \subseteq \ldots$ is a chain of proper subgroups of the group B_i with $B_i = \bigcup\limits_{n=1}^{\infty} (B_i \cap A_n)$, since $A = \bigcup\limits_{n=1}^{\infty} B_n$. We consider some intersection $B_i \cap A_n$. By the choice of the subgroup A_n, we have $A_n = Y^*$ for some nonempty subset $Y \subseteq E(A)$. We take any element $\alpha \in Y$. Since $\mathrm{Hom}(B_i, B_j) = 0$ for $i \neq j$, α induces the endomorphism α_i on the group B_i. We have $\alpha_i(B_i \cap A_n) = 0$ and $\alpha_i \neq 0$, since otherwise $B_i \cap A_n = B_i$. This means that $(B_i \cap A_n)^* \neq 0$ in $E(B_i \cap A_n)$; this contradicts Proposition 31.2(b). Therefore, $B_i \subseteq A_{n(i)}$ for some least integer $n(i)$.

Assume that the set $\{n(i) \mid i \in I\}$ is bounded by some m. We take some nonzero $\alpha \in A_m^*$ (we note that $A_m^* \neq 0$, since $A_m = A_m^{**} \neq A$). For this α, we obtain $\alpha B_i = 0$ for all $i \in I$. By assumption, $\alpha A = 0$; this is a contradiction. Now assume that the set $\{n(i) \mid i \in I\}$ is not bounded. This means that for every $m \geq 1$, there exist a subscript $i \in I$ and an element $b_i \in B_i$ with $b_i \notin A_m$. We construct a vector $a = (b_i) \in A$ such that all the elements b_i are the corresponding coordinates of a and the remaining coordinates of a are equal to zero. It is easy to verify that the assumption of the corollary implies that $\mathrm{Hom}(\prod\limits_{i \neq j} B_i, B_j) = 0$ for every $j \in I$. Therefore, the projections $\Pi B_i \to B_j$ commute with endomorphisms of the group A (i.e., the projections are contained in the center of the ring $E(A)$). This implies that the kernel subgroups A_n are invariant with respect to the projections. Therefore, if we assume that $a \in A_m$ for some m, then all elements b_i are contained in A_m; this is impossible by the choice of these elements. Thus $a \notin A_m$ for all m; this contradicts the relation $A = \bigcup\limits_{n=1}^{\infty} A_n$. This is a contradiction; Therefore, the group A is self-small.

(b) We set $A = \prod\limits_{i \in I} B_i$. We can use (a). The assumptions on the groups B_i imply $\mathrm{Hom}(B_i, B_j) = 0$ for all $i \neq j$. It is easy to verify that the factor group $\Pi B_i / \sum^{\oplus} B_i$ is divisible. Therefore, it is clear that $\alpha = 0$ for every $\alpha \in E(A)$ such that $\alpha B_i = 0$ for all $i \in I$. $\square$

We present two important criteria of the self-smallness of the group A. One of them is related to the finite topology. It is clear that the finite topology on the ring $E(A)$ is discrete (i.e., $X^* = 0$ for some finite subset $X \subseteq A$) if and only if $B^* = 0$ for some finitely generated subgroup B of the group A.

Corollary 31.4. *The group A is self-small in each of the following two cases*:

(a) *the ring $E(A)$ is countable*;
(b) *the finite topology of the ring $E(A)$ is discrete.*

Proof. (a) Assume that $E(A)$ is a countable ring but the group A is not self-small. By Proposition 31.2(d), there exists a chain $I_1 \supset \ldots \supset I_n \supset \ldots$ of nonzero annihilator left ideals of the ring $E(A)$ such that $A = \bigcup\limits_{n=1}^{\infty} I_n^*$. For every n, we choose $\alpha_n \in I_n \setminus I_{n+1}$; let $S = \{\alpha_n \mid n \in \mathbf{N}\}$. For every subset $L \subseteq N$, we define

$$\alpha_n^L = \begin{cases} \alpha_n & \text{if } n \in L; \\ 0 & \text{if } n \notin L. \end{cases}$$

We set $\alpha^L(a) = \sum\limits_{i=1}^{n} \alpha_i^L(a)$. Then α^L is a well-defined endomorphism of the group A such that for every $a \in A$, $\alpha_i^L(a) = 0$ for almost all i.

We prove that $\alpha^L = \alpha^K$ if and only if $L = K$ (in this case, the ring $E(A)$ is not countable, and we obtain a contradiction). Assume that $L \neq K$. We choose the least integer n with $\alpha_n^L \neq \alpha_n^K$; for example, $\alpha_n^L = \alpha_n$ and $\alpha_n^K = 0$. Since $\alpha_n \notin I_{n+1}$, there exists an element $a \in I_{n+1}^*$ such that $a \notin \ker \alpha_n$. Therefore, $\alpha^L(a) - \alpha^K(a) = \alpha_n(a) \neq 0$ and $\alpha^L \neq \alpha^K$.

(b) Assume that the ring $E(A)$ is discrete in the finite topology, but the group A is not self-small. By Proposition 31.2(b), there exists a chain $B_1 \subseteq \ldots \subseteq B_n \subseteq \ldots$ of proper subgroups of the group A such that $A = \bigcup\limits_{n=1}^{\infty} B_n$ and $B_n^* \neq 0$ for all n. Since the ring $E(A)$ is discrete, $B^* = 0$ for some finitely generated subgroup $B \subset A$. Then $B \subseteq B_m$ for some m and $B_m^* \subseteq B^* = 0$; this is a contradiction. Therefore, A is a self-small group. □

For a countable group A, we have a quite complete result.

Corollary 31.5. *For a countable group A, the following conditions are equivalent*:

(1) *$E(A)$ is a discrete in the finite topology ring*;
(2) *$E(A)$ is a countable ring*;
(3) *A is a self-small group.*

Proof. It is sufficient to prove only the implications (3) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (2).

(3) $\Longrightarrow$ (1). The set $\{X^* \mid X$ is an arbitrary finite subset of the group $A\}$ forms a countable basis of neighborhoods of zero for the finite topology of the ring $E(A)$. It is clear that we can choose a set $\{X_n \mid n = 1, 2, \ldots\}$ of finite subsets of the group A such that $X_1^* \supseteq X_2^* \supseteq \ldots$, $\bigcap\limits_{n=1}^{\infty} X_n^* = 0$; if V is a neighborhood of zero, then $X_m^* \subseteq V$ for some m (this follows from the property that $\{X^* \mid |X| < \aleph_0, X \subseteq A\}$ is a basis of neighborhoods of zero). If x is any element of the group A, then $X_m^* \subseteq \{x\}^*$ for some m; therefore, $x \in X_m^{**}$. Therefore, $A = \bigcup\limits_{n=1}^{\infty} X_n^{**}$. Since A is a self-small group, it follows from Proposition 31.2(d) that the existence of the obtained chain of left ideals X_n^* $(n \geq 1)$ implies $X_n^* = 0$ for some n (i.e., $E(A)$ is discrete).

(3) $\Longrightarrow$ (2). By the above argument, $E(A)$ is a discrete ring, i.e., $X^* = 0$ for some finite subset $X = \{x_1, \dots, x_n\}$ of the group A. We define

$$f : E(A) \to \sum_{i=1}^{n}{}^{\oplus} A_i \ (A_i = A), \qquad f(\alpha) = \alpha(x_1) + \cdots + \alpha(x_n),$$

where $\alpha(x_i) \in A_i$. Then f is a monomorphism and $E(A)$ is a countable ring, since $\sum\limits_{i=1}^{n}{}^{\oplus} A_i$ is a countable group. $\square$

We consider groups whose endomorphism rings satisfy the minimum condition on annihilator left ideals.

Proposition 31.6. *If the ring $E(A)$ satisfies the minimum condition on annihilator left ideals, then*
(1) *the finite topology of the ring $E(A)$ is discrete*;
(2) *the endomorphism ring of every direct summand of the group A satisfies the minimum condition on annihilator left ideals*;
(3) *the group A is a finite direct sum of indecomposable groups.*

Proof. (1) If $X^* = 0$ for some finite subset X of the group A, then the ring $E(A)$ is discrete. Otherwise, there exists a finite subset Y such that Y^* is a minimal nonzero element in the set of all X^*. If X is a finite subset of the group A, then the inclusion $(Y \cup X)^* \subseteq Y^*$ and the minimality of Y^* imply $Y^* \cap X^* = (Y \cup X)^* = Y^*$. Therefore, $Y^* \subseteq \cap X^*$, where X runs over all finite subsets of the group A. Therefore, $Y^* = 0$; this is a contradiction. Therefore, $X^* = 0$ for some finite subset X.

(2) We have $A = B \oplus C$. We show that the ring $E(B)$ satisfies the minimum condition on annihilator left ideals. We identify $E(B)$ with some subring of the ring $E(A)$ in the regular way (see Sec. 3, property (b)). In addition, $E(B) \subseteq C^* \subseteq E(A)$. Let I be an annihilator left ideal of the ring $E(B)$. We set $\Psi(I) = (I^* \oplus C)^*$. Then $\Psi(E(B)) = C^*$ and Ψ is an order-preserving mapping from the set of annihilator left ideals of the ring $E(B)$ into the set of annihilator left ideals of the ring $E(A)$. It is sufficient to prove that Ψ is an injective mapping. We take two different annihilator left ideals I and J of the ring $E(B)$. For definiteness, let $I \not\subseteq J$. We choose some $\varphi \in I \setminus J$. Then $\varphi \in \Psi(I)$ and $\varphi(J^* \oplus C) = \varphi(J^*) \neq 0$ (i.e., $\varphi \notin \Psi(J)$ and $\Psi(I) \neq \Psi(J)$).

(3) First, the group A has at least one indecomposable direct summand. Otherwise, for every n, we have the decompositions

$$A = A_1 \oplus \cdots \oplus A_n \oplus B_n \qquad \text{and} \qquad B_n = A_{n+1} \oplus B_{n+1},$$

where all $A_n \neq 0$. Then A_n^* is a nonzero annihilator left ideal and the set $\{A_n^* \mid n \geq 1\}$ has no minimal element. Therefore, $A = A_1 \oplus B_1$, where A_1 is a nonzero indecomposable group. If $B_1 \neq 0$, then we consider (2) and similarly have $B_1 = A_2 \oplus B_2$, where A_2 is a nonzero indecomposable group, and so on. The above argument implies that $B_n = 0$ for some n, and $A = A_1 \oplus \cdots \oplus A_n$ is a direct sum of indecomposable groups. $\square$

We consider torsion self-small groups and torsion-free self-small groups separately. The number of torsion self-small groups is very small. In contrast, the number of torsion-free self-small groups is quite large.

Proposition 31.7. *Every torsion self-small group is finite.*

Proof. Let A be a self-small torsion group. For every positive integer n, we set $B_n = A[n!]$. Then $B_1 \subseteq \ldots \subseteq B_n \subseteq \ldots$, $A = \bigcup\limits_{n=1}^{\infty} B_n$, and $(n!)1_A \in B_n^*$. By Proposition 31.2(b), we obtain $B_n = A$ for some n, i.e., A is a bounded group. A bounded group is a direct sum of cyclic groups. Since the group A is self-small, the sum is a finite sum. Therefore, A is a finite group. $\square$

By Corollary 31.4, a torsion-free group A with countable ring $E(A)$ is self-small. All torsion-free groups of finite rank are examples of such groups. Another example of a self-small group is given by any torsion-free group A such that every nonzero endomorphism of the group A is a monomorphism. It is obvious that the ring $E(A)$ satisfies the minimum condition on annihilator ideals in this case. The class of torsion-free groups whose nonzero endomorphisms are monomorphisms is not small. For example, the following proposition holds.

Proposition 31.8. *If the endomorphism ring $E(A)$ of some reduced torsion-free group A is a Dedekind domain, then every nonzero endomorphism of the group A is a monomorphism.*

Proof. Since the ring $E(A)$ has no nontrivial idempotents, A is an indecomposable group; therefore, A is an indecomposable $E(A)$-module. Assume that A is not a torsion-free $E(A)$-module. In the theory of modules over Dedekind domains, it is known that the $E(A)$-module A is isomorphic to the factor module $E(A)/P$, where P is a nonzero ideal of the ring $E(A)$. We take any nonzero $\alpha \in P$. Then $\alpha E(A) \subseteq P$, whence $\alpha A = 0$; this contradicts the relation $\alpha \neq 0$. Therefore, A is a torsion-free $E(A)$-module. This means that $\ker \gamma = 0$ for any $0 \neq \gamma \in E(A)$. □

We formulate several conditions in terms of its quasi-endomorphism ring $\mathcal{E}(A)$ that guarantee the self-smallness of the torsion-free group A (for the ring $\mathcal{E}(A)$, see Sec. 5). Every quasi-endomorphism of the group A can be considered as an endomorphism of the divisible hull $A \otimes \mathbf{Q}$ of the group A. We recall the identifications formulated in Secs. 4 and 5. We identify the group A and the subgroup in $A \otimes \mathbf{Q}$ consisting of the elements $a \otimes 1$ for all $a \in A$; similarly, we identify the rings $E(A)$ and $E(A) \otimes \mathbf{Q} = \mathcal{E}(A)$.

For a subset X of the torsion-free group A, we define $X_* = \{\alpha \in \mathcal{E}(A) \mid \alpha X = 0\}$, a left ideal of the ring $\mathcal{E}(A)$. If J is a left ideal of the ring $\mathcal{E}(A)$, then we define $J_* = \{a \in A \mid \alpha a = 0 \text{ for all } \alpha \in J\}$. An ideal J is called an *annihilator ideal* if $J = X_*$ for some subset X of the group A.

Proposition 31.9. *For a torsion-free group A, the mappings*

$$\begin{cases} I \to I \otimes \mathbf{Q}, \\ J \to J \cap E(A) \end{cases}$$

are mutually inverse and are order preserving correspondences between annihilator left ideals of the ring $E(A)$ and annihilator left ideals of the ring $\mathcal{E}(A)$.

The proof follows from the following remarks. For an annihilator left ideal I of the ring $E(A)$, we choose a subset X of the group A such that $I = X^*$. Then $I \otimes \mathbf{Q} = X_*$ in the ring $\mathcal{E}(A)$. Similarly, if $J = X_*$ in $\mathcal{E}(A)$, then $J \cap E(A) = X^*$ in $E(A)$. Further,

$$(I \otimes \mathbf{Q}) \cap E(A) = I \quad \text{and} \quad (J \cap E(A)) \otimes \mathbf{Q} = J$$

for annihilator left ideals I and J of the rings $E(A)$ and $\mathcal{E}(A)$, respectively. In the proof of the first relation, we use the property that annihilator ideals of the ring $E(A)$ are pure in $E(A)$. Finally, it is clear that our correspondences preserve the order on sets of annihilator ideals.

The quasi-endomorphism rings of the torsion-free group A can be provided by a topology such that annihilator left ideals of all finite subsets of the group A form a basis of neighborhoods of zero (this topology is called the *finite* topology). It follows from the proof of Proposition 31.9 that the finite topology of the ring $\mathcal{E}(A)$ induces the finite topology of the ring $E(A)$. This property and Proposition 31.9 imply the following result.

Corollary 31.10. *Let A be a torsion-free group. Then*

(1) *the ring $E(A)$ satisfies the minimum condition on annihilator left ideals if and only if the ring $\mathcal{E}(A)$ satisfies this condition*;
(2) *the ring $E(A)$ is discrete if and only if the ring $\mathcal{E}(A)$ is discrete.*

Exercise 1. Prove that if A is a self-small group, then for every positive integer n, the group A^n is self-small.

Exercises 2–6 are taken from the work of Arnold and Murley [36].

Exercise 2. There exists a self-small torsion-free group A such that the ring $E(A)$ is not discrete in the finite topology.

Exercise 3. There exists a countable torsion-free group A such that $E(A)$ is a discrete ring that does not satisfy the minimum condition on annihilator left ideals.

Exercise 4. (a) If A is a reduced torsion-free group A and there exists a p such that A/pA is a finite group and $\bigcap_{n=1}^{\infty} p^n A = 0$, then A is a self-small group.

(b) There exists a countable self-small torsion-free group A such that the factor group A/pA is infinite for every prime p.

Exercise 5. Let A be a reduced torsion-free group. If the group $B \otimes A$ is self-small for some torsion-free group B, then A is a self-small group.

Exercise 6. Assume that A is a self-small mixed group. Then for each prime integer p, the p-component A_p of the group A is finite or zero, the factor group A/pA is finite for all p with $A_p \neq 0$, and if F is a torsion-free subgroup of the group A such that A/F is a torsion group, then A/F is a p-divisible group for almost all p with $A_p \neq 0$.

The definition of a self-small group is directly extended to modules. In the work of Arnold and Murley [36], self-small modules are considered. All assertions, beginning with Lemma 31.1 and ending Proposition 31.6 (except for Corollary 31.3), hold for modules. It is possible to study groups A that are self-small left $E(A)$-modules.

Exercise 7. Let A be some R-module. If A is a self-small group, then A is a self-small R-module.

Exercise 8. Let A be an $E(R)$-module (see Definition 6.1). Then A is a self-small R-module if and only if A is a self-small group.

Exercise 9. Let A be a group A that is a self-small module over the ring $E(A)$. Prove that A cannot be an infinite direct sum of nonzero fully characteristic subgroups.

Exercise 10. Prove that any p-group A is a self-small $E(A)$-module.

Exercise 11. (a) Prove that a homogeneous separable torsion-free group A is a self-small $E(A)$-module.

(b) Find what separable torsion-free groups are self-small modules over their endomorphism rings (use Exercise 7 from Sec. 19).

A module A is said to be *small* if for any family of modules B_i $(i \in I)$, the image of every homomorphism $A \to \sum_{i\in I}^{\oplus} B_i$ is contained in a finite sum of some modules B_i.

Exercise 12. Describe small Abelian groups.

Exercise 13. If P is a projective S-module and $R = \mathrm{End}_S P$, then prove that the following conditions are equivalent:

(1) the module P is self-small;
(2) the ring R is discrete in the finite topology;
(3) the module P is finitely generated.

[Hint: use the known Kaplansky theorem on the decomposability of a projective module into a direct sum of countably generated projective modules].

32. Equivalences of Categories of Groups and Categories of Modules over Endomorphism Rings

We prove two useful theorems on the equivalence of some categories of groups related to a given group A and some categories of modules over the endomorphism ring $E(A)$ of the group A. These results have various applications. A general idea of the use of these and other similar theorems that will be obtained later is the use of one of the proved equivalences to transfer certain possible good properties of modules over endomorphism rings to the corresponding groups. These good properties often arise at the expense of a specially chosen endomorphism ring. This section contains an effective demonstration of the above idea of the application of theorems on equivalence of categories to group-theoretical problems. We indicate quite large classes of groups, where the Krull–Schmidt theorem on the isomorphism of direct decompositions holds. In this section, we obtain interesting results on the finite exchange property and the cancellation property for torsion-free groups of finite rank.

We will prove the equivalence of categories by using the basis functors $\mathrm{Hom}(A, -)$ and $(-) \otimes_R A$ defined for any module in Sec. 1. We repeat more explicitly the definitions of these functors and other accompanying notions.

Let A be a group, and let $R = E(A)$. We denote by $\mathcal{A}b$ and $\mathrm{mod} - R$ the category of all Abelian groups and the category of all right R-modules, respectively. For any group B, the group $\mathrm{Hom}(A, B)$ can be naturally considered as a right R-module if we set

$$\mathrm{Hom}(A, B) \times R \to \mathrm{Hom}(A, B), \qquad (f, \alpha) = f\alpha$$

for $f \in \mathrm{Hom}(A, B)$, $\alpha \in R$. We denote by H the covariant functor $\mathrm{Hom}(A, -) : \mathcal{A}b \to \mathrm{mod} - R$, i.e., $H(B) = \mathrm{Hom}(A, B)$ and $H(\varphi) : H(B) \to H(C)$ for every group B and each homomorphism of groups $\varphi : B \to C$, where $H(\varphi)(f) = \varphi f$ $(f \in H(B))$. On the other hand, there exists a covariant functor $(-) \otimes_R A :$ $\mathrm{mod} - R \to \mathcal{A}b$; we denote it by T. We have $T(M) = M \otimes_R A$ and $T(g) = g \otimes 1 : T(M) \to T(L)$ for every right R-module M and each homomorphism of right R-modules $g : M \to L$. Here $(g \otimes 1)(m \otimes a) = g(m) \otimes a$ for elements $m \in M$ and $a \in A$. The functor H is left exact in the sense that $H(\varphi)$ is a monomorphism for any monomorphism $\varphi : B \to C$ (we say that H maps from monomorphisms into monomorphisms). More precisely, every exact sequence of groups

$$0 \to B \xrightarrow{\varphi} C \xrightarrow{\psi} D \to 0$$

induces the exact sequence of right R-modules

$$0 \to H(B) \xrightarrow{H(\varphi)} H(C) \xrightarrow{H(\psi)} H(D).$$

The functor T is right exact. If $g : M \to L$ is an epimorphism of right R-modules, then $T(g)$ is an epimorphism (we say that T maps from epimorphisms into epimorphisms). More precisely, every exact sequence of right R-modules

$$0 \to K \xrightarrow{h} M \xrightarrow{g} L \to 0$$

induces the exact sequence of groups

$$T(K) \xrightarrow{T(h)} T(M) \xrightarrow{T(g)} T(L) \to 0.$$

There exist natural transformations $\theta : TH \to 1_{\mathcal{A}b}$ and $\phi : 1_{\mathrm{mod} - R} \to HT$, where $1_{\mathcal{A}b}$ and $1_{\mathrm{mod} - R}$ are the identity functors of categories $\mathcal{A}b$ and $\mathrm{mod} - R$, respectively. This means that for every group B, there exists a homomorphism

$$\theta_B : TH(B) = \mathrm{Hom}(A, B) \otimes_R A \to B, \quad \theta_B(f \otimes a) = f(a),\ f \in \mathrm{Hom}(A, B),\ a \in A,$$

that is natural in the sense that $\varphi\theta_B = \theta_C TH(\varphi)$ for every homomorphism $\varphi : B \to C$; this is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} TH(B) & \xrightarrow{TH(\varphi)} & TH(C) \\ \Big\downarrow{\theta_B} & & \Big\downarrow{\theta_C} \\ B & \xrightarrow{\varphi} & C \end{array} .$$

For every right R-module M, there exists an R-homomorphism

$$\phi_M : M \to \mathrm{Hom}(A, M \otimes_R A) = HT(M), \qquad [\phi_M(m)](a) = m \otimes a, \quad m \in M, \quad a \in A,$$

which is natural in the sense that $HT(g)\phi_M = \phi_L g$ for every homomorphism $g : M \to L$; this is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} M & \xrightarrow{g} & L \\ \Big\downarrow{\phi_M} & & \Big\downarrow{\phi_L} \\ HT(M) & \xrightarrow{HT(g)} & HT(L) \end{array} .$$

For the proof of the fact that the functors H and T define an equivalence between some category of groups $\mathcal{A}$ and some category $\mathcal{R}$ of right R-modules, it is sufficient to prove that

(1) $H : \mathcal{A} \to \mathcal{R}$ (i.e., H maps from objects in $\mathcal{A}$ into objects from $\mathcal{R}$) and $T : \mathcal{R} \to \mathcal{A}$;
(2) the transformations $\theta : TH \to 1_{\mathcal{A}}$ and $\phi : 1_{\mathcal{R}} \to HT$ are equivalences. This means that θ_B and ϕ_M are isomorphisms for all $B \in \mathcal{A}$ and $M \in \mathcal{R}$.

We formulate one auxiliary result of module theory. We deal with the well-known Nakayama lemma (see Faith [142, Theorem 18.4] and Bass [51, Chapter III, Proposition 2.2]).

For a right ideal I of a ring S, the following conditions are equivalent:

(1) $I \subseteq J(S)$;
(2) for every finitely generated right S-module M with $MI = M$, we have $M = 0$;
(3) if M is a finitely generated right S-module and K is a submodule of M such that $M = K + MI$, then $M = K$.

Finally, we define some classes of groups associated with a given group A. These groups play an appreciable role in this part. Finite direct sums of copies of a group A are called *finitely A-free groups* and arbitrary direct sums of copies of the group A are called *A-free groups*. Direct summands of finitely A-free groups are called *finitely A-projective groups* and direct summands of A-free groups are called *A-projective groups.*

We consider the first theorem on the equivalence of categories of groups and categories of modules over the endomorphism ring.

Theorem 32.1 (Arnold and Lady [35], Arnold and Murley [36])**.** (1) *The category of finitely A-projective groups is equivalent to the category of finitely generated projective right R-modules, and the equivalence is effected by the functors H and T.*

(2) *If A is a self-small group, then the same functors define an equivalence between the category of A-projective groups and the category of projective right R-modules.*

Proof. (1) First, we note that $H(A) = \mathrm{Hom}(A, A) = R$ and $T(R) = R \otimes_R A \cong A$. The functor H commutes with arbitrary finite direct sums of groups in the sense that there exists a canonical isomorphism of right R-modules,

$$\mathrm{Hom}\left(A, \sum_{i=1}^{n}{}^{\oplus} B_i\right) \cong \sum_{i=1}^{n}{}^{\oplus} \mathrm{Hom}(A, B_i),$$

and T commutes with arbitrary direct sums of right R-modules. Therefore, there exists the canonical isomorphism of groups

$$\Big(\sum_{i\in I}^{\oplus} M_i\Big) \otimes_R A \cong \sum_{i\in I}^{\oplus} \big(M_i \otimes_R A\big).$$

This implies that for every positive integer n, we have

$$H(A^n) = \mathrm{Hom}(A, A^n) \cong \sum_{n}^{\oplus} \mathrm{Hom}(A, A) = R^n,$$

$$T(R^n) = R^n \otimes_R A \cong \sum_{n}^{\oplus} (R \otimes_R A) \cong A^n.$$

Thus, $H(A^n) \cong H(A)^n = R^n$ and $T(R^n) \cong T(R)^n \cong A^n$. Therefore, if $B \oplus C = A^n$, then $H(B) \oplus H(C) \cong H(A^n) \cong R^n$; if $M \oplus K = R^n$, then $T(M) \oplus T(K) \cong T(R^n) \cong A^n$. Therefore, H maps from finitely A-projective groups into finitely generated projective right R-modules, and T acts in the opposite direction.

A direct calculation implies that $\theta_A : TH(A) \to A$ and $\phi_R : R \to HT(R)$ are isomorphisms. Consequently,

$$\theta_{A^n} : TH(A^n) \to A^n \qquad \text{and} \qquad \phi_{R^n} : R^n \to HT(R^n)$$

are also isomorphisms. Now assume that $G = A^n = B \oplus C$. Let $\pi_1 : G \to B$ and $\pi_2 : G \to C$ be the natural projections. There exists a commutative diagram

$$\begin{array}{ccc} TH(G) & \xrightarrow{\ \alpha\ } & TH(B) \oplus TH(C) \\ \Big\downarrow{\scriptstyle \theta_G} & & \Big\downarrow{\scriptstyle \theta_B + \theta_C} \\ G & \xrightarrow{\ 1\ } & B \oplus C \end{array},$$

where $\theta_B + \theta_C$ is the homomorphism coinciding with θ_B on $TH(B)$ and with θ_C on $TH(C)$ and $\alpha(f \otimes a) = (\pi_1 f \otimes a, \pi_2 f \otimes a)$ for $f \in H(G)$ and $a \in A$. We also define the mapping

$$\beta : TH(B) \oplus TH(C) \to TH(G), \qquad \beta(f_1 \otimes a_1, f_2 \otimes a_2) = (f_1 \otimes a_1) + (f_2 \otimes a_2).$$

Then $\alpha\beta = 1$ and $\beta\alpha = 1$. Consequently, α is an isomorphism. Since θ_G is an isomorphism, θ_B is also an isomorphism. If we have an R-module decomposition $R^n = M \oplus K$, then similar arguments imply that ϕ_M is an isomorphism. Therefore, θ and ϕ are natural equivalences. Therefore, the functors H and T define the equivalence of categories indicated in (1).

The proof of (2) is similar to the proof of (1). We use the assumption that the group A is self-small only for the proof of the property that the functor H is well defined. Indeed, we obtain that H commutes with arbitrary direct sums of copies of the group A, i.e., there exists a canonical isomorphism

$$\mathrm{Hom}\Big(A, \sum_{i\in I}^{\oplus} A_i\Big) \cong \sum_{i\in I}^{\oplus} \mathrm{Hom}(A, A_i)$$

for every set of subscripts I, where all groups $A_i \cong A$. □

If the group A satisfies additional conditions, then Theorem 32.1 admits some extension to larger categories.

A module M is said to be *locally projective* if each of its finite subsets can be embedded in a projective direct summand of the module M. Let A be a group. A group G is said to be *locally A-projective* if each of its finite subsets can be embedded in an A-projective direct summand of the group G.

For a group A, we set $R = E(A)$. We denote by $\mathcal{LP}$ the category of all right R-modules of the form $H(G)$ for all locally A-projective groups G.

Theorem 32.2 (Arnold and Murley [36]). *Assume that the endomorphism ring R of the group A is discrete in the finite topology. Then the category of locally A-projective groups is equivalent to the category $\mathcal{LP}$, and $\mathcal{LP}$ consists of locally projective R-modules.*

Proof. We first prove that if G is a locally A-projective group, then $H(G)$ is a locally projective R-module. Let $f_1, \dots, f_n \in H(G)$. Since the ring R is discrete, we can choose a finitely generated subgroup B of the group A such that $\alpha = 0$ for every $\alpha \in R$ with $\alpha B = 0$. By assumption, the sum $f_1(B) + \dots + f_n(B)$ is contained in some A-projective direct summand G_1 of the group G, where $G = G_1 \oplus E$. Then $H(G_1)$ is a projective R-module by Theorem 32.1 (consider that the group A is self-small by Corollary 31.4). It is sufficient to prove that $f_1, \dots, f_n \in H(G_1)$. Assume the contrary. For definiteness, let $f_1 \notin H(G_1)$. Then $f_1(a) \notin G_1$ for some $a \in A$. Consequently, $\pi f_1(a) \neq 0$, where $\pi : G \to E$ is a projection. The element $\pi f_1(a)$ is contained in some A-projective direct summand G_2 of the group G. Therefore, $\sigma\pi f_1(a) \neq 0$, where $\sigma : G \to G_2$ is a projection. Since G_2 is an A-projective group, it is easy to prove the existence of a homomorphism $\delta : G_2 \to A$ such that $\alpha a \neq 0$, where $\alpha = \delta\sigma\pi f_1$. It follows from $f_1(B) \subseteq G_2$ that $\pi f_1(B) = 0$ and $\alpha B = 0$; this is impossible by the choice of the subgroup B. Therefore, $f_1, \dots, f_n \in H(G_1)$, where $H(G_1)$ is a projective R-module and $H(G)$ is a locally projective R-module.

Further, we take any locally projective right R-module M. Let $y_1, \dots, y_s \in T(M) = M \otimes_R A$. We have

$$y_i = \sum_j m_{ij} \otimes a_{ij} \qquad \text{with} \quad m_{ij} \in M, \quad a_{ij} \in A \quad (i = 1, \dots, s).$$

There exists a decomposition $M = P \oplus L$, where P is a projective R-module and all elements $m_{ij} \in P$. Then $T(M) = T(P) \oplus T(L)$, where $T(P)$ is an A-projective group (Theorem 32.1) and $\{y_1, \dots, y_s\} \subseteq T(P)$. Thus, $T(M)$ is a locally A-projective group. We note that this is true for every locally projective right R-module M.

We now verify that the equivalence of categories indicated in the theorem can be obtained by using the functors H and T. The preceding two paragraphs imply that such a formulation of the question is correct. We act as in Theorem 32.1. It is sufficient only to prove that the mappings θ_G and ϕ_M are isomorphisms for every locally A-projective group G and every locally projective R-module M from the category $\mathcal{LP}$.

Let G be a locally A-projective group, and let $g \in G$. We embed the element g in some A-projective direct summand B of the group G. Now $\theta_B : TH(B) \to B$ is an isomorphism by Theorem 32.1; therefore, $g \in \operatorname{im}\theta_B$. Since θ is a natural transformation, $g \in \operatorname{im}\theta_G$. More precisely, let $\varkappa : B \to G$ be an embedding. We have the commutative diagram

$$\begin{array}{ccc} TH(B) & \xrightarrow{TH(\varkappa)} & TH(G) \\ \Big\downarrow{\scriptstyle \theta_B} & & \Big\downarrow{\scriptstyle \theta_G} \\ B & \xrightarrow{\varkappa} & G \end{array}$$

that clarifies the inclusion $g \in \operatorname{im}\theta_G$. Thus, θ_G is an epimorphism.

Assume that $y \in \ker\theta_G$, where

$$y = \sum_{i=1}^{n} f_i \otimes a_i \qquad \text{with} \quad f_i \in H(G), \quad a_i \in A \quad (i = 1, \dots, n).$$

Considering the above proof of the property that $H(G)$ is a locally projective R-module for every locally A-projective group G, we can assert that there exists an A-projective direct summand B of the group G such that $f_1, \dots, f_n \in H(B)$. We have the above commutative diagram. Then $y \in TH(B)$ (under the identification $TH(B)$ with $\operatorname{im} TH(\varkappa)$). Since θ_B and $\varkappa$ are monomorphisms and $\varkappa\theta_B(y) = 0$, we obtain $y = 0$. Therefore, $\ker\theta_G = 0$ and θ_G is an isomorphism.

We prove that $\phi_M : M \to HT(M)$ is an isomorphism for any module $M \in \mathcal{LP}$. Assume that $x \in \ker\phi_M$. We embed the element x in some projective direct summand P of the module M. Since the transformation ϕ is natural, we obtain the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\varkappa} & M \\ \Big\downarrow \phi_P & & \Big\downarrow \phi_M \\ HT(P) & \xrightarrow{HT(\varkappa)} & HT(M) \end{array}$$

where $\varkappa$ is an embedding, $HT(\varkappa)$ is a monomorphism, since P is a direct summand, and ϕ_P is an isomorphism by Theorem 32.1. It is clear that $x = 0$ and ϕ_M is a monomorphism. This part of the proof holds for every locally projective right R-module M, where as for the proof of the property that ϕ_M is an epimorphism, we use the fact that $M \in \mathcal{LP}$. The module M has the form $H(G)$ for some locally A-projective group G, and we need to prove that

$$\phi_{H(G)} : H(G) \to HTH(G)$$

is an epimorphism. By the above, $\theta_G : TH(G) \to G$ is an isomorphism. Therefore, $H(\theta_G) : HTH(G) \to H(G)$ is also an isomorphism. It can be verified that $H(\theta_G)\phi_{H(G)} = 1$ (see the remark before Lemma 34.7). Therefore, $\phi_{H(G)}$ is an isomorphism. We can also calculate $\phi_{H(G)}H(\theta_G) = 1$. □

Reformulating Theorem 32.2, we can say that the functor H effects a complete embedding of the category of locally A-projective groups in the category of locally projective right R-modules (see Theorem 30.3 concerning a complete embedding of categories).

We consider some applications of proved theorems. As usual, a principal ideal domain is a commutative domain all of whose ideals are principal.

Corollary 32.3 (Arnold and Lady [35], Arnold and Murley [36])**.** (1) *Let A be a self-small group. Then every A-projective group is A-free if and only if every projective right $E(A)$-module is free.*

(2) *If the endomorphism ring of the torsion-free group A is a principal ideal domain, then every A-projective group is A-free.*

Proof. Assertion (1) follows directly from Theorem 32.1.

(2) Let A be a torsion-free group such that the ring $E(A)$ is a principal ideal domain. Then every nonzero endomorphism of the group A is a monomorphism by Proposition 31.8; therefore, the zero ideal is a unique annihilator ideal of the ring $E(A)$. Proposition 31.2 implies that the group A is self-small. All projective modules over a principal ideal domain are free, and we can refer to (1). □

To develop Corollary 32.3 and to obtain some interesting results on direct decompositions, we prove one general theorem.

Let T be some set of subscripts. A set of groups $\mathcal{F} = \{H_t\}_{t \in T}$ is called a *semirigid system* if the set T has an order such that

$$s \le t \iff \mathrm{Hom}(A_s, A_t) \ne 0 \qquad \text{for any} \quad s, t \in T.$$

If $s \ne t$, then this implies that $\mathrm{Hom}(H_s, H_t) = 0$ or $\mathrm{Hom}(H_t, H_s) = 0$. A system of all pairwise nonisomorphic torsion-free groups of rank 1 is the standard example of a semirigid system.

We denote by $\mathcal{F}_\Sigma$ the class of all groups that are direct sums of groups from a semirigid system $\mathcal{F} = \{H_t\}_{t \in T}$. Let $G \in \mathcal{F}_\Sigma$. We fix some decomposition of the group G into a direct sum of groups H_t. For every subscript $t \in T$, let $G(t)$ be the direct sum of all summands that are isomorphic to H_t. If $R \subseteq T$, then we set

$$G(R) = \sum_{t \in R}^{\oplus} G(t), \qquad \overline{G}(R) = \sum_{t \notin R}^{\oplus} G(t).$$

Further, we set

$$S_t(G) = \sum_{s \ge t}^{\oplus} G(s), \qquad S_t^*(G) = \sum_{s > t}^{\oplus} G(s).$$

It is clear that $S_t(G) = S_t^*(G) \oplus G(t)$. Using the property that the system $\mathcal{F}$ is semirigid, it is easy to verify that the subgroups $S_t(G)$ and $S_t^*(G)$ are fully characteristic in G. Finally, for the subgroup $X \subseteq G$, we set $S_t(X) = S_t(G) \cap X$ and $S_t^*(X) = S_t^*(G) \cap X$.

Theorem 32.4 (Charles [77]). *Let $\mathcal{F} = \{H_t\}_{t\in T}$ be a semirigid system of groups and $G \in \mathcal{F}_\Sigma$. Assume that $G = A \oplus B$, where A is a countable group. Then $A = \sum_{t\in T}^{\oplus} A(t)$, where every group $A(t)$ is isomorphic to some direct summand of the group $G(t)$.*

Remark. In other words, the group $A(t)$ is H_t-projective.

Proof. For every $t \in T$, the subgroups $S_t(G)$ and $S_t^*(G)$ are fully characteristic. Therefore, we have

$$S_t(G) = S_t(A) \oplus S_t(B) \qquad \text{and} \qquad S_t^*(G) = S_t^*(A) \oplus S_t^*(B).$$

Setting

$$A_0 = A \cap [G(t) + S_t^*(B)] \qquad \text{and} \qquad B_0 = B \cap [G(t) + S_t^*(A)],$$

we obtain

$$S_t(A) = A_0 \oplus S_t^*(A),$$
$$S_t(B) = B_0 \oplus S_t^*(B),$$

and

$$S_t(G) = A_0 \oplus S_t^*(A) \oplus B_0 \oplus S_t^*(B) = A_0 \oplus B_0 \oplus S_t^*(G).$$

The last relation shows that $G(t)$ can be replaced by $A_0 \oplus B_0$ in the decomposition $G = \sum_{s\in T}^{\oplus} G(s)$. If we carry out the replacement for finitely many subscripts $t_1, \dots, t_n$, then we obtain

$$G = A_1 \oplus \dots \oplus A_n \oplus B_1 \oplus \dots \oplus B_n \oplus \overline{G}(t_1, \dots, t_n).$$

The last decomposition implies decompositions

$$A = A_1 \oplus \dots \oplus A_n \oplus \overline{A}_n \qquad \text{with} \quad \overline{A}_n = A \cap [B_1 \oplus \dots \oplus B_n \oplus \overline{G}(t_1, \dots, t_n)],$$
$$B = B_1 \oplus \dots \oplus B_n \oplus \overline{B}_n \qquad \text{with} \quad \overline{B}_n = B \cap [A_1 \oplus \dots \oplus A_n \oplus \overline{G}(t_1, \dots, t_n)],$$

and

$$G = A_1 \oplus \dots \oplus A_n \oplus B_1 \oplus \dots \oplus B_n \oplus \overline{A}_n \oplus \overline{B}_n.$$

If t is one of the subscripts $t_1, \dots, t_n$, then $\overline{A}_n \oplus \overline{B}_n \cong \overline{G}(t_1, \dots, t_n)$ implies that $S_t(\overline{A}_n \oplus \overline{B}_n) = S_t^*(\overline{A}_n \oplus \overline{B}_n)$. Therefore, if $t = t_j$ for some $j \in \{1, \dots, n\}$, then

$$S_t(G) = S_t\Big(\sum_{i=1}^{n}{}^{\oplus}(A_i \oplus B_i)\Big) \oplus S_t^*(\overline{A}_n \oplus \overline{B}_n) = A_j \oplus B_j \oplus S_t^*(G).$$

This shows that $A_j \oplus B_j$ can be replaced by $G(t_j)$ in the decomposition

$$G = A_1 \oplus \dots \oplus A_n \oplus B_1 \oplus \dots \oplus B_n \oplus \overline{A}_n \oplus \overline{B}_n;$$

therefore, we obtain the decomposition

$$G = G(t_1) \oplus \dots \oplus G(t_n) \oplus \overline{A}_n \oplus \overline{B}_n.$$

It implies a new decomposition

$$A = A \cap [G(t_1) \oplus \dots \oplus G(t_n) \oplus \overline{B}_n] \oplus \overline{A}_n.$$

The first summand, which is isomorphic to $A_1 \oplus \dots \oplus A_n$, can be represented in the form $C_1 \oplus \dots \oplus C_n$, where $C_1 \cong A_1, \dots, C_n \cong A_n$. Therefore,

$$A = C_1 \oplus \dots \oplus C_n \oplus \overline{A}_n \text{ and } A \cap [G(t_1) \oplus \dots \oplus G(t_n)] \subseteq C_1 \oplus \dots \oplus C_n.$$

We number elements of the group A as $a_1, a_2, \dots$ and consider some initial segment $a_1, \dots, a_k$. There exist subscripts $t_1, \dots, t_n$ such that $a_1, \dots a_k \in G(t_1) \oplus \cdots \oplus G(t_n)$. Constructing the groups $C_1, \dots, C_n$ for subscripts $t_1, \dots, t_n$, we obtain, $a_1, \dots, a_k \in C_1 \oplus \cdots \oplus C_n$ as above. Let b_{k+1} be the component of the element a_{k+1} in the summand A_n with respect to the decomposition $A = C_1 \oplus \cdots \oplus C_n \oplus \overline{A}_n$. Since

$$G = G(t_1) \oplus \cdots \oplus G(t_n) \oplus \overline{A}_n \oplus \overline{B}_n,$$

we apply similar arguments to the group $\overline{A}_n \oplus \overline{B}_n$ and construct subgroups $C_{n+1}, \dots, C_{n+m}$ such that the element b_{k+1} is contained in $C_{n+1} \oplus \cdots \oplus C_{n+m}$ and

$$A = \sum_{i=1}^{n+m}{}^{\oplus} C_i \oplus \overline{A}_{n+m}, \qquad \text{where} \quad a_1, \dots, a_{k+1} \in \sum_{i=1}^{n+m}{}^{\oplus} C_i\,,$$

and so on. As a result, we obtain $A = \sum_{i=1}^{\infty}{}^{\oplus} C_i$, where the group C_i is isomorphic to some direct summand of the group $G(t_i)$, since $C_i \cong A_i$ and $A_i \oplus B_i \cong G(t_i)$ by construction. □

We need one often cited result of Kaplansky. Its proof is contained, for example, in Fuchs' work [164] (Proposition 9.10). Therefore, we leave it without proof.

Proposition 32.5 (Kaplansky [235]). *Let G be a direct sum of groups G_i $(i \in I)$ of cardinality $\leq\aleph_0$. Then every direct summand of the group G is also a direct sum of groups of cardinality $\leq\aleph_0$.*

We refer the reader to Sec. 28 for the definitions of an isomorphism of direct decompositions, a refinement of a direct decomposition, and an extension of a direct decomposition. In addition, we say that the Krull–Schmidt theorem holds for a group G if any two direct decompositions of the group G with indecomposable summands are isomorphic.

Corollary 32.6 (Arnold and Lady [35], Arnold and Murley [36]). (1) *Let $\mathcal{F} = \{A_t\}_{t \in T}$ be a semirigid system of countable self-small groups A_t such that every A_t-projective group is A_t-free for every $t \in T$. Then every direct summand of the group $G \in \mathcal{F}_\Sigma$ also belongs to $\mathcal{F}_\Sigma$. In addition, if all the groups A_t are torsion-free groups of finite rank, then any two direct decompositions of the group G have isomorphic refinements. In particular, the Krull–Schmidt theorem holds for the group G.*

(2) *Let $\mathcal{F} = \{A_t\}_{t \in T}$ be a semirigid system consisting of countable torsion-free groups A_t such that $E(A_t)$ are principal ideal domains. Then last two assertions from* (1) *hold for every group $G \in \mathcal{F}_\Sigma$.*

Proof. (1) We prove that if B is a direct summand of the group $G \in \mathcal{F}_\Sigma$, then $B \in \mathcal{F}_\Sigma$ (i.e., B is a direct sum of groups that are isomorphic to A_t). By Proposition 32.5, we can assume that $|B| \leq \aleph_0$. We have $G = \sum_{t \in T}{}^{\oplus} G(t)$, where either $G(t) = 0$ or $G(t)$ is an A-free group. By Theorem 32.4, $B = \sum_{t \in T}{}^{\oplus} B(t)$, where either $B(t) = 0$ or $B(t)$ is an A_t-projective group. If the first possibility is realized, then $B(t)$ is an A_t-free group by assumption. Therefore, $B \in \mathcal{F}_\Sigma$.

We temporarily interrupt the proof of (1) and pass to (2). We take the semirigid system $\mathcal{F}$ used in (2). By Corollary 32.3(2), all A_t-projective groups are A_t-free. Therefore, the assumptions of (1) hold. We can assert that every direct summand of the group from $\mathcal{F}_\Sigma$ belongs to $\mathcal{F}_\Sigma$.

We now simultaneously prove the remaining assertions from (1) and (2). Let $\mathcal{F} = \{A_t\}_{t \in T}$ be a semirigid system of one of the following forms:

(a) all the groups A_t are torsion-free groups of finite rank and every A_t-projective group is A_t-free for every $t \in T$;

(b) all A_t are countable torsion-free groups and all rings $E(A_t)$ are principal ideal domains.

Let us have a direct decomposition of the group G. By the above argument, every direct summand of the group G is a direct sum of groups that are isomorphic to A_t. Therefore,

$$G = \sum_{t \in T}{}^{\oplus} G(t),$$

where for every $t \in T$, either $G(t) = 0$ or $G(t)$ is an A_t-free group. Before Theorem 32.4, we have defined for every $t \in T$, the groups

$$S_t(G) = \sum_{s \geq t}^{\oplus} G(s) \qquad \text{and} \qquad S_t^*(G) = \sum_{s > t}^{\oplus} G(s)$$

such that $S_t(G) = S_t^*(G) \oplus G(t)$. Therefore, $G(t) \cong S_t(G)/S_t^*(G)$. We verify that the subgroups $S_t(G)$ and $S_t^*(G)$ can be defined without dependence on the concrete decomposition of the group G into a direct sum of groups that are isomorphic to A_t (i.e., they are invariants of the group G). It is easy to prove that $S_t(G) = \sum \varphi A_s$, where φ runs over $\mathrm{Hom}(A_s, G)$ for all $s \geq t$ and $S_t^*(G) = \sum \varphi A_s$, where φ runs over $\mathrm{Hom}(A_s, G)$ for all $s > t$ ($S_t(-)$ and $S_t^*(-)$ are subfunctors of the identity functor; Charles [77] calls them socles associated with the sets of groups $\{A_s\}_{s \geq t}$ and $\{A_s\}_{s>t}$, respectively).

Assume that the group G has one more direct decomposition. It can be disintegrated to a decomposition into a direct sum of groups that are isomorphic to A_t. We have

$$G = \sum_{t \in T}^{\oplus} G(t) = \sum_{t \in T}^{\oplus} G'(t),$$

where either $G'(t) = 0$ or $G'(t)$ is an A_t-free group. By the above argument, $G'(t) \cong S_t(G)/S_t^*(G)$. Therefore, $G(t) \cong G'(t)$ for every $t \in T$. If all the groups A_t have finite rank, then their numbers in decompositions of the groups $G(t)$ and $G'(T)$ are equal. If all $E(A_t)$ are principal ideal domains, then $\mathrm{Hom}(A_t, G(t))$ and $\mathrm{Hom}(A_t, G'(t))$ are isomorphic free $E(A_t)$-modules by Theorem 32.1 (consider the self-smallness of the groups A_t following from Proposition 31.2). It is well known that the numbers of the modules $E(A_t)$ in decompositions of free $E(A_t)$-modules $\mathrm{Hom}(A_t, G(t))$ and $\mathrm{Hom}(A_t, G'(t))$ are equal. By Theorem 32.1, the number of the summand A_t in decompositions of the groups $G(t)$ and $G'(t)$ is the same. Therefore, our two direct decompositions of the group G disintegrate to isomorphic decompositions, and all summands in both decompositions are isomorphic to the groups A_t. Now it is obvious that the Krull–Schmidt theorem holds for the group $G \in \mathcal{F}_\Sigma$. $\square$

Corollary 32.6 generalizes the classical Baer–Kulikov–Kaplansky theorem on direct summands of completely decomposable groups (see Fuchs [165, Theorem 86.7]).

Before presenting other applications of Theorem 32.1, we prove three quite well-known results on rings and modules. The first result refers to the lifting of idempotents modulo a nil-ideal, and the other results establish some correspondences between finitely generated projective R-modules and finitely generated projective $R/N(R)$-modules. The functorial formulation of these results is given in the work of Bass [51, Chapter III, Proposition 2.12].

Let R be a ring, and let $N(R)$ be the nil-radical of R. We set $\overline{R} = R/N(R)$ and $\overline{M} = M/MN(R)$ for a right R-module M. It is clear that $\overline{M}$ is a right $\overline{R}$-module. In addition, $\overline{M} \cong M \otimes_R \overline{R}$.

Proposition 32.7. (1) *If $e + N(R)$ is an idempotent of the ring $\overline{R}$, then there exists an idempotent f of the ring R such that $f + N(R) = e + N(R)$ and $fe = ef$.*

(2) *If P_1 and P_2 are finitely generated projective right R-modules, then $P_1 \cong P_2 \Longleftrightarrow \overline{P}_1 \cong \overline{P}_2$.*

(3) *If $R = P \oplus P'$ as right R-modules and $\overline{P} = Q_1 \oplus Q_2$ as $\overline{R}$-modules, then $P = P_1 \oplus P_2$ as R-modules, where $\overline{P}_1 = Q_1$ and $\overline{P}_2 = Q_2$.*

Proof. (1) There exist different proofs of this classical result (see, for example, Arnold [31, Theorem 9.5]). We set $x = e$ and $y = 1 - e$. Then $xy = e - e^2 \in N(R)$ and $xy = yx$. Let $(xy)^n = 0$. We have $1 = x + y = (x + y)^{2n-1} = f + (1 - f)$, where $f = x^{2n-1} + a_1 x^{2n-2} y + \cdots + a_{n-1} x^n y^{n-1}$. Then $f(1 - f) = 0$, whence $f^2 = f$. Further, $f + N(R) = x^{2n-1} + N(R) = e^{2n-1} + N(R) = e + N(R)$ and $fe = ef$.

(2) It is clear that the relation $P_1 \cong P_2$ implies $\overline{P}_1 \cong \overline{P}_2$. Let $\overline{P}_1 \cong \overline{P}_2$. We choose some isomorphism $\bar{f} : \overline{P}_1 \to \overline{P}_2$ and its inverse isomorphism $\bar{g} : \overline{P}_2 \to \overline{P}_1$. Let $\pi : P_1 \to \overline{P}_1$ and $\sigma : P_2 \to \overline{P}_2$ be canonical mappings. Since P_1 and P_2 are projective modules, there exists a homomorphism $f : P_1 \to P_2$ with

$\sigma f = \bar{f}\pi$ and $g : P_2 \to P_1$ with $\pi g = \bar{g}\sigma$. Then $\pi g f = \bar{g}\sigma f = \overline{gf} = \pi$, whence $gf(P_1) + P_1 N(R) = P_1$. By the Nakayama lemma, $gf(P_1) = P_1$. Similarly, $fg(P_2) = P_2$. Therefore, $f : P_1 \to P_2$ is an isomorphism.

(3) We have $P = \varepsilon R$, where $\varepsilon^2 = \varepsilon \in R$. Then $\varepsilon R \varepsilon$ is a ring with identity element ε and $\varepsilon R\varepsilon/\varepsilon N(R)\varepsilon \cong \varepsilon\overline{R}\varepsilon$. However, $N(\varepsilon R\varepsilon) = \varepsilon N(R)\varepsilon$; therefore, $\varepsilon R\varepsilon/N(\varepsilon R\varepsilon) \cong \varepsilon\overline{R}\varepsilon$. In addition, $\varepsilon R\varepsilon \cong \operatorname{End}_R P$ and $\varepsilon\overline{R}\varepsilon \cong \operatorname{End}_{\overline{R}} \overline{P}$ (Sec. 3, property (b)). Consequently, the decomposition $\overline{P} = Q_1 \oplus Q_2$ gives an idempotent $\bar{e} \in \varepsilon\overline{R}\varepsilon$ with $\bar{e}\overline{P} = Q_1$. By (2), $\bar{e}$ can be lifted to some idempotent $f \in \varepsilon R\varepsilon$ with $\overline{fP} = Q_1$ and $(\overline{1-f})\overline{P} = Q_2$. If we set $P_1 = fP$ and $P_2 = (1-f)P$, then we obtain that $P = P_1 \oplus P_2$ with $\overline{P}_i = Q_i$ $(i = 1, 2)$. □

One more application of Theorem 32.1 is related to the so-called exchange property in direct sums. A group A is said to have the finite exchange property if it satisfies the following condition: if the group A is a direct summand of a group G and G is a direct sum of the subgroups B_i $(i = 1, \dots, n)$, i.e.,

$$G = A \oplus K = \sum_{i=1}^{n}{}^{\oplus} B_i,$$

then there exist subgroups D_i of the group B_i such that

$$G = A \oplus \sum_{i=1}^{n}{}^{\oplus} D_i.$$

Theorem 32.8 (Arnold and Lady [35]). *Let G be a torsion-free group of finite rank and $G = A \oplus K = B \oplus C$. If the groups A and K have no nonzero quasi-isomorphic quasi-summands, then there exist decompositions $B = B_1 \oplus B_2$ and $C = C_1 \oplus C_2$ such that $G = B_1 \oplus C_1 \oplus K = A \oplus B_2 \oplus C_2$.*

Proof. Let $H = \operatorname{Hom}(G, -)$, $R = E(G)$, and N be the nil-radical of the ring $E(G)$ (it is the largest nilpotent ideal of the ring $E(G)$, see Secs. 4 and 5). We set $\overline{R} = R/N$. If X and Y are some direct summand groups G, then by Theorem 32.1, $X \cong Y \iff H(X) \cong H(Y)$ (right R-module isomorphisms are considered). Further, $H(X) \cong H(Y) \iff \overline{X} \cong \overline{Y}$, where $\overline{X} = H(X)/H(X)N$ and $\overline{Y} = H(Y)/H(Y)N$ (as right $\overline{R}$-modules) by Proposition 32.7. Conversely, if $\overline{W}$ is a direct summand $\overline{R}$-module $\overline{X}$, then there exists a direct summand P of the R-module $H(X)$ such that $\overline{W} = P/PN$. By Theorem 32.1, a unique direct summand W of the group G corresponds to the module P (in fact, $W = PG$); therefore, $\overline{W} = H(W)/H(W)N$.

Further, we prove the following property of the summands X and Y. They are quasi-isomorphic if and only if $\mathbf{Q}\otimes\overline{X} \cong \mathbf{Q}\otimes\overline{Y}$ as right modules over the ring $\mathbf{Q}\otimes\overline{R}$ (it is clear that $\mathbf{Q}\otimes\overline{X}$ and $\mathbf{Q}\otimes\overline{Y}$ are right ideals of the ring $\mathbf{Q}\otimes\overline{R}$). First, we assume that $X \sim Y$. Let $\varepsilon : G \to X$ and $\omega : G \to Y$ be corresponding projections. By the right analog of Proposition 5.4(3), $\varepsilon(\mathbf{Q} \otimes R)$ and $\omega(\mathbf{Q} \otimes R)$ are isomorphic right $\mathbf{Q} \otimes R$-modules. However, $\varepsilon(\mathbf{Q} \otimes R) = \mathbf{Q} \otimes H(X)$ and $\omega(\mathbf{Q} \otimes R) = \mathbf{Q} \otimes H(Y)$. Thus, $\mathbf{Q} \otimes H(X)$ and $\mathbf{Q} \otimes H(Y)$ are isomorphic right $\mathbf{Q} \otimes R$-modules. Therefore, $\mathbf{Q} \otimes H(X) \otimes_R \overline{R}$ and $\mathbf{Q} \otimes H(Y) \otimes_R \overline{R}$ are isomorphic as right modules over the ring $\mathbf{Q} \otimes \overline{R}$. It is easy to verify the existence of canonical isomorphisms $H(X)/H(X)N \cong H(X) \otimes_R \overline{R}$ and $H(Y)/H(Y)N \cong H(Y) \otimes_R \overline{R}$. Therefore, we can assert that $\mathbf{Q} \otimes \overline{X} \cong \mathbf{Q} \otimes \overline{Y}$.

Conversely, let $\mathbf{Q} \otimes \overline{X}$ and $\mathbf{Q} \otimes \overline{Y}$ be isomorphic right $\mathbf{Q} \otimes \overline{R}$-modules. We have

$$\mathbf{Q} \otimes \overline{X} = \mathbf{Q} \otimes (H(X)/H(X)N) \cong (\mathbf{Q} \otimes H(X))/(\mathbf{Q} \otimes H(X))(\mathbf{Q} \otimes N).$$

A similar assertion holds for $\mathbf{Q} \otimes \overline{Y}$. In addition, $\mathbf{Q} \otimes \overline{R} \cong (\mathbf{Q} \otimes R)/(\mathbf{Q} \otimes N)$. Therefore, $\mathbf{Q} \otimes H(X) \cong \mathbf{Q} \otimes H(Y)$ by the correspondence for finitely generated projective modules (its use is possible, since $\mathbf{Q}\otimes H(X)$ and $\mathbf{Q}\otimes H(Y)$ are finitely generated projective $\mathbf{Q}\otimes R$-modules). Therefore, $\varepsilon(\mathbf{Q}\otimes R) \cong \omega(\mathbf{Q}\otimes R)$ and $X = \varepsilon G \sim \omega G = Y$ by Proposition 5.4.

Applying the proved property of the decompositions $G = A \oplus K = B \oplus C$, we obtain $\overline{G} = \overline{A} \oplus \overline{K} = \overline{B} \oplus \overline{C}$. Since $\mathbf{Q}\otimes\overline{R} \cong (\mathbf{Q}\otimes R)/(\mathbf{Q}\otimes N)$, we have that $\mathbf{Q}\otimes\overline{R}$ is a semisimple finite-dimensional $\mathbf{Q}$-algebra. Therefore, under the above correspondence $X \to \mathbf{Q} \otimes \overline{X}$, strongly indecomposable quasi-summands of groups A and K go into irreducible submodules of the $\mathbf{Q}\otimes\overline{R}$-modules $\mathbf{Q}\otimes\overline{A}$ and $\mathbf{Q}\otimes\overline{K}$, respectively. Since every nontrivial homomorphism between irreducible modules is an isomorphism, the assumptions of the theorem imply that there are no nontrivial homomorphisms between $\mathbf{Q}\otimes\overline{A}$ and $\mathbf{Q}\otimes\overline{K}$ in both directions. Consequently, there is no nontrivial homomorphism between $\overline{A}$ and $\overline{K}$ in both directions. Therefore, $\overline{A}$ and $\overline{K}$ are invariant submodules of the module $\overline{G}$. Therefore,

$$\overline{A}=(\overline{B}\cap\overline{A})\oplus(\overline{C}\cap\overline{A})\qquad\text{and}\qquad\overline{K}=(\overline{B}\cap\overline{K})\oplus(\overline{C}\cap\overline{K}).$$

Setting

$$\overline{B}_1=\overline{B}\cap\overline{A},\qquad\overline{B}_2=\overline{B}\cap\overline{K},\qquad\overline{C}_1=\overline{C}\cap\overline{A},\qquad\text{and}\qquad\overline{C}_2=\overline{C}\cap\overline{K},$$

we obtain

$$\overline{G}=\overline{B}_1\oplus\overline{B}_2\oplus\overline{C}_1\oplus\overline{C}_2,$$

where $\overline{B}_1\oplus\overline{C}_1=\overline{A}$, $\overline{B}_2\oplus\overline{C}_2=\overline{K}$, $\overline{B}_1\oplus\overline{B}_2=\overline{B}$, and $\overline{C}_1\oplus\overline{C}_2=\overline{C}$. By Proposition 32.7, the above direct decompositions of $\overline{R}$-modules $\overline{B}$ and $\overline{C}$ can be lifted to direct decompositions of the R-modules $H(B)$ and $H(C)$. By Theorem 32.1, these decompositions correspond to some direct decompositions $B=B_1\oplus B_2$ and $C=C_1\oplus C_2$ of the groups B and C. In addition, $B_1\oplus C_1\cong A$, $B_2\oplus C_2\cong K$, and

$$H(G)=H(B_1)+H(C_1)\oplus H(K)+N=H(A)+H(B_2)+H(C_2)+N.$$

By the Nakayama lemma, the ideal N can be removed from these relations. Further, we have

$$G=H(G)G=H(B_1)G+H(C_1)G+H(K)G=B_1+C_1+K.$$

Similarly, we have $G=A+B_2+C_2$. A calculation of ranks shows that these sums are direct sums (use $B_1\oplus C_1\cong A$ and $B_2\oplus C_2\cong K$). □

As a special case of Theorem 32.8, we have the following result related to the problem of cancellation in direct sums.

Corollary 32.9 (Arnold and Lady [35]). *Let A, K, and K_1 be torsion-free groups of finite rank such that $A\oplus K\cong A\oplus K_1$, and let A and K have no nonzero quasi-isomorphic quasi-summands. Then $K\cong K_1$.*

Proof. It is more convenient to assume that $A\oplus K=B\oplus K_1$ with $A\cong B$. By Theorem 32.8, there exist decompositions $B=B_1\oplus B_2$ and $K_1=C_1\oplus C_2$ such that $G=B_1\oplus C_1\oplus K=A\oplus B_2\oplus C_2$. We have $K\cong G/A\cong B_2\oplus C_2$. On the other hand, $A\cong B=B_1\oplus B_2$. By assumption on the groups A and K, we obtain $B_2=0$ and $B=B_1$. Therefore, $B\oplus K_1=B\oplus C_1\oplus K$. Since $A\oplus K=B\oplus K_1$ and $A\cong B$, the rank considerations give $r(K)=r(K_1)$. This property and the relation $B\oplus K_1=B\oplus C_1\oplus K$ imply $r(C_1)=0$ and $C_1=0$. Therefore, $B\oplus K_1=B\oplus K$ and $K_1\cong A/B\cong K$. □

One more application of Theorem 32.8 implies that the Krull–Schmidt theorem can hold for systems of groups that are not necessarily semirigid.

Corollary 32.10 (Arnold and Lady [35]). *Let $\mathcal{F}=\{A_i\}_{i\in I}$ be a set of pairwise non-quasi-isomorphic strongly indecomposable torsion-free groups of finite rank. Let $\mathcal{F}_\Sigma$ be the set of all finite direct sums of groups from $\mathcal{F}$ and let $G=B\oplus C\in\mathcal{F}_\Sigma$. Then:*

(1) *$B=\sum_{i\in I}^{\oplus}B(i)$, where every group $B(i)$ is an A_i-projective group or $B(i)=0$;*

(2) *if every projective right $E(A_i)$-module is free for all $i\in I$, then $B\in\mathcal{F}_\Sigma$ and the Krull–Schmidt theorem holds for the group G.*

Proof. (1) We have

$$C=A_1^{n_1}\oplus\cdots\oplus A_k^{n_k},$$

where $n_1,\dots,n_k$ are positive integers. We set $G_i=A_i^{n_i}$ $(i=1,\dots,k)$ and use Theorem 5.5 of Jonsson on the unique (up to quasi-isomorphism) quasi-decomposition of the torsion-free group of finite rank into a sum of strongly indecomposable groups. By this theorem, the group B (we assume that $B\neq 0$)

cannot have strongly indecomposable quasi-summands except for $A_1, \ldots, A_k$. Assume that it has a quasi-summand that is quasi-isomorphic to the group A_1 for definiteness. We set $G_2 \oplus \cdots \oplus G_k = K_1$ and obtain $G = G_1 \oplus K_1 = B \oplus C$, where the groups G_1 and K_1 have no nonzero quasi-isomorphic quasi-summands. The application of Theorem 32.8 implies the decompositions $B = B_1 \oplus B_3$ and $C = C_1 \oplus C_2$ such that $G = B_1 \oplus C_1 \oplus K_1 = G_1 \oplus B_3 \oplus C_2$. Therefore, $B_1 \oplus C_1 \cong G/K_1 \cong G_1$. Consequently, B_1 is an A_1-projective group. Further, $B_3 \oplus C_2 \cong G/G_1 \cong K_1$. Taking into account the Jonsson theorem, we obtain that the group B_3 does not contain quasi-summands that are quasi-isomorphic to the group A_1. In particular, $B_1 \neq 0$. If $B_3 \neq 0$, then we write $G = B \oplus C = B_3 \oplus (B_1 \oplus C)$ and apply a similar argument to the group B_3. Let the group B_3 have a quasi-summand that is quasi-isomorphic, for example, to the group A_2. Let $G = G_2 \oplus K_2 = B_3 \oplus (B_1 \oplus C)$, where $K_2 = \sum\limits_{i \neq 2}^{\oplus} G_i$. We have $B_3 = B_2 \oplus B_4$, where B_2 is an A_2-projective group and B_4 has no quasi-summands that are quasi-isomorphic to A_2. Since the rank of the group B is finite, we obtain

$$B = \sum_{i \in I}^{\oplus} B(i),$$

where either $B(i)$ is an A_i-projective group or $B(i) = 0$.

(2) By Corollary 32.3, A_i-projective groups coincide with A_i-free groups. Therefore, $B \in \mathcal{F}_\Sigma$. We also need to prove that any two direct decompositions of the group G with indecomposable summands are isomorphic. It is sufficient to prove that every such decomposition is isomorphic to the decomposition

$$G = A_1^{n_1} \oplus \cdots \oplus A_k^{n_k}.$$

Therefore, we take one more decomposition of the group G into a direct sum of indecomposable groups. By (1), each of these indecomposable groups belongs to $\mathcal{F}$. Consequently, the Jonsson theorem implies

$$G = D_1^{m_1} \oplus \cdots \oplus D_k^{m_k},$$

where $D_i \cong A_i$ $(i = 1, \ldots, n)$ and $m_1, \ldots, m_k$ are some positive integers. It is sufficient to prove that $m_1 = n_1, \ldots, m_k = n_k$. Applying Corollary 32.9 several times, we easily obtain $D_i^{m_i} \cong A_i^{n_i}$. Finally, $m_i = n_i$ by argument related to the ranks. ☐

Finally, we present one quite general example of a semirigid system containing all torsion-free groups of rank 1.

Proposition 32.11. *Let $S = \{A_i\}_{i \in I}$ be the set of all pairwise non-quasi-isomorphic strongly indecomposable irreducible torsion-free groups A_i of finite rank. Then S is a semirigid system.*

Proof. We prove that for any two groups A_i and A_j with $i \neq j$, either

$$\mathrm{Hom}(A_i, A_j) = 0$$

or

$$\mathrm{Hom}(A_j, A_i) = 0.$$

Assume the contrary. We choose $0 \neq \gamma : A_i \to A_j$ and $0 \neq \delta : A_j \to A_i$. Since the group A_j is irreducible, we have that for any two nonzero elements $x, y \in A_j$, there exist $\alpha \in E(A_j)$ and $n \in \mathbf{N}$ such that $\alpha x = ny$. Therefore, it is clear that we can choose δ such that $\delta\gamma \neq 0$. Thus, $\delta\gamma$ is a nonzero endomorphism of the strongly indecomposable irreducible group A_i. Therefore, $\delta\gamma$ is a monomorphism (see Corollaries 5.12 and 5.14). Therefore, γ is a monomorphism. Similarly, there exists a monomorphism $A_j \to A_i$. Therefore, $A_i \sim A_j$; this contradicts the assumption. For any subscripts $i, j \in I$, we set

$$i \leq j \quad \Longleftrightarrow \quad \mathrm{Hom}(A_i, A_j) \neq 0.$$

It follows from the above that $\leq$ is an order on the set I and S is a semirigid system. ☐

Exercise 1. Prove the Nakayama lemma.

Exercise 2. Let A be a right module over the ring S and $R = \operatorname{End}_S A$. We have a bimodule ${}_R A_S$. As in Sec. 1, we consider the functors

$$H = \operatorname{Hom}_S(A, -) : \operatorname{mod}-S \to \operatorname{mod}-R$$

and

$$T = (-) \otimes_R A : \operatorname{mod}-R \to \operatorname{mod}-S$$

and corresponding natural transformations. If we define finitely A-projective and A-projective right S-modules similarly to the group case, then Theorem 32.1 can be extended to modules without change. More precisely, prove that for any right S-module A the following assertions hold:

(1) the functors H and T define an equivalence of the category of finitely A-projective S-modules and the category of finitely generated projective right R-modules;
(2) if A is a self-small S-module, then the functors H and T define an equivalence of the category of A-projective S-modules and the category of projective right R-modules.

Exercise 3 (Arnold and Lady [35])**.** Let A be a strongly indecomposable inhomogeneous torsion-free group of rank 2. Then every A-projective group of finite rank is A-free.

Exercises 4–8 are taken from the works of Murley [333,334]; see also the remark before Theorem 19.7.

We denote by $\mathcal{E}$ the class of all reduced torsion-free groups A such that $A/pA \cong Z(p)$ for every p with $pA \neq A$ (this is equivalent to the property that A has a cyclic p-basis subgroup).

Exercise 4. (a) The torsion-free group A belongs to the class $\mathcal{E}$ if and only if A is isomorphic to some pure subgroup of the group $\prod\limits_{p=2,3,5,\ldots} I_p$.

(b) The endomorphism ring of every group $A \in \mathcal{E}$ is commutative.

Exercise 5. The endomorphism ring of an indecomposable group $A \in \mathcal{E}$ of finite rank is strongly homogeneous. This means that every element of the ring $E(A)$ is an integral multiple of some invertible element (see Sec. 19). Therefore, $E(A)$ is a principal ideal domain.

Exercise 6. If A is an indecomposable group of finite rank from $\mathcal{E}$, then $r(A) = r(E(A))r_{E(A)}(A)$, where $r_{E(A)}(A)$ is the rank of the $E(A)$-module A.

Exercise 7. For any group of finite rank $A \in \mathcal{E}$, the Krull–Schmidt theorem holds.

Exercise 8. Let $\mathcal{F}$ be a semirigid system consisting of groups of finite rank belonging to the class $\mathcal{E}$. Prove that any two direct decompositions of the group $G \in \mathcal{F}_\Sigma$ have isomorphic refinements. The Krull–Schmidt theorem holds for the group G.

Exercise 9 (Dubois [115])**.** A reduced torsion-free group is said to be cohesive if for every nonzero pure subgroup C of the group A, the factor group A/C is divisible. The following conditions (a)–(c) are equivalent:

(a) the group A is cohesive;
(b) the group A is isomorphic to some p-pure subgroup of the group I_p for every p with $pA \neq A$;
(c) $A \in \mathcal{E}$ and $p^\omega A = 0$ for every p with $pA \neq A$.

Exercise 10. Compare the definition of a cohesive group with the definition of a cohesive ring given in Exercise 3 of Sec. 6. In addition, verify that the additive group of a cohesive ring is a cohesive group and the endomorphism ring of a cohesive group is a cohesive ring.

Exercise 11 (Murley [334])**.** For a fixed integer p, let $\mathcal{F}_p = \{A \in \mathcal{E} \mid r(A) < \infty \text{ and } p^\omega A = 0\}$, and let $\mathcal{C}$ be the class of all cohesive groups of finite rank. We denote by $\mathcal{F}$ the set of all pairwise nonisomorphic groups from the class $\mathcal{C} \cup \mathcal{F}_p$. Prove that $\mathcal{F}$ is a semirigid system.

33. Faithful Groups

A faithful group is a group A such that $IA \neq A$ for every proper right ideal I of the ring $E(A)$. First, we note that it is sufficient to consider only maximal right ideals I, since every proper right ideal is contained in some maximal right ideal. Using faithful groups, we develop results on the equivalence of categories obtained in the preceding section; this implies interesting applications. Faithful groups are also useful in studying groups with hereditary endomorphism rings.

We recall that the trace $S_A(B)$ of the group A in the group B is the subgroup $\sum \alpha A$ of the group B, where α runs over the set $\mathrm{Hom}(A, B)$. The trace $S_A(B)$ (or the A-trace) is also called the A-socle. Using the trace, we can reformulate the Baer lemma in the following form, which is more available for generalizations. Let A be a torsion-free group of rank 1. Then every exact sequence of torsion-free groups $0 \to K \overset{\alpha}{\to} B \to G \to 0$, where G is an A-free group and $\alpha K + S_A(B) = B$, splits.

There are few criteria of splitting of exact sequences of torsion-free groups. The Baer lemma is the most widely used among these criteria. For this reason, there havebeen many attempts to extend the Baer lemma to some situations where the group B/K is not necessarily homogeneous and completely decomposable. In 1975, Arnold and Lady succeeded in the case of a torsion-free group B of finite rank. Albrecht developed their studies. This section is based on the corresponding works of these mathematicians.

In our subsequent considerations, we use flat modules (especially flat modules over endomorphism rings). If a group A is a flat $E(A)$-module, then the group A is called endoflat. The definition of a flat module was given in Sec. 1, and several criteria of flatness were given in Secs. 10 and 13. We repeat one of them. Let I be some right ideal of the ring R, and let M be a left R-module. The correspondence $r \otimes m \to rm$ ($r \in R$ and $m \in M$) defines the canonical epimorphism of groups

$$\Lambda : I \otimes_R M \to IM.$$

The module M is flat if and only if Λ is an isomorphism for every right ideal (finitely generated right ideal) I of the ring R. There exists one more condition of flatness related to the torsion product of modules. The theory of torsion products, induced sequences for $\otimes$, and Tor is presented in the books of Cartan and Eilenberg [75], Lambek [279], and Fuchs [164] (in the case of Abelian groups). A left R-module M is flat if and only if the torsion product $\mathrm{Tor}_R(W, M)$ is equal to zero for every right R-module W.

Every exact sequence of right R-modules

$$0 \to U \overset{\varkappa}{\to} V \overset{\sigma}{\to} W \to 0$$

implies the exact sequence of groups

$$\mathrm{Tor}_R(U, M) \xrightarrow{\varkappa'} \mathrm{Tor}_R(V, M) \xrightarrow{\sigma'} \mathrm{Tor}_R(W, M) \xrightarrow{\Delta} U \otimes_R M \xrightarrow{\varkappa \otimes 1} V \otimes_R M \xrightarrow{\sigma \otimes 1} W \otimes_R M \longrightarrow 0,$$

where $\varkappa'$ and σ' are induced homomorphisms and Δ is the connecting homomorphism. We have a similar induced sequence if in $\otimes$ and Tor the module on the first place is fixed. Therefore, if M is a flat module, then $\varkappa \otimes 1\mathrm{m}$ is a monomorphism. For an exact sequence of left R-modules $0 \to K \to M \to L \to 0$ with flat module L, we obtain an exact sequence of groups

$$0 \to V \otimes_R K \to V \otimes_R M \to V \otimes_R L \to 0$$

for every right R-module V.

From the above, it is easy to obtain the following property (Bourbaki [70, Chapter 1, § 2, Proposition 5]).

Let $0 \to K \to M \to L \to 0$ be a faithful sequence of left R-modules. Assume that the module L is flat. Then the module M is flat if and only if the module K is flat.

A monomorphism $\varkappa : B \to C$ is said to be split if $\varkappa B$ is a direct summand of the group C. This is equivalent to the existence of $\pi : C \to B$ with $\pi\varkappa = 1_B$. An epimorphism $\omega : E \to D$ is *split* if $\ker \omega$ is a direct summand of the group E; this is equivalent to the existence of $\nu : D \to E$ with $\omega\nu = 1_D$.

For a group A, we denote by H and T the functors $\mathrm{Hom}(A,-)$ and $(-)\otimes_{E(A)}A$, respectively. Further,

$$\theta_G : TH(G) \to G \qquad \text{and} \qquad \phi_M : M \to HT(M)$$

are natural homomorphisms for the group G and a right $E(A)$-module M. By definitions, we have $\operatorname{im}\theta_G = S_A(G)$ for every group G. We prove that for any right $E(A)$-module M, we always have $S_A(T(M)) = T(M)$; therefore, the natural mapping $\theta_{T(M)} : THT(M) \to T(M)$ is always an epimorphism. We take some epimorphism $\omega : F \to M$ with free $E(A)$-module F. Since $T(\omega) : T(F) \to T(M)$ is an epimorphism, it is sufficient to prove that $S_A(T(F)) = T(F)$. However, this is obvious, since $F \cong \sum^{\oplus} E(A)$. Consequently,

$$T(F) = F \otimes_{E(A)} A \cong \sum\nolimits^{\oplus}(E(A) \otimes_{E(A)} A) \cong \sum\nolimits^{\oplus} A.$$

In the following theorems, we assume for convenience of considerations that K is a subgroup of the group B and $K \to B$ is an embedding in exact sequences of the form $0 \to K \to B \to G$.

A group A is said to be *completely faithful* if $M \otimes_{E(A)} A \neq 0$ for every nonzero right $E(A)$-module M.

Theorem 33.1 (Albrecht [7]). *A self-small group A is completely faithful if and only if every exact sequence of groups $0 \to K \to B \to G \to 0$, where G is an A-projective group and $B = K + S_A(B)$, is split.*

Proof. *Necessity.* Let A be a self-small completely faithful group. We consider an exact sequence of groups

$$0 \to K \to B \xrightarrow{\beta} G \to 0, \tag{1}$$

where G is an A-projective group $B = K + S_A(B)$. It induces the exact sequence

$$0 \to H(K) \to H(B) \to M \to 0 \tag{2}$$

of right $E(A)$-modules, where $M = \operatorname{im} H(\beta)$ is a submodule of the $E(A)$-module $H(G)$. It is projective by Theorem 32.1. First, we show that (2) is split. To do this, it is sufficient to prove that $M = H(G)$. For this purpose, we define $\theta : T(M) \to G$ as $\theta(\varphi \otimes a) = \varphi a$ for all $\varphi \in M$ and $a \in A$. We consider a commutative diagram

$$\begin{array}{ccccc} TH(B) & \xrightarrow{TH(\beta)} & T(M) & \longrightarrow & 0 \\ \big\downarrow{\scriptstyle \theta_B} & & \big\downarrow{\scriptstyle \theta} & & \\ B & \xrightarrow{\beta} & G & \longrightarrow & 0 \end{array}.$$

Since $S_A(B) = \operatorname{im}\theta_B$ and $B = K + S_A(B)$, we have that $\beta\theta_B$ is an epimorphism. Therefore, $\theta TH(\beta)$ is an epimorphism and, consequently, θ is an epimorphism. Further, the inclusion mapping $i : M \to H(G)$ induces the following commutative diagram:

$$\begin{array}{ccccccc} T(M) & \xrightarrow{T(i)} & TH(G) & \longrightarrow & T(H(G)/M) & \longrightarrow & 0 \\ \| & & \big\downarrow{\scriptstyle \theta_G} & & & & \\ T(M) & \xrightarrow{\theta} & G & \longrightarrow & 0 & & \end{array},$$

where θ_G is an isomorphism by Theorem 32.1. We obtain that $T(i)$ is an epimorphism and $T(H(G)/M) = 0$. By assumption, $H(G)/M = 0$ and $M = H(G)$.

Thus, (2) is split; therefore, there exists $\eta : H(G) \to H(B)$ such that $H(B)\eta = 1_{H(G)}$. The application of the functor T implies the commutative diagram

$$\begin{array}{ccc} TH(B) & \longleftrightarrow & TH(G) \\ \Big\downarrow \theta_B & & \Big\downarrow \theta_G \\ B & \longrightarrow & G \end{array} .$$

We set $\sigma = \theta_B T(\eta)\theta_G^{-1}$. We have

$$\beta\sigma = \beta\theta_B T(\eta)\theta_G^{-1} = \theta_G TH(\beta)T(\eta)\theta_G^{-1} = \theta_G\theta_G^{-1} = 1_G.$$

Therefore, $\beta\sigma = 1_G$; this implies that the sequence (1) is split.

Sufficiency. Let M be a right $E(A)$-module with $T(M) = 0$. The module M is a homomorphic image of some free module F (i.e., there exists an exact row $0 \to U \xrightarrow{\nu} F \xrightarrow{\mu} M \to 0$). This row induces the exact sequence of groups $T(U) \xrightarrow{T(\nu)} T(F) \to 0$. We choose some epimorphism $P \xrightarrow{\delta} U$, where the $E(A)$-module P is projective. Then $T(\delta) : T(P) \to T(U)$ is an epimorphism. Since $T(\nu\delta) = T(\nu)T(\delta)$, we obtain that $T(\nu\delta) : T(P) \to T(F)$ is an epimorphism. By assumption, it is split if we consider that $S_A(T(P)) = T(P)$ (see the remark before the theorem). Consequently, the lower row of the commutative diagram

$$\begin{array}{ccccc} P & \xrightarrow{\nu\delta} & F & \longrightarrow & 0 \\ \Big\downarrow \phi_P & & \Big\downarrow \phi_F & & \\ HT(P) & \xrightarrow{HT(\nu\delta)} & HT(F) & \longrightarrow & 0 \end{array}$$

is exact and ϕ_P and ϕ_F are isomorphisms by Theorem 32.1. Therefore, $\nu\delta$ is an epimorphism, whence ν is an epimorphism. However, this is possible provided $M = 0$. □

Theorem 33.2 (Albrecht [7]). *For a self-small group A, the following conditions are equivalent*:

(1) *A is a faithful group*;
(2) *$T(M) \neq 0$ for every finitely generated right $E(A)$-module M*;
(3) *every exact sequence $0 \to K \to B \to G \to 0$ of groups, where G is a finitely A-projective group and $B = K + S_A(B)$, is split.*

Proof. (1) $\Longrightarrow$ (3). We can assume that the group G in our sequences is A-free. Indeed, assume that the assertion is proved for all sequences with A-free group G. We take an exact sequence $0 \to K \to B \xrightarrow{\sigma} G \to 0$ with any A-projective group G. There exists a group E such that $G \oplus E$ is an A-free group. We consider the exact sequence

$$0 \to K \to B \oplus E \xrightarrow{\sigma+1} G \oplus E \to 0,$$

where $(\sigma + 1)(b + e) = \sigma b + e$ for all $b \in B$ and $e \in E$; in addition, $K + S_A(B \oplus E) = B \oplus E$, since E is an A-projective group. By assumption, there exists $\omega' : G \oplus E \to B \oplus E$ such that $(\sigma + 1)\omega' = 1$. We set $\omega = \omega'_{|G}$. Then $\omega : G \to B$ and $\sigma\omega = 1_G$, which means that the original sequence is split.

Thus, we assume that G is a finitely A-free group (i.e., $G \cong A^n$ for some positive integer n). We prove the assertion by induction on n. The beginning step of induction is easily verified. Using the exact sequence $0 \to K \to B \xrightarrow{\sigma} A \to 0$, we set $I = \{\sigma\alpha \mid \alpha : A \to B\}$. Here, I is a right ideal of the ring $E(A)$. If $a \in A$, then it follows from $K + S_A(B) = B$ that there exist $b \in B$, $k \in K$, $a_i \in A$, and $\alpha_i : A \to B$ $(i = 1, \dots, m)$ such that $a = \sigma b$ and

$$b = k + \sum_{i=1}^{m} \alpha_i a_i.$$

Therefore,

$$a = \sum_{i=1}^{m} \sigma\alpha_i a \in IA.$$

Therefore, $IA = A$. However, $I = R$ by assumption. Therefore, $\sigma\omega = 1_A$ for some $\omega : A \to B$, and the sequence is split.

Assume that $0 \to K \to B \xrightarrow{\sigma} G \to 0$ is a faithful sequence with $G \cong A^n$ and (3) holds for all positive integers less than n. We have $G = \sum_{i=1}^{n}{}^{\oplus} A_i$, where $A_i \cong A$. Let $\pi_i : G \to A_i$ be projections $(i = 1, \dots, n)$. We now set $B_i = \sigma^{-1} A_i$ and verify that $K + S_A(B_i) = B_i$. To do this, we consider the sequence

$$0 \to B_i \to B \xrightarrow{(1-\pi_i)\sigma} \sum_{j \neq i}{}^{\oplus} A_j \to 0.$$

It is easy to prove that $\ker(1-\pi_i)\sigma = B_i$. Therefore, the sequence is exact, and it is split by the induction hypothesis. Therefore, B_i is a direct summand of the group B, and $K \subseteq B_i$; therefore, $K + S_A(B_i) = B_i$.

We now consider the exact sequence $0 \to K \to B_i \xrightarrow{\sigma} A_i \to 0$; it is split by the first induction step. We choose $\omega_i : A_i \to B_i$ with $\sigma\omega_i = 1_{A_i}$ and set $\omega = \sum_{i=1}^{n} \omega_i : G \to B$. Then

$$\sigma\omega = \sigma \sum_{i=1}^{n} \omega_i = \sum_{i=1}^{n} \sigma\omega_i = \sum_{i=1}^{n} 1_{A_i} = 1_A.$$

Consequently, the original sequence is split.

The proof of the implication (3) $\Longrightarrow$ (2) is similar to the proof of the suffiency in Theorem 33.1 (the only difference is that we take a finitely generated module as F).

(2) $\Longrightarrow$ (1). Let $IA = A$ for some right ideal I of the ring R $(R = E(A))$. The exact sequence of right R-modules $0 \to I \xrightarrow{\varkappa} R \to R/I \to 0$ ($\varkappa$ is an embedding) induces the upper row of the commutative diagram

$$\begin{tikzcd} T(I) \arrow[r, "T(\varkappa)"] \arrow[d, "\varepsilon_I"] & T(R) \arrow[r] \arrow[d, "\varepsilon_R"] & T(R/I) \arrow[r] & 0 \\ IA \arrow[r, equal] & A & & \end{tikzcd},$$

where ε_I and ε_R are canonical epimorphisms, $\varepsilon_I(\alpha \otimes a) = \alpha a$ for all $\alpha \in I$ and $a \in A$, ε_R is an isomorphism (see Sec. 13), and ε_R satisfies a condition that is similar to the above condition for ε_I. Therefore, $T(\varkappa)$ is an epimorphism. This implies $T(R/I) = 0$ and $R/I = 0$ by assumption. Therefore, $I = R$ and A is a faithful group. $\square$

Arnold and Lady [35] have proved Theorem 33.2 for a torsion-free group A of finite rank. For such a group A, we can write "G is an A-projective group" instead of "G is a finitely A-projective group."

Assume that A is a flat $E(A)$-module (i.e., A is an endoflat group). Let M be a right $E(A)$-module. Then $T(M) = 0$ if and only if $T(V) = 0$ for all finitely generated submodules V of the module M. Indeed, let $T(M) = 0$, and let V be a submodule of the module M. Since A is a flat $E(A)$-module, we can assume that $T(V)$ is a subgroup in $T(M)$. Therefore, $T(V) = 0$. The converse assertion always holds. Indeed, let $T(V) = 0$ for every finitely generated submodule V of the module M. Every element $z \in T(M)$ has the form $z \in \sum_i m_i \otimes a_i$, where $m_i \in M$ and $a_i \in A$ $(i = 1, \dots, s)$. Let $V = m_1 E(A) + \dots + m_s E(A)$ be the $E(A)$-submodule generated in M by the elements $m_1, \dots, m_s$. The induced mapping $T(V) \to T(M)$ maps from the zero element $\sum m_i \otimes a_i$ into the element $\sum m_i \otimes a_i = z$. Therefore, $z = 0$ and $T(M) = 0$, since z is an arbitrary element.

Theorems 33.1 and 33.2 imply that a self-small endoflat group A is faithful if and only if A is completely faithful. Therefore, we have the following corollary.

Corollary 33.3 (Albrecht [7])**.** *For a self-small endoflat group A, the following conditions are equivalent*:

(1) *A is a faithful group*;

(2) *every exact sequence $0 \to K \to B \to G \to 0$ with A-projective group G and $B = K + S_A(B)$ is split.*

Corollary 33.4. (1) *If A is a torsion-free group such that $E(A)$ is a principal ideal domain, then A is a self-small faithful endoflat group.*

(2) *Let A be a torsion-free group such that $E(A)$ is a principal ideal domain. Then every exact sequence of groups $0 \to K \to B \to \sum\limits_{\mathfrak{M}}^{\oplus} A \to 0$ with $K + S_A(B) = B$ is split.*

Proof. (1) It follows from Propositions 31.8 and 31.2 that the group A is self-small. Nonzero endomorphisms of the group A are monomorphisms (Proposition 31.8); therefore, A as a torsion-free module over the principal ideal domain is a flat module (Lambek [279, § 5.4]). Assume that $IA = A$ for some ideal I of the ring $E(A)$. We have $I = \alpha R$ for some $\alpha \in R$. Then $\alpha A = A$. Therefore, α is an epimorphism. It was noted that α is a monomorphism. Therefore, α is an invertible element of the ring $E(A)$ and $I = \alpha R = R$.

Taking into account (1), we see that assertion (2) follows directly from Corollary 33.3. □

Since the endomorphism ring of a torsion-free group A of rank 1 is a principal ideal domain, Corollary 33.4 directly implies the Baer lemma formulated at the beginning of the part. As the group A in the situation of the Baer lemma, we can take a torsion-free group A of rank 1 and of type τ. The condition that all elements from $B \setminus K$ have type τ implies that $K + S_A(B) = B$.

An interesting example of faithful groups arises in connection with E-rings.

Example 33.5. If R is an E-ring, then R^+ is a self-small faithful and endoflat group.

Proof. We can identify $E(R^+)$ with R. Let

$$\varphi : R \to \sum_{i=1}^{\infty}{}^{\oplus} R_i$$

be an additive homomorphism, where $R_i = R$ for all i. By Proposition 6.2, φ is an R-module homomorphism. Now it is clear that φR is contained in the sum of those R_i the sum of which contains $\varphi(1)$. Since $M \otimes_R R \cong M$ for every right R-module M, the group R^+ is faithful by Theorem 33.2. Obviously, R^+ is endoflat. □

We now present examples of faithful torsion-free groups of finite rank.

Proposition 33.6. *Let A be a torsion-free group of finite rank. In each of the following three cases, A is a faithful group*:

(1) (*Arnold and Lady* [35]) *$E(A)$ is a ring of principal right ideals*;

(2) *$E(A)$ is a local ring*;

(3) *the group A is an endogenerator.*

Proof. (1) Assume that the endomorphism ring of the group A is a principal right ideal ring. This means that every right ideal of the ring $E(A)$ is principal. Let $IA = A$ for some right ideal I of the ring $E(A)$. If $I = \alpha E(A)$ ($\alpha \in E(A)$), then $\alpha A = A$. Epimorphisms of the torsion-free group of finite rank are automorphisms. Therefore, α is an invertible element of the ring $E(A)$, $I = \alpha E(A) = E(A)$, and A is a faithful group.

(2) If A is a divisible group, then it is clear that $A \cong \mathbf{Q}$, $E(A) \cong \mathbf{Q}$, and A is faithful. Otherwise, A is a reduced group and there exists a unique prime integer p with $pA \neq A$ (see item (a) in Sec. 19). Let $IA = A$ for some proper right ideal I of the ring R, where $R = E(A)$. We can assume that I is a maximal right ideal. Then $I = J(R)$; therefore, $J(R)A = A$. By Lemma 21.1, $pR \subseteq J(R)$. Therefore, $(J(R)/pR)(A/pA) = A/pA$. The ring R/pR is finite; therefore, $J(R/pR)$ is a nilpotent ideal. However, $J(R)/pR = J(R/pR)$ and the relation $(J(R)/pR)(A/pA) = A/pA$ is impossible. Therefore, $IA \neq A$ for every proper right ideal I of the ring $E(A)$.

(3) In Proposition 12.7, it is proved that any group that is an endogenerator is exact. $\square$

Proposition 33.7 (Arnold and Lady [35]). (1) *Assume that A is a torsion-free group of finite rank such that either A is a strongly indecomposable group or $E(A)$ is a semiprime ring. If I is a right ideal of the ring $E(A)$ with $IA = A$, then the factor group $E(A)/I$ is finite.*

(2) *Let A be a torsion-free group of finite rank such that $A = B \oplus C$, where $S_B(C) = C$ and* $\mathrm{Hom}(C, B) = 0$. *Then there exists a two-sided ideal I of the ring $E(A)$ such that $IA = A$ and the factor group $E(A)/I$ is infinite.*

Proof. (1) If the group A is strongly indecomposable and $IA = A$, then the right ideal I should contain at least one monomorphism of the group A. Otherwise, $I \subseteq N(E(A))$ by Corollary 5.12, whence $I^n = 0$ for some positive integer n. Now we consider

$$0 = I^n A = I^{n-1}(IA) = \cdots = A,$$

which is impossible. Thus, let γ be a monomorphism of the group A contained in I. Then γ is an invertible element of the quasi-endomorphism ring $\mathcal{E}(A)$ of the group A (Corollary 5.3). We have $\gamma\delta = 1$ for some $\delta \in \mathcal{E}(A)$. Let m be a positive integer such that $m\delta \in E(A)$. Then $\gamma(m\delta) = m1 \in I$. Therefore, $mE(A) \subseteq I$ and $|E(A)/I| < \aleph_0$.

Now assume that the ring $E(A)$ is semiprime and $IA = A$. Then $(I \otimes \mathbf{Q})(A \otimes \mathbf{Q}) = A \otimes \mathbf{Q}$. Here $I \otimes \mathbf{Q}$ is a right ideal of the Artinian semisimple ring $E(A) \otimes \mathbf{Q}$ (Theorem 5.11). Consequently, $I \otimes \mathbf{Q} = \varepsilon(E(A) \otimes \mathbf{Q})$ for some idempotent $\varepsilon \in I \otimes \mathbf{Q}$. We obtain $\varepsilon(A \otimes \mathbf{Q}) = A \otimes \mathbf{Q}$. Taking a positive integer m with $m\varepsilon \in I$, we obtain that $m1 \in I$ and $|E(A)/I| < \aleph_0$.

(2) We set $I = \{\alpha \in E(A) \mid \alpha C = 0\}$. Then I is an ideal of the ring $E(A)$, since C is a fully characteristic direct summand of the group A. In addition,

$$A = B \oplus C = B \oplus S_B(C) = S_B(A) = IA.$$

Let $\pi : A \to C$ be a projection. Obviously, the element $\pi + I$ has an infinite order in $E(A)/I$. $\square$

The description of faithful torsion-free groups of finite rank is not known.

It turns out that if we begin with some countable reduced torsion-free ring R, then the Corner construction described in Theorem 29.2 leads to a self-small faithful endoflat group A.

Theorem 33.8 (Albrecht [13]). *Let R be a countable reduced torsion-free ring. If A is a countable reduced torsion-free group with $E(A) \cong R$ obtained in Theorem 29.2, then A is a self-small faithful and endoflat group.*

Proof. We consider the proof of Theorem 29.2. We assume that $E(A) = R$. By Corollary 31.4, any group with countable endomorphism ring is self-small. By construction, the group A is a pure subgroup of the $\mathbf{Z}$-adic completion $\hat{R}$ of the ring R. More precisely, the sum F of left R-modules $R\varepsilon_\alpha$ $(\alpha \in R)$ is a direct sum and A is the pure hull in $\hat{R}$ of the R-module $R \oplus F$. Since the subgroup A/R is pure in $\hat{R}/R$, we have that A/R is a divisible torsion-free group (R is a dense subring in $\hat{R}$; therefore, $\hat{R}/R$ is a divisible torsion-free group). Moreover, we have the inclusions

$$R \oplus F \subset A \subset (R \oplus F) \otimes \mathbf{Q} = (R \otimes \mathbf{Q}) \oplus (F \otimes \mathbf{Q}).$$

Let $\pi : (R \oplus F) \otimes \mathbf{Q} \to F \otimes \mathbf{Q}$ be the projection with kernel $R \otimes \mathbf{Q}$. Let $x \in (R \otimes \mathbf{Q}) \cap A$. Then $nx \in R$ for some positive integer n. Since the group A/R is torsion-free, $x \in R$. Therefore, $\ker \pi \cap A = (R \otimes \mathbf{Q}) \cap A = R$. Now we can assert that πA and A/R are isomorphic as R-modules and their additive groups are divisible. Therefore, $F \otimes \mathbf{Q} = \pi A \oplus C$ for some subgroup $C \subseteq F \otimes \mathbf{Q}$. On the other hand, if $z \in F \otimes \mathbf{Q}$ and $mz \in F$ $(m \in \mathbf{N})$, then $mz = \pi(mz) \in \pi A$. Therefore, $(F \otimes \mathbf{Q})/\pi A$ is a torsion group. Consequently, $\pi A = F \otimes \mathbf{Q}$ and

$$\pi A = F \otimes \mathbf{Q} = \Big(\sum_{\alpha \in R}^{\oplus} R\varepsilon_\alpha\Big) \otimes \mathbf{Q} \cong \sum_{\alpha \in R}^{\oplus} (R\varepsilon_\alpha \otimes \mathbf{Q}),$$

where $R\varepsilon_\alpha \cong R$ for every $\alpha \in R$. Therefore, the R-module πA is isomorphic to a direct sum of some set of modules $R \otimes \mathbf{Q}$. Since the group of rational numbers $\mathbf{Q}$ is a torsion-free group, it is the direct limit of free groups F_i ($\mathbf{Q} \cong \varinjlim F_i$). This implies

$$R \otimes \mathbf{Q} \cong R \otimes (\varinjlim F_i) \cong \varinjlim (R \otimes F_i),$$

where $R \otimes F_i$ is a free R-module. Therefore $R \otimes \mathbf{Q}$ is a flat left R-module (see the beginning of Sec. 10). In the exact sequence of left R-modules $0 \to R \to A \to \pi A \to 0$, the modules R and πA are flat. Therefore, A is also a flat R-module by the property indicated at the beginning of the section, and A is an endoflat group.

It remains to prove that the group A is faithful. Assume that $IA = A$ for some right ideal I of the ring R. First, we show that R/I is a torsion group. Assume the contrary. The pure hull J of the ideal I in R is a proper right ideal of the ring R with $JA = A$. Further, $J(\pi A) = \pi A$. Since the R-module πA is isomorphic to a direct sum of copies of the ring $R \otimes \mathbf{Q}$, we obtain $J(R \otimes \mathbf{Q}) = R \otimes \mathbf{Q}$. Since the R-module $R \otimes \mathbf{Q}$ is flat, the exact sequence $0 \to J \to R \to R/J \to 0$ induces the exact sequence

$$0 \to J \otimes_R (R \otimes \mathbf{Q}) \to R \otimes_R (R \otimes \mathbf{Q}) \to R/J \otimes_R (R \otimes \mathbf{Q}) \to 0.$$

The relation $J(R \otimes \mathbf{Q}) = R \otimes \mathbf{Q}$ implies $R/J \otimes_R (R \otimes \mathbf{Q}) = 0$ (a similar property is circumstantially studied in the proof of the implication (2) $\Longrightarrow$ (1) of Theorem 33.2). On the other hand,

$$R/J \otimes_R (R \otimes \mathbf{Q}) \cong (R/J \otimes_R R) \otimes \mathbf{Q} \cong R/J \otimes \mathbf{Q} \neq 0,$$

since R/J is a torsion-free group. Thus, R/I is a torsion group. Therefore, we can choose a positive integer k with $k1 \in I$. In this case, $kR \subseteq I$.

Since $1 \in R \subset A = IA \subseteq I\hat{R}$, we have

$$1 = \sum_{s=1}^{m} \alpha_s \beta_s \qquad \text{with} \qquad \alpha_s \in I \quad \text{and} \quad \beta_s \in \hat{R}.$$

Further, $\beta_s = \lim \delta_{s_i}$ (we consider the limit in the $\mathbf{Z}$-adic topology of the ring $\hat{R}$), where $\delta_{s_i} \in R$. Consequently,

$$1 = \sum_{s=1}^{m} \alpha_s (\lim \delta_{s_i}) = \sum_{s=1}^{m} \lim(\alpha_s \delta_{s_i}) = \lim \Big(\sum_{s=1}^{m} \alpha_s \delta_{s_i} \Big),$$

and $\sum \alpha_s \delta_{s_i} \in I$. Therefore, 1 is the limit in the $\mathbf{Z}$-adic topology of sequence $\{\rho_i\}$ of some elements of the ideal I. This means that for every positive integer n, there exists ρ_i such that $1 - \rho_i \in nR$ or $1 \in \rho_i + nR \subseteq I + nR$. In particular, $1 \in I + kR = I$ for $n = k$, since $kR \subseteq I$. We have $I = R$ and A is a faithful group. $\square$

Similarly, we obtain the following result.

Corollary 33.9. *Let R be a reduced torsion-free ring of finite rank. If A is a reduced torsion-free group of finite rank with $E(A) \cong R$ from Theorem 29.3, then A is a faithful endoflat group.*

We obtain that the existence of faithful endoflat groups considered in the next section is quite common. In addition, a strongly indecomposable torsion-free group of rank 2 is a faithful endoflat group by Corollary 13.8 and Exercise 7 below.

Exercise 1. Prove that a faithful group cannot be an infinite direct sum of nonzero groups.

Exercise 2. If A is a faithful group, then all nonzero direct summands of the group A^n are faithful for every positive integer n. In general, every finitely A-projective group is faithful.

Exercise 3. Prove that a faithful torsion group is finite.

Exercise 4. Characterize faithful finite groups and faithful finitely generated groups.

Exercise 5. Describe faithful algebraically compact groups and faithful separable groups.

Exercise 6 (Arnold and Lady [35])**.** The torsion-free group of finite rank with commutative endomorphism ring is faithful.

Exercise 7 (Arnold and Lady [35])**.** Let A be strongly indecomposable torsion-free group of rank 2. Then

(a) the group A is exact;
(b) if A is not a homogeneous group, then every A-projective group of finite rank is A-free.

Exercise 8. Prove that any group of finite rank from the class $\mathcal{E}$ (see Sec. 32, Exercises 4 and 5) is faithful and endoflat.

Exercise 9. If A is a torsion-free group, then for any maximal right ideal K of the ring $E(A)$, either the ideal K contains $pE(A)$ for some prime integer p or $E(A)/K$ is a divisible torsion-free group.

Arnold [30] calls a torsion-free group A of finite rank finitely faithful if $IA \neq A$ for every proper right ideal I of finite index of the ring $E(A)$. It is clear that it is sufficient to consider such maximal ideals I and the group A is finitely faithful provided A is faithful.

Exercise 10. For a reduced torsion-free group G of finite rank, prove that G is faithful if and only if G is finitely faithful. Nevertheless, there exists a (nonreduced) finitely faithful group that is not faithful.

Exercise 11. Characterize finitely faithful completely decomposable torsion-free groups of finite rank.

Exercise 12 (Arnold [30])**.** For a torsion-free group A of finite rank, the following conditions are equivalent:

(1) A is a finitely faithful group;
(2) $K = \mathrm{Hom}(A, KA)$ for every maximal right ideal K of finite index of the ring $E(A)$;
(3) $J_p = \mathrm{Hom}(A, J_p)$ for every prime integer p, where $J_p/pE(A) = J(E(A)/pE(A))$.

34. Faithful Endoflat Groups

This section is quite informative. The combination of exactness and endoflatness allows us to find new relations between properties of the group and properties of its endomorphism ring. After presentation of the well-known results on faithfully flat modules, we present various characterizations of endoflat and faithful endoflat groups. Then we present partial generalizations of theorems of Arnold, Lady, and Murley (Theorems 32.1 and 32.2) on the equivalence of some categories arising when starting from a given group A. These theorems transfer the study of direct decompositions of the group A to the study of direct decompositions of projective right $E(A)$-modules. Section 32 contains many remarkable examples of such a transfer. One encounters serious obstacles when using Theorems 32.1 and 32.2 for studying some other properties of the group A that are formulated in terms of the ring $E(A)$ and are not necessarily related to its direct decompositions. Theorems 34.13 and 34.14 proved in this section help us to remove many of these obstacles at the expense of considering larger categories. Indeed, we take into account not only A-projective groups but some their subgroups. Further, we apply the obtained new theorems on the equivalence of categories and classify some properties of the ring $E(A)$ in terms of the group structure of the group A. We especially consider the case of torsion-free groups (of finite rank), which is the most meaningful in the context of studies in this part. Extensions of theorems of Arnold, Lady, and Murley are sufficient for intuitively solving some difficult problems on ring properties of endomorphism rings. This gives a partial solution of Problem 14.

In this section (except for Propositions 34.1–34.3), R denotes the endomorphism ring $E(A)$ of a group A. We do not change the notation of the preceding section. For example, if A is a group, then H is the functor $\mathrm{Hom}(A, -)$ and T is the functor $(-) \otimes_R A$ and $\theta : TH \to 1$ and $\phi : 1 \to HT$ are natural transformations.

According to Sec. 1, we use the following notation. If A is a left R-module (R is not necessarily the endomorphism ring in the beginning part of this section), then $T(V) = V \otimes_R A$ for every right R-module V and $T(\alpha) = \alpha \otimes 1$ for every homomorphism α of right R-modules. In particular, the first result specifies and generalizes the relations between endoflat completely faithful and faithful groups, which were considered in Sec. 33.

Proposition 34.1 (Bourbaki [70]). *Let A be a left module over a ring R. The following conditions are equivalent*:

(1) *the sequence of right R-modules $U \xrightarrow{\alpha} V \xrightarrow{\beta} W$ is exact if and only if the sequence*
$$U \otimes_R A \xrightarrow{\alpha\otimes 1} V \otimes_R A \xrightarrow{\beta\otimes 1} W \otimes_R A$$
is exact;

(2) *the module A is flat, and for every right R-module V, the relation $V \otimes_R A = 0$ implies $V = 0$*;

(3) *the module A is flat, and for any homomorphism $\alpha : U \to V$ of right R-modules, the relation $\alpha \otimes 1 = 0$ implies $\alpha = 0$*;

(4) *the module A is flat, and for every right ideal I of the ring R, the relation $IA = A$ implies $I = R$.*

Proof. (1) $\Longrightarrow$ (2). By (1), the exact sequence of right R-modules $0 \to V \xrightarrow{\beta} W$ implies the exact sequence $0 \to V \otimes_R A \xrightarrow{\beta\otimes 1} W \otimes_R A$; therefore, the module A is flat. Assume that $T(V) = 0$ for some right R-module V. The sequence $0 \to V \to 0$ implies the exact sequence $0 \to T(V) \to 0$ by assumption. Then the sequence $0 \to V \to 0$ is exact by (1), whence $V = 0$.

(2) $\Longrightarrow$ (3). Let $\alpha : U \to V$ be a homomorphism such that $T(\alpha) = 0$. We set $X = \operatorname{im}\alpha$. Since the module A is flat, $X \otimes_R A \cong \operatorname{im} T(\alpha) = 0$. Therefore, $X = 0$ and $\alpha = 0$.

(3) $\Longrightarrow$ (4). Assume that $IA = A$. We have the commutative diagram
$$\begin{tikzcd}
0 \arrow[r] & T(I) \arrow[r] \arrow[d] & T(R) \arrow[r] \arrow[d] & T(R/I) \arrow[r] & 0 \\
 & IA \arrow[r] & A & &
\end{tikzcd},$$
where vertical arrows are canonical mappings. Since the module A is flat, the first mapping is an isomorphism; the second mapping is always an isomorphism. Therefore, $T(I) \to T(R)$ is an isomorphism and $T(R) \to T(R/I)$ is the zero mapping. Therefore, $R \to R/I$ is the zero homomorphism and $I = R$.

(4) $\Longrightarrow$ (3). Let $\alpha : U \to V$ be a homomorphism of right R-modules such that $\alpha \otimes 1 = 0$, where $\alpha \otimes 1 : U \otimes_R A \to V \otimes_R A$. We set $X \operatorname{im}\alpha$. Since the module A is flat, $T(X) = 0$ (as in (2) $\Longrightarrow$ (3)). The implication (3) $\Longrightarrow$ (4) implies that the condition "$IA = A \Longrightarrow I = R$" implies $T(K) = 0$ for all cyclic right R-modules K. Therefore, $T(M) = 0$ for every right R-module M (see the remark before Corollary 33.3 in the preceding section). Therefore, $X = 0$ and $\alpha = 0$.

(3) $\Longrightarrow$ (1). We consider a sequence of homomorphisms of left R-modules
$$U \xrightarrow{\alpha} V \xrightarrow{\beta} W \tag{1}$$
and the accompanying sequence of homomorphisms of Abelian groups
$$T(U) \xrightarrow{T(\alpha)} T(V) \xrightarrow{T(\beta)} T(W). \tag{2}$$
If sequence (1) is exact, then sequence (2) is also exact (this is directly verified by using the flatness of the module A). Conversely, if (2) is an exact sequence, then it follows from $T(\beta\alpha) = T(\beta)T(\alpha) = 0$ and the assumption that $\beta\alpha = 0$, whence $\operatorname{im}\alpha \subseteq \ker\beta$. We set $K = \ker\beta$ and $L = \operatorname{im}\alpha$. We have the exact sequence
$$0 \to L \xrightarrow{i} K \xrightarrow{q} K/L \to 0,$$

where i and q are canonical mappings. Since A is a flat module, the sequence

$$0 \longrightarrow T(L) \xrightarrow{T(i)} T(K) \xrightarrow{T(q)} T(K/L) \longrightarrow 0$$

is flat. Since the module A is flat, we can identify $T(L)$ with the image of the mapping $T(i)$ and $T(K)$ with kernel of the mapping $T(q)$. Therefore, $T(i)$ is an isomorphism and $T(q) = 0$. By assumption, $q = 0$ and $L = K$; therefore, sequence (1) is exact. ☐

A left R-module A is said to be *faithfully flat* if all four conditions of Proposition 34.1 hold.

Proposition 34.1(1) directly implies the following result.

Proposition 34.2. *Let A be a faithfully flat left R-module, and let $\alpha : U \to V$ be a homomorphism of right R-modules. The homomorphism α is injective* (*resp. surjective or bijective*) *if and only if the mapping $\alpha \otimes 1 : U \otimes_R A \to V \otimes_R A$ is a homomorphism of corresponding type.*

Proposition 34.3 (Bourbaki [70]). *Let $0 \to B \to A \to C \to 0$ be an exact sequence of left R-modules. Assume that the modules B and C are flat and at least one of these modules is faithfully flat. Then the module A also is faithfully flat.*

Proof. It is well known that the module A is flat (see the beginning of Sec. 33). We prove that A satisfies condition (2) of Proposition 34.1. Let V be a right R-module. Since the module C is flat, we have an exact sequence

$$0 \to V \otimes_R B \to V \otimes_R A \to V \otimes_R C \to 0$$

(the torsion product $\mathrm{Tor}_R(V, C)$ must stand instead of zero in the left-hand side, but it is equal to zero, since V is a flat module). If $V \otimes_R A = 0$, then $V \otimes_R B = 0 = V \otimes_R C$. At least one of the modules B and C is faithfully flat; therefore, $V = 0$. ☐

In this part, we usually proceed from some fixed group A; in so doing, other groups G participating in the study often have the property $G = S_A(G)$. In some cases, it is expedient to consider only groups G with $G = S_A(G)$. Such groups G are said to be A-generated.

Lemma 34.4. *Let A and G be two groups. The following conditions are equivalent*:

(1) *G is an A-generated group* (*i.e., $G = S_A(G)$*);
(2) *the natural mapping $\theta_G : TH(G) \to G$ is an epimorphism*;
(3) *the group G is a homomorphic image of some A-free group.*

Proof. The equivalence of (1) and (2) directly follows from the definitions of the trace and the mapping θ_G.

(1) $\Longrightarrow$ (3). We form a direct sum $\sum\limits_{\varphi}^{\oplus} A_\varphi$, where φ runs over the set $\mathrm{Hom}(A, G)$ and $A_\varphi = A$ for every $\varphi \in \mathrm{Hom}(A, G)$. We define the homomorphism

$$\Phi : \sum_{\varphi}^{\oplus} A_\varphi \to G$$

by

$$\Phi[(a_\varphi)] = \sum_{\varphi} \varphi(a_\varphi),$$

(i.e., Φ coincides with $\varphi : A \to G$ on the group A_φ). It follows from $G = S_A(G)$ that Φ is an epimorphism. The converse implication (3) $\Longrightarrow$ (1) is obvious. ☐

We note that if M is a right $E(A)$-module, then $S_A(T(M)) = T(M)$ for the group $T(M) = M \otimes_{E(A)} A$. Therefore, $T(M)$ is an A-generated group. This is shown in the preceding section.

We obtain several general criteria of endoflatness of a group A. Its endomorphism ring is denoted by R. The essence of our proofs is contained in the following remarks. We fix a positive integer n. If

$\varepsilon : A^n \to A$ is a homomorphism, then $H(\varepsilon) : H(A^n) \to H(A)$ is a homomorphism of right R-modules with the image $H(\varepsilon)(H(A^n)) = \varepsilon H(A^n)$ and the kernel

$$K = \ker H(\varepsilon) = \{\eta \in H(A^n) \mid \varepsilon\eta = 0\} = \mathrm{Hom}(A, \ker \varepsilon).$$

The exact sequence of right R-modules

$$0 \to \ker H(\varepsilon) \xrightarrow{i} H(A^n) \xrightarrow{H(\varepsilon)} \varepsilon H(A^n) \to 0$$

induces the upper exact row of the following commutative diagram:

$$\begin{tikzcd}
\ker H(\varepsilon) \otimes_R A \arrow[r, "T(i)"] & H(A^n) \otimes_R A \arrow[r, "TH(\varepsilon)"] \arrow[d, "\theta_K"'] & \varepsilon H(A^n) \otimes_R A \arrow[r] \arrow[d, "\theta_{A^n}"'] & 0 \arrow[d, "\theta_{\varepsilon A^n}"'] & \\
0 \arrow[r] & \ker \varepsilon \arrow[r, "j"] & A^n \arrow[r, "\varepsilon"] & \varepsilon A^n \arrow[r] & 0
\end{tikzcd}.$$

Here, i and j are inclusion mappings, θ_K, θ_{A^n}, and $\theta_{\varepsilon A^n}$ are natural mappings, and θ_{A^n} is an isomorphism by Theorem 32.1. Since $S_A(\ker \varepsilon) = \mathrm{im}\, \theta_K$, we have that θ_K is an epimorphism if and only if $S_A(\ker \varepsilon) = \ker \varepsilon$. It is easily verified that $\theta_{\varepsilon A^n}$ is a monomorphism if and only if θ_K is an epimorphism. This is a partial case of the well-known "Five Lemma" (Bass [51, Chapter I, Proposition 4.4]). Therefore, we obtain the following assertion.

Lemma 34.5. $S_A(\ker \varepsilon) = \ker \varepsilon$ *if and only if* $\theta_{\varepsilon A^n}$ *is a monomorphism.*

We consider the first result on endoflat groups.

Theorem 34.6 (Arnold [33], Arnold and Hausen [34])**.** *For a group A, the following conditions are equivalent*:

(1) *A is an endoflat group*;
(2) $\ker \varepsilon = S_A(\ker \varepsilon)$ *for every positive integer n and each* $\varepsilon : A^n \to A$;
(3) (a) *for all $\varepsilon \in R$,* $\ker \varepsilon = S_A(\ker \varepsilon)$
and
(b) $(I \cap J)A = IA \cap JA$ *for any two* (*finitely generated*) *right ideals I and J of the ring R.*

Proof. First, we do some preparatory work.

Let I and J be two right ideals of the ring R. There exists an exact sequence of right R-modules $0 \to I \cap J \to I \oplus J \to I + J \to 0$, where the first (nonzero) mapping is $\alpha \to (\alpha, -\alpha)$ for $\alpha \in I \cap J$ and the second mapping is $(\alpha, \beta) \to \alpha + \beta$ for $\alpha \in I$ and $\beta \in J$. Similarly, we define the mappings in the exact group sequence $0 \to IA \cap JA \to IA \oplus JA \to IA + JA \to 0$. We have the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & (I \cap J) \otimes_R A \arrow[r] \arrow[d, "\Lambda_{I\cap J}"] & (I \oplus J) \otimes_R A \arrow[r] \arrow[d, "\Lambda"] & (I + J) \otimes_R A \arrow[r] \arrow[d, "\Lambda_{I+J}"] & 0 \\
0 \arrow[r] & IA \cap JA \arrow[r] & IA \oplus JA \arrow[r] & IA + JA \arrow[r] & 0
\end{tikzcd},$$

where extreme vertical arrows are canonical mappings (see the beginning of Sec. 33) and $\Lambda = \Lambda_I + \Lambda_J$.

In addition, we prove the following property. Every finitely generated right ideal L of the ring R is equal to $\varepsilon H(A^n)$ for some positive integer n and $\varepsilon : A^n \to A$. Let $L = \varepsilon_1 R + \cdots + \varepsilon_n R$, where $\varepsilon_i \in R$. We set $A_1 = \cdots = A_n = A$. We define

$$\varepsilon : A^n = A_1 + \cdots + A_n \to A$$

such that $\varepsilon_{|A_i} = \varepsilon_i$ $(i = 1, \ldots, n)$ (i.e., $\varepsilon = \sum\limits_{i=1}^{n} \varepsilon_i$). It is clear that $L = \varepsilon H(A^n)$ for the constructed ε.

We now pass to the proof of the theorem itself.

(1) $\implies$ (2). Let $\varepsilon : A^n \to A$. Since the R-module A is flat, the canonical mapping $\theta_{\varepsilon A^n}$ from the diagram before Lemma 34.5 is an isomorphism. This lemma implies $\ker \varepsilon = S_A(\ker \varepsilon)$.

(2) $\implies$ (3). Assertion (a) follows from (2) for $n = 1$. We verify assertion (b) for arbitrary finitely generated right ideals I and J of the ring R. It was proved that $I = \varepsilon H(A^n)$ and $J = \tau H(A^m)$ for some $\varepsilon : A^n \to A$ and $\tau : A^m \to A$. It follows from the assumption and Lemma 34.5 that $\theta_{\varepsilon A^n}$ and $\theta_{\tau A^m}$ are monomorphisms. Therefore, Λ_I and Λ_J are isomorphisms and Λ_{I+J} is an isomorphism. Consequently, $\Lambda_{I\cap J}$ is an epimorphism. However, $\operatorname{im} \Lambda_{I\cap J} = (I \cap J)A$. Therefore, $(I \cap J)A = IA \cap JA$.

Now assume that I and J are arbitrary (not necessarily finitely generated) right ideals. Let

$$a = \sum_{i=1}^{s} \lambda_i b_i = \sum_{j=1}^{t} \mu_j c_j \in IA \cap JA,$$

where $\lambda_i \in I$, $\mu_j \in J$, and $b_i, c_j \in A$. We set

$$I' = \sum_{i=1}^{s} \lambda_i R \qquad \text{and} \qquad J' = \sum_{j=1}^{t} \mu_j R.$$

Then $a \in I'A \cap J'A$. By the above, $a \in (I' \cap J')A \subseteq (I \cap J)A$. Therefore, $IA \cap JA \subseteq (I \cap J)A$ and the converse inclusion always holds.

(3) $\implies$ (1). It is sufficient to prove that Λ_L is a monomorphism for every finitely generated right ideal L of the ring R. We prove this assertion by induction on the number of generating elements of the ideal L. If L is a principal right ideal, then the assertion follows from (a) and Lemma 34.5. Assume that $L = I + J$, where I and J are right ideals of the ring R whose numbers of generating elements are less than the number of generating elements of L. By the induction hypothesis, Λ_I and Λ_J are monomorphisms. Consequently, the mapping Λ in the above commutative diagram is an isomorphism. Since $\Lambda_{I\cap J}$ is an epimorphism by (b), Λ_L is a monomorphism. $\square$

We define some new notions related to a given group A (see Albrecht [13]). Using these notions, we then obtain characterizations of faithful endoflat groups and some results on their endomorphism rings.

A group G is said to be A-*soluble* if the natural mapping $\theta_G : TH(G) \to G$ is an isomorphism. We denote the class of all A-soluble groups by $\mathcal{R}_A$.

Let $\mathcal{R}$ be some class of A-generated groups. The class $\mathcal{R}$ is said to be A-*closed* if

(a) $\mathcal{R}$ is closed with respect to finite direct sums;

(b) if $G \in \mathcal{R}$ and U is an A-generated subgroup of the group G, then $U \in \mathcal{R}$;

(c) $\mathcal{R}$ is closed with respect to kernels of all homomorphisms of groups in $\mathcal{R}$.

These three conditions guarantee that $\mathcal{R}$ is an additive category with kernels if it is considered as a complete subcategory of the category of Abelian groups.

A category $\mathcal{C}$ is called a subcategory of a category $\mathcal{E}$ if the following conditions hold:

(1) all objects of the category $\mathcal{C}$ are objects of the category $\mathcal{E}$;
(2) $\operatorname{Hom}_{\mathcal{C}}(B, C) \subseteq \operatorname{Hom}_{\mathcal{E}}(B, C)$ for any $B, C \in \mathcal{C}$;
(3) the composition of morphisms in $\mathcal{C}$ is induced by their composition in $\mathcal{E}$;
(4) all identity morphisms of $\mathcal{C}$ are the identity morphisms in $\mathcal{E}$.

A subcategory $\mathcal{C}$ of the category $\mathcal{E}$ is said to be *complete* if $\operatorname{Hom}_{\mathcal{C}}(B, C) = \operatorname{Hom}_{\mathcal{E}}(B, C)$ for any objects $B, C \in \mathcal{C}$.

Additive categories were defined in Sec. 1. We do not define the general notion of kernel in a category. In the given case, this merely means that (c) holds (i.e., ordinary kernels of all homomorphisms of groups from $\mathcal{R}$ also belong to $\mathcal{R}$).

A group $E \in \mathcal{R}$ is called a projective object of the category $\mathcal{R}$ if for every epimorphism $\pi : C \to G$ and a homomorphism $\varphi : E \to G$ of groups from $\mathcal{R}$, there exists a homomorphism $\psi : E \to C$ such that $\pi\psi = \varphi$. This is equivalent to the condition that the induced mapping $\pi_* \operatorname{Hom}(E, C) \to \operatorname{Hom}(E, G)$ is an epimorphism. If E is a projective object in $\mathcal{R}$, then every epimorphism $\pi : C \to E$ is split, since there exists $\varkappa : E \to C$ such that $\pi\varkappa = 1_E$.

An A-closed class $\mathcal{R}$ is said to be A-balanced closed if the group A is a projective object of the category $\mathcal{R}$.

By $\mathcal{M}_A$, we denote the category of all right R-modules M $(R = E(A))$ such that the natural homomorphism ϕ_M is an isomorphism. The functors H and T define a category equivalence between $\mathcal{R}_A$ and $\mathcal{M}_A$; this follows from the following argument. If $M \in \mathcal{M}_A$, then

$$\theta_{T(M)}T(\phi_M)(x \otimes a) = \theta_{T(M)}(\phi_M(x) \otimes a) = [\phi_M(x)](a) = x \otimes a$$

for all $x \in M$ and $a \in A$. Consequently, $\theta_{T(M)} = [T(\phi_M)]^{-1}$ is an isomorphism. If the group $G \in \mathcal{R}_A$, then

$$[H(\theta_G)\phi_{H(G)}](\varphi)(a) = H(\theta_G)[\phi_{H(G)}(\varphi)](a) = \theta_G[\phi_{H(G)}(\varphi)](a) = \theta_G(\varphi \otimes a) = \varphi(a)$$

for all $\varphi \in H(G)$ and $a \in A$. Therefore, $\phi_{H(G)} = [H(\theta_G)]^{-1}$.

Lemma 34.7. *Let A be a group. An exact sequence $0 \to U \overset{\alpha}{\to} G \overset{\beta}{\to} E \to 0$ with A-soluble group G induces the exact sequence*

$$\mathrm{Tor}_R(M, A) \xrightarrow{\Delta} TH(U) \xrightarrow{\theta_U} U \xrightarrow{\delta} T(M) \xrightarrow{\theta_\beta} E \longrightarrow 0,$$

where $M = \operatorname{im} H(\beta)$ and $\theta_\beta : T(M) \to E$ is defined by $\theta_\beta(\alpha \otimes a) = \alpha(a)$ for all $\alpha \in M$ and $a \in A$.

Proof. We consider the induced exact sequence of R-modules

$$0 \longrightarrow H(U) \overset{H(\alpha)}{\longrightarrow} H(G) \overset{H(\beta)}{\longrightarrow} M \longrightarrow 0,$$

where $M = \operatorname{im} H(\beta)$ is a submodule in $H(E)$. The application of the functor T gives the upper row of the following commutative diagram:

$$\begin{array}{ccccccccc}
\mathrm{Tor}_R(M, A) & \xrightarrow{\Delta} & TH(U) & \xrightarrow{TH(\alpha)} & TH(G) & \xrightarrow{TH(\beta)} & T(M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \theta_U} & & \downarrow{\scriptstyle \theta_G} & & \downarrow{\scriptstyle \theta_\beta} & & \\
0 & \longrightarrow & U & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & E & \longrightarrow & 0
\end{array}$$

with exact rows, where Δ is the connecting homomorphism. We set $\delta = TH(\beta)\theta_G^{-1}\alpha$ (θ_G is an isomorphism, since the group G is A-soluble). Now we use the usual diagrammatic methods. $\square$

In particular, the last result can be used in the following situation. An exact sequence $0 \to B \to C \overset{\pi}{\to} G \to 0$ is A-balanced if the group A is projective with respect to it (i.e., $H(\pi)$ is an epimorphism). Consequently, the A-balanced exact sequence $0 \to B \to C \to G \to 0$ with $C \in \mathcal{R}_A$ induces the exact sequence

$$0 \longrightarrow TH(B) \xrightarrow{\theta_B} B \xrightarrow{\delta} TH(G) \xrightarrow{\theta_G} G \longrightarrow 0$$

if A is an endoflat group. The first application of Lemma 34.7 is the following theorem.

Theorem 34.8 (Albrecht [13]). *For a self-small group A, the following conditions are equivalent*:

(1) *A is an endoflat group*;
(2) *there exists an A-closed class $\mathcal{R}$ containing A*;
(3) *$\mathcal{R}_A$ is the largest A-closed class containing all A-projective groups.*

Proof. (1) $\Longrightarrow$ (3). At the first step, we prove that the class $\mathcal{R}_A$ is A-closed. Let B be a subgroup of some A-soluble group with $S_A(B) = B$. By Lemma 34.7, there exists an exact sequence $0 \to TH(B) \xrightarrow{\theta_B} B \to 0$, since $S_A(B) = \operatorname{im} \theta_B$. Therefore, $B \in \mathcal{R}_A$.

Further, we take some exact sequence

$$0 \to B \to C \xrightarrow{\pi} G \to 0$$

with A-soluble groups C and G. By Lemma 34.7, there is the exact sequence

$$0 \longrightarrow TH(B) \xrightarrow{\theta_B} B \xrightarrow{\delta} T(M) \xrightarrow{\theta_\pi} G \longrightarrow 0,$$

where $M = \operatorname{im} H(\pi)$ is a submodule of $H(C/B)$. We denote by i the inclusion mapping $M \to H(G)$. Since θ_π is unique, $\theta_\pi = \theta_G T(i)$. Therefore, θ_π is a monomorphism and the mapping θ_B is an isomorphism. Since $B \cong \ker \pi$, conditions (b) and (c) from the definition of an A-closed class hold. It is easy to prove that $\mathcal{R}_A$ is closed with respect to finite direct sums (see the proof of Theorem 32.1). Therefore, $\mathcal{R}_A$ is an A-closed class. Since A is a self-small group, it contains all A-projective groups by Theorem 32.1.

It remains to prove that every A-closed class $\mathcal{R}$ containing all A-projective groups is contained in $\mathcal{R}_A$. For the group $G \in \mathcal{R}$, there exists an A-balanced exact sequence

$$0 \longrightarrow U \xrightarrow{\alpha} \sum_{\mathfrak{N}}^{\oplus} A \xrightarrow{\beta} G \longrightarrow 0,$$

where $\mathfrak{N}$ is a cardinal number. Such a sequence can be obtained by using the homomorphism Φ from (1) $\Longrightarrow$ (3) of Lemma 34.4. It is clear that $H(\beta) : H\Big(\sum_{\mathfrak{N}}^{\oplus} A\Big) \to H(G)$ is an epimorphism. Since $G, \sum_{\mathfrak{N}}^{\oplus} A \in \mathcal{R}$, we obtain $S_A(U) = U$. Lemma 34.7 implies the exact sequence

$$TH(U) \xrightarrow{\theta_U} U \xrightarrow{\delta} TH(G) \xrightarrow{\theta_G} G \longrightarrow 0,$$

where θ_U is an epimorphism ($\operatorname{im} \theta_U = S_A(U)$) and $\delta = 0$; therefore, θ_G is an isomorphism.

The implication (3) $\Longrightarrow$ (2) is obvious.

(2) $\Longrightarrow$ (1). If n is a positive integer and $\varepsilon \in \operatorname{Hom}(A^n, A)$, then $\ker \varepsilon \in \mathcal{R}$. In particular, $S_A(\ker \varepsilon) = \ker \varepsilon$; therefore, the group A is endoflat by Theorem 34.6. $\square$

Taking into account the proof of the implication (1) $\Longrightarrow$ (3) of Theorem 34.8, we can obtain the following result.

Lemma 34.9. *Let A be an endoflat group. If $0 \to B \xrightarrow{\alpha} C \xrightarrow{\beta} G \to 0$ is an exact sequence of A-soluble groups and $M = \operatorname{im} H(\beta)$, then $T(H(G)/M) = 0$.*

Thus, the category $\mathcal{R}_A$ is an additive category with kernels if A is a self-small endoflat group. Projective objects in $\mathcal{R}_A$ are A-projective groups (see Corollary 34.12). In the general case, the converse assertion is not true, since we have the following example.

Example 34.10. We take the group $A = \mathbf{Z} \oplus Q_p$ for some prime integer p. The $\mathbf{Z}$-module A is a generator. Consequently, the $E(A)$-module A is projective and, in particular, flat (this follows from the Morita theorem presented at the beginning of Sec. 12). Therefore, A is an endoflat group. The group $\mathbf{Z}$ is $\mathcal{R}_A$-projective (i.e., $\mathbf{Z}$ is a projective object in $\mathcal{R}_A$). Nevertheless, the A-projective group Q_p is not $\mathcal{R}_A$-projective, since there exists a nonsplit exact sequence $\sum_{\aleph_0}^{\oplus} \mathbf{Z} \to Q_p \to 0$.

A terminology remark: if A is a faithful endoflat group, then A is a faithfully flat R-module and vice versa.

The following result characterizes self-small groups A that are endoflat and have an additional property that every A-projective group is $\mathcal{R}_A$-projective.

Theorem 34.11 (Albrecht [13])**.** *For a self-small group A, the following conditions are equivalent*:

(1) *A is a faithful endoflat group*;
(2) *there exists an A-balanced closed class $\mathcal{R}$ containing all A-projective groups*;
(3) *$\mathcal{R}_A$ is the largest A-balanced closed class containing A.*

Proof. The implication (3) $\Longrightarrow$ (2) is obvious, since every A-projective group is contained in $\mathcal{R}_A$.

(1) $\Longrightarrow$ (3). The class $\mathcal{R}_A$ is A-closed by Theorem 34.8. We consider the exact sequence

$$0 \to B \xrightarrow{\alpha} C \xrightarrow{\beta} G \to 0$$

of A-soluble groups. If $M = \operatorname{im} H(\beta)$, then $T(H(G)/M) = 0$ by Lemma 34.9. Since A is a faithful endoflat group, we obtain that $H(G)/M = 0$ (Proposition 34.1) and $M = H(G)$. Therefore, $\mathcal{R}_A$ is an A-balanced closed class.

Let $\mathcal{R}$ be some A-balanced closed class containing A. For every group $G \in \mathcal{R}$ and any $\varphi_1, \dots, \varphi_n \in H(G)$, we consider the subgroup $U = \langle \varphi_1(A), \dots, \varphi_n(A) \rangle$ of the group G. There exists an exact sequence

$$0 \to V \xrightarrow{\alpha} A^n \xrightarrow{\beta} U \to 0$$

that is A-balanced (this sequence is obtained as in the implication (1) $\Longrightarrow$ (3) of Theorem 34.8) and satisfies $V \in \mathcal{R}$. Since $S_A(V) = V$, we obtain the exact sequence $0 \to TH(U) \xrightarrow{\theta_U} U \to 0$ by Lemma 34.7. Consequently, $U \in \mathcal{R}_A$; therefore, θ_U is an isomorphism.

The embedding $\varkappa : U \to G$ induces the commutative diagram

$$\begin{tikzcd}
 & TH(U) \arrow[r, "TH(\varkappa)"] \arrow[d, "\theta_U"] & TH(G) \arrow[d, "\theta_G"] \\
0 \arrow[r] & U \arrow[r, "\varkappa"] & G
\end{tikzcd}.$$

If

$$\theta_G\Big(\sum_{i=1}^n \varphi_i \otimes a_i\Big) = 0 \qquad \text{for some} \quad a_i \in A,$$

then from

$$\theta_U\Big(\sum_{i=1}^n \varphi_i \otimes a_i\Big) = 0,$$

we obtain

$$\sum_{i=1}^n \varphi_i \otimes a_i = 0.$$

Therefore, θ_G is a monomorphism. In addition, $\operatorname{im} \theta_G = S_A(G) = G$ implies $G \in \mathcal{R}_A$.

(2) $\Longrightarrow$ (1). By Theorem 34.8, A is an endoflat group. Assume that I is a right ideal of the ring R such that $IA = A$. There exists an epimorphism $\pi : \sum_{\mathfrak{N}}^{\oplus} R \to I$ for some cardinal number $\mathfrak{N}$. If $\Lambda_I : T(I) \to A$ is the canonical homomorphism, then we obtain the exact sequence

$$T\Big(\sum_{\mathfrak{N}}^{\oplus} R\Big) \xrightarrow{\Lambda_I T(\pi)} A \to 0.$$

This sequence is A-balanced, since

$$T\Big(\sum_{\mathfrak{N}}^{\oplus} R\Big) \cong \sum_{\mathfrak{N}}^{\oplus} A \in \mathcal{R}.$$

Consequently, the diagram

$$\begin{tikzcd}
F \arrow[r, "\pi"] \arrow[d, "\phi_F"] & I \arrow[r] \arrow[d, "e_i"] & 0 \\
HT(F) \arrow[r, "H(\Lambda_I)T(\pi)"] & H(A) \arrow[r] & 0
\end{tikzcd}$$

is commutative, where F denotes $\sum_{\mathfrak{N}}^{\oplus} R$ and e_I is the evaluation map that maps an element $\alpha \in I$ into the homomorphism $A \to A$ mapping an element a into $\alpha(a)$. We obtain that e_I is an epimorphism.

Since the evaluation map is always injective, $e_I : I \to H(A)$ is an isomorphism. Therefore, the ideal I is principal; therefore, there exists $\sigma \in R$ with $\sigma R = I$. Therefore, $\sigma A = IA = A$. Consequently, the sequence $A \xrightarrow{\sigma} A \to 0$ is A-balanced exact. Since there exists a $\tau : A \to A$ such that $\sigma\tau = 1_A$, we obtain $I = R$. □

We especially mention Corollary 34.12 following from Theorem 34.11.

Corollary 34.12. *Let A be a self-small faithful endoflat group. Then projective objects of the category $\mathcal{R}_A$ coincide with A-projective groups.*

Proof. Every projective object E of the category $\mathcal{R}_A$ is an A-generated group. By Lemma 34.4, there exists an epimorphism $\pi : G \to E$ for some A-free group G. Since E is projective, this epimorphism is split and $G = \ker\pi \oplus E'$, where $E' \cong E$, whence E is an A-projective group. By Theorem 34.11, the group A is projective in $\mathcal{R}_A$. Consequently, all direct summands of direct sums of copies of the group A are projective in $\mathcal{R}_A$. The above direct summands coincide with A-projective groups. □

Therefore, if A is a self-small faithful endoflat group, then the functor $H = \mathrm{Hom}(A, -)$ is exact, i.e., it is left and right exact (see Sec. 32). We also note that for any endoflat group A, the functor $T = (-) \otimes_R A$ is also exact.

The analysis of Theorems 32.1 and 32.2 on the equivalence of categories and the proved results of this section lead to the following question. How are the category of subgroups of A-projective groups and the category of submodules of projective right R-modules related? The same for categories related to the local projectivity. Our subsequent studies answer these questions.

Let A be a group, and let $R = E(A)$. We denote by $\mathcal{P}_R$ ($\mathcal{P}_R^f$) the category of all submodules of projective (finitely generated projective) right R-modules. Further, let $\mathcal{P}_A$ ($\mathcal{P}_A^f$) be the category of all A-generated subgroups of A-projective (finitely A-projective) groups.

Theorem 34.13 (Krylov [255, 258], Faticoni [146])**.** *For an endoflat group A, the following conditions are equivalent*:

(1) *A is a faithful group*;

(2) *the functors H and T define an equivalence of the categories $\mathcal{P}_A^f$ and $\mathcal{P}_R^f$.*

In addition, if A is a self-small group, then (1) *is equivalent to*

(3) *the functors H and T define an equivalence of the categories $\mathcal{P}_A$ and $\mathcal{P}_R$.*

Proof. (1) $\Longrightarrow$ (2). Let $G \in \mathcal{P}_A^f$ and $M \in \mathcal{P}_R^f$. We prove that $H(G) \in \mathcal{P}_R^f$, $T(M) \in \mathcal{P}_A^f$, and the transformations θ and ϕ are equivalences in the sense that natural mappings θ_G and ϕ_M are isomorphisms for all G and M.

It follows from Theorem 32.1 that H maps from finitely A-projective groups into finitely generated projective right R-modules and T acts in the opposite direction. Since the functor T is left exact, we obtain $H(G) \in \mathcal{P}_A^f$. Further, since ${}_RA$ is a flat module, the functor T is exact. Therefore, $T(M)$ is a subgroup of a finitely A-projective group. The relation $T(M) = S_A(T(M))$ is always true. Therefore, $T(M)$ is an A-generated group and $T(M) \in \mathcal{P}_A^f$.

We consider the transformations θ and ϕ. Since $S_A(G) = G$, we have that θ_G is surjective, since $\operatorname{im}\theta_G = S_A(G)$. Let $i : G \to E$ be an embedding into some finitely A-projective group E. Since the module ${}_RA$ is flat, the mapping $TH(i)$ is a monomorphism. Since the transformation θ is natural, $i\theta_G = \theta_E TH(i)$. Here, θ_E is an isomorphism by Theorem 32.1; therefore, θ_G is a monomorphism. Therefore, θ_G is an isomorphism.

We prove that ϕ_M is an isomorphism. It is possible to give a short proof. We have $\phi_M : M \to HT(M)$ and consider $T(\phi_M) : T(M) \to THT(M)$ and $\theta_{T(M)} : THT(M) \to T(M)$. By the above, $\theta_{T(M)}$ is an isomorphism. The arguments before Lemma 34.7 imply that $\theta_{T(M)}T(\phi_M) = 1_{T(M)}$, whence $T(\phi_M)$ is an isomorphism. By Proposition 34.2, ϕ_M is an isomorphism.

The second proof is more complicated. Let $j : M \to P$ be an embedding, where P is a finitely generated projective right R-module. The transformation ϕ is natural; therefore, $HT(j)\phi_M = \phi_P j$. Here, ϕ_P is an isomorphism by Theorem 32.1. Consequently, ϕ_M is a monomorphism.

Let $\alpha \in HT(M)$. We set $\beta = HT(j)(\alpha)$ and $y = \phi_P^{-1}(\beta)$. We have $\beta a = y \otimes a$ for all $a \in A$. However, $\beta = (j \otimes 1)\alpha$. Therefore, $y \otimes a \in \mathrm{im}(j \otimes 1)$ for every $a \in A$. We prove that $y \in j(M)$ in this case. For convenience, we identify M with $j(M)$.

We take the submodule $\langle M, y\rangle$ generated in P by the module M and the element y. We have the exact sequence

$$M \otimes_R A \xrightarrow{j \otimes 1} \langle M, y\rangle \otimes_R A \longrightarrow \langle M, y\rangle / M \otimes_R A \longrightarrow 0.$$

By the found condition for the element y, the first mapping of the sequence is an epimorphism. By Proposition 34.2, the embedding $M \to \langle M, y\rangle$ is an epimorphism and $y \in M$. We have $HT(j)\phi_M(y) = \phi_P j(y) = \beta$ and $HT(j)(\alpha) = \beta$, where $HT(j)$ is a monomorphism, since ${}_RA$ is a flat module. Therefore, $\alpha = \phi_M(y)$ and ϕ_M is an isomorphism. We have proved the implication (1) $\Longrightarrow$ (2).

Under the assumption of the self-smallness of the group A, the proof of the implication (1) $\Longrightarrow$ (3) is similar to the proof of the implication (1) $\Longrightarrow$ (2) with the only difference that we use assertion (2) of Theorem 32.1 instead of assertion (1) of this theorem.

(2) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (1). Assume that $IA = A$ for some right ideal I of the ring R. Let $i : I \to R$ be an embedding. Since the module ${}_RA$ is flat, the canonical mapping $I \otimes_R A \to IA$ is an isomorphism. Therefore,

$$T(i) : I \otimes_R A \to R \otimes_R A$$

is an isomorphism. Since ϕ is natural, $HT(i)\phi_I = \phi_R i$. Therefore, all mappings in the relation $HT(i)\phi_I = \phi_R i$, except for i, are isomorphisms. Therefore, i is also an isomorphism, i.e., $I = R$ and A is a faithful group. □

We can similarly sharpen Theorem 32.2. Let A be a group, and let $R = E(A)$. We denote by $\overline{\mathcal{P}}_R$ and $\overline{\mathcal{P}}_A$ the category of all submodules of locally projective right R-modules and the category of all A-generated subgroups of locally A-projective groups, respectively. It is clear that $\mathcal{P}_R \subseteq \overline{\mathcal{P}}_R$ and $\mathcal{P}_A \subseteq \overline{\mathcal{P}}_A$.

Theorem 34.14. *An endoflat group A with endomorphism ring R discrete in the finite topology is faithful if and only if the functors H and T define a category equivalence between $\overline{\mathcal{P}}_A$ and $\overline{\mathcal{P}}_R$.*

Proof. Assume that A is a faithful endoflat group and has a discrete endomorphism ring. First, we prove that the functors H and T define an equivalence between the category of locally A-projective groups and the category of locally projective right R-modules (this means that the category $\mathcal{LP}$ from Theorem 32.2 is the category of all locally projective right R-modules in this situation). By Theorem 32.2, the transformation θ is an equivalence (i.e., θ_G is an isomorphism for every locally A-projective group G). Therefore, if M is a locally projective right R-module, then $\theta_{T(M)} : THT(M) \to T(M)$ is an isomorphism (the proof of Theorem 32.2 implies that $T(M)$ is a locally A-projective group). Further, $\theta_{T(M)}T(\phi_M) = 1_{T(M)}$ and as in the proof of Theorem 34.13, ϕ_M is an isomorphism.

Replacing the reference to Theorem 32.1 by the reference to the property just proved in the proof of Theorem 34.13, we obtain the proof of our theorem. □

Let $\mathcal{E}$ and $\mathcal{D}$ be two categories, and let $F : \mathcal{E} \to \mathcal{D}$ be a functor. The complete subcategory of the category $\mathcal{D}$ consisting of objects of the form $F(V)$ for all $V \in \mathcal{E}$ is called the *image* of the functor F.

The proof of the implications (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3) of Theorem 34.13 shows that the proof of the equivalence $\theta : TH \to 1$ does not use (1). The same can be said about the corresponding part of Theorem 34.14. Therefore, we have the following corollary.

Corollary 34.15. (1) *If A is an endoflat group, then the functors H and T define an equivalence between $\mathcal{P}_A^f$ and the image of the functor H (i.e., $\theta : TH \to 1$ is an equivalence). In addition, if the group A is self-small, then the same functors define an equivalence between $\mathcal{P}_A$ and the image of the functor H.*

(2) *If A is an endoflat group with the endomorphism ring discrete in the finite topology, then the category $\mathcal{P}_A$ is equivalent to the image of the functor H.*

In each of the three cases, we can say that the functor H effects a complete embedding from the categories $\mathcal{P}_A^f$, $\mathcal{P}_A$, and $\overline{\mathcal{P}}_A$ into the categories $\mathcal{P}_R^f$, $\mathcal{P}_R$, and $\overline{\mathcal{P}}_R$, respectively.

In addition, we have the following specification of Corollary 34.12.

Corollary 34.16. (1) *Let A be a self-small faithful endoflat group. Then A-projective groups coincide with projective objects of the category $\mathcal{P}_A$. In particular, the functor $H : \mathcal{P}_A \to \mathcal{P}_R$ maps from epimorphisms into epimorphisms. Every epimorphism $G \to E$, where $G \in \mathcal{P}_A$ and E is an A-projective group, is split.*

(2) *If A is an exact endoflat group whose endomorphism ring is discrete in the finite topology, then there is an analog of* (1), *where $\mathcal{P}_A$ and $\mathcal{P}_R$ are replaced by $\overline{\mathcal{P}}_A$ and $\overline{\mathcal{P}}_R$, respectively.*

Proof. The assertion on projective objects is contained in Corollary 34.12 and its proof. This property of the functor H is equivalent to the projectivity of the group A in $\mathcal{P}_A$. Finally, assume that we have an epimorphism $\pi : G \to E$. Since the group E is projective, there exists $\varkappa : E \to G$ with $\pi\varkappa = 1_E$. Therefore, $G = \ker \pi \oplus E'$ for some group $E' \cong E$. □

The application of Theorem 34.13 on the equivalence of the categories $\mathcal{P}_A^f$ and $\mathcal{P}_R^f$ to the group A and its endomorphism ring R leads to an assertion on the isomorphism of some lattices. For an arbitrary group A, the set of its A-generated subgroups is not necessarily a lattice. If A is an endoflat group and G is an A-generated subgroup of A, then G is equal to IA for some right ideal I of the ring R (see the proof below). In addition, $(I+J)A = IA+JA$ for any two right ideals I and J of the ring R. By Theorem 34.6, we have $(I \cap J)A = IA \cap JA$. Thus, in the case of the endoflat group A, we deal with the lattice of all its A-generated subgroups.

Corollary 34.17. *Assume that A is a faithful endoflat group. Then the correspondence $I \to IA$ is an isomorphism from the lattice of right ideals* (*ideals*) *of the ring R onto the lattice of A-generated* (*fully characteristic A-generated*) *subgroups of the group A. The mapping $G \to \{\alpha \in R \mid \alpha A \subseteq G\} = \mathrm{Hom}(A, G)$ is the inverse mapping.*

Proof. We prove that the following mappings Φ and Ψ from the lattices indicated in the corollary are mutually inverse mappings. Here $\Phi(I) = IA$ for a right ideal I of the ring R and $\Psi(G) = \{\alpha \in R \mid \alpha A \subseteq G\}$ for an A-generated subgroup G of the group A. We prove that $\Psi\Phi(I) = I$. It is clear that $\Psi\Phi(I) \supseteq I$. Let $\alpha \in \Psi\Phi(I)$ (i.e., $\alpha A \subseteq IA$). By Theorem 34.13, $\phi_I : I \to HT(I)$ is an isomorphism. Since the module ${}_RA$ is flat, the group $T(I)$ is canonically isomorphic to IA. Consequently, we have an isomorphism $I \to H(IA)$ acting by $\gamma \to (a \to \gamma a)$ for $\gamma \in I$ and $a \in A$ (this is the evaluation map e_I; see Theorem 34.11). In fact, $I = H(IA)$. We have $\alpha \in H(IA)$. Therefore, $\alpha \in I$ and we have proved the relation $\Psi\Phi(I) = I$. If $G \subseteq A$ and $S_A(G) = G$, then it is clear that $\Phi\Psi(G) = G$. The two proved relations and the two relations given before the corollary imply that Φ and Ψ are mutually inverse lattice isomorphisms. The partial case “in brackets” easily follows from the proved case. □

Theorem 34.13 and Corollary 34.17 are sufficient for the intuitive solution of some difficult problems. It relates the lattice of right ideals of the ring $E(A)$ to the group structure of the group A.

Let G be an A-generated group (i.e., $G = S_A(G)$). By Lemma 34.4, there exists a cardinal number $\mathfrak{N}$ and an epimorphism $\sum_{\mathfrak{N}}^{\oplus} A \to G$. The least such $\mathfrak{N}$ is called the A-rank of the group G. A group G is said to be *finitely A-generated* if there exists a finite set of homomorphisms $\varphi_i : A \to G$ $(i = 1, \dots, n)$ such that $G = \sum_{i=1}^{n} \varphi_i A$. It is clear that the group G is finitely A-generated if and only if G is an A-generated group of finite A-rank.

Corollary 34.18. *Let A be a faithful endoflat group. Then the ring R is right Noetherian if and only if every A-generated subgroup of the group A has a finite A-rank.*

Proof. Assume that the ring R is right Noetherian and G is an A-generated subgroup of the group A. By Corollary 34.17, $G = IA$ for some right ideal I of the ring R. By assumption, I is a finitely generated ideal. Therefore, there exists an epimorphism of right R-modules $\mu : R^n \to I$ for some positive integer n. Therefore, $T(\mu) : T(R^n) \to T(I)$ is an epimorphism. However, $T(R^n) \cong A^n$ and $T(I) \cong IA$, since the R-module A is flat. Therefore, there exists an epimorphism $A^n \to G$ and G has a finite A-rank.

Conversely, let all A-generated subgroups of the group G have a finite A-rank, and let I be a right ideal of the ring R. By Corollary 34.17, $I = H(IA)$. The A-generated subgroup IA has a finite A-rank; therefore, there exists an epimorphism $\lambda : A^n \to IA$ for some n. By Corollary 34.16, $H(\lambda) : H(A^n) \to H(IA)$ is an epimorphism. Therefore, there exists an epimorphism $R^n \to I$, and I is a finitely generated right ideal. $\square$

Corollary 34.19 (Faticoni [146])**.** *Assume that A is an endoflat torsion-free group of finite rank. The ring R is local if and only if A is a faithful group and there exists a unique maximal A-generated subgroup B of the group A such that $B \neq A$.*

Proof. Assume that the ring R is local. By Proposition 33.6, A is a faithful group. The local ring R has a unique maximal right ideal L. By Corollary 34.17, LA is the required subgroup B. The converse implication follows from the property that the ring R has a unique maximal right ideal by Corollary 34.17. $\square$

Further, we pass to right invariant endomorphism rings. We recall that a ring S is said to be *right invariant* if for any $x, y \in S$, there exists a $z \in S$ such that $yx = xz$. A ring S is right invariant if and only if every right ideal of the ring S is a left ideal. Item (2) of the following proposition generalizes the result of Arnold and Lady on the exactness of a torsion-free group of finite rank with commutative endomorphism ring (Sec. 33, Exercise 6). For (1), see Exercise 9 of Sec. 33.

Proposition 34.20 (Faticoni [146])**.** (1) *Let A be a reduced torsion-free group of finite rank, and let I be a maximal right ideal of the ring R. Then $pR \subseteq I$ for some prime integer p.*

(2) *The torsion-free group A of finite rank with right invariant endomorphism ring is a faithful group.*

Proof. (1) Let $N(R)$ be the nil-radical of the ring R. By Theorem 4.4, $R/N(R)$ is a quasi-summand of the ring R. In addition, R is a reduced ring, since the group A is reduced. Therefore, $R/N(R)$ is also a reduced ring. Since I is a maximal ideal, $N(R) \subseteq I$; therefore, $R/I \cong (R/N(R))/(I/N(R))$. Therefore, without loss of generality, we can assume that R is a reduced semiprime torsion-free ring of finite rank.

Now, since R/I is an irreducible right R-module, its endomorphism ring $\operatorname{End}_R R/I$ is a division ring and R/I is a vector space over this division ring. Therefore, R/I is a finite-dimensional vector space over the field of rational numbers $\mathbf{Q}$ or the residue field F_p modulo some p. Assume that the first possibility is realized. Since R is a semiprime ring, $R \otimes \mathbf{Q}$ is an Artinian semisimple ring (Proposition 4.1). Therefore, $I \otimes \mathbf{Q}$ is a direct summand in $R \otimes \mathbf{Q}$, and we have $1 = e_1 + e_2$, where $\{e_1, e_2\}$ is a system of orthogonal idempotents and $e_1(R \otimes \mathbf{Q}) = I \otimes \mathbf{Q}$. Since I is a pure ideal, $(I \otimes \mathbf{Q}) \cap R = I$. Therefore, it is clear that I is a quasi-summand of the ring R (if $ne_1, ne_2 \in R$ for some positive integer n, then $R \doteq (ne_1)R \oplus (ne_2)R$ and $(ne_1)R \doteq I$). In addition, the complemented quasi-summand should be a divisible group; this is a contradiction, since the ring R is reduced (see also Theorem 4.5). Therefore, it remains to consider the case where R/I is an F_p-space, and, therefore, $pR \subseteq I$.

(2) Since direct summands of groups with right invariant endomorphism ring are fully characteristic, either A is a reduced group or $A \cong \mathbf{Q}$. It is sufficient to consider only the case where the group A is

reduced. Assume that $IA = A$ for some maximal right ideal I of the ring R. By (1), $pR \subseteq I$ for some prime integer p. Since R/pR is a finite ring, its Jacobson radical J/pR is nilpotent. Further, $J \subseteq I$ and $I/J = \bar{e}(R/J)$ for some idempotent $1 \neq \bar{e} \in R/J$, since R/J is a semisimple Artinian ring. The ideal J/pR is nilpotent; therefore, $\bar{e}$ can be lifted to some idempotent $1 \neq e \in R/pR$, i.e., $\bar{e} = e + J/pR$ (Proposition 32.7). In addition, the right ideal $e(R/pR)$ is an ideal in R/pR, since the ring R/pR is right invariant. We note that

$$I/pR = e(R/pR) + J/pR \neq R/pR.$$

By the choice of the idempotent e, we have

$$IA/pA = (I/pR)(A/pA) = [e(R/pR) + J/pR](A/pA) = e(A/pA) + (J/pR)(A/pA).$$

Since A/pR is a finite R/pR-module and I/pR and $e(R/pR)$ are ideals of the ring R/pR, we can apply the Nakayama lemma to the relation $IA/pR = e(A/pA) + (J/pR)(A/pA)$ (it is formulated at the beginning Sec. 32). As a result, we obtain $IA/pA = e(A/pA)$. Then the nonzero element $1 - e$ annihilates IA/pA. Since A/pA is a faithful left R/pR-module, $IA/pA \neq A/pA$. Therefore, $IA \neq A$; this is a contradiction. Therefore, A is a faithful group. □

Corollary 34.21 (Faticoni [146]). (1) *Let A be a faithful endoflat group. Then R is a right invariant ring if and only if every A-generated subgroup of the group A is fully characteristic in A.*

(2) *Assume that A is an endoflat torsion-free group of finite rank. Then R is a right invariant ring if and only if A is a faithful group and every A-generated subgroup of the group A is fully characteristic in A.*

Proof. First, we note that all A-generated subgroups of the group A are fully characteristic if and only if all endomorphic images of the group A are fully characteristic.

(1) By Corollary 34.17, any right ideal I of the ring R is equal to $\{\alpha \in R \mid \alpha A \subseteq B\}$ for some A-generated subgroup B of the group A, and any A-generated subgroup B of the group A is equal to IA for some right ideal I of the ring R. It remains to use the property that a ring is right invariant if and only if every its right ideal is an ideal.

(2) By Proposition 34.20, the torsion-free group A of finite rank with right invariant ring R is faithful. All remaining assertions follow from (1). □

The end of the section is devoted to some specification of Theorems 34.13 and 34.14 in the case of torsion-free groups. We know that all self-small torsion groups are finite (see Sec. 31). Theorems 34.13 and 34.14 admit a considerable generalization for torsion-free groups at the expense of the decrease of the flatness condition of the R-module A. In this part of the text, we denote the torsion subgroup of a group G by $t(G)$.

Using a given group A and its endomorphism rings R, we denote by U the composition of the functor T with the functor of the transfer to factor groups with respect to the torsion subgroup. The role of this functor is explained by Lemma 13.1. Therefore, if M is a right R-module, then $U(M) = M \otimes_R A/t(M \otimes_R A)$, and for the homomorphism $\lambda : M \to L$ of right R-modules, $U(\lambda) : U(M) \to U(L)$ is the homomorphism induced by $\lambda \otimes 1$ on factor groups with respect to torsion subgroups. Let G be a torsion-free group, and let $\theta_G : TH(G) \to G$ be the natural homomorphism. Then $t(TH(G)) \subseteq \ker \theta_G$ and θ_G induces the homomorphism $UH(G) \to G$. It is easy to verify that this homomorphism defines a natural functor transformation $UH \to 1$. On the other hand, if M is a right R-module and $\phi_M : M \to HT(M)$, then denoting by π the canonical homomorphism $T(M) \to T(M)/t(T(M))$, we obtain a homomorphism $H(\pi)\phi_M : M \to HU(M)$ defining a natural functor transformation $1 \to HU$.

Formally, Theorem 34.22 below does not follow from Theorems 34.13 and 34.14, but its proof repeats their proofs. We need only take the functor U instead of the functor T and use Lemma 13.1 and the above transformations $UH \to 1$ and $1 \to HU$.

Theorem 34.22. (1) *Let A be a torsion-free group such that $A \otimes \mathbf{Q}$ is a flat $R \otimes \mathbf{Q}$-module. The functors $H : \mathcal{P}_A^f \to \mathcal{P}_R^f$ and $U : \mathcal{P}_R^f \to \mathcal{P}_A^f$ define a category equivalence if and only if A is a faithful group. In addition, if A is a self-small group, then a similar property holds for the categories $\mathcal{P}_A$ and $\mathcal{P}_R$.*

(2) *Let A be a torsion-free group such that its endomorphism ring R is discrete in the finite topology and $A = \otimes \mathbf{Q}$ is a flat $R \otimes \mathbf{Q}$-module. The functors $H : \overline{\mathcal{P}}_A \to \overline{\mathcal{P}}_R$ and $U : \overline{\mathcal{P}}_R \to \overline{\mathcal{P}}_A$ define a category equivalence if and only if A is a faithful group.*

We have the following analog of Corollary 34.17.

Corollary 34.23. *Assume that A is a faithful torsion-free group and $A \otimes \mathbf{Q}$ is a flat $R \otimes \mathbf{Q}$-module. The mappings*

$$\begin{cases} I \to IA, \\ G \to \operatorname{Hom}(A, G) \end{cases}$$

are mutually inverse isomorphisms from the ordered set of right ideals (ideals) of the ring R onto the ordered set of A-generated (fully characteristic A-generated) subgroups of the group A.

The proof is similar to the proof of Corollary 34.17; the only difference is that the natural mapping ϕ_I is replaced by the natural mapping $\bar{\phi}_I : I \to HU(I)$ defined before Theorem 34.22. By this theorem, $\bar{\phi}_I$ is an isomorphism. Since the $R \otimes \mathbf{Q}$-module $A \otimes \mathbf{Q}$ is flat, the group $U(I)$ is canonically isomorphic to IA by Lemma 13.1. Consequently, we have an isomorphism $I \to H(IA)$ such that $\gamma \to (a \to \gamma a)$ for all $\gamma \in I$ and $a \in A$. Therefore, $I = H(IA)$. Further, we can repeat the proof of Corollary 34.17.

For a torsion-free group A of finite rank with semiprime endomorphism ring R, we collect some proved properties (see Theorems 33.2 and 34.22 and Corollary 34.23). For such a group A, the $R \otimes \mathbf{Q}$-module $A \otimes \mathbf{Q}$ is projective, since $R \otimes \mathbf{Q}$ is an Artinian semisimple ring.

Corollary 34.24. *For a torsion-free group A with semiprime ring R, the following conditions are equivalent:*

1. *A is a faithful group;*
2. *every exact sequence of groups $0 \to K \to B \to G \to 0$, where G is a finitely A-projective group and $B = K + S_A(B)$, is split;*
3. *the functors H and U define an equivalence of the categories $\overline{\mathcal{P}}_A$ and $\overline{\mathcal{P}}_R$;*
4. *the mapping $I \to IA$ is an isomorphism from the ordered set of right ideals of the ring R onto the ordered set of A-generated subgroups of the group A.*

For torsion-free groups, we can generalize Corollaries 34.18, 34.19, and 34.21 by replacing the flatness of the R-module A by the flatness of the $R \otimes \mathbf{Q}$-module $A \otimes \mathbf{Q}$. We do not repeat these consequences in a general form and present only the following result.

Corollary 34.25. *Let A be a torsion-free group of finite rank with semiprime ring R.*

(1) *The ring R is local if and only if A is a faithful group and there exists a unique maximal A-generated subgroup B of the group A with $B \neq A$.*

(2) *The ring R is right invariant if and only if A is a faithful group and every endomorphic image of the group A is fully characteristic in A.*

We present a summary of results on faithful and endoflat torsion-free groups of finite rank. A torsion-free group A of finite rank is faithful provided one of the following conditions holds.

1. The factor ring $E(A)/pE(A)$ is semisimple for every prime integer p.
2. The factor ring $(R/pR)/J(R/pR)$ is simple for every p, where $R = E(A)$ (see Sec. 39 and Exercise 1 for (1) and (2)).
3. The ring $E(A)$ is right hereditary (this will be proved later).
4. $E(A)$ is a local or right invariant ring.

(5) A is a strongly, indecomposable strongly irreducible group. Indeed, $A \doteq E(A)$ and $E(A)$ is an E-ring by Corollary 8.10; in particular, $E(A)$ is a commutative ring.
(6) A is a strongly indecomposable group of rank 2. Using the proof of Theorem 13.7, we can prove that the ring $E(A)$ is commutative (see Exercise 7 of Sec. 33).

The following groups are endoflat.

(1) A is a group of rank 2 (Corollary 13.8).
(2) The factor ring $E(A)/pE(A)$ is semisimple for every p.
(3) $E(A)$ is a right hereditary ring.

(2) and (3) will be proved later.

Exercise 1. A torsion-free group A of finite rank is faithful in each of the following two cases:
(a) (Arnold [30]) the factor ring $E(A)/pE(A)$ is semisimple for every p;
(b) (Albrecht [7]) the factor ring $(R/pR)/J(R/pR)$ is simple for every p.

Exercise 2. Let A be strongly indecomposable torsion-free group of finite rank, and let $A = \operatorname{Soc} A$. Prove that the group A is endoflat if and only if $(I \cap J)A = IA \cap JA$ for any two right ideals I and J of the ring $E(A)$.

Exercise 3 (Albrecht [13])**.** Prove that there exists a self-small group A such that there exists an A-closed class and A is not endoflat (see Theorem 34.8).

Exercise 4 (Albrecht [13])**.** There exist A-balanced closed classes that do not contain the group A.

Exercise 5 (Albrecht [13])**.** Theorem 34.11 is not true for groups A that are not self-small.

A ring S is called a *right uniserial ring* if for any $x, y \in S$, either $xS \subseteq yS$ or $yS \subseteq xS$. A ring S is right uniserial if and only if all right ideals of the ring S form a chain.

Exercise 6 (Faticoni [146])**.** (a) Assume that A is a faithful endoflat group. Then $E(A)$ is a right uniserial ring if and only if all A-generated subgroups of the group A form a chain.

(b) Let A be a torsion-free group of finite rank. Then $E(A)$ is a right uniserial ring if and only if all A-generated subgroups of the group A form a chain. In this case, $E(A)$ is a local right invariant principal right ideal domain, and A is a faithful endoflat group.

Exercise 7 (Faticoni [146])**.** (a) There exists a noncommutative right invariant torsion-free right uniserial domain of finite rank.

(b) There exists an exact endoflat torsion-free group A of finite rank such that its A-generated subgroups form a chain and $E(A)$ is a noncommutative ring.

Exercise 8. Let A be a faithful endoflat group. Prove that $E(A)$ is a principal right ideal ring if and only if the A-rank of every A-generated subgroup of the group A is equal to 1.

A ring S is said to have a finite right Goldie dimension if S contains no infinite direct sums of right ideals.

Exercise 9. Let A be a faithful endoflat group. Prove that the ring $E(A)$ has a finite right Goldie dimension if and only if all A-generated subgroups of the group A have no infinite direct decompositions.

35. Groups with Right Hereditary Endomorphism Rings

In this section, we obtain a generalization of one classical result on completely decomposable torsion-free groups. We mean the Baer–Kolettis theorem presented in the introduction of this part. Groups with right hereditary endomorphism rings are (under some additional restrictions) acceptable objects for simultaneous extension of the Baer lemma and the Baer–Kolettis theorem. Section 33 was devoted to generalizations of the Baer lemma on the splitting of exact sequences. As with the Baer lemma, the

Baer–Kolettis theorem can be represented in another form that is more convenient for generalizations. Let A be a torsion-free group of rank 1. Then every subgroup G of an A-free group with $G = S_A(G)$ is an A-free group.

Also, we will present applications to the problem of description of groups with endomorphism rings of special form. Although we consider arbitrary groups with right hereditary endomorphism rings, it is clear that our main interest is with torsion-free groups (see also Proposition 35.11(3)).

We do not change the basic notation of the preceding sections. In particular, the symbols $\mathcal{P}_A^f$, $\mathcal{P}_R^f$, $\mathcal{P}_A$, $\mathcal{P}_R$, H, T, θ, ϕ, θ_G, and ϕ_M are used as above. We note that a group is finitely A-generated if and only if the A-rank of the group is finite.

We present some necessary definitions and results.

A ring R is said to be *right hereditary* if each of its right ideals is a projective right R-module. A left hereditary ring is defined similarly. Every principal right ideal domain R is right hereditary, since every nonzero right ideal is a module that is isomorphic to R_R. We formulate several standard properties of right hereditary rings and modules over such rings. The proofs are contained in the work of Cartan and Eilenberg [75]. First, the matrix ring R_n over a right hereditary ring R is right hereditary for every positive integer n. We also have the following general result.

Theorem. *For a ring R, the following conditions are equivalent*:

(1) *R is right hereditary*;
(2) *every submodule of a projective right R-module is projective*;
(3) *every factor module of an injective right R-module is injective.*

Item (2) holds in the following stronger form. Every submodule of a projective right module over a right hereditary ring R is isomorphic to a direct sum of right ideals of the ring R.

The well-known Kaplansky theorem states that a projective module over any ring R is a direct sum of countably generated modules (Kaplansky [235]).

A right R-module M is said to have a finite Goldie dimension (or it is finite-dimensional) if M contains no infinite direct sum of nonzero modules (Faith [141, 4.19,7.17]). It is well known that for a finite-dimensional module M, there exists a positive integer n such that every family of independent submodules of the module M contains at most n submodules. The least such n is called the (Goldie) dimension of the module M. The right dimension of the ring R is its dimension as a right R-module (see Sec. 34, Exercise 9). It is clear that every right Noetherian ring has a finite right dimension. It is well known that a right hereditary, right finite-dimensional ring is right Noetherian (see, e.g., Corollary 5.20 in [191] or Proposition 4.34 in [423]).

If we consider only finitely generated ideals of the ring R, then we obtain the following generalization of hereditary rings. A ring R is said to be *right* (*left*) *semihereditary* if each of its finitely generated right (left) ideals is projective. A ring R is right semihereditary if and only if every finitely generated submodule of a projective right R-module is projective.

We continue the section by proving some less known results on endomorphism rings of projective modules over hereditary and right semihereditary rings.

If S is a ring and $a \in S$, then the right ideal $\{x \in S \mid ax = 0\}$ is denoted by $r(a)$ (it is called the right annihilator of the element a).

Lemma 35.1. *Let S be a ring, and let $a \in S$. The principal right ideal aS is projective if and only if $r(a) = eS$ with $e^2 = e \in S$.*

Proof. Assume that aS is a projective S-module. Let $\pi : S \to aS$ be an epimorphism, where $\pi(x) = ax$, $x \in S$. Since the module aS is projective, we have $S = \ker \pi \oplus K$ for some right ideal $K \cong aS$. Here, $\ker \pi = r(a)$. Therefore, $r(a) = eS$, where e is the component of the identity element of the ring S in $\ker \pi$ with respect to the decomposition $S = \ker \pi \oplus K$. Conversely, if $r(a) = eS$ with $e^2 = e \in S$, then

$S = eS \oplus (1-e)S$, and

$$aS \cong S/\ker \pi = S/eS \cong (1-e)S.$$

Therefore, aS is a projective right ideal. □

Theorem 35.2 (Lenzing [284], Colby and Rutter [91])**.** *A ring R is right hereditary if and only if all principal right ideals are projective in the endomorphism ring of every free right R-module.*

Proof. Assume that R is a right hereditary ring and F is a free right R-module. Let $S = \mathrm{End}_R F$. Let $\alpha \in S$. By Lemma 35.1, it is sufficient to prove that $r(\alpha) = eS$, where $e^2 = e \in S$. Every submodule of the module F is projective; in particular, $\mathrm{im}\,\alpha$ is a projective module. Therefore, $F = \ker \alpha \oplus E$ for some submodule $E \cong \mathrm{im}\,\alpha$. Let e be the projection $F \to \ker \alpha$ with kernel E. We prove that $r(\alpha) = eS$. Since $\alpha e = 0$, we have $e \in r(\alpha)$ and $eS \subseteq r(\alpha)$. Let $\gamma \in r(\alpha)$. Then $\alpha\gamma = 0$, whence $\gamma F \subseteq \ker \alpha$. It is clear that $\gamma = e\gamma$ and $\gamma \in eS$. Therefore $r(\alpha) = eS$.

For the proof of the converse assertion, we take some right ideal I of the ring R. There exist a free module F and an epimorphism $F \to I$. The module F contains a submodule K that is isomorphic to I. Therefore, there exists $\alpha \in S$ such that $\alpha F = K$, where $S = \mathrm{End}_R F$. By Lemma 35.1, we have $r(\alpha) = eS$, where $e^2 = e \in S$ (by assumption, αS is a projective ideal). For the module F, we obtain the decomposition $F = eF \oplus (1-e)F$. We prove that $\ker \alpha = eF$. We have $\alpha(eF) = (\alpha(eS))F = (\alpha r(\alpha))F = 0$, whence $eF \subseteq \ker \alpha$. Therefore, $\ker \alpha = eF \oplus V$, where $V = \ker \alpha \cap (1-e)F$. Assume that $V \neq 0$. Since F is free, there exists a nonzero $\gamma \in S$ such that $\gamma F \subseteq V$. Since $r(\alpha) = \{\lambda \in S \mid \lambda F \subseteq \ker \alpha\}$, we have $\gamma \in r(\alpha)$. Therefore, $\gamma \in eS$ and $\gamma = e\mu$ with $\mu \in S$. We have $\gamma F = (e\mu)F \subseteq eF \cap (1-e)F = 0$; this is a contradiction. Thus, $\ker \alpha = eF$. Now

$$I \cong K = \alpha F \cong F/\ker \alpha = F/eF \cong (1-e)F,$$

where $(1-e)F$ is a projective R-module, is a direct summand of the free module F. We have obtained that R is a right hereditary ring. □

The proof of the following result of Small is similar to the proof of the preceding theorem.

Theorem 35.3 (Small [410])**.** *A ring R is right semihereditary if and only if for every positive integer n, all principal right ideals of the matrix ring R_n are projective.*

Before formulating the main result on endomorphism rings of projective modules over a right hereditary ring, we prove one simple lemma.

Lemma 35.4. *If all principal right ideals of the ring R are projective and $e^2 = e \in R$, then all principal right ideals of the ring eRe are also projective.*

Proof. Let $x \in eRe$. Then $exeR$ is a projective right ideal of the ring R, $x = exe$, and $exeR = xR$. We have the following exact sequence of right R-modules:

$$eR \to xR \to 0, \qquad er \to xer, \qquad r \in R;$$

it is split. Therefore, $eR = fR \oplus f'R$ and $e = f + f'$, where f and f' are orthogonal idempotents, $fR \cong xR$. Therefore,

$$xeRe = xRe \cong fRe = (efe)(eRe).$$

Further, we obtain

$$eRe = fRe \oplus f'Re = (efe)(eRe) \oplus (ef'e)(eRe)$$

(consider that $f = efe$ and $f' = ef'e$). Therefore, the principal right ideal $xeRe$ is projective, since it is isomorphic to a direct summand of the ring eRe. □

Theorem 35.5 (Lenzing [284], Colby and Rutter [91])**.** *For a ring R, the following conditions are equivalent*:

(1) *R is right hereditary*;

(2) *if P is a finitely generated projective right R-module, then* $\operatorname{End}_R P$ *is right hereditary*;
(3) *if P is a projective right R-module, then* $\operatorname{End}_R P$ *is right semihereditary.*

Proof. (1) $\Longrightarrow$ (2). Let P be a projective finitely generated right R-module. There exists a right R-module E such that $P \oplus E = F$, where F is a finitely generated free R-module. Since $\operatorname{End}_R F \cong R_n$ for some n, we have that $\operatorname{End}_R F$ is a right hereditary ring. If $\pi : F \to P$ is the projection with kernel E, then $\operatorname{End}_R P \cong \pi(\operatorname{End}_R F)\pi$. To complete the proof of the implication, it is sufficient to prove that the ring eSe is right hereditary for every right hereditary ring S and each idempotent $e \in S$.

Let G be a free right eSe-module, and let $G \cong \sum_{i\in I}^{\oplus} X_i$, where $X_i = eSe$ for all $i \in I$. We consider the free right S-module $\sum_{i\in I}^{\oplus} S_i$ with $S_i = S$ for all $i \in I$. We take the following homomorphism of right S-modules:

$$\bar{e} : \sum_{i\in I}^{\oplus} S_i \to \sum_{i\in I}^{\oplus} S_i,$$

where $\bar{e} : 1_i \to e_i$ for every $i \in I$, 1_i is the identity element of the ring S_i, and $e_i = e$. We have

$$\operatorname{End}_{eSe} G \cong \operatorname{End}_{eSe}\Big(\sum_{i\in I}^{\oplus} X_i\Big) \cong \operatorname{End}_S\Big(\sum_{i\in I}^{\oplus} e_i S_i\Big) \cong \operatorname{End}_S \bar{e}\Big(\sum_{i\in I}^{\oplus} S_i\Big) \cong \bar{e}\Big(\operatorname{End}_S\Big(\sum_{i\in I}^{\oplus} S_i\Big)\Big)\bar{e}.$$

By Theorem 35.2, all principal right ideals of the ring $\operatorname{End}_S\Big(\sum_{i\in I}^{\oplus} S_i\Big)$ are projective. By Lemma 35.4, the same assertion holds for the ring $\operatorname{End}_{eSe} G$. It follows from Theorem 35.2 that eSe is a right hereditary ring.

(2) $\Longrightarrow$ (3). Since $\operatorname{End}_R R \cong R$, we have that R is a right hereditary ring. Let F be a free right R-module, and let $T = \operatorname{End}_R F$. We prove that T is a right semihereditary ring. By Theorem 35.3, it is sufficient to prove that for every positive integer n, all principal right ideals of the ring T_n are projective. Let $F' = \sum_{i=1}^{n\,\oplus} F_i$ with $F_i = F$ for $i = 1, \dots, n$. Then F' is a free R-module and all principal right ideals of the ring $T_n \cong \operatorname{End}_R F'$ are projective by Theorem 35.2.

If P is a projective right R-module, then we can prove that $\operatorname{End}_R P$ is a right semihereditary ring using the argument of the proof of the implication (1) $\Longrightarrow$ (2) (only we use Theorem 35.3 instead of Theorem 35.2).

The implication (3) $\Longrightarrow$ (1) is contained in Theorem 35.2. $\square$

Similar arguments imply the following theorem.

Theorem 35.6 (Lenzing [284], Colby and Rutter [91]). *If P is a finitely generated projective right module over a right semihereditary ring R, then* $\operatorname{End}_R P$ *is a right semihereditary ring.*

The proof of Theorem 35.5 also implies the following corollary.

Corollary 35.7. *Let R be a right* (*left*) *hereditary ring. Then for every idempotent $e \in R$, the ring eRe is right* (*left*) *hereditary.*

Using one consequence of the following theorem, we obtain that the number of nonzero summands in any direct decomposition of the group with hereditary endomorphism ring is finite.

Theorem 35.8 (Osofsky [343]). *Let $\{e_i \mid i \in I\}$ be an infinite set of orthogonal idempotents of some ring R. Assume that for every subset $A \subseteq I$, there exists an element $m_A \in R$ such that $m_A e_i = e_i$ for all $i \in A$ and $e_i m_A = 0$ for all $i \in I \setminus A$. Then for every right R-module M containing the module R_R, the factor module $M/\Big(\sum_{i\in I} e_i R + \ker \pi\Big)$ is not injective, where $\pi : R \to \prod_{i\in I} e_i R$ is a homomorphism such that $\pi(m) = (e_i m)$ for all $m \in R$.*

Proof. Let $I = \bigcup_{A\in\mathcal{A}} A$ be a partition of the set I such that $\mathcal{A}$ is an infinite set, and for all $A, B \in \mathcal{A}$, A is an infinite set and $A \cap B \neq \varnothing \iff A = B$. By the Zorn lemma, $\mathcal{A}$ can be enlarged to some set $\mathcal{B}$ of subsets of the set I maximal with respect to the following properties:

(i) A is an infinite set for all $A \in \mathcal{B}$;
(ii) for all $A, B \in \mathcal{B}$, the relation $A \neq B$ implies that $A \cap B$ is a finite set.

Let $S = \sum_{i\in I} e_i R + \ker\pi$. Then S coincides with the set of all elements of R annihilating almost all e_i to the left. Let $A \in \mathcal{B}$, and let $r \in R$. Assume that $m_A r \notin S$. Then there are infinitely many subscripts $i \in A$ with $e_i m_A r \neq 0$. For every set $\{A_j \mid 1 \le j \le n\} \subseteq \mathcal{B} \setminus \{A\}$, the intersection $A \cap \bigcup_{j=1}^{n} A_j$ is finite. For almost all subscripts $i \in A$, we have $e_i m_{A_j} = 0$ for all j $(1 \le j \le n)$. Then

$$m_A r \notin \sum_{j=1}^{n} m_{A_j} R + S;$$

therefore, the sum

$$\sum_{A\in\mathcal{B}} (m_A R + S)/S$$

is a direct sum in M/S.

We define the homomorphism of right R-modules

$$\Psi : \sum_{A\in\mathcal{B}}{}^{\oplus} (m_A R + S)/S \to M/S$$

as follows:

$$\Psi(m_A + S) = \begin{cases} m_A + S & \text{for } A \in \mathcal{A}, \\ 0 & \text{for } A \in \mathcal{B} \setminus \mathcal{A}. \end{cases}$$

Assume that Ψ is extended to some $\overline{\Psi} : R/S \to M/S$. Let $\overline{\Psi}(1 + S) = m + S$, where $m \in M$. For all $A \in \mathcal{A}$,

$$m m_A = m_A + \sum_{s=1}^{n} e_{i_s} r_s + k,$$

where $r_s \in R$ and $k \in \ker\pi$. Therefore, the set $A' = \{a \in A \mid e_a m e_a = e_a\} \supseteq A \setminus \{i_s \mid 1 \le s \le n\}$ is infinite.

Let C be the representative set for $\{A' \mid A \in \mathcal{A}\}$ (i.e., for every $A \in \mathcal{A}$ in A', we choose one element and denote the obtained set by C). Since $\mathcal{B}$ is maximal, $C \cap D$ is infinite for some $D \in \mathcal{B}$ and D cannot belong to $\mathcal{A}$. Then $m m_D \in S$. Therefore, $e_i m m_D = 0$ for almost all subscripts $i \in I$. Therefore, $0 = e_d m m_D$ for almost all elements $d \in C \cap D$. On the other hand, $e_d = e_d m e_d = e_d m m_D e_d$ for all $d \in C \cap D$; this is a contradiction. Therefore, the homomorphism Ψ cannot be extended and the factor module M/S is not injective. $\square$

Corollary 35.9 (Osofsky [343]). *Let a ring R contain an infinite ring product $\prod_{i\in I} R_i$, where R_i is a ring with identity element e_i. Then R is not right hereditary.*

Proof. We apply the preceding theorem. We have an infinite set of orthogonal idempotents $\{e_i \mid i \in I\}$ of the ring R. For every $A \subseteq I$, we denote by m_A the element $(m_i) \in \prod_{i\in I} R_i$, where $m_i = e_i$ for all $i \in A$ and $m_i = 0$ for all $i \in I \setminus A$. By the theorem, the right R-module $\overline{R}_R/S$ is not injective, where $\overline{R}_R$ is the injective hull of the module R_R. Since all factor modules of injective modules over hereditary rings are injective, R cannot be a right hereditary ring. $\square$

Using the proved theorems, we obtain some general results on groups with hereditary endomorphism rings.

Corollary 35.10. *A direct summand of the group with a right* (*left*) *hereditary endomorphism ring has a right* (*left*) *hereditary endomorphism ring.*

Proof. Let a group A have a right (left) hereditary endomorphism ring, and let $A = B \oplus C$. If $e : A \to B$ is the projection, then $E(B) \cong eE(A)e$, and the assertion follows from Corollary 35.7. $\square$

Proposition 35.11. (1) *The group with a right or left hereditary endomorphism ring cannot be an infinite direct sum of nonzero groups. In particular, for every group A, the endomorphism ring of the group $\sum_{\mathfrak{N}}^{\oplus} A$ cannot be right or left hereditary if $\mathfrak{N} \geq \aleph_0$.*

(2) *The endomorphism ring of a self-small group A is right hereditary if and only if the endomorphism ring of every A-projective group is right semihereditary.*

(3) *Let A be a reduced group whose endomorphism ring is right or left hereditary. If the p-component A_p of the group A is nonzero for some p, then A_p is an elementary p-group of finite rank and $A = A_p \oplus B_p$ for some group B_p.*

Proof. (1) Assume that $A = \sum_{i \in I}^{\oplus} A_i$, where $A_i \neq 0$ for all $i \in I$ and any subscript set I is infinite. In this case, the ring $E(A)$ contains the infinite product $\prod_{i \in I} E(A_i)$ as a subring; this contradicts Corollary 35.9 or its left analog.

(2) The category of A-projective groups is equivalent to the category of projective right $E(A)$-modules, and both of these categories are additive (Theorem 32.1). Under the category equivalences, endomorphism rings of the corresponding objects are isomorphic to each other. Therefore, the use of Theorem 35.5 completes the proof.

(3) If the group A has a direct summand that is isomorphic to $Z(p^k)$ for some prime integer p and positive integer k, then the ring $E(Z(p^k))$ is right (resp. left) hereditary by Corollary 35.10. However, $E(Z(p^k)) \cong Z_{p^k}$, where Z_{p^k} is the residue ring modulo p^k. This is possible only if $k = 1$. Therefore, the p-component A_p of the group A is an elementary p-group. Consequently, $A = A_p \oplus B_p$ for some group B_p. It follows from (1) and Corollary 35.10 that the rank of the group A_p is finite. $\square$

We know that the group with hereditary endomorphism ring has no infinite direct decompositions. However, there exist groups with hereditary endomorphism rings that are not direct sums of indecomposable groups. We consider one case of a positive solution of the arising problem on the decomposability of the group with hereditary endomorphism ring into a direct sum of indecomposable groups.

A torsion-free group A is said to be *quasi-homogeneous* if for each of its nonzero pure subgroups B and each positive integer n, we have $nA = A$ provided $nB = B$.

Proposition 35.12. *Let A be a reduced quasi-homogeneous torsion-free group whose endomorphism ring R is right* (*left*) *hereditary. Then A is a finite direct sum of indecomposable groups.*

Proof. We consider only the right-side case. Assume the contrary. For every positive integer k, we have the decompositions

$$A = A_1 \oplus B_1 = \cdots = A_1 \oplus \cdots \oplus A_k \oplus B_k,$$

where $B_k = A_{k+1} \oplus B_{k+1}$ and $A_k \neq 0$ for all $k \geq 1$. Let $\pi_k : A \to A_k$ be the projection ($k \geq 1$). By assumption, there exists a positive integer n such that $nA_k \neq A_k$ for all k. We set $\varepsilon = n1_A$. Let L be the right ideal of the ring R generated by all elements π_k and ε. Since L is projective, we have $R^{\mathfrak{M}} = L \oplus M$ for some cardinal number $\mathfrak{M}$ and a right R-module M. The element ε is contained in a finite sum of summands R. Let $\varepsilon \in R^m$, where m is a positive integer. Then $n\pi_k = \varepsilon\pi_k \in R^m$ and $\pi_k \in R^m$ for all k. Therefore, $L \subseteq R^m$ and $R^m = L \oplus (R^m \cap M)$, whence L is a finitely generated right ideal. Expressing some finite generator system of this ideal in terms of π_k and ε, we obtain that there exists a finite set of elements $\pi_{k_1}, \dots, \pi_{k_t}$ whose combination with ε forms a generator system of the right ideal L. We fix some positive integer $k \neq k_1, \dots, k_t$ and choose $\alpha_1, \dots, \alpha_t$, $\alpha \in R$ such that $\pi_k = \pi_{k_1}\alpha_1 + \dots + \pi_{k_t}\alpha_t + \varepsilon\alpha$. Therefore, $\pi_k = \pi_k\varepsilon\alpha = n\pi_k\alpha$; this is a contradiction, since the relation $nA_k \neq A_k$ implies that π_k is not divided by n. Therefore, A is a finite direct sum of indecomposable groups. $\square$

We begin with the basic material of the section with a result extending Theorems 33.1 and 33.2. The endomorphism ring $E(A)$ of the group A is denoted by R.

Proposition 35.13 (Faticoni [146])**.** *For a group A, the following conditions are equivalent:*

(1) *$IA \neq A$ for every finitely generated proper right ideal I of the ring R;*
(2) *every exact sequence of groups $0 \to K \to B \xrightarrow{\pi} G \to 0$, where G is a finitely A-projective group, $S_A(B)$ is a finitely A-generated group, and $B = K + S_A(B)$, is split.*

Proof. (1) $\Longrightarrow$ (2). First, we show the splitting of sequences $0 \to K \to B \xrightarrow{\pi} A \to 0$, where $S_A(B)$ is a finitely A-generated group and $B = K + S_A(B)$. There exists a finitely generated right R-submodule $M \subseteq H(B)$ such that $S_A(B) = MA$. Then πM is a finitely generated right ideal of the ring R such that $(\pi M)A = \pi(K + MA) = A$. By (1), $\pi M = R$, which implies the existence of $\nu : A \to B$ with $\pi\nu = 1_A$. Therefore, the sequence $0 \to K \to B \to A \to 0$ is split.

The proof of the splitting of any sequence from (2) is similar to the proof of the implication (1) $\Longrightarrow$ (3) of Theorem 33.2.

(2) $\Longrightarrow$ (1). Assume that (1) is not true. Let $I = \sum_{i=1}^{n} \pi_i R$ be a finitely generated proper right ideal of the ring R such that $IA = A$. We define the mapping $\pi : A^n \to A$ by

$$\pi(a_1 + \dots + a_n) = \sum_{i=1}^{n} \pi_i(a_i).$$

Then π is an epimorphism, since $\sum_{i=1}^{n} \pi_i A = IA = A$. For a given homomorphism $\lambda : A \to A^n$, we have $\lambda = \lambda_1 + \dots + \lambda_n$, where λ_i is the composition of λ with the projection of the group A^n on the ith copy of the group A in A^n. Then $\pi\lambda = \sum_{i=1}^{n} \pi_i\lambda_i \neq 1_A$, since $I \neq R$. Therefore, the epimorphism π cannot split; this contradicts (2). $\square$

Theorem 35.14 (Faticoni [146] and Krylov [262])**.** *For a group A, the following conditions are equivalent:*

(1) *A is an endoflat group and R is a right semihereditary ring;*
(2) *every exact sequence of groups $0 \to K \to B \to E \to 0$, where B is a finitely A-projective group and $E \in \mathcal{P}_A^f$, is split;*
(3) *every exact sequence of groups $0 \to K \to B \to E \to 0$, where B is a finitely A-projective group and $E \subseteq A$, is split;*
(4) (a) *$IA \neq A$ for every finitely generated right ideal I of the ring R with $I \neq R$;* (b) *every group of finite A-rank $G \in \mathcal{P}_A^f$ is finitely A-projective;*
(5) (a) *$IA \neq A$ for every finitely generated right ideal I of the ring R with $I \neq R$;* (b) *every group of finite A-rank $G \subseteq A$ is finitely A-projective.*

Proof. (1) $\Longrightarrow$ (2). Let $0 \to K \to B \xrightarrow{\pi} E \to 0$ be an exact sequence of groups with finitely A-projective group B and $E \in \mathcal{P}_A^f$. We embed the group E in a finitely A-projective group P and consider the induced homomorphism $H(\pi) : H(B) \to H(E)$. We set $V = \operatorname{im} H(\pi)$. Then $V \subseteq H(E) \subseteq H(P)$. The R-module V is finitely generated as a homomorphic image of the finitely generated module $H(B)$. Since the ring R is right semihereditary and $H(P)$ is a projective R-module, V is also a projective module. Consequently, the epimorphism $H(\pi) : H(B) \to V$ is split. Since θ is natural, we have $\theta_E TH(\pi) = \pi\theta_B$, where θ_B

and θ_E are isomorphisms by Corollary 34.15. Therefore, $TH(\pi)$ is an epimorphism. Since the module ${}_RA$ is flat, we obtain $\operatorname{im} TH(\pi) = T(V)$. Therefore, $TH(E) = T(V)$. Since $H(\pi)$ is split, $TH(\pi)$ is split. Therefore, π is split.

The implication (2) $\Longrightarrow$ (3) is obvious. We prove the implication (3) $\Longrightarrow$ (1). Let I be a finitely generated right ideal of the ring R. We prove that I is a projective R-module. We fix some epimorphism $\pi : F \to I$ with finitely generated free R-module F. We denote by σ the composition of epimorphisms

$$T(F) \xrightarrow{T(\pi)} T(I) \xrightarrow{\omega} IA,$$

where ω is the canonical mapping: $\omega(r \otimes a) = ra$ for all $r \in I$ and $a \in A$. By (3), the epimorphism $\sigma : T(F) \to IA$ is split. In particular, IA is a finitely A-projective group. Since ϕ is natural, we have $H(\sigma)\phi_F = H(\omega)HT(\pi)\phi_F = H(\omega)\phi_I\pi$, where ϕ_F is an isomorphism and $H(\sigma)$ is an epimorphism. Therefore, $H(\omega)\phi_I$ is an epimorphism. However, $H(\omega)\phi_I : I \to H(IA)$ is the evaluation map e_I acting as follows: $r \to (a \to ra)$, $r \in I$, $a \in A$. The evaluation map is always a monomorphism. Therefore, $H(\omega)\phi_I$ is an isomorphism and $I \cong H(IA)$. However, $H(IA)$ is a projective R-module, since IA is a finitely A-projective group. Therefore, the ideal I is projective.

We prove that the module ${}_RA$ is flat. Let I be a finitely generated right ideal of the ring R, and let $\varkappa : I \to R$ be an embedding. As was proved above, I is a projective R-module. It is sufficient to prove that $T(\varkappa)$ is a monomorphism. We set $E = \operatorname{im} T(\varkappa)$ and $K = \ker T(\varkappa)$. We have the exact sequence

$$0 \longrightarrow K \longrightarrow T(I) \xrightarrow{T(\varkappa)} E \longrightarrow 0.$$

The group $T(I)$ is finitely A-projective, and the group E can be isomorphically embedded in A. Therefore, our sequence is split (i.e., $T(I) = K \oplus G$, where $G \cong E$). By Theorem 32.1, the decomposition $T(I) = K \oplus G$ induces the decomposition of the ideal I into a direct sum of right ideals: $I = I_1 \oplus I_2$, and $T(I_1) = K$, $T(I_2) = G$. The relation $K = \ker T(\varkappa)$ implies that the canonical mapping $T(I_1) \to I_1A$ is the zero mapping. Therefore, $I_1A = 0$, $I_1 = 0$, and $\ker T(\varkappa) = T(I_1) = 0$. Therefore, $T(\varkappa)$ is a monomorphism and ${}_RA$ is a flat module.

The equivalence of the first three conditions is proved. Let us prove (2) $\Longrightarrow$ (4). For finitely generated right ideals of the ring R, the proof is similar to the proof of (2) $\Longrightarrow$ (1) of Proposition 35.13. Let the group $G \in \mathcal{P}_A^f$ have a finite A-rank n. There exists an epimorphism $A^n \to G$ that is split by (2). Therefore, G is a finitely A-projective group.

(4) $\Longrightarrow$ (2). We consider the exact sequence $0 \to K \to B \to E \to 0$ from (2). Since the group B has a finite A-rank, E is a group of finite A-rank. By (4), E is a finitely A-projective group. By Proposition 35.13, this sequence is split (i.e., (2) holds). Therefore, conditions (2) and (4) are equivalent. The proof of the equivalence of (3) and (5) is similar to the above proof. ☐

Corollary 35.15. *Let A be an endoflat group whose endomorphism ring is right semihereditary. If A is a finite direct sum of indecomposable groups, then A is a self-small group.*

Proof. We have $A = \sum_{s=1}^{k}{}^{\oplus} B_s$, where all the groups B_s are indecomposable. It is sufficient to prove that the image of every homomorphism $A \to \sum_{i=1}^{\infty}{}^{\oplus} A_i$, where $A_i = A$ $(i \geq 1)$, is contained in a finite sum of some groups A_i. It is clear that it is sufficient to prove this assertion for the images of all homomorphisms $B_s \to \sum_{i=1}^{\infty}{}^{\oplus} A_i$ for every $s = 1, \dots, k$. Assume that there is a homomorphism $\varphi : B_s \to \sum_{i=1}^{\infty}{}^{\oplus} A_i$ such that φB_s is not contained in a finite sum of some groups A_i. Let a be an element of B_s such that $\varphi a \neq 0$ and $\varphi a \in A_1 \oplus \cdots \oplus A_n$ for some n. By assumption, there exists an element $b \in B_s$ such that $\varphi b \notin A_1 \oplus \cdots \oplus A_n$. We can choose a positive integer m such that $m > n$ and φb has a nonzero component in A_m. Let $\pi : \sum_{i=1}^{\infty}{}^{\oplus} A_i \to A_m$ be the projection. We consider the homomorphism $\pi\varphi : B_s \to A_m$. Since

$\pi\varphi a = 0$, we have $\ker \pi\varphi \neq 0$. On the other hand, $\pi\varphi \neq 0$, since $\pi\varphi b \neq 0$. By Theorem 35.14, the epimorphism $\pi\varphi : B_s \to \operatorname{im} \pi\varphi$ is split. This means that we have a decomposition $B_s \cong \ker \pi\varphi \oplus \operatorname{im} \pi\varphi$, which is nontrivial. This is a contradiction, since the group B_s is indecomposable. $\square$

An element r of some ring S is said to be *right regular* or is called *a left nondivisor of zero* if the relation $rx = 0$ implies $x = 0$ for every $x \in S$. We consider several interesting properties of groups satisfying the conditions of Theorem 35.14.

Corollary 35.16. *Let A be an endoflat group with right semihereditary endomorphism ring. Then*

(1) $\ker \alpha$ *is a direct summand of A for every $\alpha \in E(A)$*;
(2) *if γ is a right regular element of the ring $E(A)$, then γ is a monomorphism*;
(3) *if the group A is indecomposable, then each of its nonzero endomorphisms is a monomorphism* (*in particular, $E(A)$ is a domain*).

Proof. (1) We can apply Theorem 35.14 to the epimorphism $\alpha : A \to \operatorname{im} \alpha$. For some subgroup $E \subseteq A$, we have $A = \ker \alpha \oplus E$.

(2) By (1), $A = \ker \gamma \oplus E$. Let $\pi : A \to \ker \gamma$ be the projection with kernel E. Then $\gamma\pi = 0$. Therefore, $\pi = 0$ and $\ker \gamma = 0$ (i.e., γ is a monomorphism).

(3) The proof follows from (1). $\square$

After the following simple lemma, we formulate the principal result of this section.

Lemma 35.17. *Let A be a self-small group. A group G is finitely A-projective if and only if it is A-projective and has a finite A-rank.*

Proof. Let G be an A-projective group of finite A-rank. We have $G \oplus E = A^{\mathfrak{M}}$ for some group E and a cardinal number $\mathfrak{M}$. We denote by $\varkappa : G \to A^{\mathfrak{M}}$ an embedding. For some positive integer n, there exists an epimorphism $\pi : A^n \to G$ (since the A-rank of the group G is finite). We consider the homomorphism $\varkappa\pi : A^n \to G$. Since A is a self-small group, the image $\operatorname{im} \varkappa\pi$, which is equal to G, is contained in A^m for some positive integer m. Therefore, $A^m = G \oplus (A^m \cap E)$. Therefore, G is a finitely A-projective group. The converse assertion always holds. $\square$

Theorem 35.18. *If A is a group and $R = E(A)$, then the following conditions are equivalent*:

(1) *A is a self-small endoflat group and the ring R is right hereditary*;
(2) *A is a self-small faithful group and any group $G \in \mathcal{P}_A$ is A-projective*;
(3) (a) *A is a self-small group*; (b) *every exact sequence $0 \to K \to B \to E \to 0$, where $K + S_A(B) = B$ and E is a finitely A-projective group, is split*; (c) *every A-generated subgroup $G \subseteq A$ is A-projective*;
(4) (a) *the group A has no infinite direct decompositions*; (b) *every exact sequence $0 \to K \to B \to E \to 0$, where $K + S_A(B) = B$ and E is an A-projective group, is split*; (c) *every group $G \in \mathcal{P}_A$ is A-projective*;
(5) (a) *A is a self-small group*; (b) *the endomorphism ring of every A-projective group is right semihereditary*; (c) *every A-projective group is endoflat*;
(6) *A is a self-small group and the class of all A-projective groups is A-balanced closed.*

Proof. (1) $\Longrightarrow$ (2). We prove that the group A is faithful. Assume that $IA = A$ for some right ideal I of the ring R. Since R is right hereditary, I is a projective right R-module and $\phi_I : I \to HT(I)$ is an isomorphism by Theorem 32.1. We can repeat the proof of the implication (2) $\Longrightarrow$ (1) of Theorem 34.13. Thus, $I = R$ and A is a faithful group. Therefore, the conditions and item (1) of Theorem 34.13 hold. Consequently, the categories $\mathcal{P}_A$ and $\mathcal{P}_R$ from item (3) of this theorem are equivalent to one another. However, all submodules of projective right R-modules are projective, since R is right hereditary. Therefore, every group $G \in \mathcal{P}_A$ is A-projective.

(2) $\Longrightarrow$ (3). Since the group A is faithful, we have item (b) of (3) by Theorem 33.2.

(3) $\Longrightarrow$ (1). Let $0 \to K \to B \to E \to 0$ be any exact sequence of groups, where B is a finitely A-projective group and $E \subseteq A$. The group E is a homomorphic image of a finitely A-projective group; therefore, E is an A-generated group of finite A-rank and E is an A-projective group by (3). Therefore, E is a finitely A-projective group by Lemma 35.17. Consequently, the considered sequence is split. We see that item (3) of Theorem 35.14 holds and A is an endoflat group by this theorem. Further, it follows from (3) and Theorems 33.2 and 34.13 that the categories $\mathcal{P}_A$ and $\mathcal{P}_R$ are equivalent. This equivalence and (3) imply that all submodules of the module R_R are projective; this means that R is a right hereditary ring. The equivalence of conditions (1)–(3) is proved.

(1) $\Longrightarrow$ (4). A self-small group has no infinite direct decompositions. By (2), A is a faithful group. Therefore, A is a self-small faithful endoflat group. By Corollary 33.3, item (b) of (4) holds. Item (c) follows from (2).

(4) $\Longrightarrow$ (3). It is sufficient to prove that the group A is self-small. Assume that A is not a self-small group. Then there exists a homomorphism $\varphi : A \to \sum_{i=1}^{\infty}{}^{\oplus} A_i$, where $A_i \cong A$ $(i \in \mathbf{N})$, such that φA is not contained in a finite sum of groups A_i. By (c), φA is an A-projective group. Consequently, the epimorphism $\varphi : A \to \varphi A$ is split by (b). Let us prove that the module φA has an infinite direct decomposition of nonzero summands. Let $B = \varphi A$, $A^1 = 0$, and $A^k = \sum_{i<k}{}^{\oplus} A_i$ for $k > 1$. We prove that $B \cap A^{k+1}$ is an A-generated group for any k. We have

$$\sum_{i=1}^{\infty}{}^{\oplus} A_i = A^{k+1} \oplus \sum_{i>k}{}^{\oplus} A_i.$$

Denote by π the projection

$$\sum_{i=1}^{\infty}{}^{\oplus} A_i \to \sum_{i>k}{}^{\oplus} A_i.$$

Since B and πB are A-generated groups, (b) and (c) imply that the epimorphism $\pi : B \to \pi B$ is split: $B \cong \ker \pi \oplus \pi B$. Here, $\ker \pi = B \cap A^{k+1}$; therefore, $B \cap A^{k+1}$ is an A-generated group.

Let $\omega : A^{k+1} \to A_k$ be the natural projection related to the decomposition $A^{k+1} = A^k \oplus A_k$. As was proved above, $B \cap A^{k+1}$ and $\omega(B \cap A^{k+1})$ are A-generated groups. Therefore, the application of (b) and (c) to the epimorphism

$$\omega : B \cap A^{k+1} \to \omega(B \cap A^{k+1})$$

implies

$$B \cap A^{k+1} = (B \cap A^k) \oplus B_k,$$

where $B \cap A^k = \ker \omega$ and the group B_k is isomorphic to a subgroup of the group A_k. Therefore,

$$B \cap A^{k+1} = B_1 \oplus \cdots \oplus B_k \qquad \text{and} \qquad B = \sum_{i=1}^{\infty}{}^{\oplus} B_i,$$

since every element $b \in B$ is contained in A^{k+1} for some k. If $B_i = 0$ for all i exceeding some k, then $B \subseteq A_1 \oplus \cdots \oplus A_k$; this is impossible by the assumption. Therefore, $B_i \neq 0$ for an infinite set of subscripts i. The obtained decomposition $\varphi A = B = \sum_{i=1}^{\infty}{}^{\oplus} B_i$ implies an infinite direct decomposition of the group A, since $A \cong \ker \varphi \oplus \varphi A$. This contradicts (a). Consequently, A is a self-small group.

(4) $\Longrightarrow$ (5). The property that the group A is self-small is shown in the proof of (4) $\Longrightarrow$ (3). Let G be an A-projective group. We prove that the ring $E(G)$ is right semihereditary and G is an endoflat group. For this purpose, we verify that G satisfies condition (3) of Theorem 35.14. We take any exact sequence $0 \to K \to B \to E \to 0$ such that B is a finitely G-projective group and $E \subseteq G$. It is clear that B is an A-projective group and E is an A-generated group. By (4)(c), the group E is A-projective. By

(4), this sequence is split. Therefore, G satisfies condition (3) of Theorem 35.14. By Theorem 35.14, the ring $E(G)$ is right semihereditary and G is an endoflat group.

(5) $\Longrightarrow$ (1). We immediately obtain that A is an endoflat group. The category equivalence from Theorem 32.1 induces an isomorphism between endomorphism rings of A-projective groups and endomorphism rings of corresponding projective right R-modules. We obtain that the endomorphism ring of every projective right R-module is right semihereditary. By Theorem 35.5, the ring R is right hereditary.

We have proved that assertions (1)–(5) are equivalent.

(1) $\Longrightarrow$ (6). Obviously, the class of all A-projective groups is closed with respect to direct sums. By (2), the class is closed with respect to A-generated subgroups. It follows from (4)(b) that A-projective groups are closed with respect to kernels of all homomorphisms and the group A is projective with respect to all sequences of A-projective groups (see also Corollary 34.16). We can assert that the class of all A-projective groups is A-balanced closed.

(6) $\Longrightarrow$ (2). By Theorem 34.11, the group A is faithful. Since the class of all A-projective groups is closed with respect to A-generated subgroups, every group $G \in \mathcal{P}_A$ is A-projective. ☐

Arnold and Lady [35] have proved the equivalence of (1) and (2) for a torsion-free group A of finite rank (see also Arnold and Murley [36]). Then different variants and parts of Theorem 35.18 appeared in the works of Albrecht [1, 5, 13], Krylov [258, 262], and Faticoni [146].

Theorem 35.18 can be considered from different points of view. First, it can be considered as an external characterization of groups with right hereditary endomorphism rings. On the other hand, it gives a quite deep generalization of the Baer–Kolettis theorem. Finally, this theorem is the transfer to endomorphism rings of the well-known property that a ring is hereditary if and only if all submodules of projective modules over this ring are projective.

Corollary 35.19. *Let A be either a self-small group or a finite direct sum of indecomposable groups. The following conditions are equivalent*:

(1) *A is an endoflat group with right hereditary endomorphism ring*;
(2) *every exact sequence $0 \to K \to B \to E \to 0$, where $K + S_A(B) = B$ and E is an A-projective group, is split, and every A-generated subgroup of an A-projective group is A-projective.*

Proof. For a self-small group A, the equivalence of (1) and (2) follows from Theorem 35.18. Let A be a finite direct sum of indecomposable groups. If we have (1) or (2), then applying Corollary 35.15 and Theorem 35.14, we obtain that A is a self-small group in both cases. ☐

A ring S is called a *ring of free right ideals* if every right ideal of the ring S is a free S-module. A principal right ideal domain is an example of such a ring. It is clear that a ring of free right ideals is right hereditary, and over such a ring, projective right modules are free (see the theorem formulated at the beginning of the section). Using these properties, we obtain the following result.

Corollary 35.20. *For a group A, conditions* (1)–(4) *of Theorem* 35.18 *are equivalent provided we write "R is a ring of free right ideals" in* (1) *instead of "R is right hereditary" and "A-free" instead of "A-projective."*

Corollary 35.21 (Albrecht [5, 13], Faticoni [146])**.** *For a group A, the following conditions are equivalent*:

(1) *A is an endoflat group and R is a right hereditary, right Noetherian ring*;
(2) *A is a faithful group and every A-generated subgroup of the group A is finitely A-projective*;
(3) *all A-generated subgroups of the group A have no infinite direct decompositions and items* (b) *and* (c) *of Theorem* 35.18(4) *hold*;
(4) *A is a self-small group and the classes of all finitely A-projective groups and all A-projective groups are A-balanced closed.*

Proof. (1) $\Longrightarrow$ (2). Since the ring R is right Noetherian, Lemma 17.2 implies that the group A is a finite direct sum of indecomposable groups. By Corollary 35.15, A is a self-small group; therefore, we can use Theorem 35.18. We have that A is a faithful group, and every A-generated subgroup G of the group A is A-projective; in addition, G has a finite A-rank by Corollary 34.18. Finally, Lemma 35.17 implies that all A-generated subgroups of the group A are finitely A-projective.

(2) $\Longrightarrow$ (3). By Theorem 35.14, A is an endoflat group with right semihereditary endomorphism ring R. By Corollary 34.18, R is right Noetherian. Therefore, R is right hereditary and A is a self-small group as in (1) $\Longrightarrow$ (2). Therefore, assertion (1) of Theorem 35.18 holds; therefore, items (b) and (c) of assertion (4) of this theorem hold. Since right ideals of the ring R have no infinite direct decompositions, Corollary 34.17 implies the assertion on A-generated subgroups of the group A.

(3) $\Longrightarrow$ (1). We have assertion (4) of Theorem 35.18. Therefore, A is a faithful endoflat group and the ring R is right hereditary. By Corollary 34.17, R has a finite right Goldie dimension. Since R is right hereditary, R is right Noetherian (this is noted at the beginning of the section).

(1) $\Longrightarrow$ (4). We have proved that A is a self-small group. By Theorem 35.18, the class of all A-projective groups is A-balanced closed. We now verify that the class of all finitely A-projective groups is also A-balanced closed. Let B be a finitely A-projective group such that $G \subseteq B$ and $S_A(G) = G$. Then $H(G) \subseteq H(B)$. By Corollary 34.15, we have $TH(G) \cong G$. Here, $H(G)$ is a submodule of a finitely generated projective right R-module $H(B)$. Since R is a right hereditary, right Noetherian ring, $H(G)$ is a finitely generated projective right R-module. Therefore, $TH(G)$ and G are finitely A-projective groups. If K is the kernel of some homomorphism of finitely A-projective groups, then K is an A-generated group since the class of all A-projective groups is A-closed. We have shown that K is a finitely A-projective group. The group A is projective with respect to the class of all finitely A-projective groups. We have proved that the class of all finitely A-projective groups is A-balanced closed.

(4) $\Longrightarrow$ (1). By Theorem 35.18, A is an endoflat faithful group and the ring R is right hereditary. Let I be a right ideal of the ring R. Then $I \cong HT(I)$ by Theorem 34.13. The group $T(I)$ is isomorphic to some A-generated subgroup of the group A. Therefore, $T(I)$ is a finitely A-projective group. Therefore, $HT(I)$ and I are finitely generated right R-modules. Therefore, R is a right Noetherian ring. ☐

Corollary 35.22. *For a group A, the following conditions are equivalent*:

1. *A is an endoflat group and R is a domain of principal right ideals*;
2. *A is a faithful group and every nonzero A-generated subgroup of the group A is isomorphic to A*;
3. *all A-generated subgroups of the group A are indecomposable, every exact sequence $0 \to K \to B \to E \to 0$, where $K + S_A(B) = B$ and the group E is A-projective, is split, and every A-generated subgroup of an A-free group is A-free*;
4. *A is a self-small group and the classes of all finitely A-free groups and all A-projective groups are A-balanced closed.*

Proof. We do not repeat the assertions that directly follow from Corollary 35.21; we present only some fragments of the proof.

(1) $\Longrightarrow$ (2). If $G \neq 0$ is an A-generated subgroup of the group A, then $G = IA$ for some nonzero right ideal I of the ring R by Corollary 34.17. Since $I \cong R_R$, we obtain $G = IA \cong I \otimes_R A \cong R \otimes_R A \cong A$.

(2) $\Longrightarrow$ (3). By Corollary 35.21, R is a right Noetherian ring. The application of Corollary 34.17 shows that nonzero right ideals of the ring R are isomorphic to R; therefore, they are indecomposable. This implies that all A-generated subgroups of the group A are indecomposable. All A-generated subgroups of an A-free group are A-free groups by Corollary 35.20.

(3) $\Longrightarrow$ (1). Using Corollary 35.20, we obtain that R is a ring of free right ideals and nonzero ideals are indecomposable and are isomorphic to R_R (consider that all A-generated subgroups of the group A are indecomposable and use Corollary 34.17). If $xy = 0$ for some $0 \neq x, y \in R$, then the split epimorphism $R \to xR$ such that $r \to xr$ for all $r \in R$ induces a nontrivial direct decomposition of the R-module R. Therefore, R is a principal right ideal domain.

(1) $\Longrightarrow$ (4). We can use the proof of the implication (1) $\Longrightarrow$ (4) of Corollary 35.21. In this corollary, we used the projectivity of submodules of projective modules over a hereditary ring. Instead of this, we use that all submodules of free modules over a principal right ideal domain are free.

(4) $\Longrightarrow$ (1). Similar to the argument used in the implication (1) $\Longrightarrow$ (4) of Corollary 35.21, we obtain that R is a Noetherian ring of free right ideals. In fact, all right ideals of the ring R are principal and the module R_R is indecomposable. The ring R is a principal right ideal domain (see the proof of (3) $\Longrightarrow$ (1)). ☐

Corollary 35.23 (Arnold and Murley [36]). *Let A be a torsion-free group such that $E(A)$ is a principal ideal domain. Then every A-generated subgroup of any A-free group is A-free.*

Proof. By Corollary 33.4, the group A is endoflat, and we can use Corollary 35.22. ☐

If we assume that the group A from Corollary 35.22 has commutative endomorphism ring, then we obtain a characterization of groups whose endomorphism rings are principal ideal domains. We consider more explicitly the arising situation. At present, the following result is the most satisfactory description of groups whose endomorphism rings are principal ideal domains. However, we have a description only up to the description of groups with commutative endomorphism rings.

Corollary 35.24 (Albrecht [5]). *For a group A, the following conditions are equivalent*:

(a) *$E(A)$ is a principal ideal ring*;

(b) *A belongs to one of the following classes of groups*:

  (1) *$A \cong Z(p)$ for some p*;

  (2) *$A \cong Z(p^\infty)$ for some p*;

  (3) *$A \cong \mathbf{Q}$*;

  (4) *A is a reduced faithful torsion-free group, every nonzero A-generated subgroup of the group A is isomorphic to A, and $E(A)$ is a commutative ring.*
  Instead of (4), *we can write*

  (5) *A is a reduced faithful torsion-free group, every A-generated subgroup of an A-free group is A-free, and $E(A)$ is a commutative ring.*

Proof. (a) $\Longrightarrow$ (b). The ring $E(A)$ has no nontrivial idempotents; therefore, A is an indecomposable group. We obtain that A is either a torsion group or a torsion-free group. In the first case, either $A \cong Z(p^k)$ for some prime integer p and positive integer k or $A \cong Z(p^\infty)$ for some p. Since $E(Z(p^k))$ is the residue ring Z_{p^k}, it is clear that $k = 1$. For the torsion-free group A, either $A \cong \mathbf{Q}$ or A is a reduced group. The remaining assertions in (4) and (5) are verified by using Corollary 35.22.

(b) $\Longrightarrow$ (a). First, we note that $E(Z(p)) \cong F_p$, $E(Z(p^\infty)) \cong Q_p^*$, and $E(\mathbf{Q}) \cong \mathbf{Q}$. The implication (4) $\Longrightarrow$ (a) follows directly from Corollary 35.22. We now recall that endomorphic images of the group with commutative endomorphism ring are fully characteristic. Using this property and (5), it is easy to see that all A-generated subgroups of the group A are isomorphic to A. Therefore, (4) and (a) hold. ☐

Exercise 1. Let R be a right (left) semihereditary ring. Prove that the ring eRe is right (left) semihereditary for every idempotent $e \in R$.

Exercise 2. Prove that a direct summand of a group with right (left) semihereditary endomorphism ring has a right (left) semihereditary endomorphism ring.

Exercise 3. Prove that all p-components of a group with right or left semihereditary endomorphism ring are elementary groups.

Exercise 4. Using the proof of Proposition 35.12, prove that a reduced torsion-free group A with right (left) hereditary endomorphism ring is a direct sum of indecomposable groups if and only if all direct summands of the group A are divisible by some integers.

Exercise 5. Describe groups with right (left) hereditary endomorphism rings among the following groups: torsion groups, direct sums of cyclic groups, torsion-free completely decomposable groups, divisible groups, and algebraically compact groups.

Exercise 6. Prove that the endomorphism ring of the group $\mathbf{Z} \oplus \mathbf{Q}$ is left hereditary and is not right hereditary.

Exercise 7. Let A be a torsion-free group such that its quasi-endomorphism ring $\mathcal{E}(A)$ is a division ring. Prove that the following conditions are equivalent:

(1) $E(A)$ is a principal right ideal domain;

(2) A is a faithful group and every A-generated subgroup of any A-free group is A-free.

The assertions contained in Exercises 8–11 have been proved by Albrecht [1].

We say that a torsion-free group A satisfies the right (left) central condition if every essential right (left) ideal of the ring $E(A)$ contains some central monomorphism (i.e., a monomorphism contained in the center of the ring $E(A)$). A right (left) ideal of some ring is said to be *essential* if it has a nonzero intersection with each nonzero right (left) ideal of this ring.

Exercise 8. Let A be a torsion-free group with the right central condition. Then A satisfies the left central condition and $E(A)$ is a semiprime right and left Goldie ring.

A ring R is called a *right* (*left*) *Goldie ring* if

(1) R has a finite right (left) Goldie dimension;

(2) R satisfies the maximum condition on right (left) annihilators of subsets from R.

Exercise 9. For the torsion-free group A of finite rank, the following conditions are equivalent:

(a) $E(A)$ is a semiprime ring;

(b) A satisfies the right central condition.

Exercise 10. Prove that

(a) a torsion-free group with the right central condition is self-small;

(b) a torsion-free group with the right central condition and with right semihereditary endomorphism ring is endoflat.

Exercise 11. Prove that for a torsion-free group A, the following conditions are equivalent:

(a) A is a group with the right central condition and with right hereditary endomorphism ring;

(b) (i) every exact sequence $0 \to K \to B \to E \to 0$ with $K + S_A(B) = B$ and A-projective group E is split;

(ii) every A-generated subgroup of an A-projective group is A-projective;

(iii) $A = \sum_{i=1}^{n}{}^{\oplus} A_i$, where A_i is a fully characteristic subgroup in A, A_i satisfies the right central condition, and $E(A_i)$ is a prime ring for all $i = 1, \dots, n$.

The group G is called a group with the summand intersection property if the intersection of any two direct summands of the group G is a direct summand.

The remaining exercises were proved by Hausen [214].

Exercise 12. The group G satisfies the summand intersection property if and only if for every decomposition $G = E \oplus H$ and each $\varepsilon : E \to H$, we have that $\ker \varepsilon$ is a direct summand.

Exercise 13. For an indecomposable group A, prove that the following conditions are equivalent:

(1) for every cardinal number $\mathfrak{M}$, the group $\sum_{\mathfrak{M}}{}^{\oplus} A$ has the summand intersection property;

(2) if G and H are A-projective groups and $\varepsilon : G \to H$ is a homomorphism, then $\ker \varepsilon$ is a direct summand of the group G;

(3) for every A-projective group G and each $\varepsilon \in E(G)$, we have that $\ker \varepsilon$ is a direct summand;
(4) A is an endoflat group with a right hereditary endomorphism ring.

Exercise 14. For an indecomposable group A, prove that the following conditions are equivalent:

(1) for every positive integer m, the group $\sum\limits_{m}^{\oplus} A$ has the summand intersection property;
(2) if G and H are two finitely A-projective groups and $\varepsilon : G \to H$ is a homomorphism, then $\ker \varepsilon$ is a direct summand of the group G;
(3) for every finitely A-projective group G and each $\varepsilon \in E(G)$, we have that $\ker \varepsilon$ is a direct summand;
(4) A is an endoflat group with right semihereditary endomorphism ring.

Exercise 15. Let A be a group such that $E(A)$ is a principal ideal domain, and let G be an A-free group. Then

(1) G has the summand intersection property;
(2) the kernel of every endomorphism of the group G is a direct summand of the group G, which is also an A-free group.

36. Groups of Generalized Rank 1

It seems that only torsion-free groups are of real interest among groups with right hereditary endomorphism rings. In the present section, we especially study torsion-free groups with hereditary endomorphism rings. More precisely, we consider torsion-free groups with hereditary Noetherian semiprime endomorphism rings. Hereditary Noetherian semiprime rings and modules over such rings are intensively studied. It is interesting that many results obtained in this part hold for groups with hereditary Noetherian semiprime endomorphism rings. Indeed, such groups are self-small, faithful, and endoflat and their endomorphism rings are discrete in the finite topology. Therefore, it is clear that analogs of the Baer lemma and the Baer–Kolettis theorem hold for these groups. These assertions correspond to basic properties of homogeneous completely decomposable groups (i.e., A-free groups, where A is a torsion-free group of rank 1). There is one more important property of these groups that is also proved by Baer [47] in his classical work. It is formulated in the introduction of this part (item (3)). A group with hereditary Noetherian semiprime endomorphism ring is an acceptable candidate for replacement of the group A of rank 1 in the Baer theorem. We will see that a reduced torsion-free group with hereditary Noetherian semiprime endomorphism ring has fundamental properties of the group of rank 1. Therefore, we follow Albrecht [3] and call them groups of generalized rank 1.

In the beginning of the section, we formulate some properties of hereditary Noetherian semiprime rings and modules over such rings. First, we give a terminological remark. A left and right Noetherian ring is called a Noetherian ring. Similarly, a left and right hereditary ring is called a hereditary ring.

The well-known Goldie theorem states that a left Noetherian semiprime ring has the classical left ring of fractions, which is an Artinian semisimple ring (Lambek [279] and Herstein [222]).

An element r of the ring R is said to be *regular* or is called a *nondivisor of zero* if $sr \neq 0$ and $rs \neq 0$ for every $0 \neq s \in R$.

Let R be a subring of the ring S. The ring S is called the *classical left ring of fractions* of the ring R if the following conditions hold:

(1) all regular elements of the ring R are invertible in the ring S;
(2) all elements of the ring S have the form $b^{-1}a$, where $a, b \in R$ and b is a regular element of the ring R.

Similarly, we can define the classical right ring of fractions. Therefore, a Noetherian semiprime ring has the two-sided classical ring of fractions.

A left ideal of the ring R is said to be *essential* if it has nonzero intersection with every nonzero left ideal of the ring R. If R is a left Noetherian semiprime ring, then a regular element of the ring R generates an essential left ideal and an essential left ideal always contains a regular element (Herstein [222, § 7.2].

A hereditary Noetherian semiprime ring R is a finite direct product of hereditary Noetherian prime rings (see Faith [142, the Chatters theorem 20.30] and Levy [285]).

A module M is said to satisfy the restricted minimum condition if the module M/K is Artinian for every essential submodule K. If R is a hereditary Noetherian ring, then R satisfies left and right bounded minimum conditions. Moreover, every finitely generated R-module satisfies the restricted minimum condition (Faith [142, Corollary 20.29]).

For modules over hereditary Noetherian semiprime rings, there are analogs of many classical results in the theory of Abelian groups.

Let M be a left module over a hereditary Noetherian semiprime ring R. An element $m \in M$ is called a *torsion* element if $rm = 0$ for some regular element $r \in R$. The set of all torsion elements of the module M is a submodule of M. This set is called the *torsion submodule* of M; it is denoted by $T(M)$. If $T(M) = 0$, then M is called a *torsion-free module* (in the sense of Levy). This definition of a torsion-free module differs from the definition used in Sec. 6, where by a torsion-free module is meant merely a module that is a torsion-free Abelian group (see the remark before Proposition 6.9). These two notions are not completely independent (see details in exercises).

In the present and subsequent sections of this part, we consider torsion in the sense of Levy. Using the above relation between regular elements and essential left ideals, we can consider this torsion from another point of view. Let M be a left module over an arbitrary ring R. We denote by $Z(M)$ the singular submodule of the module M consisting of all elements of the module M whose annihilators are essential left ideals of the ring R. A module M is said to be *nonsingular* if $Z(M) = 0$. If R is a hereditary Noetherian semiprime ring, then we obtain $T(M) = Z(M)$, and M is a torsion-free module if and only if M is a nonsingular module.

We need a theorem on the structure of a finitely generated module and some information on flat modules over a hereditary Noetherian semiprime ring R. Every finitely generated left R-module M has the following form: $M = T(M) \oplus P$, where P is a projective module. In particular, any finitely generated torsion-free R-module is projective. Since every module is a direct limit of finitely generated modules and flat modules coincide with direct limits of finitely generated projective modules, we have the following result. A left R-module M is flat if and only if M is a torsion-free module.

We present (with the proof) one result of Small containing some condition under which right hereditary rings are left hereditary. In the proof, we use a well-known lemma, which is given without proof.

Lemma 36.1. *For a ring R, the following conditions are equivalent*:

(1) *the ring R has no infinite set of orthogonal idempotents*;
(2) *the ring R satisfies the maximum or minimum condition on right* (*left*) *ideals of the form eR* (Re), *where $e^2 = e \in R$.*

Let R be a ring, and let X be a subset of R. By $r(X)$ ($l(X)$), we denote the right (left) annihilator of a subset X in R. It is shown in Lemma 35.1 that the principal right ideal aR of R is projective if and only if $r(a) = eR$ for some idempotent $e \in R$.

Proposition 36.2 (Small [410])**.** *Let R be a ring such that every principal right ideal of R is projective and R contains no infinite set of orthogonal idempotents. Then every right and every left annihilator is generated by an idempotent. In particular, every principal left ideal is projective.*

Proof. Assume that X is a subset of the ring R and $0 \neq T = r(X)$. If $x \in X$, then $r(x) \supseteq T$. Therefore, $T \subseteq hR$, where h is an idempotent (Lemma 35.1). Now let L be any nonzero left annihilator. We have $r(L) \subseteq gR$, where $g^2 = g \in R$. Then $L = l(r(L)) \supseteq l(gR) = R(1-g)$. Consequently, every left annihilator L contains a nontrivial idempotent. Lemma 36.1 allows us to choose an idempotent $e \in L$ such that $l(e)$ is a minimal element of the set of left annihilators of idempotents in L. We assert that $l(e) \cap L = 0$. Assume the contrary. Then $l(e) \cap L$ is the left annihilator containing some nonzero idempotent f. Now $e^* = e + f - ef$ is an idempotent in L, and $e^*e = e$, $e^* \neq 0$, and $l(e^*) \subseteq l(e)$. Since $fe = 0$ and

$fe^* \neq 0$, we have $l(e^*) \neq l(e)$; this contradicts the minimality of $l(e)$. Therefore, $l(e) \cap L = 0$. If $x \in L$, then $x - xe \in L$ and $(x - xe)e = 0$. Therefore, $x - xe = 0$ and $L = Re$. Therefore, all left annihilators in the ring R are generated by idempotents. If K is a right annihilator, then $l(K) = Re$ with $e^2 = e \in R$. However, $K = r(l(K)) = (1 - e)R$. Finally, Lemma 35.1 implies the projectivity of all principal left ideals. □

Corollary 36.3 (Small [410]). *If R is a left and right Noetherian and right hereditary ring, then R is left hereditary.*

Proof. Since the ring R is left Noetherian, it is sufficient to prove that the ring R is left semihereditary. By the left analog of Theorem 35.3, it is sufficient to prove that all principal left ideals of the matrix ring R_n are projective for every positive integer n. However, the assumptions allow us to use Proposition 36.2. □

We pass to groups of generalized rank 1. A reduced torsion-free group with hereditary Noetherian semiprime endomorphism ring is called here a group of generalized rank 1. The endomorphism ring of a group A is still denoted by R. We also use some other basic notation in this part. We consider several general properties of groups of generalized rank 1.

Proposition 36.4 (Albrecht [3]). *The group A of generalized rank 1 is endoflat.*

Proof. According to the above properties of modules over hereditary Noetherian semiprime rings, it is sufficient to prove that A is a torsion-free R-module. Let φ be a regular element of the ring R. We prove that φ is a monomorphism (this directly implies the required assertion). Let $a \in \ker\varphi$, and let β be the R-module homomorphism $R \to Ra$ such that $\alpha \to \alpha a$ for all $\alpha \in R$. Then the kernel of β is the left ideal $\{\alpha \in R \mid \alpha a = 0\}$, which is denoted by $\operatorname{ann}_A a$. Therefore,

$$R/\operatorname{ann}_A a \cong Ra \subseteq A.$$

Since $\varphi \in \operatorname{ann}_A a$, we have that $\operatorname{ann}_A a$ is an essential left ideal of the ring R. Since the ring R has the restricted minimum condition, $R/\operatorname{ann}_A a$ is an Artinian left R-module. Therefore, Ra is also an Artinian module. In this case, the family of submodules $\{n(Ra) \mid n \in \mathbf{N}\}$ has a minimal element $m(Ra)$. Then $n(mRa) = mRa$ for all positive integers n (i.e., mRa is a divisible group). Since A is a reduced group, $mRa = 0$. Then $Ra = 0$, since the group Ra is torsion-free. Therefore, $\operatorname{ann}_A a = R$, $a = 0$, and $\ker\varphi = 0$ (i.e., φ is a monomorphism). □

Let R be a hereditary Noetherian semiprime ring, and let M be a left R-module such that the additive group M^+ of the module M is torsion-free. If we replace A by M in the proof of Proposition 36.4, then we obtain the following result.

Corollary 36.5. *Let M be a left module over a hereditary Noetherian semiprime ring R such that M^+ is a reduced torsion-free group. Then M is a flat R-module. In addition, if the module M is finitely generated, then M is a projective R-module.*

Proposition 36.6. (1) *Let A be a group of generalized rank 1. Then $A = \sum_{i=1}^{n}{}^{\oplus} A_i$, where A_i is a fully characteristic subgroup and $E(A_i)$ is a hereditary Noetherian prime ring ($i = 1, \dots, n$).*

(2) *Let A be a reduced torsion-free group such that $E(A)$ is a hereditary Noetherian prime ring. Then $A = \sum_{j=1}^{m}{}^{\oplus} B_j$, where B_j is an indecomposable group and $E(B_j)$ is a hereditary Noetherian domain ($j = 1, \dots, m$).*

Proof. (1) The ring R $(= E(A))$ is a hereditary Noetherian semiprime ring; therefore, $R = \prod_{i=1}^{n} R_i$, where R_i is a hereditary Noetherian prime ring $(i = 1, \dots, n)$. Therefore, we have the decomposition $A = \sum_{i=1}^{n}{}^{\oplus} A_i$, where $E(A_i) = R_i$ and A_i is a fully characteristic subgroup of the group A $(i = 1, \dots, n)$.

(2) Since the ring R is Noetherian, the group A is a finite direct sum of indecomposable groups B_j (see Lemma 17.2). Here, $E(B_j)$ is a hereditary Noetherian prime ring (see Corollary 35.10). Therefore, B_j is an endoflat group by Proposition 36.4. By Corollary 35.16, every nonzero endomorphism of the group B_j is a monomorphism. Therefore, $E(B_j)$ is a hereditary Noetherian domain. □

Taking into account the preceding proof, we obtain the following corollary.

Corollary 36.7. *For an indecomposable group of generalized rank* 1*, the endomorphism ring is a hereditary Noetherian domain.*

For groups of generalized rank 1, we have some consequences of the studies of the preceding section.

Proposition 36.8 (Albrecht [13])**.** *Let A be a group of generalized rank* 1*.*

(1) *A is a self-small faithful endoflat group and the ring R is discrete in the finite topology.*
(2) *The category $\mathcal{R}_a$ is an additive category with kernels, and projective objects of the category $\mathcal{R}_a$ coincide with A-projective groups.*
(3) *Assertion* (2) *holds for the category $\overline{\mathcal{P}}_A$.*

Proof. (1) We have proved that the group A is endoflat. We now prove that the ring R is discrete. (In this case, the group A is self-small by Corollary 31.4.) Taking into account Proposition 36.6, we assume without loss of generality that R is a prime ring. We have $A = \sum_{j=1}^{m}{}^{\oplus} B_j$, where all B_js are indecomposable groups. By Corollary 35.16, every nonzero endomorphism of the group B_j is a monomorphism. By the same corollary, every nonzero homomorphism $B_j \to B_k$ is a monomorphism for arbitrary j and k. We take a nonzero element b_j in every group B_j. Then $\alpha = 0$ for every $\alpha \in R$ with $\alpha b_1 = \cdots = \alpha b_m = 0$. Therefore, the ring R is discrete in the finite topology. By Theorem 35.18, the group A is faithful.

(2) The proof follows from (1), Corollary 34.12, and the definition of the class $\mathcal{R}_a$.

(3) The proof follows from (2), since $\overline{\mathcal{P}}_A \subseteq \mathcal{R}_a$. □

Proposition 36.9. *Let A be a group of generalized rank* 1*. Then*:

(1) *every exact sequence $0 \to K \to B \to E \to 0$, where E is an A-projective group and $K + S_A(B) = B$, is split*;
(2) *every A-generated subgroup of an A-projective group is A-projective*;
(3) *every A-projective group is a direct sum of finitely A-projective groups.*

Proof. Assertions (1) and (2) follow from Proposition 36.8 and Theorem 35.18.

(3) Let G be an A-projective group. Then $H(G)$ is a projective right R-module. Consequently, $H(G) \cong \sum_{i\in I}{}^{\oplus} P_i$, where P_i is a finitely generated right ideal of the ring R $(i \in I)$. Therefore,

$$G \cong TH(G) \cong \sum_{i\in I}{}^{\oplus} T(P_i),$$

where $T(P_i)$ is a finitely A-projective group for every $i \in I$. □

The next proposition shows that the semiprimeness condition can be omitted in the definition of the group of generalized rank 1. In fact, we prove an assertion that is far more strong.

Proposition 36.10. *Let A be a reduced torsion-free group. If the ring R is right or left semihereditary, then the nil-radical of the ring R is equal to zero.*

Proof. Let M be the nil-radical of the ring R (i.e., the sum of all its nilideals). Assume that $M \neq 0$ and the ring R is right semihereditary. Assume the contrary. We fix some nonzero element $\alpha \in M$. Since R^+ is a reduced group, $\alpha \notin tR$ for some positive integer t. We prove that the right ideal $I = tR + \alpha R$ is not projective. If I is a projective ideal, then I/IM is a projective right R/M-module. Therefore, I/IM is a torsion-free group, since R/M is torsion-free. Further, we have $IM = tM + \alpha M$. In addition, $\alpha \notin IM$, since the inclusion $\alpha \in IM$ implies $\alpha = t\beta + \alpha\gamma$, where $\beta, \gamma \in M$ and $\alpha = t\beta(1-\gamma)^{-1}$ (this is impossible by the choice of the integer t). Thus, $\alpha \notin IM$ and $t\alpha \in tM \subseteq IM$. This contradicts the property that I/IM is torsion-free. Therefore, the ideal I is not projective; this is a contradiction, since the ring R is semihereditary. Therefore, $M = 0$. $\square$

By using Proposition 36.8 and Theorem 34.14, the following important result can be verified.

Proposition 36.11. *Let A be a group of generalized rank 1. Then the functors H and T define an equivalence of the categories $\overline{\mathcal{P}}_A$ and $\overline{\mathcal{P}}_R$.*

Therefore, if A is a group of generalized rank 1, then every category property of submodules of locally projective right R-modules can be extended to the corresponding subgroups of locally A-projective groups. In this regard, the following lemmas are useful.

Lemma 36.12. *Let A be a group of generalized rank 1. The group $B \in \overline{\mathcal{P}}_A$ has a finite A-rank if and only if the R-module $H(B)$ is finitely generated. In this case, the group B is A-projective.*

Proof. Let the group B have a finite A-rank. We choose an epimorphism $A^n \to B$ for some positive integer n. By Proposition 36.8, the functor H maps from epimorphisms into epimorphisms. Therefore, $H(A^n) \to H(B)$ is an epimorphism and $H(B)$ is a finitely generated R-module, since $H(A^n) \cong R^n$. Conversely, if the module $H(B)$ is finitely generated, then there exists an epimorphism $R^n \to H(B)$ for some positive integer n. By Proposition 36.11, the natural mapping $\theta_B : TH(B) \to B$ is an isomorphism. Therefore, there exist epimorphisms

$$A^n \cong T(R^n) \to TH(B) \cong B,$$

and the group B has finite A-rank.

We prove the last assertion of the lemma. Let a group B from $\overline{\mathcal{P}}_A$ have a finite A-rank, and let $B \subseteq G$, where G is a locally A-projective group. Then $H(B)$ is a finitely generated submodule of the locally projective R-module $H(G)$. Consequently, it can be embedded in some projective direct summand of the module $H(G)$. Since the ring R is hereditary, the module $H(B)$ is projective. Therefore, B is an A-projective group, since $B \cong TH(B)$. $\square$

Lemma 36.13. *Let A be a group of generalized rank 1, and let $B \in \overline{\mathcal{P}}_A$. If the endomorphism ring of the group B is discrete in the finite topology, then the endomorphism ring $\operatorname{End}_R H(B)$ of the R-module $H(B)$ is also discrete in the finite topology.*

Proof. Considering the equivalence of the categories $\overline{\mathcal{P}}_A$ and $\overline{\mathcal{P}}_R$ (see Proposition 36.11), it is convenient to reformulate the assertion of the corollary as follows. Let M be a module in $\overline{\mathcal{P}}_R$ such that the endomorphism ring of the group $T(M)$ is discrete in the finite topology. We prove that the ring $\operatorname{End}_R M$ is also discrete in the finite topology. We note that an isomorphism of the endomorphism rings $\operatorname{End}_R M \to E(T(M))$ is given by the correspondence $\alpha \to \alpha \otimes 1$, $\alpha \in \operatorname{End}_R M$.

Since the ring $E(T(M))$ is discrete, there exists a family of elements $g_1, \ldots, g_n \in T(M)$ such that the annihilator in $E(T(M))$ of the family is equal to zero. For every i, we have

$$g_i = \sum_{j=1}^{k_i} m_{ij} \otimes a_{ij} \qquad \text{with} \quad m_{ij} \in M, \quad a_{ij} \in A.$$

We take the family of elements $\{m_{ij} \mid i = 1, \dots, n; j = 1, \dots, k_i\}$ of the module M. Assume that $\alpha \in \operatorname{End}_R M$ and $\alpha m_{ij} = 0$ for all i and j. Then

$$(\alpha \otimes 1)(m_{ij} \otimes a_{ij}) = \alpha m_{ij} \otimes a_{ij} = 0 \qquad \text{for} \quad j = 1, \dots, k_i.$$

Consequently, $(\alpha \otimes 1)g_i = 0$ for all i. By the choice of elements g_i, we have $\alpha \otimes 1 = 0$. Then $\alpha = 0$, since there is a ring isomorphism $\operatorname{End}_R M \cong E(T(M))$. We obtain that the annihilator of the family of elements $\{m_{ij} \mid i = 1, \dots, n; j = 1, \dots, k_i\}$ in the ring $\operatorname{End}_R M$ is equal to zero. Therefore, this endomorphism ring is discrete in the finite topology. □

For a group of generalized rank 1, one additional condition appears in the following lemma. It makes such a group maximally close to an ordinary group of rank 1. The work of Albrecht [14] contains the corresponding more general results. We note that every reduced torsion-free group of finite rank with right semihereditary endomorphism ring is a group of generalized rank 1, and it satisfies the additional condition of Lemma 36.14.

Lemma 36.14. *Let A be a group of generalized rank 1 such that A/V is a torsion group for every subgroup $V \subseteq A$ that is isomorphic to A. If B is a pure subgroup of some torsion-free group G, then the R-module $H(G)/H(B)$ is torsion-free.*

Proof. By Corollary 35.16, all regular elements of the ring R are monomorphisms of the group A. Now assume that $\varphi \in H(G)$ and $\varphi d \in H(B)$ for some regular element d of the ring R. Then

$$(\varphi d)A = \varphi(dA) \subseteq B.$$

Since d is a monomorphism of the group A, the group A/dA is a torsion group by assumption. Consequently, if $a \in A$, then $na \in dA$ for some positive integer n. Therefore, $n(\varphi a) = \varphi(na) \in \varphi(dA) \subseteq B$ and $\varphi a \in B$, since the subgroup B is pure. Therefore, $\varphi \in H(B)$, and the module $H(G)/H(B)$ is torsion-free. □

Lemma 36.15. *Let A be the group from Lemma* 36.14*, and let B be a pure subgroup of a locally A-projective group G. The following conditions are equivalent*:

(1) *the group B has a finite A-rank*;
(2) *the R-module $H(B)$ is finitely generated*;
(3) *the ring $E(B)$ is discrete in the finite topology*;
(4) *the ring $\operatorname{End}_R H(B)$ is discrete in the finite topology.*

Proof. The equivalence of (1) and (2) is proved in Lemma 36.12.

(2) $\Longrightarrow$ (3). It follows from Lemma 36.12 that $H(B)$ is a projective R-module and B is a finitely A-projective group. Since the ring $E(A)$ is discrete in the finite topology (Proposition 36.8), it is easy to verify that the ring $E(B)$ is also discrete in the finite topology.

The implication (3) $\Longrightarrow$ (4) is shown in Lemma 36.14.

(4) $\Longrightarrow$ (2). We denote $M = H(B)$. We choose a family of elements $\{m_1, \dots, m_t\} \subset M$ such that the annihilator of the family in the ring $\operatorname{End}_R M$ is equal to zero. Since $H(G)$ is a locally projective R-module, there exists a decomposition $H(G) = F \oplus P$ such that F is a projective module, P is an R-module, and $m_1, \dots, m_t \in F$. Taking into account the structure of projective modules over hereditary rings and the Noether property of the ring R, we can assume that the module F is finitely generated. By Lemma 36.14, the R-module $H(G)/M$ is torsion-free. Since $F/(M \cap F) \cong (M + F)/M$, the finitely generated module $F/(M \cap F)$ is torsion-free. Consequently, it is projective. Therefore, $F = (M \cap F) \oplus T$ and $H(G) = (M \cap F) \oplus T \oplus P$ for some module T. For the module M, we obtain $M = (M \cap F) \oplus (M \cap K)$, where $K = T \oplus P$. Since $m_1, \dots, m_t \in M \cap F$, $M = M \cap F$ by the choice of these elements. Therefore, $M \subseteq F$ and the module $H(B)$ equal to M is finitely generated. □

A subgroup B of a group G is said to be closed if G/B is a reduced group. It is easy to verify that the subgroup B is closed in this sense if and only if B is closed in the $\mathbf{Z}$-adic topology of the group G.

In fact, the next proposition is a part of Theorem 36.17.

Proposition 36.16 (Albrecht [13]). *Let A be a group of generalized rank* 1. *Every A-generated pure closed subgroup of the group A^n $(n \in \mathbf{N})$ is a direct summand of the group A^n.*

Proof. Let G be an A-generated pure closed subgroup of the group A^n. We have the following exact sequence of right R-modules:

$$0 \longrightarrow H(G) \xrightarrow{H(\varkappa)} H(A^n) \xrightarrow{H(\pi)} H(A^n/G),$$

where $\varkappa : G \to A^n$ is an embedding and $\pi : A^n \to A^n/G$ is the canonical homomorphism. By assumption, A^n/G is a reduced torsion-free group. Therefore, $H(A^n/G)$ and $\operatorname{im} H(\pi)$ are reduced torsion-free groups. In addition, the R-module $\operatorname{im} H(\pi)$ is finitely generated as a homomorphic image of the finitely generated module $H(A^n)$. By Corollary 36.5, we obtain that $\operatorname{im} H(\pi)$ is a projective R-module. Therefore, the epimorphism $H(\varkappa) : H(A^n) \to \operatorname{im} H(\varkappa)$ is split, whence $TH(\varkappa)$ is split. Since the transformation θ is natural, we have $\theta_{A^n} TH(\varkappa) = \varkappa\theta_G$. By Proposition 36.9, G is an A-projective group. Therefore, θ_{A^n} and θ_G are isomorphisms (Theorem 32.1). Thus, we have $\varkappa = \theta_{A^n} TH(\varkappa)\theta_G^{-1}$. Therefore, $\varkappa$ is split and G is a direct summand of the group A^n. $\square$

For a group A of generalized rank 1, the following two theorems transfer some known property of homogeneous separable torsion-free groups mentioned at the beginning of the section to locally A-projective groups.

Theorem 36.17. *For a reduced torsion-free group A with Noetherian ring R, the following conditions are equivalent*:

(1) *R is a hereditary semiprime ring*;
(2) *A is an endoflat group and if B is a pure closed subgroup of finite A-rank of any locally A-projective group G, then B is a direct summand of the group G*;
(3) *A is an endoflat group and if B is a pure closed A-generated subgroup of any finitely A-projective group G, then B is a direct summand of the group G.*

Note. In (1), the semiprimeness of the ring R follows from its hereditarity (Proposition 36.10).

Proof. (1) $\Longrightarrow$ (3). By Proposition 36.4, the group A is endoflat. The finitely A-projective group G is a direct summand of the group A^n for some positive integer n. Now (3) follows from Proposition 36.16.

(1) $\Longrightarrow$ (2). Let B be a pure closed subgroup of finite A-rank of the locally A-projective group G. By Lemma 36.12, the R-module $H(B)$ is finitely generated. Consequently, it is contained in some projective direct summand P of the locally projective R-module $H(G)$. Taking into account the structure of projective modules over hereditary rings and the Noetherianness of the ring R, we can assume that the module P is finitely generated. By Proposition 36.11, the subgroup B is contained in a finitely A-projective direct summand $T(P)$ of the group G. It was proved that the subgroup B is a direct summand of the group $T(P)$ and, consequently, of G.

(2) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (1). Let I be a right ideal of the ring R, and let

$$0 \longrightarrow L \longrightarrow F \xrightarrow{\pi} I \longrightarrow 0$$

be an exact sequence of right R-modules with the finitely generated free module F (the ideal I is finitely generated, since R is Noetherian). Since the R-module A is flat, the sequence

$$0 \longrightarrow T(L) \longrightarrow T(F) \xrightarrow{T(\pi)} T(I) \longrightarrow 0$$

is exact. Since the module ${}_RA$ is flat, $T(I) \cong IA$; therefore, the group $T(I)$ is torsion-free. Consequently, the subgroup $T(L)$ is pure and closed in $T(F)$. Since the ideal I is finitely generated, there exists an epimorphism $R^n \to I$ for some positive integer n. Therefore, $A^n \cong T(R^n) \to T(L)$ is an epimorphism and $T(L)$ has a finite A-rank. By (2) or (3), the sequence

$$0 \longrightarrow T(L) \longrightarrow T(F) \xrightarrow{T(\pi)} T(I) \longrightarrow 0$$

is split. Therefore, $HT(\pi)$ is also split (i.e., there exists a $\omega : HT(I) \to HT(F)$ such that $HT(\pi)\omega = 1$). In the relation $HT(\pi)\phi_F = \phi_I\pi$, the mapping ϕ_F is an isomorphism (Theorem 32.1) and ϕ_I is a monomorphism, since I is an ideal of the ring R. We set

$$\eta = \phi_F^{-1}\omega\phi_I : I \to F.$$

We have

$$\phi_I\pi\eta = \phi_I\pi\phi_F^{-1}\omega\phi_I = HT(\pi)\omega\phi_I = \phi_I.$$

Therefore, $\pi\eta = 1$, since ϕ_I is a monomorphism. Therefore, the epimorphism π is split, I is a projective ideal, and the ring R is right hereditary. A Noetherian, right hereditary ring is left hereditary by Corollary 36.3. □

For a subgroup B of a torsion-free group G, the pure hull of B in G is denoted by B_*. We have $B_* = \{x \in G \mid nx \in B \text{ for some positive integer } n\}$.

Theorem 36.18. *For a reduced torsion-free group A with Noetherian ring R, the following conditions are equivalent:*

(1) *R is a hereditary semiprime ring and for every subgroup $V \subseteq A$ isomorphic to A, the group A/V is a torsion group;*

(2) *if B is a subgroup of finite A-rank of some locally A-projective group G, then B_* is a direct summand of the group G;*

(3) *if B is an A-generated subgroup of some finitely A-projective group G, then B_* is a direct summand of the group G.*

Proof. (1) $\Longrightarrow$ (2). Let B be a subgroup of finite A-rank of the locally A-projective group G. By Lemma 36.12, the R-module $H(B)$ is finitely generated. Consequently, it is contained in some finitely generated projective direct summand P of the locally projective R-module $H(G)$ (see the proof of (1) $\Longrightarrow$ (2) in Theorem 36.17). Applying the functor T and Proposition 36.11, we obtain that the subgroups B and B_* are contained in the direct summand $T(P)$ of the group G (for convenience, we identify any group $C \in \overline{\mathcal{P}}_A$ and any module $M \in \overline{\mathcal{P}}_R$ with $TH(C)$ and $HT(M)$, respectively). Therefore, $H(B_*) \subseteq P$. By Lemma 36.14, the finitely generated R-module $P/H(B_*)$ is torsion-free. Therefore, it is projective and $H(B_*)$ is a direct summand in $H(G)$. Therefore, $TH(B_*)$ is a direct summand in G. Further, $B = TH(B)$ and $TH(B_*) \subseteq B_*$. Therefore, $B = TH(B) \subseteq TH(B_*) \subseteq B_*$. Since $TH(B_*)$ is a direct summand in G, we have $B_* = TH(B_*)$, and B_* is a direct summand of the group G.

(2) $\Longrightarrow$ (3). Assume that we have the subgroup $B \subseteq G$ with $B = S_A(B)$ and G is a finitely A-projective group. Then $H(B) \subseteq H(G)$, where $H(G)$ is a finitely generated module over the Noetherian ring R. Therefore, the module $H(B)$ is also finitely generated. By Lemma 36.12, the group B has a finite A-rank. Therefore, the subgroup B_* is a direct summand of the group G by (2).

(3) $\Longrightarrow$ (1). Let I be a right ideal of the ring R. We prove that I is a projective R-module. We choose the exact sequence

$$0 \longrightarrow L \longrightarrow F \xrightarrow{\pi} I \longrightarrow 0$$

of right R-modules, where F is a finitely generated free R-module. We consider the induced sequence

$$T(L) \longrightarrow T(F) \xrightarrow{T(\pi)} T(I) \longrightarrow 0.$$

Let S be the torsion part of the group $T(I)$, $W = T(I)/S$, and let $\sigma : T(I) \to W$ be the canonical homomorphism. Further, let B be the image of the group $T(L)$ in $T(F)$. Then $T(L) = S_A(T(L))$, $B = S_A(B)$, and $T(F) \cong A^n$ for some positive integer n.

The kernel of the epimorphism $\sigma T(\pi) : T(F) \to W$ is equal to B_*. By (3), $\sigma T(\pi)$ is split. Consequently, the homomorphism $H(\sigma T(\pi))$ is also split. We denote it by ρ. Since the transformation ϕ is natural and our assumptions hold, $\rho\phi_F = H(\sigma)\phi_I\pi$, where ϕ_F is an isomorphism. In the above relation, $H(\sigma)\phi_I$ is a monomorphism. Indeed, let $H(\sigma)\phi_I(r) = 0$ for some $r \in I$. This means that $r \otimes a \in S$ for all $a \in A$, since $\phi_I(r)(a) = r \otimes a$ for all $a \in A$. For a given $a \in A$, we choose a positive integer m such that $m(r \otimes a) = 0$. Using the canonical homomorphism $I \otimes_R A \to IA$, we obtain $mra = 0$. Therefore, $ra = 0$. Since a is an arbitrary element, $r = 0$. Consequently, $H(\sigma)\phi_I$ is a monomorphism.

We set

$$\eta = \phi_F^{-1}\omega H(\sigma)\phi_I : I \to F,$$

where ω splits ρ (i.e., $\rho\omega = 1$). We have

$$H(\sigma)\phi_I\pi\eta = H(\sigma)\phi_I\pi\phi_F^{-1}\omega H(\sigma)\phi_I = \rho\phi_F\phi_F^{-1}\omega H(\sigma)\phi_I = \rho\omega H(\sigma)\phi_I = H(\sigma)\phi_I.$$

Therefore, $\pi\eta = 1$, since $H(\sigma)\phi_I$ is a monomorphism. Therefore, π is split, I is a projective ideal, and R is right hereditary. By Corollary 36.3, R is left hereditary.

Finally, assume that the subgroup V of the group A is isomorphic to A and $\varphi : A \to V$ is an isomorphism. If A/V is not a torsion group, then $V_* \neq A$. By (3), $A = V_* \oplus C$ for some subgroup $C \neq 0$. Then $\pi\varphi = 0$, where $\pi : A \to C$ is the projection with kernel V_*. This is impossible. Indeed, the monomorphism φ is a one-sided nondivisor of the zero of the ring R. However, one-sided nondivisors of the zero of a Noetherian semiprime ring are two-sided nondivisors of the zero. This can be verified using the property that R has the classical ring of fractions, which is an Artinian semisimple ring. Therefore, it follows from $\pi\varphi = 0$ that $\pi = 0$; this is impossible, since $C \neq 0$. Therefore, A/V is a torsion group. $\square$

Corollary 36.19 (Albrecht [3], Krylov [263])**.** *Let A be a group of generalized rank* 1 *such that A/V is a torsion group for every subgroup $V \cong A$. If B is a pure A-generated subgroup of a locally A-projective group G such that the endomorphism ring $E(B)$ is discrete in the finite topology, then B is a direct summand in G.*

Proof. By Lemma 36.15, the group B has a finite A-rank, and we can use Theorem 36.18. $\square$

A part of the proof of the implication (3) $\Longrightarrow$ (1) of Theorem 36.18 remains valid in a more general situation.

Corollary 36.20. *If a reduced torsion-free group A satisfies the condition of Theorem* 36.18(3)*, then R is a right semihereditary semiprime ring.*

Corollary 36.21 (Arnold and Murley [36])**.** *Let A be a reduced torsion-free group such that the endomorphism ring of A is a principal ideal domain and $A/\alpha A$ is a torsion group for every $0 \neq \alpha \in E(A)$. If B is a pure A-generated subgroup of a locally A-projective group G such that the endomorphism ring $E(B)$ is discrete in the finite topology, then B is a direct summand in G.*

Other results on groups of generalized rank 1 (in particular on locally A-projective groups, where A is a group of generalized rank 1) are contained in the works of Albrecht [1, 3] and [14] (some of these results are contained in the exercises).

Exercise 1 (Albrecht [1])**.** Let A be a torsion-free group with the right central condition and right hereditary endomorphism ring (see Exercises 8–11 from the preceding section on the right central condition). Prove that the following conditions are equivalent:

(a) if U is any fully characteristic subgroup of the group A that is isomorphic to A, then A/U is a torsion group;

(b) the pure hull of each A-generated subgroup of an A-projective group of finite A-rank is a direct summand.

Exercise 2 (Albrecht [3])**.** Let A be a group of generalized rank 1, and let the group A/V be a torsion group for every subgroup $V \subseteq A$ isomorphic to A. For the group G, the following conditions are equivalent:

(a) G is a locally A-projective group;
(b) $S_A(G) = G$ and G is isomorphic to a pure subgroup of the group $S_A(\prod_{\mathfrak{N}} A)$ for some cardinal number $\mathfrak{N}$.

Exercise 3 (Albrecht [3]). Let A be the group from Exercise 2. If B is a pure A-generated subgroup of the locally A-projective group, then B is a locally A-projective group.

Exercise 4 (Albrecht [3]). Let A be a group of generalized rank 1. Then every locally A-projective group of at most countable A-rank is A-projective.

Exercise 5 (Arnold and Murley [36]). Let A be a reduced torsion-free group such that $E(A)$ is a principal ideal domain and $A\alpha A$ is a torsion group for all $0 \neq \alpha \in E(A)$. Then
(a) a group G is locally A-free if and only if $S_A(G) = G$ and G is isomorphic to a pure subgroup of the group $S_A(\prod_{\mathfrak{N}} A)$ for some cardinal number $\mathfrak{N}$;
(b) if B is a pure A-generated subgroup of a locally A-free group, then B is a locally A-free group.

An R-module M is called a *torsion-free module in the sense of Levy* if regular elements of the ring R do not annihilate nonzero elements of the module M.

Exercise 6. (a) Let M be a torsion-free module in the sense of Levy over the torsion-free ring (i.e., the additive group of the ring is torsion-free). Prove that M is a torsion-free group.

(b) Let M be a module over a hereditary Noetherian semiprime ring such that M is a reduced torsion-free group. Then M is a torsion-free module in the sense of Levy.

Exercise 7. Let M be a module over a torsion-free ring of finite rank that is a torsion-free group. In this case, M is a torsion-free module in the sense of Levy.

37. Torsion-Free Groups of Finite Rank with Hereditary Endomorphism Rings

We verify that Theorems 35.18, 36.17, and 36.18 have a particularly simple form for the torsion-free group of finite rank. In addition, we show that one more interesting property of groups of rank 1 is characteristic for groups of finite rank with hereditary endomorphism rings. We mean the following property. If A is a torsion-free group of rank 1, then every element of the group A^n $(n \in \mathbf{N})$ can be embedded in a direct summand of the group A^n that is isomorphic to A.

We denote by R the endomorphism ring of a group A. Further, T and H are the previously considered functors and $\phi : 1 \to HT$ is the natural transformation. We recall the quasi-isomorphism symbol $\sim$. We recall that a reduced torsion-free group with hereditary Noetherian semiprime endomorphism ring is called here a group of generalized rank 1.

Theorem 37.1 (Huber–Warfield [226]). (1) *Let A be a reduced torsion-free group of finite rank with right or left semihereditary endomorphism ring R. Then R is a hereditary Noetherian semiprime ring and the group A is equal to $\sum_{i=1}^{n}{}^{\oplus} A_i$, where the group A_i is fully characteristic in A and $E(A_i)$ is a prime ring for every $i = 1, \dots, n$.*

(2) *Let A be a reduced torsion-free group with hereditary prime ring R. Then*

$$A = \sum_{j=1}^{k}{}^{\oplus} B_j,$$

where $E(B_j)$ is a hereditary domain for all $j = 1, \dots, k$ and $B_1 \sim B_2 \sim \dots \sim B_k$.

Proof. (1) By Proposition 36.10, R is a semiprime ring. By Corollary 4.6, R is a right and left Noetherian ring. Therefore, A is a group of generalized rank 1 (we use the terminology of the preceding section). The application of Proposition 36.6 completes the proof of (1).

(2) Taking into account (1), we see that Proposition 36.6(2) implies $A = \sum_{j=1}^{k}{}^{\oplus} B_j$, where all the rings $E(B_j)$ are hereditary domains. Consequently, all the B_js are strongly indecomposable groups. By Corollary 7.4, $B_1 \sim B_2 \sim \ldots \sim B_k$. □

We do not have more exact relations between summands B_j from item (2) of Theorem 37.1. In the next section, we obtain them for some important class of groups of finite rank with hereditary endomorphism rings.

The proved theorem implies the following important corollary.

Corollary 37.2. *Let A be a reduced torsion-free group of finite rank whose endomorphism ring is right or left semihereditary. Then A is a group of generalized rank* 1.

Further, we write a "reduced torsion-free group of finite rank with hereditary endomorphism ring" instead of a "reduced torsion-free group of finite rank with right or left semihereditary endomorphism ring." By the above corollary, it is possible to apply all assertions of Sec. 36 related to groups of generalized rank 1 to such groups. In particular, we formulate the following result.

Corollary 37.3. *Let A be a reduced torsion-free group of finite rank with hereditary endomorphism ring. Then*:

(1) *A is a faithful endoflat group*;
(2) *the group A satisfies the assertions of Propositions* 36.8, 36.9, *and* 36.11 *and Corollary* 34.17.

Before applying some other results of Secs. 35 and 36 to groups of finite rank, we prove one lemma. Let G be one more torsion-free group. We fix the following notation: $V = G \otimes \mathbf{Q}$, $S = E(G) \otimes \mathbf{Q}$, and $K = \mathrm{End}_S V$. We assume that the $\mathbf{Q}$-space V is an $S-K$-bimodule. By a pfi-subgroup we mean a pure fully characteristic subgroup. If B is a subgroup of the group G, then B_* is its pure hull in the group G.

Lemma 37.4. *Let A be a torsion-free group of finite rank with semiprime ring R, and let $G = A^n$ for some positive integer n. Then $BK \cap G = B_*$ for every A-generated subgroup B of the group G.*

Proof. Since $\mathbf{Q} \subseteq K$, we have $B_* \subseteq B\mathbf{Q} \subseteq BK$ and $B_* \subseteq BK \cap G$.

We prove the converse inclusion. Let $w \in BK \cap G$. We have

$$w = \sum_{s=1}^{m} b_s \alpha_s,$$

where $b_s \in B$ and $\alpha_s \in K$. Since B is an A-generated group, we have

$$b_s = \sum_{i=1}^{k_s} \varphi_{s_i} a_{s_i}$$

for every $s = 1, \ldots, m$, where $\varphi_{s_i} \in \mathrm{Hom}(A, B)$ and $a_{s_i} \in A$. We assume that the group A is one of the summands A in A^n. In this case, we can assume that φ_{s_i} is an endomorphism of the group A^n if we mean that φ_{s_i} annihilates the summand of A^n that is complement to this group A. We consider the relations

$$w = \sum_{s=1}^{m} \Big(\sum_{i=1}^{k_s} \varphi_{s_i} a_{s_i} \Big) \alpha_s = \sum_{s=1}^{m} \sum_{i=1}^{k_s} \varphi_{s_i} (a_{s_i} \alpha_s).$$

By Corollary 7.5, $A \otimes \mathbf{Q}$ is a submodule of the K-module V. Since $a_{s_i} \in A$, we have $a_{s_i}\alpha_s \in A \otimes \mathbf{Q}$. We choose a positive integer t such that $t(a_{s_i}\alpha_s) \in A$ for all s_i and s. Then

$$tw = \sum_{s=1}^{m} \sum_{i=1}^{k_s} \varphi_{s_i} t(a_{s_i}\alpha_s) \in B.$$

Therefore, $w \in B_*$. □

Theorem 37.5. *For a reduced torsion-free group A of finite rank, the following conditions are equivalent*:

(1) *R is a right hereditary ring*;
(2) *A is a faithful group and every A-generated subgroup of an A-projective group is A-projective*;
(3) *A is an endoflat group and if B is a pure A-generated subgroup of finite rank of the locally A-projective group G, then B is a direct summand of the group G*;
(4) *if B is an A-generated subgroup of finite rank of the locally A-projective group G, then B_* is a direct summand of the group G*;
(5) *the ring R is semiprime and every element of each minimal pfi-subgroup of a finitely A-projective group G can be embedded in some strongly indecomposable direct summand of the group G.*

Proof. First, we note that the endomorphism ring of every torsion-free group of finite rank is discrete in the finite topology. In particular, it is self-small. Further, let V be a subgroup of the group A such that $V \cong A$. We choose some isomorphism $\omega : A \to V$. Then ω is a regular element of the Artinian ring $E(A) \otimes \mathbf{Q}$. Therefore, ω is invertible in $E(A) \otimes \mathbf{Q}$. We choose an integer n such that $n\omega^{-1} \in E(A)$. Then $\omega(n\omega^{-1}) = n1$ implies $V = \operatorname{im} \omega \supseteq nA$, and A/V is a torsion group.

The equivalence of (1) and (2) follows from Corollary 37.3 and Theorem 35.18.

(1) $\Longrightarrow$ (3). By Corollary 37.2, A is a group of generalized rank 1. By Proposition 36.4, the group A is endoflat. Now (3) can be verified by using Corollary 36.19.

(3) $\Longrightarrow$ (1). Let I be a finitely generated right ideal of the ring R, and let

$$0 \longrightarrow L \longrightarrow F \xrightarrow{\pi} I \longrightarrow 0$$

be an exact sequence of right R-modules, where F is a finitely generated free module. Since the R-module A is flat, the sequence

$$0 \longrightarrow T(L) \longrightarrow T(F) \xrightarrow{T(\pi)} T(I) \longrightarrow 0$$

is exact. Since the R-module A is flat, $T(I) \cong IA$; therefore, the group $T(I)$ is torsion-free. Consequently, the subgroup $T(L)$ is pure in $T(F)$. We also note that $T(L) = S_A(T(L))$ and $T(F) \cong A^n$ for some positive integer n. By (3), the above sequence is split. Therefore, $HT(\pi)$ is also split (i.e., there exists $\varepsilon : HT(I) \to HT(F)$ such that $HT(\pi)\varepsilon = 1$). In the relation $HT(\pi)\phi_F = \phi_I\pi$, the mapping ϕ_F is an isomorphism by Theorem 32.1 and ϕ_I is a monomorphism, since I is an ideal of the ring R. We set $\eta = \phi_F^{-1}\varepsilon\phi_I : I \to F$. We have

$$\phi_I\pi\eta = \phi_I\pi\phi_F^{-1}\varepsilon\phi_I = HT(\pi)\varepsilon\phi_I = \phi_I.$$

Therefore, $\pi\eta = 1$, since ϕ_I is a monomorphism. Therefore, π is split and I is a projective ideal. This means that R is right semihereditary. By Theorem 37.1, R is right hereditary.

(1) $\Longrightarrow$ (4). The ring R is a hereditary Noetherian semiprime ring by Theorem 37.1. Let B be an A-generated subgroup of finite rank of some locally A-projective group G. In this case, B is contained in some A-projective direct summand of the group G. By (2), the group B is A-projective. There exist a group C and cardinal number $\mathfrak{N}$ such that $A^{\mathfrak{N}} = B \oplus C$. It is clear that the group B is contained in a finite sum of groups A. Therefore, $A^n = B \oplus D$ for some positive integer n and a group D. By Lemma 35.17, B has a finite A-rank. By Theorem 36.18, B_* is a direct summand of the group G.

(4) $\Longrightarrow$ (1). By Corollary 36.20, the ring R is right semihereditary; it is right hereditary by Theorem 37.1. We have proved the equivalence of conditions (1)–(4).

(4) $\Longrightarrow$ (5). The semiprimeness of the ring R was shown above. Let G be a finitely A-projective group, and let a be a nonzero element of some minimal pfi-subgroup of the group G. We recall that $V = G \otimes \mathbf{Q}$, $S = E(G) \otimes \mathbf{Q}$, and $K = \operatorname{End}_S V$. Then aK is an irreducible submodule of the K-module V. We set $B = aK \cap G$. By Corollary 7.5, the group B is strongly indecomposable and $mG \subseteq B \oplus C \subseteq G$ for some positive integer m and a group C. Let $e : G \to B$ be a quasi-projection, i.e., e is an endomorphism of the group G such that $e_{|B} = m1$ and $eC = 0$ (see Sec. 5). Since $\langle eG \rangle_* = B$, B is a direct summand of the group G by (4). Thus, B is a strongly indecomposable direct summand of the group G and $a \in B$.

(5) $\Longrightarrow$ (1). Let n be a positive integer, $G = A^n$, and B be an A-generated subgroup of the group G. By Lemma 37.4, $BK \cap G = B_*$. Therefore, we can choose an element $b \neq 0$ contained in the intersection of B_* with some minimal pfi-subgroup of the group G. By assumption, there exists a decomposition $G = B_1 \oplus G_1$ such that the group B_1 is strongly indecomposable and $b \in B_1$. Since the group B_1 is strongly indecomposable, the K-module B_1K is irreducible (Corollary 7.5). Therefore, $B_1K \subseteq BK$, since $0 \neq b \in B_1K \cap BK$. Since $B_1 = B_1K \cap G$, we have $B_1 \subseteq BK \cap G = B_*$. Therefore, $B_* = B_1 \oplus (G_1 \cap B_*)$. Repeating a similar argument for the group $G_1 \cap B_*$, we obtain $G = B_1 \oplus B_2 \oplus G_2$ and $B_* = B_1 \oplus B_2 \oplus (G_2 \cap B_*)$. Since the rank of the group B_* is finite, we eventually obtain

$$G = \sum_{i=1}^{k}{}^{\oplus} B_i \oplus G_k \qquad \text{and} \qquad B_* = \sum_{i=1}^{k}{}^{\oplus} B_i$$

(i.e., B_* is a direct summand of the group G).

Every finitely A-projective group is a direct summand of the group A^n for some n. Therefore, the group A satisfies condition (3) of Theorem 36.18. By Corollary 36.20, the ring R is right semihereditary. Therefore, R is right hereditary. $\square$

The equivalence of conditions (1) and (2) of Theorem 37.5 is proved by Arnold and Lady [35].

The content of Theorem 37.5 can be presented as follows. In the class of all reduced groups of finite rank, groups with hereditary endomorphism rings are characterized by the circumstance that they have analogs of properties of groups of rank 1 exhibited in the main theorems on homogeneous completely decomposable and separable groups. According to this circumstance and the property that many known groups have hereditary endomorphism rings, the problem of the description of groups with hereditary endomorphism rings is important. The next section is partially devoted to this problem.

We now present some characterizations of groups of finite rank with hereditary endomorphism rings (these characterizations can be called inner).

Theorem 37.6. *For a reduced torsion-free group A of finite rank with semiprime ring R, the following conditions are equivalent:*

1. *R is right or left hereditary;*
2. *every submodule of the R-module A is flat;*
3. *the $R \otimes F_p$-module $B \otimes F_p$ is projective for every R-submodule B of A and each prime integer p (F_p is the field consisting of p elements).*

Proof. By Theorem 7.3, the ring $R \otimes \mathbf{Q}$ is classically semisimple. Therefore, the equivalence of (2) and (3) follows from Proposition 13.2.

(1) $\Longrightarrow$ (2). By Theorem 37.1, R is a hereditary Noetherian semiprime ring. The first paragraph of the proof of Theorem 37.5 shows that regular elements of the ring R are monomorphisms of the group A. Therefore, the group A and each of its R-submodules are torsion-free modules in the sense of Levy. Such modules are flat (see the beginning of the section).

(2) $\Longrightarrow$ (1). We fix a nonzero element $a \in A$. We consider an exact sequence of left R-modules $0 \to L_1 \to R \to Ra \to 0$, where $L_1 = \{\alpha \in R \mid \alpha a = 0\}$. By (2), the cyclic R-module Ra is flat. Therefore, Ra is projective, since the ring R is Noetherian. Therefore, the sequence is split:

$$R = Rf_1 \oplus Re_1, \qquad \text{where} \quad f_1^2 = f_1, \quad e_1^2 = e_1 \qquad \text{and} \qquad Re_1 \cong Ra.$$

For the group A, we have $A = f_1A \oplus e_1A$. We choose an element $a_1 \in A$ such that $f_1a_1 \neq 0$. We have the exact row

$$0 \to L_2 \to Rf_1 \to Rf_1a_1 \to 0,$$

where $L_2 = \{\alpha \in Rf_1 \mid \alpha a_1 = 0\}$. Further, we similarly obtain

$$Rf_1 = Rf_2 \oplus Re_2 \qquad \text{with} \quad f_2^2 = f_2, \quad e_2^2 = e_2, \quad Re_2 \cong Rf_1a_1, \qquad \text{and} \qquad A = f_2A \oplus e_2A \oplus e_1A.$$

If $Rf_2 \neq 0$ or, equivalently, $f_2A \neq 0$, then we repeat the argument. Since A has a finite rank, we eventually obtain

$$R = Re_1 \oplus \cdots \oplus Re_n,$$

and all the ideals Re_i are isomorphic to some cyclic submodules of the R-module A. Since R is Noetherian, condition (2) implies that all submodules of the modules Re_i are projective. Therefore, all submodules of the module ${}_RR$ are also projective (see Lambek [279, § 4.1, Proposition 7]) (i.e., the ring R is left hereditary). □

Theorem 37.6 has an interesting application in the next section.

Exercise 1. Prove that the endomorphism ring of the group $\mathbf{Z} \oplus \mathbf{Q}$ is left hereditary and is not right hereditary.

Exercise 2. Characterize completely decomposable torsion-free groups of finite rank with right (left) hereditary endomorphism rings.

Exercise 3 (Hausen [214]). For a reduced indecomposable torsion-free group A of finite rank, prove that the following conditions are equivalent:

(1) for every positive integer n, the group $\sum_n^{\oplus} A$ has the summand intersection property;

(2) for every cardinal number $\mathfrak{N}$, the group $\sum_{\mathfrak{N}}^{\oplus} A$ has the summand intersection property;

(3) the ring $E(A)$ is right hereditary.

(The information on the summand intersection property is given in Exercises 12–15 from Sec. 35).

Exercise 4. Let A be a torsion-free group of finite rank. An endomorphism α of the group A is a regular element of the ring $E(A)$ if and only if α is a monomorphism. Using this property, prove that A is a torsion-free R-module in the sense of Levy.

Exercise 5. Let A be a torsion-free group of finite rank. Prove that the quasi-endomorphism ring $E(A) \otimes \mathbf{Q}$ is the two-sided classical ring of fractions of the ring $E(A)$.

Exercise 6. Reduce the study of torsion-free groups of finite rank with right (left) hereditary endomorphism rings to the case of reduced groups.

Exercise 7. Let C be a hereditary torsion-free domain of finite rank that is not a division ring, F be a finitely generated projective right C-module, and B be a torsion-free group of finite rank such that $E(B) \cong C$. Prove that $F \otimes_C B$ is a torsion-free group of finite rank with hereditary endomorphism ring.

38. Maximal Orders as Endomorphism Rings

The representation of a torsion-free group A of finite rank with hereditary endomorphism ring obtained in Theorem 37.1 is not satisfactory, since the quasi-isomorphism condition for indecomposable summands B_j is not sufficient in Theorem 37.1(2). If $B_1, \ldots, B_k$ are pairwise quasi-isomorphic torsion-free groups of finite rank and $E(B_1), \ldots, E(B_k)$ are hereditary domains, then the group $A = B_1 \oplus \cdots \oplus B_k$ does not necessarily have a hereditary endomorphism ring. Therefore, it is desirable to specialize relations between the groups B_j. An important subclass of hereditary Noetherian prime rings is formed by the so-called Dedekind (noncommutative) rings. In the present section, we describe torsion-free groups A of finite rank, whose endomorphism rings are Dedekind rings. For such groups, the summands B_j should be almost isomorphic. This condition of almost isomorphism also implies that $E(A)$ is a Dedekind ring. Dedekind torsion-free rings of finite rank are bounded in the sense that every one-sided essential ideal of such a ring contains a two-sided essential ideal. Many properties of modules over bounded Dedekind rings are very close to the properties of Abelian groups, and bounded Dedekind rings are close to ordinary commutative Dedekind domains. It is interesting that Dedekind rings coincide with maximal orders in simple finite-dimensional rational algebras in our case. Therefore, we obtain the possibility of using some known results related to maximal orders.

The trace ideal of the module and a generator are defined in Sec. 12. We now recall some well-known definitions. First, an ideal I of a ring is called an idempotent ideal if $I^2 = I$.

A hereditary Noetherian prime ring without proper idempotent ideals is called a (not necessarily commutative) Dedekind ring.

We present some properties of Dedekind rings and modules over such rings. The next proposition contains two standard properties of Dedekind rings.

Proposition 38.1. *Let R be a Dedekind ring. Then*:

(1) *the matrix ring R_n is a Dedekind ring for every positive integer n*;
(2) *the ring eRe is a Dedekind ring for every idempotent $e \in R$.*

Proposition 38.2. *For a hereditary Noetherian prime ring R, the following conditions are equivalent*:

(1) *R is a Dedekind ring*;
(2) *every* (*finitely generated*) *projective right* (*left*) *R-module is a generator.*

Proof. (1) $\Longrightarrow$ (2). We take a projective right or left R-module P and denote by T its trace ideal. Then $T^2 = T \neq 0$ (Faith [141, Proposition 3.30]). Therefore, $T = R$ and the R-module P is a generator.

(2) $\Longrightarrow$ (1). Assume that I is a nonzero idempotent ideal of the ring R. Then I is a generator as a right (or left) R-module. Therefore, $I^n = R \oplus M$ for some positive integer n and an R-module M. Therefore,

$$I^n I = I^n = RI \oplus MI = I \oplus MI = R \oplus M,$$

whence $I = R$. Therefore, R has no proper idempotent ideals and R is a Dedekind ring. $\square$

We consider a property of modules over Dedekind rings that is well known in the case of commutative Dedekind rings.

Proposition 38.3. *If F is a finitely generated right module over a Dedekind domain C, then $F \cong C^{n-1} \oplus I$, where n is a positive integer and I is a right ideal of the ring C.*

Proof. The module F is isomorphic to a finite direct sum of right ideals of the ring C. It is clear that it is sufficient to prove the following property. Let K be a nonzero right ideal of the ring C, and let L be a finitely generated projective C-module. Then $K \oplus L \cong C \oplus P$ for some finitely generated projective C-module P. If $K = C$, then the assertion is true. Assume that $K \neq C$. Since the C-module L is a generator (Proposition 38.2), there exists a nonzero homomorphism $\varphi : L \to C/K$. Since the module L is projective, there exists a homomorphism $\psi : L \to C$ such that $\pi\psi = \varphi$, where $\pi : C \to C/K$ is the canonical homomorphism. It is clear that $\operatorname{im}\psi \not\subseteq K$. Setting $\psi_{|K} = 1_K$, we assume that ψ is a homomorphism from $K \oplus L$ in C. Since $\operatorname{im}\psi$ is a projective right ideal, $K \oplus L \cong \operatorname{im}\psi \oplus \ker\psi$. Here, $\operatorname{im}\psi$ properly contains the ideal K. Every nonzero right ideal of a Dedekind domain is essential (for essential submodules, see the beginning of Sec. 36). Therefore, $\psi L \cap K \neq 0$, whence $\ker\psi \neq 0$. We set $K_1 = \operatorname{im}\psi$ and $L_1 = \ker\psi$. We have $K \oplus L \cong K_1 \oplus L_1$, where K_1 is a right ideal of the ring C with $K \subset K_1$ and L_1 is a finitely generated projective C-module. If $K_1 \neq C$, then we can similarly obtain $K_1 \oplus L_1 \cong K_2 \oplus L_2$, where K_2 is a right ideal of the ring C with $K_1 \subset K_2$ and L_2 is a finitely generated projective C-module. Since the ring R is Noetherian, $K_s = C$ for some subscript s and $K \oplus L \cong C \oplus P$, where $P = L_s$. $\square$

We now consider Dedekind torsion-free rings of finite rank.

Let S be a simple finite-dimensional $\mathbf{Q}$-algebra, and let R be a subring in S such that $R\mathbf{Q} = S$ (this means that S/R is a torsion group). We denote by F the center of the algebra S; F is a field of algebraical numbers. Let T be a subring of the field F; T is a commutative domain.

A ring R is called a T-order in S if R is an algebra over T that is a finitely generated T-module.

A T-order R in S is called a maximal T-order in S if $R = R'$ for every T-order R' in S such that $R \subseteq R'$.

Maximal orders are studied in Arnold's work [31, §§10,11].

We recall that by Proposition 4.1, a torsion-free ring R of finite rank is prime if and only if the finite-dimensional $\mathbf{Q}$-algebra $R \otimes \mathbf{Q}$ is simple. Let R be a prime torsion-free ring of finite rank. It follows from the Pierce theorem below that R is a T-order in $R \otimes \mathbf{Q}$ for some ring T. We can take the center of the ring R as the T.

Theorem 38.4. (Arnold [31, Theorem 9.9]). *Let R be a prime torsion-free ring of finite rank. Then the center C of the ring R is a domain and R is a finitely generated torsion-free C-module.*

We formulate a result on maximal orders, which is important for our purposes.

Theorem 38.5. (Arnold [31, Theorem 11.3, Corollary 11.5]). *For a torsion-free ring R of finite rank, the following conditions are equivalent*:

(1) *R is a maximal T-order in $R \otimes \mathbf{Q}$*;

(2) *R is right and left hereditary and every ideal of the ring R is a right* (*left*) *generator as an R-module.*

The next proposition states that Dedekind torsion-free rings of finite rank coincide with the well-known classical objects.

Proposition 38.6. *A prime torsion-free ring R of finite rank is a Dedekind ring if and only if R is a maximal T-order in $R \otimes \mathbf{Q}$ for some ring T.*

Proof. Let R be a Dedekind ring. By the remark before Theorem 38.4, R is a C-order in $R\otimes\mathbf{Q}$, where C is the center of the ring R. By Proposition 38.2, all projective right (and left) R-modules are generators. By Theorem 38.5, R is a maximal C-order.

Conversely, assume that R is a maximal T-order for some T. Then R is a hereditary ring by Theorem 38.5. Therefore, R is a hereditary Noetherian prime ring (the ring R is Noetherian by Corollary 4.6). By Theorem 38.5, all nonzero ideals of the ring R are generators as right (and left) R-modules. The proof of the implication (2) $\Longrightarrow$ (1) of Proposition 38.2 shows that R has no nonzero proper idempotent ideals. Thus, R is a Dedekind ring. □

Only in this section do we need a remarkable notion which refers to torsion-free groups of finite rank and is an intermediate notion between the notions of isomorphism and quasi-isomorphism. We deal with an almost isomorphism. Almost isomorphisms were circumstantially studied by Arnold in [31]. We present only the definition and two necessary results.

Two torsion-free groups A and B of finite rank are said to be almost isomorphic if for every positive integer n, there exists a positive integer m such that m is coprime to n and there exist $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(B, A)$ with $gf = m1_A$ and $fg = m1_B$. This definition is equivalent to the following definition. Two groups A and B are said to be almost isomorphic if for every positive integer n, the group A is isomorphic to a subgroup of finite index of the group B and the index is coprime to n.

Similar to quasi-isomorphisms, almost isomorphisms can be represented as isomorphisms in some category (see Proposition 5.20 and Arnold [31, § 7]).

Proposition 38.7. (Arnold [31, Theorem 13.9, Corollary 12.8]). *Let A and B be two torsion-free groups of finite rank. Then*:

(1) *the groups A and B are almost isomorphic if and only if there exists a positive integer k with $A^k \cong B^k$*;

(2) *if the factor ring $E(A)/N(E(A))$ is a product of maximal orders, then B is almost isomorphic to A if and only if B is quasi-isomorphic to A and B is a finitely A-projective group.*

We now can prove our principal result on groups whose endomorphism rings are Dedekind rings. In the remaining part of the section, the endomorphism ring of a group A is denoted by R.

Theorem 38.8. *For a reduced torsion-free group A of finite rank, the following conditions are equivalent*:

(1) *R is a Dedekind ring*;
(2) *$A \cong F \otimes_C B$, where F is a finitely generated projective module over a Dedekind domain C and B is a torsion-free group such that $E(B) \cong C$*;
(3) *$A \cong B^{n-1} \oplus B_0$ for some positive integer n, where B is a torsion-free group such that $E(B)$ is a Dedekind domain and the group B_0 is almost isomorphic to the group B.*

Proof. First, we note that it is assumed that $B^0 = 0$ in (3) for $n = 1$.

(1) $\Longrightarrow$ (2). Let B be an indecomposable direct summand of the group A, and let $e : A \to B$ be the projection. We set $C = E(B)$. By property (b) of Sec. 3, $E(B) = E(eA) \cong eRe$. Therefore, C is a Dedekind ring (Proposition 38.1). The ring C has no zero divisors by Theorem 37.1. Therefore, C is a Dedekind domain.

We have $(ReR)^2 = ReR$, whence $ReR = R$. Further, $RB = ReA = ReRA = RA = A$ and, consequently, $A = S_B(A)$. Let $(1 - e)A \doteq B_1 \oplus B_2 \oplus \dots \oplus B_{n-1}$ for some positive integer n, where all B_i are strongly indecomposable groups. We have $A \doteq B \oplus B_1 \oplus \dots \oplus B_{n-1}$ and all summands here are strongly indecomposable groups (the group B is strongly indecomposable, since $E(B)$ is a domain). By Corollary 7.4 and the Jonsson theorem (Theorem 5.5), we have $B \sim B_1 \sim \dots \sim B_{n-1}$. We assume that the group A is a subgroup of finite index of the group B^n. Since $A = S_B(A)$ and the ring $E(B)$ is hereditary by Theorem 37.5, the group A is B-projective. In this case, Theorem 32.1 implies that $A \cong \mathrm{Hom}(B, A) \otimes_C B$. Since $A \doteq B^n$, we have $\mathrm{Hom}(B, A) \doteq \mathrm{Hom}(B, B^n)$. Therefore, $\mathrm{Hom}(B, A)$ is a finitely generated torsion-free right C-module (we recall that $C = E(B)$). Such modules are projective. Setting $F = \mathrm{Hom}(B, A)$, we have $A \cong F \otimes_C B$, where F, C, and B satisfy the conditions of (2).

(2) $\Longrightarrow$ (3). By Proposition 38.3, $F \cong C^{n-1} \oplus I$, where n is a positive integer and I is a nonzero right ideal of the ring C. Now, we have

$$A \cong F \otimes_C B \cong (C^{n-1} \oplus I) \otimes_C B \cong (C^{n-1} \otimes_C B) \oplus (I \otimes_C B) \cong B^{n-1} \oplus IB.$$

By Theorem 37.5, the C-module B is flat; therefore, $I \otimes_C B \cong IB$. The ideal I has a finite index in C (consider that $C \otimes \mathbf{Q}$ is a division ring). In particular, $I \doteq C$. Therefore, $IB \doteq B$ and $I \otimes_C B \sim B$. Since I is a projective C-module, $I \otimes_C B$ is a B-projective group. By Proposition 38.7, the groups $I \otimes_C B$ and B are almost isomorphic. Setting $B_0 = IB$, we have $A \cong B^{n-1} \oplus B_0$, where $E(B)$ is a Dedekind domain and the groups B and B_0 are almost isomorphic.

(3) $\Longrightarrow$ (1). Since B_0 and B are almost isomorphic groups, the group B_0 is B-projective by Proposition 38.7. Consequently, the group A is B-projective (i.e., $A \oplus G = B^m$ for some positive integer m and some group G). Let $e : B^m \to A$ be the natural projection related to the decomposition $A \oplus G = B^m$. We set $C = E(B)$. Since C is a Dedekind ring, the ring $E(B^m)$, which is isomorphic to the matrix ring C_m, is also a Dedekind ring (Proposition 38.1). Further, we have

$$R = E(A) \cong eE(B^m)e \cong eC_me$$

and R is a Dedekind ring by Proposition 38.1. □

Huber and Warfield [226] have constructed an example of a hereditary torsion-free domain C of finite rank that is not a Dedekind domain. By the theorem of Corner (Theorem 29.3), there exists a torsion-free group of finite rank with endomorphism ring that is isomorphic to C. We obtain that Theorem 38.8 does not describe all torsion-free groups of finite rank with hereditary endomorphism rings.

Two rings R and S are said to be Morita equivalent if the category of right R-modules is equivalent to the category of right S-modules. By the known Morita theorem (Faith [141, Theorem 12.10]), two rings R and S are equivalent if and only if there exists an S-module P that is a finitely generated projective generator with $R \cong \mathrm{End}_S P$.

Corollary 38.9. *Let A be a reduced torsion-free group of finite rank with Dedekind endomorphism ring R. Then $R \cong \operatorname{End}_C F$, where F is a finitely generated projective right module over some Dedekind domain C. The rings R and C are Morita equivalent.*

Proof. The isomorphism $\operatorname{End}_C F \cong R$ defined by $\alpha \to \alpha \otimes 1$ ($\alpha \in \operatorname{End}_C F$) follows from the isomorphism $A \cong F \otimes_C B$ proved in Theorem 38.8 and the category equivalence from Theorem 32.1. The finitely generated projective C-module F is a generator (Proposition 38.2); therefore, the rings R and C are equivalent. □

Therefore, the structure of the groups A from Theorem 38.8 is actually determined by the structure of finitely generated projective modules over Dedekind domains. For example, this is true for properties of the groups A that can be expressed by endomorphisms.

We present some characterizations of torsion-free groups of finite rank whose endomorphism rings are Dedekind rings. Different studies often contain groups whose endomorphism rings are products of Dedekind rings. First, we consider this more general situation. The theorem below is related to Theorems 37.5 and 37.6.

Theorem 38.10. *For a reduced torsion-free group A of finite rank, the following conditions are equivalent*:

1. *R is a product of Dedekind rings*:
2. *the ring R is semiprime and every faithful finitely generated submodule R-module A is a projective generator*;
3. *A is a faithful endoflat group and the group A is almost isomorphic to each of its A-generated subgroups of finite index.*

Proof. (1) $\Longrightarrow$ (2). By the definition of a Dedekind ring, R is a hereditary Noetherian semiprime ring. The R-module A is torsion-free in the sense of Levy; therefore, all its submodules are torsion-free in the sense of Levy. A finitely generated torsion-free module over a hereditary Noetherian semiprime ring is projective (see Sec. 36, the paragraphs before Lemma 36.1). We have $R = R_1 \times \cdots \times R_k$, where R_i is a prime ring ($i = 1, \dots, k$). Then $A = A_1 \oplus \cdots \oplus A_k$ with $A_i = R_iA$. If M is a faithful finitely generated R-submodule in A, then $M = M_1 \oplus \cdots \oplus M_k$ with $M_i = M \cap A_i$, where M_i is a nonzero finitely generated R_i-submodule in A_i. By Proposition 38.2, the R_i-module M_i is a generator. Consequently, the R-module M is a generator.

(2) $\Longrightarrow$ (1). Using Theorem 7.3 on the structure of the torsion-free group A of finite rank with semiprime endomorphism ring, we can prove the following assertion. If V is a finitely generated submodule R-module A, then there exists a finitely generated submodule W such that $V \cap W = 0$ and $V \oplus W$ is a faithful finitely generated submodule. Therefore, every finitely generated submodule of the R-module A is projective. Similar to the argument used in the proof of the implication (2) $\Longrightarrow$ (1) of Theorem 37.6, we can verify that the ring R is hereditary. Thus, R is a hereditary Noetherian semiprime ring. We can assume that R is a prime ring. The application of Corollary 7.4 shows that every nonzero R-submodule in A is faithful. Assume that $I^2 = I$ for some ideal $I \neq 0$ of the ring R. We fix a nonzero element $a \in A$. The ideal I is a Noetherian left R-module; consequently, Ia is also a Noetherian left R-module. In this case, the R-module Ia is a generator. Therefore, $(Ia)^n = R \oplus M$, where n is a positive integer and M is some left R-module. Further, we have

$$I(Ia)^n = (Ia)^n = IR \oplus IM = I \oplus IM = R \oplus M.$$

Therefore, $I = R$ and R is a Dedekind ring.

(1) $\Longrightarrow$ (3). The ring R is hereditary. By Theorem 37.5, A is a faithful endoflat group. Let B be an A-generated subgroup of finite index of the group A. Then $B \sim A$; in addition, the group B is A-projective by Theorem 37.5. By Proposition 38.7, the groups A and B are almost isomorphic.

(3) $\Longrightarrow$ (1). Since the group A is faithful and endoflat, the categories $\mathcal{P}_A$ and $\mathcal{P}_R$ from Theorem 34.13 are equivalent. Let B be an A-generated subgroup of the group A of finite index. Then the groups A

and B are almost isomorphic. It follows from Proposition 38.7 that B is an A-projective group. Using the indicated equivalence of categories, we obtain that every right ideal of finite index of the ring R is projective. Using the proof of Proposition 36.10, we can prove that the ring R is semiprime. We take a nonzero right ideal I of the ring R and choose a right ideal L such that $I \cap L = 0$ and $I \oplus L$ is an essential right ideal of the ring R. Then $(I \oplus L) \otimes \mathbf{Q}$ is an essential right ideal of the Artinian semisimple ring $R \otimes \mathbf{Q}$. Therefore, $(I \oplus L) \otimes \mathbf{Q} = R \otimes \mathbf{Q}$. This implies $n1 \in I \oplus L$ for some positive integer n. Therefore, the ideal $I \oplus L$ has a finite index in R. Therefore, the ideals $I \oplus L$ and I are projective. We have obtained that R is a hereditary Noetherian semiprime ring. As above, we assume that the ring R is prime. Assume that $I^2 = I$ for some ideal $I \neq 0$ of the ring R. Since $R \otimes \mathbf{Q}$ is a simple ring, we can use an argument similar to the above argument to prove that the ideal I has a finite index in R. In this case, IA is a subgroup of finite index in A. Therefore, the groups IA and A are almost isomorphic. By Proposition 38.7, $(IA)^k \cong A^k$ for some k. Since

$$IA \cong I \otimes_R A,$$

we have $(I \otimes_R A)^k \cong A^k \cong (R \otimes_R A)^k$. By the above-mentioned equivalence of categories, $I \cong R$ and the R-module I is a generator. Repeating the argument used in Proposition 38.2, we obtain that $I = R$ and R is a Dedekind ring. ☐

The equivalence of (1) and (3) is proved by Faticoni [146].

Corollary 38.11. *For a reduced torsion-free group A of finite rank with prime ring R, the following conditions are equivalent*:

(1) *R is a Dedekind ring*;
(2) *every finitely generated submodule of the R-module A is a projective generator*;
(3) *A is a faithful group that is almost isomorphic to each of its A-generated subgroups of finite index.*

We give only some remarks on the proof. In the case of a prime ring R, every nonzero finitely generated submodule of the R-module A is faithful (see Corollary 7.4). The proof of the implication (3) $\Longrightarrow$ (1) is similar to the proof of the analogous implication of Theorem 38.10 (the only difference is that the use of Theorem 34.13 should be replaced by the use of Theorem 34.22). Such a replacement is possible, since $R \otimes \mathbf{Q}$ is a prime Artinian ring and $A \otimes \mathbf{Q}$ is a flat $R \otimes \mathbf{Q}$-module.

We restrict Theorem 38.10 to the case of principal ideal domains. It is useful to compare Theorem 38.12 with Corollary 35.22.

Theorem 38.12 (Faticoni [146])**.** *For a reduced torsion-free group A of finite rank, the following conditions are equivalent*:

(1) *R is a principal right ideal ring*;
(2) *A is a faithful endoflat group and the group A is isomorphic to each of its A-generated subgroups of finite index*;
(3) *R is a semiprime ring, A is a faithful group, and every A-generated subgroup of the group A is an endomorphic image of A.*

Proof. (1) $\Longrightarrow$ (2). Let I be a right ideal of finite index of the ring R. We have $I = \alpha R$, where $\alpha \in R$. Using arguments related to the finiteness of the rank of the ring R, it is easy to prove that the homomorphism of right R-modules $R \to \alpha R$, $\xi \to \alpha\xi$, is an isomorphism. Thus, $I \cong R$. We have obtained that all right ideals of finite index of the ring R are projective. It follows from the proof of Proposition 36.10 that the ring R is semiprime. All essential right ideals of the ring R have a finite index, and we obtain that R is a hereditary Noetherian semiprime ring (see the proof of the implication (3) $\Longrightarrow$ (1) of Theorem 38.10). The group A is a faithful endoflat group. Consequently, the categories $\mathcal{P}_A$ and $\mathcal{P}_R$ are equivalent categories (Theorem 34.13). Let G be an A-generated subgroup of finite index of the group A. Then $H(G)$ is a right ideal of finite index of the ring R. By the above, $H(G) \cong R = H(A)$. Taking into account the equivalence of categories $\mathcal{P}_A$ and $\mathcal{P}_R$, we obtain $G \cong A$.

(2) $\Longrightarrow$ (1). We use the proof of the implication (3) $\Longrightarrow$ (1) of Theorem 38.10. Let I be a nonzero right ideal of the ring R. Then $I \oplus L$ is an essential right ideal for some right ideal L. Therefore, the ideal $I \oplus L$ has finite index in R and the subgroup $(I \oplus L)A$ has a finite index in A. Therefore,

$$(I \oplus L) \otimes_R A \cong (I \oplus L)A \cong R \otimes_R A \qquad \text{and} \qquad I \oplus L \cong R,$$

whence I is a cyclic R-module, (i.e., I is a principal right ideal of the ring R).

(1) $\Longrightarrow$ (3). Taking into account the equivalence of (1) and (2) and the proof of the implication (1) $\Longrightarrow$ (2), it is sufficient to prove the assertion for A-generated subgroups. Let G be an A-generated subgroup of the group A. Then $H(G) = \alpha R$ for some $\alpha \in R$. Therefore,

$$G \cong H(G) \otimes_R A = \alpha R \otimes_R A \cong \alpha RA = \alpha A,$$

i.e., G is an endomorphic image of the group A. (Note: using Corollary 34.17 on lattice isomorphisms, we have $G = H(G)A = \alpha RA = \alpha A$.)

(3) $\Longrightarrow$ (1). We can apply the equivalence of categories from Theorem 34.22 to the group A. Let G be an A-generated subgroup of finite index of the group A. There exists an element $\alpha \in R$ such that $\alpha A = G$. Since $r(A) < \infty$, it is clear that α is a monomorphism and $G \cong A$. If I is a right ideal of finite index of the ring R, then $I \cong HU(I)$ by Theorem 34.22. Here $U(I) \cong IA$, where IA is an A-generated subgroup of finite index of the group A. Therefore, $IA \cong A$, whence $I \cong R$. Let I be a right ideal of the ring R. Taking a right ideal L such that $I \cap L = 0$ and $I \oplus L$ is an essential right ideal, we obtain that $I \oplus L \cong R$. Therefore, I is a principal right ideal of the ring R and R is a principal right ideal ring. $\square$

We consider the method for obtaining groups studied in this section from maximal orders in fields of algebraical numbers. (We recall that a finite extension of the field of rational numbers $\mathbf{Q}$ is called a field of algebraical numbers.) Let K be a field of algebraical numbers distinct from $\mathbf{Q}$, and let J be the ring of algebraical integers of this field. We set $C = \bigcap_{P \in \Pi} J_P$, where Π is some set of prime ideals of the ring J and J_P is the localization of the ring J with respect to Π (i.e., $J_P = \{x/y \mid x \in J, y \in J \setminus P\}$). We choose a torsion-free group B of finite rank such that $E(B) \cong C$. Let F be a finitely generated projective right C-module. In the theory of maximal orders in fields of algebraical numbers, it is well known that the ring C is a Dedekind (commutative) ring (Zariski and Samuel [450, Chapter V]). Since the module F is isomorphic to a direct summand of some free C-module, $\operatorname{End}_C F$ is a Dedekind ring by Proposition 38.1. We set $A = F \otimes_C B$. By Theorem 32.1, we have $E(A) \cong \operatorname{End}_C F$. Thus, the endomorphism ring of the torsion-free group A of finite rank is a (not necessarily commutative) Dedekind ring.

The final result of the section shows which additional information can be obtained for an endofinite group with hereditary endomorphism ring. Corollary 38.13 is related to the study of endogenerators given in Sec. 12. In this corollary, C denotes the center of the endomorphism ring R of the group A.

Corollary 38.13. *Let A be a reduced endofinite torsion-free group of finite rank. The following conditions are equivalent*:

(1) *R is right hereditary*;
(2) *the group A is an endogenerator and R is a product of Dedekind rings*;
(3) *R is semiprime and C is a hereditary ring*;
(4) *$A \cong P$, where P is a finitely generated projective module over some hereditary torsion-free E-ring of finite rank*;
(5) *C is a product of Dedekind E-rings and $A \cong J_1 \oplus \cdots \oplus J_n$, where $J_1, \ldots, J_n$ are some ideals of the ring C.*

Proof. (1) $\Longrightarrow$ (2). The group A is a faithful endoflat group by Theorem 37.5. Therefore, the endofinite endoflat group A is endoprojective by Proposition 12.1, and the faithful endoprojective group A is an endogenerator by Proposition 12.7. On the other hand, the group A is a finitely generated torsion-free module over the hereditary Noetherian semiprime ring R. Since $C = \operatorname{End}_R A$, we have that C is also a hereditary Noetherian semiprime ring (see Theorem 7.3 and Corollary 35.7). In this case, $C \otimes \mathbf{Q}$ is a product of fields and C is a product of Dedekind (commutative) rings. In turn, the group A is a finitely generated projective C-module (Proposition 12.7). Since $R = \operatorname{End}_C A$, we have that R is also a product of Dedekind rings (see Proposition 38.1).

(2) $\Longrightarrow$ (3). Since R is a product of Dedekind rings, R is semiprime. In proving (1) $\Longrightarrow$ (2), it is proved that the ring C is hereditary.

(3) $\Longrightarrow$ (4). We have the following property: A is an endofinite group and the ring R is semiprime. By Theorem 11.4, C is an E-ring and A is a finitely generated torsion-free C-module. As in the implication (1) $\Longrightarrow$ (2), we obtain that C is a hereditary ring and A is a finitely generated projective C-module.

(4) $\Longrightarrow$ (5). Let S be the hereditary E-ring mentioned in (4). We can assume that P is a faithful S-module; see the proof of Corollary 12.10. By this corollary, $C \cong S$ and the R-module A is a generator. The assertion on the group A follows from Theorem 12.9. The ring C is a product of hereditary Noetherian prime rings, and, consequently, C is a product of Dedekind E-rings (see the proof of the implication (1) $\Longrightarrow$ (2)).

(5) $\Longrightarrow$ (1). By Theorem 12.9, the group A is an endogenerator. By Proposition 12.7, A is a finitely generated projective C-module. Using $R = \operatorname{End}_C A$, we prove that the ring R is hereditary. □

In the next section, we consider Dedekind endomorphism rings in one special case.

Exercise 1. Prove Proposition 38.1.

Exercise 2. Prove Theorem 38.4 of Pierce.

Exercise 3 (Arnold [31])**.** For a prime torsion-free ring R of finite rank, prove that the following conditions are equivalent:

(1) R is a maximal T-order in $R \otimes \mathbf{Q}$;
(2) if I is a nonzero ideal of the ring R, then $II^{-1} = I^{-1}I = R$, where $I^{-1} = \{x \in R \otimes \mathbf{Q} \mid IxI \subseteq I\}$;
(3) if I is a proper ideal of the ring R, then I is uniquely represented as a product of prime ideals of the ring R; if P_1 and P_2 are prime ideals of the ring R, then $P_1P_2 = P_2P_1$;
(4) if I is a nonzero ideal of the ring R, then there exists an ideal J of the ring R such that $IJ = JI = nR$ for some positive integer n.

Exercise 4. Let A be a torsion-free group of finite rank with prime endomorphism ring, and let B be a fully characteristic subgroup of finite index of the group A. If $E(A)$ is a hereditary (Dedekind) ring, then $E(B)$ is a hereditary (Dedekind) ring.

Exercise 5. Using the example of Huber and Warfield of a hereditary domain C that is not a Dedekind ring (Huber–Warfield [226]), prove the existence of indecomposable torsion-free groups B and B_0 of finite rank such that $B_0 \sim B$, the group B_0 is B-projective, the endomorphism ring of the group $B \oplus B_0$ is hereditary, and the groups B_0 and B are not almost isomorphic.

39. p-Semisimple Groups

In this section, we encounter one interesting phenomenon. We will see that some superficially weak condition on the factor rings $E(A)/pE(A)$ guarantees the heredity of the ring $E(A)$ for the torsion-free group A of finite rank. Moreover, $E(A)$ is a Dedekind ring in the prime case. The mentioned condition can be easily verified. The corresponding groups (the so-called groups p-semisimple for every p) often appear in different questions. They form a very important class of groups with hereditary endomorphism rings.

In this section, A denotes a reduced torsion-free group of finite rank and R is the endomorphism ring of A. For a prime integer p, we fix the following notation: $A_p = A \otimes F_p$ and $R_p = R \otimes F_p$, where F_p is the field consisting of p elements. We identify A_p with A/pA and R_p with R/pR, respectively. We assume that the group A_p is a left R_p-module. We note that A_p is a faithful R_p-module (i.e., $x = 0$ for any $x \in R_p$ with $xA_p = 0$).

Since $\dim_{F_p} A/pA \leq r(A)$ and $\dim_{F_p} R/pR \leq (\dim_{F_p} A/pA)^2$ (see Sec. 2), the group A_p and the ring R_p are finite for every p. In connection with this property, the following definitions are natural.

A group A is said to be *p-semisimple* (resp., *p-simple*) if R_p is a semisimple (resp., simple) ring. The group A is said to be *p-irreducible* if A_p is an irreducible R_p-module.

The Jacobson radical of a ring S is denoted by $J(S)$, and S_n is the ring of $n \times n$-matrices over S.

Thus, the p-semisimplicity of the group A means that $J(R_p) = 0$. If A is a p-simple group, then the ring R_p has no proper nonzero ideals. Since the ring R_p is finite and the group A is p-semisimple (p-simple), R_p is an Artinian semisimple (Artinian prime) ring. The following assertions show that it is possible to give uniform definitions of p-semisimple, p-simple, or p-irreducible groups.

Lemma 39.1. (1) *A group A is p-semisimple if and only if the R_p-module A_p is completely reducible.*

(2) *A group A is p-simple if and only if A_p is a homogeneous completely reducible R_p-module.*

Proof. (1) Every module over an Artinian semisimple ring is completely reducible (i.e., it is a direct sum of irreducible modules). Assume that A_p is a completely reducible R_p-module and $x \in J(R_p)$. The radical $J(R_p)$ is equal to the intersection of annihilators of all irreducible R_p-modules. Therefore, $xA_p = 0$ implies $x = 0$ and $J(R_p) = 0$.

(2) Every module over an Artinian simple ring is homogeneous completely reducible (i.e., it is equal to a direct sum of pairwise isomorphic irreducible modules). Conversely, if A_p is a homogeneous completely reducible R_p-module, then R_p is an Artinian semisimple ring as in (1). Therefore, R_p is a product of simple rings. Since the R_p-module A_p is faithful, the ring R_p is simple. □

Proposition 39.2. (1) *Let A be a p-semisimple group. Then every direct summand of the group A and the group A^n for every positive integer n are p-semisimple groups.*

(2) *A reduced torsion-free group B of finite rank that is almost isomorphic to the p-semisimple group A is p-semisimple.*

(3) *Assertions* (1) *and* (2) *remain valid if we replace p-semisimple groups by p-simple or p-irreducible groups.*

Proof. (1) Assume that we have a direct summand G of the group A; let $e : A \to G$ be a projection. It follows from relations

$$E(G)/pE(G) = eRe/peRe \cong \bar{e}(R/pR)\bar{e} \quad (\bar{e} = e + pR)$$

that the ring $E(G)/pE(G)$ is semisimple. Now we have isomorphisms

$$E(A^n)/pE(A^n) \cong R_n/pR_n \cong (R/pR)_n$$

and obtain the required semisimplicity.

(2) By Proposition 38.7, $A^k \cong B^k$ for some positive integer k. It remains to cite (1).

(3) In the case of p-simple groups, the proof coincides with the proof in the case of p-semisimple groups.

Now assume that we have a direct summand G of a p-irreducible group A; let $e : A \to G$ be the projection. Since the R/pR-module A/pA is irreducible, the $\bar{e}(R/pR)\bar{e}$-module $\bar{e}(A/pA)$ is irreducible. We can naturally identify

$$eRe/peRe \quad \text{with} \quad \bar{e}(R/pR)\bar{e}, \qquad eA/peA \quad \text{with} \quad \bar{e}(A/pA).$$

Since $E(G) = eRe$ and $G = eA$, we obtain that G/pG is an irreducible $E(G)/pE(G)$-module, i.e., G is a p-irreducible group. □

Proposition 39.3 (Arnold [30])**.** (1) *The group A is p-simple for every p if and only if every ideal of finite index of the ring R is equal to nR for some positive integer n.*

(2) *The group A is p-irreducible for every p if and only if every fully characteristic subgroup of finite index of the group A is equal to nA for some positive integer n.*

Proof. (1) Let A be a group that is p-simple for every p, and let I be an ideal of finite index of the ring R. Let n be the least positive integer such that $nR \subseteq I$, and let p be some prime divisor of the integer n. Then $I + pR = R$ or $I + pR = pR$, since R/pR is a simple ring. In the first case, $(n/p)R \subseteq I$; this contradicts the minimality of n. In the second case, $(n/p)R \subseteq (1/p)I \subseteq R$; therefore, $(1/p)I = (n/p)R$ by induction on n. Therefore, $I = nR$.

Conversely, if I/pR is an ideal of the ring R/pR, then $I/pR = 0$ or $I/pR = R/pR$, whence the ring R/pR is simple.

The proof of (2) is similar to the proof of (1); we only replace the ring R and the ideal I by the group A and some fully characteristic subgroup G of it of finite index, respectively. $\square$

We pass to the presentation of the main results of the section. The structure of a group A that is p-semisimple for every p is described by the following two theorems. The first theorem reduces the study of the group A to the prime case and shows that R is a Dedekind ring in the prime case. Therefore, the application of Theorem 38.8 to the group A is possible; this is done in Theorem 39.5.

Theorem 39.4. *Let A be a reduced torsion-free group of finite rank, and let A be p-semisimple for every p. Then $A = \sum_{i=1}^{n} \oplus A_i$, where all the summands A_i are fully characteristic in A, all $E(A_i)$ are Dedekind rings, and the groups A_i are p-semisimple for every p.*

Proof. Let $N(R)$ be the prime radical of the ring R. For every p, we have

$$(N(R) + pR)/pR \subseteq J(R/pR) = 0.$$

Therefore, $N(R) \subseteq pR \cap N(R) = pN(R)$ and $pN(R) = N(R)$; this means the divisibility of the ideal $N(R)$ as a group. However, R is a reduced ring, since A is a reduced group. Therefore, $N(R) = 0$ and R is a semiprime ring.

All the rings R_p are Artinian semisimple. Therefore, condition (3) of Theorem 37.6 holds. By condition (1) of this theorem and Theorem 37.1, R is a hereditary Noetherian semiprime ring and the group A is equal to $\sum_{i=1}^{n} \oplus A_i$, where the summand A_i is fully characteristic in A and $E(A_i)$ is a prime ring for every $i = 1, \dots, n$. By Proposition 39.2, all the groups A_i are p-semisimple for every p.

It remains to prove that $E(A_i)$ are Dedekind rings ($i = 1, \dots, n$). For convenience, we omit the subscript i. Thus, let A be a group such that A is p-semisimple for every p and the ring R is prime. We recall that all nonzero ideals of the ring R have a finite index (use the simplicity of the ring $R \otimes \mathbf{Q}$; see also the proof of (3) $\Longrightarrow$ (1) of Theorem 38.10). Taking into account the proved properties, it is sufficient to prove that R has no proper nonzero idempotent ideals.

Assume the contrary. There exists a proper nonzero ideal I of the ring R such that $I^2 = I$. We choose a positive integer t such that $I \supseteq tR$. We consider the case where t is a product of different prime integers: $t = p_1 \cdot \ldots \cdot p_s$. Then

$$R/tR \cong R/p_1R \times \cdots \times R/p_sR$$

is an Artinian semisimple ring (this isomorphism is a consequence of the Chinese remainder theorem; see Faith [142, Theorem 18.30]). Let J/tR be an ideal of the ring R/tR that is a complement to I/tR:

$$R/tR = I/tR \oplus J/tR.$$

We prove that $JI = tR$. The inclusion $JI \subseteq tR$ is obvious. For the proof of the converse inclusion, we choose elements $a \in J$, $b \in I$, and $c \in R$ such that $1 = a + b + tc$. For any $r \in R$, we have $tr = a(tr) + b(tr) + (tc)(tr)$. Here $a \in J$ and $tr \in I$. Therefore, $a(tr) \in JI$. Further,

$$b(tr) \in tI = (tR)I \subseteq JI \qquad \text{and} \qquad (tc)(tr) \in (tR)(tR) \subseteq JI.$$

Therefore, $tr \in JI$ and $JI = tR$. We have

$$tI = tRI = JII = JI = tR \qquad \text{and} \qquad I = R;$$

this contradicts the choice of the ideal I.

We study the remaining case, where $I \supseteq nR$, $n \in \mathbf{N}$, $n = p_1^{k_1} \cdot \ldots \cdot p_s^{k_s}$, and $k_i \neq 1$ for at least one i. We assume that n is the minimum number with such a property. We denote $t = p_1 \cdot \ldots \cdot p_s$; let k be the largest element among the exponents $k_1, \ldots, k_s$. By the above, there exists an ideal J of the ring R such that $J(I + tR) = tR$. Using the relations

$$JI = J(I + tI) = J(I^2 + tRI) = J(I + tR)I = tRI = tI,$$

we verify that $JI = tI$. Further, since $I \supseteq nR \supseteq t^k R$, we have

$$(I + tR)^k = I + t^k R = I.$$

Now we have

$$J(I + tR)^k = J(I + tR)(I + tR)^{k-1} = tR(I + t^{k-1}R) = tI + t^k R.$$

On the other hand,

$$J(I + tR)^k = JI = tI.$$

Therefore,

$$tI + t^k R = tI,$$

whence $t^k R \subseteq tI$ and $t^{k-1}R \subseteq I$. We have

$$nR + t^{k-1}R \subseteq I.$$

However, $nR + t^{k-1}R = mR$, where $m = (n, t^{k-1})$. Therefore, $I \supseteq mR$. By the choice of our numbers, $m < n$; this is impossible, since the integer n is minimal. Consequently, such an n does not exist, the considered case is impossible, and we have proved that R has no proper idempotent ideals. Therefore, R is a Dedekind ring. $\square$

Theorem 39.5. *Let A be a reduced torsion-free group of finite rank with prime ring R. The following conditions are equivalent*:

(1) *A is p-semisimple for every p*;
(2) *$A \cong F \otimes_C B$, where F is a finitely generated projective module over the domain C such that the ring C/pC is semisimple for every p and B is a torsion-free group with $E(B) \cong C$*;
(3) *$A \cong B^{n-1} \oplus B_0$ for some positive integer n, where B is an indecomposable group that is p-semisimple for every p and the group B_0 is almost isomorphic to the group B.*

Proof. (1) $\Longrightarrow$ (2). By the preceding theorem, R is a Dedekind ring. Theorem 38.8 implies $A \cong F \otimes_C B$, where F is a finitely generated projective module over the domain C and $E(B) \cong C$. The group B is chosen as a direct summand of the group A. Therefore, the group B is p-semisimple for every p (Proposition 39.2). Therefore, C/pC is a semisimple ring for every p.

(2) $\Longrightarrow$ (3). By Theorem 39.4, C is a Dedekind ring. Therefore, (2) $\Longrightarrow$ (3) follows directly from the similar implication of Theorem 38.8.

(3) $\Longrightarrow$ (1). By Proposition 38.7, $B^k \cong B_0^k$ for some positive integer k. Consequently,

$$A^k \cong B^{k(n-1)} \oplus B_0^k \cong B^{k(n-1)} \oplus B^k = B^{kn}.$$

Proposition 39.2 guarantees the p-semisimplicity of the group A for every p. $\square$

We apply Theorems 39.4 and 39.5 to groups that are p-simple for every p. For a group G, we set $\Pi(G) = \{p \mid pG \neq G\}$.

Corollary 39.6 (Arnold [30]). (1) *Let A be a reduced torsion-free group of finite rank that is p-simple for every p. Then $A = \sum_{i=1}^{n} {}^{\oplus} A_i$, where $E(A_i)$ is a Dedekind ring, A_i is p-simple for every p ($i = 1, \ldots, n$), and $\Pi(A_i) \cap \Pi(A_j) = \varnothing$ for all i, j with $i \neq j$.*

(2) *Theorem 39.5 remains valid after the replacement of the p-semisimplicity by the p-simplicity in the theorem.*

Proof. (1) Since $\Pi(A_i) \cap \Pi(A_j) = \varnothing$ for all $i \neq j$, the summands A_i are fully characteristic. Let p be a prime integer such that $pA \neq A$. We have

$$R = E(A_1) \times \cdots \times E(A_n)$$

and

$$R/pR = E(A_1)/pE(A_1) \times \cdots \times E(A_n)/pE(A_n).$$

Since the ring R/pR is simple, only one of the factor rings standing to the right is nonzero. This argument shows that we obtain the required condition for the sets $\Pi(A_1), \ldots, \Pi(A_n)$.

(2) See the proof of Theorem 39.5. □

We consider the following question. How can we obtain all torsion-free groups of finite rank that are p-semisimple (p-simple) for every p?

Corollary 39.7. *Let C be a torsion-free domain of finite rank such that C is not a division ring and all the rings C/pC are semisimple* (*simple*), *F be a finitely generated projective right C-module, and B be a torsion-free group of finite rank such that $E(B) \cong C$. Then the group $A = F \otimes_C B$ is p-semisimple* (*p-simple*) *for every p and its endomorphism ring is prime. Every torsion-free group of finite rank that is p-semisimple* (*p-simple*) *for every p and has prime endomorphism ring can be obtained in such a manner.*

Proof. The first assertion is actually shown in Theorem 39.5, and the second assertion also follows from this theorem. The group B indicated in the corollary always exists by Theorem 29.3. □

Let A be a torsion-free group of finite rank that is p-semisimple for every p. In this case, the ring $E(A)$ is a product of Dedekind rings by Theorem 39.4. In particular, $E(A)$ is a hereditary Noetherian semiprime ring and A is a group of generalized rank 1. It is fair to assert that the group A has many good properties. We present some of these properties.

Corollary 39.8. *Let A be a reduced torsion-free group of finite rank that is p-semisimple for every p. Then*

(1) *A is indecomposable if and only if A is strongly indecomposable* (*this follows from Theorem* 37.1);
(2) *for the group A, conditions* (1) *and* (2) *of Corollary* 37.3 *hold*;
(3) *the group A satisfies conditions* (2)–(5) *of Theorem* 37.5;
(4) *in addition, if the ring $E(A)$ is prime, then for the group A, there exists the isomorphism from Corollary* 38.9.

We continue the presentation begun in Sec. 38 of the method for obtaining groups with Dedekind endomorphism rings from maximal orders in fields of algebraical numbers. When does this method lead to groups that are p-semisimple for every p? For a prime integer p, the ring C/pC is semisimple if the relation

$$pJ = P_1^{e_1} \cdot \ldots \cdot P_n^{e_n} \quad (e_i \in \mathbf{N})$$

with different prime ideals P_i implies that $e_i = 1$ provided $P_i \in \Pi$ (the number e_i is called the ramification index of the ideal P_i). In this case, A is a p-semisimple group. It is known that there are only finitely many integers p such that at least one $e_i > 1$ (i.e., ramifying prime integers) (Zariski and Samuel [450, Chapter V]). It is clear that all the rings C/pC are semisimple if and only if Π contains no ideals with ramification index >1 (there are only finitely many such ideals). In this case, the group $A = F \otimes_C B$ is p-semisimple for every p (see Corollary 39.7). The ring C can be chosen to have a nonprincipal ideal I. Since the ideal I is projective, $C \oplus C \cong I \oplus P$ for some C-module P. Therefore,

$$(C \oplus C) \otimes_C B \cong (I \otimes_C B) \oplus (P \otimes_C B).$$

Since $I \ncong C$, we have $I \otimes_C B \ncong C \otimes_C B$ (Theorem 32.1). This gives an example of nonisomorphic direct decompositions of the p-semisimple for every p group $(C \oplus C) \otimes_C B$. In addition, we see that there exists a group A such that $E(A)$ is a Dedekind ring and A is not p-semisimple for some p.

The group A is p-simple for every p if and only if the ring C/pC is simple for every p (Corollary 39.7). The ring C/pC is simple if only one ideal P_i in the relation

$$pJ = P_1^{e_1} \cdot \ldots \cdot P_n^{e_n}$$

is contained in Π and $e_i = 1$ for this $P_i \in \Pi$. Therefore, it is clear that the class of groups that are p-semisimple for every p is larger than the class of all groups that are p-simple for every p.

Exercise 1. Which completely decomposable torsion-free groups of finite rank are p-semisimple, p-simple, and p-irreducible for every p?

Exercise 2. Let A be a reduced torsion-free group of finite rank such that $r_p(A) \leq 1$ for every p. Prove that the group A is p-irreducible for every p.

Exercise 3 (Arnold [30]). For a reduced torsion-free group of finite rank that is p-irreducible for every p, formulate and prove a result similar to Corollary 39.6.

The content of the remaining exercises is taken from the work of Arnold [30]. In these exercises, A is a reduced torsion-free group of finite rank and $R = E(A)$.

Exercise 4. Let A be a group such that A is p-simple for every p and the ring is prime, and let C be the center of the ring R. Then:

(a) C is a strongly homogeneous ring (strongly homogeneous rings are defined before Proposition 19.9);
(b) R is a maximal C-order in $C \otimes \mathbf{Q}$;
(c) R is a free C-module.

The group A is called an *S-group* if each of its subgroups of finite index is A-generated. The group A is called a *J-group* if A is isomorphic to every subgroup of finite index in A.

Exercise 5. For a group A, prove that the following conditions are equivalent:

(a) $r_p(R) = r_p(A)^2$ for every p;
(b) $R/pR \cong (F_p)_{m_p}$ for every p with $pA \neq A$, where $m_p = r_p(A)$;
(c) A is a faithful S-group;
(d) A is a faithful group and it is almost isomorphic to each of its subgroups of finite index;
(e) the ring R is semiprime and the group A is almost isomorphic to each of its subgroups of finite index.

Exercise 6. Let A be a faithful S-group. Then:

(a) A is p-irreducible for every p;
(b) if $A \sim B$, then B is a faithful S-group.

Exercise 7. (a) The group A is a faithful S-group if and only if $A = \sum_{i=1}^{n} {}^{\oplus} A_i$, where A_i is a faithful S-group with prime ring $E(A_i)$ and $\Pi(A_i) \cap \Pi(A_j) = \varnothing$ for all i and j with $i \neq j$.

(b) The group A is a faithful S-group with prime ring R if and only if $A \cong B^{n-1} \oplus B_0$ for some positive integer n, where B is an indecomposable faithful S-group and the group B_0 is almost isomorphic to the group B. In this case, $C/pC \cong F_p$ for every p with $pA \neq A$, where C is the center of the ring R.

Exercise 8. Prove that the following conditions are equivalent:

(a) A is a faithful J-group;
(b) A is a faithful S-group and every right ideal of finite index of the ring R is principal;
(c) A is a J-group and the ring R is semiprime.

Remarks

After publication of the work of Baer [47] in 1937, which began the general theory of torsion-free groups, there were many attempts to obtain structural theorems for some classes of Abelian groups besides the classes of completely decomposable and separable groups. The traditional proof of Theorems (1)–(3) formulated in the introduction to this part is strongly related to the well-known lattice of subgroups of the group of rational numbers $\mathbf{Q}$. It is natural to study the dependence of these results on the use of subgroups of the group $\mathbf{Q}$. The first results in this direction were obtained by Murley [334], Arnold and Lady [35], and Arnold and Murley [36]. Instead of a group A of rank 1, they took a torsion-free group A of finite rank with right hereditary endomorphism ring; they proved that Theorems (1)–(3) remain valid (in the case of Theorem (3), it is assumed that the endomorphism ring of the group A is a principal ideal domain).

The ideas and methods of the works of Arnold, Lady, and Murley were used in subsequent works of different authors. The transition to groups of infinite rank was made in the works of Albrecht [1, 3, 5], Krylov [256, 258, 260, 262, 263], and Faticoni [146, 147]. The works of Albrecht are the most complete and systematical.

In [35], Arnold and Lady defined the notion of a faithful group and proved the category equivalence from Theorem 32.1 for a group A of finite rank. (For a group A of rank 1, this was previously done by Warfield [442].) Albrecht [13] defined and studied A-soluble groups and A-closed classes (see also Albrecht [12]). Faticoni [146] explicitly considered correspondences between the lattice of A-generated subgroups of the group A and the lattice of right ideals of the ring $E(A)$.

It is easy to verify that the formulations and proofs of some general results do not use the specificity of the ring of integers and are module-theoretical. In particular, this is true for Theorems 32.1, 32.2, 34.13, and 34.14 on the equivalence of categories and Theorems 35.14 and 35.18 on groups with right semihereditary and right hereditary endomorphism rings. The definitions of an A-projective group, the categories $\mathcal{P}_A$ and $\mathcal{P}_R$, and so on admit obvious generalizations, and the above theorems can be extended to arbitrary modules with corresponding corrections. For example, there are interesting applications to endomorphism rings of projective modules (see Exercises 7–13 of Sec. 31 and Exercise 2 of Sec. 32 and also the work of Faticoni [149]).

Problem 31. Let B_i $(i \in I)$ be a family of self-small groups. Find necessary and sufficient conditions that guarantee the self-smallness of the product $\prod_{i\in I} B_i$ (Corollary 31.3).

Problem 32. Describe self-small groups in different classes of groups. In particular, describe self-small separable torsion-free groups and self-small vector groups.

Problem 33. Study groups that are small (self-small) modules over their endomorphism rings (see Exercises 7–13 of Sec. 31).

Problem 34. Characterize mixed self-small groups up to torsion-free factor groups (see Exercise 6 of Sec. 31). (We note that for a self-small mixed group A, its homomorphic image $A/T(A)$ is a self-small torsion-free group.)

Problem 35 (Arnold and Lady)**.** Describe in group terms faithful torsion-free groups of finite rank. Is a strongly indecomposable torsion-free group of finite rank faithful?

Problem 36. Describe faithful endoflat torsion-free groups of finite rank in terms of the groups.

For Problems 35 and 36, see the summary of results at the end of Sec. 34.

Problem 37. Find the structure of torsion-free groups of finite rank with right (left) hereditary endomorphism rings (see Theorem 37.1). For example, is it true that every reduced torsion-free group of finite rank with hereditary prime endomorphism ring is isomorphic to a group of the form $F \otimes_C B$, where F is a finitely generated projective right module over the hereditary domain C and B is a torsion-free group such that $E(B) \cong C$?

For a group with Dedekind endomorphism ring, the answer to the second part of the problem is positive (Theorem 38.8).

Problem 38. Reduce the study of arbitrary groups with right (left) hereditary endomorphism rings to the study of torsion-free groups with right (left) hereditary endomorphism rings (see Proposition 35.11(3)).

Problem 39. Characterize groups with right (left) semihereditary endomorphism rings among separable and vector torsion-free groups.

Part 7

COMPLETELY TRANSITIVE GROUPS AND ENDOMORPHISM RINGS

Completely transitive groups appeared in Secs. 22 and 25. Here we study them in particular. We focus our efforts on studying the structure of such groups. Completely transitive groups can be called groups with many endomorphisms; this warrants the consideration of this theme in this work.

The study of completely transitive groups is interesting from different points of view. The class of completely transitive groups contains all algebraically compact groups and all homogeneous separable groups that play a fundamental role in the theory of Abelian groups and other fields of mathematics. This class also contains quasi-pure injective and strongly homogeneous groups. On the other hand, many subclasses of completely transitive groups are quite large and consist of groups that were not considered before. Finally, completely transitive torsion-free groups admit a meaningful study in this case.

Objects close to completely transitive groups will be defined later in terms of automorphisms; these objects are transitive groups. Since the theory of transitive groups is similar to the theory of completely transitive groups, we consider only the case of completely transitive groups (however, some results on transitive groups are contained in the exercises).

All groups considered in this part are torsion-free. We use well-known terms and notation (in particular, a homogeneous group, an irreducible group, and an idempotent type). Let G be a torsion-free group, and let $a \in G$. Then $h_p(a)$ $(t(a))$ is the p-height [the type] of the element a in G, $\operatorname{Aut} G$ is the automorphism group of the group G, $\mathcal{E}(G)$ is the quasi-endomorphism ring of G, and $\mathcal{E}(G) = E(G) \otimes \mathbf{Q}$. Further, $\prod(G) = \{p \mid pG \neq G\}$ and $\langle M\rangle_*$ is a pure subgroup generated in G by the subset $M \subseteq G$. For a homogeneous group G, $t(G)$ denotes the type of the group G. We note that in this case $t(\langle a\rangle_*) = t(G)$ for every nonzero $a \in G$. A pfi-subgroup is a pure fully characteristic subgroup. In addition, $\mathcal{T}(G)$ denotes the set of types of all nonzero elements of the group G.

40. Homogeneous Completely Transitive Groups

We recall the definition of a completely transitive group. A torsion-free group G is said to be *completely transitive* if for any two nonzero elements a and $b \in G$ such that $\mathcal{X}(a) \leq \mathcal{X}(b)$, there exists an endomorphism $\alpha \in E(G)$ such that $\alpha a = b$.

Section 22 contains the following two examples of completely transitive torsion-free groups:

(a) homogeneous separable groups;
(b) algebraically compact groups. A similar argument shows that these groups are transitive in the sense of the following definition.

A torsion-free group G is said to be *transitive* if for any two nonzero elements a and b of G with $\chi(a) = \chi(b)$, there exists an automorphism $\alpha \in \operatorname{Aut}(G)$ such that $\alpha a = b$.

The proof of the following result is similar to the proofs of Lemmas 25.1 and 25.2(1).

Lemma 40.1. (1) *For a torsion-free group G, the following conditions are equivalent*:
(a) *G is a transitive group*;
(b) *every isomorphism between any two isomorphic pure subgroups of rank* 1 *of the group G is extended to an automorphism of the group G*;

(c) *for every pure subgroup A of rank 1 of the group G, all homomorphisms $A \to G$ preserving characteristics of elements can be extended to automorphisms of the group G (the preservation of characteristics means that $\chi(a) = \chi(\beta a)$ for every element $a \in A$ and a homomorphism $\beta : A \to G$).*

(2) *A homogeneous torsion-free group G is transitive if and only if for any two of its pure subgroups A and B of rank 1, there exists an automorphism $\alpha \in \mathrm{Aut}(G)$ such that $\alpha A = B$.*

Following specialists, we call homogeneous transitive groups strongly homogeneous groups. The original definition of strongly homogeneous groups is the following. A torsion-free group G is said to be *strongly homogeneous* if for any two of its pure subgroups A and B of rank 1, there exists an automorphism $\alpha \in \mathrm{Aut}(G)$ such that $\alpha A = B$. Therefore, the group G is strongly homogeneous if and only if the group $\mathrm{Aut}(G)$ acts transitively on the set of all pure subgroups of rank 1 of the group G.

Some relations between completely transitive and transitive groups will be presented later. These two notions are independent. We note that a homogeneous transitive (= strongly homogeneous) group is completely transitive by Lemma 25.2(1) and Lemma 40.1.

We now pass to the main theme of the section (i.e., homogeneous completely transitive groups). It follows from Lemma 25.2(2) that a completely transitive torsion-free group is homogeneous if and only if it is irreducible. The group G is irreducible if and only if the $E(G) \otimes \mathbf{Q}$-module $(G) \otimes \mathbf{Q}$ is irreducible. It is convenient to rewrite Proposition 25.3.

Proposition 40.2. *A homogeneous torsion-free group G is completely transitive if and only if $G \cong F \otimes A$, where F is a homogeneous completely transitive torsion-free group of idempotent type, A is a torsion-free group of rank 1, and for every prime integer p, the relation $pA = A$ implies $pF = F$. In this case, the mapping $\alpha \to \alpha \otimes 1$ ($\alpha \in E(F)$) defines a ring isomorphism $E(F) \to E(G)$.*

Considering the proof of Proposition 25.3, it is easy to see that for a strongly homogeneous group G, an analog of this proposition holds.

In the structural theory of completely transitive groups, strongly homogeneous rings play an important role. Every element of such a ring is an integral multiple of some invertible element of the ring. Subrings of the field $\mathbf{Q}$ and the ring of p-adic integers are the simplest examples of strongly homogeneous rings (also see Theorem 19.10, Corollary 19.11, Exercise 9 of Sec. 40, and Exercise 1 of Sec. 44). Proposition 19.9 states that one-sided ideals of the strongly homogeneous ring T coincide with ideals of the form mT for all integers m. Therefore, T is a principal left ideal domain and a principal right ideal domain. A commutative strongly homogeneous ring is a principal ideal domain. There is no conventional terminology related to strongly homogeneous rings. In Benabdallah's work [61], they are called special rings. We consider the term "special ring" in Sec. 44, where a special ring is a strongly homogeneous ring with one additional condition. Our choice of the term is substantiated by the following property. The additive group T^+ of the strongly homogeneous torsion-free ring T is a strongly homogeneous group. Indeed, let A and B be two pure subgroups in T^+ of rank 1. There exist invertible elements $u \in A$ and $v \in B$. The left multiplication of the ring T by the element $w = vu^{-1}$ is an automorphism of the group T^+ and $wA = B$. Consequently, T^+ is a strongly homogeneous group.

We take a strongly homogeneous torsion-free ring C. As usual, we identify every element $x \in C$ with the element $x \otimes 1 \in C \otimes \mathbf{Q}$ and assume that C is a subring in $C \otimes \mathbf{Q}$. We have $0 \neq x = nv$, where n is a nonzero integer and v is an invertible element of the ring C. The element $v^{-1} \otimes \frac{1}{n}$ is the inverse element for the element $x = v \otimes n$ in $C \otimes \mathbf{Q}$. We obtain that $C \otimes \mathbf{Q}$ is a division ring. For a torsion-free left C-module M in such a situation, we can define the C-rank $\mathrm{rank}_C M$ of the module M as the dimension of the $C \otimes \mathbf{Q}$-space $M \otimes \mathbf{Q}$.

A torsion-free C-module M is said to be $\aleph_1$-free if each of its submodules of finite or countable rank is free. For the commutative ring C, the definitions of C-rank and $\aleph_1$-free module were considered in Sec. 8 (see the paragraphs before or after Theorem 8.5). The following theorem states that a homogeneous completely transitive group has a good "local" structure.

Theorem 40.3. *Let G be a homogeneous completely transitive torsion-free group. Then the center C of the ring $E(G)$ is a strongly homogeneous ring and $G \cong F \otimes A$, where F is an $\aleph_1$-free C-module and A is a torsion-free group of rank 1 of type $t(G)$.*

Proof. The group G is irreducible by Lemma 25.2(2). Consequently, every nonzero endomorphism of the group G contained in C is a monomorphism (see the remark before Theorem 8.5). We prove that the ring C is strongly homogeneous.

Let B be a pure subgroup of rank 1 of the group G. Let $0 \neq \alpha \in C$. We set $E = \langle \alpha B \rangle_*$. Since G is a homogeneous group and α is a monomorphism, $\alpha B \cong B \cong E$. The rank of the group E is equal to 1; therefore, $\alpha B = nE$ for some positive integer n. We prove that $\alpha G = nG$. Let X be any pure subgroup of the group G of rank 1. We choose endomorphisms $\varphi, \psi \in E(G)$ such that $\varphi B = X$ and $\psi E = X$ (Lemma 25.2). We have

$$\alpha X = \alpha(\varphi B) = \varphi(\alpha B) = \varphi(nE) = n(\varphi E) \subseteq nG.$$

Therefore, $\alpha G \subseteq nG$, since the subgroup X is arbitrary. Further, we have

$$nX = n(\psi E) = \psi(nE) = \psi(\alpha B) = \alpha(\psi B) \subseteq \alpha G.$$

Therefore, $nG \subseteq \alpha G$, whence $\alpha G = nG$. We define an automorphism β of the group G as follows. If $a \in G$, then $\alpha a = nb$ for a unique element $b \in G$. We set $\beta a = b$. Since $\alpha G = nG$ and α is a monomorphism, it is clear that β is an automorphism of the group G and $\alpha = n\beta$. In addition, $\beta, \beta^{-1} \in C$, and we have proved that the ring C is strongly homogeneous.

Let F and A be the groups from Proposition 40.2. Then $G \cong F \otimes A$, $E(G) \cong E(F)$, F is a homogeneous completely transitive torsion-free group of idempotent type, and A is a torsion-free group of rank 1 of the type $t(G)$ ($t(A) = t(G)$ holds by the choice of the group A; see the proof of Proposition 25.3). By Lemma 25.2(2), F is an irreducible endocyclic group. Since the center of the ring $E(F)$ is canonically isomorphic to C, F is an $\aleph_1$-free C-module by Corollary 8.6. □

In the case of a strongly homogeneous group G, Theorem 40.3 was proved by Krylov [250]. Then Hausen and Krylov observed that the proof for strongly homogeneous groups is applicable for any homogeneous completely transitive group (Hausen [213] and Krylov [255]).

Theorem 40.3 has various corollaries and applications. Taking into account Corollary 8.6, we obtain the following result:

Corollary 40.4. *If G is a homogeneous completely transitive torsion-free group and $\mathrm{rank}_C\, G \leq \infty$, then $G \cong F \otimes A$, where F is a free C-module. Therefore, $G \cong \sum_{\mathfrak{N}}^{\oplus}(C \otimes A)$, where $\mathfrak{N} \leq \aleph_0$. In addition, if G has an idempotent type, then G is a free C-module.*

The succeeding series of corollaries is related to countable homogeneous completely transitive groups.

Corollary 40.5. *For a countable torsion-free group G, the following conditions are equivalent:*

1. *G is a strongly homogeneous group;*
2. *G is a homogeneous completely transitive group;*
3. *$G \cong F \otimes A$, where F is a finitely or countably generated free module over some countable strongly homogeneous torsion-free E-ring T, A is a group of rank 1, and if $pA = A$, then $pF = F$. In this case, the center of the ring $E(G)$ is canonically isomorphic to T;*
4. *$G \cong \sum_{\mathfrak{N}}^{\oplus} H$ $(\mathfrak{N} \leq \aleph_0)$, where H is an indecomposable strongly homogeneous group.*

Proof. The implication (1) $\Longrightarrow$ (2) is always true.

(2) $\Longrightarrow$ (3). The application of Corollary 40.4 implies $G \cong F \otimes A$, where F is a free module over the strongly homogeneous ring C and A is a group of rank 1. The assertion related to prime integers p follows from Proposition 40.2. Since F can be isomorphically embedded in G, we have $|F| = \aleph_0$. Therefore, F is a finitely or countably generated C-module. We fix a nonzero element $g \in G$. The mapping $C \to C_g$ such

that $x \to xg$ for all $x \in C$ is an isomorphism of C-modules (consider that all nonzero elements of C are monomorphisms of the group G). Therefore, C is a countable ring. By Proposition 40.2, $E(G) \cong E(F)$. Therefore, we can assume that the ring C is the center of the ring $E(F)$ and C is an E-ring by Theorem 8.7 applied to the group F. Therefore, the ring C can play the role of the ring T from (3).

(3) $\Longrightarrow$ (4). Since

$$F \cong \sum_{\mathfrak{N}}{}^{\oplus} T,$$

we have

$$G \cong \sum_{\mathfrak{N}}{}^{\oplus} (T \otimes A).$$

It is simpler to assume that $G = \sum_{\mathfrak{N}}^{\oplus} (T \otimes A)$. We set $H = T \otimes A$. The group T^+ is strongly homogeneous; therefore, the group H is strongly homogeneous (see the remark after Proposition 40.2). By this proposition, $E(H) \cong E(T^+) \cong T$. Therefore, $E(H)$ has no nontrivial idempotents and the group H is indecomposable. It remains to prove that the center C of the ring $E(G)$ is isomorphic to T. The center of the ring $E(G)$ is canonically isomorphic to the center of the ring $E(T \otimes A)$ (considering Theorem 14.3, we have that property (h) from Sec. 3 holds for every infinite number of summands A_i). By Proposition 40.2, $E(T \otimes A) \cong E(T^+) \cong T$ and $C \cong T$, since the ring T is commutative (property (e) from Sec. 6). Taking into account the isomorphisms $E(G) \cong E(T \otimes A) \cong E(T^+) \cong T$, we obtain that the center of the ring $E(G)$ consists of multiplications of the group $G = \sum_{\mathfrak{N}}^{\oplus} (T \otimes A)$ by elements of the ring T. This identification is the canonical isomorphism of (3).

(4) $\Longrightarrow$ (1). Let T be the center of the ring $E(H)$. Since the group H is countable and indecomposable, Theorem 40.3 implies $H \cong T \otimes A$, where A is a group of rank 1 and of type $t(G)$. Then $G \cong F \otimes A$, where F is a free module over the principal ideal domain T. Every element of F can be embedded in a direct summand that is isomorphic to T. Since the group T^+ is strongly homogeneous, it is clear that the group F is strongly homogeneous. Consequently, the group G is strongly homogeneous (i.e., we obtain (1)). □

Corollary 40.6. *For a countable indecomposable torsion-free group G, the following conditions are equivalent*:

(1) *G is a strongly homogeneous group*;
(2) *G is a homogeneous completely transitive group*;
(3) *$G \cong R \otimes A$, where R is a countable strongly homogeneous torsion-free E-ring, A is a group of rank* 1, *and $pR = R$ provided $pA = A$. In addition, $E(G) \cong R$.*

The representation of a countable homogeneous completely transitive group obtained in Corollaries 40.5 and 40.6 gives us quite complete information on such groups. It allows us to solve almost all problems related to them or to reduce the problems to some problems on countable strongly homogeneous E-rings. The following two results confirm this statement.

Corollary 40.7. *A countable homogeneous completely transitive torsion-free group G is equal to the direct sum of some of its indecomposable subgroups B_i $(i \in I, |I| \leq \aleph_0) : G = \sum_{i \in I}^{\oplus} B_i$, and $B_i \cong B_j$ and $E(B_i)$ is a strongly homogeneous ring for all $i, j \in I$.*

Corollary 40.8. (1) *Let G be a countable homogeneous completely transitive torsion-free group, and let $G \cong \sum_{\mathfrak{M}}^{\oplus} B$, where B is an indecomposable group. If H is a nonzero direct summand of the group G, then $H \cong \sum_{\mathfrak{N}}^{\oplus} B$, where $\mathfrak{N} \leq \mathfrak{M}$ and H is a strongly homogeneous group.*

(2) *If G and G_1 are countable homogeneous completely transitive torsion-free groups and C and C_1 are the centers of the rings $E(G)$ and $E(G_1)$, respectively, then $G \cong G_1$ if and only if $t(G) = t(G_1)$, $C \cong C_1$, and $\operatorname{rank}_C G = \operatorname{rank}_{C_1} G_1$.*

Proof. (1) It is easy to verify that the direct summand H of the completely transitive group G is a completely transitive group. In addition, the center of the ring $E(H)$ is isomorphic to C. By Corollary 40.5, $H \cong \sum\limits_{\mathfrak{N}}^{\oplus}(C \otimes A)$ with the group A of rank 1. In particular, $B \cong (C \otimes A)$ and $H \cong \sum\limits_{\mathfrak{N}}^{\oplus} B$.

(2) By Corollary 40.5,

$$G \cong \sum_{\mathfrak{N}}{}^{\oplus}(C \otimes A) \qquad \text{and} \qquad G_1 \cong \sum_{\mathfrak{N}_1}{}^{\oplus}(C_1 \otimes A_1),$$

where A and A_1 are groups of rank 1 of types $t(G)$ and $t(G_1)$. If $t(G) = t(G_1)$, then $A \cong A_1$ and $C \otimes A \cong C_1 \otimes A_1$. Since $\operatorname{rank}_C(C \otimes 1) = 1 = \operatorname{rank}_{C_1}(C_1 \otimes A_1)$, we have

$$\mathfrak{N} = \operatorname{rank}_C G = \operatorname{rank}_{C_1} G_1 = \mathfrak{N}_1 \qquad \text{and} \qquad G \cong G_1.$$

□

Our information on countable homogeneous completely transitive groups is sufficient for constructing concrete semirigid systems of groups and, consequently, for developing the Baer–Kulikov–Kaplansky theorem on direct summands of completely decomposable groups.

Assume that G and H are two countable indecomposable homogeneous completely transitive torsion-free groups such that $\operatorname{Hom}(G,H) \neq 0$ and $\operatorname{Hom}(H,G) \neq 0$. In such a situation, we can choose homomorphisms $\varphi : G \to H$ and $\psi : H \to G$ such that $\varphi\psi \neq 0$. Since the ring $E(H)$ is strongly homogeneous (see, for example, Corollary 40.6), we have $\varphi\psi = n\omega$, where n is a positive integer and $\omega \in \operatorname{Aut} H$. Therefore, it is clear that $\operatorname{im}\varphi \supseteq nH$, whence $S_G(H) \supseteq nH$. In the proof of Proposition 22.4(1), we have proved that every homogeneous completely transitive group is p-irreducible for every p (we use the terminology of Sec. 39). Since the trace $S_G(H)$ is a fully characteristic subgroup of the group H, the application of an obvious generalization of Proposition 39.3(2) implies $S_G(H) = mH$ for some positive integer m. However, $mH \cong H$ and $S_G(H) = H$. Similarly, we have $S_H(G) = G$. The relation $S_G(H) = H$ implies the existence of the epimorphism $f : G^{\mathfrak{M}} \to H$ for some cardinal number $\mathfrak{M}$ (Lemma 34.4). Since $E(H)$ is a principal ideal domain and $S_H(G^{\mathfrak{M}}) = G^{\mathfrak{M}}$, the epimorphism f is split by Corollary 33.4(2). Therefore, the group H is isomorphic to a direct summand of the group $G^{\mathfrak{M}}$. By Corollary 40.8, $H \cong G^{\mathfrak{N}}$, where $\mathfrak{N} \leq \mathfrak{M}$. Therefore, $H \cong G$, since the group H is indecomposable.

Assume that E is one more countable indecomposable homogeneous completely transitive torsion-free group, $\operatorname{Hom}(E,G) \neq 0$, and $\operatorname{Hom}(G,H) \neq 0$. Then it is clear that $\operatorname{Hom}(E,H) \neq 0$.

We take some set $\mathcal{P} = \{G_i \mid i \in I\}$ of pairwise nonisomorphic countable homogeneous indecomposable completely transitive groups. It follows from the above that we obtain an order on the subscript set I by $i \leq j \iff \operatorname{Hom}(G_i, G_j) \neq 0$ for any $i, j \in I$. The following result holds (cf. Proposition 32.11).

Corollary 40.9. (1) *Let $\mathcal{P}$ be a set of pairwise nonisomorphic countable homogeneous indecomposable completely transitive torsion-free groups. Then $\mathcal{P}$ is a semirigid system of groups.*

(2) *We denote by $\mathcal{P}_\Sigma$ the class of all groups that are direct sums of groups from the system $\mathcal{P}$. Then the assertions of Corollary* 32.6 *hold for every group from the class $\mathcal{P}_\Sigma$.*

We know that a homogeneous completely transitive torsion-free group G of idempotent type is irreducible and endofinite (Lemma 25.2) and the center C of the ring $E(G)$ is a strongly homogeneous ring. In particular, C is a principal ideal domain. Therefore, conditions (1), (2), (3′), and (4) of Theorem 8.7 are equivalent for the group G. If one of these conditions holds, then C is an E-ring and G is a strongly homogeneous group (see the proof of the implication (4) $\Longrightarrow$ (1) of Corollary 40.5 concerning the strong homogeneity). Assuming that the group G is strongly homogeneous, we can formulate a more meaningful result.

A module M is said to be locally free if every finite family of its elements can be embedded in a free direct summand of the module M (see the definition of a locally projective module given in Sec. 32).

Theorem 40.10. *Let G be a strongly homogeneous torsion-free group of idempotent type, and let C be the center of the ring $E(G)$. The following conditions are equivalent*:

(1) *G is a locally free C-module*;
(2) *G is an endoprojective group*;
(3) *the ring $\mathcal{E}(G)$ has a minimal left ideal*;
(4) *the group G has an endomorphic image of at most countable C-rank*;
(5) *there exist a strongly homogeneous torsion-free E-ring T and a locally free T-module M such that $G \cong M$. In this case, $C \cong T$.*

Proof. The implications (3) $\Longleftrightarrow$ (4) and (1) $\Longrightarrow$ (3) follow directly from Theorem 8.7.

(1) $\Longrightarrow$ (2). It is obvious that the locally free C-module G is a generator. Using the fact that $E(G) = \operatorname{End}_C G$ and the Morita theorem presented at the beginning of Sec. 12, we obtain that the group G is endoprojective.

(2) $\Longrightarrow$ (5). We have a finitely generated projective $E(G)$-module G and the relations $E(G) = \operatorname{End}_C G$ and $C = \operatorname{End}_{E(G)} G$. By the above-mentioned Morita theorem, the C-module G is a generator. Consequently, there exists a C-homomorphism $0 \neq \eta : G \to C$ and $\eta G = nC$ for some positive integer n. Since $nC \cong C$, we can assume that $\eta G = C$. The homomorphism η is split. Therefore, the C-module G has a decomposition $G = C \oplus H$ with some module H. We fix an element $0 \neq a \in C$. Let g be a nonzero element of G. We choose an automorphism $\alpha \in \operatorname{Aut} G$ such that $\alpha(\langle a \rangle_*) = \langle g \rangle_*$. We have a decomposition $G = \alpha C \oplus \alpha H$ of the C-module G, where $\alpha C \cong C$ and $g \in \alpha C$. We obtain that every element of the C-module G can be embedded in a direct summand that is isomorphic to C. Therefore, it is clear that G is a locally free C-module. Considering Theorem 8.7, we can take C as the ring T.

(5) $\Longrightarrow$ (1). Without loss of generality, we assume that $G = M$. It is sufficient to prove the isomorphism $C \cong T$. More precisely, we prove that this isomorphism is canonical. This means that the center of the ring $E(G)$ coincides with multiplications by elements of the ring T. The module M can be embedded in a product of copies of the ring T. Let π_i $(i \in I)$ be all projections of the module M on summands that are isomorphic to T. We define the mapping

$$f : M \to \prod_{i \in I} \pi_i M$$

as $f(x) = (\pi(x))$ for every $x \in M$. Here f is a monomorphism of T-modules and $\pi_i M \cong T$ $(i \in I)$. By Corollary 6.3 and properties (b) and (c) from Sec. 6, we can assert that M is an $E(T)$-module and $\operatorname{End}_{\mathbf{Z}} M = \operatorname{End}_T M$. Since the T-module M is a generator, the ring T is canonically isomorphic to $\operatorname{End}_R M$ with $R = \operatorname{End}_T M$. However, $\operatorname{End}_T M = \operatorname{End}_{\mathbf{Z}} M$. Therefore, $\operatorname{End}_R M$ is the center of the ring $E(M)$.

(3) $\Longrightarrow$ (1). Condition (3) implies that condition (3′) from Theorem 8.7 holds. Therefore, we can use the argument from (2) $\Longrightarrow$ (5). $\square$

Hausen [213] proved the equivalence of conditions (1) and (4).

Since all torsion-free groups of finite rank are countable, it is possible to apply Corollaries 40.5–40.9 to these groups. We present some results of Arnold [29]. These results practically do not require additional comments. Arnold proved Theorem 40.11 for strongly homogeneous groups. We modify his result.

Theorem 40.11. *Let G be a homogeneous torsion-free group of finite rank. The group G is completely transitive if and only if $G \cong T^n \otimes A$, where n is a positive integer, T is a strongly homogeneous torsion-free E-ring of finite rank and A is a group of rank 1, and if $pA = A$, then $pT = T$. In this case, the center of the ring $E(G)$ is canonically isomorphic to T and the group G is strongly homogeneous.*

Corollary 40.12. *An indecomposable homogeneous group G of finite rank is completely transitive if and only if $G \cong T \otimes A$, where T and A satisfy the conditions of Theorem* 40.11. *In this case, $E(G)$ and T are canonically isomorphic rings.*

We present the basic properties of homogeneous completely transitive groups of finite rank.

Corollary 40.13. *For a homogeneous completely transitive group G of finite rank, the following assertions hold*:

(1) *$G \cong H^n$, where H is an indecomposable completely transitive group*;
(2) *the group G is indecomposable if and only if G is strongly indecomposable*;
(3) *if B is a direct summand of the group G, then $B \cong H^k$, where $k \leq n$*;
(4) *if G_1 is another homogeneous completely transitive group of finite rank, then $G \sim G_1 \Longleftrightarrow G \cong G_1$*;
(5) *if $\mathcal{P}$ is some system of pairwise nonisomorphic homogeneous indecomposable completely transitive groups of finite rank, then $\mathcal{P}$ is a semirigid system and Corollary* 32.6 *holds for every group in $\mathcal{P}_\Sigma$.*

Proof. We only give some explanations of (4). Assume that $G \sim G_1$. As in (1), we have $G = H^n$ and $G_1 = H_1^m$, where n and m are positive integers and H and H_1 are indecomposable groups. Taking into account (2) and Jonsson theorem 5.5, we obtain that $n = m$ and $H \sim H_1$. Therefore, we can assume that the groups G and G_1 are indecomposable. It follows from $G \sim G_1$ that $\mathrm{Hom}(G, G_1) \neq 0$ and $\mathrm{Hom}(G_1, G) \neq 0$. Since the system $\mathcal{P}$ is semirigid, we obtain $G \cong G_1$. The converse assertion always holds. □

Arnold [29] showed how one can obtain all commutative strongly homogeneous torsion-free rings of finite rank (see Exercise 9). Beaumont and Pierce [54] established what rings T from Exercise 9 are E-rings. Therefore, we have a complete description of the rings T from Theorem 40.11, and it can be assumed that the classification of homogeneous completely transitive groups of finite rank is completed.

Thus, homogeneous completely transitive groups of finite rank coincide with strongly homogeneous groups of finite rank. Richman [376] called a group G special if G is a strongly homogeneous group of idempotent type and $G/pG = 0$ or $G/pG \cong Z(p)$ for all p (see Corollary 44.5 and Exercise 1 of Sec. 44). Strongly homogeneous groups of rank 2 were characterized by Arnold, Vinsonhaler, and Wickless [42]. Now special groups are called Richman special groups. Then Arnold [29] described strongly homogeneous groups of finite rank; Reid [373] and Krylov [250] studied strongly homogeneous groups of arbitrary rank.

Theorem 40.3, the corollaries given after Theorem 40.3, and Theorem 40.10 call forth some questions. First, is every homogeneous completely transitive group transitive? A locally free module over a strongly homogeneous torsion-free ring is a homogeneous transitive group. Is every homogeneous transitive or even completely transitive group a locally free module over a ring with such a property? We also consider the following two questions. Is it true that any indecomposable homogeneous transitive (completely transitive) group has the form $T \otimes A$ (see Corollary 40.6)? Is it true that the center of the endomorphism ring of a homogeneous transitive (completely transitive) group is always an E-ring? Accepting the constructibility axiom of Gödel, Dugas and Shelah [135] have proved a theorem that gives a negative answer to all these questions (this theorem can be called an existence theorem). Dugas and Hausen [130] have constructed an example of a homogeneous transitive group that is not a locally free module.

Exercise 1. Prove that a direct summand of a completely transitive torsion-free group is a completely transitive group.

Exercise 2. The complete transitivity of a torsion-free group is equivalent to the complete transitivity of its reduced part.

Exercise 3 (Hausen [213])**.** The torsion-free group G is said to be E-transitive if for any pure subgroups X and Y of the group G of rank 1, there exists $\alpha \in E(G)$ such that $\alpha X = Y$. Prove that E-transitive groups coincide with homogeneous completely transitive groups.

Exercise 4 (Grinshpon–Misyakov [196]). Every direct product of copies of a homogeneous completely transitive torsion-free group is a completely transitive group (regarding a direct sum see Exercise 12 of Sec. 44).

Exercise 5. Let A be a strongly homogeneous torsion-free group, and let G be an A-free group. Every element of the group G can be embedded in a direct summand of it that is isomorphic to A. In addition, the complement is an A-free group.

Exercise 6. (1) Every direct sum of copies of a strongly homogeneous group is a strongly homogeneous group.

(2) Let A be a strongly homogeneous group of idempotent type, and let G be a direct product of copies of the group A. Then every element of the group G can be embedded in a direct summand of it that is isomorphic to A. The group G is strongly homogeneous.

Exercise 7 (Reid [373]). The torsion-free group G is strongly homogeneous if and only if for any two one-dimensional subspaces X and Y of the $\mathbf{Q}$-space $G \otimes \mathbf{Q}$, there exists $\alpha \in \operatorname{Aut} G$ such that $\alpha X = Y$.

Exercise 8 (Reid [373]). Let G be a strongly homogeneous group. Then every $E(G)$-submodule of $G \otimes \mathbf{Q}$ is a strongly homogeneous group.

The notation used in Exercise 9 was defined in Secs. 38 and 39.

Exercise 9 (Arnold [29]). Let T be a commutative domain such that its field of fractions K is a field of algebraical numbers. Then T is a strongly homogeneous ring if and only if $T = \bigcap_{P \in \Pi} J_P$ and if p is a prime integer and $pJ = P_1^{e_1} \cdot \ldots \cdot P_n^{e_n}$ is a product of powers of different prime ideals in J, then at most one ideal P_i is contained in Π and $e_i = 1$ for this $P_i \in \Pi$.

41. Completely Transitive Groups Whose Quasi-Endomorphism Rings are Division Rings

This section and the subsequent two sections are devoted to arbitrary (inhomogeneous) completely transitive groups. Inhomogeneous groups G such that $\mathcal{E}(G)$ is a division ring are the simplest examples of such groups.

We somewhat reformulate the definition of a quasi-homogeneous group (see Proposition 35.12). A group G is said to be *quasi-homogeneous* if $\Pi(A) = \Pi(G)$ for every nonzero pure subgroup A of the group G.

Proposition 41.1. *Let G be a torsion-free group such that $\mathcal{E}(G)$ is a division ring. Then the following conditions are equivalent*:

(1) *the group* $\operatorname{Aut} G$ *acts transitively on the set of all pure subgroups of rank* 1 *of some minimal pfi-subgroup of the group G*;

(2) $E(G)$ *is a strongly homogeneous ring.*

In addition, if G is a quasi-homogeneous group, then we can add the following condition (3) *to conditions* (1)–(2).

(3) $\operatorname{Hom}(G, G)$ *acts transitively on the set of all pure subgroups of rank* 1 *of some minimal pfi-subgroup of the group G.*

Proof. (1) $\Longrightarrow$ (2). Let $0 \neq \gamma \in E(G)$. We choose some pure subgroup of rank 1 in the minimal pfi-subgroup mentioned in (1). We set $B = \langle \gamma A \rangle_*$. By (1), $\alpha A = B$, where $\alpha \in \operatorname{Aut} G$. We fix a nonzero element $a \in A$. Since elements γa and αa are contained in the subgroup B of rank 1, there exist nonzero integers m and n such that $m(\gamma a) = n(\alpha a)$ and $(m, n) = 1$. Since nonzero endomorphisms of the group G are monomorphisms and $(m\gamma - n\alpha)a = 0$, we have $m\gamma = n\alpha$. We have $\operatorname{im}(m\gamma) \subseteq mG$ and $\operatorname{im}(n\alpha) = nG$, whence $nG \subseteq mG$. Since $(m, n) = 1$, we have $G = mG + nG = mG$. Therefore, $m1$ is an automorphism of the group G. Therefore, $\gamma = n(\alpha/m)$, where $\alpha/m \in \operatorname{Aut} G$. We have obtained that γ is an integral multiple of an automorphism of the group G, which implies (2).

(2) $\Longrightarrow$ (1). Let H be a minimal pfi-subgroup of the group G, A and B be pure subgroups of rank 1 from H, $0 \neq a \in A$, and $0 \neq b \in B$. Then $H \otimes \mathbf{Q}$ is a minimal submodule $\mathcal{E}(G)$-module $G \otimes \mathbf{Q}$. Consequently, $\gamma a = b$, where $\gamma \in \mathcal{E}(G)$. We choose a positive integer m such that $\beta = m\gamma \in E(G)$. By assumption, $\beta = n\alpha$, where $n \in N$, and $\alpha \in \operatorname{Aut} G$. Since $(n\alpha)a = (m\gamma)a = mb \in B$, we have $\alpha a \in B$. Since α is an automorphism of the group G, we have $\alpha A = B$. We have proved that $\operatorname{Aut} G$ acts transitively on the set of all pure subgroups of rank 1 of the minimal pfi-subgroup H.

We assume that $\Pi(A) = \Pi(G)$ for every pure subgroup $A \neq 0$ of the group G. It is sufficient to prove (3) $\Longrightarrow$ (1). Let H be a minimal pfi-subgroup of the group G such that $\operatorname{Hom}(G, G)$ acts transitively on the set of all its pure subgroups of rank 1. By (3), $\alpha A = B$ for some $\alpha \in E(G)$. We prove that $\alpha \in \operatorname{Aut} G$. By (3), $\beta B = A$, where $\beta \in E(G)$. Then $(\beta\alpha)A = A$. We fix an element $0 \neq a \in A$. Since $(\beta\alpha)a \in A$, we have $n(\beta\alpha)a = ma$, where n and m are nonzero integers and $(n, m) = 1$. The restrictions $\alpha_{|A}$ and $\beta_{|B}$ are isomorphisms. Therefore, $\chi(a) = \chi((\beta\alpha)a)$. This relation and $n(\beta\alpha)a = ma$ imply that $nA = A = mA$. Since $\Pi(A) = \Pi(G)$, we have $nG = G = mG$. Therefore, the multiplication by n/m is an automorphism of the group G. Thus, $(n/m)\beta\alpha(a) = a$ and $(n/m)\beta\alpha = 1_G$ (i.e., α has a left inverse endomorphism). By symmetry, α has a right inverse endomorphism. Therefore, α is an automorphism of the group G and $\alpha A = B$. Therefore, $\operatorname{Aut} G$ acts transitively on the set of all pure subgroups of rank 1 of the minimal pfi-subgroup H. $\square$

In the following theorem, we characterize inhomogeneous completely transitive groups whose quasi-endomorphism rings are division rings. If T is a strongly homogeneous torsion-free ring, then $T \otimes \mathbf{Q}$ is a division ring (see the paragraphs before Theorem 40.3). The T-rank of some T-module M is the dimension of the $T \otimes \mathbf{Q}$-space $M \otimes \mathbf{Q}$.

Theorem 41.2. *Let G be a torsion-free group such that $\mathcal{E}(G)$ is a division ring. The following conditions are equivalent*:

(1) *G is an inhomogeneous completely transitive group*;
(2) *one of the conditions* (1)–(2) *of Proposition* 41.1 *holds, G is a quasi-homogeneous group, the types of different minimal pfi-subgroups of the group G are incomparable, and $|\mathcal{T}(G)| \geq \aleph_0$*;
(3) *there exists a strongly homogeneous ring T such that*
   (a) *G is a torsion-free left T-module*;
   (b) *if X and Y are different pure T-submodules in G of T-rank* 1*, then* $\operatorname{Hom}_T(X, Y) = 0$;
   (c) $\Pi(A) = \Pi(T)$ *for every pure subgroup $A \neq 0$ in G*;
   (d) $|\mathcal{T}(G)| \geq \aleph_0$. *In this case, $E(G) \cong T$ canonically. In addition, such a group G is transitive.*

Proof. (1) $\Longrightarrow$ (2). We take some minimal pfi-subgroup H of the group G. Let X and Y be two pure subgroups in H of rank 1. Since the group H is homogeneous, we can choose nonzero elements $x \in X$ and $y \in Y$ with equal characteristics. Since the group G is completely transitive, $\alpha x = y$, where $\alpha \in E(G)$. It is clear that $\alpha X \subseteq Y$ and since $\chi(x) = \chi(\alpha x)$, we have that α maps from X onto Y (i.e., $\alpha X = Y$). Proposition 41.1(3) holds.

Let A be a pure subgroup in G of rank 1. We prove that $\Pi(A) = \Pi(G)$. We have $\Pi(A) \subseteq \Pi(G)$. Assume that $pA = A$. Then A has the endomorphism $(1/p)1_A$. By (1), there exists $\theta \in E(G)$ such that $\theta_{|A} = (1/p)1_A$ or $p\theta_{|A} = 1_A$. Since nonzero endomorphisms of the group G are monomorphisms, $p\theta = 1_G$. Therefore, $G = p\theta G \subseteq pG \subseteq G$ and $G = pG$. We obtain $\Pi(G) \subseteq \Pi(A)$ and $\Pi(A) = \Pi(G)$.

If H_1 and H_2 are two different minimal pfi-subgroups of the group G and $t(H_1) \leq t(H_2)$, then there exist nonzero elements $a \in H_1$ and $b \in H_2$ such that $\chi(a) \leq \chi(b)$. Then $\alpha a = b$ for some $\alpha \in E(G)$; this contradicts the property that the subgroup H_1 is fully characteristic (we note that $H_1 \cap H_2 = 0$). Consequently, $t(H_1)$ and $t(H_2)$ are incomparable.

For the one-dimensional $\mathcal{E}(G)$-subspace W in $G \otimes \mathbf{Q}$, the intersection $W \cap G$ is a minimal pfi-subgroup of the group G. Since G is an inhomogeneous group, the preceding paragraph implies

$$\dim_{\mathcal{E}(G)}(G \otimes \mathbf{Q}) \geq 2.$$

Therefore, G has infinitely many minimal pfi-subgroups. Since their types are pairwise incomparable, $|\mathcal{T}(G)| \geq \aleph_0$.

(2) $\Longrightarrow$ (3). By Proposition 41.1, $E(G)$ is a strongly homogeneous ring. We set $T = E(G)$. Then (a) holds, since nonzero endomorphisms of the group G are monomorphisms. Let X be a pure T-submodule in G of T-rank 1. Then $X \otimes \mathbf{Q}$ is an $\mathcal{E}(G)$-subspace in $G \otimes \mathbf{Q}$ of dimension 1. Since $X = (X \otimes \mathbf{Q}) \cap G$, we have that X is a minimal pfi-subgroup in G. Therefore, all pure T-submodules of T-rank 1 in G are minimal pfi-subgroups of the group G. The converse assertion is also true. If Y is another pure T-submodule of T-rank 1 in G, then $\mathrm{Hom}_T(X, Y) = 0$, since the types of minimal pfi-subgroups are incomparable. Since $\Pi(G) = \Pi(E(G))$, we have

$$\Pi(A) = \Pi(G) = \Pi(T)$$

for every pure subgroup $A \neq 0$ in G.

(3) $\Longrightarrow$ (1). The T-module G is torsion-free; therefore, the action of a nonzero element of T on G is a nonzero endomorphism of the group G. Therefore, we obtain an embedding of the ring T in $E(G)$. Let A be a fixed pure subgroup in G of rank 1. We verify that the subgroup TA is pure in G. Since $TA \cong T \otimes A$ and A has rank 1, every element from TA has the form ta for some elements $t \in T$ and $a \in A$.

Now assume that $ng = ta$, where n is a positive integer and $g \in G$. We have $t = mv$, where m is a positive integer and v is an invertible element in T. Then $ng = mva \in vA$. The element v is invertible in $E(G)$. Consequently, the subgroup vA is pure in G and $g \in vA \subseteq TA$; therefore, TA is pure.

We prove that the subgroup TA is fully characteristic in G. Let $\alpha \in E(G)$, $t \in T$, $a \in A$, $\alpha(ta) \neq 0$, $B = \langle ta \rangle_*$, and let $C = \langle \alpha ta \rangle_*$. The homomorphism $\alpha : B \to C$ induces the T-homomorphism $1 \otimes \alpha : T \otimes B \to T \otimes C$. Using $1 \otimes \alpha$ and the canonical isomorphisms $TB \cong T \otimes B$ and $TC \cong T \otimes C$, we obtain a nonzero homomorphism of pure T-submodules $TB \to TC$. By assumption, $TC = TB \subseteq TA$. Therefore, $\alpha(TA) \subseteq TA$, which means that TA is fully characteristic in G.

We prove that the ring embedding $T \to E(G)$, which is discussed at the beginning of the proof of the implication, is an isomorphism. Let $\alpha \in E(G)$, and let the subgroup TA be as above. We fix a nonzero element $a \in A$ and identify TA with $T \otimes A$. Since $\Pi(A) = \Pi(T)$, Proposition 40.2 implies $\alpha(1 \otimes a) = t \otimes a$ for some element $t \in T$. The multiplication by t coincides with α on the group G. Indeed, $t(1 \otimes a) = t \otimes a$. Therefore, $\alpha(1 \otimes a) = t(1 \otimes a)$ and $\alpha = t$ (all nonzero endomorphisms of the group G are monomorphisms). Therefore, $E(G) = T$.

If X and Y are two different minimal pfi-subgroups and $t(X) \leq t(Y)$, then choosing in X and Y two pure subgroups of rank 1, A and B, respectively, we obtain, as in the preceding paragraph, a nonzero homomorphism $TA \to TB$ (consider that there exists a nonzero homomorphism $A \to B$, since $t(A) \leq t(B)$). By the above, TA and TB are pure T-submodules in G (i.e., pfi-subgroups of the group G). By assumption, $TA = TB$. However, $TA = X$, $TB = Y$. Therefore, $X = Y$. Consequently, different minimal pfi-subgroups have incomparable types.

We prove the complete transitivity of the group G. If a and b are two nonzero elements of G and $\chi(a) \leq \chi(b)$, then the elements a and b are contained in a common minimal pfi-subgroup H. We set $A = \langle a \rangle_*$ and $B = \langle b \rangle_*$. Since the ring $E(G)$ is strongly homogeneous, it follows from Proposition 41.1 that $\alpha A = B$, where $\alpha \in \mathrm{Aut}\, G$. If we choose nonzero integers n and m such that $(n, m) = 1$ and $n(\alpha a) = mb$, then $mY = Y$, since $\chi(\alpha a) = \chi(a) \leq \chi(b)$. Therefore $mG = G$ and the group G has the endomorphism $(1/m)1_G$. Then $b = (n\alpha/m)a$ and the group G is completely transitive. $\square$

The above theorem does not give the method for constructing groups considered in the theorem. Examples of groups from Theorem 41.2 are contained in the works of Arnold, O'Brien, and Reid [38] and Dobrusin [110].

Dobrusin [110] constructed an example of an inhomogeneous transitive group of finite rank that is not completely transitive. Its quasi-endomorphism ring is a division ring (also see Exercises 1 and 2).

Chekhlov [85] presented a transitive group G such that G is not completely transitive $r_p(G) \leq 1$ for every p.

Exercise 1. Let G be a torsion-free group such that $\mathcal{E}(G)$ is a division ring. Prove that the following conditions are equivalent:

(a) G is an inhomogeneous transitive group;
(b) G satisfies an analog of condition (2) of Theorem 41.2, where words "are incomparable" are replaced by "are not equal";
(c) G satisfies an analog of condition (3) of Theorem 41.2, where (b) is replaced by the following condition: if X and Y are different pure T-submodules in G of T-rank 1, then the types $t(X)$ and $t(Y)$ are not equal. In the case (c), $E(G) \cong T$.

Exercise 2. Prove that we can add the following condition (4) to conditions (1)–(3) of Theorem 41.2.

(4) G is a transitive group and the types of different minimal pfi-subgroups of the group G are incomparable.

Exercise 3. Let G and H be two quasi-isomorphic groups of finite rank whose endomorphism rings are strongly homogeneous. Prove the following assertions:

(a) the group H is isomorphic to some fully characteristic subgroup of finite index of the group G; the converse assertion is also true;
(b) $E(G) \cong E(H)$;
(c) $G \cong H$ if and only if $G \oplus H$ is a p-simple for every p group;
(d) the group G is isomorphic to any group that is quasi-isomorphic to G and has strongly homogeneous endomorphism ring if and only if the group G is p-irreducible for every p.

In the next exercise, we consider groups that are inhomogeneous analogs of strongly homogeneous groups. For example, such groups arise in studying inhomogeneous completely transitive groups (Theorem 41.2).

Exercise 4. Let G be a torsion-free group such that the $E(G) \otimes \mathbf{Q}$-module $G \otimes \mathbf{Q}$ is a homogeneous completely reducible module, and let $R = E(G)$. Assume that the following conditions hold:

(a) the group G is generated by its minimal pfi-subgroups;
(b) the group $\operatorname{Aut} G$ acts transitively on the set of all pure subgroups of rank 1 of some minimal pfi-subgroup of the group G.

Then there exists a cyclic left R-module F such that the ring $C = \operatorname{End}_R F$ is strongly homogeneous and F is an $\aleph_1$-free right C-module. Further, there exists a torsion-free left C-module A such that $Z(C)$ and $\operatorname{End}_C A$ are canonically isomorphic rings. In addition:

(1) $G \cong F \otimes_C A$;

(2) the canonical mapping of rings $R \to \operatorname{End}_C F$ is a topological isomorphism (endomorphism rings are endowed with the finite topology).

In a certain sense, the above result reduces the study of the group G to the study of strongly homogeneous groups.

Different questions contain groups of finite rank whose endomorphism rings are matrix rings over strongly homogeneous rings or products of such matrix rings. Exercise 4 implies a characterization of such groups.

Exercise 5. Let G be a group of finite rank with prime ring $E(G)$. Prove that the following conditions are equivalent:

(1) the group G is A-free for some group A and $E(A)$ is a strongly homogeneous ring;
(2) $E(G) \cong T_n$, where n is a positive integer and T is a strongly homogeneous ring;
(3) (a) the group G is generated by its minimal pfi-subgroups and

(b) the group $\operatorname{Aut} G$ acts transitively on the set of all pure subgroups of rank 1 of some minimal pfi-subgroup of the group G;

(4) G is p-simple for every p and (b) from (3) holds;

(5) $E(G)$ is a right hereditary ring and (b) from (3) holds.

42. Completely Transitive Groups Coinciding with Their Pseudosocles

In the present section, we study a completely transitive torsion-free group G coinciding with its pseudosocle $\operatorname{Soc} G$. In the next section, we consider a completely transitive group G with quite large pseudosocle $\operatorname{Soc} G$. This allows us to consider all actually interesting meaningful situations. Theorem 42.2 provides us with some canonical direct decomposition of a completely transitive group $G = \operatorname{Soc} G$; this reduces the study of G to the homogeneous case. This theorem and Theorem 43.5 are the basic structural results on inhomogeneous completely transitive groups.

We recall some notation: $S = \mathcal{E}(G) = E(G) \otimes \mathbf{Q}$, $V = G \otimes \mathbf{Q}$, and $K = \operatorname{End}_S V$. For groups G_1 and G_2, we also note that $\Pi(G_1) \cap \Pi(G_2) = \varnothing \iff pG_2 = G_2$ for every p with $pG_1 \neq G_1$.

Lemma 42.1. *Let G be a torsion-free group such that $G = \sum\limits_{i\in I}^{\oplus} G_i$, and let $\Pi(G_i) \cap \Pi(G_j) = \varnothing$ for all different $i, j \in I$. Then the complete transitivity of the group G is equivalent to the complete transitivity of all summands G_i.*

Proof. A direct summand of a completely transitive group is always completely transitive. Indeed, let G be a completely transitive group, and let $G = H \oplus E$. Let $\pi : G \to H$ and $\varkappa : H \to G$ be the projection and the embedding, respectively. If a and b are nonzero elements of H and $\chi(a) \leq \chi(b)$ (in the group H), then $\chi(a) \leq \chi(b)$ (in the group G). Therefore, $\alpha a = b$, where $\alpha \in E(G)$. Then $\pi\alpha\varkappa \in E(H)$ and $(\pi\alpha\varkappa)a = b$.

Now assume that all groups G_i are completely transitive. Let a and b be two nonzero elements of G with $\chi(a) \leq \chi(b)$. Since $G = \sum\limits_{i\in I}^{\oplus} G_i$, we have the corresponding decompositions

$$a = \sum_{s=1}^{k} a_s \qquad \text{and} \qquad b = \sum_{t=1}^{m} b_t.$$

The condition $\Pi(G_i) \cap \Pi(G_j) = \varnothing$ $(i \neq j)$ implies $k \geq m$ and $s_1 = 1, \dots, s_m = m$, where $1 \leq s_1 \leq \dots \leq s_m \leq k$ and $\chi(a_{s_l}) \leq \chi(b_l)$ $(l = 1, \dots, m)$. Since the group G_i is completely transitive, $\alpha_l(a_{s_l}) = b_l$ for some $\alpha_l \in E(G_{s_l})$ $(l = 1, \dots, m)$. The endomorphism of the group G coinciding with α_l on G_{s_l} $(l = 1, \dots, m)$ and mapping G_i into 0 for $i \neq s_1, \dots, s_m$ maps from the element a into the element b. $\square$

The proof of the following theorem substantially uses Theorem 7.1 and Corollary 7.2. We note that $G = \operatorname{Soc} G$ if and only if the $E(G) \otimes \mathbf{Q}$-module $G \otimes \mathbf{Q}$ is completely reducible (Proposition 5.8).

Theorem 42.2. *For a reduced torsion-free group G, the following conditions are equivalent:*

(1) *G is a completely transitive group and $G = \operatorname{Soc} G$;*

(2) *$G = \sum\limits_{i\in I}^{\oplus} G_i$, where $\Pi(G_i) \cap \Pi(G_j) = \varnothing$ for any different subscripts $i, j \in I$ and either G_i is a homogeneous completely transitive group or G_i is an inhomogeneous completely transitive group such that $\mathcal{E}(G_i)$ is a division ring.*

Proof. (2) $\Longrightarrow$ (1). By Lemma 42.1, the group G is completely transitive. We prove that $G = \operatorname{Soc} G$. Let $i, j \in I$ with $i \neq j$, and let $0 \neq \varphi : G_i \to G_j$. If $pG_i \neq G_i$, then $pG_j = G_j$ by assumption. Let $pG_i = G_i$. Then $p(\varphi G_i) = \varphi G_i$ and $p\langle \varphi G_i\rangle_* = \langle \varphi G_i\rangle_*$. Therefore, if G_j is a homogeneous group, then $pG_j = G_j$. If G_j is an inhomogeneous group, then $pG_j = G_j$ by Theorem 41.2. We have obtained that G_j is a divisible group. This is a contradiction, since the group G is reduced. Therefore, $\operatorname{Hom}(G_i, G_j) = 0$ for all different $i, j \in I$, which implies that the summands G_i are fully characteristic in the group G.

Moreover, all the G_i are pfi-subgroups of the group G. If G_i is a homogeneous group, then it is irreducible (Lemma 25.2); therefore, G_i is a minimal pfi-subgroup of the group G. If G_i is an inhomogeneous group, then $\mathcal{E}(G_i)$ is a division ring and $G_i \otimes \mathbf{Q}$ is a completely reducible $\mathcal{E}(G_i)$-module. Therefore, $G_i = \operatorname{Soc} G_i$ (Proposition 5.8). Every pfi-subgroup of the group G_i is pfi-subgroup of the group G, since G_i is pure and fully characteristic. Therefore, the group G_i is generated by minimal pfi-subgroups of the group G. Finally, we obtain that the group G is generated by its minimal pfi-subgroups, i.e., $G = \operatorname{Soc} G$.

(1) $\Longrightarrow$ (2). Since $G = \operatorname{Soc} G$, we have that the S-module V is completely reducible (Proposition 5.8). We have $V = \sum_{i\in I}^{\oplus} V_i$, where V_i are homogeneous components of the S-module V and I is some set of subscripts. For $i \in I$, we set $G_i = V_i \cap G$.

We take two different subscripts $i, j \in I$ and assume that $qG_i \neq G_i$ and $qG_j \neq G_j$ for some prime integer q. We choose a cyclic K-submodule W_i in V_i and a cyclic K-submodule W_j in V_j; their generating elements are contained in some irreducible submodules of the S-module V. Let $A = W_i \cap G$, $B = W_j \cap G$, and $C = (W_i \oplus W_j) \cap G$. By Corollary 7.2, $C \doteq A \oplus B$. We verify that $qA \neq A$. To do this, we need to consider that the sum of all such cyclic submodules of the K-module V_i coincides with V_i. Intersections of these submodules with the group G are pairwise quasi-isomorphic (Corollary 7.2) and the factor group of the group G_i with respect to the sum of all intersections is a torsion group. Therefore, if we assume that $qA = A$, then all considered intersections are divided by q by the quasi-isomorphism. Since the factor group of the group G_i with respect to the sum of all intersections is a torsion group, we obtain that the group G_i is also q-divisible. This is a contradiction, since $qG_i \neq G_i$. Therefore, $qA \neq A$; similarly, $qB \neq B$.

We choose elements $x \in A$ and $y \in B$ such that $h_q(x) = h_q(y) = 0$. We consider the elements $x + qy$ and $qx + y$. It is clear that $h_q(x + qy) = h_q(qx + y)$, and these elements have equal characteristics in the group $A \oplus B$. Since $C \doteq A \oplus B$, we have $mC \subseteq A \oplus B$ for some positive integer m. Therefore, the p-heights of the elements $x + qy$ and $qx + y$ in C are equal for all prime integers $p \nmid m$ and the p-heights of the elements $x + qy$ and $qx + y$ in G are equal for all prime integers $p \nmid m$ (C is pure in G). Therefore, p-heights can be distinct only for prime integers $p \mid m$ and $|h_p(x + qy) - h_p(qx + y)| \leq l$, where l is the maximal exponent such that $p^l | m$. Therefore, we can choose positive integers s and t such that $(t, q) = 1$ and $\mathcal{X}(sx + sqy) = \mathcal{X}(tqx + ty)$. Since the group G is completely transitive, there exists $\alpha \in E(G)$ such that

$$\alpha(sx + sqy) = \alpha(sx) + \alpha(sqy) = tqx + ty.$$

Since $\alpha(sx), tqx \in V_i$ and $\alpha(sqy), ty \in V_j$, we obtain $ty = \alpha(sqy) = q(s\alpha y)$. Therefore, $h_q(ty) > 0$. Since $(t, q) = 1$, we have $h_q(y) = h_q(ty) > 0$. This contradicts the choice of the element y.

We have proved that $pG_j = G_j$ for any different subscripts $i, j \in I$ and every prime integer p with $pG_i \neq G_i$. Since

$$G_i \otimes \mathbf{Q} = (V_i \cap G) \otimes \mathbf{Q} = V_i,$$

we have

$$\Big(\sum_{i\in I}^{\oplus} G_i\Big) \otimes \mathbf{Q} = V.$$

For the subgroup $\sum_{i\in I}^{\oplus} G_i$, we have that $G / \sum_{i\in I}^{\oplus} G_i$ is a torsion group. For an element $z \in G$, we choose the minimum positive integer n such that $nz \in \sum_{i\in I}^{\oplus} G_i$. We prove that $n = 1$. We have $nz = x_1 + \cdots + x_k$, where $x_i \in G_i \quad (i = 1, \ldots, k)$. Assume that a prime integer p divides n and $n = pm$. If $pG_i = G_i$ for all $i = 1, \ldots, k$, then $x_i = py_i$, where $y_i \in G_i$ $(i = 1, \ldots, k)$. Therefore, $mz = y_1 + \cdots + y_k$; this contradicts the minimality of n. Therefore, $pG_s \neq G_s$ for some s. As was proved above, $pG_i = G_i$ for all $i \neq s$. As before, $x_i = py_i$, where $y_i \in G_i$ and $i \neq s$. We obtain

$$pmz = p \sum_{i\neq s} y_i + x_s \qquad \text{and} \qquad x_s = p(mz - \sum_{i\neq s} y_i).$$

Taking into account that the subgroup G_s is pure, we have $x_s = py_s$, where $y_s \in G_s$. We obtain $pmz = py_1 + \cdots + py_k$ and $mz = y_1 + \cdots + y_k$, where $y_i \in G_i$. This contradicts the choice of n. It follows from the above argument that $n = 1$ and $z \in \sum_{i\in I}^{\oplus} G_i$. Therefore, $G = \sum_{i\in I}^{\oplus} G_i$ and if $pG_i \neq G_i$ for some i and prime integer p, then $pG_j = G_j$ for all remaining subscripts $j \in I$.

The groups G_i are completely transitive by Lemma 42.1. Since $G_i = V_i \cap G$, where V_i is a submodule of the S-module V, the subgroups G_i are fully characteristic in G. Therefore

$$S = \mathcal{E}(G) = \prod_{i\in I} \mathcal{E}(G_i).$$

We recall that $V = \sum_{i\in I}^{\oplus} V_i$ and $G_i = V_i \cap G$ $(i \in I)$. Therefore, $V_i = G_i \otimes \mathbf{Q}$ $(i \in I)$. Since every V_i is a homogeneous completely reducible S-module and the product of rings $\prod_{i\in I} \mathcal{E}(G_i)$ acts on the sum $\sum_{i\in I}^{\oplus} V_i$ componentwise, every module V_i is a homogeneous completely reducible $\mathcal{E}(G_i)$-module.

Now we assume that some group G_i is not homogeneous and prove that its quasi-endomorphism ring $\mathcal{E}(G_i)$ is a division ring. To simplify the presentation, we assume that the group G is an inhomogeneous completely transitive group and V is a homogeneous completely reducible S-module. We prove that S is a division ring.

We choose a cyclic K-submodule W_1 of V generated by an element contained in some irreducible submodule of the S-module V. If $V \neq W_1$, then V contains the cyclic K-submodule W_2 generated by an element of some irreducible S-submodule in V such that $W_1 \cap W_2 = 0$. We set $A_i = W_i \cap G$ and $A = (W_1 + W_2) \cap G$. By Corollary 7.2, $A \doteq A_1 \oplus A_2$; therefore, $kA \subseteq A_1 \oplus A_2$ for some positive integer k.

We choose two different irreducible submodules V_1 and V_2 of the S-module V. This is possible, since otherwise the module V is an irreducible S-module and the group G is an irreducible group and, in particular, is homogeneous. It is clear that the intersections $V_i \cap W_j$ are nonzero. Consequently, the intersections $V_i \cap A_j$ are nonzero $(i, j = 1, 2)$.

There exist nonzero elements $x_1 \in V_1 \cap A_1$, $x_2 \in V_2 \cap A_1$, $y_1 \in V_2 \cap A_2$, and $y_2 \in V_1 \cap A_2$ such that $\chi(x_1) = \chi(y_2)$ and $\chi(x_2) = \chi(y_1)$. This is possible, since the subgroups $V_1 \cap G$ and $V_2 \cap G$ are minimal pfi-subgroups in G; therefore, they are homogeneous groups. Further, there exists a prime integer p such that $h_p(x_1) = h_p(x_2) = 0$. Assume that such a number does not exist for every pair of different irreducible S-submodules in V. Indeed, $V = \sum_{j\in J}^{\oplus} V_j$, where V_j is an irreducible S-module $(j \in J)$. Setting $B_j = V_j \cap G$ and taking into account the homogeneity of the groups B_j, we use the assumption to obtain the following property. For every prime integer p and any different $n, m \in J$, we have $pB_m = B_m$ provided $pB_n \neq B_n$. Since $G/\sum_{j\in J}^{\oplus} B_j$ is a torsion group, we have $G = \sum_{j\in J}^{\oplus} B_j$, where the summands B_j are fully characteristic in G. Therefore, $S = \prod_{j\in J} \mathcal{E}(B_j)$. The ring S acts on the space $V = \sum_{j\in J}^{\oplus} V_j$ componentwise. This contradicts the fact that V is a homogeneous completely reducible S-module. Therefore, there exists a prime integer p such that $h_p(x_1) = h_p(x_2) = 0$.

We consider the elements $x_1 + py_1, x_2 + py_2 \in A_1 \oplus A_2$. These elements have equal characteristics in the group $A_1 \oplus A_2$, since $h_q(x_1) = h_q(py_2)$ and $h_q(py_1) = h_q(x_2)$ for $q \neq p$ and $h_p(x_1 + py_1) = h_p(x_2 + py_2) = 0$. The subgroup A is pure in G and $kA \subseteq A_1 \oplus A_2$. As in the first part of the proof, there exist positive integers s and t such that $\chi(sx_1 + spy_1) = \chi(tx_2 + tpy_2)$ and $(t, p) = 1$. Since the group G is completely transitive, we can choose $\alpha \in E(G)$ such that

$$\alpha(sx_1 + spy_1) = \alpha(sx_1) + \alpha(spy_1) = tx_2 + tpy_2.$$

Here $\alpha(sx_1), tpy_2 \in V_1$, and $\alpha(spy_1), tx_2 \in V_2$. Since $V_1 \cap V_2 = 0$, we have $\alpha(spy_1) = p(s\alpha y_1) = tx_2$ and $h_p(tx_2) > 0$. Therefore, $h_p(x_2) > 0$, since $(t, p) = 1$; this contradicts the choice of x_2.

The obtained contradiction means that the module V coincides with W_1 (i.e., the K-module V is generated by an element of some irreducible submodule in V). By Corollary 7.2, $M(G)/N(G)$ is a division ring. In addition, $N(G) = 0$ and $M(G) = \mathcal{E}(G)$. Therefore, $\mathcal{E}(G)$ is a division ring. $\square$

We consider more particularly the representation $G = \sum\limits_{i\in I}^{\oplus} G_i$ obtained in the theorem. The condition $\Pi(G_i) \cap \Pi(G_j) = \varnothing$ $(i \neq j)$ implies that for every $i \in I$ and any prime integer p, the following assertion holds: if $pG_i \neq G_i$, then $pG_j = G_j$ for all $j \neq i$. Therefore, it is clear that $|I| \leq \aleph_0$ and the summands G_i can be numbered by prime integers (this is done in Sec. 43).

We can assert that the study of completely transitive groups coinciding with their pseudosocles breaks down to the study of homogeneous completely transitive groups and the study of inhomogeneous completely transitive groups whose quasi-endomorphism rings are division rings. Since the groups of the last type are described in Theorem 41.2, Theorem 42.2 reduces the study of completely transitive groups $G = \operatorname{Soc} G$ to the study of homogeneous completely transitive groups. These groups are studied in Sec. 40.

For any completely transitive group, we can now formulate the following result.

Corollary 42.3. *If G is a completely transitive torsion-free group, then $\operatorname{Soc} G$ is a completely transitive group and $\operatorname{Soc} G = \sum\limits_{i\in I}^{\oplus} G_i$, where G_i are groups that satisfy the conditions of Theorem 42.2.*

Proof. We denote $P = \operatorname{Soc} G$. It is clear that all pfi-subgroups of a completely transitive group are completely transitive. Therefore, P is a completely transitive group. If H is a minimal pfi-subgroup of the group G, then H is also a minimal pfi-subgroup in the group P. Therefore, P is purely generated by its minimal pfi-subgroups, $P = \operatorname{Soc} P$, and we can apply Theorem 42.2 to the group P. $\square$

Exercise 1. Prove an analog of Theorem 42.2 for transitive groups.

Exercise 2. Prove that a pfi-subgroup of a completely transitive group is completely transitive.

Exercise 3. Assume that G_i $(i \in I)$ are completely transitive groups and $\Pi(G_i) \cap \Pi(G_j) = \varnothing$ for all different $i, j \in I$. Prove that $\prod\limits_{i\in I} G_i$ is a completely transitive group.

Exercise 4 (Grinshpon–Misyakov [196])**.** Let G be a product of separable torsion-free groups. The group G is completely transitive if and only if $\Pi(A) \cap \Pi(B) = \varnothing$ for any nonisomorphic direct summands A and B of rank 1 of the group G.

43. Completely Transitive Groups with Restrictions on Types of Elements

For a completely transitive group G, the condition that G coincides with the pseudosocle $\operatorname{Soc} G$ under some restrictions can be replaced by the maximum condition on the set $\mathcal{T}(G)$ of all different types of its nonzero elements. This maximum condition is common in the theory of torsion-free groups.

In this section, R denotes the endomorphism ring of a group G.

We will use the density Chevalley–Jacobson theorem (see Sec. 1) in the following situation. If T is a minimal pfi-subgroup of the torsion-free group G, then $T \otimes \mathbf{Q}$ is an irreducible submodule of the left $R \otimes \mathbf{Q}$-module $G \otimes \mathbf{Q}$ (Lemma 5.6). Therefore, D is a division ring, where $D = \operatorname{End}_{R\otimes\mathbf{Q}}(T \otimes \mathbf{Q})$ and $T \otimes \mathbf{Q}$ is a right vector space over D. By the density theorem, $(R \otimes \mathbf{Q})/A$ is a dense ring of linear transformations of the D-space $T \otimes \mathbf{Q}$, where $A = \operatorname{Ann}(T \otimes \mathbf{Q}) = \{\alpha \in R \otimes \mathbf{Q} \mid \alpha(T \otimes \mathbf{Q}) = 0\}$.

Proposition 43.1. *Let G be a completely transitive torsion-free group, T be some minimal pfi-subgroup of it, and $C = \operatorname{End}_R T$. Then C is a strongly homogeneous ring and $T \cong F \otimes A$, where F is an $\aleph_1$-free right C-module and A is a group of rank 1 of type $t(T)$. In addition, $C \otimes \mathbf{Q} = D$, where $D = \operatorname{End}_{R\otimes\mathbf{Q}}(T \otimes \mathbf{Q})$.*

The proof is based on the considerations used in proving Theorem 40.3 and is omitted (also see Lemma 8.4). We note only that the ring $\operatorname{End}_R T$ plays the role of the center C from Theorem 40.3 ($C = \operatorname{End}_R G$!).

In the situation of Proposition 43.1, for any pure submodule H of the C-module T, there exists a pure submodule M of the module F such that $H \cong M \otimes A$. We also note that a submodule H of the C-module T is C-pure if and only if H is $\mathbf{Z}$-pure (the C-purity H in T means that for any two nonzero elements $t \in T$ and $c \in C$, the relation $tc \in H$ implies $t \in H$). Indeed, since $\mathbf{Z} \subseteq C$, the C-purity implies the purity. Conversely, let a submodule H be pure in T as a subgroup. Assume that $tc = h \in H$, where $t \in T$ and $0 \neq c \in C$. We have $c = nv$, where n is a positive integer and v is an invertible element. Further, $tc = n(tv) = h \in H$. Since H is pure, we obtain $tv = g \in H$. Therefore, $t = gv^{-1} \in H$ and H is C-pure.

Lemma 43.2. *Let G be a completely transitive torsion-free group, T be a minimal pfi-subgroup of G, and $C = \operatorname{End}_R T$. Further, let $a \in G \setminus T$ with $t(a) < t(T)$, and let H be a pure C-submodule in T of finite C-rank. Then $\langle a, H\rangle_* \doteq B \oplus F$ for some subgroup B and C-submodule F in T of C-rank 1.*

Proof. If $\langle a, H\rangle_* = \langle a\rangle_* \oplus H$, then the assertion is true, since H is a direct sum of pure C-submodules of C-rank 1 by Proposition 43.1. Otherwise, there exist a prime integer p and elements $x \in \langle a\rangle_*$, $b \in H$, and $c \in \langle a, H\rangle_*$ such that $pc = x + b$ and $h_p(x) = h_p(b) = 0$. Without loss of generality, we assume that $x = a$. Since $t(a) < t(b)$, there exists a positive integer n such that $(n, p) = 1$ and $\chi(a) < \chi(nb)$. Then $\alpha a = nb$ for some $\alpha \in R$ and $\alpha b \neq 0$. Otherwise, $\alpha(pc) = \alpha a + \alpha b = \alpha a = nb$. Since $(n, p) = 1$, we have $h_p(nb) = h_p(b) = 0$. On the other hand, $h_p(nb) = h_p(p\alpha c) \geq 1$. Therefore, $\alpha a = nb$ and $\alpha b \neq 0$.

Let F be the pure C-submodule in T generated by the element b, and let $F = \langle bC\rangle_*$. We prove that we can choose the above α with $\alpha b \in F$. Assume that $\alpha b \notin F$. Then b and αb are two linearly independent elements of the $C \otimes \mathbf{Q}$-space $T \otimes \mathbf{Q}$. By the density theorem, there exists $\gamma \in R \otimes \mathbf{Q}$ such that $\gamma b = b$ and $\gamma\alpha b = b$. We set $\alpha' = m\gamma\alpha$, where m is a positive integer such that $m\gamma \in R$. Then

$$\alpha' a = m\gamma\alpha a = m\gamma nb = mnb \qquad \text{and} \qquad \alpha' b = m\gamma\alpha b = mb.$$

Assuming that α is the α' and b is the mnb, we obtain $\alpha a = b \in F$, $\alpha F \subseteq F$, and $\alpha F \neq 0$.

By Proposition 43.1, $H = M \otimes A$ and $F = E \otimes A$, where A is a group of rank 1 of type $t(T)$, M and E are free finitely generated C-modules, and E is C-pure in M. Since C is a principal right ideal domain, $M = E \oplus M'$ for some module M' (consider that M/E is projective as a finitely generated torsion-free module). Therefore, $H = (E \otimes A) \oplus (M' \otimes A) = F \oplus Y$, where $Y = M' \otimes A$. Let $X = \langle \alpha Y\rangle_*$. As above, $X = (F \cap X) \oplus X'$ for some submodule X'. Since $F \otimes \mathbf{Q}$ and $X' \otimes \mathbf{Q}$ are finite-dimensional subspaces of the $C \otimes \mathbf{Q}$-space $T \otimes \mathbf{Q}$, we use the density theorem and choose $\delta \in R$ and $m \in \mathbf{N}$ such that $\delta_{|F} = m1$ and $\delta X' = 0$. We denote $\alpha' = \delta\alpha$. Then $\alpha' a = \delta\alpha a = \delta b = mb$, $\alpha' F = \delta\alpha F \subseteq F$, and $\alpha' F \neq 0$. Further,

$$\alpha' Y = \delta\alpha Y \subseteq \delta X = \delta(F \cap X) \subseteq \delta F \subseteq F \qquad \text{and} \qquad \alpha' H \subseteq F.$$

Replacing α' by α, we have $0 \neq \alpha a \in F$ and $\alpha H \subseteq F$, whence $\alpha\langle a, H\rangle_* \subseteq F$.

Since the C-module F has rank 1, we have that $F \otimes \mathbf{Q}$ is a $C \otimes \mathbf{Q}$-space of dimension 1. Applying the density theorem, we obtain $\beta \in R$ and $k \in \mathbf{N}$ such that $\beta\alpha_{|F} = k1$. We set $\pi = (\beta\alpha)_{|\langle a,H\rangle_*}$. Then $\pi\langle a, H\rangle_* \subseteq F$, $\pi_{|F} = k1$, and $\pi^2 = k\pi$ (i.e., π is a quasi-projection of the group $\langle a, H\rangle_*$). Therefore, $k\langle a, H\rangle_* \subseteq \ker\pi \oplus \operatorname{im}\pi \subseteq \ker\pi \oplus F$. We set $B = \ker\pi$ and obtain $\langle a, H\rangle_* \doteq B \oplus F$. □

Lemma 43.3. *Let G be a completely transitive torsion-free group, and let T be some minimal pfi-subgroup of it of finite C-rank, where $C = \operatorname{End}_R T$. Assume that a is an element of $G \setminus T$ with $t(a) < t(T)$. Then the residue class $a + T$ contains a unique element c such that $\langle a, T\rangle_* = \langle c\rangle_* \oplus T$ and $\Pi(\langle c\rangle_*) \cap \Pi(T) = \varnothing$.*

Proof. Assume that $t(na + x) < t(T)$ for every nonzero integer n and any element $x \in T$. We show that this assumption leads to a contradiction. By the assumption, $\operatorname{rank}_C T < \infty$; therefore, Lemma 43.2 implies $\langle a, T\rangle_* \doteq B \oplus F$ for some pure subgroup B and C-submodule F of C-rank 1. We have $B = \langle a', H\rangle_*$, where $H = T \cap B$ and $a' \in na + T$ for some nonzero integer n. The proof of Lemma 43.2 implies that $T \cap B$ is a C-submodule. Since $t(a') < t(T)$ by assumption, Lemma 43.2 can be applied to a' and H; therefore, $B \doteq B_1 \oplus F_1$, where B_1 is a pure subgroup and F_1 is a C-submodule of rank 1. Therefore,

$\langle a,T\rangle_* \dot{=} B_1 \oplus F \oplus F_1$. We repeat the above argument for the subgroup B_1. Since the C-rank of T is finite, we eventually obtain $\langle a,T\rangle_* \dot{=} \langle a_1\rangle_* \oplus T$, where $a_1 \in na+T$ and n is a nonzero integer; therefore, $t(a_1) < t(T)$.

Assume that we take a positive integer k such that $k\langle a,T\rangle_* \subseteq \langle a_1\rangle_* \oplus T$. Since $t(a_1) < t(T)$, we can choose a prime integer p that does not divide $\langle a_1\rangle_*$ and T. Further, we choose elements $x \in \langle a_1\rangle_*$ and $y \in T$ of zero p-height. Let $k = p^t l$, where $(l,p) = 1$. Considering the relation $h_p(p^{t+1}x + y) = h_p(x + py) = 0$ and the quasi-decomposition $\langle a,T\rangle_* \dot{=} \langle a_1\rangle_* \oplus T$, we obtain that there exist positive integers m and n such that $\chi(mp^{t+1}x + my) = \chi(nx + npy)$ and $(n,p) = 1$. Since the group G is completely transitive, $\alpha(mp^{t+1}x + my) = nx + npy$, where $\alpha \in R$. Therefore,

$$\alpha(mp^{t+1}x) = nx + npy - \alpha(my) \in \langle a_1\rangle_* \oplus T.$$

Therefore, $\alpha((mp^{t+1}x) = x_1 + y_1$, where $x_1 \in \langle a_1\rangle_*,\ y_1 \in T$, and $h_p(x_1) \geq 1$. Comparing this with the preceding relation, we obtain

$$x_1 - nx = npy - \alpha(my) - y_1 \in \langle a_1\rangle_* \cap T = 0 \quad \text{and} \quad x_1 = nx.$$

Since $(n,p) = 1$, we have $0 = h_p(x) = h_p(nx) = h_p(x_1) \geq 1$; this is a contradiction.

Therefore, our assumption is not true (i.e., there exist a nonzero integer n and an element $y \in T$ such that $t(na + y) \not< t(T)$). We set $F = \langle yC\rangle_*$. By Lemma 43.2, $\langle a,F\rangle_* \dot{=} \langle a_1\rangle_* \oplus F$ for some element a_1. Let $z = na + y$. Then $z \in \langle a,F\rangle_*$. Therefore, $mz = x + y_1$, where $x \in \langle a_1\rangle_*,\ y_1 \in F$, and m is a positive integer. The element $y_1 = 0$, otherwise $t(z) = t(mz) \leq t(y_1) = t(T)$; this contradicts the choice of z. Therefore, $mz = x$ and $z \in \langle a_1\rangle_*$. We have $\langle a,F\rangle_* \dot{=} \langle z\rangle_* \oplus F$, where $t(z) \not< t(T)$.

We prove that $\Pi(\langle z\rangle_*) \cap \Pi(F) = \varnothing$. Assume the contrary (i.e., a prime integer p does not divide $\langle z\rangle_*$ and F). We choose elements $v \in \langle z\rangle_*$ and $w \in F$ of zero p-height. Our quasi-decomposition allows us to choose a positive integer k such that $\chi(v + pw) \leq \chi(kw)$ and $(k,p) = 1$. Let $\alpha \in R$ map from $v + pw$ into kw. We have $\alpha v = (k1 - p\alpha)w \in T$. If $\alpha v \neq 0$, then $t(z) = t(v) < t(\alpha v) = t(T)$; this is a contradiction. Therefore, $\alpha v = 0$ and $kw = p\alpha w$; this is impossible, since $(k,p) = 1$ and $h_p(w) = 0$. Therefore, $\Pi(\langle z\rangle_*) \cap \Pi(F) = \varnothing$.

The group T is homogeneous; therefore, $\Pi(T) = \Pi(F)$ and $\Pi(\langle z\rangle_*) \cap \Pi(T) = \varnothing$. Therefore, $\langle a,T\rangle_* = \langle z,T\rangle_* = \langle z\rangle_* \oplus T$. We have $a = c + u$, where $c \in \langle z\rangle_*$ and $u \in T$. Then

$$\langle a,T\rangle_* = \langle c\rangle_* \oplus T, \qquad \Pi(\langle c\rangle_*) \cap \Pi(T) = \varnothing,$$

and $c \in a + T$. We prove that c is unique. If $c_1 \in a + T$ is one more such element, then $t(c_1) \not\leq t(T)$. We have $c_1 = c_2 + x$, where $c_2 \in \langle c\rangle_*$ and $x \in T$. Then $x = 0$ (otherwise $t(c_1) \leq t(x) = t(T)$). Therefore, $c_1 \in \langle c\rangle_*,\ c - c_1 \in \langle c\rangle_* \cap T = 0$, and $c = c_1$. □

We define some finiteness condition for a group. A torsion-free group G is said to have a pseudosocle of finite type if every irreducible submodule W of the $R \otimes \mathbf{Q}$-module $G \otimes \mathbf{Q}$ is finite-dimensional over the division ring D, where $D = \operatorname{End}_{R\otimes\mathbf{Q}} W$.

Now assume that the group G is completely transitive. By Corollary 42.3, $\operatorname{Soc} G = \sum_{p\in\Pi}^{\oplus} G_p$, where Π is some set of prime integers, $\Pi(G_p) \cap \Pi(G_q) = \varnothing$ for $p \neq q,\ p,q \in \Pi$, and either G_p is a homogeneous completely transitive group or G_p is an inhomogeneous completely transitive group such that $\mathcal{E}(G_p)$ is a division ring (beginning with this moment, we change the subscripts of the summands G_i from Theorem 42.2 according to the remark after the proof of this theorem). The subgroups G_p are uniquely determined by the group G. We call them components of the completely transitive group G.

If W is a submodule as above, then $T = W \cap G$ is a minimal pfi-subgroup of the group G. By Proposition 43.1, $D = C \otimes \mathbf{Q}$, where $C = \operatorname{End}_R T$. Therefore, $\operatorname{rank}_C T = \dim_D W$. We see that the group G has a pseudosocle of finite type if and only if $\operatorname{rank}_C T < \infty$ for every minimal pfi-subgroup T in G. The subgroup T is contained in some component G_p of the group G. If G_p is an inhomogeneous group, then $\operatorname{rank}_C T = 1$, since $\mathcal{E}(G_p)$ is a division ring. This substantiates the following definition.

We say that the component G_p of a completely transitive group G has a finite type if either G_p is an inhomogeneous group or G_p is a homogeneous group and $\operatorname{rank}_C G_p < \infty$, where $C = \operatorname{End}_R G_p$.

If a completely transitive group G has a pseudosocle of finite type, then each of its components has finite type and vice versa.

Theorem 43.4. *Let G be a completely transitive torsion-free group, and let G_p be its component of finite type. Then $G = G_p \oplus H$ for some pfi-subgroup H, $\Pi(G_p) \cap \Pi(H) = \varnothing$, and H is uniquely determined.*

Proof. Let H be a pfi-subgroup of the group G such that H is the maximal among the pfi-subgroups having zero intersection with G_p. For an element $a \in G \backslash (G_p \oplus H)$, we set $E = \langle Ra \rangle_*$. Then $(G_p \oplus H) \cap E \neq 0$ and $(G_p \oplus H) \cap E \neq H \cap E$. Indeed, $G_p \cap (H + E) \neq 0$ by the choice of H. Let $0 \neq x = y + z$, where $x \in G_p, y \in H$, and $z \in E$. Since $G_p \cap H = 0$, we have $z \neq 0$ and $z = x - y \in (G_p \oplus H) \cap E$. However, $z \notin H \cap E$, since $x \neq 0$.

We set $G' = \langle G_p \oplus H \rangle_*$. Then G' is a completely transitive group and G_p is the component of G'. We define a mapping $f : G' \to G_p$ as follows. Let $a \in G'$. If $Ra \cap G_p = 0$, then we set $f(a) = 0$. If $Ra \cap G_p \neq 0$, then $Ra \cap T \neq 0$ for some minimal pfi-subgroup $T \subseteq G_p$. By Lemma 43.3, $\langle a, T \rangle_* = \langle c \rangle_* \oplus T$ for a unique element $c \in a + T$ and $\Pi(\langle c \rangle_*) \cap \Pi(T) = \varnothing$. Let $a = c + b$, and let $b \in T$. We set $f(a) = b$. It is easy to prove that $f(na) = nf(a)$ for any $a \in G'$ and $n \in \mathbf{N}$. The mapping f acts identically on G_p and annihilates H. Consequently, the restriction f to $G_p \oplus H$ is a homomorphism. Assume that we have taken $a, b \in G'$ and $n \in \mathbf{N}$ such that $na, nb \in G_p \oplus H$. We have

$$nf(a + b) = f(na + nb) = f(na) + f(nb) = nf(a) + nf(b)$$

and

$$f(a + b) = f(a) + f(b).$$

Therefore, f is an epimorphism $G' \to G_p$, and $f^2 = f$, whence $G' = \operatorname{im} f \oplus \ker f = G_p \oplus H$ (the subgroup $\ker f$ is fully characteristic in G'; therefore, it coincides with H).

Thus, the subgroup $G_p \oplus H$ is pure in G. Assume that there exists an element $a \in G \setminus (G_p \oplus H)$. As was proved above, there exists $\alpha \in R$ such that $0 \neq \alpha a \in G_p \oplus H$ and $\alpha a = b + d$, where $b \in G_p,\ d \in H$, and $b \neq 0$. Considering that $G_p \oplus H$ is pure in G, we obtain $\chi(a) \leq \chi(\alpha a) \leq \chi(b)$. As in the preceding paragraph, Lemma 43.3 implies $\langle a, T \rangle_* = \langle c \rangle_* \oplus T$ for some minimal pfi-subgroup $T \subseteq G_p$, where $c \in a + T$ and $\Pi(\langle c \rangle_*) \cap \Pi(T) = \varnothing$. By Theorem 41.2, $\Pi(G_p) = \Pi(T)$. Therefore, $\Pi(\langle c \rangle_*) \cap \Pi(G_p) = \varnothing$ and $\langle a, G_p \rangle_* = \langle c \rangle_* \oplus G_p$. We have $a = c + g$, where $g \in G_p$. We consider the element c. Since $a \notin G_p \oplus H$, we have $c \notin G_p \oplus H$. Therefore, there exists $\beta \in R$ such that $\beta c = x + y,\ x \in G_p,\ y \in H$, and $x \neq 0$. Then $\chi(c) \leq \chi(\beta c) \leq \chi(x)$; this is impossible, since $\Pi(\langle c \rangle_*) \cap \Pi(G_p) = \varnothing$. This contradiction implies $G = G_p \oplus H$. The summand H is unique, since H is fully characteristic in G. If $0 \neq a \in H$ and $0 \neq b \in G_p$, then

$$\langle a + b, G_p \rangle_* = \langle a, G_p \rangle_* = \langle c \rangle_* \oplus G_p,$$

where $c \in a + H$ and $\Pi(\langle c \rangle_*) \cap \Pi(G_p) = \varnothing$. Then $\langle c \rangle_* \subseteq H$ and $a \in \langle c \rangle_*$. Therefore, $\Pi(\langle a \rangle_*) \cap \Pi(G_p) = \varnothing$ and $\Pi(G_p) \cap \Pi(H) = \varnothing$, since a is arbitrary. □

In the following theorem only, a direct product of groups is denoted by $\sum^*$.

Theorem 43.5. *For a completely transitive torsion-free group G with pseudosocle of finite type, the following conditions are equivalent*:

1. *for every $v \in \mathcal{T}(G)$, there exists a maximal element $w \in \mathcal{T}(G)$ such that $v \leq w$ and $\Pi(\operatorname{Soc} G) = \Pi(G)$*;
2. *the subgroup $\operatorname{Soc} G$ is dense in G with respect to the $\mathbf{Z}$-adic topology*;
3. *the group G can be represented in the form $\sum\limits_{p \in \Pi}^{\oplus} G_p \subseteq G \subseteq \sum\limits_{p \in \Pi}^{*} G_p$, where G is pure in $\sum\limits_{p \in \Pi}^{*} G_p$, Π is some set of prime integers, $\Pi(G_p) \cap \Pi(G_q) = \varnothing$ for $p \neq q$ $(p, q \in \Pi)$, and the groups G_p satisfy the conditions that are satisfied by the groups G_i in Theorem 42.2.*

Proof. (1) $\Longrightarrow$ (3) and (2) $\Longrightarrow$ (3). Let G_p ($p \in \Pi$) be the set of all different components of the group G (i.e., $\operatorname{Soc} G = \sum_{p\in\Pi}^{\oplus} G_p$). By Theorem 43.4, $G = G_p \oplus H_p$ and $\Pi(G_p) \cap \Pi(H_p) = \varnothing$ for every $p \in \Pi$. We denote $H = \bigcap_{p\in\Pi} H_p$. Assume that $H \neq 0$. Since $\Pi(G_p) \cap \Pi(H) = \varnothing$ for every $p \in \Pi$, we have $\Pi(\operatorname{Soc} G) \cap \Pi(H) = \varnothing$. For an element $0 \neq a \in H$, we set $v = t(a)$. In case (1), there exists $w \in \mathcal{T}(G)$ such that $v < w$. We choose $b \in G$ with $w = t(b)$. Since $t(b)$ is maximal, it can be verified that the pfi-subgroup $\langle Rb\rangle_*$ is minimal in G (see Exercise 1). Therefore, $b \in G_p$ for some p. Then the inequality $t(a) < t(b)$ contradicts the fact that $\Pi(G_p) \cap \Pi(H) = \varnothing$. Considering (2), we have $a = \lim a_i$ (the limit in the **Z**-adic topology), where $a_i \in \operatorname{Soc} G$. Let p be a prime integer such that $h_p(a) < \infty$. Then $h_p(a_i) < \infty$ for some i. We have obtained that $p \operatorname{Soc} G \neq \operatorname{Soc} G$ and $pH \neq H$; this contradicts the fact that $\Pi(\operatorname{Soc} G) \cap \Pi(H) = \varnothing$. Therefore, $H = 0$.

For $p \in \Pi$, we denote by ε_p the projection $G \to G_p$ with the kernel H_p. We define the homomorphism

$$f : G \to \sum_{p\in\Pi}^{*} G_p$$

by $f(g) = \langle \varepsilon_p(g)\rangle$, where $g \in G$ and $\langle \varepsilon_p(g)\rangle$ is a vector in $\sum_{p\in\Pi}^{*} G_p$ with the element $\varepsilon_p(g)$ at the pth place. Since $\bigcap_{p\in\Pi} H_p = 0$, we have that f is a monomorphism and it does not move elements of the subgroup $\sum_{p\in\Pi}^{\oplus} G_p$. We can assume that

$$\sum_{p\in\Pi}^{\oplus} G_p \subseteq G \subseteq \sum_{p\in\Pi}^{*} G_p.$$

We prove that G is pure in $\sum_{p\in\Pi}^{*} G_p$. Let $g = qz$, where $g \in G, z \in \sum_{p\in\Pi}^{*} G_p$, and q is a prime integer. If there exists $p \in \Pi$ such that $qG_p \neq G_p$, then $qH_p = H_p$ in the decomposition $G = G_p \oplus H_p$. Let $g = a + b$, where $a \in G_p$ and $b \in H_p$. Then $b = qc$ for some $c \in H_p$, $qz = a + qc$, and $a = q(z - c)$. Therefore, $z - c \in G_p$, since G_p is pure and $z \in G_p \oplus H_p = G$. If $qG_p = G_p$ for all $p \in \Pi$, then $q \operatorname{Soc} G = \operatorname{Soc} G$. In case (1), $qG = G$ by the assumption. In case (2), $qG = G$, since $\operatorname{Soc} G$ is dense in G. Therefore, $g = qa$, where $a \in G$. Therefore, $qz = qa$, where $z = a \in G$. Therefore, G is pure in $\sum_{p\in\Pi}^{*} G_p$.

(3) $\Longrightarrow$ (1). We have $\operatorname{Soc} G = \sum_{p\in\Pi}^{\oplus} G_p$. For an element $0 \neq g \in G$, there exists $p \in \Pi$ such that it has the nonzero component a in G_p with respect to the decomposition $G = G_p \oplus H_p$. Then $t(g) \leq t(a)$ and it remains to note that the type $t(a)$ is maximum in $\mathcal{T}(G)$. Indeed, either G_p is a homogeneous group or $\mathcal{E}(G_p)$ is a division ring. In the first case, $|\mathcal{T}(G_p)| = 1$. In the second case, $\mathcal{T}(G_p)$ consists of pairwise incomparable elements (Theorem 41.2). Therefore, it is obvious that the type of a is maximum in $\mathcal{T}(G)$ and the first condition in (1) holds. Further, we have

$$\Pi(\operatorname{Soc} G) \subseteq \Pi(G) \subseteq \Pi\Big(\sum_{p\in\Pi}^{*} G_p\Big) = \Pi\Big(\sum_{p\in\Pi}^{\oplus} G_p\Big) = \Pi(\operatorname{Soc} G)$$

and

$$\Pi(\operatorname{Soc} G) = \Pi(G).$$

(3) $\Longrightarrow$ (2). A subgroup H of G is dense in the **Z**-adic topology of the group G if and only if the factor group G/H is divisible. It is easy to verify that the relations $\Pi(G_p) \cap \Pi(G_q) = \varnothing$ ($p \neq q$) imply that the group $\sum_{p\in\Pi}^{*} G_p / \sum_{p\in\Pi}^{\oplus} G_p$ is divisible. Since G is pure in $\sum_{p\in\Pi}^{*} G_p$, the group $G/\operatorname{Soc} G$ is also divisible (we recall that $\operatorname{Soc} G = \sum_{p\in\Pi}^{\oplus} G_p$). Consequently, the subgroup $\operatorname{Soc} G$ is dense in G. ☐

The proof of the theorem implies the following corollary.

Corollary 43.6. *For a completely transitive torsion-free group G with pseudosocle of finite type, the following conditions are equivalent*:

(1) *for every $v \in \mathcal{T}(G)$, there exists a maximal element $w \in \mathcal{T}(G)$ such that $v \leq w$;*

(2) *the group G can be represented in the form $\sum_{p\in\Pi}^{\oplus} G_p \subseteq G \subseteq \sum_{p\in\Pi}^{*} G_p$, where Π is some set of prime integers, $\Pi(G_p) \cap \Pi(G_q) = \varnothing$ for $p \neq q$ $(p, q \in \Pi)$, and the groups G_p satisfy the conditions that are satisfied by the groups G_i in Theorem* 42.2.

Corollary 43.7. *If G is a torsion-free group and the pseudosocle of G has a finite type, then the following conditions are equivalent*:

(1) *G is a completely transitive group and $\mathcal{T}(G)$ satisfies the maximum condition*;

(2) *G is a completely transitive group and $G = \operatorname{Soc} G$*;

(3) *item* (2) *of Theorem* 42.2 *holds.*

Proof. (2) and (3) are always equivalent (Theorem 42.2).

(1) $\Longrightarrow$ (3). By Corollary 43.6,

$$\sum_{p\in\Pi}^{\oplus} G_p \subseteq G \subseteq \sum_{p\in\Pi}^{*} G_p,$$

where $\Pi(G_p) \cap \Pi(G_q) = \varnothing$ for $p \neq q$. For every $p \in \Pi$, $G = G_p \oplus H_p$, where H_p is a subgroup and $G_q \subseteq H_p$ for all $q \neq p$. Similar to the argument used at the end of the proof of Theorem 43.4, we obtain $\Pi(G_p) \cap \Pi(H_p) = \varnothing$. Then it is clear that $G = \sum_{p\in\Pi}^{\oplus} G_p$; otherwise, the maximum condition does not hold in the set $\mathcal{T}(G)$.

(3) $\Longrightarrow$ (1). We prove that the maximum condition holds in the set $\mathcal{T}(G)$. Considering the relations $\Pi(G_p) \cap \Pi(G_q) = \varnothing$ $(p \neq q)$, it is sufficient to prove that the maximum condition holds in each of the sets $\mathcal{T}(G_p)$. This is obvious if G_p is a homogeneous group. If G_p is an inhomogeneous group, then $\mathcal{T}(G_p)$ consists of pairwise incomparable elements by Theorem 41.2. $\square$

Using the preceding results of this part, we obtain a complete classification of completely transitive groups from Corollary 43.7.

Corollary 43.8. *Let G be a completely transitive torsion-free group with pseudosocle of finite type. If $\mathcal{T}(G)$ satisfies the maximum condition, then*

$$G = \sum_{p\in\Pi_1}^{\oplus} G_p \oplus \sum_{p\in\Pi_2}^{\oplus} G_q,$$

where Π_1 and Π_2 are some disjoint sets of prime integers and $\Pi(G_p) \cap \Pi(G_q) = \varnothing$ for $p \neq q$ $(p, q \in \Pi_1 \cup \Pi_2)$. Every group G_p $(p \in \Pi_1)$ is a homogeneous completely transitive group and $G_p \cong (\sum_{n_p}^{\oplus} C_p) \otimes A_p$, where n_p is a positive integer, C_p is a strongly homogeneous E-ring that is the center of the ring $E(G_p)$, and A_p is a group of rank 1 of type $t(G_p)$. Every group G_q $(q \in \Pi_2)$ is an inhomogeneous completely transitive group such that $\mathcal{E}(G_q)$ is a division ring. Such groups G_q are characterized in Theorem 41.2.

Proof. The group G has the indicated decomposition by Corollary 43.7 and Theorem 42.2. If $p \in \Pi_1$, then we set $C_p = \operatorname{End}_R G_p$ for the group G_p. Then

$$\operatorname{rank}_{C_p} G_p = \dim_{C_p \otimes \mathbf{Q}}(G_p \otimes \mathbf{Q}) < \infty.$$

The required representation of the group G_p follows from Corollary 40.4 and Theorem 40.10. $\square$

Corollary 43.7 describes the structure of the component G_p in Theorems 43.4 and 43.5. The corollary implies the Dobrusin theorem on completely transitive groups of finite rank, since such a group G has the pseudosocle of finite type and $\mathcal{T}(G)$ satisfies the maximum condition.

Corollary 43.9 (Dobrusin [110])**.** *The group G of finite rank is completely transitive if and only if*

$$G = G_1 \oplus \cdots \oplus G_k \oplus G_{k+1} \oplus \cdots \oplus G_{k+l},$$

where $\Pi(G_i) \cap \Pi(G_j) = \varnothing$ for $i \neq j$ $(i, j = 1, 2, \dots, k+l)$. For $i = 1, \dots, k$, the group G_i is homogeneous completely transitive, and for $i = k+1, \dots, k+l$, the group G_i is an inhomogeneous completely transitive group and $\mathcal{E}(G_i)$ is a division ring. The structure of the groups G_i is known in both cases (*Corollary* 43.8). *Such groups G are transitive.*

Exercise 1. Prove the following assertion. Let G be a completely transitive group. If the type $t(g)$ of an element $0 \neq g \in G$ is maximum in $\mathcal{T}(G)$, then $\langle E(G)g\rangle_*$ is a minimal pfi-subgroup of the group G. Conversely, if T is a minimal pfi-subgroup in G, then $t(T)$ is a maximal element in $\mathcal{T}(G)$. Therefore, $\operatorname{Soc} G = 0$ if and only if $\mathcal{T}(G)$ does not have maximal elements.

Exercise 2. Extend Theorem 43.5 to transitive groups.

Exercise 3. Prove that if G is a completely transitive group and $r_p(G) < \infty$ for every p, then G has a pseudosocle of finite type.

Let A be a pure subgroup of the torsion-free group G. An element $a \in G \setminus A$ is said to be proper with respect to A if $\chi_G(a) = \chi_{G/A}(a + A)$. A subgroup A is said to be balanced if every residue class of the group G with respect to the subgroup A contains a proper element.

Exercise 4. Let G be completely transitive group, H be a pfi-subgroup of G, and a be an element proper with respect to H. Then $\langle a, H\rangle_* = \langle a\rangle_* \oplus H$, where $\Pi(\langle a\rangle_*) \cap \Pi(H) = \varnothing$.

Exercise 5. For a completely transitive group G and a balanced pfi-subgroup H of G, there exists a direct decomposition $G = H \oplus E$ such that the summand E is uniquely determined and $\Pi(H) \cap \Pi(E) = \varnothing$.

Exercise 6 (Dobrusin [110])**.** Let G be a reduced completely transitive group, and let Ω be a subset of $\mathcal{T}(G)$ such that for every $\tau \in \mathcal{T}(G)$, there exists $\omega \in \Omega$ such that $\tau \leq \omega$. Then any two endomorphisms coinciding on the subset $\bigcup\limits_{\omega \in \Omega} G(\omega)$ are equal to one another, where $G(\omega) = \{x \in G \mid t(x) \geq \omega\}$.

Exercise 7. Let A be a group of rank 1 of nonidempotent type (see Exercise 4 of Sec. 42), and let $G = \prod\limits_{\aleph_0} A$. Prove that G is a completely transitive group, G does not have apseudosocle of finite type, and G satisfies condition (1) of Theorem 43.5 and does not satisfy conditions (2) or (3) of Theorem 43.5.

44. Torsion-Free Groups of p-Rank ≤ 1

In this section, we consider reduced torsion-free groups G such that the p-rank $r_p(G)$ of the group G does not exceed 1 for every prime integer ($r_p(G) \leq 1$). The class of all such groups G is denoted by $\mathcal{E}$. Murley [333, 334] systematically studied groups of finite rank in $\mathcal{E}$; now these groups are called Murley groups. In this section, we present some examples and apply the results of Secs. 7, 40, and 41 to groups in the class $\mathcal{E}$. The set of all prime integers is denoted by P.

The class $\mathcal{E}$ was already considered in the exercises of Sec. 32. Now we briefly prove some of the main properties of groups from $\mathcal{E}$ formulated previously in the exercises.

Let G be a torsion-free group. Using the definition of a p-basis subgroup (Sec. 2), we obtain $r(B) = r_p(G)$, where B is any p-basis subgroup of the group G. Therefore, we obtain $G \in \mathcal{E}$ if and only if a p-basis subgroup of the group G is cyclic for every p with $pG \neq G$ (i.e., it is isomorphic to the group of integers $\mathbf{Z}$).

Proposition 44.1. (1) *If a group G belongs to the class $\mathcal{E}$, then the completion of G with respect to the* $\mathbf{Z}$*-adic topology is isomorphic to* $\prod\limits_{p \in \Pi(G)} I_p$.

(2) *A group belongs to $\mathcal{E}$ if and only if it is isomorphic to some pure subgroup of the group* $\prod\limits_{p \in P} I_p$.

(3) *The endomorphism ring $E(G)$ of the group G from $\mathcal{E}$ is isomorphic to a pure dense (in the* **Z***-adic topology) subring of the ring* $\prod_{p\in\Pi(G)} Q_p^*$. *The ring $E(G)$ is an E-ring. In particular, $E(G)$ is a commutative ring.*

Proof. (1) The information on the **Z**-adic completion and the p-adic completion of groups and rings is contained in Sec. 29. We have

$$\widehat{G} \cong \prod_{p\in\Pi(G)} \widehat{G}_p,$$

where $\widehat{G}[\widehat{G}_p]$ is the **Z**-adic (p-adic) completion of the group G. For every $p \in \Pi(G)$, the p-basis subgroup of the group G is isomorphic to **Z**. Therefore, $\widehat{G}_p \cong \widehat{Z}_p \cong I_p$, since the p-basis subgroup is p-pure and dense in the p-adic topology (see Proposition 29.1).

(2) Since $r_p(\prod_{p\in P} I_p) = 1$ for every p, we have $r_p(G) \leq 1$ for every pure subgroup G of the group $\prod_{p\in P} I_p$ and $G \in \mathcal{E}$. Any reduced torsion-free group can be isomorphically embedded as a pure subgroup in its **Z**-adic completion; therefore, the converse assertion follows from (1).

(3) By (1), we assume that G is a pure dense subgroup of the group $\prod_{p\in\Pi(G)} I_p$. Identifying all endomorphisms of the group G with their unique extensions to endomorphisms of the group $\prod_{p\in\Pi(G)} I_p$, we obtain a pure dense embedding of the ring $E(G)$ in the endomorphism ring of the group $\prod_{p\in\Pi(G)} I_p$ that is isomorphic to $\prod_{p\in\Pi(G)} Q_p^*$. It is sufficient to prove that every pure dense subring R of the ring $\prod_{p\in S} Q_p^*$ is an E-ring, where $S \subseteq P$. First, we note that all Q_p^* are E-rings (Example 3.5). Therefore, $\prod_{p\in S} Q_p^*$ is also an E-ring. We take some $\alpha \in E(R^+)$. Let $\hat{\alpha}$ be its extension to an endomorphism of the group $\prod_{p\in S} Q_p^*$. There exists $a \in \prod_{p\in S} Q_p^*$ such that $\hat{\alpha}(x) = ax$ for all $x \in \prod_{p\in S} Q_p^*$. Therefore, $\alpha(1) = \hat{\alpha}(1) = a1 = a \in R$. We obtain that α is the multiplication of the ring R by the element a and R is an E-ring. $\square$

The class $\mathcal{E}$ contains all pure subgroups of the group $\prod_{p\in P} Q_p$ (where $Q_p = \{s/t \in \mathbf{Q} \mid (t,p) = 1\}$) and pure subgroups of groups of p-adic integers J_p considered in Sec. 19 in connection with the study of local endomorphism rings (see Theorem 19.10).

Quasi-homogeneous groups contained in $\mathcal{E}$ are called cohesive Dubois groups. Exercises 9 and 10 from Sec. 32 contain the original definition of a cohesive group given by Dubois [115] and some properties of cohesive groups.

We call a strongly homogeneous torsion-free ring R special if $R/pR \cong F_p$ for every p such that $pR \neq R$. It can be verified that special rings coincide with strongly homogeneous pure subrings of the rings $\prod_{p\in S} Q_p^*$ for all $S \subseteq P$ (see Proposition 44.1(2)).

Theorem 44.2. *Let G be a group such that $G \in \mathcal{E}$ and $G = \operatorname{Soc} G$. Then $G = \sum_{i\in I}^{\oplus} G_i$, where every ring $E(G_i)$ is special.*

Proof. Let $V = G \otimes \mathbf{Q}$, $S = E(G) \otimes \mathbf{Q}$, and $K = \operatorname{End}_S V$. By Proposition 5.8, the S-module V is completely reducible. Let $V = \sum_{i\in I}^{\oplus} V_i$, where V_i is a homogeneous component of the S-module V ($i \in I$) and I is a set of subscripts. We set $G_i = V_i \cap G$ for every $i \in I$.

We take two different subscripts $i, j \in I$. Assume that $pG_i \neq G_i$ and $pG_j \neq G_j$ for some p. We choose a cyclic K-submodule W_i in V_i and a cyclic K-submodule W_j in V_j; their generating elements are contained in some irreducible submodules of the S-module V. We set $A = W_i \cap G$, $B = W_j \cap G$, and $C = (W_i \oplus W_j) \cap G$. By Corollary 7.2, $C \doteq A \oplus B$. Similar to the argument used in proving the implication (1) $\Longrightarrow$ (2) of Theorem 42.2, we obtain $pA \neq A$ and $pB \neq B$. Therefore, $r_p(A), r_p(B) > 0$ and $r_p(A \oplus B) > 1$. Since $C \doteq A \oplus B$, we have $r_p(C) > 1$. However, this is impossible, since the subgroup C is pure in G and $r_p(C) \leq r_p(G) \leq 1$.

The obtained contradiction means that the following assertion holds for any different subscripts $i, j \in I$ and every p: if $pG_i \neq G_i$, then $pG_j = G_j$. Therefore, $G = \sum_{i \in I}^{\oplus} G_i$ (see the proof of Theorem 42.2).

We now prove that the quasi-endomorphism ring $\mathcal{E}(G_i)$ is a division ring for every $i \in I$. For this purpose, we verify that the K-module V_i is cyclic. It is sufficient to prove that V_i is a homogeneous completely reducible module. Assume the contrary. Then V_i contains two nonzero cyclic submodules W_1 and W_2 with zero intersection. We set $A = (W_1 \oplus W_2) \cap G$ and $A_i = W_i \cap G$, $i = 1, 2$. By Corollary 7.2, $A \doteq A_1 \oplus A_2$ and $A_1 \sim A_2$. We choose a prime integer p such that $pA_1 \neq A_1$. Then $pA_2 \neq A_2$ and $r_p(A) = r_p(A_1 \oplus A_2) > 1$; this contradicts the purity of A in G. Therefore, V_i is a cyclic K-module. Then the factor ring $M(G_i)/N(G_i)$ is a division ring by Corollary 7.2. However, G_i is a direct summand in G; therefore, $M(G_i)/N(G_i) \cong \mathcal{E}(G_i)$. Therefore, $\mathcal{E}(G_i)$ is a division ring. In particular, nonzero endomorphisms of the group G_i are monomorphisms.

We can now prove that the ring $E(G_i)$ is special. For convenience, we omit the subscript i. Let φ be a nonzero endomorphism of the group G. We choose some minimal pfi-subgroup H of the group G and a nonzero element z of H. There exist an endomorphism $\xi \in E(G)$ and a positive integer k such that $\xi(\varphi z) = kz$. We consider the endomorphism $\omega\xi\varphi - k1$. If we assume that $\omega \neq 0$, then $\omega(z) = 0$ implies $\ker \omega \neq 0$, which is impossible, since all nonzero endomorphisms of the group G are monomorphisms. Therefore, $\omega = 0$ and $\varphi\xi = \xi\varphi = k1$. Therefore, $\operatorname{im}\varphi \supseteq kG$. Since all groups G/pG are cyclic, G/kG is a cyclic group. Therefore,

$$\operatorname{im}\varphi/kG = n(G/kG) = nG/kG$$

for some divisor n of the integer k. Therefore, $\operatorname{im}\varphi = nG$. We define an automorphism ψ of the group G as follows. If $x \in G$, then $\varphi x \in nG$; therefore, $\varphi x = ny$ for a unique element $y \in G$. Setting $\psi x = y$, we obtain an automorphism ψ such that $\varphi = n\psi$. We obtain that every element of the ring $E(G)$ is an integral multiple of some invertible element; therefore, the ring $E(G)$ is strongly homogeneous. Since $r_p(E(G)) \leq r_p(G)^2 \leq 1$, either $pE(G) = E(G)$ or $E(G)/pE(G) \cong F_p$. Consequently, the ring $E(G)$ is special. □

The ring $\prod_{p \in P} Q_p^*$ has no nonzero nilpotent elements. Consequently, they are absent in the ring $E(G)$ for every group $G \in \mathcal{E}$. Therefore, if $r(G) < \infty$, then $G = \operatorname{Soc} G$ by Theorem 5.11. Therefore, we obtain the following result.

Corollary 44.3 (Murley [333]). *A group $G \in \mathcal{E}$ of finite rank is equal to $\sum_{i=1}^{n\,\oplus} G_i$, where all the rings $E(G_i)$ are special.*

Corollary 44.4. *The endomorphism ring of an indecomposable group $G \in \mathcal{E}$ is special if and only if $G = \operatorname{Soc} G$.*

Proof. If the ring $E(G)$ is special, then $E(G) \otimes \mathbf{Q}$ is a field. Therefore, the $E(G) \otimes \mathbf{Q}$-module $G \otimes \mathbf{Q}$ is completely reducible and $G = \operatorname{Soc} G$. The converse assertion is contained in Theorem 44.2. □

Corollary 44.5. *For a group $G \in \mathcal{E}$, the following conditions are equivalent:*

(1) *G is a strongly homogeneous group;*
(2) *G is an irreducible group;*
(3) *$G \cong T \otimes A$, where T is a special ring and A is a group of rank 1 such that $pT = T$ if $pA = A$. In addition, $E(G) \cong T$.*

Proof. (1) $\Longrightarrow$ (2) always holds.

(2) $\Longrightarrow$ (3). Any irreducible group is homogeneous. Therefore, if $G = F \oplus H$ with $F, H \neq 0$ and $pG \neq G$, then $pF \neq F$ and $pH \neq H$. Therefore, $r_p(G) = r_p(F) + r_p(H) = 2$; this is a contradiction. We obtain that the group G is indecomposable. By Corollary 44.4, the ring $E(G)$ is special. By Proposition 41.1, the group $\operatorname{Aut} G$ acts transitively on the set of all pure subgroups of rank 1 of the group G; therefore, G is strongly homogeneous. For any nonzero element $g \in G$ and the fully characteristic subgroup $E(G)g$, the group $G/E(G)g$ is a torsion group (since G is irreducible). Therefore, G is an $E(G)$-module of rank 1. By Corollary 40.4, $G \cong E(G) \otimes A$, where $r(A) = 1$ and $t(A) = t(G)$.

(3) $\Longrightarrow$ (1). The additive group T^+ of the strongly homogeneous ring T is a strongly homogeneous group. The group G is strongly homogeneous. Then $E(G) \cong E(T^+) \cong T$ (Proposition 40.2; see also the proof of the assertion on the center in Corollary 40.5(3), which does not use the countability of the group G). $\square$

The above corollary reduces the study of strongly homogeneous groups from the class $\mathcal{E}$ to the study of special rings. After some preparation, we consider one simple ring construction used for obtaining some description of special rings (this is also important in connection with Theorem 44.2). Another method for constructing special rings of finite rank is presented in Exercise 1.

Lemma 44.6. *For a torsion-free ring R, the following conditions are equivalent:*

1. *for every $p \in \Pi(R)$ and any $a, b \in R$, if $ab \,\vdots\, p$, then either $a \,\vdots\, p$ or $b \,\vdots\, p$;*
2. *$h_p(ab) = h_p(a) + h_p(b)$ for every $p \in \Pi(R)$ and any $a, b \in R$ or, equivalently, $\chi(ab) = \chi(a) + \chi(b)$;*
3. *the ring R/pR is a domain for all $p \in \Pi(R)$.*

Proof. (1) $\Longrightarrow$ (2). If $a = p^k a'$ and $b = p^l b'$, where $k, l \in \mathbf{N}$ and $a', b' \in R$, then $ab = p^{k+l} a' b'$. Therefore, $h_p(ab) \geq h_p(a) + h_p(b)$. Assume that $h_p(ab) = n$, $h_p(a) = k$, $h_p(b) = l$ and $n > k + l$. We have $\dfrac{a}{p^k} \cdot \dfrac{b}{p^l} = \dfrac{ab}{p^{k+l}} \,\vdots\, p$, but $\dfrac{a}{p^k} \not{\vdots}\, p$ and $\dfrac{b}{p^l} \not{\vdots}\, p$; this contradicts (1). Therefore, (2) holds.

(2) $\Longrightarrow$ (3). Assume that there exists $a, b \in R \backslash pR$ such that $ab \in pR$. This is equivalent to $h_p(ab) > 0$ and $h_p(a) = 0 = h_p(b)$; this is impossible by (2).

The implication (3) $\Longrightarrow$ (1) is obvious. $\square$

Proposition 44.7. *Let R be a commutative torsion-free domain such that all rings R/pR are domains and R^+ is a quasi-homogeneous nondivisible group. Let $\widehat{R}$ be the field of fractions of the ring R, and let $\bar{R} = \{b/a \in \widehat{R} \mid a, b \in R, \chi(a) \leq \chi(b)\}$. Then $\bar{R}$ is a subring in $\widehat{R}$ such that $\bar{R}^+$ is a completely transitive group and R is a pure subring in $\bar{R}$.*

Proof. Assume that $b/a = d/c$ and $b/a \in \bar{R}$. Then $bc = ad$ and $\chi(b) + \chi(c) = \chi(a) + \chi(d)$ by Lemma 44.6. Since $\chi(a) \leq \chi(b)$, we have $\chi(c) \leq \chi(d)$. Therefore, $d/c \in \bar{R}$ and the definition of $\bar{R}$ is correct. If $a \in R$, then $\chi(1) \leq \chi(a)$ and $a = a/1 \in \bar{R}$ (i.e., $R \subseteq \bar{R}$). Let $b/a, d/c \in \bar{R}$. Then $\chi(a) \leq \chi(b)$ and $\chi(c) \leq \chi(d)$. Therefore, $\chi(ac) = \chi(a) + \chi(c) \leq \chi(b) + \chi(d) = \chi(bd)$ and $b/a \cdot d/c \in \bar{R}$. Further, we have $\chi(ac) = \chi(a) + \chi(c) \leq \chi(b) + \chi(c) = \chi(bc)$. Similarly, we have $\chi(ac) \leq \chi(ad)$. Therefore,

$$h_p(ac) \leq \min(h_p(bc), h_p(ad)) \leq h_p(bc + ad)$$

for all $p \in P$ and $\chi(ac) \leq \chi(bc + ad)$. Therefore,

$$b/a + d/c = (bc + ad)/ac \in \bar{R}$$

and $\bar{R}$ is a subring in $\widehat{R}$.

We prove that R is a pure subring in $\bar{R}$. Let $b/a \in \bar{R}$, $p \in P$, and $p(b/a) = c \in R$, and also $pR \neq R$. Canceling $h_p(a)$ out of the elements b and a, we can assume that $h_p(a) = 0$. Then $pb = ac$, whence $h_p(ac) \geq 1$. However, $h_p(ac) = h_p(a) + h_p(c) = h_p(c)$. Therefore, $h_p(c) \geq 1$ and $c = pc'$, where $c' \in R$. Therefore, $p(b/a) = c = pc'$ and $(b/a) = c' \in R$; therefore, R is pure in $\bar{R}$.

The following property will be useful later. If $b/a \in \bar{R}$, then $\chi(b/a) = \chi(b) - \chi(a)$, where the difference is well defined componentwise, since $h_p(a) \leq h_p(b)$ for all $p \in P$. We fix $p \in P$ with $pR \neq R$ and prove that $h_p(b/a) = h_p(b) - h_p(a)$. As before, we assume that $h_p(a) = 0$. Therefore, we prove that $h_p(b/a) = h_p(b)$. Let $b = p^n b'$, where $n \in \mathbf{N}$ and $b' \in R$ with $h_p(b') = 0$. If $b'/a = p(d/c)$ and $h_p(c) = 0$ (we note that $\chi(a) \leq \chi(b')$ and $b'/a \in \bar{R}$), then $b'c = pad$ and

$$h_p(b') = h_p(b') + h_p(c) = h_p(b'c) \geq 1;$$

this contradicts the choice of b'. This shows that $h_p(b'/a) = 0$. Since $b'/a = p^n(b'/a)$, we have $h_p(b/a) = n = h_p(b)$. Therefore, $\chi(b/a) = \chi(b) - \chi(a)$.

It remains to prove that $\bar{R}^+$ is a completely transitive group. Let $0 \neq b/a, d/c \in \bar{R}$ with $\chi(b/a) \leq \chi(d/c)$. Then

$$\chi(b) - \chi(a) = \chi(b/a) \leq \chi(d/c) = \chi(d) - \chi(c),$$
$$\chi(b) + \chi(c) \leq \chi(a) + \chi(d),$$

or $\chi(bc) \leq \chi(ad)$. Therefore, $ad/bc \in \bar{R}$ and $b/a \cdot ad/bc = d/c$. Therefore, the multiplication of the ring $\bar{R}$ by the element ad/bc is an endomorphism of the group $\bar{R}^+$ mapping from b/a into d/c, whence $\bar{R}^+$ is a completely transitive group. $\square$

We need one generalization of injective groups. A group G is said to be *quasi-pure injective* (or a QPI-group) if every homomorphism $A \to G$, where A is a pure subgroup of the group G, is induced by an endomorphism of the group G. It follows directly from Lemma 25.1 that all torsion-free QPI-groups are complete transitive.

We now consider some subrings of the rings $\prod_{p\in S} Q_p^*$, where $S \subseteq P$. A quasi-homogeneous pure subring of the ring $\prod_{p\in S} Q_p^*$ is said to be *cohesive* (Gardner and Stewart [173]). A ring is cohesive if and only if its additive group is cohesive. We note that the original definition of Gardner and Stewart [173] differs from the above definition (see Exercise 3 of Sec. 6). Special rings are cohesive. More exact relations between these notions are proved in the following proposition.

Proposition 44.8. *A cohesive ring R satisfies all the conditions of Proposition* 44.7. *In addition, $\bar{R}$ is a cohesive ring. If R^+ is a homogeneous group, then $\bar{R}$ is a special ring. Otherwise, $\bar{R}^+$ is a cohesive* QPI-*group without maximal elements in $\mathcal{T}(\bar{R})$.*

Proof. If $R \neq pR$, then $R/pR \cong F_p$ and R satisfies the conditions of Proposition 44.7 by the definition of a cohesive ring.

We denote by $\mathcal{A}_p$ the field of p-adic numbers. We assume that R is a pure subring in $\prod_{p\in\Pi(R)} Q_p^*$. Since the ring R is quasi-homogeneous, all p-components of every nonzero element from R are nonzero for $p \in \Pi(R)$. Therefore, nonzero elements of the ring R are invertible in $\prod_{p\in\Pi(R)} \mathcal{A}_p$. We can assume that the field of fractions $\widehat{R}$ is contained in $\prod_{p\in\Pi(R)} \mathcal{A}_p$. Under this condition, we verify that $\bar{R} \subset \prod_{p\in\Pi(R)} Q_p^*$. Indeed, assume that $b/a \in \bar{R}$, $b = \langle b_p \rangle$, and $a = \langle a_p \rangle$. Then $b_p = p^{m_p} u_p$ and $a_p = p^{n_p} v_p$, where u_p and v_p are invertible elements in Q_p^*. Since $\chi(a) \leq \chi(b)$, we have $h_p(a_p) \leq h_p(b_p)$ for every p and $m_p \geq n_p$. Therefore,

$$b_p/a_p = p^{m_p - n_p} u_p v_p^{-1} \in Q_p^* \qquad \text{and} \qquad b/a = \langle b_p/a_p \rangle \in \prod_{p\in\Pi(R)} Q_p^*.$$

We now prove that the subring $\bar{R}$ is pure in $\prod_{p\in\Pi(R)} Q_p^*$. Let $z \in \prod_{p\in\Pi(R)} Q_p^*$, $b/a \in \bar{R}$, and $pz = b/a$ for some p. We can assume that $p \in \Pi(R)$. Then $b = pza$ and $b = pb'$ for some element $b' \in R$, since R is pure in $\prod_{p\in\Pi(R)} Q_p^*$. We have $pz = p(b'/a)$ and $z = b'/a$. Since $b = pza$, we have $h_p(a) < h_p(b)$ and

$h_p(a) \le h_p(b')$. If $q \in \Pi(R)$ and $q \neq p$, then $h_q(b') = h_q(b) \ge h_q(a)$, since $b = pza$. Therefore, $\chi(a) \le \chi(b')$ and $z = b'/a \in \bar{R}$; therefore, the subring $\bar{R}$ is pure. Further, if $\bar{R} \neq p\bar{R}$, then $R \neq pR$. Then $\bigcap_{n\ge1} p^n R = 0$. Since $h_p(b/a) = h_p(b) - h_p(a)$ for every element $b/a \in \bar{R}$, we have $h_p(b/a) \le h_p(b) < \infty$ and $\bigcap_{n\ge1} p^n \bar{R} = 0$. We have proved that the ring $\bar{R}$ is cohesive.

Assume that R^+ is a homogeneous group. Let $0 \neq b/a \in \bar{R}$. Since $\chi(a) \le \chi(b)$ and $\bar{R}^+$ is a homogeneous group of idempotent type (the characteristic $\chi(1)$ consists of 0 and ∞), we have $b = nc$, where n is a positive integer, $c \in R$, and $\chi(a) = \chi(c)$. Therefore, $c/a, a/c \in \bar{R}$, and the element c/a is invertible in $\bar{R}$. Since $b/a = n(c/a)$, we obtain that the ring $\bar{R}$ is strongly homogeneous. Since the ring $\bar{R}$ is cohesive, $\bar{R}/p\bar{R} \cong F_p$ for $p\bar{R} \neq \bar{R}$; this implies that the ring $\bar{R}$ is special.

The group $\bar{R}^+$ is cohesive as the additive group of a cohesive ring. Let A be a nonzero pure subgroup of the group $\bar{R}^+$, and let $0 \neq \varphi : A \to \bar{R}^+$ be a homomorphism. We choose a nonzero element $a \in A$. Since the group $\bar{R}^+$ is completely transitive (Proposition 44.7), there exists $\alpha \in E(\bar{R}^+)$ such that $\alpha a = \varphi a$. Since $(\alpha - \varphi)a = 0$, we have $(\alpha - \varphi)A = 0$, since A is a cohesive group and every nonzero homomorphism $A \to \bar{R}^+$ is a monomorphism (see Exercise 5). Therefore, α induces φ and $\bar{R}^+$ is a QPI-group.

Assume that the set $\mathcal{T}(\bar{R})$ has maximal elements. Then $\operatorname{Soc} \bar{R}^+ \neq 0$ (see Exercise 1 of Sec. 43). Similar to the argument at the end of the proof of Theorem 44.2, we can prove that $E(\bar{R}^+)$ is a strongly homogeneous ring. However, $\bar{R} \cong E(\bar{R}^+)$. Therefore, $\bar{R}^+$ and R^+ are homogeneous groups. Consequently, if R^+ is an inhomogeneous group, then $\mathcal{T}(\bar{R})$ has no maximal elements. $\square$

Let R be a cohesive ring. It is directly verified that $\bar{\bar{R}} = \bar{R}$ and $\bar{R} = R$ for the special ring R. Proposition 44.8 states that all special rings can be obtained from cohesive rings by the application of the construction from Proposition 44.7. In addition, the use of Proposition 44.8 completes the description of strongly homogeneous groups begun in Corollary 44.5.

Example 44.9. There exists a cohesive ring R such that R^+ is an inhomogeneous group. Consequently, there exists a cohesive QPI-group G without maximal elements in $\mathcal{T}(G)$.

Proof. For every $p \in P$, we choose an invertible element $\sigma_p \in Q_p^*$ that is not an algebraical number. Then we set $a_n = \langle p^n \sigma_p^n \rangle \in \prod_{p\in P} Q_p^*$ for every $n = 0, 1, 2, \ldots$. Let R be a pure subring generated by the set $\{a_n \mid n \ge 0\}$ in $\prod_{p\in P} Q_p^*$. Assume that R has a nonzero element of infinite p-height for some p. This means that the p-component of this element in $\prod_{p\in P} Q_p^*$ is equal to 0. Therefore, there are positive integers $m_1, \ldots, m_s$ and $n_1, \ldots, n_s$ such that $m_1 p^{n_1} \sigma_p^{n_1} + \cdots + m_s p^{n_s} \sigma_p^{n_s} = 0$ (consider that $a_m a_n = a_{m+n}$). This contradicts the choice of σ_p. Therefore, $\bigcap_{n\ge1} p^n R = 0$ and the ring R is quasi-homogeneous; therefore, R is cohesive. By construction, R^+ is an inhomogeneous group. Setting $G = \bar{R}^+$, we obtain the required group G by Proposition 44.8. $\square$

The existence of a QPI-group G without maximal elements in $\mathcal{T}(G)$ is very important for the structural theory of QPI-groups.

A group G is said to be *superdecomposable* if it has no nonzero indecomposable direct summands.

Theorem 44.10. *There exists a countable superdecomposable group G such that G is the additive group of some E-ring that is a pure subring of the ring $\prod_{p\in P} Q_p$.*

Proof. Let $V = \prod_{p\in P} Q_p$, and let e_{00} be the identity element of the ring V. We note that $e_{00} = \langle 1_p \rangle$, where 1_p is the identity element of the ring Q_p. We have $e_{00} = e_{10} + e_{11}$, where e_{10} and e_{11} are some orthogonal idempotents in V containing an infinite number of nonzero components. In general, if e_{nm} is defined, then we choose some orthogonal idempotents $e_{n+1,2m} + e_{n+1,2m+1}$ in V containing an infinite number of nonzero components and such that $e_{nm} = e_{n+1,2m} + e_{n+1,2m+1}$. The obtained set $E = \{e_{nm} \mid n = 0, 1, 2, \dots ; 0 \le m < 2^n\}$ can be called a complete binary tree with root e_{00} (see Fig. 1):

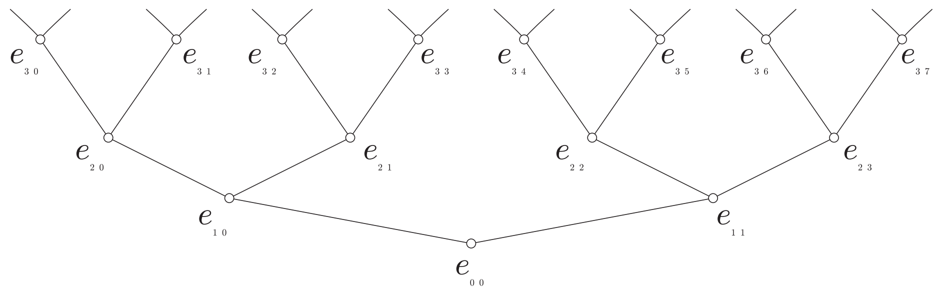

Fig. 1

Let B be the subgroup in V generated by the set E. Then B is a subring in V. Indeed, let $b, c \in B$, where

$$b = \sum_{i=1}^{k} s_i e_{n_i m_i} \quad \text{and} \quad c = \sum_{j=1}^{l} t_j e_{n'_j m'_j} \quad \text{with} \quad s_i, t_j \in \mathbf{Z}.$$

We can assume that all elements $e_{n_i m_i}$ and $e_{n'_j m'_j}$ have the same first subscript. Assume the contrary. We choose a positive integer $n \ge \max\{n_1, \dots , n_k, n'_1, \dots , n'_l\}$ and express all $e_{n_i m_i}$ and $e_{n'_j m'_j}$ in terms of the elements e_{nm} $(0 \le m < 2^n)$. We have

$$b = \sum_{m=0}^{2^n-1} s_m e_{nm} \qquad \text{and} \qquad c = \sum_{m=0}^{2^n-1} t_m e_{nm},$$

where s_m and t_m are integers. Now it is clear that $bc \in B$, since for a fixed integer n, the set $\{e_{nm} \mid 0 \le m < 2^n\}$ consists of pairwise orthogonal idempotents. Let R be the pure subring in V generated by B (equivalently, by E). For every $r \in R$, there exists a positive integer n such that $nr \in B$. We set $G = R^+$. Since R is an E-ring (Proposition 44.1), we have that $E(G)$ is canonically isomorphic to R.

It remains to prove that G is a superdecomposable group. To do this, it is sufficient to prove that every idempotent $e \ne 0$ of the ring R is equal to the sum of some e_{nm}, which directly implies that the group G is superdecomposable. We choose positive integers s and n and an integer t_m $(0 \le m < 2^n)$ such that $se = \sum t_m e_{nm}$ or $e = \sum (t_m/s) e_{nm}$ in the ring V. Since $e^2 = e$, we have $(t_m/s)^2 e_{nm} = (t_m/s) e_{nm}$ and $(t_m/s)^2 = (t_m/s)$ for all m. Therefore, either $t_m = 0$ or $t_m = s$. Therefore, the idempotent e is equal to the sum of some e_{nm} for a fixed integer n. $\square$

The existence of superdecomposable groups was proved by Corner [93] (see the end of Sec. 29). The group from Theorem 44.10 has many interesting properties. We prove only one of them; other properties are formulated in Exercise 4.

Corollary 44.11. *The superdecomposable group G constructed in Theorem* 44.10 *is a completely transitive group with zero pseudosocle.*

Proof. For every $n \ge 0$, R_n denotes the pure subring of the ring R generated by the set $\{e_{nm} \mid 0 \le m < 2^n\}$. Then R_n is an E-ring and the additive group R_n is a completely decomposable group of rank 2^n. It is clear that $G = \bigcup_{n \ge 0} G_n$, where $G_n = R_n^+$.

Let $\alpha : A \to G$, where A is a pure subgroup of rank 1 of the group G. The subgroups A and αA are contained in some subgroup G_n. The group G_n is completely transitive (Corollary 43.9); therefore, α is extended to an endomorphism of the group G_n. Since $G_n = R_n^+$, where R_n is an E-ring, this endomorphism is the multiplication of the ring R_n by some element $r \in R_n$. Then the multiplication of the ring R by the element r is an endomorphism of the group G extending α.

Assume that there exists a minimal pfi-subgroup H of the group G. Considering the structure of the group G, we can choose a nonzero element $a \in H$ such that

$$a = \sum_{0 \le m < 2^n} t_m e_{nm} \quad (t_m \in \mathbf{Z})$$

and at least two coefficients t_m are nonzero. Let t_{m_1} and t_{m_2} be the above coefficients. Since H is fully characteristic, the subgroup H is closed with respect to projections. Therefore, $t_{m_1} e_{nm_1}, t_{m_2} e_{nm_2} \in H$. There exists $p \in P$ such that $h_p(e_{nm_1}) = 0$ and $h_p(e_{nm_2}) = \infty$. However, this is impossible, since H is a homogeneous group. Therefore, G has no minimal pfi-subgroups and $\operatorname{Soc} G = 0$. $\square$

Exercise 1 (Richman [376]). A ring T is a special ring of finite rank if and only if $T = \bigcap_{P \in \Pi} J_P$, and if $pJ = P_1^{e_1} P_2^{e_2} \cdot \ldots \cdot P_n^{e_n}$, then Π contains at most P_i, and for this P_i, we have $e_i = 1$ and $J/P_i \cong F_p$ (J,J_P, Π, and P are defined in Exercise 9 of Sec. 40).

Exercise 2 (Chekhlov). For a cohesive group G, the following conditions are equivalent:

(a) G is a QPI-group;
(b) G is a completely transitive group;
(c) for the characteristic χ of every nonzero element of the group G, there is an isomorphism $E(G)^+ \cong G(\chi)$ and $E(G) = \overline{E(G)}$;
(d) G can be turned into a torsion-free module over a cohesive ring T such that $T = \bar{T}$, $\Pi(G) = \Pi(T)$, and any two linearly independent over T elements of the module G have incomparable types.

Exercise 3 (Chekhlov). For a cohesive group G, the following conditions are equivalent:

(a) G is a QPI-group and $G = \operatorname{Soc} G$;
(b) $E(G)^+ \cong G(\chi)$ for the characteristic χ of every nonzero element of G and $E(G)$ is a special ring;
(c) G can be turned into a torsion-free module over a special ring T such that $\Pi(G) = \Pi(T)$, and any two linearly independent over T elements of the module G have incomparable types.

Exercise 4. Prove that the superdecomposable group G from Theorem 44.10 has the following properties:

(a) G is a transitive group;
(b) G is not a QPI-group and for every pure subgroup A of finite rank of G, each homomorphism $A \to G$ is extended to an endomorphism of the group G;
(c) the closure in the $\mathbf{Z}$-adic topology of each pure subgroup of finite rank of G is a direct summand of G;
(d) $G = \ker \alpha \oplus \langle \operatorname{im} \alpha \rangle_*$ for every $\alpha \in E(G)$;
(e) the ring $E(G)$ is hereditary, and each of its ideals is principal (cf. Proposition 35.12).

Exercise 5. Prove that every nonzero homomorphism from a cohesive group into a reduced torsion-free group is a monomorphism.

Remarks

The notions of completely transitive or transitive torsion-free groups appeared due to Kaplansky. In [234], Kaplansky defined completely transitive and transitive modules over a complete discrete valuation ring. Completely transitive and transitive primary Abelian groups were primarily studied. The definitions of Kaplansky were extended to torsion-free groups. Then different authors naturally suggested definitions of a completely transitive (transitive) group. These definitions are formulated in terms of height matrices of elements of the group; in so doing, it is possible to consider only indicators in the primary case, and in the torsion-free case, it is possible to consider only characteristics of elements.

Any reduced group G is called completely transitive (transitive) if for any two nonzero elements a and $b \in G$ such that $H(a) \le H(b)$ $(H(a) = H(b))$, there exists an endomorphism $\alpha \in E(G)$ (an automorphism $\alpha \in \operatorname{Aut} G$) such that $\alpha a = b$ (here $H(x)$ is the height matrix of $x \in G$ considered in Sec. 27). Completely transitive groups are considered in the works of Grinshpon [193], Grinshpon and Misyakov [194–196], and Misyakov [329]. For example, they found criteria for the complete transitivity of direct sums and some direct products of groups.

The study of QPI-groups is formulated as Problem 17 in Fuchs's work [164]. These groups are of great interest to specialists. A number of mathematicians developed the structural theory of torsion-free QPI-groups (see Arnold, O'Brien, and Reid [38], Dobrusin [108,109], Krylov [259,260], and Chekhlov [79–85]). Chekhlov ([82] and [86–88]) defined and studied several classes of groups close to QPI-groups. In particular, he considered CS-groups and QCPI-groups. A group G is called a CS-*group* if every closed (in the $\mathbf{Z}$-adic topology) pure subgroup of the group G is a direct summand of G. A group G is called a QCPI-*group* if for every closed pure subgroup A of the group G, each homomorphism $A \to G$ is extended to an endomorphism of the group G.

Problem 40. Let G be a completely transitive torsion-free group such that the set $\mathcal{T}(G)$ satisfies the maximum condition. When does $G = \operatorname{Soc} G$ hold?

It seems to be of interest to establish conditions under which the well-known general constructions lead to transitive (strongly homogeneous) groups.

Problem 41. (1). Is the class of transitive (strongly homogeneous) torsion-free groups closed with respect to direct summands?

(2). Under which conditions for a transitive (strongly homogeneous) torsion-free group A is the group $\sum\limits_{\mathfrak{M}}^{\oplus} A$ transitive (strongly homogeneous)? The same question for $\prod\limits_{\mathfrak{M}} A$ ($\mathfrak{M}$ is a cardinal number).

The case of a strongly homogeneous group A was considered in Exercises 5 and 6 of Sec. 40. Direct sums and products of torsion, torsion-free, or mixed completely transitive groups are studied in the works of Grinshpon and Misyakov [194–196] and Misyakov [329].

Problem 42. Assume that G and H are strongly homogeneous (homogeneous completely transitive) groups. Are the groups $G \otimes G$, $G \otimes H$, $\operatorname{Hom}(G, G)$, or $\operatorname{Hom}(H, G)$ strongly homogeneous (homogeneous completely transitive)?

Problem 43. If G is a torsion-free QPI-group, then $G = \overline{\operatorname{Soc} G} \oplus H$ for some group H, where $\overline{\operatorname{Soc} G}$ is the closure in the $\mathbf{Z}$-adic topology of the subgroup $\operatorname{Soc} G$ (this can be obtained from works of Dobrusin [108] and Chekhlov [83]). For which completely transitive torsion-free groups G does a similar result hold?

A torsion-free group G is said to be *weakly transitive* if for any nonzero elements a and b of G with $\chi(a) = \chi(b)$, there exists $\alpha \in E(G)$ such that $\alpha a = b$.

Problem 44. Do there exist weakly transitive groups that are not transitive or completely transitive?

The remaining three problems are related to arbitrary transitive or completely transitive groups.

Problem 45 (Grinshpon and Misyakov)**.** For some groups G_i, find necessary and sufficient conditions for the complete transitivity (transitivity) of the direct sum $\sum^{\oplus} G_i$ and the product ΠG_i.

Problem 46. Reveal when is the factor group $\Pi G_i / \sum^{\oplus} G_i$ a QPI-group (transitive, completely transitive), where G_i $(i \in I)$ is an arbitrary family of groups.

It is known that if $|I| = \aleph_0$, then the above factor group is pure-injective (Fuchs [164, Corollary 42.2]).

Problem 47. Is every QPI-group transitive?

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