Electromagnetic Induction within the Earth

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Abstract. A method is given by which we can calculate response characteristics of a model earth whose electric conductivity is a given function of r (distance from the center of the earth) to exterior periodic magnetic variations. The essence of the method is to solve numerically a fundamental differential equation of electromagnetic induction by using electronic computers. Numerical examples are worked out for a model earth proposed by T. Rikitake. In the case of impulsive magnetic variations, we must make a Fourier integral representation of observed data to make comparisons with the above theoretical results. An example of this interpretation is given for the D_{st} variations studied by S. Chapman and A. T. Price. Using variational calculus, we can derive formulas relating variations of response characteristics of the sphere and variations of its conductivity

Introduction. Electromagnetic induction in a conducting sphere and its applications to geophysical problems have been studied by many authors [*Lamb,* 1883, 1889; *Chapman and Whitehead,* 1923; *Chapman and Price,* 1930; *Price,* 1930, 1932; *Lahiri and Price,* 1939; *Terada,* 1939, 1948; *Rikitake,* 1950, 1951]. Computational difficulties restricted the model earths used in the above papers to those for which the solutions of the problem are given by known and tabulated functions. It is our purpose in the present paper to give a systematic method for handling the problem with electronic computers. In this way, we can be free from the above restrictions and other computational troubles. Thus our problem will be to get amplitude and phase lag characteristics for a model earth whose electric conductivity is a given function of the distance r from the center of the sphere. We assume a sinusoidal time variation e^{-ipt}. This type of time variation is suited to the analysis of S_q, but it does not appear to be suitable to that of storm time variations, D_{st}. In fact, in all analyses of D_{st} previously made, the only mathematical tool used was the Laplace transform. In that kind of analysis, we must first get an approximate operational form for the input function $e_n(t)$. After getting a corresponding operational form for the output function $i_n(t)$, we must interpret this in t- (time) space. We encounter complicated residue calculations here, and we do not know how many poles we must take into account to get perfect convergence. Furthermore, the interpretation can be done for very small and very large values of t, but not for intermediate t. We can avoid all these difficulties by assuming the sinusoidal time variation. In return, we must make a Fourier integral representation of D_{st} data for our present method. As we shall see, we can do this easily.

Mathematical formulation. The fundamental equation for the present problem is

$$\frac{d}{dr}\left(r^2 \frac{df}{dr}\right) = (N^2 - iP\sigma r^2)f \qquad (1)$$

where $f(r)$ is a radial function of the r component of the magnetic field, r is the distance from the center of the sphere in units of its radius a, σ is the electric conductivity as a function of r, and

$$N^2 = n(n + 1) \quad P = 4\pi a^2 \sigma_0 p \qquad (2)$$

In (2), n is the azimuthal wave number and σ_0 is a representative conductivity in emu. Angular parts of magnetic field components are functions of surface spherical harmonics, and electric conductivity σ is in units of σ_0. Since radial parts of the r, θ, and ϕ components of the magnetic field are $f(r)$, df/dr, and df/dr, respectively, we can satisfy the boundary conditions that these components must be continuous at discontinuity surfaces, if any, by requiring that $f(r)$ and $f'(r) = df/dr$ be continuous there. As we did in the theory of free oscillations of the earth [*Alterman et al.,* 1959], we can write (1) in the following form:

$$y_l'(r) = \sum_m \widehat{lm}(r) y_m(r) \qquad l,\ m = 1, 2, 3, 4 \qquad (3)$$

TABLE 1. Theoretical Results for the Rikitake Model

n	m	$(\alpha^2+\beta^2)^{1/2}$	$\tan^{-1}(\beta/\alpha)$	r_0
2	1	2.345	−7°4′	0.76
3	2	2.357	−7°16′	0.82
4	3	2.517	−7°47′	0.82
5	4	2.758	−8°26′	0.84
1	1	2.615	−4°15′	0.76
4	1	2.795	−12°41′	0.76

where

$$f = y_1 + iy_3, \quad f' = y_2 + iy_4 \tag{4}$$

and

$$\widehat{12} = 1, \widehat{21} = \frac{N^2}{r^2}, \widehat{22} = -\frac{2}{r}, \widehat{23} = P\sigma,$$

$$\widehat{34} = 1, \widehat{41} = -P\sigma, \widehat{43} = \frac{N^2}{r^2}, \widehat{44} = -\frac{2}{r},$$

$$\widehat{lm} = 0 \quad \text{otherwise}$$

The boundary conditions require that the y's be continuous at discontinuity surfaces. Equation 3 can easily be integrated numerically for given initial values of y. In order to get appropriate initial values, we assume that the conductivity, σ, is constant and very large for $r \leq r_0$, where r_0 is chosen appropriately. In the studies referred to above, it is concluded that the electrical conductivity in the deeper part of the earth is very large, so the assumption is justified. For $r \leq r_0$, we get $P \to \infty$ and $f \to 0$, so we may assume rather tentatively that

$$y_1 = y_2 = y_3 = 0$$
$$\text{and} \quad y_4 = 1 \quad \text{at} \quad r = r_0 \tag{5}$$

Alternatively, we might assume that $y_1 = y_2 = 0$ and $y_3 = y_4 = 1$ at $r = r_0$. The solution thus obtained will differ by a complex constant from that obtained for (5). However, final adjustment of the constant will be made in (11) and (12).

Let us now examine boundary conditions at the earth's surface, $r = 1$. The functions $f(r)$ and $f'(r)$ for $r \geq 1$ are usually given as

$$f = \frac{e_n r^n}{n+1} - \frac{i_n r^{-(n+1)}}{n}$$

$$f' = \frac{n}{n+1} e_n r^{n-1} + \frac{n+1}{n} i_n r^{-(n+2)} \tag{6}$$

where $e_n(P)$ and $i_n(P)$ are constants denoting exterior and interior potentials, respectively. The boundary conditions require that the expressions for f and f' in (4) be equal to those in (6) at $r = 1$. The equations thus obtained,

$$e_n = \frac{n+1}{2n+1}\{[(n+1)y_1 + y_2] + i[(n+1)y_3 + y_4]\}$$

$$i_n = \frac{n}{2n+1}\{(-ny_1 + y_2) + i(-ny_3 + y_4)\} \tag{7}$$

will determine a complex amplitude ratio, $I_n(P)$, as follows:

$$I_n(P) = \frac{i_n}{e_n} = \frac{n}{n+1} \cdot \frac{(-ny_1 + y_2) + i(-ny_3 + y_4)}{\{(n+1)y_1 + y_2\} + i\{(n+1)y_3 + y_4\}}$$

The real and imaginary parts of $I_n(P)$ will be denoted by α and β, respectively; that is, $I_n(P) = \alpha + i\beta$. The absolute value of the amplitude ratio and the phase angle are given by $(\alpha^2 + \beta^2)^{1/2}$ and $\tan^{-1}(\beta/\alpha)$, respectively. Thus our problem is reduced to calculating $I_n(P)$ for a given earth model and given values of n and P.

Example. Our way of attacking the problem will now be demonstrated in an example. We adopt the model described by *Rikitake* [1951] as the earth model in our numerical example. In his model, the conductivity is 10^{-15} emu for $0.94 \leq r \leq 1.0$, and for $r \leq 0.94$ it is given by $1.0 \times 10^{-12}\, r^{-11}$ emu. In our computation we put $\sigma_0 = 1.0 \times 10^{-15}$ emu in (2), so we have

$$\sigma = 1.0 \quad \text{for} \quad 0.94 \leq r \leq 1.0$$
$$\text{and} \quad \sigma = 10^3 r^{-11} \quad \text{for} \quad r \leq 0.94 \tag{8}$$

We first calculate $I_n(P)$ for $n = 2$, $m = 1$; $n = 3$, $m = 2$; $n = 4$, $m = 3$; $n = 5$, $m = 4$; $n = 1$, $m = 1$; and $n = 4$, $m = 1$. Having S_q-type variations in mind, we put $p = (2\pi m)/(24 \times 60 \times 60)$. Results of the calculations are in Table 1.

The most difficult point in the computations seems to be the choice of r_0 in (5). Too large a value for r_0 will not be justified from a geophysical point of view, and too small a value will give unreliable numerical results owing to the

TABLE 2. Spherical Harmonic Analyses of S_q Field

		Chapman		Hasegawa		Benkova	
n	m	$(\alpha^2+\beta^2)^{1/2}$	$\tan^{-1}(\beta/\alpha)$	$(\alpha^2+\beta^2)^{1/2}$	$\tan^{-1}(\beta/\alpha)$	$(\alpha^2+\beta^2)^{1/2}$	$\tan^{-1}(\beta/\alpha)$
2	1	2.8	−13°	2.30	−9°	2.34	−5°
3	2	2.2	−18°	2.43	−10°	2.30	−5°
4	3	2.5	−21°	2.25	−14°		
5	4	2.7	−23°				
1	1	2.5	−7°	1.98	−2°	2.15	3°

exponential decay for small r. Too small a value can be avoided by repeating the numerical integrations and successively halving integration steps to see whether convergent results are obtained. The results in Table 1 turned out to be quite independent of the choice of r_0 thus made. The r_0 values in Table 1 are shown only for the reader's convenience. A slightly different choice of r_0 will not change the results in the table at all. The reason for this will be given later.

For his model, Rikitake found $(\alpha^2 + \beta^2)^{1/2} =$ 2.34, 2.35, and 2.51 and $\tan^{-1}(\beta/\alpha) = -7.0°$, $-7.4°$, and $-7.7°$ for $n = 2$, $m = 1$; $n = 3$, $m = 2$ and $n = 4$, $m = 3$, respectively. Our results in Table 1 agree well with his and justify our method. In Table 2 the results of spherical harmonic analyses of the S_q field by *Chapman* [1919], *Benkova* [1940], and *Hasegawa* [1943] are shown. For $n = 2$, $m = 1$ and $n = 3$, $m = 2$, the results in Table 1 agree well with mean values of those by Benkova and Hasegawa. For $n = 4$, $m = 3$; $n = 5$, $m = 4$; and $n = 1$, $m = 1$, amplitude ratios in Table 1 agree well with those obtained by Chapman.

Let us now examine whether Rikitake's model works well for storm time variations, D_{st}. The characteristics of D_{st} are that $n = 1$ and $m = 0$, and the period T is 1 to 5 days. Results of our calculations for $n = 1$, $m = 0$ and $T = 1$, 2, and 3 days are shown in Table 3. In Figure 1 the famous result by *Chapman and Price* [1930] is given, showing time variations of $e(t)$ and $i(t)$ for a magnetic storm. In order to get data compared with those in Table 3, we must make a Fourier analysis of $e(t)$ and $i(t)$:

$$e(P) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} e(t)e^{ipt}\, dt$$

$$i(P) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} i(t)e^{ipt}\, dt$$

Chapman and Price's analysis stopped 48 hours after the beginning of the storm. We first made the analysis assuming $e(t)$ and $i(t)$ are zero after that time and got the amplitude ratios $e(P)/i(P)$ in the second row in Table 4. It will certainly be better to make the analysis by using the extrapolation formulas $e(t)$, $i(t) \sim e^{-\omega t}$, $\omega = 3.2 \times 10^{-6}$ sec^{-1} by *Lahiri and Price* [1939] for t larger than 48 hours. Values of $e(P)/i(P)$ thus obtained are shown in the third row in Table 4. It is interesting that there is almost no difference between the results in the second and third rows and that, except for $T = 2$ days, values of $e(P)/i(P)$ in Table 4 agree well with the theoretical values in Table 3. Thus we see that Rikitake's model is successful in explaining the storm time variations. Theoretical results for shorter periods ($T = 3$ hours, 1 hour, and 3 minutes) are also shown in Table 3. They may be useful in interpreting bay-type disturbances and sudden commencements.

Variational principles. In making the calculations in Table 3, we find that α and β, and accordingly $(\alpha^2 + \beta^2)^{1/2}$ and $\tan^{-1}(\beta/\alpha)$, are stationary for small perturbations in the mode functions, y. The reason for this is understood as follows. Multiplying (1) by f and integrating the result with respect to r, we get

$$r^2ff' = \int \{(rf')^2 + N^2f^2\}\, dr - iP \int \sigma r^2 f^2\, dr = I \qquad (9)$$

The left-hand side of (9) must be calculated at the upper and lower boundaries of each continuous layer, and the difference between the sums for upper and lower boundaries must be taken. In making variations of f and determin-

TABLE 3. Theoretical Results for the Rikitake Model

Period	$(\alpha^2 + \beta^2)^{1/2}$	$\tan^{-1}(\beta/\alpha)$	r_0
3 days	2.772	$-6°45'$	0.70
2 days	2.704	$-5°43'$	0.70
1 day	2.615	$-4°15'$	0.76
3 hours	2.479	$-1°40'$	0.86
1 hour	2.447	$-1°6'$	0.90
3 minutes	2.384	$-2°12'$	0.94

ing what will occur in (9), we shall always satisfy the boundary conditions at discontinuity surfaces. With this understanding and using the fundamental equation (1), we get from (9)

$$\delta\{(r^2 ff')_{r=1}\} = 0$$
$$= \delta\left\{\left(\frac{e_n}{n+1} - \frac{i_n}{n}\right)\left(\frac{ne_n}{n+1} + \frac{n+1}{n} i_n\right)\right\} \quad (10)$$

Without loss of generality, we may specify that

$$e_n = 1 + 0i \quad (11)$$

in our calculation. Referring to (7), we see that we can do this by multiplying the y's obtained from the numerical integrations by the following constant:

$$\frac{(2n+1)(A - iB)}{(n+1)(A^2 + B^2)} \quad (12)$$

where $A = \{(n+1)y_1 + y_2\}_{r=1}$ and $B = \{(n+1)y_3 + y_4\}_{r=1}$. From (10) and (11) we get

$$\delta\left\{\left(\frac{1}{n+1} - \frac{1}{n} I_n\right)\left(\frac{n}{n+1} + \frac{n+1}{n} I_n\right)\right\} = 0$$

TABLE 4. Fourier Integral Interpretations of D_{st} Variations in Figure 1

Period, days	1	2	3	4	5
$e(P)/i(P)$	2.81	2.27	2.61	2.68	2.71
	2.74	2.07	2.60	2.68	2.71

which shows the stationary characters of α and β found 'experimentally.' Similarly, differentiating (9) with respect to $\sigma(r)$, we get

$$\left\{-\frac{2(n+1)\alpha}{n^2} + \frac{1}{n(n+1)}\right\}\frac{\partial\alpha}{\partial\sigma'} + \frac{2(n+1)\beta}{n^2}\frac{\partial\beta}{\partial\sigma'} = Pr^2 \,\mathrm{Im}\,(f^2)$$
$$-\frac{2(n+1)\beta}{n^2}\frac{\partial\alpha}{\partial\sigma'} + \left\{-\frac{2(n+1)\alpha}{n^2} + \frac{1}{n(n+1)}\right\}\frac{\partial\beta}{\partial\sigma'} = -Pr^2 \,\mathrm{Re}\,(f^2) \quad (13)$$

where $\Delta\sigma'$ ($= \partial\sigma'$ in equation 13) is the variation of σ ($\Delta\sigma$) multiplied by Δr over which the variation takes place. These equations can be used to estimate variations of α and β, or amplitude ratio and phase angle, resulting from a variation of $\sigma(r)$.

The current density **C** is given by

$$\mathbf{C} = iap\sigma f\left(0, \frac{\partial S_n^m}{\sin\theta\partial\phi}, \frac{\partial S_n^m}{\partial\theta}\right)$$

and the r dependent part of $|\mathbf{C}|$, $\sigma|f|$, is shown in Table 5 together with $(\partial/\partial\sigma')$ $(\alpha^2 + \beta^2)^{1/2}$ and $(\partial/\partial\sigma')$ (β/α) for the case $n = 2$ and $m = 1$ in Table 1. Note that f is normalized as in (11) and (12). Suppose we have $\Delta\sigma = 10^8$,

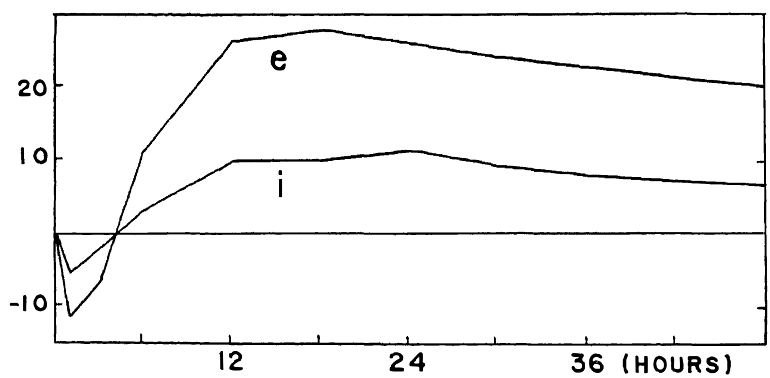

Fig. 1. Variations of $e(t)$ and $i(t)$ during a magnetic storm.

TABLE 5. Partial Derivatives of Response Characteristics with Respect to $\sigma(r)$

r	$\sigma\|f\|$	$\partial/\partial\sigma'(\alpha^2+\beta^2)^{1/2}$	$\partial/\sigma'(\beta/\alpha)$
1.00	0.125	-79.4×10^{-4}	218×10^{-4}
0.98	0.0993	−56.0	115
0.96	0.0742	−35.5	41.9
0.94	0.0508	−17.8	−2.48
	100.3	−17.8	−2.48
0.92	81.4	−4.52	−12.8
0.90	63.1	53.0×10^{-6}	−5.78
0.88	46.3	74.0	-58.7×10^{-6}
0.86	31.9	14.0	38.0
0.84	20.3	−2.82	9.60
0.82	11.9	-79.5×10^{-8}	−1.19
0.80	6.30	9.99	-25.5×10^{-8}
0.78	3.12	1.05	3.95
0.76	0.0	0	0

$\Delta r = 0.01$, and $\Delta\sigma' = \Delta\sigma\Delta r = 10$ immediately under $r = 0.94$. Since σ and r are in units of 10^{-15} emu and a (= 6370 km), respectively, and σ in this region is 2.0×10^{-12} emu, the above $\Delta\sigma'$ corresponds to a 50 per cent increase of σ over the 63.7-km range immediately under $r = 0.94$ (at a depth of 400 km). From Table 5, the corresponding amplitude ratio variation will be $-0.00178 \times 10 = -0.0178$, a little less than a 1 per cent decrease of $(\alpha^2 + \beta^2)^{1/2} = 2.345$ (Table 1). This is the way we use Table 5 in geophysical problems.

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