EFFICIENT SYMPLECTIC INTEGRATION OF SATELLITE ORBITS

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(Received: 9 March 1999; accepted: 22 July 1999)

Abstract. The use of the extended phase space and time transformations for constructing efficient symplectic methods for computing the long term behavior of perturbed two-body systems are discussed. Main applications are for artificial satellite orbits. The methods suggested here are efficient also for large eccentricities.

Key words: symplectic integration, satellite orbits

1. Introduction

Since the publication of the Wisdom–Holman mapping method (Wisdom and Holman, 1991; Kinoshita et al., 1991), symplectic methods have gained increasing popularity in Solar System studies. Recently Mikkola (1997) suggested the use of time transformation to improve the treatment of highly eccentric orbits. This has been further studied by Rauch and Holman (1999) and found to be an essential improvement in some respects. Recently interest has arisen in the application of symplectic methods for artificial satellite orbits (Mikkola et al., 1999).

The problem of computing satellite orbits differ in some respects from Solar System calculations. The oblateness potential is usually the most important perturbation and this is strong near the earth. Also air drag may be present. Drag forces are non-canonical and the equations of motion are not symplectic any more, but this is inconsequential if the drag is weak (Mikkola, 1998). Thus the most important new feature in the problem is the presence of strong perturbing potential near pericentre.

The original Wisdom–Holman method and the time regularized version suggested by Mikkola use mean anomaly and the eccentric anomaly as independent variable, respectively. (Although this is accurate only for unperturbed motion, we will use this notion for its easy understandability.) In this paper the various possibilities for computing (eccentric) satellite orbits using time transformations are discussed. A new feature is the possibility of using the true anomaly as the new independent variable. Furthermore all the possibilities, mean, eccentric and true anomaly are encoded into a single time-transformation function. One can then choose between them, or mix them, simply by adjusting numerical parameters.

Celestial Mechanics and Dynamical Astronomy **74:** 275–285, 1999.

2. Time Transformations for Perturbed Two-Body Motion

Recently I considered (Mikkola, 1997) the use of time transformation in constructing symplectic integration methods for few body problem. For the two-body problem the suggested transformation takes the form $\mathrm{d}t = r\mathrm{d}s$, where s is the new independent variable, r the distance and t the physical time. This is approximately equivalent to using the eccentric anomaly as the independent variable. This transformation has favorable properties for eccentric orbits (Rauch and Holman, 1999). However, if we consider eccentric orbits of artificial satellites, then almost all the perturbation occurs near pericentre. Thus, it may be worth considering, e.g., the use of the true anomaly instead of the eccentric anomaly. The actual time transformation would thus be $\mathrm{d}t = r^2\mathrm{d}s$. Since in symplectic integration the steplength in the independent variable must be kept constant, this alternative would concentrate the perturbation evaluations near pericentre and improvement in precision (for the same number of perturbation evaluations per period) is likely.

It is possible to parameterize the time transformation so as to include all the alternatives, mean, eccentric and true anomaly, as well as any linear combination in one expression:

$$\mathrm{d}s = \frac{\mathrm{d}t}{\mu(r)}, \tag{1}$$

$$\mu(r) = \frac{1}{B_0 + B_1/r + B_2/r^2}, \tag{2}$$

where B_0, B_1, B_2 are adjustable parameters (constants). Let us consider the Hamiltonian $H = K + R$, where $K = \mathbf{p}^2/2 - m/r$ is the Keplerian part and $R = R(\mathbf{r}, t)$ is the perturbing potential. Following Mikkola (1997) we apply the Poincare transformation to obtain the time-transformed Hamiltonian Γ in the form

$$\Gamma = \mu(r)(K + p_0 + R) = \mu(r)(K + p_0) + \mu(r)R(\mathbf{r}, t), \tag{3}$$

where p_0 is the momentum conjugate to the physical time, which now is a coordinate while s is the independent variable. For p_0 the initial value $p_0 = -(K + R)_{/t=0}$ must be used to make the Hamiltonian $\Gamma(t) = 0$ along the orbit and consequently ensure the correctness of the Hamiltonian equations of motion:

$$\mathbf{r}' = \frac{\partial \Gamma}{\partial \mathbf{p}}, \quad \mathbf{p}' = -\frac{\partial \Gamma}{\partial \mathbf{r}}, \quad t' = \frac{\partial \Gamma}{\partial p_0} \quad \text{and} \quad p_0' = -\frac{\partial \Gamma}{\partial t},$$

(prime denoting derivative with respect to the new independent variable s).

If the generalized leapfrog method is used, an integration step may be written as

$$\Gamma_0(\tfrac{1}{2}h_s)\Gamma_1(h_s)\Gamma_0(\tfrac{1}{2}h_s), \tag{4}$$

where $\Gamma_0 = \mu(r)(\mathbf{p}^2/2 - m/r + p_0)$ represents the motion of the system according to the 'unperturbed' part of the Hamiltonian (3) and $\Gamma_1 = \mu(r)R(\mathbf{r}, t)$ represents the kick of the momenta due to the perturbation. The argument tells the length of the motion in terms of the independent variable step h_s. Note that when steps are repeated we can combine the half length Γ_0 steps and thus more generally the leapfrog algorithm should be written as

$$\Gamma_0(\tfrac{1}{2}h_s)\Gamma_1(h_s)\Gamma_0(h_s)\Gamma_1(h_s)\cdots\Gamma_1(h_s)\Gamma_0(\tfrac{1}{2}h_s), \tag{5}$$

and half steps only appear at the beginning and at the end (at output).

The Γ_1 step here is simple, but the motion under the Hamiltonian Γ_0 is not completely trivial. To find the solution it is convenient to transform back to the physical time system. Like in Mikkola (1997) we transform back to the physical time system using once again the Poincare transformation. The result is

$$H_0 = \frac{1}{\mu(r)}[\mu(r)(K + p_0) - \gamma_0] = K + p_0 - \frac{\gamma_0}{\mu(r)}, \tag{6}$$

where γ_0 is the constant (over one step) numerical value of Γ_0 computed at the beginning of the step. When we now substitute the suggested expression for the time transformation function $\mu(r)$ we get

$$\begin{aligned} H_0 &= \frac{\mathbf{p}^2}{2} - \frac{m}{r} + p_0 - \gamma_0\left(B_0 + \frac{B_1}{r} + \frac{B_2}{r^2}\right) \\ &= \frac{\mathbf{p}^2}{2} - \frac{m + \gamma_0 B_1}{r} - \frac{\gamma_0 B_2}{r^2} + p_0 - \gamma_0 B_0. \end{aligned} \tag{7}$$

We note that if $B_2 = 0$, this has the form of a two-body Hamiltonian with a modified mass. However, when $B_2 \neq 0$ a small extra term with $1/r^2$ dependence appears. This could be handled using symplectic approximation, but it appears that the analytical solution is fairly simple and can be anyway done using the well-known two-body formulae: if we use the fact that the motion is planar and transform to polar coordinates (r, θ) the above Hamiltonian can be written

$$H_0 = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2 - 2\gamma_0 B_2}{r^2}\right) - \frac{\widetilde{m}}{r}, \tag{8}$$

where the unnecessary constant term has been dropped and $\widetilde{m} = m + \gamma_0 B_1$ is the modified mass. The generating function $S = \psi p_\psi = \psi p_\theta\sqrt{1 - 2\gamma_0 B_2/p_\theta^2}$ gives the canonical transformation $p_\psi^2 = p_\theta^2 - 2\gamma_0 B_2$, $\psi = \theta/\sqrt{1 + 2\gamma_0 B_2/p_\psi^2}$. After this the Hamiltonian has again the two-body form

$$H_0 = \frac{1}{2}\left(p_r^2 + \frac{p_\psi^2}{r^2}\right) - \frac{\widetilde{m}}{r}. \tag{9}$$

Let us consider the f, g formulation of the two-body motion. We have $\mathbf{r} = f\mathbf{r}_0 + g\mathbf{v}_0$, where $f = f(r_0, v_0^2, \dot{r}_0, t)$ and $g = g(r_0, v_0^2, \dot{r}_0, t)$ depend only on the initial distance, its time derivative and the square of velocity. Thus, in the modified motion the only change in the computation of these quantities is that the square of velocity is to be taken from the modified motion $w^2 = \dot{r}_0^2 + p_\psi^2/r_0^2$, while all other quantities are unaffected. The motion of the distance r can be now written in universal variables (Stumpff, 1962) as $r = r_0 + \eta h c_1(\beta h^2) + \zeta h^2 c_2(\beta h^2)$, where $h = \int \mathrm{d}t/r$, $\eta = \mathrm{d}r/\mathrm{d}h$, $\zeta = \mathrm{d}^2r/\mathrm{d}h^2$ and $\beta = 2\widetilde{m}/r_0 - w^2$. To obtain the position, however, we cannot apply the f, g-formulation anymore because the angular velocity in the modified ellipse is $\dot{\psi} = p_\psi/r^2$ and not $\dot{\theta} = p_\theta/r^2$. Thus, we use the more general formulation in terms of the rotation angle θ (understood to have the value 0 at the beginning of the step) and the angular momentum vector $\mathbf{p}_\theta = \mathbf{r}_0 \times \mathbf{v}_0$.

$$\mathbf{r} = \frac{r}{r_0}\left(\cos(\theta)\mathbf{r}_0 + \frac{\sin(\theta)}{p_\theta}\mathbf{p}_\theta \times \mathbf{r}_0\right), \qquad \mathbf{v} = \frac{\dot{r}}{r}\mathbf{r} + \frac{1}{r^2}\mathbf{p}_\theta \times \mathbf{r}, \tag{10}$$

which apply to any central motion. The constant stepsize of the independent variable s gives the equation

$$h_s = \int \left(B_0 + \frac{B_1}{r} + \frac{B_2}{r^2}\right) \mathrm{d}t = B_0\Delta t + B_1 h + \frac{B_2\psi}{p_\psi}, \tag{11}$$

which has the role of Kepler's equation in this formulation. It must be solved for h when h_s is given. As function of h the physical timestep has the form $\Delta t = r_0 h + \eta h c_2(\beta h^2) + \zeta h^2 c_3(\beta h^2)$, while from the relation $\tan(\psi) = |\mathbf{r} \times \mathbf{r}_0|/(\mathbf{r} \cdot \mathbf{r}_0)$ one can derive the result

$$\psi = \mathrm{atan}\left(\frac{p_\psi g}{r_0^2 f + \eta g}\right). \tag{12}$$

In Equation (11) we actually need the quantity $y_\psi = \psi/p_\psi$, and in (10) the angle $\theta = p_\theta y_\psi$ is required instead of ψ. If the orbit has very low (or even zero) angular momentum, it may happen that $p_\psi^2 < 0$, which causes problems in the computation of y_ψ as well as in evaluation of $\sin(\theta)/p_\theta$ in the expression for $\mathbf{r}$ in (10). However, all the formulae are still valid and they can be 'universalized' by means of the following modifications: Write

$$\xi = \frac{g}{r_0^2 f + \eta g}, \tag{13}$$

then

$$y_\psi = \frac{\mathrm{atan}(p_\psi \xi)}{p_\psi} = \xi\ \mathrm{at}_1(p_\psi^2 \xi^2), \tag{14}$$

where the new function $\mathrm{at}_1(z)$ is defined as

$$\begin{aligned}\mathrm{at}_1(z) &= \frac{\mathrm{atan}(\sqrt{z})}{\sqrt{z}} \\ &= \frac{1}{2}\log\left(\frac{1+\sqrt{-z}}{1-\sqrt{-z}}\right)\Big/\sqrt{-z} \\ &= 1-\frac{z}{3}+\frac{z^2}{5}-\frac{z^3}{7}+\frac{z^4}{9}-\frac{z^5}{11}+-\cdots,\end{aligned} \tag{15}$$

and the series expansion is used when the argument z is small (the normal situation), while for larger absolute values of z one or the other of the closed form expressions are chosen depending on the sign of the argument.

The expression $\sin(\theta)/p_\theta$ in the formula for $\mathbf{r}$ has the form $0/0$ if $p_\theta = 0$. This can be avoided by use of Stumpff's c-functions, which give $\cos(\theta) = c_0(\theta^2)$, $\sin(\theta)/p_\theta = y_\psi c_1(\theta^2)$. Thus, we get for the position vector the universal expression

$$\mathbf{r} = \frac{r}{r_0}\left(c_0(\theta^2)\mathbf{r}_0 + y_\psi c_1(\theta^2)\mathbf{p}_\theta \times \mathbf{r}_0\right). \tag{16}$$

Strictly, the modification of the computation of $\cos(\theta)$ is not necessary, but it is simpler to use c_0 if the Stumpff functions are computed anyway.

3. Collection of Formulae

In the above section we have obtained for the motion under the Hamiltonian Γ_0 an algorithm which we now present in detail.

Let $\mathbf{r}_0$, $\mathbf{v}_0$ and t_0 be the values of position vector, velocity vector and time, respectively, in the beginning of a Γ_0 step, so that the equations below can be used to obtain new values of these quantities at the end of the step, which has the length h_s in the independent variable. Note that the variable p_0 does not change during this step. The special function at_1 is defined in (15), and the c_ν's are the Stumpff functions.

$$\mu = \frac{1}{B_0 + B_1/r_0 + B_2/r_0^2}, \tag{17}$$

$$\gamma_0 = \left(\frac{\mathbf{v}_0^2}{2} - \frac{m}{r_0} + p_0\right)\mu, \tag{18}$$

$$\widetilde{m} = m + \gamma_0 B_1, \tag{19}$$

$$\epsilon = \gamma_0 B_2, \tag{20}$$

$$\mathbf{p}_\theta = \mathbf{r}_0 \times \mathbf{v}_0, \tag{21}$$

$$p_\theta^2 = |\mathbf{p}_\theta|^2, \tag{22}$$

$$\eta = \mathbf{r}_0 \cdot \mathbf{v}_0, \tag{23}$$

$$\dot{r}_0 = \frac{\eta}{r_0}, \tag{24}$$

$$p_\psi^2 = p_\theta^2 - 2\epsilon, \tag{25}$$

$$w^2 = \dot{r}_0^2 + \frac{p_\psi^2}{r_0^2}, \tag{26}$$

$$\beta = \frac{2\widetilde{m}}{r_0} - w^2, \tag{27}$$

$$\zeta = \widetilde{m} - r_0\beta. \tag{28}$$

Starting value for h

$$h = \frac{h_s}{B_0 r_0 + B_1 + B_2/r_0}. \tag{29}$$

Beginning of iteration loop (for h)

$$G_\nu = h^\nu c_\nu(\beta h^2), \tag{30}$$

$$r = r_0 + \eta G_1 + \zeta G_2, \tag{31}$$

$$f = 1 - \frac{\widetilde{m} G_2}{r_0}, \tag{32}$$

$$g = r_0 G_1 + \eta G_2, \tag{33}$$

$$\xi = \frac{g}{r_0^2 f + \eta g}, \tag{34}$$

$$y_\psi = \frac{\operatorname{atan}(p_\psi \xi)}{p_\psi} = \xi \ \mathrm{at}_1(p_\psi^2 \xi^2), \tag{35}$$

$$\Delta t = r_0 h + \eta G_2 + \zeta G_3, \tag{36}$$

$$d = B_0 r + B_1 + \frac{B_2}{r}, \tag{37}$$

$$h = h - \frac{B_0 \Delta t + B_1 h + B_2 y_\psi - h_s}{d}. \tag{38}$$

End of iteration loop (exit at convergence)

$$\dot{r} = \frac{\eta G_0 + \zeta G_1}{r}, \tag{39}$$

$$\theta^2 = p_\theta^2 y_\psi^2, \tag{40}$$

$$\mathbf{r} = \frac{r}{r_0}(c_0(\theta^2)\mathbf{r}_0 + y_\psi c_1(\theta^2)\mathbf{p}_\theta \times \mathbf{r}_0), \tag{41}$$

$$\mathbf{v} = \frac{\dot{r}}{r}\mathbf{r} + \frac{1}{r^2}\mathbf{p}_\theta \times \mathbf{r}, \tag{42}$$

$$t = t_0 + \Delta t. \tag{43}$$

Finally, what remains to be explained here is the advancement of the system under the perturbation part $\mu(r)R(\mathbf{r}, t)$ of the Hamiltonian. Since this depends only on coordinates (time is also a coordinate now), the motion is linear in momentum space and coordinates remain constant. We thus have the momentum jumps

$$\delta\mathbf{v} = -h_s \frac{\partial(\mu(r)R(\mathbf{r}, t))}{\partial \mathbf{r}}, \qquad \delta p_0 = -h_s \frac{\partial(\mu(r)R(\mathbf{r}, t))}{\partial t}, \tag{44}$$

which are to be added to the most recent values of $\mathbf{v}$ and p_0.

4. Numerical Experiments

Here we first shortly discuss about the simple improvements to the basic leapfrog method and then describe numerical experiments in a toy model. One of these improved leapfrog methods (the symplectic Simpson rule) was used in the calculations described in the latter subsection. It was also confirmed that the Gaussian rule gives similar accuracy.

4.1. SYMPLECTIC SIMPSON RULE AND OTHER ALTERNATIVES

McLahlan (1995) and Koseleff (1993) have shown that instead of the basic leapfrog one may consider algorithms in which the steplengths of both the unperturbed motion and the perturbing jumps vary in certain sequences. Two simplest examples of such methods (beyond the leapfrog) are

$$H_0(X_1 h_s)H_1(h_s)H_0(X_2 h_s)H_1(h_s)H_0(X_1 h_s), \qquad 2X_1 + X_2 = 2 \tag{45}$$

or

$$H_1(W_1 h_s)H_0(h_s)H_1(W_2 h_s)H_0(h_s)H_1(W_1 h_s), \qquad 2W_1 + W_2 = 2, \tag{46}$$

where the parameters are positive and determined such that the accuracy is the best possible for first order perturbations. This means that one considers H_1 to be a

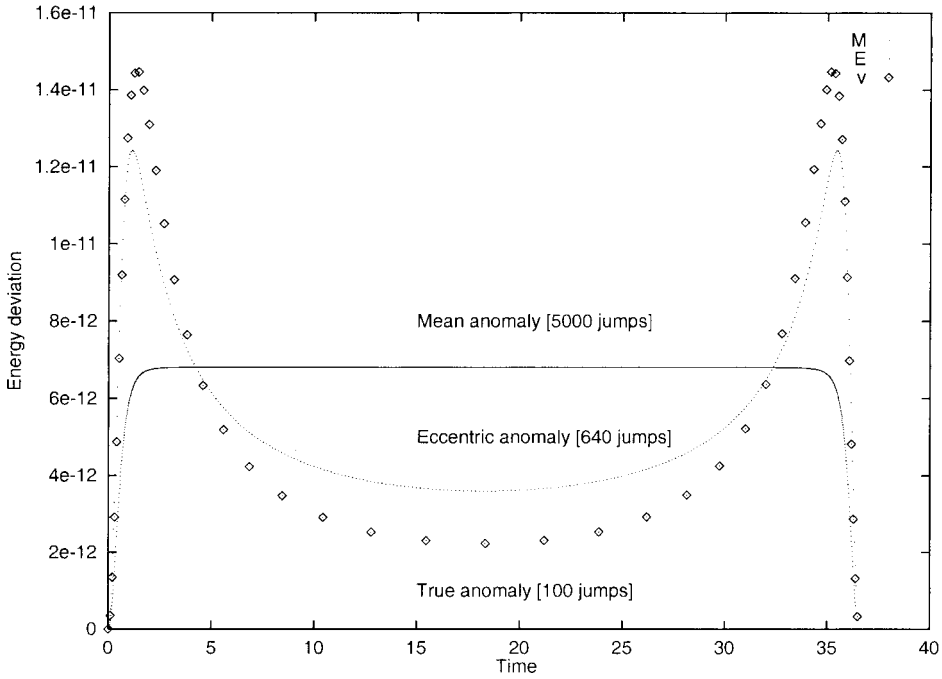

Figure 1. Energy errors for the different time transformations over one satellite period. The perturbing potential used was $0.5 \times 10^{-3}/r^3$. For this potential the true anomaly is clearly the most efficient transformation. Units are those of the example orbit given in the main text.

very small perturbation and the results are expanded in powers of the perturbation. In the case, we discuss here, the order higher than the first in the perturbation are considered negligible. It turns out that the best choice for the first example above corresponds to a two-point Gaussian integration formula $X_1 = 1 - 1/\sqrt{3},\ X_2 = 2/\sqrt{3}$, while in the latter example the result is simply the Simpson's rule: $W_1 = 1/3;\ W_2 = 4/3$. In this sense the original leapfrog can be seen as the midpoint rule of numerical integration. If the leapfrog is started with a half-step of perturbation, one gets an algorithm corresponding to the trapezoidal rule. One explanation for the excellent performance of the simple leapfrog in Solar System integrations is the fact that the solutions are generally quasi-periodic and the midpoint rule (as well as the trapezoidal) is known to have very good accuracy in (over a period) integration of periodic functions.

If the perturbing Hamiltonian has the order of magnitude $H_1 = \mathrm{O}(\epsilon)$, then the error of the above methods has the form

$$\mathrm{O}(\epsilon h_s^4) + \mathrm{O}(\epsilon^2 h_s^2) \tag{47}$$

and it is often not necessary to use more complicated sequences, because the order (with respect to h_s) of the ϵ^2-term cannot be increased without using negative timesteps (Yoshida, 1993). Because in accurate computations the stepsize (in practical problems) is to be taken anyway short enough so that the first order (in ϵ)

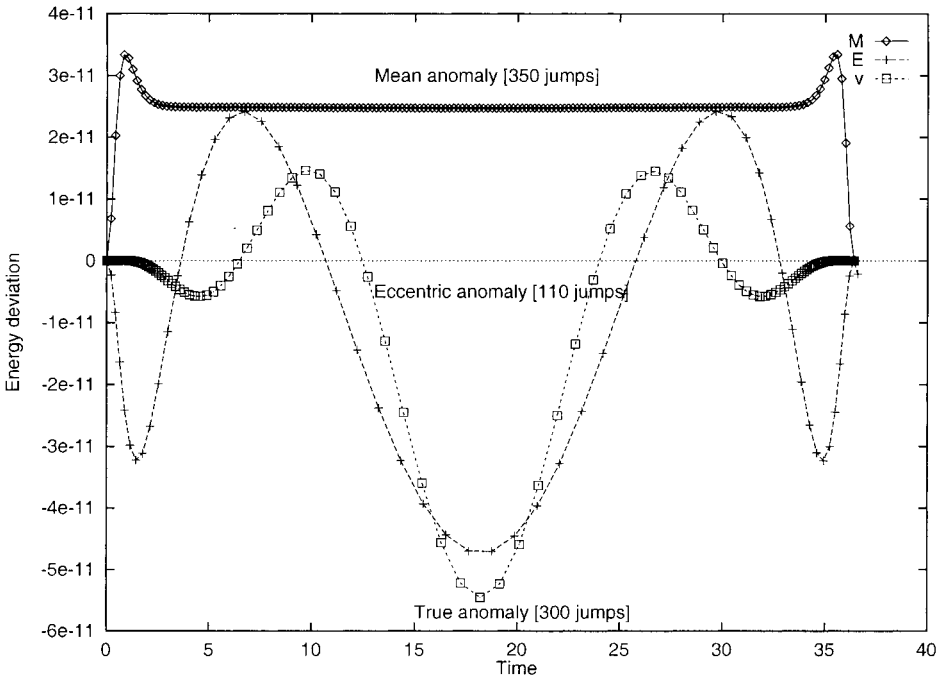

Figure 2. Energy errors in case of a tidal potential $-10^{-6}(r^2 - 3(\mathbf{r} \cdot \mathbf{u})^2)$, where $\mathbf{u} = (\cos(\omega t), \sin(\omega t))$. For this case the eccentric anomaly seems the most efficient one. Units are same as in the previous figure.

error has already died out and the method is in practice of order $O(\epsilon^2 h_s^2)$. There are other ways to increase further the precision by using a modified potential (Koseleff, 1993), but this is beyond the scope of this paper.

4.2. TOY MODEL TESTS

Here we demonstrate in a simple way the effect of various time transformations to the computing accuracy. We used the symplectic Simpson rule to obtain the results illustrated in Figure 1. In this figure we plot the energy errors in an equatorial satellite orbit with eccentricity $e = 0.69$, and the perigee distance $q = 1$, computed over (approximately) one period. Only the so-called J_2 term was taken into account. Thus, in this case, the perturbing potential was simply $J_2/(2r^3)$ ($J_2 = 0.001$). The stepsizes in the different computations were adjusted so as to get roughly similar error levels, this resulted into the values: 5000 jumps for using no time transformation (labeled Mean anomaly), 640 jumps for $\mathrm{d}t = r\mathrm{d}s$ (labeled Eccentric anomaly) and only 100 jumps for $\mathrm{d}t = r^2\mathrm{d}s$ (labeled True anomaly). However, in symplectic integration error is not necessarily error in the usual sense, because the coordinates cannot be interpreted as physical coordinates, but as auxiliary quantities which can be reduced, afterwards, into the physical coordinates by using symplectic cor-

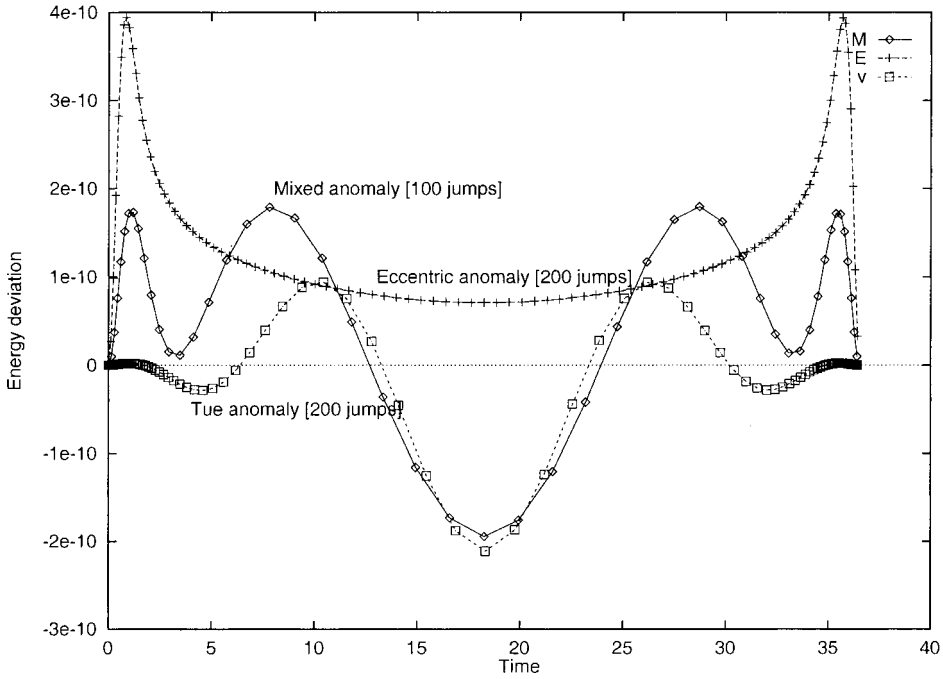

Figure 3. Energy errors in case of a combination of flattening and tidal potential $0.5\times10^{-3}/r^3-10^{-6}(r^2-3(\mathbf{r}\cdot\mathbf{u})^2)$. In this case a mixed anomaly defined by $\mu(r) = 0.5q/r+q^2/r^2$ seems most efficient.

rectors (Wisdom et al., 1997). As we can see, the errors in all the integration are periodic and return to a very small value after a period and thus they are likely to be removable using the symplectic correctors. Keeping this in mind we note the enormous difference in the number of steps needed, with the different time transformations.

Another interesting special case is the tidal perturbation. In Figure 2 an experiment with the test potential $-10^{-6}(r^2 - 3(\mathbf{r}\cdot\mathbf{u}))$ is illustrated. The direction vector $\mathbf{u}$ was taken to be $\mathbf{u} = (\cos(\omega t), \sin(\omega t))$ and $\omega = 1/500$ to roughly mimic the lunar perturbations on the earth's satellite (although the potential is stronger than the true lunar one to make the effect more clearly visible). One notes that in this case the eccentric anomaly seems the most efficient choice.

If both perturbations are combined, we expect that some mixed anomaly (more than one of the B-coefficients in time transformation nonzero) would do best. This indeed is the case as illustrated in Figure 3. Here the time transformation defined by $B_0 = 0,\ B_1 = 0.5q,\ B_2 = q^2$ (q = perigee distance) gives comparable precision with only half number of steps per period.

5. Conclusion

The use of time transformations, other than the traditional eccentric anomaly, are promising alternatives in the symplectic integration of eccentric orbits of satellites. In the case of J_2 perturbations only, the true anomaly offers the highest efficiency, while in very elongated orbits with tidal perturbation only, the eccentric anomaly is the best alternative. In the realistic cases the best choice is likely to be some intermediate one. Our new method offers the possibility of any linearly combined transformation. The problem of determining which transformation is best for a given orbit needs further investigation. Also the problem of how to determine, in advance, the stepsize for a pre-given accuracy level remains an open question.

Finally it must be stressed that in real satellite orbit integrations the perturbing force is complicated and computationally costly. Thus the possibility of decreasing the necessary number of force evaluations (velocity jumps) is important.

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