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Multidimensional linearized nonsteady infiltration with prescribed boundary conditions at the soil surface

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Abstract. The present work explores a class of analytical solutions of moisture movement in unsaturated porous media characterized by an exponential dependence of the hydraulic conductivity and the moisture content on water pressure. The Green's function method is used to derive a general analytical model pertaining to multidimensional nonsteady infiltration in a semi-infinite flow domain with arbitrary initial conditions, boundary conditions, and root-uptake forcing functions and for various simple source geometry. The general solution is expressed in integral form from which particular analytical solutions pertaining to cases of surface and subsurface irrigation, evaporation, root uptake, and moisture redistribution can be easily deduced from the general analytical model. The model offers the analyst significant flexibility in deriving results and analyzing infiltration phenomena of practical interest. New explicit solutions have been obtained for one-dimensional infiltration under various prescribed time-dependent flux boundary conditions and for two- and three-dimensional moisture redistribution. For constant initial or boundary conditions, the multidimensional solution is essentially the product of two or three time-dependent terms with each term being a function of only one space variable.

1. Introduction

The analysis of nonsteady infiltration has largely been carried out using numerical methods mainly because of the difficulty in solving analytically the governing nonlinear differential equation. Although numerical methods are powerful in solving complex nonlinear problems, analytical results provide general insights and concisely identify the relationships among the variables from which rational approximations and simplifications can be derived. They can also be useful for checking numerical schemes. During the past few decades, many analytical solutions for steady and nonsteady infiltration have been developed. Most of these solutions use the exponential hydraulic conductivity model first proposed by *Gardner* [1958]:

$$k(\psi) = k_0 \exp(\alpha\psi) \tag{1}$$

The exponential dependence of k on ψ coupled with Kirchhoff's transformation [*Kirchhoff*, 1894] linearizes the nonlinear steady state unsaturated flow equation, thus allowing the derivation of analytical solutions for steady infiltration. For the nonsteady case, linearization is possible only if the log linear variation of k is also linearly related to the moisture content [*Warrick*, 1974]. This additional assumption simplifies the transient partial differential equation to a form similar to the advection-dispersion equation.

A complete review of the quasi-linear theory of infiltration is given by *Pullan* [1990]; however, for completeness a review of the nonsteady solutions is presented here. *Braester* [1973] solved the one-dimensional infiltration towards the water table. Solutions for constant flux from point [*Warrick*, 1974], line [*Lomen and Warrick*, 1974], strip, and disc-shaped surface sources [*Warrick and Lomen*, 1976] have also been obtained.

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The point and line source solutions have been extended to include evaporation at the land surface [*Lomen and Warrick*, 1978a]. *Warrick* [1975] published the one-dimensional nonsteady infiltration for arbitrary initial and boundary conditions. *Lomen and Warrick* [1978b] extended it to include water extraction. *Batu* [1982, 1983] derived the formal solution for two-dimensional infiltration problem with a nonuniform flux at the land surface using a combination of separation of variables and Laplace transforms. *Warrick and Lomen* [1983] obtained the results for transient infiltration with water uptake by roots over two-dimensional flow domain. *Philip* [1986] published a multidimensional solution for the unsteady infiltration from buried cylindrical and spherical sources under a prescribed potential. Most previous analytical studies have either neglected the effect of root uptake or considered simple root extraction patterns with an impermeable land surface [*Warrick and Lomen*, 1983].

The present work explores further the class of analytical solutions, which is characterized by the log linear form of the hydraulic conductivity and the moisture content functions. The Green's function method is used to derive a general analytical model for multidimensional nonsteady infiltration into a semi-infinite medium. It is shown to be the most effective approach for handling arbitrary root-uptake functions, initial conditions, and time-dependent boundary conditions. Particular solutions in closed or integral forms pertaining to cases of surface and subsurface irrigation, root uptake, moisture redistribution, or isothermal evaporation can be obtained easily by a simple integration. However, the general solution is limited to a single type of boundary condition on the soil surface, either a prescribed head or a prescribed flux boundary condition. The problem of infiltration with mixed boundary conditions, one with a prescribed potential at the ponded surface and a prescribed flux boundary condition at the nonponded one, requires a more sophisticated mathematical treatment. The flexibility of the present model offers the analyst the advantage of deriving solutions and developing models pertaining to practical problems of interest in infiltration.

2. Theory

The most widely applied model for water flow in the unsaturated zone consists of Richards' equation [*Richards*, 1931]:

$$\nabla \cdot [k(\nabla\psi - \nabla z)] = \frac{\partial\theta}{\partial t} + S \tag{2}$$

Richards derived this partial differential equation by assuming that the equation of motion is governed by the Darcy-Buckingham law [*Buckingham*, 1907] and that the change in saturation of the pores accounts for almost the entire storage of water in soils. The equation is a parabolic differential equation and is highly nonlinear owing to the dependence of the hydraulic conductivity and saturation on the pressure in the water phase. To obtain a particular solution, (2) must be subjected to initial and boundary conditions. The initial condition consists of the specification of ψ in the whole flow domain at $t = 0$:

$$\psi(x, y, z, 0) = h_i(x, y, z) \qquad h_i \leq 0 \tag{3}$$

The boundary conditions may be of two physical types: prescribed pressure head, to model ponding, on the soil surface,

$$\psi(x, y, 0, t) = h_0(x, y, 0, t) \qquad h_0 \leq 0 \tag{4}$$

and prescribed vertical flux, to model rainfall infiltration or sprinkler irrigation whose intensities are lower than the infiltration capacity of the soil,

$$-k(\psi)\left[\frac{\partial\psi}{\partial z} - 1\right] = q(x, y, t) \qquad z = 0 \tag{5}$$

Where q is positive for infiltration and negative for evaporation.

The root uptake functions S can have the following form,

$$S = a \exp[-bz] \tag{6}$$

which models an exponentially decreasing uptake as well as a constant uptake with $b = 0$ over a defined depth $z \leq d$.

Analytical solutions to (2) can be obtained only for some special functions of the hydraulic conductivity. Such a special function is the exponential form of the hydraulic conductivity (1), where α, the sorptive number, is a measure of the importance of gravity relative to capillarity. It is small in fine-textured soils where capillarity is dominant and large in coarse-textured soils where gravity is dominant [*Philip*, 1969]. *Philip* [1984] stated that α^{-1} ranges between 0.2 and 5 m; and fitted values of k_0 and α for some 17 soils is given by *Bresler* [1978].

In order to linearize the right-hand side of the differential equation, a similar assumption on the variation of the moisture content must be taken:

$$\theta = \theta_r + (\theta_s - \theta_r)\exp(\alpha\psi) \tag{7}$$

Comparing (1) with (7), one deduces that a linear variation of the hydraulic conductivity with moisture content is assumed, $\theta = a + bk$, where a and b could also be fitted parameters. Although expressions (1) and (7) do not fit most of the experimental data very well over the entire range of ψ observed, they are applicable to field situations when the moisture variations are relatively small such in the case of high-frequency irrigation [*Warrick*, 1974]. Using the above relationships, the water diffusivity $D = k\partial\psi/\partial\theta$ becomes constant and equal to $D = 1/\alpha b$. The parameters b or D can then be chosen such that small time values match the ones obtained from the general nonlinear absorption equation [*Philip*, 1986; *Pullan*, 1992].

By using a particular case of the Kirchhoff's transformation,

$$u = \exp[\alpha\psi] \tag{8}$$

and introducing the following dimensionless numbers,

$$\frac{X}{x} = \frac{Y}{y} = \frac{Z}{z} = \frac{\alpha}{2} \qquad \frac{T}{t} = \frac{\alpha}{4b} \qquad \bar{S} = \frac{S}{\alpha k_0} \tag{9}$$

(2) becomes

$$\nabla^2 u - 2\frac{\partial u}{\partial Z} = \frac{\partial u}{\partial T} + 4\bar{S} \tag{10}$$

By making use of the following transformation [*Farlow*, 1982, p. 58],

$$u = w \exp[Z - T] \tag{11}$$

(10) can then be written in the form of a heat diffusion equation

$$\nabla^2 w = \frac{\partial w}{\partial T} + 4\bar{S}\exp[-Z + T] \tag{12}$$

subject to the following transformed initial and boundary conditions:

Initial state

$$w(x, y, z, 0) = \exp[\alpha\psi_i(x, y, z) - Z] = u_i \exp[-Z] \tag{13}$$

Prescribed head

$$w(x, y, 0, t) = \exp[\alpha\psi_0(x, y, t) + T] = u_0 \exp[T] \tag{14}$$

Prescribed flux

$$-\frac{\partial w}{\partial Z} + w\bigg|_{Z=0} = 2\bar{q}_0 \exp[T] \tag{15}$$

where u_0 and $\bar{q}_0 = q_0/k_0$ are the dimensionless transformed potential and flux distribution at the land surface $Z = 0$, respectively, and u_i is the transformed potential of the initial condition at $T = 0$.

The dimensionless horizontal and vertical fluxes are given by

$$\begin{aligned} \bar{q}_X &= \frac{q_X}{k_0} = -\frac{1}{2}\frac{\partial u}{\partial X} \\ \bar{q}_Z &= \frac{q_Z}{k_0} = -\frac{1}{2}\frac{\partial u}{\partial Z} + u \end{aligned} \tag{16}$$

where q_X and q_Z are the actual fluxes. It should be noted that u is a dimensionless number bounded between 0 and 1 and is related to the matric flux potential ϕ of previous investigators [*Philip*, 1969; *Warrick*, 1974] by $\phi = k_0 u/\alpha$. It is assumed that the soil at large distances is relatively dry such that at infinity $u = 0$. For the case of a nonzero condition at infinity, $u_\infty = \exp[\alpha\psi_\infty]$, the dependent variable u can still be normalized such that the boundary condition at infinity vanishes, that is, by defining $u' = u - u_\infty$. The flux is then given by (15) but with $\bar{q}_0$ replaced by $\bar{q}_0 - u_\infty$.

Notice that (10) is mathematically analogous to the equation governing the physical process of diffusion and advection of heat, pollutants, or sediments. However, the linearized partial differential equation possesses a special property arising from strong gravity effects, which transforms the flux boundary conditions from the usual second type (Neumann) to a third type (Fourier), that is, from a boundary condition that involves only the derivative of the variable into one that linearly relates the dependent variable with its derivative (15). This makes the mathematics much more tedious. In similar processes such as heat advection and diffusion, the equivalent advection term can be neglected at the boundary, thereby simplifying the boundary condition from the third type to the second type.

3. Green's Function Method of Solution

The Green's function method allows the derivation of an analytical model of general applicability. The model is expressed in integral form in terms of familiar closed-form functions, which offer more insight into the overall behavior of the solution and its peculiarities than infinite series. The derived integral solution is in the real domain as opposed to a transformed domain as might result from application of the method of integral transforms. The Green's function method can be considered as an extension of the use of sources and sinks whereby it handles the presence of boundaries, while the method of singularities applies to infinite domains. It is also closely related to the boundary integral method since both methods are derived from the same equation and use the same assumptions, such as linearity of the governing equation, homogeneity, and isotropy of the porous medium. However, the boundary integral method solves the aforementioned integral equation numerically, and hence it can handle irregular source and domain geometry while the Green's function method is suitable only for simple idealized geometry. The distinct advantage the Green's function approach has over the other analytical methods is when solutions are sought for assorted boundary conditions and forcing functions, because then the particular solutions can be deduced easily from the general integral solution without any further derivation, while the other mathematical methods would normally require the complete derivation of the solution for every new set of conditions.

The method of Green's function is the most effective approach for handling arbitrary forcing functions, initial conditions, and time-dependent boundary conditions. Its thrust lies in the capability of the method to convert a difficult problem into simpler ones. Once the transformed problem is solved, the solution of the original mathematical statement can be easily retrieved by a simple integration. Particular solutions to any specific problem can then be obtained easily in closed or integral forms.

In order to facilitate the method of solution, the Green's function problem is split into two parts [*Greenberg*, 1971]: (1) the fundamental solution, which models the effect of an instantaneous buried point source or sink in an infinite medium, and (2) the regular solution, which accounts for the existence of the boundaries and can sometimes be obtained readily through the method of images.

Once the Green's function is obtained, the solution of the original problem becomes [*Greenberg*, 1971]

$$w = \int_0^T \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[w \frac{\partial G}{\partial Z_s} - G \frac{\partial w}{\partial Z_s} \right]_{Z_s=0} dX_s\, dY_s\, dT_s$$

$$+ \int_0^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [Gw]_{T_s=0}\, dX_s\, dY_s\, dZ_s$$

$$- 4 \int_0^T \int_0^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp\,[-Z_s + T_s] G \bar{S}\, dX_s\, dY_s\, dZ_s\, dT_s \tag{17}$$

where the first integral on the right-hand side accounts for the boundary conditions on the soil surface, the second integral pertains to the initial conditions, and the last integral is for the source or sink functions in the domain.

3.1. Fundamental Solution

The fundamental solution deals with an instantaneous source injection in an infinite domain. It satisfies the following partial differential equation:

$$\nabla_s^2 G + \frac{\partial G}{\partial T_s} = -\delta(X_s - X)\delta(Y_s - Y)\delta(Z_s - Z)\delta(T_s - T) \tag{18}$$

For the one-dimensional diffusion equation, the solution is

$$G_{1\mathrm{D}} = \frac{1}{\sqrt{4\pi(T - T_s)}} \exp\left[-\frac{(Z - Z_s)^2}{4(T - T_s)} \right] \tag{19}$$

and for the two-dimensional (X, Z) domain

$$G_{2\mathrm{D}} = \frac{G_{1\mathrm{D}}}{\sqrt{4\pi(T - T_s)}} \exp\left[-\frac{(X - X_s)^2}{4(T - T_s)} \right] \tag{20}$$

and in three dimensions

$$G_{3\mathrm{D}} = \frac{G_{2\mathrm{D}}}{\sqrt{4\pi(T - T_s)}} \exp\left[-\frac{(Y - Y_s)^2}{4(T - T_s)} \right] \tag{21}$$

3.2. General Solution

The general solution is the sum of the fundamental and regular solution, which accounts for the existence of boundaries and their corresponding conditions. For a prescribed flux boundary condition at the soil surface

$$-\frac{\partial G_{n\mathrm{D}}^{\mathrm{f}}}{\partial Z_s} + G_{n\mathrm{D}}^{\mathrm{f}} = 0 \tag{22}$$

$G_{n\mathrm{D}}^{\mathrm{f}}$ can be expressed in terms of an image system. The derivation can be found in *Sommerfeld* [1949, p. 67]

$$G_{1\mathrm{D}}^{\mathrm{f}} = \frac{1}{\sqrt{4\pi(T - T_s)}} \left\{ \exp\left[-\frac{(Z - Z_s)^2}{4(T - T_s)} \right] \right.$$

$$+ \exp\left[-\frac{(Z + Z_s)^2}{4(T - T_s)} \right]$$

$$\left. - 2 \exp(Z_s) \int_{Z_s}^{\infty} \exp\left[-\lambda - \frac{(Z + \lambda)^2}{4(T - T_s)} \right] d\lambda \right\} \tag{23}$$

The third term in (23) is actually a continuous source with variable strength placed from Z_s to ∞ and can be analytically evaluated in terms of the error function [*Abramowitz and Stegun*, 1972, equation 7.4.32]

$$G_{1D}^{f} = \frac{1}{\sqrt{4\pi\tau}}\left\{\exp\left[-\frac{(Z-Z_s)^2}{4\tau}\right]+\exp\left[-\frac{(Z+Z_s)^2}{4\tau}\right]\right\} - \exp\left[Z+Z_s+\tau\right]\operatorname{erfc}\left[\frac{Z+Z_s}{\sqrt{4\tau}}+\sqrt{\tau}\right] \quad (24)$$

For the two- or three-dimensional cases Green's function is obtained from (24) and (20) or (21), respectively.

The Green's function for the prescribed head case $G_{nD}^{h} = 0$ is adapted from known results in the literature [*Haberman*, 1987, chap. 10]:

$$G_{1D}^{h} = \frac{1}{\sqrt{4\pi\tau}}\left\{\exp\left[-\frac{(Z-Z_s)^2}{4\tau}\right]-\exp\left[-\frac{(Z+Z_s)^2}{4\tau}\right]\right\} \quad (25)$$

And, similarly, Green's function for the two- or three-dimensional cases is obtained from (25) and (20) or (21), respectively.

Since there are only two kinds of boundary conditions, (17) could be simplified by rewriting the integrand of the surface integral in terms of (22). Using (11), the general solution of the original problem is then

$$u = \int_0^T\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\left[Z-\tau\right]F\left.\frac{\partial G}{\partial Z_s}\right|_{Z_s=0} dX_s\, dY_s\, d\tau + \int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\left[Z-Z_s-T\right]u_iG|_{T_s=0}\, dX_s\, dY_s\, dZ_s - 4\int_0^T\int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\left[Z-Z_s-\tau\right]G\bar{S}\, dX_s\, dY_s\, dZ_s\, d\tau \quad (26)$$

where F takes into account the boundary condition at the soil surface, u_i is the initial condition given in (13), and $\tau = T - T_s$. For the prescribed flux case $F = 2\bar{q}_0$, while for the prescribed head boundary condition $F = u_0$. The derivative of the appropriate Green's function can be derived from (24) for the prescribed flux case

$$\left.\frac{\partial G_{1D}^{f}}{\partial Z_s}\right|_{Z_s=0} = \frac{1}{\sqrt{\pi\tau}}\exp\left[-\frac{Z^2}{4\tau}\right] - \exp\left[Z+\tau\right]\operatorname{erfc}\left[\frac{Z}{\sqrt{4\tau}}+\sqrt{\tau}\right] \quad (27)$$

and from (25) for the prescribed head case

$$\left.\frac{\partial G_{1D}^{h}}{\partial Z_s}\right|_{Z_s=0} = \frac{Z}{\tau\sqrt{4\pi\tau}}\exp\left[-\frac{Z^2}{4\tau}\right] \quad (28)$$

For the two- or three-dimensional cases the derivative of G is simply the product of (27) or (28) with (20) or (21), respectively. The pressure head anywhere in the semi-infinite domain can then be evaluated from (8) or $\alpha\psi = \ln u$. Once the distribution of ψ is determined, the moisture content distribution can then be evaluated using (7).

4. Application and Results

Equation (26) is the general analytical model for multidimensional linearized nonsteady infiltration, evaporation, moisture redistribution, and water extraction by plants. It is used as the basis for obtaining solutions to some particular problems of infiltration that either have already been published or have been hitherto unsolved. In this way the inclusive character and advantages of the general solution is most apparent. Although the present model can handle arbitrary flux distribution, moisture distribution, and plant uptake function, simple mathematical forms have been assumed in order to facilitate the evaluation of the integrals. Explicit one-dimensional solutions with time-dependent flux boundary conditions have been derived. Closed-form solutions for the multidimensional moisture redistribution have also been obtained which clearly show the effect of gravity and capillarity on the postinfiltration water movement.

Depending on the problem at hand, some of the integrals cannot be evaluated analytically, and numerical integration is then carried out using Gauss-Legendre quadrature. Multidimensional singular integrals can be broken up into several regions to isolate the singularity and to perform on each region a separate numerical integration with an appropriate number of Gauss points [*Press et al.*, 1986, chap. 4]. For infinite integrals the transformation of the variable of integration through $\xi = 1/x$ transforms the infinite domain $[a, \infty]$ into a finite one $[1/a, 0]$ and allows the use of Gauss-Legendre quadrature. In this study most of the integrals were one-dimensional and bounded between 0 and T, and the numerical integration scheme used 16 Gauss points with recursive refinements until the error was within an error tolerance level. The reader is remined that the whole analysis hinges on the assumption that the soil is homogeneous, isotropic, and nowhere becomes saturated; that the water is incompressible; that hysteresis is negligible; and that the domain is infinite in the horizontal extent and bounded by a horizontal soil surface. Unless specified, the initial condition and the transformed potential at infinity is assumed to be zero.

4.1. One-Dimensional Flow

The following solutions model the one-dimensional infiltration process under a prescribed head or flux, with and without root-uptake, and for a uniform and nonuniform initial distribution. These solutions can help quantify the cumulative infiltration or achieve a better quantification of the infiltration component in catchment hydrology models. The relevant equations are the one-dimensional form of (26) with G and its derivatives given by (24) and (27) for the prescribed flux case, and by (25) and (28) for the prescribed head case.

4.1.1. Prescribed Flux Boundary Condition

4.1.1.1. Nonsteady infiltration rate. A prescribed flux boundary condition applies when the sprinkler is operating at a rate lower than the critical one so that no water spreading or ponding occurs. For a general infiltration rate $\bar{q}_0(T_s)$, the solution is

$$u = 2\int_0^T \bar{q}_0(T-\tau)g_f(Z,\tau)\,d\tau \quad (29)$$

where $g_f(Z, \tau)$ is given by

$$g_f(Z,\tau) = \frac{1}{\sqrt{\pi\tau}}\exp\left(Z-\frac{Z^2}{4\tau}-\tau\right) - \exp(2Z)\operatorname{erfc}\left(\frac{Z}{\sqrt{4\tau}}+\sqrt{\tau}\right) \quad (30)$$

For a constant rate at the soil surface $\bar{q}_0 = \bar{q}_\iota$, the solution can be obtained by integrating (30) to give

$$u_{un} = \frac{\bar{q}_\iota}{2}\left[\operatorname{erfc}\left(\frac{Z}{\sqrt{4T}} - \sqrt{T}\right) - (1 + 2Z + 4T)\exp(2Z)\operatorname{erfc}\left(\frac{Z}{\sqrt{4T}} + \sqrt{T}\right) + 4\sqrt{\frac{T}{\pi}}\exp\left(Z - T - \frac{Z^2}{4T}\right)\right] \quad (31)$$

which is similar to (9) of *Braester* [1973] and (17) of *Lomen and Warrick* [1978b]. Equation (31) can be used to evaluate the ponding time at which the soil at the surface becomes saturated for a given flux, that is, $u = 1$ at $Z = 0$,

$$1 = \bar{q}_\iota\left[(1 + 2T)\operatorname{erf}(\sqrt{T}) - 2T + 2\sqrt{\frac{T}{\pi}}\exp(-T)\right] \quad (32)$$

The solution of (32) gives the time to ponding assuming that $\bar{q}_\iota > 1$ for ponding to occur.

The solution for a cyclic input can also be obtained from (29) by subdividing the integral over the respective flux segments. The solution is as shown by *Warrick* [1975]. For a periodic input, $\bar{q}_0 = \bar{q}_\iota[1 + \mu\cos(aT_s)]$, $a = 2\pi/T_p$, where T_p is the period and $\mu \le 1$. Equation (29) can be integrated using known results [*Gradshtein and Ryzhik*, 1980, equations 3.926-2, 2.663-1, 2.663-3] to obtain the steady state solution for $\tau \to \infty$:

$$\frac{u_{pe}}{\bar{q}_\iota} = 1 + 2\mu\exp[-(b - 1)Z]\frac{(1 + b)\cos v + c\sin v}{(1 + b)^2 + c^2} \quad (33)$$

where

$$v = aT - cZ \qquad b = \sqrt{\frac{\sqrt{a^2 + 1} + 1}{2}} \qquad c = \sqrt{\frac{\sqrt{a^2 + 1} - 1}{2}} \quad (34)$$

Equation (33) shows that the transformed potential distribution is also periodic with a period of T_p and a lag of cZ/a. Notice that the lag and the amplitude are depth-dependent, the latter decreasing exponentially.

For Horton's infiltration model, $\bar{q}_0 = \bar{q}_f + (\bar{q}_i - \bar{q}_f)\exp(-cT_s)$, $c \le 1$, the potential distribution is

$$u = u_{un} + u_{ex} \quad (35)$$

where u_{un} is given in (31) with $\bar{q}_\iota = \bar{q}_f$ and u_{ex} is obtained from (29) with $\bar{q}_0 = (\bar{q}_i - \bar{q}_f)\exp(-cT_s)$ using known analytic integrations [*Beck et al.*, 1992, p. 426]:

$$\frac{u_{ex}}{\bar{q}_\iota - \bar{q}_f} = \frac{1}{2}\left[\frac{1}{d} - \frac{1}{c}\left(\frac{1}{d} - 1\right)\right]\exp[Z(1 - d) - cT] \cdot \operatorname{erfc}\left(\frac{Z}{\sqrt{4T}} - d\sqrt{T}\right) - \frac{1}{2}\left[\frac{1}{d} - \frac{1}{c}\left(\frac{1}{d} + 1\right)\right] \cdot \exp[Z(1 + d) - cT]\operatorname{erfc}\left(\frac{Z}{\sqrt{4T}} + d\sqrt{T}\right) - \frac{1}{c}\exp[2Z]\operatorname{erfc}\left(\frac{Z}{\sqrt{4T}} + \sqrt{T}\right) \quad (36)$$

$$d = \sqrt{1 - c}$$

4.1.1.2. Moisture redistribution. For a given nonuniform initial distribution of potential $u_i(Z_s)$ the resulting potential at a later time is given by

$$u = \int_0^\infty u_i\exp[Z - Z_s - T]G_{1D}^f|_{T_s=0}\,dZ_s \quad (37)$$

where G_{1D}^f is the Green's function (24) evaluated at $T_s = 0$. For a uniform distribution $u_\iota = u_{\iota 0}$ over a depth $Z \le Z_0 = \alpha D/2$, the solution is

$$u = u_{\iota 0}[I_{ZT}(Z_0) - I_{ZT}(0)] \quad (38)$$

where

$$I_{ZT}(A) = \tfrac{1}{2}[f_1(A) + (1 + 2Z + 4T)f_2(A)] + \sqrt{\frac{4T}{\pi}}f_3(A) - f_4(A) \quad (39)$$

and

$$f_1(A) = \operatorname{erf}\left[\sqrt{T} - \frac{Z - A}{\sqrt{4T}}\right]$$
$$f_2(A) = \exp(2Z)\operatorname{erf}\left[\sqrt{T} + \frac{Z + A}{\sqrt{4T}}\right]$$
$$f_3(A) = \exp\left[2Z - \left(\sqrt{T} + \frac{Z + A}{\sqrt{4T}}\right)^2\right] \quad (40)$$
$$f_4(A) = A\exp(2Z)\operatorname{erfc}\left[\sqrt{T} + \frac{Z + A}{\sqrt{4T}}\right]$$

4.1.1.3. Root uptake. For a given water extraction pattern $\bar{S}(Z_s, T_s)$ the resulting potential is

$$u = -4\int_0^T\int_0^\infty \bar{S}G_{1D}^f\exp[Z - Z_s - \tau]\,dZ_s\,d\tau \quad (41)$$

which must be added to the potential associated with infiltration for the case of infiltration with root uptake. For an exponential plant uptake $\bar{S} = A\exp[-BZ_s]$, integration of (41) yields [*Beck et al.*, 1992, p. 426]

$$\frac{u}{A} = -\frac{2}{B} + \frac{4\exp[-BZ]}{B(B + 2)} - \frac{2u_1}{B(B + 2)} - \frac{2u_2}{B^2} + \frac{u_3}{B} + \frac{u_4}{B} \cdot [4T + 2Z - 1] - \frac{4\sqrt{T}}{B\sqrt{\pi}}\exp\left[2Z - \left(\sqrt{T} + \frac{Z}{\sqrt{4T}}\right)^2\right] \quad (42)$$

where

$$u_1 = \exp[-BZ + B(B + 2)T]\operatorname{erfc}\left[(1 + B)\sqrt{T} - \frac{Z}{\sqrt{4T}}\right]$$
$$u_3 = \operatorname{erfc}\left[\sqrt{T} - \frac{Z}{\sqrt{4T}}\right] \quad (43)$$
$$u_2 = \exp[(B + 2)(Z + BT)]\operatorname{erfc}\left[(1 + B)\sqrt{T} + \frac{Z}{\sqrt{4T}}\right]$$
$$u_4 = e^{2Z}\operatorname{erfc}\left[\sqrt{T} + \frac{Z}{\sqrt{4T}}\right]$$

At large times (42) reduces to the first two terms as previously found for the steady state case [*Basha*, 1994].

4.1.2. Prescribed Head Boundary Condition

4.1.2.1. Nonsteady pressure head. The prescribed pressure head condition is applicable when the sprinkler is operating at a rate higher than the critical one so that water spreading or ponding occurs under a constant pressure head. For a prescribed pressure head at the surface, the potential distribution in the domain is

$$u = \int_0^T u_0(T - \tau) g_h(Z, \tau) \, d\tau \tag{44}$$

where $g_h(Z, \tau)$ is given by

$$g_h(Z, \tau) = \frac{Z}{\tau\sqrt{4\pi\tau}} \exp\left(Z - \frac{Z^2}{4\tau} - \tau\right) \tag{45}$$

For a constant potential $u = u_w$, (44) can be integrated to give

$$u = \frac{u_w}{2}\left[\operatorname{erfc}\left(\frac{Z}{\sqrt{4T}} - \sqrt{T}\right) + \exp(2Z)\operatorname{erfc}\left(\frac{Z}{\sqrt{4T}} + \sqrt{T}\right)\right] \tag{46}$$

which has the same mathematical form as the classic solution published by *Ogata and Banks* [1961] for the transport of a tracer in a semi-infinite column. The flux at the land surface can be obtained from (16) and (46):

$$\bar{q}_{z=0} = \frac{1}{2}\left[1 + \operatorname{erf}(\sqrt{T}) + \frac{\exp(-T)}{\sqrt{\pi T}}\right] \tag{47}$$

Equation (47) can give an estimate of the postponding infiltration assuming that the depth of ponding is negligible so that the soil just beneath the surface is still unsaturated.

4.2. Two-Dimensional Flow

The following solutions are useful for some aspects of agricultural engineering such as the design of sprinkler and furrow irrigation systems. The water supply at the land surface is assumed to be contained within a uniform width and stretching a great length in the perpendicular horizontal direction. Evidently, the geometry of the problem is appropriate for modeling irrigation of row crops. The solutions herein pertain only to the prescribed flux boundary condition. The solution for a discontinuous head boundary condition has been shown to be nonphysical as it yields a singularity in the total flux [*Weir*, 1986]; however, it can provide the outer asymptotic expansion of the mixed boundary condition problem, which is valid only in the wetted regions for a flow dominated completely by gravity [*Basha*, 1994].

4.2.1. Prescribed Flux Boundary Condition

4.2.1.1. Nonuniform nonsteady infiltration rate. For an arbitrary flux boundary condition at the soil surface $\bar{q}_0(X_s, \tau)$ the distribution of u in the semi-infinite domain is

$$u = \int_0^T \int_{-\infty}^{+\infty} \frac{2\bar{q}_0}{\sqrt{4\pi\tau}} \exp\left[-\frac{(X - X_s)^2}{4\tau}\right] g_f(Z, \tau) \, dX_s \, d\tau \tag{48}$$

For a constant infiltration $\bar{q}_0 = \bar{q}_i$ beneath an infinite strip $|X| \leq X_0$, (48) can be integrated over X_s to yield

$$u = 2\bar{q}_i \int_0^T \mathrm{I}_{X\tau}(X_0) g_f(Z, \tau) \, d\tau \tag{49}$$

where

$$\mathrm{I}_{X\tau}(X_0) = \frac{1}{2}\left[\operatorname{erf}\left(\frac{X + X_0}{\sqrt{4\tau}}\right) - \operatorname{erf}\left(\frac{X - X_0}{\sqrt{4\tau}}\right)\right] \tag{50}$$

which is similar to (14) of *Warrick and Lomen* [1976] and (73) of *Batu* [1982]. For a constant evaporation $-\bar{q}_e$ between $X_0 \leq |X| \leq X_1$, (48) gives

$$u = -2\bar{q}_e \int_0^T [\mathrm{I}_{X\tau}(X_1) - \mathrm{I}_{X\tau}(X_0')] g_f(Z, \tau) \, d\tau \tag{51}$$

The solution for a nonuniform flux distribution at the soil surface can be obtained directly from (48) or the continuous flux distribution can be approximated as a discrete one with constant values and integrated accordingly as in (49). For a given two-dimensional water extraction pattern $\bar{S}(X_s, Z_s, T_s)$ the corresponding potential is obtained from the quadruple integral in (26). For a one-dimensional exponential uptake, (42) can be added to any of the above solutions.

4.2.1.2. Moisture redistribution. For a given initial distribution of potential $u_i(X_s, Z_s)$ the resulting potential at a later time is given by

$$u = \int_0^\infty \int_{-\infty}^{+\infty} u_i \exp[Z - Z_s - T] G_{2D}^f|_{T_s=0} \, dX_s \, dZ_s \tag{52}$$

where G_{2D}^f is the two-dimensional Green's function evaluated at $T_s = 0$. The moisture redistribution of a wetted strip $[-X_0, X_0]$ over a depth $[0, Z_0]$ with $u_i = u_{i0}$ is therefore

$$u = u_{i0}\mathrm{I}_{XT}(X_0)[\mathrm{I}_{ZT}(Z_0) - \mathrm{I}_{ZT}(0)] \tag{53}$$

where I_{ZT} and I_{XT} are given in (39) and (50), respectively.

Figures 1a and 1b and 2a and 2b show the moisture redistribution as given by (53) for $s = 0.1$ and $s = 1$ at various times $\bar{T}$ where $s = \alpha L/2$. The time variable has been scaled with respect to s by defining $\bar{T} = T/s$. The axis coordinates are also normalized with respect to the source dimensions, that is, $\bar{X} = x/L$ and $\bar{Z} = z/L$, which are related to the dimensionless variables X and Z through $\bar{X} = X/s$ and $\bar{Z} = Z/s$. Choosing $L = 1$, the soil parameter is then $\alpha = 0.2$ for $s = 0.1$ and $\alpha = 2$ for $s = 1$. Small s corresponds to capillary-dominated flow, while large s is for gravity dominated flow. In both figures the initial distribution is an initial saturated state $u_{i0} = 1$ in a square zone with side $\bar{X}_0 = \bar{Z}_0 = 1$. Comparing Figures 1a and 1b with Figures 2a and 2b, one notices that the moisture front expands faster with small s than with large s and that the effect of gravity is more pronounced in the latter. The moisture movement for small s is almost uniformly diffusing in all directions, while for large s, Figure 2b shows a significant downward movement of the wetted front. It should be remembered that hysteresis is neglected in the above solutions.

For an initial distribution discrete in space

$$u = \begin{Bmatrix} u_{i0} & 0 \leq |X| \leq X_0 & 0 \leq Z \leq Z_0 \\ u_{i1} & X_0 \leq |X| \leq X_1 & 0 \leq Z \leq Z_0 \\ u_{i2} & 0 \leq |X| \leq X_1 & Z_0 \leq Z \leq Z_1 \end{Bmatrix} \tag{54}$$

(52) yields

$$u = [I_{ZT}(Z_0) - I_{ZT}(0)][u_{i0}I_{XT}(X_0) + u_{i1}[I_{XT}(X_1) - I_{XT}(X_0)]] + u_{i2}I_{XT}(X_1)[I_{ZT}(Z_1) - I_{ZT}(Z_0)] \quad (55)$$

The solution for a nonuniform initial distribution can therefore be obtained from (52) by approximating the continuous distribution as a discrete one and integrating accordingly as for (55).

4.3. Three-Dimensional Flow

The geometry of the source is assumed to be defined by the circular cylindrical coordinates system. Two practical problems

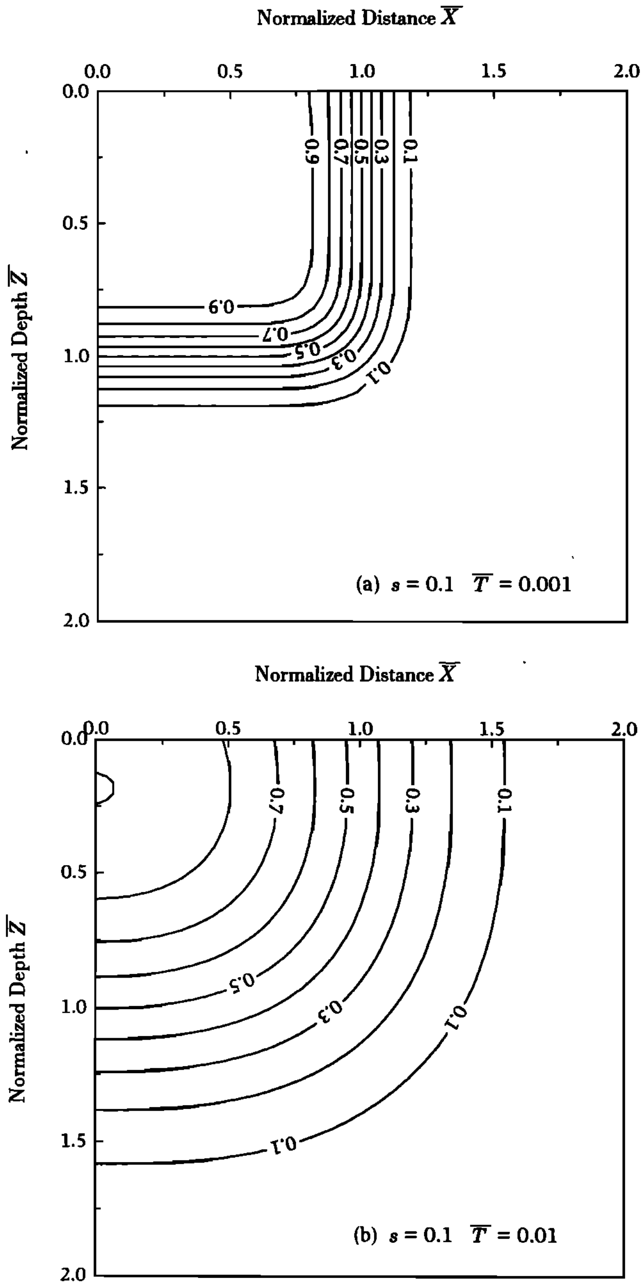

Figure 1. Transformed potential redistribution for capillary dominated flow with $s = 0.1$ at two different dimensionless times: (a) $\bar{T} = 0.001$ and (b) $\bar{T} = 0.01$. The initial distribution consists of a saturated state $u_{i0} = 1$ in a square zone with side $\bar{X}_0 = \bar{Z}_0 = 1$.

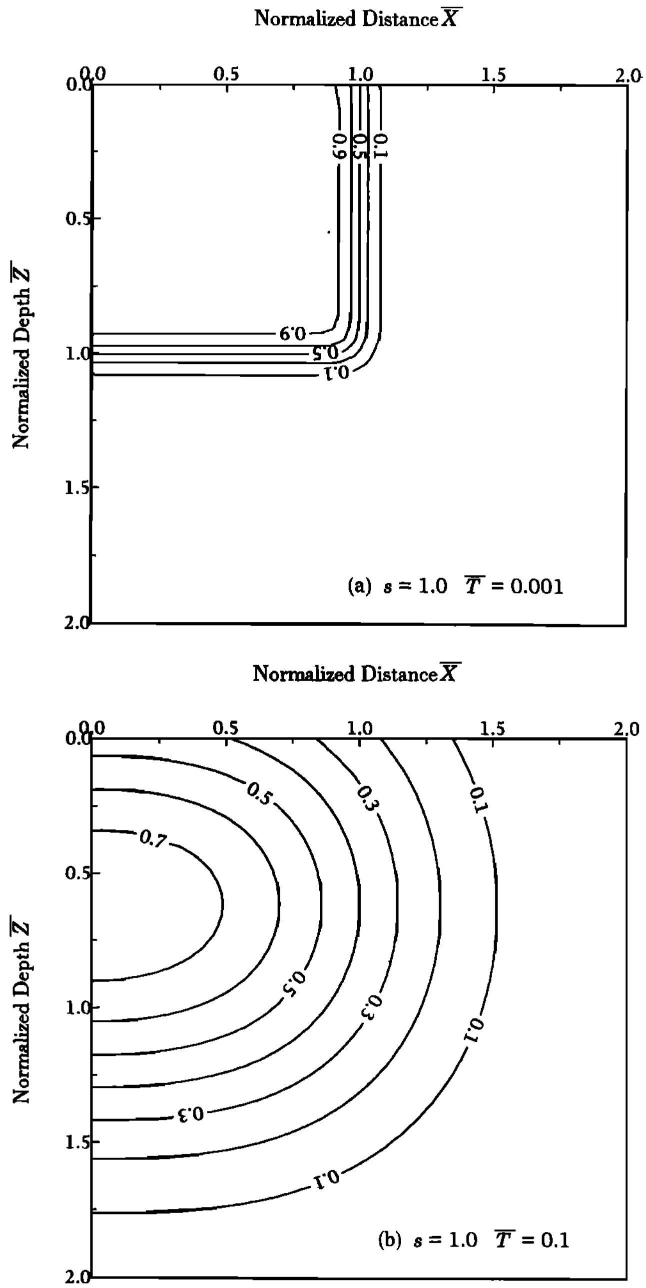

Figure 2. Transformed potential redistribution for gravity dominated flow with $s = 1.0$ at (a) $\bar{T} = 0.001$ and (b) $\bar{T} = 0.1$. The initial distribution consists of a saturated state $u_{i0} = 1$ in a square zone with side $\bar{X}_0 = \bar{Z}_0 = 1$.

of interest in this case are ring infiltrometers and trickle irrigation. Such a physical phenomena could be idealized by assuming a circular pond of negligible depth which lies on the horizontal land surface of a semi-infinite medium.

4.3.1. Prescribed Flux Boundary Condition

4.3.1.1. Nonuniform nonsteady infiltration rate. If the flow rate is lower than the critical rate, a flux boundary condition is assumed. Defining

$$\rho^2 = R^2 + R_s^2 - 2RR_s \cos(\theta - \theta_s) \quad (56)$$

the solution in cylindrical coordinates becomes

$$u = \int_0^T \int_0^{2\pi} \int_0^\infty \frac{\bar{q}_0}{2\pi\tau} \exp\left[-\frac{\rho^2}{4\tau}\right] g_f(Z, \tau) R_s \, dR_s \, d\theta_s \, d\tau \quad (57)$$

Equation (57) can be used to obtain particular solutions for infiltration from surface sources under a prescribed flux. For an axisymmetrical flux distribution $\bar{q}_0 = \bar{q}_i$ beneath the circular pond $R \leq R_0 = \alpha L/2$, (57) can be integrated over θ_s [*Gröbner and Hofreiter*, 1957, equation 337.10a] to obtain

$$u = \int_0^T \int_0^{R_0} \frac{\bar{q}_0}{\tau} \mathrm{I}_0\left(\frac{RR_s}{2\tau}\right) \exp\left(-\frac{R^2 - R_s^2}{4\tau}\right) g_f(Z, \tau) R_s \, dR_s \, d\tau \quad (58)$$

where I_0 is the modified Bessel function of the first kind and of order 0. For a uniform flux distribution $\bar{q}_i$ at $R \leq R_0$, (58) can be integrated over R_s to give

$$u = 2\bar{q}_i \int_0^T \mathrm{P}\left(\frac{R_o}{\sqrt{2\tau}}, \frac{R}{\sqrt{2\tau}}\right) g_f(Z, \tau) \, d\tau \quad (59)$$

where P is the off-set circular probability function and defined as

$$\mathrm{P}\left(\frac{a}{\sigma}, \frac{r}{\sigma}\right) = \frac{1}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) \int_0^a \exp\left(-\frac{r'^2}{2\sigma^2}\right) \mathrm{I}_o\left(\frac{rr'}{\sigma^2}\right) r' \, dr' \quad (60)$$

Approximations for $a/\sigma < 1$ and for $a/\sigma > 1$ are presented by *Abramowitz and Stegun* [1972, equations 26.3.25, 26.3.26] and further approximations for small and large times are given by *Beck et al.* [1992, pp. 434–435]. Equation (59) can be shown to be equivalent to (25) of *Warrick and Lomen* [1976] by rewriting (60) in terms of the Bessel functions of the first kind [*Carslaw and Jaeger*, 1959, p. 260], interchanging the order of integration in (59), and integrating using the integral representation of erfc (x) and other known results [*Abramowitz and Stegun*, 1972, equations 7.4.32, 7.4.33, 7.4.36].

For a constant evaporation $-\bar{q}_e$ between $R_o \leq R \leq R_1$, (58) gives

$$u = -2\bar{q}_e \int_0^T \left[\mathrm{P}\left(\frac{R_1}{\sqrt{2\tau}}, \frac{R}{\sqrt{2\tau}}\right) - \mathrm{P}\left(\frac{R_o}{\sqrt{2\tau}}, \frac{R}{\sqrt{2\tau}}\right)\right] g_f(Z, \tau) \, d\tau \quad (61)$$

which can be superposed to the potential resulting from infiltration. Similarly, for a given three-dimensional water extraction pattern $\bar{S}(X_s, Y_s, Z_s, T_s)$ the resulting potential is obtained from the corresponding integral in (26) and can be added to the above solutions.

4.3.1.2. Moisture redistribution. For a given initial distribution of potential $u_i(R_s, \theta_s, Z_s)$, the potential at a later time is given by

$$u = \int_0^{+\infty} \int_0^{2\pi} \int_0^{+\infty} u_i \exp\left[Z - Z_s - T\right] G_{3\mathrm{D}}^f|_{T_s=0} R_s \, dR_s \, d\theta_s \, dZ_s \quad (62)$$

where $G_{3\mathrm{D}}^f$ is the three-dimensional Green's function evaluated at $T_s = 0$. The moisture redistribution of a wetted cylinder of radius R_o and height Z_o with initial transformed potential $u_i = u_{i0}$ is then

$$u = u_{i0} \mathrm{P}\left(\frac{R_o}{\sqrt{2T}}, \frac{R}{\sqrt{2T}}\right) [\mathrm{I}_{ZT}(Z_o) - \mathrm{I}_{ZT}(0)] \quad (63)$$

where I_{ZT} and P are given in (39) and (60), respectively. The moisture redistribution of a wetted cube centered at the origin with sides $(2X_o, 2Y_o, Z_o)$ with $u_i = u_{i0}$ is

$$u = u_{i0} \mathrm{I}_{XT}(X_o) \mathrm{I}_{YT}(Y_o) [\mathrm{I}_{ZT}(Z_o) - \mathrm{I}_{ZT}(0)] \quad (64)$$

where I_{XT} and I_{ZT} are given in (50) and (39), respectively, and I_{YT} is given by

$$\mathrm{I}_{YT}(Y_0) = \frac{1}{2}\left[\mathrm{erf}\left(\frac{Y + Y_0}{\sqrt{4T}}\right) - \mathrm{erf}\left(\frac{Y - Y_0}{\sqrt{4T}}\right)\right] \quad (65)$$

The solution for a nonuniform three-dimensional initial distribution can be obtained from (62) by approximating the continuous distribution as a discrete one and integrating accordingly as for the two-dimensional case (55).

5. Concluding Remarks

The Green's function method has been shown to be an effective analytical tool for studying and deriving a class of analytical solutions characterized by a log linear form of the hydraulic conductivity and the moisture content. The power of the Green's function method is such that a number of new problems are amenable to analysis and that the general solution includes in itself many solutions previously derived in the literature. Some of these solutions are derived in a form better suited for numerical computation involving closed-form functions rather than infinite series, which converged slowly, especially for large values of the soil parameter [*Philip*, 1984]. The model developed constitutes then an extension and a generalization of the existing literature and handles nonsteady infiltration into a three-dimensional semi-infinite flow domain with arbitrary initial conditions and time-dependent boundary conditions, root-uptake forcing functions, and for various simple source geometry. However, the general solution is restricted to boundary conditions at the soil surface of a homogeneous type and cannot thereby handle the mixed boundary condition problem. For constant initial or boundary conditions the multidimensional solutions are essentially integrals of two or three terms: I_{XT}, I_{YT}, or P, which are functions of the horizontal coordinates and dependent on the source geometry, and g_f or g_h, which are functions of the vertical coordinate and dependent on the boundary condition at the soil surface. The particular function g_f describes the evolution of linearized multidimensional unsteady infiltration and includes, for the special case of $Z = 0$, the $g(\tau)$ function previously obtained by *Philip* [1986] and *Pullan* [1992].

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References

Abramowitz, M., and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1972.

Basha, H. A., Multidimensional steady infiltration with prescribed boundary conditions at the soil surface, *Water Resour. Res.*, *30*, 2105–2118, 1994.

Batu, V., Time-dependent linearized two-dimensional infiltration and evaporation from nonuniform and nonperiodic strip sources, *Water Resour. Res.*, *18*, 1725–1733, 1982.

Batu, V., Time-dependent linearized two-dimensional infiltration and evaporation from nonuniform and periodic strip sources, *Water Resour. Res.*, *19*, 1523–1529, 1983.

Beck, J. V., K. D. Cole, A. Haji-Sheikh, and B. Litkouhi, *Heat Conduction Using Green's Functions*, Hemisphere, Bristol, Pa., 1992.

Braester, C., Moisture variation at the soil surface and advance of wetting front during infiltration at constant flux, *Water Resour. Res.*, *9*, 687–694, 1973.

Bresler, E., Analysis of trickle irrigation with application to design problems, *Irrig. Sci.*, *1*, 3–17, 1978.

Buckingham, E., Studies on the movement of soil moisture, *U.S. Dep. Agric. Bur. Soils Bull. 38*, 1907.

Carslaw, H. S., and J. C. Jaeger, *Conduction of Heat in Solids*, 2nd ed., 510 pp., Oxford Univ. Press, New York, 1959.

Farlow, S., *Partial Differential Equations for Scientists and Engineers*, John Wiley, New York, 1982.

Gardner, W. R., Some steady-state solutions of the unsaturated moisture flow equation with application to evaporation from water table, *Soil Sci.*, *85*, 228–232, 1958.

Gradshteyn, I. S., and I. M. Ryzhik, *Tables of Integrals, Series and Products*, Academic, San Diego, Calif., 1980.

Greenberg, M. D., *Application of Green's Functions in Science and Engineering*, Prentice-Hall, Englewood Cliffs, N. J., 1971.

Gröbner, W., and N. Hofreiter, *Integraltafel*, Springer-Verlag, New York, 1957.

Haberman, R., *Elementary Applied Partial Differential Equations*, 2nd ed., 547 pp., Prentice-Hall, Engelwood Cliffs, N. J., 1987.

Kirchhoff, G., *Vorlesungen über die Theorie der Wärme*, Barth, Leipzig, 1894.

Lomen, D. O., and A. W. Warrick, Time-dependent linearized infiltration, II, Line sources, *Soil Sci. Soc. Am. J.*, *38*, 568–572, 1974.

Lomen, D. O., and A. W. Warrick, Linearized moisture flow with loss at the soil surface, *Soil Sci. Soc. Am. J.*, *42*, 396–400, 1978a.

Lomen, D. O., and A. W. Warrick, Time-dependent solutions to the one-dimensional linearized moisture flow equation with water extraction, *J. Hydrol.*, *39*, 59–67, 1978b.

Ogata, A., and R. B. Banks, A solution of the differential equation of longitudinal dispersion in porous media, *U.S. Geol. Survey, Prof. Pap 411-A*, 1961.

Philip, J. R., Theory of infiltration, *Adv. Hydrosci.*, *5*, 215–296, 1969.

Philip, J. R., Steady infiltration from circular cylindrical cavities, *Soil Sci. Soc. Am. J.*, *48*, 270–278, 1984.

Philip, J. R., Linearized unsteady multidimensional infiltration, *Water Resour. Res.*, *22*, 1717–1727, 1986.

Press, W. H., B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes: The Art of Scientific Computing*, Cambridge Univ. Press, New York, 1986.

Pullan, A. J., The quasilinear approximation for unsaturated porous media flow, *Water Resour. Res.*, *26*, 1219–1234, 1990.

Pullan, A. J., Linearized time-dependent infiltration from a shallow pond, *Water Resour. Res.*, *28*, 1041–1046, 1992.

Richards, L. A., Capillary conduction of liquids through porous medium, *Physics*, *1*, 318–333, 1931.

Sommerfeld, A., *Partial Differential Equations in Physics*, Academic, San Diego, Calif., 1949.

Warrick, A. W., Time-dependent linearized infiltration, I, Point sources, *Soil Sci. Soc. Am. J.*, *38*, 383–386, 1974.

Warrick, A. W., Analytical solutions to the one-dimensional linearized moisture flow equation for arbitrary input, *Soil Sci.*, *120*, 79–84, 1975.

Warrick, A. W., and D. O. Lomen, Time-dependent linearized infiltration, III, Strip and disc sources, *Soil Sci. Soc. Am. J.*, *40*, 639–643, 1976.

Warrick, A. W., and D. O. Lomen, Linearized moisture flow with root extraction over two-dimensional zones, *Soil Sci. Soc. Am. J.*, *47*, 869–872, 1983.

Weir, G. J., Steady infiltration from large shallow ponds, *Water Resour. Res.*, *22*, 1462–1468, 1986.

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