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Parallel time scales and two-dimensional manufacturer and individual customer warranties

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We deal with a system whose failures depend on several parallel effects, such as the time in use $\mathscr{L}$ and the mileage $\mathscr{H}$. Manufacturer warranties are typically described by a two-dimensional region in the $(\mathscr{L}, \mathscr{H})$-plane. A proper determination of the warranty limits must be based on a two-dimensional distribution of time to failure on this plane. The aim of this paper is to demonstrate the possibility of designing individual warranties for a "nontypical" customer who has a very low or very high usage rate $b = H/L$, and to show a simple way to calculate warranty limits by minimizing the lifetime coefficient of variation. The latter is carried out by introducing the "best" combined time scale in the form $\mathscr{K} = (1-\epsilon)\mathscr{L} + \epsilon\mathscr{H}$ which provides the minimal lifetime coefficient of variation.

1. Introduction

In engineering practice, the choice of a Warranty Period (WP) must take into consideration the most relevant time scales in which the system is operating and in which its failures are recorded, as is discussed in Farewell and Cox [1], Kordonsky and Gertsbakh [2–6], Singpurwalla and Wilson [7], Murthy *et al.* [8] and Oakes [9]. Typical areas of interest are aircraft, jet engines and cars. There are three main parallel time scales for aircraft: the calendar time (time in use), the total time in the air, and the number of flights, [4,5]. There are two significant time scales for aircraft jet engines: the total operational time and the number of operational cycles. Two time scales are in common use for cars: calendar time and mileage [10].

Depending on the time scales used, the WP can be defined in several dimensions. For example, two-dimensional rectangular warranty regions have been considered by Singpurwalla and Wilson [7] and Murthy *et al.* [8]. We will consider in this paper the choice of WP in two scales which will be denoted as $\mathscr{L}$ and $\mathscr{H}$.

The mathematical aspects of multidimensional warranties have been considered by Singpurwalla and Wilson [7] and also Murthy *et al.* [8]. Their starting point for a formal analysis is a two-dimensional distribution of the time-to failure in $(\mathscr{L}, \mathscr{H})$. Statistical data analysis in the presence of parallel time scales has been examined in the papers of Oakes [9], Lawless *et al.* [10] and Kordonsky and Gertsbakh [2,6].

Farewell and Cox [1] and Oakes [9], have introduced the concept of a lifetime in a certain "optimal" time scale $\mathscr{K}$ which is defined as a linear combination of $\mathscr{L}$ and $\mathscr{H}$. Kordonsky and Gertsbakh [2–6] have introduced a linear combination of the form $\mathscr{K} = \mathscr{L} + g\mathscr{H}$, or for computational convenience $\mathscr{K} = (1-\epsilon)\mathscr{L} + \epsilon\mathscr{H}$. The parameter ϵ is chosen in such a way that the lifetime coefficient of variation attains its *minimal* value in the scale $\mathscr{K}$.

The choice of the WP by the manufacturer must take into account the demands of potential customers and any traditions of the market place. For example, a car manufacturer would not consider selling cars with a 2 year and 20 000 mile warranty because there are already cars on the market sold with a 5 year and 50 000 mile warranty. However, the warranty limits in terms of years and mileage can't be too high because this would lead to very high losses on warranty claims, thus making the car production unprofitable. So, the key issue in choosing the WP is finding a compromise between the interests of the manufacturer and the customer. In practice, after a market study and forecasting exercise, the manufacturer formulates his marketing policy so as to take into account, among other factors, reliability and quality parameters. Very often a highly competitive market place dictates the required values for the product quality and reliability. An overall quality and reliability characteristic that significantly determines the WP is the percentage of failed products during the WP.

The choice of warranty limits L_w and H_w must take into account the rate of use $b = H/L$, i.e., the average mileage per year for a typical customer. If for example, most customers drive 20 000 miles each year then a warranty of 50 000 miles and 5 years is a bad choice. Such a WP is targeted at customers doing on the average about 10 000 miles per year.

Three types of customer are present in the family of the second author. The first one drives 10 000 miles yearly and represents the "average" type. The second one drives about 20 000 miles per year, a figure which is significantly above the average, and the third one drives only 5000–6000 miles per year. Suppose that all three users have similar warranties (5 years, 50 000 miles). Then obviously they are in nonequal conditions. The second user would not be satisfied with his warranty while the first one would consider his warranty as a "fair" one.

Lawless *et al.* [10] have presented data on the distribution of the rate of use, H/L, over a large population of customers. The median rate of use is about 13 000 miles per year, with 60% of all customers doing between 8000 and 17 000 miles yearly, 20% of all customers doing less than 8000 miles and the rest – above 17 000 miles in a year. The warranty limits (5 years, 65 000 miles) are well-suited for the main body of customers. At the same time, the "marginal" customers with a very small or very large mileage per year are aware as soon as they buy their car that it will be impossible for them to fully use the warranty limits. For these customers a two-dimensional warranty region is irrelevant, the true warranty limit for them will be either the maximum mileage or the maximum calendar time.

This situation is shown on Fig. 1. Each customer, at the instant of purchase, could determine to which category A, B or C he/she belongs. It is quite obvious that category A customers know in advance that for them the *mileage* H_w will be the true warranty limit, while for category C customers, the *time* limit L_w is the relevant one.

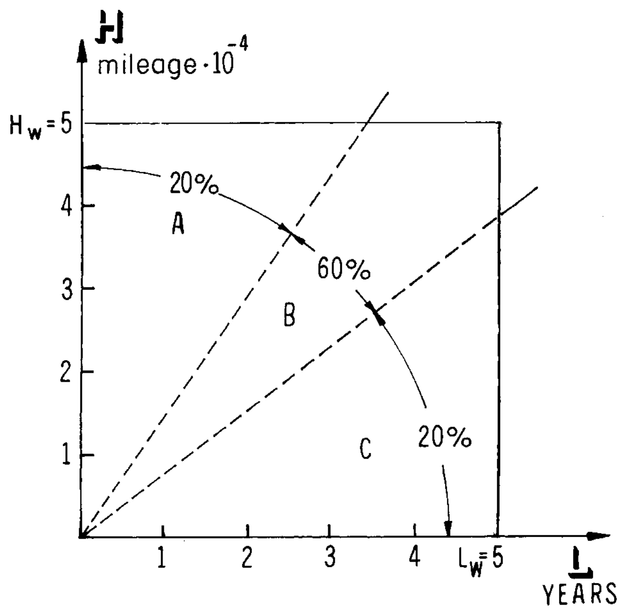

Fig. 1. Three categories of car users.

It becomes obvious from the above reasoning that there might be in fact two types of WP's. The first we call the *market* or *manufacturer* WP. It is a two-dimensional rectangular region which is suggested by the manufacturer, as shown on Fig. 1. Market competition forces the manufacturer to come up with this type of WP. On the other hand, there is a second type of warranty which we call *individual* WP which is meant for customers with either relatively high or relatively low driving rates. Selling individual warranties "tailored" to customer specific needs may increase the number of customers and provide higher profits for the manufacturer.

Figure 2 presents market and individual WP for aircraft jet engines. The authors have data on 543 engines that were used by Aeroflot during the period 1980–85. The market WP is a rectangle with $L_w = 6000$ hours and $H_w = 2000$ operational cycles. About 92% of all engines have a use rate $b = H/L$ which falls in the narrow sector B. The total number of claims from the 543 engines was 21, 17 failures were recorded on the engines in the B sector. In addition to the market WP, Fig. 2 also shows two individual WP's. These are the rectangular regions with vertices M_1, M_2. Thus, a "nontypical" customer with a high rate of use is given an option to choose one of these warranty policies. For example, a company which carries out short and frequent flights might prefer to buy the M_1 individual warranty (i.e., 3800 operating cycles and 1000 flight hours).

An interesting problem is the influence of warranty policy on the distribution of customers. Selling individual WP's may attract "nontypical" customers, and thus increase the number of customers with a high usage ratio among all other customers.

Choosing individual warranties raise two principal questions:

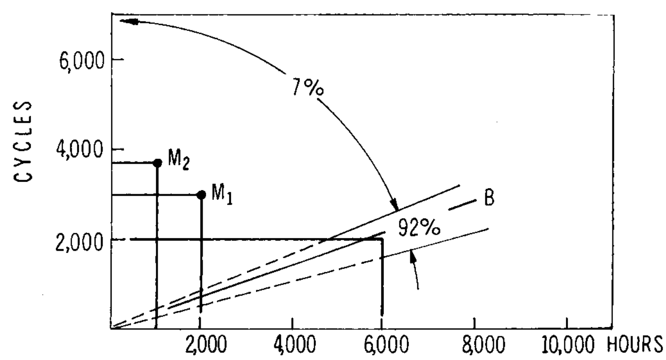

Fig. 2. Ninety-two percent of all (L, H) points are in region B. The rectangle (6000, 2000) is the manufacturer's warranty for "regular" users. Rectangles with vertices M_1, M_2 are the individual warranties for users with high b values.

(1) Could the individual warranties stimulate an increase in the number of customers?
(2) How do the individual warranties influence the warranty claims?

This paper is entirely devoted to answering the second question.

The next section is devoted to the lifetime distribution in the $(\mathscr{L}, \mathscr{H})$-plane. Our approach is based on introducing the conditional lifetime distribution for a fixed value of the usage rate $b = H/L$. In connection with the lifetime distribution along a fixed *operational trajectory* (OT), i.e., for fixed b, we investigate the so-called *impact curve* which describes the dependence of the mean $\mathscr{H}$-time or the mean $\mathscr{L}$-time on b. Section 3 is devoted to finding the "best" lifetime scale. According to our approach, the best time scale corresponds to the linear combination of $\mathscr{L}$ and $\mathscr{H}$ which provides the minimum lifetime coefficient of variation. Geometrically, creating the best scale is similar to projecting lifetimes along OT's on an axis $\mathscr{K}_0$. Then, under appropriate conditions, the Δ-quantile point on the best scale determines a line whose intersections with the OT's provide Δ-quantiles for each particular b value. This serves as a key for choosing individual warranties, as is shown in Section 4.

From a formal point of view, the use of the optimal linear combination of $\mathscr{L}$ and $\mathscr{H}$ allows us to overcome difficulties which always arise in working with two-dimensional lifetime distributions. The mathematics greatly simplifies because it becomes possible to use the Δ-quantile line of the optimal time scale for determining warranty regions for various types of users by preserving the same risk of a warranty claim.

To make the exposition more transparent and to explain the results we used geometric notions. A more formal exposition is presented elsewhere [2,6].

2. Operational conditions, operational trajectories and lifetime distribution

Identical systems are used by individual users under different Operational Conditions (OC). Theoretically, the OC might be described by a complex, multidimensional random process reflecting random alterations in time of operation regimes and external loading factors. For a car, for example, the OC could include trip duration, road conditions and idle periods, acceleration and breaking histories, engine loading parameters, weather and climate data, etc. It would be an extremely difficult and practically hopeless task to present an adequate formal description of the OC process acting on a single car. On the other hand, it would be a gross error to ignore differences in OC's for different examples in predicting possible failures and warranty claims. This situation is resolved by incorporating only the most important components of the OC. These components must satisfy two conditions: (1) they have the greatest impact on the failure development; and (2) they must be easily measurable. The best way of characterizing the OC is to describe for each example its individual *trajectory* in the $(\mathscr{L}, \mathscr{H})$-plane. Figure 3 depicts realizations of two OC's for aircraft jet engines on the $(\mathscr{L}, \mathscr{H})$-plane. They will be further termed as *Operational Trajectories* (OT's). These OT's are stepwise random fuctions. It will be assumed that for each single example these functions can be approximated by straight lines [3–5]. Similarly, it has been assumed in the literature that each car has a stationary pattern of use and does approximately the same mileage per unit of calendar time [10].

The possibility of approximating an OT by a straight line is an important and useful simplification [8].

Formally, the whole set of OT's can be represented in the form

$$\mathscr{H} = b\mathscr{L}, \quad b \in (0, \infty), \tag{1}$$

where b is the slope of an OT. Formally, we view b as a random variable over the whole emsemble of OT's.

For the data on aircraft engines, see Fig. 2, the random variable b with a probability of 0.92 lies in a relatively narow range of [0.36, 0.41] . It is natural to assume that b is a continuous random variable with some density $f_b(x)$.

Probabilistic validation of the WP is based on the possibility of estimating the probability of failure leading to a warranty claim for every fixed OT (i.e., for every fixed b).

Consider a fixed OT with a given value of b. It determines its own time scale $\mathscr{W}$ along OT. Let W be the corresponding lifetime, see Fig. 4. Denote by $f_W(w|b)$ the lifetime density for W. Let $f_b(\cdot)$ be the density of the slope b. It is obvious that the joint density of W, b will be $f_{W,b}(w, v) = f_W(w|b) \times f_b(v)$.

Variations of initial product quality and the influence of various random loading factors which act under a fixed rate of use are responsible for the variability of the

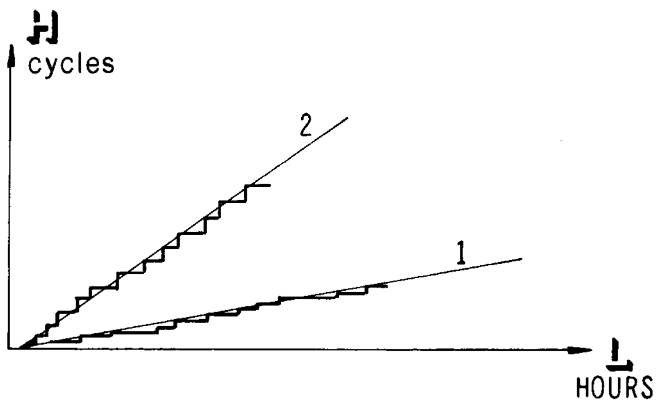

Fig. 3. OT **1** corresponds to long-distance flights whilst OT **2** corresponds to short distance flights.

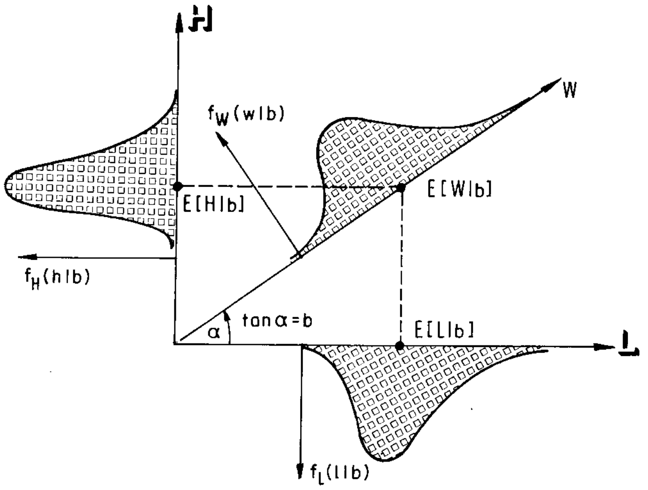

Fig. 4. The density $f_W(w|b)$ in the direction W along a fixed OT and its projections $f_L(l|b)$ and $f_H(h|b)$.

lifetime along a fixed OT. Lawless *et al.* [10] have assumed that $f_W(w|b)$ belongs to the Weibull family. This seems a reasonable approximation to reality, especially when the failure along a fixed OT is the minimum time to failure of several systems subject to mechanical fatigue and damage.

From the practical point of view, it is more convenient to use the two-dimensional distribution of L and b, instead of W and b, because the lifetime in scale $\mathscr{L}$ is simple to measure.

Let

$$\phi = \tan^{-1} b. \tag{2}$$

Note that

$$L = W \cos \phi, \tag{3}$$

and

$$H = W \sin \phi. \tag{4}$$

For a fixed b, and therefore for a fixed ϕ, (3) and (4) define a scale transformation of W into L and H, see Fig. 4. Therefore, for a Weibull lifetime along a fixed OT, the lifetime L in the $\mathscr{L}$ scale has distribution

$$F_L(l;\theta(b)|b) = 1 - \exp\left\{-\left(\frac{l}{\theta_1(b)}\right)^{\theta_2(b)}\right\}. \tag{5}$$

It is often assumed that

$$\theta_2(b) = \text{Const.}, \quad b \in (0, \infty), \tag{6}$$

which means that the lifetime coefficient of variation is the same for all b, [6,9]. This assumption is also confirmed by some real-life data sets.

A very important issue for further analysis is the correct modeling of the relationship between θ_1 and b.

Since

$$E[L|b] = \theta_1(b) \times \Gamma(1 + 1/\theta_2), \tag{7}$$

the problem is to determine how $E[L|b]$ depends on b. This function will be termed the *impact curve*. Different rates of b reflect different loading patterns. For example, a high rate of b for jet engines corresponds to an intensive temperature loading. The impact curve (7) gives the connection between the average lifetime and the loading and is similar to the so-called Wohler curve which describes how the average fatigue life depends on the level of the load, [11]. Fatigue experts are familiar with the difficulties associated with determining its analytic form.

Following the methodology of fatigue analysis, we assume the simplest functional relationship for $E[L|b]$:

$$E[L|b] = \frac{\alpha}{(\gamma + \delta b)^{\beta}}, \tag{8}$$

where α, β, γ and δ are positive constants that are independent of b.

Let us note that it is always implied in the literature [4–7,10] that $Cov[L,H] < 0$. Physically, this condition means that the operation in both time scales $\mathscr{L}$ and $\mathscr{H}$ creates an accumulation of damage and shortens system lifetime. It is not difficult to see that $Cov[L,H] < 0$ implies $Cov[E[L|b], E[H|b]] < 0$.

Figure 4 shows that the point $E[W|b]$ has coordinates $(E[L|b], E[H|b])$. When b varies, this point describes a curve on the $(\mathscr{L}, \mathscr{H})$-plane. Denote this curve by

$$m_H = \psi(m_L). \tag{9}$$

To make $Cov[L,H] < 0$, it is enough to demand that $\psi' < 0$.

Now let us express b from (8) as a function of $E[L|b]$ and use the fact that $E[H|b] = bE[L|b]$. Then we obtain the curve (9) as

$$E[H|b] = \delta^{-1}(\alpha^{1/\beta} \times E[L|b]^{1-1/\beta} - \gamma E[L|b]). \tag{10}$$

To provide a negative derivative of the right-hand side, β must be less or equal to 1. When $\beta = 1$, we have

$$E[H|b] = \delta^{-1}(\alpha - \gamma E[L|b]), \tag{11}$$

a linear relationship between $E[L|b]$ and $E[H|b]$!

Figure 5 shows an example of an impact curve, see **1**. A remarkable fact is that the segment AB of this curve between two OT's **3** and **4** is quite close to a straight line. For aircraft engine data, more than 97% of all OT's lie between **3** and **4**.

If the impact curve $E[L|b]$ is known, then we know $\theta_1(b)$ and the joint density of L and b can be written as

$$f_{L,b}(l,b) = f_L(l;\theta_1(b),\theta_2) f_b(b), \tag{12}$$

where $f_L(l;\theta_1(b),\theta_2)$ is the Weibull density for a fixed b. Knowing (12), it is possible in principle to calculate the probability of a warranty claim during the warranty period.

On the other hand, using (12) is a complex procedure since it is difficult to estimate the impact curve parame-

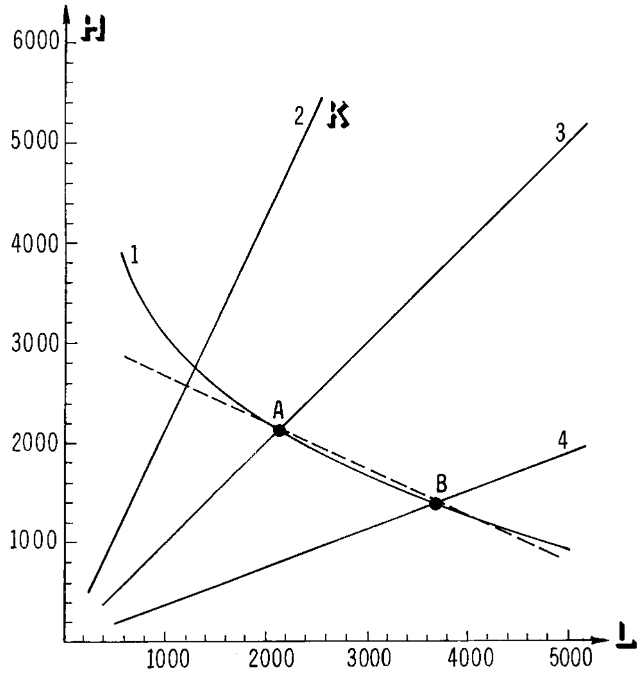

Fig. 5. **1** is an impact curve (10) for $\delta = 5$, $\alpha = 8100$, $\gamma = 1$ and $\beta = 0.75$. Most of the OT's are located between the OT **3** and the OT **4** ($b = 0.3$). The curved segment AB of **1** is close to a straight line orthogonal to the $\mathscr{K}$-scale **2**.

ters. We will describe an alternative approach which seems to be simpler in implementation.

3. The best lifetime scale and the failure probability

The scale transformation, i.e., the change of units for measuring the lifetime should not influence the failure probability. Therefore, the distribution of lifetime T must satisfy the relationship:

$$P(T \leq t) = F(t; \bar{\theta}) = F_0\left(\frac{t}{\theta_1}; \bar{\theta}_0\right), \tag{13}$$

where θ_1 is the scale parameter which has the dimension of time and $\bar{\theta}_0$ is dimensionless, [6]. Introduce a new variable

$$u = \frac{t}{\theta_1}. \tag{14}$$

Then (13) becomes

$$F(t; \bar{\theta}) = F_0(u; \bar{\theta}). \tag{15}$$

Note that the moments of the distribution $F_0(u; \bar{\theta}) = P(U \leq u)$ do not depend on θ_1. Therefore,

$$E[T] = \theta_1 E[U]. \tag{16}$$

It will be assumed in the continuing exposition that (13) takes place and that the only parameter which depends on the slope b of the OT is θ_1. This assumption is also made by other authors [9,10]. Its statistical validation can be made by checking whether the coefficient of variation (c.v.) of T remains constant for different b values. It is well known from fatigue test practice that a decrease of loading may double or even triple the mean lifetime while the lifetime c.v. shows only a 5–10% increase. Typically, the c.v. is close to constant for a wide range of b values.

Suppose, there are only two OT's with probabilistic weights p_1 and p_2, $p_1 + p_2 = 1$. Denote by b_1 and b_2 their slopes. Let $f_W(w/\theta_w(b_i); \bar{\theta}_W)$, $i = 1, 2$, be the corresponding density functions of the lifetimes along these two OT's. According to (3) and (4), the transition to the scales $\mathscr{L}$ and $\mathscr{H}$ means only that the scale parameters $\theta_W(b_i)$ will change. The resulting lifetime distribution in the $\mathscr{L}$ and $\mathscr{H}$ scales are *mixtures* of the respective densities. It follows from (16) that

$$E[L|b] = \theta_1 E[U]. \tag{17}$$

One can take $E[L|b_1]$ and $E[L|b_2]$ as scale parameters in the $\mathscr{L}$ scale. The mixture in the $\mathscr{L}$ scale has the density

$$f_L(x) = p_1 f_L(x/E[L|b_1]; \bar{\theta}_W) + p_2 f_L(x/E[L|b_2]; \bar{\theta}_W). \tag{18}$$

Similarly, for the $\mathscr{H}$ scale we have

$$f_H(x) = p_1 f_H(x/E[H|b_1]; \bar{\theta}_W) + p_2 f_H(x/E[H|b_2]; \bar{\theta}_W). \tag{19}$$

Figure 6 illustrates the situation. One can see that the mixture in $\mathscr{L}$ is bimodal while in $\mathscr{H}$ it appears to be unimodal since $E[H|b_1]$ and $E[H|b_2]$ are close to each other. The values of $E[H|b_1]$ and $E[H|b_2]$ are *projections* of the corresponding mean values from the OT's. It is clear that if $\mathscr{K}$ is a linear combination of $\mathscr{L}$ and $\mathscr{H}$, then $E[K|b]$ is a linear combination of $E[L|b]$ and $E[H|b]$. A natural idea would be to project the lifetime values from different OT's onto a single axis, i.e., on a time scale $\mathscr{K}$,

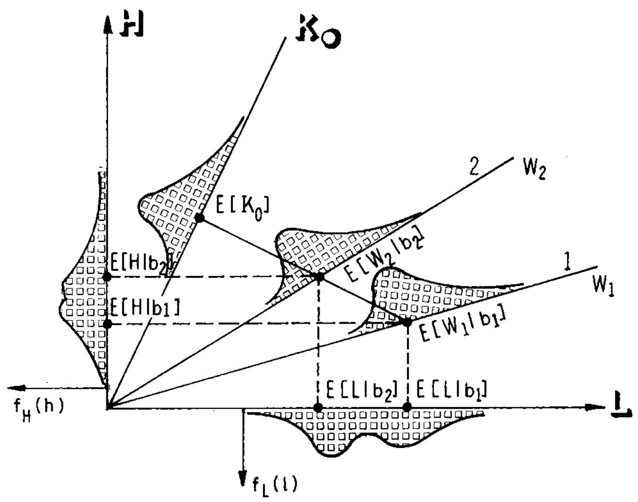

Fig. 6. $E[W_1|b_1]$ and $E[W_2|b_2]$ are the conditional means along the OT's **1** and **2**. $f_L(l)$ and $f_H(h)$ are the mixtures in $\mathscr{L}$ and $\mathscr{H}$. $\mathscr{K}_0$ is the direction of the "best" scale.

in such a way that $E[K|b_1]$ and $E[K|b_2]$ would be as close as possible. For only two OT's, one can design a scale $\mathscr{K}_0$ in which the means coincide, i.e., for which $E[K_0|b_1] = E[K_0|b_2] = E[K_0]$. Figure 6 demonstrates that the "best" scale $\mathscr{K}_0$ is orthogonal to the segment which connects the means $E[W|b_1]$ and $E[W|b_2]$.

When the number of OT's exceeds two, one can hardly expect that it would be possible to construct a time scale $\mathscr{K}$ in which all "projected" densities from the OT's would coincide. It is however possible to design a time scale in which the densities $f(K|b)$ are close to each other.

Following [1,2–6], we are looking for a linear combination of scales $\mathscr{L}$ and $\mathscr{H}$ in the form

$$\mathscr{K} = (1-\epsilon)\mathscr{L} + \epsilon\mathscr{H}, \quad \epsilon \in [0,1], \tag{20}$$

where ϵ is chosen to provide the *minimal coefficient of variation* (c.v.) for the lifetime K in the $\mathscr{K}$ scale. Formally, we are looking for the time scale $\mathscr{K}_0$ in which the lifetime $\mathscr{K}_0$ satisfies

$$\min_{0 \le \epsilon \le 1} \text{c.v.}[K] = \text{c.v.}[K_0]. \tag{21}$$

Suppose that our "best" time scale has the form

$$\mathscr{K}_0 = (1-\epsilon_0)(\mathscr{L} + g\mathscr{H}), \quad g = \frac{\epsilon_0}{1-\epsilon_0}. \tag{22}$$

This formula establishes that 1 unit of $\mathscr{H}$ time is equivalent, in terms of damage accumulation, to g units of $\mathscr{L}$ time. The time scale $\mathscr{K}_0$ is in a sense a generalization of the Miner scale, [12]. The principle defined by (22) and its connection with the Miner scale is discussed in detail elsewhere is the literature [4–6].

Let us note that (21) gives a scale $\mathscr{K}_0$ in which the lifetimes K are minimally dependent on the usage rate b.

Equation (8) does not describe the whole variety of operational conditions but we will use it to demonstrate the meaning of the $\mathscr{K}_0$ scale. Equation (11) corresponds to an ideal situation in which (9) is a straight line. It is easy to show that in the latter case the "best" scale is orthogonal to this line, and for all $b \in (0,\infty)$, $E[K_0|b] = E[K_0]$. The equation (21) means that the "best" scale is orthogonal to a line which approximates a certain curved part of the impact curve. Figure 5 illustrates this situation.

If the $\mathscr{K}_0$-scale has been found and we know the density $f_{K_0}(v)$, then we are able to calculate the failure probability for each OT and for each point (l,h) on it, $h/l = b$. This probability is equal to $F_{K_0}((1-\epsilon_0)l + \epsilon_0 h)$, where $F_{K_0}(\cdot)$ is the distribution function of the lifetime K_0 in the "best" scale.

Obviously, the "best" scale coincides with the line

$$\mathscr{H} = \frac{\epsilon_0}{1-\epsilon_0}\mathscr{L}, \tag{23}$$

in the $(\mathscr{L},\mathscr{H})$-plane. The Δ-quantile is defined as

$$\Delta = P(K_0 \le k_\Delta) = F_{K_0}(k_\Delta). \tag{24}$$

The equation of the quantile line is, therefore,

$$k_\Delta = (1-\epsilon_0)l^*_\Delta + \epsilon_0 h^*_\Delta, \tag{25}$$

or

$$h^*_\Delta = -\frac{1-\epsilon_0}{\epsilon_0}l^*_\Delta + \frac{k_\Delta}{\epsilon_0}. \tag{26}$$

Obviously, the quantile line (25) is orthogonal to the $\mathscr{K}_0$-scale. Its intersection with the OT's determines the corresponding quantiles for the densities $f_W(w|b)$.

If (9) is not a straight line then the quantiles calculated from (25) might be considered as an approximation to the true ones. It is believed that for the largest part of the b-range this approximation is reasonably accurate.

An important issue is a statistical data analysis procedure to find the "best" scale $\mathscr{K}_0$, [4–6]. The fact that data on failures are available only for those items which have failed during the warranty period makes it necessary to use right censored data. The most appropriate way of incorporating censored observations is an assumption that the lifetime K in time scale $\mathscr{K}$ can be approximated by a single density $f_K(k)$ which has a known functional form. This density represents a mixture of distributions obtained from projecting lifetime distributions along different OT's on a common $\mathscr{K}$ axis, [4–6,8].

The respective likelihood function takes the form:

$$Lik = \prod_{j=1}^{n}(\bar{F}_K((1-\epsilon)L + \epsilon H))^{1-\delta_j} \times f_K((1-\epsilon)L + \epsilon H))^{\delta_j}, \tag{27}$$

where $\delta_j = 1$ for a failure and $\delta_j = 0$ for a censored observation. The parameters were estimated from this expression for each value of ϵ on the grid [0 (0.01) 1], and ϵ_0 was that value on this grid which provided the minimal c.v. If, for example, $F_K(v)$ is a Weibull distribution of the type in (5), then the search for ϵ_0 reduces to finding the value ϵ which provides the maximal value of the shape parameter θ_2.

We have processed the data obtained on 543 aircraft engines that was highlighted in Fig. 2. For a failed engine, the information was in the form (l_f, h_f), i.e., the failure times in both scales. For a nonfailed engine, it was known that it did not fail during l^* hours and h^* cycles. We assumed that the corresponding OT had a slope $b^* = h^*/l^*$. It was assumed that

$$\bar{F}_K(v) = \exp\left(-\left(\frac{v}{\theta_1}\right)^{\theta_2}\right). \tag{28}$$

Processing the data according to the above described procedure, gave the following results:

$\mathscr{L}$ scale: $\epsilon = 0$, $\hat{\theta}_1 = 21\,620$, $\hat{\theta}_2 = 2.288$. c.v. $= 0.463$.

$\mathscr{H}$ scale: $\epsilon = 1$, $\hat{\theta}_1 = 8054$, $\hat{\theta}_2 = 2.58$. c.v. $= 0.416$.

$\mathscr{K}_0$ scale: $\epsilon_0 = 0.831$, $\hat{\theta}_1 = 8007$, $\hat{\theta}_2 = 3.09$. c.v. $= 0.354$.

3.1. *Example: computing failure probabilities in the best scale*

A typical OT for a jet engine has $b = 0.375$, see Fig. 2. Let us compute the failure probability in the $\mathscr{K}_0$ scale for $L = 6000$ hours and $H = 0.375 \times 6000 = 2250$ cycles. According to (25), this probability is equal to

$$P(K_0 \leq (0.169 \times 6000 + 0.831 \times 2250) \\ = 1 - \exp(-(2883/8007)^{3.09}) = 0.0417.$$

Suppose now that the OT has $b = 4$, i.e., it corresponds to a high number of cycles per hour. Suppose that we are interested in failure probability during the first 2000 cycles. Then, similarly to the above computations, we obtain that for 500 hours and 2000 cycles the failure probability equals

$$P(K_0 \leq (0.169 \times 500 + 0.831 \times 2000)) = 0.009.$$

If the rate of use of an engine is small, e.g., the OT has a slope $b = 0.166$, the failure probability for 6000 hours and 1000 cycles will be

$$P(K_0 \leq (0.169 \times 6000 + 0.831 \times 1000)) = 0.011.$$

We see therefore, that the probability of a warranty claim is different for different WP's. For most of the users with a usage rate close to $b = 0.375$, the failure probability will be close to 0.04. This number corresponds to the WP (6000 hours, 2250 cycles). For a "nontypical" customer with $b = 4$, there is a possibility to extend the warranty far beyond 2250 cycles. For example, consider an individual warranty (750 hours, 3000 cycles) for a customer with $b = 4$. The failure probability will be $P(K_0 \leq (0.169 \times 750 + 0.831 \times 3000)) = 0.031$.

4. Manufacturer and individual customer warranties

Choosing warranty periods is a complex decision-making process based on both a market study and on an engineering reliability analysis. Statistical methods play an important role in determining the expected costs associated with failures during the warranty period. We will discuss this issue by considering two examples – aircraft engines and cars.

4.1. *Warranty policy for aircraft engines*

Choosing warranty periods for aircraft jet engines is a relatively simple problem. Warranty claims are mainly the result of failures in the hot part of the engine (the turbines, combustion chamber). These failures are typically swirler cracks, cracks of blades, etc. The failure mechanism and failure type justify the use of a Weibull distribution for the lifetime description.

An important feature of the appearance of failure in jet engines is that both the operation time L and the operation cycles (warmups) H are highly relevant to the damage accumulation process. There are no failures for which only the operation time (and not the operation cycles) would be responsible. Similarly, there are no failures whose appearance would solely depend on the number of cycles and not on the operation time.

The cost structure associated with the failures, is also relatively simple. These costs typically reflect engine replacement and penalty costs for aircraft idle time.

Figure 7 shows the 0.03-quantile line **2** for the $\mathscr{K}_0$-scale. About 93% of all engines have their OT's within the range $b \in [0.35, 0.40]$, between the OT's **3** and **4**. Thus the manufacturer's WP of 5500 hours and 1800 cycles would be appropriate for most customers and would guarantee failure probability within the warranty region not exceeding 0.03.

For a customer with a high usage rate, say with $b = 2.5$, see **1** on Fig. 7, a more attractive policy would be an individual WP with 1200 hours and 2800 cycles which also guarantees for the manufacturer the same failure probability of 0.03 within the individual warranty region.

Suppose there are many potential users having b near 0.2. These might be, for example, aircraft companies carrying out long flights. For them, an individual warranty region (1500 cycles, 8000 hours) is more attractive than the "standard" one of (1800 cycles and 5500 hours). At the same time, the failure probability is about the same, 0.03, for this individual warranty, meaning that for the manufacturer the risk of a warranty claim remains the same. Summing up, individual warranties are capable of

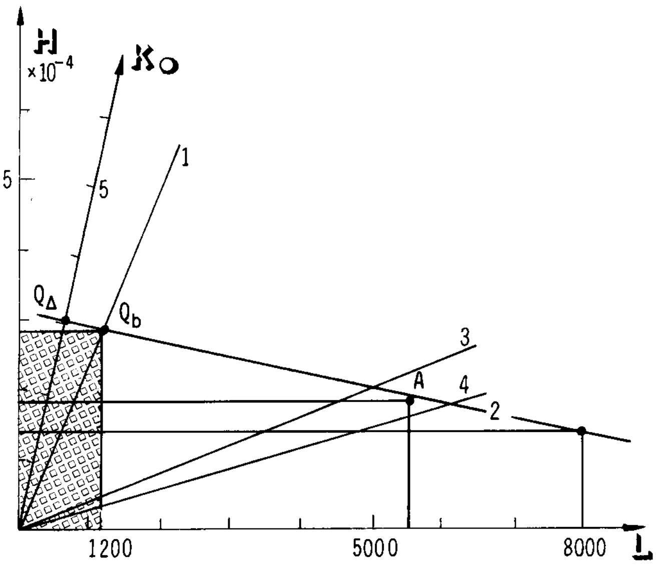

Fig. 7. The majority of users have their OT's between **3** and **4**. Q_Δ is the 0.03-quantile in the "best" scale $\mathscr{K}_0$. **2** is the 0.03-quantile line. The rectangle with a vertex Q_b is the individual warranty region for users with $b = 2.5$. The rectangle with vertex A is the warranty region for "regular" users.

attracting more customers without increasing the risk of warranty claims (and the respective warranty costs).

4.2. *Warranty policy for passenger cars*

Here the overall situation is considerably more complicated. There are failures which predominantly solely depend on the mileage and hence do not depend on the time in use. For example, failures in the braking system. On the other hand, many failures result from damage accumulation in calendar time and only slightly depend on the mileage, as is the case of rusting of the car body. Generally speaking, there are failures for which only one of the scales $\mathscr{L}$ or $\mathscr{H}$ is responsible. Therefore, all failures can be classified into three groups:

Group I: failures solely predicted in the scale $\mathscr{L}$.
Group II: failures solely predicted in the scale $\mathscr{H}$.
Group III: failures predicted jointly by both $\mathscr{L}$ and $\mathscr{H}$.

Another complication is caused by the complex cost structure of the warranty claims created by the fact that different failures can have large differences in costs associated with part replacement and labor. We will assume that the manufacturer controls the costs of warranty claims by a using an "adjustable" warranty policy for different parts and systems and that the main concern for the manufacturer is preserving a certain limiting value Δ_{lim} for a warranty claim probability during the warranty period. The presence of such a limit is also attractive for customers because their judgment about the product and their desire to acquire it is often based on the overall notion that "it is reliable and has very few failures".

Typically, $\Delta_{\text{lim}} \leq 0.10$ for passenger cars. Denote the failure probabilities during the WP in the above groups by $\Delta_I, \Delta_{II}, \Delta_{III}$, respectively. Assuming that these numbers are small, the overall failure probability Δ_M during the WP equals

$$\Delta_M \simeq \Delta_I + \Delta_{II} + \Delta_{III}. \tag{29}$$

Based on statistical analysis of failure data, three scales must be constructed for the above three groups. The first two scales will coincide with $\mathscr{L}$ and $\mathscr{H}$, respectively, and only the third scale $\mathscr{K}_0^{III}$ is a combination of $\mathscr{L}$ and $\mathscr{H}$ of type (20). Let $F_L(l), F_H(h)$ and $F_K(k)$ be the lifetime distributions for the corresponding failure groups.

A possible scenario for choosing the WP's might be the following. Choose a "typical" OT which represent the "median" part of all customers, see Fig. 8. Take a point A on this OT and consider the warranty rectangle L_A, H_A. Point A corresponds to the point

$$K_A = (1-\epsilon)L_A + \epsilon H_A, \tag{30}$$

on the $\mathscr{K}_0^{III}$ scale. Now calculate the failure probabilities

$$\Delta_I^A = F_L(L_A),\ \Delta_{II}^A = F_H(H_A),\ \Delta_{III}^A = F_K(K_A). \tag{31}$$

Moving the point A towards the origin, find a point A_0 such that

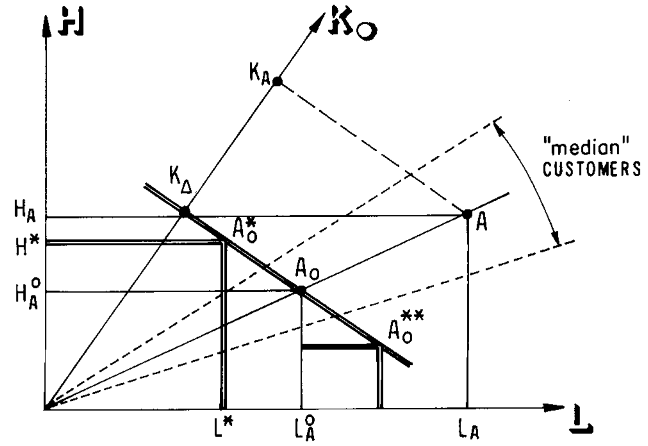

Fig. 8. Choosing individual warranty regions subject to the probability constraint (33).

$$\Delta_I^{A_0} + \Delta_{II}^{A_0} + \Delta_{III}^{A_0} \leq \Delta_{\text{lim}}. \tag{32}$$

Draw the corresponding quantile line through the point A_0. Use the points on this line as the vertices for the individual warranty regions. For example, check the individual warranty region with vertex $A_0^* = (L^*, H^*)$, see Fig. 8. If

$$F_L(L^*) + F_H(H^*) + F_K(K_\Delta) \leq \Delta_{\text{lim}}, \tag{33}$$

then this warranty region is acceptable and may be suggested for customers with usage rate above the "typical" one. A similar check should be done for individual warranty region with vertex A^{**}, for customers with a low usage rate.

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