Journal of Low Temperature Physics, Vol. 126, Nos. 3/4, February 2002 (© 2002)

Orbital Magnetism in Confined Two-Dimensional Systems

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The way how the Landau diamagnetism emerges in finite two-dimensional systems, disc and ring, has been clarified by examining both the field dependences of the magnetization and the spatial distribution of the persistent currents induced by the magnetic field.

1. INTRODUCTION

Orbital magnetism in solids, which is due to the persistent electric current induced by the external magnetic field, is completely quantum-mechanical in its origin as exemplified by the Bohr-van Leeuven theorem.[1,2] The finite orbital magnetic susceptibility has been first derived by Landau[3] for the simple case of electron gas and is now called the Landau diamagnetism. In solids the presence of periodic potential results in various contributions[4,5] to the magnetization and sometimes, *e.g.* typically in graphite[6] and bismuth[7] which are the solid-state version of neutrinos and the Dirac electrons, the interband effects have major contributions[8,9] invalidating such one-band description like Peierls formula[10] and the general understanding is still far from complete. Even for the case of electron gas, where the Landau formula is expected to apply, the understanding of orbital magnetism is challenged by the theoretical finding of the anomalously large fluctuations as the temperature is lowered.[11–15] In pursuing this issue it turns

0022-2291/02/0200-1067/0

out to be important to take account of the boundary effects explicitly since the currents in the presence of the magnetic field flow perpendicularly to the potential gradient due to the surface boundary.[16] In this context it is interesting to note that the conventional boundary conditions for the wave-functions in deriving the Landau formula based on the so-called Landau gauge, fast decaying in one direction and periodic in another direction on the plane perpendicular to the field, are not realistic but that the Landau formula is still expected to be valid for macroscopic systems because of the irrelevance of the particular forms of the boundary condition for macroscopic observables. Hence it is important to clarify theoretically the conditions of the validity of the Landau diamagnetism in finite systems. At the same time there will be of actual importance in the understanding of the orbital magnetism in meso- and nano- systems.[17] In this context it should be noted that Gurevich and Shapiro[18] had found that in weak magnetic field there are three distinct temperature regimes (microscopic, mesoscopic and macroscopic) and that in the last case the Landau diamagnetism is observed whereas there exist particular types of fluctuations in other cases. Moreover Hajdu and Shapiro,[19] by considering the case of the harmonic groove with confining potential in the x-direction, found that the qualitative features of the field dependence of the magnetic moment changes once the thermal coherence length $l_c = v_F/k_BT$ gets longer than the width of the groove. Based on these findings the purpose of this paper is to review our recent studies on this basic question to clarify the range of the validity of the Landau diamagnetism based on the exactly solvable two-dimensional models and to present new results on the model which has two boundaries, inner and outer, *i.e.* a ring. We take $\hbar = k_B = 1$.

2. ORBITAL MAGNETISM IN TWO-DIMENSIONAL HARMONIC CONFINING POTENTIAL

In order to clarify the dependences on magnetic field and temperature of the orbital magnetic moments we considered[15,17] the following well-known model with harmonic confining potential whose eigenfunction and eigenvalues are exactly known,

$$\mathcal{H} = \frac{1}{2m}\left(\boldsymbol{p} + \frac{e}{c}\boldsymbol{A}\right)^2 + \frac{1}{2}m\omega_0^2\boldsymbol{r}^2, \tag{1}$$

where $\boldsymbol{p}$ and $\boldsymbol{r}$ are two-dimensional vector, m is the electron mass and $(-e)$ is the electron charge. The radius of the system, R, is defined effectively as

$$\frac{1}{2}m\omega_0^2 R^2 = \mu, \tag{2}$$

where μ is the chemical potential.

By taking a symmetric gauge $\boldsymbol{A} = \frac{1}{2}\boldsymbol{H} \times \boldsymbol{r}$, we can obtain an eigenfunction diagonal with respect to the angular momentum α as

$$\begin{aligned} \psi_{n\alpha}(\boldsymbol{r}) &= \frac{\mathrm{e}^{\mathrm{i}\alpha\theta}}{\sqrt{2\pi}} R_{n\alpha}(r) \\ R_{n\alpha}(r) &= \frac{1}{l}\sqrt{\frac{n\,!}{(\,n+|\alpha|\,)\,!}} \exp\left[-\frac{r^2}{4l^2}\right]\left(\frac{r}{\sqrt{2}l}\right)^{|\alpha|} L_n^{(|\alpha|)}\left[\frac{r^2}{2l^2}\right] \end{aligned} \tag{3}$$

where polar coordinates (r, θ) are used, and $n = 0, 1, 2, \ldots$, $\alpha = 0, \pm 1, \pm 2, \ldots$ and $l = \sqrt{\hbar/m\omega}$, $\omega = \sqrt{\omega_c^2 + (2\omega_0)^2}$, $\omega_c = eH/mc$ being the cyclotron frequency and $L_n^{(\alpha)}$ is the Laguerre polynomial. The eigenenergy of this state, $E_{n\alpha}$, is given by

$$E_{n\alpha} = \hbar\omega_c\,\frac{\alpha}{2} + \hbar\omega\left(n + \frac{|\alpha|+1}{2}\right). \tag{4}$$

Especially under a extremely strong field ($\omega_c \gg \omega_0$), this eigenenergy $E_{n\alpha}$ becomes like a Landau series, $\hbar\omega_c(n + 1/2)$, for negative α.

2.1. The Field Dependence of the Magnetization

The magnetization can be determined by the thermodynamic potential Ω given by

$$\Omega = -\frac{1}{\beta}\sum_{n=0}^{\infty}\sum_{\alpha=-\infty}^{\infty} \log\left[\,1 + \mathrm{e}^{-\beta(E_{n\alpha}-\mu)}\right], \tag{5}$$

where $\beta = 1/T$ and μ is the chemical potential adjusted to fix the average electron number to N_0 at each values of H and T. From the thermodynamic potential, the magnetic moment M is given as

$$\begin{aligned} M &= -\left(\frac{\partial\Omega}{\partial H}\right)_{\mu} \\ &= \sum_{n\alpha}\left(-\frac{\partial E_{n\alpha}}{\partial H}\right) f(E_{n\alpha}) \end{aligned} \tag{6}$$

where $f(E)$ is the Fermi distribution function.

Based on this, the field dependence of the magnetic moment is calculated, which turns out to be classified into three regions; " Mesoscopic Fluctuation (MF)"," Landau Diamagnetism (LD) " and " de Haas-van Alphen (dHvA)", as is shown in Fig. 1. In the following we will mainly concerned with the low field region, where "MF" corresponds to the region as $T \lesssim \hbar\omega_- \equiv \frac{\hbar(\omega-\omega_c)}{2}$, which implies $T/\hbar\omega_0 \lesssim 1$ under a weak field

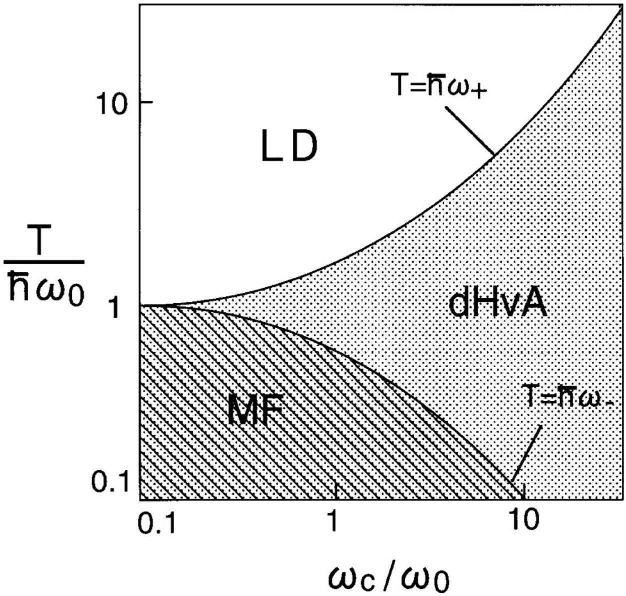

Fig. 1. Phase diagram showing characteristic regions of the field dependence of the magnetic moment at various temperatures. (After Y. Ishikawa *et al.*: *J. Phys. Soc. Jpn.* **68** (1999) 2405.)

($\omega_c/\omega_0 \lesssim 1$) and $T/\hbar\omega_0 \lesssim (\omega_c/\omega_0)^{-1}$ under a strong field ($\omega_c/\omega_0 \gtrsim 1$), while "LD" corresponds to the region as $T \gtrsim \hbar\omega_+ \equiv \frac{\hbar(\omega+\omega_c)}{2}$, corresponding to $T/\hbar\omega_0 \gtrsim 1$ under a weak field ($\omega_c/\omega_0 \lesssim 1$) and $T/\hbar\omega_0 \gtrsim \omega_c/\omega_0$ under a strong field ($\omega_c/\omega_0 \gtrsim 1$). Figs. 2(a), 2(b) show field dependences of magnetic moment at various temperatures under a weak field ($\omega_c \lesssim \omega_0$) . N_0 is set at 5000 in these calculation. Fig. 2(a) corresponds to "MF" and Fig. 2(b) ranges from "MF" to "LD". At low temperature as $T \lesssim \hbar\omega_0$ corresponding to "MF", the magnetic moment shows a large fluctuation with respect to the field, as Yoshioka and Fukuyama[15] have found. In this "MF" region, all the oscillatory terms in Ω_{osc} contribute to the magnetic moment leading to such a large fluctuation, in addition to $\Omega_L = -1/2 \cdot \chi_L H^2$ ($\chi_L = -1/3 \cdot D_0 \mu_B^2$, the Landau diamagnetic susceptibility, where $D_0 = \mu/(\hbar\omega_0)^2$, the density of states at Fermi energy). Particularly at much lower temperature ($T \ll \hbar\omega_0$) and under a much weaker field ($\omega_c \ll \omega_0$), the magnetic moment shows a strong paramagnetism, which was first noticed by Meier and Wyder[20] and discussed by Budzin *et al.*[21] This strong paramagnetism is attributed to the rotational symmetry of the system. Hence, in the presence of weak disorder, a large spatial variation of magnetic moment, either paramagnetic or diamagnetic, is expected, which fact is the cause of a large variance of orbital susceptibility in the limit of weak magnetic field at low temperature.[11–14] As the temperature is raised ($T \gtrsim \hbar\omega_0$) under such a weak field ($\omega_c \lesssim \omega_0$), the fluctuation of magnetic moment is reduced and the magnetic moment becomes linearly dependent on the field, the slope of which gives the Landau diamagnetic susceptibilities χ_L corresponding to "LD".

2.2. The Spatial Distribution of the Current

The spatial distribution of current in the system is given as follows,

$$\begin{aligned} \boldsymbol{J}(\boldsymbol{r}) &= \mathrm{Re}\left\langle \hat{\psi}^\dagger(\boldsymbol{r}) \frac{(-e)}{m} \left(\hat{\boldsymbol{p}}(\boldsymbol{r}) + \frac{e}{c}\boldsymbol{A}(\boldsymbol{r}) \right) \hat{\psi}(\boldsymbol{r}) \right\rangle \\ &\equiv J_\theta(r)\, \boldsymbol{e}_\theta, \end{aligned} \tag{7}$$

where $< \cdots >$ denotes the thermal average and $\hat{\psi}(\boldsymbol{r})$ is the field operator and $\boldsymbol{e}_\theta = \frac{\partial \boldsymbol{r}}{\partial \theta} / |\frac{\partial \boldsymbol{r}}{\partial \theta}|$ as shown in Fig. 3. By use of the eigenfunctions in eq. (3) as the basis, $J_\theta(r)$ is given as

$$J_\theta(r) = (-e)v_0 \sum_{n\alpha} \left[\alpha \left(\frac{r}{\xi}\right)^{-1} + \frac{\omega_c}{2\omega_0}\left(\frac{r}{\xi}\right) \right] R_{n\alpha}(r)^2\, f(E_{n\alpha}), \tag{8}$$

where $\xi = \sqrt{\hbar/m\omega_0}$, the characteristic length, and $v_0 = \omega_0 \xi$, the characteristic velocity of electrons.

This local current $\boldsymbol{J}(\boldsymbol{r})$ induces a magnetic moment $\boldsymbol{M}(\boldsymbol{r})$ given as follows,

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{r}) &= \frac{1}{2c} \boldsymbol{r} \times \boldsymbol{J}(\boldsymbol{r}) \\ &= \frac{1}{2c}\, r\, J_\theta(r)\, \boldsymbol{e}_z \\ &\equiv M_z(r)\, \boldsymbol{e}_z. \end{aligned} \tag{9}$$

We focus on the properties of the spatial distribution of current in the weak field region ($\omega_c \lesssim \omega_0$). At low temperature ($T \lesssim \hbar\omega_0$, *i.e.* in "MF"), the magnetic moment shows a large fluctuation as a function of the field. The spatial distribution of current in such a situation is shown in Fig. 4(a), which indicates that $J_\theta(r)$ in the bulk region (mainly, $r \lesssim R$) can be either positive or negative, *i.e.* paramagnetic or diamagnetic, respectively. This large bulk currents are sensitive to the strength of the field and lead

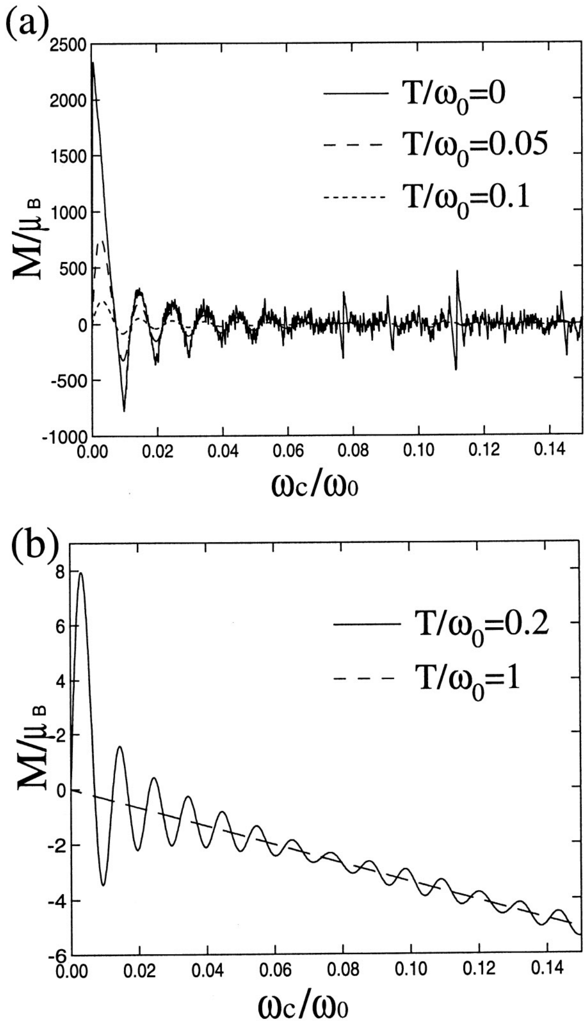

Fig. 2. Magnetic field dependences of magnetic moment at various temperatures; (a) corresponds to "MF" and (b) ranges from "MF" to "LD". In (b) the broken straight line corresponds to the Landau diamagnetism. (After Y. Ishikawa *et al.*: *J. Phys. Soc. Jpn.* **68** (1999) 2405.)

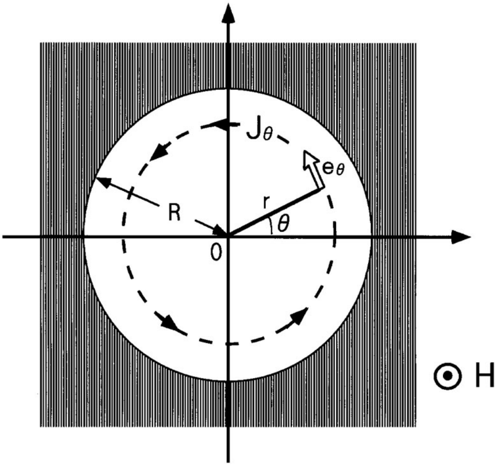

Fig. 3. A schematic illustration of the current flowing in the system. $J_\theta(r)$ is defined positive in the direction as here. (After Y. Ishikawa *et al.*: *J. Phys. Soc. Jpn.* **68** (1999) 2405.)

to the large fluctuation of magnetic moment. This behavior of $J_\theta(r)$ is a characteristic feature of local currents in the region of "MF".

As the temperature is raised ($T \gtrsim \hbar\omega_0$) under such a weak field, however, the fluctuation of magnetic moment disappears and the magnetic moment shows the Landau diamagnetism as discussed in the previous section. According with this change of magnetic moment, the spatial distribution of current $J_\theta(r)$ is changed as shown in Fig. 4(b); the fluctuating large bulk currents are immediately reduced and finally only the diamagnetic current flowing along the edge ($r \simeq R$) survives.

2.3. Conditions for the Validity of the Landau Diamagnetism

Based on the Wigner representation, Kubo[16] derived the analytic form of a current distribution proportional to a magnetic field and leading to the Landau diamagnetism in the system under a confining potential $V(r)$, which is assumed to be slowly varying in space compared to the electron wave length. For the present model $T/\hbar\omega_0 \gtrsim (R/\xi)^{2/3}$ and $T/\hbar\omega_0 \gtrsim \omega_c/\omega_0$ are required for the validity of the expansion in the Wigner representation, which are actually satisfied in the region "LD" in Fig. 1

At low temperature as $T \lesssim \hbar\omega_0$, on the other hand, the contributions of higher order terms in magnetic field become larger and the above formula of

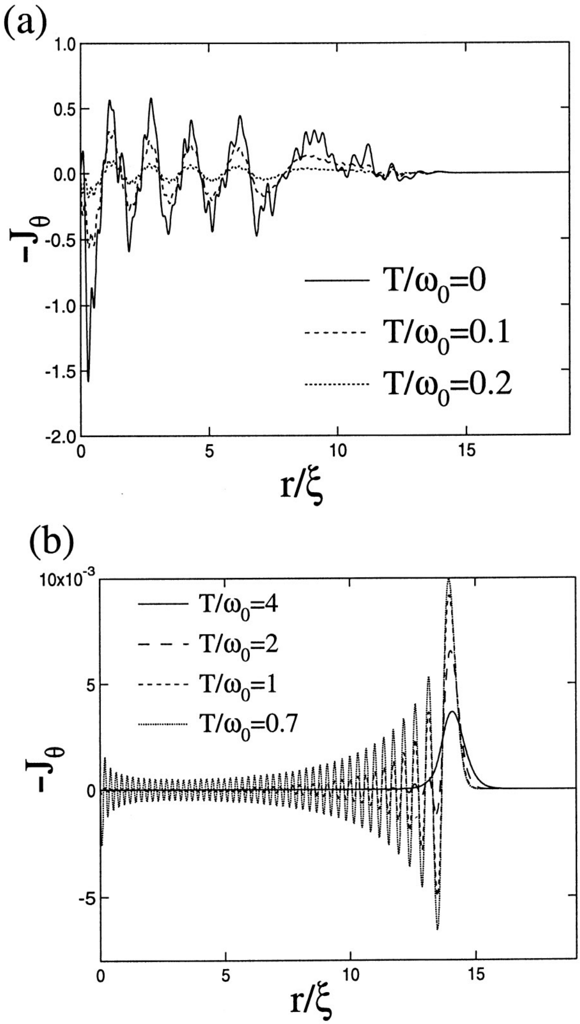

Fig. 4. Spatial distribution of current at various temperatures under a weak field ($\omega_c/\omega_0 = 0.1$), where (a) corresponds to "MF" and (b) ranges from "MF" to "LD". (After Y. Ishikawa *et al.*: *J. Phys. Soc. Jpn.* **68** (1999) 2405.)

current density is not valid. Then, the current distribution changes dramatically at around the temperature $T = \hbar\omega_0$ under a weak field ($\omega_c \lesssim \omega_0$) as is shown in Figs. 4(a) and 4(b) and this change is clearly reflected in a magnetic moment of the system. Hajdu and Shapiro[19] studied a two-dimensional system under a confining potential $V(\boldsymbol{r}) = m\omega_0^2 x^2/2$ (*i.e.* harmonic groove) with a width L_x and an arbitrary long length L_y by imposing a periodic boundary condition in the y-direction, and pointed out that the temperature defined by $T = \hbar\omega_0$, below which magnetic moment shows a large fluctuation under a weak field ($\omega_c \lesssim \omega_0$), corresponds to $\hbar/\tau_{tr}$ where $\tau_{tr} = L_x/v_F$, time of propagation for electrons at the Fermi energy across the groove. In the present model, the condition $l_c \gtrsim L(\equiv 2\pi R)$ corresponds to $T \lesssim \hbar\omega_0$ i.e. "MF". As shown in Fig. 4 (a), it is seen that this multiple reflection induces large currents irregularly distributed paramagnetically or diamagnetically in the bulk region. On the other hand, under the condition $l_c \lesssim L$ ($T \gtrsim \hbar\omega_0$) the effect of the multiple reflection by the boundary potential wall is reduced, and this is considered to lead to the suppression of the bulk currents and the recovery of the Landau diamagnetism.

2.4. Extension to the Three-Dimensional Case

A similar studies on the three-dimensional cases with harmonic confining potential have recently carried out by Suzuki *et al.*[22] who reached similar conclusions as in Fig.1 with slight numerical differences reflecting those of the eigenvalues and their degeneracy.

3. ORBITAL MAGNETISM IN A TWO-DIMENSIONAL RINGS

It is interesting to note[23] that the following model, which mimics the shape of the ring with both inner and outer boundaries, is also exactly solvable.

$$H = \frac{(\boldsymbol{p} + \frac{e}{c}\boldsymbol{A})^2}{2m} + V(r) \tag{10}$$

$$V(r) = \frac{1}{2}m\omega_0^2 r^2 + \frac{1}{2}B\frac{1}{r^2} \quad (B > 0) \tag{11}$$

The particular form of this confining potential, $V(r)$, makes this model exactly solvable. The eigenfunction and eigenvalues are

$$\begin{aligned} \psi_{n\alpha}(\boldsymbol{r}) &= \frac{\mathrm{e}^{\mathrm{i}\alpha\theta}}{\sqrt{2\pi}} R_{n\alpha}(r) \\ R_{n\alpha}(r) &= \frac{1}{l}\sqrt{\frac{n\,!}{\Gamma(\,n+|\gamma|\,+1)}}\ \exp\left[-\frac{r^2}{4l^2}\right]\left(\frac{r}{\sqrt{2}l}\right)^{|\gamma|} L_n^{(|\gamma|)}\left[\frac{r^2}{2l^2}\right] \end{aligned} \tag{12}$$

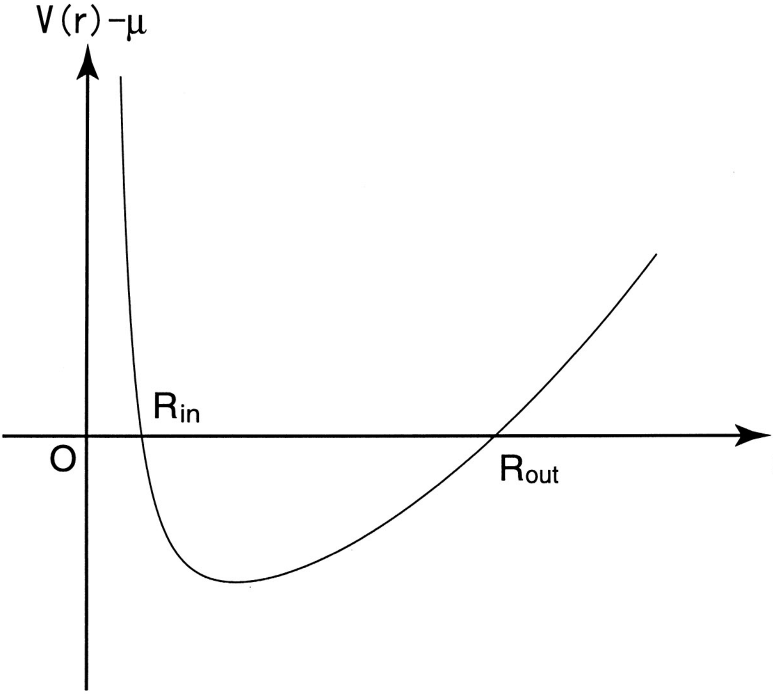

Fig. 5. The spatial dependence of the potential, $V(r)$, for a model of ring with the inner and outer boundaries R_{in} and R_{out}.

$$E_{n\alpha} = \frac{\hbar\omega_c}{2}\alpha + \frac{\hbar\omega}{2}\left(\gamma + 2n + 1\right), \tag{13}$$

where $\gamma = \sqrt{(\alpha^2 + \frac{mB}{\hbar^2})}$. For a system with a fixed average number of electron, these eigenstates are occupied up to the chemical potential as shown in Fig. 5 where the energy is taken relative to the chemical potential and then the inner and outer boundaries, R_{in} and R_{out}, are naturally defined. Following the same procedures as in Sec. 2 the total magnetization, M, and the spatial pattern of the orbital current, $J(r)$, are calculated similarly to eqs. (6) and (8). In Fig. 6 the magnetic field dependence of the magnetization are shown for several choices of temperature, T, while the spatial distribution of orbital currents, $J(r)$, are shown in Fig. 7 for choices of magnetic field and temperature. The results of Figs. 6 and 7 can be understood easily based on the findings of Sec. 2. For a fixed value of relatively weak magnetic field, there are three different regions of temperature, T, characterized by $T_{in} \equiv \frac{\hbar v_F}{L_{in}}$ and $T_{out} \equiv \frac{\hbar v_F}{L_{out}}$, where $L_{in}$and L_{out} are the length of the inner and

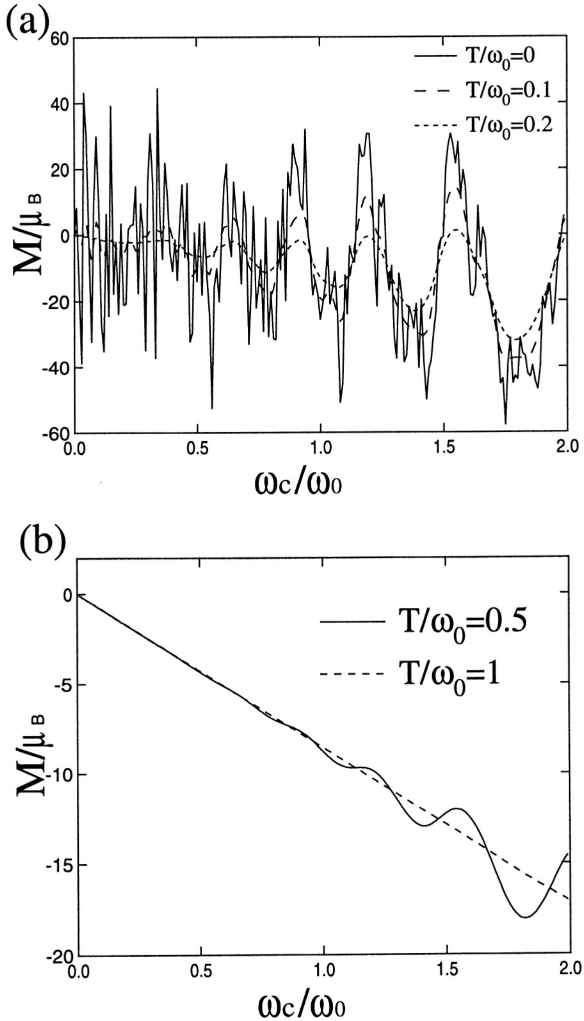

Fig. 6. Magnetic field dependences of magnetic moment at various temperatures in a ring. The dependences under a weak field ($\omega_c \lesssim \omega_0$) are shown in (a) and (b), where (a) corresponds to "MF" and (b) ranges from "MF" to "LD".

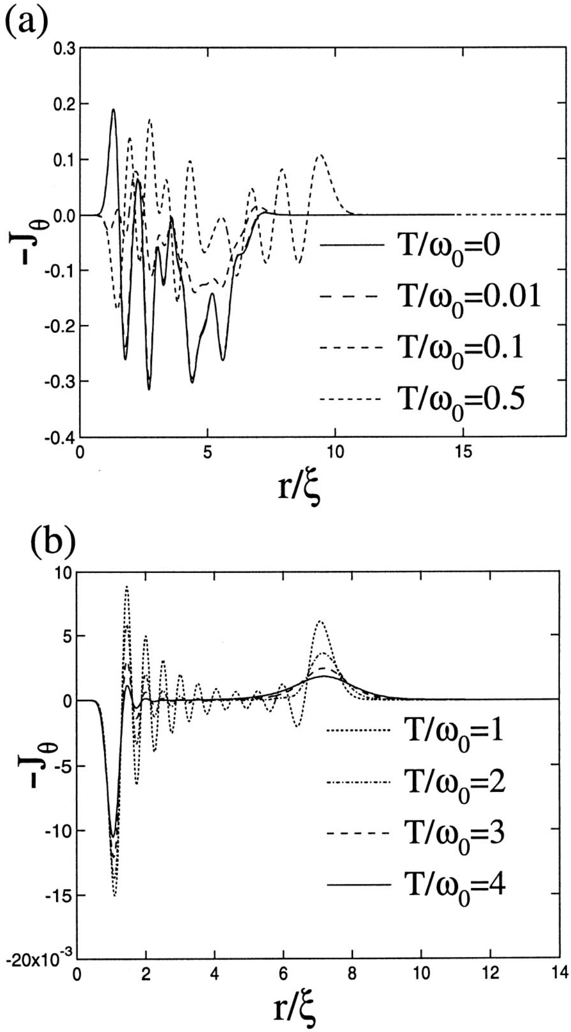

Fig. 7. Spatial distribution of current at various temperatures in a ring under a weak field ($\omega_c/\omega_0 = 0.1$), where (a) corresponds to "MF" and (b) ranges from "MF" to "LD".

outer circumference. For $T < T_{out}(T_{out}/\omega_0 \sim 0.5)$, all the orbits are coherent through the whole system and then mesoscopic fluctuations overwhelm. For $T_{out} < T < T_{in}(T_{in}/\omega_0 \sim 3.6)$, the currents flowing along the inner boundary can maintain the coherence, though those along the outer one can not. For $T_{in} < T$, only incoherent current are present along the both boundaries leading to magnetization given by the Landau diamagnetism corresponding to the area of a ring defined by R_{in} and R_{out}. In closer look at the field dependence of the magnetization, there exist distinct features in this case of a ring different from those in a disc, which needs further analysis.

4. CONCLUSION AND DISCUSSION

In this paper the microscopic understanding of the orbital magnetism, which is due to the persistent current induced by the external magnetic field in the presence of the boundary potential, has further been developed by studying the spatial distribution of the current and the magnetic field dependence of the resultant total magnetization. By extending our former studies on the confined two-dimensional electron gas of a shape of a disc to the exactly solvable model of a ring. It has been indicated that the Landau diamagnetism is realized only above some characteristic temperature where the coherency of the current flow along the inner and outer perimeter is lost. A similar problem of the induced orbital magnetism by the superconducting proximity effect is of a great interest.

ACKNOWLEDGMENTS

One of the authors, H. Fukuyama, is happy to thank Professor Peter Wölfle for many illuminating discussions and suggestions on various subjects in the condensed matter physics for the past more than two decades.

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