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TECHNICAL NOTE

On Lexicographic Vector Equilibrium Problems[1]

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Abstract. We consider vector equilibrium problems using the lexicographic order. We show that several classes of inverse lexicographic optimization problems can be reduced to lexicographic vector equilibrium problems. Some approaches to solve such problems are also suggested.

Key Words. Vector equilibrium problems, lexicographic order, inverse optimization problems.

1. Introduction

The scalar equilibrium problem has numerous applications in mathematical physics and game theory and includes optimization, fixed-point problems, and variational inequality problems. Inspired by the notion of vector variational inequality and vector optimization, equilibrium problems have been extended to vector-valued functions and have been studied by many researchers; see e.g. Konnov (Ref. 1), Hadjisavvas and Schaible (Ref. 2), Giannessi (Ref. 3), and references therein. Each vector problem is clearly based on the corresponding preference relation in an estimation space, which substitutes the real line in the scalar case. Most works on vector variational inequality and vector equilibrium problems are based on orders induced by convex closed cones; i.e., they use various extensions of the Pareto order. However, it is known from the theory of vector optimization that the set of Pareto-optimal points is usually too large, so that one needs certain additional rules to reduce it. One of the possible approaches is to

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utilize the lexicographic order, which was investigated in connection with its applications in optimization and decision making theory; see e.g. Fishburn (Ref. 4), Podinovskii and Gavrilov (Ref. 5), and Martinez-Legaz (Ref. 6).

This work is devoted to lexicographic vector equilibrium problems and their relationships with some other vector problems. Namely, we establish such relationships with several classes of inverse lexicographic optimization problems having rather broad areas of applications. Also, we suggest possible approaches to solve such problems.

We now fix our notation. Set $M=\{1,\dots,m\}$. For an arbitrary pair of points $a,b\in R^m$, the notation $a>b\,[a<b]$ means that a is better [worse, respectively] than b in the Pareto sense; i.e., we have $a_i\geq b_i[a_i\leq b_i]$ for all $i\in M$ and $a\neq b$. Hence, the notation $a>/b\ [a\nless b]$ means that a is not better [not worse, respectively] than b in the Pareto sense; i.e., we have either $a=b$ or $a_i<b_i[a_i>b_i]$ for some $i\in M$. Next, the notation $a\leq_L b\,[a\geq_L b]$ means that a is not lexicographically better (worse) than b; i.e., we have either $a=b$ or $a_k<b_k[a_k>b_k]$ for $k=\min\{i\in M\,|a_i\neq b_i\}$. For any element $a\in R^m$ and for any index $k\in M$, we denote by $[a]^k$ the vector from R^k such that $[a]^k_i=a_i$ for $i=1,\dots,k$. Given a set $X\subseteq R^n$ and a map $f\colon X\to R^m$, we denote by

$$\min\to_L \{f(x)|x\in X\}$$

the lexicographic optimization problem of finding a point $x^*\in X$ such that

$$f(x^*)\leq_L f(x),\qquad \text{for all } x\in X.$$

2. Inverse Lexicographic Optimization Problems

Let $F\colon R^n\to R^{m\times s}$ and $b\colon R^n\to R^l$ be continuous mappings; let $S\colon R^s\to R^m$ and $H\colon R^s\to R^l$ be continuous mappings with convex components $S_i\colon R^s\to R$ and $H_i\colon R^s\to R$, respectively. Fix $x\in R^n$, where $n=s+l$, and consider the following lexicographic vector optimization problem:

$$\text{(LVOP)}\quad \min\to_L \{F(x)u+S(u)|H(u)\leq b(x),\ u\in R^s_+\}. \tag{1}$$

We denote by $U(x)$ the solution set of this problem. Since $n=s+l$, for each $x\in R^n$ we have $x=(u,v)$, where $u\in R^s$ and $v\in R^l$. Now, we can define the inverse lexicographic vector optimization problem (ILVOP for short), which is to find $x^*=(u^*,v^*)\in R^n$ such that u^* is a solution to the LVOP (1) associated to x^*, i.e., $u^*\in U(x^*)$. We denote by X^* the solution set of this problem.

Given the sets $U\subseteq R^s$, $V\subseteq R^l$, and a bifunction N, we say that a pair $x^*=(u^*,v^*)\in U\times V$ constitutes a lexicographic saddle point of N if

$$N(u^*, v) \leq_L N(u^*, v^*) \leq_L N(u, v^*), \qquad \forall u \in U, \forall v \in V. \tag{2}$$

In Ref. 7, an approach to convert inverse Pareto vector optimization problems into vector equilibrium problems was suggested. Now, we intend to apply this approach to lexicographic problems.

First, we introduce the vector Lagrange function of the LVOP (1):

$$\tilde{N}(u, v, x) = F(x)u + S(u) - \langle v, b(x) - H(u) \rangle e,$$

where $e = (1, \ldots, 1)^T \in R^m$.

Proposition 2.1. If $x^* = (u^*, v^*)$ solves problem (2) with $U = R^s_+$, $V = R^l_+$, and if $N(u, v) = \tilde{N}(u, v, x^*)$, then $x^* \in X^*$.

Proof. If $x^* = (u^*, v^*)$ is a lexicographic saddle point of $N(u, v) = \tilde{N}(u, v, x^*)$, and if $U = R^s_+$, $V = R^l_+$, then the left inequality in (2) yields

$$\langle v - v^*, b(x^*) - H(u^*) \rangle \geq 0, \qquad \forall v \in R^l_+.$$

It follows that $b(x^*) \geq H(u^*)$; i.e., u^* is a feasible point. Next, we must have also

$$v_i^*(b_i(x^*) - H_i(u^*)) = 0, \qquad \forall i = 1, \ldots, l.$$

Take an arbitrary point $u \in R^s_+$ such that $H(u) \leq b(x^*)$. Then, using the right inequality in (2), we have

$$F(x^*)u + S(u) \geq_L \tilde{N}(u, v^*, x^*) \geq_L \tilde{N}(u^*, v^*, x^*) = F(x^*)u^* + S(u^*);$$

i.e., $u^* \in U(x^*)$. It means that $x^* \in X^*$. □

Let us now consider the lexicographic vector equilibrium problem (LVEP for short), which is to find a point $x^* \in K$ such that

$$\Phi(x^*, y) \geq_L 0, \qquad \forall y \in K, \tag{3}$$

where

$$\Phi(x, y) = \tilde{N}(u', v, x) - \tilde{N}(u, v', x), \tag{4a}$$

$$K = R^n_+ = R^s_+ \times R^l_+, \tag{4b}$$

$$x = (u, v), \qquad y = (u', v'). \tag{4c}$$

Note that

$$\Phi(x, x) = 0, \qquad \text{for every } x \in K.$$

Theorem 2.1. If x^* solves the LVEP (3)–(4), then $x^* \in X^*$.

Proof. By definition, if x^* solves the LVEP (3)–(4), then

$$\tilde{N}(u^*, v', x^*) \leq_L \tilde{N}(u', v^*, x^*), \qquad \forall u' \in R^s_+, \forall v' \in R^l_+.$$

Hence, $x^* = (u^*, v^*)$ is a lexicographic saddle point of the function

$$N(u, v) = \tilde{N}(u, v, x^*).$$

Now, the desired result follows from Proposition 2.1. □

So, we can replace any ILVOP, which is based on the LVOP (1), with the LVEP (3)–(4). Note that the LVEP (3) reduces to the scalar equilibrium problem in the case $m = 1$. In the general case where $m > 1$, we can find a solution to the LVEP (3) via its reduction to a system of vector equilibrium problems using the Pareto order. This approach is based on the fact, which was noticed and proved in Ref. 1, Lemma 5.1, that the lexicographic order is equivalent to a system of reverse Pareto orders. To make the paper more self-contained, we give now the proof of this assertion.

Proposition 2.2. Let a and b be arbitrary elements of R^m. Then,

$$a \geq_L b \Leftrightarrow [a]^s \not< [b]^s, \qquad \text{for all } s = 1, \ldots, m.$$

Proof. The case $a = b$ is trivial. Next, suppose that $a \geq_L b$ and $a \neq b$. Then, there exists an index $j \leq m$ such that $a_j > b_j$ and $a_i = b_i$ for all $i < j$. Hence, $[a]^i = [b]^i$ for $i < j$ and $[a]^i \not< [b]^i$ for all $i \geq j$. Conversely, let $a \leq_L b$ and $a \neq b$. Then, by analogy, $a_j < b_j$ for some $j \leq m$ and $a_i = b_i$ for $i < j$; hence, $[a]^j \neq [b]^j$ and $[a]^j_i \leq [b]^j_i$ for all $i \leq j$; i.e., $[a]^j < [b]^j$. It means that the system

$$[a]^s \not< [b]^s, \qquad \text{for all } s = 1, \ldots, m,$$

implies $a \geq_L b$. The proof is complete. □

The assertion of this proposition differs essentially from similar reduction results in Refs. 5 and 6, since it does not involve any parameters depending on a and b. Note also that it is based on a reverse Pareto ordering, rather than the parametric system of scalar ones. Although the results concerning the representation of the Pareto order with the help of the lexicographic order are more usual (see e.g. Ref. 8), the assertion above enables us to replace the LVEP with the VEP, as the following corollary states.

Corollary 2.1. The LVEP (3) is equivalent to the problem of finding a point $x^* \in K$ such that

$$[\Phi(x^*, y)]^s \not< [0]^s, \qquad \text{for all } s = 1, \ldots, m \text{ and for all } y \in K. \tag{5}$$

Note that each separate problem in the system (5) corresponds to the usual vector equilibrium problem (VEP); see e.g. Yang and Goh (Ref. 9). Additionally, an iterative method to find a solution of such systems was suggested in Ref. 1.

Combining Theorem 2.1 and Corollary 2.1, we obtain the following substitution result.

Theorem 2.2. If x^* solves the system (5), where Φ and K are given in (4), then $x^* \in X^*$.

3. Some Specializations

In this section, we give additional examples of problems which are particular cases of the ILVOP from Section 2.

Example 3.1. Let us consider problem (1) where $S \equiv 0$, b is a constant mapping, and F depends on u only; i.e., $F: R^s \to R^{m \times s}$. Then, the corresponding ILVOP can be formulated as follows: Find $u^* \in D$ such that

$$F(u^*)u \geq_L F(u^*)u^*, \qquad \forall u \in D, \tag{6}$$

where

$$D = \{u \in R^s_+ | H(u) \leq b\}.$$

For instance, this problem can be interpreted as the vector cost minimization problem under the resource constraints in the case where the scalar costs of each unit are dependent of the volume of supply.

If $m = 1$, problem (6) is nothing but the usual variational inequality. In the general case, (6) is the lexicographic vector variational inequality (LVVI), which is clearly a particular case of the LVEP (3). In fact, if we set $K = D$ and

$$\Phi(u, v) = F(u)(v - u),$$

then the LVEP (3) reduces to the LVVI (6). Using the results of Corollary 2.1, we can replace also problem (6) with a system of vector variational inequalities of the form (5).

Example 3.2. Let us now consider the problem (1) in the case where $F \equiv 0$, H is affine, and b is a constant mapping. Then, (1) can be rewritten as the usual LVOP of the form

$$\min \to_L \{S(u) | Hu \leq b, u \in R^s_+\},$$

where H is an $l \times s$ matrix. It is well known that the solution set of this problem can be determined equivalently as the set D_m, where

$$D_k = \{w \in D_{k-1} | S_k(w) \leq S_k(u), \forall u \in D_{k-1}\}, \qquad \text{for } k = 1, \ldots, m,$$

with $D_0 = D$. Next, following the Karush–Kuhn–Tucker theorem, we can determine also the set D_1 with the help of the corresponding scalar saddle-point problem; i.e., $\tilde{u} \in D_1$ if and only if there exists $\tilde{x} = (\tilde{u}, \tilde{v}) \in R_+^s \times R_+^l$ such as

$$\begin{aligned} &S_1(\tilde{u}) + \langle v, H\tilde{u} - b\rangle \\ &\leq S_1(u) + \langle \tilde{v}, H\tilde{u} - b\rangle \\ &\leq S_1(u) + \langle \tilde{v}, Hu - b\rangle, \qquad \forall u \in R_+^s, \forall v \in R_+^l. \end{aligned} \tag{7}$$

This approach to the characterization of the solution set D_m of the constrained lexicographic optimization problem can be regarded as an alternative to the usual lexicographic vector Lagrangian approach or to the parametric Lagrangian approach from Ref. 8.

Note that the initial LVOP can be interpreted as the vector cost minimization problem under the resource constraints. Now, suppose that the inventory of resources is dependent of their shadow prices with respect to the first cost function S_1; i.e., let $b = b(\tilde{v})$, where $\tilde{v}$ is defined from the saddle-point problem: Find $\tilde{u} \in R_+^s$, $\tilde{v} \in R_+^l$ such that

$$\begin{aligned} &S_1(\bar{u}) + \langle v, H\tilde{u} - b(\bar{v})\rangle \\ &\leq S_1(\tilde{u}) + \langle \tilde{v}, H\bar{u} - b(\tilde{v})\rangle \\ &\leq S_1(u) + \langle \tilde{v}, Hu - b(\tilde{v})\rangle, \qquad \forall u \in R_+^s, \forall v \in R_+^l. \end{aligned} \tag{8}$$

In other words, we consider ILVOP of the form

$$\min \to_L \{S(u) | Hu \leq b(\tilde{v}), u \in R_+^s\}. \tag{9}$$

Clearly, this problem is equivalent to the problem of finding an element of the set $\tilde{D}_m$, where

$$\tilde{D}_k = \{w \in \tilde{D}_{k-1} | S_k(w) \leq S_k(u), \forall u \in \tilde{D}_{k-1}\}, \qquad \text{for } k = 2, \ldots, m;$$

$\tilde{D}_1$ is the solution set of the inverse scalar optimization problem

$$\min \to \{S_1(u) | Hu \leq b(\tilde{v}), u \in R_+^s\},$$

whose solutions are given in (8).

Note that a dual approach to finding points $\tilde{u}$ and $\tilde{v}$ satisfying (8) under rather general assumptions was proposed recently in Ref. 10. Thus, the solution of the ILVOP (9) can be computed with the help of iterative solution methods of convex minimization. It was also noticed in Ref. 10 that the solution of the basic problem (8) can be simplified essentially in the case where S_1 is affine. In fact, we then can find a solution $\tilde{v}$ of the dual problem

$$\min \to \{\langle b(\tilde{v}), v\rangle | S_1 + H^T v \geq 0, v \geq 0\},$$

which is nothing but the scalar variational inequality problem with affine constraints. Afterward, we can fix the vector $b = b(\tilde{v})$ and solve the following primal linear programming problem:

$$\min \to \{\langle S_1, u\rangle | Hu \leq b, u \in R_+^s\}.$$

The dimensionality of this problem is reduced essentially if we use the complementarity conditions. So, this basic problem can be solved in a finite number of iterations.

Both problems are very natural modifications and specializations of the LVEP. Therefore, together with other kinds of VEPs, the theory and solution methods of the LVEPs deserve detailed investigations.

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