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A NEW WAY OF SOLVING THE WHEELER–DEWITT EQUATION AS APPLIED TO POINT SOURCES

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A new way of solving the Wheeler–DeWitt equation is proposed which is based on quantization over free parameters of metrics satisfying the Einstein equations. This technique is applied to two point sources described in the classical case by the Tangherlini metric (in an n-dimensional space) and the Reissner–Nordström metric (in the case of the presence of a charge). The results obtained clarify the sense of the Wheeler hypothesis about statistical weights of small dimensionalities and make possible a new approach to the problem of variation of fundamental constants.

INTRODUCTION

Over a period of three decades, the Wheeler–DeWitt equation [1] remains one of the most reliable tools in quantum gravitation. This equation is generally solved in a minisuperspace, making it possible to reduce it to a Schrödinger-type equation (in the one-dimensional case) or a Klein–Gordon equation (in the two-dimensional case), for instance, for the case of Friedmann cosmological models without or with a scalar field, respectively [2]. Kuchař [3] has proposed to use for the vacuum Schwarzschild metric the Arnowitt–Deser–Misner formalism, which allows one to quantize the mass of a black hole. The quasi-classical approximation is also widely used in quantum gravitation. Some researchers even believe that quantum cosmology is valid only in the quasi-classical approximation, where, the notion of time can be introduced [4].

In this paper, we propose a method making possible to reduce the Wheeler–DeWitt equation in variational derivatives to an equation in a minisuperspace with respect to parameters entering in a corresponding classical metric. This method is applied to an n-dimensional noncharged point source (Tangherlini metric) and a 3-dimensional charged point source (Reissner–Nordström metric). In the first case, the space dimensionality is the parameter to be quantized. In the second case, the gravitation and classical radii of the source are quantized, allowing an approach to the problem of variation of fundamental constants.

1. THE METHOD

Let us consider the Wheeler–DeWitt equation in the form [1]

$$\{G_{ijkl}\frac{\delta^2}{\delta\gamma_{ij}\delta\gamma_{kl}} + {}^3R\}\psi = 0, \tag{1}$$

with the supermetric

$$G_{ijkl} = \frac{1}{2\gamma}(\gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk} - \gamma_{ij}\gamma_{kl}), \tag{2}$$

and ${}^3R$ being the scalar curvature of the 3-geometry, to formulate a quasi-classical procedure of going from Eq. (1) to an equation in a minisuperspace [5]. Let us find a solution to Eq. (1) in the class of metrics satisfying the Einstein equations

$$R_{ik} - \frac{1}{2}Rg_{ik} = \frac{8\pi G}{c^4}T_{ik} + \Lambda g_{ik}, \tag{3}$$

since, according to the constructive interference principle, a wave packet in a superspace propagates along the classical trajectory [6]. The space metric γ_{ik} is related to g_{ik} by the formula

$$\gamma_{ik} = -g_{ik} + \frac{g_{0i}g_{0k}}{g_{00}} \tag{4}$$

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and simultaneously depends on a free parameter λ, which determines the degree of freedom to be quantized. Using the formula

$$\frac{\delta}{\delta\gamma_{ik}} = \frac{\frac{\partial}{l\lambda}}{\frac{\partial\gamma_{ik}}{\partial\lambda}}, \tag{5}$$

we reduce Eq. (1) to the Wheeler–DeWitt equation in a minisuperspace with respect to the parameter λ:

$$\psi'' + f(\lambda)\psi = 0, \tag{6}$$

where $' \equiv \dfrac{\partial}{\partial\lambda}$ and

$$f(\lambda) = {}^3R\sum_{ijkl}\frac{(\partial\gamma_{ij}/\partial\lambda)(\partial\gamma_{kl}/\partial\lambda)}{G_{ijkl}} \tag{7}$$

(we use a symmetric arrangement of the operators).

The quasi-classical limit for a wave function is given by [7]

$$\psi = Ae^{i\frac{S}{\hbar}}, \tag{8}$$

where

$$S = \hbar\int\sqrt{f}\,d\lambda \tag{9}$$

is the action satisfying the Hamilton–Jacobi equation

$$\frac{1}{\hbar^2}G_{ijkl}\frac{\delta S}{\delta\gamma_{ij}}\frac{\delta S}{\delta\gamma_{kl}} - {}^3R = 0. \tag{10}$$

This approach, using the derivative of γ_{ik} with respect to a parameter, resembles the Wheeler method, where the 4- geometry is completely determined by γ_{ik} and $\dfrac{\partial\gamma_{ik}}{\partial\lambda}$ at a given point and by the Einstein equations [8].

Below, we generalize Eq. (6) to the case of a discrete parameter and the case of two parameters.

2. A POINT SOURCE IN AN *N*-DIMENSIONAL SPACE

Wheeler et al. has supposed [9] that there exists some probability for points to be collected in a space of a given dimensionality. The goal of the present work is to derive the distribution of the probabilities that a space will have a given dimensionality from the Wheeler–DeWitt equation of a special form, i.e., to find a corresponding wave function and a dimensionality at which the squared absolute value of the wave function has a maximum. It appears that this can be done in the neighborhood of a singularity.

Generalizing Eq. (1) to the case of an n-dimensional space, we obtain [10]

$$\{G_{ijkl}\frac{\delta^2}{\delta\gamma_{ik}\delta\gamma_{kl}} + {}^{(n)}R\}\psi = 0, \tag{11}$$

where ${}^{(n)}R$ is the scalar curvature of the n-geometry.

Formulas (5)–(7) remain valid with $\lambda = n$ (with n being the space dimensionality). The Tangherlini metric for a point source in an n-dimensional space generalizing the Schwartszchild metric for a point source in a three-dimensional space is given by [11]

$$dl^2 = \frac{dr^2}{1 - \frac{r_g}{r^{n-2}}} + r^2 d\Omega^2, \tag{12}$$

where we put $r_g = 1$, assuming that r is measured in the units of r_g, and

$$d\Omega^2 = dx_2{}^2 + \sin^2 x_2 dx_3{}^3 + \ldots + \prod_{l=2}^{n-1}\sin^2 x_l dx_n{}^2. \tag{13}$$

Thus we have

$$^{(n)}R = \frac{n(n-3)}{2r^n}. \tag{14}$$

Now we generalize Eq. (6) for integer n to the case of a differential-difference equation with $\frac{\partial^2\psi}{\partial n^2}$ replaced by $\psi_{n+2} - 2\psi_{n+1} + \psi_n$ and $\frac{\partial\gamma_{11}}{\partial n}$ replaced by $\gamma_{11}(n+1) - \gamma_{11}(n)$. Let $f = {}^{(n)}RF$; then $F = \frac{[\gamma_{11}(n+1)-\gamma_{11}(n)]^2}{G_{1111}}$, where $G_{1111} = \frac{\gamma_{11}}{2r^2}$ and $\gamma_{11} = \frac{1}{1-\frac{1}{r^{n-2}}}$. Thus we get

$$F = 2r^2 \frac{(\frac{1}{r^{n-1}} - \frac{1}{r^{n-2}})^2}{(1-\frac{1}{r^{n-1}})^2(1-\frac{1}{r^{n-2}})}.$$

For $\frac{1}{r^{n-2}} \gg 1$ ($n > 2$, $r \to 0$, $n < 2$, $r \to \infty$) we have $F = -2r^n$ and $f = -n(n-3)$. Returning to the continuous variable x equal to n for integer values, we arrive at the equation

$$\psi'' - x(x-3)\psi = 0. \tag{15}$$

Let us rewrite this equation as

$$\psi'' - (x-\frac{3}{2})^2\psi + \frac{9}{4}\psi = 0, \tag{16}$$

which is the quantum oscillator equation [7]

$$\psi'' + \frac{2m}{\hbar^2}(E - \frac{m\omega^2 x^2}{2}\psi) = 0 \tag{17}$$

with $E = (N+\frac{1}{2})\hbar\omega$ shifted by $\frac{3}{2}$.

Identifying (16) and (17), we obtain $\frac{m^2\omega^2}{\hbar^2} = 1$. Thus, Eq. (16) is close to the oscillator equation with $N = 1$. To restore the oscillator for any N, we should replace $\frac{9}{4}$ in Eq. (16) by $2(N+\frac{1}{2})$ and then we get

$$f = -(x - \frac{3}{2} + \sqrt{2}\sqrt{N+\frac{1}{2}})(x - \frac{3}{2} - \sqrt{2}\sqrt{N+\frac{1}{2}}),$$

$f = 0$ for $x \approx 0,23$ and $x \approx 3,23$ for $N = 1$ (cf. with $x = 0$ and $x = 3$ in Eq. (15)).

The restored oscillator has the following wave functions [12]:

$$\psi_0 = \frac{1}{\pi^{\frac{1}{4}}} e^{-\frac{(x-\frac{3}{2})^2}{2}}, \tag{18}$$

$$\psi_1 = \frac{\sqrt{2}}{\pi^{\frac{1}{4}}}(x-\frac{3}{2})e^{-\frac{(x-\frac{3}{2})^2}{2}}. \tag{19}$$

$|\psi_0|^2$ has a maximum at $x = \frac{3}{2}$ and $|\psi_1|^2$ has maxima at $x = 2.5$ and $x = 0.5$. For $r \to 0$ and $n > 2$, a single maximum remains, which corresponds to the most probable dimensionality in the neighborhood of the singularity, i.e., at $x = 2.5$. This result means that the Wheeler hypothesis that the lowest dimensionalities have the highest statistical weights [9] is valid in the neighborhood of the singularity of the Tangherlini metrics.

3. A CHARGED POINT SOURCE IN A 3-DIMENSIONAL SPACE

The Reissner–Nordström metric for a charged point source in a 3-dimensional space is given by [13, 14]

$$dl^2 = \frac{dr^2}{1 - \frac{r_g}{r} + \frac{r_g r_q}{2r^2}} + r^2 d\Omega^2, \tag{20}$$

where $r_g = \frac{2GM}{c^2}$, $r_q = \frac{q^2}{Mc^2}$, and M and q are the mass and the charge of the field source, respectively.

Since Eq. (20) contains two free parameters, formula (5) is generalized to

$$\frac{\delta}{\delta\gamma_{ik}} = \frac{1}{\frac{\partial\gamma_{ik}}{\partial\alpha}}\frac{\partial}{\partial\alpha} + \frac{1}{\frac{\partial\gamma_{ik}}{\partial\beta}}\frac{\partial}{\partial\beta}, \tag{21}$$

where $\alpha = r_g$ and $\beta = r_q$. From Eq. (20) we obtain

$$^3R = -\frac{r_g r_q}{r^4}. \tag{22}$$

For $r_g \gg r$ and $r_q \gg r$ in the neighborhood of extreme points $(\frac{\partial\psi}{\partial r_g} = \frac{\partial\psi}{\partial r_q} = 0)$, Eq. (6) is reduced to

$$r_g{}^2\frac{\partial^2\psi}{\partial r_g{}^2} + r_q{}^2\frac{\partial^2\psi}{\partial r_q{}^2} - \frac{r_g{}^2 r_q{}^2}{r^4}\psi = 0. \tag{23}$$

Introducing a new variable

$$y = \frac{r_g}{r_g{}^{(0)}} = \frac{r_q}{r_q{}^{(0)}},$$

where $r_g{}^{(0)}$ and $r_q{}^{(0)}$ are observables, depending on the fundamental constants, c, $\hbar$, G, e, and m_e, for the problem on the motion of an electron in the Schwarszchild field of a planckeon $(r_g = 2\sqrt{\frac{\hbar G}{c^3}})$, which is reduced to the problem on the motion of a neutral particle whose mass is equal to the electron mass in the Reissner–Nordström field with a charge $q = e\sqrt{\frac{m_{pl}}{m_e}}$ $(r_q = \frac{e^2}{m_e c^2})$ [15], we reduce Eq. (23) to

$$\frac{d^2\psi}{dy^2} - by^2\psi = 0, \tag{24}$$

where $b^2 = \frac{r_g^{(0)2} r_q^{(0)2}}{2r^4} \gg 1$.

This equation has an exact solution [16]:

$$\psi = \sqrt{y} Z_{\frac{1}{4}}(i\sqrt{b}\frac{y^2}{2}). \tag{25}$$

For $y \ll 1$ and $y \gg 1$, ψ tends to zero. This means that the quantity $|\psi|^2$, which determines the probability of y taking one or another value is a maximum at $y \sim 1$, i.e., the fundamental constants are close to their observables. At fixed constants of quantum electrodynamics, such as e, $\hbar$, and c, the variations of Planck's parameters are reduced to the variations of the gravitation constant G, and the variations of the classical electron radius $\frac{e^2}{m_e c^2}$ are reduced to the variations of the electron mass m_e.

CONCLUSIONS

We have considered the Wheeler–DeWitt equation in the neighborhood of a singularity for two different superspaces. It turned out that small dimensionalities are most probable for the case of a point source in an n-dimensional space, and this is in agreement with the Wheeler hypothesis about the highest statistical weights of the lowest dimensionalities (dimensionalities $n > 2$ are meant). In treating the problem on the motion of an electron in the field of a planckeon, the variations of Planck's parameters depending on $G, \hbar$, and c are reduced to the variations of the gravitational constant G, and the variations of the classical electron radius $r_e = \frac{e^2}{m_e c^2}$ are reduced to the variations of the electron mass m_e with the constants of quantum electrodynamics $e, \hbar$, and c being fixed. The proposed procedure, yielding reasonable results in analyzing the parameter of a source in the neighborhood of a singularity, can be applied for solving problems other than those treated in this work. The principal conclusion is that in some extreme cases, the variations and the most probable values of free parameters of classical metrics can be calculated in terms of quantum geometrodynamics.

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