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A closure model for intermittency in three-dimensional incompressible turbulence

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A simplified Lagrangian closure for the Navier–Stokes equation is used to study the production of intermittency in the inertial range of three-dimensional turbulence. This is done using localized wave packets following the fluid rather than a standard Fourier basis. In this formulation, the equation for the energy transfer acquires a noise term coming from the fluctuations in the energy content of the different wave packets. Assuming smallness of the intermittency correction to scaling allows the adoption of a quasi-Gaussian approximation for the velocity field, provided a cutoff on small scales is imposed and a finite region of space is considered. In these approximations, the amplitude of the local energy transfer fluctuations can be calculated self-consistently in the model. Definite predictions on anomalous scaling are obtained in terms of the modified structure functions: $\langle\langle E(l,a)\rangle_R^q\rangle$, where $\langle E(l,a,\mathbf{r},t)\rangle_R$ is the part of the turbulent energy coming from Fourier components in a band $(a-1)k$ around $k\sim l^{-1}$, spatially averaged over a volume of size $R\sim l/(a-1)$ around $\mathbf{r}$. © *1995 American Institute of Physics.*

I. INTRODUCTION

The statistics of large Reynolds numbers three-dimensional (3-D) turbulence is characterized by scaling behaviors of the structure functions: $S(l,q)=\langle v_l^q\rangle$, $\mathbf{v}_l(\mathbf{r},t)=\mathbf{v}(\mathbf{r}+\mathbf{l},t)-\mathbf{v}(\mathbf{r},t)$, which, in first approximation, appear to follow the Kolmogorov relation: $S(l,q)\ =\ \propto l^{\zeta_q^0}$, $\zeta_q^0\simeq q/3$.[1] However, both experiments[2] and numerical simulations[3] show the presence of corrections to Kolmogorov scaling, which become more and more pronounced for higher-order moments. These corrections go in a direction of increasing non-Gaussianity of v_l as $l\to 0$: $S(l,q)\ \propto\ l^{\zeta_q}$, with $\delta\zeta_q=\zeta_q-\zeta_q^0$, satisfying the convexity condition: $d^2\ \delta\zeta_q/dq^2<0$. Of course, the relations above hold only for scales corresponding to the inertial range (and a part of the viscous range[4] if only ratios of moments are considered), so that for finite Reynolds numbers: Re$<\infty$, the generalized kurtosis $K(2q)=[S(l,2q)/S(l,2)^q]_{l\to 0}$ is finite. However, a situation with scaling corrections in the form just described, persisting for Re$\to\infty$, would imply that $K(q>2)\to\infty$ in this limit, corresponding to infinitely intermittent small-scale velocity fluctuations.

The traditional approach to the issue of intermittency dates back to the remarks by Landau[5] on the presence of spatial fluctuations in the energy transfer between scales and its effect on the turbulent dynamics. From the Refined Similarity hypothesis of Kolmogorov,[6] down the line to the Beta Model,[7,8] the main ingredient is the assumption of a turbulent dynamics acting locally in scale, so that the energy dissipation at a given point in space is a product of independent, fluctuating coefficients, describing the transfer of energy between successive, contiguous scales. In this picture, intermittency is produced locally in scale, over the whole of the inertial range, so that the resulting corrections to scaling are Reynolds number independent. This mechanism can be interpreted as a rule for the construction of a multifractal,[9,10] in which the measure is given by $d\mu_l(\mathbf{r},t)=l\langle(\partial_x v_x)^2\rangle_l(\mathbf{r},t)$, where $\langle\ \rangle_l(\mathbf{r},t)$ indicates a spatial average taken at time t in an interval of length l along the x axis, centered around $\mathbf{r}$. In this way, the multifractal dimensions D_q, defined by the relation $\langle d\mu_l^q\rangle\sim l^{(q-1)D_q}$, are given by $(q-1)(1-D_q)=q\zeta_2-\zeta_{2q}$.

Recently, an alternative explanation for the presence of scaling corrections has been proposed, namely that which is observed is a finite size, i.e., a finite Reynolds number effect.[11,12] This point of view is supported both by the smallness of the corrections and the strongly nonlocal character of finite size effects with respect to scale, as shown in numerical simulations of both Navier–Stokes dynamics[2] and reduced wave vector models.[11] In Ref. 12 a dynamical explanation for this effect was presented, based on closure arguments. The basic idea is that the strong intermittency of the viscous range, due to the interplay between dissipation and nonlinearity in the dynamics of small vortices,[13] may be enough to generate, in the inertial range, scaling corrections of the appropriate size for all Reynolds numbers of practical interest. In this picture, the total intermittency, as parametrized by $K(q)$, would remain finite, while the scaling corrections would tend to zero in the infinite Reynolds number limit.

In this situation, it would be interesting to have some quantitative assessment of the magnitude of the finite size effects, relative to the amount of intermittency produced in the inertial range by energy transfer fluctuations. However, while Refs. 11 and 12 can afford some quantitative prediction on the size of the first effect, all models dealing with the second are heavily phenomenological and have no connection with real Navier–Stokes dynamics. An exception is the very recent paper by Yakhot,[14] in which quantitative predictions have been obtained after approximating the turbulent energy dynamics with that of a passive scalar.

The purpose of the present paper is to obtain quantitative predictions on the Re-independent part of anomalous scaling (if present at all), following the route of evaluating the energy transfer fluctuations from statistical closure of the

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Navier–Stokes equation. The basic difficulty in using this technique is the real space nature of the quantities that have to be calculated: the moments $S(l,q)$,[15] while the main component of the turbulent dynamics, the energy transfer toward smaller scales, is better described in a Fourier basis. Although a real space closure of the Navier–Stokes equation was derived in Ref. 16, we found it very difficult to extend the model to study energy fluctuations, especially when trying to separate contributions at different scales.

Here we prefer to follow the idea of Nakano[17] and, in a different context, of Eggers and Grossmann,[18] of using a localized wave packet representation of the Fourier–Weierstrass type rather than a global Fourier basis or a purely real space one. In this way, energy fluctuations on a scale R, instead of being buried in the phase relationship between Fourier modes separated by $\Delta k \sim R^{-1}$, are described by the space dependence of the wave packets. Now, if the wave packets are centered around wave vectors lying in shells of radii $k_n = a^n k_0$, the spatial extension of a wave packet at $k \sim l^{-1}$ will be $R(l) \sim l/(a-1)$. This means that a new object is being used in place of $S(l,q)$ to give a measure of intermittency: the generalized structure function, $S(l,a,2q) = \langle\langle E(l,a)\rangle_R^q\rangle$, where $E(l,a,) = E(l,a;\mathbf{r},t)$ is the total energy of Fourier modes in the shell $k_n \sim l^{-1}$, and $R = R(l,a) = l/(a-1)$. More, in general, we shall consider the situation in which also the quantity $c_w = l/(a-1)R$ is treated as a free parameter, resulting in the definition of still another structure function, $S(l,a,c_w,q)$, interpolating between the intermittency-free limit $S(l,a,0,q)$ and the original case: $S(l,a,c_w^{\mathrm{MAX}},q) \equiv S(l,a,q)$.

Here the shell width $(a-1)k$ plays a crucial role; this can be understood better by looking at how energy is transferred between shells as a rule for the construction of a multifractal. In this picture, a small value for $a-1$ corresponds to multipliers between one scale l and the next l/a being constant on domains $R(l) > l$ (in typical examples like the various "middle third" Cantor sets,[10] one has $R(l) \simeq l$, i.e., the fluctuation scale and that of the geometrical structures is the same). In terms of turbulent dynamics, this has the following interpretation: $R(l)$ is the scale of the eddies that contribute the most to the straining of those of size l; the parameter $l^{-1}R(l) = (a-1)^{-1}$ therefore gives the degree of nonlocality of the nonlinear interaction. In principle, there will be fluctuations in the transfer of energy between eddies of size l also at a scale smaller than $R(l)$, and this effect will result in correlations between phases of different wave packets separated by $\Delta k > R^{-1}$. This effect contributes to the scaling of $S(l,q)$, but not to that of $S(l,a,q)$, and the last one is likely to underestimate the actual value of $\delta\zeta_q$.

Now, a recent analysis performed by Domaradzki and Rogallo[19] on numerical simulations and statistical closures of the Quasi-Normal Markovian type[20] has shown that there is indeed a separation of scales between straining flow and strained eddies, so that fluctuations in the transfer, occurring at scales similar to those of the modes exchanging energy, are not expected to be large, implying $S(l,q) \sim S(l,a,q)$, even for a close to 1. More importantly, it is this separation of scales that allows us to consider meaningful wave packets, extending over several characteristic wavelengths, rather than having to deal with the usual eddy breaking picture, which is very attractive on the grounds of simplicity, but does not allow us to make any contact with the Navier–Stokes dynamics.

There is a second conceptual difficulty in using closures to study multifractal intermittency. This is the contradiction between an ansatz of quasi-Gaussianity and the "infinitely non-Gaussian" character of a multifractal with no cutoffs, as attested by the equation $K(q>2) = \infty$. A quasi-Gaussian hypothesis becomes meaningful, however, when studying limited regions of space and limited ranges of scales, which is possible if the degree of nonlocality of the nonlinear interactions is not too high and the intermittency correction $\delta\zeta_q$ is sufficiently small. In this way, although the distribution of values of, say $\partial_x v_x$, over different averaging volumes V_i is infinitely intermittent (if the total volume $V_{\mathrm{tot}} = \cup V_i$ is infinite itself), the moments of $\partial_x v_x$ in each V_i will be close to Gaussian [if the ultraviolet cutoff scale r is not too small: $(rV_i^{-1/3})^{\delta\zeta_q/q} - 1 \ll 1$]. This is equivalent to study the statistics of $\partial_x v_x$ conditional to ϵ_i, and the quasi-Gaussianity of the conditional $\partial_x v_x$ is an extension of the quasi-Gaussianity observed in Ref. 21, in the case $r \sim V_i^{1/3}$. Of course, in order for the statistical sample to be significant, it is necessary that the range of scales $r < l < V_i^{1/3}$ be large enough to accommodate all relevant interactions: $(a-1)V_i^{1/3}/r > 1$, and that enough wave packets be present in the volume V_i: i.e., $[k/(a-1)]^3 V_i \gg 1$. With a degree of nonlocality $(a-1)^{-1}$ of at most ten (see Ref. 19 and results in the next section), and $\delta\zeta_q/q$ in the range of a few percents,[2] these conditions can be assumed to be satisfied, and all statistical averages, indicated by $\langle\ \rangle$, will be understood here to be carried on in space, over a single large but finite volume V_i.

The closure that is going to be used in the next sections is of the Quasi-Lagrangian type,[22] in which fluid structures at scale l are studied in a reference frame moving with a speed given by the average fluid velocity in a volume of radius l/λ, $\lambda < 1$; the free parameter λ is then adjusted to lead, in a mean field theory, to values of the Kolmogorov constant, in agreement with experiments. This mean field theory is obtained using still a global Fourier basis. Notice that here l/λ is not the wave packet size, and, in this respect, the present approach differs from that of Nakano.[17] Of course, our approximation is completely uncontrolled, in that sweep effects from scales between l and l/λ are still present, while part of the strain from scales larger than l/λ is lost. However, it is still an improvement upon using infrared cutoffs in the expression for the energy transfer, especially in view of the fact that this last operation would not preserve the nonlocal character of the nonlinear interaction and would lead to transfer profiles in disagreement with Ref. 19.

The fluctuating dynamics is obtained studying the energy equation for the wave packets (which, after angular integration, becomes the energy equation for the shells $k_n < k < k_{n+1}$). This is obtained by substituting the statistical averages, which would lead to a mean field closure, with partial spatial averages over wave packets volumes. This leads to an energy equation in the same form (in Fourier space) as the original mean field one, plus a noise term that is essentially $\langle v^{(0)}(v^{(0)}\nabla)v^{(0)}\rangle_R(\mathbf{r},t)$. Here $v^{(0)}$ is the Gaussian random field, which gives the lowest-order approximation for the fluid velocity. The analysis is carried on as an expansion to lowest order in the two parameters $a-1$ and c_w, and the end result is an equation for the energy content of shells at a certain position in space, which is in a form very close to the stochastic chains studied by Eggers in Refs. 23–24.

This paper is organized as follows. The closure technique is going to be described in Sec. II and equations for the energy balance in the laboratory, as well as in the Lagrangian frame, are derived in the mean field approximation. In Sec. III, this closure is applied to the analysis of the energy transfer fluctuations. In Sec. IV, an energy balance equation in terms of shells is derived, and its solution is used to obtain the intermittency corrections to scaling. Section V is devoted to discussion of the results and to concluding remarks. Most of the calculation details have been confined to the appendices.

II. MEAN ENERGY TRANSFER IN LAGRANGIAN REFERENCE FRAME

A. Closure outline

In the inertial range of fully developed 3-D turbulence, the dynamics obeys the Euler equation:

$$\partial_t \mathbf{v}(r,t)+(\mathbf{v}\cdot\nabla)\mathbf{v}(\mathbf{r},t)+\nabla P(\mathbf{r},t)=0, \tag{1}$$

while the presence of dissipation is accounted for by the energy flux toward small scales; the pressure P is calculated through the incompressibility condition $\nabla\cdot\mathbf{v}=0$. In the laboratory frame, the time decay of the velocity correlations is dominated by the effect of sweep by large scales. In order to eliminate this effect, at least locally in space, it is necessary to shift to a Lagrangian frame. In the reference frame of a Lagrangian tracer that at time t_0 was at $\mathbf{r}_0$, Eq. (1) takes the form

$$D_t\mathbf{v}(\mathbf{z}_t+\mathbf{r},t)=\{[\mathbf{v}(\mathbf{z}_t,t)-\mathbf{v}(\mathbf{z}_t+\mathbf{r},t)]\cdot\nabla\}\mathbf{v}(\mathbf{z}_t+\mathbf{r},t)$$
$$-\nabla P(\mathbf{z}_t+\mathbf{r},t)=0, \tag{2}$$

where $\mathbf{z}_t=\mathbf{z}_t(\mathbf{r}_0,t_0)$ is the position of the tracer at time t and $D_t=\partial_t+\mathbf{v}(\mathbf{z}_t,t)\cdot\nabla$ is the material derivative along the trajectory $\mathbf{z}_t$.

The first assumption of the model is that the velocity field can be taken to lowest order to obey Gaussian statistics. In particular, velocity correlations between pair of points, of which the initial one lies on the tracer trajectory, are assumed to decay, following the law (valid at least for large enough time separations)

$$\begin{aligned} C_k^{\alpha\gamma}(t) &\simeq \int d^3r\, e^{i\mathbf{k}\cdot\mathbf{r}}C^{\alpha\gamma}(\mathbf{r},t;0,0)\\ &=2\pi^2 C_{\mathrm{Kol}}\bar{\epsilon}^{2/3}P^{\alpha\gamma}(\mathbf{k})k^{-11/3}\exp(-\eta_k t),\\ C_{\alpha\gamma}(\mathbf{r},t;\mathbf{r}_1,t_1)&\equiv\langle v_\alpha^{(0)}(\mathbf{z}_t+\mathbf{r},t)v_\gamma^{(0)}(\mathbf{z}_{t_1}+\mathbf{r}_1,t_1)\rangle, \end{aligned} \tag{3}$$

where $t>0$, C_{Kol} is the Kolmogorov constant, $\bar{\epsilon}$ is the mean energy dissipation (i.e., the inertial range energy flux to small scales), and $P^{\alpha\gamma}(\mathbf{k})=\delta^{\alpha\gamma}-k^\alpha k^\gamma/k^2$ the transverse projector; $\eta_k=\rho\bar{\epsilon}^{1/3}k^{2/3}$ is then an eddy turnover frequency at scale k^{-1}, with ρ a dimensionless constant. The statistical average $\langle\ \rangle$ taken in Eq. (3) can be understood as an average over the initial position $\mathbf{r}_0$, which is taken in a large but finite volume, as discussed in the Introduction.

The first non-Gaussian correction $\mathbf{v}^{(1)}$ is obtained from (2); this is integrated, keeping second-order terms, in the Direct Interaction Approximation (DIA),[25–27] rather than following Ref. 16. The result then contains a Green's function: $G(t,\mathbf{r}|t_1,\mathbf{r}_1)$, which gives the effect of a source, which at time t_1 was in $\mathbf{z}_{t_1}+\mathbf{r}_1$, on a point $\mathbf{z}_t+\mathbf{r}$ at time t:

$$v_\alpha^{(1)}(\mathbf{r}_0+\mathbf{r},0)=\int_{-\infty}^0 dt\int d^3s\; G_\alpha^\rho(0,\mathbf{r}|t,\mathbf{r}+\mathbf{s})[v_\sigma^{(0)}(\mathbf{z}_t+\mathbf{r}+\mathbf{s},t)-v_\sigma^{(0)}(\mathbf{z}_t,t)]\partial^\sigma v_\rho^{(0)}(\mathbf{z}_t+\mathbf{r}+\mathbf{s},t). \tag{4}$$

The time decay of the (retarded) Green's function introduced above is assumed to be the same as that of the correlation function,[27] and the following approximation is adopted:

$$G(t,\mathbf{r}|t_1,\mathbf{r}_1)\simeq G(t,\mathbf{r}-\mathbf{r}_1|t_1,0)\equiv G(\mathbf{r}-\mathbf{r}_1,t-t_1), \tag{5}$$

which means that the initial and final position in G are shifted together until the first lies on the Lagrangian trajectory $\mathbf{z}_t$ $(\mathbf{r}_0,t_0)$. Equations (5) allow a great simplification in taking Fourier transforms, in that all correlations become diagonal: $\langle v_\mathbf{p}v_\mathbf{q}\rangle\propto\delta(\mathbf{p}-\mathbf{q})$. In particular, it becomes possible to write time correlations in the form $C_k^{\alpha\gamma}(t)=G_{\beta,k}^\alpha(|t|)C_k^{\beta\gamma}(0)$, where

$$G_k^{\alpha\gamma}(t)=\int d^3r\, e^{i\mathbf{k}\cdot\mathbf{r}}G^{\alpha\gamma}(\mathbf{r},t)=P^{\alpha\gamma}(\mathbf{k})\exp(-\eta_k t),\quad t>0. \tag{6}$$

The meaning of this approximation is that the divergence of Lagrangian trajectories is disregarded. In particular, the statistical average over the initial position $\mathbf{r}_0$ coincides identically with the spatial average at the given time. We also have the result that, in this approximation, the advanced Green's function coincides with the "transpose," with respect to the space slots, of the retarded one; in the following sense: for $t_1>t_2$,

$$\begin{aligned} &\langle v_\alpha^{(0)}(\mathbf{z}_{t_1}+\mathbf{r}_1,t_1)v_\gamma^{(0)}(\mathbf{z}_{t_2}+\mathbf{r}_2,t_2)\rangle\\ &\quad=\int d^3s\; G_\gamma^\sigma(t_1,\mathbf{r}_2+\mathbf{s}|t_2,\mathbf{r}_2)C_{\alpha\sigma}(\mathbf{r}_2+\mathbf{s}-\mathbf{r}_1,0)\\ &\quad=\int d^3s\; G_\alpha^\sigma(t_1,\mathbf{r}_1|t_2,\mathbf{r}_1+\mathbf{s})C_{\sigma\gamma}(\mathbf{r}_1+\mathbf{s}-\mathbf{r}_2,0). \end{aligned} \tag{7}$$

Further discussion of these points is contained in Appendix A.

The final assumption is necessary in order to separate sweeping from straining scales, and is that, inside averages, one has to carry on the following substitution:

$$\langle\cdots[\mathbf{v}(\mathbf{z}+\mathbf{r})-\mathbf{v}(\mathbf{z})]\cdot\nabla\cdots\rangle_k$$
$$\to\langle\cdots[\mathbf{v}(\mathbf{z}+\mathbf{r})-\hat{w}(\lambda,k)\mathbf{v}(\mathbf{z})]\cdot\nabla\cdots\rangle_k, \tag{8}$$

where $\hat{w}(\lambda,k)$ is a smoothing operator acting on $\mathbf{v}(\mathbf{z})$ by filtering out Fourier modes above λk. The hypothesis here is that, although all velocity components are integrated along a single Lagrangian trajectory $\mathbf{z}_t(r_0,t_0)$, when these components are large-scale ones, small scale details of $\mathbf{z}_t$ are assumed not to contribute in averages. One may question whether this last step is really necessary, in that the difference $\mathbf{v}(\mathbf{z}_t+\mathbf{r})-\mathbf{v}(\mathbf{z}_t)$ in the nonlinearity of the Lagrangian frame Navier–Stokes equation, should be already enough to suppress sweep. Unfortunately, including in the Lagrangian sweep, contributions from scales smaller than r, leads to unphysical, negative values for the eddy turnover frequency η_k, and this is one of the points in which the limitations of the Quasi-Lagrangian closure procedure of integrating along a single Lagrangian trajectory become manifest. The quantity λ will be the only free parameter of the theory, that will be fixed in terms of the Kolmogorov constant C_{Kol}.

B. Mean field analysis

In order to study the fluctuation dynamics in the energy transfer T, we will have to work in a Lagrangian frame. However, in order to fix the free parameter λ introduced above, it is necessary to match theoretical predictions with the experimental data of the Kolmogorov constant that are taken in a fixed laboratory frame. For this reason, the first step in the analysis is the derivation of closure equations for Eulerian correlations.

The calculations to obtain the energy equation in the laboratory frame are standard:[27] by multiplying Eq. (1) by $\mathbf{v}(\mathbf{r}_0,0)$ and taking the average, the pressure term drops off because of incompressibility, and one is left with

$$\partial_t C^{\alpha\gamma}(\mathbf{r},t)|_{t=0}=-\langle v^\alpha(0,0)v^\beta(\mathbf{r},0)\partial_\beta v^\gamma(\mathbf{r},0)\rangle, \qquad (9)$$

which, of course, is equal to zero at steady state. Actually, this is the equation for the Eulerian two-time correlation at zero time separation; the energy equation is obtained from (9) by multiplying its RHS (right-hand side) by 2. Expansion to first order in $v^{(1)}$ and use of Eqs. (3)–(4) and (7), leads to the same expression for the energy equation as in DIA (both Eulerian and Lagrangian[25–27]), and in "Quasi-Normal-Markovian" closures:[20]

$$\left.\frac{\partial C_k(t)}{\partial t}\right|_{t=0}=\left(\frac{\pi}{k}\right)^2 T(k)$$
$$=\frac{1}{4\pi^2}\int_\Delta dp\ dq\ kpq\,\theta_{kpq}b_{kpq}C_q(C_p-C_k), \qquad (10)$$

where $C_k^{\alpha\gamma}(t)=P^{\alpha\gamma}(\mathbf{k})C_k(t)$; $C_k\equiv C_k(0)$; Δ is the domain defined by the triangle inequalities: $p>0$ and $|k-p|<q<k+p$; $\theta_{kpq}=(\eta_k+\eta_p+\eta_q)^{-1}$ is the relaxation time; and $b_{kpq}=(p/k)(xy+z^3)$ is the geometric factor, in which x, y, and z are cosines of the angles opposite, respectively, to k, p, and q in a triangle with sides kpq. The terms associated with integrating along $\mathbf{z}$ are uniform in space, they do not couple with the others, and are shown in Appendix A not to contribute to the final result: in this model, the choice of integrating back in time along a Lagrangian path is felt only in the eddy turnover time θ_{kpq}.

The Green's function G is a Lagrangian object and is obtained by multiplying this time in Eq. (2) by $\mathbf{v}(\mathbf{r}_0,0)$ and then taking averages. The resulting triplet term is in the form $\langle v^\alpha(\mathbf{r}_0,0)[v^\beta(\mathbf{z}_t+\mathbf{r},t)-v^\beta(\mathbf{z}_t,t)]\partial_\beta v^\gamma(\mathbf{z}_t+\mathbf{r},t)\rangle$; now, $v^\beta(\mathbf{z}_t,t)$, that is, the term coming from the shift to a Lagrangian frame, contributes to the final expression and is responsible for the cancellation of the sweep terms. Substituting Eqs. (3)–(4) and (7) in the new triplet, and expanding again to first order in $v^{(1)}$, we obtain at steady state the following equation:

$$\frac{DC_k(t)}{Dt}=\frac{1}{4\pi^2}\int_0^t d\tau\int_\Delta dp\,dq[b_{kpq}G_p(t-\tau)C_k(\tau)C_q(t-\tau)$$
$$+w_p(\lambda,k)b_{kpq}^{(1)}G_p(t-\tau)C_k(t-\tau)C_q(\tau)$$
$$-w_q(\lambda,k)b_{kpq}^{(2)}G_k(t-\tau)C_p(\tau)C_q(t-\tau)], \qquad (11)$$

where $G_k^{\alpha\gamma}=P^{\alpha\gamma}G_k$, $b_{kpq}^{(1)}=(p/2k)[xy(1-2z^2)-y^2z]$ and $b_{kpq}^{(2)}=(p/2k)[xy+z(1+z^2-y^2)]$ are new geometric terms, and $w_q(\lambda,k)$ gives the effect in k space of the cutoff operator $\hat{w}(\lambda,k)$ The term in Eq. (11), which cancels the divergence of the integral for $q\to 0$, is the one in $b_{kpq}^{(2)}$. Integrating from $t=0$ to $t=\infty$, and using Eqs. (3) and (6), we obtain the result

$$\frac{\rho^2}{C_{\mathrm{Kol}}}=\frac{1}{2}\int_\Delta dp\ dq\ pq^{-8/3}\left(\frac{b_{kpq}}{p^{2/3}+q^{2/3}}+\frac{w_pb_{kpq}^{(1)}}{(1+p^{2/3})q^{2/3}}\right.$$
$$\left.-\frac{w_qb_{kpq}^{(2)}}{(1+q^{2/3})p^3}\right), \qquad (12)$$

which gives a first equation connecting C_{Kol} and ρ with λ. A second equation connecting C_{Kol} and ρ, given an energy balance equation in the form of (10), was derived by Kraichnan:[28]

$$\rho/C_{\mathrm{Kol}}^2\simeq 0.19. \qquad (13)$$

The constant C_{Kol} considered in this section is a well-defined quantity, provided the average volume V_i and the range of scales k are not too large; in this sense we have, locally, $C_k(V_i)\propto C_{\mathrm{Kol}}\bar{\epsilon}_i^{2/3}k^{-11/3}$, even though for $V_i\to\infty$, anomalous corrections become important, and C_{Kol} ceases to have a clear meaning. Imposing that the Kolmogorov constant matches the experimentally observed value $C_{\mathrm{Kol}}\simeq 1.5$ (we are assuming here that the finite size effects and intermittency corrections that affect the experimental C_{Kol} are indeed small), and using a Gaussian profile for the cutoff w: $w_p(\lambda,k)=\exp[-(p/\lambda k)^2]$, Eqs. (12) and (13) set $\lambda=0.9$. In the following, we will therefore fix

$$w_p(\lambda,k)\to w_p(k)=\exp[-1.23(p/k)^2]. \qquad (14)$$

The same steps leading from Eqs. (1)–(10) can be repeated, starting from Eq. (2). The result is the following equation for velocity correlations in a Lagrangian frame:

$$\left.\frac{DC_k(t)}{Dt}\right|_{t=0}=\left(\frac{\pi}{k}\right)^2 T(\lambda',k)$$
$$=\frac{1}{4\pi}\int_\Delta dp\ dq\ kpq\,\theta_{kpq}[b_{kpq}$$
$$+w_q(\lambda',k)b_{kpq}^{(3)}]C_q(C_p-C_k), \qquad (15)$$

where $b_{kpq}^{(3)}=(p/k)\{1+xy+z[z^2-(x^2+y^2)/2]\}$. Again, the corresponding energy equation is obtained by multiplying

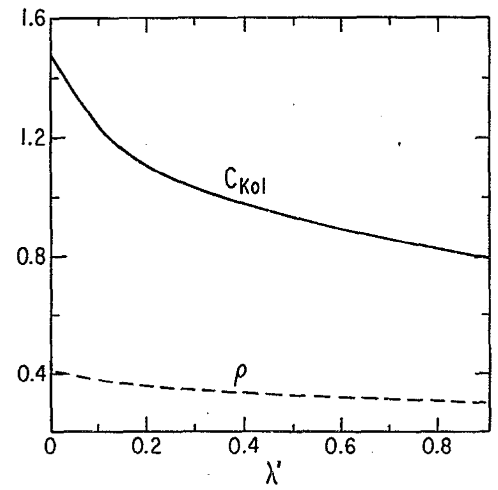

FIG. 1. Kolmogorov constant C_{Kol} and dimensionless parameter ρ, as measured from the energy flux through scale k, in a reference frame moving with velocity $\mathbf{v}(\lambda',k)=\hat{w}(\lambda',k)\mathbf{v}$, $0<\lambda'<0.9$, with $\lambda=0.9$, fixed [see Eq. (14)].

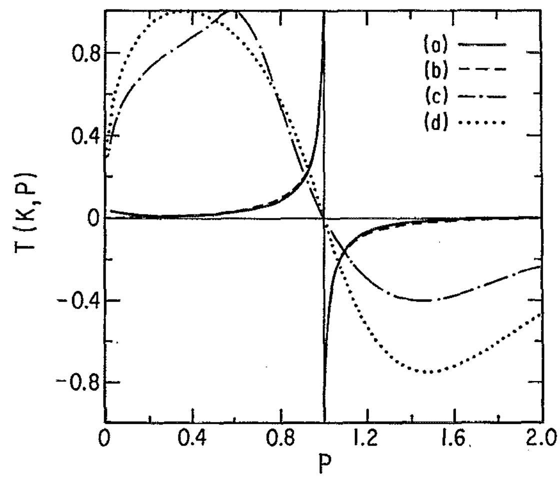

FIG. 2. Energy transfer profiles $T(k,p)$ for $k=1$. (a) Lagrangian frame using Eq. (15); $\lambda'=0.9$. (b) Laboratory reference frame, using Eq. (10). (c) Lagrangian frame using simplified closure with Eq. (17) and $w_{p,q}=w_{p,q}(0.7,k)$. (d) The same as (c), but with $w_p=w_p$ [0.7, min(k,q)] and $w_q=w_p$ [0.7, min(k,p)].

the RHS of (15) by 2 and understanding the t on the LHS (left-hand side), not as a time separation in a two-time correlation, but as the t dependence in a nonstationary one-time correlation.[29]

Notice that the parameter λ' entering Eq. (15) does not have to coincide with $\lambda=0.9$, which is fixed and separates between sweeping and straining scales. This allows us to calculate the Kolmogorov constant and the parameter ρ in different reference frames, which corresponds to express the velocity correlation C_k, defined through Eq. (3), in terms of the Lagrangian frame expression for the energy flux to small scales. This quantity contains a contribution from the shift to a reference frame moving with a nonconstant velocity, which comes from the difference in the work along the Lagrangian trajectory, done by the various Fourier components of the fluid velocity. (This effect is a consequence of the correlation between the velocity of the moving frame and the fluid velocity itself.) The analysis carried on here predicts a value of the energy flux to small scales along a Lagrangian trajectory, approximately three times larger than the value in the fixed frame; this same result is shown in Fig. 1, in terms of C_{Kol} and ρ. Notice that this calculation is carried on, keeping fixed C_k and η_k, which are defined in terms of spatial averages. The same approach will be followed in the next section in the analysis of the fluctuations on C_k. This means that, despite the shift to a Lagrangian frame, and the consequent change in the expression of the energy transfer, we are always dealing with the same reference frame-independent quantity; in this last case, essentially $\langle v^4\rangle$, which, like C_k, is defined through a global spatial average. In the next section, this independence will become apparent through the dependence of the coefficients in Eq. (43) on combinations of C_{Kol} and ρ, which are invariant under reference frame shifts.

The structure of the energy transfer is studied, following Domaradzki and Rogallo,[19] by decomposing $T(\lambda,k)$ in its contribution from different scales:

$$T(\lambda,k)=\int dp\ T(\lambda,k,p). \tag{16}$$

It appears that the present closure is able to maintain the features of large-scale straining observed in Ref. 19 also in a Lagrangian frame, as it should be expected. It is clear instead, from Fig. 2, that simpler closures based on a Navier–Stokes nonlinearity in k space, amputated of the large-scale convection contributions:

$$[k_\alpha P_{\beta\gamma}(\mathbf{k})+k_\gamma P_{\beta\alpha}(\mathbf{k})]v_{\mathbf{p}}^\alpha v_{\mathbf{q}}^\gamma$$
$$\to[k_\alpha(1-w_q)P_{\beta\gamma}(\mathbf{k})+k_\gamma(1-w_p)P_{\beta\alpha}(\mathbf{k})]v_{\mathbf{p}}^\alpha v_{\mathbf{q}}^\gamma, \tag{17}$$

would lead to transfer profiles almost without any exchange of energy between nearby scales.

III. ENERGY TRANSFER FLUCTUATIONS

Although the analysis in Sec. II disregarded fluctuations, all averages were implicitly dependent on the position (through $\bar{\epsilon}$), at the scale of the volumes V_i. In this section, we calculate the same averages over balls of radius R and the variations of the result from ball to ball are used to study the fluctuations of the energy transfer. If the ratio $RV_i^{-1/3}$ is not too large, these fluctuations are going to be small and the analysis can be carried on in perturbation theory.

In principle, it would be nice to carry on the calculations in the laboratory frame, however, the statistics of the velocity field is ill-defined there, due to the effect of sweep. The balls are then imagined to move rigidly along Lagrangian trajectories passing through their centers at time t_0, while, thanks to the simplifying assumptions of Eqs. (6) and (7), the average over the initial position is substituted by a spatial average inside the balls. As discussed in the Introduction, we associate to each scale l, a certain radius $R(l)\sim l/(a-1)$; this automatically induces a basis of wave packets of width $\Delta k\sim R^{-1}$. In particular, we obtain a partition of Fourier space in shells $\mathbf{S}_n$ of radii $k_n = a^n k_0$, each containing $\sim(k_n/\Delta k)^2$ wave packets, associated with the different orientations of the wave vector $\mathbf{k}$.[17] Clearly, a further degeneracy in the wave packets is produced by their different location in real space.

In the limit of $a-1 \sim l/R$ small, the main interactions occur among overlapping balls and are associated with local transfer of energy toward small scales. Therefore we consider a sequence of nested balls and study the transfer of energy among them. Notice that a derivation of deterministic equations for a shell model of the type considered here, would require some dynamical analog of the statistical assumptions of Eqs. (5)–(7); at this point it is more natural to follow the route of statistical closure to the end.

Let us indicate with $\langle\ \rangle_m$ the spatial average over $\mathbf{B}_m$: the ball of radius R_m associated with the mth shell; once we have fixed the origin of the axis of the moving frame on $\mathbf{z}_t$, we can write $\langle\ \rangle_m$ in terms of a kernel $W(r,m)$:

$$\langle\Psi\rangle_m(\mathbf{r}) = \int d^3r'\ W(r',m)\Psi(\mathbf{r}+\mathbf{r}') = \int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{r}} W_k(m)\Psi_k.$$

In this way, the energy density in $\mathbf{B}_m$ reads as

$$E(m) = \int dk\ E_k(m);$$

$$E_k(m) = 2\pi k^2 \int \frac{d^3p}{(2\pi)^3}\, W_p(m)\mathbf{v}_{\mathbf{p}/2+\mathbf{k}}\cdot\mathbf{v}_{\mathbf{p}/2-\mathbf{k}}. \tag{18}$$

To fix our ideas we shall consider Gaussian wave packets:

$$W_k(m) = \exp[-(k/\Delta k_m)^2], \quad \Delta k_m = c_w(a-1)k_m, \tag{19}$$

with c_w relating the wave packet thickness Δk_m and the shell spacing $k_{m+1}-k_m=(a-1)k_m$. For each choice of shell spacing (i.e., for a given value of a), we can still consider arbitrarily thin wave packets, or in the limit, even Fourier modes. Of course, it is the thickest wave packet for a given a (that is the maximum value of c_w, i.e., the minimum spatial extension of the wave packets compatible with the uncertainty on k), which will be able to catch most of the fluctuation dynamics. In the following, we will have, however, to carry on our analysis as an expansion for c_w small.

In the limit of $k_n R_m$ large, $W_p(m) \propto R_m^{-3}\delta(\mathbf{p})$, and we can approximate

$$E_k(m) \simeq [1+\phi_n(m)]E_k^{(0)} \equiv [1+\phi_n(m)]C_{\mathrm{Kol}}\bar{\epsilon}^{2/3}k^{-5/3}, \tag{20}$$

with $E_k^{(0)} = C_k k^2/2\pi^2$ the spectral energy density in V_i, $n=n(\mathbf{k})$ the shell of the wave vector $\mathbf{k}$, and $\phi_n(m)=\phi_n(m,t)$ fluctuating and small.

The energy $E(m)$ can be expressed also as a sum of contributions from different shells:

$$E(m) = \sum_n E_n(m); \quad E_n(m) = \int_n dk\ E_k(m). \tag{21}$$

The term $E_n(m) \sim \langle E(k_n^{-1},a,\mathbf{z}_t,t)\rangle_m$ can be seen as the instantaneous total energy of wave packets in $\mathbf{S}_n$, lying in the volume $\mathbf{B}_m$ at $\mathbf{z}_t$; however, if $\mathbf{B}_m$ becomes smaller than the wave packets, i.e. (for c_w maximal) $n>m$, the only average taking place will be over wave vector orientations and will be independent of m; hence, we obtain $\phi_n(m<n)=\phi_n(n)$.

We can write the equation for the energy contained in this shells in the form

$$D_t E_n(m) = T_n(m) + f_n(m), \tag{22}$$

with $T_n(m)$, representing the contribution to the transfer to $\mathbf{S}_n$, from terms schematically in the form: $\langle v^{(0)}v^{(0)}v^{(1)}\rangle_m$; these gave rise, in the previous section, to the mean field energy transfer in a Lagrangian frame $T(\lambda,k)$ [see Eq. (15)]. The term $T_n(m)$ will appear to play the role of a relaxation, giving the response of the system to fluctuations in the energy content of the various shells. In terms of Fourier components,

$$T_n(m) = \int_n dk\ T'(k). \tag{23}$$

To lowest order in $a-1$, Fourier modes take the place of the wave packets and $\phi_n(m)=0$, so that, by definition, $T'(k)$ coincides with $T(\lambda,k)$. To next order, $C_k \to [1+\phi_n(m)]C_k$ in $T(\lambda,k)$, and the spectral thickness of the wave packets produces discreteness corrections in the $dp\,dq$ integrals. However, if we focus on the fluctuations, to lowest order, the discreteness corrections do not contribute; writing formally: $T'(k) \simeq T(\lambda,k) + \int dp[\delta T(\lambda,k)/\delta E_p^{(0)}]E_p^{(0)}\phi_{n(p)}(m)$; in the following, this calculation will be carried on explicitly. Clearly, when $c_w \to c_w^{\mathrm{MAX}}$, similar discreteness effects are produced in the dk integral of Eq. (23), and it is for this reason that we are forced to consider the limit of c_w small, together with $a-1$ small.

The second term on the RHS of Eq. (22), to lowest order in the expansion around Gaussian statistics, has two components. The first comes from the pressure, $\langle\mathbf{v}\cdot\nabla P\rangle_m$, and results in a surface integral over the boundary of $\mathbf{B}_m$. The second, schematically in the form $\langle v^{(0)}v^{(0)}v^{(0)}\rangle_m$, did not contribute in the mean field analysis of the previous section, because of the Gaussianity of $\mathbf{v}^{(0)}$; it plays a role here, however, by acting as a source of fluctuations in Eq. (22). Now, since we are considering a situation in which kR is large, the pressure contribution can be neglected, and we are left with

$$f_n(m) = -\langle\hat{h}(n)[\mathbf{v}^{(0)}(\mathbf{z}_t,t)\cdot\{[\mathbf{v}^{(0)}(\mathbf{z}_t+\mathbf{r},t)-\mathbf{v}^{(0)}(\mathbf{z}_t,t)]\cdot\nabla\}\mathbf{v}^{(0)}(\mathbf{z}_t+\mathbf{r},t)]\rangle_m, \tag{24}$$

where $\hat{h}(n)$ is a bandpass filter for the modes in $\mathbf{S}_n$: $\hat{h}(n)f(\mathbf{r}) \equiv \int d^3r'\ h(n,r')f(\mathbf{r}+\mathbf{r}') = \int[d^3k/(2\pi)^3]h_k(n)f_\mathbf{k} \times e^{i\mathbf{k}\cdot\mathbf{r}}$, with $h_k(n)=H(k-k_n)-H(k-k_{n+1})$; $H(x)$ is the Heaviside step function. This is precisely the point where a Lagrangian closure becomes necessary. In the laboratory frame, $f_n(m)$ would be dominated by the largest-scale components, with $\langle f^2\rangle$ divergent for $\mathrm{Re}\to\infty$, and time correlations on the scale of the sweeping time. Even if the volume $\mathbf{B}_m$ were allowed to follow the fluid with the instantaneous velocity $\langle\mathbf{v}\rangle_m$, the noise $f_n(m)$ would be dominated in an Eulerian closure, by contributions at scales $\sim R_m$, that, for large values of $k_n R_m$, would be associated mainly with sweep. In the present model instead, different regions of size k_n^{-1} in the volume $\mathbf{B}_m$ are studied in their individual Lagrangian frame, and sweep by scales $\sim R_m$ is therefore absent.

Treating $f_n(m)$ as an external (non-Gaussian) noise, Eq. (22) becomes a stochastic differential equation for the $\phi_n(m)$s. Following Eggers,[23–24] we express $f_n(m)$ as the difference between the fluctuations in the energy flux across k_n and k_{n+1}:

$$f_n(m)=g_n(m)-g_{n+1}(m), \tag{25}$$

where g_n and g_{n+1} are associated, one with each of the step functions entering the definition of $h_k(n)$.

A. Fluctuation source

The statistics of $g_n(m)$ can be calculate explicitly from Eqs. (3), (24), and (25). Here we shall content ourselves with the analysis of the two-point correlations. Already, this calculation requires the evaluation of some 15 contractions of the product:

$$v^{\alpha}(\mathbf{z}_1)[v^{\beta}(\mathbf{z}_1+\mathbf{r}_1)-v^{\beta}(\mathbf{z}_1)]\partial_{\beta}v^{\alpha}(\mathbf{z}_1+\mathbf{r}_1)v^{\gamma}(\mathbf{z}_2)[v^{\sigma}(\mathbf{z}_2+\mathbf{r}_2)-v^{\sigma}(\mathbf{z}_2)]\partial_{\sigma}v^{\gamma}(\mathbf{z}_2+\mathbf{r}_2); \tag{26}$$

these contractions are listed in Tables I–III in Appendix B. Of these only the six contained in Table III contribute; one example is (B3.2) in Table III:

$$\begin{aligned}\langle g_{n_1}(m_1,t_1)g_{n_2}(m_2,t_2)\rangle^{(3.2)} &= \int d^3r_1\ d^3r_2\ H(r_1,n_1)H(r_2,n_2)\\ &\times\int d^3z_1\ d^3z_2\ W(z_1,m_1)W(z_2,m_2)C^{\alpha\gamma}(\mathbf{z}_1-\mathbf{z}_2,t)\\ &\times\partial_{\sigma}[C^{\beta\gamma}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2-\mathbf{r}_2,t)-\hat{w}C^{\beta\gamma}(\mathbf{z}_1-\mathbf{z}_2-\mathbf{r}_2,t)]\\ &\times\partial_{\beta}[C^{\alpha\sigma}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2-\mathbf{r}_2,t)-\hat{w}C^{\alpha\sigma}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2,t)];\end{aligned} \tag{27}$$

notice the cutoff operators $\hat{w}$ signaling a term coming from working in a Lagrangian reference frame. In terms of Fourier components, we obtain

$$\begin{aligned}\langle g_{n_1}(m_1,t_1)g_{n_2}(m_2,t_2)\rangle^{(3.2)} &= \int\frac{d^3k}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}C_kC_pC_q\\ &\times\exp[-(\eta_k+\eta_p+\eta_q)|t|]\\ &\times W_{\mathbf{p}+\mathbf{q}-\mathbf{k}}(m_1)W_{\mathbf{p}+\mathbf{q}-\mathbf{k}}(m_2)pkyz(y+xz)\\ &\times\{[2-w_p(k)]H(k-k_{n_1})-w_p(k)H(q-k_{n_1})\}\\ &\times\{[2-w_q(k)]H(k-k_{n_2})-w_q(k)H(p-k_{n_2})\}.\end{aligned} \tag{28}$$

[The wave vectors entering the two last lines of Eq. (28) can be tracked back to Eq. (26) and (B3.2): the p and k in $w_p(k)$ come from v_p^{β} and v_k^{α}, while the k and q in $H(k-k_{n_1})$ and $H(q-k_{n_1})$ come from v_k^{α} and v_q^{α}.] It has already been mentioned that, for large kR, the averaging kernel W is proportional in k space to a Dirac delta. Also, the product $W(m_1)W(m_2)$ is a Dirac delta:

$$W_{\mathbf{k}}(m_1)W_{\mathbf{k}}(m_2)\simeq[c_w(a-1)]^3\left(\frac{\pi k_{m_1}^2k_{m_2}^2}{k_{m_1}^2+k_{m_2}^2}\right)^{3/2}\delta(\mathbf{k}), \tag{29}$$

where use has been made of Eq. (19). Equation (29) allows the simplification of Eq. (28) by means of the bipolar integral formula[27] $\int d^3p\ d^3q\ \delta(\mathbf{p}+\mathbf{q}-\mathbf{k})=2\pi pq/k\int_{\Delta}dp\ dq$. Repeating the calculations leading to (28) with all the other contractions, we obtain the result

$$\begin{aligned}\langle g_{n_1}(m,t_1)g_{n_2}(m,t_2)\rangle &= \frac{\pi^{1/2}\bar{\epsilon}^2}{2^{9/2}}[C_{\mathrm{Kol}}c_w(a-1)k_m]^3\int_0^{\infty}dk\int_{\Delta}dp\ dq(kpq)^{-8/3}\\ &\times\exp[-(\eta_k+\eta_p+\eta_q)|t|][H_1B_1(kpq)\\ &+H_2B_2(kpq)+H_3B_3(kpq)].\end{aligned} \tag{30}$$

The terms B_i, $i=1,2,3$ are geometrical factors similar to the b_{kpq} terms entering the expression for the transfer function. The factors $H_i=H_i(n_{1,2};k,p,q)$ are expressed in terms of step functions and restrict the integrals to the appropriate domains for the calculation of Lagrangian frame energy fluxes; they have the same origin as the two last lines of Eq. (28). The exact form of both the B_i and H_i functions is given in Eqs. (B4)–(B7) of Appendix B.

For $t=0$, Eq. (30) can be reduced to double integrals using the standard change of variables,[27] $k=k_n/u$, $p=k_nv/u$, $q=k_nw/u$, and exploiting the similarity of the integrand with respect to u. Similarly, the following expression for the noise correlation time:

$$\bar{\eta}_n(m)^{-1}=\langle f_n(m)^2\rangle^{-1}\int_0^{\infty}dt\langle f_n(m,t)f_n(m,0)\rangle,$$

can be reduced, after explicit calculation of the time integral, to a double integral. Numerical evaluation then gives the results

$$\langle g_n(m)^2\rangle\simeq 0.15[C_{\mathrm{Kol}}c_w(a-1)]^3a^{l(n,m)}\bar{\epsilon}^2; \tag{31}$$

and

$$\bar{\eta}_n(m)=\bar{\rho}\bar{\epsilon}^{1/3}k_n^{2/3},\quad \bar{\rho}\simeq 0.6, \tag{32}$$

where $l(n,m)=\min(0,m-n)$ for c_w maximal. The decay of the correlation with respect to scale is shown in Fig. 3. Notice that $\bar{\rho}\simeq 2\rho_{\lambda=0.9}$ (see Fig. 1), which is what one would obtain, identifying, naively, $\langle g(0)g(t)\rangle\propto\langle v(0)v(t)\rangle^2$. Similarly, the decay of $\langle g_ng_{n'}\rangle$ with respect to $(n-n')\ln a$ is approximately the same as that of the autocorrelation for $|T(k,p)|:A(q)=\int dp|T(k,p)T(k+q,p+q)|$, which is due to the fact that the energy flux at scale k is roughly equal to $\int dp|T(k,p)|$.

B. Relaxation term

The shell energy equation (22) is in the form of a nonlinear stochastic differential equation. Since for $a-1$ small, (large ratio of partial averaging scale to the characteristic fluctuation scale) ϕ is small, we can linearize $T_n(m)$. This has to be done, including the frequencies $\eta_k\sim k^{3/2}E_k^{1/2}$ entering the term θ_{kpq}, and the result is one that is similar to

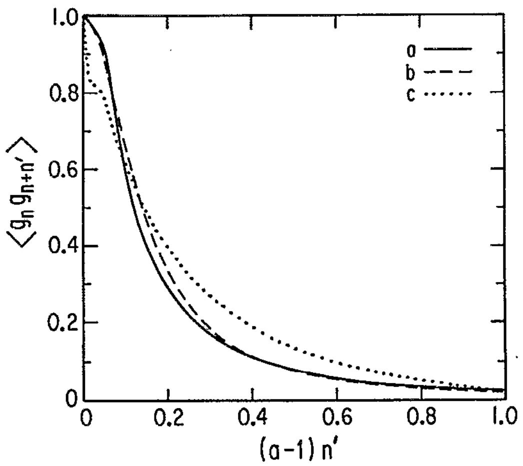

FIG. 3. Normalized noise correlation $\langle g_n g_{n+n'}\rangle/\langle g_n^2\rangle$ versus shell separation $x=(a-1)n'$. (a) Calculation from Eq. (30). (b) Fit of (a) by $F(x)=(1+50x^2)^{-1}$. (c) Profile for $A(x)=\int dp|T(k,p)T(k+x,p+x)|$, normalized to $A(0)$.

the stochastic model of Eggers.[23] One difference is the possibility here of energy transfer between nonadjacent shells. To analyze this issue, we write the transfer $T_n(m)$ as a sum of contributions in which the wave vectors k, p, and q are, respectively, in $\mathbf{S}_n$, $\mathbf{S}_r$, and $\mathbf{S}_s$:

$$T_n(m)=\frac{C_{\mathrm{Kol}}^2\bar{\epsilon}}{\rho}\sum_{r,s}T_{nrs}(m);$$

$$T_{nrs}=c_{nrs}\phi_r+c_{nsr}\phi_s-d_{nrs}\phi_n, \tag{33}$$

where, from Eqs. (15) and (20),

$$c_{nrs}=\int_n dk\int_{\Delta_{rs}}dp\ dq\ \tilde{\theta}_{kpq}\bigg[\tilde{a}_{kpq}k^2(pq)^{-5/3} -\bigg(\frac{2}{3}p^{2/3}\tilde{\theta}_{kpq}[\tilde{a}_{kpq}k^2(pq)^{-5/3}-\tilde{b}_{kpq}p^2(kq)^{-5/3} -\tilde{b}_{kqp}q^2(kp)^{-5/3}]+\tilde{b}_{kpq}q^2(kp)^{-5/3}\bigg)\bigg], \tag{34}$$

and

$$d_{nrs}=\int_n dk\int_{\Delta_{rs}}dp\ dq\ \tilde{\theta}_{kpq}\bigg(\frac{2}{3}k^{2/3}\tilde{\theta}_{kpq}[\tilde{a}_{kpq}k^2(pq)^{-5/3} -\tilde{b}_{kpq}p^2(kq)^{-5/3}-\tilde{b}_{kqp}q^2(kp)^{-5/3}]+\tilde{b}_{kpq}p^2 \times(kq)^{-5/3}+b_{kqp}q^2(kp)^{-5/3}\bigg). \tag{35}$$

The quantities appearing on the RHS of Eqs. (33)–(34) are defined as follows: Δ_{rs} is the restriction of Δ to $p\in\mathbf{S}_r$; $q\in\mathbf{S}_s$; $\tilde{b}_{kpq}=b_{kpq}+w_q(k)b_{kpq}^{(3)}$; $\tilde{a}_{kpq}=\tilde{b}_{kpq}+\tilde{b}_{kqp}$ and $\tilde{\theta}_{kpq}=(k^{2/3}+p^{2/3}+q^{2/3})^{-1}$. The RHS of Eqs. (33)–(34) can be reduced to double integrals using the same method of Eq. (30), and are evaluated numerically. We can rearrange the sum in Eq. (33) in the following form:

$$\sum_{r,s}T_{nrs}(m)=\sum_r a_r(m)\phi_r(m)=\sum_{r=1}^{\infty}A_r(m)\Delta^r\phi_n(m), \tag{36}$$

where Δ is the finite difference operator acting as follows:

$$\Delta^{2r+1}\phi_n=\tfrac{1}{2}(\Delta^{2r}\phi_{n+1}-\Delta^{2r}\phi_{n-1});$$
$$\Delta^{2r}\phi_n=\Delta^{2(r-1)}\phi_{n+1}+\Delta^{2(r-1)}\phi_{n-1}-2\Delta^{2(r-1)}\phi_n, \tag{37}$$

so that the coefficients $A_r(m)$ are expressed in terms of the $a_r(m)$'s through the relation

$$A_{2r+1}=a_{n+2r-1}-a_{n-2r+1};\ A_{2r}=\frac{a_{n+2r-1}+a_{n-2r+1}}{2}. \tag{38}$$

The fact that only differences between ϕ_n enter the expression above is due to the fact that we are expanding around a Kolmogorov spectrum, for which $T_n=0$. For the same reason, the coefficients c_{nrs} and d_{nrs} are invariant under the transformation: $\{n,r,s\}\rightarrow\{n+j,r+j,s+j\}$, which explains the fact that the finite difference coefficients A_r do not depend on the shell index n.

In Fig. 4 these coefficients are plotted in terms of the shell constant a. Notice that finite differences of order higher than 2 appear to be negligible for most choices of a. We therefore obtain the basic result that the shell dynamics obeys, for most values of a, a (discrete) heat equation forced by a random noise, with an advection term proportional to Δ.

The physical picture corresponding to this result is not new.[30] Fluctuations over finite volumes $\mathbf{B}_m$ moving with the flow, in the energy transfer to eddies at scale k_n, generate fluctuations in the energy content of these volumes and scales. The eddies are stretched by the turbulent flow, so that their energy (together with its fluctuation ϕ_n) is transferred toward smaller scales; the term responsible for this effect is the advection $A_1\ \Delta\phi_n$. At the same time, the randomness of

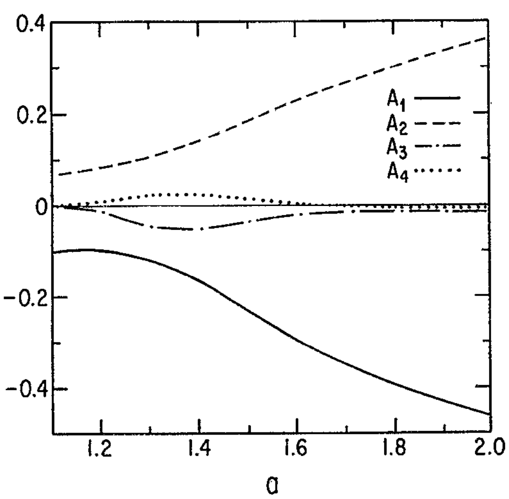

FIG. 4. Finite difference coefficients A_r vs a [see Eqs. (33), (36), and (37)]. Notice that higher-order differences are always dominated by the advection-diffusion part.

the turbulent flow causes some eddies to be stretched more and some less, resulting in the end in a diffusion of energy over different scales.

IV. ANOMALOUS SCALING ESTIMATES

The fluctuations in the energy of wave packets at different space locations is what is responsible for intermittency in this model. Instead of studying the scaling behaviors of the structure functions, $S(l,q)=\langle v^q\rangle$, we focus on the modified structure functions, defined through

$$S(k_n^{-1},a,c_w,2q)=\langle\langle E_n\rangle_n^q\rangle=E_n^{(0)}\langle[1+\phi_n(n)]^q\rangle. \quad (39)$$

Identifying naively the space separation l in $S(l,q)$ with the wavelength k_n^{-1} in Eq. (39), the two definitions of structure function coincide, for an appropriate choice of a, and for $c_w = c_w^{\mathrm{MAX}}$. Given a large enough averaging volume V_i, the intermittency correction is small and we can linearize in n; in this way, using also $k_n=k_0a^n$, we have

$$\zeta_{2q}-q\zeta_2\simeq-\frac{1}{\ln a}\frac{d}{dn}\langle[1+\phi_n(n)]^q\rangle, \quad (40)$$

where $\zeta_q=\zeta_q(a,c_w)$ will indicate, from now on, the scaling exponents associated with the generalized structure functions $S(l,a,c_w,q)$. If we confine ourselves to the lowest-order moments, large fluctuations of ϕ do not contribute too much, so that we can expand $[1+\phi_n(n)]^q\simeq1+q(q-1)/2\langle\phi_n(n)^2\rangle$. Imposing the Kolmogorov relation for the third-order structure function, $S(l,a,3)\propto l$, fixes the value for $\delta\zeta_2$, leading to the lognormal statistics result,[6]

$$\delta\zeta_q\simeq-\frac{q(q-3)}{4\ln a}\frac{d\langle\phi_n(n)^2\rangle}{dn}; \quad (41)$$

we see then that the presence of anomalous scaling is associated with secular behavior of the fluctuations $\phi_n(n)$.

Unfortunately, the energy balance for the wave packets, Eq. (22), is not an equation for $\phi_n(n)$, but one for $\phi_n(m)$ for fixed m. We can rewrite Eq. (22) in a more explicit form, by approximating the relaxation term $T_n(m)$ with the advection-diffusion operator introduced through Eqs. (35)–(37), and by rewriting the noise term f_n as a difference of energy flux fluctuations at different scales, as from Eq. (25). After adequate rescaling, we are left with the equation

$$[\exp(-\gamma n)\partial_t-D\Delta^2+V\Delta)]\phi_n(m,t)=\hat\Delta[F_n(m)^{1/2}\xi], \quad (42)$$

where $\hat\Delta\xi_n=\xi_{n+1}-\xi_n$ and

$$\hat t=t\bar\rho\bar\epsilon^{1/3}k_0^{2/3};\quad \gamma=\frac{2}{3}\ln a;\quad D=\frac{C_{\mathrm{Kol}}}{\rho\bar\rho(1-a^{-2/3})}A_2;$$

$$V=\frac{C_{\mathrm{Kol}}}{\rho\bar\rho(1-a^{-2/3})}A_1;\ F\simeq0.83\frac{[c_w(a-1)]^3C_{\mathrm{Kol}}a^{l(m,n)}}{[\bar\rho(1-a^{-2/3})]^2}; \quad (43)$$

$$\langle\xi_{n+n'/2}(m,\hat t)\xi_{n-n'/2}(m,0)\rangle\simeq\frac{\exp(-e^{\gamma n}|t|)}{1+112.5(\gamma n')^2}.$$

Notice that the Kolmogorov and time scale constants C_{Kol} and ρ enter here in the combination $C_{\mathrm{Kol}}/\bar\rho^2$ or $C_{\mathrm{Kol}}/\rho\bar\rho$, which are both invariant with respect to change in λ'; the fact that we are working in a Lagrangian frame is felt only in the coefficients A_1,A_2, due to the modified form of the balance equation (15), and in the prefactor 0.83 entering the noise amplitude F (in an Eulerian frame this quantity would have been divergent). The dependence of the noise correlation $\langle\xi\xi\rangle$ on $n-n'$ in the formula above is a fit of the result of numerical integration shown in Fig. 3.

Clearly, Eq. (42) does not lead to intermittent behaviors; the factor $a^{l(m,n)}=a^{(m-n)}$ in the noise amplitude, which goes to zero at small scales, prevents it. Passing from Eq. (42) to an equation for $\phi_n(n)$ requires the introduction of corrections due to the fact that now wave packets associated with different scales do not overlap exactly in real space. There are two such contributions:

$$\begin{aligned}(-D\Delta^2+V\Delta)\phi_n(m)|_{m=n}&-(-D\Delta^2+V\Delta)\phi_n(n)\\&=(-D+V/2)[\phi_{n+1}(n+1)-\phi_{n+1}(n)]\end{aligned} \quad (44a)$$

and

$$\begin{aligned}\hat\Delta[F^{1/2}(n,m)\xi_n(m)]|_{n=m}&-\hat\Delta[F^{1/2}(n,n)\xi_n(n)]\\&=F^{1/2}(n,n)[\xi_{n+1}(n+1)-\xi_{n+1}(n)].\end{aligned} \quad (44b)$$

The shell equation for $\phi_n\equiv\phi_n(n,t)$ then takes the form (to simplify notations, the overcaret on the rescaled time $\hat t$ will be dropped in the following):

$$[\exp(-\gamma n)\partial_t-D\Delta^2+V\Delta]\phi_n=F^{1/2}(\hat\Delta\xi+\delta\xi), \quad (45)$$

with $F^{1/2}\,\delta\xi$, which is equal to the sum of the RHSs of Eqs. (44a)–(44b), providing a new fluctuation source beside the original term $F^{1/2}\,\hat\Delta\xi$.

At this point we are in the condition to identify the terms in the shell energy equation, which are responsible for the generation of intermittency. For the sake of simplicity, and to make contact with the model of Eggers,[23] let us adopt for a moment the approximation $\xi_n\simeq e^{-\gamma n/2}\hat\xi$ with $\langle\hat\xi(t)\hat\xi(t')\rangle=\delta(t-t')$, and a similar equation for $\delta\xi$. The RHS of Eq. (45) is then proportional to

$$\hat\Delta\hat\xi-\gamma\hat\xi/2+\delta\hat\xi. \quad (46)$$

We see then that Eq. (45), at statistical equilibrium, is very similar to a random walk equation in which the role of the time is played by n/V and that of the random kicking by $-\gamma\hat\xi/2+\delta\hat\xi$. This noise term continuously pumps into the system fluctuations, which are dissipated at very large n, by viscosity. It is the random walk character of the process that leads to the linear growth of $\langle\phi_n^2\rangle$, with respect to n, already observed in Ref. 23. The two terms providing the source of intermittency have different physical origin. The term $\delta\xi$, which comes from Eqs. (44a) and (44b), is due to the competition in the energy transfer between eddies at different locations, characteristic of the Random Beta Model.[8] The term $\gamma\hat\xi/2$, instead, comes from the mismatch in the characteristic time scales of the energy transfer between different shells, and was responsible for the production of intermittency in the stochastic model of Eggers.[23–24]

The first term in Eq. (46), which is the derivative of a random noise, produces the Gaussian part of the fluctuations in the energy content of the wave packets. Notice that this same quantity can be calculated directly from finite volume averages of $|\mathbf{v}^{(0)}|^2$, given the expression for the correlation given by Eq. (3), together with Eq. (18). This fact will be used to provide a check on the goodness of the approximations used to arrive at Eq. (45). Notice finally how in this approach, the smallness of the anomalous corrections is associated with the smallness of the parameter $a-1$, and with the fact that the amplitude of the intermittency source term is second order in this quantity, with respect to the source term of the Gaussian fluctuations.

A. Solution of the shell energy equation

It is possible to solve Eq. (45) either following,[23] by diagonalizing the LHS of (45), considered as a matrix equation, or using the multiplier technique adopted in Ref. 24. Here, the smallness of the parameter $a-1$ allows us to consider the continuous limit of Eq. (45) and to use a multiple scale expansion in which, to lowest order, the dependence on n produced by the $\exp(\gamma n)$ terms is neglected on the scale of the fluctuations. This allows a solution of the problem in terms of the Green's functions, in which no approximation on the form of the noise correlation is required. The Green's function for Eq. (45) is

$$g(n,n'-n,t)=\frac{\exp[-(n-n'-Ve^{\gamma n}t)^2/4De^{\gamma n}t+\gamma n/2]}{\sqrt{4\pi Dt}},\quad t>0. \tag{47}$$

The first quantity that we are going to calculate is the Gaussian part of the fluctuations:

$$\langle\phi_G^2\rangle=F\int\frac{dk}{2\pi}\frac{d\omega}{2\pi}|g_{k\omega}(n)|^2k^2\langle|\xi_{k\omega}|^2\rangle=\frac{F}{2D}\int\frac{dk}{2\pi}\frac{(1+Dk^2)\exp[-0.14/(a-1)|k|]}{1+(V^2+2D)k^2+D^2k^4}, \tag{48}$$

where $g_{k\omega}(n)=[-i(\omega e^{-\gamma n}-Vk)+Dk^2]^{-1}$ is the Fourier transform with respect to n' and t of $g(n,n',t)$. The expression for $\langle\phi_G^2\rangle$ obtained above descends from the Navier–Stokes dynamics via statistical closure and the neglect of the intermittency source $\sim-\gamma\hat{\xi}/2+\delta\xi$ in Eq. (46). However, $\langle\phi_G^2\rangle$ itself does not have any dynamic content, being simply the kurtosis of a Gaussian random field, in our case, the lowest-order approximation for the velocity field $\mathbf{v}^{(0)}$. From the relation $\langle\phi_G^2\rangle=\langle\langle|\mathbf{v}_n^{(0)}|^2\rangle_n^2\rangle/\langle|\mathbf{v}_n^{(0)}|^2\rangle^2$, where $\mathbf{v}_n$ indicates the contribution to $\mathbf{v}$ from Fourier modes in $\mathbf{S}_n$, and using Eq. (3), we obtain then, for small $a-1$,

$$\langle\phi_G^2\rangle=\frac{2}{E_n^{(0)2}}\int_n\frac{d^3k}{(2\pi)^3}C_k^2\int_n\frac{d^3q}{(2\pi)^3}|W_q(n)|^2\simeq 0.032\frac{(1-a^{-13/3})[c_w(a-1)]^3}{(1-a^{-2/3})^2}. \tag{49}$$

Clearly, the dynamical result of Eq. (48) and the kinematic one of Eq. (49) should coincide, if the quasi-Gaussian approximation for $\mathbf{v}$ that lies at the heart of the closure is valid. The two expressions for $\langle\phi_G^2\rangle$ are plotted in Fig. 5 against a; the best agreement, though still rather rough, is obtained for

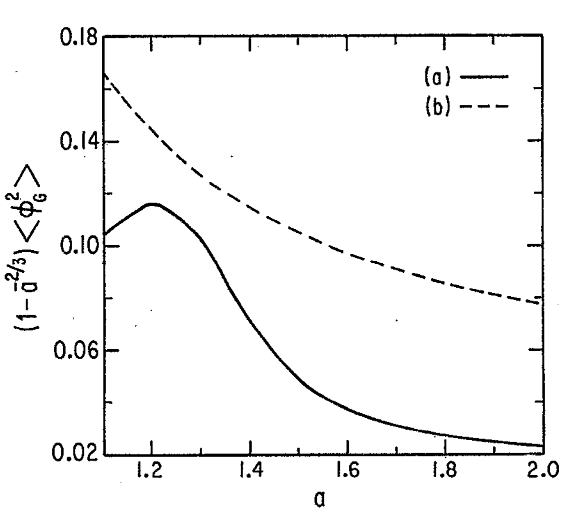

FIG. 5. Comparison of the Gaussian part of the fluctuation amplitude $\langle\langle v_l^2\rangle_k^2\rangle-\langle\langle v_l^2\rangle^2\rangle$, obtained from integration of Eq. (48) (a), and from the direct average of $v^{(0)}$ carried on in Eq. (49) (b).

the range $1.2\leq a\leq 1.3$, which is consistent with what would be expected by looking at the energy transfer profile (Fig. 2) and at the one for the correlation of the energy flux fluctuations (Fig. 3). In the same way, it is possible to calculate the correlation time for ϕ_G:

$$\tau_c=\frac{1}{\langle\phi_G^2\rangle}\int_0^\infty dt\langle\phi_G(0)\phi_G(t)\rangle=\frac{-iF}{\langle\phi_G^2\rangle}\int\frac{dk}{2\pi}\frac{d\omega}{2\pi}\frac{k^2}{\omega}|g_{k\omega}(n)|^2\langle|\xi_{k\omega}|^2\rangle. \tag{50}$$

The prediction from direct calculation, obtained from the generalization of Eq. (49) to two-time correlations, is $\tau_c(k_0)\simeq 1$, corresponding, in nonrescaled units, to $\tau_c(k)^{-1}=2\eta_k$; here we find that $0.9\leq\tau_c(k_0)\leq 1$ in the whole range $1.1\leq a\leq 2$.

Next we turn to the calculation of the intermittent part of the fluctuation ϕ. Care must now be taken due to the divergent nature of correlations at large n, which forbids, in particular, the use of the Fourier representation adopted in Eq. (48).

The equation giving the growth of $\langle\phi_n^2\rangle$ at large n is obtained by multiplying Eq. (45) by ϕ and taking the average,

$$\langle\phi_n[\exp(-\gamma n)\partial_t-D\Delta^2+V\Delta]\phi_n\rangle=F^{1/2}\langle\phi_n(\hat{\Delta}\xi_n+\delta\xi_n)\rangle. \tag{51}$$

We subtract from Eq. (51) the Gaussian part of the fluctuations, as given by Eq. (48); then the finite difference $\hat{\Delta}$ acts only on the part of the noise variation due to the scaling of the correlation time, which is of order $a-1$. The RHS of Eq. (51) can be calculated, expressing $\phi(n,t)$ as a convolution in terms of g and $\hat{\Delta}\xi+\delta\xi$, and keeping only the lowest order in $a-1$. This calculation is carried on explicitly in Appendix C, leading to the following result:

$$\Xi(a,c_w)=F\int_0^\infty dt\int dn'[g(n,n',t)[\gamma^2(t^2-t)+\beta] + g(n,n'-1,t)\beta' t]\langle\xi_n(0)\xi_{n+n'}(t)\rangle + \beta(D-V/2)\langle\phi_G^2\rangle, \tag{52}$$

where β and β' are $\mathscr{O}[(a-1)^2]$ coefficients whose expression is given by Eqs. (C4) and (C6). The various terms entering the expression for Ξ have the following origin.

(a) The one in γ^2 comes from the slow part of the noise derivative $\hat{\Delta}\xi$ (the source of intermittency in the model of Eggers).

(b) The term in β inside the integral comes from the contribution to $\delta\xi$ due to Eq. (44b) [part of the Beta Model effect from the noise term in Eq. (42)].

(c) The term in β' comes from cross-correlation between (a) and (b).

(d) The remaining term in β comes from the contribution to $\delta\xi$ from Eq. (44a) (relaxation part of the Beta Model effect).

Next, turn to the LHS of Eq. (51). The time derivative term is equal to zero at steady state, while it is shown in Appendix C that the intermittent part of the fluctuations does not contribute to the $\langle\phi\,\Delta^2\phi\rangle$ term. We are then left with $\langle\phi\,\Delta\phi\rangle\simeq\frac{1}{2}\Delta\langle\phi^2\rangle$, so that we obtain the result for the kurtosis scaling exponent:

$$2\zeta_2-\zeta_4\simeq\frac{1}{\ln a}\frac{d\langle\phi^2\rangle}{dn}=\frac{2\Xi(a,c_w)}{V(a)\ln a}. \tag{53}$$

The dependence of $2\zeta_2-\zeta_4$ on a for fixed c_w is shown in Fig. 6;[31] notice the saturation occurring at $a\simeq1.3$, suggesting that the bulk of intermittency production occurs at scales of the order of three to ten times the size of the eddies in exam. It should be kept in mind, however, that the approximations get worse as a becomes larger, so that the plateau for $a>1.3$ in Fig. 6 should be interpreted only as a qualitative indication. From inspection of Eqs. (43), (52), (53), (C5), and (C7), we see that, for small $a-1$, $2\zeta_2-\zeta_4=\mathscr{O}[c_w^3(a-1)^3]$, so that we obtain a direct connection between the smallness of the intermittency corrections and their being proportional to a rather large power of the small parameter $a-1$. Although the various contributions to $2\zeta_2-\zeta_4$ are all of the same order in $(a-1)c_w$, it appears that those associated with the Beta Model effect dominate the others, with (b) accounting for roughly 65% of $2\zeta_2-\zeta_4$, (d) for 25%, and the remaining (a) and (c), each for approximately 5% of the total.

The intermittency exponents defined here depend sensitively on the parameter c_w; due to the difficulty in determining c_w^{MAX} with precisions, it is therefore problematic to make accurate predictions on the scaling of $S(l,q)$ starting from $S(l,a,c_w,q)$. A "reasonable" guess could be to set $c_w^{\mathrm{MAX}}\simeq 2$, corresponding to the condition that the noise granularity be equal to the shell thickness $(a-1)k_n$, i.e., from Eqs. (19) and (29): $W^2_{\mathbf{p}+\mathbf{q}-\mathbf{k}}\sim\exp[-|\mathbf{p}+\mathbf{q}-\mathbf{k}|^2/2(k_{n+1}-k_n)^2]$. This leads to a value of the kurtosis exponent: $2\zeta_2-\zeta_4\simeq0.016$. However, choices as reasonable as the one just considered, lead to results differing from one another by up to an order of magnitude, so that the above formula should not be taken too seriously.

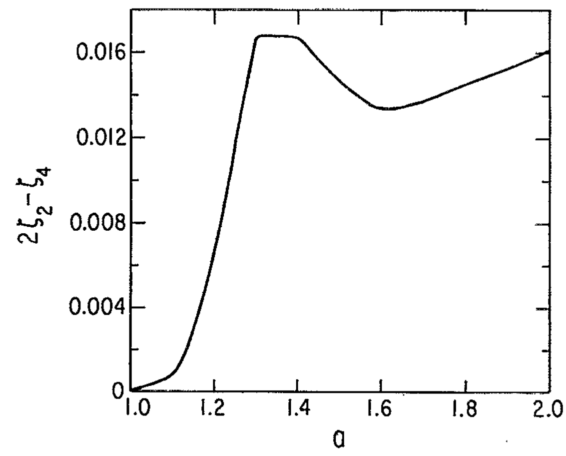

FIG. 6. Scaling correction to $S(l,a,c_w,4)$, as a function of a for $c_w=2$; notice the saturation occurring at $a\simeq1.3$.

V. SUMMARY AND CONCLUSIONS

We have described a model for the production of intermittency in the inertial range of three-dimensional turbulence, based on statistical closure of the Navier–Stokes equation. A connection between Navier–Stokes dynamics and phenomenological models like the Random Beta Model,[8] and the stochastic chains considered in,[23,24] is, in this way, established. Although this connection may be rather tenuous, because of the assumptions adopted in deriving the closure, it is still pleasing that the results in this paper are obtained as lowest-order corrections to a mean field approximation, which by itself would produce Kolmogorov scaling. It should also be mentioned that the present approach is able to produce dynamically, values of nonintermittent part of the energy fluctuations that are consistent with the assumption of quasi-Gaussian statistics of the velocity field; this is a bonus, which provides an indirect check on the goodness of the closure technique adopted. It also provides a suggestion for an alternative derivation of scaling corrections, in which the noise amplitude is determined by requiring that it produce a nonintermittent part of the shell energy fluctuation, which obeys Gaussian statistics.

The use of wave packets, rather than Fourier modes, is the reason why a perturbative treatment of intermittency has been possible here. Indeed there has been an intriguing aspect in this subject, namely, the difficulty in associating, to the smallness of the anomalous corrections, a small parameter in which to carry on perturbation theory. One of the results of the present model is the identification of this parameter with the physical quantity $\Delta k/k$, i.e., the ratio between eddy size and the scale of the characteristic flow straining the eddy. Scaling corrections appear to be of third order in this quantity, with the contributions from Random Beta Model kind of effects,[8] appearing to be dominant [although of the same order in $(a-1)c_w\sim\Delta k/k$], with respect to the mechanism of intermittency production of the model studied by Eggers.[23,24]

The main result of this paper justifies *a posteriori* the use of wave packets in the problem: the saturation in the a dependence of the generalized structure functions $S(l,a,c_w,q)$, for $a\geqslant 1.3$, which is consistent with a separation of scales between the production of intermittency and energy transfer. This is a definite prediction of the model, which, together with predictions on the actual magnitude of scaling corrections, could be tested by direct analysis of experimental data in terms of the generalized structure functions $S(l,a,c_w,q)$. This would extend similar studies, carried on by Meneveau[32] using wavelet analysis. It is important to stress that the shell spacing and wave packet extension parameters a and c_w do not play in this theory the role of adjustable constants: different choices of a and c_w are associated with the measurement of different physical: quantities the generalized structure functions $S(l,a,c_w,q)$ and the corresponding scaling exponents $\zeta_q=\zeta_q(a,c_w)$.

We have not been able yet to establish a quantitatively accurate relation between the two structure function definitions, $S(l,a,c_w,q)$ and $S(l,q)$, the reason being the sensitive dependence of $S(l,a,c_w,q)$ on $c_w=\Delta k/(a-1)k$, the ratio of wave packet to shell thickness in k space. Preliminary estimates hint toward a value for the scaling corrections smaller by a factor of the order of 5 than the experimental one. In the lognormal approximation, $\delta\zeta_q^{\text{theo}}/q(q-3)\sim-0.002$, which should be compared with the lognormal fit of experimental values: $\delta\zeta_q/q(q-3)\sim-0.01$.[2] Due to the problems just mentioned, it is not possible from our results to state whether inertial range processes are as important for intermittency production as finite size corrections, or if they are themselves just corrections to dominant finite size effects. We reiterate, however, that such an answer could be obtained analyzing experimental data in terms of $S(l,a,c_w,q)$ instead of $S(l,q)$.

Turning our attention to more formal issues, it is interesting to look for similarities between the model described in this paper and the various phenomenological approaches that have been used to study infinite Reynolds number intermittency. The basic ingredient here is the partition of Fourier space in shells of exponentially increasing radii. For this reason, our approach has a lot in common with the deterministic Shell Models studied, among others, by Yamada and Ohkitani[33] and by Jensen, Paladin, and Vulpiani.[34] An interesting interpretation of these models, particularly clear in Zimin,[35] comes from looking at the dynamical variable in each shell as the velocity of nested eddies, all located at the same space position, and interacting with one another locally in both scale and space. In our approach, such a dramatic restriction of phase space does not take place, in that the behavior of the various wave packets in a given shell is treated in an average sense; it is then possible that our model may underestimate the amount of intermittency produced in the inertial range. It remains to be seen how important this difference is; in principle, one should compare the present model with some closure for the Yamada–Ohkitani model or, *vice versa*, look for a deterministic dynamical system, whose closure coincides with the one described in this paper, and then compare with the Yamada–Ohkitani system.

The connection with the stochastic model of Eggers[23,24] is clearly simpler. In both cases one ends up with the same stochastic chain, the only difference being here the not exact conservation of energy transferred between shells. If this effect, due to the not exact overlapping of wave packets at different scales were neglected, the present model and the one of Eggers would coincide.

Going back to the possibility of turbulent Navier–Stokes dynamics taking place in a very limited region of phase space, this constitutes the main conceptual limitation of our model. The possibility of the preferential transfer of energy between individual wave packets would correspond to intermittency being associated with locally anisotropic fluctuations. It should be mentioned that this looks somewhat unlikely, due to the separation of scales described earlier and the consequent large number of eddies in one straining region Δk^{-1}. In any case, this is a possibility that could lead, in a suggestive way, to a mechanism for the creation of coherent structures. In Ref. 36, she derived anomalous exponents, based on assumptions of the shape of the most intense and rare dissipation regions, so that intermittency and coherent structures may be connected. It is difficult to see now shell models of either the deterministic or the stochastic variety, could be used to study such coherent structures, given the very distant scales involved in their dynamics. In any case, perhaps this is not a big problem, since, although vortex tubes have been observed in many numerical simulations,[3] they do not seem to contribute much to energy dissipation and their relevance to the turbulent dynamics has been questioned recently in Ref. 37.

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APPENDIX A: QUASI-LAGRANGIAN CLOSURE

The basic difficulty in dealing with Quasi-Lagrangian closures is that studying the Navier–Stokes dynamics in a reference frame moving with a single Lagrangian tracer, neglects the divergence of trajectories of different tracers. In a more refined closure scheme, this effect would be included by associating each point in a correlation function to a different tracer trajectory, like in the Lagrangian History DIA (LHDIA) of Kraichnan.[26] However, also in that theory, abridgements of the closure equations were necessary in the end, which were similar in their effect to neglecting the divergence of Lagrangian trajectories.

Here we try to implement this approximation in as much a consistent way as possible. Notice first that correlations are defined, starting from an initial case in which the initial point lies on the Lagrangian trajectory $\mathbf{z}_t$, so that Eq. (3) basically describes the decorrelation of points at distance r from the Lagrangian trajectory $\mathbf{z}_t$, with respect to points lying on it. Since decorrelation occurs forward in time, this is the optimal choice; choosing $\mathbf{z}_t$ as the final point and $\mathbf{z}_0+\mathbf{r}$ as the initial one, would lead, in particular, to no decorrelation, due to the advection term being zero in Eq. (2).

The approximation of Eq. (5) in which the initial point $\mathbf{z}_{t_1}+\mathbf{r}_1$ of the Green's function $G(t,\mathbf{r}|t_1,\mathbf{r}_1)$ is shifted on the Lagrangian trajectory, is justified with the previous choice. We try to make it more appealing by giving the next order in the expansion around $G(t,\mathbf{r}|t_1,\mathbf{r}_1)=G(\mathbf{r}-\mathbf{r}_1,t-t_1)$. First we have

$$\int d^3r\, e^{-i\mathbf{kr}}\langle v^\alpha(\mathbf{r}_0,0)v^\gamma[\mathbf{z}_t(\mathbf{r}_0,0)+\mathbf{r},t]\rangle$$
$$=\int d^3r\, e^{-i\mathbf{kr}}\int d^3s\, G^{\gamma\rho}(0,\mathbf{r}+\mathbf{s}|t,\mathbf{r})C^\alpha_\rho(\mathbf{r}-\mathbf{s})$$
$$=\int d^3r\int\frac{d^3k_1}{(2\pi)^3}\,G^{\gamma\rho}_{\mathbf{k}_1}(\mathbf{r},t)C^\alpha_{\mathbf{k}_1,\rho}e^{i(\mathbf{k}_1-\mathbf{k})\mathbf{r}},\tag{A1}$$

where $G^{\gamma\rho}_{\mathbf{k}_1}(\mathbf{r},t)$ is the Fourier transform of $G^{\gamma\rho}(0,\mathbf{r}+\mathbf{s}|t,\mathbf{r})$ with respect to $\mathbf{s}$. Expanding the argument of the integral in the last line of Eq. (A1) in $\mathbf{k}-\mathbf{k}_1$ and $\mathbf{r}$, we then obtain

$$\int d^3r\, e^{-i\mathbf{kr}}\langle v^\alpha(\mathbf{r}_0,0)v^\gamma[\mathbf{z}_t(\mathbf{r}_0,0)+\mathbf{r},t)\rangle$$
$$=G^{\gamma\rho}_{\mathbf{k}}(0,t)C^\alpha_{\mathbf{k},\rho}-\frac{1}{2}\frac{\partial}{\partial k^\phi}\frac{\partial}{\partial k^\psi}\frac{\partial}{\partial r_\phi}\frac{\partial}{\partial r_\psi}\times[G^{\gamma\rho}_{\mathbf{k}}(\mathbf{r},t)C^\alpha_{\mathbf{k},\rho}]_{\mathbf{r}=0}+\cdots.\tag{A2}$$

More, in general, we would obtain

$$\int d^3r\, e^{-i\mathbf{kr}}\langle v^\alpha[\mathbf{z}_{t_1}(\mathbf{r}_0,0)+\mathbf{r}_1,t_1]v^\gamma[\mathbf{z}_{t_1+t}(\mathbf{r}_0,0)+\mathbf{r}_1+\mathbf{r},t_1+t)\rangle=G^{\gamma\rho}_{\mathbf{k}}(\mathbf{r}_1,t)C^\alpha_{\mathbf{k},\rho}-\frac{1}{2}\frac{\partial}{\partial k^\phi}\frac{\partial}{\partial k^\psi}\frac{\partial}{\partial r_\phi}\frac{\partial}{\partial r_\psi}\times[G^{\gamma\rho}_{\mathbf{k}}(\mathbf{r}_1+\mathbf{r},t)C^\alpha_{\mathbf{k},\rho}]_{\mathbf{r}=0}+\cdots,\tag{A3}$$

which can be then be expanded in Taylor series around $\mathbf{r}_1=0$. We see that no small parameters are present in this expansion, and for this reason the present closure cannot be expected to lead to quantitatively accurate results.

We turn next to the derivation of the energy balance equation in the laboratory frame, Eq. (10). We consider just one term:

$$-\frac{1}{(2\pi)^3}\int d^3r_1\,d^3r_2\,d^3r_3\,e^{-i(\mathbf{kr}_1-\mathbf{pr}_2-\mathbf{qr}_3)}\langle v^{(0)\alpha}(\mathbf{r}_1,0)v^{(0)}_\beta(\mathbf{r}_2,0)\partial^\beta v^{(1)}_\alpha(\mathbf{r}_3,0)\rangle$$
$$=\frac{1}{(2\pi)^3}\int d^3r_1\,d^3r_2\,d^3r_3\,e^{-i(\mathbf{kr}_1-\mathbf{pr}_2-\mathbf{qr}_3)}\int d^3s\int_{-\infty}^0 d\tau\, G^\rho_\alpha(\mathbf{s},-\tau)\langle v^\alpha(\mathbf{r}_1,0)\mathbf{v}_\beta(\mathbf{r}_2,0)\partial^\beta[v^\sigma(\mathbf{z}_\tau+\mathbf{r}_3+\mathbf{s},\tau)-\hat{w}v^\sigma(\mathbf{z}_\tau,\tau)]\partial_\sigma v_\rho(\mathbf{z}_\tau+\mathbf{r}_3+\mathbf{s},\tau)\rangle.\tag{A4}$$

Splitting the four-point correlation into two-point ones, we obtain, in terms of Fourier transforms,

$$2\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\int d^3r_2\,d^3r_3\int d^3k_1\,d^3k_2\,d^3k_3\,\delta(\mathbf{k}+\mathbf{p}+\mathbf{k}_2-\mathbf{k}_3)\theta_{k_1k_2k_3}C_{k_2}C_{k_3}\{B_4(k_1k_2k_3)[\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\delta(\mathbf{k}+\mathbf{q}+2\mathbf{k}_2+\mathbf{k}_3)-w\,\delta(\mathbf{k}_1-\mathbf{k}_3)\delta(\mathbf{k}+\mathbf{q}+\mathbf{k}_2+\mathbf{k}_3)]+B_2(k_1k_2k_3)[\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\delta(\mathbf{k}+q+2\mathbf{k}_2+\mathbf{k}_3)-w\,\delta(\mathbf{k}_1-\mathbf{k}_2)\delta(\mathbf{k}+\mathbf{q}+2\mathbf{k}_2)]\},\tag{A5}$$

where

$$B_2(k_1k_2k_3)=P^{\alpha\rho}(k_1)P_{\alpha\rho}(k_2)k_1^\beta P_{\beta\sigma}(k_3)k_2^\sigma=k_1k_2(1+z^2)(z+xy)\tag{A6}$$

and

$$B_4(k_1k_2k_3)=k_2^\sigma P_{\sigma\alpha}(k_1)P^{\alpha\rho}(k_2)P_{\rho\beta}(\mathbf{k}_3)k_1^\beta=-k_1k_2xz(x+yz).\tag{A7}$$

The various terms in Eq. (A5) are tracked back to Eq. (A4) as follows: the terms in B_4 come from the contraction $\langle v^\alpha(v^\sigma-wv^\sigma)\rangle\partial^\beta G^\rho_\alpha\langle v_\beta\,\partial_\sigma v_\rho\rangle$; those in B_2, from the remaining contraction; the terms in w come from the shift to the Lagrangian frame. Notice now that, from Eqs. (A6)–(A7): $B_2(k_1k_1k_3)=B_4(k_1k_2k_1)=0$. Thus, the terms in w in Eq. (A5) disappear, and one is left with the same expression that would be obtained from the Eulerian DIA,[25] but with the Lagrangian response time θ. Repeating the same calculation with the other two choices for $v^{(1)}$, we see that no terms in w contribute and we obtain the standard result of Eq. (10).

We pass to the calculation of the two-time Lagrangian correlation: at steady state the part of the time integrals from $\tau<0$ do not contribute, and we have

$$-\int d^3r\,e^{-i\mathbf{kr}}\langle v^\alpha(\mathbf{r}_0,0)[v^\beta(\mathbf{z}_t+\mathbf{r},t)-\hat{w}v^\beta(\mathbf{z}_t,t)]\partial_\beta v_\alpha(\mathbf{z}_t+\mathbf{r},t)\rangle$$
$$=\int d^3r\,e^{-i\mathbf{kr}}\int_0^t d\tau\int d^3s(G^\beta_\rho(\mathbf{s},t-\tau)\{[C^{\alpha\sigma}(-\mathbf{r}-\mathbf{s},\tau)-\hat{w}C^{\alpha\sigma}(0,\tau)]\partial_\sigma\,\partial_\beta C^\rho_\alpha(-\mathbf{s},t-\tau)+\partial_\beta[C^\sigma_\alpha(-\mathbf{s},t-\tau)-\hat{w}C^\sigma_\alpha(\mathbf{r},t-\tau)]\partial_\sigma C^{\rho\alpha}(-\mathbf{r}-\mathbf{s},\tau)\}-\hat{w}G^\beta_\rho(\mathbf{s},t-\tau)\{[C^{\alpha\sigma}(-\mathbf{s},\tau)-\hat{w}C^{\alpha\sigma}(0,\tau)]\partial_\sigma\,\partial_\beta C^\rho_\alpha(\mathbf{r}-\mathbf{s},t-\tau)+\partial_\beta[C^\sigma_\alpha(\mathbf{r}-\mathbf{s},t-\tau)-\hat{w}C^\sigma_\alpha(\mathbf{r},t-\tau)]\partial_\sigma C^{\rho\alpha}(-\mathbf{s},\tau)\}+\partial_\beta G_{\alpha\rho}\{[C^{\alpha\sigma}(-\mathbf{r}-\mathbf{s},\tau)-\hat{w}C^{\alpha\sigma}(0,\tau)]\partial_\sigma[C^\beta_\rho(-\mathbf{s},t-\tau)-\hat{C}^\beta_\rho(-\mathbf{r}-\mathbf{s},t-\tau)]$$

$$+\partial_\sigma C^{\alpha\rho}(-\mathbf{r}-\mathbf{s},\tau)[C^{\beta\sigma}(-\mathbf{s},t-\tau)-\hat{w}C^{\beta\sigma}(\mathbf{r},t-\tau)-\hat{w}C^{\beta\sigma}(-\mathbf{r}-\mathbf{s},t-\tau)+\hat{w}\hat{w}C^{\beta\sigma}(0,t-\tau)]). \quad (A8)$$

Of the terms on the RHS of Eq. (A8), there is a group that does not depend on the integration variable **s**; these terms come from integrating along the Lagrangian trajectory and lead, after Fourier transform, to triads in which one of the wave vectors is zero; as before, they do not contribute to the final result. The remaining terms in $\hat{w}$ come from the $\hat{w}$ on the LHS of the equation, which express the fact that we are dealing with a Lagrangian correlation. These terms remain and are responsible for the cancellation of the sweep part of the correlation decay. After Fourier transform, we obtain the result

$$\int_0^t d\tau \int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\,\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\{B_3(kpq)G_p(t-\tau)C_k(\tau)C_q(t-\tau)-w_pB_3(q,-p,k)G_p(t-\tau)C_k(t-\tau)C_q(\tau)]$$
$$+[B_1(kpq)G_p(t-\tau)C_k(\tau)C_q(t-\tau)-w_pB_1(q,-p,k)G_p(t-\tau)C_k(t-\tau)C_q(\tau)]+[B_4(kpq)G_p(t-\tau)C_k(\tau)C_q(t-\tau)$$
$$-w_qB_4(p,k,-q)G_k(t-\tau)C_p(\tau)C_q(t-\tau)]+B_2(kpq)G_p(t-\tau)C_k(\tau)C_q(t-\tau)-w_qB_2(p,k,-q)G_k(t-\tau)C_p(\tau)C_q(t$$
$$-\tau)]\}, \quad (A9)$$

where

$$B_1(kpq)=p_\sigma P^{\sigma\alpha}(q)P_{\alpha\rho}(k)P^{\rho\beta}(p)k_\beta=-kpyz(y+xz) \quad (A10)$$

and

$$B_3(kpq)=k_\beta P^{\beta\rho}(p)P_{\rho\alpha}(q)P^{\alpha\sigma}(k)p_\sigma=kpxy(1-z^2). \quad (A11)$$

(The same notation of Leslie[27] is used here for the functions B_i, $i=1,...,4$.) The terms on the RHS of Eq. (A9) are ordered as on the RHS of Eq. (A8), once the terms of the last one, which are equal to zero, are eliminated. Using the following relations:

$$B_3(q,-p,k)=-B_1(kpq);\quad B_1(q,-p,k)=-B_3(kpq),$$

$$B_4(p,k,-q)=B_1(kpq);\quad B_2(p,k,-q)=B_2(kpq), \quad (A12)$$

and the definitions

$$b_{kpq}=\frac{1}{2k^2}\sum_i B_i(kpq);\quad b^{(1)}_{kpq}=\frac{1}{2k^2}[B_1(kpq)+B_3(kpq)];$$

$$b^{(2)}_{kpq}=\frac{1}{2k^2}[B_1(kpq)+B_2(kpq)], \quad (A13)$$

and substituting into Eq. (A9), we obtain immediately the result of Eq. (11).

Finally, we derive the energy equation in the moving reference frame. Using conservation of energy triad by triad (which is preserved, together with incompressibility, when passing to the Lagrangian reference frame), we can write

$$D_tC_k(t)|_{t=0}=\frac{1}{4\pi}\int_\Delta dp\,dq\,kpq\,\theta_{kpq}[-H(kpq)+H(pkq)+H(qkp)], \quad (A14)$$

where

$$H(kpq)=[b_{kpq}+w_pb^{(1)}_{kpq}]C_kC_q-w_qb^{(2)}_{kpq}C_pC_q. \quad (A15)$$

We immediately obtain

$$D_tC_k(t)|_{t=0}=\frac{1}{4\pi}\int_\Delta dp\,dq\,kpq\,\theta_{kpq}\left\{(b_{kpq}+w_qb^{(3)}_{kpq})C_pC_q-\left[b_{kpq}+\left(\frac{p}{k}\right)^2w_qb^{(3)}_{p,k,-q}\right]C_kC_q\right\}, \quad (A16)$$

where

$$b^{(3)}_{kpq}=\frac{1}{2k^2}[2B_2(kpq)+B_1(kpq)+B_4(kpq)]. \quad (A17)$$

A form of Eq. (A15), which is manifestly zero at equipartition, can be obtained using the relations

$$B_2(kpq)=B_2(p,k,-q)\quad\text{and}\quad B_1(kpq)=B_4(p,k,-q). \quad (A18)$$

Substituting Eqs. (A16)–(A17) into Eq. (A15), we finally obtain the result shown in Eq. (15).

APPENDIX B: NOISE CORRELATION

In order to calculate the amplitude of the energy flux fluctuation $g_n(m)$, we must evaluate the contractions of the product:

$$v^\alpha(\mathbf{z}_1)[v^\beta(\mathbf{z}_1+\mathbf{r}_1)-v^\beta(\mathbf{z}_1)]\partial_\beta v^\alpha(\mathbf{z}_1+\mathbf{r}_1)v^\gamma(\mathbf{z}_2)$$
$$\times[v^\sigma(\mathbf{z}_2+\mathbf{r}_2)-v^\sigma(\mathbf{z}_2)]\partial_\sigma v^\gamma(\mathbf{z}_2+\mathbf{r}_2), \quad (B1)$$

which is in the form: $X\,\Delta Y\,ZX'\,\Delta Y'\,Z'$. (In the equation above, $\mathbf{v}(\mathbf{z}_{1,2}+\mathbf{r}_{1,2})$ is shorthand for $\mathbf{v}[\mathbf{z}_{t_{1,2}}(\mathbf{r}_0,t_0)+\mathbf{r}_{1,2},t_{1,2}]$.) These contractions are divided into three groups and are listed in Tables I–III. In this section, we shall adopt the notation $\psi_\alpha\equiv\partial_\alpha\psi$.

Only the contractions in the third group appear to contribute to the noise amplitude. The ones in the first group (Table I) are identically zero because of incompressibility. To prove that the ones in Table II do not contribute either, we have to work a little bit more. These contractions are in the form $\langle g(0)g(t)\rangle\to\langle v(0)v(0)\rangle\langle v(0)v(t)\rangle\langle v(t)v(t)\rangle$, where $\langle v(0)v(t)\rangle$ is associated with velocity fluctuations on the scale of the wave packet radius R. These terms come from noise components of g in the form $v\langle(v\nabla)v\rangle$, which are identically zero, and $(v\nabla)\langle v^2/2\rangle$, which are associated with sweep and do not contribute to energy transfer. It is possible to show explicitly that these contractions do not contribute to $\langle g^2\rangle$. Consider, for example, contraction (B2.2) in Table II:

TABLE I. First group of contractions for the product (B1); these terms are identically zero because of incompressibility.

$$\langle X\,\Delta Y\rangle\langle ZX'\rangle\langle\Delta Y'\,Z'\rangle =[C^{\alpha\beta}(\mathbf{r}_1 0)-C^{\alpha\beta}(00)]C^{\alpha\gamma}_{\beta}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2-\mathbf{r}_2,t)\times[C^{\sigma\gamma}_{\sigma}(00)-C^{\sigma\gamma}_{\sigma}(\mathbf{r}_2 0)]; \tag{B1.1}$$

$$\langle XZ\rangle\langle\Delta Y\,X'\rangle\langle\Delta Y'\,Z'\rangle =C^{\alpha\alpha}_{\beta}(\mathbf{r}_1 0)[C^{\beta\gamma}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2,t)-C^{\beta\gamma}(\mathbf{z}_1-\mathbf{z}_2,t)]\times[C^{\alpha\gamma}_{\sigma}(0,0)-C^{\alpha\gamma}_{\sigma}(\mathbf{r}_2 0)]; \tag{B1.2}$$

$$\langle XX'\rangle\langle\Delta Y\,Z\rangle\langle\Delta Y'\,Z'\rangle =C^{\alpha\gamma}(\mathbf{z}_1-\mathbf{z}_2,t)[C^{\beta\alpha}_{\beta}(\mathbf{r}_1 0)-C^{\beta\alpha}_{\beta}(00)][C^{\sigma\gamma}_{\sigma}(00)-C^{\sigma\gamma}_{\sigma}(\mathbf{r}_2 0)]; \tag{B1.3}$$

$$\langle X\,\Delta Y'\rangle\langle\Delta Y\,Z\rangle\langle X'Z'\rangle =[C^{\alpha\sigma}(\mathbf{z}_1-\mathbf{z}_2-\mathbf{r}_2,t)-C^{\alpha\sigma}(\mathbf{z}_1-\mathbf{z}_2,t)]\times[C^{\beta\alpha}_{\beta}(00)-C^{\beta\alpha}_{\beta}(\mathbf{r}_1 0)]C^{\gamma\gamma}_{\sigma}(\mathbf{r}_2 0); \tag{B1.4}$$

$$\langle XZ'\rangle\langle\Delta Y\,Z\rangle\langle X'\,\Delta Y'\rangle =C^{\alpha\gamma}_{\sigma}(\mathbf{z}_1-\mathbf{z}_2-\mathbf{r}_2,t)[C^{\beta\alpha}_{\beta}(00)-C^{\beta\alpha}_{\beta}(\mathbf{r}_1 0)][C^{\gamma\sigma}(\mathbf{r}_2 0)-C^{\gamma\sigma}(00)]. \tag{B1.5}$$

TABLE II. Second group of contractions for the product (B1); these terms are associated with sweep and appear not to contribute to the noise amplitude.

$$\langle X\,\Delta Y\rangle\langle ZZ'\rangle\langle X'\,\Delta Y'\rangle =[C^{\alpha\beta}(\mathbf{r}_1 0)-C^{\alpha\beta}(00)]C^{\alpha\gamma}_{\beta\sigma}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2-\mathbf{r}_2,t)\times[C^{\sigma\gamma}(\mathbf{r}_2 0)-C^{\sigma\gamma}(00)]; \tag{B2.1}$$

$$\langle X\,\Delta Y\rangle\langle Z\,\Delta Y'\rangle\langle X'Z'\rangle \times[C^{\alpha\beta}(\mathbf{r}_1 0)-C^{\alpha\beta}(00)][C^{\alpha\sigma}_{\beta}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2-\mathbf{r}_2,t)-C^{\alpha\sigma}_{\beta}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2,t)]C^{\gamma\gamma}_{\sigma}(\mathbf{r}_2 0); \tag{B2.2}$$

$$\langle XZ\rangle\langle\Delta Y\,\Delta Y'\rangle\langle X'Z'\rangle =C^{\alpha\alpha}_{\beta}(\mathbf{r}_1 0)[C^{\beta\sigma}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2-\mathbf{r}_2,t)-C^{\beta\sigma}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2,t)-C^{\beta\sigma}(\mathbf{z}_1-\mathbf{z}_2-\mathbf{r}_2,t)+C^{\beta\sigma}(\mathbf{z}_1-\mathbf{z}_2,t)]C^{\gamma\gamma}_{\sigma}(\mathbf{r}_2,0); \tag{B2.3}$$

$$\langle XZ\rangle\langle\Delta Y\,Z'\rangle\langle X'\,\Delta Y'\rangle =C^{\alpha\alpha}_{\beta}(\mathbf{r}_1 0)[C^{\beta\gamma}_{\sigma}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2-\mathbf{r}_2,t)-C^{\beta}_{\sigma}(\mathbf{z}_1-\mathbf{z}_2-\mathbf{r}_2,t)][C^{\gamma\sigma}(\mathbf{r}_2 0)-C^{\gamma\sigma}(00)]. \tag{B2.4}$$

$$\begin{aligned}
&\langle g_{n_1}(m_1,t_1)g_{n_2}(m_2,t_2)\rangle^{(2.2)}\\
&=\int d^3r_1\,d^3r_2\,H(r_1,n_1)H(r_2,n_2)\int d^3z_1\,d^3z_2\,W(z_1,m_1)W(z_2,m_2)[C^{\alpha\beta}(\mathbf{r}_1,0)-\hat{w}C^{\alpha\beta}(0,0)][C^{\alpha\sigma}_{\beta}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2-\mathbf{r}_2,t)\\
&\quad-\hat{w}C^{\alpha\sigma}_{\beta}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2,t)]C^{\gamma\gamma}_{\sigma}(\mathbf{r}_2,0)\\
&=2\int\frac{d^3k}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\,C_kC_pC_q\exp(-\eta_p|t|)W_{\mathbf{p}}(m_1)W_{\mathbf{p}}(m_2)\left(\frac{(\mathbf{kp})^2(\mathbf{pq})}{k^2p^2}-\frac{(\mathbf{kp})(\mathbf{kq})}{k^2}\right)\{[2-w_k(k+p)]\\
&\quad\times H(k+p-k_{n_1})-w_k(k+p)H(p-k_{n_1})\}\{[2-w_p(p+q)]H(p+q-k_{n_2})-w_p(k)H(q-k_{n_2})\}.
\end{aligned}\tag{B2}$$

The geometric term in curly brackets is antisymmetric in $\mathbf{q}$, so that that only nonzero contributions are proportional to $H(p+q-k_{n_2})$. We change first variables, $q+p\to q$, so that $H(p+q-k_{n_2})\to H(q-k_{n_2})$. Because of the factor $W_{\mathbf{p}}(m_1)W_{\mathbf{p}}(m_2)$, we can expand in a power series in p,

$$-k_2^\alpha\partial_{k_3^\alpha}\left[(kq)^{-5/3}\left(\frac{(\mathbf{kp})^2(\mathbf{pq})}{k^2p^2}-\frac{(\mathbf{kp})(\mathbf{kq})}{k^2}\right)\right]=-\frac{5}{3}(kq)^{-5/3}p^2(xyz-x^2z^2). \tag{B3}$$

Indicating by u the cosine of the angle between the planes $\mathbf{kp}$ and $\mathbf{kq}$, we have

$$xyz-x^2z^2=uxz\sqrt{(1-x^2)(1-z^2)},$$

so that, substituting Eq. (B3) into (B2), gives zero. Going to next order produces a factor p^2x^2, which gives zero again, while still the next order leads to a contribution, which is by a factor $(kR)^{2/3}$ smaller than the terms in Table III.

Table III lists all the other contractions, which are in the form $\langle v(0)v(t)\rangle\langle v(0)v(t)\rangle\langle v(0)v(t)\rangle$, act on the same scale of k_1 and k_2, and give a finite contribution to the variance of g. The calculation of the terms in this group is very tedious but straightforward and follows the same lines of Eqs. (28) and (B2). We conclude by giving the exact form of the coefficients entering the expression for the fluctuation amplitude $\langle g^2\rangle$ [Eq. (30)]:

$$B_1(kpq)=-pkyz(y+xz),$$

$$B_2(kpq)=kp(1+z^2)(z+xy),B_3(kpq)=kpxy(1-z^2); \tag{B4}$$

$$\begin{aligned}
H_1=&\{[2-w_p(k)]H(k-k_{n_1})-H(q-k_{n_1})w_p(k)\}\\
&\times([[2-w_q(k)]H(k-k_{n_2})-H(p-k_{n_2})w_q(k)]\\
&-\{[2-w_q(p)]H(p-k_{n_2})-H(k-k_{n_2})w_q(p)\});
\end{aligned}\tag{B5}$$

TABLE III. Group of contractions for the product (B1) contributing to the noise correlation $\langle g_n(m)^2\rangle$.

$$\langle XX'\rangle\langle\Delta Y\,\Delta Y'\rangle\langle ZZ'\rangle=C^{\alpha\gamma}(\mathbf{z}_1-\mathbf{z}_2,t)[C^{\beta\sigma}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2-\mathbf{r}_2,t)-C^{\beta\sigma}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2,t)-C^{\beta\sigma}(\mathbf{z}_1-\mathbf{z}_2-\mathbf{r}_2,t)+C^{\beta\sigma}(\mathbf{z}_1-\mathbf{z}_2,t)]C^{\alpha\gamma}_{\beta\sigma}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2-\mathbf{r}_2,t); \tag{B3.1}$$

$$\langle XX'\rangle\langle\Delta Y\,Z'\rangle\langle Z\,\Delta Y'\rangle=C^{\alpha\gamma}(\mathbf{z}_1-\mathbf{z}_2,t)[C^{\beta\gamma}_{\sigma}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2-\mathbf{r}_2,t)-C^{\beta\gamma}_{\sigma}(\mathbf{z}_1-\mathbf{z}_2-\mathbf{r}_2,t)][C^{\alpha\sigma}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2-\mathbf{r}_2,t)-C^{\alpha\sigma}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2,t)]; \tag{B3.2}$$

$$\langle X\,\Delta Y'\rangle\langle\Delta Y\,X'\rangle\langle ZZ'\rangle=[C^{\alpha\sigma}(\mathbf{z}_1-\mathbf{z}_2-\mathbf{r}_2,t)-C^{\alpha\sigma}(\mathbf{z}_1-\mathbf{z}_2,t)][C^{\beta\gamma}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2,t)-C^{\beta\gamma}(\mathbf{z}_1-\mathbf{z}_2,t)]C^{\alpha\gamma}_{\beta\sigma}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2-\mathbf{r}_2,t); \tag{B3.3}$$

$$\langle X\,\Delta Y'\rangle\langle\Delta Y\,Z'\rangle\langle ZX'\rangle=[C^{\alpha\sigma}(\mathbf{z}_1-\mathbf{z}_2-\mathbf{r}_2,t)-C^{\alpha\sigma}(\mathbf{z}_1-\mathbf{z}_2,t)][C^{\beta\gamma}_{\sigma}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2-\mathbf{r}_2,t)-C^{\beta\gamma}_{\sigma}(\mathbf{z}_1-\mathbf{z}_2-\mathbf{r}_2,t)]C^{\alpha\gamma}_{\beta}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2,t); \tag{B3.4}$$

$$\langle XZ'\rangle\langle\Delta Y\,X'\rangle\langle Z\,\Delta Y'\rangle=C^{\alpha\gamma}_{\sigma}(\mathbf{z}_1-\mathbf{z}_2-\mathbf{r}_2,t)[C^{\beta\gamma}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2,t)-C^{\beta\gamma}(\mathbf{z}_1-\mathbf{z}_2,t)][C^{\alpha\sigma}_{\beta}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2-\mathbf{r}_2,t)-C^{\alpha\sigma}_{\beta}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2,t)]; \tag{B3.5}$$

$$\langle XZ'\rangle\langle\Delta Y\,\Delta Y'\rangle\langle ZX'\rangle=C^{\alpha\gamma}_{\sigma}(\mathbf{z}_1-\mathbf{z}_2-\mathbf{r}_2,t)[C^{\beta\sigma}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2-\mathbf{r}_2,t)-C^{\beta\sigma}(\mathbf{z}_1-\mathbf{z}_2-\mathbf{r}_2,t)-C^{\beta\sigma}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2,t)+C^{\beta\sigma}(\mathbf{z}_1-\mathbf{z}_2,t)]C^{\alpha\gamma}_{\beta}(\mathbf{z}_1+\mathbf{r}_1-\mathbf{z}_2,t). \tag{B3.6}$$

$$H_2=\{[2-w_p(k)]H(k-k_{n_1})-H(q-k_{n_1})w_p(k)\}$$
$$\times(\{[2-w_p(k)]H(k-k_{n_2})-H(q-k_{n_2})w_p(k)\}$$
$$-\{[2-w_p(q)]H(q-k_{n_2})-H(k-k_{n_2})w_p(q)\}); \tag{B6}$$

$$H_3=\{[2-w_p(k)]H(k-k_{n_1})-H(q-k_{n_1})w_p(k)\}$$
$$\times(\{[2-w_k(q)]H(q-k_{n_2})-H(p-k_{n_2})w_k(q)\}$$
$$-\{[2-w_k(p)]H(p-k_{n_2})-H(q-k_{n_2})w_k(p)\}). \tag{B7}$$

APPENDIX C: COEFFICIENTS FOR THE SHELL ENERGY EQUATION

The first quantity that we are going to study is the RHS of Eq. (51). Substituting ϕ by its expression in terms of the Green's function $g(n,n',t)$ and $\hat{\Delta}\xi+\delta\xi$, we can write

$$F^{1/2}\langle\phi(\hat{\Delta}\xi+\delta\xi)\rangle=F\int_0^\infty dt\int dn'\ g(n,n'-n,t)\langle[\hat{\Delta}\xi_{n'}(t)+\delta\xi_{n'}(t)][\hat{\Delta}\xi_n(0)+\delta\xi_n(0)]\rangle, \tag{C1}$$

and, following Eqs. (44a)–(44b), we decompose $\delta\xi=\delta\xi^a+\delta\xi^b$, with

$$\delta\xi_n^a=F^{-1/2}(-D+V/2)[\phi_{n+1}(n+1)-\phi_{n+1}(n)], \tag{C2a}$$

$$\delta\xi_n^b=\xi_{n+1}(n+1)-\xi_{n+1}(n). \tag{C2b}$$

We have to study Eq. (C1) for small n, in order for the intermittent part of ϕ to be small compared to ϕ_G. There are several terms in Eq. (C1). The dominant one comes from the part of $\langle\hat{\Delta}\xi\,\hat{\Delta}\xi\rangle$ due to the fast variation of ξ, which has the form $-\partial_{n'}^2\langle\xi_n\xi_{n+n'}\rangle$. This term, which was taken into account by Eq. (48), is not associated with the secular behavior of $\langle\phi_n^2\rangle$, and is subtracted out of Eq. (C1). The part due to the slow variation of ξ is responsible for the kind of intermittency found in Ref. 23, and, from Eq. (43), has the form $\partial_n^2\langle\xi_n(0)\xi_{n+n'}(t)\rangle=\gamma^2(t^2-|t|)\langle\xi_n(0)\xi_{n+n'}(t)\rangle$. The corresponding contribution to Eq. (C1) is therefore

$$F\int_0^\infty dt\int dn'\ g(n,n'-n,t)\gamma^2(t^2-t)\langle\xi_n(0)\xi_{n'}(t)\rangle. \tag{C3}$$

The first piece of the contribution due to Beta Model effects comes from the correlation $\langle\delta\xi^b\ \delta\xi^b\rangle$, and has the form, from Eq. (C2b),

$$F\int_0^\infty dt\int dn'\ g(n,n'-n,t)\langle[\xi_n(n,0)-\xi_n(n-1,0)]\times[\xi_{n'}(n',t)-\xi_{n'}(n'-1,t)]\rangle$$
$$=F\beta\int_0^\infty dt\int dn'\ g(n,n'-n,t)\langle\xi_n(0)\xi_{n'}(t)\rangle; \tag{C4}$$

where β is obtained using Eq. (29):

$$\beta=1+a^{-3}-2^{5/2}(1+a^2)^{-3/2}. \tag{C5}$$

The next term is the cross-correlation between the Beta Model effect and the time scale mismatch of different shells; it comes from $\langle\hat{\Delta}\xi\,\delta\xi^b\rangle$ after elimination of the fast part of $\hat{\Delta}$:

$$2F\int_0^\infty dt\int dn'\ g(n,n',t)\partial_n\langle\xi_{n+n'-1}(n,t)\times[\xi_n(n,0)-\xi_n(n-1,0)]\rangle$$
$$=F\beta'\int_0^\infty dt\int dn'\ g(n,n'-n,t)\langle\xi_n(0)\xi_{n'-1}(t)\rangle, \tag{C6}$$

where, again from Eq. (29),

$$\beta'=2\gamma\left[1-\left(\frac{1+a^{-2}}{2}\right)^{3/2}\right]. \tag{C7}$$

Finally, there is the correction to the relaxation terms of the shell equation, from the Beta Model effect, which comes from the product $\langle\delta\xi^a\ \delta\xi^a\rangle$ in Eq. (C1); using Eq. (C2a),

$$\left(D-\frac{V}{2}\right)\langle[\phi_n(n,0)-\phi_n(n-1,0)][\phi_n(n,t)-\phi_n(n-1,t)]\rangle$$
$$=\beta\left(D-\frac{V}{2}\right)\langle\phi^2\rangle. \tag{C8}$$

For small n, the intermittent part of ϕ has not yet had the time to build up, and one can approximate $\langle\phi^2\rangle\simeq\langle\phi_G^2\rangle$. Putting together Eqs. (C3), (C4), and (C6) then leads to the result of Eq. (52). All terms containing a single factor $\hat{\Delta}^{\text{fast}}\xi$ can be shown not to contribute to Eq. (C1). For the same reason, no cross-correlations between $\delta\xi^a$ and the other terms are present to lowest order, due to the origin of ϕ_G from $\partial_{n'}\xi$ rather than from ξ.

The result of Eq. (52) can be further simplified by calculating explicitly the time integral. Using the formula $\int_0^\infty dt\exp(-b/t^2-ct^2)=\sqrt{\pi/4c}\exp(-2\sqrt{bc})$,[38] the integral term in Eq. (52) can be rewritten in the form

$$\int dn'\left\{\langle\xi_0(0)\xi_{n'}(0)\rangle\left[\gamma^2\left(\frac{\partial^2}{\partial c^2}+\frac{\partial^2}{\partial c}\right)+\beta\right]+\beta'\langle\xi_0(0)\xi_{n'-1}(0)\rangle\frac{\partial}{\partial c}\right\}\sqrt{\frac{\pi a'^2}{4c^2}}\exp(d-2\sqrt{bc}), \tag{C9}$$

where

$$a'=\frac{1}{\sqrt{\pi D}};\quad b=\frac{n'^2}{4D};$$
$$c=\frac{V}{4D}+\exp(\gamma n');\quad d=\frac{n'V}{2D}. \tag{C10}$$

Writing explicitly,

$$\int dn'\left\{\left[\gamma^2\left(\frac{32D^2}{(V^2+4D)^2}+\frac{6D|n'|}{(V^2+4D)^{3/2}}+\frac{n'^2-|n'|\sqrt{4D+V^2}-4D}{V^2+4D}\right)+\beta\right]\frac{1}{1+50(a-1)^2n'^2}\right.$$

$$+\beta' \frac{4D+|n'|\sqrt{4D+V^2}}{V^2+4D} \frac{1}{1+50(a-1)^2(n'-1)^2}\Bigg\}$$
$$\times \frac{F}{\sqrt{V^2+4D}} \exp\left(\frac{1}{2D}(n'-|n'|\sqrt{V^2+4D})\right). \quad \text{(C11)}$$

The final item on the agenda is the evaluation of $\langle \phi\, \Delta^2 \phi\rangle$ on the LHS of Eq. (51). The part of this term associated with intermittency does not receive contributions from noise derivatives with respect to n, and is in the form

$$\langle \phi(n,t) \Delta^2 \phi(n,t)\rangle$$
$$=\beta'' F \int_{-\infty}^{t} dt_1\, dt_2 \int dn_1\, dn_2\, g(n-n_1,t-t_1)\partial_n^2 g$$
$$\times (n-n_2,t-t_2)\langle \xi_{n_1}(t_1)\xi_{n_2}(t_2)\rangle, \quad \text{(C12)}$$

where β'' is an $\mathcal{O}[(a-1)^2]$ coefficient in the same form as γ^2, β, and β'. Writing the expression for $\langle\xi\xi\rangle$, shown in the last line of Eq. (43) as $\langle \xi_{n_1}(t_1)\xi_{n_2}(t_2)\rangle = (\beta'' F)^{-1} X(n_1 - n_2) Y(t_1-t_2)$, Eq. (C12) leads, after the Taylor expansion of $g(n-n_2,t-t_2)$ around $t_2=t_1$, to the equation

$$\langle \phi\, \Delta^2 \phi\rangle = \sum_{m=0}^{\infty} \frac{Y_m}{m!} \int_0^{\infty} dt \int dn_1\, dn_2\, X(n_1$$
$$-n_2) g(n_1,t)\partial_t^m\, \partial_{n_2}^2 g(n_2,t), \quad \text{(C13)}$$

where

$$Y_m \simeq \int_0^{\infty} dt\, t^m Y(t).$$

Using the defining relation for g, $(\partial_t - D\partial_n^2 + V\partial_n) g(n,t) = \delta(n)\delta(t)$ (in the continuum limit) and the condition $g(n,0)=g(n,\infty)=0$, we can write

$$\langle \phi\, \Delta^2 \phi\rangle = \sum_{m=0}^{\infty} \frac{Y_m}{m!} \int_0^{\infty} dt \int dn_1\, dn_2\, X(n_1-n_2)$$
$$\times g(n_1,t_1)(D\partial_{n_2}^2 - V\partial_{n_2})^m\, \partial_{n_2}^2 g(n_2,t_2)$$
$$= \sum_{m=0}^{\infty} \frac{Y_m}{m!} \int dn_1\, dn_2\, X(n_1$$
$$-n_2)(D\partial_{n_2}^2 - V\partial_{n_2})^m \frac{n_1^2-n_2^2}{2D(n_1^2+n_2^2)^2}. \quad \text{(C14)}$$

Integrating by part we obtain a series in the form

$$\langle \phi\, \Delta^2 \phi\rangle = \sum_{m=0}^{\infty} \frac{Y_m}{m!} \int dn_1\, dn_2 \frac{n_1^2-n_2^2}{2D(n_1^2+n_2^2)^2} (a_m \partial_{n_2}^{2m}$$
$$+ b_m \partial_{n_2}^{2m-1}) X(n_1-n_2). \quad \text{(C15)}$$

Given the form of the correlation X [see Eq. (43)], the action on it of the operators $(a_m \partial_{n_2}^{2m} + b_m \partial_{n_2}^{2m-1})$ will lead to terms in the form

$$\frac{c_1 + c_2(n_1-n_2)}{[1+50(a-1)^2(n_1-n_2)^2]^p}. \quad \text{(C16)}$$

Substituting back into Eq. (C15), we see that terms in c_1 do not contribute because of the antisymmetry of the integrand under the transformation $n_1 \to n_2$, $n_2 \to n_1$. Similarly, for the terms in c_2, due to the antisymmetry under a simultaneous change of sign of n_1 and n_2. Hence, the term $\langle \phi\, \Delta^2 \phi\rangle$ does not contribute to Eq. (51).

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[29]*In this case, the LHS of Eq. (15) is understood as* $[\partial_t+\mathbf{v}(\mathbf{z}_1)\cdot\nabla_2+\mathbf{v}(\mathbf{z}_2)\cdot\nabla_1]\langle v(\mathbf{r}_1,t)v(\mathbf{r}_2,t)\rangle$; *each point in the velocity correlation is followed along the Lagrangian trajectory of the other. Notice that the shift to a Lagrangian frame does not introduce cutoffs in the integrals at small pq, either in Eq. (11) or in (15); what occurs is that while in the Eulerian DIA, the Green's function expression is associated with the sink term* C_qC_k *in the energy equation, here, the new terms generated in the Lagrangian reference frame are associated with contributions to the source* C_pC_q *in (15).*
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