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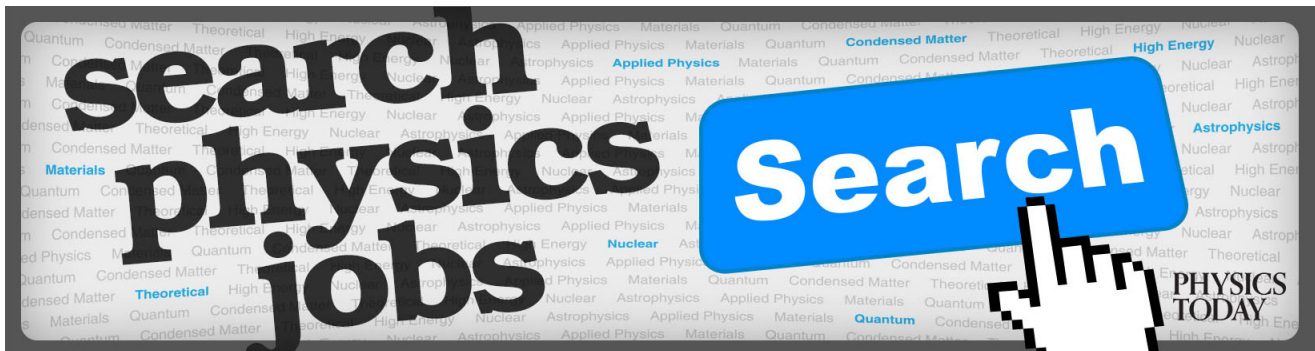
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## Histories electromagnetism

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Working within the HPO (History Projection Operator) Consistent Histories formalism, we follow the work of Savvidou on (scalar) field theory [J. Math. Phys. **43**, 3053 (2002)] and that of Savvidou and Anastopoulos on (first-class) constrained systems [Class. Quantum Grav. **17**, 2463 (2000)] to write a histories theory (both classical and quantum) of Electromagnetism. We focus particularly on the foliation-dependence of the histories phase space/Hilbert space and the action thereon of the two Poincaré groups that arise in histories field theory. We quantize in the spirit of the Dirac scheme for constrained systems. © 2004 American Institute of Physics. [DOI: 10.1063/1.1723702]

### I. INTRODUCTION

Our aim in this paper is to demonstrate the application of certain ideas and techniques that have been developed within the HPO (History Projection Operator) histories formalism over recent years to the theory of Electromagnetism. Specifically, we follow up on two pieces of work which are naturally combined therein:

**Field theory.** In Ref. 1, Savvidou describes the histories theory of the (classical and quantum) scalar field. This has the important feature that there exist two distinct Poincaré groups. The “internal” group is simply the histories analog of that of the standard theory, but there also exists an “external” group that explicitly performs changes of the foliation. This is important as it provides a way of relating quantities that are defined with respect to different foliations. These groups arise as a consequence of one of the most powerful and interesting features of histories theories, namely that there exist two distinct types of time transformation each of which represents a distinct *quality* of time: (a) the internal time (“time of becoming”), which is related to the dynamics of the particular system in question, and (b) the external time (“time of being”), which is related to the causal ordering of events, i.e., the kinematics. (For a detailed exposition of the HPO continuous time histories program, the reader is referred to Ref. 2.)

**Constrained systems.** In Ref. 3, Savvidou and Anastopoulos describe an algorithm for working with systems with first-class constraints within the HPO formalism. They focused specifically on parametrized systems, i.e., those systems whose Hamiltonian is itself a first-class constraint, as a natural precursor to understanding “histories” general relativity, and demonstrated that the histories on the reduced phase space *retained* their intrinsic temporal ordering. The quantization algorithm is in the spirit of the Dirac scheme for constrained systems. (For progress with this enterprise, see Refs. 4–6.)

The theory of Electromagnetism, as a field theory with first-class constraints, thus perfectly combines the above pieces of work, but also brings something new to each when studied within the histories framework. In the first instance we shall see explicitly how the histories phase space and reduced phase space depend on the foliation and discuss the importance of the external boost in this respect. Second, we will have to deal with the fact (not tackled in detail in Ref. 3) that our constraints have continuous spectra, and thus the physical Hilbert space cannot be a true linear subspace of the full (unconstrained) Hilbert space.

The outline of the paper is as follows: in Sec. II we give a brief account of those aspects of the

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Histories program most relevant to our needs, and then we present the histories theory of Electromagnetism, starting with the classical theory in Sec. III, and then its quantization in Sec. IV. We conclude in Sec. V.

Finally we note that the classical history theory of vector fields has been studied by Nolting<sup>7</sup> as well as their BRST quantization,<sup>8</sup> though he follows a fundamentally different approach which centers on defining five component vector fields to incorporate the two times.

## II. THE HISTORIES PROGRAM

The consistent histories version of quantum mechanics was originally developed in the 1980s by Griffiths<sup>9</sup> and then built on (each with different emphases) by Omnès<sup>10</sup> and then Gell-Mann and Hartle<sup>11</sup> and Hartle.<sup>12</sup> The main aim (particularly of the latter) was to develop a quantum mechanics of closed systems.

As formulated by Gell-Mann and Hartle, a history,  $\alpha$ , is represented by a class operator,  $C_\alpha$ , that is a product of Heisenberg picture projection operators. Dynamic information is contained in the decoherence functional, defined on a pair of histories as

$$d(\alpha, \beta) = \text{tr}(C_\alpha^\dagger \rho_0 C_\beta), \quad (2.1)$$

where  $\rho_0$  is the density matrix describing the initial state of the system. If a history is part of a “consistent” set, then probabilities (in the usual Kolmogorov sense) may be assigned to the individual histories according to  $p(\alpha) = d(\alpha, \alpha)$ .

The HPO formalism was developed initially by Isham<sup>13</sup> and Isham and Linden<sup>14</sup> who sought a histories version of single-time quantum logic. To this end they re-defined the class operator as a *tensor* product of *Schrödinger* picture operators, so it would be a genuine projection operator on some suitable “history” Hilbert space. This formalism was extended to the case of continuous time histories by Isham and co-workers,<sup>15,16</sup> in which the “history group”—analogous to the usual Heisenberg–Weyl group—was introduced. However, this structure lacked any clear notion of time evolution. It was only with Savvidou’s introduction of the action operator—the quantum analog of the classical Hamilton–Jacobi action functional—that the temporal structure of histories theory was established in the form as it is used now (see Ref. 17). It is these two—the history group and action operator—that are the key elements of any history theory.

### A. The history group

By introducing the history group, a HPO theory may be seen as seeking a suitable representation of a certain algebra, e.g., for a (nonrelativistic) particle moving on the real line (see Ref. 2) and a continuous time label,  $t \in \mathbb{R}$ , the (nonzero) commutation relation is ( $\hbar = 1$ )

$$[x_t, p_{t'}] = i \delta(t - t'). \quad (2.2)$$

This algebra is isomorphic to that of a field theory in one spatial dimension, and field-theoretic techniques are usefully employed to find a suitable representation. Following Araki,<sup>18</sup> the proper representation of this algebra is selected by requiring that the Hamiltonian exist as a self-adjoint operator, and it will come as no surprise that a Fock representation provides the necessary “history” Hilbert space.

In histories quantum scalar field theory (see Ref. 1), after foliating Minkowski space with a unit timelike vector,  $n_\mu$ , we have

$$[\phi(X), \pi(X')] = i \delta^4(X - X'), \quad (2.3)$$

where we are using a “pseudo-covariant” notation  $X = n \cdot t + x_n$  ( $x_n$  is a four vector such that  $n \cdot x_n = 0$ ). [We use the metric signature  $(+, -, -, -)$ .] A representation of this algebra is found in terms of creation and annihilation,  $b^\dagger(X)$  and  $b(X)$ , on the (history) Fock space,  $\mathcal{V}_{\text{scalar}} = \exp(L^2(\mathbb{R}^4, d^4X))$ . Indeed, it is found that all foliation dependent representations exist *on the same* Fock space.

## B. The action operator

The other key element of a histories theory is the action operator,  $S(\gamma)$  (see Ref. 17). It is this that is the generator of time transformations of a HPO theory, combining the Liouville operator,  $V(\gamma)$ , which generates time translations in the external (kinematical) time label, and the Hamiltonian,  $H(\gamma)$ , which generates time translations in the internal (dynamical) time label. For our nonrelativistic particle, the action would be written as

$$S(\gamma) = V(\gamma) - H(\gamma) = \int dt [p_t \dot{x}_t - H_t(p_t, x_t)](\gamma), \quad (2.4)$$

where  $H_t$  is a one parameter family of Hamiltonians.

In the field theory case, we have two Poincaré groups, with the Hamiltonian being the time translations generator of the internal group, and the Liouville being the time translations generator of the external group. The generators of the internal group are time-averaged versions of the generators of the standard group, but it is the external group that is the novel object, as the boost of this group generates *changes of the foliation* as well. Its action on the foliation-dependent scalar field is given by

$${}^{\text{ext}}U(\Lambda)^n \phi(X) {}^{\text{ext}}U(\Lambda)^{-1} = \Lambda^n \phi(\Lambda^{-1}X) \quad (2.5)$$

[where  $U(\Lambda)$  is the unitary operator that generates the Lorentz transformation] and thus we have a way of relating quantities defined with respect to different foliations.

Finally, we note, without going into great detail, that there is an analogous formalism for classical histories which we will use to write the classical history theory of EM below (see Chap. 5 of Ref. 2 for further details). This involves thinking of a history as a map from the real line into the classical phase space,  $\Gamma$ . A natural symplectic structure can be defined on  $\Pi$  (the history phase space), giving rise to the Poisson algebra. The equations of motion can be expressed by saying that, for any function  $F$  on  $\Pi$ , their solutions,  $\gamma_{\text{cl}}$ , will satisfy

$$\{F, S\}(\gamma_{\text{cl}}) = 0. \quad (2.6)$$

## C. The constrained systems algorithm

The theory of constrained systems was extensively studied by Dirac,<sup>19</sup> though we primarily use Refs. 20 and 21. In essence, a first-class constraint,  $\phi(x, p) = 0$ , is to be seen as a generator of gauge transformations which partitions the phase space,  $\Gamma$  (and, thus, the constraint surface) into orbits. The reduced phase space,  $\Gamma_{\text{red}}$ , is then isomorphic to the space of orbits. There exists a unique “reduction” of a function  $F$  on  $\Gamma$  to a function  $\tilde{F}$  on  $\Gamma_{\text{red}}$  if  $F$  has a weakly vanishing Poisson bracket with the constraint. Dirac quantization proceeds by constructing the unconstrained Hilbert space,  $\mathcal{H}$ , writing the constraint as an operator, and then defining the physical Hilbert space as that linear subspace (modulo considerations of the constraint spectrum) of  $\mathcal{H}$  which is spanned by those eigenvectors of the constraint whose corresponding eigenvalue is zero.

In histories theory<sup>3</sup> we write the (time-averaged) constraint as  $\Phi_\lambda(\gamma) = \int dt \lambda(t) \phi(x_t, p_t) \times (\gamma)$ . As above, the action of the constraint will partition  $\Pi$  (the history phase space) into orbits, and we can define  $\Pi_{\text{red}}$  (the reduced phase space) as the space of equivalence classes of histories on the constraint surface,  $C_h$ . (Histories will be equivalent if they lie on the same orbit.) Again, there will be a unique “reduction” of any function,  $F$ , on  $\Pi$  to a function,  $\tilde{F}$ , on  $\Pi_{\text{red}}$  if  $\{F, \Phi_\lambda\} \approx 0$ .

The quantization algorithm for a histories theory follows the spirit of the Dirac scheme, briefly described above. It is implemented, once the constraint is suitably defined as an operator, by first observing that we require

$$d(e^{i\Phi_\lambda} \alpha e^{-i\Phi_\lambda}, e^{i\Phi_\lambda} \beta e^{-i\Phi_\lambda}) = d(\alpha, \beta). \quad (2.7)$$

To meet this requirement (modulo, as above, issues concerning the nature of the constraint spectrum) we define a projector,  $E$ , onto the closed linear subspace of the (unconstrained) history space,  $\mathcal{V}$ , corresponding to the zero eigenvalue of  $\Phi_\lambda$  and then substitute  $\alpha$  for  $E\alpha E$  in the expression for the decoherence functional (evidently  $e^{i\Phi_\lambda}E=E$ ).

We are now in a position to put these ideas into practice, writing the classical histories theory of Electromagnetism in the next section, and its quantization in the subsequent one.

### III. ELECTROMAGNETISM—CLASSICAL

We begin with a brief review of the standard theory.

#### A. Basics

The EM Action is

$$S = -\frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu}, \quad (3.1)$$

where  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ . The equations of motion are computed from setting  $\delta S = 0$ , and we get

$$\partial^\mu F_{\mu\nu} = 0. \quad (3.2)$$

To write this in Hamiltonian form we first define the momentum conjugate to the vector potential:

$$\pi^\mu = \frac{\delta L}{\delta \dot{A}_\mu} = F^{\mu 0}. \quad (3.3)$$

So  $\pi^i = \partial^i A^0 - \partial^0 A^i = -[(\nabla A^0)^i + \dot{A}^i] = (\underline{E})_i$ , and we have the following constraints:

$$\pi^0 = 0 \quad (3.4)$$

(this is a primary, first-class constraint), and the Gauss Law constraint (this is a secondary, first-class constraint),

$$\partial_i \pi^i = 0, \quad (3.5)$$

derived from the consistency condition that the primary constraint be conserved in time, i.e.,  $\{H, \pi^0\} = 0$  (where  $H$  is the canonical Hamiltonian given below). In terms of the observable fields,  $\underline{E}$  and  $\underline{B} = \nabla \times \underline{A}$  the canonical Hamiltonian is written as

$$H = \int d^3x \left( \frac{1}{2} \underline{E}^2 + \frac{1}{2} \underline{B}^2 - A^0 \nabla \cdot \underline{E} \right). \quad (3.6)$$

#### 1. Poincaré invariance in the standard theory

The generators of the Poincaré group of the standard theory are “taken over” (in time-averaged form) to the histories theory as the “internal” group. Following Ref. 22, these are derived from the energy–momentum tensor (to which a total divergence has to be added):

$$\tilde{\Theta}^{\mu\nu} = -F^{\mu\rho} \partial^\nu A_\rho + \frac{1}{4} \eta^{\mu\nu} F^2 + \partial_\rho (F^{\mu\rho} A^\nu). \quad (3.7)$$

The ten generators of the Poincaré group are then written as

$$P^\alpha = \int d^3x \tilde{\Theta}^{0\alpha}, \quad (3.8)$$

$$M^{\alpha\beta} = \int d^3x (x^\alpha \Theta^{0\beta} - x^\beta \Theta^{0\alpha}). \quad (3.9)$$

From these, we deduce the explicit, canonical form of the Hamiltonian, linear momentum, angular momentum and boost generators to be, respectively,

$$H = \int d^3x \left( -\frac{1}{2} \pi_i \pi^i + \frac{1}{2} (\nabla \times \underline{A})^2 - A^0 \partial_i \pi^i \right), \quad (3.10)$$

$$P_i = \int d^3x \pi^j \partial_i A_j, \quad (3.11)$$

$$J^i = \epsilon^{ijk} \int d^3x (\pi^l x_j \partial_k A_l + \pi_j A_k), \quad (3.12)$$

$$K_i = \int d^3x x_i \left( -\frac{1}{2} \pi_j \pi^j + \frac{1}{2} (\nabla \times \underline{A})^2 - A^0 \partial_j \pi^j \right) \quad (3.13)$$

(where we have chosen  $x^0=0$  in the expression for the boost). This algebra closes only weakly, i.e., subject to the Gauss Law constraint, a fact that will be of significance when we come to the quantization.

We now turn to the histories formulation of classical electromagnetism.

## B. The histories phase space, $\Pi$

The phase space of canonical EM is  $\Gamma = (A_i(\underline{x}), \pi_j(\underline{x}))$ , and a history is defined to be a path:

$$\gamma: \mathbb{R} \rightarrow \Gamma, \quad (3.14)$$

$$t \mapsto (A_i(t, \underline{x}), \pi_j(t, \underline{x})). \quad (3.15)$$

The space of histories,  $\Pi$ , is defined to be the space of all such smooth paths  $\gamma$ . As explained before we can use a “covariant-like” notation, writing  $X = n \cdot t + x_n$  where  $n_\mu$  is a unit time-like vector, so  $t = n \cdot X$  and  $x_n$  is “ $n$ -spatial,” i.e.,  $n \cdot x_n = 0$ .

However, we wish to find a representation of the time-averaged canonical expressions on the phase space coordinated by  $(A_\mu(X), \pi^\nu(X))$ . This is so we can write a representation of both the internal *and* external groups on the same space. This is achieved using the  $n$ -spatial projector,  $P_{\mu\nu}$ , introduced earlier, along with the foliating timelike vector,  $n_\mu$ . In this notation, we write the foliation dependent (canonical) fields  ${}^n A_\mu := P_\mu^\nu A_\nu$  (and likewise for the conjugate momenta). The Hamiltonian is written as

$$H_n = \int d^4X \left( \frac{1}{2} [P^{\mu\nu} \pi_\mu \pi_\nu + (\nabla_\sigma^\mu A^\sigma)(\nabla_\mu^\delta A^\delta)] + n^\rho A_\rho P^{\mu\nu} \partial_\mu \pi_\nu \right) \chi(n \cdot X) \quad (3.16)$$

(where the subscript “ $n$ ” refers to the particular foliation, and we have introduced the notation  $\nabla_{\mu\sigma} A^\sigma \equiv \epsilon_{\mu\nu\rho\sigma} n^\nu \partial^\rho A^\sigma$ ). This is the generator of time translations of the internal group. We then define the Liouville operator (the generator of time translations in the external group):

$$V_n = \int d^4X \pi^\mu n^\rho \partial_\rho A_\mu, \quad (3.17)$$

and thus can write the action functional:

$$S_n = V_n - H_n. \quad (3.18)$$

It is the action functional that is to be understood as the “true” generator of time translations of the theory, naturally intertwining the two modes of time represented by the Hamiltonian and Liouville operators. The fundamental Poisson brackets are now

$$\{A_\mu(X), A_\nu(Y)\} = 0 = \{\pi^\mu(X), \pi^\nu(Y)\} \quad (3.19)$$

and

$$\{A_\mu(X), \pi^\nu(Y)\} = \delta_\mu^\nu \delta^4(X - Y). \quad (3.20)$$

We now turn to the central issue of Poincaré invariance.

### C. The Poincaré groups

As was the case for the scalar field, we seek representations for two Poincaré groups on the history space: one associated with the internal time label, and one associated with the external time label. The generators for spatial translations and spatial rotations will be the same for each group, so we focus our attentions on the time translation and boost generators in each case.

#### 1. The internal Poincaré group

The generators of the internal Poincaré group will be time-averaged versions of the generators of the standard theory [Eqs. (3.10)–(3.13)]. The time translation generator is, of course, just the Hamiltonian of Eq. (3.16) and we define the boost at  $s=0$  as

$$\text{int} K(m) = -m_\mu \int d^4X \ X^\mu \left( \frac{1}{2} [P^{\mu\nu} \pi_\mu \pi_\nu + (\nabla_\sigma^\mu A^\sigma)(\nabla_\mu^\delta A^\delta)] + n^\rho A_\rho P^{\mu\nu} \partial_\mu \pi_\nu \right), \quad (3.21)$$

where  $m_\mu$  is a space-like vector, i.e.,  $n \cdot m = 0$ , parametrizing the boost. Given a function  $A$  on  $\Pi$ , we can denote the one parameter group of transformations it generates as  $s \mapsto T_A(s)$  and its action on the algebra of functions,  $B$ , as

$$T_A(s)[B] = \sum_n \frac{s^n}{n!} \underbrace{\{A, \{A, \dots \{A, B \dots\}\}}_{n\text{-times}} \quad (3.22)$$

So, we first define the classical analog of the Heisenberg picture fields by

$$T_H(s)[{}^n A_\mu(X)] = {}^n A_\mu(X, s), \quad (3.23)$$

$$T_H(s)[{}^n \pi^\mu(X)] = {}^n \pi^\mu(X, s), \quad (3.24)$$

and can now see explicitly the sense in which the Hamiltonian generates time translations in the internal time label by looking at its action on the “Heisenberg” picture fields:

$$T_H(\tau)[{}^n A_\mu(X, s)] = {}^n A_\mu(X, s + \tau), \quad (3.25)$$

$$T_H(\tau)[{}^n \pi^\mu(X, s)] = {}^n \pi^\mu(X, s + \tau). \quad (3.26)$$

The internal boost generator will mix the internal time parameter, “ $s$ ,” with the spatial coordinates

$$T_{\text{int} K(m)}[{}^n A_\mu(X, s)] = {}^n A_\mu(\Lambda^{-1}(X, s)), \quad (3.27)$$

$$T_{\text{int} K(m)}[{}^n \pi^\mu(X, s)] = {}^n \pi^\mu(\Lambda^{-1}(X, s)), \quad (3.28)$$



where  $\Lambda^{-1}(X, s)$  is related to  $(X, s)$  (the time label “ $t$ ” is, of course, constant) by the Lorentz boost parametrized by  $m^\mu$ , i.e., the velocity of the moving frame is given by

$$v^i = c \frac{\tanh|m|m^i}{|m|}. \quad (3.29)$$

## 2. The external Poincaré group

In contrast to the definition of the generators of the internal group, we use the covariant fields,  $(A_\mu(X), \pi^\mu(X))$  in the definition of the generators for the external Poincaré group. These are

$$P^\mu = \int d^4X \pi^\nu \partial^\mu A_\nu \quad (3.30)$$

and

$$M^{\mu\nu} = \int d^4X [\pi^\rho (X^\mu \partial^\nu - X^\nu \partial^\mu) A_\rho] + \sigma^{\mu\nu}, \quad (3.31)$$

where  $\sigma^{\mu\nu}$  is the spin term, given by

$$\sigma^{\mu\nu} = \int d^4X (\pi^\mu A^\nu - \pi^\nu A^\mu). \quad (3.32)$$

As before, we are particularly interested in the actions of the time translation generator  $V = P^0$  and the boosts generator  $K(m) = n_\mu m_\nu M^{\mu\nu}$ . These are therefore written as

$$V = \int d^4X \pi^\mu n^\nu \partial_\nu A_\mu \quad (3.33)$$

and

$${}^{\text{ext}}K(m) = m_\mu \int d^4X [(n \cdot X) \pi^\nu \partial^\mu A_\nu - X^\mu \pi^\rho n^\nu \partial_\nu A_\rho] + n_\mu m_\nu \sigma^{\mu\nu}. \quad (3.34)$$

The effect of the Liouville functional is to generate the following algebra automorphisms, in which we can clearly see that it generates time translation in the external time label:

$$T_V(\tau)[A_\mu(X, s)] = e^{-\tau n_\sigma \partial^\sigma} A_\mu(X, s) = A_\mu(X', s), \quad (3.35)$$

$$T_V(\tau)[\pi^\mu(X, s)] = e^{-\tau n_\sigma \partial^\sigma} \pi^\mu(X, s) = \pi^\mu(X', s), \quad (3.36)$$

where  $X'$  is the point in  $\mathcal{M}$  associated with the pair  $(x, t + \tau)$ .

Let us now turn to the transformations generated by the external boosts. These will mix the external time parameter, “ $t$ ,” with the spatial coordinates. The finite transformations can be written as

$$T_{\text{ext}_{K(m)}}[A_\mu(X, s)] = \Lambda_\mu^\nu A_\nu(\Lambda^{-1}(X), s), \quad (3.37)$$

$$T_{\text{ext}_{K(m)}}[\pi^\mu(X, s)] = \Lambda_\nu^\mu \pi^\nu(\Lambda^{-1}(X), s). \quad (3.38)$$

As previously stated, the role of the external group is an interesting one, and it is this that is one of the novel features of histories field theory. The effect of the external boosts is to mix the



spatial coordinate with the external time label “ $t$ ” and, as the phase space has an implicit foliation dependence, it will also boost the foliation vector itself, thus generating transformations between different foliation-dependent representations.

#### D. The reduced phase space, $\Pi_{\text{red}}$

Our next task is to follow the algorithm of Ref. 3 to ascertain a suitable description of the reduced phase space,  $\Pi_{\text{red}}$ , on which the true degrees of freedom of the theory are defined. To this end, we are interested in the actions of the constraints on the phase space (and in particular the history constraint surface,  $C_h$ ) because, by examining their action, we can define suitable coordinates (i.e., ones constant along the orbits) for the reduced phase space  $\Pi_{\text{red}}$ .

We write the time-averaged analogs of the constraints of the standard theory as follows:

$$\Psi_\lambda = \int d^4X \lambda(X) n_\mu \pi^\mu \approx 0, \quad (3.39)$$

$$\Phi_\lambda = \int d^4X \lambda(X) P^{\mu\nu} \partial_\mu \pi_\nu \approx 0, \quad (3.40)$$

and consider their action on the coordinates of  $\Pi$ . Under  $\Psi_\lambda$  we have

$$(A_\mu(X), \pi^\mu(X)) \rightarrow (A_\mu(X) - \lambda(X) n_\mu, \pi^\mu(X)). \quad (3.41)$$

Under  $\Phi_\lambda$  we have

$$(A_\mu(X), \pi^\mu(X)) \rightarrow (A_\mu(X) + P_\mu^{\rho} \partial_\rho \lambda(X), \pi^\mu(X)). \quad (3.42)$$

Evidently  $\pi^\mu(X)$  is constant along the orbits, so we just seek a quantity associated with the vector potential that is gauge invariant.

Equations (3.41) and (3.42) tell us that the transverse components of the vector potential remain constant along the orbits of the constraints and are thus good coordinates for  $\Pi_{\text{red}}$ , whereas the scalar and longitudinal components correspond to the degenerate directions of  $\Psi_\lambda$  and  $\Phi_\lambda$ , respectively. (This state of affairs is more clearly seen if we use a Fourier transform and work in momentum space.) If we combine this knowledge with a look at the constraints themselves, which (if we were to Fourier transform them) readily show us that the constraint surface,  $C_h$ , is defined by  $\pi^0 = \pi^L = 0$ , where these are, respectively, the scalar and longitudinal components of the conjugate momentum, we can deduce that  $\Pi_{\text{red}}$  is suitably coordinated by  $(A_\mu^\perp(X), \pi_\mu^\perp(X))$ , where the superscript “ $\perp$ ” indicates the transverse components, and these are defined by

$$A_\mu^\perp(X) = \left( \frac{{}^n \partial_\mu {}^n \partial^\nu}{{}^n \Delta} - P_\mu^\nu \right) A_\nu(X) \quad (3.43)$$

(and likewise for  $\pi^\perp$ ) and where  ${}^n \partial_\mu$  is shorthand for  $P_\mu^\alpha \partial_\alpha$  and the (invertible) partial differential operator  ${}^n \Delta$  is defined as

$$({}^n \Delta f_\rho)(X) = (P^{\mu\nu} \partial_\mu \partial_\nu) f_\rho(X). \quad (3.44)$$

The (nonzero) Poisson bracket relation on the reduced phase space is given by

$$\{A_\mu^\perp(X), \pi^\perp{}^\nu(X')\} = T_\mu^\nu \delta^4(X - X'), \quad (3.45)$$

where

$$(T_\mu^\nu f_\nu)(X) \equiv \left( \frac{{}^n \partial_\mu {}^n \partial^\nu}{{}^n \Delta} - P_\mu^\nu \right) f_\nu(X). \quad (3.46)$$

We are now in a position to examine whether or not we can write a representation of the two Poincaré groups on  $\Pi_{\text{red}}$ .

### E. The reduced Poincaré algebras

As explained in Sec. II C, for a function on the whole phase space to reduce to a corresponding function on the reduced phase space, it is necessary that its Poisson bracket with the constraints is weakly zero. We expect to find that the generators of the *internal* Poincaré group reduce to  $\Pi_{\text{red}}$ . However, we do not expect to find a full representation of the *external* Poincaré group on the reduced phase space. In Ref. 1 the foliation dependence of the phase space was emphasized but not explicit. In the case of EM we shall see this dependence explicitly as the action of the external boost will affect the definition of  $\Pi_{\text{red}}$  and so we do not expect to find a reduced version of this generator. We now turn to the explicit results.

As before, we are only interested in the time translation and boost generators of each Poincaré group and thus we need only compute the Poisson brackets of  $S$ ,  $^{\text{int}}K(m)$  and  $^{\text{ext}}K(m)$  with the constraints. We find the following results (recall that  $\Psi_\lambda$  is the “ $\pi^0$ ” constraint and  $\Phi_\lambda$  the Gauss Law constraint):

$$\{S, \Psi_\lambda\} = \Psi_\lambda - \Phi_\lambda \approx 0 \quad (3.47)$$

and

$$\{S, \Phi_\lambda\} = \Phi_\lambda \approx 0. \quad (3.48)$$

So the action functional weakly commutes with both constraints and so can be reduced to a functional  $\tilde{S}$  acting on  $\Pi_{\text{red}}$ .

The internal boost generator has the following Poisson brackets with the constraints:

$$\{^{\text{int}}K(m), \Psi_\lambda\} = \Phi_{-m_\alpha X^\alpha \lambda} \approx 0 \quad (3.49)$$

and

$$\{^{\text{int}}K(m), \Phi_\lambda\} = 0. \quad (3.50)$$

This is in line with what we expected, i.e., that the generators of the internal Poincaré group commute with the constraints and thus we have a representation of the internal group on  $\Pi_{\text{red}}$ . (Of course, something would be quite amiss if we did not have this as the internal group is the histories analog of the Poincaré group of standard Maxwell theory.)

The external boost generator forms the following Poisson brackets with the constraints:

$$\{^{\text{ext}}K(m), \Psi_\lambda\} = \Psi_{(n_\beta X^\beta m_\alpha \partial^\alpha - m_\beta X^\beta n_\alpha \partial^\alpha) \lambda} - \int d^4 X \lambda(X) m_\alpha \pi^\alpha \quad (3.51)$$

and

$$\{^{\text{ext}}K(m), \Phi_\lambda\} = \Psi_{m_\alpha \partial^\alpha \lambda} + \int d^4 X (n_\alpha \partial^\alpha \lambda(X)) m_\beta \pi^\beta. \quad (3.52)$$

Neither of these are weakly zero, and so the external boost generator cannot be reduced to  $\Pi_{\text{red}}$ .

For those functions that *can* be reduced, we use the coordinates for the reduced phase space that we worked out in the previous section. The Hamiltonian and Liouville functionals on the reduced phase space are written as follows:

$$\tilde{H} = \int d^4 X \frac{1}{2} (\pi^{\perp \mu} \pi_\mu^\perp + A^{\perp \mu} \Delta A^{\perp \mu}), \quad (3.53)$$

$$\tilde{V} = \int d^4X \pi^{\perp\mu} n_\nu \partial^\nu A_\mu^\perp, \quad (3.54)$$

where we have used, in the expression for the Hamiltonian,

$$A_\mu{}^n \Gamma^{\mu\nu} A_\nu = A_\mu^\perp{}^n \Delta A^\perp{}^\mu, \quad (3.55)$$

with  ${}^n\Delta$  defined in Eq. (3.44). Thus the action functional on  $\Pi_{\text{red}}$  is written as

$$\tilde{S} = \tilde{V} - \tilde{H}, \quad (3.56)$$

and the classical paths which are solutions to the equations of motion are those which satisfy

$$\{\tilde{S}, \tilde{F}\}(\gamma_{\text{cl}}) = 0, \quad (3.57)$$

for all functions  $\tilde{F}$  defined on  $\Pi_{\text{red}}$ .

#### IV. ELECTROMAGNETISM—QUANTIZATION

For the quantization of the theory we continue to follow the algorithm laid down by Savvidou and Anastopoulos, which, as outlined in Sec. II C, essentially follows the Dirac scheme. We define the history space,  $\mathcal{V}$ , by consideration of the history group, and define the constraints thereon. However, as we mentioned, the constraints have continuous spectra, and thus the physical Hilbert space,  $\mathcal{V}_{\text{phys}}$ , will not be a genuine subspace of the history Hilbert space. This will be explicitly demonstrated. And so we are led to a creative implementation of the algorithm (the central idea here is due to Savvidou<sup>25</sup>), in which the physical Hilbert space is defined separately, based on an analysis, in terms of coherent states, of how the constraints act on  $\mathcal{V}$ . Appropriate mappings are then defined between  $\mathcal{V}$  and  $\mathcal{V}_{\text{phys}}$  such that objects on one can be related to objects on the other.

##### A. The History Hilbert space $\mathcal{V}$

So the first stage is to define the History Hilbert space. Following the methods of Refs. 15 and 1, we start by defining the History Algebra:

$$[A_\mu(X), A_\nu(X')] = 0, \quad (4.1)$$

$$[\pi_\mu(X), \pi_\nu(X')] = 0, \quad (4.2)$$

$$[A_\mu(X), \pi^\nu(X')] = i \delta_\mu^\nu \delta^4(X - X'), \quad (4.3)$$

or, in its more rigorous, smeared form:

$$[A_\mu(f^\mu), A_\nu(f'^\nu)] = 0, \quad (4.4)$$

$$[\pi_\mu(h^\mu), \pi_\nu(h'^\nu)] = 0, \quad (4.5)$$

$$[A_\mu(f^\mu), \pi^\nu(h_\nu)] = i \int d^4X \delta_\mu^\nu f^\mu(X) h_\nu(X), \quad (4.6)$$

where  $f^\mu(X)$ ,  $h_\mu(X)$  are elements of a suitable space of smearing functions which we will leave unspecified beyond saying that it must at least be a subspace of  $\oplus_{i=1\dots 4} L^2_{\mathbb{R}}(\mathbb{R}^4, d^4X)_i$ . Let us denote this space  $\mathcal{T}_{\mathbb{R}}$ . It is natural to seek a Fock representation of this algebra, and this is achieved by first taking the complexification of the space of smearing functions, i.e.,  $\mathcal{T}_{\mathbb{C}} = \mathcal{T}_{\mathbb{R}} \oplus i\mathcal{T}_{\mathbb{R}}$  and then exponentiating the resulting space to give  $\mathcal{V} = e^{\mathcal{T}_{\mathbb{C}}}$ . The Fock space thus defined will carry a natural representation of the above History Algebra, which we seek explicitly below, in terms of creation and annihilation operators:

$$[b_\mu(X), b^{\dagger\nu}(X)] = \delta_\mu^\nu \delta^4(X - X'). \quad (4.7)$$

### 1. The representation of $(A, \pi)$ in terms of $(b^\dagger, b)$

We can easily write a representation of the fully covariant fields  $(A_\mu(X), \pi^\mu(X))$ :

$$A_\mu(X) = \frac{1}{\sqrt{2}}(b_\mu(X) + b_\mu^\dagger(X)), \quad (4.8)$$

$$\pi_\mu(X) = -\frac{i}{\sqrt{2}}(b_\mu(X) - b_\mu^\dagger(X)). \quad (4.9)$$

However, what we require in order to define the Hamiltonian is foliation-dependent fields. So we start from a normal-ordered analog of the classical unconstrained Hamiltonian:

$${}^n H =: \frac{1}{2} \int d^4 X (P^{\mu\nu} {}^n \pi_\mu {}^n \pi_\nu + {}^n A_\mu {}^n \Gamma^{\mu\nu} {}^n A_\nu): \quad (4.10)$$

and may think, at first, to define

$${}^n A_\mu(X) = \frac{1}{\sqrt{2}}({}^n \Gamma_\mu^\nu)^{-1/4}(b_\nu(X) + b_\nu^\dagger(X)), \quad (4.11)$$

$${}^n \pi_\mu(X) = -\frac{i}{\sqrt{2}}({}^n \Gamma_\mu^\nu)^{1/4}(b_\nu(X) - b_\nu^\dagger(X)). \quad (4.12)$$

However, there is a problem here, as the operator  $\Gamma^{\mu\nu}$  has zero eigenvalues, and is, therefore, not invertible. To see this, it is easier to use “canonical notation,” i.e.,  $\Gamma^{ij} = \partial^i \partial^j - \delta^{ij} \partial_k \partial^k$ . We then examine the action of this operator on an element,  $f_i(x)$ , of the smearing function space—which we split into its transverse and longitudinal components,  $f_i = f_i^\perp + f_i^\parallel$ —and find

$$\Gamma^{ij} f_i(x) = \Gamma^{ij} (f_i^\perp(x) + f_i^\parallel(x)) = \Gamma^{ij} f_i^\perp(x). \quad (4.13)$$

So the longitudinal components of the smearing functions are the zero eigenvectors of  $\Gamma^{ij}$ . If we now split the Hamiltonian into its transverse and longitudinal parts, we find (reverting to the full “histories” notation, and dropping the  $n$  superscript for ease),

$$H = \frac{1}{2} \int d^4 X (\pi_\mu^\perp \pi^{\perp\mu} + A_\mu^\perp {}^n \Delta A^{\perp\mu} + \pi_\mu^\parallel \pi^{\parallel\mu}), \quad (4.14)$$

where the operator  ${}^n \Delta$  was defined in Eq. (3.44). And now we see that the transverse part of the Hamiltonian is, in essence, that of the usual “harmonic oscillators,” whereas the longitudinal part is that of a “free particle.” This form now prompts us towards the correct definition of the fields in terms of the creation and annihilation operators:

$${}^n A_\mu(X) = \frac{1}{\sqrt{2}} {}^n \Delta^{-1/4} (b_\mu(X) + b_\mu^\dagger(X)), \quad (4.15)$$

$${}^n \pi_\mu(X) = -\frac{i}{\sqrt{2}} {}^n \Delta^{1/4} (b_\mu(X) - b_\mu^\dagger(X)). \quad (4.16)$$

Now, on the one hand, we can consider the Fock space in terms of the orthonormal basis obtained by a continual application of the creation operator on a translationally invariant vacuum state (defined by  $b_\mu|0\rangle=0$ ). However, it has also proved very useful to consider the Fock space in terms of coherent states—indeed, in Ref. 15, these were vital to the demonstration that there exists a natural isomorphism between an exponential Hilbert space and the “continuous tensor product” of Hilbert spaces so vital to the Histories program. In the next section, we make use of the technology of coherent states (an excellent reference is Ref. 23) as we seek to define the Physical Hilbert space.

## B. The Physical Hilbert space $\mathcal{V}_{\text{phys}}$

Recall that the constraints are written as

$$\Psi_\lambda = \int d^4X \lambda(X) n_\mu \pi^\mu, \quad (4.17)$$

$$\Phi_\lambda = \int d^4X \lambda(X) P^{\mu\nu} \partial_\mu \pi_\nu. \quad (4.18)$$

Substituting for the fields in terms of the creation and annihilation operators, these become

$$\Psi_\lambda = \frac{-i}{2} \int d^4X \lambda(X) n_\mu {}^n\Delta^{1/4} (b^\mu - b^{\dagger\mu}), \quad (4.19)$$

$$\Phi_\lambda = \frac{-i}{2} \int d^4X \lambda(X) P^{\mu\nu} \partial_\mu {}^n\Delta^{1/4} (b_\nu - b_\nu^\dagger). \quad (4.20)$$

It is clear that these constraints are self-adjoint operators on  $\mathcal{V}$ , and also that they have continuous spectra, so the physical Hilbert space will not be a genuine subspace. However, by consideration of the Fock space,  $\mathcal{V}$ , in terms of coherent states we are led naturally to the correct definition of  $\mathcal{V}_{\text{phys}}$ , and explicitly show in what sense the latter is not a true subspace of the former.

The Weyl operator which generates the (overcomplete) set of coherent states is written as

$$U[f, h] = \exp[i({}^nA_\mu(f^\mu) - {}^n\pi_\nu(h^\nu))]|0\rangle, \quad (4.21)$$

$$= \exp[b_\mu^\dagger(z^\mu) - b_\mu(z^{*\mu})]|0\rangle, \quad (4.22)$$

with  $z_\mu(X) = (1/\sqrt{2})({}^n\Delta^{1/4}h_\mu(X) + i{}^n\Delta^{-1/4}f_\mu(X))$ . The un-normalized coherent states on  $\mathcal{V}$  are defined for each  $z^\mu(X) \in \mathcal{T}_\mathbb{C}$  as

$$|\exp z\rangle = e^{b_\mu^\dagger(z^\mu)}|0\rangle. \quad (4.23)$$

Their overlap is given by

$$\langle \exp z | \exp z' \rangle = e^{\langle z, z' \rangle} \quad (4.24)$$

[where the inner product is  $\langle z, z' \rangle = \int d^4X z_\mu^*(X) z'^\mu(X)$ ] and there exists a measure,  $d\sigma[z]$ , such that

$$1 = \int |\exp z\rangle \langle \exp z| d\sigma[z]. \quad (4.25)$$

(That this measure exists was demonstrated in Ref. 15.) With the aid of this resolution of unity, we can thus define an integral representation of  $\mathcal{V}$  in terms of wave functionals,  $\psi[z]$ :

$$|\psi\rangle = \int |\exp z\rangle \langle \exp z | \psi \rangle d\sigma[z] = \int \psi[z] |\exp z\rangle d\sigma[z]. \quad (4.26)$$

Furthermore, we can write a differential representation of a general operator,  $O$  on  $\mathcal{V}$ :

$$\langle \exp z | : O(b_\mu^\dagger, b_\mu) : | \psi \rangle = O\left(z_\mu^*, \frac{\delta}{\delta z_\mu^*}\right) \psi[z]. \quad (4.27)$$

Given this last construction, we can now rewrite the constraint operators as follows:

$$\langle \exp z | \Psi_\lambda | \psi \rangle = \int d^4 X g_\mu \left( \frac{\delta}{\delta z^{*\mu}} - z^{*\mu} \right) \psi[z] \quad (4.28)$$

[where  $g_\mu(X) = (-i/2) \lambda(X) {}^n\Delta^{1/4} n_\mu$ ] and, similarly,

$$\langle \exp z | \Phi_\lambda | \psi \rangle = \int d^4 X w_\mu \left( \frac{\delta}{\delta z^{*\mu}} - z^{*\mu} \right) \psi[z] \quad (4.29)$$

[where  $w_\mu(X) = (i/2) P_\mu^\nu \partial_\nu \lambda(X) {}^n\Delta^{1/4}$ ].

So now we can consider the action of the constraints on a general wave functional  $\psi[z]$ , finding

$$e^{i\Psi_\lambda} \psi[z] = e^{-(1/2)\langle g, g \rangle - i\langle z, g \rangle} \psi[z + ig], \quad (4.30)$$

and, similarly,

$$e^{i\Phi_\lambda} \psi[z] = e^{-(1/2)\langle w, w \rangle - i\langle z, w \rangle} \psi[z + iw]. \quad (4.31)$$

We can now explicitly see that  $\mathcal{V}_{\text{phys}}$  will not be a subspace of  $\mathcal{V}$  as we require the subspace to be invariant under the action of the constraints, and are thus essentially looking for solutions to the pair of equations

$$\psi[z] = \psi[z + ig], \quad (4.32)$$

$$\psi[z] = \psi[z + iw]. \quad (4.33)$$

The solutions to these will be  $\psi[z^\perp]$ , where  $z_\mu^\perp(X)$  are only the transverse components of  $z_\mu(x)$  defined as

$$z_\mu^\perp(X) = \left( \frac{{}^n\partial_\mu {}^n\partial^\nu}{{}^n\Delta} - P_\mu^\nu \right) z_\nu(X) \quad (4.34)$$

(where  ${}^n\partial_\mu$  is just shorthand for  $P_\mu^\rho \partial_\rho$ ). However, it is clear that the corresponding wave-functionals,  $\psi[z^\perp]$ , will not be square integrable. To see this, we need only consider  $\int \int \int |\psi[z^\perp]|^2 d\sigma[z^\perp] d\sigma[z^0] d\sigma[z^\parallel]$  which will be infinite on account of the contributions from the integrations over the scalar and longitudinal parts. This leads us to the conclusion that what we need to do is to construct the physical Hilbert space *separately* so that the wave-functionals,  $\psi[z^\perp]$ , are square-integrable, and then define a suitable mapping from  $\mathcal{V}$  to  $\mathcal{V}_{\text{phys}}$ .

Equipped with what we know from the classical theory, and what we have ascertained from the analysis above, we construct  $\mathcal{V}_{\text{phys}}$  in the usual way—first by positing the algebra

$$[A_\mu(f^{\perp\mu}), A_\nu(f'^{\perp\nu})] = 0, \quad (4.35)$$

$$[\pi_\mu(h^{\perp\mu}), \pi_\nu(h'^{\perp\nu})] = 0, \quad (4.36)$$

$$[A_\mu(f^{\perp\mu}), \pi^\nu(h_\nu^\perp)] = i \int d^4X \delta_\mu^\nu f^{\perp\mu}(X) h_\nu^\perp(X), \quad (4.37)$$

where the smearing functions belong to  $\mathcal{T}_R^\perp = L_R^2(\mathbb{R}^4, d^4X) \oplus L_R^2(\mathbb{R}^4, d^4X)$ . We then take the complexification of this space  $\mathcal{T}_C^\perp = \mathcal{T}_R^\perp \oplus \mathcal{T}_R^\perp$  and exponentiate the resulting space to give  $\mathcal{V}_{\text{phys}} = e^{\mathcal{T}_C^\perp}$ . We can then write a representation of the transverse fields in terms of the creation and annihilation operators of this Fock space:

$${}^n A_\mu^\perp(X) = \frac{1}{\sqrt{2}} {}^n \Delta^{-1/4} (b_\mu^\perp(X) + b_\mu^{\perp\dagger}(X)), \quad (4.38)$$

$${}^n \pi_\mu^\perp(X) = -\frac{i}{\sqrt{2}} {}^n \Delta^{1/4} (b_\mu^\perp(X) - b_\mu^{\perp\dagger}(X)), \quad (4.39)$$

where

$$[b_\mu(z^{\perp\mu}), b^{\dagger\nu}(z'^{\perp\nu})] = \langle z^\perp, z'^\perp \rangle, \quad (4.40)$$

and  $z_\mu^\perp(X) = (1/\sqrt{2}) ({}^n \Delta^{1/4} h_\mu^\perp(X) + i {}^n \Delta^{-1/4} f_\mu^\perp(X))$ .

In direct analogy to  $\mathcal{V}$ , we can consider  $\mathcal{V}_{\text{phys}}$  in terms of the un-normalized coherent states defined by

$$|\exp z^\perp\rangle_{\mathcal{V}_{\text{phys}}} = e^{b^\mu(z_\mu^\perp)} |0\rangle, \quad (4.41)$$

and these will admit a resolution of unity:

$$1 = \int |\exp z^\perp\rangle \langle \exp z^\perp| d\sigma[z^\perp], \quad (4.42)$$

and thus an integral representation for  $|\psi\rangle \in \mathcal{V}_{\text{phys}}$ :

$$|\psi\rangle = \int \psi[z^\perp] |\exp z^\perp\rangle_{\mathcal{V}_{\text{phys}}} d\sigma[z^\perp]. \quad (4.43)$$

We now define a mapping between  $\mathcal{V}$  and  $\mathcal{V}_{\text{phys}}$ :

$$L: \mathcal{V} \rightarrow \mathcal{V}_{\text{phys}},$$

$$|\exp z\rangle_{\mathcal{V}} \mapsto L(|\exp z\rangle_{\mathcal{V}}) \equiv |\exp z^\perp\rangle_{\mathcal{V}_{\text{phys}}}, \quad (4.44)$$

where  $|\exp z^\perp\rangle_{\mathcal{V}_{\text{phys}}}$  is defined as in Eq. (4.41). We define the (continuous) dual mapping:

$$L^\dagger: \mathcal{V}_{\text{phys}}^* \rightarrow \mathcal{V}^* \quad (4.45)$$

by

$${}_{\mathcal{V}_{\text{phys}}} \langle \exp z^\perp | L^\dagger | \exp w \rangle_{\mathcal{V}} = {}_{\mathcal{V}_{\text{phys}}} \langle \exp z^\perp | L | \exp w \rangle_{\mathcal{V}}. \quad (4.46)$$

We can now use these maps (and the fact that, due to the Riesz Lemma (see, e.g., Ref. 24), there is an isomorphism between a Hilbert space,  $\mathcal{H}$ , and the space of continuous linear functionals,  $\mathcal{H}^*$ , from  $\mathcal{H}$  to  $\mathbb{C}$ ) to relate objects on  $\mathcal{V}_{\text{phys}}$  to objects on  $\mathcal{V}$ :

$$b_{\mathcal{V}_{\text{phys}}} = L b_{\mathcal{V}} L^\dagger. \quad (4.47)$$



Having now established the relationship between the “full” Hilbert space,  $\mathcal{V}$ , and the “physical” Hilbert space,  $\mathcal{V}_{\text{phys}}$ , we can now turn to the issue of Poincaré invariance and use Eq. (4.47) to define the action operator on  $\mathcal{V}_{\text{phys}}$ .

### C. The Poincaré groups

In the case of classical histories electromagnetism, we proved the existence of the two Poincaré groups on the histories phase space,  $\Pi$ , and analyzed their “reduction” to the reduced phase space,  $\Pi_{\text{red}}$ , by considering their compatibility with the constraints. We demonstrated the existence of the internal group on  $\Pi$ , finding that the algebra closed only weakly, i.e., was only satisfied on the constraint surface. We then proved the existence of a “reduced” internal Poincaré group on  $\Pi_{\text{red}}$ , with the generators written in terms of the transverse components of the fields. We also demonstrated the existence of the external Poincaré on  $\Pi$ , but found that the external boost generator did not commute with the constraints, and thus could not be represented on  $\Pi_{\text{red}}$ . This, as we shall see in greater detail in the quantum case below, results from the fact that  $\Pi$  and  $\Pi_{\text{red}}$  are foliation dependent, and that the external boost boosts the foliation vector as well. So let us now discuss the issue of Poincaré invariance in the quantum theory.

#### 1. The internal Poincaré group

Our starting point for the internal Poincaré group is (a normal ordered version of) the unconstrained Hamiltonian given in Sec. IV A 1 and repeated here:

$$H = \frac{1}{2} \int d^4X (\pi_\mu^\perp \pi^{\perp\mu} + A_\mu^\perp \Delta A^{\perp\mu} + \pi_\mu^\parallel \pi^{\parallel\mu}). \quad (4.48)$$

In terms of the creation and annihilation operators [Eqs. (4.38)–(4.39)], this reads as

$$H = \int d^4X \left[ b_\mu^{\perp\dagger} \Delta^{1/2} b^{\perp\mu} - \frac{1}{4} ((b_\mu^\parallel - b_\mu^{\parallel\dagger}) \Delta^{1/2} (b^{\parallel\mu} - b^{\parallel\dagger\mu})) \right]. \quad (4.49)$$

However, while the transverse part can easily be shown to exist in the usual way, the longitudinal part does not generate automorphisms which are unitarily implementable on account of the presence of terms quadratic in  $b_\mu$  and  $b_\mu^\dagger$ . And so the Hamiltonian does not exist on  $\mathcal{V}$  as a self-adjoint operator. Of course, this is no tragedy and we half-expected it anyway as we had already seen in the classical case that the algebra of the internal group closed only weakly.

What is important is that a representation of the internal group can be found on  $\mathcal{V}_{\text{phys}}$ . This is straightforward. The generators are taken straight from the classical case, suitably ordered and then written in terms of  $b_\mu^\perp$  and  $b_\mu^{\perp\dagger}$  using Eqs. (4.38)–(4.39). They are

$$\tilde{H} = \int d^4X b_\mu^{\perp\dagger} \Delta^{1/2} b^{\perp\mu}, \quad (4.50)$$

$$\tilde{P}(m) = i m_\nu \int d^4X b_\mu^{\perp\dagger} \partial^\nu b^{\perp\mu}, \quad (4.51)$$

$$\tilde{J}(m) = i \epsilon_{\mu\nu\rho\sigma} n^\mu m^\nu \int d^4X (b_\alpha^{\perp\dagger} X^\rho \partial^\sigma b^{\perp\alpha} + b^{\perp\dagger\rho} b^{\perp\sigma}), \quad (4.52)$$

$$\tilde{K}(m) = m_\nu \int d^4X b_\mu^{\perp\dagger} \Delta^{1/4} X^\nu \Delta^{1/4} b^{\perp\mu}, \quad (4.53)$$

where we have used an obvious shorthand for operators on  $\mathcal{V}_{\text{phys}}$  [see Eq. (4.47)], i.e.,

$$b^{\perp\mu} \equiv b_{\mathcal{V}_{\text{phys}}}^\mu = L b_\mathcal{V}^\mu L^\dagger. \quad (4.54)$$

The analysis of this group is essentially the same as the classical case. We define the Heisenberg picture fields:

$$b_{\mu}^{\perp}(X, s) = e^{is\tilde{H}} b_{\mu}^{\perp}(X) e^{-is\tilde{H}}, \quad (4.55)$$

$$b_{\mu}^{\perp\dagger}(X, s) = e^{is\tilde{H}} b_{\mu}^{\perp\dagger}(X) e^{-is\tilde{H}}. \quad (4.56)$$

The Hamiltonian generates transformations in the internal time label “ $s$ ,” and the internal boost mixes the internal time parameter with the spatial coordinates. These transformations all happen at constant “ $t$ ” (where “ $t$ ” is the external time parameter).

## 2. The external Poincaré group

As in the case of the scalar field, one of the novel features of histories theories is the existence of a second Poincaré group—the external group—that is associated with the external time label, “ $t$ .” Again we start from the classical expressions, suitably ordered:

$$P^{\mu} =: \int d^4X \pi^{\nu} \partial^{\mu} A_{\nu} : \quad (4.57)$$

$$M^{\mu\nu} =: \int d^4X [\pi^{\rho} (X^{\mu} \partial^{\nu} - X^{\nu} \partial^{\mu}) A_{\rho} + (\pi^{\mu} A^{\nu} - \pi^{\nu} A^{\mu})] : \quad (4.58)$$

Note that these expressions use the covariant fields defined in Eqs. (4.8)–(4.9), and thus we write

$$P^{\mu} = i \int d^4X b^{\dagger\nu} \partial^{\mu} b_{\nu}, \quad (4.59)$$

$$M^{\mu\nu} = i \int d^4X [b^{\dagger\rho} (X^{\mu} \partial^{\nu} - X^{\nu} \partial^{\mu}) b_{\rho} + (b^{\dagger\mu} b^{\nu} - b^{\dagger\nu} b^{\mu})]. \quad (4.60)$$

As in the classical case, the Liouville operator,  $V = n_{\mu} P^{\mu}$ , generates translations in the external time parameter. And it is the external boost generator,  ${}^{\text{ext}}K(m) = n_{\mu} m_{\nu} M^{\mu\nu}$  that is of the most importance as we can see in its action on foliation dependent objects:

$$U(\Lambda)^n A_{\mu}(X) U(\Lambda)^{-1} = \Lambda_{\mu}^{\nu} \Lambda^n A_{\nu}(\Lambda^{-1}X), \quad (4.61)$$

where  $U(\Lambda) = e^{iK(m)}$ . The crucial point here is that it generates Lorentz transformations on the foliation vector as well. Let us now analyze this issue in a bit more detail.

Though the set of all coherent states is independent of the foliation vector,  $n_{\mu}$  (they are eigenstates of the annihilation operator), the definition of them in terms of the Weyl generator, Eqs. (4.21)–(4.22) is clearly not. It is thus that the Fock space,  $\mathcal{V}$ , depends upon the choice of foliation. Now, as in the case of the scalar field, all the foliation-dependent representations of the history algebra exist on the same Fock space,  $\mathcal{V}$ , and  ${}^{\text{ext}}K(m)$  relates the objects defined with respect to a foliation “ $n$ ,” with those same objects defined with respect to the foliation “ $\Lambda n$ .” For example, under  ${}^{\text{ext}}K(m)$ , the constraint operators will transform

$${}^n\Psi_{\kappa} \xrightarrow{{}^{\text{ext}}K(m)} {}^{\Lambda n}\Psi_{\kappa}, \quad (4.62)$$

$${}^n\Phi_{\kappa} \xrightarrow{{}^{\text{ext}}K(m)} {}^{\Lambda n}\Phi_{\kappa}. \quad (4.63)$$

Now the map  $L$  from  $\mathcal{V}$  to  $\mathcal{V}_{\text{phys}}$  is also evidently  $n$ -dependent and thus  $\mathcal{V}_{\text{phys}}$  also depends on the choice of foliation used to define  $\mathcal{V}$ . However, whereas all foliation-dependent representations can

exist on  $\mathcal{V}$  (and we can thus talk about transformations between them), the physical Hilbert spaces,  ${}^n\mathcal{V}_{\text{phys}}$  and  ${}^{\Lambda n}\mathcal{V}_{\text{phys}}$  (where, we trust, the point of the added superscript is self-evident), are clearly different. This is why there will be no representation of  ${}^{\text{ext}}K(m)$  on  $\mathcal{V}_{\text{phys}}$ . Mathematically (and analogously to the classical case) this situation is represented by the fact that the external boost does not (weakly) commute with either of the constraints.

Of course, we can still relate the important quantities on  ${}^n\mathcal{V}_{\text{phys}}$  such as the action,  ${}^nS$ , to those same quantities on  ${}^{\Lambda n}\mathcal{V}_{\text{phys}}$  via the prescription given at the end of Sec. IV B, i.e., by mapping back to  $\mathcal{V}$ , boosting there, and then mapping to  ${}^{\Lambda n}\mathcal{V}_{\text{phys}}$ .

The other generators will be represented on  $\mathcal{V}_{\text{phys}}$ , we just make use of Eq. (4.47) to define them. The most important of these is the Liouville operator, and this will be defined on the physical Hilbert space as

$$\tilde{V} = i \int d^4X b_v^{\perp \dagger} n_\mu \partial^\mu b^{\perp \nu}. \quad (4.64)$$

This will generate time transformations in the external time label, “ $t$ ,” on the physical Hilbert space. We thus arrive, using Eqs. (4.50) and (4.64), at the definition of the action operator on the physical Hilbert space:

$$\tilde{S} = \tilde{V} - \tilde{H}. \quad (4.65)$$

## V. CONCLUSION

Our aim in this paper has been to construct a histories theory of Electromagnetism working in the HPO consistent histories framework. As a vector field theory with two first class constraints, we have built on the work of Savvidou<sup>1</sup> on scalar field theory, as well as demonstrating an application of the constrained systems algorithm developed by Savvidou and Anastopoulos.<sup>3</sup>

Classically, we defined the histories phase space and the two Poincaré groups that are a feature of histories field theories. The constraints partition the constraint surface (and indeed the whole phase space) into orbits, and by defining coordinates that are constant on each orbit, we defined the reduced phase space that carries the physical degrees of freedom of the theory. We stressed the importance of the foliation dependence of the phase space (and thus the reduced phase space) focusing particularly on the action of the external boost generator which transforms between different foliations.

Quantizing within the Dirac scheme, we first constructed the Hilbert space of the unconstrained theory ( $\mathcal{V}$ ), motivated by finding a suitable representation of the History algebra. We then defined the constraints as operators, and, making use of the technology of coherent states, sought to define the physical Hilbert space ( $\mathcal{V}_{\text{phys}}$ ). As the constraints have continuous spectra, this was not going to be a true linear subspace of the full Hilbert space. We got around this issue by analyzing  $\mathcal{V}$  in terms of coherent states, which led to a definition of  $\mathcal{V}_{\text{phys}}$  in terms of just the transverse components of the vector field and their conjugate momenta (or, more strictly, the space of test functions). We then defined a suitable mapping from  $\mathcal{V}$  to  $\mathcal{V}_{\text{phys}}$  and used this to define the action operator on  $\mathcal{V}_{\text{phys}}$ .

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