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NUCLEAR MODELS

The spherical shell, the deformed shell, "infinite nuclear matter," the droplet—each of these models provides a simplification that can be useful, in appropriate circumstances, to the study of the nucleus.

DAVID R. INGLIS

To UNDERSTAND the behavior of the atomic nucleus we resort to models, for the complexity of the problems that we meet demands simplification if progress is to be made.

We can treat each nucleon as having an independent existence in the potential well of the nucleus as a whole (the "independent-nucleon shell model"), a method patterned on the Bohr model of the atom. We can introduce interaction among individual nucleons, by short-range forces. Basis wavefunctions with ellipsoidal equipotential surfaces are appropriate when we try to account for strongly deformed nuclei with quadruple moments and rotational spectra (the "deformed-shell model"). Scattering nucleons from each other tells us about the effective interaction between them, and allows us to work out the properties of "infinite nuclear matter," which we assume are similar to those of the centers of heavy nuclei. For fission we can use the "droplet model."

A major problem is that of developing one model sufficiently to predict all the interesting properties—for example, an independent-nucleon model that includes a fission barrier, as in the droplet model, but with two humps to account for some recent observations.

What is a model?

The concept of a model varies from one profession to another. In the graphic arts, a model is the real thing of which the artist makes an imitation. In physics, and in technology in general, a model is an imitation of the real thing. A model-boat builder can see the real boat and incorporate as many of its features into the model as the smaller scale permits. The physicist cannot see the atomic nucleus, but through various inadequate probes he can sense some its important properties. He finds them so elusive, and in some cases almost self-contradictory, that he uses several models; one model may emphasize relationships between some of the properties while another emphasizes different properties. One may portray the most essential characteristics of one set of nuclei, another those of another set.

A model of a molecule or a crystal can be built of sticks and balls, but a model of a nucleus is seldom a physical portrayal of this sort. It is instead a mathematical treatment of a simplified and imaginary mechanical system. If the results of the analysis correspond rather well with results of observations on the nucleus, then the properties assumed for the model are believed to have some relationship to the properties of the actual nucleus. Our concept of the nucleus develops as a synthesis of the properties of several successful models, often with one model preponderant.

Nuclei are so complicated that we will never know all about them. Each of the various models brings out some of the main features and leaves room for the physicist to fill in further details as the work progresses. Some models lead to calculations of great complexity. The test of their merit comes in their ability to correlate experimental results, preferably simply and elegantly, and to suggest meaningful new experiments.

When a model boat sails it is guided by the same classical mechanics as the full-size boat. The nuclear model must, like the nucleus, follow quantum mechanics and thus cannot be just a pictorial replica on a different scale. Nevertheless, my survey here of some

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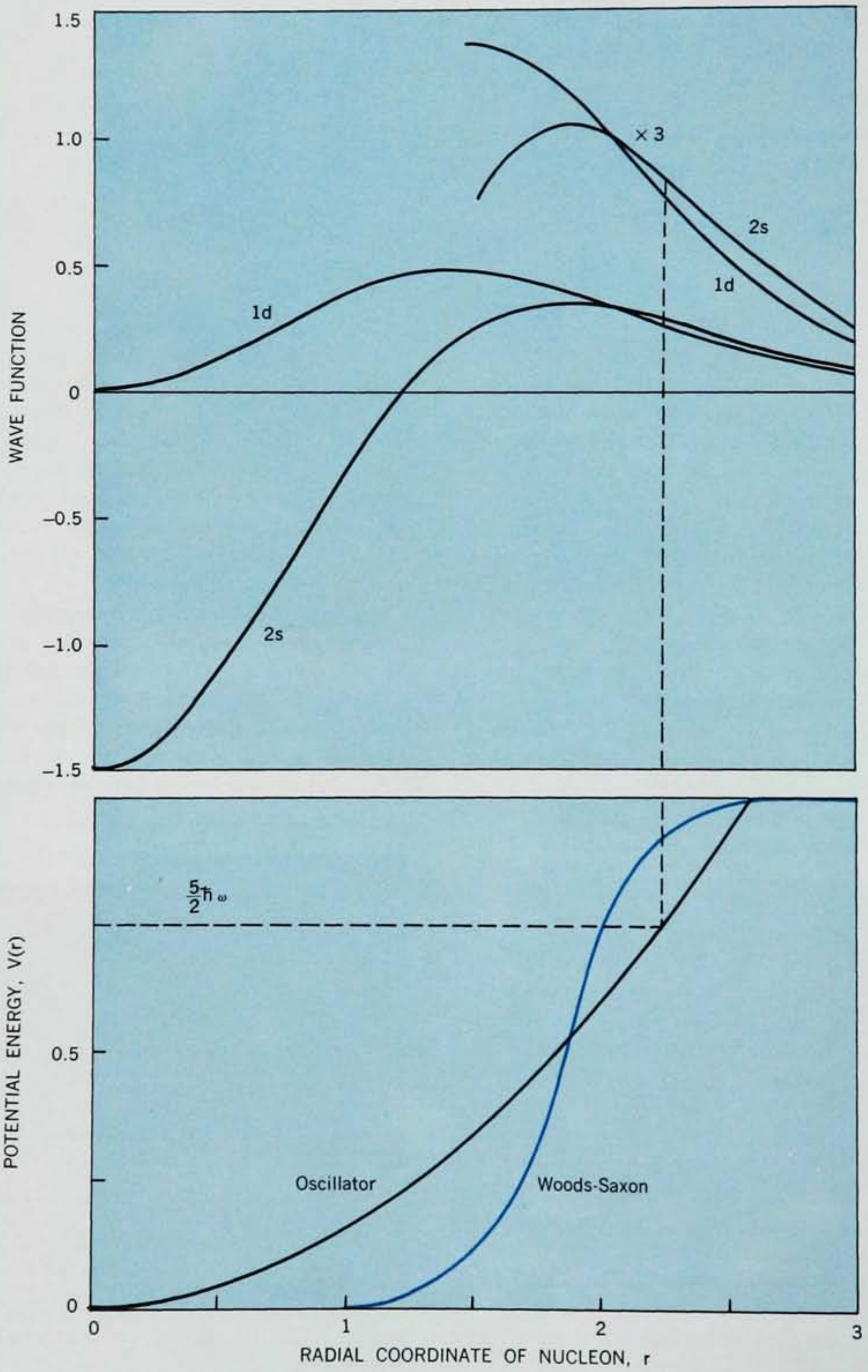
of the important nuclear models takes a pictorial approach in the spirit of the correspondence principle, in the knowledge that there is at least a rough correspondence between quantum mechanics and quantized classical physics that gets better as quantum numbers get larger.

Independent-nucleon shell model

The Bohr model of the atom, as translated into wave mechanics by Erwin Schrödinger and others, worked so well that it was natural to use a very similar model, the shell model, when energy spectra showed up in nuclei. In a typical atom the potential felt by a single electron is very steep and deep near the center, and flat toward the edges. The low single-particle levels are consequently widely separated, and the higher ones are squeezed together toward the zero-binding limit. The sharp depression at the center arises of course from the electric attraction of the central nucleus. In nuclei there is no such big boss at the center. An atom is a dictatorship but a nucleus is an ideal democracy where all nucleons have an equal opportunity to influence the shape of things. The consequence is a great leveling toward a common norm. The potential felt by each nucleon, due to the short-range attraction toward all the others, is a rather flat-bottomed well with fairly steep sides, and the single-nucleon energy levels are fairly evenly spaced. For mathematical convenience, the potential is usually approximated by that of a three-dimensional harmonic oscillator, in which the groups of levels are exactly evenly spaced.

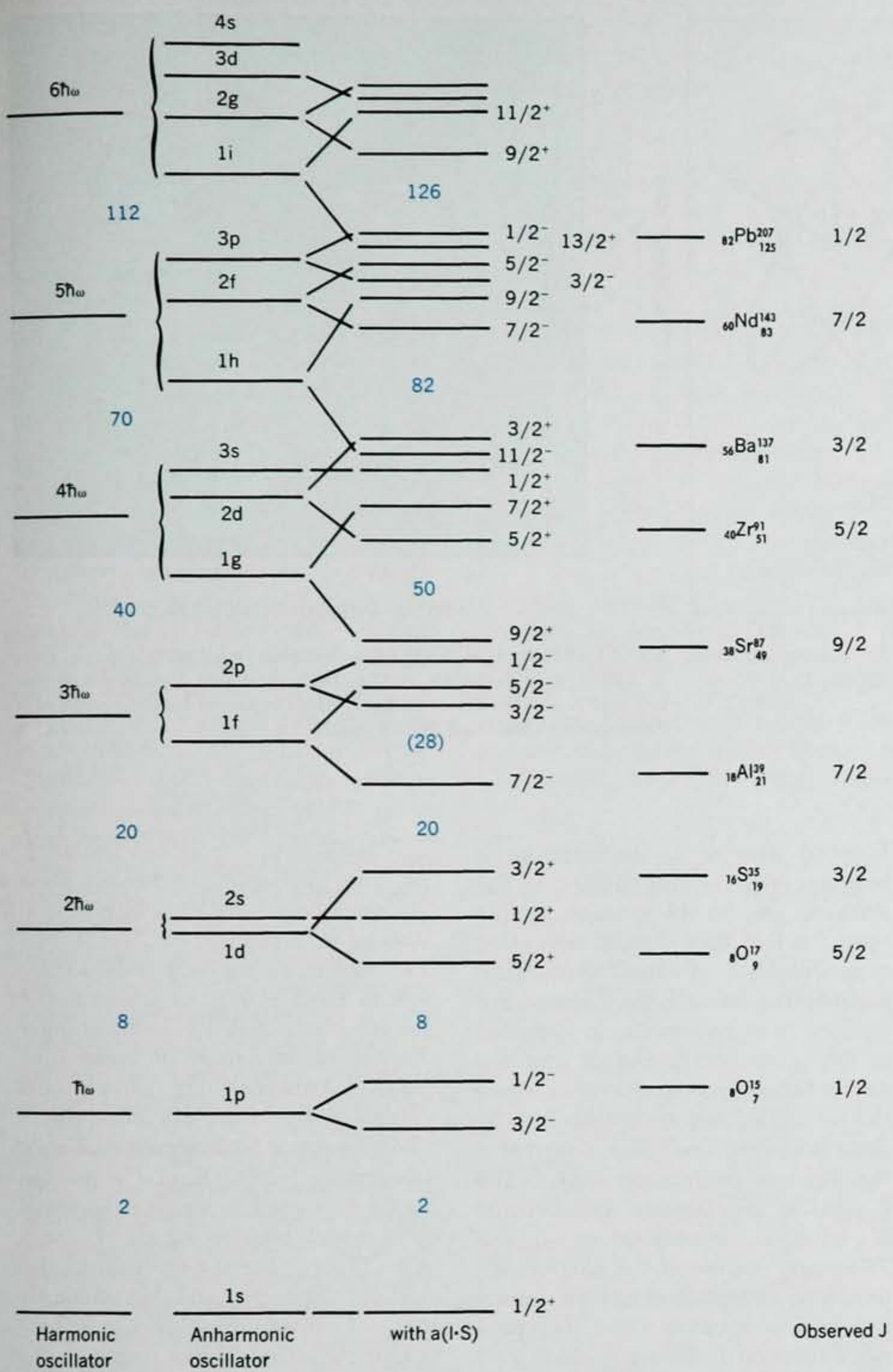


TWO SHELL-MODEL POTENTIALS (below): the oscillator or parabolic well and the Woods–Saxon approximation. Some wavefunctions, 2s and 1d, for the oscillator potential are shown above. —FIG. 1

This approximation may be visualized by thinking of a two-dimensional harmonic oscillator in terms of a small ball rolling about in a round bowl that has a parabolic cross section. If the motion is quantized, the energy levels climb in equal steps like the rungs of a ladder but with several states of motion corresponding to each rung.

These states have various integral values of the nucleon's orbital angular momentum, which is indicated (in units of Planck's $\hbar$) by the quantum number l but has orientation in space represented by the vector **l** perpendicular to the plane of the orbital motion. With the three-dimensional harmonic-oscillator potential, each rung of the ladder may have states of several values of l, either all odd or all even, and the states are said to have odd or even parity. The states are denoted by the letters s, p, d, f, g, . . ., for $l = 0, 1, 2, 3, 4,$ . . ., and the third rung of the ladder has the two states called 1d and 2s, the first state with $l = 2$ and the second state with $l = 0$ (the first being on the lowest rung).

The actual potential is somewhat more flat-bottomed than a parabola; in figure 1 we see a comparison of a parabolic well with D. S. Saxon's and R. D. Woods's approximation to the nuclear potential well. The actual potential lies below the oscillator potential in the region where, for example, the 1d wave function is large and the 2s function has a node. The nucleon in a 1d state thus has a higher probability of being where the potential energy is lower, and the 1d state has a lower energy than the 2s state.



SINGLE-NUCLEON ENERGY LEVELS in increasingly realistic models, compared with observed data (on right). Magic numbers are in color. —FIG. 2

The flatter bottom and steeper sides thus pull down the levels of higher angular momentum within the group of levels at each rung of the ladder. The energy levels thus deviate from the uniform harmonic-oscillator spacing, as shown in the first two columns of figure 2.

Spin-orbit coupling

But each nucleon also has its spin angular momentum represented by a vector **s** of length 1/2 (again in units of $\hbar$). The spin and orbit are coupled energetically in such a way that the energy of the state depends not only on **l** but also on the angle between **s** and **l**. The vector sum of **s** and **l** of each nucleon is called **j**, and when the nucleon is not too much mixed up with the vectors of other nucleons, j is quantized to be either $l + 1/2$ or $l - 1/2$. These two states differ quite substantially in energy when the spin-orbit coupling is large. The state with larger j has lower energy, the opposite of the situation in atoms.

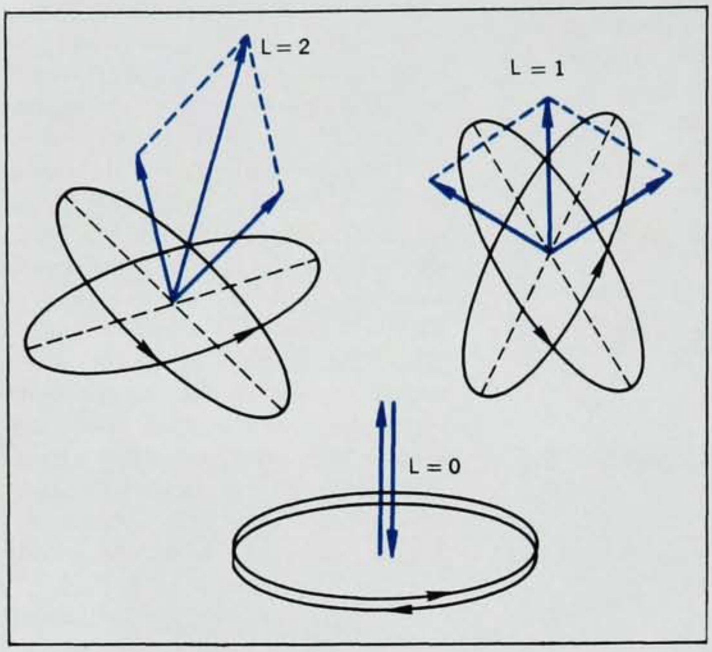
Spin-orbit coupling is larger for large angular momenta than for small angular momenta, and the consequent splitting of the levels of various orbital angular momentum l into those with $j = l \pm 1/2$ gives rise to the spectrum of single-particle levels shown in the third column of figure 2. They are grouped with energy gaps at the famous "magic numbers." When the number of protons in a nucleus is one of the "magic numbers," they form a filled shell, and likewise for neutrons. The closed-shell nuclei are particularly stable—a nucleon added just beyond the magic number is relatively weakly bound.

When the nucleons are thus almost independent of one another, with **l** and **s** coupled together to give the total angular momentum vector **j** for each nucleon, then if there are two or more neutrons outside of closed shells their **j**'s are added to give a grand total **J** (also quantized) for the whole nucleus. Since it contains two j's, it is called the "jj coupling" scheme. A "closed shell" consists of as many nucleons as the law (or Pauli principle) allows crowded into one energy level, with their vectors pointing all ways to yield a total angular momentum of zero.

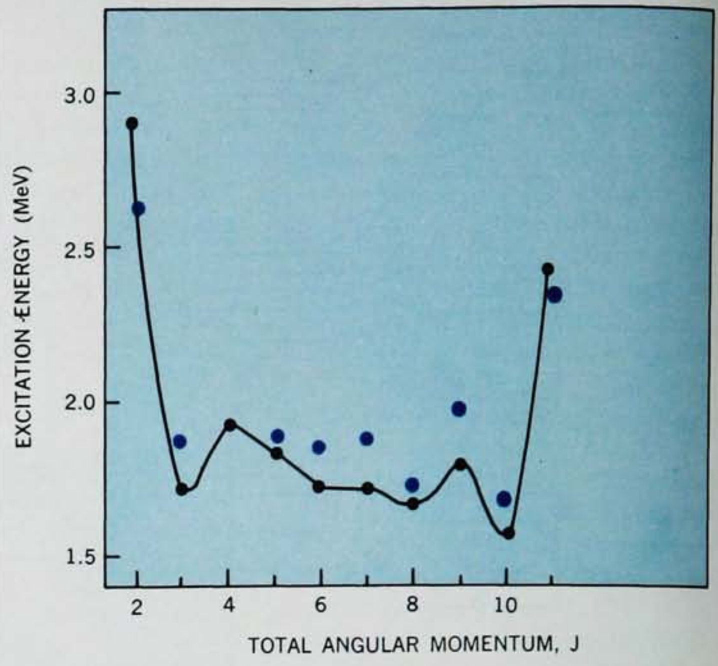
This *jj*-coupling shell model for the single-nucleon states is remarkably successful in describing properties of the nuclei very near the magic numbers. The last two columns of figure 2 indicate that the observed angular momenta of the nuclei either lacking one nucleon from a closed shell or with one nucleon in addition to a closed shell (and with an even number of nucleons of the other kind) agree beautifully with the model.

Nucleons in a light nucleus

When we take into account the interaction among the nucleons by a short-range and predominantly attractive force, the story becomes more complicated. In the light nuclei, in particular in the first p shell from lithium to oxygen, the interaction is so strong, compared to the spin–orbit coupling, that the *jj*-coupling scheme is no longer valid. Indeed, for lithium we find the opposite *LS*-coupling scheme. In this version of the shell model, the orbital angular momenta $\mathbf{l}_i$ couple together to make a vector sum **L**, and similarly for the spins. The effective force between nucleons de-



ORIENTATION of the orbital planes for three two-nucleon states; the angle between the planes is the same for L = 2 and L = 1 and is zero for L = 0. —FIG. 3



EXCITATION ENERGY in a bismuth multiplet, $(\nu i_{13/2})^{-1}\pi h_{9/2}$, in $_{83}Bi^{208}_{125}$, as a function of the angle between the j's for a proton and a neutron hole. Observed data are in black; results of a delta-force calculation are in color. —FIG. 4

pends on whether the space wave function is symmetric or antisymmetric in the space coöordinates, and also somewhat on the spin orientations.

In spite of these complications an important part of the energy spacing of the two-nucleon levels in Li^6, for example, depends on the simple geometric considerations sketched in figure 3. The angular momentum vectors l_1 and l_2 of the two nucleons add up to $L = 0$, 1 or 2. With $L = 0$, the two are exactly antiparallel, the orbital planes coincide, and the average interaction energy is large (and negative) so the energy of the S-state is low. Classically, one would think that the $L = 2$ state would also involve parallel l's and coincident orbital planes, but when one substitutes the proper expression $\sqrt{L(L+1)}$ for the length L of the total angular momentum vector, and similarly for the l's, there is a 60-deg angle between the l's and between the orbital planes. With such small quantum numbers, the angles are far removed from their classical values. The uncertainty principle will not let us define parallelism of the l's more closely than that. It happens in this case that the angle between the orbital planes is the same for $L = 2$ and for $L = 1$, as shown in figure 3. These D and P states, as they are called, are thus higher than the S state (which has $L = 0$) because of the large angle between the orbital planes. The nucleons are, on the average, farther apart, so that they attract each other more weakly. Another geometrical effect is that, because the P-state wave function is antisymmetric in exchange of the space coördinates of two nucleons and is thus equal to zero where they coincide, they are not likely to be close together, and this too makes the average interaction weak. The P state is the highest also because of what is sometimes called the "Majorana" nature of the effective interaction (which changes an attraction into a repulsion when the space wave function is antisymmetric).

A simple heavy nucleus

With the big new tandem accelerators, detailed investigation of nuclear spectra, by means of d,p reactions and so on, have been extended to the very heavy nuclei where quantum numbers are large enough to make the semiclassical pictures more nearly valid. An interesting new example is that of $_{83}Bi^{208}_{125}$, as observed at Rochester.[1] On looking back to figure 2, we see that this nucleus has one proton outside the 82 shell and lacks one neutron from the 126 shell. The fairly low excited state with the largest quantum numbers has the proton in an $h_{9/2}$ state, with $l = 5$ and $j = 9/2$, and the neutron "hole" in the $i_{13/2}$ state, with $l = 6$ and $j = 13/2$. In the jj-coupling scheme, these momenta couple together to give total angular momentum $J = 2, 3, 4, \ldots 11$, varying all the way from $13/2 - 9/2$ to $13/2 + 9/2$. The energies of the ten states of this jj multiplet differ because of the interaction. This is repulsive between the particle and "hole," rather than attractive, for a "hole" is just a bookkeeping device for accounting for the lack of a particle. In figure 4 you can see that the nearly antiparallel position of the j's, with $J = 2$, gives the strongest interaction and highest energy and that the nearly parallel orientation is also exceptionally high. The colored points on the figure represent the result of calculations with an infinitely short-range (or "delta-function") force,[1] and follow the trend quite well.

Although the simple concept of closeness of the orbital planes is important here, we are still rather far from the classical limit, and some of the complication in this relatively simple case is suggested in figure 5. In these pictures of the jj-coupling scheme, the precession of each l_i and s_i to give a nearly constant j_i is indicated by a little ellipse suggesting the cone of precession of l_i. In the nearly antiparallel case $J = 2$, the two cones come very close to having a straight line in common (figure 5c). The angle between the orbital planes (normal to the l_i vectors) thus varies through a small range that almost includes zero, and the average interaction is strong. The situation is not quite so favorable for a strong interaction in the case of the maximal J, $J = 11$, because of the way the substitution for J of $\sqrt{J(J+1)}$ affects the cosine of an angle that varies slowly with the angle (figure 5b).

The classical concept of an orbital plane is of course an oversimplification. With it, a delta-function force would give zero interaction except when the planes are actually coincident. What takes its place is suggested in figure 5d, for the simple case of a j-vector with a projection m_j along an axis. The wave function here is a sum of two terms. The first, which more closely resembles the jj picture with its spin down, enters with a large coefficient C and has a spatial distribution resembling a fuzzed-out orbital plane. Some of the probability-amplitude distribution lies outside the extreme position of the classical orbital plane that is indicated by a broken black line. The small term, with the coefficient c, has positive and negative parts of its spatial wave functions. In the more complicated coupling scheme at the top of figure 5, such terms can cause cancellations to make average interactions small when the small coefficients become larger in cases of intermediate vector orientations.

Intermediate coupling

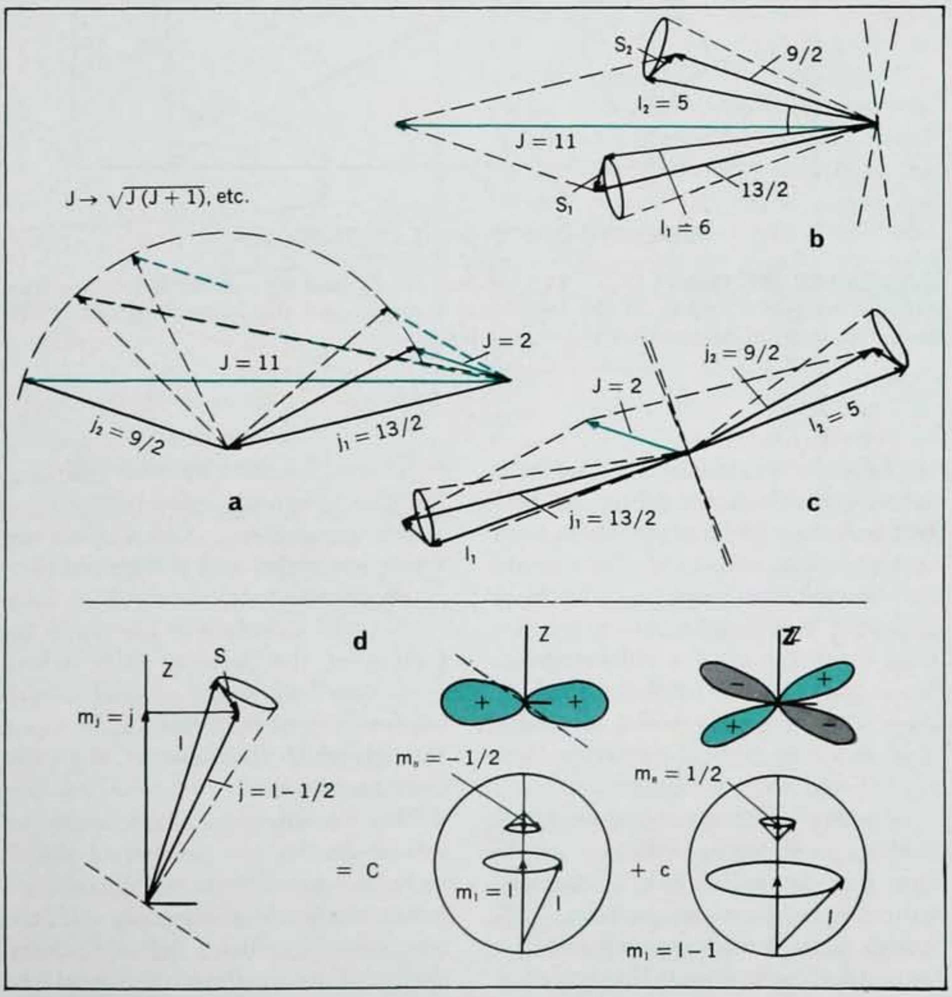
One of the complications in the shell model for several nucleons is the competition among various types of forces, such as that between the central attractive force linking two nucleons and the spin-orbit coupling of each nucleon. The simplest case of this is found in Li^6 and He^6, as shown in figure 6. There are four nucleons in a closed s shell and two nucleons coupled in the p shell. On the left side there are double superscripts. The simple symbol ^3P means "triplet P," with total spin $S = 1$, as in atomic spectra. The additional preceding superscript in ^{33}P means isospin $T = 1$ as well. This addition simply means that the symmetry of the state is such that the Pauli principle will allow both nucleons to be neutrons, that is, the state exists in He^6 as well as Li^6, whereas the state ^{13}D, for example, is a ^3D state with $T = 0$ that exists only in Li^6. Among the $T = 1$ states, the S state is lowest, and the S and D states lie well below the P state, as we have seen they should. The parameter a measures the strength of spin-orbit coupling in the energy term

$$a\mathbf{l}_i \cdot \mathbf{s}_i$$

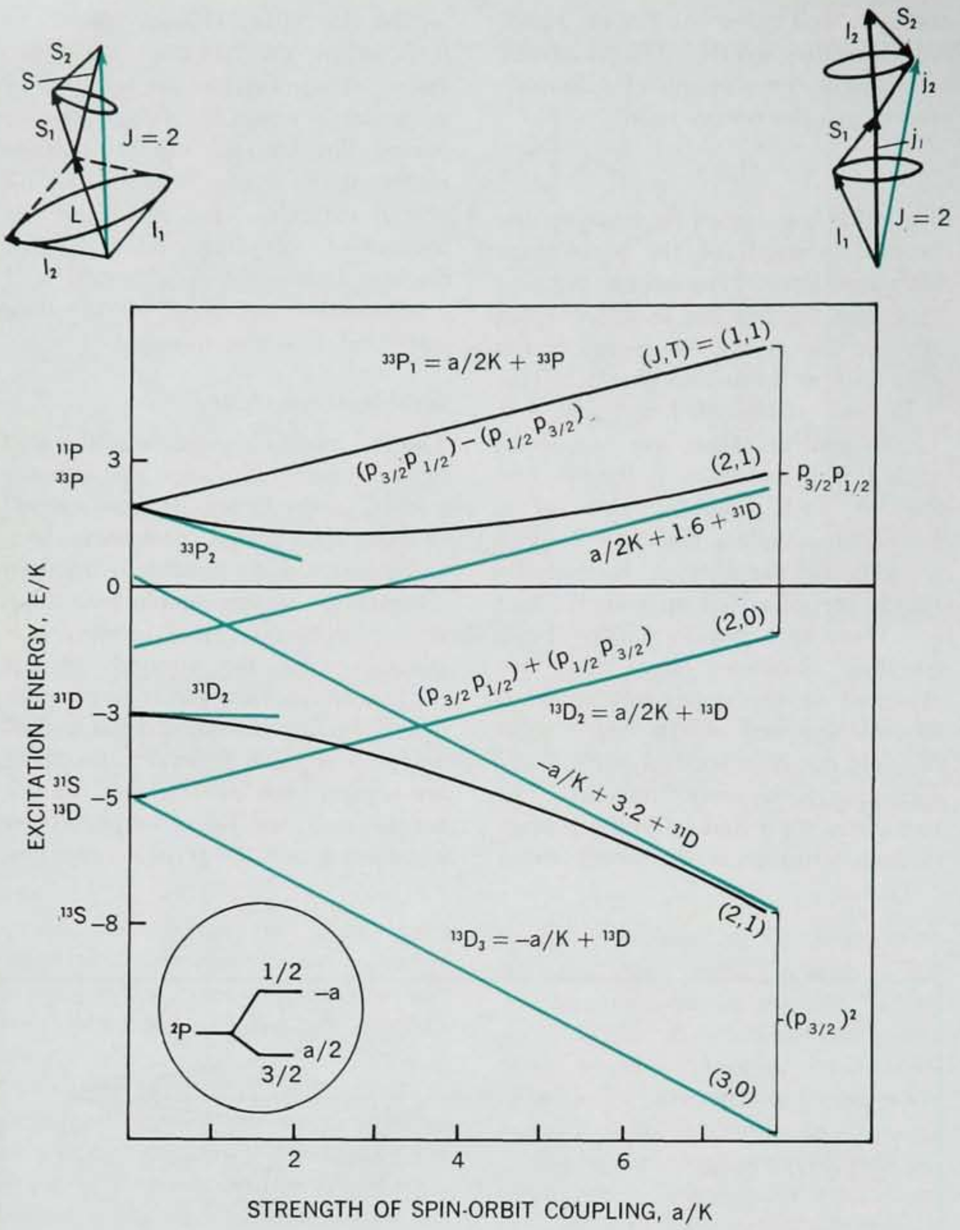
The other parameter, K, measures the average strength of the short-range attractive force. The energy is linear in a, and the top line in figure 6 is a straight line giving the energy of the $^{33}P_1$ term as a function of a/K. This is the only state with $J = 1$ and $T = 1$. In contrast, there are two states with $J = 2$ and $T = 1$, the $^3P_2$ and the $^1D_2$ with different values of S. Spin-orbit coupling does not leave S a constant of the motion. Instead, the two s_i get all mixed up with the two l_i. There is a transformation of the coupling scheme from the one sketched on the upper left to the jj scheme sketched at the upper right of figure 6, with a corresponding transformation of the wave functions. The two states get mixed up in the process of diagonalization of the energy matrix within the little "Hilbert space," as it is called, consisting of these two states. Their energies are solutions of a quadratic equation giving the two curved lines shown. In the middle region of the figure, between the LS and jj extremes, one speaks of intermediate coupling. Incidentally, this simple calculation with $a/K \approx 1$ predicted the spectrum of Li^6 very nicely before it was observed.

Increased complexity

This is a miniscule example of the kind of shell-model calculation that is being pushed to the larger "Hilbert spaces" of more complicated cases these days, an extension made possible by modern computers. In this simple case there was only one integral K to serve as a parameter for the strength of the nucleon interaction, and it serves with little variation throughout the p-shell nuclei. In more general cases there are several such parameters. In this simple case we have simplified by considering only the p shell, whereas a



ANGULAR-MOMENTUM VECTOR orientation in the states of highest energy in figure 4. Parts a, b and c of the figure show the jj coupling scheme, and part d suggests that the sign changes in wave functions also affect the resulting energy. —FIG. 5



INTERMEDIATE COUPLING. This scheme, for Li^6 and He^6, shows the competition between spin-orbit energy of the individual nucleons and the interaction energy between nucleons in determining the energy levels. —FIG. 6

second-order calculation or diagonalization within a larger portion of Hilbert space would have put states with one p nucleon and one f nucleon into the mixture, for example. We have neglected this "configuration interaction" by truncating the Hilbert space. Such truncation is severely used to keep the larger problems manageable, for even present-day computers are not infinite in their capacity.

The single interaction parameter K in the p shell corresponds to a simple attraction dependent only on the distance r_{ij} between the particles. Although this simple approximation is quite good, refinements show that a much more complicated interaction should be assumed, involving at least tensor terms and a more complicated expression for the spin-orbit coupling, and this complexity increases the number of parameters. Calculations are then meaningful only if they correlate many experimental data. The large shell-model calculations are made for regions of the periodic table where there is a long string of nuclei, such as, for example, all the tin isotopes, through which the same set of parameters may be used.

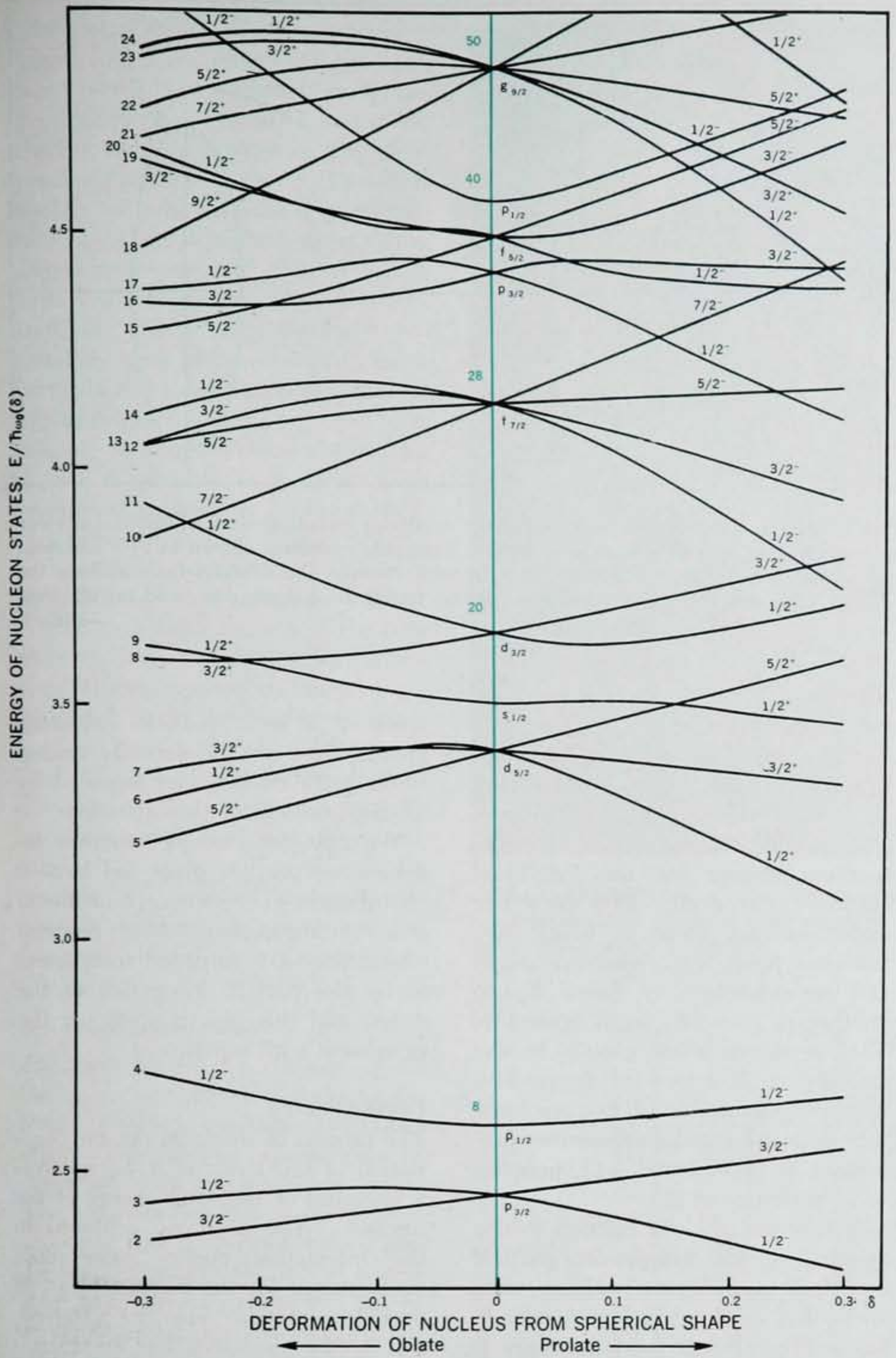
The size of the Hilbert space involved—that is, the number of simple wave functions that get mixed up—increases surprisingly rapidly with the number of particles outside closed shells. Four particles are about the limit. Even so, matrices as large as 500×500 are sometimes involved. In spite of all this complexity, we find very interesting correlations of the experimental data, although frequently one detailed property, the electric quadrupole moment, turns out to be systematically too small. The shell-model states do not contain enough collective distortion—it takes several nucleons moving in concert to make large electric effects. The wave functions of the spherical-shell model, when properly combined, may possibly describe a distorted charge distribution; even the distributions implied by our picture of Li^6 states in figure 3 have a door-knob shape. The truncation of the Hilbert space, however, may prevent this from going far enough.

Deformed-shell model

Nuclei with nucleon numbers far from the closed shells are strongly deformed. This is apparent not only from their quadrupole moments but also from the observed rotational spectra, which are very much like those observed in molecules. Such nuclei are best described by starting out with basis wave functions suited for the purpose, those defined by a three-dimensional harmonic-oscillator potential that is not spherical but rather has ellipsoidal equipotential surfaces. This procedure, along with spin-orbit coupling, leads to Sven Gösta Nilsson's well known model for the single-nucleon intrinsic states.[2] An axis of symmetry is assumed, and the projection of angular momentum along this axis is conserved. States with the same projection mix, and the lines giving energies of states as functions of distortion are curved for about the same reason as we encountered in the discussion of intermediate coupling. Some of these are shown in figure 7, with increasing prolate distortions on the right side (positive deformation parameter δ) and oblate on the left. One can fill the states starting at the bottom with pairs of protons and neutrons and add up their energies. As the distortion proceeds, with the nuclear volume kept constant, one finds a minimal total energy and thus an equilibrium distortion.

Rotation

The spacing in energy of the rotational states of such a nucleus goes as $J(J+1)/\mathcal{g}$. The effective moment of inertia $\mathcal{g}$ is not like that of a rigid flywheel, or even of a molecule, for the nucleus is not a rigid structure.



THE NILSSON DIAGRAM shows the energies of single-nucleon states as functions of the deformation of the nucleus. A correction for constant nuclear volume has still to be applied; it would make the unit $\hbar\omega_0$ vary with deformation δ in such a way as to make corresponding levels curve upward. —FIG. 7

Instead, its moment of inertia depends on modifications of the single-particle energy levels caused by the rotation. Here again we can think of the principal effect in terms of classical orbits. Without rotation the periodic motion of a nucleon (without spin) in the distorted oscillator potential is a simple harmonic motion along a principal axis, as in figure 8a. If the initial motion is more complicated, the motion is a Lissajous figure, as in figure 8b, because the two periods of motion along the y' and z' axes are different. With rotation, if the motion of figure 8a were to start at the lower end, the effect of the Coriolis force, due to the rotation, would be to deflect the path as in figure 8c. This force superposed on the force that brings the motion too quickly back to the z' axis in figure 8b leads to the periodic motion of figure 8d. Correspondingly, basis wave functions may be formed of the lowest states in the unfilled shell that have "forward" particle angular momentum. This angular momentum contributes to the effective moment of inertia, because moment of inertia may be defined as a measure of the amount of angular momentum for a given angular speed. In filled shells the contributions of these "forward" motions are approximately cancelled by "backward" contributions from the higher states that correspond to elliptical paths near the y' axis.

If you ever want to get really graphic with nuclear models you can make working models for the lecture table. I once rigged up a sand-dripping pendulum of the sort with which one demonstrates Lissajous figures, with different pendulum lengths for the two directions, put it on a turntable with a crank as in figure 9, and played with getting the initial conditions about right. One picture in figure 10 is a Lissajous figure made without rotation, and the other is a crude approximation to an ellipse made with rotation.

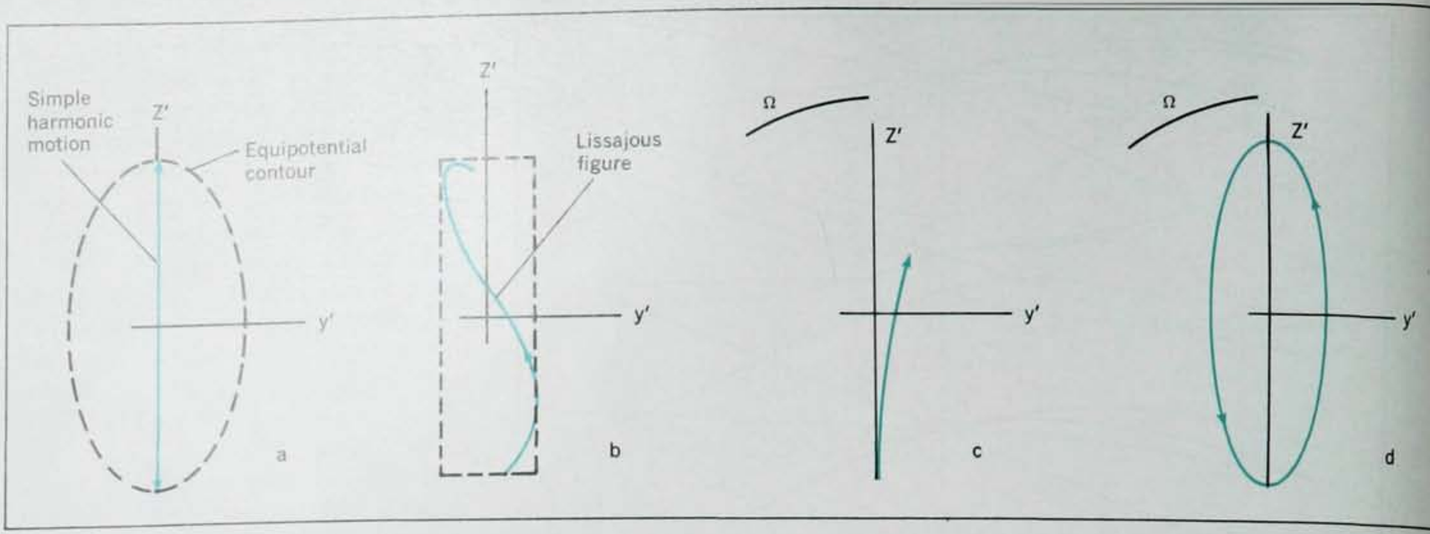
These contributions from the lower states of the unfilled shell are too large and make the effective moment of inertia essentially the rigid-rotation value, which is larger than observed. But in such simple calculations, based on the individual-nucleon states of the Nilsson model, we have again neglected configuration interaction; taking it into account greatly improves the agreement with experimental moments of inertia.

Quasi-nucleons

The configuration interaction is caused by the interaction of pairs of particles, and in many-nucleon problems this is quite difficult to take into account explicitly. We introduce the concept of a quasi-particle to do this in a convenient approximate way.

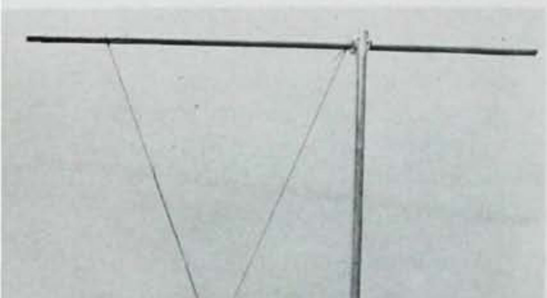
We have seen already in figures 3 and 5 that there is a geometric reason for the tendency for two nucleons to have their lowest energy when paired with opposite angular momenta. There is a second-order effect of configuration interaction that makes this pairing tendency even stronger, and this can be a dominant effect of configuration interaction.

For any closed-shell nucleus, or more generally any nucleus with even numbers N of neutrons and Z of protons, the lowest state in the independent-particle model would have



MOTION OF A PARTICLE in a two-dimensional harmonic oscillator potential with ellipsoidal contour lines, corresponding to the stretching of a nucleus. Simple harmonic motion is shown at a; a Lissajous figure with different periods on the x' and y' axes is at b; with rotation the Coriolis force deflects the path as shown at c, and with rotation and the right start these trends are balanced to yield the elliptical path at d. —FIG. 8

the occupation density of the levels represented by figure 11a, with all states occupied up to the "Fermi level" E_F. The interaction between pairs of particles, which can be calculated by a perturbation theory, modifies this occupation density ρ as shown in figure 11b. The appropriate wave-mechanical formulas are such that, as appears rather natural, the excitations of excited states are strongest if rather little energy change is involved, so that $\rho(E)$ tapers off gradually at higher energies. The excitations are also strongest if little or no angular-momentum change for the individual nucleons is involved. Thus the strong excitations are those in which two nucleons jump from what we might call one macrolevel α, below E_F, to another, β, above E_F, as in figure 11c. This is shown more clearly in the magnifying-glass view of figure 11d, between microlevels of two nucleons with opposite angular-momentum projections m (here 1 and -1), jumping without change of m.



A STRING-AND-SEALINGWAX nuclear model. This compound pendulum demonstrates approximately elliptical orbits in rotating coördinates. —FIG. 9

If now we add one neutron to this nucleus, in the independent-particle model it goes into a definite state above E_F, as shown schematically in figure 11e. But if pairing energy is taken into account, the distribution to be modified is that of figure 11b, which should be drawn separately for m and $-m$, and the rather remarkable result is shown in figure 11f and g. The states at that energy above E_F are already partly occupied by a pair of neutrons that get there by means of their interaction with each other, both in that state and in states below E_F. They have lifted each other up, so to speak. The added neutron in the state $+m$ cuts off this pair excitation by excluding excitation into $+m$ so that the state $-m$ at that level is left empty, as in figure 11g. A quasi-particle is thus partly a particle in state m and partly a "hole" in state $-m$, or we might think of it as a particle swimming around in a sea of partially excited states, with its behavior impeded by its displacement of the excitations.

We saw, for example, that an independent particle gives too large a contribution to the moment of inertia of a rotating nucleus. With pairwise interactions it is impeded from doing so by the partial occupation of the states, and this greatly improves the agreement with experiment.

Pairing energy

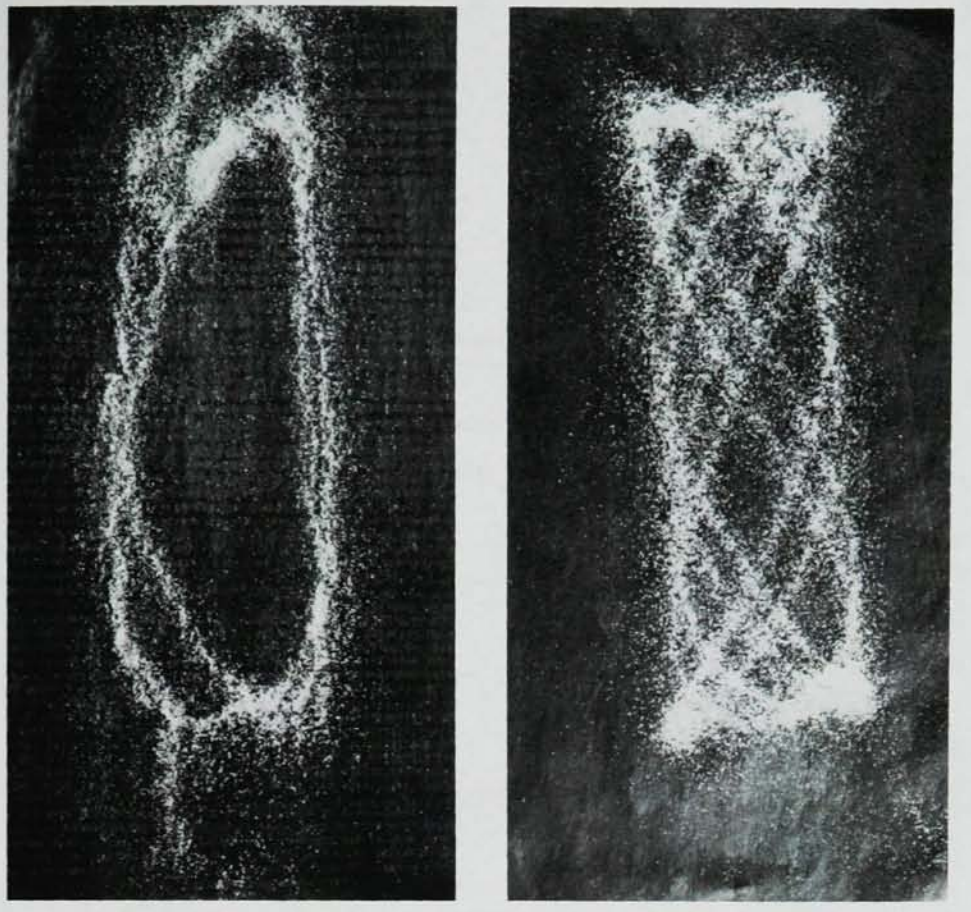
The process of smearing out the occupation of the levels near E_F involves a lowering of the total energy of the nucleus. The economy achieved in the interaction energy more than compensates for the increased cost in particle-excitation energy. Indeed, this is the incentive for the excitation—the system tends to settle into a low-energy situation. Thus the original nucleus, with A nucleons, has a lower energy than it would have if such excitation were not permitted. When another neutron is added to make a quasi-particle, the lowering of energy is not as great as it otherwise would be, because some of that earlier energy-lowering excitation is destroyed in the process. That is, the added neutron has a smaller binding energy than it would have in the independent-particle model of figure 11e.

Now let us add another neutron. If we add it in just any state, it becomes another quasi-particle with a binding energy similar to that of the first. But if we add it instead in the very special state $-m$ of macrolevel β, it goes into a completely empty state, from which second-order excitation has already been excluded, and it has the full binding energy of the independent-particle model. The first particle we add to make a quasi-particle has a small binding energy; the second that we pair with it has a large binding energy. Thus the second-order effects of configuration interaction give the quasi-particles a particularly large pairing energy. This is important in many ways. For example it simplifies calculations by making it possible to treat an even number of protons as a closed shell, even when it is not, thus making it permissible to calculate only the coupling scheme of the neutrons.

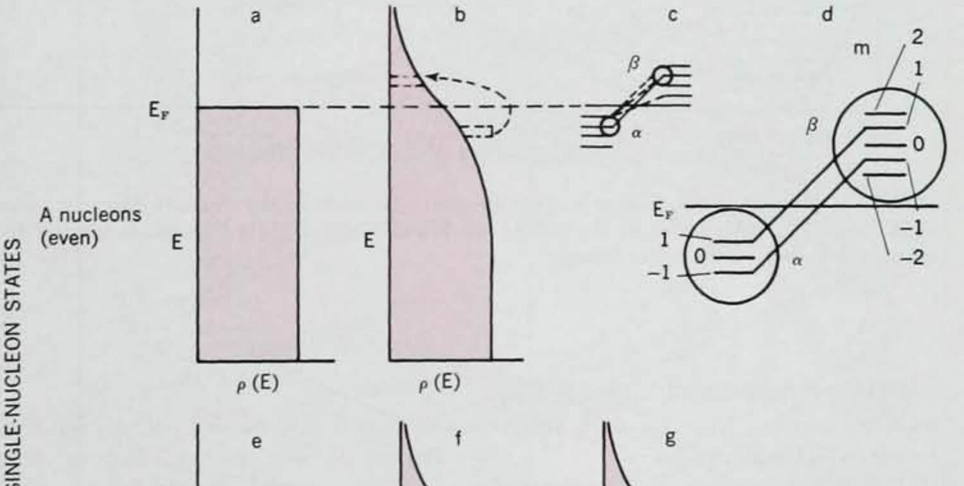
Shell-model calculations of the coupling of several particles with truncated Hilbert spaces are surprisingly successful partly because they really apply to quasi-particles, not particles. That is, much of the neglected effect from outside the space is systematically associated with the particle coördinates. The large pairing energy, for example, is included automatically when the parameters are determined empirically by fitting part of the available data.

Hard-core interactions

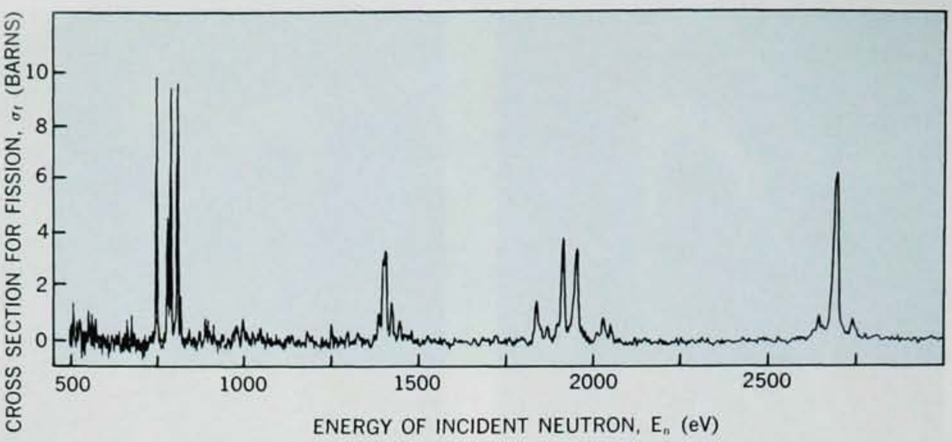
The most direct observations of the forces acting between nucleons are made by scattering them from each other. The analysis of scattering experiments does not lead to a unique description of nucleon–nucleon interactions in free space, but it does limit the possibilities to a few sets of reasonable assumptions. However, the effective interactions most useful in nuclear models—the interactions between quasi-particles—may be quite different from those observed between nucleons in free space. Extensive calculations have been made to derive them from the accepted descriptions of free-space interactions observed by scattering. The real interactions are usually described as having a strong and very short-range repulsion, followed by a longer-range attraction. The attraction is the important part, being only partly canceled out by the repulsion. The hard repulsive core would disrupt a perturbation theory. It bends the wave functions sharply at short distances, but the range of this bending is itself sufficiently short that the nucleons manage to swim around among each other, as they



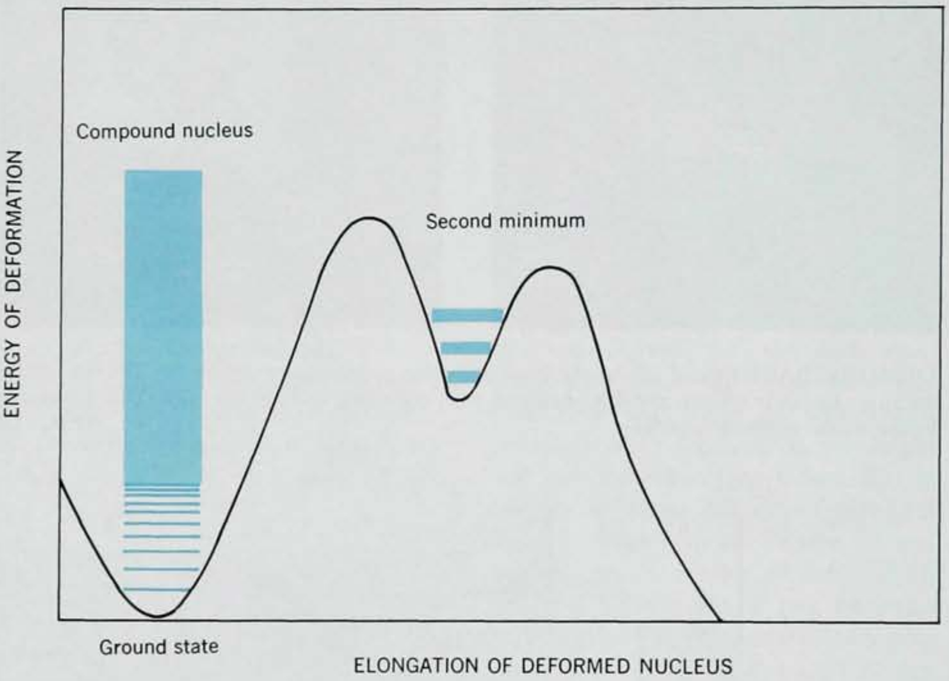
DEMONSTRATIONS of orbits obtained with the apparatus of figure 9. On the left is an approximately elliptical orbit obtained with rotation, and on the right is a Lissajous figure made without rotation. —FIG. 10



GRADUALLY CHANGING OCCUPATION of states near the "Fermi energy" E_F as a result of internucleon interaction. When a nucleon is added to a partially filled state in this region, it creates a "quasi-particle." —FIG. 11



NEUTRON-INDUCED FISSION. This observation of clusters of resonances, for the nucleus Pu^{240}, was made at Geel in Belgium. —FIG. 12



DOUBLE HUMP of the fission barrier, proposed to explain the clusters of resonances as in figure 12. The states in the saddle are broad because their lifetime is limited by easy penetration of the outer hump. —FIG. 13

would in a shell model, with effective interactions that have no such singularities as a hard core.

The calculations are so complicated that they are made not for a nucleus at all, but for hypothetical stuff known as "infinite nuclear matter," for which analysis is simpler. The central region of a heavy nucleus is presumably quite uniform, and its properties can be deduced from observed regularities among nuclei. One therefore assumes that it can be considered to be a small sample of nuclear matter having the same bulk properties. The binding energy and density, calculated in this way from scattering data as a great tour de force, come out about right.

Fission

So much for models of nuclei that remain in one piece. Let us now look at a model of a nucleus coming apart in the fission process. Fission involves quite extreme deformation, and it has been customary to bypass the complications by invoking the droplet model. We see the nucleus as a droplet of nuclear matter not unlike an electrically charged water droplet, with electrostatic repulsion encouraging deformation and surface tension opposing it. During the deformation, as it proceeds up the inside of the fission potential barrier, the energy of the system increases with increasing deformation. The droplet stretches against surface tension until the neck becomes so narrow that the electric repulsion turns the energy curve sharply downward, and the neck breaks. General features can be understood in this way; neutron-induced fission occurs by going over the barrier and spontaneous fission by tunneling through it in the peculiar way that quantum mechanics permits.

But some fascinating new measurements indicate that the barrier is more complicated. Figure 12 shows the fission yield as a function of neutron energy for the nucleus Pu^{240}, with 94 protons and 146 neutrons, according to measurements at Geel in Belgium.[3] Neutron energies are well resolved by time-of-flight techniques, and fission is observed essentially only at narrow clusters of very sharp resonances. The explanation as discussed by V. M. Strutinsky and others[4] is that the fission barrier must be a two-humped camel, as in figure 13. In the saddle between the humps are some fairly widely spaced states that are broad because their lifetime is limited by easy penetration of the little outer hump. By contrast, the states in the compound-nucleus region inside the double hump are very sharp and also very closely spaced because of the many modes of excitation so high above the ground state. Actually, the outer broad states and the inner sharp states are not completely separate; they combine into clusters of sharp states by penetration of the inner hump. These combinations lead to the observed clusters of resonances as well as to observed short-lived fission isomers induced by d,p reactions.

Why two humps?

From the point of view of nuclear models, the really interesting question is, "Why are there two humps?" The single-nucleon energy levels, as portrayed in the Nilsson diagram,[2] suggest an important part of the answer,[5] but not all of it.

The higher part of the originally published energy-level diagram is shown in figure 14. Although this diagram goes only out to the deformation $\delta = 0.3$, not far enough to include the top of the fission barrier, it contains examples of the type of level crossing needed. Counting up ten states beyond the magic number 126, we find that for small deformations to the right of center the last pair of the 146 neutrons occupies the last but two of the levels that slope upward towards the right, but for larger deformations these would transfer into the last downward-sloping level that happens to be shown in this diagram. Similarly, the last of the 94 protons are in upward-sloping states at small deformations.

In the schematic extension of the diagram shown in figure 15b, the topmost occupied levels are indicated by small circles that may refer separately to either neutrons or protons. The highest circles represent the top of the "Fermi Sea." As the deformation is increased, the topmost nucleons transfer from upgoing levels to downgoing levels at a crossover region. It is clear that this can cause a downward curvature of the curve in figure 15a giving the sum of all the single-nucleon levels as a function of deformation. This downward curvature, in turn, corresponds to the first hump in the fission barrier.

For still greater deformation, there is a region where the downgoing levels at the top of the Fermi Sea cross over with some upgoing levels, but here both sets of levels remain filled, and there is no transfer to cause a drastic curvature. Thus one can get a downward curvature of the total energy but not an upward curvature out of such a set of crossing levels. At yet greater deformation, another crossover region sketched in the figure provides the downward curvature of the second hump, but we are left with the question, "Whence the upward curvature between the two humps?"

The way the Nilsson diagram is conventionally plotted, the lines seem to curve downward about as frequently as upward so as to contribute very little net curvature to the sum. However, as an energy diagram, this is misleading, for the energy scale depends on deformation, by the requirement of volume conservation within an equipotential surface. The scale factor of figures 7 and 15

$$\omega_0(\delta) \approx \mathring{\omega}_0(1 + \tfrac{2}{9}\delta^2)$$

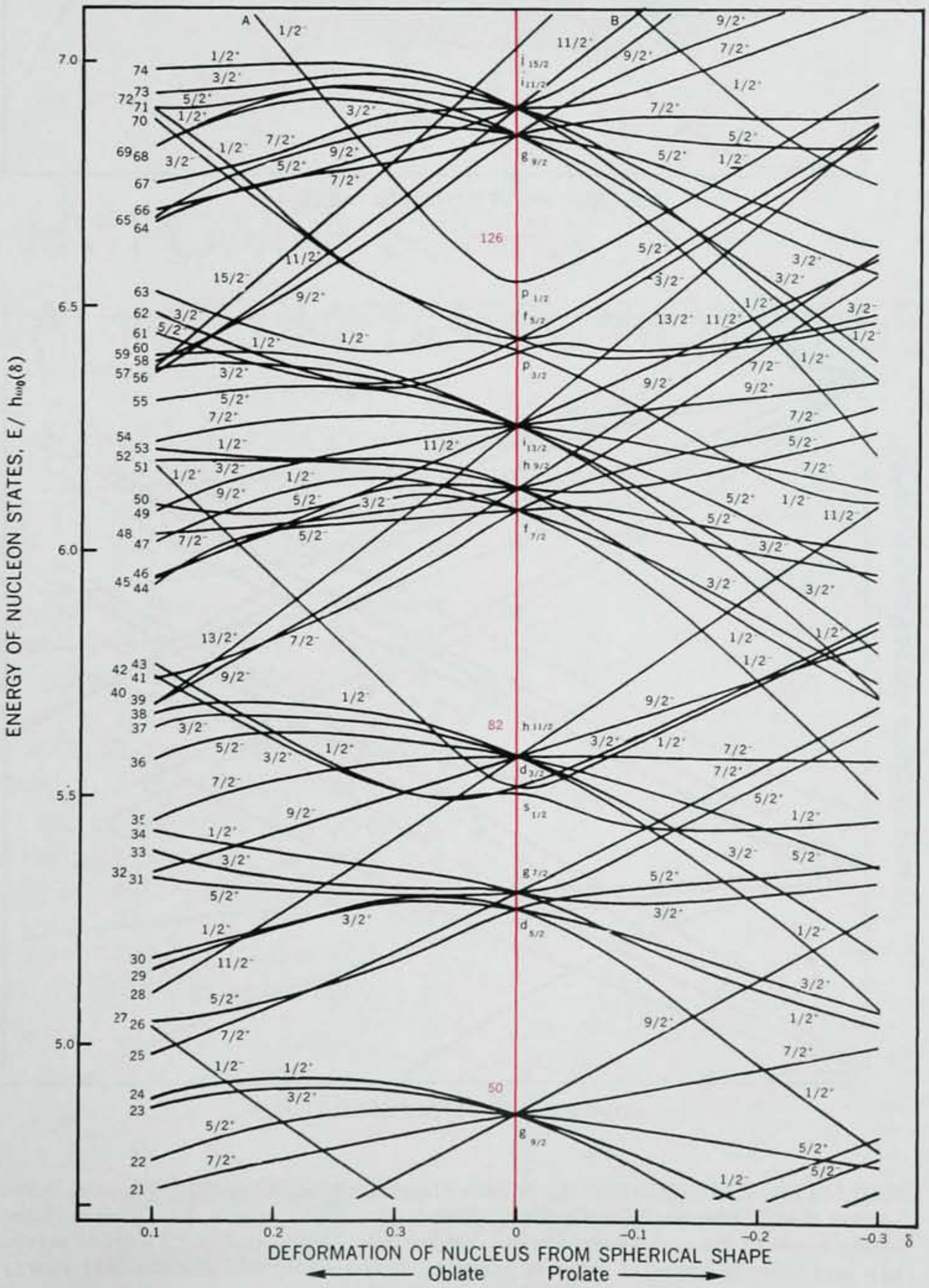
(Nilsson's equation 4 in reference 2) shows that energy, which is quadratic in the momenta, gains more from the squeezing of the lateral wavelengths than it loses from the stretching of the longitudinal wavelengths when the deformation is carried out to conserve volume with fixed occupation of the states. This effect disappears, on the average, if readjustment of the occupation of states for minimal energy is made at crossovers of energy levels; in the classical limit the readjustments become continuous to keep the energy per unit volume constant.

This means that all the curves would tend to curve upward, slightly more than shown in figure 15b, if they were plotted on a constant energy scale. For each individual-nucleon energy level in figure 15b this additional upward curvature is very slight, but the upper solid curve in figure 15a represents the sum of the individual-nucleon energies over all the occupied levels, and the sum of many small curvatures can add up to the appreciable upward curvature shown.

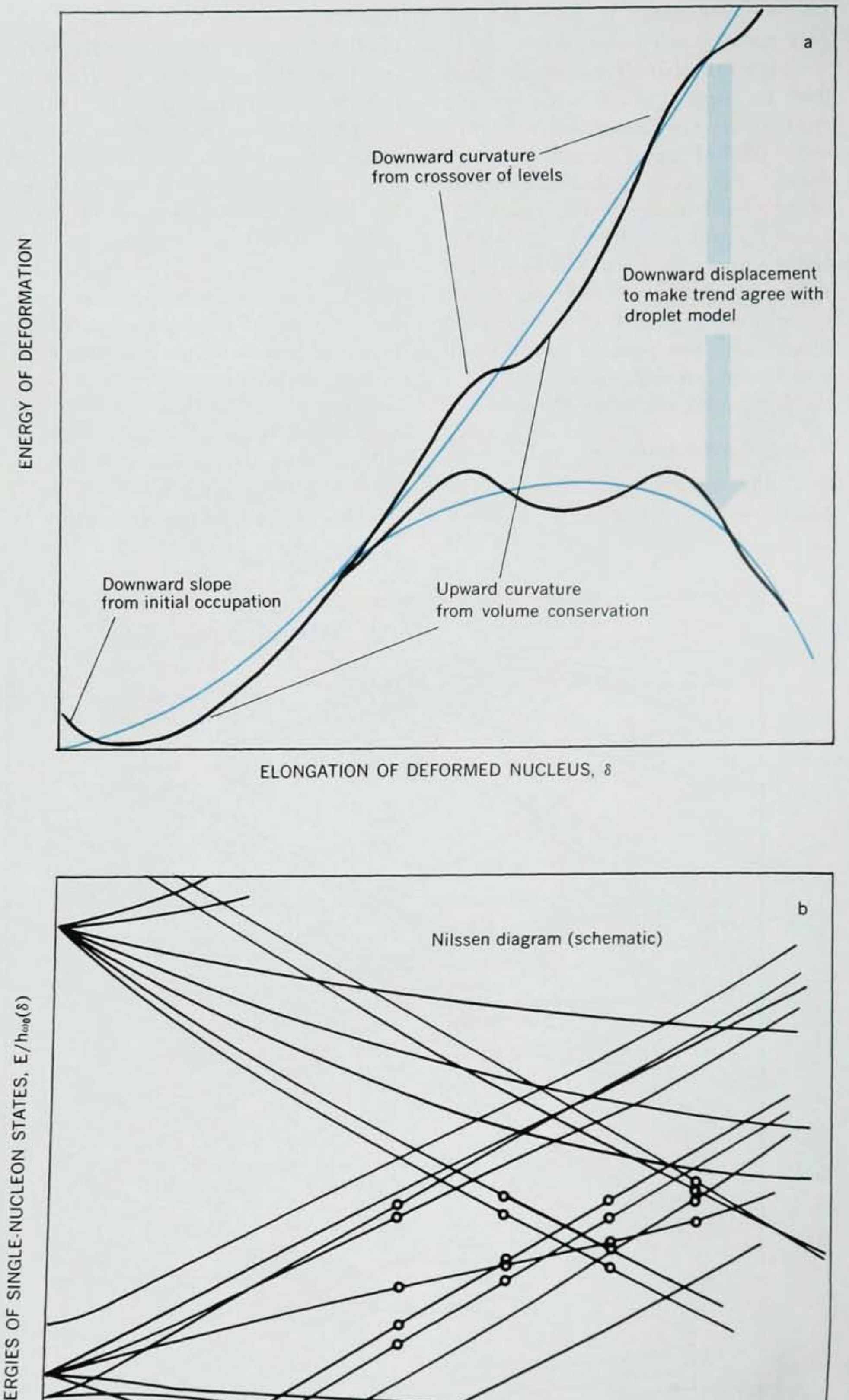
Fission barrier is lacking

Thus the net result of adding all the single-nucleon energies is a curve that, as a general characteristic of the oscillator aspect of the model with constant volume, sweeps upward with increasing deformation. It curves sharply upward where there are no crossovers and downward in the regions of influence of crossovers at the top of the Fermi Sea. Thus it wavers about an average upward curve shown as a colored line in the figure. Even if Coulomb energy were included in the calculation, it would presumably not be powerful enough to turn the average curve downward, and no satisfactory analytic way has been found to make the curve actually calculated in the oscillator model resemble more closely a fission barrier.

What is lacking in the model is



CONTINUATION of the Nilsson diagram shown in figure 7. —**FIG. 14**



SCHEMATIC EXTENSION of the Nilsson diagram, in lower part (*b*) of figure, shows features that help to explain the energy curves (*a*) at the top of the figure and the double hump of figure 13. Small open circles indicate the topmost occupied levels; they may refer to either neutrons or protons. Initial downward curvature of energy in *a* (top) corresponds to the first hump of the fission barrier; thereafter the curve wavers upward and downward about an average curve shown in color. —FIG. 15

something to correspond to the way, in the droplet model, the neck between two nascent fission fragments can narrow down, so as to reduce the rate of increase of surface as the deformation proceeds. This effect is what makes it possible, in the droplet model, for the Coulomb repulsion between the two fragments to overcome the cohesive effect of the surface energy and make the total-energy curve turn downward.

The present theory of the two-humped barrier is an artificial mixture of the best features of both theories. In recognition of this shortcoming of the individual-nucleon treatment, the results of this theory are systematically modified to make the average trend agree with that given by the droplet theory for large deformations.[4,5] In this way we obtain a satisfactory fission barrier, as shown by the lower solid curve of figure 15a.

It is thus entirely plausible that there should be a two-humped fission barrier, as the observations appear to require. The barrier as a whole is to be understood as usual in terms of the competition between surface energy and Coulomb repulsion in the droplet model, but the two-hump feature is to be understood as a modification introduced by detailed consideration of the individual-nucleon energy levels: The downward curvature at the humps arises from the crossover of levels at the top of the Fermi Sea, and the upward curvature between them comes from the lateral squeezing of the nucleus at constant volume when the occupied states remain occupied.

* * *

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