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Citation: Physics of Plasmas **1**, 3676 (1994); doi: 10.1063/1.870902
View online: http://dx.doi.org/10.1063/1.870902
View Table of Contents: http://scitation.aip.org/content/aip/journal/pop/1/11?ver=pdfcov
Published by the AIP Publishing

Microwave signal amplification and Pierce instability on radial electron flows in cylindrical and spherical diodes

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(Received 10 January 1994; accepted 2 August 1994)

Linear space charge perturbations of focused electron beams flowing between cylindrical and spherical electrodes on convergent or divergent trajectories are studied, and the amplification of high-frequency signals when the flow is modulated at one electrode is computed. It is shown that divergent beams give the largest amplification effect. The instability of electron beams drifting through grounded grids (Pierce instability in cylindrical or spherical diodes) is also considered. The instability threshold occurs at higher critical currents when the curvature of the electrodes is large. Results for planar electrodes are recovered in the limit of zero curvature devices. Spherical configurations have better signal amplification and stability properties than similar planar or cylindrical systems.

I. INTRODUCTION

Beams of charged particles have great importance for the understanding of the physical properties of plasmas, as well as for their technological applications. High power beams are used in plasma heating, electrostatic confinement, and inertial fusion. This paper deals with two distinct problems of the theory of electron beams with a fixed neutralizing background. We study (i) the properties of fast and slow waves carried by one beam, and their application to small signal amplification; (ii) the conditions for the instability of the beam flowing between two grounded grids, and the question of the limiting current. For electron beams formed by a bundle of parallel trajectories, i.e., for a planar beam geometry, these subjects have been extensively studied for many decades. Here we present an extension of these topics to flows formed by convergent or divergent electron trajectories, focused through a center. We consider systems formed with cylindrical or spherical grids (or, perhaps, with angular sectors of cylinders or spheres also known as Pierce electrodes[1]).

A motivation for this research is of theoretical nature. We have noted[2–5] that a cylindrical or a spherical focusing of the beams substantially modifies the spectrum of the counterstreaming (two beam) instability. For instance, the analysis of the counterstreaming instability for a spherical system with radial trajectories focused through a center (so that radially opposed flows are produced) has shown that, for radial modes, the instability is suppressed.[4,5] Thus, we were led to enquire on the basic oscillatory properties of a monoenergetic beam, in the case of cylindrical and spherical radial flows, as a step toward the understanding of multiple beam configurations.

Further incitement for this work comes from experimental and theoretical research on particle beams and Pierce diodes recently published. The problem of microwave amplification is closely related to the physics of electron beams: an account of the evolution of this field can be found in Dunn.[1] Lau *et al.*[6] investigated the nonlinear properties of space charge waves on relativistic electron beams for planar electrodes. An experimental and technological review of systems with electrodes of various shapes has been presented by Dorodnov and Petrosov.[7] Vorob'ev and Zhukov[8] have carried out experimental studies on flows between convex electrodes. Nikonov *et al.*[9] have studied cylindrical configurations. An investigation of the radial flow of relativistic electrons in spherical diodes was given by Chetvertkov.[10] The work of Kadish[11] is devoted to nonlinear solutions, using the method of characteristics to analyze perturbations of relativistic flows in plane, cylindrical, and spherical geometry.

The planar Pierce diode literature has increased in the last decade. A review of this topic by Kuhn[12] contains a list of about 150 references. An analysis of the Pierce modes, beyond the threshold of the instability, was given by Cary and Lemon.[13] Detailed studies on Pierce instability, as well as on the role of boundary effects in plasmas have been presented by Kuhn and co-workers.[14,15] The increasing interest in the nonlinear behavior of Pierce diodes by theoretical and computational simulations can be followed in the papers of Godfrey[16] and Crystal *et al.*[17] It should be mentioned that Birdsall and co-workers have produced an extension to cylindrical and spherical geometries of their particle simulation codes.[18,19]

Particle beams in spherical geometries are of current interest in fusion research also. Spherical systems have been studied for electrostatic confinement schemes.[20] The Pierce limit for ion beams in spherical geometries for fusion applications has been considered by Poukey *et al.*[21] More recently, Miley and co-workers[22] have obtained interesting experimental results in spherical chambers.

The existence of fast and slow waves in a beam allows, by superposition, the formation of a stationary wave anchored on an electrode or any other exciting element such as a metallic grid, that acts as a modulator of the velocity of the particles. A model of this type illustrates the principle of the klystron or microwave amplifier.[1] In Sec. III we study the amplification properties for a beam flowing between cylindrical and spherical electrodes. We discuss the modes

 1070-664X/94/1(11)/3676/10/$6.00

equivalent to the fast and slow waves in a convergent or a divergent beam, and the possibility of generating a stationary wave using a modulating electrode.

When a beam flows in a region between grids that are at the same potential an instability may develop.[23] This effect is interesting because in the absence of boundaries a monoenergetic beam is stable. Let Φ be the accelerating potential such that $mv^2/2=e\Phi$. The beam (with density n_0) drifts henceforth through a region free of fields between two grounded planar electrodes separated by a distance L. When the current density of the beam $j_0=en_0v$ is increased, a critical value is reached when the electrons start to be repelled from the drift region.[24] The theoretical condition for the threshold of the Pierce instability, assuming a neutralizing background of fixed ions, is

$$\alpha \equiv \omega_{pb}L/v=(2s+1)\pi, \quad s=0,1,2,..., \tag{1}$$

where ω_{pb} is the plasma frequency defined with the beam density. Taking $s=0$ one may easily obtain

$$|j_c|=\pi\left(\frac{e}{2m}\right)^{1/2}\frac{\Phi^{3/2}}{L^2}\approx 10^{-4}\frac{\Phi^{3/2}}{L^2}, \tag{2}$$

where the units are A/cm^2, volts, and cm. This result reasonably reproduces the characteristics observed for the critical current.[25]

In Sec. IV we study the stability of a convergent or divergent electron flow between cylindrical and spherical grids. The conditions of the instability threshold are associated to the existence of nontrivial equilibrium solutions, with space charge and zero frequency. A comparison with the plane case when the curvature of the electrodes tends to zero is also given.

Summarizing, Sec. II gives the basic equations for the oscillations of the radial flow of electrons. Section III contains the extension of the small signal problem to cylindrical and spherical geometry. Finally, Sec. IV deals with Pierce instability for a beam drifting between grounded cylindrical and spherical grids. Section V contains our conclusions.

II. BASIC EQUATIONS FOR RADIAL PERTURBATIONS OF CYLINDRICAL AND SPHERICAL RELATIVISTIC ELECTRON FLOWS

We address the problem of a monoenergetic electron beam flowing radially through a neutralizing background of fixed ions. In the case of a convergent beam, the electrons are collected near the origin to ensure that there is no crossing of beams at $r=0$. The electron dynamics can be treated relativistically to include the case of high energy injection. This system may be described by the equations

$$\frac{d}{dt}(\gamma\mathbf{v})=-\frac{e}{m}\mathbf{E}, \tag{3}$$

$$\frac{\partial n}{\partial t}+\mathrm{div}(n\mathbf{v})=0, \tag{4}$$

$$\frac{\partial \mathbf{E}}{\partial t}+4\pi\mathbf{j}=c\ \mathrm{rot}\ \mathbf{B}. \tag{5}$$

We use cgs Gaussian units everywhere, except for a few instances explicitly indicated, c denotes the speed of light in vacuum. The symbols $-e$, m, $\mathbf{v}$, and n denote the electron charge, rest mass, velocity, and particle density, respectively; $\mathbf{E}$ and $\mathbf{B}$, the electric and magnetic field, and $\gamma(v)\equiv(1-v^2/c^2)^{-1/2}$.

In a system with cylindrical symmetry ($l=1$) or spherical symmetry ($l=2$), where the velocity is radial, we have for the steady state

$$\mathbf{v}=-v_0\mathbf{e}_r, \tag{6}$$

$$\frac{\partial}{\partial r}(r^l v_0 n_0)=0, \tag{7}$$

$$4\pi e v_0 n_0=c(\mathrm{rot}\ \mathbf{B}_0)_r. \tag{8}$$

Here the subscript zero denotes steady state values and $\mathbf{e}_r$ is the unit radial vector in cylindrical or spherical coordinates. The neutralizing ion background has density n_0 also and, therefore, the steady-state electric field is zero. The velocity v_0 is constant, and positive for a convergent beam. The density n_0 may be expressed as

$$n_0=n_e\frac{R_e^l}{r^l}, \tag{9}$$

where n_e is the density at a given radius R_e.

Note that the steady state $\mathbf{B}_0$ is not an external field but the one generated by the unperturbed current density $\mathbf{j}_0=en_0v_0\mathbf{e}_r$. Thus, $\mathbf{B}_0$ cannot be zero if Maxwell's current equation is to be satisfied. On the other hand, this magnetic field is neglected in the equations of motion by assuming that the typical length of the system transversal to the beam is much smaller than c/ω_p, which gives an estimate of the length where the effects of the Lorentz force would be noticeable.

A description of cylindrical and spherical electrodes is given in Ref. 1. Birdsall[19] has found in numerical simulations indications of ion relaxation toward the $1/r$ density distribution, starting with a uniform hot plasma between grounded cylindrical grids.

For radial perturbations of the system, we define

$$\mathbf{v}\equiv(-v_0+\delta v)\mathbf{e}_r, \quad n\equiv n_0+\delta n, \quad \mathbf{j}\equiv(j_0+\delta j)\mathbf{e}_r,$$

$$\mathbf{E}\equiv E\mathbf{e}_r, \quad \gamma\equiv\gamma_0+\delta\gamma, \quad \gamma_0\equiv\gamma(v_0).$$

Then, the linearized equations for the perturbations are

$$\left(\frac{\partial}{\partial t}-v_0\frac{\partial}{\partial r}\right)(\gamma_0\delta v-v_0\delta\gamma)=-\frac{e}{m}E, \tag{10}$$

$$\frac{\partial \delta n}{\partial t}+\frac{1}{r^l}\frac{\partial}{\partial r}[r^l(n_0\delta v-v_0\delta n)]=0, \tag{11}$$

$$r^l\left(\frac{\partial E}{\partial t}+4\pi\delta j\right)=f(t), \tag{12}$$

where $f(t)$ is a function of time to be determined by the boundary conditions of the problem.[11] Taking into account, from Poisson's equation, that

$$4\pi\delta j=-4\pi e n_0\delta v-v_0\frac{1}{r^l}\frac{\partial}{\partial r}(r^l E), \tag{13}$$

Eqs. (10)–(12) may be written as

$$\left(\frac{\partial}{\partial t}-v_0\frac{\partial}{\partial r}\right)\delta v=-\frac{e}{m}\frac{1}{\gamma_0^3}E, \tag{14}$$

$$\left(\frac{\partial}{\partial t}-v_0\frac{\partial}{\partial r}\right)(r^l\delta n)=-n_e R_e^l\frac{\partial}{\partial r}\delta v, \tag{15}$$

$$\left(\frac{\partial}{\partial t}-v_0\frac{\partial}{\partial r}\right)(r^l E)=4\pi e n_e R_e^l\delta v+f(t). \tag{16}$$

The preceding system may be expressed in terms of nondimensional quantities by setting, for $l=1,2$,

$$\mathscr{V}\equiv\frac{\delta v}{v_0},\quad \mathscr{N}\equiv\frac{r^l\delta n}{R_e^l n_e},\quad \mathscr{E}\equiv\frac{r^l E}{4\pi e n_e R_e^{l+1}},$$

$$x\equiv\frac{r}{R_e},\quad \tau\equiv\frac{v_0 t}{R_e},\quad \mu\equiv\frac{\omega_p R_e}{v_0},$$

$$\mathscr{F}(\tau)\equiv\frac{f(R_e\tau/v_0)}{4\pi e v_0 n_e R_e^l},$$

where $\omega_p^2\equiv 4\pi e^2 n_e/\gamma_0^3 m$ is the square of the plasma frequency at $r=R_e$. With these definitions, we have

$$\left(\frac{\partial}{\partial\tau}-\frac{\partial}{\partial x}\right)\mathscr{V}=-\frac{\mu^2}{x^l}\mathscr{E}, \tag{17}$$

$$\left(\frac{\partial}{\partial\tau}-\frac{\partial}{\partial x}\right)\mathscr{N}=-\frac{\partial\mathscr{V}}{\partial x}, \tag{18}$$

$$\left(\frac{\partial}{\partial\tau}-\frac{\partial}{\partial x}\right)\mathscr{E}=\mathscr{V}+\mathscr{F}(\tau). \tag{19}$$

We may also note that Poisson's equation takes the form

$$\frac{\partial\mathscr{E}}{\partial x}=-\mathscr{N}, \tag{20}$$

and that the perturbed current density δj may be expressed in terms of a nondimensional variable $\mathscr{J}$,

$$\mathscr{J}\equiv\frac{r^l\delta j}{R_e^l e n_e v_0},$$

so that Maxwell's current equation is equivalent to

$$\frac{\partial\mathscr{E}}{\partial\tau}+\mathscr{J}=\mathscr{F}(\tau). \tag{21}$$

We note that $\mathscr{J}(x,\tau)$ is proportional to the perturbation current that flows through a surface of radius $r=xR_e$. From Eqs. (17) and (19) we derive an equation for $\mathscr{E}$,

$$\left(\frac{\partial}{\partial\tau}-\frac{\partial}{\partial x}\right)^2\mathscr{E}+\frac{\mu^2}{x^l}\mathscr{E}=\frac{\partial}{\partial\tau}\mathscr{F}(\tau). \tag{22}$$

We may proceed to find the normal modes of the system by taking

$$\mathscr{F}(\tau)=\mathscr{F}_0 e^{\nu\tau},$$

with $\mathscr{F}_0$ constant and setting

$$\mathscr{E}(x,\tau)\equiv\mathscr{E}(x)e^{\nu\tau},$$

and, thus, obtain a general expression for $\mathscr{E}(x)$ by solving

$$\left(\nu-\frac{d}{dx}\right)^2\mathscr{E}+\frac{\mu^2}{x^l}\mathscr{E}=\nu\mathscr{F}_0. \tag{23}$$

Once $\mathscr{E}(x)$ is known, the expressions for

$$\mathscr{V}(x,\tau)\equiv\mathscr{V}(x)e^{\nu\tau},\quad \mathscr{N}(x,\tau)\equiv\mathscr{N}(x)e^{\nu\tau},$$

$$\mathscr{J}(x,\tau)\equiv\mathscr{J}(x)e^{\nu\tau},$$

are derived from

$$\mathscr{V}(x)=\left(\nu-\frac{d}{dx}\right)\mathscr{E}-\mathscr{F}_0, \tag{24}$$

$$\mathscr{N}(x)=-\frac{d\mathscr{E}}{dx}, \tag{25}$$

$$\mathscr{J}(x)=-\nu\mathscr{E}+\mathscr{F}_0. \tag{26}$$

The equations for the planar case are also contained in the previous deductions taking $l=0$.

A. Solutions for cylindrical systems

We want to find the general solution of Eq. (23) for $l=1$. The solution of the homogeneous equation, $\mathscr{E}_h$, may be expressed in terms of Bessel and Neumann functions as

$$\mathscr{E}_h(x)=e^{\nu x}\sqrt{x}[c_1 J_1(2\mu\sqrt{x})+c_2 Y_1(2\mu\sqrt{x})], \tag{27}$$

where c_1,c_2 are constants. From the solutions of the homogeneous equation one may construct a particular solution $\mathscr{E}_p$ of the inhomogeneous equation of the form

$$\mathscr{E}_p(x)=\pi\nu\mathscr{F}_0 e^{\nu x}\sqrt{x}\left(Y_1(2\mu\sqrt{x})\int_1^x e^{-\nu u}\sqrt{u}J_1(2\mu\sqrt{u})du-J_1(2\mu\sqrt{x})\int_1^x e^{-\nu u}\sqrt{u}Y_1(2\mu\sqrt{u})du\right). \tag{28}$$

Therefore,

$$\mathscr{E}(x)=e^{\nu x}\sqrt{x}[c_1 J_1(2\mu\sqrt{x})+c_2 Y_1(2\mu\sqrt{x})]+\mathscr{E}_p, \tag{29}$$

$$\mathscr{V}(x)=\mu e^{\nu x}[c_1 J_0(2\mu\sqrt{x})+c_2 Y_0(2\mu\sqrt{x})]+\left(\nu-\frac{d}{dx}\right)\mathscr{E}_p-\mathscr{F}_0, \tag{30}$$

$$\mathscr{N}(x)=-e^{\nu x}\{c_1[\nu\sqrt{x}J_1(2\mu\sqrt{x})+\mu J_0(2\mu\sqrt{x})]+c_2[\nu\sqrt{x}Y_1(2\mu\sqrt{x})+\mu Y_0(2\mu\sqrt{x})]\}-\frac{d}{dx}\mathscr{E}_p, \tag{31}$$

$$\mathscr{J}(x)=-\nu e^{\nu x}\sqrt{x}[c_1 J_1(2\mu\sqrt{x})+c_2 Y_1(2\mu\sqrt{x})]-\nu\mathscr{E}_p+\mathscr{F}_0, \tag{32}$$

where

$$\frac{d}{dx}\mathscr{E}_p(x)=\nu\mathscr{E}_p+\pi\mu\nu\mathscr{F}_0e^{\nu x}\bigg(Y_0(2\mu\sqrt{x})\times\int_1^x e^{-\nu u}\sqrt{u}J_1(2\mu\sqrt{u})du-J_0(2\mu\sqrt{x})\times\int_1^x e^{-\nu u}\sqrt{u}Y_1(2\mu\sqrt{u})du\bigg). \tag{33}$$

B. Solutions for spherical systems

We now give the solutions of (23) when $l=2$. The solution of the homogeneous equation is, for $\delta\neq 0$,

$$\mathscr{E}_h(x)=e^{\nu x}(c_1x^{(1/2)+\delta}+c_2x^{(1/2)-\delta}), \tag{34}$$

with

$$\delta\equiv\sqrt{\tfrac{1}{4}-\mu^2}.$$

A particular solution $\mathscr{E}_p$ of the inhomogeneous equation may be expressed as

$$\mathscr{E}_p(x)=\frac{\nu\mathscr{F}_0}{2\delta}e^{\nu x}\bigg(x^{(1/2)+\delta}\int_1^x e^{-\nu u}u^{(1/2)-\delta}\,du-x^{(1/2)-\delta}\int_1^x e^{-\nu u}u^{(1/2)+\delta}\,du\bigg). \tag{35}$$

We then obtain,

$$\mathscr{E}(x)=e^{\nu x}(c_1x^{(1/2)+\delta}+c_2x^{(1/2)-\delta})+\mathscr{E}_p, \tag{36}$$

$$\mathscr{V}(x)=-e^{\nu x}\bigg[c_1\bigg(\frac{1}{2}+\delta\bigg)x^{-(1/2)+\delta}+c_2\bigg(\frac{1}{2}-\delta\bigg)x^{-(1/2)-\delta}\bigg]+\bigg(\nu-\frac{d}{dx}\bigg)\mathscr{E}_p-\mathscr{F}_0, \tag{37}$$

$$\mathscr{N}(x)=-e^{\nu x}\bigg[c_1\bigg(\frac{1}{2}+\delta+\nu\bigg)x^{-(1/2)+\delta}+c_2\bigg(\frac{1}{2}-\delta+\nu\bigg)x^{-(1/2)-\delta}\bigg]-\frac{d}{dx}\mathscr{E}_p, \tag{38}$$

$$\mathscr{J}(x)=-\nu e^{\nu x}(c_1x^{(1/2)+\delta}+c_2x^{(1/2)-\delta})-\nu\mathscr{E}_p+\mathscr{F}_0. \tag{39}$$

with

$$\frac{d}{dx}\mathscr{E}_p(x)=\nu\mathscr{E}_p+\frac{\nu\mathscr{F}_0}{2\delta}e^{\nu x}\bigg[\bigg(\frac{1}{2}+\delta\bigg)x^{-(1/2)+\delta}\times\int_1^x e^{-\nu u}u^{(1/2)-\delta}\,du-\bigg(\frac{1}{2}-\delta\bigg)x^{-(1/2)-\delta}\times\int_1^x e^{-\nu u}u^{(1/2)+\delta}\,du\bigg]. \tag{40}$$

III. AMPLIFICATION OF SPACE CHARGE MODULATIONS

In this section we consider the amplification of a signal by a beam due to a modulation of the velocity of electrons produced at a given radius $r=R_e$, of the form

$$\delta v(R_e,t)=v_g e^{-i\omega t},$$

where ω is the frequency of the oscillation and v_g its amplitude. The electrons are accelerated in a region of dimensions much smaller than the typical wavelength of the system, thus producing no modulation of the current at the location of the exciting grids. Therefore, the following boundary condition for the perturbed current density δj holds

$$\delta j(R_e,t)=0.$$

The values of $\mathscr{E}(x)$ and $\mathscr{J}(x)$ given by Eqs. (29) and (32) for cylindrical systems, or by Eqs. (36) and (39) for spherical systems, may be applied with the boundary conditions

$$\mathscr{V}(1)=\frac{v_g}{v_0},\quad \mathscr{J}(1)=0, \tag{41}$$

and in the absence of additional boundary conditions we may take $\mathscr{F}(t)=0$, so that $\mathscr{F}_0=0$ and $\mathscr{E}_p=0$. Since ν is pure imaginary for this problem, we define a real nondimensional quantity $q\equiv i\nu=\omega R_e/v_0$. In this case the value of $\mu=\omega_pR_e/v_0$, is defined using the unperturbed electron density at the position of the modulating electrode, $n_0(r=R_e)=n_e$, and the relativistic plasma frequency ω_p.

A. Signal amplification by a cylindrical flow

Applying the boundary conditions (41), in the case of a cylindrical system we find

$$c_1J_0(2\mu)+c_2Y_0(2\mu)=\frac{v_g}{v_0}\frac{e^{iq}}{\mu}, \tag{42}$$

$$c_1J_1(2\mu)+c_2Y_1(2\mu)=0. \tag{43}$$

Thus, the constants c_1, c_2 are given by

$$c_1=-\pi\frac{v_g}{v_0}e^{iq}Y_1(2\mu),\quad c_2=\pi\frac{v_g}{v_0}e^{iq}J_1(2\mu),$$

so that

$$\begin{aligned}\mathscr{J}(x,\tau)&=iq\mathscr{E}(x,\tau)\\&=-i\frac{\omega}{\omega_p}\frac{v_g}{v_0}e^{-iq(x-1+\tau)}\\&\quad\times\pi\mu\sqrt{x}[Y_1(2\mu)J_1(2\mu\sqrt{x})\\&\quad-J_1(2\mu)Y_1(2\mu\sqrt{x})].\end{aligned} \tag{44}$$

Equation (44) corresponds to a sinusoidal wave modulated by an envelope $A(x,\mu)$, that describes the amplification properties of the system, defined as

$$\begin{aligned}A(x,\mu)\equiv{}&\pi\mu\sqrt{x}[Y_1(2\mu)J_1(2\mu\sqrt{x})\\&-J_1(2\mu)Y_1(2\mu\sqrt{x})].\end{aligned}$$

The maximum of amplification takes place at points x_m where

$$Y_1(2\mu)J_0(2\mu\sqrt{x_m})-J_1(2\mu)Y_0(2\mu\sqrt{x_m})=0. \tag{45}$$

Note that $\mathscr{J}$ is proportional to the total current flowing in the device at the position x, since it incorporates the factor r. Hence A can be compared directly with the corresponding value for planar devices where a similar function $A_{\rm pl}(x,\mu)$ may be defined as

$$A_{\rm pl}(x,\mu)=\sin \mu x, \tag{46}$$

taking R_e as an arbitrary scaling length in this case, so that the amplification is maximum for $x_m=\pi/2\mu$, where $A_{\rm pl}=1$.

An estimation of the amplification for $2\mu\sqrt{x}\gg 1$ and $2\mu\gg 1$ in (44) is obtained taking into account that in this case $\mathscr{J}$ may be expressed as

$$\mathscr{J}(x,\tau)\approx -i\,\frac{\omega}{\omega_p}\,\frac{v_g}{v_0}\,e^{-iq(x-1+\tau)}\,\sqrt[4]{x}\,\sin[2\mu(1-\sqrt{x})], \tag{47}$$

and, thus,

$$A(x,\mu)=\sqrt[4]{x}\,\sin[2\mu(1-\sqrt{x})]. \tag{48}$$

In this case a maximum value of $|A|$ is attained at $x_m=1\pm\pi/2\mu$, where the plus sign holds when the beam is divergent ($v_0<0,x\geqslant 1$) and the minus sign for a convergent beam ($v_0>0,x\leqslant 1$), with $A(x_m,\mu)\approx 1\pm\pi/8\mu$, so that the amplification factor A for a divergent beam is slightly greater than in the planar case, while for a convergent beam it is smaller than the planar one.

On the other hand, when $2\mu\ll 1$,

$$A(x,\mu)\approx -\sqrt{x}[J_1(2\mu\sqrt{x})+\pi\mu^2 Y_1(2\mu\sqrt{x})]. \tag{49}$$

Large values of $|A(x,\mu)|$ are attained when $2\mu\sqrt{x}\approx 2$. In fact, $|A(x,\mu)|\approx 0.6/\mu$, for $\sqrt{x}=1/\mu$. For example, when $\mu=0.1$, we have $|A|\approx 6$ at $x=100$. Therefore, the amplification of signals in a cylindrical configuration can take, in principle, considerably higher values than in the planar case, for larger drifting intervals (in planar geometry, e.g., $|A|=1$ at $x_m=\pi/2\mu=15.7$, when $\mu=0.1$).

In the neighborhood of a point x_0, Eq. (47) can be expressed as

$$\mathscr{J}(x,\tau)\approx -i\,\frac{\omega}{\omega_p}\,\frac{v_g}{v_0}\,e^{-iq(x-1+\tau)} \times\sqrt[4]{x}\,\sin\left[2\mu\left(1-\frac{\sqrt{x_0}}{2}\right)-\mu\,\frac{x}{\sqrt{x_0}}\right], \tag{50}$$

which is of the form

$$\mathscr{J}(x,\tau)=\sqrt[4]{x}\left\{A\,\exp\left[-i\left(q+\frac{\mu}{\sqrt{x_0}}\right)x-iq\tau\right] +B\,\exp\left[-i\left(q-\frac{\mu}{\sqrt{x_0}}\right)x-iq\tau\right]\right\},$$

where A and B are constants. This shows that in this case the solution can be interpreted in terms of a "fast" and a "slow" wave (as in the planar case) whose phase velocities are given by

$$v_{f,s}(x_0)=\frac{\omega}{\omega\pm\omega(x_0)}\,v_0, \tag{51}$$

where $\omega^2(x_0)\equiv 4\pi e^2 n_e/\gamma_0^3 m x_0$ is the square of a local plasma frequency defined using the unperturbed density $n_0(x_0)$ at the point x_0, $n_0(x_0)=n_e/x_0$. When the curvature of the modulating electrode is small, solution (47) can be approximated taking the relative distance $|1-x|\ll 1$. Then, one obtains the zero curvature limit of $\mathscr{J}$,

$$\mathscr{J}(x,\tau)=-i\,\frac{\omega}{\omega_p}\,\frac{v_g}{v_0}\,e^{-iq(x-1+\tau)}\,\sin[\mu(1-x)], \tag{52}$$

which coincides with the expression for the planar case, taking into account that $\mu(1-x)=(\omega_p/v_0)(R_e-r)$, and that $|R_e-r|$ is the distance from point r to the modulating electrode.

B. Perturbations amplified by a spherical flow

In a spherical system the boundary conditions (41) lead to

$$\frac{1}{2}(c_1+c_2)+\delta(c_1-c_2)=-\frac{v_g}{v_0}\,e^{iq}, \tag{53}$$

$$c_1+c_2=0, \tag{54}$$

and, therefore, $c_2=-c_1=(e^{iq}/\sqrt{1-4\mu^2})(v_g/v_0)$, since $\delta=\sqrt{\frac{1}{4}-\mu^2}$. We then obtain

$$\mathscr{J}(x,\tau)=i\,\frac{\omega}{\omega_p}\,\frac{v_g}{v_0}\,e^{-iq(x-1+\tau)}\sqrt{x}\,\frac{\mu}{\sqrt{1-4\mu^2}} \times\left(-x^{\sqrt{(1/4)-\mu^2}}+x^{-\sqrt{(1/4)-\mu^2}}\right). \tag{55}$$

From the previous expression we note that we may define an amplification factor $A(x,\delta)$ as

$$A(x,\delta)\equiv -\sqrt{\frac{1}{4}-\delta^2}\,\sqrt{x}\,\frac{\sinh(\delta\ln x)}{\delta}. \tag{56}$$

Since $\mathscr{J}$ already includes a factor r^2, it is proportional to the total current so that $A(x,\delta)$ can be compared with $A_{\rm pl}(x,\mu)$, the amplification factor of the planar case, Eq. (46).

Taking $\mu=0.1$ and $x=100$ we find $|A|\approx 9.6$ for the amplification factor. As μ approaches the value 1/2 from below, larger values of $|A|$ can be obtained. For instance, when $\mu\approx 0.49$ ($\delta=0.1$) and $x=100$, we get $|A|=23$.

For $\delta\ll 1/2$ (i.e., $\mu\to 1/2$) and $\delta\ln x\ll 1$, we find that $A(x,\delta)=-(1/2)\sqrt{x}\ln x$. Thus, for a convergent beam, A attains a maximum at $x_0=e^{-2}$ where $A(x_0,\delta\ll 1/2)=1/e$. Alternatively, when δ approaches the value 1/2, then $A(x,\delta\approx 1/2)=\sqrt{1/4-\delta^2}(1-x)$, so that the amplification tends to zero with increasing δ (i.e., $\mu\to 0$).

For $\mu>1/2$, the amplification factor may be written in terms of $\lambda\equiv -i\delta$ as

$$A(x,\lambda)=-\sqrt{1+\frac{1}{4\lambda^2}}\,\sqrt{x}\,\sin(\lambda\ln x). \tag{57}$$

For a given λ, we find a maximum of $|A|$ at $x_m=e^{\pm\pi/2\lambda}$, where $A(x_m,\lambda)=\sqrt{1+1/(4\lambda^2)}e^{\pm\pi/4\lambda}$. The plus sign corresponds to a divergent beam, and the minus sign to a convergent one. For a divergent beam, $A(x_m,\lambda)$ is a decreasing function of λ, which is always greater than one. When $\mu=0.6$, and $x=100$, for example, the amplification factor $A(x_m,\lambda)$ is near 178. Thus, the amplification exceeds that of the planar or cylindrical configurations. For a convergent beam, $A(x_m,\lambda)$ increases with λ, but its value is always less than 1. When $\lambda\rightarrow\infty$, $A(x_m,\lambda)\rightarrow 1$, which is the value of the planar case.

Asymptotic expressions for the spherical problem may be readily found taking $\ln x\approx x-1$ in (57) so that

$$\mathscr{J}(x)\approx-\frac{\omega}{\omega_p}\frac{\sqrt{1/4+\lambda^2}}{2\lambda}\frac{v_g}{v_0}\sqrt{x}\times\left(e^{i(q-\lambda)(1-x)}-e^{i(q+\lambda)(1-x)}\right), \tag{58}$$

which shows the presence of "fast" and "slow" waves as in the planar and cylindrical case, when $\mu>1/2$. From this equation we can immediately see that as the curvature of the modulating electrode tends to zero, the results of the planar case are obtained. When $\mu<1/2$, the equivalent of "fast" and "slow" waves do not exist, although there is a space charge modulation carried by the flow.

The case $\delta=0$ may be treated similarly to the previous ones. No new features appear in the results for amplification which, on the other hand, could also be extracted from the limiting case $\delta\rightarrow 0$ discussed previously. The expression for $\mathscr{J}(x)$ is in this case given by

$$\mathscr{J}(x,\tau)=iq\mathscr{E}(x,\tau)=-i\frac{\omega}{\omega_p}\frac{v_g}{v_0}e^{-iq(x-1+\tau)}\frac{1}{2}\sqrt{x}\ln x. \tag{59}$$

Finally, we remark that the results of this section are valid for relativistic energies, since the square of the classical plasma frequency has been defined as $\omega_p^2\equiv 4\pi e^2 n_e/\gamma_0^3 m$, where γ_0 is the (constant) relativistic mass dilation factor.

IV. INSTABILITY OF THE RADIAL ELECTRON FLOW IN GROUNDED GRIDS

We shall study the conditions under which a beam of electrons, that have been accelerated to an energy $mv_0^2/2$ and enter a drift region between grids (transparent to the beam) that are at the same potential, may develop an instability. This instability has been studied for plane geometry, and we shall extend the analysis to the cases of cylindrical and spherical symmetry. Classically, a space charge perturbation of the beam induces an instantaneous redistribution of charges on the boundaries that keeps the electric field in the conductors equal to zero. When v_0 approaches the speed of light, the time delay for the traveling of signals that lead to this process becomes relevant. The application of an equipotential boundary condition (characteristic of the Pierce problem) requires that the vacuum wavelength, λ, of the time-dependent electromagnetic fields be much larger than the interelectrode distance, L. When this condition is not satisfied the electrodes cannot be treated as equipotential surfaces and the electric field cannot be derived from a scalar potential alone. For the Pierce instability the transit time L/v_0 is of the order of $1/\omega_p$ (see the Introduction) and, therefore, $L/\lambda\approx c/v_0$. For this reason we shall restrict the treatment of the Pierce problem to classical energies, i.e., velocity of the beam much smaller than the speed of light. The electric field is then considered as a quasipotential field, as assumed in Sec. II. In some relativistic studies of the planar Pierce diode (e.g., Refs. 26 and 27) this limitation has not been noticed.

The Pierce instability develops in the vicinity of steady states, when a critical current density has been reached. For the boundary conditions of the Pierce problem, there exist other steady states besides the one with uniform velocity $v(x)=v_0$. In particular, for certain critical values of the current density there are states which are small amplitude modulations of those with velocity v_0, described by nontrivial solutions of the linearized perturbation equations presented in Sec. II. The boundary conditions to be applied to the perturbation variables for a beam which has velocity $-v_0\mathbf{e}_r$ and density n_e at the position $r=R_e$, flowing through electrodes that are at the same potential and located at $r=R_e$ and $r=R_i$, are

$$\mathscr{V}(1)=0, \tag{60}$$

$$\mathscr{N}(1)=0, \tag{61}$$

$$\int_{x_i}^{1}\frac{\mathscr{E}(x)}{x^l}\,dx=0, \tag{62}$$

where $x_i\equiv R_i/R_e$ and 1 are the position of the electrodes in nondimensional units, and $l=1,2$ correspond to cylindrical and spherical diodes, respectively. After introducing these conditions in the general solutions, a relationship between the velocity v_0 and the density n_e is obtained as a condition for the existence of the modulated steady states. These values of velocity and density define the critical current density j_c of the Pierce problem.

A. Pierce instability for cylindrical diodes

Given the solutions, Eqs. (29)–(32) for the cylindrical problem the boundary conditions (60)–(62) have the following expressions:

$$C_1J_0(2\mu)+C_2Y_0(2\mu)=0, \tag{63}$$

$$-\nu e^{\nu}[C_1J_1(2\mu)+C_2Y_1(2\mu)]=\mu\mathscr{F}_0, \tag{64}$$

$$\int_{x_i}^{1}\frac{e^{\nu x}}{\sqrt{x}}\left[\pi\mu^2\mathscr{F}_0\left(J_1(2\mu\sqrt{x})\int_1^x e^{-\nu u}Y_0(2\mu\sqrt{u})du-Y_1(2\mu\sqrt{x})\int_1^x e^{-\nu u}J_0(2\mu\sqrt{u})du\right)+C_1J_1(2\mu\sqrt{x})+C_2Y_1(2\mu\sqrt{x})\right]dx=0. \tag{65}$$

A nontrivial solution for $C_1,C_2,\mathscr{F}_0$ exists when the determinant of this system of linear algebraic equations is zero. Thus the dispersion equation for ν is given by

$$J_0(2\mu)I_2-Y_0(2\mu)I_1+\nu e^{\nu}I_0=0 \tag{66}$$

with

$$I_0=\int_{x_i}^{1}\frac{e^{\nu x}}{\sqrt{x}}\left(J_1(2\mu\sqrt{x})\int_1^x e^{-\nu u}Y_0(2\mu\sqrt{u})du\right.$$
$$\left.-Y_1(2\mu\sqrt{x})\int_1^x e^{-\nu u}J_0(2\mu\sqrt{u})du\right),$$

$$I_1=\int_{x_i}^{1}\frac{e^{\nu x}}{\sqrt{x}}J_1(2\mu\sqrt{x})dx,$$

$$I_2=\int_{x_i}^{1}\frac{e^{\nu x}}{\sqrt{x}}Y_1(2\mu\sqrt{x})dx.$$

Steady-state solutions are determined by taking $\nu=0$ [in Eqs. (29)–(32) and in the boundary conditions]. Integrating, with $\nu=0$, we find that there are an infinite number of pairs μ, x_i such that[28]

$$J_0(2\mu)Y_0(2\mu\sqrt{x_i})-Y_0(2\mu)J_0(2\mu\sqrt{x_i})=0. \tag{67}$$

For these values, steady-state solutions with nonuniform velocity exist. This means that, given the velocity v_0 of the electrons and the position R_e of an electrode, there exist infinite pairs R_i, n_e such that the system admits modulated steady states, when the electrodes are at the same potential. On the other hand, given R_i, R_e, and v_0, there is a minimum value of the density below which there are no solutions of (67). For example, for $\sqrt{x_i}=1.25$, there are no zeros for $\mu<6.3$. As x_i approaches 1, the minimum value of μ becomes increasingly larger.

Given a ratio x_i between the radii of the electrodes, let $2\mu=D$ stand for a zero of (67). Since $D^2=16\pi e^2 n_e R_e^2/(mv_0^2)$, the value of the critical current density j_c is given by

$$j_c=en_e v_0=\frac{D^2}{4\pi}\left(\frac{e}{2m}\right)^{1/2}\frac{\Phi^{3/2}}{R_e^2}\approx 10^{-4}\frac{D^2}{4\pi^2}\frac{\Phi^{3/2}}{R_e^2}, \tag{68}$$

in terms of the acceleration potential $\Phi=mv_0^2/2e$, where the units of the last expression are A/cm^2, volts, and cm. The total critical current per unit height, I_c, is then

$$I_c=2\pi R_e j_c=\frac{D^2}{2}\left(\frac{e}{2m}\right)^{1/2}\frac{\Phi^{3/2}}{R_e}. \tag{69}$$

For fixed values of R_e, n_e, and v_0, we observe that if the pair x_i, D is a solution of (67), then the pair $1/x_i$, $D\sqrt{x_i}$ is also a solution. This implies that for fixed R_e and v_0, the current is larger for an outgoing beam with electrode ratio x_i (>1) than for an incoming one with electrode ratio $1/x_i$ (<1). As an example, for $\sqrt{x_i}=1.25$, that is for a divergent beam with $R_i=(25/16)R_e$ we have $D=12.6$, and for a convergent beam with $R_i=(16/25)R_e$, $D=12.6\times1.25=15.7$.

If we define the distance between electrodes, L, as $L=|R_e-R_i|=|x_i-1|R_e$, we may express the current density in terms of L as,

$$j_c=\frac{D^2}{4\pi}\left(\frac{e}{2m}\right)^{1/2}\frac{\Phi^{3/2}}{L^2}(x_i-1)^2$$
$$\approx 10^{-4}\frac{D^2}{4\pi^2}\frac{\Phi^{3/2}}{L^2}(x_i-1)^2. \tag{70}$$

Comparing this expression with Eq. (2) of the planar case for the same values of L and v_0, we find that they differ by a factor $(D^2/4\pi^2)(x_i-1)^2$. Using the values of $\sqrt{x_i}$ and D of the previous example we see that $j_c=0.8\times10^{-4}\Phi^{3/2}/L^2$ for the convergent case, and $j_c=1.3\times10^{-4}\Phi^{3/2}/L^2$ for the divergent case. Thus, taking the same values of L and v_0 for the plane and the cylindrical structures, we find that the current density of the divergent beam is higher than that of the planar case, and the planar case one is, in turn, higher than the density of the convergent beam.

As the distance between electrodes increases, the critical current may be higher in the cylindrical case than the equivalent critical current (per unit area) for the planar geometry, Eq. (2). Consider, for example, a configuration where $|2\mu|\gg1$ and $x_i>1$. Then, using an asymptotic expression for the Bessel functions, Eq. (67) may be approximated by

$$\sin[2\mu(1-\sqrt{x_i})]=0,$$

which shows that $D=\pi/(1-\sqrt{x_i})$. Introducing this value in (68) we have

$$j_c=\frac{\pi}{4}(1+\sqrt{x_i})^2\left(\frac{e}{2m}\right)^{1/2}\frac{\Phi^{3/2}}{L^2}. \tag{71}$$

When $R_i\gg R_e$ we find that

$$j_c=\frac{\pi}{4R_e}\left(\frac{e}{2m}\right)^{1/2}\frac{\Phi^{3/2}}{L}, \tag{72}$$

so that $I_c\propto 1/L$, while in the plane case $I_c\propto1/L^2$. We may also note that small R_e values favor higher critical currents, if the interelectrode distance is fixed. Thus, at small radii the properties of the instability threshold are quite different from the planar case.

For the modulated steady states the perturbation variables are proportional to

$$\mathscr{E}(x)=-\frac{1}{\mu}\sqrt{x}[Y_0(2\mu)J_1(2\mu\sqrt{x})$$
$$-J_0(2\mu)Y_1(2\mu\sqrt{x})], \tag{73}$$

$$\mathscr{V}(x)=\mathscr{N}(x)=Y_0(2\mu)J_0(2\mu\sqrt{x})$$
$$-J_0(2\mu)Y_0(2\mu\sqrt{x}), \tag{74}$$

$$\mathscr{J}(x)=0. \tag{75}$$

The corresponding expression for the electrostatic potential $\phi(x)$ may be derived by integrating the electric field. This gives

$$\phi(x)-\phi(1)=-\frac{m}{e}v_0^2\mathscr{V}(x)=-\frac{m}{e}v_0\delta v(x). \tag{76}$$

A small change in the parameters of a steady state (in velocity, density, or radius of the electrodes) may generate an instability. To show this, one may calculate the frequency $\nu\equiv\epsilon$ that corresponds to a state whose parameters are near the steady state. Putting such values in the boundary conditions, a first order in ϵ estimate of the frequency gives

$$\epsilon=\frac{2\mu^2(b-a)}{(2/\mu\pi)-b-a}\Delta, \tag{77}$$

where Δ denotes the relative change in μ ($\Delta \ll 1$) and we have defined

$$a \equiv J_1(2\mu)Y_0(2\mu\sqrt{x_i}) - Y_1(2\mu)J_0(2\mu\sqrt{x_i}),$$

$$b \equiv \sqrt{x_i}[Y_0(2\mu)J_1(2\mu\sqrt{x_i}) - J_0(2\mu)Y_1(2\mu\sqrt{x_i})],$$

with a and b evaluated at the steady-state values. Equation (77) holds under the condition that $(2/\mu\pi) - b - a \neq 0$. Since the expression for ϵ is real and the sign of Δ can always be chosen to make the growth rate $\epsilon > 0$, we have shown the existence of Pierce's instability in a cylindrical system.

When the radius of curvature of the electrodes is very large, one would expect to find a connection between the cylindrical and plane solutions. In fact, we may evaluate the asymptotic behavior of the cylindrical solutions by considering that the radius of the outer electrode is very large, the relative distance between electrodes small, and keeping the density at the external electrode at a fixed value. In terms of nondimensional quantities this is stated as $x_i \approx 1$, $\mu \gg 1$. Using an asymptotic expression for Bessel functions[28] we find that Eq. (67) for the steady state reduces to $2\mu(1 - \sqrt{x_i}) = n\pi$, alternatively expressed as

$$\frac{\omega_p(R_e - R_i)}{v_0} = n\pi \quad (n = 1,2,\ldots) \tag{78}$$

which is the condition for the steady states of the planar case, Eq. (1), when n is odd. The asymptotic value of the growth rate may be estimated taking into account that in this case $\sqrt{x_i}a = b = (1/\pi\mu)(-1)^n x_i^{1/4}$, so that

$$\epsilon = -\frac{\mu^2}{2}(\sqrt{x_i} - 1)\Delta, \tag{79}$$

where the value of a has to be taken with n odd to avoid a zero of the denominator of ϵ in Eq. (77). We also note that since $2\mu(1 - \sqrt{x_i}) \approx \mu(1 - x_i)$, close to a steady state we may write

$$\mu(1 - x_i) \equiv n\pi + \zeta \quad (n = 1,3,\ldots),$$

defining a parameter $\zeta \ll 1$. As a consequence $\epsilon = n\pi/[4(1 - x_i)]\zeta$ $(n = 1,3,\ldots)$. Therefore the growth rate Γ is

$$\Gamma = \frac{n\pi v_0}{4(R_e - R_i)}\zeta \quad (n = 1,3,\ldots) \tag{80}$$

which coincides with the values of the planar case.

B. Pierce instability for spherical diodes

For spherical systems the boundary conditions (60)–(62), derived from Eqs. (36)–(39), are the following:

$$e^{\nu}[c_1 + c_2(\tfrac{1}{2} - \delta)] + \mathscr{F}_0 = 0, \tag{81}$$

$$c_1(\tfrac{1}{2} + \delta - \nu) + c_2(\tfrac{1}{2} - \delta - \nu) = 0, \tag{82}$$

$$\int_{x_i}^{1} e^{\nu x}\Bigg[c_1 x^{-(3/2)+\delta} + c_2 x^{-(3/2)-\delta} + \frac{\nu\mathscr{F}_0}{2\delta}\left(x^{-(3/2)+\delta}\int_1^x e^{-\nu u}u^{(1/2)-\delta}\,du - x^{-(3/2)-\delta}\int_1^x e^{-\nu u}u^{(1/2)+\delta}\,du\right)\Bigg]dx = 0. \tag{83}$$

The dispersion relation for the spherical Pierce instability is obtained by equating to zero the determinant of this system of homogeneous linear algebraic equations,

$$(\tfrac{1}{2} + \delta - \nu)I_2 - (\tfrac{1}{2} - \delta - \nu)I_1 + I_0[(\tfrac{1}{2} + \delta)(\tfrac{1}{2} - \delta - \nu) - (\tfrac{1}{2} - \delta)(\tfrac{1}{2} + \delta - \nu)] = 0, \tag{84}$$

where the following definitions have been used:

$$I_0 = \frac{\nu}{2\delta}\int_{x_i}^{1} e^{\nu x}\left(x^{-(3/2)+\delta}\int_1^x e^{-\nu u}u^{(1/2)-\delta}\,du - x^{-(3/2)-\delta}\int_1^x e^{-\nu u}u^{(1/2)+\delta}\,du\right)dx,$$

$$I_1 = \int_{x_i}^{1} e^{\nu x}x^{-(3/2)+\delta}\,dx,$$

$$I_2 = \int_{x_i}^{1} e^{\nu x}x^{-(3/2)-\delta}\,dx.$$

For the steady state, $\nu = 0$, (84) reduces to $x_i^{2\delta} = 1$. Hence we find that a steady state exists when

$$\delta \ln x_i = \pm n\pi i \quad (n = 1,2,\ldots). \tag{85}$$

Since δ must be pure imaginary, we see that

$$\mu^2 > 1/4, \tag{86}$$

or, equivalently, the density must be greater than a value

$$n_{\min} = \frac{mv_0^2}{16\pi e^2 r^2}. \tag{87}$$

Below this density Pierce instability is absent, for any value of the interelectrode separation L, a result markedly different from the planar case, where the critical density is proportional to $1/L^2$ [see (2)], or from the cylindrical case where there is also a critical density function of L.

The value for the critical current density, derived from (85) is

$$j_c = \pi\left(\frac{e}{2m}\right)^{1/2}\left(\frac{1}{[\ln(R_i/R_e)]^2} + \frac{1}{4\pi^2}\right)\frac{\Phi^{3/2}}{R_e^2} \approx 10^{-4}\left(\frac{1}{[\ln(R_i/R_e)]^2} + \frac{1}{4\pi^2}\right)\frac{\Phi^{3/2}}{R_e^2}, \tag{88}$$

where the units are A/cm^2, volts, and cm, and Φ is the accelerating potential. The critical current is, therefore,

$$I_c \approx 10^{-4}\left(\frac{4\pi}{[\ln(R_i/R_e)]^2} + \frac{1}{\pi}\right)\Phi^{3/2}. \tag{89}$$

We note that for a fixed value of R_e, the value of the critical current tends to $I_c \approx 10^{-4}\Phi^{3/2}/\pi$ when R_i goes to zero or infinity. On the other hand, when L tends to zero with constant R_e values, then $I_c \propto \Phi^{3/2}/L^2$ as in the planar case.

When the system is slightly displaced from the steady state the existence of an instability may be shown by writing (84) with $\nu = \epsilon \ll 1$ to first order in ϵ. The expression derived is

$$\epsilon = -\frac{\mu^2 \ln x_i}{[1-(-1)^n \sqrt{x_i}]^2} \Delta, \tag{90}$$

where Δ stands for the relative variation in $\delta \ln x_i$ as the parameters of the system are slightly modified and x_i satisfies (85).

In spherical geometry, the analogy with the plane case exists when $x_i \approx 1$. From (85) we see that x_i near one is equivalent to $\mu \gg \pi$, which may be expressed as a condition on the unperturbed density n_0, that is, $n_0 \gg m v_0^2/(4e^2 r^2)$. When this condition is fulfilled, the nondimensional growth rate ϵ in the first-order approximation is

$$\epsilon = -\frac{\mu^2 \ln x_i}{2[1-(-1)^n]} \Delta, \tag{91}$$

so that the expression is valid for odd values of n only. Taking n odd we obtain the growth rates of the planar case. If the condition $x_i \approx 1$ does not hold, the situation is different from the planar case, and ϵ is given by Eq. (90), which is valid for n even or odd. An approximation to the growth rate like Eq. (90) does not exist in the planar case for n even.

V. CONCLUSIONS

We have given the small amplitude oscillatory properties of a single neutralized electron beam, flowing radially in cylindrical and spherical configurations. Solutions for steady-state and time-dependent perturbations have been derived, and applied to the problem of amplification of an electric signal in the first place, and then separately to the study of Pierce instability for convergent and divergent beams in cylindrical and spherical diodes. The slow and fast space charge waves carried by the radial beams show considerable amplitude modulation and variable local wavelengths at small radii (due to a singularity at $r=0$), while in the regions of small curvature their properties approach those of the planar case, as expected. The parameter $\mu = \omega_p R_e / v_0$ plays a crucial role in radial beam systems in both geometries. Slow and fast space charge waves with properties similar to the planar case exist also in cylindrical configurations. In spherical systems a new qualitative feature appears: slow and fast space charge waves, with phase velocity distinct from v_0 exist only when $\mu > 1/2$. For $\mu < 1/2$, instead, there are space charge modulations only, transported by the beam with velocity v_0. Note that μ may decrease by relativistic effects, i.e., when $\gamma_0 \gg 1$. These peculiar features of the oscillatory spectrum of a single beam in cylindrical and spherical geometries are the basis of the markedly different results obtained here in matters of signal amplification and beam critical current, compared to the planar case.

To obtain better signal amplification than in the planar case we found that one must work with divergent beams. Thus, the beams are modulated at an inner electrode and the amplified signal is picked up at a larger radius. In the comparison with the planar case the total current, proportional to $r^l \delta j$ with $l=1$ or 2, is examined (as the generator of an amplified signal), and the amplification factor $A(x,\mu)$ defined in Sec. III contains the characteristic properties of cylindrical and spherical systems. It turns out that this factor can be larger than 1, while in the planar case $|A|$ can be at most equal to 1. In cylindrical systems by exploiting the singularity of Bessel functions of the second kind, amplification factors noticeably larger than 1 ($|A| \approx 6$) can be achieved at low values of μ with respect to 1, if we allow for ratios of radii as large as 100. In the spherical case we obtain, in general, much better gains than in the other two geometries: planar and cylindrical. Values of $|A|$ much larger than 1 are possible here (allowing for R_i/R_e as large as 100) both for μ smaller or larger than the limit value 0.5, but the best range for μ is slightly above 0.5, in the physical regime where the slow and fast waves exist. For $\mu = 0.6$, an amplification factor as large as 178 times that of the planar case is obtained. Finally, as a control of the given results, the formulas of the planar geometry have been recovered in the limit of small curvature of the system.

While the signal amplification formulas of Sec. III can be extended to relativistic beam energies, the section on Pierce instability is limited to classical electron speeds. This is due to the fact that the ordinary equipotential boundary conditions of Sec. IV cannot be applied straightforwardly when v_0 becomes comparable to c. The known planar threshold of Pierce instability is strongly modified in the radial flow cases when curvature effects are dominant. At a fixed acceleration potential, the critical current is proportional to $1/L^2$ in the planar case, where L is the interelectrode distance. In the cylindrical case, when $|2\mu| \gg 1$ and $R_i \gg R_e$, we find that it is proportional to $1/L$. In the spherical case, however, for fixed R_e the critical current I_c is proportional to $1/L^2$ when $L \ll R_e$ but, as L increases, I_c decreases toward a limit value that is independent of R_e. Moreover, in spherical configurations a limit density always exists such that for beam densities below this value the system is stable against Pierce instability, no matter how large the interelectrode separation. The growth rate of the instability near the marginal states has been derived for cylindrical and spherical systems. The connection between the planar Pierce instability and the small curvature limit of the cylindrical and spherical cases has been controlled also. It is important to note that for focused beams the instability depends on two parameters μ and x_i, while in the planar case the spectrum is characterized by one parameter, α, only.

From the point of view of realism we should recall that the model treated here is subject to several limitations. Among other questions, we have ignored the steady-state self-induced magnetic field of the beam, so that a constraint on transversal dimensions must be introduced. Also, the mass of the neutralizing ions has been taken as infinite, while in fact an electron–ion, two-stream instability should be taken into account. This sets a limit on the longitudinal size of the system (to prevent development of the electron–ion interaction).

Collective processes in bounded systems of charged particles differ considerably from the predictions of infinite models, as we had occasion to confirm here also. In particular, care must be exerted when both Poisson's equation and Ampère's current equation are applied to bounded system problems [see Sec. II, Eqs. (12), (13), and (16) and their consequences]. Ampère's equation is treated differently in the amplification configuration $[f(t)=0]$ than in Pierce's problem $[f(t)\neq 0]$. To conclude, let us emphasize the remarkable changes in the theoretical properties of the oscillations of a single beam introduced by convergence or divergence of the trajectories. It would also be interesting to learn about changes in other collective processes, like Landau damping for instance, in a synthetic (i.e., *ad hoc* generated) plasma formed by convergent and/or divergent particle trajectories. Comparisons of our theoretical results with computational experiments for cylindrical and spherical configurations could be developed by adapting the code of Birdsall and co-workers[19] mentioned previously, to include fixed ions with a constant $1/r^l$ density distribution, and electrons with an initial density varying as $1/r^l$, with $l=1,2$.

ACKNOWLEDGMENTS

Thanks are due to the anonymous referee for pointing out the limitations of the Pierce boundary conditions to classical speeds of the beam.

Both authors are members of the Argentine National Research Council (CONICET). This work has been supported by CONICET Grant No. PIDBID 594/92 on Dynamics of Plasmas and Fluids and by the Institute for Plasma Physics (INFIP–CONICET).

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