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# Classical formulations of the electromagnetic self-force of extended charged bodies

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**Abstract.** Several formulations of the classical electrodynamics of charged particles, as have been developed in the course of the twentieth century, are compared. The mathematical equivalence of the various dissimilar expressions for the electromagnetic self-force is demonstrated explicitly by deriving these expressions directly from one another. The new connections that are established present the previously published results on a common basis, thereby contributing to a coherent historical picture of the development of charged particle models.

## 1 Introduction

The self-consistent description of charged particles is no doubt the longest-standing fundamental problem in classical electrodynamics. The conceptual difficulties, which invariably arise from the interaction of the particle with its self-generated electromagnetic fields, depend on the particle model that is postulated [Jiménez and Campos 1999]. In line with modern particle physics, often a structureless point charge model is adopted, leading to the well-known Lorentz-Abraham-Dirac (LAD) equation of motion for elementary charged particles [Dirac 1938; Landau and Lifshitz 1975]. As is also well known [Panofsky and Phillips 2005], however, the LAD equation is plagued by irreconcilable deficiencies manifested by runaway solutions, in which the particle momentum grows exponentially toward infinity, or by preacceleration solutions, where the particle starts to accelerate even before the onset of any external force. A popular remedy is to express in the LAD equation the particle acceleration perturbatively in terms of the external force [Landau and Lifshitz 1975], which renders the equation stable. However, arguably this procedure does nothing to improve the LAD equation intrinsically. It has been put forward that the stabilized equation may be considered as more fundamental than the LAD equation itself, but the argument usually requires abandoning the point particle limit proper by ascribing some structure to the particle [Ford and O'Connell 1991; Rohrlich 2002; Ares de Parga 2006]. Meanwhile, alternative classical electrodynamics of point particles continue to be proposed [Hammond 2013; Kholmetskii 2006; Ares de Parga 2006; Gill et al. 2001; Oliver 1998; Villarroel 2006; Bosanac 2001].

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Another well-studied possibility in the development of a consistent charged particle electrodynamics is to dispense with the point charge model and associated problems altogether, and picture charged particles as extended charged bodies. In fact, this was historically the first charged particle model that was investigated [Janssen and Mecklenburg 2006; Lorentz 1916; Abraham 1902]. If the particle size is larger than a very small but finite critical length (which is comparable to the classical electron radius), the particle dynamics is free of the unphysical runaway solutions [Wildermuth 1955; Moniz and Sharp 1977; Medina 2006]. This continues to motivate detailed calculations of the electromagnetic fields and forces inside accelerated extended charged bodies [Roa-Neri and Jiménez 1993, 2002; Medina 2006; Yaghjian 2006]. Moreover, extended charged particles keep open the possibility of electromagnetic interpretations of inertia [Martins and Pinheiro 2008], as originally proposed in pre-relativistic times [Lorentz 1916; Abraham 1902]. Furthermore, the unphysical prediction of singular or preacceleration behavior of charged particles at a sudden onset of applied forces can be successfully removed by taking into account the finite propagation velocity of signals traversing the extended particle [Yaghjian 2006]. The latter effect is the classical analogue of the concept of "self-dressing" familiar from quantum electrodynamics [Compagno and Persico 2002], and plays a role in the measurability of the electromagnetic field [Hnizdo 2000]. In addition, extended charged particle models provide an important way to access point particle models by taking the appropriate limit corresponding to vanishing particle radius. Consequently, the physical and mathematical consistency of this limiting procedure is itself subject of active research [Gralla et al. 2009; Aguirregabiria et al. 2006; Noja and Posilicano 1999]. Finally, as is nicely summarized by Barut and Dowling [1987], there exists an intimate relation between the phenomenon of zero-point fluctuations of the electromagnetic field in quantum electrodynamics and the classical notion of the electromagnetic self-field. Fermi [1927] was already able to arrive at the Einstein $A$ coefficient by adding a self-field term to the Schrödinger equation. Later, it was indeed shown that the decay of an excited quantum mechanical state can alternatively be interpreted as being caused by zero-point fluctuations or by the reaction of the particle to its self-field [Ackerhalt et al. 1973; Senitzky 1973; Milonni et al. 1973]. Non-perturbative reformulation of quantum electrodynamics based on inclusion of the electromagnetic self-field even proved to successfully reproduce well-known results for spontaneous emission, the electron $g$-factor, and the Lamb shift [Barut and Dowling 1987, 1989; Barut et al. 1992].

For the above reasons, a thorough understanding of the electrodynamics of extended charged bodies is a prerequisite for the development of classical charge particle models of both the point charge and the extended charge variety, and may contribute to developments in quantum electrodynamics. Spread out over a century, however, many formulations of these dynamics have appeared in often dissimilar forms, making a broad comparison of results difficult. In this paper, we aim to contribute to a more coherent historical picture of extended particle models by comparing several published expressions for the self-force of a rigid charged body, that were derived in the course of the twentieth century [Herglotz 1903; Schott 1908; Sommerfeld 1904a, 1904b; Bohm and Weinstein 1948; Jackson 1999]. We demonstrate the equivalence of these dissimilar expressions by deriving them directly from one another. The central concept in electrodynamic models of charged particles is the electromagnetic self-force, which is the resultant Lorentz force experienced by the particle as a consequence of the electromagnetic fields produced by the particle itself. Figure 1 shows schematically the various techniques applied in the considered publications to calculate this self-force. In addition, the figure indicates the contributions of this paper in relation to these publications. It should be stressed that the figure represents only a very small fraction of the available literature on the subject; correspondingly this paper

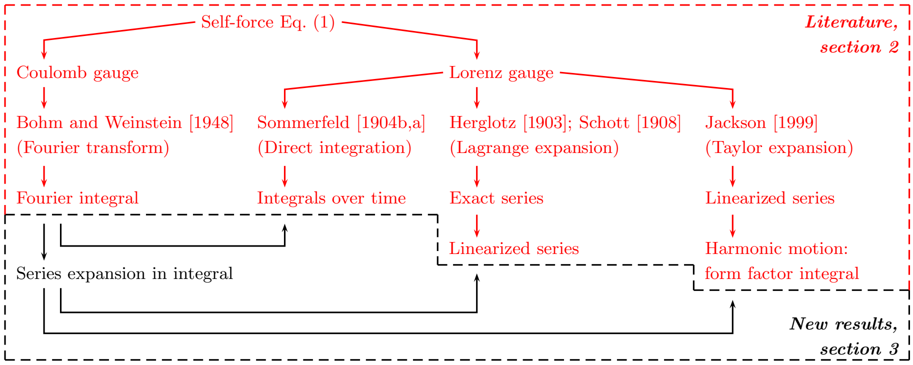


**Fig. 1.** The content of this paper in relation to existing results in literature.

is not meant as a comprehensive review. Rather, the new connections that will be established here complete the literature shown in Figure 1, and present the results on a common basis. This paper is organized as follows. Section 2 gives an overview of the considered existing self-force expressions. In Section 3 it is shown how the various expressions follow directly from each other. Because of their content, these sections have a decidedly mathematical character. Section 4 discusses the applicability of the self-force formulations, and puts them in the wider context of classical charge particle models. Section 5 summarizes and concludes the paper.

## 2 Existing self-force derivations

The self-force of a rigid charged body reads in noncovariant form

$$\boldsymbol{F} = \int \left(\rho \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B}\right) d^3\boldsymbol{x} = \int \left[-\rho \left(\nabla \phi + \frac{\partial \boldsymbol{A}}{\partial t}\right) + \boldsymbol{J} \times \left(\nabla \times \boldsymbol{A}\right)\right] d^3\boldsymbol{x}, \qquad (1)$$

where $\rho$ is the charge density, $\boldsymbol{J}$ is the current density, $\boldsymbol{E}$ is the electric field, $\boldsymbol{B}$ is the magnetic field, $\phi$ and $\boldsymbol{A}$ are the electromagnetic potentials, and the integration is over the extent of the charged body. The self-force calculations that will be considered here take equation (1) as a starting point, and have the form of series expansions [Herglotz 1903; Schott 1908], definite integrals over retarded time [Sommerfeld 1904a, 1904b], and Fourier integrals [Bohm and Weinstein 1948; Jackson 1999]. Like the primary equation (1), our formulation will be noncovariant throughout. The evaluation of equation (1) involves a calculation of the electromagnetic potentials $\phi$ and $\boldsymbol{A}$, which necessitates a choice of gauge. The potentials $\phi^{(L)}$ and $\boldsymbol{A}^{(L)}$ in the Lorenz gauge and the vector potential $\boldsymbol{A}^{(C)}$ in the Coulomb gauge satisfy the wave equations

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) \Psi(\boldsymbol{x}, t) = \mu_0 c \Pi(\boldsymbol{x}, t). \qquad (2)$$

Here, $\Psi \equiv \left\{\phi^{(L)}, c\boldsymbol{A}^{(L)}, c\boldsymbol{A}^{(C)}\right\}$ and $\Pi \equiv \{c\rho, \boldsymbol{J}, \boldsymbol{J}_T\}$, with $\boldsymbol{J}_T$ the divergenceless part of the current density [Jackson 1999]. The scalar potential in the Coulomb gauge is not relevant to the problem, as will be discussed below. In terms of the causal Green's function for the wave equation, which equals [Jackson 1999]

$$G(\boldsymbol{x}, \boldsymbol{x}', t, t') = \frac{\delta\left(t - t' - \left|\boldsymbol{x} - \boldsymbol{x}'\right|/c\right)}{\left|\boldsymbol{x} - \boldsymbol{x}'\right|}, \qquad (3)$$

with $\delta$ the Dirac delta function, the relevant particular solution of equation (2) is given by

$$\Psi(\boldsymbol{x},t)=\frac{\mu_0 c}{4\pi}\iint G(\boldsymbol{x},\boldsymbol{x}',t,t')\Pi(\boldsymbol{x}',t')d^3\boldsymbol{x}'dt'. \tag{4}$$

For any given charge distribution and given history of the motion of the charged body, $\Pi(\boldsymbol{x}',t')$ is known, so that in principle the potentials can be evaluated with equation (4), after which the self-force can be determined via equation (1). The calculations available in literature where this program is followed differ in the order in which the integrations in equation (4) are carried out. In view of the delta function in equation (3), it is tempting to start with the integration with respect to $t'$. This immediately yields the well-known retarded integral expressions [Jackson 1999] for the potentials, which indeed are the starting point for the self-force calculations presented in Sections 2.1 and 2.2 below. However, integrating first with respect to $t'$ in equation (4) is not the only possibility. For certain charge distributions, it is advantageous to start with the integration over $\boldsymbol{x}'$, as will be described in Section 2.3. Still another possibility is to Fourier transform equation (4), that is, to integrate with respect to the coordinates $\boldsymbol{x}$; this is shown in Section 2.4.

### 2.1 Taylor expansion

Adopting the Lorenz gauge, integration of equation (4) with respect to $t'$ yields the retarded integral expressions

$$\Psi(\boldsymbol{x},t)=\frac{\mu_0 c}{4\pi}\int\frac{\Pi(\boldsymbol{x}',t_{ret})}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3\boldsymbol{x}', \tag{5}$$

where now $\Psi\equiv\{\phi,c\boldsymbol{A}\}$ and $\Pi\equiv\{c\rho,\boldsymbol{J}\}$. In equation (5), the integration is complicated by the fact that $\Pi$ must be evaluated at the retarded time $t_{ret}\equiv t-|\boldsymbol{x}-\boldsymbol{x}'|/c$, which is different for each volume element $d^3\boldsymbol{x}'$. Jackson [1999] approaches this problem by expanding $\Pi$ in a Taylor series around the current time $t$,

$$\Pi(\boldsymbol{x}',t_{ret})=\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right)^n\frac{\partial^n\Pi(\boldsymbol{x}',t)}{\partial t^n}. \tag{6}$$

Substitution of equation (6) in equation (5) expresses the potential in terms of quantities evaluated at the current time only. Using the result in equation (1) gives, after some manipulations [Jackson 1999], the electric part of the self-force

$$\begin{aligned}\boldsymbol{F}=-\int d^3\boldsymbol{r}\,\rho(\boldsymbol{r},t)\Bigg[&\frac{\mu_0}{4\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^n}\\ &\times\int R^{n-1}\frac{\partial^{n+1}}{\partial t^{n+1}}\left(\frac{n+1}{n+2}\boldsymbol{J}(\boldsymbol{r}',t)-\frac{n-1}{n+2}\frac{[\boldsymbol{J}(\boldsymbol{r}',t)\cdot\boldsymbol{R}]\,\boldsymbol{R}}{R^2}\right)d^3\boldsymbol{r}'\Bigg].\end{aligned} \tag{7}$$

Here, the integration variables have been changed to $\boldsymbol{r}=\boldsymbol{x}-\boldsymbol{\xi}(t),\boldsymbol{r}'=\boldsymbol{x}'-\boldsymbol{\xi}(t)$ where $\boldsymbol{\xi}(t)$ is the trajectory of the center of the charged body, and $\boldsymbol{R}=\boldsymbol{r}-\boldsymbol{r}'$. For a spherically symmetric rigid charge distribution, equation (7) simplifies to

$$\boldsymbol{F}=-\frac{\mu_0}{6\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^n}\frac{d^{n+2}\boldsymbol{\xi}}{dt^{n+2}}\iint\rho(\boldsymbol{r})\rho(\boldsymbol{r}')\left|\boldsymbol{r}-\boldsymbol{r}'\right|^{n-1}d^3\boldsymbol{r}d^3\boldsymbol{r}'. \tag{8}$$

In equation (8), the magnetic part corresponding to the last term in square brackets in equation (1) has been neglected, so that equation (8) is the self-force linearized in $\boldsymbol{\xi}$ and its time derivatives. In case of harmonic motion $\boldsymbol{\xi} = \boldsymbol{\xi}_0 \exp(-i\omega t) \equiv \widetilde{\boldsymbol{\xi}}$, the series in equation (8) can be readily summed, and is proportional to $\exp(i\omega\,|\boldsymbol{r}-\boldsymbol{r}'|\,/c)$. Furthermore, writing in equation (8) the charge distributions $\rho$ in terms of their spatial Fourier transforms, and integrating the resulting expression, it is found that [Jackson 1999]

$$\boldsymbol{F} = \frac{8\pi\omega^2}{3\epsilon_0 c^2}\,\widetilde{\boldsymbol{\xi}}\,\lim_{\lambda\downarrow 0}\int_0^\infty \frac{k^2\,|\rho_k|^2}{k^2-(\omega/c+i\lambda)^2}dk. \tag{9}$$

Here, the symmetrical convention for Fourier transformed quantities is adopted, i.e. $\boldsymbol{Y}_k \equiv (2\pi)^{-3/2}\int \boldsymbol{Y}(\boldsymbol{r})\exp(-i\boldsymbol{k}\cdot\boldsymbol{r})d^3\boldsymbol{r}$. The quantity $\rho_k$ is often called the form factor of the charge distribution.

## 2.2 Lagrange expansion

As mentioned above, equation (8) is a linearized approximation to the exact self-force due to the neglect of the magnetic term in equation (1). However, the derivation in Section 2.1 is inexact for another reason. Namely, by making use of a predefined rigid charge distribution $\rho$ throughout the derivation (or more precisely, using the distribution in the proper frame), it is implied that the potentials are generated by a total charge $\int \rho(\boldsymbol{x}', t_{ret})d^3\boldsymbol{x}'$. The latter is in general not equal to the true charge of the body $\int \rho(\boldsymbol{x}', t)d^3\boldsymbol{x}' \equiv q$, but rather depends on the body's state of motion. To correct for this inconsistency, either the quantity $\Pi$ should be defined in a relativistically covariant way, or else the integral (5) should be modified to leave the total charge invariant. The latter, however, is precisely how the Liénard-Wiechert potentials for a moving point charge were devised, as is explained clearly by Panofsky and Phillips [2005], Section 19.1. Accordingly, the charged body may be regarded as a collection of infinitesimal particles moving with the trajectory $\boldsymbol{\xi}(t)+\boldsymbol{r}'$ and having a fixed charge $\rho(\boldsymbol{r}')d^3\boldsymbol{r}'$ with $\rho$ the proper frame distribution. The corresponding potentials are thus given by

$$\Psi(\boldsymbol{x},t) = \frac{\mu_0 c}{4\pi}\int \left.\frac{\{c,\boldsymbol{v}\}}{R-\boldsymbol{R}\cdot\boldsymbol{v}/c}\right|_{t=t_{ret}}\rho(\boldsymbol{r}')d^3\boldsymbol{r}', \tag{10}$$

where $\boldsymbol{v}(t) = d\boldsymbol{\xi}/dt$ is the velocity of the charged body, and $\boldsymbol{R}(t) \equiv \boldsymbol{x}-\boldsymbol{\xi}(t)-\boldsymbol{r}'$. An important difference between equation (5) and the Liénard-Wiechert formulation equation (10), apart from the different denominator, is that in the former the retarded time was known explicitly in terms of the coordinates $\boldsymbol{x}$ and $\boldsymbol{x}'$, while in the latter it is only defined implicitly by the retardation condition $t_{ret} = t - R(t_{ret})/c$. This complicates the derivation of the self-force significantly. Herglotz [1903] and Schott [1908] proceeded by expanding retarded quantities $Y$ in series using Lagrange's reversion theorem [Whittaker and Watson 1962],

$$Y(t_{ret}) = Y(t) + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!\,c^n}\frac{d^{n-1}}{dt^{n-1}}\left[R(t)^n\frac{dY(t)}{dt}\right]. \tag{11}$$

Note that differentiation of the quantity $R^n$ in equation (11) produces factors of the velocity $\boldsymbol{v}$ and derivatives thereof, so that the Taylor series (6) is in fact a linearization of (11) in which all terms nonlinear in $\boldsymbol{v}$ and its derivatives have been neglected. Likewise, the potentials (5) are linearizations of the Liénard-Wiechert potentials (10).

Now let $b$ be the characteristic size of the charged body. On working out the first few terms of equation (11), and noting that $R \sim b$ for relevant field points $\boldsymbol{x}$, it becomes apparent that these linearizations are good approximations provided that

$$\left|\frac{b^n}{c^n}\frac{d^n}{dt^n}\boldsymbol{v}\right| \ll |\boldsymbol{v}| \tag{12}$$

for $n \geq 1$. Roughly speaking, this means that the motion of the body should not change significantly on the time scale necessary for light to travel across the body, which is the time scale at which self-forces are communicated. This condition is known as quasi-stationary motion [Erber 1961]. It indicates the range of validity of the form factor integral (9), in addition to the condition $|\boldsymbol{v}| \ll c$ associated with the neglect of magnetic forces.

Substitution of equation (11) in equation (10) expresses the potentials in terms of quantities evaluated at the current time only. Using the result in equation (1), and performing all integrations, gives a series expansion for the self-force. This series has been evaluated explicitly up to cubic terms in the velocity for a homogeneously charged sphere of radius $b$ by Herglotz [1903]. The linear terms are

$$\boldsymbol{F} = -\frac{6\mu_0 q^2}{\pi b}\sum_{n=0}^{\infty}\frac{(n+1)(n+4)(-2b/c)^n}{(n+5)!}\frac{d^{n+2}\boldsymbol{\xi}}{dt^{n+2}}, \tag{13}$$

and dominate the nonlinear terms in case of quasi-stationary motion (12). For a homogeneously charged sphere in rectilinear motion $\boldsymbol{\xi}(t) = \xi(t)\boldsymbol{e}_z$, Schott [1908] derived the following closed-form expression including terms up to arbitrary order:

$$\boldsymbol{F} = -\frac{36q^2}{\pi\epsilon_0 b}\,\boldsymbol{e}_z\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(m+1)(n+1)(n+4)(-2b)^n}{(2m+1)(2m+3)!(n+5)!}$$
$$\times\,\frac{\partial^{2m+n+2}}{\partial u^{2m+n+2}}\left[\xi(t+u/c)-\xi(t)\right]^{2m+1}\Big|_{u=0}, \tag{14}$$

which reduces to equation (13) when truncated at $m = 0$.

### 2.3 Direct evaluation

Sommerfeld [1904a, 1904b] has evaluated the self-generated potentials of a charged body by integrating equation (4) with respect to $\boldsymbol{x}'$. For a homogeneously charged sphere, this leaves a one-dimensional integral over $t'$ [Sommerfeld 1904a]:

$$\Psi(\boldsymbol{x},t) = -\frac{3q}{16\pi^2 b^3}\int_{-\infty}^{t}\frac{\{c,\boldsymbol{v}(t')\}}{R_c(t')}\chi\,dt', \tag{15}$$

where $R_c(t') = |\boldsymbol{x} - \boldsymbol{\xi}(t')|$ is the distance to the center of the sphere, and

$$\chi = \begin{cases} 4c(t-t')R_c & c(t-t') < b - R_c \\ b^2 - [c(t-t') - R_c]^2 & b - R_c < c(t-t') < b + R_c \\ 0 & c(t-t') > b + R_c. \end{cases} \tag{16}$$

In virtue of the delta function in equation (4), times between $t'$ and $t' + dt'$ in equation (15) correspond to the contribution to the potentials $\Psi(\boldsymbol{x},t)$ generated by the charge located within a shell with radius $c(t-t')$ and thickness $cdt'$ centered around the field point $\boldsymbol{x}$. Depending on $R_c$ and $t'$, this shell may fall completely within the charged sphere, or only partially, or not at all, each case leading to a different factor $\chi$ as given by equation (16). Using the potentials (15) in equation (1) results in the self-force [Sommerfeld 1904b]

$$\boldsymbol{F} = -\frac{3q^2}{32\pi\epsilon_0 b^4 c}\left(\int_0^{\tau^+} G^+(\tau)\,d\tau - \int_0^{\tau^-} G^-(\tau)\,d\tau\right), \tag{17}$$

in which the integrations are over the time difference $\tau \equiv t - t'$, and

$$G^{\pm}(\tau) = \left[c^2 - \boldsymbol{v}(t)\cdot\boldsymbol{v}(t-\tau)\right]\frac{\boldsymbol{s}}{s}\frac{\partial}{\partial s}\frac{g(c\tau \pm s)}{s} + \frac{\partial}{\partial t}\frac{\boldsymbol{v}(t-\tau)g(c\tau \pm s)}{s};$$
$$g(y) = \frac{y^5}{20b^2} - y^3 + 2by^2 - \frac{8b^3}{5}.$$

Here, $\boldsymbol{s} = \boldsymbol{\xi}(t) - \boldsymbol{\xi}(t-\tau)$ is the displacement of the charged sphere during the time interval $\tau$. The upper integration limits in equation (17) are the roots of the equations $c\tau^{\pm} \pm s(\tau^{\pm}) = 2b$. These limits demarcate different stages in the communication of electromagnetic signals between the parts of the charged sphere that lead to the self-force at the current time $t$. For subluminal motion, the trailing end of the sphere receives electromagnetic signals at time $t$ that were emitted by the other parts of the sphere at times between $t-\tau^+$ and $t$. The signals received by the leading end at time $t$ were emitted by the other parts during the slightly longer interval between $t-\tau^-$ and $t$. Signals emitted at still earlier at times before $t-\tau^-$ do not arrive at any other part of the sphere at time $t$, so that the domain $\tau > \tau^-$ does not contribute to the self-force (17) at all.

### 2.4 Fourier transform

Bohm and Weinstein [1948] have adopted the Coulomb gauge to evaluate equation (1). The benefit of this gauge choice for the calculation of the self-force of a rigid charged body is that the scalar potential $\phi$ equals the electrostatic potential corresponding to the instantaneous distribution of charge. Since for any pair of charge elements $de_1$ and $de_2$ the instantaneous electrostatic force on $de_1$ due to $de_2$ is the negative of the electrostatic force on $de_2$ due to $de_1$, the contribution of $\phi$ to the self-force $\boldsymbol{F}$ integrates to zero identically. Therefore only the vector potential has to be taken into account, which is given by equation (4) as before. It can be shown [Bohm and Weinstein 1948] that a Fourier transformation of this equation from the spatial domain $\boldsymbol{x}$ to the wave vector domain $\boldsymbol{k}$ yields the potential

$$\boldsymbol{A}_k(\boldsymbol{k},t) = \frac{\mu_0 c}{k}\int_{-\infty}^{t} \boldsymbol{J}_{T,k}(\boldsymbol{k},t)\sin\left[ck(t-t')\right]dt'. \tag{18}$$

Notice that the integration in equation (18) extends to the upper boundary $t$, so that the potential at time $t$ depends only on currents at past times $t' < t$, that is, equation (18) is causal as it should be. Using in the self-force (1) the inverse Fourier transform $\boldsymbol{A} \equiv (2\pi)^{-3/2}\int \boldsymbol{A}_k \exp(i\boldsymbol{k}\cdot\boldsymbol{x})d^3\boldsymbol{k}$, and substituting equation (18), gives [Bohm and Weinstein 1948]

$$\begin{aligned}\boldsymbol{F} = &-\frac{1}{\epsilon_0}\int_{-\infty}^{t} dt' \int d^3\boldsymbol{k}\,|\rho_k|^2 \exp\left(i\boldsymbol{k}\cdot\boldsymbol{s}\right)\\ &\times\left(\frac{\boldsymbol{k}\times[\boldsymbol{v}(t')\times\boldsymbol{k}]}{k^2}\cos ck\tau - \frac{\boldsymbol{v}(t)\times[\boldsymbol{k}\times\boldsymbol{v}(t')]}{ck} i\sin ck\tau\right).\end{aligned} \tag{19}$$

Note that the second term in large braces is proportional to and perpendicular to the current velocity $\boldsymbol{v}(t)$ of the charged body, and therefore represents the magnetic component of the self-force. The first term gives the electric component. For a spherically symmetric charge distribution, $\rho_k(\boldsymbol{k})$ is a function of the magnitude of $\boldsymbol{k}$ but not of its direction. In this case, equation (19) can be straightforwardly integrated over angles in $\boldsymbol{k}$-space. This reduces equation (19) to

$$\boldsymbol{F} = -\frac{4\pi}{\epsilon_0}\int_{-\infty}^{t} dt' \int_0^\infty dk\, k^2 \left|\rho_k\right|^2 \left[\left(\boldsymbol{v}(t') - \frac{[\boldsymbol{v}(t')\cdot\boldsymbol{s}]\boldsymbol{s}}{s^2}\right) j_0\left(ks\right)\cos ck\tau \right.$$
$$\left. - \left(\boldsymbol{v}(t') - \frac{3[\boldsymbol{v}(t')\cdot\boldsymbol{s}]\boldsymbol{s}}{s^2}\right)\frac{j_1\left(ks\right)}{ks}\cos ck\tau + \frac{\boldsymbol{v}(t)\times[\boldsymbol{s}\times\boldsymbol{v}(t')]}{cs} j_1\left(ks\right)\sin ck\tau\right], \tag{20}$$

where $j_n$ denotes the spherical Bessel function of the first kind and order $n$ [Abramowitz and Stegun 1965]. In equation (20), the first two terms in large square brackets represent the electric component of the force and are given by Bohm and Weinstein [1948]; the last term gives the magnetic component. The integral over $k$ containing Bessel function kernels has the typical form of an inverse Fourier transform in spherical coordinates [Stratton 1941]. In Section 3, we will derive the other self-force representations given in Sections 2.1 to 2.3 from this Fourier integral.

## 3 Equivalence of the self-force expressions

### 3.1 Fourier integral and integral over time

Sommerfeld [1904a] derived for the piecewise function $\chi$ given by equation (16) the integral representation

$$\chi = \frac{8b^2}{\pi}\int_0^\infty \frac{j_1(kb)\sin(kR_c)\sin[ck(t-t')]}{k}\, dk. \tag{21}$$

Substituting this representation in equation (15), and using the result in equation (1), gives the self-force [Sommerfeld 1904a]

$$\boldsymbol{F} = -\frac{9q^2}{2\pi^2\epsilon_0 b^2 c}\int_0^\infty d\tau \int_0^\infty dk\, [j_1(kb)]^2$$
$$\times\left(\frac{1}{k}\frac{\partial}{\partial t}\left[\boldsymbol{v}(t-\tau) j_0(ks)\sin ck\tau\right] - \left[c^2 - \boldsymbol{v}(t)\cdot\boldsymbol{v}(t-\tau)\right]\frac{\boldsymbol{s}}{s} j_1(ks)\sin ck\tau\right). \tag{22}$$

Performing the integration over $k$ indeed yields the force (17). We now show that equation (22) is equivalent to equation (20) that was derived by Fourier analysis of the potentials in the Coulomb gauge. Note that the integrands of both equations already have a similar structure due to the form of the integral representation (21). Performing the differentiation $\partial/\partial t$ in equation (22) using the property

$$\frac{\partial j_0(ks)}{\partial t} = -kj_1(ks)\frac{\partial s}{\partial t} = -kj_1(ks)\frac{[\boldsymbol{v}(t)-\boldsymbol{v}(t-\tau)]\cdot\boldsymbol{s}}{s}$$

gives, after changing the integration variable back to $t'$ and rearranging,

$$\boldsymbol{F} = -\frac{9q^2}{2\pi^2\epsilon_0 b^2 c}\int_{-\infty}^{t} dt' \int_0^\infty dk\, \left[j_1(kb)\right]^2 \sin ck(t-t')\left[\frac{c^2\boldsymbol{s}}{s} j_1(ks)\right.$$
$$\left. -\frac{1}{k}\frac{d\boldsymbol{v}(t')}{dt'} j_0(ks) + \left(\frac{\{[\boldsymbol{v}(t)-\boldsymbol{v}(t')]\cdot\boldsymbol{s}\}\,\boldsymbol{v}(t')}{s} - \frac{[\boldsymbol{v}(t)\cdot\boldsymbol{v}(t')]\,\boldsymbol{s}}{s}\right) j_1(ks)\right]. \tag{23}$$

Next we integrate by parts the first two terms in the large square brackets with respect to $t'$, choosing for the differentiated factors respectively $f_1(t') = \boldsymbol{s} j_1(ks)/s$ and $f_2(t') = j_0(ks)\sin ck\tau$. To carry out this integration unambiguously, it is necessary to replace the lower integration limit $t' = -\infty$ by $t' = -a$, and take the limit $a \to \infty$ afterwards. With the help of the relations

$$\begin{aligned}\frac{\partial f_1}{\partial t'} &= \left(\frac{3j_1(ks)}{ks} - j_0(ks)\right)\frac{k\left[\boldsymbol{v}(t')\cdot\boldsymbol{s}\right]\boldsymbol{s}}{s^2} - \frac{\boldsymbol{v}(t')}{s} j_1(ks);\\ \frac{\partial f_2}{\partial t'} &= \frac{k\boldsymbol{v}(t')\cdot\boldsymbol{s}}{s} j_1(ks)\sin ck\tau - ckj_0(ks)\cos ck\tau,\end{aligned}$$

the resulting self-force is

$$\boldsymbol{F} = \frac{9q^2}{2\pi^2\epsilon_0 b^2}\lim_{a\to\infty}\int_0^\infty \left[j_1(kb)\right]^2 (B + I\,)\,dk, \tag{24}$$

where

$$\begin{aligned}B &= \left[\frac{\boldsymbol{v}(t')}{ck} j_0(ks)\sin ck\tau - \frac{\boldsymbol{s}}{ks} j_1(ks)\cos ck\tau\right]_{t'=-a}^{t};\\ I &= \int_{-a}^{t}\left[\left(\boldsymbol{v}(t') - \frac{[\boldsymbol{v}(t')\cdot\boldsymbol{s}]\boldsymbol{s}}{s^2}\right) j_0\,(ks)\cos ck\tau\right.\\ &\quad\left. - \left(\boldsymbol{v}(t') - \frac{3[\boldsymbol{v}(t')\cdot\boldsymbol{s}]\boldsymbol{s}}{s^2}\right)\frac{j_1\,(ks)}{ks}\cos ck\tau + \frac{\boldsymbol{v}(t)\times[\boldsymbol{s}\times\boldsymbol{v}(t')]}{cs} j_1\,(ks)\sin ck\tau\right] dt'.\end{aligned}$$

Taking in equation (24) the limit $a \to \infty$ of $I$ presents no difficulties, and yields precisely equation (20), specialized to a homogeneous sphere that has the form factor

$$\rho_k = \frac{3q}{(2\pi)^{3/2}}\frac{j_1(kb)}{kb}. \tag{25}$$

Therefore equation (24) is equivalent to equation (20), provided that the boundary term $B$ vanishes. This can be shown to be the case as follows. $B$ evaluated at $t' = t$ vanishes since $\sin ck\tau = 0$ and $s(t') = 0$ at $t' = t$. In the limit $t' \to -\infty$, the first term of $B$ is zero trivially when $\boldsymbol{v}(-\infty) = \boldsymbol{0}$. When $\boldsymbol{v}(-\infty) \neq \boldsymbol{0}$, it must be that $s(t') \to \infty$ and hence $j_0(ks) \to 0$ as $t' \to -\infty$, so that the first term does not contribute in this case either. The second term of $B$, on the other hand, vanishes at $t \to -\infty$ only when $\boldsymbol{v}(-\infty) \neq \boldsymbol{0}$. Namely, when $\boldsymbol{v}(-\infty) = \boldsymbol{0}$ it is possible that $s(-\infty) \equiv S$ has a finite value. In that case, the boundary term makes a contribution to equation (24) proportional to

$$\lim_{a\to\infty}\int_0^\infty h(k)\cos ck(t-a)dk, \tag{26}$$

where $h(k) = \left[j_1(kb)\right]^2 j_1(kS)/k$. However, equation (26) evaluates to zero by the Riemann-Lebesgue lemma [Whittaker and Watson 1962]. Hence $B = 0$ for all possible $\boldsymbol{v}(-\infty)$, so that the force (24) is indeed identical to the force (20) that was derived by Fourier analysis of the potentials in the Coulomb gauge.

## 3.2 Fourier integral and form factor integral

As discussed above, the self-force (9) in terms of a form factor integral is valid for quasi-stationary motion (12) and $|\boldsymbol{v}| \ll c$, and for the special case of harmonic motion.

In order to compare equation (9) with the self-force derived in Section 2.4, the latter should be specialized accordingly. This may be effected by expanding the integrand of equation (19) in a Taylor series around $t' = t$, and linearizing the result by neglecting all terms nonlinear in $\boldsymbol{v}$ and its derivatives. The extremely involved full expansion, in which all nonlinear terms have been kept, is given by Roa-Neri and Jiménez [1993]. Formally, such use of a Taylor series to describe the integrand on the infinite interval $-\infty < t' < t$ is questionable because the series may have a finite radius of convergence. However, for subrelativistic motion electromagnetic signals are communicated between parts of the charged body on a time scale $\sim b/c$, so that only the small interval $t - b/c \lesssim t' < t$ significantly contributes to the integral in equation (19). This can be seen by noting in equation (19) that the integrand only contributes in the domain $|\boldsymbol{k}| \lesssim b^{-1}$ because the form factor $|\rho_k|^2 \approx 0$ elsewhere, and that the integral over this domain averages out due to the sinusoidal functions unless $ck\tau \lesssim \pi/2$, that is, unless $t - b/c \lesssim t' < t$. Proceeding on this basis by Taylor-expanding, neglecting nonlinear terms, and integrating over angles in $\boldsymbol{k}$-space, yields

$$\boldsymbol{F} = -\frac{8\pi}{3\epsilon_0}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{d^n\boldsymbol{v}}{dt^n}\int_0^{\infty}\!\!\int_0^{\infty} k^2\,|\rho_k|^2\,\tau^n\cos ck\tau\,d\tau dk. \tag{27}$$

Writing $\tau^{2n}\cos ck\tau = (-1)^n c^{-2n}(d/dk)^{2n}\cos ck\tau$ and $\tau^{2n+1}\cos ck\tau = (-1)^n c^{-2n-1}(d/dk)^{2n+1}\sin ck\tau$, as is suggested by Roa-Neri and Jiménez [1993], and integrating by parts with respect to $k$ repeatedly, gives

$$\begin{aligned}\boldsymbol{F} = &-\frac{8\pi}{3\epsilon_0}\sum_{n=0}^{\infty}\frac{(-1)^n}{c^{2n}(2n)!}\frac{d^{2n}\boldsymbol{v}}{dt^{2n}}\int_0^{\infty}\left(B_n^{(e)}+I_n^{(e)}\right)d\tau\\ &-\frac{8\pi}{3\epsilon_0}\sum_{n=0}^{\infty}\frac{(-1)^n}{c^{2n+1}(2n+1)!}\frac{d^{2n+1}\boldsymbol{v}}{dt^{2n+1}}\int_0^{\infty}\left(B_n^{(o)}+I_n^{(o)}\right)d\tau,\end{aligned} \tag{28}$$

in which

$$\begin{aligned}B_n^{(e)} &= \sum_{m=0}^{2n-1}(-1)^m\frac{d^m}{dk^m}k^2\,|\rho_k|^2\,\frac{d^{2n-m-1}}{dk^{2n-m-1}}\cos ck\tau\bigg|_{k=0}^{\infty};\\ B_n^{(o)} &= \sum_{m=0}^{2n}(-1)^{m+1}\frac{d^m}{dk^m}k^2\,|\rho_k|^2\,\frac{d^{2n-m}}{dk^{2n-m}}\sin ck\tau\bigg|_{k=0}^{\infty};\\ I_n^{(e)} &= \int_0^{\infty}\frac{d^{2n}}{dk^{2n}}\left(k^2\,|\rho_k|^2\right)\cos ck\tau\,dk;\\ I_n^{(o)} &= \int_0^{\infty}\frac{d^{2n+1}}{dk^{2n+1}}\left(k^2\,|\rho_k|^2\right)\sin ck\tau\,dk.\end{aligned}$$

All boundary terms $B_n^{(e)}$ and $B_n^{(o)}$ vanish identically. At $k = \infty$, this is because $\rho_k(\infty) = 0$ for any finite charge distribution. At $k = 0$, the terms with odd $m$ are zero because $k^2\,|\rho_k|^2$ is an even function, and those with even $m$ vanish because they contain $\sin ck\tau$ as a factor. The quantity $\sqrt{2/\pi}I_n^{(e)} \equiv J_n^{(e)}$ can be interpreted as the symmetric cosine transform of the function $j_{2n}(k) = (d/dk)^{2n}k^2\,|\rho_k|^2$; likewise, $\sqrt{2/\pi}I_n^{(o)} \equiv J_n^{(o)}$ is the symmetric sine transform of $j_{2n+1}(k)$. Therefore the double integrals in equation (28) reduce to the single integrals

$$\int_0^\infty I_n^{(e)}\,d\tau = \frac{\pi}{2c}\left(\sqrt{\frac{2}{\pi}}\int_0^\infty J_n^{(e)}\cos kx\,dx\right)_{k=0} = \frac{\pi}{2c}j_{2n}(0); \tag{29}$$

$$\begin{aligned}\int_0^\infty I_n^{(o)}\,d\tau &= \frac{1}{c}\sqrt{\frac{2}{\pi}}\left(\int_0^\infty \frac{\sin kx}{k}\,dk\right)\left(\int_0^\infty J_n^{(o)}\,dx\right)\\ &= \int_0^\infty\left(\sqrt{\frac{2}{\pi}}\int_0^\infty J_n^{(o)}\sin kx\,dx\right)\frac{dk}{ck} = \int_0^\infty \frac{j_{2n+1}(k)}{ck}\,dk.\end{aligned} \tag{30}$$

Here, the identity $\int_0^\infty k^{-1}\sin kx\,dk = \pi/2$ and the variable $x = c\tau$ have been used. With the help of equations (29) and (30), the force (28) reduces to

$$\begin{aligned}\boldsymbol{F} = \frac{8\pi}{3\epsilon_0 c}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{c^{2n}(2n)!}&\left[\frac{\pi}{2}\left(\frac{d^{2n}}{dk^{2n}}k^2\left|\rho_k\right|^2\right)_{k=0}\frac{d^{2n}\boldsymbol{v}}{dt^{2n}}\right.\\ &\left.+\frac{1}{(2n+1)c}\left(\int_0^\infty\frac{d^{2n+1}}{dk^{2n+1}}k^2\left|\rho_k\right|^2\frac{dk}{k}\right)\frac{d^{2n+1}\boldsymbol{v}}{dt^{2n+1}}\right].\end{aligned} \tag{31}$$

This expression now has the manageable form of a series in terms of the derivatives of the current velocity, with coefficients that are readily calculated from the form factor of the charge distribution. In the next section, we will specialize this result to a homogeneously charged sphere, and show that it is equivalent to the series expansion (13) obtained by application of Lagrange's reversion theorem. Here, we apply equation (31) to the case of harmonic motion, for which $\boldsymbol{v} = -i\omega\boldsymbol{\xi}_0\exp(-i\omega t) \equiv -i\omega\widetilde{\boldsymbol{\xi}}$. Since $(d/dt)^n\boldsymbol{v} = (-i\omega)^n\boldsymbol{v}$, equation (31) then becomes the sum of two ordinary power series in the quantity $\omega/c$. The series corresponding to the first line of equation (31) may be interpreted as the even part of the Taylor series of the function $p(\kappa) = \kappa^2\left|\rho_k(\kappa)\right|^2$ around $\kappa = 0$, evaluated at $\kappa = \omega/c$. Similarly, the series in the second line may be identified with the odd part of the Taylor series of $p(\kappa)$ around $\kappa = k$, evaluated at $\kappa = k + \omega/c$. Summing these two series therefore results in

$$\boldsymbol{F} = \frac{4\pi\omega}{3\epsilon_0 c}\widetilde{\boldsymbol{\xi}}\left(\int_0^\infty\left[p(k+\omega/c) - p(k-\omega/c)\right]\frac{dk}{k} + \frac{i\pi}{2}\left[p(\omega/c) + p(-\omega/c)\right]\right). \tag{32}$$

Noting that $p(k)$ is an even function, the integral in equation (32) may be recognized as the Hilbert transform of $p(k)$ in a less common notation [Zygmund 1968]. Accordingly, by changing variables it may be shown [Zygmund 1968] that

$$\boldsymbol{F} = \frac{8\pi\omega^2}{3\epsilon_0 c^2}\widetilde{\boldsymbol{\xi}}\left(⨍_0^\infty\frac{k^2\left|\rho_k(k)\right|^2}{k^2-\omega^2/c^2}\,dk + i\pi\operatorname*{Res}_{k=\omega/c}\frac{k^2\left|\rho_k(k)\right|^2}{k^2-\omega^2/c^2}\right), \tag{33}$$

where $⨍$ denotes the Cauchy principal value. Here, the second line of equation (32) has been interpreted as a residue. Equation (33) is identical to the force (9) derived from a Taylor expansion of the retarded integrals for the potentials.

### 3.3 Fourier integral and Lagrange expansion

In the previous section, we derived the series expansion (31) that expresses the linearized self-force in terms of the derivatives of the current velocity of the charged body, for a general spherically symmetric charge distribution. We will now specialize this result to a homogeneously charged sphere, and show that this yields the self-force (13) that was obtained from series expansion of the retarded potentials. Evaluation of equation (31) using the form factor of a homogeneous sphere (25) requires determination of the quantities

$$S_n = \left.\frac{d^{2n}\,[j_1(x)]^2}{dx^{2n}}\right|_{x=0}; \qquad T_n = \int_0^\infty \frac{d^{2n+1}\,[j_1(x)]^2}{dx^{2n+1}}\,\frac{dx}{x}. \tag{34}$$

The first of these equals $(2n)!$ times the coefficient of $x^{2n}$ in the Taylor series of $[j_1(x)]^2$ around $x=0$. By squaring the ascending power series of the Bessel function [Abramowitz and Stegun 1965], it is thus found that

$$S_n = \sum_{m=0}^{n-1} \frac{(2n)!\,\left(-\frac{1}{2}\right)^{n-1}}{m!\,(n-m-1)!\,(2m+3)!!\,(2n-2m+1)!!}. \tag{35}$$

Writing factorials in terms of Pochhammer symbols $(p)_q \equiv \Gamma(p+q)/\Gamma(q)$ with $\Gamma$ the Gamma function [Abramowitz and Stegun 1965], equation (35) becomes

$$S_n = \frac{\pi(2n)!\left(-\frac{1}{4}\right)^{n+1}}{\Gamma\left(\frac{5}{2}\right)\Gamma(n)\,\Gamma\left(n+\frac{3}{2}\right)} \sum_{m=0}^{n-1} \frac{(1-n)_m\left(-\frac{1}{2}-n\right)_m}{m!\left(\frac{5}{2}\right)_m}. \tag{36}$$

Here, it has been used that $(p)_{-q} = (-1)^q/(1-p)_q$ [Hansen 1975]. The series in equation (36) defines a Gauss hypergeometric function with unit argument [Abramowitz and Stegun 1965]. Evaluating this hypergeometric function, and converting Gamma functions to factorials, results in

$$S_n = -\frac{n(-4)^n}{(n+1)(n+2)(2n+1)}. \tag{37}$$

Establishing $T_n$ is more involved. The squared Bessel function $[j_1(x)]^2$ can be expanded in a series of Bessel functions with doubled argument [Watson 1966]. This gives

$$[j_1(x)]^2 = \sum_{m=0}^{\infty} \frac{2m+2}{(2m+1)(2m+3)}\,\frac{J_{4m+3}(2x)+J_{4m+5}(2x)}{x}, \tag{38}$$

where $J$ denotes the cylindrical Bessel function of the first kind [Abramowitz and Stegun 1965]. The factor $x$ in the denominator can be removed with the help of the recurrence relation $2pJ_p(z)/z = J_{p-1}(z)+J_{p+1}(z)$. Subsequently, the integrand of $T_n$ in equation (34) is found by application of the expansion [Luke 1962]

$$\frac{d^p J_q(z)}{dz^p} = \frac{1}{2^p}\sum_{u=0}^{p}(-1)^u \binom{p}{u} J_{q-p+2u}(z). \tag{39}$$

This gives

$$\begin{aligned}\frac{1}{x}\,\frac{d^{2n+1}\,[j_1(x)]^2}{dx^{2n+1}} = &\sum_{m=0}^{\infty}\sum_{u=0}^{2n+1} \frac{(2m+2)(-1)^u}{(2m+1)(2m+3)}\binom{2n+1}{u}\\ &\times\Big[C_0 J_{4m-2n+2u}(2x) + C_2 J_{4m-2n+2u+2}(2x)\\ &+ C_4 J_{4m-2n+2u+4}(2x) + C_6 J_{4m-2n+2u+6}(2x)\Big],\end{aligned} \tag{40}$$

where

$$C_0 = \frac{1}{(4m+3)(4m-2n+2u+1)};$$
$$C_6 = \frac{1}{(4m+5)(4m-2n+2u+5)};$$
$$C_2 = C_0 + \frac{2(4m+4)}{(4m+3)(4m+5)(4m-2n+2u+3)};$$
$$C_4 = C_6 + \frac{2(4m+4)}{(4m+3)(4m+5)(4m-2n+2u+3)}.$$

Substituting this expansion in equation (34), the integral $T_n$ can be evaluated trivially because $\int_0^\infty J_p(z)dz = 1$ for arbitrary $p > -1$ [Abramowitz and Stegun 1965]. Therefore $T_n$ is given by equation (40) if each Bessel function is replaced by $1/2$. The remaining double series can be summed in closed form. The sums over $u$ of the various terms have been tabulated [Hansen 1975]; together they evaluate to

$$T_n = \frac{\pi(-1)^n(2n+1)!}{4\,\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{9}{2}\right)} \times \sum_{m=0}^{\infty}\left(4+\frac{1}{m+\frac{1}{2}}-\frac{1}{m+\frac{3}{2}}\right)\frac{(1)_m\left(\frac{1}{4}-\frac{n}{2}\right)_m\left(\frac{3}{4}-\frac{n}{2}\right)_m}{m!\left(\frac{9}{4}+\frac{n}{2}\right)_m\left(\frac{11}{4}+\frac{n}{2}\right)_m}. \tag{41}$$

The series in the second line of equation (41) is derivable from the series

$$U_n(z) = \sum_{m=0}^{\infty}\frac{(1)_m\left(\frac{1}{4}-\frac{n}{2}\right)_m\left(\frac{3}{4}-\frac{n}{2}\right)_m z^m}{m!\left(\frac{9}{4}+\frac{n}{2}\right)_m\left(\frac{11}{4}+\frac{n}{2}\right)_m}, \tag{42}$$

which defines the generalized hypergeometric function [Slater 1966]

$$U_n(z) = {}_3F_2\left[\begin{matrix}1 & \frac{1-2n}{4} & \frac{3-2n}{4} \\ & \frac{11+2n}{4} & \frac{9+2n}{4}\end{matrix};z\right]. \tag{43}$$

Comparing equations (41) and (42), it is found that

$$T_n = \frac{\pi(-1)^n(2n+1)!}{4\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{9}{2}\right)}\left(4U_n(1)+\int_0^1 U_n(z)D(z)dz\right), \tag{44}$$

with $D(z) = z^{-1/2} - z^{1/2}$. The integral in equation (44) is equal to [Wolfram 2012a]

$$\int_0^1 U_n(z)D(z)dz = \frac{4}{3}\,{}_4F_3\left[\begin{matrix}1 & \frac{1}{2} & \frac{1-2n}{4} & \frac{3-2n}{4} \\ & \frac{5}{2} & \frac{11+2n}{4} & \frac{9+2n}{4}\end{matrix};1\right]. \tag{45}$$

Equations (43)–(45) define $T_n$ in terms of two generalized hypergeometric functions of unit argument; for both functions closed form expressions in terms of Gamma functions exist [Wolfram 2012b]. Writing these expressions in terms of factorials yields, after considerable reduction,

$$T_n = \frac{\pi(2n+1)(-4)^n}{(n+1)(2n+3)(2n+5)}. \tag{46}$$

Finally, having the quantities $S_n$ and $T_n$ at our disposal, the self-force (31) can be evaluated. Combining equations (31), (25), (37) and (46) gives

$$\boldsymbol{F} = \frac{3q^2}{\pi\epsilon_0 cb^2}\sum_{n=0}^{\infty}\left(\frac{(2n)(2n+3)(-2b)^{2n}}{(2n+4)!}\frac{1}{c^{2n}}\frac{d^{2n}\boldsymbol{v}}{dt^{2n}}\right.$$
$$\left.+\frac{(2n+1)(2n+4)(-2b)^{2n+1}}{(2n+5)!}\frac{1}{c^{2n+1}}\frac{d^{2n+1}\boldsymbol{v}}{dt^{2n+1}}\right). \tag{47}$$

Taking the two terms in large braces together by relabeling the summation index gives precisely equation (13). It has thus been shown that the self-force obtained by Fourier analysis of the potentials in the Coulomb gauge is equivalent to the force derived by Lagrange expansion of the potentials in the Lorenz gauge.

## 4 Discussion

In this paper, we have demonstrated the equivalence of a number of published expressions for the self-force of a rigid charged body. These included the form factor integral (9) of Jackson [1999] based on a Taylor expansion technique, the series (13) and (14) of Herglotz [1903] and Schott [1908] resulting from a Lagrange expansion, the integral over retarded time (17) of Sommerfeld [1904a] obtained by direct integration of the fundamental equation, and the Fourier integral (20) of Bohm and Weinstein [1948] derived by Fourier analysis adopting the Coulomb gauge. To this list we may add our result (31), which is intermediate between an integral and a series representation.

### 4.1 Generality of the self-force expressions

The various expressions differ in their degree of generality. The Fourier type results of Jackson [1999] and of Bohm and Weinstein [1948] allow calculation of the self-force for arbitrary spherically symmetric charge distributions, whereas the direct integration of Sommerfeld [1904a] and Lagrange expansion methods of Herglotz [1903] and Schott [1908], which stay in the spatial domain throughout, require specialization to a particular charge distribution at an early stage. In addition, several approximations implied in the derivations may be discerned. The neglect of magnetic forces in equation (7), and of nonlinear terms in the series (13), restrict the validity of the corresponding self-forces to nonrelativistic velocities. More subtle is the improper use of the charge distribution at the retarded time, as discussed at the start of Section 2.2, which further constrains the results based on Taylor rather than Lagrange expansions to the quasi-stationary motion defined by equation (12). This restriction does not apply to the more correct self-force expansion (14). On the other hand, numerical evaluation of this series by truncation is only possible when the series converges sufficiently rapidly, which is when $(b/c)^{(n+1)}(d/dt)^{(n+1)}\xi \ll (b/c)^n(d/dt)^n\xi$, a condition even stronger than quasi-stationary motion. In contrast, the Fourier technique leading to equation (20) does not involve the explicit evaluation of retarded source densities in the first place, so that the issue is avoided altogether. Therefore, equation (20) may be regarded as the most general self-force expression in this paper, in the sense that this result does not neglect any magnetic or other nonlinear terms, and is not restricted to quasi-stationary motion or any particular charge distribution. At the same time, equation (20) is suitable for numerically stable evaluation as it does not involve truncated series.

### 4.2 Point particle limits

As emphasized in the introduction of this paper, the electrodynamics of extended charged bodies is important in relation to point charge models of elementary particles, since the two concepts are connected by an appropriate limiting procedure. It is therefore illustrative to see how some well-known point particle results follow from the self-force expressions discussed in this paper. This connection is most straightforward in case of the series expansion (13), which is valid for nonrelativistic velocities. In the point particle limit $b \to 0$, the first terms become dominant and evaluate to the well-known nonrelativistic self-force [Panofsky and Phillips 2005, chap. 21]

$$\boldsymbol{F} \to -\frac{4U_{es}}{3c^2}\frac{d\boldsymbol{v}}{dt} + \frac{q^2}{6\pi\epsilon_0 c^3}\frac{d^2\boldsymbol{v}}{dt^2} + \ldots, \tag{48}$$

with $U_{es} = 3q^2/(20\pi\epsilon_0 b)$ the electrostatic energy. The first term on the right of equation (48) has the appearance of an inertial term; correspondingly the quantity $4U_{es}/(3c^2)$ is usually interpreted as an electromagnetic contribution to the mass of the particle. However, the factor 4/3 violates relativistic mass-energy equivalence, and its removal by more sophisticated arguments has been the subject of much discussion [Janssen and Mecklenburg 2006; Poincaré 1906; Rohrlich 2007; Medina 2006; Ori and Rosenthal 2004]. The inertial term is commonly absorbed in an effective mass of the particle, leaving the second term on the right of equation (48) as the radiation reaction force usually featuring in the equation of motion of the classical point particle, in the nonrelativistic limit.

The derivation based on Taylor expansion in Section 2.1 produces the limit (48) as well. The first two terms of equation (8) are

$$n = 0: \qquad -\frac{4}{3c^2}\left(\frac{1}{8\pi\epsilon}\iint\frac{\rho(\boldsymbol{r})\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3\boldsymbol{r}d^3\boldsymbol{r}'\right)\frac{d\boldsymbol{v}}{dt}; \tag{49}$$

$$n = 1: \qquad \frac{1}{6\pi\epsilon_0 c^3}\left(4\pi\int_0^\infty \rho(r)r^2 dr\right)^2\frac{d^2\boldsymbol{v}}{dt^2}. \tag{50}$$

The quantity in large parentheses in the $n = 0$ term is just $U_{es}$, so that equation (49) equals the inertial term of equation (48). The quantity in parentheses in the $n = 1$ term is the total charge, and equation (50) reduces to the radiation reaction term of equation (48). Moreover, these observations are independent of the particular charge distribution one chooses to model the particle with [Panofsky and Phillips 2005, chap. 21][1].

Less obvious is the point particle limit of the Fourier integral representation of the self-force (20), which, as we have shown, is equivalent to the series (13), and should therefore reduce to equation (48) as well. However, this is difficult to see from the Fourier integral directly. In contrast, from our intermediate result (31) the limit (48) is readily obtained. Namely, the term $n = 0$ in the first line of equation (31) vanishes for any finite $\rho_k$. The term $n = 0$ in the second line gives, after integration by parts, the force

$$-\frac{2}{3\epsilon_0 c^2}\frac{d\boldsymbol{v}}{dt}\int\frac{|\rho_k|^2}{k^2}d^3\boldsymbol{k}. \tag{51}$$

[1] Equations (49) and (50) do imply spherical symmetry; the general case of an arbitrary charge distribution generally involves self-force components that are no longer collinear with the particle velocity, necessitating a tensorial description of the electromagnetic mass and radiation reaction force terms [Fermi 1921].

By writing out $\rho_k$ as the Fourier transform of the charge distribution $\rho(\boldsymbol{r})$, and simplifying the result by using the identity $\int \exp\left[-i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')\right] k^{-2} d^3\boldsymbol{k} = 2\pi^2/\left|\boldsymbol{r}-\boldsymbol{r}'\right|$ [Morse and Feshbach 1953], equation (51) is reduced to equation (49) and hence to the inertial term of equation (48). The radiation reaction force, in turn, is generated by the term $n=1$ in the first line of equation (31). Exploiting in the latter spherical symmetry by writing $\rho_k = \sqrt{2/\pi}\int_0^\infty \rho(r) j_0(kr) r^2 dr$, performing the differentiations with respect to $k$ and evaluating the result for $k=0$, yields the force (50). None of the other terms of equation (31) contribute in the point particle limit. This is because $\rho_k$ will tend to a constant as the support of $\rho(\boldsymbol{r})$ shrinks to a point, and therefore the differentiations with respect to $k$ will make these terms vanish. This confirms that also the Fourier integral (31) correctly reduces to the well-known self-force (48).

### 4.3 Relativistic rigidity

An important aspect of the relativistic dynamics of an extended body, left out of the discussion so far, is the way is which the body maintains or changes its shape while being accelerated. On the one hand relativity theory requires a velocity dependent Lorentz contraction; on the other hand instantaneous contractions are impossible due to the finite velocity at which information about velocity changes can propagate through the extended body. Accelerated motion, therefore, necessitates some notion of relativistic rigidity, the precise formulation of which is still being studied [Lyle 2010; Epp et al. 2009]. Often the approximation of Born rigidity is applied [Born 1909], valid for adiabatic velocity changes, where it is assumed that the accelerated body always maintains its shape in its continuously changing instantaneous rest frame [Pierce 2007][2]. The corresponding Born rigid extended electron model was advocated by Lorentz [1916]. A significant strength of this model is that it provides a natural way to cure the 4/3 problem described above, by assuming a negative pressure inside the electron that balances the Coulomb repulsion of the distributed charge [Poincaré 1906]. The work done by this pressure during Lorentz contractions removes the factor 4/3 precisely [Yaghjian 2006; Medina 2006]. This relation between the 4/3 problem and lack of Born rigidity was demonstrated by Fermi [1923] in the context of Lagrangian mechanics[3]. Still, the nature of the necessarily non-electromagnetic internal negative pressure has remained highly speculative. One particularly interesting proposal was made by Casimir [1953], who suggested that the force compensating the Coulomb repulsion could be explained in terms of the quantum electromagnetic zero-point energy by the effect that now bears Casimir's name. However, detailed calculation later showed that the Casimir force on a spherical shell is actually repulsive [Boyer 1968; Milton et al. 1978], thereby invalidating this idea. Nevertheless, variations on Casimir's electron model continue to be pursued [Milton 1980; Puthoff 2007; Leonardt and Simpson 2011], partly driven by the fact that the assumed balance between electrostatic and Casimir forces potentially yields a numeric prediction of the fine structure constant [Casimir 1953].

In spite of the attractive features of Lorentz's electron model, none of the self-force derivations discussed in this paper actually corresponds to it. Rather, the Abraham model is implied [Abraham 1902], in which the charged body maintains its shape in the frame of an observer at rest. Differences between the self-forces according to each model do not yet appear in the linear terms, however, so that the limit (48) is

[2] [Pierce 2007] describes interesting examples of motion where Born rigidity cannot apply.

[3] Quite another possibility to resolve the 4/3 problem is to adopt a manifestly covariant definition of four-momentum of systems involving electromagnetic fields [Rohrlich 2007; Ori and Rosenthal 2004; Janssen and Mecklenburg 2006].

model independent. In the Lagrange series (14), on the other hand, the sum over $m$ of the nonlinear terms $n = 0$ indeed agrees [Schott 1908] with the model of Abraham [1902] and not with that of Lorentz [1916]. Similarly, the time integral (15), and therefore also the equivalent Fourier integral (20), coincide with the Abraham model [Sommerfeld 1904b]. Since the contracting Lorentz model seems more in line with special relativity, it would be valuable for classical charged particle theories to derive a Born rigid version of the Fourier representation (20) of the self-force. A covariant formulation based on formal cut-off procedures has been developed in this direction [Prigogine and Henin 1962]. Yet, it should be realized that it has not been established that a consistent extended body model of elementary particles, if any, should necessarily respect rigidity. Therefore alternative postulates, such as the Abraham model or other non-rigid models [Aguirregabiria et al. 2006], may continue to prove their value. What is more, the current state of technology is starting to enable experimental conditions in which the electromagnetic self-force of macroscopic charged systems, such as high-density electron bunches [Smorenburg et al. 2010] and ultracold plasma bunches [Smorenburg et al. 2013], becomes significant. It would be interesting to see to what extent the self-force formulations in this paper can model these evidently non-rigid systems.

## 5 Conclusion

The self-consistent classical formulation of charged particle dynamics has been attempted throughout the twentieth century. Many formulations of these dynamics have appeared in dissimilar mathematical forms. In this paper, we have demonstrated the equivalence of a number of these published formulations by deriving them directly from one another, thereby aiming to contribute to a more coherent historical picture of charged particle models. This topic continues to be subject of active research, not only due to its relevance for related fundamental topics in modern physics such as electromagnetic inertia, self-dressing, and mass renormalization, but also because experimental conditions start to enter regimes where the effects of the electromagnetic self-force might become observable.

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### References

Abraham, M. 1902. Principien der Dynamik des Elektrons. *Ann. Phys. (Leipzig)* **315**: 105-179

Abramowitz, M. and I.A. Stegun. 1965. *Handbook of Mathematical Functions*. Dover, New York

Ackerhalt, J.R., P.L. Knight and J.R. Eberly. 1973. Radiation reaction and radiative frequency-shifts. *Phys. Rev. Lett.* **30**: 456-460

Aguirregabiria, J.M., J. Llosa and A. Molina. 2006. Motion of a classical particle. *Phys. Rev. D* **73**: 125015

Ares de Parga, G. 2006. A physical deduction of an equivalent Landau-Lifshitz equation of motion in classical electrodynamics. A new expression for the large distance radiation rate of energy. *Found. Phys.* **36**: 1474-1510

Barut, A.O. and J.P. Dowling. 1987. Quantum electrodynamics based on self-energy: spontaneous emission in cavities. *Phys. Rev. A* **36**: 649-654

Barut, A.O. and J.P. Dowling. 1989. QED based on self-fields: a relativistic calculation of *g*-2. *Z. Naturforsch. A* **44**: 1051-1055

Barut, A.O., J. Kraus, Y. Salamin and N. Ünal. 1992. Relativistic theory of the Lamb shift in self-field quantum electrodynamics. *Phys. Rev. A* **45**: 7740-7745

Bohm, D. and M. Weinstein. 1948. The self-oscillations of a charged particle. *Phys. Rev.* **74**: 1789-1798

Born, M. 1909. Die Theorie des starren Elektrons in der Kinematik des Relativitätsprinzips. *Ann. Phys. (Leipzig)* **335**: 1-56

Bosanac, S.D. 2001. General classical solution for the dynamics of charges with radiation reaction. *J. Phys. A* **34**: 473-490

Boyer, T.H. 1968. Quantum electromagnetic zero-point energy of a conducting spherical shell and the Casimir model for a charged particle. *Phys. Rev.* **174**: 1764-1776

Casimir, H.B.G. 1953. Introductory remarks on quantum electrodynamics. *Physica* **19**: 846-849

Compagno, G. and F. Persico. 2002. Self-dressing and radiation reaction in classical electrodynamics. *J. Phys. A* **35**: 3629-3645

Dirac, P.A.M. 1938. Classical theory of radiating electrons. *Proc. R. Soc. London A* **167**: 148-169

Epp, R.J., R.B. Mann and P.L. McGrath. 2009. Rigid motion revisited: rigid quasilocal frames. *Classical Quant. Grav.* **26**: 035015

Erber, T. 1961. The classical theories of radiation reaction. *Fortschr. Phys.* **9**: 343-392

Fermi, E. 1921. Sulla dinamica di un sistema rigido di cariche elettriche in moto traslatorio. *Nuovo Cimento* **22**: 199-207

Fermi, E. 1923. Correzione di una contraddizione tra la teoria elettrodinamica e quella relativistica delle masse elettromagnetiche. *Nuovo Cimento* **25**: 159-170

Fermi, E. 1927. Sul meccanismo dell'emissione nella meccanica ondulatoria. *Rend. Lincei* **5**: 795-800

Ford, G.W. and R.F. O'Connell. 1991. Radiation reaction in electrodynamics and the elimination of runaway solutions. *Phys. Lett. A* **157**: 217-220

Gill, T.L., W.W. Zachary and J. Lindesay. 2001. The classical electron problem. *Found. Phys.* **31**: 1299-1355

Gralla, S.E., A.I. Harte and R.M. Wald. 2009. Rigourous derivation of electromagnetic self-force. *Phys. Rev. D* **80**: 024031

Hammond, R.T. 2013. Electrodynamics and radiation reaction. *Found. Phys.* **43**: 201-209

Hansen, E.R. 1975. *A Table of Series and Products.* Prentice-Hall, Englewood Cliffs. pp. 5 and 124

Herglotz, G. 1903. Zur Elektronentheorie. *Nachr. Ges. Wiss. Göttingen* **1903**: 357-382

Hnizdo, V. 2000. The electromagnetic self-force on a charged spherical body slowly undergoing a small, temporary displacement from a position of rest. *J. Phys. A* **33**: 4095-4103

Jackson, J.D. 1999. *Classical Electrodynamics,* 3rd edn. Wiley, New York

Janssen, M. and M. Mecklenburg. 2006. From classical to relativistic mechanics: electromagnetic models of the electron. In: V.F. Hendricks, K.F. Jørgenson, J. Lützen and S.A. Pedersen (eds.) *Interactions: Mathematics, Physics and Philosophy, 1860-1930*, Springer, Dordrecht, pp. 65-134

Jiménez, J.L. and I. Campos. 1999. Models of the classical electron after a century. *Found. Phys. Lett.* **12**: 127-146

Kholmetskii, A.L. 2006. On "gauge renormalization" in classical electrodynamics. *Found. Phys.* **36**: 715-744

Landau, L.D. and E.M. Lifshitz. 1975. *Classical Theory of Fields,* 4th rev. edn. Pergamon, Oxford. Section 76

Leonardt, U. and W.M.R. Simpson. 2011. Exact solution for the Casimir stress in a spherically symmetric medium. *Phys. Rev. D* **84**: 081701(R)

Lorentz, H.A. 1916. *The Theory of Electrons,* 2nd edn. Teubner, Leipzig

Luke, Y.L. 1962. *Integrals of Bessel Functions.* McGraw-Hill, London. p. 28

Lyle, S.N. 2010. *Self-Force and Inertia.* Springer, Berlin. Chap. 12

Martins, A.A. and M.J. Pinheiro. 2008. On the electromagnetic origin of inertia and inertial mass. *Int. J. Theor. Phys.* **47**: 2706-2715

Medina, R. 2006. Radiation reaction of a classical quasi-rigid extended particle. *J. Phys. A* **39**: 3801-3816

Milonni, P.W., J.R. Ackerhalt and W.A. Smith. 1973. Interpretation of radiative corrections in spontaneous emission. *Phys. Rev. Lett.* **31**: 958-960

Milton, K.A. 1980. Semiclassical electron models: Casimir self-stress in dielectric and conducting balls. *Ann. Phys. (New York)* **127**: 49-61

Milton, K.A., L.L. DeRaad Jr., and J. Schwinger. 1978. Casimir self-stress on a perfectly conducting spherical shell. *Ann. Phys. (New York)* **115**: 388-403

Moniz, E.J. and D.H. Sharp. 1977. Radiation reaction in nonrelativistic quantum mechanics. *Phys. Rev. D* **15**: 2850-2865

Morse, P.M. and H. Feshbach. 1953. *Methods of Theoretical Physics*. McGraw-Hill, New York. Part 2, p. 1255

Noja, D. and A. Posilicano. 1999. On the point limit of the Pauli-Fierz model. *Ann. Inst. Henri Poincaré A* **71**: 425-457

Oliver, M.A. 1998. Classical electrodynamics of a point particle. *Found. Phys. Lett.* **11**: 61-82

Ori, A. and E. Rosenthal. 2004. Calculation of the self force using the extended-object approach. *J. Math. Phys.* **45**: 2347-2364

Panofsky, W.K.H. and M. Phillips. 2005. *Classical Electricity and Magnetism,* 2nd edn. Dover, Mineola

Pierce, E. 2007. The lock and key paradox and the limits of rigidity in special relativity. *Am. J. Phys.* **75**: 610-614

Poincaré, M.H. 1906. On the dynamics of the electron. *Rend. Circ. Mat. Palermo* **21**: 129-176

Prigogine, I. and F. Henin. 1962. Motion of a relativistic charged particle. *Physica* **28**: 667-688

Puthoff, H.E. 2007. Casimir vacuum energy and the semiclassical electron. *Int. J. Theor. Phys.* **46**: 3005-3008

Roa-Neri, J.A.E. and J.L. Jiménez. 1993. On the classical dynamics of non-rotating extended charges. *Nuovo Cimento B* **108**: 853-869

Roa-Neri, J.A.E. and J.L. Jiménez. 2002. An alternative approach to the classical dynamics of an extended charged particle. *Found. Phys.* **32**: 1617-1634

Rohrlich, F. 2002. Dynamics of a classical quasi-point charge. *Phys. Lett. A* **303**: 307-310

Rohrlich, F. 2007. *Classical Charged Particles,* 3rd edn. World Scientific, New Jersey. Section 6-3

Schott, G.A. 1908. Über den Einfluß von Unstetigkeiten bei der Bewegung von Elektronen. *Ann. Phys. (Leipzig)* **330**: 63-91

Senitzky, I.R. 1973. Radiation-reaction and vacuum-field effects in Heisenberg-picture quantum electrodynamics. *Phys. Rev. Lett.* **31**: 955-958

Slater, L.J. 1966. *Generalized Hypergeometric Functions.* Cambridge University Press, Cambridge

Smorenburg, P.W., L.P.J. Kamp, G.A. Geloni and O.J. Luiten. 2010. Coherently enhanced radiation reaction effects in laser-vacuum acceleration of electron bunches. *Laser Part. Beams* **28**: 553-562

Smorenburg, P.W., L.P.J. Kamp and O.J. Luiten. 2013. Ponderomotive manipulation of cold subwavelength plasmas. *Phys. Rev. E* **87**: 023101

Sommerfeld, A. 1904a. Simplified deduction of the field and forces of an electron moving in any given way. *Proc. K. Akad. Wet. Amsterdam, Sect. Sci.* **7**: 346-367. (English translation)

Sommerfeld, A. 1904b. Zur Elektronentheorie. *Nachr. Ges. Wiss. Göttingen* **1904**: 99-439

Stratton, J.A. 1941. *Electromagnetic Theory.* McGraw-Hill, London. Section 7.8

Villarroel, D. 2006. Enlarged Lorentz-Dirac equations. *J. Phys. A* **39**: 8543-8556

Watson, G.N. 1966. *A Treatise on the Theory of Bessel Functions,* 2nd edn. Cambridge University Press, Cambridge. p. 528

Whittaker, E.T. and G.N. Watson. 1962. *A Course of Modern Analysis,* 4th edn. Cambridge University Press, Cambridge

Wildermuth, K. 1955. Zur physikalischen Interpretation der Elektronenselbstbeschleunigung. *Z. Naturforsch. A* **10**: 450-459

Wolfram. 2012a. http://functions.wolfram.com/HypergeometricFunctions/Hypergeometric-3F2/21/01/02/01/01/01/ and .../Hypergeometric4F3/17/02/01/0002/. Accessed 12 November 2012

Wolfram. 2012b. http://functions.wolfram.com/HypergeometricFunctions/Hypergeometric-3F2/03/02/03/0003/ and .../Hypergeometric4F3/03/02/05/0002/. Accessed 12 November 2012

Yaghjian, A.D. 2006. *Relativistic Dynamics of a Charged Sphere,* 2nd edn. Springer, New York

Zygmund, A. 1968. *Trigonometric Series,* 2nd edn. Cambridge University Press, Cambridge. Vol. 2, p. 243