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A UNIQUE INFORMATIONALLY EFFICIENT ALLOCATION MECHANISM IN ECONOMIES WITH CONSUMPTION EXTERNALITIES*

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This article investigates the informational requirements of resource allocation processes in pure exchange economies with consumption externalities. It is shown that the distributive Lindahl mechanism has a minimal informational size of the message space, and thus it is informationally the most efficient allocation process that is informationally decentralized and realizes Pareto-efficient allocations over the class of economies that include nonmalevolent economies. Furthermore, it is shown that the distributive Lindahl mechanism is the unique informationally efficient decentralized mechanism that realizes Pareto-efficient and individually rational allocations over a certain class of nonmalevolent economies.

1. INTRODUCTION

This article studies the informational requirements of resource allocation mechanisms that select Pareto-optimal allocations for economies with consumption externalities (interdependent preferences). A formal study on the informational requirements and informational optimality of resource allocation processes was initiated by Hurwicz (1960). The interest in such a study was greatly stimulated by the "socialist controversy"—the debate over the feasibility of central planning between Mises–Hayek and Lange–Lerner (von Hayek, 1935, 1945; Lange, 1936–7, 1944; Lerner, 1944). In line with the prevailing tradition, interest in this area was focused on the design of nonwasteful and privacy-preserving mechanisms, i.e., the mechanisms that result in Pareto-efficient allocations and use informationally decentralized decision-making processes. Allocative efficiency and informational efficiency are two highly desired properties for an economic system to have. Pareto optimality requires resources be allocated efficiently whereas informational efficiency requires an economic system to have the minimal informational cost of operation. The informational requirements depend upon two basic components: the class and types of economic environments over which a mechanism is supposed to operate and the particular outcomes that a mechanism is required to realize.

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A mechanism can be viewed as an abstract planning procedure; it consists of a message space in which communication takes place, rules by which the agents form messages, and an outcome function that translates messages into outcomes (allocations of resources). Mechanisms are imagined to operate iteratively. Attention, however, may be focused on mechanisms that have stationary or equilibrium messages for each possible economic environment. A mechanism realizes a prespecified welfare criterion (also call performance, social choice rule, or social choice correspondence) if the outcome given by the outcome function agrees with the welfare criterion at the stationary messages. The realization theory studies the question of how much communication must be provided to realize a given performance, or more precisely, studies the minimal informational cost of operating a given performance in terms of the size of the message space and determines which economic system or social choice rule is informationally the most efficient in the sense that the minimal informational cost is used to operate the system. Since the pioneering work of Hurwicz (1960), there has been a lot of work on studying the informational requirements of decentralized resource allocation mechanisms over various classes of economies such as those in Calsamiglia (1977), Calsamiglia and Kirman (1993), Hurwicz (1972, 1977, 1999), Hurwicz, Reiter, and Saari (1985), Mount and Reiter (1974), Sato (1981), Tian (1990, 1994, 2000a, 2000b) among others.

One of the well-known results in this literature established the minimality of the competitive (Walrasian) mechanism in using information for pure exchange economies with only private goods. Hurwicz (1972, 1986), Mount and Reiter (1974), Walker (1977) among others proved that, for pure exchange private goods economies, the competitive allocation process is the most informationally efficient process in the sense that any smooth informationally decentralized allocation mechanism that achieves Pareto-optimal allocations must use information at least as large as the competitive mechanism. Thus, the competitive allocation process has a message space of a minimal dimension among a certain class of resource allocation processes that are privacy preserving and nonwasteful.[2] For brevity, this result has been referred to as the Efficiency Theorem. Jordan (1982) and Calsamiglia and Kirman (1993) further provided the Uniqueness Theorem for pure exchange economies. Jordan (1982) proved that the competitive allocation process is uniquely informationally efficient among mechanisms that realize Pareto-efficient and individually rational allocations. Calsamiglia and Kirman (1993) proved that the equal income Walrasian mechanism is uniquely informationally efficient among all resource allocation mechanisms that realize fair allocations. Recently, Tian (2000a) further proved the informational optimality and uniqueness of the competitive mechanism in using information efficiently for convex private ownership production economies. These efficiency and uniqueness results are of fundamental importance from the point of view of a political economy.

[2] A mechanism is called *smooth* if the stationary message correspondence is either locally threaded or if the inverse of the stationary message correspondence has a Lipschizian-continuous selection in the subset. This terminology was used by Hurwicz (1999). I will give the definition of the local threadedness below.

They show the uniqueness of the competitive market mechanism in terms of allocative efficiency and informational efficiency for private goods economies. These informational efficiency and uniqueness results on the competitive mechanism, however, are true only for economies with convex production sets. Calsamiglia (1977) showed that in production economies with unbounded increasing returns to scale there exists no smooth privacy-preserving and nonwasteful process that uses a finite-dimensional message space.

When a consumer's level of preference also depends on the consumption of others, not every competitive equilibrium is necessarily Pareto optimal, and thus one needs to adopt other types of resource allocation mechanisms. For the cases wherein the externalities arise out of the provision of public goods and the case of production externalities, the problem of designing informationally efficient processes has been investigated in the literature. For a certain class of public goods economies, Sato (1981) obtained a similar result showing that the Lindahl allocation process has a minimal size of message space among the class of resource allocation processes that are privacy-preserving and nonwasteful. For the class of public goods economies allowing the presence of by-products, Tian (1994) showed that the generalized ratio process is informationally the most efficient process among privacy-preserving and nonwasteful resource allocation processes, and the Lindahl allocation process needs more information than the generalized ratio process in the smuggling information case of by-products. However, Sato (1981) only dealt with the class of economies with just a single producer, and in such a case, a special class of economies is constructed using a class of linear production sets for the producer. Tian (2000b) proved the informational optimality of the Lindahl mechanism for the general class of public goods economies with any number of producers and general convex production sets, and further proved that the Lindahl mechanism is the unique informationally efficient and decentralized process that realizes Pareto-efficient and individually rational allocations for public goods economies.

As for economies with production externalities, only impossibility results on the design of informationally decentralized mechanisms have been given in the literature. Hurwicz (1972, 1999) showed that for a certain class of economies with production externalities, either processes that realize Pareto-efficient allocations may not be informationally decentralized or there does not exist a smooth finite-dimensional message space mechanism that realize Pareto-efficient allocations. As we will discuss in somewhat more detail in the concluding remarks section, Hurwicz's impossibility results are drawn on a class of economies in which externalities result in nonconvex production sets. A conclusion drawn from the present article is that the prospects for designing the privacy-preserving and nonwasteful mechanisms over economies with important categories of nonmalevolent preferences are encouraging.

The purpose of this article is thus threefold. First, we establish a lower bound of information, as measured by the size of the message space that is required to guarantee an informationally decentralized mechanism to realize Pareto-efficient allocations over the class of pure exchange economies with widespread interdependent preferences. Theorem 1 shows that any smooth informationally decentralized mechanism that realizes Pareto-efficient allocations over the class of pure exchange economies with benevolent (nonmalevolent) consumption externalities has a message space of dimension no smaller than $I(I+L-2)$, where I is the number of agents and L is the number of commodities.

Second, we establish the informational optimality of the distributive Lindahl mechanism, a solution principle that was introduced by Bergstrom (1970) to handle economies with interdependent preferences, and can accommodate the existence of altruism and income redistribution without resorting to arbitrary equity assessments. Theorem 2 shows that the lower bound is exactly the size of the message space of the distributive Lindahl mechanism, and thus any smooth informationally decentralized mechanism that realizes Pareto-efficient allocations has a message space whose topological size is greater than or equal to the one for the distributive Lindahl allocation mechanism. Thus, the distributive Lindahl mechanism is an informationally efficient process among privacy-preserving and nonwasteful resource allocation mechanisms over the class of economies where distributive Lindahl equilibria exist.

Third, we show that the distributive Lindahl mechanism is the unique informationally efficient and decentralized process that realizes Pareto-efficient and individually rational allocations for the class of pure exchange economies considered in the article. Theorem 3 shows that any informationally decentralized, individually rational, and nonwasteful mechanism with the $I(I+L-2)$-dimensional message space and a continuous single-valued stationary message function is essentially the distributive Lindahl mechanism on the test family. As a result, Theorem 4 shows that any nondistributive Lindahl mechanism that has a continuous single-valued stationary message function and is informationally decentralized, individually rational, and nonwasteful must use a larger message space for any superset of the test family. Thus, any other informationally decentralized economic institution that achieves Pareto-efficient and individually rational allocations must use a message space whose informational size is larger than that of the distributive Lindahl mechanism so that the mechanism necessarily has strictly larger informational requirements.

The remainder of this article is organized as follows. In order to help readers understand the proofs of the main results in the article, Section 2 outlines the logic of arguments that underlie these results. In Section 3, we provide a description of the general framework. We specify economic environments with nonmalevolent externalities, and give notation and definitions on resource allocation, distributive Lindahl equilibrium, social choice correspondence, outcome function, allocation mechanism, etc. Section 4 establishes a lower bound of the size of the message space that is required to guarantee that an informationally decentralized mechanism realizes Pareto-efficient allocations on the class of pure exchange economies with consumption externalities. Section 5 gives an Efficiency Theorem on the allocative efficiency and informational efficiency of the distributive Lindahl mechanism for the class of pure exchange economies with nonmalevolent consumers. Section 6 provides the Uniqueness Theorems that show that the distributive Lindahl mechanism is the unique informationally efficient mechanism that realizes Pareto-efficient and individually rational allocations in the class of pure exchange economies that include nonmalevolent consumption externalities. Concluding remarks are presented in Section 7.

2. THE LOGIC OF THE ARGUMENTS

It may be helpful to indicate the logic of the arguments that underlie the main results obtained in the article. First, to establish a lower bound of the informational requirement as stated in Theorem 1, we adopt a standard approach that is widely used in the mechanism design literature. This approach states that, for a set of admissible economies and a smooth informationally decentralized mechanism realizing a social choice correspondence, if one can find a (parametrized) subset (test family) of the set such that the subset is of dimension n, and the stationary message correspondence is injective, that is, if the inverse of the stationary message correspondence is single-valued, then the size of the message space required for an informationally decentralized mechanism to realize the social choice correspondence cannot be lower than n on the subset. Thus, it cannot be lower than n for any superset of the subset, and in particular, for the entire class of economies. It is this result that was used by Calsamiglia and Kirman (1993), Hurwicz (1977), Mount and Reiter (1974), Sato (1981), Tian (2000a, 2000b), and Walker (1977) among others to show the minimal dimension and thus informational efficiency of the competitive mechanism, Lindahl mechanism, and the equal-income Walrasian mechanism over the various classes of economic environments. It is also this result that was used by Calsamiglia (1977) and Hurwicz (1999) to show the nonexistence of a smooth finite-dimensional message space mechanism that realizes Pareto-efficient allocations in certain economies with increasing returns and economies with production externalities that result in nonconvex production sets. It is the same result that will be used in the present article to establish the lower bound of the size of the message space required for an informationally decentralized and nonwasteful smooth mechanism on the test family that we will specify, and consequently over the entire class of economies with interdependent preferences.

To establish the informational efficiency of the Lindahl distributive mechanism as stated in Theorem 2, we then need to show that the lower bound is exactly the size of the message space of the distributive Lindahl mechanism. Since it will be seen that the upper bound dimension of the message space of the distributive Lindahl mechanism is $I(I + L - 2)$, which is the lower bound of message spaces for any smooth privacy-preserving and nonwasteful mechanism, it suffices to show that this upper bound can be actually reached on the test family of economies for the distributive Lindahl mechanism. Hence, the size of the message space of the distributive Lindahl mechanism is $I(I + L - 2)$ and thus the distributive Lindahl mechanism is informationally efficient among all resource allocation mechanisms that are informationally decentralized and nonwasteful over any subclass of economies where the distributive Lindahl mechanism is well defined and that include the test family as a subset.

To show the distributive Lindahl mechanism is uniquely informationally efficient, we need to impose an additional requirement that the mechanism results in individually rational allocations, and then show that any informationally decentralized wasteful and individually rational mechanism that has the same size of message space is topologically equivalent to the distributive Lindahl mechanism. Using the same techniques as in proving Theorems 1 and 2, we first establish the uniqueness result on the test family. Since the test family is a subset of the entire class of economies under consideration, any informationally decentralized and nondistributive Lindahl mechanism that achieves Pareto-efficient and individually rational allocations must use a larger message space for any superset of the test family, and in particular, for the entire class of pure exchange economies with interdependent preferences. The results are summarized in Theorems 3 and 4.

We should point out that the complicated setting of pure exchange economies with widespread consumption externalities has caused two technical difficulties in establishing our main results. One is how to find a suitable subset of economies so that it guarantees the injectiveness of the stationary message correspondence of any privacy-preserving and nonwasteful mechanism, and the other is how to construct a privacy-preserving stationary message correspondence for the distributive Lindahl process due to the redistribution of the initial endowments. These two issues are of fundamental importance in establishing the informational efficiency and uniqueness results.

One essential step in establishing the results on the lower bound of the information requirement, informational efficiency, and uniqueness of the Lindahl distributive process is to specify a subset (test family) of economies with consumption externalities on which the stationary message correspondence of any informationally decentralized and nonwasteful mechanism is injective and there is a unique distributive Lindahl equilibrium. However, in pure exchange economies with interdependent preferences, the first-order conditions for Pareto-efficiency will lead to equations where the weighted sum of all agents' marginal utility of goods is equal to the Lagrangian multipliers associated to the corresponding goods (see (11) below). Since these equations are with respect to the summation over all agents, any change in one agent's characteristic will likely lead to changes in weights and the Lagrangian multipliers, and consequently, lead to a whole new set of equations, which, in general, do not have too much to do with the original marginal conditions. As such, the standard technique of showing injectiveness by relying on the fact that a characteristic change in one agent does not change the marginal value of the others may not be applied for economies with widespread consumption externalities. Thus, we need to construct a suitable set of utility functions with special structures so that changes in agents' characteristics will keep the weights and Lagrangian multipliers unchanged when an allocation is unchanged. Thus, it enables us to establish the injectiveness of the stationary message correspondence of a privacy-preserving and nonwasteful mechanism. Further, in order to show the informational efficiency of the distributive Lindahl process and the uniqueness of the distributive Lindahl process in informational efficiency, the test family of utility functions should also be constructed in a way that every economy in the test family has a unique distributive Lindahl equilibrium. As such, we first find such a test family of economies and then, in Lemmas 1 and 2, we show that the stationary message correspondence of any privacy-preserving and nonwasteful mechanism is injective and there exists a unique distributive Lindahl equilibrium for every economy in the set.

The other technical difficulty is to establish the privacy-preserving property of the distributive Lindahl process. Since the actual initial wealth of each individual is in general dependent on not only his own initial endowment but also other agents' initial endowments in the definition of the distributive Lindahl process, the standard approach of using the constrained demand correspondence of each individual to construct each individual stationary message correspondence does not work. Thus we need to modify the standard approach in a suitable way in order to construct a privacy-preserving stationary message correspondence for the distributive Lindahl mechanism.

3. FRAMEWORK

In this section we will give notation, definitions, and a model that will be used in the article.

3.1. *Economic Environments with Interdependent Preferences.* Consider pure exchange economies with $L \geqq 2$ private goods and $I > 2$ consumers who are characterized by their preferences and endowments.[3] Let $N = \{1, \ldots, I\}$ denote the set of consumers. Throughout this article, subscripts are used to index consumers and superscripts are used to index goods unless otherwise stated. For the ith consumer, his characteristic is denoted by $e_i = (w_i, R_i)$, where $w_i \in \mathbb{R}^L_{++}$ is his initial endowments of the goods, and, as we are considering economies with preference externalities, R_i is a preference ordering defined on $\mathbb{R}^{IL}_+$. Let P_i be the strict preference (asymmetric part) of R_i. We assume that all R_i are locally nonsatiated. A pure exchange economy with preference externalities is thus the full vector $e = (e_1, \ldots, e_I)$ and the set of all such pure exchange economies is denoted by E. We assume that E is endowed with the product topology.

Before formalizing the notion of nonmalevolence that was introduced by Bergstrom (1970), we give the concept of separability of preferences.

The preference relations R_i of consumer i are said to be *separable between consumers* if for all $j \in N$ and all z and y in $\mathbb{R}^{IL}_+$ such that $z_k = y_k$ for all $k \neq j$, zR_iy implies $z'R_iy'$ for any z' and y' such that $z'_j = z_j$, $y'_j = y_j$, and $z'_k = y'_k$ for all $k \neq j$.

In other words, preferences are separable between individuals if each consumer's preference between any two allocations that contain the same commodity bundles for all consumers except some consumer j is unaffected by what consumers other than j consume, so long as in each of the two allocations compared, the amount consumed by the others is the same. It rules out such effects as the desire to imitate the consumption of others or a desire for a commodity solely because of its scarcity but does allow persons to be concerned about the consumption of others. This is the notion of separability that is familiar in consumption theory.

[3] As usual, vector inequalities are defined as follows: Let $a, b \in \mathbb{R}^m$. Then $a \geqq b$ means $a_s \geqq b_s$ for all $s = 1, \ldots, m$; $a \geq b$ means $a \geqq b$ but $a \neq b$; $a > b$ means $a_s > b_s$ for all $s = 1, \ldots, m$.

When R_i are represented by a continuous utility function, then they are separable between consumers if and only if they can be represented by a utility function of the form $u_i(g_1(z_1), \ldots, g_I(z_I))$, where $g_j(\cdot)$ is a continuous real-valued function and z_j is the consumption bundle received by consumer j.

When preferences are separable between individuals, one can define the notion of a private preference ordering $\succeq_i$ for consumer i on his individual consumption set $\mathbb{R}^L_+$ as follows: For z_i and $y_i \in \mathbb{R}^L_+$, we say that $z_i \succeq_i y_i$ if and only if uR_iv whenever $u_i = z_i$, $v_i = y_i$, and $u_j = v_j$ for all $j \in N$ with $j \neq i$. It is clear that $\succeq_i$ is a complete preordering on $\mathbb{R}^L_+$ if preferences of i are separable between individuals.

We are now in the position to define the concept of nonmalevolence. Consumer i is said to be *nonmalevolently* (*benevolently*) related to consumer j if preferences of i and j are separable between individuals and for any z and y in $\mathbb{R}^{IL}_+$ such that $z_k = y_k$ for $k \neq j$ and $z_j \succeq_j y_j$ implies that zR_iy.

In other words, consumer i is nonmalevolently related to consumer j if for any two allocations z and y, which contain the same bundles for everyone except j such that j privately prefers his bundle in z to his bundle in y, consumer i respects j's private preference to the extent that he prefers z to y. Nonmalevolence rules out the possibility that i disagrees with j about what kind of goods j should consume. When each consumer is nonmalevolently related to every other consumer, we say that the economy is characterized by nonmalevolent preference externalities.

When R_i are represented by a continuous utility function, then consumer i is nonmalevolently related to all consumers if and only if R_i can be represented by a utility function of the form $u_i(f_1(z_1), \ldots, f_I(z_I))$, where $f_j(\cdot)$ is a continuous real-valued function that represents the private preferences of consumer j.

3.2. *Distributive Lindahl Allocations.* Let x_i denote the net increment in commodity holdings (net trade) by consumer $i \in N$. x_i is said to be *individually feasible* if $x_i + w_i \in \mathbb{R}^L_+$. Let $x = (x_1, \ldots, x_I)$, which is called a net distribution.

For an economy $e \in E$, *an allocation* x is a net distribution $x \in \mathbb{R}^{IL}$. An allocation x is said to be *balanced* if $\sum_{i=1}^I x_i = 0$. An allocation x is said to be *feasible* if it is balanced and individually feasible for every individual.

An allocation x in an economy $e \in E$ is said to be *Pareto-efficient* if it is feasible and there does not exist another feasible allocation x' such that $(x' + w)R_i\,(x + w)$ for all $i \in N$ and $(x' + w)P_i\,(x + w)$ for some $i \in N$. Denote by $P(e)$ the set of all such allocations.

An allocation x in economy $e \in E$ is said to be *individually rational* if $(x + w)R_i w$ for all $i \in N$. Note that the individual rationality defined here reduces to the conventional individual rationality when there are no consumption externalities. Denote by $\mathcal{I}(e)$ the set of all such allocations.

Let

$$A = \left\{ \alpha = [\alpha_{ij}] : \alpha_{ij} \geqq 0 \quad \text{for all } i, j \in N \text{ and } \sum_{i \in N} \alpha_{ij} = 1 \right\}$$

For pure exchange economies with nonmalevolent preference externalities, it is possible to define an equilibrium concept called the distributive Lindahl equilibrium that was introduced by Bergstrom (1970) and described as follows. Let $p \in R^L$ be a normalized price vector, and let $\alpha = [\alpha_{ij}]$ be an $I \times I$ share matrix system in the admissible share space A where α_{ij} are the shares of the cost of consumer j's consumption to be borne by consumer i that are assigned to each $i \in N$ for each $j \in N$. For a commodity price vector p, a share matrix α, and the initial endowment vector $w = (w_1, \ldots, w_I)$, each consumer has an initial wealth distribution of property rights that is determined by the institutions of the economy. Each consumer $i \in N$ then states the allocation that he likes best among those allocations that he can afford if each consumer j pays the fraction α_{ij} of the cost of any bundle allocated to consumer j. At an equilibrium set of prices and shares, all consumers agree on the same allocation and there are no excess demands or supplies in any commodity market.

Formally, for a price vector $p \in R^L$ and a share matrix $\alpha \in A$, an *initial wealth distribution* W is a function $W(p,\alpha) = (W_1(p,\alpha), \ldots, W_I(p,\alpha))$ whose value depends on prices, shares, and the initial distribution of property rights for redistributing the initial endowments, such that for any price–share system $(p, \alpha) \in R^L \times A$, $W_i(p, \alpha)$ are homogenous of degree 1 in p for all $i \in N$, and $\sum_{i=1}^{I} W_i(p, \alpha) = \sum_{i=1}^{I} p \cdot w_i$. For each price–share system $(p, \alpha) \in \mathbb{R}^L \times A$, the budget set $B_i(p, \alpha, w)$ of the ith consumer is given by

$$\left\{ x = (x_1, \ldots, x_I) \in \mathbb{R}^{IL} : (x_i + w_i) \in \mathbb{R}^L_+ \& \sum_{j=1}^{I} \alpha_{ij}[p \cdot (x_j + w_j)] = W_i(p, \alpha) \right\}$$

Note that the initial wealth distribution function $W(p, \alpha) = (W_1(p, \alpha), \ldots, W_I(p,\alpha))$ given in the present article is more general and includes the initial wealth distribution function $W(p) = (W_1(p), \ldots, W_I(p))$ given by Bergstrom (1970) as a special case since Bergstrom's initial wealth distribution function $W(p)$ depends only on prices, but not on α. It will be seen that the initial wealth distribution function $W(p, \alpha)$ defined in the article has some advantages that the one given in Bergstrom (1970) does not share.

An allocation $x = (x_1, x_2, \ldots, x_I) \in \mathbb{R}^{IL}$ is a *distributive Lindahl allocation* with respect to a given initial wealth distribution function $W(\cdot) = (W_1(\cdot), \ldots, W_I(\cdot))$ for an economy e if it is feasible and there is a price vector $p \in R^L$ and the Lindahl share system $\alpha \in A$ such that

(1) $x \in B_i(p, \alpha, w)$ for all $i = 1, \ldots, I$;
(2) for all $i = 1, \ldots, I$, $(x' + w)P_i\,(x + w)$ implies $\sum_{j \in N} \alpha_{ij}[p \cdot (x'_j + w_j)] > W_i(p, \alpha)$

The price–share allocation (p, α, x) is then called a distributive Lindahl equilibrium. Thus, at distributive Lindahl equilibrium, there is unanimous agreement about what the consumption bundle of each consumer should be, given that each i must pay α_{ij} of the cost of the bundle consumed by each j.

Theorem 1 of Bergstrom (1970) has shown that there exists a distributive Lindahl equilibrium, if the economy is characterized by nonmalevolent preference externalities and there exists an initial allocation of resource ownership

$w' = (w'_1, \ldots, w'_I)$, such that the initial wealth distribution function has a special form given by $W_i(p,\alpha) = p \cdot w'_i$ and $\sum_{i=1}^I w'_i = \sum_{i=1}^I w_i$, and if the other usual conditions (such as the convexity, local nonsatiation, and continuity of preferences) are satisfied.[4]

Let $DL(e)$ denote the set of distributive Lindahl allocations and by $\mathcal{DL}(e)$ the set of distributive Lindahl equilibria. Let E^{DL} denote the set of all pure exchange economies with preference externalities such that $DL(e) \neq \emptyset$ for all $e \in E^{DL}$.

It may be remarked that, by the local nonsatiation of preferences, every distributive Lindahl allocation is Pareto-efficient (cf. Theorem 2 in Bergstrom, 1970).

In general, a distributive Lindahl allocation may not be individually rational, and further there may be no relationship with the so-called Pigouvian equilibrium for a general wealth distribution function W. However, when the initial wealth distribution function is given by $W_i(p, \alpha) = \sum_{j \in N} \alpha_{ij}[p \cdot w_j]$, then, every distributive Lindahl allocation with such a spatial form of initial wealth distribution is clearly individually rational so that $DL(e) \subset \mathcal{I}(e) \cap \mathcal{P}(e)$ for all $e \in E^{DL}$, and further it is a Pigouvian allocation.[5] To see this, let us first define the Pigouvian equilibrium.

Let $t \in \mathbb{R}^{LI^2}$ be a transfer system that satisfies the condition

$$\sum_{j=1}^{I} t_{ij} = 0$$

For each price–transfer system $(p, t) \in \mathbb{R}^L \times \mathbb{R}^{LI^2}$, the budget set $B_i(p, t)$ of the ith consumer is given by

$$\left\{ x = (x_1, \ldots, x_I) \in \mathbb{R}^{IL} : (x_i + w_i) \in \mathbb{R}^L_+ \& \sum_{j=1}^{I} t_{ij}(x_j + w_j) \right.$$
$$\left. + p(x_i + w_i) = \sum_{j=1}^{I} t_{ij} w_j + p w_i \right\}$$

An allocation $x = (x_1, x_2, \ldots, x_I) \in \mathbb{R}^{IL}$ is a *Pigouvian allocation* for an economy e if it is feasible and there is a price vector $p \in R^L$ and a transfer system $t \in \mathbb{R}^{LI^2}$ such that

(1) $x \in B_i(p, t)$ for all $i = 1, \ldots, I$
(2) for all $i = 1, \ldots, I$, $(x' + w)\ P_i\ (x + w)$ implies $\sum_{j \in N} t_{ij}(x'_j + w_j) + p(x'_i + w_i) > \sum_{j \in N} t_{ij} w_j + p w_i$

The price–transfer-allocation (p, t, x) is then called the *Pigouvian equilibrium*. Nakamura (1988) has shown that there is always a Pigouvian equilibrium under

[4] Two examples for the initial wealth distribution function $W_i(p, \alpha)$ having such a special form can be given: (1) $W_i(p, \alpha) = p w_i$ by letting $w'_i = w_i$. Such an initial wealth distribution function has been used by Asdrubali (1996) to argue coalitional instability of the distributive Lindahl equilibrium. (2) $W_i(p, \alpha) = \sum_{j \in N} \alpha_{ij}[p w_j]$ by letting $w'_i = \sum_{j \in N} \alpha_{ij} w_j$. We guess under such a specification about the initial wealth function, Asdrubali's conclusion may not be true.

[5] Because of these good properties, it motivates us to generalize Bergstrom's initial wealth distribution function to the general one given in the present article.

the regular conditions of convexity and continuity of preferences as well as interior endowments. It is clear that every Pigouvian is Pareto-efficient and individually rational.

Now, for $i = 1, \ldots, I$ and $j = 1, \ldots, I$, if we let $t_{ij} = \alpha_{ij} p$ for $i \neq j$ and $t_{ii} = (\alpha_{ii} - 1)p$, then we have $\sum_{i=1}^{I} t_{ij} = 0$ so that it is a transfer system, and thus every distributive Lindahl allocation with respect to the initial wealth distribution function $W_i(p, \alpha) = \sum_{j \in N} \alpha_{ij}[pw_j]$ is a Pigouvian allocation, but the reverse may not be true.

3.3. *Allocation Mechanisms.* Let $X = \{x \in \mathbb{R}^{IL} : \sum_{i=1}^{I} x_i = 0\}$ and let F be a social choice rule, i.e., a performance correspondence from E to X. Following Mount and Reiter (1974), a *message process* is a pair $\langle M, \mu \rangle$, where M is a set of abstract messages and is called a message space, and $\mu : E \to \to M$ is a stationary or equilibrium message correspondence that assigns to every economy e the set of stationary (equilibrium) messages.

An *allocation mechanism or process* is a triple $\langle M, \mu, h \rangle$ defined on E, where $h : M \to X$ is the outcome function that assigns every equilibrium message $m \in \mu(e)$ to the corresponding trade $x \in X$.

An allocation mechanism $\langle M, \mu, h \rangle$, defined on E, *realizes* a social choice rule F, if for all $e \in E$, $\mu(e) \neq \emptyset$ and $h(m) \in F(e)$ for all $m \in \mu(e)$.

In this article, the social choice rule is restricted to the one that yields Pareto-efficient outcomes. Let $\mathcal{P}(e)$ be a subset of Pareto-efficient allocations for $e \in E$. An allocation mechanism $\langle M, \mu, h \rangle$ is said to be *nonwasteful* on E with respect to $\mathcal{P}$ if for all $e \in E$, $\mu(e) \neq \emptyset$ and $h(m) \in \mathcal{P}(e)$ for all $m \in \mu(e)$. If an allocation mechanism $\langle M, \mu, h \rangle$ is nonwasteful on E with respect to P, the set of all Pareto-efficient outcomes, then it is said simply to be nonwasteful on E.

An allocation mechanism $\langle M, \mu, h \rangle$ is said to be *privacy-preserving or informationally decentralized* on E, if there exist correspondence $\mu_i : E_i \to \to M$, one for each i, such that $\mu(e) = \bigcap_{i=1}^{n} \mu_i(e_i)$ for all $e \in E$.

Thus, when a mechanism is privacy preserving, each individual's messages are dependent on environments only through the characteristics of the individual. Therefore, the individual does not need to know the characteristics of the other individuals.

Remark 1. This important feature of the communication process implies that the so-called "crossing condition" has to be satisfied. Mount and Reiter (1974, Lemma 5) have shown that an allocation mechanism $\langle M, \mu, h \rangle$ is privacy preserving on E if and only if for every i and every e and e' in E, $\mu(e) \cap \mu(e') = \mu(e'_i, e_{-i}) \cap \mu(e_i, e'_{-i})$, where $(e'_i, e_{-i}) = (e_1, \ldots, e_{i-1}, e'_i, e_{i+1}, \ldots, e_I)$ where the ith element of e is replaced by e'_i. Thus, if two economies have the same equilibrium message, then any "crossed economy" in which one agent from one of the two initial economies is "switched" with the agent from the other must have the same equilibrium message. Hence, for a given mechanism, if two economies have the same equilibrium message m, the mechanism leads to the same outcome for both, and further, this outcome must also be the outcome of the mechanism for any of the crossed economies because of the crossing condition.

Let $\langle M, \mu, h \rangle$ be an allocation mechanism on E. The stationary message correspondence μ is said to be locally threaded at $e \in E$ if it has a locally continuous single-valued selection at e. That is, there is a neighborhood $N(e) \subset E$ and a continuous function $f : N(e) \to M$ such that $f(e') \in \mu\,(e')$ for all $e' \in N(e)$. The stationary message correspondence μ is said to be locally threaded on E if it is locally threaded at every $e \in E$.

The notion of local threadedness was first introduced into the realization literature by Mount and Reiter (1974). This regularity condition is used mainly to exclude the possibility of intuitive smuggling information.[6] Many continuous selection results have been given in the mathematics literature since Michael (1956).

3.4. *The Distributive Lindahl Process.* We now give a privacy-preserving process that realizes the distributive Lindahl correspondence DL on E^{DL}, and in which messages consist of prices, shares, and trades of all agents.

Define the trade (demand) correspondence of consumer $i \in N$ $D_i : R^L \times A \times \mathbb{R}^{IL}_{++} \times E_i$ by

(1)

$$D_i(p, \alpha, w, e_i) = \Bigg\{ x = (x_1, \ldots, x_I) : (x + w) \in \mathbb{R}^{IL}_{+}, \sum_{j=1}^{I} \alpha_{ij}[p(x_j + w_j)] = W_i(p, \alpha)$$

$$(x' + w) P_i (x + w) \text{ implies } \sum_{j=1}^{I} \alpha_{ij}[p(x'_j + w_j)] > W_i(p, \alpha) \Bigg\}$$

Note that (p, α, x) is a distributive Lindahl equilibrium for economy $e \in E^{DL}$ if $p \in R^L$, $\alpha \in A$, $x \in D_i(p, \alpha, w, e_i)$ for $i = 1, \ldots, I$, and the allocation x is balanced.

The distributive Lindahl process $\langle M_{DL}, \mu_{DL}, h_{DL} \rangle$ is defined as follows:

Define $M_{DL} = \mathbb{R}^L \times A \times X \times \mathbb{R}^{IL}_{++}$.
Define $\mu_{DL} : E^{DL} \to\to M_{DL}$ by

$$\mu_{DL}(e) = \bigcap_{i=1}^{I} \mu_{DLi}(e_i) \tag{2}$$

where $\mu_{DLi} : E_i \to\to M_{DL}$ is defined by

$$\mu_{DLi}(e_i) = \Bigg\{ (p, \alpha, x, v) : p \in \mathbb{R}^{L-1}, \alpha \in A, x \in D_i(p, \alpha, v, e_i), v \in \mathbb{R}^{IL}_{++}, \tag{3}$$

$$v_i = w_i, \,\&\, \sum_{i=1}^{I} x_i = 0 \Bigg\}$$

for $i = 1, \ldots, I$. Thus, we have $\mu_{DL}(e) = \mathcal{DL}(e)$ for all $e \in E^{DL}$.

[6] Hurwicz (1972) has provided examples to show that there exist encoding procedures by which a simple message conveys an unlimited amount of information. As Hurwicz notes, such procedures use, in a certain sense, "tricks" not inherent to the allocation mechanism.

Finally, the distributive Lindahl outcome function $h_{DL}: M_{DL} \to X$ is defined by

$$(4) \qquad h_{DL}(p, \alpha, x, v) = x$$

which is an element in $DL(e)$ for all $e \in E^{DL}$.

The distributive Lindahl process can be viewed as a formalization of resource allocation, which is nonwasteful. The distributive Lindahl message process is privacy preserving by the construction of the distributive Lindahl process.

Remark 2. Since an element, $m = (p, \alpha, x_1, \ldots, x_I, v) \in \mathbb{R}^L \times A \times X \times \mathbb{R}^{IL}$, of the distributive Lindahl message space M_{DL} satisfies the normalization condition for prices, $\sum_{i=1}^I \alpha_{ij} = 1$ for $j = 1, \ldots, I$, $\sum_{j=1}^I x_i = 0$, $\sum_{j=1}^I \alpha_{ij}[p(x_j + w_j)] = W_i(p, \alpha)$, $v_i = w_i$ for $i = 1, \ldots, I$, and one of these equations is not independent by Walras' Law, any distributive Lindahl message is contained within a Euclidean space of dimension $(L + I^2 + IL + IL) - (1 + I + I + L + IL) + 1 = I(I + L - 2)$ and thus, an upper bound on the Euclidean dimension of M_{DL} is $I(I + L - 2)$.

3.5. *Informational Size of Message Spaces.* The notion of informational size can be considered as a concept that characterizes the relative sizes of topological spaces that are used to convey information in the resource allocation process. It would be natural to consider that a space, say S, has more information than the other space T whenever S is topologically "larger" than T. This suggests the following definition, which was introduced by Walker (1977).

Let S and T be two topological spaces. The space S is said to have as much information as the space T by the Fréchet ordering, denoted by $S \geqq_F T$, if T can be embedded homeomorphically in S, i.e., if there is a subspace of S' of S that is homeomorphic to T.

Let S and T be two topological spaces and let $\psi: T \to S$ be a correspondence. The correspondence ψ is said to be injective if $\psi(t) \cap \psi(t') \neq \emptyset$ implies $t = t'$ for any $t, t' \in T$. That is, the inverse, $(\psi)^{-1}$, of ψ is a single-valued function.

A topological space M is a *k-dimensional manifold* if it is locally homeomorphic to $\mathbb{R}^k$.

An informationally decentralized and nonwasteful mechanism $\langle M, \mu, h\rangle$ is said to be *informationally efficient* on E if the size of its message space M is the smallest among all other informationally decentralized and nonwasteful mechanisms.

3.6. *A Special Class of Nonmalevolent Economies (Test Family).* We should also find it necessary to work with a special class of pure exchange economies with nonmalevolent preference externalities, denoted by $E^c = \prod_{i=1}^I E_i^c$ and called the test family, where preference orderings are characterized by Cobb–Douglas-type utility functions.

For $i = 1, \ldots, I$, consumer i's admissible economic characteristics in E_i^c are given by the set of all $e_i = (w_i, R_i)$ such that $w_i \in \mathbb{R}^L_{++}$, and R_i is represented by a utility function of form $u_i(\cdot, \beta_i)$ such that

$$u_i(x+w,\beta_i)=\sum_{j=1}^{I}\sum_{l=1}^{L-1}\beta_{ij}^l\log\left(x_j^l+w_j^l\right)+\sum_{j\neq i}\log\left(x_j^L+w_j^L\right) \\ +2\log\left(x_i^L+w_i^L\right) \tag{5}$$

for $i=1,\ldots,I$, where $\beta_i=[\beta_{ij}^l]$ is an $I\times(L-1)$ matrix such that

$$\beta_{ij}^l=\begin{cases}\frac{1}{I}\left[1+\left(\sigma_{ij}-\frac{1}{I}\right)\gamma_i^l+\left(\rho_{ij}-\frac{1}{2}\right)\left(\gamma_i^l-\frac{1}{L-1}\right)\right] & \text{if } j=i,i+1\\ \frac{1}{I}\left[1+\left(\sigma_{ij}-\frac{1}{I}\right)\gamma_i^l\right], & \text{otherwise}\end{cases} \tag{6}$$

with $\sigma_{ij}\in\mathbb{R}_{++}$, $\sigma_{ii}+\sigma_{i,i+1}\neq\frac{2}{I}$, $\sum_{j=1}^{I}\sigma_{ij}=1$, $\gamma_i^l\in\mathbb{R}_{++}$, $\gamma_i^l\neq\frac{1}{L-1}$, $\sum_{l=1}^{L-1}\gamma_i^l=1$, $\rho_{ij}\in\mathbb{R}_{++}$ for $j=i,\ i+1$, and $\rho_{ii}+\rho_{i,i+1}=1$.[7] Note that, by construction, we have $\sum_{j=1}^{I}\beta_{ij}^l=1$ for all $l=1,\ldots,L$.

It is clear that any economy e in E^c is fully specified by the parameters $\sigma=(\sigma_1,\ldots,\sigma_I)$, $\gamma=(\gamma_1,\ldots,\gamma_I)$, and $\rho=(\rho_1,\ldots,\rho_I)$.

REMARK 3. The proofs of all the main results on the lower bound of informational requirements, the informational efficiency, and uniqueness of the distributive Lindahl mechanism will be based on the above special class of pure exchange economies with nonmalevolent preference externalities that are represented by the Cobb–Douglas utility functions.

Define a subset E_0^c of E^c by $E_0^c=\{e\in E^c: w_i=\dot{w}_i\}$. That is, w_i is constant over E_0^c.

A topology is introduced to the class E_0^c as follows. Let $\|\cdot\|$ be the usual Euclidean norm. For each consumer $i\in N$, define a metric δ on E_i^c by $\delta[e_i,\bar{e}_i]=\|\sigma_i-\bar{\sigma}_i\|+\|\gamma_i-\bar{\gamma}_i\|+\|\rho_i-\bar{\rho}_i\|$, where $\sigma_i=(\sigma_{i1},\ldots,\sigma_{iI})$, $\gamma_i=(\gamma_i^1,\ldots,\gamma_i^{L-1})$, and $\rho_i=(\rho_{ii},\rho_{i,i+1})$. Thus, this defines a topology on E_i^c. We may endow E_0^c with the product topology of the E_i^c $(i=1,\ldots,I)$ and we call this the parameter topology, which is denoted by $\mathcal{T}_p$. Then it is clear that the topological space $(E_0^c,\mathcal{T}_p)$ is homeomorphic to the $I(I+L-2)$-dimensional Euclidean space $\mathbb{R}^{I(I+L-2)}$.

4. THE LOWER BOUND OF INFORMATIONAL REQUIREMENTS OF ALLOCATION MECHANISMS

In this section we establish a lower bound (the minimal amount) of information, as measured by the size of the message space, that is required to guarantee that an informationally decentralized mechanism realizes Pareto-efficient allocations on, E, the class of pure exchange economies under consideration.

To make the problem nontrivial, as usual, the assumption of interiority has to be made.[8] Indeed, a mechanism that gives everything to a single individual

[7] When $i=I$, $I+1$ is regarded as 1.

[8] A stronger condition that can guarantee interior outcomes is that a mechanism is individually rational.

yields Pareto-efficient outcomes and no information about prices is needed. Thus, given a class of economies $\tilde{E} \subset E$, which includes E^c, we define an optimality correspondence $\mathcal{P} : \tilde{E} \rightarrow\rightarrow Z$ such that the restriction $\mathcal{P} \mid E^c$ associates with $e \in E^c$ the set $\mathcal{P}(e)$ of all the Pareto-efficient allocations that assign strictly positive consumption to every consumer.

The following lemma, which is based on the special class of pure exchange nonmalevolent economies E_0^c specified in the above section, is central in finding the lower bound of informational requirements of resource allocation processes.

LEMMA 1. *Suppose $\langle M, \mu, h\rangle$ is an allocation mechanism on the special class of pure exchange nonmalevolent economies $E_0^c \subset E$ such that*

(i) it is informationally decentralized;
(ii) it is nonwasteful with respect to $\mathcal{P}$.

Then, the stationary message correspondence μ is injective on E_0^c. That is, its inverse is a single-valued mapping on $\mu(E_0^c)$.

PROOF. Suppose that there is a message $m \in \mu(e) \cap \mu(\bar{e})$ for $e, \bar{e} \in E_0^c$. It will be proved that $e = \bar{e}$. Since μ is a privacy-preserving correspondence,

$$\mu(e) \cap \mu(\bar{e}) = \mu(\bar{e}_i, e_{-i}) \cap \mu(e_i, \bar{e}_{-i}) \tag{7}$$

for all $i \in N$ by Remark 1, and hence, in particular,

$$m \in \mu(e) \cap \mu(\bar{e}_i, e_{-i}) \tag{8}$$

for all $i \in N$. Let $x = h(m)$. Since the process $\langle M, \mu, h\rangle$ is nonwasteful with respect to $\mathcal{P}$, $x = h(m)$ and (8) imply that $x \in \mathcal{P}(e) \cap \mathcal{P}(\bar{e}_i, e_{-i})$. Since x is Pareto-efficient, then it can be obtained as a solution to the maximization problem

$$W(x) = \sum_{i=1}^{I} a_i u_i(x + w, \beta_i) \tag{9}$$

subject to

$$\sum_{i=1}^{I} x_i = 0$$

for $a = (a_1, \ldots, a_I) \in \mathbb{R}_+^I$ with $\sum_{i=1}^{I} a_i = 1$. Since $x \in \mathcal{P}(e)$ is an interior point, by the usual Lagrangian method of constrained maximization, it must satisfy the following first-order conditions:

$$\sum_{i=1}^{I} a_i \frac{\partial u_i(x + w)}{\partial x_j^l} - \lambda^l = 0 \quad l = 1, \ldots, L,\ j = 1, \ldots, I \tag{10}$$

where λ^l are the Lagrangian multipliers. The above equation implies that the Pareto-efficiency condition is that the weighted sum of all individuals' marginal utility of x_j^l equals a constant across j.

With the Cobb–Douglas utility functions given by

$$u_i(x+w,\beta_i)=\sum_{j=1}^{I}\sum_{l=1}^{L-1}\beta_{ij}^l\log\left(x_j^l+w_j^l\right)+\sum_{j\neq i}\log\left(x_j^L+w_j^L\right)+2\log\left(x_i^L+w_i^L\right)$$

(10) becomes

$$\sum_{i=1}^{I}a_i\frac{\beta_{ij}^l}{\left(x_j^l+w_j^l\right)}=\lambda^l \tag{11}$$

for $l=1,\ldots,L-1,\ j=1,\ldots,I$, and

$$\sum_{i\neq j}^{I}a_i\frac{1}{\left(x_j^L+w_j^L\right)}+2a_j\frac{1}{\left(x_j^L+w_j^L\right)}=\lambda^L \tag{12}$$

for $j=1,\ldots,I$.

Since $\sum_{j=1}^{I}\sigma_{ij}=1$ and $\rho_{ii}+\rho_{i,i+1}=1$, we have, by (6), $\sum_{j=1}^{I}\beta_{ij}^l=1$ for all $i=1,\ldots,I$. Thus, multiplying $(x_j^l+w_j^l)$ on both sides of (11), making summations over j, noting that $\sum_{i=1}^{I}a_i=1$, and solving for λ^l, we have

$$\lambda^l=\frac{1}{\sum_{j=1}^{I}w_j^l}\quad l=1,\ldots,L-1 \tag{13}$$

which means $\lambda^l, l=1,\ldots,L-1$, are constant over E_0^c since $w_i=\dot{w}_i$ for all $w_i\in E_0^c$ and $i=1,\ldots,I$. Rearranging (12) and writing it in matrix form, we have

$$A(a/\lambda^L)=x^L+w^L \tag{14}$$

where

$$A=\begin{pmatrix}2&1&\ldots&1&1\\1&2&\ldots&1&1\\\ldots&\ldots&\ldots&\ldots&\ldots\\1&1&\ldots&2&1\\1&1&\ldots&1&2\end{pmatrix} \tag{15}$$

is an $I\times I$ coefficient matrix, $a/\lambda^L=(a_1,\ldots,a_I)/\lambda^L$ is an $I\times 1$ column vector, $(x_1^L,\ldots,x_I^L)$ is an $I\times 1$ column vector, and $(w_1^L,\ldots,w_I^L)$ is an $I\times 1$ column vector. Since the determinant of matrix A, $|A|$, equals $I+1$, the solution a/λ^L is unique for given x^L+w^L. Thus, for any $\bar{a}/\bar{\lambda}$ resulting from $(\bar{e}_i,e_{-i})$ for any i, by

$x \in \mathcal{P}(\bar{e}_i, e_{-i}) \cap \mathcal{P}(e)$, we have

$$a_j/\lambda^L = \bar{a}_j/\bar{\lambda}^L$$

for $j = 1, \ldots, I$. Making summations over j and noting that $\sum_{j=1}^{I} a_j = 1$ and $\sum_{j=1}^{I} \bar{a}_j = 1$, we have

$$\lambda^L = \bar{\lambda}^L$$

and consequently we have

$$a_j = \bar{a}_j$$

for $j = 1, \ldots, I$.

Thus, by $x \in \mathcal{P}(\bar{e}_s, e_{-s})$, we have

(16) $$\sum_{i \neq s} \frac{a_i \beta_{ij}^l}{\left(x_j^l + w_j^l\right)} + \frac{a_s \bar{\beta}_{sj}^l}{\left(x_j^l + w_j^l\right)} = \lambda^l$$

for $l = 1, \ldots, L-1$, $j = 1, \ldots, I$. From Equations (11) and (16), we have

(17) $$\frac{a_s \beta_{sj}}{\left(x_j^l + w_j^l\right)} = \frac{a_s \bar{\beta}_{sj}}{\left(x_j^l + w_j^l\right)}$$

so that

(18) $$\beta_{sj} = \bar{\beta}_{sj} \quad s = 1, \ldots, I,\ j = 1, \ldots, I$$

which means, by (6),

(19) $$\begin{aligned}\left(\sigma_{sj} - \frac{1}{I}\right)\gamma_s^l + \left(\rho_{sj} - \frac{1}{2}\right)\left(\gamma_s^l - \frac{1}{L-1}\right) \\ = \left(\bar{\sigma}_{sj} - \frac{1}{I}\right)\bar{\gamma}_s^l + \left(\bar{\rho}_{sj} - \frac{1}{2}\right)\left(\bar{\gamma}_s^l - \frac{1}{L-1}\right)\end{aligned}$$

for $j = s, s+1$, and

(20) $$\left(\sigma_{sj} - \frac{1}{I}\right)\gamma_s^l = \left(\bar{\sigma}_{sj} - \frac{1}{I}\right)\bar{\gamma}_s^l$$

for $j \neq s, s+1$. Making summations over $l = 1, \ldots, L-1$ in the above equations and noting that $\sum_{l=1}^{L-1} \gamma_s^l = 1$, we have

$$\left(\sigma_{sj} - \frac{1}{I}\right) = \left(\bar{\sigma}_{sj} - \frac{1}{I}\right) \tag{21}$$

and thus

$$\sigma_{sj} = \bar{\sigma}_{sj} \tag{22}$$

for $j = 1, \ldots, I$ and $s = 1, \ldots, I$.

Making summations over $j = s, s+1$ in (19) and noting that $\rho_{ss} + \rho_{s,s+1} = 1$, we have

$$\left(\sigma_{ss} + \sigma_{s,s+1} - \frac{2}{I}\right)\gamma_s^l = \left(\sigma_{ss} + \sigma_{s,s+1} - \frac{2}{I}\right)\bar{\gamma}_s^l \tag{23}$$

and thus

$$\gamma_s^l = \bar{\gamma}_s^l, \quad i = 1, \ldots, L-1, s = 1, \ldots, I \tag{24}$$

by noting that $\sigma_{ss} + \sigma_{s,s+1} \neq \frac{2}{I}$.

Again, by (19), we have

$$\left(\rho_{sj} - \frac{1}{2}\right)\left(\gamma_s^l - \frac{1}{L-1}\right) = \left(\bar{\rho}_{sj} - \frac{1}{2}\right)\left(\gamma_s^l - \frac{1}{L-1}\right) \tag{25}$$

and therefore

$$\rho_{sj} = \bar{\rho}_{sj} \tag{26}$$

for $j = s, s+1$ by noting that $\gamma_s^l \neq \frac{1}{L-1}$. Thus, we have proved that $\sigma_s = \bar{\sigma}_s, \gamma_s = \bar{\gamma}_s$, and $\rho_s = \bar{\rho}_s$ for all $s = 1, \ldots, I$. Consequently, the inverse of the stationary message correspondence, $(\mu)^{-1}$ is a single-valued mapping from $\mu(E_0^c)$ to E_0^c. ■

The following theorem establishes a lower bound informational size of messages spaces of any allocation mechanism that is informationally decentralized and nonwasteful over any subclass of economies that includes E_0^c as a subset.

THEOREM 1 (INFORMATIONAL BOUNDEDNESS THEOREM). *Let $\tilde{E} \subset E$ be a subclass of pure exchange economies that includes the special class of pure exchange nonmalevolent economies E_0^c as a subset. Suppose that $\langle M, \mu, h\rangle$ is an allocation mechanism defined on $\tilde{E}$ such that*

(i) it is informationally decentralized;
(ii) it is nonwasteful with respect to $\mathcal{P}$;

(iii) M is a Hausdorff topological space;
(iv) μ is locally threaded at some point $e \in E_0^c$.

Then, the size of the message space M is at least as large as $\mathbb{R}^{I(I+L-2)}$, that is, $M \geqq_F \mathbb{R}^{I(I+L-2)}$.

PROOF. As was noted above, E_0^c is homeomorphic to the $I(I+L-2)$-dimensional Euclidean space $\mathbb{R}^{I(I+L-2)}$. Hence, it suffices to show $M \geqq_F E_0^c$.

By the injectiveness of Lemma 1, we know that the restriction $\mu \mid E_0^c$ of the stationary message correspondence μ to E_0^c is an injective correspondence. Since μ is locally threaded at $e \in E_0^c$, there exists a neighborhood $N(e)$ of e and a continuous function $f: N(e) \to M$ such that $f(e') \in \mu(e')$ for all $e' \in N(e)$. Then f is a continuous injection from $N(e)$ into M. Since μ is an injective correspondence from E_0^c into M, thus f is a continuous one-to-one function on $N(e)$.

Since E_0^c is homeomorphic to the $I(I+L-2)$-dimensional Euclidean space $\mathbb{R}^{I(I+L-2)}$, there exists a compact set $\bar{N}(e) \subset N(e)$ with a nonempty interior point. Also, since f is a continuous one-to-one function on $N(e)$, f is a continuous one-to-one function from the compact space $\bar{N}(e)$ onto a Hausdorff topological space $f(\bar{N}(e))$. Hence, it follows that the restriction $f|_{\bar{N}(e)}$ is a homeomorphic imbedding on $\bar{N}(e)$ by Theorem 5.8 in Kelley (1955, p. 141). Choose an open ball $\mathring{N}(e) \subset \bar{N}(e)$. Then $\mathring{N}(e)$ and $f(\mathring{N}(e))$ are homeomorphic by a homeomorphism $f \mid \mathring{N}(e) : \mathring{N}(e) \to f(\mathring{N}(e))$. This, together with the fact that E_0^c is homeomorphic to its open ball $\mathring{N}(e)$, implies that E_0^c is homeomorphic to $f(\mathring{N}(e)) \subset M$, implying that E_0^c can be homeomorphically imbedded in $\mu(\tilde{E})$. Hence, it follows that $M \geqq_F E_0^c =_F \mathbb{R}^{I(I+L-2)}$. ∎

5. INFORMATIONAL EFFICIENCY OF DISTRIBUTIVE LINDAHL PROCESS

In the previous section, we have found that the dimension of the Euclidean space $\mathbb{R}^{I(I+L-2)}$ is a lower bound for informational size of message spaces for allocation mechanisms that are privacy preserving and nonwasteful over any subclass of pure exchange economies $\tilde{E}$. In this section we assert that the lower bound is exactly the size of the message space of the distributive Lindahl mechanism, and thus the distributive Lindahl mechanism is informationally efficient among all resource allocation mechanisms that are informationally decentralized and nonwasteful over any subclass of pure exchange economies $\tilde{E} \subset E^{DL}$.

From Remark 2, we know that the upper bound of the informational size of the message space for the distributive Lindahl mechanism over $\tilde{E}$ is also the $I(I+L-2)$-dimensional Euclidean space $\mathbb{R}^{I(I+L-2)}$ for any given initial wealth distribution function $W(\cdot) = (W_1(\cdot), \ldots, W_I(\cdot))$. As a result, if we can show that this upper bound can be reached on the restriction of the message space of the distributive Lindahl mechanism with some initial wealth distribution function $W(\cdot)$ to the test family of pure exchange nonmalevolent economies E_0^c, i.e., if we can show that, for a given initial wealth distribution function $W_i(\cdot)$, $\mu_{DL}|E_0^c$ is homeomorphic to the $I(I+L-2)$-dimensional Euclidean space $\mathbb{R}^{I(I+L-2)}$, then we know that this upper bound is exactly the size of the message space of the distributive Lindahl mechanism, and thus we have the informational efficiency of the distributive Lindahl process on any subclass of $\tilde{E} \subset E^{DL}$.

We will first state the following lemmas that will be used to derive our main results on the informational efficiency of the distributive Lindahl mechanism and the uniqueness of the distributive Lindahl mechanism in informational efficiency. Again, these lemmas are based on the special class of pure exchange nonmalevolent economies E_0^c specified in the above section.

Denote $L-1$-dimensional unit simplex by

$$\Delta^{L-1} = \left\{ p^l \in \mathbb{R}^L_{++} : \sum_{l=1}^{L} p^l = 1 \right\}$$

Lemma 2. *For every pure exchange nonmalevolent economy $e \in E^c$ and any given initial wealth distribution function $W(p, \alpha) = (W_1(p, \alpha), \ldots, W_I(p, \alpha))$, there exists a distributive Lindahl equilibrium. Furthermore, when the initial wealth distribution function $W(p, \alpha)$ has forms of either $W_i(p, \alpha) = W_i(p)$ with $\frac{\partial W_i}{\partial p^l} > 0$ for all $l = 1, \ldots, L$ or $W_i(p, \alpha) = \sum_{j \in N} \alpha_{ij}[p \cdot w_j]$, there exists a unique distributive Lindahl equilibrium, i.e., $\mathcal{DL}$ is a single-valued mapping from E^c to $\Delta^{L-1} \times A \times X$.*

Proof. To show the existence of a distributive Lindahl equilibrium for every $e \in E^c$ and every $W_i(p, \alpha)$, we first derive the net demand functions of consumers.

Consumer $i \in N$ chooses a consumption bundle $(x_{i1}, x_{i2}, \ldots, x_{iI})$ so as to maximize his utility function, $u(\cdot, \beta_i)$, subject to his budget constraint, $\sum_{j \in N} \alpha_{ij}[p(x_j + w_j)] = W_i(p, \alpha)$. That is,

$$\max \sum_{j=1}^{I} \sum_{l=1}^{L-1} \beta_{ij}^l \log\left(x_{ij}^l + w_j^l\right) + \sum_{j \neq i} \log\left(x_{ij}^L + w_j^L\right) + 2 \log\left(x_{ii}^L + w_i^L\right)$$

subject to

$$\sum_{l=1}^{L} \sum_{j=1}^{I} \alpha_{ij} p^l \left(x_{ij}^l + w_j^l\right) = W_i(p, \alpha)$$

Since all utility functions are forms of the Cobb–Douglas functions and $\alpha_{ij} p^l$ can be regarded as the price of commodity x_{ij}^l, the net demand functions for goods x_{ij}^l are given by

$$x_{ij}^l(p, \alpha) = \begin{cases} \frac{\beta_{ij}^l}{\alpha_{ij} p^l (L+I)} W_i(p, \alpha) - w_j^l, & \text{if } l = 1, \ldots, L-1 \\ \frac{2}{\alpha_{ij} p^L (L+I)} W_i(p, \alpha) - w_j^l, & \text{if } j = i \text{ and } l = L \\ \frac{1}{\alpha_{ij} p^L (L+I)} W_i(p, \alpha) - w_j^l, & \text{if } j \neq i \text{ and } l = L \end{cases} \tag{27}$$

which means

$$(28)\qquad \alpha_{ij} p^l\left[x_{ij}^l(p,\alpha)+w_j^l\right]=\begin{cases}\frac{\beta_{ij}^l}{L+I}W_i(p,\alpha), & \text{if } l=1,\ldots,L-1\\ \frac{2}{L+I}W_i(p,\alpha), & \text{if } j=i \text{ and } l=L\\ \frac{1}{L+I}W_i(p,\alpha), & \text{if } j\neq i \text{ and } l=L\end{cases}$$

for $i=1,\ldots,I$.

Summing over i on both sides of (28), and noting that $x_{1j}^l=x_{2j}^l=\cdots=x_{Ij}^l=:x_j^l$ for $j=1,\ldots,I$ and $l=1,\ldots,L$ at the distributive Lindahl equilibrium and $\sum_{i=1}^I \alpha_{ij}=1$ for $i=1,\ldots,I$, we have

$$(29)\quad x_j^l(p)=\begin{cases}\frac{1}{p^l(L+I)}\sum_{i=1}^I \beta_{ij}^l W_i(p,\alpha)-w_j^l, & \text{if } l=1,\ldots,L-1\\ \frac{1}{p^L(L+I)}\left[\sum_{i=1}^I p\cdot w_i+W_j(p,\alpha)\right]-w_j^L, & \text{if } l=L\end{cases}$$

for $j=1,\ldots,I$ by noting that $\sum_{i=1}^I W_i(p,\alpha)=\sum_{i=1}^I pw_i$.

Substituting (29) into (28) and summing over $l=1,\ldots,L$, we have

$$(30)\qquad \alpha_{ij}=\begin{cases}\frac{\left[\sum_{l=1}^{L-1}\beta_{ij}^l+1\right]W_i(p,\alpha)}{\sum_{t=1}^I\sum_{l=1}^{L-1}\beta_{tj}^l W_t(p,\alpha)+\sum_{t=1}^I p\cdot w_t+W_j(p,\alpha)} & \text{if } j\neq i\\ \frac{\sum_{l=1}^{L-1}\left[\beta_{ij}^l+1\right]W_i(p,\alpha)+W_j(p,\alpha)}{\sum_{t=1}^I\sum_{l=1}^{L-1}\beta_{tj}^l W_t(p,\alpha)+\sum_{t=1}^I p\cdot w_t+W_j(p,\alpha)} & \text{if } j=i\end{cases}$$

for $i,\, j=1,\ldots,I$.

Summing the demand functions in Equation (29) over j and noting that $\sum_{j=1}^I \beta_{ij}=1$ and $\sum_{j=1}^I W_i(p,\alpha)=p\sum_{j=1}^I w_j$ by definition, we have the aggregate excess demand function $\hat{z}:\Delta^{L-1}\to\mathbb{R}$ defined by

$$(31)\quad \hat{z}^l(p)=:\sum_{j=1}^I x_j^l(p)=\begin{cases}\frac{1}{p^l(L+I)}\sum_{i=1}^I p^l w_i^l-w_j^l, & \text{if } l=1,\ldots,L-1\\ \frac{I+1}{p^L(L+I)}\sum_{i=1}^I p^l w_i^l-w_j^L, & \text{if } l=L\end{cases}.$$

Since the aggregate excess demand function $\hat{z}$ is continuous on Δ^{L-1} and satisfies Walras' law $p\hat{z}(p)=0$, we know that there is a price vector $p\in\Delta^{L-1}$ such that $\hat{z}(p)=0$ (cf. Varian, 1992, p. 321). Also, since the function g, defined by

$$(32)\qquad g_{ij}(\alpha)=\begin{cases}\frac{\left[\sum_{l=1}^{L-1}\beta_{ij}^l+1\right]W_i(p,\alpha)}{\sum_{t=1}^I\sum_{l=1}^{L-1}\beta_{tj}^l W_t(p,\alpha)+\sum_{t=1}^I p\cdot w_t+W_j(p,\alpha)} & \text{if } j\neq i\\ \frac{\sum_{l=1}^{L-1}\left[\beta_{ij}^l+1\right]W_i(p,\alpha)+W_j(p,\alpha)}{\sum_{t=1}^I\sum_{l=1}^{L-1}\beta_{tj}^l W_t(p,\alpha)+\sum_{t=1}^I p\cdot w_t+W_j(p,\alpha)} & \text{if } j=i\end{cases}$$

for $i,\ j=1,\ldots,I$, is a continuous mapping from the I^2-1-dimensional unit simplex Δ^{I^2-1} to itself, by the Brouwer fixed-point theorem, there is some $\alpha\in\Delta^{I^2-1}$ such that $\alpha=g(\alpha)$. Thus, the price–share system $(p,\,\alpha)$ is a distributive

Lindahl price–share equilibrium system, and therefore $(p, \alpha, x(\cdot))$ is a distributive Lindahl equilibrium for every economy $e \in E^c$.

We now show that every economy $e \in E^c$ has a unique distributive Lindahl equilibrium. There are two cases to be considered: (1) $W_i(p, \alpha)) = W_i(p)$, and (2) $W_i(p, \alpha) = \sum_{j \in N} \alpha_{ij}[p \cdot w_j]$.

For the case of $W_i(p, \alpha)) = W_i(p)$, since the demand functions and cost-sharing functions given by (29) and (30) are single-valued in all parameters and variables, we only need to show that there is a unique equilibrium price vector p for every economy $e \in E^c$. Indeed, this is true by noting that all goods are gross substitutes at any price $p \in \mathbb{R}^L_{++}$, i.e., $\frac{\partial x^l_j(p)}{\partial p^s} = \frac{1}{Lp^l} \sum_{i=1}^I \beta^l_{ij} \frac{\partial W_i(p)}{\partial p^s} > 0$ for $s \neq l$, and thus the distributive Lindahl equilibrium price vector must be unique for every economy e (cf. Varian, 1992, p. 395), and consequently, by (29) and (30), distributive Lindahl equilibrium consumption x and share system α_{ij} are unique for every economy $e \in E^c$.

For the case of $W_i(p, \alpha) = \sum_{j \in N} \alpha_{ij}[pw_j]$, since $q^l_{ij} =: \alpha_{ij} p^l$ can be regarded as the price of commodity x^l_{ij}, by (27), the net demand functions for goods x^l_{ij} can be written as

$$x^l_{ij}(q) = \begin{cases} \frac{\beta^l_{ij}}{q_{ij}(I+L)} \sum_{l=1}^L \sum_{t=1}^I q^l_{it} w^l_t - w^l_j, & \text{if } l = 1, \ldots, L-1 \\ \frac{2}{q^L_j(I+L)} \sum_{l=1}^L \sum_{t=1}^I q^l_{it} w^l_t - w^l_j, & \text{if } j = i \text{ and } l = L \\ \frac{1}{q^L_j(I+L)} \sum_{l=1}^L \sum_{t=1}^I q^l_{it} w^l_t - w^l_j, & \text{if } j \neq i \text{ and } l = L \end{cases} \tag{33}$$

Similarly, we can show that the aggregate excess demand functions $\hat{z}^l(q) = \sum_{i=1}^I \sum_{j=1}^I x^l_{ij}(q)$ and $\hat{z}^l(p, \alpha) = \sum_{i=1}^I \sum_{j=1}^I x^l_{ij}(p, \alpha)$ are gross substitutes in q and p, respectively, the equilibrium price vectors q and p with $\hat{z}(q) = 0$ and $\hat{z}(p, \alpha) = 0$ are uniquely determined and thus $\alpha_{ij} = \frac{q^l_{ij}}{p^l}$ are uniquely determined. Thus, for both forms of $W_i(p, \alpha)$, we have shown that the distributive Lindahl equilibrium is unique for every pure exchange nonmalevolent economy in E^c. ■

Thus, in the remainder of this section, we assume that the initial wealth distribution function $W(p, \alpha)$ is given either by $W_i(p, \alpha) = W_i(p)$ with $\frac{\partial W_i}{\partial p^l} > 0$ for all $l = 1, \ldots, L$, or by $W_i(p, \alpha) = \sum_{j \in N} \alpha_{ij}[p \cdot w_j]$ so that there exists a unique distributive Lindahl equilibrium.

LEMMA 3. *Let μ^c_{DL} be the distributive Lindahl equilibrium message correspondence on E^c. Then, μ^c_{DL} is a continuous function.*

PROOF. By Lemma 2, we know $\mu^c_{DL} = (p, \alpha, x, w)$ is a (single-valued) function. Also, from (29) and (30), we know that the net demand function $x(p, \xi)$ and the share system $\alpha(p, \xi)$ are continuous in p and $\xi := (\sigma, \gamma, \rho)$. w is clearly continuous in ξ. So we only need to show that the price vector p is a continuous function on E^c. Since the demand function $x(p; \xi)$ and share system $\alpha(p; \xi)$ are homogeneous of degree zero in p, we can normalize the price system as an element in the compact simplex set $\bar{\Delta}^{L-1} = \{p \in \mathbb{R}^L_+ : \sum_{l=1}^{L} p^i = 1\}$.

Let $\{e(k)\}$ be a sequence in E^c and $e(k) \to e \in E^c$. Since any economy in E^c is fully specified by the parameter vector ξ, $e(k) \to e$ implies $\xi(k) \to \xi$.

Let $\mu^c_{DL} = (p, \alpha(p;\xi), x(p;\xi), w)$ and $\mu^c_{DL}(k) = (p(k), \alpha(p(k);\xi(k)), x(p(k);\xi(k)), w(k))$. Then we have

$$\sum_{i=1}^{I} x_i(p;\xi) = 0 \tag{34}$$

Since the sequence $\{p(k)\}$ is contained in the compact set $\bar{\Delta}^{L-1}$, there exists a convergent subsequence $\{p(k_t)\}$ that converges to, say, $\bar{p} \in \bar{\Delta}^{L-1}$ and $\hat{x}(p(k_t);\xi(k_t)) =: \sum_{i=1}^{I} x_i(p(k_t);\xi(k_t)) = 0$. Since $\alpha_{ij}(p(k);\xi(k))$ and $x_i(p(k);\xi(k))$ are continuous in ξ, $\hat{x}(p;\xi)$ is continuous in ξ and thus we have $\hat{x}(p(k_t);\xi(k_t)) \to \hat{x}(\bar{p};\xi)$ as $k_t \to \infty$ and $\xi(k_t) \to \xi$. However, since every $e \in E^c$ has the unique distributive Lindahl equilibrium price vector p that is completely determined by $\hat{x}(p;\xi) = 0$, so we must have $\bar{p} = p$. ∎

Lemma 4. *Let μ_{DL} be the distributive Lindahl equilibrium message correspondence on $\tilde{E} \subset E$ that includes E^c_0 as a subset. Then $\mu_{DL}(\tilde{E})$ is homeomorphic to E^c_0.*

Proof. By Lemma 2, we know that μ^c_{DL} is the restriction of μ_{DL} to E^c_0. We first prove that the inverse of μ^c_{DL}, $(\mu^c_{DL})^{-1}$ is a function.

Let $m \in \mu_{DL}(E^c_0)$ and let $e, e' \in (\mu_{DL})^{-1}(m)$. Then $m \in \mu^c_{DL}(e) \cap \mu^c_{DL}(e') = \mu^c_{DL}(e'_i, e_{-i}) \cap \mu^c_{DL}(e_i, e'_{-i})$ for all $i = 1, \ldots, N$ by Remark 1. Let $x = h^c_{DL} \in DL(E^c_0)$ be the distributive Lindahl outcome function. Since u_i is monotonically increasing on E^c_0, we know x is Pareto-efficient by Theorem 2 of Bergstrom (1970). Then, the allocation process $\langle M_{DL}, \mu^c_{DL}, h^c_{DL}\rangle$ is privacy preserving and nonwasteful over E^c_0 with respect to $\mathcal{P}$. Furthermore, by Lemma 1, $e = e'$ and thus $(\mu^c_{DL})^{-1}$ is a function. Also, by Lemma 3, μ^c_{DL} is a continuous function. Therefore, μ^c_{DL} is a continuous one-to-one function on E^c_0.

Since every e is fully characterized by $(w, \sigma, \gamma, \rho) \in \mathbb{R}^{I(I+L-2)}_{++}$, the topological space $(E^c_0, \mathcal{T}_p)$ is homeomorphic to the finite-dimensional Euclidean space $\mathbb{R}^{I(I+L-2)}$. Thus, it must be homeomorphic to any open ball centered on any of its points, and also locally compact. It follows that for any $e \in E^c_0$, we can find a neighborhood $N(e)$ of e and a compact set $\bar{N}(e) \subset N(e)$ with a nonempty interior point. Since μ^c_{DL} is a continuous one-to-one function on $N(e)$, μ^c_{DL} is a continuous one-to-one function from the compact space $\bar{N}(e)$ onto the Euclidean (and hence Hausdorff topological) space $\mu^c_{DL}(\bar{N}(e))$. Hence, it follows that the restriction μ^c_{DL} restricted to $N(e)$ is a homeomorphic imbedding on $\bar{N}(e)$ by Theorem 5.8 in Kelley (1955, p. 141). Choose an open ball $\mathring{N}(e) \subset \bar{N}(e)$. Then $\mathring{N}(e)$ and $\mu_c(\mathring{N}(e))$ are homeomorphic by a homeomorphism $\mu^c_{DL} \mid \mathring{N}(e) : \mathring{N}(e) \to \mu^c_{DL}(\mathring{N}(e))$. This, together with the fact that E^c_0 is homeomorphic to its open ball $\mathring{N}(e)$, implies that

E_0^c is homeomorphic to $\mu_{DL}^c(\mathring{N}(e)) \subset M$, implying that $\mu_{DL}^c(\tilde{E})$ can be homeomorphically imbedded in $\mu_{DL}(\tilde{E})$.

Finally, by Remark 2, the distributive Lindahl message space M_{DL} is contained within the Euclidean space of dimension $I(I + L - 2)$. This necessarily implies that M_{DL} and thus $\mu_{DL}(E_0^c)$ is homeomorphic to $\mathbb{R}^{I(I+L-2)}$ because his restriction $\mu_{DL}^c(E_0^c)$ is homeomorphic to $\mathbb{R}^{I(I+L-2)}$, and consequently, $\mu_{DL}(\tilde{E})$ is homeomorphic to E_0^c. ■

From the above lemmas and Theorem 1, we have the following theorem that establishes the informational efficiency of the distributive Lindahl mechanism within the class of all resource allocation mechanisms that are informationally decentralized and nonwasteful over any subclass of pure exchange economies $\tilde{E} \subset E^{DL}$.

THEOREM 2 (INFORMATIONAL EFFICIENCY THEOREM). *Suppose that $\langle M, \mu, h\rangle$ is an allocation mechanism on any subclass of pure exchange nonmalevolent economies $\tilde{E}$ with $E^c \subset \tilde{E} \subset E^{DL}$ such that*

(i) it is informationally decentralized;
(ii) it is nonwasteful with respect to $\mathcal{P}$;
(iii) M is a Hausdorff topological space;
(iv) μ is locally threaded at some point $e \in E_0^c$.

Then, the size of the message space M is at least as large as that of the distributive Lindahl mechanism, that is, $M \geqq_F M_{DL} =_F \mathbb{R}^{I(I+L-2)}$.

PROOF. Let $\langle M_{DL}, \mu_{DL}, h_{DL}\rangle$ be the distributive Lindahl mechanism and let E_0^c be the special class of economies defined in the previous section. Since $DL(e) \neq \emptyset$ for all $e \in \tilde{E} \subset E^{DL}$, the distributive Lindahl mechanism is well defined. Furthermore, since u_i are locally nonsatiated on E by assumption, we know x is Pareto-efficient by Theorem 2 of Bergstrom (1970). Then, the distributive Lindahl process $\langle M_{DL}, \mu_{DL}, h_{DL}\rangle$ is privacy preserving and nonwasteful over $\tilde{E}$.

Also, as was noted above, E_0^c is homeomorphic to $\mathbb{R}^{I(I+L-2)}$, and by Lemma 4, M_{DL} is homeomorphic to E_0^c. Thus, by Theorem 1, we have $M \geqq_F M_{DL} =_F \mathbb{R}^{I(I+L-2)} =_F E_0^c =_F M_{DL}$. ■

REMARK 4. In particular, if we let $\tilde{E} = E^B$, which is the set of pure exchange economies satisfying Bergstrom's sufficient conditions for the existence of distributive Lindahl equilibrium, we have $E^c \subset E^B \subset E^{DL}$, and thus, by Theorem 2, the distributive Lindahl mechanism is informationally efficient within the class of all resource allocation mechanisms that are informationally decentralized and nonwasteful on E^B.

REMARK 5. It is of interest to compare the dimension of the distributive Lindahl process with that of the Lindahl process. If we regard the consumption bundles of all consumers $x_1, \ldots, x_I$ as a consumption bundle of IL public goods $x = (x_1, \ldots, x_I) \in \mathbb{R}^{IL}$, then the dimension of the Lindahl mechanism is $I(IL-1)$. As $I(IL-1) - I(I+L-2) = I(L-1)(I-1) \geqq 0$, with strict inequality if $I > 2$ and $L > 1$, it follows that the message space of the distributive Lindahl is informationally smaller than that of the Lindahl mechanism if $L > 1$ and $I > 2$. Thus, for economies including nonmalevolent preference externalities, this is consistent with Theorem 2. As for the Pigouvian mechanism, since the transfer system t is an element in $\mathbb{R}^{LI^2}$ whereas the Lindahl share system α is an element in $\mathbb{R}^{I^2}$, the message space of the Pigouvian is much larger than that of the distributive Lindahl mechanism.

6. THE UNIQUENESS THEOREM

In order to guarantee that a distributive Lindahl allocation is individually rational, in this section, we assume the initial wealth distribution function is given by $W_i(p, \alpha) = \sum_{j \in N} \alpha_{ij}[p \cdot w_j]$ for $i = 1, \ldots, I$. We will establish that the size of the message space of any other informationally decentralized mechanism that achieves Pareto-optimal and individually rational allocations is larger than that of the distributive Lindahl mechanism over the class of pure exchange nonmalevolent economies. As a result, the distributive Lindahl allocation process with respect to the initial wealth distribution function $W_i(p, \alpha)$ with $W_i(p, \alpha) = \sum_{j \in N} \alpha_{ij}[p \cdot w_j]$ is the unique informationally efficient decentralized mechanism among mechanisms that achieve Pareto-optimal and individually rational allocations over the class of pure exchange nonmalevolent economies.

We first want to indicate that a Pareto-efficient allocation can be characterized by the following notion:

A price–share system $(p, \alpha) \in \Delta^{L-1} \times A$ is called an *efficiency price–share system* for a Pareto-optimal allocation x if $\sum_{j=1}^{I} \alpha_{ij} p x_j \leq \sum_{j=1}^{I} \alpha_{ij} p x'_j$ for all x' such that $x' + w \in \mathbb{R}^{IL}_{+}$ and $x' R_i x$ for all $i = 1, \ldots, I$.

We say that x is an equilibrium relative to the price–share system (p, α). When preference orderings are characterized by Cobb–Douglas-type utility functions, all the conditions imposed in Lemma 5 of Bergstrom (1970) are satisfied, and thus we know that every Pareto-optimal allocation x in E^c has an efficiency price–share associated with it under the assumptions we have imposed in the article.

We first show the following lemmas.

LEMMA 5. *Suppose that $\langle M, \mu, h\rangle$ is an allocation mechanism on the special class of nonmalevolent economies E^c such that*

(i) it is informationally decentralized;
(ii) it is nonwasteful with respect to $\mathcal{P}$;
(iii) it is individually rational.

Then, there is a function $\phi : \mu(E^c) \to \Delta^{L-1} \times A \times X$ defined by $\phi(m) = (p, \alpha, x)$ where $x = h(m)$ and (p, α) is an efficiency price–share system for the allocation x. In particular, $p\alpha_i$ is proportional to $Du_i(x + w, \beta_i)$ for consumer $i = 1, \ldots, I$ and for each $e \in E^c$ with $m \in \mu(e)$. Here $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{iI})$ is the $1 \times I$ row vector *and p is the $L \times 1$ column vector so that the product $p\alpha_i$ is a $L \times I$ matrix, and $Du_i(x + w, \beta_i)$ is the $L \times I$ matrix with $\frac{\partial u_i}{\partial x_j^l}$ as its element in the* lth *row and* jth *column.*

Proof. Let $e \in E^c$, let $m \in \mu(e)$, and let $x = h(m)$. Since x is Pareto optimal, there exists an efficiency price–share system $(p, \alpha) \in \Delta^{L-1} \times A$ so that $\phi(m) = (p, \alpha, x)$ is an equilibrium relative to the price–share system (p, α). Also, since x is individually rational and utility functions $u_i(x)$ are Cobb–Douglas, we have $(x_i + w_i) \in \mathbb{R}^L_{++}$. Therefore, $p\alpha_i$ must be proportional to $Du_i(x + w, \beta_i)$ for $i = 1, \ldots, I$. Let $e' \in E^c$ be any other environment with $m \in \mu(e')$. Since $\langle M, \mu, h\rangle$ is privacy preserving, by Remark 1, we have $m \in \mu(e'_i, e_{-i})$ for each $i \in N$. Therefore, $x \in h[\mu(e'_i, e_{-i})]$ for each $i \in N$, and thus $p\alpha_i$ is proportional to $Du_i(x + w, \beta'_i)$ for $i = 1, \ldots, I$. Thus, ϕ is well defined. ■

Next we want to show that the equilibrium message cannot reveal any more than the allocation x and the efficiency price–share system (p, α). In other words, if the message space M has minimal dimension and μ is a continuous function, then ϕ is a one-to-one mapping.

Lemma 6. *Suppose that $\langle M, \mu, h\rangle$ is an allocation mechanism on the class of pure exchange nonmalevolent economies E^c such that*

(i) it is informationally decentralized;
(ii) it is nonwasteful with respect to $\mathcal{P}$;
(iii) it is individually rational;
(iv) M is a $I(I + L - 2)$-dimensional manifold;
(v) μ is a continuous function on E^c.

Let $\bar{e} \in E^c$ and let $(\bar{p}, \bar{\alpha}, \bar{x}) = \phi[\mu(\bar{e})]$, where ϕ is defined as Lemma 5. If e^* is any environment such that $w_i^* + \bar{x}_i > 0$, and $Du_i(\bar{x} + w^*, \beta_i^*)$ is proportional to $(\bar{p}, \bar{\alpha}_i)$ for consumer $i = 1, \ldots, I$, then $\mu(e^*) = \mu(\bar{e})$. In particular, ϕ is a one-to-one function.

Proof. For $\bar{e} \in E^c$, define a subset E_0^c of E^c by $E_0^c = \{e \in E^c : w_i = \bar{w}_i)\}$. Note that, by Lemma 1, for each $e, e' \in E_0^c$, if for some $x \in X$, $Du_i(x + w, \beta_i)$ is proportional to $Du_i(x + w, \beta'_i)$ for consumer $i = 1, \ldots, I$, $\sigma = \sigma'$, $\gamma = \gamma'$, and $\rho = \rho'$. Thus, by Lemma 5, $\phi\mu$ is one-to-one on E_0^c and so μ is one-to-one on E_0^c.

Since E_0^c is homeomorphic to $\mathbb{R}^{I(I+L-2)}$, we can consider E_0^c as an open subset of $\mathbb{R}^{I(I+L-2)}$. Let $N(\bar{e})$ be an open neighborhood of $\bar{e}$ in E_0^c and let $\bar{m} = \mu(\bar{e})$. Since M is a $I(I + L - 2)$-dimensional manifold, M and E_0^c are manifolds of the same dimension. Also, since μ is a one-to-one function on $N(\bar{e})$, $V(\bar{m}) \equiv \mu(N(\bar{e}))$ is an open neighborhood of $\bar{m}$ and μ is a homeomorphism on $N(\bar{e})$ to $V(\bar{m})$ by Exercise 18.10 in Greenberg (1967, p. 82). Let $n = I(I + L - 2)$, and let $H_n(M, M - \bar{m})$ denote the nth singular homology module of M relative to $M - \bar{m}$, with integral coefficients. Then $H_n(M, M - \bar{m})$ is isomorphic to Z (denoted by $H_n(M, M - \bar{m}) \cong Z$), the set of integers, and the homomorphism $i_* : H_n(V(\bar{m}), V(\bar{m}) - \bar{m}) \to H_n(M, M - \bar{m})$

induced by the inclusion $i : (V(\bar{m}), V(\bar{m}) - \bar{m}) \to (M, M - \bar{m})$ is isomorphism (see Greenberg, 1967, p. 111).

Also, by the hypothesis, $Du_i(\bar{x} + w^*, \beta_i)$ is proportional to $Du_i(\bar{x} + \bar{w}, \bar{\beta}_i)$ for consumer $i = 1, \dots, I$, so there is some $\lambda_i^* > 0$, one for each i, such that for consumer $i = 1, \dots, I$

$$\frac{\beta_{ij}^*}{\bar{x}_j^l + w_j^{*l}} = \lambda_i^* \frac{\bar{\beta}_{ij}}{\bar{x}_j^l + \bar{w}_j^l}, \quad j = 1, \dots, I, l = 1, \dots, L-1 \tag{35}$$

$$\frac{1}{\bar{x}_i^L + w_i^{*L}} = \lambda_i^* \frac{1}{\bar{x}_i^L + \bar{w}_i^L} \tag{36}$$

Define the function $G: N(\bar{e}) \times [0, 1] \to E^c$ by $G((\bar{w}, \sigma, \gamma, \rho), t) = (w', \sigma', \gamma', \rho')$, where

$$w_i'^l = t w_i^{*l} + (1 - t)\bar{w}_i^l, \quad l = 1, \dots, L-1$$

$$w_i'^L = \frac{\bar{x}_i^L + \bar{w}_i^L}{t\lambda_i^* + (1 - t)} - \bar{x}_i^L$$

$$\gamma_i' = t\gamma_i^* + (1 - t)\gamma_i$$

$$\rho_i' = t\rho_i^* + (1 - t)\rho_i$$

$$\sigma_{ij}'^l = \begin{cases} \frac{1}{\gamma_i'^l}\Big\{I\big[\beta_{ij}^l\big(t\bar{\xi}_{ij}^l + (1 - t)\xi_{ij}^l\big)\big]\frac{\bar{x}_j + w_j'^l}{\bar{x}_j^l + \bar{w}_j^l}\big[t\lambda_i^* + (1 - t)\big] - 1 \\ \qquad\qquad - \big(\rho_{ij}' - \frac{1}{2}\big)\big(\gamma_i'^l - \frac{1}{L-1}\big)\Big\} - \frac{1}{I} & \text{if } j = i, i+1 \\ \frac{1}{\gamma_i'^l}\Big\{I\big[\beta_{ij}^l\big(t\bar{\xi}_{ij}^l + (1 - t)\xi_{ij}^l\big)\big]\frac{\bar{x}_j + w_j'^l}{\bar{x}_j^l + \bar{w}_j^l}\big[t\lambda_i^* + (1 - t)\big] - 1\Big\} - \frac{1}{I} & \text{otherwise} \end{cases}$$

for $i, j = 1, \dots, I, l = 1, \dots, L-1$, so that, by the construction of β_{ij}^l given in (6), we have

$$\beta_{ij}'^l = \beta_{ij}^l\big(t\bar{\xi}_{ij}^l + (1 - t)\xi_{ij}^l\big)\frac{\bar{x}_j + w_j'^l}{\bar{x}_j^l + \bar{w}_j^l}\big[t\lambda_i^* + (1 - t)\big] \tag{37}$$

where $\xi_{ij}^l = (\sigma_{ij}^l, \gamma_{ij}^l, \rho_{ij}^l)$ and $\beta_{ij}^l(t\bar{\xi}_{ij}^l + (1 - t)\xi_{ij}^l)$ is defined by substituting $t\bar{\xi}_{ij}^l + (1 - t)\xi_{ij}^l$ into (6).

Then, $G(\cdot, 0)$ is the inclusion map by noting that $G((\bar{w}, \sigma, \gamma, \rho, 0) = (\bar{w}, \sigma, \gamma, \rho)$ for all $(\bar{w}, \sigma, \gamma, \rho) \in N(\bar{e})$, and $G(\cdot, 1)$ is the constant map on $N(\bar{e})$ to e^* by noting that $G((\bar{w}, \sigma, \gamma, \rho), 1) = (w^*, \sigma^*, \gamma^*, \rho^*)$ for all $(\bar{w}, \sigma, \gamma, \rho) \in N(\bar{e})$. Let $t \in [0, 1]$, $(\bar{w}, \sigma, \gamma, \rho) \in N(\bar{e})$ with $(\bar{w}, \sigma, \gamma, \rho) \neq \bar{e}$, and $(w', \sigma', \gamma', \rho') = G((\bar{w}, \sigma, \gamma, \rho), t)$. Then $\frac{\beta_{ij}'^l}{\bar{x}_j^l + w_j'^l}$ is proportional to $\frac{\beta_{ij}^l(t\bar{\xi}_{ij}^l + (1-t)\xi_{ij}^l)}{\bar{x}_j^l + \bar{w}_j^l}$ for $i, j = 1, \dots, I$, and $l = 1, \dots, L-1$, and $\frac{1}{\bar{x}_j^L + w_j'^L}$ is proportional to $\frac{1}{\bar{x}_i^L + \bar{w}_i^L}$ for $i = 1, \dots, I$. Then environment $(\bar{w}, t\bar{\sigma} + (1 - t)\sigma, t\bar{\gamma} + (1 - t)\gamma, t\bar{\rho} + (1 - t)\rho) \in E_0^c$, so by the argument in the first paragraph, for each $t < 1$, $\mu[G(e, t)] = \bar{m}$ only if $e = \bar{e}$.

Now, suppose by way of contradiction that $\mu(e^*) \neq \bar{m}$. Then $\mu[G(e,t)] \neq \bar{m}$ whenever $e \neq \bar{e}$. Define $G' : (V(\bar{m}), V(\bar{m}) - \bar{m}) \times [0,1] \to (M, M - \bar{m})$ by $G'(m, t) = \mu(G[\mu^{-1}(m), t])$. Then G' is a homotopy between the inclusion $i : (V(\bar{m}), V(\bar{m}) - \bar{m}) \to (M, M - \bar{m})$ and the constant map $j : (V(\bar{m}), V(\bar{m}) - \bar{m}) \to (m^*, M - \bar{m})$, where $m^* = \mu(e^*)$. Hence, $i_* = j_*$ and consequently i_* is the zero homomorphism, which contradicts the fact that i_* is isomorphic to Z. ■

THEOREM 3 (THE EQUIVALENCE THEOREM). *Suppose that $\langle M, \mu, h\rangle$ is an allocation mechanism on the class of pure exchange nonmalevolent economies E^c such that*

(i) it is informationally decentralized;
(ii) it is nonwasteful with respect to $\mathcal{P}$;
(iii) it is individually rational;
(iv) M is a $I(I + L - 2)$ dimensional manifold;
(v) μ is a continuous function on E^c.

Let $\langle M_{DL}, \mu_{DL}, h_{DL}\rangle$ be the distributive Lindahl mechanism with the initial wealth distribution function given by $W_i(p, \alpha) = \sum_{j\in N} \alpha_{ij}[p \cdot w_j]$. Then, there is a homeomorphism ϕ on $\mu(E^c)$ to M_{DL} such that

(a) $\mu_{DL} = \phi \cdot \mu$;
(b) $h_{DL} \cdot \phi = h$.

The conclusion of the theorem is summarized in the following commutative homeomorhpism diagram:

$$\begin{array}{ccc} E & \xrightarrow{\mu_c} & M_c \\ \Big\downarrow \mu & \phi \nearrow \swarrow \phi^{-1} & \Big\downarrow h_c \\ \mu(E) & \xrightarrow{h} & Z \end{array}$$

PROOF. Let $\phi : M \to \Delta^{L-1} \times A \times X$ be the function defined as Lemma 5. We first show that $\mu_{DL} = \phi \cdot \mu$. Suppose, by way of contradiction, that $\mu_{DL}(e) \neq \phi[\mu(e)]$ for some $e \in E^c$. Let $(p, \alpha, x) = \phi[\mu(e)]$. Then x is not a distributive Lindahl equilibrium allocation, i.e., $x \neq x \in DL(e)$. Then, for some i, we must have $\sum_{j=1}^{I} \alpha_{ij} p[x_j + w_j] < \sum_{j=1}^{I} \alpha_{ij} p w_j$, and consequently we have $\sum_{j=1}^{I} \alpha_{ij} p x_j < 0$. Let $w_i^* = (1/t)(w_i + x_i) - x_i$ for $i = 1, \ldots, I$, $e_i^* = (w_i^*, \beta_i)$, let $e^* = (e_1^*, \ldots, e_I^*)$, and let $g(t) = u_i[(1-t)x + w] - u_i(w + x)$ where $0 < t < 1$. Then, $g(t) \to 0$ as $t \to 0$, and $dg/dt = \sum_{j=1}^{I} \sum_{l=1}^{L} \frac{\partial u_i[(1-t)x + w)]}{\partial x_j^l}(-x_j^l) \to \sum_{j=1}^{I} \sum_{l=1}^{L} \frac{\partial u_i(x+w)}{\partial x_j^l}(-x_j^l) = -\lambda \sum_{j=1}^{I} \alpha_{ij} p \cdot x_j > 0$ as $t \to 0$ by noting that $Du_i[w + x]$ is proportional to $p\alpha_i$ and $\sum_{j=1}^{I} \alpha_{ij} p x_j < 0$. Thus we have $g(t) = u_i[(1-t)x + w] - u_i(w + x) > 0$, i.e., $u_i[(1-t)x + w] > u_i(w + x)$ when t is a sufficiently small positive number by noting that $g(\cdot)$ is continuous and $g(t) \to 0$ as $t \to 0$. Then, multiplying $1/t$ on the both sides of $u_i[(1-t)x + w] > u_i(w + x)$ and by the homogeneity of preferences represented by $u_i(\cdot)$, we have $u_i[(1/t)(w + x)] < u_i[(1/t)(w + x) - x]$ when t is a sufficiently small positive number. Thus, since $\langle M, \mu, h\rangle$ is individually rational, we must have $\mu(e^*) \neq \mu(e)$; otherwise we have $u_i[w^* + x] = u_i[(1/t)(w + x)] < u_i[(1/t)(w + x) - x] = u_i[w^*]$, which contradicts the hypothesis that $x = h[\mu(e^*)]$ is individually rational. However, on the other hand, x is Pareto optimal for e^*, and $Du_i(w^* + x)$ is proportional to $p\alpha_i$, Lemma 6 implies that $\mu(e^*) = \mu(e)$, which is a contradiction. So we must have $\mu_{DL} = \phi \cdot \mu$. Furthermore, since h_{DL} is the projection $(p, \alpha, x, w) \to x$, it follows that $h_{DL} \cdot \phi = h$.

Now we show that ϕ is a homeomorphism on $\mu(E^c)$ to M_{DL}. By Lemma 6, ϕ is a one-to-one mapping. Also, since $h_{DL} \cdot \phi = h$, the range of ϕ is M_{DL}. So it only remains to be shown that ϕ and ϕ^{-1} are continuous. To show that ϕ^{-1} is continuous, let $\{m(k)\}$ be a sequence in M_{DL}, which converges to some $m \in M_{DL}$ with $m(k) = \mu(e(k))$ for all k and $m \in \mu(e)$. Since $\phi^{-1} \cdot \mu_{DL} = \mu$, $\phi^{-1}(m(k)) = \mu(e(k))$ for all k and $\phi^{-1}(m) = \mu(e)$. Since μ is continuous, $\mu(e(k))$ converges to $\mu(e)$, so ϕ^{-1} is continuous. Since M_{DL} and M are manifolds of the same dimension, ϕ^{-1} is a homeomorphism on M_{DL} to $\phi^{-1}(M_{DL}) = \mu(E^c)$ by Exercise 18.10 in Greenberg (1967, p. 82). ∎

The above theorem is based on the class of Cobb–Douglas utility functions. As a direct corollary of Theorems 2 and 3, we have the following theorem that shows that the distributive Lindahl mechanism is the unique informationally decentralized and informationally efficient process that realizes Pareto-optimal and individually rational allocations over any subclass of pure exchange nonmalevolent economies $\tilde{E}$ with $E^c \subset \tilde{E} \subset E^{DL}$.

THEOREM 4 (THE UNIQUENESS THEOREM). *Suppose that $\langle M, \mu, h\rangle$ is a nondistributive Lindahl allocation mechanism on any subclass of pure exchange economies $E^c \subset \tilde{E} \subset E^{DL}$ such that:*

(i) it is informationally decentralized;
(ii) it is nonwasteful with respect to $\mathcal{P}$;
(iii) it is individually rational;
(iv) μ is a continuous function on E^c.

Then, the size of the message space M is larger than that of the distributive Lindahl mechanism, that is, $M >_F M_{DL} =_F \mathbb{R}^{I(I+L-2)}$. Consequently, the distributive Lindahl mechanism with the initial wealth distribution function given by $W_i(p, \alpha) = \sum_{j \in N} \alpha_{ij}[p \cdot w_j]$ is the unique and informationally decentralized and informationally efficient decentralized mechanism that realizes Pareto-efficient and individually rational allocations on $\tilde{E}$.

PROOF. By Theorem 2, we know that the distributive Lindahl mechanism is informationally the most efficient decentralized mechanism that realizes Pareto-efficient allocations over any subclass of pure exchange nonmalevolent economies $\tilde{E}$ such that $M \geqq_F M_{DL} =_F \mathbb{R}^{I(I+L-2)}$. Furthermore, by Theorem 3, the distributive Lindahl mechanism with the initial wealth distribution function given by $W_i(p, \alpha) = \sum_{j \in N} \alpha_{ij}[pw_j]$ is the unique informationally efficient and decentralized mechanism that realizes Pareto-efficient and individually rational allocations on E^c. Thus, the size of the message space M is larger than that of the distributive Lindahl mechanism, that is, $M >_F M_{DL} =_F \mathbb{R}^{I(I+L-2)}$ on E^c. Therefore, any mechanism that realizes Pareto-efficient and individually rational allocations on E must use a larger message space than the distributive Lindahl mechanism, i.e., $M >_F M_{DL} =_F \mathbb{R}^{I(I+L-2)}$, by noting that $E^c \subset E^{DL}$. Finally, since every distributive Lindahl allocation with the initial wealth distribution function given by $W_i(p, \alpha) = \sum_{j \in N} \alpha_{ij}[pw_j]$ is individually rational, the distributive Lindahl mechanism with $W_i(p, \alpha) = \sum_{j \in N} \alpha_{ij}[pw_j]$ is the unique informationally decentralized and informationally efficient decentralized mechanism that realizes Pareto-efficient and individually rational allocations. ■

As a corollary of Theorem 4, when $\tilde{E} = E^B$, we know that the distributive Lindahl mechanism with the initial wealth distribution function given by $W_i(p, \alpha) = \sum_{j \in N} \alpha_{ij}[pw_j]$ is the unique and informationally decentralized and informationally efficient decentralized mechanism that realizes Pareto-efficient and individually rational allocations on the set of nonmalevolent economies that satisfy the Bergstrom's sufficient conditions for the existence of distributive Lindahl equilibrium.

REMARK 6. The above Uniqueness Theorems, unlike the Efficiency Theorem, are based on the assumption of individual rationality. As in Jordan (1982), a similar example may be constructed to show that this assumption cannot be dispensed with for the distributive Lindahl mechanism to be the unique informationally efficient and nonwasteful mechanism.

7. CONCLUDING REMARKS

In this article, it has been shown that the minimal informational size of the message space for a privacy-preserving and nonwasteful resource allocation process over the class of nonmalevolent economies is the size of the distributive Lindahl message space that equals $I(I + L - 2)$, and thus the distributive Lindahl mechanism is informationally the most efficient decentralized mechanism. Furthermore, it is shown that the distributive Lindahl mechanism is the unique informationally efficient and decentralized mechanism that realizes Pareto-efficient and individually rational allocations over the class of nonmalevolent economies. In particular, since the Pigouvian equilibrium principle can be defined as an informationally decentralized mechanism that realizes Pareto-efficient and individually rational allocations, it has a larger size of the message space. This optimality and uniqueness result on the distributive Lindahl mechanism for nonmalevolent economies is a kind of impossibility theorem: It implies that there exists no other privacy-preserving, individually rational, and nonwasteful resource allocation mechanism that uses a message space whose informational size is smaller than, or the same as, that of the distributive Lindahl message space.

Our results thus indicate that the impossibility results in the presence of externalities may not be as severe as suggested by Hurwicz (1972, 1999). Hurwicz's impossibility result involves the nonconvexity of both the victim's externality effect set and the production transformation set. We think it is the absence of convexities rather than the presence of externalities that posts the greater problem. Convexity and continuity of preferences, together with certain other conditions of a technical nature, permit us to define a distributive Lindahl process even when externalities are present, and such a process is privacy-preserving, nonwasteful, and uses also a finite-dimensional (in fact minimal dimensional) message space. On the other hand, even in the absence of externalities, nonconvexities may cause the nonexistence of a smooth informationally decentralized and nonwasteful mechanism that has a finite-dimensional message space as suggested by Calsamiglia (1977). In fact, as noted by Hurwicz (1999), Baumol and Oates (1988, p. 129) have shown by diagram that the externality transformation set under strong nonconvexities is equivalent to a Calsamiglia increasing returns transformation set. It therefore seems natural that the Calsamiglia type of impossibility would have its counterpart in economies with detrimental externalities. It is the presence of certain nonconvex production sets among admissible economies that enables Hurwicz (1999) to obtain the impossibility result. Hence, the basis for Hurwicz's impossibility result is the presence of nonconvexity of the relevant sets on the production side, and therefore there is no contradiction between Hurwicz's impossibility result and the possibility results of the finite-dimensional message spaces and informational efficiency of the distributive Lindahl mechanism in the present article. Thus, it is convexity that is needed to make it possible that this guarantee be realized through a mechanism using a finite-dimensional message space, in particular, the distributive Lindahl mechanism, thereby avoiding informational complexity of an infinite-dimensional computation.

For the simplicity of exposition, attention has only been confined to pure exchange economies and consequently, to purely consumption externalities in the present article. However, the results may likely be extended to economies with convex production technologies by extending the notion of the distributive Lindahl equilibrium to convex production economies and adopting similar techniques used by Tian (2000a, 2000b). Indeed, as long as externalities do not result in nonconvexities, such an extension is possible. Of course, a question that remains, as pointed out by a referee of this article, is how frequent or rare the convex production sets in the presence of detrimental externalities are. Starrett (1972) argued that such nonconvexities are fundamental in economies with producer firms. As shown by Hurwicz (1995), analogous nonconvexity phenomena can arise due to certain satiation effects. Nevertheless, since there is no empirical evidence concerning the prevalence of situations modeled by Starrett, we may regard the question of frequency as open to debate.

It may be worth pointing out that the reason why this article, like most work in the realization literature, has focused on informational size is assumed to reflect information cost. However, this approach may overlook the role and cost of informational quality, complexity, and incomplete information transmission. As a result, an opaque economic environment, which makes very difficult the access to complex information, may be more cheaply served by a resource allocation mechanism with larger but simpler messages, or messages that represent a finer partition of the set of states of nature. Also, the notion of a stationary message correspondence ignored the issues of its existence, stability, and convergence, but it simply is obtained through ad hoc assumptions. Jordan (1987) showed that, if the properties are considered, the size of the message space of a mechanism has to increase.[9]

Also, in this article, we did not study the incentive aspects of the distributive Lindahl mechanism. Like the Lindahl mechanism, the distributive Lindahl mechanism has the so-called "free-rider" problem, and thus it is not incentive compatible, and thus one needs to propose an incentive mechanism that implements the distributive Lindahl allocations with a solution concept of self-interested behavior. Recently, Tian (2003) amended the definition of the distributive Lindahl equilibrium by allowing free disposal and considered the implementation problem of distributive Lindahl allocations in Nash and strong Nash equilibria when preferences, individual endowments, and coalition patterns among individuals are unknown to the planner. The mechanism constructed in Tian (2003) is a market-type mechanism that implements the constrained distributive Lindahl allocations in Nash and strong Nash equilibria. Although this mechanism implements interior distributive Lindahl allocations, the size of the message space of the mechanism is much larger than that of the distributive Lindahl mechanism, and thus it is not informationally efficient. This is not surprising since the well-known result given by Reichelstein and Reiter (1988) shows that a Nash implementation typically increases the size of the message space of the mechanism.

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[9] I thank an anonymous referee for pointing out these observations.

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