Density: Leading to an Inductive Lesson With Rational Numbers

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In an introductory course of mathematics for elementary school teachers, the density of the rational numbers was being explored. The class was searching for a rational number "between" $\frac{5}{6}$ and $\frac{6}{7}$. After some exploration a student suggested $\frac{11}{13}$ which indeed satisfied the criterion $\frac{5}{6} < \frac{11}{13} < \frac{6}{7}$. However, the interesting question arose from the observation that $\frac{11}{13} = \frac{5+6}{6+7}$ and the question, "Will $\frac{a+c}{b+d}$ always be a rational number between rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, where $\frac{a}{b} \neq \frac{c}{d}$?"

Since the instructor had not addressed this question before, an exploration ensued. The investigation continued that evening as an assignment for the class to (1) try to find a counterexample or (2) check a number of examples that would support the conclusion. From these examples the students were to determine the level of their faith in the "surprising" outcome.

By the next day, several students were convinced of the validity of the outcome, and the author had sketched the proof based on an earlier theorem stating "If $\frac{a}{b}$, $\frac{c}{d}$ are rational numbers where b, d are positive integers, then $\frac{a}{b} < \frac{c}{d}$ if and only if ad < bc".

Theorem: If $\frac{a}{b}$, $\frac{c}{d}$ are rational numbers where b, d are positive integers and $\frac{a}{b} < \frac{c}{d}$, then $\frac{a+c}{b+d}$ is a rational number and $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

Proof: (1) Show $\frac{a+c}{b+d}$ is a rational number. Since a, c are integers then $a + c$ is an integer (integers closed under addition). Since b, d are positive integers, then $b + d$ is a positive integer (positive integers closed under addition). Hence, $\frac{a+c}{b+d}$ is a rational number.

(2) Show $\frac{a}{b} < \frac{a+c}{b+d}$.

Assume $\frac{a}{b} \geq \frac{a+c}{b+d}$. Since $b + d > 0$ and $b > 0$, then $a(b + d) \geq (a + c)b$.

or

$$ab + ad \geq ab + bc$$
$$ad \geq bc.$$

However, since $\frac{a}{b} < \frac{c}{d}$, it is known that ad < bc. This contradiction confirms $\frac{a}{b} < \frac{a + c}{b + d}$.

(3) Show $\frac{a + c}{b + d} < \frac{c}{d}$

By similar argument as (2) above. Therefore, $\frac{a + c}{b + d}$ is a rational number such that $\frac{a}{b} < \frac{a + c}{b + d} < \frac{c}{d}$ where $\frac{a}{b}, \frac{c}{d}$ properly restricted.

Though it may be true that this result is of interest primarily to "discoverers", this reason in itself may provide value to other teachers. The result obtained here was by serendipity, but the outcome could be effectively developed in an inductive fashion. The teacher can set the stage for student discovery by first showing and then having students show several examples of $\frac{a + c}{b + d}$ being between $\frac{a}{b}$ and $\frac{c}{d}$. These examples with appropriate guidance should lead to the discovery of the pattern. After the formulation of the generalization, the students (or the teacher) could proceed to justify the result or outline the proof.

Essentially, of course, an intuitive basis for the validity of the conclusion is based upon the notion of "averaging" the numerators and denominators to locate a rational number "between" the given distinct rationals; i.e., $\frac{a + c}{2} \div \frac{b + d}{2} = \frac{a + c}{b + d}$, or more generally $\frac{a + c}{n} \div \frac{b + d}{n} = \frac{a + c}{b + d} (n \neq 0)$.

There *is* a warning that should be noted by a teacher who plans to use this notion with mathematically immature students. This danger is the tendency for a number of students to try to add rational numbers as follows: $\frac{a}{b} + \frac{c}{d} = \frac{a + c}{b + d}$. Inadvertantly, the result mentioned above *could* reinforce this incorrect algorithm for addition. Of course with appropriate teacher direction it could convince students that such a procedure *is* incorrect (at least for non-negative rational numbers) since the result is "between" the two addends.