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Finite-temperature Casimir effect in piston geometry and its classical limit

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Abstract We consider the Casimir force acting on a d-dimensional rectangular piston due to a massless scalar field with periodic, Dirichlet and Neumann boundary conditions and an electromagnetic field with perfect electric-conductor and perfect magnetic-conductor boundary conditions. The Casimir energy in a rectangular cavity is derived using the cut-off method. It is shown that the divergent part of the Casimir energy does not contribute to the Casimir force acting on the piston, thus renders an unambiguously defined Casimir force acting on the piston. At any temperature, it is found that the Casimir force acting on the piston increases from $-\infty$ to 0 when the separation a between the piston and the opposite wall increases from 0 to ∞. This implies that the Casimir force is always an attractive force pulling the piston towards the closer wall, and the magnitude of the force gets larger as the separation a gets smaller. Explicit exact expressions for the Casimir force for small and large plate separations and for low and high temperatures are computed. The limits of the Casimir force acting on the piston when some pairs of transversal plates are large are also derived. An interesting result regarding the influence of temperature is that in contrast to the conventional result that the leading term of the Casimir force acting on a wall of a rectangular cavity at high temperature is the Stefan–Boltzmann (or black-body radiation) term which is of order T^{d+1}, it is found that the contributions of this term from the two regions separating the piston cancel with each other in the case of piston. The high-temperature leading-order term of the Casimir force acting on the piston is of order T, which shows that the Casimir force has a nontrivial classical $\hbar \to 0$ limit. Explicit formulas for the classical limit are computed.

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1 Introduction

In 1948, Casimir predicted the existence of an attractive force between two perfectly conducting parallel plates, which is due to the zero-point fluctuations of the electromagnetic field between the plates [1, 2]. Nowadays, the Casimir effect is generally referred to in a similar effect due to any quantum fields. Since 1948, thousands of papers due to the Casimir effect have appeared in the literature, and it has gained more and more attraction from physicists and even engineers. Many experiments have been designed to verify the existence of the Casimir force. For example, Mohideen et al. [3, 4] used atomic force microscopy to confirm (within a few percent error) the existence of a Casimir force for plate–sphere separations between 100 nm to 900 nm. At these length scales, the Casimir force becomes non-negligible and therefore people working in nanoscience and nanotechnology start to show interest in this effect. Research has been done on how to use the Casimir force to drive a nanodevice, as well as to eliminate unpleasant effects due to the Casimir force, such as adhesion or stiction.

In the conventional calculation of the Casimir energy, one sums over different modes of zero-point energies in the presence of boundaries:

$$E_{\text{Cas}}^0 = \frac{1}{2}\sum_{\alpha}\omega_{\alpha}. \tag{1.1}$$

However, this sum is divergent and regularization methods are used to obtain a finite result. One way of regularization is by subtracting the vacuum energy due to the infinite background space. However, this procedure does not always lead to a finite quantity, due to the possible hypersurface divergence. As has been pointed out by Deutsch and Candelas [5] and Baacke and Krüsemann [6], it is impossible to compute the Casimir energy with boundary conditions to approximate the interaction of the vacuum fluctuations with the boundary material. This has been further discussed by Graham and Jaffe et al. in a series of papers [7–10]. They argued that the surface divergences cannot be removed by any renormalization of the physical parameters of the theory. In 2004, Cavalcanti [11] showed that despite the presence of the divergence in the Casimir energy, it is possible to obtain an unambiguous finite Casimir force for a special geometric setup known as a piston. He showed that for a two-dimensional massless scalar field theory in a rectangular piston, the surface divergent terms of the Casimir force on the piston due to the two regions divided by the piston cancel each other and the resulting Casimir force acting on the piston is always attractive. Since the original work of Cavalcanti, piston geometry and its variants have attracted considerable interest. In [12, 13], it was shown that for an electromagnetic field with perfect electric-conductor conditions inside a three-dimensional rectangular piston, the surface divergent terms are also canceled and the resulting Casimir force acting on the piston is attractive. The Casimir piston for a three-dimensional electromagnetic field with prefect conductor conditions are studied further in [14–16], where pistons with arbitrary cross sections are considered. In [17], the rectangular piston for a massless scalar field with Dirichlet boundary conditions in three dimensions was discussed in detail. In [18], it was proved that for a massless scalar field with Dirichlet and Neumann boundary conditions, the hypersurface divergent terms of the Casimir force due to the two regions separating a d-dimensional rectangular piston always cancel each other, and the Casimir force is always attractive. The Casimir force on a rectangular piston due to an electromagnetic field with perfect magnetic-conductor conditions are computed and discussed by Edery and Marachevsky [19]. In another recent work [20], the Casimir force for a massless scalar field with Dirichlet boundary conditions on a rectangular piston in the spacetime with extra compactified dimensions was discussed. In [21], it was shown that the Casimir force between two (nonmagnetic) dielectric bodies which are related by reflection is always attractive. This result was generalized by Bachas [22] who showed that reflection positivity implies that the force between any mirror pair of charge-conjugate probes of the quantum vacuum is attractive. The attractive nature of the Casimir force will create undesirable effects such as the collapse of a nano device—an effect known as stiction [23, 24]. Therefore, it becomes desirable to search for circumstances in which the Casimir force can be made less attractive, or even repulsive. In [25], Barton showed that for a thin piston with weakly reflecting dielectrics, the Casimir force at small separations is attractive, but turns to repulsive as the separation increases. In [26], it was shown that in one, two or three dimensions, if one surface assumes the Dirichlet boundary condition and the other assumes the Neumann boundary condition, then the Casimir force on a rectangular piston is repulsive. Another scenario that leads to a repulsive Casimir force was discussed in [27]. We would like to remark that a scenario similar to the piston geometry has been considered by Reuter and Dittrich in 1985 [28] in the context of a mechanism to regularize the Casimir energy. Recently, a similar mechanism has been used to compute the Dirichlet Casimir effect for ϕ^4 theory in $(3+1)$ dimensions [29].

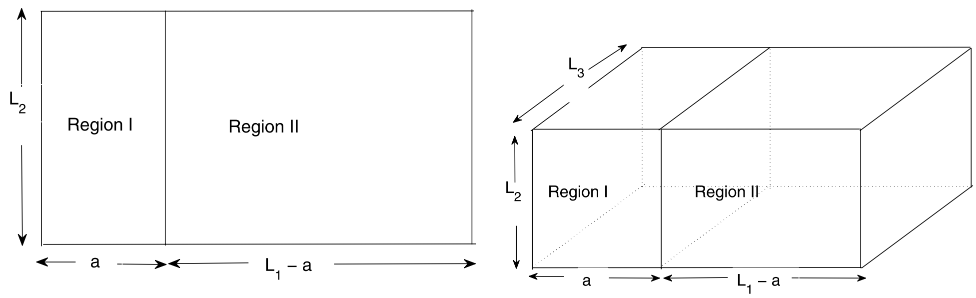
In this paper, we study the Casimir effect of a massless scalar field and electromagnetic field in a d-dimensional rectangular piston at finite temperature. Different boundary conditions are considered including periodic, Dirichlet and Neumann boundary conditions for the massless scalar field and perfect electric-conductor (PEC) and perfect magnetic-conductor (PMC) boundary conditions for the electromagnetic field. In this paper, we propose an alternative method to calculate the cut-off dependent Casimir energy. We notice that the divergent part of the Casimir energy is linear in each of the variables $L_1, \ldots, L_d$—the lengths of the rectangular cavity. This immediately implies that the divergent terms do not contribute to the Casimir force acting on the piston. Therefore, the Casimir force acting on the piston is finite and unambiguous without any renormalization. The thermal effect of the Casimir force has so far only been considered for an electromagnetic field with PEC boundary conditions in three dimensions [13, 15, 16]. However, [13] does not give a correct high-temperature behavior of the Casimir force. In this paper, we derive for the d-dimensional piston geometry explicit and exact formulas of the Casimir force, which can be used to study the high-temperature and low-temperature behaviors of the Casimir force. We show that in the high-temperature regime, the black-body radiation contributions to the Casimir force from the two regions separating the piston cancel with each other. The leading-order term of the Casimir force is of order T, which shows that the Casimir force has a classical limit.

In this paper, we choose units where $\hbar = c = k = 1$.

2 Casimir energy by exponential cut-off method

Consider a d-dimensional rectangular piston, which is a rectangular cavity $[0, L_1] \times [0, L_2] \times \cdots \times [0, L_d]$ separated by a hyperplane $x_1 = a$ (the piston) into two regions: region I of dimension $[0, a] \times [0, L_2] \times \cdots \times [0, L_d]$ and region II of dimension $[a, L_1] \times [0, L_2] \times \cdots \times [0, L_d]$. We are mainly concerned with the case where $L_1 \to \infty$, which implies that region II is open. The case where L_1 remains finite will be discussed in a later section. The two- and three-dimensional rectangular pistons are illustrated in Fig. 2.1. The Casimir force acting on the piston due to the massless scalar field with periodic (P), Dirichlet (D) and Neumann (N) boundary conditions (b.c.), as well as an electromagnetic field with perfect electric-conductor (PEC) b.c. and perfect magnetic-conductor (PMC) b.c. will be computed.

Fig. 2.1 The two- and three-dimensional rectangular pistons



We first recall some basic definitions. The finite-temperature Casimir energy is naively the sum of the zero-temperature Casimir energy (1.1) and a thermal correction:

$$\begin{aligned}E_{\mathrm{Cas}} &= E^0_{\mathrm{Cas}} + \Delta E_{\mathrm{Cas}}\\ &= E^0_{\mathrm{Cas}} + T\sum_{\omega_\alpha\neq 0}\log\bigl(1-\mathrm{e}^{-\omega_\alpha/T}\bigr)\\ &= -T\log\prod_{\omega_\alpha\neq 0}\frac{\mathrm{e}^{-\omega_\alpha/(2T)}}{1-\mathrm{e}^{-\omega_\alpha/T}}.\end{aligned}$$

For the piston system, the Casimir energy is the sum of the Casimir energies of regions I and II and the Casimir energy of the region outside the rectangular cavity:

$$E_{\mathrm{Cas}} = E^{\mathrm{I}}_{\mathrm{Cas}} + E^{\mathrm{II}}_{\mathrm{Cas}} + E^{\mathrm{out}}_{\mathrm{Cas}}. \tag{2.1}$$

Although each of these terms is divergent, we are going to see below that the Casimir force acting on the piston defined by

$$F_{\mathrm{Cas}} = -\frac{\partial}{\partial a}E_{\mathrm{Cas}} = -\frac{\partial}{\partial a}\bigl(E^{\mathrm{I}}_{\mathrm{Cas}} + E^{\mathrm{II}}_{\mathrm{Cas}}\bigr) \tag{2.2}$$

is finite without any renormalization. The term $E^{\mathrm{out}}_{\mathrm{Cas}}$ has dropped out from the last equality of (2.2), since it is independent of a.

Notice that $E^{\mathrm{I}}_{\mathrm{Cas}} = E_{\mathrm{Cas}}(a, L_2, \dots, L_d)$ and $E^{\mathrm{II}}_{\mathrm{Cas}} = E_{\mathrm{Cas}}(L_1 - a, L_2, \dots, L_d)$, where $E_{\mathrm{Cas}}(L_1, L_2, \dots, L_d)$ is the Casimir energy of the rectangular cavity $[0, L_1] \times [0, L_2] \times \cdots \times [0, L_d]$. As was proved in [30, 31], the Casimir energy $E_{\mathrm{Cas}}(L_1, L_2, \dots, L_d)$ for a massless scalar field with Pb.c., Db.c. and Nb.c. and for an electromagnetic field with PEC b.c. and PMC b.c. are related to each other by the following *linear* relations:

$$\begin{aligned}&E^{\mathrm{D/N}}_{\mathrm{Cas}}(L_1, \dots, L_d)\\ &\quad= 2^{-d}\sum_{j=1}^{d}(\mp 1)^{d-j}\\ &\qquad\times\sum_{1\leq m_1<\cdots<m_j\leq d}E^{\mathrm{P}}_{\mathrm{Cas}}(2L_{m_1}, \dots, 2L_{m_j}),\end{aligned} \tag{2.3}$$

$$\begin{aligned}&E^{\mathrm{PEC}}_{\mathrm{Cas}}(L_1, \dots, L_d)\\ &\quad= (d-1)E^{\mathrm{D}}_{\mathrm{Cas}}(L_1, \dots, L_d)\\ &\qquad+\sum_{j=1}^{d}E^{\mathrm{D}}_{\mathrm{Cas}}(L_1, \dots, L_{j-1}, L_{j+1}, \dots, L_d),\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}&E^{\mathrm{PMC}}_{\mathrm{Cas}}(L_1, \dots, L_d)\\ &\quad= \sum_{j=2}^{d}(j-1)\\ &\qquad\times\sum_{1\leq m_1<\cdots<m_j\leq d}E^{\mathrm{D}}_{\mathrm{Cas}}(s; L_{m_1}, \dots, L_{m_j}).\end{aligned} \tag{2.5}$$

Notice that when the number of dimensions d is equal to 1, there are no electromagnetic fields satisfying either PEC b.c. or PMC b.c.; the Casimir energy for a massless scalar field with Db.c. and Nb.c. is related to the Casimir energy for a massless scalar field with Pb.c. by the simple relation $E^{\mathrm{D/N}}_{\mathrm{Cas}}(a) = (1/2)E^{\mathrm{P}}_{\mathrm{Cas}}(2a)$. In the following discussions, when we mention a massless scalar field with Db.c. and Nb.c. or an electromagnetic field with PEC b.c. and PMC b.c., we implicitly assume that the number of space dimensions d is larger than 2.

For a massless scalar field with Pb.c. in a rectangular cavity $[0, L_1] \times [0, L_2] \times \cdots \times [0, L_d]$, the eigenfrequencies are

$$\sqrt{\sum_{j=1}^{d}\left(\frac{2\pi k_j}{L_j}\right)^2}, \qquad \mathbf{k} = (k_1, \dots, k_d) \in \mathbb{Z}^d.$$

Let

$$\begin{aligned}&E^{\mathrm{P},0}_{\mathrm{Cas}}(\lambda; L_1, \dots, L_d)\\ &\quad= \frac{1}{2}\sum_{\mathbf{k}\in\mathbb{Z}^d\setminus\{0\}}\sqrt{\sum_{j=1}^{d}\left(\frac{2\pi k_j}{L_j}\right)^2}\exp\left(-\lambda\sqrt{\sum_{j=1}^{d}\left(\frac{2\pi k_j}{L_j}\right)^2}\right)\\ &\quad= -\frac{1}{2}\frac{\partial}{\partial\lambda}K(\lambda; L_1, \dots, L_d)\end{aligned} \tag{2.6}$$

be the λ-dependent zero-temperature Casimir energy. Here the function $K(\lambda; L_1, \dots, L_d)$ is defined by

$$K(\lambda; L_1, \dots, L_d) = \sum_{\mathbf{k}\in\mathbb{Z}^d\setminus\{0\}} \exp\left(-\lambda\sqrt{\sum_{j=1}^d \left(\frac{2\pi k_j}{L_j}\right)^2}\right).$$

Using the formula

$$e^{-z} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(w) z^{-w}\, dw, \quad \mathrm{Re}\, c > 0,$$

we find that

$$\begin{aligned} &K(\lambda; L_1, \dots, L_d) \\ &\quad= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \sum_{\mathbf{k}\in\mathbb{Z}^d\setminus\{0\}} \Gamma(w)\lambda^{-w}\left(\sum_{j=1}^d \left[\frac{2\pi k_j}{L_j}\right]^2\right)^{-\frac{w}{2}} dw \\ &\quad= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(w)\lambda^{-w} Z_d\left(\frac{w}{2}; \frac{2\pi}{L_1}, \dots, \frac{2\pi}{L_d}\right), \\ &\quad\mathrm{Re}\, c > \frac{d}{2}, \end{aligned} \tag{2.7}$$

where $Z_d(s; a_1, \dots, a_d)$ is the homogeneous Epstein zeta function [32, 33] defined by the infinite series

$$Z_d(s; a_1, \dots, a_d) = \sum_{\mathbf{k}\in\mathbb{Z}^d\setminus\{0\}} \left(\sum_{j=1}^d [a_j k_j]^2\right)^{-s} \tag{2.8}$$

when $\mathrm{Re}\, s > d/2$. The function $Z_d(\frac{w}{2}; \frac{2\pi}{L_1}, \dots, \frac{2\pi}{L_d})$ has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $w = d$ with residue

$$\frac{1}{2^{d-1}\pi^{d/2}\Gamma(\frac{d}{2})}\prod_{j=1}^d L_j.$$

By shifting the contour of integration in (2.7) from the line $\mathrm{Re}\, w = c$, $c > d/2$ to a line $\mathrm{Re}\, w = -2 - \varepsilon$, $\varepsilon > 0$, we find that

$$\begin{aligned} &E_{\mathrm{Cas}}^{\mathrm{P},0}(\lambda; L_1, \dots, L_d) \\ &\quad= \frac{\Gamma(d+1)}{2^d\pi^{d/2}\Gamma(\frac{d}{2})}\left[\prod_{j=1}^d L_j\right]\lambda^{-d-1} \\ &\qquad+ \frac{1}{2} Z_d\left(-\frac{1}{2}; \frac{2\pi}{L_1}, \dots, \frac{2\pi}{L_d}\right) + O(\lambda). \end{aligned} \tag{2.9}$$

Using the functional equation (see e.g. [32–36])

$$\begin{aligned} &\pi^{-s}\Gamma(s) Z_d(s; a_1, \dots, a_d) \\ &\quad= \frac{\pi^{s-\frac{d}{2}}}{[\prod_{j=1}^d a_j]}\Gamma\left(\frac{d}{2} - s\right) Z_d\left(\frac{d}{2} - s; \frac{1}{a_1}, \dots, \frac{1}{a_d}\right), \end{aligned} \tag{2.10}$$

we find that the cut-off dependent finite-temperature Casimir energy is given by

$$\begin{aligned} &E_{\mathrm{Cas}}^{\mathrm{P}}(\lambda; L_1, \dots, L_d) \\ &\quad= \frac{\Gamma(d+1)}{2^d\pi^{d/2}\Gamma(\frac{d}{2})}\left[\prod_{j=1}^d L_j\right]\lambda^{-d-1} \\ &\qquad+ E_{\mathrm{Cas,\,reg}}^{\mathrm{P}}(L_1, \dots, L_d) + O(\lambda), \end{aligned} \tag{2.11}$$

where the regularized finite-temperature Casimir energy $E_{\mathrm{Cas,\,reg}}^{\mathrm{P}}(L_1, \dots, L_d)$ is given by [30]

$$\begin{aligned} &E_{\mathrm{Cas,\,reg}}^{\mathrm{P}}(L_1, \dots, L_d) \\ &\quad= -\frac{\Gamma(\frac{d+1}{2})}{2\pi^{\frac{d+1}{2}}}\left[\prod_{j=1}^d L_j\right] Z_d\left(\frac{d+1}{2}; L_1, \dots, L_d\right) \\ &\qquad+ T\sum_{\mathbf{k}\in\mathbb{Z}^d\setminus\{0\}} \log\left(1 - \exp\left(-\frac{2\pi}{T}\sqrt{\sum_{j=1}^d\left(\frac{k_j}{L_j}\right)^2}\right)\right) \\ &\quad= -\frac{T}{2} Z_{d+1}'\left(0; T, \frac{1}{L_1}, \dots, \frac{1}{L_d}\right) - T\log\frac{2\pi}{T}. \end{aligned} \tag{2.12}$$

The cut-off dependent finite-temperature Casimir energy for a massless scalar field with Db.c. and Nb.c. and an electromagnetic field with PEC b.c. and PMC b.c. can be computed using the formulas (2.3), (2.4) and (2.5). It is easy to deduce from (2.11), (2.3), (2.4) and (2.5) that in all cases the cut-off dependent Casimir energy can be written as a sum of the $\lambda \to 0^+$ divergent term and the regularized term:

$$\begin{aligned} E_{\mathrm{Cas}}(\lambda; L_1, \dots, L_d) &= E_{\mathrm{Cas,\,div}}(\lambda; L_1, \dots, L_d) \\ &\quad+ E_{\mathrm{Cas,\,reg}}(L_1, \dots, L_d) + O(\lambda). \end{aligned}$$

Although the divergent term for the massless scalar field with Pb.c. only depends on the bulk volume, we can see from (2.3), (2.4) and (2.5) that for a massless scalar field with Db.c. and Nb.c. and an electromagnetic field with PEC b.c. and PMC b.c., the divergent terms depend on the area of lower-dimensional hypersurfaces. As mentioned in the introduction, it has been argued by several authors [5–10] that these hypersurface divergences cannot be removed by renormalization of physical parameters. However, we notice that, regarded as a function of L_1, *the divergent term is linear in* L_1, i.e.

$$\begin{aligned} E_{\mathrm{Cas,\,Div}}(\lambda; L_1, \dots, L_d) &= e_1(\lambda, L_2, \dots, L_d) L_1 \\ &\quad+ e_0(\lambda; L_2, \dots, L_d). \end{aligned} \tag{2.13}$$

Here we want to remark that the above calculation is an idealization of the more physical description of the interaction between the quantum field and an external potential, which goes to zero away from the boundary surfaces. For a more rigorous treatment, one may consider the approach of [7], where the imposed boundary conditions are approximated by adding a δ-type background potential to the quantum field, which is concentrated on the boundaries. In our idealization, where the piston is assumed to be a perfect rigid rectangular cavity partitioned into two regions by a perfect rigid piston with negligible thickness, the Casimir force acting on the piston would be independent of the approach used to compute the Casimir energy, as has already been pointed out by [11]. However, we would like to emphasize that we do not claim to have renormalized the Casimir energy in a physical way. Instead, our point is that for an idealized piston the divergence terms, which plague the calculations of the Casimir energy, can be ignored, since they would not contribute to the Casimir force acting on the piston (see next section). But this only holds in the idealized situation. For more details of physical situations that would invalidate our assumptions, one may refer to [7] and the more recent preprint [37].

3 Casimir force on d-dimensional rectangular pistons

Since the $\lambda \to 0^+$ divergent term of the Casimir energy for a d-dimensional rectangular cavity is linear in L_1, see (2.13), it follows that for the Casimir energy of the piston system, (2.1), the $\lambda \to 0^+$ divergent term depends only on L_1, and not on a. Therefore, the $\lambda \to 0^+$ divergent part of the Casimir energy does not contribute to the Casimir force acting on the piston. Consequently, the Casimir force can be computed by using only the regularized Casimir energy:

$$\begin{aligned}
&F_{\mathrm{Cas}}(a; L_2, \dots, L_d)\\
&\quad = -\lim_{L_1\to\infty} \frac{\partial}{\partial a}\big\{E_{\mathrm{Cas,reg}}(a, \dots, L_d)\\
&\qquad + E_{\mathrm{Cas,\,reg}}(L_1 - a, \dots, L_d)\big\}.
\end{aligned} \tag{3.1}$$

For a massless scalar field with Pb.c., we obtain from (A.3) that the regularized Casimir force acting on the piston due to the vacuum fluctuation of the field in region II is given by

$$\begin{aligned}
&F_{\mathrm{Cas}}^{\mathrm{P,II}}(a; L_2, \dots, L_d)\\
&\quad = \pi T Z_d\left(-\frac{1}{2}; \frac{1}{L_2}, \dots, \frac{1}{L_d}, T\right)\\
&\quad = -\frac{\Gamma(\frac{d+1}{2})}{2\pi^{\frac{d+1}{2}}}\left[\prod_{j=2}^{d} L_j\right] Z_d\left(\frac{d+1}{2}; L_2, \dots, L_d, \frac{1}{T}\right);
\end{aligned} \tag{3.2}$$

and the total Casimir force acting on the piston is

$$\begin{aligned}
&F_{\mathrm{Cas}}^{\mathrm{P}}(a; L_2, \dots, L_d)\\
&\quad = -\frac{T}{a} - 2\pi T \sum_{k_1=1}^{\infty} \sum_{(k_2,\dots,k_d,l)\in\mathbb{Z}^d\setminus\{0\}}\\
&\qquad \times \left(\sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2 + (lT)^2\right)^{\frac{1}{2}}\\
&\qquad \times \exp\left(-2\pi k_1 a \sqrt{\sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2 + (lT)^2}\right)
\end{aligned} \tag{3.3}$$

for a massless scalar field with Pb.c. For a massless scalar field with Db.c. and Nb.c., (2.2) and (2.3) give

$$\begin{aligned}
&F_{\mathrm{Cas}}^{\mathrm{D/N}}(a; L_2, \dots, L_d)\\
&\quad = 2^{-d+1} \sum_{j=1}^{d} (\mp)^{d-j}\\
&\qquad \times \sum_{2\le m_1<\cdots<m_{j-1}\le d} F_{\mathrm{Cas}}^{\mathrm{P}}(2a; 2L_{m_1}, \dots, 2L_{m_{j-1}}).
\end{aligned} \tag{3.4}$$

For $j = 1$, the second summation is a single term, $F_{\mathrm{Cas}}^{\mathrm{P}}(2a)$. Using (3.3), we find that the Casimir force on the piston for a massless scalar field with Db.c. and Nb.c. can be written, respectively, as

$$\begin{aligned}
&F_{\mathrm{Cas}}^{\mathrm{D}}(a; L_2, \dots, L_d)\\
&\quad = -\pi T \sum_{k_1=1}^{\infty} \sum_{(k_2,\dots,k_d)\in\mathbb{N}^{d-1}} \sum_{l=-\infty}^{\infty}\\
&\qquad \times \left(\sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2 + (2lT)^2\right)^{\frac{1}{2}}\\
&\qquad \times \exp\left(-2\pi k_1 a \sqrt{\sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2 + (2lT)^2}\right)
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&F_{\mathrm{Cas}}^{\mathrm{N}}(a; L_2, \dots, L_d)\\
&\quad = -\frac{T}{2a} - \pi T \sum_{k_1=1}^{\infty} \sum_{(k_2,\dots,k_d,l)\in[(\mathbb{N}\cup\{0\})^{d-1}\times\mathbb{Z}]\setminus\{0\}}\\
&\qquad \times \left(\sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2 + (2lT)^2\right)^{\frac{1}{2}}
\end{aligned}$$

$$\times \exp\left(-2\pi k_1 a \sqrt{\sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2 + (2lT)^2}\right). \tag{3.6}$$

For an electromagnetic field under PEC b.c. and PMC b.c., (2.4) and (2.5) give

$$\begin{aligned} &F_{\mathrm{Cas}}^{\mathrm{PEC}}(a; L_1, \dots, L_d) \\ &\quad = (d-1) F_{\mathrm{Cas}}^{\mathrm{D}}(a; L_2, \dots, L_d) \\ &\qquad + \sum_{j=2}^{d} F_{\mathrm{Cas}}^{\mathrm{D}}(a; L_2, \dots, L_{j-1}, L_{j+1}, \dots, L_d) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} &F_{\mathrm{Cas}}^{\mathrm{PMC}}(a; L_1, \dots, L_d) \\ &\quad = \sum_{j=2}^{d} (j-1) \sum_{2 \le m_1 < \cdots < m_{j-1} \le d} F_{\mathrm{Cas}}^{\mathrm{D}}(a; L_{m_1}, \dots, L_{m_{j-1}}). \end{aligned} \tag{3.8}$$

Using these formulas and the formula (3.5), we conclude that

$$\begin{aligned} &F_{\mathrm{Cas}}^{\mathrm{PEC/PMC}}(a; L_2, \dots, L_d) \\ &\quad = -\pi T \sum_{k_1=1}^{\infty} \sum_{(k_2,\dots,k_d)\in(\mathbb{N}\cup\{0\})^{d-1}} \Lambda^{\mathrm{PEC/PMC}}(k_2, \dots, k_d) \\ &\qquad \times \sum_{l=-\infty}^{\infty} \left(\sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2 + (2lT)^2\right)^{\frac{1}{2}} \\ &\qquad \times \exp\left(-2\pi k_1 a \sqrt{\sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2 + (2lT)^2}\right) \\ &\qquad - \delta^{\mathrm{PEC/PMC}} \frac{T}{2a}, \end{aligned} \tag{3.9}$$

where

$$\Lambda^{\mathrm{PEC}}(k_2, \dots, k_d) = \begin{cases} d-1, & \text{if all } k_i, i = 2, \dots, d \text{ are nonzero}, \\ 1, & \text{if exactly one of the } k_i \text{ is zero}, \\ 0, & \text{if more than one of the } k_i \text{ is zero}; \end{cases} \tag{3.10}$$

$$\Lambda^{\mathrm{PMC}}(k_2, \dots, k_d) = j, \quad \text{if exactly } j \text{ of the } k_i, i = 2, \dots, d \text{ is nonzero}; \tag{3.11}$$

and $\delta^{\mathrm{PEC}} = 1$ if and only if $d = 2$ and $\delta^{\mathrm{PMC}} \equiv 0$.

Notice that since the Bessel function $K_\nu(z)$ is positive for any positive z, we immediately obtain from (3.3), (3.5), (3.6) and (3.9) that *for a massless scalar field with periodic, Dirichlet and Neumann boundary conditions and for an electromagnetic field with PEC and PMC boundary conditions, the Casimir force acting on the piston always has a negative sign, and therefore it is attractive. This holds for any dimension and size of the piston as well as for any temperature.*

By taking the derivative of the Casimir force with respect to a, one finds from (3.3), (3.5), (3.6) and (3.9) that the derivative of the Casimir force with respect to a is always positive. Therefore, we can conclude that *for either a massless scalar field with periodic, Dirichlet or Neumann boundary conditions or an electromagnetic field with PEC and PMC boundary conditions, the Casimir force is always an increasing function of a*. Since the Casimir force is always negative, this implies that *the magnitude of the Casimir force is always decreasing when a is increased.* To the best of our knowledge, this is the first time that such results are obtained analytically.

4 Asymptotic behaviors of the Casimir force for large and small plate separations

The formulas (3.3), (3.5), (3.6) and (3.9) are ideal for studying the large-a and small-L_j, $2 \le j \le d$, behaviors of the Casimir force. In particular, the leading-order terms of the Casimir force when $a \to \infty$ are given respectively by

$$\begin{aligned} F_{\mathrm{Cas}}^{\mathrm{P}}(a; L_2, \dots, L_d) &\sim -\frac{T}{a} - 4\pi T \min\{L_2^{-1}, \dots, L_d^{-1}, T\} \\ &\qquad \times e^{-2\pi a \min\{L_2^{-1}, \dots, L_d^{-1}, T\}}, \\ F_{\mathrm{Cas}}^{\mathrm{D}}(a; L_2, \dots, L_d) &\sim -\pi T \sqrt{\sum_{j=2}^{d} L_j^{-2}}\, e^{-2\pi a \sqrt{\sum_{j=2}^{d} L_j^{-2}}}, \\ F_{\mathrm{Cas}}^{\mathrm{N}}(a; L_2, \dots, L_d) &\sim -\frac{T}{2a} - \pi T \min\{L_2^{-1}, \dots, L_d^{-1}, 2T\} \\ &\qquad \times e^{-2\pi a \min\{L_2^{-1}, \dots, L_d^{-1}, 2T\}}, \end{aligned} \tag{4.1}$$

$$\begin{aligned} &F_{\mathrm{Cas}}^{\mathrm{PEC}}(a; L_2, \dots, L_d) \\ &\quad \sim -\delta^{\mathrm{PEC}} \frac{T}{2a} - \pi T \min\left\{\sqrt{\sum_{\substack{l=2 \\ l \ne j}}^{d} L_l^{-2}}\right\}_{j=2}^{d} \\ &\qquad \times \exp\left(-2\pi a \min\left\{\sqrt{\sum_{\substack{l=2 \\ l \ne j}}^{d} L_l^{-2}}\right\}_{j=2}^{d}\right), \end{aligned} \tag{4.2}$$

$$\begin{aligned} F_{\mathrm{Cas}}^{\mathrm{PMC}}(a; L_2, \dots, L_d) &\sim -\pi T \min\{L_2^{-1}, \dots, L_d^{-1}\} \\ &\qquad e^{-2\pi a \min\{L_2^{-1}, \dots, L_d^{-1}\}}. \end{aligned} \tag{4.3}$$

The case that $T=0$ will be considered in the next section. These asymptotic behaviors imply that when $a\to\infty$, the magnitude of the Casimir force tends to zero. Moreover, for a massless scalar field with Pb.c. and Nb.c. and electromagnetic field with PEC b.c. in $d=2$ dimensions, the Casimir force tends to zero polynomially, in the order a^{-1}. However, for a massless scalar field with Db.c. and for an electromagnetic field with PEC b.c. and PMC b.c., the magnitude of the Casimir force decays to zero exponentially fast.

For the Casimir force when the plate separation a is small, (A.4) and (3.2) give

$$\begin{aligned}
&F_{\mathrm{Cas}}^{\mathrm{P}}(a; L_2,\ldots,L_d)\\
&\quad=-\frac{d\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}a^{d+1}}\left[\prod_{j=2}^{d}L_i\right]\zeta_R(d+1)\\
&\qquad+\pi T Z_d\left(-\frac{1}{2};\frac{1}{L_2},\ldots,\frac{1}{L_d},T\right)\\
&\qquad+\frac{4\pi}{a^{\frac{d+4}{2}}}\left[\prod_{j=2}^{d}L_i\right]\sum_{k_1=1}^{\infty}\sum_{(k_2,\ldots,k_d,l)\in\mathbb{Z}^d\setminus\{0\}}k_1^{\frac{d+2}{2}}\\
&\qquad\times\left(\sum_{j=2}^{d}(k_jL_j)^2+\left(\frac{l}{T}\right)^2\right)^{-\frac{d-2}{4}}\\
&\qquad\times K_{\frac{d-2}{2}}\left(\frac{2\pi k_1}{a}\sqrt{\sum_{j=2}^{d}(k_jL_j)^2+\left(\frac{l}{T}\right)^2}\right).
\end{aligned}\tag{4.4}$$

When $a\to 0^+$, the first term tends to infinity, of order a^{-d-1}. The second term is $O(a^0)$ and the third term tends to 0 exponentially fast. In other words, when the plate separation a is small, the leading term of the Casimir force for a massless scalar field with Pb.c. is

$$\begin{aligned}
&F_{\mathrm{Cas}}^{\mathrm{P}}(a; L_2,\ldots,L_d)\\
&\quad\sim-\frac{d\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}a^{d+1}}\left[\prod_{j=2}^{d}L_i\right]\zeta_R(d+1)+O(a^0).
\end{aligned}\tag{4.5}$$

For a massless scalar field with Db.c. and Nb.c. and an electromagnetic field with PEC b.c. and PMC b.c., the corresponding expression for the Casimir force when the separation a is small can be found by substituting (4.4) into (3.4), (3.7) and (3.8). In particular, we find that when a is small, the asymptotic behaviors of the Casimir force are given respectively by

$$\begin{aligned}
&F_{\mathrm{Cas}}^{\mathrm{D/N}}(a; L_2,\ldots,L_d)\\
&\quad\sim-\frac{1}{2^{d+1}}\sum_{j=1}^{d}(\mp)^{d-j}\frac{j\Gamma(\frac{j+1}{2})\zeta_R(j+1)}{\pi^{\frac{j+1}{2}}a^{j+1}}S_j+O(a^0),
\end{aligned}$$

and

$$\begin{aligned}
&F_{\mathrm{Cas}}^{\mathrm{PEC/PMC}}(a; L_2,\ldots,L_d)\\
&\quad\sim-\frac{1}{2^{d+1}}\sum_{j=1}^{d}(\mp)^{d-j}(2j-d-1)\\
&\qquad\times\frac{j\Gamma(\frac{j+1}{2})\zeta_R(j+1)}{\pi^{\frac{j+1}{2}}a^{j+1}}S_j+O(a^0);
\end{aligned}$$

here

$$S_j=\sum_{2\le m_1<\cdots<m_{j-1}\le d}L_{m_1}\cdots L_{m_{j-1}}.\tag{4.6}$$

Therefore, for a massless scalar field with Db.c. or Nb.c., the small-a leading term of the Casimir force is

$$-\frac{d\Gamma(\frac{d+1}{2})\zeta_R(d+1)}{2^{d+1}\pi^{\frac{d+1}{2}}a^{d+1}}\left[\prod_{j=2}^{d}L_i\right],$$

which is 2^{d+1} times smaller than the leading term for the massless scalar field with Pb.c. For the electromagnetic field with PEC b.c. and PMC b.c., the small-a leading term is

$$-(d-1)\frac{d\Gamma(\frac{d+1}{2})\zeta_R(d+1)}{2^{d+1}\pi^{\frac{d+1}{2}}a^{d+1}}\left[\prod_{j=2}^{d}L_i\right],$$

which is $\frac{2^{d+1}}{d-1}$ times smaller than the leading term for the massless scalar field with Pb.c. and $d-1$ times larger than the leading term for the massless scalar field with Db.c.

Notice that these leading-order terms are independent of the temperature T. Therefore, we conclude that *at low temperature, the effect of the temperature on the Casimir force is insignificant when the plate separation a is small.* On the other hand, the leading-order terms also show that the Casimir force tends to $-\infty$ when $a\to 0^+$. Combining this with the fact that the Casimir force tends to 0 as $a\to\infty$ and the Casimir force is an increasing function of a, we conclude that *for a massless scalar field with Pb.c., Db.c. and Nb.c. and an electromagnetic field with PEC b.c. and PMC b.c., the Casimir force always increases from $-\infty$ to 0 as the separation a increases from 0 to ∞.*

5 Low-temperature and high-temperature expansions of the Casimir force

As discussed in [30], the low-temperature expansion of the regularized Casimir energy inside a rectangular cavity is just the zero-temperature Casimir energy plus the temperature correction. In the case of a massless scalar field with Pb.c., it is given explicitly by (2.12). Together with (3.2) and (A.1), we find that the low-temperature expansion of the Casimir force acting on the piston is given by

$$\begin{aligned}
&F_{\mathrm{Cas}}^{\mathrm{P}}(a; L_2, \ldots, L_d)\\
&\quad= F_{\mathrm{Cas}}^{\mathrm{P},T=0}(a; L_2, \ldots, L_d) + \frac{4\pi}{a^3}\sum_{l=1}^{\infty}\sum_{k_1=1}^{\infty}\sum_{(k_2,\ldots,k_d)\in\mathbb{Z}^{d-1}} k_1^2\\
&\qquad\times\left(\left(\frac{k_1}{a}\right)^2 + \sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2\right)^{-1/2}\\
&\qquad\times\exp\left(-\frac{2\pi l}{T}\sqrt{\left(\frac{k_1}{a}\right)^2 + \sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2}\right)\\
&\qquad-\frac{\pi T^2}{6} - 2T\sum_{l=1}^{\infty}\sum_{(k_2,\ldots,k_d)\in\mathbb{Z}^{d-1}\setminus\{0\}} l^{-1}\sqrt{\sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2}\\
&\qquad\times K_1\left(\frac{2\pi l}{T}\sqrt{\sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2}\right)
\end{aligned} \tag{5.1}$$

for massless scalar field with Pb.c., and by

$$\begin{aligned}
&F_{\mathrm{Cas}}(a; L_2, \ldots, L_d)\\
&\quad= F_{\mathrm{Cas}}^{T=0}(a; L_2, \ldots, L_d)\\
&\qquad+\frac{\pi}{a^3}\sum_{l=1}^{\infty}\sum_{k_1=1}^{\infty}\sum_{(k_2,\ldots,k_d)\in(\mathbb{N}\cup\{0\})^{d-1}} k_1^2\\
&\qquad\times\Lambda(k_2,\ldots,k_d)\left(\left(\frac{k_1}{a}\right)^2 + \sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2\right)^{-1/2}\\
&\qquad\times\exp\left(-\frac{\pi l}{T}\sqrt{\left(\frac{k_1}{a}\right)^2 + \sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2}\right) - \delta\frac{\pi T^2}{6}\\
&\qquad-T\sum_{l=1}^{\infty}\sum_{(k_2,\ldots,k_d)\in(\mathbb{N}\cup\{0\})^{d-1}\setminus\{0\}}\Lambda(k_2,\ldots,k_d)l^{-1}\\
&\qquad\times\sqrt{\sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2}K_1\left(\frac{\pi l}{T}\sqrt{\sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2}\right)
\end{aligned} \tag{5.2}$$

for a massless scalar field with Db.c. and Nb.c. and an electromagnetic field with PEC b.c. and PMC b.c. Here

$\Lambda^{\mathrm{D}}(k_2,\ldots,k_d) = 1$

if and only if all k_i, $i = 2,\ldots,d$ are nonzero;

$\Lambda^{\mathrm{N}}(k_2,\ldots,k_d) \equiv 1$ and $\Lambda^{\mathrm{PEC}}(k_2,\ldots,k_d)$ and $\Lambda^{\mathrm{PMC}}(k_2,\ldots,k_d)$ are given by (3.10) and (3.11), respectively; $\delta^{\mathrm{N}} \equiv 1$; $\delta^{\mathrm{D}} = \delta^{\mathrm{PMC}} \equiv 0$ and $\delta^{\mathrm{PEC}} = 1$ if and only if $d = 2$. These formulas show that for a massless scalar field with Pb.c. and Nb.c., the temperature correction is of order $O(T^2)$ when $T \ll 1$; but for a massless scalar field with Db.c. and for an electromagnetic field with PEC b.c. and PMC b.c., the temperature-correction terms go to zero exponentially fast when $T \to 0^+$. The main contribution to the low-temperature Casimir force comes from the first term of (5.1) or (5.2), which is the zero-temperature contribution. For a massless scalar field with Pb.c., it is given by

$$\begin{aligned}
&F_{\mathrm{Cas}}^{\mathrm{P},T=0}(a; L_1, \ldots, L_d)\\
&\quad= -\frac{d\Gamma(\frac{d+1}{2})}{a^{d+1}\pi^{\frac{d+1}{2}}}\left[\prod_{j=2}^{d}L_j\right]\zeta_R(d+1)\\
&\qquad-\frac{\Gamma(\frac{d+1}{2})}{2\pi^{\frac{d+1}{2}}}\left[\prod_{j=2}^{d}L_j\right]Z_{d-1}\left(\frac{d+1}{2}; L_2,\ldots,L_d\right)\\
&\qquad+\frac{4\pi[\prod_{j=2}^{d}L_j]}{a^{\frac{d+4}{2}}}\\
&\qquad\times\sum_{k_1=1}^{\infty}\sum_{(k_2,\ldots,k_d)\in\mathbb{Z}^{d-1}\setminus\{0\}} k_1^{\frac{d+2}{2}}\left(\sum_{j=2}^{d}(k_jL_j)^2\right)^{-\frac{d-2}{4}}\\
&\qquad\times K_{\frac{d-2}{2}}\left(\frac{2\pi k_1}{a}\sqrt{\sum_{j=2}^{d}(k_jL_j)^2}\right).
\end{aligned}$$

The first term is the leading term, and it is of order $O(a^{-d-1})$ when $a \to 0^+$. We have seen in the previous section that for any finite temperature, this is still the leading term when $a \to 0^+$. The second term is $O(a^0)$ and the last term goes to zero exponentially fast when $a \to 0^+$. For a massless scalar field with Db.c. and Nb.c. and an electromagnetic field with PEC b.c. and PMC b.c., the leading behavior of the zero-temperature Casimir force when $a \to 0^+$ is also the same as the finite-temperature case.

An alternative expression for the zero-temperature Casimir force acting on the piston which can be used to study the large-a behavior is given by

$$\begin{aligned}
&F_{\mathrm{Cas}}^{\mathrm{P},T=0}(a; L_2, \ldots, L_d)\\
&\quad= -\frac{\pi}{6a^2} - \frac{2}{a}\sum_{k_1=1}^{\infty}\sum_{(k_2,\ldots,k_d)\in\mathbb{Z}^{d-1}\setminus\{0\}} k_1^{-1}\sqrt{\sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2}\\
&\qquad\times K_1\left(2\pi k_1 a\sqrt{\sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2}\right)
\end{aligned}$$

$$-4\pi\sum_{k_1=1}^{\infty}\sum_{(k_2,\dots,k_d)\in\mathbb{Z}^{d-1}\setminus\{0\}}\left(\sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2\right)\times K_0\left(2\pi k_1 a\sqrt{\sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2}\right)$$

for a massless scalar field with Pb.c. For a massless scalar field with Db.c. and Nb.c. and an electromagnetic field with PEC b.c. and PMC b.c., (3.4), (3.7) and (3.8) give immediately

$$\begin{aligned}&F_{\text{Cas}}^{T=0}(a;L_2,\dots,L_d)\\&\quad=-\delta\frac{\pi}{24a^2}\\&\qquad-\frac{1}{2a}\sum_{k_1=1}^{\infty}\sum_{(k_2,\dots,k_d)\in(\mathbb{N}\cup\{0\})^{d-1}\setminus\{0\}}\Lambda(k_2,\dots,k_d)k_1^{-1}\\&\qquad\times\sqrt{\sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2}K_1\left(2\pi k_1 a\sqrt{\sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2}\right)\\&\qquad-\pi\sum_{k_1=1}^{\infty}\sum_{(k_2,\dots,k_d)\in(\mathbb{N}\cup\{0\})^{d-1}\setminus\{0\}}\Lambda(k_2,\dots,k_d)\\&\qquad\times\left(\sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2\right)K_0\left(2\pi k_1 a\sqrt{\sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2}\right).\end{aligned}$$

Using the fact that

$$K_\nu(z)\sim\sqrt{\frac{\pi}{2z}}e^{-z}\quad\text{as } z\to\infty,$$

we find that when $a\gg 1$, the leading-order terms of the zero-temperature Casimir force are given respectively by

$$\begin{aligned}&F_{\text{Cas}}^{\text{P},T=0}(a;L_2,\dots,L_d)\\&\quad\sim-\frac{\pi}{6a^2}-4\pi a^{-\frac12}\left(\min\{L_2^{-1},\dots,L_d^{-1}\}\right)^{\frac32}\\&\qquad\times e^{-2\pi a\min\{L_2^{-1},\dots,L_d^{-1}\}},\end{aligned}$$

$$F_{\text{Cas}}^{\text{D},T=0}(a;L_2,\dots,L_d)\sim-\pi a^{-\frac12}\left(\sum_{j=2}^{d}L_j^{-2}\right)^{\frac34}\times\exp\left(-2\pi a\sqrt{\sum_{j=2}^{d}L_j^{-2}}\right),$$

$$\begin{aligned}&F_{\text{Cas}}^{\text{N},T=0}(a;L_2,\dots,L_d)\\&\quad\sim-\frac{\pi}{24a^2}-\pi a^{-\frac12}\left(\min\{L_2^{-1},\dots,L_d^{-1}\}\right)^{\frac32}\\&\qquad\times e^{-2\pi a\min\{L_2^{-1},\dots,L_d^{-1}\}},\end{aligned}$$

$$\begin{aligned}&F_{\text{Cas}}^{\text{PEC},T=0}(a;L_2,\dots,L_d)\\&\quad\sim-\delta\frac{\pi}{24a^2}-\pi a^{-\frac12}\left(\min\left\{\sum_{\substack{l=2\\l\neq j}}^{d}L_l^{-2}\right\}_{j=2}^{d}\right)^{\frac34}\\&\qquad\times\exp\left(-2\pi a\min\left\{\sqrt{\sum_{\substack{l=2\\l\neq j}}^{d}L_l^{-2}}\right\}_{j=2}^{d}\right),\end{aligned}$$

$$\begin{aligned}F_{\text{Cas}}^{\text{PMC}}(a;L_2,\dots,L_d)&\sim-\pi a^{-\frac12}\left(\min\{L_2^{-1},\dots,L_d^{-1}\}\right)^{\frac32}\\&\quad\times e^{-2\pi a\min\{L_2^{-1},\dots,L_d^{-1}\}}.\end{aligned}$$

We notice a considerable difference between the $a\to\infty$ leading behavior of the Casimir force in the zero-temperature case and the finite-temperature case. At zero temperature, the leading-order terms of the Casimir force for a massless scalar field with Pb.c. and Nb.c. are of order a^{-2}, in contrast to the order a^{-1} when the temperature is nonzero. For massless scalar field with Db.c. and for electromagnetic field with PEC b.c. and PMC b.c., the zero-temperature Casimir force also tends to zero exponentially fast when $a\to\infty$; but it tends to zero faster than at finite temperature.

Now we consider the high-temperature behavior of the Casimir force. From (A.5), we find that for a massless scalar field with Pb.c., the contribution from region I to the Casimir force acting on the piston is given by

$$\begin{aligned}&F_{\text{Cas}}^{\text{P,I}}(a;L_2,\dots,L_d)\\&\quad=\frac{\Gamma(\frac{d+1}{2})\zeta_R(d+1)}{\pi^{\frac{d+1}{2}}}\left[\prod_{j=2}^{d}L_j\right]T^{d+1}\\&\qquad+\frac{T}{2}\frac{\partial}{\partial a}Z_d'\left(0;\frac1a,\frac{1}{L_2},\dots,\frac{1}{L_d}\right)\\&\qquad+2\left[\prod_{j=2}^{d}L_j\right]T^{\frac{d+2}{2}}\sum_{l=1}^{\infty}\sum_{\mathbf{k}\in\mathbb{Z}^d\setminus\{0\}}l^{\frac d2}\\&\qquad\times\left((k_1a)^2+\sum_{j=2}^{d}(k_jL_j)^2\right)^{-\frac d4}\\&\qquad\times K_{\frac d2}\left(2\pi lT\sqrt{(k_1a)^2+\sum_{j=2}^{d}(k_jL_j)^2}\right)\\&\qquad-8\pi a\left[\prod_{j=2}^{d}L_j\right]T^{\frac{d+4}{2}}\sum_{l=1}^{\infty}\sum_{k_1=1}^{\infty}\sum_{(k_2,\dots,k_d)\in\mathbb{Z}^{d-1}}l^{\frac{d+2}{2}}k_1^2\\&\qquad\times\left((k_1a)^2+\sum_{j=2}^{d}(k_jL_j)^2\right)^{-\frac{d+2}{4}}\end{aligned}$$

$$\times K_{\frac{d+2}{2}}\left(2\pi l T\sqrt{(k_1 a)^2+\sum_{j=2}^{d}(k_j L_j)^2}\right). \tag{5.3}$$

The leading term is the force due to black-body radiation, and it is independent of a. For the contribution from region II to the Casimir force acting on the piston, notice that

$$\begin{aligned}F_{\text{Cas}}^{\text{P,II}}(a;L_2,\dots,L_d)&=-\lim_{L_1\to\infty}F_{\text{Cas}}^{\text{P,I}}(L_1-a;L_2,\dots,L_d)\\&=-\lim_{a\to\infty}F_{\text{Cas}}^{\text{P,I}}(a;L_2,\dots,L_d).\end{aligned}$$

Therefore, we find that the black-body radiation contributions from regions I and II cancel with each other, and the Casimir force acting on the piston is given by

$$\begin{aligned}&F_{\text{Cas}}^{\text{P}}(a;L_2,\dots,L_d)\\&\quad=\frac{T}{2}\frac{\partial}{\partial a}Z_d'\left(0;\frac{1}{a},\frac{1}{L_2},\dots,\frac{1}{L_d}\right)\\&\qquad-\frac{T}{2}\lim_{a\to\infty}\frac{\partial}{\partial a}Z_d'\left(0;\frac{1}{a},\frac{1}{L_2},\dots,\frac{1}{L_d}\right)\\&\qquad+4\left[\prod_{j=2}^{d}L_j\right]T^{\frac{d+2}{2}}\sum_{l=1}^{\infty}\sum_{k_1=1}^{\infty}\sum_{(k_2,\dots,k_d)\in\mathbb{Z}^{d-1}}l^{\frac{d}{2}}\\&\qquad\times\left((k_1a)^2+\sum_{j=2}^{d}(k_jL_j)^2\right)^{-\frac{d}{4}}\\&\qquad\times K_{\frac{d}{2}}\left(2\pi l T\sqrt{(k_1a)^2+\sum_{j=2}^{d}(k_jL_j)^2}\right)\\&\qquad-8\pi a^2\left[\prod_{j=2}^{d}L_j\right]T^{\frac{d+4}{2}}\\&\qquad\times\sum_{l=1}^{\infty}\sum_{k_1=1}^{\infty}\sum_{(k_2,\dots,k_d)\in\mathbb{Z}^{d-1}}l^{\frac{d+2}{2}}k_1^2\\&\qquad\times\left((k_1a)^2+\sum_{j=2}^{d}(k_jL_j)^2\right)^{-\frac{d+2}{4}}\\&\qquad\times K_{\frac{d+2}{2}}\left(2\pi l T\sqrt{(k_1a)^2+\sum_{j=2}^{d}(k_jL_j)^2}\right).\end{aligned} \tag{5.4}$$

Notice that the high-temperature leading term is linear in T, given by the sum of the first two terms. It is called the *classical term* of the Casimir force. The remaining (last two) terms decays exponentially when $T\to\infty$.

Since the sign of the Stefan–Boltzmann term in (5.3) is positive, the contribution to the Casimir force from region I will become very repulsive at high temperature. However, the black-body radiation of region II creates a counterforce to cancel this repulsive force and we find that *the leading-order term of the Casimir force acting on the piston at high temperature is an attractive force* (see (5.5)) *of order* T.

Now let us explicitly compute the classical term of the Casimir force for a massless scalar field with Pb.c. Using (A.2), we find that the classical term of the Casimir force acting on the piston for a massless scalar field with Pb.c. is

$$\begin{aligned}&F_{\text{Cas}}^{\text{P,classical}}(a;L_2,\dots,L_d)\\&\quad=-T\left(\frac{1}{a}+2\pi\sum_{k_1=1}^{\infty}\sum_{(k_2,\dots,k_d)\in\mathbb{Z}^{d-1}\setminus\{0\}}\sqrt{\sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2}\right.\\&\qquad\left.\times\exp\left(-2\pi k_1 a\sqrt{\sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2}\right)\right).\end{aligned} \tag{5.5}$$

A drawback of formula (5.5) is that it does not make manifest the small-a behavior of the classical term. Using (A.2) again, we obtain an alternative formula for the classical term:

$$\begin{aligned}&F_{\text{Cas}}^{\text{P,classical}}(a;L_2,\dots,L_d)\\&\quad=-T\left\{\frac{(d-1)}{\pi^{\frac{d}{2}}}\Gamma\left(\frac{d}{2}\right)\zeta_R(d)\left[\prod_{j=2}^{d}L_j\right]a^{-d}-4\pi a^{-\frac{d+3}{2}}\right.\\&\qquad\times\left[\prod_{j=2}^{d}L_j\right]\sum_{k_1=1}^{\infty}\sum_{(k_2,\dots,k_d)\in\mathbb{Z}^{d-1}\setminus\{0\}}\\&\qquad\times\left(\sum_{j=2}^{d}(k_jL_j)^2\right)^{-\frac{d-3}{4}}k_1^{\frac{d+1}{2}}\\&\qquad\times K_{\frac{d-3}{2}}\left(\frac{2\pi k_1}{a}\sqrt{\sum_{j=2}^{d}(k_jL_j)^2}\right)\\&\qquad\left.+\frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}}\left[\prod_{j=2}^{d}L_j\right]Z_{d-1}\left(\frac{d}{2};L_2,\dots,L_d\right)\right\}\end{aligned} \tag{5.6}$$

for a massless scalar field with Pb.c., which is suitable for studying the small-a behavior.

The high-temperature expansion for the Casimir force acting on the piston due to a massless scalar field with Db.c. and Nb.c. and an electromagnetic field with PEC b.c. and PMC b.c. can be obtained using (3.4), (3.7) and (3.8). For the contribution from region I, (3.4), (3.7), (3.8) and (5.3) show that the leading-order terms of the Casimir force from

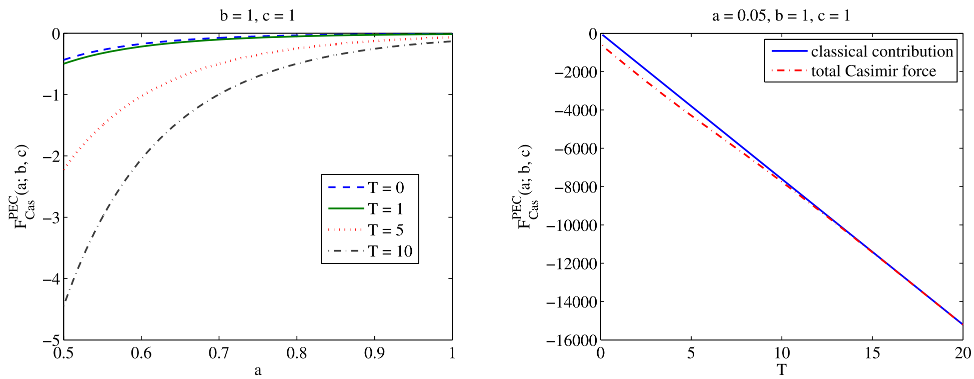


Fig. 5.1 *Left*: the dependence of the Casimir force $F^{\mathrm{PEC}}_{\mathrm{Cas}}(a;b,c)$ on the plate separation a when $T=0,1,5,10$. *Right*: the deviation of the Casimir force from the classical term when $a=0.05$, $b=1$ and $c=1$

region I are given by

$$\begin{aligned}&F^{\mathrm{D/N,I}}_{\mathrm{Cas}}(a;L_2,\dots,L_d)\\&\quad=\sum_{j=1}^{d}\left[(\mp)^{d-j}\frac{\Gamma(\frac{j+1}{2})\zeta_R(j+1)}{2^{d-j}\pi^{\frac{j+1}{2}}}S_j\right]T^{j+1}+O(T)\end{aligned}$$

and

$$\begin{aligned}&F^{\mathrm{PEC/PMC,I}}_{\mathrm{Cas}}(a;L_2,\dots,L_d)\\&\quad=\sum_{j=1}^{d}\left[(\mp)^{d-j}(2j-d-1)\frac{\Gamma(\frac{j+1}{2})\zeta_R(j+1)}{2^{d-j}\pi^{\frac{j+1}{2}}}S_j\right]T^{j+1}\\&\qquad+O(T)\end{aligned}$$

respectively, where S_j is the partial hypersurface area defined by (4.6). Notice that in contrast to the Pb.c. case, now we have temperature corrections of order T^l for all $2\le l\le d$ besides the leading Stefan–Boltzmann term of order T^{d+1}. However, all these terms are independent of a, and therefore they cancel with the corresponding terms from region II. The high-temperature leading-order term of the Casimir force acting on the piston is still the classical term of order T and is given explicitly by

$$\begin{aligned}&F^{\mathrm{classical}}_{\mathrm{Cas}}(a;L_2,\dots,L_d)\\&\quad=-T\left(\frac{\delta}{2a}+\pi\sum_{k_1=1}^{\infty}\sum_{(k_2,\dots,k_d)\in(\mathbb{N}\cup\{0\})^{d-1}\setminus\{0\}}\Lambda(k_2,\dots,k_d)\right.\\&\qquad\left.\times\sqrt{\sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2}\exp\left(-2\pi k_1 a\sqrt{\sum_{j=2}^{d}\left(\frac{k_j}{L_j}\right)^2}\right)\right).\end{aligned}\tag{5.7}$$

By taking the derivative of (5.5) and (5.7) with respect to a, we find that the classical term of the Casimir force is also an increasing function of a. In other words, the magnitude of the classical term decreases with increasing a. An alternative expression suitable for studying the classical term of the Casimir force at small plate separation can be derived using (3.4), (3.7), (3.8) and (5.6).

We would like to remark that the thermal Casimir effect for electromagnetic field in a three-dimensional rectangular cavity is considered in [40]. In that paper, the authors obtain a Casimir force that depends linearly in T in the high-temperature regime by subtracting the terms

$$\sum_{j=1}^{d}\left[(\mp)^{d-j}(2j-d-1)\frac{\Gamma(\frac{j+1}{2})\zeta_R(j+1)}{2^{d-j}\pi^{\frac{j+1}{2}}}S_j\right]T^{j+1},\tag{5.8}$$

so that in the large-volume limit, the Casimir force tends to zero. In our approach, no subtraction is required since the contribution from the terms (5.8) automatically cancel.

Before ending this section, we would like to comment on the nomenclature 'classical term' for the high-temperature leading term. To restore the constants $\hbar$, c and k into the expressions for the Casimir force, we replace T by $kT/(\hbar c)$ everywhere and multiply the overall expression by $\hbar c$. Notice that a term with order T^j will be accompanied with $\hbar^{1-j}$. Since in the high-temperature limit the Casimir force acting on the piston is $O(T)$, it has a finite classical $\hbar\to 0^+$ limit. Moreover, the leading term linear in T gives the classical limit and therefore it is called the classical term.

In Appendix B, we compute the explicit expressions for the zero-temperature Casimir force when $d=1,2$ and 3 from the formulas in this section. The results are found to agree with the existing results in [11–13, 17–19]. Explicit formulas for the low- and high-temperature expansions of the Casimir force are also given. In Fig. 5.1, we show the dependence of the Casimir force $F^{\mathrm{PEC}}_{\mathrm{Cas}}(a;b,c)$ on the plate separation a and the temperature T for three-dimensional electromagnetic field with PEC b.c.

6 Comment on piston inside a closed cavity

Consider the case that the piston is confined in a closed rectangular cavity. More precisely, we consider the scenario in which the $L_1\to\infty$ limit is not taken. In showing that the Casimir force is divergence free in the beginning of Sect. 3, we only used the fact that the divergent part of the Casimir energy in a rectangular cavity depends linearly on L_1 (without passing to the limit $L_1 \to \infty$). Therefore in the present case, where L_1 is finite, it is still true that the Casimir force acting on the piston $\hat{F}_{\text{Cas}}(a; L_1, \ldots, L_d)$ is free of a divergence. On the other hand, the derivation of (3.3) from (A.3) shows that after some cancellations, the Casimir force $\hat{F}_{\text{Cas}}(a; L_1, \ldots, L_d)$ for finite L_1 case can be written in terms of the Casimir force $F_{\text{Cas}}(a; L_2, \ldots, L_d)$ of the infinite-L_1 case by

$$\hat{F}_{\text{Cas}}(a; L_1, \ldots, L_d) = F_{\text{Cas}}(a; L_2, \ldots, L_d) - F_{\text{Cas}}(L_1 - a; L_2, \ldots, L_d). \quad (6.1)$$

This is true for either a massless scalar field with Pb.c., Db.c. or Nb.c. or an electromagnetic field with PEC b.c. or PMC b.c. A trivial consequence of (6.1) is that the Casimir force acting on a piston placed in a closed rectangular cavity is equal to zero when the piston is placed exactly in the middle of the cavity. On the other hand, since we have shown that in all the cases we consider, $F_{\text{Cas}}(a; L_2, \ldots, L_d)$ is an increasing function of a, this implies that if the piston is placed closer to the left-hand side (i.e. $a < L_1 - a$), then the Casimir force on the piston $\hat{F}_{\text{Cas}}(a; L_1, \ldots, L_d)$ is negative and tends to pull it to the left. In other words, the Casimir force acting on a piston which is placed inside a closed rectangular cavity always is trying to collapse the piston to the nearer end.

The low- and high-temperature expansions of the Casimir force $\hat{F}_{\text{Cas}}(a; L_1, \ldots, L_d)$ in the case of finite L_1 can easily be computed using the formula (6.1) and the results for the case of infinite L_1 in the previous sections. Therefore, we omit them here.

7 Casimir force density when the distances of some pairs of transversal plates are large

In this section, we want to study the asymptotic behavior of the Casimir force when $d - p > 0$ pairs of the transversal plates are large, i.e., $a, L_1, \ldots, L_p \ll L_{p+1} = \cdots = L_d$ for some $p \geq 1$. More precisely, we are going to derive the limit

$$\mathcal{F}_{\text{Cas}}(p, d; a; L_2, \ldots, L_p) = \lim_{L_{p+1}=\cdots=L_d\to\infty} \frac{F_{\text{Cas}}(a; L_1, \ldots, L_d)}{L_{p+1}\cdots L_d},$$

which we call the Casimir force density.[1] We consider the low-temperature expansion and high-temperature expansion separately. The computations are similar as in the previous sections. We only write down the final answers. Details can be found in the preprint version of this paper posted in [41].

7.1 Low-temperature expansion

In the low-temperature regime, the Casimir force density acting on the piston is given by

$$\begin{aligned}
&\mathcal{F}^{\text{P}}_{\text{Cas}}(p, d; a, L_2, \ldots, L_d)\\
&= \mathcal{F}^{\text{P},T=0}_{\text{Cas}}(p, d; a, L_2, \ldots, L_d) - 2T^{\frac{d-p+2}{2}}\\
&\quad \times \sum_{(k_2,\ldots,k_p)\in\mathbb{Z}^{p-1}\setminus\{0\}} \sum_{l=1}^{\infty} \left(\sum_{j=2}^{p}\left(\frac{k_j}{L_j}\right)^2\right)^{\frac{d-p+2}{4}} l^{-\frac{d-p+2}{2}}\\
&\quad \times K_{\frac{d-p+2}{2}}\left(\frac{2\pi l}{T}\sqrt{\sum_{j=2}^{p}\left(\frac{k_j}{L_j}\right)^2}\right)\\
&\quad + \frac{8\pi}{a^3} T^{\frac{d-p-1}{2}} \sum_{k_1=1}^{\infty} \sum_{(k_2,\ldots,k_p)\in\mathbb{Z}^{p-1}} \sum_{l=1}^{\infty} \left(\left(\frac{k_1}{a}\right)^2\right.\\
&\quad \left. + \sum_{j=2}^{p}\left(\frac{k_j}{L_j}\right)^2\right)^{\frac{d-p-1}{4}} l^{-\frac{d-p-1}{2}} k_1^2\\
&\quad \times K_{\frac{d-p-1}{2}}\left(\frac{2\pi l}{T}\sqrt{\left(\frac{k_1}{a}\right)^2 + \sum_{j=2}^{p}\left(\frac{k_j}{L_j}\right)^2}\right)\\
&\quad - \frac{\Gamma(\frac{d-p+2}{2})}{\pi^{\frac{d-p+2}{2}}} T^{d-p+2} \zeta_R(d-p+2)
\end{aligned}$$

for a massless scalar field with Pb.c. The expressions of the zero-temperature Casimir force density which are suitable for studying the small-a and large-a behaviors are given by

$$\begin{aligned}
&\mathcal{F}^{\text{P},T=0}_{\text{Cas}}(p, d; a, L_2, \ldots, L_d)\\
&= -\frac{d\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}}\left[\prod_{j=2}^{p} L_j\right] a^{-d-1} - \frac{\Gamma(\frac{d+1}{2})}{2\pi^{\frac{d+1}{2}}}\left[\prod_{j=2}^{p} L_j\right]\\
&\quad \times Z_{E,p-1}\left(\frac{d+1}{2}; L_2, \ldots, L_p\right)\\
&\quad + \frac{4\pi}{a^{\frac{d+4}{2}}}\left[\prod_{j=2}^{p} L_j\right] \sum_{k_1=1}^{\infty} \sum_{(k_2,\ldots,k_p)\in\mathbb{Z}^{p-1}\setminus\{0\}} k_1^{\frac{d+2}{2}}\\
&\quad \times \left(\sum_{j=2}^{p}(k_j L_j)^2\right)^{-\frac{d-2}{4}}\\
&\quad \times K_{\frac{d-2}{2}}\left(\frac{2\pi k_1}{a}\sqrt{\sum_{j=2}^{p}(k_j L_j)^2}\right)
\end{aligned}$$

[1] When $p = 1$, this is actually the pressure on the piston when the cross section of the piston becomes an infinite hyperplane.

and

$$\begin{aligned}
&\mathcal{F}_{\mathrm{Cas}}^{\mathrm{P},T=0}(p,d;a,L_2,\dots,L_d)\\
&\quad=-\frac{d-p+1}{\pi^{\frac{d-p+2}{2}}}\Gamma\left(\frac{d-p+2}{2}\right)\zeta_R(d-p+2)a^{-d+p-2}\\
&\qquad-2(d-p+1)a^{-\frac{d-p+2}{2}}\\
&\qquad\times\sum_{k_1=1}^{\infty}\sum_{(k_2,\dots,k_p)\in\mathbb{Z}^{p-1}\setminus\{0\}}\left(\sum_{j=2}^{p}\left(\frac{k_j}{L_j}\right)^2\right)^{\frac{d-p+2}{4}}k_1^{-\frac{d-p+2}{2}}\\
&\qquad\times K_{\frac{d-p+2}{2}}\left(2\pi k_1 a\sqrt{\sum_{j=2}^{p}\left(\frac{k_j}{L_j}\right)^2}\right)-4\pi a^{-\frac{d-p}{2}}\\
&\qquad\times\sum_{k_1=1}^{\infty}\sum_{(k_2,\dots,k_p)\in\mathbb{Z}^{p-1}\setminus\{0\}}\left(\sum_{j=2}^{p}\left(\frac{k_j}{L_j}\right)^2\right)^{\frac{d-p+4}{4}}\\
&\qquad\times k_1^{-\frac{d-p}{2}}K_{\frac{d-p}{2}}\left(2\pi k_1 a\sqrt{\sum_{j=2}^{p}\left(\frac{k_j}{L_j}\right)^2}\right),
\end{aligned}$$

respectively. For a massless scalar field with Db.c. and Nb.c. and electromagnetic field with PEC b.c. and PMC b.c., the results can be obtained from that for a massless scalar field with Pb.c. by using

$$\begin{aligned}
&\mathcal{F}_{\mathrm{Cas}}^{\mathrm{D/N}}(p,d;a;L_2,\dots,L_p)\\
&\quad=2^{-p+1}\sum_{j=1}^{p}(\mp)^{p-j}\\
&\qquad\times\sum_{2\le m_1<\dots<m_{j-1}\le p}\mathcal{F}_{\mathrm{Cas}}^{\mathrm{P}}(j,j+d-p;2a;2L_{m_1},\dots,2L_{m_{j-1}});
\end{aligned}\tag{7.1}$$

$$\begin{aligned}
&\mathcal{F}_{\mathrm{Cas}}^{\mathrm{PEC}}(p,d;a;L_2,\dots,L_p)\\
&\quad=(d-1)\mathcal{F}_{\mathrm{Cas}}^{\mathrm{D}}(p,d;a;L_2,\dots,L_p)\\
&\qquad+\sum_{j=2}^{p}\mathcal{F}_{\mathrm{Cas}}^{\mathrm{D}}(p-1,d-1;a;L_2,\dots,L_{j-1},L_{j+1},\dots,L_p);
\end{aligned}\tag{7.2}$$

$$\begin{aligned}
&\mathcal{F}_{\mathrm{Cas}}^{\mathrm{PMC}}(p,d;a;L_2,\dots,L_p)\\
&\quad=\sum_{j=1}^{p}(d-p+j-1)\\
&\qquad\times\sum_{2\le m_1<\dots<m_j\le p}\mathcal{F}_{\mathrm{Cas}}^{\mathrm{D}}(j,j+d-p;a;L_{m_1},\dots,L_{m_j}),
\end{aligned}\tag{7.3}$$

which can be derived from (3.4), (3.7) and (3.8) [30]. We notice that when $p\ge 2$, the leading-order terms of the temperature correction to the Casimir force density is of order T^{d-p+2} when $T\ll 1$ for a massless scalar field with Pb.c. and Nb.c. However, for a massless scalar field with Db.c. and an electromagnetic field with PEC b.c. and PMC b.c., the temperature correction to the Casimir force density decays to zero exponentially fast when $T\to 0$. The main contribution to the low-temperature Casimir force density comes from the zero-temperature Casimir force density which, at small plate separation a, has leading order proportional to $-a^{-d-1}$, going to negative infinity. At large a, the leading-order term is proportional to $-a^{-d+p-2}$ for a massless scalar field with Pb.c. and Nb.c.; and it decays exponentially for a massless scalar field with Db.c. and an electromagnetic field with PEC b.c. and PMC b.c.

In the particular case that $p=1$, i.e. when the piston becomes a pair of infinite parallel hyperplanes, we find that the low-temperature expansion of the Casimir pressure acting on the piston is given by

$$\begin{aligned}
\mathcal{P}_{\mathrm{Cas}}^{\mathrm{P}}(d;a)&=-\frac{d\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}}\zeta_R(d+1)a^{-d-1}\\
&\quad-\frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}}T^{d+1}\zeta_R(d+1)\\
&\quad+\frac{8\pi}{a^{\frac{d+4}{2}}}T^{\frac{d-2}{2}}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}k^{\frac{d+2}{2}}l^{-\frac{d-2}{2}}\\
&\quad\times K_{\frac{d-2}{2}}\left(\frac{2\pi lk}{Ta}\right)
\end{aligned}$$

for a massless scalar field with Pb.c. For a massless scalar field with Db.c. and Nb.c., the Casimir pressure is equal to $\mathcal{P}_{\mathrm{Cas}}^{\mathrm{P}}(d;2a)$, i.e.

$$\begin{aligned}
\mathcal{P}_{\mathrm{Cas}}^{\mathrm{D/N}}(d;a)&=-\frac{d\Gamma(\frac{d+1}{2})}{(2\pi)^{\frac{d+1}{2}}}\zeta_R(d+1)a^{-d-1}\\
&\quad-\frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}}T^{d+1}\zeta_R(d+1)\\
&\quad+\frac{\pi}{2^{\frac{d-2}{2}}a^{\frac{d+4}{2}}}T^{\frac{d-2}{2}}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}k^{\frac{d+2}{2}}l^{-\frac{d-2}{2}}\\
&\quad\times K_{\frac{d-2}{2}}\left(\frac{\pi lk}{Ta}\right).
\end{aligned}$$

For an electromagnetic field with PEC b.c. and PMC b.c., the Casimir pressure is $(d-1)$ times larger than that of a massless scalar field with Db.c., i.e.

$$\mathcal{P}_{\mathrm{Cas}}^{\mathrm{PEC/PMC}}(d;a)=(d-1)\mathcal{P}_{\mathrm{Cas}}^{\mathrm{D/N}}(d;a).$$

Notice that the zero-temperature Casimir pressure on a pair of infinite parallel plates is equal to

$$\mathcal{P}_{\mathrm{Cas}}^{\mathrm{P},T=0}(d;a)=-\frac{d\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}}\zeta_R(d+1)a^{-d-1}$$

for a massless scalar field with Pb.c. It is 2^{-d-1} times weaker for a massless scalar field with Db.c. and Nb.c., i.e. $\mathcal{P}^{\mathrm{D/N},T=0}_{\mathrm{Cas}}(d;a) = 2^{-d-1}\mathcal{P}^{\mathrm{P},T=0}_{\mathrm{Cas}}(d;a)$. For an electromagnetic field with PEC b.c. or PMC b.c., the zero-temperature Casimir pressure on a pair of infinite parallel plates is

$$\mathcal{P}^{\mathrm{PEC/PMC}}_{\mathrm{Cas}}(d;a) = -\frac{d(d-1)}{(2\pi)^{\frac{d+1}{2}}}\Gamma\left(\frac{d+1}{2}\right)\zeta_R(d+1)a^{-d-1}.$$

These agree with the well-known results. For the temperature correction, we find that the leading term is of order T^{d+1} when $T \ll 1$. In particular, the leading thermal correction to the Casimir force for a massless scalar field with Dirichlet boundary condition is

$$-\frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}}T^{d+1}\zeta_R(d+1),$$

in agreement with the result in [42].

7.2 High-temperature expansion

In the high-temperature regime, we obtain from (5.4) the following expansion for the Casimir force density for a massless scalar field with Pb.c.:

$$\begin{aligned}
&\mathcal{F}^{\mathrm{P}}_{\mathrm{Cas}}(p,d;a;L_2,\ldots,L_p)\\
&\quad = B^{\mathrm{P}}(a;L_2,\ldots,L_p)T\\
&\qquad + 4\left[\prod_{j=2}^p L_j\right]T^{\frac{d+2}{2}}\sum_{l=1}^\infty\sum_{k_1=1}^\infty\sum_{(k_2,\ldots,k_p)\in\mathbb{Z}^{p-1}} l^{\frac{d}{2}}\\
&\qquad \times\left((k_1a)^2+\sum_{j=2}^p(k_jL_j)^2\right)^{-\frac{d}{4}}\\
&\qquad \times K_{\frac{d}{2}}\left(2\pi lT\sqrt{(k_1a)^2+\sum_{j=2}^p(k_jL_j)^2}\right)\\
&\qquad - 8\pi a^2\left[\prod_{j=2}^p L_j\right]T^{\frac{d+4}{2}}\\
&\qquad \times\sum_{l=1}^\infty\sum_{k_1=1}^\infty\sum_{(k_2,\ldots,k_p)\in\mathbb{Z}^{p-1}} l^{\frac{d+2}{2}}k_1^2\\
&\qquad \times\left((k_1a)^2+\sum_{j=2}^p(k_jL_j)^2\right)^{-\frac{d+2}{4}}\\
&\qquad \times K_{\frac{d+2}{2}}\left(2\pi lT\sqrt{(k_1a)^2+\sum_{j=2}^p(k_jL_j)^2}\right).
\end{aligned}$$

In the high-temperature limit, the last two terms decay exponentially, while the leading term is the classical term linear in T with coefficient $B^{\mathrm{P}}(a;L_2,\ldots,L_p)$, which can be computed from (5.6). Using (A.1), one gets

$$\begin{aligned}
&B^{\mathrm{P}}(a;L_2,\ldots,L_p)\\
&\quad = -\frac{(d-1)}{\pi^{\frac{d}{2}}}\Gamma\left(\frac{d}{2}\right)\zeta_R(d)\left[\prod_{j=2}^p L_j\right]a^{-d} + 4\pi a^{-\frac{d+3}{2}}\\
&\qquad \times\left[\prod_{j=2}^p L_j\right]\sum_{k_1=1}^\infty\sum_{(k_2,\ldots,k_p)\in\mathbb{Z}^{p-1}\setminus\{0\}}\left(\sum_{j=2}^p(k_jL_j)^2\right)^{-\frac{d-3}{4}}\\
&\qquad \times k_1^{\frac{d+1}{2}}K_{\frac{d-3}{2}}\left(\frac{2\pi k_1}{a}\sqrt{\sum_{j=2}^p(k_jL_j)^2}\right)\\
&\qquad - \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}}\left[\prod_{j=2}^p L_j\right]Z_{p-1}\left(\frac{d}{2};L_2,\ldots,L_p\right).
\end{aligned}$$

The corresponding results for a massless scalar field with Db.c. and Nb.c. and an electromagnetic field with PEC b.c. and PMC b.c. can be obtained using (7.1), (7.2) and (7.3). Again, we find that in the high-temperature limit, the leading-order term of the Casimir force density is the classical term of order T. It may be very negative when the plate separation a is small.

Consider the particular case that $p=1$. The high-temperature expansion of the Casimir pressure for the massless scalar field with Pb.c. is given by

$$\begin{aligned}
\mathcal{P}^{\mathrm{P}}_{\mathrm{Cas}}(d;a) = {}& -\frac{(d-1)}{\pi^{\frac{d}{2}}}\Gamma\left(\frac{d}{2}\right)\zeta_R(d)a^{-d}T\\
& + 4T^{\frac{d+2}{2}}a^{-\frac{d}{2}}\sum_{l=1}^\infty\sum_{k=1}^\infty l^{\frac{d}{2}}k^{-\frac{d}{2}}\\
& \times K_{\frac{d}{2}}(2\pi lkTa) - 8\pi a^{-\frac{d-2}{2}}T^{\frac{d+4}{2}}\\
& \times\sum_{l=1}^\infty\sum_{k=1}^\infty l^{\frac{d+2}{2}}k^{-\frac{d-2}{2}}K_{\frac{d+2}{2}}(2\pi lkTa).
\end{aligned}$$

For a massless scalar field with Db.c. and Nb.c., it is given by

$$\begin{aligned}
\mathcal{P}^{\mathrm{D/N}}_{\mathrm{Cas}}(d;a) = {}& -\frac{(d-1)}{2^d\pi^{\frac{d}{2}}}\Gamma\left(\frac{d}{2}\right)\zeta_R(d)a^{-d}T\\
& + 2^{-\frac{d-4}{2}}T^{\frac{d+2}{2}}a^{-\frac{d}{2}}\sum_{l=1}^\infty\sum_{k=1}^\infty l^{\frac{d}{2}}k^{-\frac{d}{2}}\\
& \times K_{\frac{d}{2}}(4\pi lkTa) - 2^{-\frac{d-6}{2}}\pi a^{-\frac{d-2}{2}}T^{\frac{d+4}{2}}\\
& \times\sum_{l=1}^\infty\sum_{k=1}^\infty l^{\frac{d+2}{2}}k^{-\frac{d-2}{2}}K_{\frac{d+2}{2}}(4\pi lkTa). \qquad (7.4)
\end{aligned}$$

For an electromagnetic field with PEC b.c. and PMC b.c., it is $(d-1)$ times (7.4). The leading term in (7.4) agrees with those given in [42].

8 Conclusion

We consider a finite-temperature massless scalar field with Pb.c., Db.c. and Nb.c. and an electromagnetic field with PEC b.c. and PMC b.c. for any number of space dimensions d. For a rectangular piston, which can be considered as a one-sided open rectangular cavity divided into two regions, it is shown that one can obtain a finite unambiguous Casimir force acting on the piston. Different exact expressions of the Casimir force which are suitable for studying the small and large plate separation limits and low- and high-temperature limits are derived. It is verified analytically that for all the cases we considered, although the regularized Casimir force acting on a wall of a rectangular cavity can be attractive or repulsive, depending on the relative size of the cavity, the Casimir force acting on the piston is always an attractive force. Moreover, the magnitude of the Casimir force decreases as the separation distance between the piston and the opposite wall increases. Another interesting result obtained in this paper is that, at high temperature, the magnitude of the Casimir force is found to grow linearly in temperature T. This is in contrast to the result for rectangular cavities, where at high temperature the leading term of the Casimir force is the Stefan–Boltzmann term of order T^{d+1}, an order much larger than T. It also shows that the Casimir force has a classical $\hbar\to 0$ limit. On the other hand, we also establish that at low temperature, the effect of temperature to the magnitude of the Casimir force is insignificant when the plate separation is small.

We have derived exact expressions for the Casimir force acting on the piston, which are suitable for studying low- and high-temperature, and small and large plate separation limits. A more detailed numerical study of the results will be considered in a future work. The methods used in this paper can easily be extended to other quantum fields as well as pistons with an arbitrary cross section. In particular, it will be interesting to consider a massive field, a fermionic field or a massless field with mixed boundary conditions. The latter will be a possible candidate for a repulsive Casimir force. Another possible direction for further research is to study the Casimir piston made of dielectric and magnetic materials.

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Appendix A: Formulas for Epstein zeta function and Casimir energy

Here we gather the formulas we need for computing the Epstein zeta function (2.8). The Chowla–Selberg formula [34, 35, 38, 39] for the Epstein zeta function tells us that

$$\begin{aligned}
&Z_d(s;a_1,\dots,a_d)\\
&=Z_p(s;a_1,\dots,a_p)+\frac{\pi^{p/2}\Gamma(s-\frac{p}{2})}{[\prod_{j=1}^p a_j]\Gamma(s)}\\
&\quad\times Z_{d-p}\left(s-\frac{p}{2};a_{p+1},\dots,a_d\right)+\frac{2\pi^s}{[\prod_{j=1}^p a_j]\Gamma(s)}\\
&\quad\times\sum_{\mathbf{k}\in(\mathbb{Z}^p\setminus\{0\})\times(\mathbb{Z}^{d-p}\setminus\{0\})}\left(\frac{\sum_{j=1}^p(\frac{k_j}{a_j})^2}{\sum_{j=p+1}^d(k_ja_j)^2}\right)^{\frac{2s-p}{4}}\\
&\quad\times K_{s-\frac{p}{2}}\left(2\pi\sqrt{\left(\sum_{j=1}^p\left(\frac{k_j}{a_j}\right)^2\right)\left(\sum_{j=p+1}^d(k_ja_j)^2\right)}\right).
\end{aligned}\tag{A.1}$$

Taking the derivative at $s=0$, we have

$$\begin{aligned}
&Z'_{d+1}(0;a_1,\dots,a_d)\\
&=Z'_p(0;a_1,\dots,a_p)+\frac{\pi^{\frac{p}{2}}\Gamma(-\frac{p}{2})}{[\prod_{j=1}^p a_j]}\\
&\quad\times Z_{d+1-p}\left(-\frac{p}{2};a_{p+1},\dots,a_{d+1}\right)+\frac{2}{[\prod_{j=1}^p a_j]}\\
&\quad\times\sum_{\mathbf{k}\in(\mathbb{Z}^p\setminus\{\mathbf{0}\})\times(\mathbb{Z}^{d+1-p}\setminus\{\mathbf{0}\})}\left(\frac{\sum_{j=1}^p[\frac{k_j}{a_j}]^2}{\sum_{j=p+1}^{d+1}[a_jk_j]^2}\right)^{-\frac{p}{4}}\\
&\quad\times K_{\frac{p}{2}}\left(2\pi\sqrt{\left(\sum_{j=1}^p\left[\frac{k_j}{a_j}\right]^2\right)\left(\sum_{j=p+1}^{d+1}[a_jk_j]^2\right)}\right).
\end{aligned}\tag{A.2}$$

From this, we obtain the following alternative expressions for the regularized Casimir energy, $E_{\text{Cas, reg}}^{\text{P}}(L_1,\dots,L_d)$, see (2.12), for a massless scalar field with Pb.c.:

$$\begin{aligned}
&E_{\text{Cas, reg}}^{\text{P}}(L_1,\dots,L_d)\\
&=T\log L_1+T\log T+\pi TL_1Z_d\left(-\frac{1}{2};\frac{1}{L_2},\dots,\frac{1}{L_d},T\right)\\
&\quad-T\sum_{k_1=1}^{\infty}\sum_{(k_2,\dots,k_d,l)\in\mathbb{Z}^d\setminus\{0\}}k_1^{-1}\\
&\quad\times\exp\left(-2\pi k_1L_1\sqrt{\sum_{j=2}^d\left(\frac{k_j}{L_j}\right)^2+(lT)^2}\right).
\end{aligned}\tag{A.3}$$

$$\begin{aligned}
&E^{\mathrm{P}}_{\text{Cas, reg}}(L_1,\ldots,L_d)\\
&\quad=-\frac{T}{2}Z'_{E,d}\left(0;T,\frac{1}{L_2},\ldots,\frac{1}{L_d}\right)-T\log\frac{2\pi}{T}\\
&\qquad-\frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}L_1^d}\left[\prod_{j=2}^d L_i\right]\zeta_R(d+1)\\
&\qquad-2L_1^{-\frac{d}{2}}\left[\prod_{j=2}^d L_i\right]\sum_{k_1=1}^\infty\sum_{(k_2,\ldots,k_d,l)\in\mathbb{Z}^d\setminus\{0\}}k_1^{\frac{d}{2}}\\
&\qquad\times\left(\sum_{j=2}^d(k_jL_j)^2+\left(\frac{l}{T}\right)^2\right)^{-\frac{d}{4}}\\
&\qquad\times K_{\frac{d}{2}}\left(\frac{2\pi k_1}{L_1}\sqrt{\sum_{j=2}^d(k_jL_j)^2+\left(\frac{l}{T}\right)^2}\right).
\end{aligned}\tag{A.4}$$

$$\begin{aligned}
&E^{\mathrm{P}}_{\text{Cas, reg}}(L_1,\ldots,L_d)\\
&\quad=-\frac{\Gamma(\frac{d+1}{2})\zeta_R(d+1)}{\pi^{\frac{d+1}{2}}}\left[\prod_{j=1}^d L_j\right]T^{d+1}-T\log\frac{2\pi}{T}\\
&\qquad-\frac{T}{2}Z'_d\left(0;\frac{1}{L_1},\ldots,\frac{1}{L_d}\right)\\
&\qquad-2\left[\prod_{j=1}^d L_j\right]T^{\frac{d+2}{2}}\sum_{l=1}^\infty\sum_{\mathbf{k}\in\mathbb{Z}^d\setminus\{0\}}l^{\frac{d}{2}}\\
&\qquad\times\left(\sum_{j=1}^d(k_jL_j)^2\right)^{-\frac{d}{4}}K_{\frac{d}{2}}\left(2\pi lT\sqrt{\sum_{j=1}^d(k_jL_j)^2}\right).
\end{aligned}\tag{A.5}$$

Appendix B: One-, two- and three-dimensional pistons

In this appendix, we present the results we obtained in the previous sections to the special cases that $d=1$, 2 and 3.

B.1 $d=1$

As we mentioned in Sect. 2, there are no electromagnetic fields in the case of number of dimensions $d=1$. For a massless scalar field with Pb.c., we find from (5.1) that the low-temperature expansion of the Casimir force is equal to

$$F^{\mathrm{P}}_{\mathrm{Cas}}(a)=F^{\mathrm{P},T=0}_{\mathrm{Cas}}(a)-\frac{\pi T^2}{6}+\frac{4\pi}{a^2}\sum_{l=1}^\infty\sum_{k=1}^\infty k\exp\left(-\frac{2\pi kl}{Ta}\right),\tag{B.1}$$

where $F^{\mathrm{P},T=0}_{\mathrm{Cas}}(a)$ is the zero-temperature Casimir force given by

$$F^{\mathrm{P},T=0}_{\mathrm{Cas}}(a)=-\frac{\pi}{6a^2}.$$

For the high-temperature expansion of the Casimir force, (5.4) give

$$F^{\mathrm{P}}_{\mathrm{Cas}}(a)=-\frac{T}{a}-4\pi T^2\sum_{k=1}^\infty\sum_{l=1}^\infty l\mathrm{e}^{-2\pi klTa}.\tag{B.2}$$

This formula can also directly be derived from (3.3). Equation (B.2) shows that the classical limit of the Casimir force due to a massless scalar field with Pb.c. is given by $-T/a$. For a massless scalar field with Db.c. and Nb.c., we have

$$F^{\mathrm{D/N}}_{\mathrm{Cas}}(a)=F^{\mathrm{P}}_{\mathrm{Cas}}(2a).$$

It is interesting to note that (B.1) and (B.2) give us the identity

$$\begin{aligned}
&-\frac{\pi}{6}-\frac{\pi z^2}{6}+4\pi\sum_{k=1}^\infty\sum_{l=1}^\infty\mathrm{e}^{-\frac{2\pi kl}{z}}\\
&\quad=-z-4\pi z^2\sum_{k=1}^\infty\sum_{l=1}^\infty l\mathrm{e}^{-2\pi klz}.
\end{aligned}$$

B.2 $d=2$

In $d=2$ dimensions, the Casimir force acting on a piston for a massless scalar field with Db.c. and an electromagnetic field with PMC b.c. coincide; and the Casimir force for a massless scalar field with Nb.c. and an electromagnetic field with PEC b.c. coincide. Moreover,

$$\begin{aligned}
F^{\mathrm{D/PMC}}_{\mathrm{Cas}}(a;L_2)&=\frac{1}{2}\big(F^{\mathrm{P}}_{\mathrm{Cas}}(2a;2L_2)-F^{\mathrm{P}}_{\mathrm{Cas}}(2a)\big);\\
F^{\mathrm{N/PEC}}_{\mathrm{Cas}}(a;L_2)&=\frac{1}{2}\big(F^{\mathrm{P}}_{\mathrm{Cas}}(2a;2L_2)+F^{\mathrm{P}}_{\mathrm{Cas}}(2a)\big).
\end{aligned}\tag{B.3}$$

Denote L_2 by b. Using (5.1), we find that the low-temperature expansion of the Casimir force acting on the piston for a massless scalar field with Pb.c. is

$$\begin{aligned}
F^{\mathrm{P}}_{\mathrm{Cas}}(a;b)&=F^{\mathrm{P},T=0}_{\mathrm{Cas}}(a;b)\\
&\quad+\frac{4\pi}{a^3}\sum_{l=1}^\infty\sum_{k_1=1}^\infty\sum_{k_2=-\infty}^\infty\\
&\quad\times k_1^2\left(\sqrt{\left(\frac{k_1}{a}\right)^2+\left(\frac{k_2}{b}\right)^2}\right)^{-1}\\
&\quad\times\exp\left(-\frac{2\pi l}{T}\sqrt{\left(\frac{k_1}{a}\right)^2+\left(\frac{k_2}{b}\right)^2}\right)
\end{aligned}$$

$$-\frac{\pi T^2}{6}-\frac{4T}{b}\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}kl^{-1}K_1\left(\frac{2\pi kl}{Tb}\right).$$

When a is small, the zero-temperature Casimir force has the representation

$$F_{\text{Cas}}^{\text{P},T=0}(a;b)=-\frac{\zeta_R(3)}{\pi}\frac{b}{a^3}-\frac{\zeta_R(3)}{2\pi}\frac{1}{b^2}+\frac{8\pi b}{a^3}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}k_1^2K_0\left(\frac{2\pi k_1k_2b}{a}\right).$$

When a is large, it has another representation:

$$F_{\text{Cas}}^{\text{P},T=0}(a;b)=-\frac{\pi}{6a^2}-\frac{4}{ab}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\frac{k_2}{k_1}K_1\left(\frac{2\pi k_1k_2a}{b}\right)-\frac{8\pi}{b^2}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}k_2^2K_0\left(\frac{2\pi k_1k_2a}{b}\right).$$

The high-temperature expansion of the Casimir force can be computed by (5.4), which gives

$$\begin{aligned}&F_{\text{Cas}}^{\text{P}}(a;b)\\&=-T\left\{\frac{\pi b}{6a^2}+\frac{\pi}{6b}-\frac{4\pi b}{a^2}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}k_1e^{-\frac{2\pi k_1k_2b}{a}}\right\}\\&\quad+4bT^2\sum_{l=1}^{\infty}\sum_{k_1=1}^{\infty}\sum_{k_2=-\infty}^{\infty}l\\&\quad\times\left((k_1a)^2+(k_2b)^2\right)^{-\frac{1}{2}}K_1\left(2\pi lT\sqrt{(k_1a)^2+(k_2b)^2}\right)\\&\quad-8\pi a^2bT^3\sum_{l=1}^{\infty}\sum_{k_1=1}^{\infty}\sum_{k_2=-\infty}^{\infty}l^2k_1^2\\&\quad\times\left((k_1a)^2+(k_2b)^2\right)^{-1}\\&\quad\times K_2\left(2\pi lT\sqrt{(k_1a)^2+(k_2b)^2}\right).\end{aligned}$$

The first term gives the classical limit of the Casimir force for a massless scalar field with Pb.c.

For an electromagnetic field with PEC b.c. or PMC b.c., we can use (B.3) to obtain the low-temperature expansion and the high-temperature expansion from the corresponding expansions for a massless scalar field with Pb.c. For a low temperature, we have

$$\begin{aligned}&F_{\text{Cas}}^{\text{PEC/PMC}}(a;b)\\&=F_{\text{Cas}}^{\text{PEC/PMC},T=0}(a;b)+\frac{\pi}{a^3}\sum_{l=1}^{\infty}\sum_{k_1=1}^{\infty}\sum_{k_2\in(\mathbb{N}\cup\{0\})/\mathbb{N}}k_1^2\\&\quad\times\left(\sqrt{\left(\frac{k_1}{a}\right)^2+\left(\frac{k_2}{b}\right)^2}\right)^{-1}\\&\quad\times\exp\left(-\frac{\pi l}{T}\sqrt{\left(\frac{k_1}{a}\right)^2+\left(\frac{k_2}{b}\right)^2}\right)-\frac{\pi T^2}{6}\delta\\&\quad-\frac{T}{b}\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}kl^{-1}K_1\left(\frac{\pi kl}{Tb}\right).\end{aligned}$$

Here $\delta^{\text{PEC}}=1$ and $\delta^{\text{PMC}}=0$, and the zero-temperature Casimir force is given by

$$\begin{aligned}&F_{\text{Cas}}^{\text{PEC/PMC},T=0}(a;b)\\&=\frac{1}{8}F_{\text{Cas}}^{\text{P},T=0}(a;b)\mp\frac{\pi}{48a^2}\\&=-\frac{\zeta_R(3)}{8\pi}\frac{b}{a^3}-\frac{\zeta_R(3)}{16\pi}\frac{1}{b^2}\\&\quad+\frac{\pi b}{a^3}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}k_1^2K_0\left(\frac{2\pi k_1k_2b}{a}\right)\mp\frac{\pi}{48a^2},\end{aligned}\tag{B.4}$$

or

$$\begin{aligned}&F_{\text{Cas}}^{\text{PEC/PMC},T=0}(a;b)\\&=-\delta\frac{\pi}{24a^2}-\frac{1}{2ab}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\frac{k_2}{k_1}K_1\left(\frac{2\pi k_1k_2a}{b}\right)\\&\quad-\frac{\pi}{b^2}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}k_2^2K_0\left(\frac{2\pi k_1k_2a}{b}\right).\end{aligned}\tag{B.5}$$

For an electromagnetic field with PEC b.c. or equivalently a massless scalar field with Nb.c., (B.4) and (B.5) agree with the corresponding formulas in [18]. For an electromagnetic field with PMC or equivalently a massless scalar field with Db.c., they agree with the results of [11, 19].

For high temperature,

$$\begin{aligned}&F_{\text{Cas}}^{\text{PEC/PMC}}(a;b)\\&=-T\left\{\pm\frac{1}{4a}+\frac{\pi b}{24a^2}+\frac{\pi}{24b}\right.\\&\quad\left.-\frac{\pi b}{a^2}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}k_1e^{-\frac{2\pi k_1k_2b}{a}}\right\}+2bT^2\\&\quad\times\sum_{l=1}^{\infty}\sum_{k_1=1}^{\infty}\sum_{k_2=-\infty}^{\infty}l\left((k_1a)^2+(k_2b)^2\right)^{-\frac{1}{2}}\\&\quad\times K_1\left(4\pi lT\sqrt{(k_1a)^2+(k_2b)^2}\right)-8\pi a^2bT^3\sum_{l=1}^{\infty}\sum_{k_1=1}^{\infty}\\&\quad\times\sum_{k_2=-\infty}^{\infty}l^2k_1^2\left((k_1a)^2+(k_2b)^2\right)^{-1}\\&\quad\times K_2\left(4\pi lT\sqrt{(k_1a)^2+(k_2b)^2}\right)\end{aligned}$$

$$\mp 2\pi T^2\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} l e^{-4\pi klTa}.$$

The first term is the classical limit of the Casimir force for electromagnetic field.

In the $b\to\infty$ limit, we find that the pressure on the infinite piston $x_1=a$ due to a massless scalar field with Pb.c. is

$$\begin{aligned}\mathcal{P}_{\mathrm{Cas}}^{\mathrm{P}}(a) &= -\frac{\zeta_R(3)}{\pi a^3}-\frac{T^3}{2\pi}\zeta_R(3)\\ &\quad+\frac{8\pi}{a^3}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}k^2K_0\left(\frac{2\pi kl}{Ta}\right),\end{aligned}$$

or

$$\begin{aligned}\mathcal{P}_{\mathrm{Cas}}^{\mathrm{P}}(a) &= -\frac{\pi}{6}\frac{T}{a^2}+\frac{4T^2}{a}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}lk_1^{-1}K_1(2\pi klTa)\\ &\quad-8\pi T^3\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}l^2K_2(2\pi klTa).\end{aligned}$$

For a massless scalar field with Db.c. or Nb.c., $\mathcal{P}_{\mathrm{Cas}}^{\mathrm{D/N}}(a)=\mathcal{P}_{\mathrm{Cas}}^{\mathrm{P}}(2a)$.

B.3 $d=3$

When $d=3$, the Casimir force for the electromagnetic field with PEC b.c. and PMC b.c. coincide. Moreover,

$$\begin{aligned}&F_{\mathrm{Cas}}^{\mathrm{D/N}}(a;L_2,L_3)\\ &\quad=\frac{1}{4}\big(F_{\mathrm{Cas}}^{\mathrm{P}}(2a;2L_2,2L_3)\mp F_{\mathrm{Cas}}(2a;2L_2)\\ &\qquad\mp F_{\mathrm{Cas}}^{\mathrm{P}}(2a;2L_3)+F_{\mathrm{Cas}}^{\mathrm{P}}(2a)\big),\\ &F_{\mathrm{Cas}}^{\mathrm{PEC}}(a;L_2,L_3)\\ &\quad=2F_{\mathrm{Cas}}^{\mathrm{D}}(a;L_2,L_3)+F_{\mathrm{Cas}}^{\mathrm{D}}(a;L_2)+F_{\mathrm{Cas}}^{\mathrm{D}}(a;L_3)\\ &\quad=\frac{1}{2}\big(F_{\mathrm{Cas}}^{\mathrm{P}}(2a;2L_2,2L_3)-F_{\mathrm{Cas}}^{\mathrm{P}}(2a)\big).\end{aligned}\tag{B.6}$$

Setting $L_2=b$ and $L_3=c$, we find from (5.1) that in the low-temperature limit the Casimir force acting on the piston for a massless scalar field with Pb.c. is

$$\begin{aligned}&F_{\mathrm{Cas}}^{\mathrm{P}}(a;b,c)\\ &\quad=F_{\mathrm{Cas}}^{\mathrm{P},T=0}(a;b,c)+\frac{4\pi}{a^3}\sum_{l=1}^{\infty}\sum_{k_1=1}^{\infty}\sum_{(k_2,k_3)\in\mathbb{Z}^2}k_1^2\\ &\qquad\times\left(\sqrt{\left(\frac{k_1}{a}\right)^2+\left(\frac{k_2}{b}\right)^2+\left(\frac{k_3}{c}\right)^2}\right)^{-1}\\ &\qquad\times\exp\left(-\frac{2\pi l}{T}\sqrt{\left(\frac{k_1}{a}\right)^2+\left(\frac{k_2}{b}\right)^2+\left(\frac{k_3}{c}\right)^2}\right)-\frac{\pi T^2}{6}\\ &\qquad-2T\sum_{l=1}^{\infty}\sum_{(k_2,k_3)\in\mathbb{Z}^2\setminus\{0\}}l^{-1}\sqrt{\left(\frac{k_2}{b}\right)^2+\left(\frac{k_3}{c}\right)^2}\\ &\qquad\times K_1\left(\frac{2\pi l}{T}\sqrt{\left(\frac{k_2}{b}\right)^2+\left(\frac{k_3}{c}\right)^2}\right)\end{aligned}$$

and for a massless scalar field with Db.c. and Nb.c. and an electromagnetic field with PEC b.c. and PMC b.c. it is

$$\begin{aligned}&F_{\mathrm{Cas}}(a;b,c)\\ &\quad=F_{\mathrm{Cas}}^{T=0}(a;b,c)\\ &\qquad+\frac{\pi}{a^3}\sum_{l=1}^{\infty}\sum_{k_1=1}^{\infty}\sum_{(k_2,k_3)\in(\mathbb{N}\cup\{0\})^2}k_1^2\Lambda(k_2,k_3)\\ &\qquad\times\left(\sqrt{\left(\frac{k_1}{a}\right)^2+\left(\frac{k_2}{b}\right)^2+\left(\frac{k_3}{c}\right)^2}\right)^{-1}\\ &\qquad\times\exp\left(-\frac{\pi l}{T}\sqrt{\left(\frac{k_1}{a}\right)^2+\left(\frac{k_2}{b}\right)^2+\left(\frac{k_3}{c}\right)^2}\right)-\delta\frac{\pi T^2}{6}\\ &\qquad-T\sum_{l=1}^{\infty}\sum_{(k_2,k_3)\in(\mathbb{N}\cup\{0\})^2\setminus\{0\}}l^{-1}\Lambda(k_2,k_3)\\ &\qquad\times\sqrt{\left(\frac{k_2}{b}\right)^2+\left(\frac{k_3}{c}\right)^2}K_1\left(\frac{\pi l}{T}\sqrt{\left(\frac{k_2}{b}\right)^2+\left(\frac{k_3}{c}\right)^2}\right).\end{aligned}$$

Here $\Lambda^{\mathrm{D}}(k_2,k_3)=1$ if and only if both k_2 and k_3 are nonzero; $\Lambda^{\mathrm{N}}(k_2,k_3)=1$ for all $(k_2,k_3)\in(\mathbb{N}\cup\{0\})^2$, $\Lambda^{\mathrm{PEC}}(k_2,k_3)=2$ if both k_2 and k_3 are nonzero, $\Lambda^{\mathrm{PEC}}(k_2,k_3)=1$ if exactly one of the k_2 or k_3 is zero and $\Lambda^{\mathrm{PEC}}(k_2,k_3)=0$ if $k_2=k_3=0$. The zero-temperature Casimir force for a massless scalar field with Pb.c. can be expressed as

$$\begin{aligned}&F_{\mathrm{Cas}}^{\mathrm{P},T=0}(a;b,c)\\ &\quad=-\frac{\pi^2bc}{30a^4}-\frac{bc}{2\pi^2}Z_2(2;b,c)\\ &\qquad+\frac{2\pi bc}{a^3}\sum_{k_1=1}^{\infty}\sum_{(k_2,k_3)\in\mathbb{Z}^2\setminus\{0\}}k_1^2\big((k_2b)^2+(k_3c)^2\big)^{-1/2}\\ &\qquad\times\exp\left(-\frac{2\pi k_1}{a}\sqrt{(k_2b)^2+(k_3c)^2}\right)\\ &\quad=-\frac{\pi^2bc}{30a^4}-\frac{\pi^2}{90}\frac{c}{b^3}-\frac{\zeta_R(3)}{2\pi}\frac{1}{c^2}\\ &\qquad-\frac{4}{b^{\frac{3}{2}}c^{\frac{1}{2}}}\sum_{k_2=1}^{\infty}\sum_{k_3=1}^{\infty}\left(\frac{k_2}{k_3}\right)^{\frac{3}{2}}K_{\frac{3}{2}}\left(\frac{2\pi k_2k_3c}{b}\right)\end{aligned}$$

$$+\frac{2\pi bc}{a^3}\sum_{k_1=1}^{\infty}\sum_{(k_2,k_3)\in\mathbb{Z}^2\setminus\{0\}}k_1^2\big((k_2b)^2+(k_3c)^2\big)^{-1/2}$$
$$\times\exp\left(-\frac{2\pi k_1}{a}\sqrt{(k_2b)^2+(k_3c)^2}\right),$$

or

$$F_{\text{Cas}}^{\text{P},T=0}(a;b,c)$$
$$=-\frac{\pi}{6a^2}-\frac{2}{a}\sum_{k_1=1}^{\infty}\sum_{(k_2,k_3)\in\mathbb{Z}^2\setminus\{0\}}k_1^{-1}\sqrt{\left(\frac{k_2}{b}\right)^2+\left(\frac{k_3}{c}\right)^2}$$
$$\times K_1\left(2\pi k_1a\sqrt{\left(\frac{k_2}{b}\right)^2+\left(\frac{k_3}{c}\right)^2}\right)$$
$$-4\pi\sum_{k_1=1}^{\infty}\sum_{(k_2,k_3)\in\mathbb{Z}^2\setminus\{0\}}\left(\left(\frac{k_2}{b}\right)^2+\left(\frac{k_3}{c}\right)^2\right)$$
$$\times K_0\left(2\pi k_1a\sqrt{\left(\frac{k_2}{b}\right)^2+\left(\frac{k_3}{c}\right)^2}\right),$$

which are suitable for studying the small-a and large-a behaviors of the Casimir force. For a massless scalar field with Db.c. and Nb.c., the corresponding formulas are given respectively by

$$F_{\text{Cas}}^{\text{D/N},T=0}(a;b,c)$$
$$=-\frac{\pi^2bc}{480a^4}\pm\frac{b+c}{16\pi a^3}\zeta_R(3)-\frac{\pi^2}{1440}\frac{c}{b^3}$$
$$\pm\frac{\zeta_R(3)}{32\pi}\left(\frac{1}{b^2}+\frac{1}{c^2}\right)-\frac{\zeta_R(3)}{32\pi}\frac{1}{c^2}-\frac{\pi}{96a^2}$$
$$-\frac{1}{4b^{\frac32}c^{\frac12}}\sum_{k_2=1}^{\infty}\sum_{k_3=1}^{\infty}\left(\frac{k_2}{k_3}\right)^{\frac32}K_{\frac32}\left(\frac{2\pi k_2k_3c}{b}\right)$$
$$\mp\left\{\frac{\pi b}{2a^3}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}k_1^2K_0\left(\frac{2\pi k_1k_2b}{a}\right)+b\longleftrightarrow c\right\}$$
$$+\frac{\pi bc}{8a^3}\sum_{k_1=1}^{\infty}\sum_{(k_2,k_3)\in\mathbb{Z}^2\setminus\{0\}}k_1^2\big((k_2b)^2+(k_3c)^2\big)^{-1/2}$$
$$\times\exp\left(-\frac{2\pi k_1}{a}\sqrt{(k_2b)^2+(k_3c)^2}\right)\tag{B.7}$$

and

$$F_{\text{Cas}}^{\text{D/N},T=0}(a;b,c)$$
$$=-\delta\frac{\pi}{24a^2}-\frac{1}{2a}\sum_{k_1=1}^{\infty}\sum_{(k_2,k_3)\in(\mathbb{N}\cup\{0\})^2\setminus\{0\}}k_1^{-1}$$
$$\times\Lambda(k_2,k_3)\sqrt{\left(\frac{k_2}{b}\right)^2+\left(\frac{k_3}{c}\right)^2}$$
$$\times K_1\left(2\pi k_1a\sqrt{\left(\frac{k_2}{b}\right)^2+\left(\frac{k_3}{c}\right)^2}\right)$$
$$-\pi\sum_{k_1=1}^{\infty}\sum_{(k_2,k_3)\in(\mathbb{N}\cup\{0\})^2\setminus\{0\}}\Lambda(k_2,k_3)$$
$$\times\left(\left(\frac{k_2}{b}\right)^2+\left(\frac{k_3}{c}\right)^2\right)$$
$$\times K_0\left(2\pi k_1a\sqrt{\left(\frac{k_2}{b}\right)^2+\left(\frac{k_3}{c}\right)^2}\right).\tag{B.8}$$

Here $\delta^{\text{D}}=0$ and $\delta^{\text{N}}=1$. Equation (B.7) agrees with the results of [17, 18]. For an electromagnetic field with PEC b.c. or PMC b.c., the large-a expansion for the Casimir force is still given by (B.8) with $\delta^{\text{PEC}}=0$, whereas the small-a expansion is

$$F_{\text{Cas}}^{\text{PEC},T=0}(a;b,c)$$
$$=-\frac{\pi^2bc}{240a^4}-\frac{\pi^2}{720}\frac{c}{b^3}-\frac{\zeta_R(3)}{16\pi}\frac{1}{c^2}+\frac{\pi}{48a^2}$$
$$-\frac{1}{2b^{\frac32}c^{\frac12}}\sum_{k_2=1}^{\infty}\sum_{k_3=1}^{\infty}\left(\frac{k_2}{k_3}\right)^{\frac32}K_{\frac32}\left(\frac{2\pi k_2k_3c}{b}\right)$$
$$+\frac{\pi bc}{4a^3}\sum_{k_1=1}^{\infty}\sum_{(k_2,k_3)\in\mathbb{Z}^2\setminus\{0\}}k_1^2\big((k_2b)^2+(k_3c)^2\big)^{-1/2}$$
$$\times\exp\left(-\frac{2\pi k_1}{a}\sqrt{(k_2b)^2+(k_3c)^2}\right).$$

This agrees with the results of [12, 13, 19].

For the high-temperature expansion of the Casimir force acting on the piston due to a massless scalar field with Pb.c., we obtain from (5.4) the formula

$$F_{\text{Cas}}^{\text{P}}(a;b,c)$$
$$=-T\left\{\frac{1}{a}+2\pi\sum_{k_1=1}^{\infty}\sum_{(k_2,k_3)\in\mathbb{Z}^2\setminus\{0\}}\sqrt{\left(\frac{k_2}{b}\right)^2+\left(\frac{k_3}{c}\right)^2}\right.$$
$$\left.\times\exp\left(-2\pi k_1a\sqrt{\left(\frac{k_2}{b}\right)^2+\left(\frac{k_3}{c}\right)^2}\right)\right\}$$
$$+4bcT^{\frac52}\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\sum_{(k_2,k_3)\in\mathbb{Z}^2}l^{\frac32}$$
$$\times\big((k_1a)^2+(k_2b)^2+(k_3c)^2\big)^{-\frac34}$$

$$\times K_{\frac{3}{2}}\left(2\pi l T\sqrt{(k_1a)^2+(k_2b)^2+(k_3c)^2}\right)$$
$$-8\pi a^2bcT^{\frac{7}{2}}\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\sum_{(k_2,k_3)\in\mathbb{Z}^2} l^{\frac{5}{2}}k_1^2$$
$$\times\left((k_1a)^2+(k_2b)^2+(k_3c)^2\right)^{-\frac{5}{4}}$$
$$\times K_{\frac{5}{2}}\left(2\pi l T\sqrt{(k_1a)^2+(k_2b)^2+(k_3c)^2}\right). \qquad \text{(B.9)}$$

The Bessel functions $K_{3/2}(z)$ and $K_{5/2}(z)$ can be re-expressed by elementary functions using the formulas

$$K_{\frac{3}{2}}(z)=\sqrt{\frac{\pi}{2z}}\left(1+\frac{1}{z}\right),$$
$$K_{\frac{5}{2}}(z)=\sqrt{\frac{\pi}{2z}}\left(1+\frac{3}{z}+\frac{3}{z^2}\right).$$

The high-temperature expansion of the Casimir force for a massless scalar field with Db.c. and Nb.c. and an electromagnetic field with PEC b.c. or PMC b.c. can be obtained using (B.6). The first term in (B.9) gives the classical term of the Casimir force for a massless scalar field with Pb.c. It has an alternative expression given by (5.6)

$$F_{\text{Cas}}^{\text{P,classical}}(a;b,c)$$
$$=-T\left\{\frac{\zeta_R(3)}{\pi}\frac{bc}{a^3}+\frac{bc}{4\pi}Z_2\left(\frac{3}{2};b,c\right)\right.$$
$$-\frac{4\pi bc}{a^3}\sum_{k_1=1}^{\infty}\sum_{(k_2,k_3)\in\mathbb{Z}^2\setminus\{0\}}k_1^2$$
$$\left.\times K_0\left(\frac{2\pi k_1}{a}\sqrt{(k_2b)^2+(k_3c)^2}\right)\right\}$$
$$=-T\left\{\frac{\zeta_R(3)}{\pi}\frac{bc}{a^3}+\frac{\zeta_R(3)}{2\pi}\frac{c}{b^2}+\frac{\pi}{6c}\right.$$
$$+\frac{4}{b}\sum_{k_2=1}^{\in fty}\sum_{k_3=1}^{\infty}\frac{k_2}{k_3}K_1\left(\frac{2\pi k_2k_3c}{b}\right)$$
$$-\frac{4\pi bc}{a^3}\sum_{k_1=1}^{\infty}\sum_{(k_2,k_3)\in\mathbb{Z}^2\setminus\{0\}}k_1^2$$
$$\left.\times K_0\left(\frac{2\pi k_1}{a}\sqrt{(k_2b)^2+(k_3c)^2}\right)\right\}. \qquad \text{(B.10)}$$

For a massless scalar field with Db.c. and Nb.c., the classical term of the Casimir force is

$$F_{\text{Cas}}^{\text{D/N,classical}}(a;b,c)$$
$$=-T\left\{\frac{\zeta_R(3)}{8\pi}\frac{bc}{a^3}\mp\frac{\pi(b+c)}{48a^2}+\frac{1}{8a}\right.$$
$$+\frac{\zeta_R(3)}{16\pi}\frac{c}{b^2}+\frac{\pi}{48c}\mp\frac{\pi}{48}\left(\frac{1}{b}+\frac{1}{c}\right)$$
$$\pm\frac{\pi b}{2a^2}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}k_1\mathrm{e}^{-\frac{2\pi k_1k_2b}{a}}$$
$$\pm\frac{\pi c}{2a^2}\sum_{k_1=1}^{\infty}\sum_{k_3=1}^{\infty}k_1\mathrm{e}^{-\frac{2\pi k_1k_3c}{a}}$$
$$+\frac{1}{2b}\sum_{k_2=1}^{\infty}\sum_{k_3=1}^{\infty}\frac{k_2}{k_3}K_1\left(\frac{2\pi k_2k_3c}{b}\right)$$
$$-\frac{\pi bc}{2a^3}\sum_{k_1=1}^{\infty}\sum_{(k_2,k_3)\in\mathbb{Z}^2\setminus\{0\}}k_1^2$$
$$\left.\times K_0\left(\frac{2\pi k_1}{a}\sqrt{(k_2b)^2+(k_3c)^2}\right)\right\}.$$

For an electromagnetic field with PEC b.c. or PMC b.c., the classical term is given by

$$F_{\text{Cas}}^{\text{PEC/PMC,classical}}(a;b,c)$$
$$=-T\left\{\frac{\zeta_R(3)}{\pi}\frac{bc}{4a^3}-\frac{1}{4a}+\frac{\zeta_R(3)}{8\pi}\frac{c}{b^2}+\frac{\pi}{24c}\right.$$
$$+\frac{1}{b}\sum_{k_2=1}^{\infty}\sum_{k_3=1}^{\infty}\frac{k_2}{k_3}K_1\left(\frac{2\pi k_2k_3c}{b}\right)$$
$$-\frac{\pi bc}{a^3}\sum_{k_1=1}^{\infty}\sum_{(k_2,k_3)\in\mathbb{Z}^2\setminus\{0\}}k_1^2$$
$$\left.\times K_0\left(\frac{2\pi k_1}{a}\sqrt{(k_2b)^2+(k_3c)^2}\right)\right\}.$$

In the infinite parallel-plate limit, i.e. $b=c\to\infty$, the Casimir pressure on the plates due to a massless scalar field with Pb.c. is

$$\mathcal{P}_{\text{Cas}}^{\text{P}}(a)=-\frac{\pi^2}{30a^4}-\frac{\pi^2T^4}{90}$$
$$+\frac{4\pi T}{a^3}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}k^2l^{-1}\exp\left(-\frac{2\pi lk}{Ta}\right)$$

or

$$\mathcal{P}_{\text{Cas}}^{\text{P}}(a)=-\frac{\zeta_R(3)T}{\pi a^3}+4\frac{T^{\frac{5}{2}}}{a^{\frac{3}{2}}}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}l^{\frac{3}{2}}k^{-\frac{3}{2}}K_{\frac{3}{2}}(2\pi klTa)$$
$$-8\pi\frac{T^{\frac{7}{2}}}{a^{\frac{1}{2}}}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}l^{\frac{5}{2}}k^{-\frac{1}{2}}K_{\frac{5}{2}}(2\pi klTa).$$

For a massless scalar field with Db.c. or Nb.c., $\mathcal{P}_{\text{Cas}}^{\text{D/N}}(a)=\mathcal{P}_{\text{Cas}}^{\text{P}}(2a)$. For an electromagnetic field with PEC b.c. or

PMC b.c., $\mathcal{P}^{\rm PEC}_{\rm Cas}(a)=2\mathcal{P}^{\rm P}_{\rm Cas}(2a)$ or more precisely

$$\mathcal{P}^{\rm PEC}_{\rm Cas}(a)=-\frac{\pi^2}{240a^4}-\frac{\pi^2T^4}{45}+\frac{\pi T}{a^3}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}k^2l^{-1}\exp\left(-\frac{\pi lk}{Ta}\right),$$

or

$$\mathcal{P}^{\rm P}_{\rm Cas}(a)=-\frac{\zeta_R(3)T}{4\pi a^3}+2\sqrt{2}\frac{T^{\frac52}}{a^{\frac32}}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}l^{\frac32}k^{-\frac32}\times K_{\frac32}(4\pi klTa)-8\sqrt{2}\pi\frac{T^{\frac72}}{a^{\frac12}}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}l^{\frac52}k^{-\frac12}\times K_{\frac52}(4\pi klTa).$$

In particular, in the low-temperature limit, we find that the leading-order term of the Casimir pressure on a pair of infinitely conducting parallel plates is the zero-temperature Casimir pressure given by the well-known result of Casimir [1, 2], i.e.

$$\mathcal{P}^{{\rm PEC},T=0}_{\rm Cas}(a)=-\frac{\pi^2}{240a^4}.$$

The leading term of the temperature correction is

$$-\frac{\pi^2T^4}{45},$$

agreeing with the result of [42]. In the high-temperature limit, the leading term of the Casimir pressure is the classical term

$$\mathcal{P}^{\rm PEC,classical}_{\rm Cas}(a)=-\frac{\zeta_R(3)T}{4\pi a^3},$$

which agrees with the results of [42–44].

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