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A 29-root soluble model for the nonlinear calculation of a four-body propagator

B. G. Giraud, S. Kessal,[a] and A. Weiguny[b]
Service de Physique Théorique, DPH-G, CEN Saclay, F91191, Gif-sur-Yvette, France

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The time-independent mean-field theory of collisions is applied to the collision of four one-dimensional particles. Their two-body interactions are taken as separable, with Lorentzian form factors. This allows completely analytical solutions. A unique and satisfactory physical branch emerges out of 29 candidate solutions. It is very stable when the strength of the interaction is modified.

I. INTRODUCTION

The N-body problem can often be formalized, in quantum mechanics, by the calculation of a propagator $G=(E-H)^{-1}$, where the Hamiltonian $H=\mathbb{T}+\mathbb{V}$ reads, in obvious notation,

$$H=\sum_{i=1}^{N} t_i+\sum_{i>j}^{N} v_{ij}, \tag{1.1}$$

and E is a complex parametric energy.

This is formally, a trivial linear problem, with a unique solution when $\mathrm{Im}\,E>0$. In practice, as soon as there are more than three particles, and Faddeev equations are no longer available, this would-be linear problem amounts to the formidable inversion of a huge matrix in a large-dimensional space. The numerical accuracy demanded by the intricacies of the on-shell limit, $\mathrm{Im}\,E=0$, is out of reach.

This is why we have proposed a nonlinear approximation[1] to the calculation of a matrix element such as

$$D\equiv\langle\chi|(E-H)^{-1}|\chi\rangle\,, \tag{1.2}$$

where χ is a generic product of single-particle orbitals

$$\chi(\mathbf{r}_1,\ldots,\mathbf{r}_N)=\chi_1(\mathbf{r}_1)\cdots\chi_N(\mathbf{r}_N)\,. \tag{1.3}$$

For that purpose, it is trivial to show that D is the stationary value of the functional F of ϕ,

$$F\equiv(\phi|\chi)^2/(\phi|(E-H)|\phi)\,, \tag{1.4}$$

where round brackets $(\ |\ )$ denote the absence of complex conjugation for bras. An ansatz for ϕ similar to Eq. (1.3) for χ,

$$\phi=\varphi_1\cdots\varphi_N, \tag{1.5}$$

provides the variational, nonlinear equations

$$(\eta_i-h_i)\varphi_i=\chi_i\,, \tag{1.6}$$

with the complex, single-particle self-energies

$$\eta_i\equiv E-\sum_{j\neq i}\frac{\theta_j}{n_j}-\sum_{\substack{j>k\\ j\neq i\\ k\neq i}}\frac{w_{jk}}{n_j n_k}, \tag{1.7}$$

where

$$n_i=(\varphi_i|\varphi_i)\,, \tag{1.8}$$

$$\theta_i=(\varphi_i|t|\varphi_i)\,, \tag{1.9}$$

$$w_{jk}=(\varphi_j\varphi_k|v|\varphi_j\varphi_k)\,, \tag{1.10}$$

$$h_i=t_i+U_i\,, \tag{1.11}$$

and finally

$$\langle\mathbf{p}'|U_i|\mathbf{p}\rangle=\int d\mathbf{p}''\,d\mathbf{p}'''\,\langle\mathbf{p}'\mathbf{p}''|v|\mathbf{p}\mathbf{p}'''\rangle \times\sum_{j\neq i} n_j^{-1}\varphi_j(\mathbf{p}'')\varphi_j(\mathbf{p}''')\,. \tag{1.12}$$

This set of equations, Eqs. (1.6)–(1.12), is plain Hartree theory, except for the right-hand side of Eq. (1.6), the source term χ_i. Antisymmetrization is trivial and has been published elsewhere.[2]

This problem has been investigated in the framework of a soluble model[3,4] where only the kinetic energy is present. It may be pointed out here that, although T is a trivial one-body operator, the bare propagator $(E-\mathrm{T})^{-1}$ is a nontrivial, many-body operator. It was thus a significant result that the approximation generated by Eq. (1.6) turned out to be excellent[3] and that one out of its five competing solutions could easily be selected as the unique, physical approximation[4] for outgoing waves.

The purpose of the present paper is to generalize the soluble model[3] into a model including interactions. As will be shown, an analytical control of the whole algebra can be obtained all the way. It will turn out, once again, that only one of the many solutions (29 in that case) generated by nonlinearity is physically acceptable.

In Sec. II we describe the model and the basic equations that make it soluble. The technical, analytical solution of the model is explained in Sec. III. Then we study the main properties of the solution in Sec. IV, where numerical results are displayed. Our conclusion is proposed in Sec. V.

II. THE MODEL

Consider the symmetry group S of a polyhedron in a d-dimensional space, and a rotation invariant wave packet $\chi(\mathbf{p})$ centered at the origin in that space. A vector $\mathbf{K}$ will be associated to the first particle, via the wave packet

$$\chi_1(\mathbf{p}_1)\equiv\chi(\mathbf{p}_1-\mathbf{K})\,. \tag{2.1}$$

[a] Permanent address: Institut de Physique, Université des Sciences et de la Technologie Houari Boumediène, PB 32 El Alia, Alger, Algeria.
[b] Permanent address: Institut für Theoretische Physik, Universität Münster, Domagkstrasse, Münster, Federal Republic of Germany.

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Then we let the group S act upon $\mathbf{K}$ and generate $\mathbf{K}_2,\ldots,\mathbf{K}_N$. We define the wave packets

$$\chi_i(\mathbf{p}_i)\equiv\chi(\mathbf{p}_i-\mathbf{K}_i),\quad i=2,\ldots,N, \tag{2.2}$$

and consider the N-body amplitude $D=\langle\chi|G|\chi\rangle$, where $\chi=\Pi_{i=1}^{N}\chi_i$.

It is clear that S is a symmetry group for an exact solution of this problem and that the mean-field approximation can take great advantage of this symmetry. As already discussed in Ref. 3, the mean-field orbitals φ_i span a representation of S, provided of course the two-body interaction commutes with S.

In the following we take dimension $d=1$, and our potential is defined by

$$\langle\mathbf{p}'\mathbf{p}''|v|\mathbf{p}\mathbf{p}'''\rangle=-\lambda_0 f(\mathbf{p}')f(\mathbf{p}'')f(\mathbf{p})f(\mathbf{p}'''), \tag{2.3}$$

where λ_0 is a strength constant and the form factor f is given as a Lorentzian with width γ,

$$f(\mathbf{p})=(2/\pi)^{1/2}\gamma(p^2+\gamma^2)^{-1}, \tag{2.4}$$

which allows easy analytical calculations. Many other choices and generalizations are obviously available. For the sake of definitiveness, the width γ is taken as $\gamma=1$ fm^{-1}, a typical value for nuclear physics as an example.

Finally we set

$$\chi(p-K)=(2/\pi)^{1/2}\gamma[(p-K)^2+\gamma^2]^{-1}\quad(\text{fm}), \tag{2.5}$$

a boosted Lorentzian with the same width γ. In the following we list dimensions after each new symbol. We obtain dimension fm^{-4} for λ_0, so that the matrix element $\langle p'p''|v|pp'''\rangle$ becomes dimensionless.

The symmetry group S and the separability of v induce a degeneracy of all the quantities n, θ, and w listed in Eqs. (1.8)–(1.10). Hence it is trivial to omit the subscript i and write Eq. (1.6) as

$$(\eta-p^2)\varphi(p)=\chi(p-K)+\int dp'\langle p|U|p'\rangle\varphi(p'), \tag{2.6}$$

where $\hbar^2/2m=1$ (hence the dimension of E is fm^{-2}), and

$$\eta=E-(N-1)\theta/n+\Lambda\alpha^4/n^2\quad(\text{fm}^{-2}), \tag{2.7}$$

$$\Lambda=\tfrac{1}{2}(N-1)(N-2)\lambda_0\quad(\text{fm}^{-4}), \tag{2.8}$$

$$\alpha=(\varphi|f)\quad(\text{fm}^3), \tag{2.9}$$

$$\langle p|U|p'\rangle=-\lambda(\alpha^2/n)f(p)f(p')\quad(\text{fm}^{-1}), \tag{2.10}$$

$$\lambda=(N-1)\lambda_0\quad(\text{fm}^{-4}). \tag{2.11}$$

Hence the self-consistency demanded by the variational principle reduces to the determination of the three parameters θ (or η), α, and n of the mean-field orbital

$$\varphi(p)=\frac{\chi(p-K)}{\eta-p^2}-\lambda\frac{\alpha^3}{n}\frac{f(p)}{\eta-p^2}\quad(\text{fm}^3). \tag{2.12}$$

As will be shown below, this self-consistency condition will reduce to only the search for the roots of a polynomial.

For that purpose we define the three basic integrals

$$I_1(\eta)\equiv\int dp\,\frac{\chi^2(p-K)}{\eta-p^2}\quad(\text{fm}^3), \tag{2.13a}$$

$$I_2(\eta)\equiv\int dp\,\frac{\chi(p-K)f(p)}{\eta-p^2}\quad(\text{fm}^3), \tag{2.13b}$$

$$I_3(\eta)\equiv\int dp\,\frac{f^2(p)}{\eta-p^2}\quad(\text{fm}^3), \tag{2.13c}$$

and their derivatives, $I'\equiv dI/d\eta$. It is easy to insert Eq. (2.12) into Eqs. (1.8), (1.9), and (2.9), respectively, and obtain

$$n=-I_1'+2\lambda\alpha^3I_2'n^{-1}-\lambda^2\alpha^6I_3'n^{-2}\quad(\text{fm}^5), \tag{2.14}$$

$$\theta=\eta n-I_1+2\lambda\alpha^3I_2n^{-1}-\lambda^2\alpha^6I_3n^{-2}\quad(\text{fm}^3), \tag{2.15}$$

$$\alpha=I_2-\lambda\alpha^3I_3n^{-1}\quad(\text{fm}^3). \tag{2.16}$$

Through the integrals I,I', these three equations, Eqs. (2.14)–(2.16), are parametrized by η, Eq. (2.7). This closes the self-consistency algebra.

A numerical iterative procedure thus seems to be in order, and easy. A fully analytical solution is, however, available. We first define the auxiliary quantity

$$A\equiv\alpha^2/n, \tag{2.17}$$

and obtain from Eq. (2.16)

$$\alpha=I_2/(1+\lambda AI_3). \tag{2.18}$$

Hence Eqs. (2.14) and (2.15) can be written in terms of A only,

$$n=-I_1'+2\lambda A\frac{I_2}{1+\lambda AI_3}I_2'-\lambda^2A^2\left(\frac{I_2}{1+\lambda AI_3}\right)^2I_3', \tag{2.19}$$

$$\theta=\eta n-I_1+2\lambda A\frac{I_2}{1+\lambda I_3}I_2-\lambda^2A^2\left(\frac{I_2}{1+\lambda AI_3}\right)^2I_3. \tag{2.20}$$

Since by definition $A=\alpha^2/n$, we find from Eqs. (2.18) and (2.19)

$$A=I_2^2/[-(1+\lambda AI_3)^2I_1'+2\lambda A(1+\lambda AI_3)I_2I_2'-\lambda^2A^2I_2^2I_3'], \tag{2.21}$$

where A and η are present. Returning to the definition of η, Eq. (2.7), we find

$$\eta=E-(N-1)\eta-(N-1)\frac{(1+\lambda AI_3)^2I_1-2\lambda A(1+\lambda AI_3)I_2^2+\lambda^2A^2I_2^2I_3}{(1+\lambda AI_3)^2I_1'-2\lambda A(1+\lambda AI_3)I_2I_2'+\lambda^2A^2I_2^2I_3'}+\Lambda A^2, \tag{2.22}$$

which simplifies, through Eq. (2.21), into

$$0=E-N\eta+(N-1)A[(1+\lambda AI_3)^2I_1-2\lambda A(1+\lambda AI_3)I_2^2+\lambda^2A^2I_2^2I_3]/I_2^2+\Lambda A^2, \tag{2.23}$$

another condition between only A and η.

It is remarkable that, as far as A is concerned, Eqs. (2.21) and (2.23) are only cubic and read

$$aA^3+bA^2+cA+d=0, \tag{2.21'}$$

$$a'A^3 + b'A^2 + c'A + d' = 0\,, \tag{2.23'}$$

with

$$a = \lambda^2(I_2^2 I_3' - 2I_2 I_2' I_3 + I_1' I_3^2)\,, \tag{2.24a}$$

$$b = 2\lambda(I_1' I_3 - I_2 I_2')\,, \tag{2.24b}$$

$$c = I_1'\,, \tag{2.24c}$$

$$d = I_2^2\,, \tag{2.24d}$$

$$a' = \lambda^2(N-1)I_3(I_1 I_3 - I_2^2)\,, \tag{2.24e}$$

$$b' = \lambda\{2(N-1)I_1 I_3 - [(3N-2)/2]I_2^2\}, \tag{2.24f}$$

$$c' = (N-1)I_1\,, \tag{2.24g}$$

$$d' = (E - N\eta)I_2^2\,. \tag{2.24h}$$

The traditional compatibility condition between two such cubic equations reads

$$\det \equiv \begin{bmatrix} a & a' & 0 & 0 & 0 & 0 \\ b & b' & a & a' & 0 & 0 \\ c & c' & b & b' & a & a' \\ d & d' & c & c' & b & b' \\ 0 & 0 & d & d' & c & c' \\ 0 & 0 & 0 & 0 & d & d' \end{bmatrix} = 0 \tag{2.25}$$

which is an equation where only η is left.

An elementary manipulation, after inspection of Eqs. (2.24), shows that det is a sum of three terms, of order λ^6, λ^5, and λ^4, respectively,

$$\det_6 = (ad' - da')^3\,, \tag{2.26a}$$

$$\det_5 = (ab' - ba')[(bd' - db')^2 - 2\det_0(ad' - da')] - (ac' - ca')(ad' - da')(bd' - db')\,, \tag{2.26b}$$

$$\det_4 = \det_0 \det_2\,, \tag{2.26c}$$

with

$$\det_0 = cd' - dc'\,, \tag{2.26d}$$

$$\det_2 = (ac' - ca')^2 - (ab' - ba')(bc' - cb')\,. \tag{2.26e}$$

As is now obvious from the definitions of the integrals I, Eqs. (2.13), and the explicit forms of f and χ, Eqs. (2.4) and (2.5), the coefficients $a, b, \ldots, d'$, and the determinant det are rational fractions with respect to the auxiliary variable $\omega \equiv \eta^{1/2}$. Hence the solution of Eq. (2.25) boils down to the search for the roots of a polynomial. Our model is thus analytical. Its properties are the subject of Sec. III.

III. ANALYTICAL PROPERTIES OF THE SOLUTION

A straightforward contour integration in the upper half of the complex p plane provides

$$I_1 = -\frac{1}{z^2 + K^2}\left[1 + \frac{2z^2}{(z-1)(z^2+K^2)}\right], \tag{3.1a}$$

$$I_2 = -\frac{2}{K^2+4}\,\frac{2z(z+1) + K^2}{z(z-1)(z^2+K^2)}\,, \tag{3.1b}$$

$$I_3 = -\frac{z+1}{z^2(z-1)}\,, \tag{3.1c}$$

hence,

$$I_1' = -\frac{z[z^4 + z^3 + (K^2-1)z^2 - 3K^2 z + 3K^2]}{[(z-1)(z^2+K^2)]^3}\,, \tag{3.1d}$$

$$I_2' = -\frac{1}{K^2+4} \times \frac{4z^5 + 4z^4 + 4(K^2-1)z^3 + K^2 z^2 + 2K^4 z - K^4}{z^2(z-1)^3(z^2+K^2)^2}\,, \tag{3.1e}$$

$$I_3' = -\frac{z^2 + z - 1}{[z(z-1)]^3}\,, \tag{3.1f}$$

where we have defined z from

$$\omega = i(z - \gamma)\,, \quad \gamma = 1\,, \tag{3.2}$$

and ω from

$$\eta = \omega^2, \quad \mathrm{Im}\,\omega > 0\,. \tag{3.3}$$

This condition, $\mathrm{Im}\,\omega > 0$, is linked to the calculation of our contour integrals, see Eqs. (2.13), in the upper half of the complex p plane.

Insertion of Eqs. (3.1) into Eqs. (2.24) gives $a, b, \ldots, d'$ as rational fractions of z (or ω). We notice that all of these coefficients $a, b, \ldots, d'$ depend on K. Also most of them depend on N, if only because λ depends on N. The only coefficient that depends on E, however, is d'. For the sake of definiteness, we set $\mathrm{Re}\,E = NK^2$, and $N = 4$. The symmetry group S is realized if, for instance, $K_1 = K_2 = K$ and $K_3 = K_4 = -K$. These specifications leave λ, Im E, and K as the only parameters for our analysis of the model.

Before going into a detailed discussion of Eq. (2.25), a remark on the behavior of the solution as a function of λ is in order. On one hand, when $\lambda = 0$, corresponding to a pure kinetic model similar to that studied in Ref. 3, the coefficients a, b, a', b' vanish. There only $\det_0$, Eq. (2.26d), makes sense. As a matter of fact, $\det_4$ is the product of $\det_0$ and $\det_2$. When λ takes on nonvanishing values, it is clear that those roots of det that are continuations of roots of $\det_0$ are likely to be more physical than the roots of det that are continuations of the roots of $\det_2$.

On the other hand, the model should also accommodate a "potential" limit, corresponding to the situation where $|\lambda|$ is very large. There, the dominant term of det is $\det_6$, which leads to threefold degenerate roots. We must thus search, and keep track of, "physical" roots ω of det, which are continuous functions of λ, and which should interpolate between roots of $\det_0$ and roots of $\det_6$.

The straightforward, but tedious calculation of det, or rather of its polynomial numerator, demanded our use of symbolic language AMP, since an explicit listing of this numerator demands ten pages of computer output. This polynomial cannot be published here, naturally, but is available on request. It turns out to be a polynomial of degree 46. Out of it a factor $(2z^2 + 2z + K^2)^4(z-1)^6 z^3$ can be factorized and at once discarded, for the roots of this factor depend neither on λ nor on Im E, and are thus unphysical. This leaves a polynomial of degree 29.

For the small λ limit, the numerator of $\det_0$ provides a polynomial of degree 7, while the numerator of $\det_2$ provides a polynomial of degree 16 and a factor z^6. As already noticed, we have less interest in the latter 22 (16 + 6) roots, for they do not depend on Im E. The numerical investigation, whose final results are abstracted in Sec. IV, further revealed that these 22 roots failed to satisfy the condition $\mathrm{Im}\,\omega > 0$. An

analytical, rather than numerical, proof of this failure could not be found, but in the following we focus on those branches of roots of det that are continuations of the seven roots of $\det_0$.

For the large λ limit, the numerator of $\det_6$ provides a polynomial of degree 42, which includes a trivial factor $(2z^2+2z+K^2)^6(z-1)^6z^3$ rather than $(2z^2+2z+K^2)^4 \times (z-1)^6z^3$ like det. This leaves seven triply degenerate roots as potentially "physical" candidates.

The difference between the total degree (42) obtained from $\det_6$, and the total degree (46) obtained from det or $\det_4$ corresponds thus to four additional roots ω that diverge when λ is very large and are seemingly of little interest.

We have not found an elegant physical justification for the triple degeneracy of the seven roots of $\det_6$ we retain for our analysis, but the numerical and physical results are quite unambiguous. As a matter of fact, we have studied numerically the 29 nontrivial roots of the full det as functions of λ, Im E, and K, and studied their connection with the "interesting" roots of $\det_0$ and the "interesting" roots of $\det_6$. It was critical that only one of these 29 roots be acceptable, for the linear problem at the beginning of our theory has a unique physical answer. Satisfactorily enough, the following selection and classification scheme for roots turned out to be available: a root ω is acceptable if and only if

(i) $\mathrm{Im}\,\omega > 0$,

(ii) $\mathrm{Im}\,\overline{D} < 0$,

when $\mathrm{Im}\,E > 0$, where $\overline{D}$ is that approximate amplitude generated by ω and the corresponding φ via Eq. (1.4),

(iii) $\mathrm{Im}\,\eta(\equiv \mathrm{Im}\,\omega^2) > 0$,

(iv) $\omega^2 \simeq K^2 + \mathrm{Im}\,E$, when $\mathrm{Im}\,E \gg K^2$,

(v) $\mathrm{Re}\,A > 0$ and $\mathrm{Im}\,A > 0$.

The first two rules, (i) and (ii), are mathematically rigorous, as seen from our choice of "upper" contour integrals and our search for retarded diagonal amplitudes, respectively.

The third rule is almost rigorous, in the sense that many-body retardation should be translated into single-particle retardation, hence a self-energy η with a positive imaginary part.

The fourth rule, (iv), is empirical, but fairly intuitive insofar as $\mathrm{Re}\,\eta$ ($\equiv \mathrm{Re}\,\omega^2$) is concerned for the source wave packet χ and the variational wave packet φ should have similar energies. For the imaginary part, this rule hints that retardation effects are similar for the single-particle and the many-particle propagations.

The first four rules were sufficient to eliminate all roots but one. The survivor root turns out also to be almost independent of λ, which almost decouples the self-energy from the interaction strength, an interesting approximation. The last rule is an empirical, and apparently specific property, of this survivor root. More details are given in Sec. IV.

IV. NUMERICAL RESULTS

We have solved Eq. (2.25) for a large range of parameters, namely λ ranging from 0.001 to 120 fm^{-4}, K from 1 to 5 fm^{-1} (which means, with traditional units, Re E from 80 to 2000 MeV), and Im E from 0 to 2000 MeV. Most roots are at once eliminated because they violate rule (i) and/or rule (ii). This leaves at most four to five roots to be tracked when the various parameters λ, Im E, and K evolve.

The "on-shell" case, namely Im $E = 0$, is of special interest, since it corresponds to a physical collision. As χ has been chosen a real wave function and since the functional F, Eq. (1.4), is formally real, the analyticity and formal reality of ϕ with respect to the parameters η and α generate pairs of conjugate solutions $(\omega, -\omega^*)$, hence pairs of conjugate self-energies (η, η^*). In other words, the variational principle systematically generates both an advanced and a retarded solution in this on-shell limit. Any advanced solution is eliminated under rules (i) and/or (ii), of course. Real roots for η (purely imaginary for ω), which do not appear in pairs, are more delicate to analyze, for they may be candidates for a principal part amplitude. We have scrutinized them one by one.

In order to sort the survivor roots, it is very useful to study the pure kinetic energy case, namely $\lambda = 0$. We know already from our past experience[3] that this model exhibits a well-defined physical root ω_1, which is a smooth function of K and Im E. A striking property of this root is that it fits rule (iv) very closely. An even more striking property of ω_1 is that the bare (kinetic) propagator amplitudes D (exact) and $\overline{D}$ (approximate) are very close to each other, while the $\overline{D}$ generated by any other competitor root in this kinetic model generates a much poorer approximation. Hence there is no doubt that in this kinetic limit ω_1 is that unique physical root to be found.

Our strategy is thus very simple.

First we study the "kinetic" model, namely the seven roots of $\det_0$ in the present case, as functions of only K and Im E. It is easy to isolate the physical root there. Its behavior is shown in Fig. 1, where we have also shown, for the sake of completeness, branches $(-\omega_2^*)$ of an "advanced" root that merge with the branches of the physical root ω_1 when Im $E = 0$. There is little comment to add, except that both Ref. 3, where there were five roots, and the present model

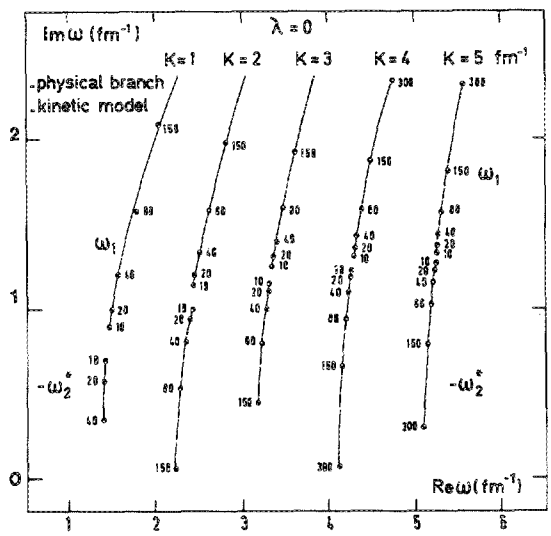

FIG. 1. Kinetic model (no interaction). Each point is labeled by Im E, in a system where $\hbar^2/2m = 20$ MeV fm^2. The upper branches correspond to the physical solution. The lower branches correspond to the complex conjugation of the opposite of another root.

show that there is no branch crossing as long as $K \geqslant 2\gamma$, namely that the average momentum exceeds the fluctuations in the wave packets χ_i.

Then we freeze $\mathrm{Im}\, E = 0$, which is the physical situation, and let λ increase. We follow that root which is a continuation of ω_1 and also any other root whose branch may interfere. A typical situation is shown in Fig. 2, with $K = 3$. The root ω_1 hardly depends on λ, while two other roots tend to merge with ω_1 when λ increases. This merging remarkably confirms the triple degeneracy expected from our formal analysis in Sec. III. For a value of K ($= 3$) which is sufficiently large compared to that of γ ($= 1$), we find no branch crossing that could interfere with the identification of ω_1.

Some of the roots plotted on Fig. 2 are shown between parentheses, in order to indicate that they violate one or several of the rules (i)–(v). The physical branch seems to be the only one for which these rules are never violated.

Finally we freeze K and λ, the latter to a large value in order to have a nontrivial potential influence, and study the behavior of the roots as functions of Im E. This is shown in Fig. 3, in a typical situation $K = 3$ again. The root ω_1 is very well approximated by its qualitative estimate, rule (iv), competing branches show some illegal points (between parentheses), and again we find no branch crossing.

To summarize our long numerical analysis, whose details cannot be published extensively, there is indeed only one physical branch provided by this many-solution model.

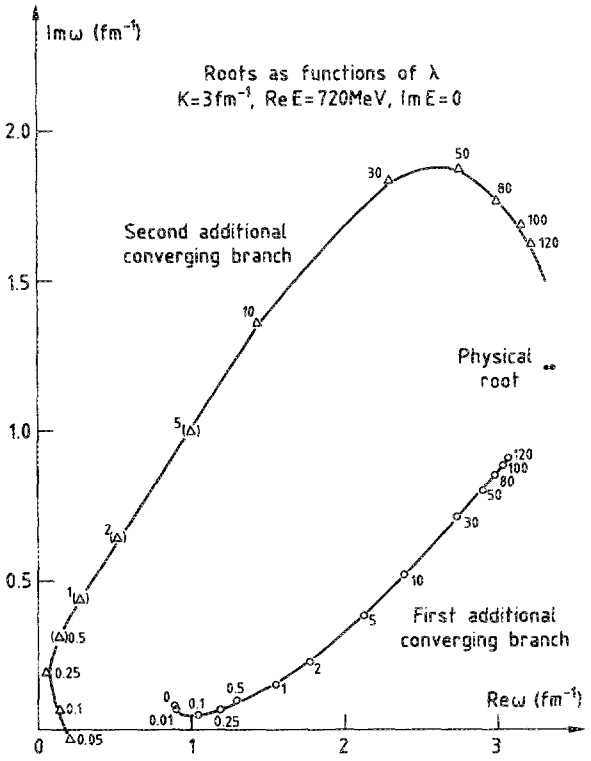

FIG. 2. Behavior of the roots as functions of the strength constant λ. The physical branch is reduced to a minute line element since ω_1 hardly depends on λ. Points are labeled by the value of λ. Some points, between parentheses, explicitly violate rule (ii) described at the end of Sec. III. Most points on the two "additional" branches violate rule (v).

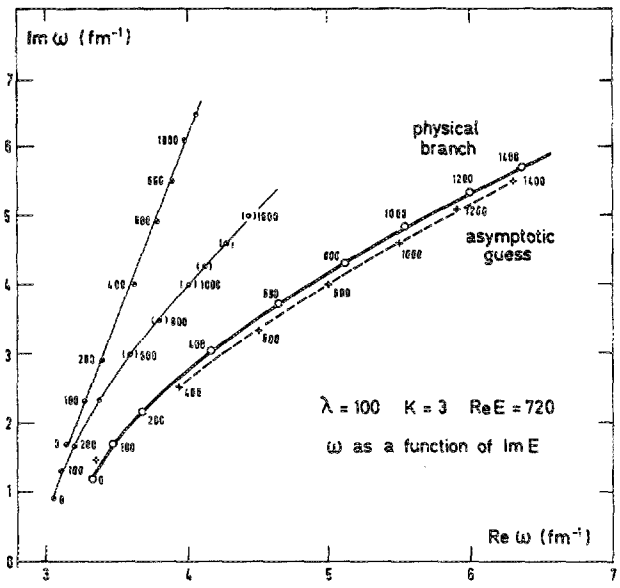

FIG. 3. Behavior of the roots as functions of their label, Im E. The asymptotic guess is generated by rule (iv).

V. DISCUSSION AND CONCLUSION

Our work would be complete if we could compare, for $\lambda \neq 0$, the exact amplitude D with the approximate $\bar{D}$ generated by ω_1. We have indeed easily calculated $\bar{D}$, but the calculation of D is another matter. This four-body problem, however, is not completely out of reach, for the separability of v leads to great simplifications. We are considering the comparison of D and $\bar{D}$ for a further paper.

The main result of the present paper is thus the rich nature of the mean-field approximation in this model, because of the occurrence of 29 competing solutions. Even though most of them can be rapidly ruled out, one day there should be a physical interpretation for most of them.

It is extremely satisfactory that finally only one among these roots survives as the undisputed physical solution, with great stability and simplicity properties (in particular, no branching). This gives great confidence in the validity of the variational method in those practical cases where no analytical solution is available and the mean-field equations, Eqs. (1.6), have to be solved via only a numerical iteration.

It may be pointed out, incidentally, that the present model could also be used to increase the efficiency of various iteration schemes and then define the attraction basins of the various solutions. In the case of Ref. 3 and in the present model, we have established that ω_1 has in general good stability under various iterative approximation schemes, but the solution $\omega_2 \simeq -\omega_1^*$ also has a non-negligible stability domain, hence rules (i)–(v) always retain their interest.

In summary, there exists in our mean-field theory of collisions a nontrivial and soluble model that provides all desirable tests for the validity of our theory.

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