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Wake-free waves in one and three dimensions

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A recent paper by Gottlieb [J. Math. Phys. **29**, 2434 (1988)] provides examples of acoustic wave equations, in various dimensions, that have nontrivial families of solutions that are progressing waves of order 1, and relates this to whether or not these equations satisfy Huygens' principle. A statement made in that paper related to Huygens' principle in one space dimension is clarified, and it is shown in this connection that, in general, the relationship between the possession of progressing wave solutions and the satisfaction of Huygens' principle is more complex than the situation described by Gottlieb. In addition, the attractive properties of progressing waves of order 1 are retained by progressing waves of any finite order, and we use this to generalize in several ways Gottlieb's results on "wake-free" solutions of the acoustic equation in three dimensions.

I. INTRODUCTION

An acoustic equation in n space dimensions,

$$c^2(\mathbf{x},t)\nabla^2_{(n)} U = U_{tt}, \tag{1.1}$$

is said to satisfy Huygens' principle if and only if the supports of its fundamental solutions, that is those generated by a unit source localized at a space-time event, lie on the light cone emanating from that event.[1-3] A Huygens' principle equation is described more intuitively in the preface to Ref. 4 as one such that, if a spatially localized source abruptly stops sending, any distant observer subsequently abruptly stops receiving.

A particular class of waveforms that can be useful in the description of solutions of such field equations are the *relatively undistorted progressing waves*,[1] also called *progressing waves of order 1*,[2] which are those of the form

$$U = g(\mathbf{x},t)S[\phi(\mathbf{x},t)], \tag{1.2}$$

where g and ϕ are fixed functions on space-time, and S is an arbitrary function of one variable (the "wave profile"). In a recent paper, Gottlieb[4] considers some examples of acoustic wave equations (1.1) that have significant families of solutions of the form (1.2). His calculations are correct, and the examples useful, but he also asserts that for a subset of these equations, "If a source signal ceases, the disturbance at a finitely distanced point will also cease in a finite time, ...," that is, that certain of these equations satisfy Huygens' principle. In particular, it said that if $c(\mathbf{x},t)$ is constant in (1.1), then "traveling waves resulting from general localized sources in odd space dimensions are sharp, while those in even dimensions are not." We shall show this to be insufficiently precise for the case $n = 1$ in the next paragraph; this is of course well known, since it would be tantamount to the incorrect statement that the ordinary wave operator in one dimension, $\partial_t^2 - \partial_x^2$, is a Huygens' principle operator.[1] We shall also briefly discuss at that point the tenuous connection between the existence of progressing wave solutions to homogeneous wave equations, and whether or not the corresponding wave operators satisfy Huygens' principle; this is a matter on which there seems still to be some confusion.

In Sec. II, in a more constructive spirit, we consider the natural generalization of (1.2) to *progressing waves of order N*, which are those of the form[1,2]

$$U = \sum_{i=1}^{N} g_i(\mathbf{x},t)S^{(i)}[\phi(\mathbf{x},t)], \tag{1.3}$$

where $\{g_i(\mathbf{x},t)\}_{i=1,\dots,N}$ and $\phi(\mathbf{x},t)$ are fixed functions on space-time, and the wave profile function $S^{(0)}$ is any sufficiently differentiable function of one variable, with $S^{(i)}$ its ith derivative. These waves share with those of the form (1.2) the property of being "wake-free" when the wave profile $S^{(0)}$ has compact support. We shall thus find a large variety of acoustic equations in three space dimensions with spherically fronted "wake-free" solutions to supplement those given in Ref. 4.

Returning now to the case of (1.1) with $n = 1$ and $c(\mathbf{x},t) = 1$, it is easy to see that the retarded fundamental solution U_R, which satisfies

$$\begin{aligned} U_{tt} - U_{xx} &= \delta(x)\delta(t) \\ &= 2\delta(t-x)\delta(t+x), \quad U = 0 \quad \text{if } t<0, \end{aligned} \tag{1.4}$$

is given by

$$U_R = H(t-x)H(t+x), \quad H(z) = \begin{cases} 0, & \text{if } z<0, \\ 1, & \text{if } z \geqslant 0. \end{cases} \tag{1.5}$$

The support of U_R is the future light cone of the source at (0,0) *and* its interior, so we are *not* dealing with a Huygens' principle wave equation. In the language of Ref. 4, the source is as local as it could possibly be, however, once an observer detects the signal, it will *never* stop for him, and so the constant velocity acoustic equation in one space dimension does not have the property claimed for it there. It is enlightening to consider this simple example further. The *general* solution of (1.1) with $n = 1$ and $c(x,t) = 1$ is given by

$$U(x,t) = f(t-x) + g(t+x), \tag{1.6}$$

where f and g are arbitrary. This is not inconsistent with (1.5), since by choosing $f - H = g = 0$ for $t + x > 0$, $f = g - H = 0$ for $t - x > 0$, and $f = g = 0$ for $t + x < 0$ and $t - x < 0$, we can express (1.5) in the form (1.6) in any region of the (x,t) plane, excluding of course the origin, where the source sits. The point is that the field generated by a particular source and satisfying particular boundary conditions is everywhere expressible in the form (1.6) with the *appropriate choice* of f and g, but in this case those choices must be functions that do not "turn off" after a finite time. Progressing waves are *capable* of describing "wake-free" solutions, but do not do so for all choices of wave profile.

This example hints at the uncertain connection between equations that possess solutions comprising progressing waves, and equations that satisfy Huygens' principle. In the simple case just analyzed, the general solution of the homogeneous equation is made of two progressing waves of order 1, while the retarded fundamental solution "has a wake," and Huygens' principle is violated. The plot thickens if we consider (1.1) with $n \geqslant 2$. For example, if $n = 2$ Huygens' principle is violated for any $c(\mathbf{x},t)$ (Ref. 3), but there exist (variable) velocity functions for which the circularly symmetrical solutions of the sourceless equation are progressing waves, as was shown in Ref. 4. In three space dimensions with $c(\mathbf{x},t) = 1$ Huygens' principle does hold and *all* solutions with *spherical* wave fronts are progressing waves of finite order,[5] with the spherically symmetrical waves being of order 1. The situation concerning progressing wave solutions of (1.1) with $n = 3$ and $c = 1$ but with nonspherical wave fronts is surprisingly complicated, and has been thoroughly worked out in a remarkable paper by Friedlander.[6]

Wave operators that satisfy Huygens' principle are known to be very rare, while those with nontrivial families of progressing wave solutions are much more common and also of interest, both because the solutions can be written in an explicit closed form, and because of their physical interpretation. Because of the latter, they have something in common with Huygens' principle operators, but there may be no simple connection between the two classes of equations.

II. EQUATIONS IN THREE DIMENSIONS WITH WAKE-FREE SOLUTIONS

In 1968, Kundt and Newman[7] used a generalization of the classical Darboux map to give an indirect characterization of probably all linear wave equations in one space dimension, whose general solution can be expressed in terms of progressing waves of finite order; they referred to these equations as satisfying a "modified Huygens' principle." More recently,[8] probably all the wave equations that satisfy their criterion have been constructed, so a probably maximal family of linear wave equations in one space dimension of the desired type is known. The specialization of these results to acoustic equations has been worked out for the self-adjoint case.[9]

It is clear from Friedlander's 1946 paper that the existence of progressing wave solutions in higher dimensions is dependent on the shape of wave front demanded. In what follows, we shall restrict ourselves to solutions of self-adjoint acoustic equations (1.1), with $n = 3$ and wave speeds depending on the time t and the radial coordinate r, and ask for progressing wave solutions with spherical wave fronts. The three-dimensional results of Ref. 1 refer to special cases of equations of this kind, and will therefore be included here.

Following the usual procedure for separating radial and angular coordinates by the ansatz $U(\mathbf{x},t) = w(r,t)Y(\theta,\varphi)$, we find that the angular part of the wave is a spherical harmonic Y_{lm}, while the radial part satisfies

$$[1/c^2(r,t)]W_{tt} - W_{rr} + [l(l+1)/r^2]W = 0, \quad (2.1)$$

with $W := rw$ and l a non-negative integer. It has been shown[9] that (2.1), and thus (1.1) in the case being considered here, is a self-adjoint equation if and only if the velocity function is of the form $c(r,t) = \alpha(r)/\beta(t)$, and in precisely this case it is always possible to recast (2.1) into the form

$$\psi_{\bar{t}\bar{t}} - \psi_{\bar{r}\bar{r}} + J(\bar{r},\bar{t})\psi = 0. \quad (2.2)$$

We have thus traded a nonconstant wave speed for a unit speed and a possibly changed potential term, and (2.2) is a special case of the normal form commonly used[7-9] in looking for progressing wave solutions. This trade is done by a combined factor and coordinate transformation, given by

$$\psi = W/\sqrt{\alpha\beta}, \quad (2.3)$$

and

$$\frac{d\bar{r}}{dr} = \frac{1}{\alpha(r)}, \quad \frac{d\bar{t}}{dt} = \frac{1}{\beta(t)}, \quad (2.4)$$

and the result is

$$J(\bar{r},\bar{t}) = F(\bar{r}) - G(\bar{t}) + l(l+1)\alpha^2\beta^2/r(\bar{r})^2, \quad (2.5)$$

where

$$F(\bar{r}) = -\alpha^{3/2}(\sqrt{\alpha})_{rr} = \sqrt{\alpha}(1/\sqrt{\alpha})_{\bar{r}\bar{r}}, \quad (2.6a)$$

$$G(\bar{t}) = -\beta^{3/2}(\sqrt{\beta})_{tt} = \sqrt{\beta}(1/\sqrt{\beta})_{\bar{t}\bar{t}}. \quad (2.6b)$$

In Ref. 4, Gottlieb showed that in three dimensions

$$c(r) = (Ar + B)^2 \quad (2.7)$$

gives progressing waves for all values of A and B when $l = 0$. By using (2.5) and known results, we can generalize Gottlieb's results in three different ways. The first generalization is to point out that with the choice $B = 0$, (2.7) actually gives progressing waves not only for $l = 0$, but for *all* $l > 0$ as well. This is shown by the fact that, when

$$c(r) = A^2r^2, \quad (2.8)$$

integration of (2.4) gives

$$\bar{r} = -1/A^2r, \quad \bar{t} = t, \quad (2.9)$$

and (2.2) becomes

$$\psi_{tt} - \psi_{\bar{r}\bar{r}} + [l(l+1)/\bar{r}^2]\psi = 0. \quad (2.10)$$

This is the radial wave equation for a scalar field in Minkowski space, and is well known to have progressing wave solutions of order l, given by

$$\psi = \sum_{m=0}^{l} \frac{c_{lm}}{\bar{r}^m}[\partial_u^{l-m}P(u) + (-1)^m\partial_v^{l-m}Q(v)], \quad (2.11)$$

$$c_{lm} = (l+m)!/m!(l-m)!, \quad 2u = t - \bar{r}, \ 2v = t + \bar{r},$$

where $P(u)$ and $Q(v)$ are arbitrary sufficiently differentiable functions. Since it follows from (2.3) that W and U are progressing waves whenever ψ is, we see that the acoustic equation (1.1) has progressing wave solutions for all $l \geqslant 0$ when $c(r) = A^2r^2$.

All the cases in Ref. 4 which yield progressing waves can be related by transformation to the standard PDE (2.10) with $l = 0$, and in the generalization just discussed it was again transformed into (2.10), with arbitrary $l \geqslant 0$. We shall see in what follows that one can obtain an infinite set of acoustic equations which have progressing wave solutions, most of which are *not* equivalent, via (2.3) and (2.4), to (2.10). In all of the remaining discussion, except the next to last paragraph, we shall be referring only to the situation $l = 0$ in Eqs. (2.1) and (2.5), i.e., to spherically symmetrical waves.

A large class $\bar{\mathscr{S}}$ of functions of one variable is known such that, whenever F and G in (2.5) are each any member of $\bar{\mathscr{S}}$, Eq. (2.2) has progressing wave solutions. The set $\bar{\mathscr{S}}$ is large in the sense that, for any non-negative integer N, the elements of $\bar{\mathscr{S}}$ which yield progressing waves of order N can be parametrized by no less than $2N$ real, positive parameters. All the nonsingular members of $\bar{\mathscr{S}}$ have been given in a well-known paper by Kay and Moses.[10] An infinite subset of $\bar{\mathscr{S}}$ with especially useful dependence on special parameters is given in Ref. 5, and probably all the functions $J(\bar{r},\bar{t})$, which give progressing waves are given in Ref. 8, with the specialization to single variable functions (elements of $\bar{\mathscr{S}}$) easily obtained from those given there.

As our goal is to find wave speeds for which (2.1) has progressing wave solutions, we can start with functions F, $G \in \bar{\mathscr{S}}$, and, inverting the transformations (2.4) and (2.6), find the corresponding $c(r,t)$, i.e., generate a new large class $\mathscr{S}$ of functions of one variable, such that (2.2) has progressing waves whenever $\alpha \in \mathscr{S}$, $\beta \in \mathscr{S}$, and $c = \alpha(r)/\beta(t)$. More specifically, it follows from (2.2) that, for any F in $\bar{\mathscr{S}}$ and any G in $\bar{\mathscr{S}}$, (2.6) is always exactly solvable in closed form for α in terms of $\bar{r}$ and β in terms of $\bar{t}$. Then, by integration of (2.4), a wave speed $c(r,t) = \alpha(r)/\beta(t)$ can be found corresponding to each choice of F and G in $\bar{\mathscr{S}}$. To stay within the class of time-independent wave speeds, we just need to choose $G = 0$ $(\beta = 1)$, with F any element of $\bar{\mathscr{S}}$. This gives an infinite set of $c(r)$'s with progressing waves, for which in general (2.1) does not transform into (2.10), and it is our second generalization of Gottlieb's results.

A nontrivial example for which the calculations are relatively simple [but for which the normal form *is* (2.10)] can be based on the functions $F(\bar{r}) = k(k+1)/\bar{r}^2$, where k is any integer, which we know are included in $\bar{\mathscr{S}}$. If we follow the construction described in the previous paragraph, we obtain

$$c(r) = (Ar+B)^2\{[1 - b(Ar+B)]/(Ar+B)\}^{2k/(2k+1)}, \tag{2.12}$$

where A, B, and b are any real numbers. In this case, (2.4) gives

$$\bar{r} = -\frac{2k+1}{A}\left(\frac{1-b(Ar+B)}{Ar+B}\right)^{1/(2k+1)}, \quad \bar{t} = t, \tag{2.13}$$

and the normal form of (2.1) is of course

$$\psi_{tt} - \psi_{\bar{r}\bar{r}} + [k(k+1)/\bar{r}^2]\psi = 0. \tag{2.14}$$

Thus, for positive k, we have progressing waves given by (2.11) with order k instead of order l. For negative k, we notice that (2.14) is invariant under $k \mapsto -k-1$; it follows that a wave speed (2.12) with $k < 0$ yields the same progressing waves as the one with the corresponding positive k. The actual form of the acoustic progressing waves for $c(r)$ given by (2.12) is easily obtained by performing the transformation (2.13) to the waves (2.11).

In Ref. 4, Gottlieb remarks that, when the wave speed is given by (2.7), the frequency spectrum for spherical waves is harmonic. It is easy to see that this is true, from the fact that the spectrum of $\psi_{tt} - \psi_{\bar{r}\bar{r}} = 0$ is harmonic, and the transformations (2.3)–(2.4) cannot change the spectrum when c is a function of r alone. We can similarly say that the spectrum of (2.1) with either wave speed (2.8) and arbitrary l, or with $l = 0$ and wave speed (2.12), will be the same as that of (2.10) or (2.14), but in general this spectrum does not have a simple form.

The third generalization is that, as noted above, an infinite set of wave speeds simultaneously dependent on t and r, $c(r,t) = \alpha(r)/\beta(t)$, which give progressing waves (when $l = 0$) can be constructed by taking $F \in \bar{\mathscr{S}}$, $G \in \bar{\mathscr{S}}$, $G \neq 0$. Progressing waves for arbitrary $l \geqslant 0$ may be obtained by restricting the wave speed to depend on t alone $(F = 0)$ and taking $G \in \bar{\mathscr{S}}$. In both of these situations however, our remark regarding the spectrum of the acoustic equation no longer holds.

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