Some properties of rotating coherent structures in a non-neutral plasma column

J. J. Aly

Citation: Physics of Plasmas **1**, 2401 (1994); doi: 10.1063/1.870568
View online: http://dx.doi.org/10.1063/1.870568
View Table of Contents: http://scitation.aip.org/content/aip/journal/pop/1/7?ver=pdfcov
Published by the AIP Publishing

Some properties of rotating coherent structures in a non-neutral plasma column

J. J. Aly
CEA, Service d'Astrophysique, CE Saclay, 91191 Gif sur Yvette Cédex, France

(Received 7 February 1994; accepted 23 March 1994)

Coherent structures rotating at the angular velocity ω in a two-dimensional pure electron plasma confined inside a conducting grounded cylinder are considered. These structures are described by a streamfunction ψ, obeying a nonlinear boundary value problem. It is shown, in particular, that (i) this problem has a minimal cylindrically symmetric solution if it has a solution at all. (ii) For a nonsymmetric solution, 2ω is necessarily comprised between the minimum and maximum values taken by the local diocotron frequency in the cylinder. (iii) Bifurcation of a symmetric radially decreasing solution to a nonsymmetric one can occur only when 2ω is equal to the mean diocotron frequency.

Consider a cold pure electron plasma confined by a uniform magnetic field $\mathbf{B}=B\hat{\mathbf{z}}$ (with $B>0$ a given constant), and surrounded by a conducting grounded cylinder of radius R (here we use a standard system of cylindrical coordinates $\{r,\phi,z\}$). If spatial variations along the z axis are negligible and the plasma is assumed to be in a stationary state in a frame rotating at the angular velocity ω around $\hat{\mathbf{z}}$, then its structure is determined by a streamfunction,

$$\psi(r,\phi):=-c\Phi(r,\phi)/B+\omega r^2/2, \tag{1}$$

obeying the equations[1,2]

$$-\nabla^2\psi=\omega_d(\psi)-2\omega, \quad \text{in } \Omega, \tag{2}$$

$$\psi(R,\phi)=\psi_\omega:=\omega R^2/2. \tag{3}$$

Here, Ω is the disk $\{r<R;z=0\}$; Φ is the electrostatic potential, vanishing on the wall (then ψ is equal, up to a factor $-c/B$, to the electrostatic potential in the rotating frame); and $\omega_d(\psi):=4\pi ecn(\psi)/B\geqslant 0$ is the local diocotron frequency associated to the electron density $n(\psi)$ ($-e$ is the electron charge and c is the speed of light; equilibrium imposes n to be constant along the equipotential lines $\{\psi(r,\phi)=\mu\}$, and then to be a function of ψ). Here ω_d will be assumed to be a non-negative continuously differentiable function of ψ, depending possibly on some parameters λ_i, and ψ will be in $c^2(\bar{\Omega})$. Two situations are generally considered in the literature: (i) The values of the parameters λ_i and ω are explicitly given. In that case, Eqs. (2)–(3) define a standard semilinear elliptic boundary value problem (BVP1, hereafter). (ii) The values of λ_i and ω are fixed implicitly by a set of external constraints $G_k[\psi(\lambda_i,\omega)]=\mu_k$ (in general, of the integral type), in which case the problem at hand (BVP2 hereafter) has a more complicated mathematical structure. Of course, solutions to BVP1 and solutions to BVP2 correspond to each other in an obvious way. To illustrate the difference between both problems, let us choose

$$\omega_d(\psi)=e^{-\lambda_1-\lambda_2\psi}. \tag{4}$$

Then, BVP1 corresponds to the case where the values of ω and $\lambda_{1,2}$ are fixed in a direct way, while BVP2 is obtained, for instance, if we impose the solutions to have given particles number, energy, and angular momentum—which results in three supplementary equations for the three unknown ω, λ_1, and λ_2: $N[\psi(\omega,\lambda_1,\lambda_2)]=\mu_1$, $E[\psi(\omega,\lambda_1,\lambda_2)]=\mu_2$, and $J[\psi(\omega,\lambda_1,\lambda_2)]=\mu_3$. This last situation is obtained in the thermodynamical approach developed in Ref. 3, where the plasma is assumed to have relaxed to the state of maximal entropy compatible with the only three constraints on N, E, and J.

There has been recently[3–7] some particular interest in noncylindrically symmetric solutions to BVP1/2, as they could be used to model the long-lived large-amplitude coherent structures, which have been observed to develop in some circumstances in both laboratory and numerical experiments.[8,9] It is the aim of this Brief Communication to show how standard mathematical theorems can be used to derive straightforwardly a few useful properties of nonsymmetric solutions (the word "cylindrically" will be understood from now on). It is worth noticing that the first two results reported below do fully answer questions recently addressed in Ref. 6.

Our first general result deals with the existence of a symmetric solution to BVP1: *If BVP1 does admit a solution (for given values of ω and λ_i), then it admits a minimal, symmetric solution $\hat{\psi}$* (by definition, a minimal solution $\hat{\psi}$ satisfies $\hat{\psi}<\psi$ in Ω for any other solution ψ). This theorem was conjectured in Ref. 6 to hold true when $\omega_d(\psi)$ is a monotonic function of ψ—a restriction that appears here to be unnecessary. From a physical point of view, it should be emphasized that the parameter ω has no intrinsic meaning, with respect to the solution whose existence is claimed here, as the latter corresponds to a configuration of the system that is stationary in all rotating frames of reference.[6]

Let us turn to the proof. As a preliminary remark, we note that any solution ψ (actually either to BVP1 or to BVP2) satisfies

$$\psi>\underline{\psi}:=\omega r^2/2. \tag{5}$$

This inequality just results from Eq. (1) and the relation $\Phi<0$ in Ω, the latter being an immediate consequence of the well-known maximum principle for elliptic equations[10] [remember that $(c/B)\nabla^2\Phi=\omega_d\geqslant 0$ (and $\neq 0$, some electrons are present!) and $\Phi=0$ on the boundary].

 1070-664X/94/1(7)/2401/3/$6.00

Now, we remark that

$$-\nabla^2\underline{\psi}-\omega_d(\underline{\psi})+2\omega=-\omega_d(\underline{\psi})\leq 0, \quad \text{in } \Omega, \tag{6}$$

$$\underline{\psi}(R,\phi)\leq\psi_\omega, \tag{7}$$

and then $\underline{\psi}$ is a "subsolution" of BVP1 [i.e., a function that satisfies the system of inequations obtained by substituting "$\leq$" for "$=$" in Eqs. (2)–(3).[11,12] Moreover, BVP1 admits any solution $\bar{\psi}$ (by assumption, there is at least one) as a "supersolution" (defined as a subsolution, but with "$\geq$" substituted for "$\leq$"[11,12]), and this supersolution is ordered with $\underline{\psi}$ as a consequence of Eq. (5). By a standard theorem,[11,12] the existence of such an ordered couple of sub/supersolution implies the existence of a solution $\hat{\psi}$ of BVP1, which does not depend on the particular $\bar{\psi}$ that is chosen to play the role of supersolution, and satisfies $\underline{\psi}<\hat{\psi}\leq\bar{\psi}$ (with the last inequality being strict if $\hat{\psi}\not\equiv\bar{\psi}$)—therefore a minimal solution indeed. Here ψ is obtained explicitly as the limit when $n\to\infty$ of the sequence $\{\psi^{(n)}\}$ defined by the iterative scheme:

$$-\nabla^2\psi^{(n+1)}+k\psi^{(n+1)}=\omega_d(\psi^{(n)})-2\omega+k\psi^{(n)}, \quad \text{in } \Omega, \tag{8}$$

$$\psi^{(n+1)}(R,\phi)=\psi_\omega, \tag{9}$$

$$\psi^{(0)}=\underline{\psi}, \tag{10}$$

where $k:=\sup_I|d\omega_d/d\psi|$, the supremum of ω_d being taken on the interval $I:=(\inf_\Omega\underline{\psi},\sup_\Omega\bar{\psi})$ [adding the term $k\psi$ on each side of Eq. (2) is a technical trick giving to the right-hand side of this equation the pleasant property to be, in any case, a monotonically increasing function of ψ, while conserving the elliptic character of the operator on its left-hand side. It is that trick that allows one to bypass the assumption[6] that ω_d is a monotonic function of ψ]. Moreover, it is clear that $\hat{\psi}$ is necessarily symmetric—for otherwise we could obtain by a mere rotation of $\hat{\psi}$ (owing to the rotational invariance of the problem) another solution that would be smaller than $\hat{\psi}$ at some points, in contradiction with $\hat{\psi}$ being minimal. We could also note here that the iterative process defined above preserves symmetry at each step. Q.E.D.

Our second result is as follows. *Suppose that a nonsymmetric solution to BVP1/2 does exist indeed. Then the associated density pattern rotates at an angular velocity that is necessarily larger (smaller) than half the minimum (maximum) value ω_d^- (ω_d^+) of the local diocotron frequency ω_d in* Ω, i.e.,

$$\omega_d^-<2\omega<\omega_d^+. \tag{11}$$

It must be insisted here upon the fact that the two numbers $\omega_d^\pm$ depend on the solution under consideration, and then are not known *a priori*. In many cases, however, bounds necessarily satisfied by any solution ψ and associated $\omega_d(\psi)$ can be derived on general grounds, and then the theorem can be used from a practical point of view to restrict the range of values of ω in which one should look for nonsymmetric solutions. The simplest situation arises when we deal with BVP1 and $\omega_d(\psi)$ is chosen to be a bounded function, with $0\leq\omega_{dm}\leq\omega_d(\psi)\leq\omega_{dM}<\infty$ for any arbitrary value of ψ. In that case, it is clear from Eq. (11) that a nonsymmetric solution can exist only if

$$\omega_{dm}<2\omega<\omega_{dM}. \tag{12}$$

To prove the necessity of condition (11), we show that a solution is symmetric if either $2\omega\leq\omega_d^-$ or $\omega_d^+\leq 2\omega$. If $\omega_d^+=\omega_d^-$, this last statement is obvious [the right-hand side of Eq. (2) is constant, and the solution is thus unique and symmetric]. In the opposite case, we use a recent powerful theorem[13] (Theorem S, hereafter), which asserts that, in the disk Ω, any *positive* solution $u>0$ of a nonlinear Laplace equation with homogeneous Dirichlet's boundary conditions [i.e., $-\nabla^2 u=g(u)$ in Ω and $u(R,\phi)=0$] is symmetric and radially decreasing. Then, let us first assume that $\omega_d^-<\omega_d^+\leq 2\omega$. This implies that the right-hand side of Eq. (2) is nonpositive and $\not\equiv 0$, whence $\psi-\psi_\omega<0$ in Ω by the maximum principle.[10] But then we can conclude at once by applying Theorem S to $\psi_\omega-\psi$ that ψ is a radial increasing function—i.e., $\psi=\psi(r)$ and $d\psi/dr>0$ for $0<r<R$. Similarly, we obtain that ψ is a radial decreasing function if $2\omega\leq\omega_d^-<\omega_d^+$. Q.E.D.

In Ref. 6, the upper bound in Eq. (11) was derived by a different argument for the particular case, where ω_d is an increasing function of ψ, and it was also physically argued that $\omega>0$ is necessary for the existence of a nonsymmetric solution. As pointed out by a referee, this last condition coincides with our lower bound in Eq. (11) in most cases of practical interest, as in laboratory conditions, the plasma needs to be separated from the wall by a finite vacuum gap—which implies $\omega_d^-=0$. Our new bound, however, turns out to be interesting in connection with some theoretical studies in which this last condition is not satisfied [see, e.g., Ref. 3, where ω_d is taken to be of the form (4)].

Next, we consider the possibility of *symmetry breaking*, i.e., the possibility for a branch of nonsymmetric solutions to result from the bifurcation of a branch of symmetric ones (a branch corresponding to a continuum of solutions obtained by changing the value of one of the parameters appearing in BVP1/2). Here we shall consider a branch of radial decreasing solutions $\psi(r)$, for which then $\psi(r)>\psi_\omega$ in Ω and $(d\psi/dr)(R)\leq 0$. We have the following necessary (but by no means sufficient) condition: *ψ can be a symmetry breaking point only if*

$$\frac{d\psi}{dr}(R)=0, \tag{13}$$

or equivalently if

$$2\omega=\bar{\omega}_d:=4\pi ec\bar{n}/B. \tag{14}$$

Also, we need to have

$$2\omega\geq\omega_d(\psi_\omega). \tag{15}$$

Here $\bar{\omega}_d$ is the mean diocotron frequency in Ω associated with ψ—i.e., the diocotron frequency corresponding to the mean density $\bar{n}:=N/\pi R^2$, with N the total number of electrons per unit of z length for the solution ψ. Note that Eq. (15) bears the same relation to Eq. (14), as does Eq. (12) to Eq. (11), in that sense that it can be used *a priori* for practical purposes when dealing with BVP1. Of course, the conditions of the theorem have already been quoted in the plasma literature [for instance, a referee noted that Eq. (14) has been known for years in the context of analyses of small-amplitude diocotron modes about azimuthally symmetric equilibria[1]. But we introduce them here in a completely general framework, and give straightforward proof of them.

We first establish Eq. (13). If we deal with BVP1, it can be obtained from a thorough analysis of the linearized equation associated with Eq. (2). Here we just need to follow the arguments given in Ref. 14, to which we refer the interested reader (this reference contains theorems more far reaching than the one above, giving, in particular, sufficient conditions for symmetry breaking to occur, conditions that are unfortunately not easy to check in any particular situation). More generally, Eq. (13) can be also directly connected to Theorem S by the following simple argument: If the decreasing function ψ satisfies $(d\psi/dr)(R)<0$, it is clear that any other solution χ of BVP1/2 sufficiently close to ψ ("close" meaning here, in particular, that the maximum values of $|\chi-\psi|$ and of $|\nabla\chi-\nabla\psi|$ are small), takes larger values in Ω than on the boundary, and then is also a symmetric decreasing function by Theorem S. Therefore, ψ can be a symmetry breaking point indeed only if $(d\psi/dr)(R)=0$.

For deducing Eq. (14) from Eq. (13), we integrate Eq. (2) over Ω and apply Gauss' theorem. As for Eq. (15), it is also an obvious consequence of Eq. (13) in the case where ω_d is an increasing function of ψ (this is the case generally encountered in practical applications); but then it coincides with the first inequality appearing in Eq. (11), and it does not bring anything new. In the general case, we note that the condition $(d\psi/dr)(R)=0$ can be satisfied only if $\omega_d(\psi)<2\omega$ for values of r arbitrarily close to R [whence Eq. (15) indeed]—for otherwise the so-called Hopf's boundary principle[10] would imply $(d\psi/dr)(R)<0$. Q.E.D.

As it results from the arguments used to prove the second theorem, nonsymmetric solutions have regions with $\psi<\psi_\omega$, as well as regions with $\psi>\psi_\omega$, and the contour level $\{\psi=\psi_\omega\}$, separating them is expected, on general grounds, to not consist of closed lines contained inside Ω, but to be formed of lines connecting two points on the boundary. Thus, near bifurcation, the nonsymmetric solution should have in Ω a large "off-axis" region, in which ψ reaches its maximum near the axis, but also a small adjacent region, stuck between the previous one and the boundary, in which ψ assumes its minimum. The existence of two such regions appears clearly in Fig. 5 of Ref. 6—but it should be emphasized here that our results have been proven under the assumption that $\omega_d(\psi)$ is a regular function, and then do not rigorously apply to the situation considered in Ref. 6, where $\omega_d(\psi)$ is chosen to be discontinuous.

As our last point, we argue that conditions ensuring *a priori* that BVP1/2 has a unique solution, provide at once obstructions to the existence of nonsymmetric solutions (if there is a unique solution indeed, it is clearly symmetric and nonbifurcating—and minimal!), and then supplementary necessary conditions for their existence. Such a condition preventing nonsymmetric solutions to BVP1 $[d\psi_d(\psi)/d\psi\leq 0$, i.e., ω_d is a nonincreasing function of $\psi]$ has already been discussed recently.[4] Here we give another condition, which is relevant when $\omega\geq 0$ [which implies by Eq. (5) that all the solutions are positive in Ω]:

$$\frac{d}{d\psi}\frac{\omega_d(\psi)-2\omega}{\psi}<0. \tag{16}$$

If under these conditions there are two solutions, $\psi_1>0$ and $\psi_2>0$ to BVP1, indeed, we obtain, after some straightforward algebra,[15]

$$\begin{aligned}0&\leq\int_\Omega|\psi_2\nabla\psi_1-\psi_1\,\nabla\psi_2|^2(\psi_1^{-2}+\psi_2^{-2})d\mathbf{r}\\&=\int_\Omega\left(-\frac{\nabla^2\psi_1}{\psi_1}+\frac{\nabla^2\psi_2}{\psi_2}\right)(\psi_1^2-\psi_2^2)d\mathbf{r}\\&=\int_\Omega\left(\frac{\omega_d(\psi_1)-2\omega}{\psi_1}-\frac{\omega_d(\psi_2)-2\omega}{\psi_2}\right)(\psi_1^2-\psi_2^2)d\mathbf{r}\leq 0,\end{aligned} \tag{17}$$

which clearly implies $\psi_1=\psi_2$. Q.E.D.

Note that Eq. (16) may allow us to eliminate the possibility of nonsymmetric solutions in some interesting situations, where ω_d is an increasing function of ψ, and thus complements in a useful way the criterion of Ref. 4.

An example illustrating all the results reported in this paper, as well as some of those obtained in Ref. 16 in a context that is physically different, but mathematically similar, will be presented in a forthcoming paper.

ACKNOWLEDGMENT

We thank an anonymous referee for helpful comments.

[1] R. C. Davidson, *An Introduction to the Physics of Nonneutral Plasmas* (Addison–Wesley, Reading, MA, 1990).
[2] R. C. Davidson, H. W. Chan, C. Chen, and S. M. Lund, Rev. Mod. Phys. **63**, 341 (1991).
[3] R. A. Smith and T. M. O'Neil, Phys. Fluids B **2**, 2961 (1990).
[4] R. A. Smith, T. M. O'Neil, S. M. Lund, J. J. Ramos, and R. C. Davidson, Phys. Fluids B **4**, 1373 (1992).
[5] T. M. O'Neil and R. A. Smith, Phys. Fluids B **4**, 2720 (1992).
[6] S. M. Lund, J. J. Ramos, and R. C. Davidson, Phys. Fluids B **5**, 19 (1993).
[7] S. M. Lund and R. C. Davidson, Phys. Fluids B **5**, 1421 (1993).
[8] K. S. Fine, C. F. Driscoll, and J. H. Malmberg, Phys. Rev. Lett. **63**, 2232 (1989).
[9] J. W. Poukey and J. R. Freeman, Phys. Fluids **24**, 2376 (1981).
[10] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience–Wiley, New York, 1962), Vol. II, pp. 326–327.
[11] See Ref. 10, pp. 367–374.
[12] H. Amman, SIAM Rev. **18**, 620 (1992).
[13] B. Gidas, W. M. Ni, and L. Nirenberg, Comments Math. Phys. **68**, 209 (1979).
[14] J. A. Smoller and A. G. Wasserman, Comments Math. Phys. **105**, 415 (1986).
[15] H. Brézis and S. Kamin, Manuscr. Math. **74**, 87 (1992).
[16] J. J. Aly, Phys. Fluids B **5**, 281 (1993).