A Note on the Optical Property of the Parabola

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1. The occasion for the following is a paper by Baron B. Hicken in the February, 1969 issue of School Science and Mathematics entitled: Some Interesting Mathematical and Optical Properties of Parabolas and Parabolic Surfaces. In this paper the author speaks of the difficulty he encountered in proving the optical property of the parabola to his students who had not studied the calculus.

An examination of the method used by the author shows that it would be more properly described in terms of analytic geometry rather than just algebra and geometry as stated. Moreover, a generalization of the usual calculus definition of a tangent is tacitly assumed: a tangent to a conic at a given point is the limiting position of a line having two distinct points in common with the conic when these points approach coincidence in the given point.

The method about to be described is, I think, more in keeping with the traditional content of high school plane geometry and is the result of a search for a proof of the parabola's optical property in a course in College Geometry without the use of the calculus.

2. *Definition.* A tangent to a parabola is a line having a single point in common with the curve but which does not cut across it.

The *optical property* of the parabola may then be stated: a tangent to a parabola at a point makes equal angles with the focal radius of the point and a line drawn through the point parallel to the axis of the parabola. Thus, in Fig. 1 the tangent T at P makes equal angles, α and β, with FP and PA, where PA is parallel to the axis.

Next, to prove the optical property we first prove

Theorem 1. Any point outside a parabola is nearer the directrix than the focus, while any point inside the parabola is nearer the focus than the directrix.

In Fig. 2 P is a point outside the parabola, and Q is inside. PR is perpendicular to the directrix, and N is the intersection of RP with the parabola. Similar statements may be made for the points Q, S, and M. Then

$$FP > FN - PN;$$

so

$$FP > RN - PN = RP,$$

since $FN = RN$ from the properties of the parabola. Thus the first part of Theorem 1 is proved.

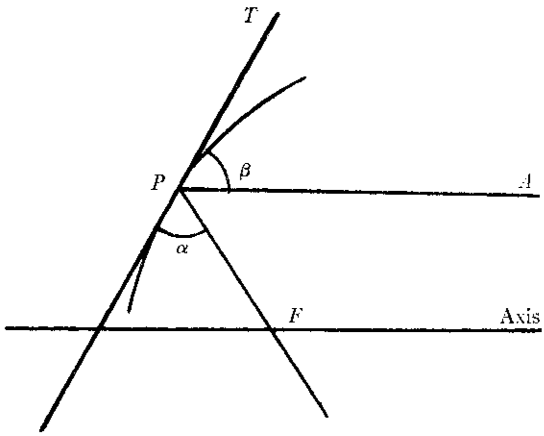

FIG. 1

Again,

$$FQ < MQ + MF;$$

whence

$$FQ < MQ + SM = SQ.$$

So the second part of Theorem 1 is proved.

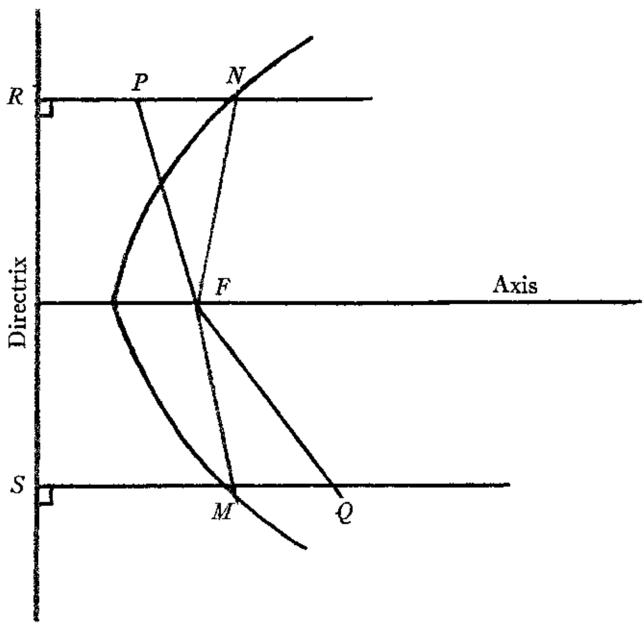

FIG. 2

From Theorem 1 it is easy to see that we may assert

Theorem 2. A point is outside or inside a parabola according as its distance from the directrix is less than or greater than its distance from the focus.

For if a point is nearer the directrix than the focus, it cannot be inside the parabola since this would contradict Theorem 1. Also, it cannot be on the parabola, for then its distances from the focus and directrix would be equal. It must therefore be outside the parabola.

The second part of Theorem 2 may be treated similarly.

Our next step is to prove

Theorem 3. In Fig. 3 let PB bisect the angle between PF, the focal radius for P, and PS, parallel to the axis of the parabola. Then PB is tangent to the parabola at P.

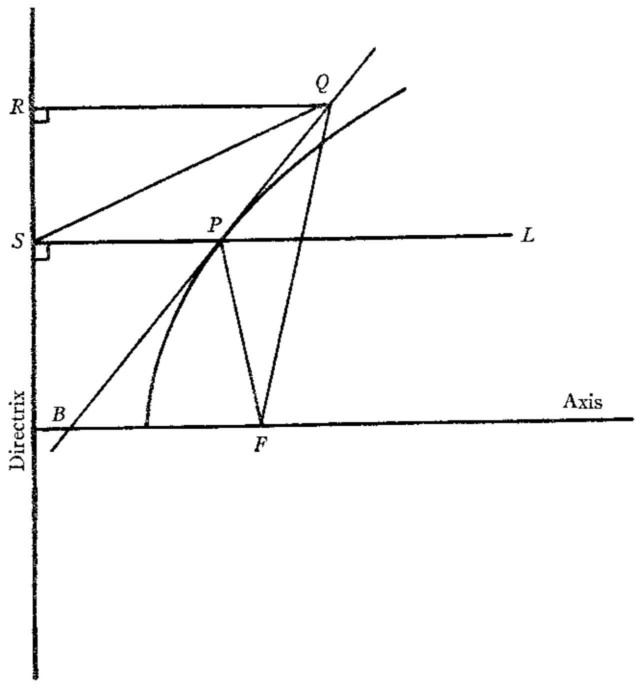

FIG. 3

Let Q be any point on PB other than P. Draw QR perpendicular to the directrix, and draw QF and QS. In triangles SQP and FQP

$SP = FP$	(definition of the parabola)
$QP = QP$	(identity)
$\angle SPQ = \angle FPQ$	(supplements of equal angles are equal)

Hence

$$\Delta SPQ \cong \Delta FPQ.$$

So

$$SQ = QF.$$

But

$$SQ > RQ,$$

Therefore $RQ < QF$, and Q must be outside the parabola by Theorem 2.

Thus the bisector of $\angle SPF$ has the point P in common with the parabola, and all of its other points are outside the curve. It must therefore be tangent to the curve by the definition of a tangent to the parabola.

The optical property is then immediate, for we have $\angle SPB = \angle QPL$, so that $\angle BPF$ also equals $\angle QPL$.

3. *Some Added Comments.* The above exposition necessarily lacks sophistication and relies almost wholly on intuitive geometric notions. This is due to the purpose expressed in section 1. Thus Theorem 3 shows that PB of Fig. 3 is a tangent to the parabola. But it does not show that this tangent line is the only one to the parabola at the point P. Here we take the viewpoint of elementary plane geometry which, for example, tacitly assumes the existence of a unique tangent at any point of a circle.

Another criticism lies in the use of the terms "inside" and "outside" with respect to the parabola, which is an open curve. Here, too, we rely on intuitive notions.

This last criticism could be met by taking the content of Theorems 1 and 2 and using this content as a basis for the following definition.

Definition. A point is outside (inside) a parabola if and only if its distance from the directrix is less than (greater than) its distance from the focus.

The proof of Theorem 3 would then follow by making use of this definition rather than the content of Theorem 2.
