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Division with Fractional Numbers: Invert and Multiply?

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It is generally agreed that the development of a topic in mathematics in a meaningful way promotes understanding and favors retention. However, the fact that a process is not performed in a rote, mechanical way, and that reasons are given for the steps taken, does not insure understanding. In fact, we often find that rationalization results in widespread confusion. The writers were discussing this question the other day after having observed children in the sixth grade struggling with the rationale underlying a method for dividing a number by a fractional number (hereafter referred to as a fraction).

The method is one described with some variations in several modern textbooks. The problem under consideration was $18 \div 2/3$. The textbook invoked such concepts as multiplicative inverse and the identity element. If the pupil recalls these principles plus the additional facts that $18 \div 2/3$ may be rewritten as $\frac{18}{2/3}$ and that $\frac{3/2}{3/2}$ is another way of writing 1, it is clear that $\frac{18 \times 3/2}{2/3 \times 3/2}$ becomes $18 \times 3/2$ which is negotiable, as $18 \div 2/3$ was not. After going through the steps several times, most pupils generalize that all we need do is to invert the fractional divisor and multiply. The method is mathematically logical, even ingenious. It is, in fact, its very ingenuity that accounts for its ability to perplex. To perform this series of steps, the student must know exactly where he wants to go and be so familiar with the mathematical terrain that he can traverse to and fro without losing sight of the relation of these moves to his final goal. The mathematics teacher cannily uses this principle and that, manipulates terms, disposes of them, and simplifies until he has cleared away all obstacles and almost magically comes upon the solution. Many students, led step by step through the procedure, can understand mathematically and can verbalize what is being done at each step but do not seem to really understand what is going on in a meaningful way. If asked to demonstrate with diagrams or to illustrate with a situation, they are unable to do so.

We are reminded in listening to them of a story told by the Gestalt psychologist, Prof. Max Wertheimer. The story tells of an auto salesman in the early days of the automobile who tried to sell a car to a skeptical farmer. He explained the mechanism and the principle involved in each step of the working of a car but the farmer still looked uncertain. "Perhaps," the salesman suggested, "Something is not clear," to which the farmer responded, "Everything is clear, but one thing. Where do you attach the horse?"

Similarly the pupil is very likely to know that $18 \div 2/3$ can be rewritten as $\frac{18}{2/3}$. He will recall or will at least agree, if reminded about multiplicative inverses, that $2/3 \times 3/2$ yields the multiplicative identity 1. There is little difficulty, too, in understanding that if this multiplication is performed in the denominator, it must also be performed in the numerator in order to avoid changing the value of the fraction. The only question is—what do all of these clever operations have to do with how many two-thirds are in 18. There is no perceptible connection between the mathematical steps involved in rationalizing the "invert and multiply" rule and any physical counterpart of the question, how many 2/3's are there in 18. This is precisely the point that must be kept in mind. To pupils of elementary school age mathematical procedures are rarely understood unless they can be related to a situation or problem which is structured in such a way that the very structure lends itself to a step by step derivation of the solution.

An example of the type of situation which does lend itself to a meaningful evolution of the steps is that in which we have 18 inches of ribbon and want to know how many strips 2/3″ long, to be used in making badges, can be cut from the 18″ piece. It now becomes possible for the pupil to work experientially. A 2/3″ strip can be fitted repeatedly into the 18″ piece. There is no conceptual break in this approach from the one the pupil has previously used in division problems such as $18 \div 3$. In fact, it should be expected that there will be situations in which we have to know how many 2/3's there are in 18 as well as those in which we have to know how many

2's or 3's or 6's. With this approach, it is rather easy to lead to the understanding that if we can find out how many 2/3's are in 1", then we can find out how many 2/3's there are in any number of inches. One need simply multiply the number of 2/3's in 1 by that number. It is helpful to have the pupil diagram the situation under discussion.

The unit or 1 is divided into thirds since we want to find out how many 2/3's there are in it.

1/3	1/3	1/3

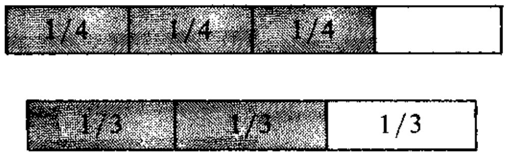
It is readily seen that there is one 2/3 with something, 1/3 to be specific, left over.

1/3	1/3	1/3

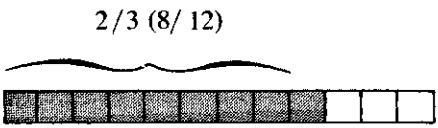
A question arises at this point about the "something" left over. Remember that what we want to know is how many *2/3's* are in 1 and we can see that the answer is 1 and a piece which was 1/3 of the whole unit. By holding the 2/3" strip alongside the 1/3" piece, it is perceptually clear that the 1/3" is half of the 2/3". Thus the number of 2/3's in 1 unit is 1-1/2 or 3/2. Therefore, the number of 2/3's in 18 would be 18 × 1-1/2 or 18 × 3/2. The number of 2/3's in 2 would be 2 × 3/2, etc. leading to the generalization: the number of 2/3's in n is $n \times 3/2$ regardless of the value of n.

Done in this way, the pupil is not only able to accept the correctness of each step. He can relate it to a situation in which he can physically verify each step and in which he can grasp perceptually the origin of the inversion. It may happen that a child will continue to use the mixed number rather than the fraction for a while in multiplying—e.g. 18 × 1-1/2 instead of 18 × 3/2. The answer, of course, will be correct. We find, however, that children notice rather quickly that the equivalent "improper" fraction is always the reciprocal of the fraction they started with and this is an exciting discovery.

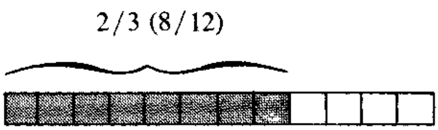
Moving ahead to the division of a fraction by a fraction does not present any difficulty using this method. The pupil now knows that to find the number of 2/3's (or any other fraction) in n, we must multiply n by the number of 2/3's in 1. It does not matter what the value of n is; the answer will be $n \times 3/2$. If n is 3/4, the answer will be 3/4 × 3/2. At this stage, physical verification is no longer necessary inasmuch as the procedure does not involve anything more than the pupil has always done in simple division and multiplication. However, using a diagram for the problem 3/4 ÷ 2/3 provides a challenge for the more motivated pupils and deepens the understanding of all pupils. Let us consider how it might proceed. Our question is how many 2/3's are in 3/4.



By inspection, it can be seen that there is one 2/3 and again something left over—a piece of one of the fourths. But what part of 2/3 is it? How can we compare that piece of 1/4 with 2/3? We can approximate it by inspection. If we hold the piece against the 2/3 strip, it is less than half of it, less than a fourth. To compare more exactly, we'd have to talk in a unit common to both. This is quite easy for children in grades 5 and 6. We return to our problem now but we have a reason to use twelfths rather than fourths and thirds.



Our 3/4 is 9/12



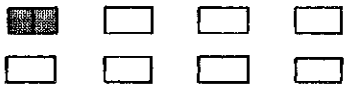
Our 2/3 is 8/12

If we cut the 2/3 (8/12) strip and hold it against the 3/4 (9/12) strip, how many of the 2/3 strips are in the 3/4. It is now possible to be exact, 1 and one piece left over. The one piece was 1/12 of the whole unit but it is only 1/8 of the 2/3. The pupil has been able to see and to verify that 3/4 ÷ 2/3 is 1-1/8.

A less frequent type of situation is one in which the pupil is called upon to divide a fraction by a whole number. When this problem does appear, it is usually more meaningfully expressed as the multiplication of a fraction by a fraction. For example, 1/2 ÷ 8 can be read as 1/2 divided into 8 parts.

This situation may be considered similar to the problem in which half a pie is to be divided among 8 children. In this case, 1/8 of 1/2 would be the usual way of expressing it. However, the problem may also be interpreted as what part of 8 is in 1/2, e.g. I have to clean 8 windows and, so far, have finished half a window. What part of the job have I done? It is possible to apply the same method that was used in dividing a whole number by a fraction or a fraction by a fraction.

Following the procedure we used before, we determine what part of 8 units is in 1 unit, an easily perceptible task, as illustrated below.



The whole job consists of cleaning 8 windows. Each window is readily seen as $1/8$ of the job. As before, $n \times$ the number in 1 unit will yield the answer. Since n in this case is $1/2$, then $1/2 \times 1/8$ is derived by the same method used in dividing a whole number or a fraction by a fraction. Again, we are led to "invert and multiply."

The major advantage of this approach is that it is perceptually meaningful (capable of visualization as well as rationalization). This method is more appropriate for the maturity level of intermediate grade children.

RADIOACTIVE WASTE DISPOSAL IN THE ATLANTIC UNDER INTERNATIONAL SUPERVISION

An internationally supervised operation for the disposal of packaged solid radioactive wastes into the deep Atlantic Ocean, organized under the auspices of the OECD Nuclear Energy Agency (NEA), took place during the months of June and July this year, in accordance with the principles of the Convention on the Prevention of Marine Pollution by Dumping of Wastes and Other Matter (London Convention). This operation was carried out, as previous ones, under the close supervision of international escorting officers designated by the NEA.

Some 4,500 tons of concreted and bituminized low-activity wastes from nuclear energy establishments in Belgium, the Netherlands, Switzerland and the United Kingdom, packed in metal drums, disposed of during this operation. The transport of the waste from the originating establishments to the dumping site was done in accordance with national and international regulations for the safe transport of radioactive materials.

The dumping site, located in a deep sea area of the North Eastern Atlantic, is represented by a circle of 35 nautical miles radius centred at 46° 15′ N and 17° 25′ W. This site fulfills the requirements recommended by the International Atomic Energy Agency for the implementation of the London Convention in the case of radioactive materials. It has been used already for past operations and is approximately 4.5 kilometres deep.