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Wendell Horton, Jr., R. Estes, H. Kwak, and Duk-In Choi

Fusion Research Center and Department of Physics, The University of Texas at Austin, Austin, Texas 78712
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Modified drift wave dispersion relations and shear stabilization criteria are derived from an analysis of the radial drift wave equations coupled in azimuthal mode numbers through the toroidal effects of ion magnetic drifts and trapped electrons.

I. INTRODUCTION

In a recent article, Taylor[1] has questioned the applicability of previous shear stabilization criteria for drift modes in toroidal systems. For tokamaks with circular magnetic surfaces, however, the validity of his criticism is not altogether evident. In the present work we give further consideration to the modifications that arise in the drift wave dispersion relation for the shear localized modes when the toroidal effects that couple drift modes centered on neighboring rational surfaces are taken into account. From the analysis we obtain solutions to the normal mode equation that take into account the toroidal magnetic drifts of the ions and the azimuthal variation of the number of trapped electrons. The dispersion relations derived provide conditions that relate the strength of the toroidal drifts, the shear parameter and the azimuthal wavelength that occur when shear stabilization is lost. Although the actual conditions are found to be considerably more complicated than given by Taylor, we conclude that shear stabilization is often lost for the azimuthal wavelengths and geometry parameters of general interest in tokamaks.

In our previous studies of the drift wave radial modes[2,3] the toroidal mode coupling effects were considered to be weak and subsequently eliminated by averaging the mode equation over the magnetic surface. Recent investigations by Rewoldt *et al.*[4] on the dissipative trapped electron mode in tokamaks show that this is not generally a good approximation. A similar conclusion is reached by Cordey and Hastie[5] who obtain two-dimensional Gaussian shaped localized modes under certain localization conditions. The dispersion relation obtained under these conditions by Cordey and Hastie is shown to be closely related to the dispersion relations obtained in the present analysis. Earlier work[6-8] in which the radial location is not treated explicitly also suggests the importance of modes localized or strongly modulated along the magnetic field line. Consistency conditions obtained in our previous studies[2,7] also indicated that the neglect of ballooning of the mode on the magnetic surface from toroidal effects is generally only justified for sufficiently long azimuthal wavelengths. At shorter azimuthal wavelengths and for weak shear, ballooning modes driven by either the ion grad B drift[6-9] or the variation in the trapped electron population[10-12] were reported earlier. The conditions where these earlier approximate solutions remain valid are indicated in the following analysis. For small values of the shear parameter $\xi = rq'(r)/q(r)$, where $q(r)$ is the toroidal safety factor, we find that the ion toroidal drifts cause the shear stabilizing radial anti-well to become a radial well which localizes the mode away from the regions of ion Landau damping. As a consequence, we find that the shear stabilization of the modes is essentially lost for azimuthal wavelengths in a certain range of practical interest. For these same wavelengths, but for stronger shear values, where $\xi = rq'/q > \frac{1}{2}$, we find a modified form of the earlier shear stabilization criterion that is a function of the toroidal curvature parameter.

For axisymmetric systems the problem of the drift wave modes in the presence of shear and finite toroidal curvature is described by a set of linearly coupled radial mode equations governing the fluctuating potentials $\varphi_m(r)$ belonging to the different azimuthal harmonics $\exp(im\theta)$. For the case of weak toroidal coupling, a perturbation expansion in the coupling parameter gives solutions for the $m = m_0$, $m_0 \pm 1$ components which determine the first correction to the dispersion relation. For the important case of strong coupling where a significant number of toroidal modes are linked together, Taylor[1] has developed a procedure for solving the coupled equations. Here, we follow his procedure to obtain a new dispersion relation and wavefunctions for the toroidal problem.

Determining the spatial structure of the drift wave modes in axisymmetric toroidal geometry with shear in the magnetic field is a complicated problem. Early work on the problem by Coppi *et al.*[9,13] and Hastie and Connor[14] established the general analytical form which the doubly periodic wavefunction must take in order to maintain the long parallel wavelength required for instability in drift modes in a sheared magnetic field. In Sec. III the normal mode wavefunctions implied by the Taylor quasi-mode solutions of the coupled differential equations are examined. The wavefunctions are shown to be a particular case of the general toroidal wavefunctions considered by Coppi.[9] The toroidal normal mode composed of a superposition of strongly localized quasimodes is now broad in its radial dependence. The fact that the toroidal modes are radially broad partially justifies *a posteriori* the local radial analysis assumed in the earlier studies[6-9] of the mode localization along the magnetic field line. Some of the dispersion relations obtained in those investigations are recovered here as the limit where shear stabilization is lost from toroidal mode coupling effects.

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In Sec. IV the quasi-mode equation for the trapped electron mode is formulated. In the limit of vanishing shear the trapped electron effect induces the usual ballooning of the mode to the outside of the torus given approximately by a $1+\cos\theta$ variation on the magnetic surface. For strong shear, $\xi = rq'/q \sim 1$, the situation is considerably more complicated as already reported by Rewoldt *et al.*[4] and Ross and Miner.[12] Here, we develop analytical results for this problem which show the enhancement of the growth rate and the loss of the usual shear stabilization for these modes. In Sec. V some conclusions are drawn from the analysis presented.

II. COUPLED EQUATIONS FROM ION TOROIDAL DRIFTS

For the toroidal equilibrium we consider simple circular magnetic surfaces described by $R = R_0 + r\cos\theta$, where R_0 is the major radius of the magnetic axis, r is the minor radius, and θ is the angle measured from the outside of the horizontal plane of symmetry. In addition, we approximate the elliptic integrals that describe the trapped particle orbits at small $\epsilon = r/R_0$ by the corresponding sinusoidal orbits. This approximation for the orbits was studied earlier and shown to be reasonably accurate for the trapped electron mode.[2] The sinusoidal bounce orbit approximation allows substantially more analytic work to be carried out than is possible with the elliptic integral orbits.

Let us begin by considering the low ion temperature fluid equations to describe the ion dynamics. The equation for the total ion density $N(\mathbf{x}t)$ as determined by the electrostatic potential $\varphi(\mathbf{x}t)$ is

$$\frac{\partial N}{\partial t} + \nabla\cdot\left(\frac{cN\mathbf{B}\times\nabla\varphi}{B^2}\right) - \nabla_\perp\cdot\left(\frac{c^2 m_i N}{eB^2}\nabla_\perp\frac{\partial\varphi}{\partial t}\right) - \mathbf{B}\cdot\nabla\left(\frac{eN}{m_i B^2}\mathbf{B}\cdot\nabla\int^t \varphi(\mathbf{x}t')dt'\right) = 0\ . \qquad (1)$$

We linearize Eq. (1) about the toroidal equilibrium with density profile $n_0(r)$ by writing $N(\mathbf{x}t) = n_0(r) + n(\mathbf{x}) \times \exp(-i\omega t)$ and $\varphi(\mathbf{x}t) = \varphi(\mathbf{x})\exp(-i\omega t)$ and dropping terms of order $n(\mathbf{x})\varphi(\mathbf{x})$. For simple concentric circular magnetic surfaces, the toroidal coupling arises from

$$\nabla\cdot\left(\frac{cn_0(r)}{B^2}\mathbf{B}\times\nabla\varphi\right) = \nabla n_0(r)\cdot\frac{c\mathbf{B}\times\nabla\varphi}{B^2} + cn_0\mathbf{B}\times\nabla\varphi\cdot\nabla\left(\frac{1}{B^2}\right)$$
$$= -\frac{c}{B}\frac{dn_0}{dr}\hat{e}_\perp\cdot\nabla\varphi - \frac{2cn_0}{BR}\left(\sin\theta\frac{\partial\varphi}{\partial r} + \frac{\cos\theta}{r}\frac{\partial\varphi}{\partial\theta}\right)\ . \qquad (2)$$

In this section the linear electron response is written as $n_e(\mathbf{x}) = n_0(e\varphi/T_e)(1-i\delta)$, where δ describes the dissipative part of the oscillating electron density distribution. In the following section we introduce the trapped electron formula for δ. We define $r_n^{-1} = -d\ln n_0/dr$, $\epsilon_n = r_n/R_0$, $h = c_s/r_n$, $c_s = (T_e/m_i)^{1/2}$, $\rho = c_s/\omega_{ci} = c(m_iT_e)^{1/2}/eB$, and the derivatives $\partial_\perp = \hat{e}_\perp\cdot\nabla$ and $\partial_\parallel = \hat{n}\cdot\nabla$, where $\hat{n}$ is the unit vector along the magnetic field and $\hat{e}_\perp$ is the unit vector perpendicular to the magnetic field in the magnetic surface. Using these definitions and quasineutrality we obtain, from Eqs. (1) and (2),

$$-\left[\frac{\rho^2}{n_0 r}\frac{\partial}{\partial r}\left(rn_0\frac{\partial\varphi}{\partial r}\right) + \rho^2(\hat{e}_\perp\cdot\nabla)^2\varphi\right] + (1-i\delta)\varphi + \frac{ih\rho}{\omega}\left[\hat{e}_\perp\cdot\nabla\varphi - 2\epsilon_n\left(\sin\theta\frac{\partial\varphi}{\partial r} + \frac{\cos\theta}{r}\frac{\partial\varphi}{\partial\theta}\right)\right] + \frac{c_s^2}{\omega^2}(\hat{n}\cdot\nabla)^2\varphi = 0\ , \qquad (3)$$

where $\hat{e}_\perp\cdot\nabla = (B_\phi/Br)\partial/\partial\theta - (B_\theta/BR)\partial/\partial\phi$ and $\hat{n}\cdot\nabla = (B_\theta/Br)\partial/\partial\theta + (B_\phi/BR)\partial/\partial\phi = (1/qR)(\partial/\partial\theta + q\partial/\partial\phi)$. As noted in the review by Tang[15] the mode coupling Eq. (3) agrees with the cold ion fluid limit of the Vlasov mode coupling equation investigated by Rewoldt *et al.*[4] For modes with $\langle k_x^2\rangle \ll k_y^2$ the Vlasov modifications to Eq. (3) are readily derived and are given in Appendix B.

Let us consider the solution of Eq. (3) which in terms of the spectrum of azimuthal mode numbers is centered about the mode m_0. For such a solution we write that

$$\varphi(\mathbf{x}) = \varphi(r,\theta,\phi) = \exp(im_0\theta - il\phi)\sum_n \varphi_n(r)\exp(in\theta)\ . \qquad (4)$$

In the limit of sufficiently small ϵ_n we have, by the definition of m_0, that $|\varphi_{\pm1}(r)/\varphi_0(r)| \sim \epsilon_n \ll 1$. For the drift waves of principal interest we have that $m_0 \gg 1$ and the toroidal spread of azimuthal mode numbers Δn can be expected to be small compared with m_0. Here, Δn is a measure of the number of dominant terms in the sum (4) where for definiteness we define $\Delta n^2 = \sum_n n^2|\varphi_n|/\sum_n|\varphi_n|$.

For modes with $m_0 \gg \Delta n$ we may, as a first approximation, neglect the toroidal spread induced in the perpendicular wavenumber $k_\perp = -i\hat{e}_\perp\cdot\nabla\ln\varphi \cong (m_0+n)/r$ and define, in slab analog notation, $k_y = m_0/r$. Since in the limit of small ϵ_n the radial eigenmode corresponding to Eq. (4) is localized about the rational surface $k_\parallel(m_0, l, r_0) = 0$, defined by $q(r_0) = m_0/l$, it follows that even for small spreads $\Delta n \ll m_0$ it is important to retain the toroidal shifts induced in $k_\parallel(m, l, r)$. Expanding $q(r)$ about its value at the rational surface we obtain

$$k_\parallel(m, r) = \frac{m_0 + n - l[q(r_0) + (r-r_0)q'(r_0)]}{qR} = \frac{1}{qR}\left[n - \frac{m_0}{r_0}\frac{r_0 q'(r_0)}{q(r_0)}(r-r_0)\right]\ , \qquad (5)$$

where $q(r) = rB_0/RB_\theta(r)$. As in our previous studies we find it convenient to measure distance from the rational surface in terms of ρ by writing $x = (r-r_0)/\rho$ and k_y in units of ρ^{-1} by writing $k_y\rho$. From Eq. (5) we note that the distance between neighboring rational surfaces is $\Delta x = 1/(k_y\rho\xi_0)$, where $\xi_0 = r_0q'(r_0)/q(r_0)$. In the following, we shall omit the subscript zero leaving the evaluation of quantities at the principal rational surface implied.

With these approximations and definitions Eqs. (3), (4), and (5) reduce to

$$[\partial_x^2 - (k_y\rho)^2 - 1 + i\delta]\varphi_n(x) + \frac{hk_y\rho}{\omega}\times\left[\varphi_n - \epsilon_n(\varphi_{n+1} + \varphi_{n-1}) - \frac{\epsilon_n}{k_y\rho}(\partial_x\varphi_{n+1} - \partial_x\varphi_{n-1})\right] + \left(\frac{c_s}{\omega qR}\right)^2(n - k_y\rho\xi x)^2\varphi_n(x) = 0 \qquad (6)$$

with the boundary conditions that $\sum_n |\varphi_n|$ be finite and that $\varphi_n(x)$ for fixed n and $x\to\pm\infty$ represent the outgoing wave energy propagation. In writing Eq. (6) we neglected the variation of $n_0(r)$ over the width of the radial mode. In some cases of practical interest this is a poor approximation so that in the numerical studies[16] we retain the self-adjoint equation in the form $(\partial_x n_0 \partial_x - n_0 Q)[\varphi(x)] = 0$.

A. Weak coupling approximation

For sufficiently small ϵ_n we truncate the harmonic expansion at $n=\pm 1$. The first order equations are

$$[\partial_x^2 - (k_y\rho)^2 - 1 + i\delta]\varphi_{\pm 1} + \left[\frac{hk_y\rho}{\omega} + \left(\frac{c_s}{\omega q R}\right)^2 (\pm 1 - k_y\rho\xi x)^2\right]\varphi_{\pm 1} = \frac{\epsilon_n h}{\omega}[k_y\rho\varphi_0(x) \mp \partial_x\varphi_0(x)], \tag{7}$$

where the normalized solution $\varphi_0(x)$ of the lowest order equation is $\varphi_0(x) = (i\sigma/\pi)^{1/4}\exp(-i\sigma x^2/2)$ with $\sigma = hk_y\rho\epsilon_n\xi/\omega q$. The reaction of the first harmonic components $\varphi_{\pm 1}(x)$ on the fundamental mode is followed by substituting the solution of Eq. (7) into the $n=0$ equation which gives

$$1 + (k_y\rho)^2 - i\delta + i\sigma - \frac{hk_y\rho}{\omega} + \frac{\epsilon_n h}{\omega}\left[k_y\rho\int_{-\infty}^{+\infty}\varphi_0(\varphi_1 + \varphi_{-1})dx + \int_{-\infty}^{+\infty}\varphi_0\partial_x(\varphi_1 - \varphi_{-1})dx\right] = 0\,. \tag{8}$$

Clearly, there are two cases (i) where $k_y\rho\xi\sigma^{-1/2} < 1$ and the overlap is weak and (ii) where $k_y\rho\xi\sigma^{-1/2} > 1$ and the overlap is strong. In case (i) the detailed solution gives a ϵ_n^2 a toroidal correction to the dispersion. In the second case the coupling is strong, and for further analytic progress we are led to the ansatz that a large number of azimuthal modes are coupled.

In case (i) where $k_y\rho\xi\sigma^{-1/2} < 1$ the approximate solution of Eq. (7) is

$$\varphi_{\pm 1}(x) = -\frac{\epsilon_n h}{\omega D_+(\omega)}[k_y\rho\varphi_0(x) \mp \partial_x\varphi_0(x)]\,,$$

where

$$D_+(\omega) = 1 + (k_y\rho)^2 - i\delta - \frac{hk_y\rho}{\omega} - \left(\frac{c_s}{\omega q R}\right)^2.$$

Substituting this result into Eq. (8) and using that $\int_{-\infty}^{+\infty}(\partial\varphi_0/\partial x)^2 dx = i\sigma/2$, we obtain

$$1 + (k_y\rho)^2 - i\delta + i\sigma - \frac{hk_y\rho}{\omega} + \left(\frac{c_s}{\omega R}\right)^2 \frac{[-2(k_y\rho)^2 + i\sigma]}{D_+(\omega)}\,.$$

Although it is straightforward to work out the roots of this dispersion relation, it becomes evident that for frequencies $\omega > c_s/qR$ the modes are degenerate with $\varphi_{\pm 1}(x) \sim \varphi_0(x)$, which implies that the coupling to higher order $\varphi_n(x)$ is important.

B. Strong coupling approximation

In cases where the weak coupling approximation made in Sec. II A fails by an appreciable margin, we assume that a significant number Δn of azimuthal modes are coupled by the equilibrium toroidal variations. In this case where $m_0 \gg \Delta n \gg 1$ we consider replacing the discrete set of functions $\varphi_n(x)$ by the continuous functions $\varphi(x,n)$ where we write that

$$\varphi_{n+1}(x) - \varphi_{n-1}(x) = 2\frac{\partial\varphi(x,n)}{\partial n}\,, \tag{9}$$

$$\varphi_{n+1}(x) + \varphi_{n-1}(x) = 2\varphi(x,n) + \frac{\partial^2\varphi(x,n)}{\partial n^2}\,. \tag{10}$$

Substituting Eqs. (9) and (10) into Eq. (6) we obtain the following partial differential equation

$$[\partial_x^2 - (k_y\rho)^2 - 1 + i\delta]\varphi(x,n) + \frac{k_y\rho h}{\omega}\left[(1-2\epsilon_n)\varphi - \epsilon_n\frac{\partial^2\varphi}{\partial n^2} - \frac{2\epsilon_n}{k_y\rho}\frac{\partial^2\varphi}{\partial x\partial n}\right] + \left(\frac{c_s}{\omega q R}\right)^2(n - k_y\rho\xi x)^2\varphi = 0 \tag{11}$$

for the continuous function $\varphi(x,n)$. Equation (11) has solutions which are functions of the single variable $\chi = x - n/(k_y\rho\xi)$. Rewriting Eq. (11) for solutions of the form $\varphi(x,n) = \varphi(\chi)$ we obtain

$$\left[1 + \frac{\epsilon_n h(2\xi - 1)}{\omega k_y\rho\xi^2}\right]\frac{\partial^2\varphi}{\partial\chi^2} + \left(\frac{k_y\rho h\epsilon_n\xi}{\omega q}\right)^2\chi^2\varphi = \left[1 + (k_y\rho)^2 - i\delta - \frac{k_y\rho h(1-2\epsilon_n)}{\omega}\right]\varphi \tag{12}$$

which we write more compactly as

$$\mathfrak{L}\varphi = \lambda\varphi\,, \tag{13}$$

where $\mathfrak{L} = \eta\partial\chi^2 + (k_y\rho h\epsilon_n\xi/\omega q)^2\chi^2$, $\lambda = 1 + (k_y\rho)^2 - i\delta - k_y\rho h(1-2\epsilon_n)/\omega$, and $\eta = 1 + \epsilon_n h(2\xi - 1)/(\omega k_y\rho\xi^2)$. The solutions of Eq. (12) or Eq. (13) that are localized in χ are $\varphi(\chi) = u_p(\chi)$ with $\lambda = \lambda_p$ for $p = 0, 1, 2, \ldots$. For $\eta > 0$ and $\sigma = k_y\rho h\epsilon_n\xi/(\omega q\sqrt{\eta})$ we have

$$u_p(\chi) = (i^{1/2}\sigma^{1/2}/2^p p!\,\pi^{1/2})^{1/2} H_p(i^{1/2}\sigma^{1/2}\chi)\exp(-i\sigma\chi^2/2)$$

and (14)

$$\lambda_p = -i\eta\sigma(1+2p)\,.$$

For $\eta < 0$ and $\sigma = k_y\rho h\epsilon_n\xi/[\omega q(-\eta)^{1/2}]$ we have

$$u_p(\chi) = (\sigma^{1/2}/2^p p!\,\pi^{1/2})^{1/2} H_p(\sigma^{1/2}\chi)\exp(-\sigma\chi^2/2)$$

and (15)

$$\lambda_p = -\eta\sigma(1+2p)\,.$$

In the case where $\eta < 0$, the toroidal effects from the ion magnetic drifts reverse the character of the radial propagation potential from an anti-well to a well with reflection occurring at $\chi_{tp} = (1+2p)^{1/2}(-\eta)^{1/2}(\omega_q/k_y\rho h\epsilon_n\xi)^{1/2}$.

The new dispersion relation, which clearly reduces to the previous shear stabilization results for $|\epsilon_n h(2\xi-1)/\omega k_y\rho\xi^2| \ll 1$, is for $\eta > 0$,

$$1 + k_y^2\rho^2 - i\delta - \frac{k_y\rho h(1-2\epsilon_n)}{\omega} + i\left(\frac{k_y\rho h\epsilon_n\xi}{\omega q}\right)\left(1 + \frac{\epsilon_n h(2\xi-1)}{\omega k_y\rho\xi^2}\right)^{1/2}(1+2p) = 0 \tag{16}$$

and for $\eta < 0$

$$1 + k_y^2\rho^2 - i\delta - \frac{k_y\rho h(1-2\epsilon_n)}{\omega} - \left(\frac{k_y\rho h\epsilon_n\xi}{\omega q}\right)\left(\frac{\epsilon_n h(1-2\xi)}{\omega k_y\rho\xi^2} - 1\right)^{1/2}(1+2p) = 0 \ . \tag{17}$$

In the case where $\eta(\omega)$ may be estimated by its value in the limit of small ϵ_n, namely at $\omega_0 = k_y\rho h(1-2\epsilon_n)/(1 + k_y^2\rho^2 - i\delta)$, we write the complex mode frequency as

$$\omega = \frac{k_y\rho h(1 - 2\epsilon_n - iS\eta_0^{1/2}}{1 + k_y^2\rho^2 - i\delta} \tag{18}$$

with $\eta_0^{1/2} \to i(-\eta_0)^{1/2}$ for $\eta_0 < 0$ where

$$\eta_0^{1/2} = \left(1 + \frac{1 + (k_y\rho)^2}{(k_y\rho)^2}\ \frac{2\xi - 1}{\xi^2}\ \frac{\epsilon_n}{1 - 2\epsilon_n}\right)^{1/2} \tag{19}$$

and $S = \epsilon_n\xi/q$.

For $\eta_0 > 0$, the shear stabilization from outgoing wave energy remains, and the condition for marginal stability is

$$(1 - 2\epsilon_n)\delta = (1 + k_y^2\rho^2)S\eta_0^{1/2} \ , \tag{20}$$

where δ is evaluated at $\omega = k_y\rho h(1-2\epsilon_n)/(1 + k_y^2\rho^2)$. For $\eta_0 < 0$, the shear stabilization is lost and all modes with $\delta > 0$ are unstable.

The simplest case where the shear stabilization is lost is for weak shear, $\xi \ll \frac{1}{2}$, and small $(k_y\rho)^2$. The resulting mode is found to be the same as described by Cheung and Horton in their Eqs. (52)–(53). For $\xi \ll \frac{1}{2}$, the geodesic component of the ion magnetic drift dominates and for small $(k_y\rho)^2$ the mode equation[7] becomes approximately $[(c_s^2/\omega_*^2)\partial_s^2 + (1 - \omega_*/\omega) + 2\epsilon_n\cos(s/qR)]\varphi(s) \cong 0$ where $s = qR\theta$ is the coordinate along the magnetic field. From Eq. (19) we have that

$$iS\eta_0^{1/2} \simeq \frac{-\epsilon_n^{3/2}}{q\,|k_y\rho|(1 - 2\epsilon_n)^{1/2}} \ ,$$

and the frequency shifts below ω_{*e} from toroidal effects for $k_y\rho > [\epsilon_n/(1-2\epsilon_n)]^{1/2}/2q \simeq \epsilon_n^{1/2}/2q$ as reported earlier.[7] This condition on $k_y\rho$ for the frequency to shift below ω_{*e} is the same as requiring a significant amount of mode ballooning from the inside $s = \pm qR$ to the outside $s = 0$ of the toroidal surface. For this mode the frequency shift due to $S\eta_0^{1/2}$ is small, and the complex mode frequency is approximately $\omega \cong k_y\rho h(1-2\epsilon_n)/(1 + k_y^2\rho^2 - i\delta)$.

Combining the conditions for the loss of shear stabilization and the applicability of the strong coupling approximation yields the upper limit, $\xi < q(1 - 2\xi)$, to the shear parameter for the occurrence of the absolute ballooning instability. This condition is also obtained by Cordey and Hastie[5] for the occurrence of a Gaussian shaped mode localized to the outside of the tokamak. Cordey and Hastie observe, however, that for $\xi > \frac{1}{2}$ a new mode occurs which is localized to the inside of the tokamak. They argue that to include modes ballooning at $\theta = \theta_0$ within the strong coupling theory, explicit account of the phase coherence at $\theta = \theta_0$ must be recognized in the expansion for $\varphi(\theta)$ by writing $\varphi_n(x) \to \varphi_n(x) \times \exp(-in\theta_0)$ in Eq. (4). Proceeding with this generalization leads to the conclusion that for $\xi > \frac{1}{2}$, a faster growing ballooning mode occurs localized about $\theta_0 = \pi$ with the factor $\epsilon_n(2\xi - 1)$ in the shear stabilization term in Eq. (16) replaced by $\epsilon_n(1 - 2\xi)$. Thus, allowing for the ballooning either to the inside or the outside of the tokamak, the shear stabilization criterion is lost for $\xi < q|1 - 2\xi|$ which is generally satisfied for ξ not too close to 0.5. With this condition satisfied, unstable ballooning modes occur for k_y in the range $\epsilon_n^{1/2}/q|1-2\xi|^{1/2} < k_y\rho < \epsilon_n^{1/2} \times |1 - 2\xi|^{1/2}/\xi$. In Appendix A the dispersion relations are re-derived using the method of Cordey and Hastie.[5] Clearly, the trapped electron effect, which enhance the driving term δ for modes localized in the region of maximum electron trapping, considerably modifies the characteristics of the quasi-modes as shown in Sec. IV.

Another mode occurs when the shear is weak ($\xi < \frac{1}{2}$) and when $\omega k_y\rho \ll \epsilon_n h/\xi^2$. For this case the radial propagation is in a well, and an eigenmode is formed by balancing the adiabatic electron response with the ion magnetic drifts. For $\eta \ll -1$ we obtain from Eq. (17)

$$1 + k_y^2\rho^2 - i\delta - \frac{k_y\rho h(1-2\epsilon_n)}{\omega} - \left(\frac{\epsilon_n h}{\omega}\right)^{3/2}\frac{(k_y\rho)^{1/2}}{q}(1+2p) = 0 \tag{21}$$

which for $\omega > k_y\rho h(1-2\epsilon_n)$ has the root with $\omega/k_y > 0$ given by

$$\omega = \frac{\epsilon_n h(k_y\rho)^{1/3}(1+2p)^{2/3}}{q^{2/3}(1 + k_y^2\rho^2 - i\delta)^{2/3}} \tag{22}$$

which is approximately independent of ξ and r_n. The threshold for instability is determined by ion Landau damping $\omega > v_i/qR$ which gives the critical wavenumber condition

$$k_y\rho(1+2p)^2 > (2T_i/T_e)^{3/2}/q \ , \tag{23}$$

and we require that $\omega/k_y\rho > h(1 - 2\epsilon_n)$ for neglecting the density gradient which gives

$$\frac{\epsilon_n}{1 - 2\epsilon_n} > \frac{2T_i}{T_e}\ \frac{1}{(1+2p)^2} \tag{24}$$

as the condition for sufficiently strong toroidal drifts. The growth rate that follows from Eq. (22) is

$$\gamma = \frac{2}{3}\ \frac{c_s}{R}\ \frac{\delta(k_y\rho)^{1/3}}{q^{2/3}(1 + k_y^2\rho^2)^{5/3}} \tag{25}$$

which reaches a maximum at $k_y\rho = \frac{1}{3}$ when ϵ_n is large enough to permit the neglect of the density gradient. The maximum growth rate is given by

$$\gamma_{max} \simeq \frac{2}{3^{4/3}}\ \frac{c_s\delta}{Rq^{2/3}} \tag{26}$$

which provides a fast growing instability for not too large a major radius. The corresponding frequency is $c_s/3^{1/3}Rq^{2/3}$, and the neglect of ion bounce frequency damping requires $T_e/T_i > 2(3/q)^{2/3}$.

As an example of the behavior of the unstable modes predicted by the theory we show in Figs. 1 and 2, the complex roots $\omega = \omega_k + i\gamma_k$, in units of $h = c_s/r_n$, that are obtained from the dispersion relation (16) or (17) that govern modes localized to $\theta = 0$, labeled outside modes, and the equivalent dispersion relation given in Appendix

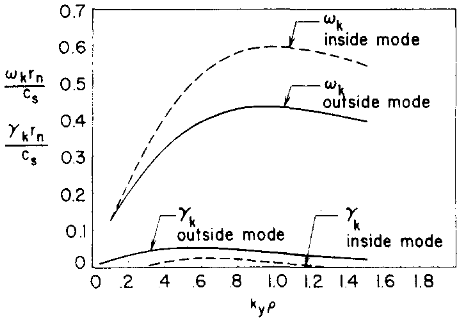

FIG. 1. The wave frequency, ω_k, and growth rate, γ_k, versus $k_y\rho$ for a low shear case ($\xi \approx 0.2$) as obtained from numerical solutions of the dispersion relation (16). The dimensionless system parameters are $\epsilon_n = 0.1$, $\xi = 0.2$, $q = 0.4$, $S = 0.05$ and the nonadiabatic electron response parameter $\delta = 0.2$.

A for modes localized to $\theta = \pi$, and labeled inside modes. As is evident from the analysis the modes localized to the outside are Doppler shifted by the ion drifts to lower frequencies and those localized to the inside are Doppler shifted to higher frequencies. For the weak shear case ($\xi = 0.2$) in Fig. 1 the outside mode dominates with a growth rate approximately twice that of the inside mode. In the strong shear ($\xi = 1$) case in Fig. 2 the inside mode dominates. These results apply in the absence of the trapped electron modifications introduced in Sec. IV which requires the collisionality of the plasma to be sufficiently high that the trapped electron contribution is negligible.

Returning to the condition that the continuum approximations made in Eqs. (9) and (10) be valid we note that at a position near the principal rational surface $r = r_0$, the spectrum of azimuthal mode numbers is given by $\varphi(0, n) = A\exp(-n^2/2\Delta n^2)$, where $\Delta n^2 = \omega(k_y\rho)q\xi(-\eta)^{1/2}/\epsilon_n h$, and for $\eta < 1$, we have $\Delta n^2 \simeq q(\omega k_y\rho/\epsilon_n h)^{1/2}$. Thus, for modes with $k_y\rho$ of order unity the continuum approximation $\Delta n^2 > 1$ is generally valid, and for long wavelength modes where $\omega \simeq k_y\rho h$ the approximation is valid for $k_y\rho > \epsilon_n^{1/2}/q$.

III. TWO-DIMENSIONAL WAVEFUNCTION

It is interesting to consider in more detail the form of the wavefunctions implied by the strong coupling approximation introduced in Sec. II B. For a solution $\varphi_n(x)$ that is a function of $\chi = x - n/k_y\rho\xi$ we may rewrite Eq. (3) according to

$$\begin{aligned}\varphi(\mathbf{x}) &= \exp(im_0\theta - il\phi)\sum_n \varphi(x, n)\exp(in\theta)\\ &= \exp(im_0\theta + ik_y\rho\xi x\theta - il\phi)\sum_n \varphi(x - n/k_y\rho\xi)\\ &\quad\times\exp[i(n - k_y\rho\xi x)\theta]\ . \end{aligned} \tag{27}$$

In the second form of Eq. (27) the first phase factor outside the summation sign expresses the variation of the phase on the magnetic surface in the direction perpendicular to the local magnetic field. This follows by observing that the local perpendicular coordinate in the magnetic surface is $q(r)\theta - \varphi$ and that a wave propagating in that direction with the specified toroidal symmetry is $\exp\{il[q(r)\theta - \phi]\}$ which upon expanding $q(r) = q(r_0) + (r - r_0)q'(r_0)$ and using the definitions of k_y, ξ, and x gives the result as stated. The perpendicular wave $\exp\{il[q(r)\theta - \phi]\}$ is not periodic in θ, and to restore periodicity Coppi *et al.*[8,9,13] and Hastie and Connor[14] introduce the additional phase factor $\exp\{il[q(r_0) - q(r)]\times F(\theta)\}$, where $F(\theta)$ is a general monotonic function which increases by 2π for each revolution in θ, that is $F(\theta + 2\pi) = F(\theta) + 2\pi$. Thus, in the Coppi–Rewoldt notation the general wave field in a torus is

$$\varphi(\mathbf{x}) = \tilde{\varphi}(\theta)\exp\{il[q(r)\theta - \phi] + il[q(r_0) - q(r)]F(\theta)\}\ , \tag{28}$$

where $\tilde{\varphi}(\theta + 2\pi) = \tilde{\varphi}(\theta)$ and $F(\theta) = \int_0^\theta G(\theta')\,d\theta'$, where $G(\theta + 2\pi) = G(\theta)$. In the present analysis in the strong coupling case $G = 1$ or $F(\theta) = \theta$ and $q(r) - q(r_0) = (r - r_0)\times q'(r_0)$. Comparing the field in Eq. (27) with Eq. (28) we observe that the periodic function $\tilde{\varphi}(\theta)$ is

$$\tilde{\varphi}(\theta) = \sum_n \varphi(x, n)e^{in\theta}\ .$$

It is also important to consider the variation of the function along the field line at a given radial position. Letting s be the distance along the field line with $s = 0$ corresponding to $\theta = 0$ at the outside of the torus where the field line crosses the horizontal plane of symmetry, we observe that the variation along the field line is

$$\begin{aligned}\tilde{\varphi}(\theta)\exp\{il[q(r_0) - q(r)]\theta\} &\cong \sum_n \varphi(x, n)\exp[i(n - k_y\rho\xi x)\theta]\\ &\cong \sum_n \varphi(x, k_\parallel)\exp(ik_\parallel s)\end{aligned} \tag{29}$$

where

$$k_\parallel(n) = (n - k_y\rho\xi x)/qR\ ,\qquad n = 0, \pm 1, \pm 2, \ldots \tag{30}$$

and $s = qR\theta$. Thus, in the strong coupling limit $\varphi_n(x)$ reduces to a function of the local $k_\parallel(x, n)$. The wave field in Eq. (27) may now be understood as a wave with toroidal symmetry l propagating across the magnetic field on the magnetic surface multiplied by the function $\sum_{k_\parallel}\varphi(k_\parallel)\exp(ik_\parallel s)$ that varies with positions along the field. This product function for the fluctuating field is doubly periodic in the toroidal angles.

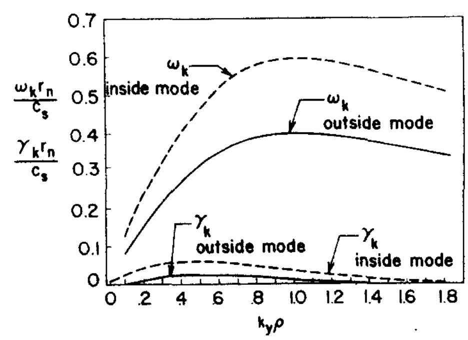

FIG. 2. The same information as in Fig. 1 except for a high shear case ($\xi = 1.0$). The dimensionless system parameters are $\epsilon_n = 0.1$, $\xi = 1.0$, $q = 2.0$, $S = 0.05$, and $\delta = 0.2$.

IV. TRAPPED ELECTRON MODE COUPLING EFFECTS

The trapped electron contribution to the fluctuating electron density is peaked on the outside of the torus in the region of maximum magnetic trapping and thus is an additional mechanism for coupling the azimuthal modes. For the pure trapped electron mode, described with a local approximation for the radial variation, the mode coupling result in the $\xi = rq'(r)/q(r) = 0$ limit is a ballooning,[10] given approximately by $\tilde{\varphi}(\theta) = 1 + \cos\theta$, of the mode to the outside of the torus with nearly zero amplitude on the inside. Further calculations that include the effect of the ion-acoustic wave propagation along the magnetic field which tends to reduce the ballooning of the mode are given by Liewer *et al.*[11]

Re-examining the $\xi = 0$ ballooning calculation with sinusoidal and elliptic integral trapped particle orbits, Ross and Miner[12] compare the results obtained for $\tilde{\varphi}(\theta)$ and the enhancement factor g that gives the increase in the trapped particle contribution over the flute-like mode value of $(\epsilon/2)^{1/2}$ to the ballooning mode value of $g(\epsilon/2)^{1/2}$. For sinusoidal orbits they obtain $\tilde{\varphi}(\theta) = 1 + 1.225\cos\theta$ and $g = 1.707$, where as for elliptic integral orbits they obtain $\tilde{\varphi}(\theta) = 1 + 0.8954\cos\theta$ and $g = 1.653$. In the case where the shear parameter ξ is appreciable the mode is forced to develop a finite $k_\parallel$ variation between the rational surfaces, and consequently, the effect of coupling between modes localized on neighboring rational surfaces also requires the formation of a quasi-mode. We now consider this coupling problem with the sinusoidal orbit approximation.

In view of the comparison given in the preceding paragraph we formulate the trapped electron orbits in terms of the approximate sinusoidal motion in the magnetic well as used previously in Horton *et al.*[2] and Liewer *et al.*[11] From a straightforward calculation we obtain the following expression for the nonadiabatic contribution to the oscillating electron density written as $i\delta\varphi_n(x)$ in Sec. II:

$$i\delta\varphi_n(x) - \left(\frac{\epsilon}{2}\right)^{1/2} Y(\omega) \times \sum_{n'} J_0[\theta_0(n - k_y\rho\xi x)]\, J_0[\theta_0(n' - k_y\rho\xi x)]\, \varphi_{n'}(x) , \tag{31}$$

where θ_0 is a mean value of the angle at the turning point of the trapped electron orbits. The function $Y(\omega)$ is the Maxwellian average of the energy dependent trapped electron response[2] given approximately by $\{\omega - \omega_*[1 + \eta_e(E/T_e - \frac{3}{2})]\}/[\omega - \omega_{De}(E/T_e) + i\nu_{eff}(T_e/E)^{3/2}]$. Substituting Eq. (31) for $i\delta\varphi_n(x)$ into Eq. (6) gives the set of coupled differential equations that describe the mode when trapped electrons are significant. As already noted, finite shear ($\xi \neq 0$) is expected to permit a strong coupling between modes on neighboring rational surfaces so that we again seek solutions of the form $\varphi_n(x) = \varphi(\chi)$ and obtain the following integral-differential equation where the kernel is degenerate:

$$\mathcal{L}\varphi(\chi) - \lambda\varphi(\chi) + \left(\frac{\epsilon}{2}\right)^{1/2} Y(\omega) k_y\rho\xi J_0(\alpha\chi) \times \int_{-\infty}^{+\infty} J_0(\alpha\chi')\varphi(\chi')d\chi' = 0 , \tag{32}$$

where $\alpha = \theta_0(k_y\rho\xi)$ and $dn' = -k_y\rho\xi\, d\chi'$. The differential operator $\mathcal{L}$ and the function λ are defined in Eq. (13). There appears to be no simple solution of the integral-differential Eq. (32). We consider two limiting approximate solutions of the equation.

In the case where the trapped electron term is a perturbation to the eigenvalue problem defined previously, for example when only the high energy tail of the electron distribution is trapped, the solutions of the integral equation are readily expressed in terms of the eigenfunctions of $\mathcal{L}$. Namely, expanding $\varphi(x)$ and $J_0(\alpha x)$ according to

$$\begin{aligned} \varphi(x) &= \sum_{p=0}^{\infty} a_p u_p(x) , \\ J_0(\alpha x) &= \sum_{p=0}^{\infty} b_p u_p(x) , \end{aligned} \tag{33}$$

where $\mathcal{L}u_p = \lambda_p u_p$, we obtain the symmetric complex matrix equation for a_p

$$\sum_{q=0}^{\infty} \left[(\lambda - \lambda_p)\delta_{p,q} - \left(\frac{\epsilon}{2}\right)^{1/2} Y(\omega) k_y\rho\xi b_p b_q\right] a_q = 0 . \tag{34}$$

Computing the lowest order coefficients b_q, we obtain

$$\begin{aligned} b_0 &= (\pi/i\sigma)^{1/4} 2^{1/2} J_0(\alpha^2/4\sigma)\exp(i\alpha^2/4\sigma) , \\ b_2 &= 2^{-1/2}[1 + (\alpha^2/\sigma)(i + J_0'/J_0)]b_0 , \end{aligned} \tag{35}$$

and $b_{2p+1} = 0$. It appears that the basic result for $\alpha^2/\sigma \sim 1$ occurs at the lowest order where we obtain from Eqs. (31), (34), and (35) that for $\eta > 0$ the dispersion relation is

$$1 + (k_y\rho)^2 - \frac{k_y\rho h(1 - 2\epsilon_n)}{\omega} + i\frac{k_y\rho h\epsilon_n\xi\eta^{1/2}}{\omega q} - \left(\frac{\epsilon}{2}\right)^{1/2} \frac{2\pi^{1/2}Y(\omega)k_y\rho\xi}{(i\sigma)^{1/2}} J_0^2\left(\frac{\alpha^2}{4\sigma}\right)\exp\left(-\frac{\alpha^2}{2i\sigma}\right) = 0 , \tag{36}$$

with $\alpha = \theta_0(k_y\rho\xi)$ and $\sigma = k_y\rho h\epsilon_n\xi/\omega_q\eta^{1/2}$. For $\eta < 0$, the quantity $\eta^{1/2}$ in Eq. (36) becomes $\eta^{1/2} \to i(-\eta)^{1/2}$ as explained in more detail in Eqs. (14)–(17).

The modification to the trapped electron contribution represents the average effect of (1) reducing the trapped particle contribution which is proportional to $\oint ds\, \varphi(s)/|v_\parallel|$ due to phase oscillations from the finite $k_\parallel qR$ between the rational surfaces and of (2) increasing the trapped electron contribution by the number of rational surfaces, $\delta N = k_y\rho\xi\langle\Delta x^2\rangle^{1/2} = k_y\rho\xi/(i\sigma)^{1/2} \gtrsim 1$, covered by the radial eigenmode.

Since the mode coupling effects from the trapped electron terms given by Eq. (31) are not limited to the two neighboring rational surfaces, as are the coupling effects from the ion magnetic drifts, but rather extend over many rational surfaces, we may expect solutions $\varphi(\chi)$ that are considerably broader than the $u_0(\chi)$ Gaussian modes obtained in the absence of the trapped electron effects. Consider a broader quasi-mode for which we assume

$$\left|\frac{1}{\varphi}\frac{\partial^2\varphi}{\partial\chi^2}\right| \ll \sigma^2\chi^2 ,$$

then Eq. (32) reduces to a simple integral equation which has the solution

$$\varphi(\chi) = \frac{(\epsilon/2)^{1/2} Y(\omega) J_0(\alpha\chi) C}{\lambda - (hk_y\rho\epsilon_n\xi/\omega q)^2\chi^2} \tag{37}$$

where the constant $C = \int_{-\infty}^{+\infty} dn' J_0(\alpha\chi')\varphi(\chi')$ and as in Sec. II B

$$\lambda = 1 + (k_y\rho)^2 - i\delta - \frac{hk_y\rho(1-2\epsilon_n)}{\omega} \tag{38}$$

with $i\delta$ including dissipative effects other than those arising from the trapped electrons. Evaluating the normalization integral C yields the dispersion relation

$$1 = \left(\frac{\epsilon}{2}\right)^{1/2} Y(\omega) \int_{-\infty}^{+\infty} \frac{k_y\rho\xi\, dx\, J_0^2(\alpha x)}{\lambda - (hk_y\rho\epsilon_n\xi/\omega q)^2 x^2} . \tag{39}$$

Expanding the ω dependence about the approximate root $\omega_k = hk_y\rho(1-2\epsilon_n)/[1+(k_y\rho)^2 - i\delta]$ by writing $\omega = \omega_k + \Delta\omega_k$ we obtain the approximate solution of Eq. (39) as

$$\Delta\omega_k = \frac{\omega_k}{1+(k_y\rho)^2}\left(\frac{\epsilon}{2}\right)^{1/2} Y(\omega)\int_{-N}^{N} dn' J_0^2(\theta_0 n') , \tag{40}$$

where the cutoff at $|n'| = N$ is given approximately by

$$\frac{c_s^2 N^2}{\omega^2 q^2 R^2} \cong \Delta\lambda = \frac{\Delta\omega_k[1+(k_y\rho^2)]}{\omega_k} . \tag{41}$$

Clearly, N defines the maximum number of rational surfaces about the central surface for which the frequency shift is not dominated by the ion-acoustic wave propagation. The growth rate, $\gamma_k = \mathrm{Im}(\Delta\omega_k)$, increases with increasing N until $\theta_0 N \gg 1$. The maximum growth rate enhancement occurring in Eq. (40) is estimated by

$$\int_{-N}^{N} dn\, J_0^2(\theta_0 n) \cong \frac{2}{\pi\theta_0}\ln\left(\frac{\theta_0\omega q R\Delta\lambda^{1/2}}{c_s}\right)$$

for $\theta_0 N \gg 1$.

The enhancement of the trapped electron contribution found here, as well as the modification to the shear stabilization term found in Sec. II due to the ion magnetic drifts, are features that are embodied in the numerical analysis of the toroidal eigenmode problem investigated by Rewoldt *et al.*[4] The increase in the growth rate with torodial mode coupling effects as reported in the studies of Rewoldt *et al.*[4] appears consistent with the enhancement to the growth rate obtained here. The weaker variation than in the slab model of the growth rate with variation in the shear parameter $\xi = rq'/q$ is also consistent with their results. Further, remarks concerning the comparison of the results in the two studies are made in the review article by Tang.[15] In practical applications of the stability theory to tokamak systems, it is often important to include the finite ion gyroradius effects. These effects modify the dispersion relation in a relatively straightforward manner that follows from modifying the coefficients in the present equation according to the substitutions given in Appendix B.

V. CONCLUSIONS

An analysis is presented of the toroidal eigenmode problem for circular cross section tokamaks with equations written in the low temperature ion fluid approximation. In the analysis we make use of the strong coupling theory recently introduced by Taylor[1] for a similar toroidal mode coupling problem. We also obtain the quasi-mode and dispersion relation by the alternate method developed by Cordey and Hastie[5] and given here in Appendix A.

For the case with only ion magnetic drifts, which occurs when the collisionality of the system is sufficiently high so that only a small fraction of high energy electrons form trapped orbits, the physical mode coupling effects occur through (1) the normal component of magnetic curvature that produces guiding center drifts in the magnetic surface and (2) the geodesic component of the magnetic curvature which produces radial guiding center drifts. For weak shear defined by $\xi = rq'(r)/q(r) < \frac{1}{2}$, the effect of the drift in the magnetic surface dominates and produces a mode ballooning to the outside of the torus. The surface drift produces a downward shift in the frequency of the mode and a contraction of each local mode $u_0(x - x_k)$. The potential of the quasi-mode Eq. (12) changes character from an anti-well to a well when $\xi k_y\rho < [\epsilon_n/(1-2\epsilon_n)]^{1/2}$ for $\xi < \frac{1}{2}$. Thus, for typical values of $\epsilon_n = r_n/R$ and for weak shear $\xi < \frac{1}{2}$ the effect of shear stabilization is lost for most values of $k_y\rho$ of interest. In this case, the quasi-mode frequency is $\omega = \omega_{*e}(1-2\epsilon_n)/[1+(k_y\rho)^2 - i\delta_k]$. In contrast, the effect of the radial ion guiding center drift competes against the surface drift in regard to the loss of shear stabilization. The effect of the two drifts balance at $\xi = \frac{1}{2}$, and for $\xi > \frac{1}{2}$ the effective potential of the quasi-mode equation remains an anti-well even for large values of ϵ_n. In this case the shear stabilization term is enhanced as given by the dispersion relation (16).

An investigation of the toroidal modification to shear stabilization using the Princeton two-dimensional drift mode code is reported in the second part of Ref. 4. In their Fig. 11 and the associated text, the variation of the growth rate with the shear parameter ξ is analyzed. For values of ξ less than unity the growth rate is approximately independent of ξ and for ξ greater than unity an approximately linear decrease of the growth rate is found. The variation is reported to be consistent with the dispersion relation obtained in the present article when the effect of the finite values of $k_y\rho_i$ and $k_r\rho_i$ are considered. The possible competing effect from the shear dependence of the trapped electron magnetic drift frequency was also investigated and shown to be a subdominant effect to those considered here.

We also consider the case where a modulation on the magnetic surface arises from the variation of the trapped electron density from its minimum on the inside of the torus to its maximum on the outside. Here, the quasi-mode Eq. (32) is an integral-differential equation. A formal solution of the equation is obtained by expanding the degenerate kernel in the eigenfunction $u_p(\chi)$ of the problem in the absence of the trapped electron effects while including the magnetic drifts. The lowest order truncation of this matrix gives an approximate perturbation formula for the shift in the quasi-mode frequency and growth rates. In addition, since the kernel of the trapped electron mode equation couples many rational surfaces, an approximate broad quasi-mode solution $u_{te}(\chi)$ is also obtained which represents the sum over many of the Gaussian quasi-modes $u_p(\chi)$. The broad quasi-mode $u_{te}(\chi)$ has a somewhat complicated dispersion relation given in Eq. (39). An approximate solution of this dispersion relation yields the usual trapped electron mode growth rate enhanced by a factor proportional to the number of rational surfaces contained in the quasi-mode and weighted by the square of the bounce orbit averaged phase of the wave along the magnetic field. The enhancement of the growth rate is calculated approximately in Eq. (40). The effect of shear stabilization is weak in this regime. The effect of the bounce orbit averaging of the parallel phase of the mode is also recently reported by Koechlin and Samain.[17]

An implication of these results for tokamaks is that the strength of ion Landau damping in the drift modes is less than previously assumed. In terms of anomalous transport this has the effect of increasing the fluctuation level and the associated transport rate above that previously reported.[2,3] With the analytic formulae derived here it becomes feasible to modify the anomalous radial transport codes to take into account these toroidal effects. In radial regions where strongly growing drift modes are presently calculated in a transport code the toroidal modifications developed here are not expected to produce large effects. Other radial regions previously calculated to be stable may now be unstable which principally acts to extend the anomalous transport over a broader radial region.

In addition to the limiting cases investigated in the present work where analytic solutions are obtained, we are developing a numerical procedure for computing the two-dimensional mode structure and eigenfrequencies for parameters where the approximations fail. The numerical algorithm used in the computations follows from extending to the toroidal problem previous time dependent initial value solutions obtained for the non-uniform slab.[16] Using the approximate solutions obtained here as initial data the wave functions and complex wave frequencies are rapidly computed for the fastest growing modes.

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APPENDIX A

Here, we consider the alternate method[5] of solving the two-dimensional mode equation obtained from the finite difference Eq. (6) for $\varphi_n(x)$. Transforming Eq. (6) back to an equation for $\varphi(x,\theta)$ we obtain, in dimensionless units where $\rho = h = 1$,

$$-\frac{\partial^2\varphi}{\partial x^2} + \left(1 + k^2 - i\delta - \frac{k}{\omega}\right)\varphi - \frac{2i\epsilon_n}{\omega}\sin\theta\,\frac{\partial\varphi}{\partial x} + \frac{2\epsilon_n k}{\omega}\varphi\cos\theta + \left(\frac{\epsilon_n}{\omega q}\right)^2\left(\frac{\partial}{\partial\theta} - ik\xi x\right)^2\varphi = 0\,. \tag{A1}$$

We now consider modes of the form

$$\varphi(x,\theta) = \exp(-\tfrac{1}{2}\sigma x^2 + ibx\theta - \tfrac{1}{2}\gamma\theta^2) \tag{A2}$$

localized about $\theta = 0$. Expanding the trigonometric functions in Eq. (A1) about $\theta = 0$ and substituting the ansatz (A2) into Eq. (A1) we obtain the conditions,

$$x^2:\quad \sigma^2 = -\left(\frac{\epsilon_n}{\omega q}\right)^2 (b - \xi k)^2\,, \tag{A3}$$

$$x\theta:\quad \sigma\left(b + \frac{\epsilon_n}{\omega}\right) = \left(\frac{\epsilon_n}{\omega q}\right)^2 \gamma(b - \xi k)\,, \tag{A4}$$

$$\theta^2:\quad \gamma^2\left(\frac{\epsilon_n}{\omega q}\right)^2 = \frac{\epsilon_n k}{\omega} - b\left(b + \frac{2\epsilon_n}{\omega}\right)\,, \tag{A5}$$

and the generalized dispersion relation is

$$1 + k^2 - i\delta - \frac{k(1-2\epsilon_n)}{\omega} + \sigma - \left(\frac{\epsilon_n}{\omega q}\right)^2\gamma = 0\,. \tag{A6}$$

Now in the limit where $S = \epsilon_n\xi/q$ is finite and $\epsilon_n \to 0$ the solutions of Eq. (A3)–(A6) are

$$\sigma = i\,\frac{\epsilon_n\xi k}{\omega q} = i\,\frac{kS}{\omega}\,,\qquad \gamma = b = 0\,. \tag{A7}$$

The dispersion relation for $\epsilon_n = 0$ reduces to

$$1 + k^2 - i\delta - (k/\omega) + i\,(kS/\omega) = 0 \tag{A8}$$

which has as its solution $\omega(k) = k(1 - iS)/(1 + k^2 - i\delta)$.

With $\epsilon_n \neq 0$, however, the solution becomes $b = \xi k$, $\sigma = 0$ and

$$\gamma = -i\,\frac{\omega q}{\epsilon_n}\,|b|\left(1 + \frac{2\epsilon_n}{\omega b} - \frac{\epsilon_n k}{\omega b^2}\right)^{1/2} = -i\,\frac{\omega q k\xi}{\epsilon_n}\left(1 + \frac{\epsilon_n(2\xi - 1)}{\omega k\xi^2}\right)^{1/2} \tag{A9}$$

with the expansion condition that $|\gamma| > 1$. The new dispersion relation obtained from Eqs. (A5) and (A9) is

$$1 + k^2 - \frac{k(1-2\epsilon_n)}{\omega} - i\delta + i\,\frac{\epsilon_n k\xi}{\omega q}\left(1 + \frac{\epsilon_n(2\xi-1)}{\omega k\xi^2}\right)^{1/2} = 0\,. \tag{A10}$$

Marginal stability now occurs for

$$\omega = k(1 - 2\epsilon_n)/(1 + k^2)\,,$$

and

$$\delta = \left|\frac{\epsilon_n\xi/q}{1 - 2\epsilon_n}\right|(1 + k^2)\left[1 + \frac{1+k^2}{k^2}\left(\frac{\epsilon_n}{1-2\epsilon_n}\right)\frac{2\xi - 1}{\xi^2}\right]^{1/2}.$$

For the mode $k^2 = \frac{1}{2}$ we have the critical value of δ for instability

$$\delta_c = \frac{3}{2}\left|\frac{\xi\epsilon_n/q}{1 - 2\epsilon_n}\right|\left[1 + \frac{3(2\xi - 1)}{\xi^2}\left(\frac{\epsilon_n}{1 - 2\epsilon_n}\right)\right]^{1/2}.$$

Accordingly, for modes localized at $\theta = 0$ and for $\xi < \frac{1}{2}$ we lose shear stabilization when $\xi^2 k^2 < \epsilon_n(1 - 2\xi)(1 + k^2)/$

$(1-2\epsilon_n)$. The modes are broad in x and the local theory applies. Returning to Eq. (A1) and assuming the density gradient term becomes $(k/\omega)(1-x^2/\bar{r}_n^2)$ we obtain as a perturbation that $\sigma=(k/\omega)^{1/2}|\bar{r}_n|^{-1}$. Thus, these are quasi-modes with the radial distribution determined by $\omega_*(x)$.

To examine modes centered at $\theta=\pi$ we return to expression (A2) and let $\theta\to\theta-\pi$. From Eq. (A1) we then obtain equations identical to (A3)–(A6) with $\epsilon_n\to-\epsilon_n$. The dispersion relation becomes

$$1+k^2-\frac{k(1+2\epsilon_n)}{\omega}-i\delta+i\,\frac{\epsilon_n k\xi}{\omega q}\left(1+\frac{\epsilon_n(1-2\xi)}{\omega k\xi^2}\right)^{1/2}=0\ . \tag{A11}$$

Now, marginal stability occurs for

$$\omega=k(1+2\epsilon_n)/(1+k^2)\ ,$$

and

$$\delta=\frac{\epsilon_n\xi(1+k^2)}{q(1+2\epsilon_n)}\left[1+\frac{1+k^2}{k^2}\left(\frac{\epsilon_n}{1+2\epsilon_n}\right)\left(\frac{1-2\xi}{\xi^2}\right)\right]^{1/2}\ .$$

Shear stabilization for modes localized at $\theta=\pi$ is lost for $\xi>\frac{1}{2}$ and $\xi^2k^2<\epsilon_n(2\xi-1)(1+k^2)/(1+2\epsilon_n)$. In the small ϵ_n limit this reduces to the condition $\xi k<\epsilon_n^{1/2}(2\xi-1)^{1/2}$ as reported earlier by Cordey and Hastie.[5]

APPENDIX B

For practical evaluations of the drift wave dispersion relation in tokamak systems is not justified to neglect the finite ion gyroradius effects. The finite ion gyroradius effects occur through the parameter $b=(k_y\rho)^2\times(T_i/T_e)=(k_y\rho_i)^2$ where $\rho_i=c(m_iT_i)^{1/2}/eB$ and, although it is straightforward to derive them from the collisionless ion kinetic equation, they have been omitted in the text for clarity. For completeness, we give the modification of the basic Eq. (6) that occurs when these effects are included. These kinetic effects are also given by Rewoldt *et al.*[4] Defining $\eta_i=d\ln T_i/d\ln n_e$ and $\omega_{*i}=k_y(cT_i/eB)d\ln n_e/dx$, we find that, following the terms in Eq. (6) sequentially, the modifications are as follows:

$$\partial_x^2\to\left((I_0-I_1)e^{-b}-\frac{\omega_{*i}}{\omega}\{(I_0-I_1)e^{-b}+\eta_i[I_0+2b(I_1-I_0)]e^{-b}\}\right)\partial_x^2 \underset{b\to0}{\simeq}\left(1-\frac{\omega_{*i}(1+\eta_i)}{\omega}\right)\partial_x^2\ , \tag{B1}$$

$$1+(k_y\rho)^2\to\left[1+\frac{T_e}{T_i}(1-I_0e^{-b})\right]\underset{b\to0}{\simeq}1+\frac{T_e}{T_i}\,b\ , \tag{B2}$$

$$\frac{hk_y\rho}{\omega}\to[I_0e^{-b}+\eta_ib(I_1-I_0)e^{-b}]\,\frac{hk_y\rho}{\omega}\underset{b\to0}{\simeq}[1-b(1+\eta_i)]\,\frac{hk_y\rho}{\omega}\ , \tag{B3}$$

$$2\epsilon_n\to\left\{[(2-b)I_0+bI_1]e^{-b}-\frac{\omega_{*i}}{\omega}[(2-b)I_0+bI_1+\eta_i(2-4b+2b^2)I_0+(3b-2b)^2I_1]e^{-b}\right\}\epsilon_n \underset{b\to0}{\simeq}\left(1-\frac{\omega_{*i}(1+\eta_i)}{\omega}\right)2\epsilon_n\ , \tag{B4}$$

$$\left(\frac{c_s}{\omega qR}\right)^2\to\left(I_0e^{-b}-\frac{\omega_{*i}}{\omega}\{I_0e^{-b}+\eta_i[I_0e^{-b}+b(I_1-I_0)e^{-b}]\}\right)\left(\frac{c_s}{\omega qR}\right)^2 \underset{b\to0}{\simeq}\left[1-\frac{\omega_{*i}(1+\eta_i)}{\omega}\right]\left(\frac{c_s}{\omega qR}\right)^2\ . \tag{B5}$$

With the new frequency dependences that occur through the ion diamagnetic pressure drift terms, $(1-\omega_{*pi}/\omega)$, where $\omega_{*pi}=\omega_{*i}(1+\eta_i)$, the relatively simple dispersion relations obtained in the text become considerably more complicated.

[1] J. B. Taylor, in *Plasma Physics and Controlled Nuclear Fusion Research* (International Atomic Energy Agency, Vienna, 1977), Vol. II, p. 323.

[2] W. Horton, Jr., D. W. Ross, W. M. Tang, H. L. Berk, E. A. Frieman, R. E. LaQuey, R. V. Lovelace, S. M. Mahajan, M. N. Rosenbluth, and P. H. Rutherford, in *Plasma Physics and Controlled Nuclear Fusion Research* (International Atomic Energy Agency, Vienna, 1975), Vol. I, p. 541.

[3] N. T. Gladd and W. Horton, Jr., Phys. Fluids **16**, 879 (1973); W. Horton, Jr., *ibid*. **19**, 711 (1976).

[4] G. Rewoldt, W. M. Tang, and E. A. Frieman, Phys. Fluids **20**, 402 (1977); and Princeton Plasma Physics Report PPPL-1387 (1977).

[5] J. G. Cordey and R. J. Hastie, Nucl. Fusion **17**, 523 (1977).

[6] J. C. Adam, G. Laval, and R. Pellat, Nucl. Fusion **13**, 47 (1973).

[7] L. Cheung and W. Horton, Jr., Ann. Phys. (N.Y.) **81**, 201 (1973).

[8] B. Coppi, R. Pozzoli, G. Rewoldt, and T. J. Schep, in *Plasma Physics and Controlled Nuclear Fusion Research* (International Atomic Energy Agency, Vienna, 1975), Vol. I, p. 549.

[9] B. Coppi and G. Rewoldt, in *Advances in Plasma Physics*, edited by A. Simon and W. B. Thompson (Wiley, New York, 1978), Vol. 6 (to be published).

[10] B. B. Kadomtsev and O. P. Pogutse, in *Reviews of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1970), Vol. 5, p. 327.

[11] P. C. Liewer, W. M. Manheimer, and W. M. Tang, Phys. Fluids **19**, 276 (1976).

[12] D. W. Ross and W. Miner, Bull. Am. Phy. Soc. **21**, 1152 (1976) and University of Texas, Fusion Research Center Report No. 128 (1977).

[13] B. Coppi and E. Minardi, Phys. Fluids **16**, 1021 (1973).

[14] J. W. Connor and R. J. Hastie, Plasma Phys. **17**, 97 (1975) and in *Proceedings of the Sixth European Conference on Controlled Fusion and Plasma Physics* (Joint Institute for Nuclear Research, Moscow, 1973), Vol. I, p. 631.

[15] W. M. Tang, Nucl. Fusion (to be published).

[16] W. Horton, Jr., H. Kwak, R. Estes, and D. I. Choi, Phys. Fluids **20**, 1476 (1977).

[17] F. Koechlin and A. Samain, Association Euratom-CEA, Report EUR-CEA-FC-910 (1977).