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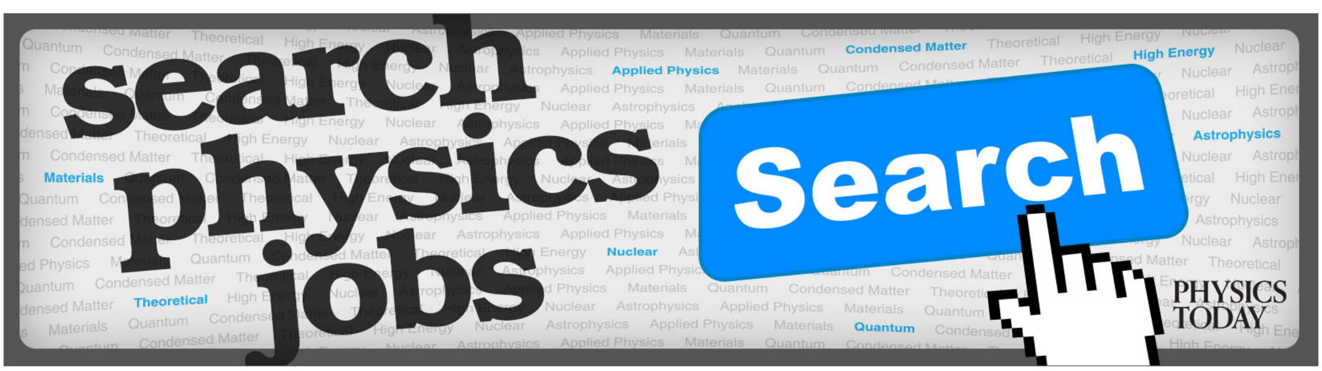

Quantum conservation laws for charged particle systems in magnetic fields

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The conservation laws in the exact operator form are derived for the two-component system of interacting charged particles in magnetic and electric fields. These laws represent the equations of motion for the mass-, current-, and energy-density operators, where both the pressure tensor and energy-current operators are calculated exactly.

1. INTRODUCTION

The purpose of the present paper is to obtain equations of motion for the density operators for the interacting charged Fermi (or Bose) particle system in magnetic and electric fields. These equations are the differential conservation laws for the charge-, current-, and energy-density operators, where the stress tensor and energy-current operators are calculated exactly, it means, without any approximations concerned with a range of the interparticle forces.

The quantum hydrodynamic (QHD) equations were first introduced to one-particle quantum mechanics by Madelung.[1] In the many-body case, under the influence of Landau's phenomenological theory of helium, the quantum-statistical hydrodynamic approach was successfully used to calculate Green's functions (or correlation functions) for the normal and superfluid bosons by Bogolubov,[2] Kadanoff and Martin,[3] Hohenberg and Martin,[4] and for the fermions by Galasiewicz.[5] The derivations of QHD equations in operator form from the many-body Hamiltonian were given by Puff and Gillis,[6] Kugler,[7] and recently by Robertson.[8] In the last five years the QHD method has been studied extensively, especially by the Fröhlich School[9] (structure of the equations of motion and its application to superfluids), and by the group of authors: Bierter, Garrison, Morrison, and Wong[10] (mathematical structure of the algebra of density operators and its physical consequences).

However the QHD description of a many-charged-particles system in a magnetic field has not been considered, although, e.g., an electron gas and an ion lattice seem to be an interesting subject for application of such a theory. This is done in the present paper. Starting from the Hamiltonian for the interacting charged particles in the magnetic (and electric) field the exact equations and conservations laws are derived in Sec. 2, and in the Appendix.

2. CONSERVATION LAWS IN OPERATOR FORM

Switching on the magnetic field generated by vector potential $\mathbf{A}(\mathbf{r},t)$ leads to the change (we put $\hbar = 1 = c$)

$$\partial_\alpha \to \partial_\alpha - ieA_\alpha$$

so we introduce the operation denoted by semicolon as follows:

$$\psi_{;\alpha} \equiv (\psi_{,\alpha} - ieA_\alpha \psi) \equiv (\partial_\alpha - ieA_\alpha)\psi, \tag{1}$$

for destruction operators, and

$$\psi^{+}_{;\alpha} \equiv \psi^{+}_{,\alpha} + ieA_\alpha \psi^{+}, \tag{2}$$

for creation operators of the fermion (or boson) field.

In this notation the Hamiltonian of the two-component system to be considered here is

$$H = \sum_{i=1,2.} H^i,$$

$$H^i = (2m^i)^{-1} \int d\mathbf{r}\, \overset{+}{\psi}{}^{i}_{;\alpha}(\mathbf{r}t)\psi^{i}_{;\alpha}(\mathbf{r}t) + \int d\mathbf{r}\, e^i n^i(\mathbf{r}t)\, U(\mathbf{r}t)$$
$$+ \tfrac{1}{2} \sum_{k=1,2} \int d\mathbf{r}\, d\mathbf{r}'\, V^{ik}(|\mathbf{r}-\mathbf{r}'|)\overset{+}{\psi}{}^{i}(\mathbf{r}t)\, n^k(\mathbf{r}'t)\, \psi(\mathbf{r}t), \tag{3}$$

where we assume a summation over repeated indexes α. $U(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$ are the time-and space-dependent external scalar and vector potentials. The charge-density operators satisfy the neutrality condition

$$\sum_{i=1,2} \int d\mathbf{r}\, e^i n^i(\mathbf{r},t) = 0,$$

which insures the stability of the system. [In (3) we omit the spin.]

The charge- (or mass-), current-, and energy-density operators are defined in the symmetric form

$$\rho^e \equiv \sum_i^2 e^i n^i, \quad \rho^m \equiv \sum_i^2 m^i n^i,$$
$$n^i(\mathbf{r}t) \equiv \overset{+}{\psi}{}^{i}(\mathbf{r}t)\, \psi^i(\mathbf{r}t), \tag{4}$$

$$j^e_\alpha \equiv \sum_i j^i_\alpha e^i, \quad j^m_\alpha \equiv \sum_i j^i_\alpha,$$
$$j^k_\alpha(\mathbf{r}\mathrm{t}) \equiv \tfrac{1}{2} i(\overset{+}{\psi}{}^{k}_{;\alpha}\psi^k - \overset{+}{\psi}{}^{k}\psi^k_{;\alpha})(\mathbf{r}t), \tag{5}$$

and

$$(\rho\epsilon)(\mathbf{r}t) \equiv -\sum_i (4m^i)^{-1}(\overset{+}{\psi}{}^{i}_{;\alpha\alpha}\psi^i + \overset{+}{\psi}{}^{i}\psi^i_{;\alpha\alpha})(\mathbf{r}t)$$
$$+ \sum_i e^i n^i(\mathbf{r}t)\, U(\mathbf{r}t) \tag{6}$$
$$+ \tfrac{1}{2} \sum_{i,k} \overset{+}{\psi}{}^{i}(\mathbf{r}t) \int V^{ik} |\mathbf{r}-\mathbf{r}'|\, n^k(\mathbf{r}'t)\, d\mathbf{r}' \psi^i(\mathbf{r}t),$$

respectively.

From the commutation relations with the Hamiltonian (3) it follows that

$$i\partial_t \psi^k(\mathbf{r}t) = [\psi^k, H] = -(2m^k)^{-1}\psi^k_{;\alpha\alpha}(\mathbf{r}t) + e^k \psi^k(\mathbf{r}t)\, U(\mathbf{r}t)$$
$$+ \sum_l \int V^{kl} |\mathbf{r}-\mathbf{r}'|\, n^l(\mathbf{r}'t)\, d\mathbf{r}'\, \psi^k(\mathbf{r}t). \tag{7}$$

The equations of motion for the density operators (4)–(6) can be calculated with the aid of (7) and have the following form (see Appendix):

$$\partial_t \rho^m = -j^m_{\alpha,\alpha}, \quad \partial_t \rho^e = -\sum_i (e^i/m^i)\, j^i_{\alpha,\alpha}, \tag{8}$$

$$\partial_t j^e_\alpha = -T_{\alpha\beta,\beta} + \sum_i [(e^i)^2/m^i][\mathbf{j}^i \times \mathrm{rot}\mathbf{A}]_\alpha$$
$$- \sum_i (e^i)^2 n^i (U_{,\alpha} + \partial_t A_\alpha), \tag{9}$$

$$\partial_t(\rho\epsilon) = -I_{\alpha,\alpha} - \sum_i (e^i/m^i) j_\alpha^i(\partial_\alpha U + \partial_t A_\alpha) + \partial_t \sum_i e^i n^i U. \tag{10}$$

The stress tensor operator, $T_{\alpha\beta}$, is symmetric and gauge-invariant, and is given by

$$T_{\alpha\beta} \equiv T'_{\alpha\beta} + T''_{\alpha\beta},$$

$$T'_{\alpha\beta} = \sum_{i=1}^{2} \frac{e^i}{2m^i}(\overset{+}{\psi}{}^i_{;\alpha}\psi^i_{;\beta} + \overset{+}{\psi}{}^i_{;\beta}\psi^i_{;\alpha}) - \sum_i \frac{e^i}{4m^i} n^i_{,\alpha\beta}, \tag{11a}$$

$$T''_{\alpha\beta} = -\frac{1}{4}\int d\mathbf{R}\,\frac{R_\alpha R_\beta}{R}\int_{-1}^{+1} d\lambda \sum_{ik} \frac{\partial V^{ik}(R)}{\partial R}\,\overset{+}{\psi}{}^i(x_+) e^k n^k(x_-)\,\psi^i(x_+), \tag{11b}$$

where

$$x_\pm = [\mathbf{r} + \tfrac{1}{2}(\lambda \pm 1)\mathbf{R}, t].$$

The energy-current operator is given by

$$I_\beta \equiv I'_\beta + I''_\beta,$$

$$I'_\beta = -(i/4)\sum_k (1/m_k^2)(\overset{+}{\psi}{}^k_{;\beta}\psi^k_{;\alpha\alpha} - \overset{+}{\psi}{}^k_{;\alpha\alpha}\psi^k_{;\beta}) - \sum_k (4m_k^2)^{-1} j^k_{\alpha,\alpha\beta} + \frac{1}{2}\sum_{i,k}(m_k)^{-1}\int V^{ik}(\mathbf{r}-\mathbf{r}')\,\overset{+}{\psi}{}^i(\mathbf{r}'t) j^k_\beta(\mathbf{r}t)\,\psi^i(\mathbf{r}'t)\,d\mathbf{r}', \tag{12a}$$

$$I''_\beta = -\frac{1}{4}\int d\mathbf{R}\frac{R_\alpha R_\beta}{R}\int_{-1}^{+1} d\lambda \sum_{ik}\frac{1}{m_k}\frac{\partial V^{ik}(R)}{\partial R}\,\overset{+}{\psi}{}^i(x_+) j^k_\alpha(x_-)\,\psi^i(x_+). \tag{12b}$$

Note, that the "semicolon notation" (1)–(2) underlines the external similarity of Eqs. (8)–(12) to those found for the case $A \equiv 0$ (see, e.g., Ref. 6). The Lorentz term $(e/m)\,[\mathbf{j}\times\mathcal{H}] + \rho\mathbf{E}$ has a form identical with the classical one, but in (9) ρ and $\mathbf{j}$ are operators.

3. DISCUSSION

We want to emphasize that the operators (11) and (12) are calculated exactly (see Appendix), and then the formulas (8)–(10) are valid for the various types of two-particle interactions. Since these conservation laws are written in operator form and, consequently, the method of averaging can be arbitrary, they can be applied also to systems far away from equilibrium.

The exact methods of closure of the set of QHD equations were given by Zubarev[11] and Robertson[8] for neutral particles. These methods can be adapted to the case of charged particles in a magnetic field and then apply to an electron gas and/or an ion lattice. This is the object of our current investigation. The approximate approach has been made[12] with the aid of the Bogolubov method[6,9] (hydrodynamic and acoustic approximation) and the Gorkov quasiclassical ansatz.[13]

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APPENDIX

In this appendix the formula (11) for $T''_{\alpha\beta}$ is derived. Putting (7) into $\partial_t j_\alpha$, where j_α is given by (5), we obtain by simple but tedious calculations the right-hand side of (9) with the following expression for $T''_{\alpha\beta}$:

$$\sum_\alpha T''_{\alpha\beta,\beta} = -\int d\mathbf{r}\sum_{i,k}^{1,2}\frac{\partial V^{ik}(|\mathbf{r}-\mathbf{r}'|)}{\partial r_\alpha}\overset{+}{\psi}{}^i(\mathbf{r}t)\,n^k(\mathbf{r}'t)\,\psi^i(\mathbf{r}t)\,e^k. \tag{A1}$$

It is convenient to pass to a new integration variable $\mathbf{R} = \mathbf{r} - \mathbf{r}'$:

$$\sum_\beta T''_{\alpha\beta,\beta} = \frac{1}{2}\int d\mathbf{R}\,\frac{R_\alpha}{R}\sum_{i,k}\frac{\partial V^{ik}(R)}{\partial R}\,\overset{+}{\psi}{}^i(\mathbf{r}t)[n^k(\mathbf{r}+\mathbf{R},t) - n^k(\mathbf{r}-\mathbf{R},t)]\,\psi^i(\mathbf{r}t)\,e^k. \tag{A2}$$

We use, in analogy to Ref. 6, the relations

$$i[\psi^k(\mathbf{r},0), \mathbf{P}(t)] = [\nabla_r - (ie^k/c)\,\mathbf{A}(\mathbf{r}t)]\,\psi^k(\mathbf{r},0),$$

$$\exp[-i\mathbf{R}\mathbf{P}(t)]\,n^k(\mathbf{r},0)\exp[i\mathbf{R}\mathbf{P}(t)] = n^k(\mathbf{r}+\mathbf{R},0), \tag{A3}$$

where $\mathbf{P}$ is the total momentum:

$$P_\alpha(t) \equiv \int d\mathbf{r}\sum_k\{(i/2)[\overset{+}{\psi}{}^k_{,\alpha}(\mathbf{r},0)\,\psi^k(\mathbf{r},0) - \overset{+}{\psi}{}^k(\mathbf{r},0)\,\psi^k_{,\alpha}(\mathbf{r},0)] - c^{-1}e^k n^k(\mathbf{r},0)A_\alpha(\mathbf{r},t)\}.$$

With the aid of (A3) we have

$$iU^+(t)[\overset{+}{\psi}{}^k(\mathbf{r})\,n^l(\mathbf{r}-\mathbf{R})\,\psi^k(\mathbf{r}),\ P_\alpha(t)]U(t) = \frac{\partial}{\partial r_\alpha}[\overset{+}{\psi}{}^k(\mathbf{r}t)\,n^l(\mathbf{r}-\mathbf{R},t)\,\psi^k(\mathbf{r}t)]. \tag{A4}$$

[$U(t)$ is the time evolution operator.]

Consider now the integral

$$J(t\mathbf{r}) \equiv U^+(t)\int_{-1}^{+1} d\lambda\frac{d}{d\lambda}\exp[-i(1+\lambda)\mathbf{R}\mathbf{P}/2]\,\overset{+}{\psi}{}^i(\mathbf{r})\,n^k(\mathbf{r}-\mathbf{R})\,\psi^i(\mathbf{r}) \times\exp[i(1+\lambda)\mathbf{R}\mathbf{P}/2]\,U(t)$$
$$= -\{\overset{+}{\psi}{}^i(\mathbf{r}t)\,n^k(\mathbf{r}-\mathbf{R},t)\,\psi^i(\mathbf{r}t) - \overset{+}{\psi}{}^i(\mathbf{r}+\mathbf{R},t)\,n^k(\mathbf{r}t)\,\psi^i(\mathbf{r}+\mathbf{R},t)\}. \tag{A5}$$

On the other hand, by differentiation of the integrand, this integral takes the form

$$J(tr) = (i/2)\mathbf{R}\int_{-1}^{1} d\lambda\, U^+(t)\exp[-i(\lambda+1)\mathbf{R}\mathbf{P}/2] \times[\overset{+}{\psi}{}^i(\mathbf{r})\,n^k(\mathbf{r}-\mathbf{R})\,\psi^i(\mathbf{r}), \mathbf{P}]\exp[i(\lambda+1)\mathbf{R}\mathbf{P}/2]\,U(t)$$
$$= \nabla_r\mathbf{R}\tfrac{1}{2}\int_{-1}^{1} d\lambda\,\overset{+}{\psi}{}^i[\mathbf{r}+\tfrac{1}{2}(\lambda+1)\mathbf{R},t]\,n^k[\mathbf{r}+\tfrac{1}{2}(\lambda-1)\mathbf{R},t] \times\psi^i[\mathbf{r}+\tfrac{1}{2}(\lambda+1)\mathbf{R},t]. \tag{A6}$$

By substituting (A5) and (A6) into (A2) we find the final result given by Eq. (11b).

In an analogous way one can prove the validity of Eq. (12b) for I''_β.

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