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On the accidental degeneracy of the n-dimensional anisotropic harmonic oscillator. II

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In an earlier paper, a group, G, associated with the n-dimensional anisotropic harmonic oscillator was shown to be embedded in a semidirect product, L, of the Weyl group N and the symplectic group $\mathrm{Sp}(2n,\mathrm{R})$. A particular representation R^ν of L, when restricted to G, was proved to be unitarily equivalent to $\oplus_s d_{\omega,s}\, U_G^{\nu,-(\mathrm{sgn}\,\nu)s}$, where $d_{\omega,s}$ is the degeneracy of the energy level $E_{\omega,s}$ of the n-dimensional anisotropic harmonic oscillator with frequencies $(\omega_1,\omega_2,\ldots,\omega_n)=\omega$, $U_G^{\nu,-(\mathrm{sgn}\,\nu)s}$ is an irreducible representation of G and s may be regarded as indexing all distinct energy levels of the system. In the present paper, the representation R^ν of L is shown to be unitarily equivalent to the representation $U_N^\nu W\otimes \overline{W}$ of L, where U_N^ν is an irreducible representation of N, W is the projective representation of $\mathrm{Sp}(2n,\mathrm{R})$ which intertwines the representations U_N^ν and SU_N^ν of N [where $S\in\mathrm{Sp}(2n,\mathrm{R})$], and $\overline{W}$ is the complex conjugate of W. This alternative form for the representation R^ν of L enables it to be decomposed, into two irreducible representations.

1. INTRODUCTION

In Ref. 1, a group G associated with the n-dimensional anisotropic harmonic oscillator was constructed: G is essentially a group generated by the position and momentum observables, the identity operator, and the Hamiltonian of the system.

G was shown to be embedded in a group, L, which is a semidirect product of the Weyl group, N, and the symplectic group, $\mathrm{Sp}(2n,\mathrm{R})$.[2] A particular representation R^ν of L, when restricted to G, was proved to be unitarily equivalent to $\oplus_s d_{\omega,s} U_G^{\nu,-(\mathrm{sgn}\,\nu)s}$, where $d_{\omega,s}$ is precisely the degeneracy of the energy level $E_{\omega,s}$ of the n-dimensional anisotropic harmonic oscillator with frequencies $(\omega_1,\omega_2,\ldots,\omega_n)$, $U_G^{\nu,-(\mathrm{sgn}\,\nu)s}$ is an irreducible representation of G and the summation may be regarded as over all distinct energy levels $E_{\omega,s}$ of the system.[2]

In the present paper, the representation R^ν of L is studied in greater detail. R^ν is shown to be unitarily equivalent to the representation $U_N^\nu W\otimes\overline{W}$ of L, where U_N^ν is an irreducible representation of N, W is the projective representation of $\mathrm{Sp}(2n,\mathrm{R})$ which intertwines the representations U_N^ν and SU_N^ν of N [where $S\in\mathrm{Sp}(2n,\mathrm{R})$], and $\overline{W}$ is the complex conjugate of W. This alternative form for the representation R^ν enables it to be decomposed, into two irreducible representations.

In Secs. 2–4, the quantum mechanical irreducible representations of L are obtained, using Mackey's theory of induced representations (summarized in Ref. 1). In Sec. 5, various informal arguments which suggest the unitary equivalence of R^ν and $U_N^\nu W\otimes\overline{W}$ are given. The result is proved in Sec. 6. The decomposition of the projective representation W of $\mathrm{Sp}(2n,\mathrm{R})$ is found in Sec. 7, and, from this, the decomposition of the representation R^ν of L is obtained.

2. THE LITTLE GROUP OF U_N^ν

From Ref. 2, Eq. (7), the group law of L is given by

$$(\sigma',\beta',\alpha',S')(\sigma,\beta,\alpha,S)=(\sigma'',\beta'',\alpha'',S''), \tag{1}$$

where

$$\begin{aligned}\sigma''&=\sigma'+\sigma+\tfrac{1}{2}(A'\alpha.C'\alpha+B'\beta.D'\beta)\\&\quad+B'\beta.C'\alpha+\alpha'.(D'\beta+C'\alpha),\\ \beta''&=D'\beta+C'\alpha+\beta',\\ \alpha''&=B'\beta+A'\alpha+\alpha',\\ S''&=S'S,\end{aligned}$$

with $S'=\left(\begin{smallmatrix}A' & B'\\ C' & D'\end{smallmatrix}\right)\in\mathrm{Sp}(2n,\mathrm{R})$ and $S\in\mathrm{Sp}(2n,\mathrm{R})$.

Every quantum mechanical irreducible representation of N is unitarily equivalent to one of the form [Ref. 1, Eq. (15)]

$$[U_N^\nu(\sigma,\beta,\alpha,I)\psi](x)=\exp i\nu(\sigma-x.\beta)\psi(x-\alpha), \tag{2}$$

where $\nu(\neq 0)\in\mathrm{R}$, $\psi\in L^2(\mathrm{R}^n,\mathrm{C})$.

The action of $S=\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)\in\mathrm{Sp}(2n,\mathrm{R})$ on U_N^ν is, by definition [Ref. 1, Eq. (11)],

$$\begin{aligned}&[(SU_N^\nu)(\sigma,\beta,\alpha,I)\psi](x)\\&=[U_N^\nu((0,0,0,S)^{-1}(\sigma,\beta,\alpha,I)(0,0,0,S))\psi](x)\\&=\exp i\nu(\sigma+\tfrac{1}{2}(-D^T\alpha.C^T\alpha-B^T\beta.A^T\beta)+B^T\beta.C^T\alpha\\&\qquad -x.(A^T\beta-C^T\alpha))\psi(x+B^T\beta-D^T\alpha),\end{aligned} \tag{3}$$

with T denoting transpose, using $\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)^{-1}=\left(\begin{smallmatrix}D^T & -B^T\\ -C^T & A^T\end{smallmatrix}\right)$, together with (1) and (2).

SU_N^ν is an irreducible representation of N. Restricted to the subgroup $\{(\sigma,0,0,I)\}$, it is just the phase $\exp i\nu\sigma$. Hence SU_N^ν must be unitarily equivalent to U_N^ν, since U_N^ν is the only irreducible representation of N which has the required form on restriction to the subgroup $\{(\sigma,0,0,I)\}$.

Therefore, there exists a unitary operator $W(S)$, dependent on S, such that

$$SU_N^\nu=W(S)^{-1}U_N^\nu W(S)\quad\text{for each } S\in\mathrm{Sp}(2n,\mathrm{R}). \tag{4}$$

The orbits of the quantum mechanical part of $\hat{N}$ under the action of $\mathrm{Sp}(2n,\mathrm{R})$ thus consist of single points U_N^ν. The little group of U_N^ν under the action of $\mathrm{Sp}(2n,\mathrm{R})$ is the whole of $\mathrm{Sp}(2n,\mathrm{R})$, and so the isotropy group of U_N^ν is $N \circledS \mathrm{Sp}(2n,\mathrm{R})$ $(=L)$.

3. THE OPERATOR W FOR THE GROUP $\mathrm{Sp}(2n, \mathrm{R})$

A. General approach

In Ref. 3, a method was developed for finding the operator W for the group $\mathrm{SU}(n)$ [which can be regarded as a subgroup of $\mathrm{Sp}(2n, \mathrm{R})$]. This method depends partly on the fact that $\mathrm{SU}(n)$ has no nontrivial multipliers, and so W can be taken to be an ordinary representation of $\mathrm{SU}(n)$. Now, the group $\mathrm{Sp}(2n,\mathrm{R})$ possesses nontrivial multipliers, and so it is possible that W is a projective representation; hence the method of Ref. 3 cannot be applied directly in the present case. Nevertheless, since $\mathrm{Sp}(2n,\mathrm{R})$ is a connected semisimple Lie group, every multiplier is locally trivial;[4] it follows that the (possibly) projective representation W of $\mathrm{Sp}(2n,\mathrm{R})$ can be chosen in such a way that the corresponding representation w of the Lie algebra of $\mathrm{Sp}(2n,\mathrm{R})$ is ordinary.

Suppose $h_i(\tau)$ is a one-dimensional Lie subgroup of $\mathrm{Sp}(2n,\mathrm{R})$. For each one-parameter subgroup $\mu(t)$ of N, let $Z_{\mu i}(\tau)$ be the unique skew-adjoint operator defined by

$$(h_i(\tau)U_N^v)(\mu(t)) = \exp tZ_{\mu i}(\tau) \text{ [cf. Ref. 3, Eq. (9)]}. \quad (5)$$

Then [cf. Ref. 3, Eq. (11)] the skew-adjoint operator $dW(h_i(\tau))/d\tau\,|_{\tau=0}$ is determined up to an arbitrary imaginary constant ξ_i by

$$\frac{dZ_{\mu i}(\tau)}{d\tau}\bigg|_{\tau=0} = \left[Z_{\mu i}(0),\ \frac{dW(h_i(\tau))}{d\tau}\bigg|_{\tau=0}\right]$$

$$\text{for each } \mu \in \{\sigma, \beta_1, \ldots, \beta_n, \alpha_1, \ldots, \alpha_n\}. \quad (6)$$

[The proof of this result is similar to the corresponding one of Ref. 3, Sec. 3 A, with $\exp\tau J$ replaced by $W(h_i(\tau))$.]

Since the operators $dW(h_i(\tau))/d\tau\,|_{\tau=0}$ can be chosen to give an ordinary representation w of the Lie algebra of $\mathrm{Sp}(2n,\mathrm{R})$, there are constraints on the constants ξ_i; as will be shown, these are sufficient to determine the constants ξ_i uniquely.

Since w is an ordinary representation of the Lie algebra of the connected group $\mathrm{Sp}(2n,\mathrm{R})$, w exponentiates to an ordinary representation, $\widetilde{W}$ say, of the connected, simply connected, covering group $\widetilde{\mathrm{Sp}(2n,\mathrm{R})}$ of $\mathrm{Sp}(2n,\mathrm{R})$.[5] $\mathrm{Sp}(2n,\mathrm{R})$ is the image of $\widetilde{\mathrm{Sp}(2n,\mathrm{R})}$ under a homomorphism δ whose kernel is a discrete central subgroup of $\widetilde{\mathrm{Sp}(2n,\mathrm{R})}$: $\mathrm{Sp}(2n,\mathrm{R}) \approx \widetilde{\mathrm{Sp}(2n,\mathrm{R})}/\ker\delta$. It is possible to show that $\widetilde{W}$ maps $\ker\delta$ into the unit circle, and hence $\widetilde{W}$ determines the required projective representation W of $\mathrm{Sp}(2n,\mathrm{R})$.[6] However, for what follows, it is not necessary to know W explicitly; it is sufficient that the ordinary representation w of the Lie algebra of $\mathrm{Sp}(2n,\mathrm{R})$ is known in detail.

B. Detailed calculation

The symplectic group $\mathrm{Sp}(2n,\mathrm{R})$ is the set of matrices $S \in \mathrm{GL}(2n,\mathrm{R})$ for which $S^TJS = J$, where $J = \sum_{j=1}^n (E_{j,j+n} - E_{j+n,j})$ [with E_{jk} a $2n \times 2n$ matrix having 1 in the (j,k) position and zeros elsewhere]. Let $S(\tau)$ be an analytic curve in $\mathrm{Sp}(2n,\mathrm{R})$, and suppose $S(0) = I$. Then, differentiating $S(\tau)^TJS(\tau) = J$ with respect to τ, and putting $\tau = 0$, gives

$$S'(0)^TJ + JS'(0) = 0,$$

where $'$ denotes differentiation with respect to τ. Hence the Lie algebra of $\mathrm{Sp}(2n,\mathrm{R})$ is the algebra of all matrices of the form $\left(\begin{smallmatrix} A & B \\ C & -A^T\end{smallmatrix}\right)$, where A, B, and C are $n \times n$ real matrices, with A arbitrary, $B = B^T$, and $C = C^T$; this algebra will be denoted by $\mathrm{sp}(2n,\mathrm{R})$.

A basis for $\mathrm{sp}(2n,\mathrm{R})$ is

$$X_{jk} = \begin{pmatrix} E_{jk} & 0 \\ 0 & -E_{kj}\end{pmatrix} \quad (1 \le j, k \le n),$$

$$Y_{jk} = \begin{pmatrix} 0 & E_{jk}+E_{kj} \\ 0 & 0\end{pmatrix} \quad (1 \le j \le k \le n), \quad (7)$$

$$Z_{jk} = \begin{pmatrix} 0 & 0 \\ E_{jk}+E_{kj} & 0\end{pmatrix} \quad (1 \le j \le k \le n)$$

(where E_{lm} is an $n \times n$ matrix having 1 in the (l,m) position and zeros elsewhere).

If

$$h(\tau) = \begin{pmatrix} a_{jk}(\tau) & b_{jk}(\tau) \\ c_{jk}(\tau) & d_{jk}(\tau)\end{pmatrix}$$

is a one-dimensional Lie subgroup of $\mathrm{Sp}(2n,\mathrm{R})$, then, from (3) and (5), the representatives of the generators of N in the ordinary representation $h(\tau)U_N^v$ of N are

$$Z_\sigma(\tau) = iv,$$

$$Z_{\beta_j}(\tau) = -iv\sum_{k=1}^n a_{jk}(\tau)x_k + \sum_{k=1}^n b_{jk}(\tau)\frac{\partial}{\partial x_k}, \quad (8)$$

$$Z_{\alpha_j}(\tau) = iv\sum_{k=1}^n c_{jk}(\tau)x_k - \sum_{k=1}^n d_{jk}(\tau)\frac{\partial}{\partial x_k}.$$

From (6),

$$dW(h(\tau))/d\tau\,|_{\tau=0} = w(h'(0))$$

must satisfy,
when $\mu = \sigma$,

$$0 = [iv, w(h'(0))],$$

which is satisfied by any $w(h'(0))$;
when $\mu = \beta_j$,

$$-iv\sum_{k=1}^n a'_{jk}(0)x_k + \sum_{k=1}^n b'_{jk}(0)\frac{\partial}{\partial x_k} = [-ivx_j, w(h'(0))]; \quad (9)$$

when $\mu = \alpha_j$,

$$iv\sum_{k=1}^n c'_{jk}(0)x_k - \sum_{k=1}^n d'_{jk}(0)\frac{\partial}{\partial x_k} = \left[-\frac{\partial}{\partial x_j}, w(h'(0))\right] \quad (10)$$

(where $'$ denotes differentiation with respect to τ).

The operator $w(X_{lm})$

Suppose, with the above notation, that X_{lm} is the tangent at the identity of a one-dimensional Lie subgroup

$$h(\tau) = \begin{pmatrix} a_{jk}(\tau) & b_{jk}(\tau) \\ c_{jk}(\tau) & d_{jk}(\tau)\end{pmatrix}$$

in $\mathrm{Sp}(2n,\mathrm{R})$. Then

$$X_{lm} = h'(0) = \begin{pmatrix} a'_{jk}(0) & b'_{jk}(0) \\ c'_{jk}(0) & d'_{jk}(0)\end{pmatrix}.$$

In this case, the only nonzero elements of $h'(0)$ are

$$a'_{lm}(0) = 1, \quad d'_{ml}(0) = -1.$$

Hence the conditions (9) and (10) become,

for $j \neq l$ or m,

$$0=[-ivx_j, w(X_{lm})] \text{ and } 0=\left[-\frac{\partial}{\partial x_j}, w(X_{lm})\right];$$

for $j=l$,

$$-ivx_m=[-ivx_l, w(X_{lm})] \text{ and } \delta_{lm}\frac{\partial}{\partial x_l}=\left[-\frac{\partial}{\partial x_l}, w(X_{lm})\right];$$

for $j=m$,

$$-iv\delta_{lm}x_m=[-ivx_m, w(X_{lm})] \text{ and } \frac{\partial}{\partial x_l}=\left[-\frac{\partial}{\partial x_m}, w(X_{lm})\right].$$

If $l\neq m$, these relations are all satisfied by the skew-adjoint operator

$$\tilde{w}(X_{lm})=-x_m\frac{\partial}{\partial x_l}+\alpha_{lm},$$

where α_{lm} is an arbitrary imaginary constant.

If $l=m$, these relations are all satisfied by

$$\tilde{w}(X_{mm})=-x_m\frac{\partial}{\partial x_m}+\epsilon_m,$$

where ϵ_m is an arbitrary constant. Now the adjoint of $\tilde{w}(X_{mm})$ is $\tilde{w}(X_{mm})^*=(-\partial/\partial x_m)^*x_m^*+\bar{\epsilon}_m=(\partial/\partial x_m)x_m+\bar{\epsilon}_m$ $=x_m(\partial/\partial x_m)+1+\bar{\epsilon}_m$. Thus $\tilde{w}(X_{mm})$ is a skew-adjoint operator provided $\epsilon_m=-\frac{1}{2}+\alpha_{mm}$, where α_{mm} is an arbitrary imaginary constant. Therefore, for $1\leq l, m\leq n$,

$$\tilde{w}(X_{lm})=-x_m\frac{\partial}{\partial x_l}-\tfrac{1}{2}\delta_{lm}+\alpha_{lm}. \tag{11a}$$

The operator $w(Y_{lm})$ $(l<m)$

Similarly, the operator $w(Y_{lm})$ must satisfy the relations,

for $j\neq l$ or m,

$$0=[-ivx_j, w(Y_{lm})] \text{ and } 0=\left[-\frac{\partial}{\partial x_j}, w(Y_{lm})\right];$$

for $j=l$ $(\neq m)$,

$$\frac{\partial}{\partial x_m}=[-ivx_l, w(Y_{lm})] \text{ and } 0=\left[-\frac{\partial}{\partial x_l}, w(Y_{lm})\right];$$

for $j=m$ $(\neq l)$,

$$\frac{\partial}{\partial x_l}=[-ivx_m, w(Y_{lm})] \text{ and } 0=\left[-\frac{\partial}{\partial x_m}, w(Y_{lm})\right].$$

These relations are all satisfied by the skew-adjoint operator

$$\tilde{w}(Y_{lm})=\frac{1}{iv}\frac{\partial^2}{\partial x_l\partial x_m}+\beta_{lm}, \tag{11b}$$

where β_{lm} is an arbitrary imaginary constant.

The operator $w(Y_{mm})$

Similarly, the operator $w(Y_{mm})$ must satisfy the relations,

for $j\neq m$,

$$0=[-ivx_j, w(Y_{mm})] \text{ and } 0=\left[-\frac{\partial}{\partial x_j}, w(Y_{mm})\right];$$

for $j=m$,

$$2\frac{\partial}{\partial x_m}=[-ivx_m, w(Y_{mm})] \text{ and } 0=\left[-\frac{\partial}{\partial x_m}, w(Y_{mm})\right].$$

These relations are all satisfied by the skew-adjoint operator

$$\tilde{w}(Y_{mm})=\frac{1}{iv}\frac{\partial^2}{\partial x_m^2}+\beta_{mm}, \tag{11c}$$

where β_{mm} is an arbitrary imaginary constant.

The operator $w(Z_{lm})$ $(l<m)$

Similarly, the operator $w(Z_{lm})$ must satisfy the relations,

for $j\neq l$ or m,

$$0=[-ivx_j, w(Z_{lm})] \text{ and } 0=\left[-\frac{\partial}{\partial x_j}, w(Z_{lm})\right];$$

for $j=l$ $(\neq m)$,

$$0=[-ivx_l, w(Z_{lm})] \text{ and } ivx_m=\left[-\frac{\partial}{\partial x_l}, w(Z_{lm})\right];$$

for $j=m$ $(\neq l)$,

$$0=[-ivx_m, w(Z_{lm})] \text{ and } ivx_l=\left[-\frac{\partial}{\partial x_m}, w(Z_{lm})\right].$$

These relations are all satisfied by the skew-adjoint operator

$$\tilde{w}(Z_{lm})=-ivx_lx_m+\gamma_{lm}, \tag{11d}$$

where γ_{lm} is an arbitrary imaginary constant.

The operator $w(Z_{mm})$

Similarly, the operator $w(Z_{mm})$ must satisfy the relations,

for $j\neq m$,

$$0=[-ivx_j, w(Z_{mm})] \text{ and } 0=\left[-\frac{\partial}{\partial x_j}, w(Z_{mm})\right];$$

for $j=m$,

$$0=[-ivx_m, w(Z_{mm})] \text{ and } 2ivx_m=\left[-\frac{\partial}{\partial x_m}, w(Z_{mm})\right].$$

These relations are all satisfied by the skew-adjoint operator

$$\tilde{w}(Z_{mm})=-ivx_m^2+\gamma_{mm}, \tag{11e}$$

where γ_{mm} is an arbitrary imaginary constant.

C. Calculation of the values of the constants $\alpha_{lm}, \beta_{lm}, \gamma_{lm}$

It may be shown that the commutation relations of $\mathrm{sp}(2n,\mathrm{R})$, in terms of the basis X_{jk}, Y_{jk}, Z_{jk} [see (7)] are

$$[X_{jk}, X_{lm}]=\delta_{kl}X_{jm}-\delta_{jm}X_{lk}, \tag{12a}$$

$$[X_{jk}, Y_{lm}]=\delta_{kl}Y_{jm}+\delta_{km}Y_{jl}, \tag{12b}$$

$$[X_{jk}, Z_{lm}]=-\delta_{jl}Z_{km}-\delta_{jm}Z_{kl}, \tag{12c}$$

$$[Y_{jk}, Y_{lm}]=0, \tag{12d}$$

$$[Y_{jk}, Z_{lm}]=\delta_{km}X_{jl}+\delta_{kl}X_{jm}+\delta_{jm}X_{kl}+\delta_{jl}X_{km}, \tag{12e}$$

$$[Z_{jk}, Z_{lm}]=0, \tag{12f}$$

where, for convenience of notation, define

$$Y_{jk}=Y_{kj}, \quad Z_{jk}=Z_{kj}, \quad \text{for } j>k.$$

Since $\tilde{w}$ is to be an ordinary representation of $\mathrm{sp}(2n,\mathrm{R})$, $\tilde{w}(X_{jk})$, $\tilde{w}(Y_{jk})$, and $\tilde{w}(Z_{jk})$ must satisfy the commutation relations of $\mathrm{sp}(2n,\mathrm{R})$. Explicit calculation shows that this implies

$$-\delta_{kl}\alpha_{jm}+\delta_{jm}\alpha_{lk}=0, \tag{13a}$$

$$-\delta_{kl}\beta_{jm}-\delta_{km}\beta_{jl}=0, \tag{13b}$$

$$\delta_{jl}\gamma_{km}+\delta_{jm}\gamma_{kl}=0, \tag{13c}$$

$$-\delta_{km}\alpha_{jl}-\delta_{kl}\alpha_{jm}-\delta_{jm}\alpha_{kl}-\delta_{jl}\alpha_{km}=0, \tag{13d}$$

where, for convenience of notation, define

$$\beta_{jk}=\beta_{kj}, \quad \gamma_{jk}=\gamma_{kj}, \quad \text{for } j>k.$$

Condition (13a) gives that $\alpha_{jm}=0$ for $j\neq m$ (putting $k=l$). Condition (13d) gives that $\alpha_{jj}=0$ for $1\leq j\leq n$ (putting $j=k=l=m$). From condition (13b), it follows that $\beta_{jm}=0$ for $1\leq j, m\leq n$ (putting $k=l\neq m$, if $n>1$, and $j=k=l=m=1$, if $n=1$). Similarly, from condition (13c), $\gamma_{km}=0$ for $1\leq k, m\leq n$. Hence, $\tilde{w}$ is an ordinary representation w of $\mathrm{sp}(2n,\mathrm{R})$ if and only if the α's, β's, and γ's are all zero. Therefore, from (11), the ordinary representation w of $\mathrm{sp}(2n,\mathrm{R})$ is determined by

$$\begin{aligned}
w(X_{jk})&=-x_k\frac{\partial}{\partial x_j}-\tfrac{1}{2}\delta_{jk} \quad (1\leq j,k\leq n),\\
w(Y_{jk})&=-\frac{i}{v}\frac{\partial^2}{\partial x_j\partial x_k} \quad (1\leq j\leq k\leq n),\\
w(Z_{jk})&=-ivx_jx_k \quad (1\leq j\leq k\leq n).
\end{aligned} \tag{14}$$

4. THE IRREDUCIBLE ORDINARY REPRESENTATIONS OF L

It now follows from Mackey's theory for semidirect product groups that every quantum mechanical irreducible ordinary representation of $L=N \circledS \mathrm{Sp}(2n,\mathrm{R})$ is unitarily equivalent to one of the form

$$U_N^v W\otimes\eta:(\sigma,\beta,\alpha,S) \to \exp iv(\sigma-x.\beta)\exp\left(-\sum_{j=1}^{n}\alpha_j\frac{\partial}{\partial x_j}\right)W(S)\otimes\eta(S), \tag{15}$$

where $(\sigma,\beta,\alpha,S)\in L$, W is the projective representation of $\mathrm{Sp}(2n,\mathrm{R})$ determined up to trivial multipliers by (14), η is an irreducible projective representation of $\mathrm{Sp}(2n,\mathrm{R})$ with multiplier inverse to that of W, and the operators in the first part of the inner Kronecker product are defined on a dense subspace of $L^2(\mathrm{R}^n,\mathrm{C})$.

5. HEURISTIC ARGUMENTS SUGGESTING THAT $(R^v\downarrow L)\simeq U_N^v W\otimes\overline{W}$

To simplify the notation denote $R^v\downarrow L$ by R_L^v. The result stated above was found through an attempt to decompose R_L^v into irreducible representations of L, of the form (15). It was hoped that the decomposition of R_L^v would be suggested by that of $R_L^v\downarrow(N\circledS\mathrm{SU}(n))$. The decomposition of $R_L^v\downarrow(N\circledS\mathrm{SU}(n))$ may be obtained as follows. From Ref. 2, Sec. 5,

$$R_L^v\downarrow(N\circledS\mathrm{SU}(n))=(D(H)\uparrow L)\downarrow(N\circledS\mathrm{SU}(n)),$$

where D is the representation of the subgroup $H=\{(\sigma,0,0,I)\}\times\mathrm{Sp}(2n,\mathrm{R})$ of L, given by

$$D:(\sigma,0,0,S)\to\exp iv\sigma.$$

Hence $R_L^v\downarrow(N\circledS\mathrm{SU}(n))$ is the restriction of an induced representation of L. Since $N\circledS\mathrm{SU}(n)$ and H are closed subgroups of the separable locally compact group L, and L can be expressed as a single double coset of $N\circledS\mathrm{SU}(n)$ and $H: L=N\circledS\mathrm{SU}(n)(0,0,0,I)H$ [from (1)], Mackey's subgroup theorem[7] may be applied,

$$(D(H)\uparrow L)\downarrow(N\circledS\mathrm{SU}(n))=D((N\circledS\mathrm{SU}(n))\cap H)\uparrow(N\circledS\mathrm{SU}(n)).$$

Therefore,

$$R_L^v\downarrow(N\circledS\mathrm{SU}(n))=D(C\times\mathrm{SU}(n))\uparrow(N\circledS\mathrm{SU}(n)), \tag{16}$$

where $C=\{(\sigma,0,0,I)\}$.

The group $N\circledS\mathrm{SU}(n)$ is locally compact, but not compact, so the Frobenius reciprocity theorem[7] does not necessarily hold for it. However, a formal application of the theorem may, nevertheless, give the correct decomposition of $R_L^v\downarrow(N\circledS\mathrm{SU}(n))$.

Suppose $U_N^t W\otimes\eta$ is an arbitrary quantum mechanical irreducible representation of $N\circledS\mathrm{SU}(n)$, where W is now regarded as an ordinary representation of $\mathrm{SU}(n)$ (Ref. 3, Sec. 4). Then, formally,

$$\begin{aligned}
&\langle R_L^v\downarrow(N\circledS\mathrm{SU}(n)),\, U_N^t W\otimes\eta\rangle_{N\circledS\mathrm{SU}(n)}\\
&=\langle D(C\times\mathrm{SU}(n))\uparrow(N\circledS\mathrm{SU}(n)),\, U_N^t W\otimes\eta\rangle_{N\circledS\mathrm{SU}(n)}\\
&=\langle(U_N^t W\otimes\eta)\downarrow(C\times\mathrm{SU}(n)),\, D(C\times\mathrm{SU}(n))\rangle_{C\times\mathrm{SU}(n)}.
\end{aligned} \tag{17}$$

The representation $D:(\sigma,0,0,U)\to\exp iv\sigma$ of $C\times\mathrm{SU}(n)$ is contained in the representation $U_N^t W\otimes\eta:(\sigma,0,0,U)\to\exp it\sigma W(U)\otimes\eta(U)$ of $C\times\mathrm{SU}(n)$ only when $t=v$; in this case, the number of times that D is contained in $U_N^t W\otimes\eta$ [as representations of $C\times\mathrm{SU}(n)$] equals the frequency of the one-dimensional identity representation $I_{\mathrm{SU}(n)}$ in the representation $U\to W(U)\otimes\eta(U)$ of $\mathrm{SU}(n)$.

Now, from Ref. 3, Eq. (23),

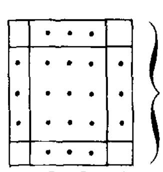
$$W\otimes\eta=\left(\bigoplus_{a=0}^{\infty}W_a\right)\otimes\eta=\bigoplus_{a=0}^{\infty}(W_a\otimes\eta). \tag{18}$$

It thus remains to determine, for each a, how often $I_{\mathrm{SU}(n)}$ occurs in $W_a\otimes\eta$. A general method for decomposing inner Kronecker products of irreducible representations of $\mathrm{SU}(n)$ is given in Ref. 8.

When $v>0$, from Ref. 3, Eq. (29a),

$$W_a=\underbrace{\boxed{\;\cdots\;}}_{a\text{ boxes}}=(a,0,\ldots,0).$$

It follows that, when $v>0$, $W_a\otimes\eta$ contains $I_{\mathrm{SU}(n)}$ if and only if

$\eta=$  $n-1$ rows $=(a,\ldots,a,0)$.

a columns

When $v<0$, from Ref. 3, Eq. (29b), $W_a=(a,\ldots,a,0)$. It follows that, when $v<0$, $W_a\otimes\eta$ contains $I_{\mathrm{SU}(n)}$ if and only if $\eta=(a,0,\ldots,0)$.

Now the representations $(a, 0, \ldots, 0)$ and $(a, \ldots, a, 0)$ are mutually contragredient,[8] so, since W_a is unitary, the representation contragredient to W_a is just the complex conjugate $\overline{W}_a$ of W_a.

So, for any $v(\neq 0)$, $I_{\mathrm{SU}(n)}$ occurs in $W_a \otimes \eta$ if and only if $\eta = \overline{W}_a$, and then it occurs exactly once. Hence, using (18), D is contained in $U_N^t W \otimes \eta$ [as representations of $C \times \mathrm{SU}(n)$] if and only if $t = v$ and $\eta = \overline{W}_a$ for some a; in this case, D occurs exactly once.

A formal application [(17)] of the Frobenius reciprocity theorem therefore suggests that the decomposition of $R_L^v \downarrow (N \circledS \mathrm{SU}(n))$ into irreducible representations is

$$R_L^v \downarrow (N \circledS \mathrm{SU}(n)) \simeq \bigoplus_{a=0}^{\infty} (U_N^v W \otimes \overline{W}_a).$$

Now

$$\bigoplus_{a=0}^{\infty} (U_N^v W \otimes \overline{W}_a) = U_N^v W \otimes \Big(\bigoplus_{a=0}^{\infty} \overline{W}_a \Big) = U_N^v W \otimes \overline{W}.$$

This indicates that, perhaps, the representation R_L^v of L is unitarily equivalent to $U_N^v W \otimes \overline{W}$, where W is now regarded as a projective representation of $\mathrm{Sp}(2n, \mathrm{R})$. $U_N^v W \otimes \overline{W}$ is an ordinary representation of L, since $\overline{W}$ has multiplier inverse to that of W.

From Ref. 1, Eq. (26), and Ref. 2, Eqs. (27) and (44), it follows that the restrictions of R_L^v and $U_N^v W \otimes \overline{W}$ to the subgroup G of L are certainly unitarily equivalent,

$$V^{-1}(R_L^v \downarrow G) V = (U_N^v W \otimes \overline{W}) \downarrow G, \tag{19}$$

where V is the operator defined in Ref. 2, Eq. (43).

The operator V was chosen so that the representations of the generators of the subgroup N of G transformed in the required manner. Once V had been chosen in this way, it happened that the representations of the remaining generator of G also transformed in the required manner.

These heuristic arguments suggest that, perhaps, $V^{-1} R_L^v V = U_N^v W \otimes \overline{W}$ as representations of L.

6. STATEMENT AND PROOF OF RESULT

Theorem:

$$V^{-1} R_L^v V = U_N^v W \otimes \overline{W} \quad (v \neq 0), \tag{20}$$

where $R_L^v = R^v \downarrow L$ is van Hove's representation of L [Ref. 2, Eq. (12)], U_N^v is a quantum mechanical irreducible representation of N [(2)], V is a unitary operator defined on $L^2(\mathrm{R}^{2n}, \mathrm{C})$ [Ref. 2, Eq. (43)] and W is a projective representation of $\mathrm{Sp}(2n, \mathrm{R})$ [Sec. 3].

Proof:

Method: Every element $l \in L = N \circledS \mathrm{Sp}(2n, \mathrm{R})$ can be expressed uniquely in the form $l = n'S$, where $n' \in N$, $S \in \mathrm{Sp}(2n, \mathrm{R})$. From (19), the restriction of the theorem to N is certainly true. Hence, since $V^{-1} R_L^v V$ is an ordinary representation of L, it is sufficient to show that the theorem holds for $\mathrm{Sp}(2n, \mathrm{R})$.

Since $\mathrm{Sp}(2n, \mathrm{R})$ is a connected Lie group, every element may be expressed as a product of elements of one-parameter subgroups.[9] As $(V^{-1} R_L^v V) \downarrow \mathrm{Sp}(2n, \mathrm{R})$ is an ordinary representation of $\mathrm{Sp}(2n, \mathrm{R})$, it is thus enough to prove the theorem for a set of independent one-parameter subgroups which generate $\mathrm{Sp}(2n, \mathrm{R})$.

If $h(\tau)$ is a one-parameter subgroup of $\mathrm{Sp}(2n, \mathrm{R})$, then $(V^{-1} R_L^v V) \downarrow h(\tau)$ and $(W \otimes \overline{W}) \downarrow h(\tau)$ are unitary ordinary representations of the one-parameter subgroup $h(\tau)$, to which Stone's theorem may be applied. Hence, it is sufficient to show that V transforms the representation r_L^v of $\mathrm{sp}(2n, \mathrm{R})$ into the representation $w \otimes \overline{w}$ of $\mathrm{sp}(2n, \mathrm{R})$ [where r_L^v, $w \otimes \overline{w}$ are the representations of $\mathrm{sp}(2n, \mathrm{R})$ corresponding to the representations R_L^v, $W \otimes \overline{W}$ of $\mathrm{Sp}(2n, \mathrm{R})$, respectively].

The representation $w \otimes \overline{w}$

Suppose W acts on functions $[\in L^2(\mathrm{R}^n, \mathrm{C})]$ dependent on $x \in \mathrm{R}^n$, and that $\overline{W}$ acts on functions $[\in L^2(\mathrm{R}^n, \mathrm{C})]$ dependent on $y \in \mathrm{R}^n$. Let $h(\tau)$ be a one-parameter subgroup of $\mathrm{Sp}(2n, \mathrm{R})$. Then

$$(w \otimes \overline{w})(h'(0)) = w(h'(0)) + \overline{w}(h'(0)) \tag{21}$$

(where $'$ denotes differentiation with respect to τ). $\overline{w}(h'(0))$ may be obtained from $w(h'(0))$ by replacing x_m by y_m $(m = 1, 2, \ldots, n)$, and taking the complex conjugate.

From (14) and (21), it follows that

$$\begin{aligned}
(w \otimes \overline{w})(X_{jk}) &= -x_k \frac{\partial}{\partial x_j} - y_k \frac{\partial}{\partial y_j} - \delta_{jk} \quad (1 \le j, k \le n), \\
(w \otimes \overline{w})(Y_{jk}) &= \frac{i}{v} \left(-\frac{\partial^2}{\partial x_j \partial x_k} + \frac{\partial^2}{\partial y_j \partial y_k} \right) \quad (1 \le j \le k \le n), \\
(w \otimes \overline{w})(Z_{jk}) &= iv(-x_j x_k + y_j y_k) \quad (1 \le j \le k \le n).
\end{aligned} \tag{22}$$

Explicit expression for $r_L^v(h'(0))$

By definition [Ref. 2, Eq. (12)], $R_L^v \downarrow \mathrm{Sp}(2n, \mathrm{R})$ is given by

$$[R_L^v(\gamma)\phi](q, p) = \exp iv\pi_\gamma(\gamma^{-1}(q, p))\phi(\gamma^{-1}(q, p)),$$

where

$$\gamma = \left(0, 0, 0, \begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) \in \mathrm{Sp}(2n, \mathrm{R}),$$

$$\pi_\gamma(q, p) = \tfrac{1}{2} Ap.Cp + \tfrac{1}{2} Bq.Dq + Bq.Cp,$$

and $\phi \in L^2(\mathrm{R}^{2n}, \mathrm{C})$. Therefore,

$$\begin{aligned}
[R_L^v(\gamma)\phi](q, p) = \exp iv[&\tfrac{1}{2}(-AB^Tq + AD^Tp).(-CB^Tq + CD^Tp) \\
&+ \tfrac{1}{2}(BA^Tq - BC^Tp).(DA^Tq - DC^Tp) \\
&+ (BA^Tq - BC^Tp).(-CB^Tq + CD^Tp)] \\
&\times \phi(A^Tq - C^Tp, -B^Tq + D^Tp).
\end{aligned} \tag{23}$$

Suppose $h(\tau) = \begin{pmatrix} A(\tau) & B(\tau) \\ C(\tau) & D(\tau) \end{pmatrix}$ is a one-parameter subgroup of $\mathrm{Sp}(2n, \mathrm{R})$. Then, from (23), since $A(0) = I_n = D(0)$ and $B(0) = 0 = C(0)$,

$$\begin{aligned}
[r_L^v(h'(0))\phi](q, p) &= \frac{d}{d\tau}[R_L^v(h(\tau))\phi](q, p)\Big|_{\tau=0} \\
&= \Big[\frac{iv}{2}(p.C'(0)p + B'(0)q.q) \\
&\quad + \sum_{m=1}^{n}\Big((A'(0)^Tq - C'(0)^Tp)_m \frac{\partial}{\partial q_m} + (-B'(0)^Tq \\
&\quad + D'(0)^Tp)_m \frac{\partial}{\partial p_m}\Big)\Big]\phi(q, p).
\end{aligned} \tag{24}$$

The action of V

From the definition of V [Ref. 2, Eq. (43)],

$$\begin{aligned} V^{-1}q_jV &= \frac{i}{v}\left(\frac{\partial}{\partial x_j}-\frac{\partial}{\partial y_j}\right), \\ V^{-1}p_jV &= x_j+y_j, \\ V^{-1}\frac{\partial}{\partial q_j}V &= ivx_j, \\ V^{-1}\frac{\partial}{\partial p_j}V &= \frac{\partial}{\partial y_j}. \end{aligned} \tag{25}$$

Evaluation of $V^{-1}r_L^v(X_{jk})V$

In this case, from (7),

$$A'(0)^T=E_{kj},\quad B'(0)=0,$$

$$C'(0)=0,\quad D'(0)^T=-E_{jk}.$$

Hence, from (24),

$$r_L^v(X_{jk})=q_j\frac{\partial}{\partial q_k}-p_k\frac{\partial}{\partial p_j}.$$

Therefore, using (25),

$$\begin{aligned} V^{-1}r_L^v(X_{jk})V &= (V^{-1}q_jV)\left(V^{-1}\frac{\partial}{\partial q_k}V\right)-(V^{-1}p_kV)\left(V^{-1}\frac{\partial}{\partial p_j}V\right) \\ &= -x_k\frac{\partial}{\partial x_j}-y_k\frac{\partial}{\partial y_j}-\delta_{jk} \\ &= (w\otimes\overline{w})(X_{jk}) \text{ from (22).} \end{aligned}$$

Evaluation of $V^{-1}r_L^v(Y_{jk})V$

From (7) and (24),

$$r_L^v(Y_{jk})=ivq_jq_k-q_k\frac{\partial}{\partial p_j}-q_j\frac{\partial}{\partial p_k}.$$

Therefore, using (25),

$$\begin{aligned} V^{-1}r_L^v(Y_{jk})V &= -\frac{i}{v}\left(\frac{\partial}{\partial x_j}-\frac{\partial}{\partial y_j}\right)\left(\frac{\partial}{\partial x_k}-\frac{\partial}{\partial y_k}\right)-\frac{i}{v}\left(\frac{\partial}{\partial x_k}\right. \\ &\quad \left.-\frac{\partial}{\partial y_k}\right)\left(\frac{\partial}{\partial y_j}\right)-\frac{i}{v}\left(\frac{\partial}{\partial x_j}-\frac{\partial}{\partial y_j}\right)\left(\frac{\partial}{\partial y_k}\right) \\ &= (w\otimes\overline{w})(Y_{jk}) \text{ from (22).} \end{aligned}$$

Evaluation of $V^{-1}r_L^v(Z_{jk})V$

From (7) and (24),

$$r_L^v(Z_{jk})=ivp_jp_k-p_k\frac{\partial}{\partial q_j}-p_j\frac{\partial}{\partial q_k}.$$

Therefore, using (25),

$$\begin{aligned} V^{-1}r_L^v(Z_{jk})V &= iv(x_j+y_j)(x_k+y_k)-(x_k+y_k)(ivx_j)-(x_j+y_j)(ivx_k) \\ &= (w\otimes\overline{w})(Z_{jk}) \text{ from (22).} \end{aligned}$$

This completes the proof of the theorem.

7. THE DECOMPOSITION OF THE PROJECTIVE REPRESENTATION W OF $\mathrm{Sp}(2n,\mathbb{R})$

From Ref. 3, Eq. (28),

$$W(\mathrm{Sp}(2n,\mathrm{R}))\downarrow \mathrm{SU}(n)=\bigoplus_{a=0}^{\infty}W_a,$$

where W_a is an irreducible ordinary representation of $\mathrm{SU}(n)$ defined on the subspace Ω_a of $L^2(\mathrm{R}^n,\mathrm{C})$ (Ref. 3, Sec. 5).

Suppose Ω is a subspace of $L^2(\mathrm{R}^n,\mathrm{C})$ which is invariant and irreducible under the projective representation W of $\mathrm{Sp}(2n,\mathrm{R})$. Then, *a priori*, Ω is invariant, although not necessarily irreducible, under $W(\mathrm{Sp}(2n,\mathrm{R}))\downarrow\mathrm{SU}(n)$. Hence Ω is a direct sum of closed subspaces of $L^2(\mathrm{R}^n,\mathrm{C})$ which are invariant and irreducible under $W(\mathrm{Sp}(2n,\mathrm{R}))\downarrow\mathrm{SU}(n)$. Since the irreducible representations W_a of $\mathrm{SU}(n)$ have different dimensions, they are inequivalent, and hence, by the uniqueness (up to unitary equivalence) of the decomposition of $W(\mathrm{Sp}(2n,\mathrm{R}))\downarrow\mathrm{SU}(n)$, it follows that Ω must be a direct sum of subspaces of the form Ω_a.

Suppose Ω contains the subspace Ω_b spanned by the set $\{\psi_m:\sum_{j=1}^n m_j=b\}$. Since Ω is invariant under the projective representation W of $\mathrm{Sp}(2n,\mathrm{R})$, the space generated by Ω_b, $w(X_{jk})\Omega_b$ $(1\le j,k\le n)$, $w(Y_{jk})\Omega_b$ and $w(Z_{jk})\Omega_b$ $(1\le j\le k\le n)$ must be contained in Ω. Now, from (14) and Ref. 3, Eq. (21), for $j\neq k$,

$$\begin{aligned} &w(X_{jk})\psi_m(x) \\ &= -\left[\left(\frac{m_k}{2}\right)^{1/2}\psi_{m_k-1}(x_k)+\left(\frac{m_k+1}{2}\right)^{1/2}\psi_{m_k+1}(x_k)\right] \\ &\quad\times\left[\left(\frac{m_j}{2}\right)^{1/2}\psi_{m_j-1}(x_j)-\left(\frac{m_j+1}{2}\right)^{1/2}\psi_{m_j+1}(x_j)\right] \\ &\quad\times\prod_{l\neq j,k}\psi_{m_l}(x_l). \end{aligned}$$

Therefore, Ω contains elements which are linear combinations of elements belonging to Ω_{b-2}, Ω_b, and Ω_{b+2}. Hence, since Ω is the direct sum of subspaces of the form Ω_a, Ω must contain the subspaces Ω_{b-2}, Ω_{b+2} as well as Ω_b. By induction, it follows that Ω contains all the subspaces Ω_a for which a has the same parity as b.

Hence, the projective representation W of $\mathrm{Sp}(2n,\mathrm{R})$ splits into at most two irreducible projective representations.

From the form of the representation w of the remaining basis elements of $\mathrm{sp}(2n,\mathrm{R})$ [(14)], it follows, again using Ref. 3, Eq. (21), that the subspaces

$$\Omega_{\text{even}},\quad\text{spanned by }\{\psi_m:\sum_{j=1}^n m_j\text{ is even}\},\tag{26a}$$

and

$$\Omega_{\text{odd}},\quad\text{spanned by }\{\psi_m:\sum_{j=1}^n m_j\text{ is odd}\},\tag{26b}$$

are each invariant under the representation w of $\mathrm{sp}(2n,\mathrm{R})$.

Therefore, the decomposition of the projective representation W of $\mathrm{Sp}(2n,\mathrm{R})$ is

$$W=W_{\text{even}}\oplus W_{\text{odd}},\tag{27}$$

where W_{even}, W_{odd} are irreducible projective representations of $\mathrm{Sp}(2n,\mathrm{R})$ (namely the restriction of W to the subspaces Ω_{even}, Ω_{odd} respectively).

8. THE DECOMPOSITION OF THE REPRESENTATION $R^v \downarrow L$

$$\begin{aligned} R^v \downarrow L &\simeq U_N^v W \otimes \overline{W} \\ &= U_N^v W \otimes (\overline{W_{\text{even}}} \oplus \overline{W_{\text{odd}}}) \\ &= (U_N^v W \otimes \overline{W_{\text{even}}}) \oplus (U_N^v W \otimes \overline{W_{\text{odd}}}) . \end{aligned} \tag{28}$$

Now W_{even} and W_{odd} are irreducible projective representations each of which has the same multiplier as W; so $\overline{W_{\text{even}}}$ and $\overline{W_{\text{odd}}}$ are irreducible projective representations each of which has multiplier inverse to that of W. Hence, from (15), $U_N^v W \otimes \overline{W_{\text{even}}}$ and $U_N^v W \otimes \overline{W_{\text{odd}}}$ are irreducible ordinary representations of L.

It follows that the representation $R^v \downarrow L$ $(v \neq 0)$ splits into two irreducible ordinary representations of L.

9. CONCLUSION

As far as the anisotropic harmonic oscillator is concerned, almost all reference to van Hove's paper[10] can be removed. For clarity, let $W_{\text{Sp}(2n,\text{R})}$ denote the projective representation W of $\text{Sp}(2n,\text{R})$ determined by (14). Van Hove's representation $R^v \downarrow L$ [defined in Ref. 2, Eq. (12)] can now be replaced by the unitarily equivalent representation $U_N^v W_{\text{Sp}(2n,\text{R})} \otimes \overline{W_{\text{Sp}(2n,\text{R})}}$; all that is then required from van Hove's paper is the action of $\text{Sp}(2n, \text{R})$ on N in the definition of $N Ⓢ \text{Sp}(2n,\text{R})$. The main results obtained can then be summarized as follows.

Main results

I. A group G intrinsically related to the anisotropic harmonic oscillator has been constructed: G is essentially a group generated by the position and momentum observables, the identity operator, and the Hamiltonian of the system. G can be regarded as a subgroup of the group $L = N Ⓢ \text{Sp}(2n,\text{R})$ (where N is the Weyl group). Let $W_{\text{Sp}(2n,\text{R})}$ be the projective representation of $\text{Sp}(2n,\text{R})$ which intertwines the irreducible representations U_N^v and $S U_N^v$ of N [where $S \in \text{Sp}(2n,\text{R})$]. Then the degeneracies of the energy levels of the anisotropic harmonic oscillator occur in the following way (whether the frequencies are rationally related or not).

$$(U_N^v W_{\text{Sp}(2n,\text{R})} \otimes \overline{W_{\text{Sp}(2n,\text{R})}} \downarrow G = \bigoplus_s d_{\omega,s} U_G^{v,-(\text{sgn} v)s} ,$$

where $d_{\omega,s}$ is the degeneracy of the energy level $E_{\omega,s}$ of the n-dimensional anisotropic harmonic oscillator with frequencies $(\omega_1, \omega_2, \ldots, \omega_n)$, $U_G^{v,-(\text{sgn} v)s}$ is an irreducible representation of G, and the summation may be regarded as over all distinct energy levels $E_{\omega,s}$ of the system.

II. Every quantum mechanical irreducible (ordinary) representation of L is unitarily equivalent to one of the form $U_N^v W_{\text{Sp}(2n,\text{R})} \otimes \eta_{\text{Sp}(2n,\text{R})}$, where $\eta_{\text{Sp}(2n,\text{R})}$ is an irreducible projective representation of $\text{Sp}(2n,\text{R})$, with multiplier inverse to that of $W_{\text{Sp}(2n,\text{R})}$.

III. The (ordinary) representation $U_N^v W_{\text{Sp}(2n,\text{R})} \otimes \overline{W_{\text{Sp}(2n,\text{R})}}$ of L (see I) splits into two irreducible (ordinary) representations of L.

IV. Denote $W_{\text{Sp}(2n,\text{R})} \downarrow \text{SU}(n)$ by $W_{\text{SU}(n)}$. Then $U_N^v W_{\text{SU}(n)}$ is an irreducible (ordinary) representation of $N Ⓢ \text{SU}(n)$; when $U_N^v W_{\text{SU}(n)}$ is restricted to $\text{SU}(n)$, which is a degeneracy group for the n-dimensional isotropic harmonic oscillator, its decomposition is

$$(U_N^v W_{\text{SU}(n)}) \downarrow \text{SU}(n) = \bigoplus_{a=0}^{\infty} W_a ,$$

where W_a is an irreducible (ordinary) representation of $\text{SU}(n)$, of dimension equal to the degeneracy of the $(a+1)$th energy level of the n-dimensional isotropic harmonic oscillator. Hence, $N Ⓢ \text{SU}(n)$ is a noninvariance group for the n-dimensional isotropic harmonic oscillator.

This alternative expression of the results illustrates more clearly the structure of the representation of L which yields the degeneracies of the anisotropic harmonic oscillator (see I). The Weyl group N, together with its irreducible representation U_N^v, is also seen to be important. Lastly, the alternative expression emphasizes the significant role played by the projective representation $W_{\text{Sp}(2n,\text{R})}$.

Note: The original parametrization of G was chosen in such a way that it would immediately fit in with that used by van Hove.[10] Now that the connection with van Hove's representation R^v is no longer required, the group L can be parametrized in other ways. One alternative parametrization which involves $\text{Sp}(2n,\text{R})$ in a more intrinsic way is obtained as follows.

The Weyl form of the commutation relations $[\hat{Q}_j, \hat{P}_k] = i\delta_{jk} I$ is

$$U(\alpha)V(\beta) = \exp i\alpha.\beta V(\beta)U(\alpha) , \tag{29}$$

where $\alpha \to U(\alpha)$, $\beta \to V(\beta)$ are unitary representations of the additive groups of momentum space ($\simeq \text{R}^n$) and configuration space ($\simeq \text{R}^n$) respectively.[11]

Putting $(\sigma, \beta, \alpha) = \exp i\sigma V(\beta)U(\alpha)$, with $\sigma \in \text{R}$, yields the group law of the Weyl group N in the form used earlier [Ref. 1, Eq. (12)]

$$(\sigma', \beta', \alpha')(\sigma, \beta, \alpha) = (\sigma' + \sigma + \alpha'.\beta, \beta' + \beta, \alpha' + \alpha) .$$

Alternatively, let M = configuration space $\oplus$ momentum space ($\simeq \text{R}^{2n}$). Define the nondegenerate skew-symmetric bilinear form $[\ ,\ ]$: $M \times M \to \text{R}$ by $[\gamma_1, \gamma_2] = \alpha_1.\beta_2 - \alpha_2.\beta_1$, where $\gamma_i = (\beta_i, \alpha_i) \in M$. Now put $\langle \sigma, \gamma \rangle = \exp i\sigma Z(\gamma)$, where $\sigma \in \text{R}$, and $Z(\gamma) = \exp(i/2)\alpha.\beta V(\beta)U(\alpha)$. Using (29), the group law of N then takes the form

$$\langle \sigma', \gamma' \rangle \langle \sigma, \gamma \rangle = \langle \sigma' + \sigma + \tfrac{1}{2}[\gamma', \gamma], \gamma' + \gamma \rangle .$$

It is easily verified that the group $\text{Sp}(2n,\text{R})$ can be characterized as the set of all $S \in \text{GL}(2n,\text{R})$ which satisfy $[S\gamma_1, S\gamma_2] = [\gamma_1, \gamma_2]$, for any $\gamma_i \in \text{R}^{2n}$.

The group law of $L = N Ⓢ \text{Sp}(2n,\text{R})$ can then be taken as

$$\langle \sigma', \gamma', S' \rangle \langle \sigma, \gamma, S \rangle = \langle \sigma' + \sigma + \tfrac{1}{2}[\gamma', S'\gamma], \gamma' + S\gamma, S'S \rangle .$$

The connection between the two parametrizations of L is $\langle \sigma, \beta, \alpha, S \rangle = (\sigma + \frac{1}{2}\alpha.\beta, \beta, \alpha, S)$. This change in parametrization means that the relation $(\sigma, \beta, \alpha, I) = (\sigma, 0, 0, I)(0, \beta, 0, I)(0, 0, \alpha, I)$ is replaced by $\langle \sigma, \beta, \alpha, I \rangle = \langle \sigma + \frac{1}{2}\alpha.\beta, 0, 0, I \rangle \langle 0, \beta, 0, I \rangle \langle 0, 0, \alpha, I \rangle$, with corresponding minor modifications in several places. The characterization of $\text{Sp}(2n,\text{R})$ as a group leaving $[\ ,\ ]$ invariant results in easier calculations in one or two places, but does not lead to any overall simplifications.

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