NONLINEAR OPTIMAL CONTROL: A CONTROL LYAPUNOV FUNCTION AND RECEDING HORIZON PERSPECTIVE

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ABSTRACT

Two well known approaches to nonlinear control involve the use of control Lyapunov functions (CLFs) and receding horizon control (RHC), also known as model predictive control (MPC). The on-line Euler-Lagrange computation of receding horizon control is naturally viewed in terms of optimal control, whereas researchers in CLF methods have emphasized such notions as inverse optimality. We focus on a CLF variation of Sontag's formula, which also results from a special choice of parameters in the so-called pointwise min-norm formulation. Viewed this way, CLF methods have direct connections with the Hamilton-Jacobi-Bellman formulation of optimal control. A single example is used to illustrate the various limitations of each approach. Finally, we contrast the CLF and receding horizon points of view, arguing that their strengths are complementary and suggestive of new ideas and opportunities for control design. The presentation is tutorial, emphasizing concepts and connections over details and technicalities.

KeyWords: Nonlinear optimal control, control Lyapunov function, receding horizon control, predictive control.

I. INTRODUCTION

The optimal control of nonlinear systems is one of the most challenging and difficult subjects in control theory. It is well known that the nonlinear optimal control problem can be reduced to the Hamilton-Jacobi-Bellman partial differential equation [3], but due to difficulties in its solution, this is not a practical approach. Instead, the search for nonlinear control schemes has generally been approached on less ambitious grounds than requiring the exact solution to the Hamilton-Jacobi-Bellman partial differential equation.

In fact, even the problem of stabilizing a nonlinear system remains a challenging task. Lyapunov theory, the most successful and widely used tool, is a century old. Despite this, there still do not exist systematic methods for obtaining Lyapunov functions for general nonlinear systems. Nevertheless, the ideas put forth by Lyapunov nearly a century ago continue to be used and exploited extensively in the modern theory of control for nonlinear systems. One notably successful use of the Lyapunov methodology is its generalization to control systems, known as a control Lyapunov function (CLF) [29,30,7,10,16,9,8]. The knowledge of such a function is sufficient to design stabilizing control schemes. Once again, there do not exist systematic techniques for finding CLFs for general nonlinear systems, but this approach has been applied successfully to many classes of systems for which CLFs can be found (feedback linearizable, strict feedback and feed-forward systems, etc. [16,9,7]).

In contrast to the emphasis on guaranteed stability that is the primary goal of CLFs, another class of nonlinear control schemes that go by the names receding horizon, moving horizon, or model predictive control place importance on optimal performance [18,17,20,12,15]. These techniques apply a so-called receding horizon implementation in an attempt to approximately solve the optimal control problem through on-line computation. For systems under which on-line computation is feasible, receding horizon control (RHC) has proven quite successful [28,27]. But both stability concerns and practical implementation issues remain a major research focus [20,21].

In this paper we consider both techniques, CLFs and receding horizon control, in the context of the nonlinear optimal control problem. Important connections between these techniques and approaches to the nonlinear optimal control problem are extracted and highlighted.

Thus we hope that this paper will both introduce new insights into the methods and provide a tutorial on their strengths and limitations. In addition, the strengths of these approaches are found to be complementary, and offer new perspectives and opportunities for control design.

The organization is as follows. Section 2 presents a brief review of the standard approaches to the nonlinear optimal control problem, emphasizing the difference between the global aspects of the Hamilton-Jacobi-Bellman solution and the local properties of Euler-Lagrange techniques. Control Lyapunov functions are introduced in Section 3, where connections between a variation on Sontag's formula, Hamilton-Jacobi-Bellman equations, and pointwise min-norm formulations are explored. Section 4 introduces receding horizon control (RHC), contrasting the philosophy behind this approach with that which underlies the control Lyapunov function approach. Section 5 explores the opportunities created by viewing CLFs and receding horizon control in this framework, while Section 6 summarizes.

II. NONLINEAR OPTIMAL CONTROL

The reals will be denoted by $I\!R$, with $I\!R_+$ indicating the set of nonnegative real numbers. For notational convenience, the gradient of a function V with respect to x will be denoted by V_x (i.e. $V_x = \dfrac{\partial V}{\partial x} = \left[\dfrac{\partial V}{\partial x_1}, \ldots, \dfrac{\partial V}{\partial x_n} \right]$).

We consider the nonlinear dynamics,

$$\dot{x} = f(x) + g(x)u \, , \ f(0) = 0 \tag{1}$$

with $x \in I\!R^n$ denoting the state, $u \in I\!R^m$ the control and $f(x): I\!R^n \to I\!R^n$ and $g(x): I\!R^n \to I\!R^{n \times m}$ continuously differentiable in all arguments. Our emphasis will be on the infinite horizon nonlinear optimal control problem stated below:
(Optimal Control Problem)

$$\inf_{u(\cdot)} \int_0^{\infty} (q(x) + u^T u)dt$$
$$\text{s.t. } \dot{x} = f(x) + g(x)u \tag{2}$$

for $q(x)$ continuously differentiable, positive semi-definite and $[f,q]$ zero-state detectable, with the desired solution being a *state-feedback* control law.

2.1 Hamilton-Jacobi-Bellman equations

A standard dynamic programming argument reduces the above optimal control problem to the Hamilton-Jacobi-Bellman partial differential equation (HJB) [3],

$$V_x^* f - \frac{1}{4}(V_x^* g g^T V_x^{*T}) + q(x) = 0 \tag{3}$$

where V^* is commonly referred to as the *value function* and can be though of as the minimum cost to go from the current state $x(t)$, i.e.,

$$V^*(x(t)) = \inf_{u(\cdot)} \int_t^{\infty} (q(x(\tau)) + u^T(\tau)u(\tau))d\tau \, . \tag{4}$$

If there exists a continuously differentiable, positive definite solution to the HJB equation (3), then the optimal control action is given by,

$$u^* = -\frac{1}{2}g^T V_x^{*T} \, . \tag{5}$$

At this point, we note the following properties of the HJB approach to the solution of the optimal control problem. The HJB partial differential equation solves the optimal control problem for every initial condition all at once. In this sense it is a *global* approach, and provides a *closed-loop* (state-feedback) formula for the optimal control action (eqn. 5).

Unfortunately, the HJB partial differential equation (3) is extremely difficult to solve and in general precludes any hope of an exact global solution to the nonlinear optimal control problem.

2.2 Euler-Lagrange equations

An alternate approach to optimal control takes a *local*, *open-loop* viewpoint and is more suited to trajectory optimization. Consider the following problem:

$$\inf_{u(\cdot)} \int_0^T (q(x) + u^T u)dt + \varphi(x(T))$$
$$\text{s.t. } \dot{x} = f(x) + g(x)u \tag{6}$$
$$x(0) = x_0 \, .$$

In contrast to the optimal control problem (2), in this formulation an initial condition is explicitly specified and it does not require that the solution be in the form of a state-feedback control law. Moreover, the use of a finite horizon means that the solution to this problem will coincide with the solution to the infinite horizon problem only when the *terminal weight* $\varphi(\cdot)$ is chosen as the value function V^*.

A *calculus of variations* approach results in the following necessary conditions, known as the Euler-Lagrange equations [3]:

$$\dot{x} = H_{\lambda}(x, u^*, \lambda)$$
$$\dot{\lambda} = -H_x(x, u^*, \lambda)$$
$$u^* = \arg\min_u H(x, u, \lambda)$$

where $H(x, u, \lambda) = q(x) + u^T u + \lambda^T(f(x) + g(x)u)$ is referred to as the *Hamiltonian*. These equations are solved subject to the initial and final condition:

$$x(0) = x_0 , \quad \lambda(T) = \varphi_x^T(x(T)) .$$

This is referred to as a two-point boundary value problem, which, although is a difficult problem in the context of ordinary differential equations, is magnitudes of order easier than solving the HJB equation. Furthermore, while the classical solution is through these Euler-Lagrange equations, many modern numerical techniques attack the problem (6) directly.

We emphasize once again that this approach solves the optimal control problem from a single initial condition and generates an *open-loop* control assignment. It is often used for trajectory optimization and in this sense is a local approach requiring a new solution at every new initial condition. By itself, the Euler-Lagrange approach solves a much more restricted problem than we are interested in, since we ultimately desire a state-feedback controller which is valid for the entire state space. Yet, as will be clarified later, the receding horizon methodology exploits the simplicity of this formulation to provide a state-feedback controller. For an extensive explanation of the connections and differences between the HJB and Euler-Lagrange approaches see ([22], Chpt 1).

While an exact solution to the nonlinear optimal control problem is currently beyond the means of modern control theory, the problem has motivated a number of alternate approaches to the control of nonlinear systems.

III. CONTROL LYAPUNOV FUNCTIONS (CLFs)

A control Lyapunov function (CLF) is a continuously differentiable, proper, positive definite function $V: I\!R^n \to I\!R_+$ such that:

$$\inf_u [V_x(x)f(x) + V_x(x)g(x)u] < 0 \tag{7}$$

for all $x \neq 0$ [1,29,30]. This definition is motivated by the following consideration. Assume we are supplied with a positive definite function V and asked whether this function can be used as a Lyapunov function for a system we would like to stabilize. To determine if this is possible we would calculate the time derivative of this function along trajectories of the system, i.e.,

$$\dot{V}(x) = V_x(f(x) + g(x)u) .$$

If it is possible to make the derivative negative at every point by an appropriate choice of u, then we have achieved our goal and can stabilize the system with V a Lyapunov function under those control actions. This is exactly the condition given in (7).

Given a general system of the form (1), it may be difficult to find a CLF or even to determine whether one exists. Fortunately, there are significant classes of systems for which the systematic construction of a CLF is possible (feedback linearizable, strict feedback and feed-forward systems, etc.). This has been explored extensively in the literature ([16,9,7] and references therein). We will not concern ourselves with this question. Instead, we will pay particular attention to techniques for designing a stabilizing controller once a CLF has been found, and their relationship to the nonlinear optimal control problem.

3.1 Sontag's formula

It can be shown that the existence of a CLF for the system (1) is equivalent to the existence of a globally asymptotically stabilizing control law $u = k(x)$ which is continuous everywhere except possibly at $x = 0$, [1]. Moreover, one can calculate such a control law k explicitly from f, g and V. Perhaps the most important formula for producing a stabilizing controller based on the existence of a CLF was introduced in [30] and has come to be known as Sontag's formula. We will consider a slight variation of Sontag's formula (which we will continue to refer to as Sontag's formula with slight abuse), originally introduced in [10]:

$$u_{\sigma_s} =$$

$$\begin{cases} -\left[\dfrac{V_x f + \sqrt{(V_x f)^2 + q(x)(V_x g g^T V_x^T)}}{V_x g g^T V_x^T} \right] g^T V_x^T & V_x g \neq 0 \\[4ex] 0 & V_x g = 0 \end{cases} \tag{8}$$

(The use of the notation u_{σ_s} will become clear later.) While this formula enjoys similar continuity properties to those for which Sontag's formula is known, (i.e. for $q(x)$ positive definite it is continuous everywhere except possibly at $x = 0$, [30]), for us its importance lies in its connection with optimal control. At first glance, one might note that the cost parameter associated with the state, $q(x)$ (refer to eqn (2)), appears explicitly in (8). In fact, the connection runs much deeper and our version of Sontag's formula has a strong interpretation in the context of Hamilton-Jacobi-Bellman equations.

Sontag's formula, in essence, uses the directional information supplied by a CLF, V, and scales it properly to solve the HJB equation. In particular, if V has level curves that agree in shape with those of the value function, then Sontag's formula produces the optimal controller [10]. To see this, assume that V is a CLF for the system (1) and, for the sake of motivation, that V possesses the same shape level curves as those of the value function V^*. Even though in general V would not be the same as V^*, this does imply a relationship between their gradients. We may assert that there exists a scalar function $\lambda(x)$ such that $V_x^* = \lambda(x)V_x$ for

every x (i.e. the gradients are co-linear at every point). In this case, the optimal controller (5) can also be written in terms of the CLF V,

$$u^* = -\frac{1}{2}g^T V_x^{*T} = -\frac{\lambda(x)}{2}g^T V_x^T . \qquad (9)$$

Additionally, $\lambda(x)$ can be determined by substituting $V_x^* = \lambda(x)V_x$ into the HJB partial differential equation, (3):

$$\lambda(x)V_x f - \frac{\lambda(x)^2}{4}(V_x gg^T V_x^T) + q(x) = 0 . \qquad (10)$$

Solving this pointwise as a quadratic equation in $\lambda(x)$, and taking only the positive square root gives,

$$\lambda(x) = 2\left(\frac{V_x f + \sqrt{[V_x f]^2 + q(x)[V_x gg^T V_x^T]}}{V_x gg^T V_x^T} \right). \qquad (11)$$

Substituting this value into the controller u^* given in (9) yields,

$$u^* =$$

$$\begin{cases} -\left[\dfrac{V_x f + \sqrt{(V_x f)^2 + q(x)(V_x gg^T V_x^T)}}{V_x gg^T V_x^T} \right]g^T V_x^T & V_x g \neq 0 \\\\ \qquad\qquad 0 & V_x g = 0 \end{cases}$$

which is exactly Sontag's formula, $u_{\sigma s}$ (8). In this case, Sontag's formula will result in the optimal controller.

For an arbitrary CLF V, we may still follow the above procedure which results in Sontag's formula. Hence Sontag's formula may be though of as using the direction given by the CLF (i.e. V_x), which, by the fact that it is a CLF will result in stability, but pointwise scaling it by $\lambda(x)$ so that it will satisfy the HJB equation as in (3). Then $\lambda(x)V_x$ is used in place of V_x^* in the formula for the optimal controller u^*, (5). Hence, we see that there is a strong connection between Sontag's formula and the HJB equation. In fact, Sontag's formula just uses the CLF V as a substitute for the value function in the HJB approach to optimal control.

In the next section, we introduce the notion of pointwise min-norm controllers ([8,7,9]), and demonstrate that Sontag's formula is the solution to a specific pointwise min-norm problem. It is from this framework that connections with optimal control have generally been emphasized.

3.2 Pointwise min-norm controllers

Given a CLF, $V > 0$, by definition there will exist a

control action u such that $\dot{V} = V_x(f + gu) < 0$ for every $x \neq 0$. In general there are many such u that will satisfy $V_x(f + gu) < 0$. One method of determining a specific u is to pose the following optimization problem [8,7,9]:
(Pointwise Min-Norm)

$$\min_u \quad u^T u \qquad (12)$$

$$\text{s.t.} \quad V_x(f + gu) \leq -\sigma(x) \qquad (13)$$

where $\sigma(x)$ is some positive definite function (satisfying $V_x f \leq -\sigma(x)$ whenever $V_x g = 0$) and the optimization is solved pointwise (i.e. for each x). This formula pointwise minimizes the control energy used while requiring that V be a Lyapunov function for the closed-loop system and decrease at a rate of $\sigma(x)$ at every point. The resulting controller can be solved for off-line and in closed form (see [7] for details).

In [8] it was shown that every CLF, V, is the value function for some meaningful cost function. In other words, it solves the HJB equation associated with a meaningful cost. This property is commonly referred to as being "inverse optimal'. Note that a CLF, V, does not uniquely determine a control law because it may be the value function for many different cost functions, each of which may have a different optimal control. What is important is that under proper technical conditions the pointwise min-norm formulation always produces one of these inverse optimal control laws [8].

It turns out that Sontag's formula results from a pointwise min-norm problem by using a special choice of $\sigma(x)$. To derive this $\sigma(x)$, assume that Sontag's formula is the result of the pointwise min-norm optimization. It should be clear that for $V_x g \neq 0$, the constraint will be active, since the norm of u will be reduced as much as possible. Hence, from (13) we can write that:

$$-\sigma_s = V_x(f + gu_{\sigma_s})$$

$$= V_x f + V_x g\left(-\frac{V_x f + \sqrt{(V_x f)^2 + q(x)(V_x gg^T V_x^T)}}{V_x gg^T V_x^T} \right)g^T V_x^T$$

$$= -\sqrt{(V_x f)^2 + q(x)(V_x gg^T V_x^T)} .$$

Hence, choosing σ to be

$$\sigma_s = \sqrt{(V_x f)^2 + q(x)(V_x gg^T V_x^T)} \qquad (14)$$

in the pointwise min-norm scheme (12) results in Sontag's formula. This connection, first derived in [10], provides us with an important alternative method for viewing Sontag's formula. It is the solution to the above pointwise min-norm problem with parameter σ_s.

We have seen that these CLF based techniques share much in common with the HJB approach to nonlinear

optimal control. Nevertheless, the strong reliance on a CLF, while providing stability, can lead to suboptimal performance when applied naively, as demonstrated in the following example.

3.3 Example

Consider the following 2d nonlinear oscillator:

$$
\begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = -x_1(\frac{\pi}{2} + \arctan(5x_1)) - \dfrac{5x_1^2}{2(1 + 25x_1^2)} + 4x_2 + 3u
\end{cases}
$$

with performance index:

$$
\int_0^\infty (x_2^2 + u^2)dt .
$$

This example was created using the so-called converse HJB method [6] so that the optimal solution is known. For this problem, the value function is given by:

$$
V^* = x_1^2(\frac{\pi}{2} + \arctan(5x_1)) + x_2^2
$$

which results in the optimal control action:

$$
u^* = -3x_2.
$$

A simple technique for obtaining a CLF for this system is to exploit the fact that it is feedback linearizable. In the feedback linearized coordinates, a quadratic may be chosen as a CLF. In order to ensure that this CLF will at least produce a locally optimal controller, we have chosen a quadratic CLF that agrees with the quadratic portion of the true value function (this can be done without knowledge of the true value function by performing Jacobian linearization and designing an LQR optimal controller for the linearized system). This results in the following[1]:

$$
V = \frac{\pi}{2}x_1^2 + x_2^2 .
$$

We will compare Sontag's formula using this CLF to the performance of the optimal controller. Figure 1 is a plot of the level curves of the true value function, V^*, versus those of the CLF, V. Clearly, these curves are far from the

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[1]This function is actually not a CLF in the strict sense in that there exist points where V may only be made equal to zero and not stictly lss than zero. This is sometimes referred to as a weak CLF. Nevertheless, we will use this CLF since it is the only quadratic function that locally agrees with our value function (which itself is not even a strict CLF for this system).
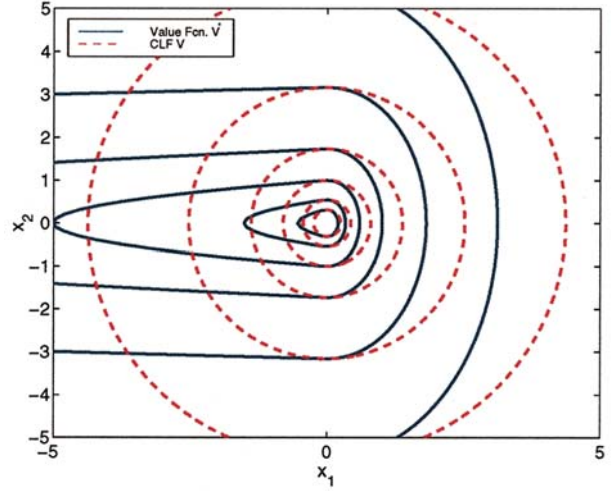

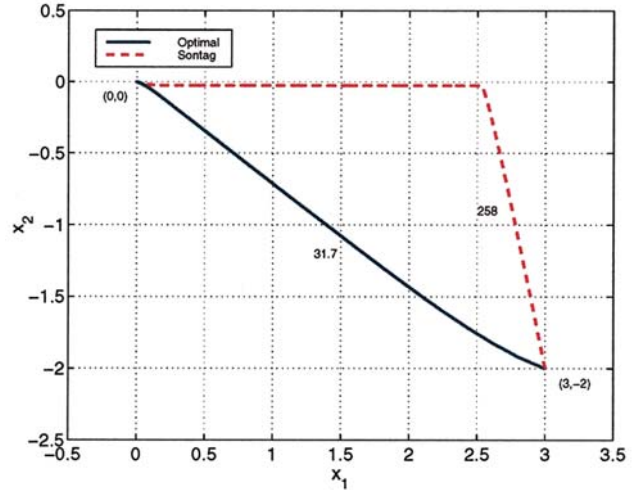Fig. 1. Contours of the Value function (solid) and CLF (dashed).

Fig. 2. Phase Portrait: Optimal (solid), Sontag's (dashed).

level curves of a quadratic function. Since Sontag's formula uses the directions provided by the CLF, one might suspect that Sontag's formula with the quadratic CLF given above will perform poorly on this system.

This is indeed the case, as shown in Fig. 2, where Sontag's formula (dotted) accumulates a cost of over 250 from the initial condition [3,–2]. One might note that we have naively utilized the CLF methodology without though as to how to better craft a more suitable and sensible CLF for the this problem. In this simple example it is not too difficult to iterate on the selection of parameters and find a controller that performs admirably. Nevertheless, we have constructed this example to illustrate the specific point that these controllers rely on the directions offered by the CLF, which often deviate from those of the value function.

The costs achieved by Sontag's formula and the optimal controller from the initial condition [3,–2] are summarized in Table 1. This example shows that CLF based designs can be particularly sensitive to differences

Table. 1. Cost of Sontag's formula vs. the optimal controller from initial condition [3, –2].

Controller	Cost
Sontag	258
Optimal	31.7

between the CLF and the value function, even for a technique such as Sontag's formula that directly incorporates information from the optimal control problem into the controller design process.

IV. RECEDING HORIZON CONTROL (RHC)

We now switch from the HJB approach embodied in CLF methods to an Euler-Lagrange philosophy introduced in methods known as receding horizon, moving horizon, or model predictive control (cf. [18,17,12,20,15,12]). these techniques are based upon using on-line computation to repeatedly solve optimal control problems emanating from the current measured state. To be more specific, the current control at state x and time t is obtained by determining on-line the optimal control solution u over the interval $[t, t + T]$ respecting the following objective:
(Receding Horizon Control)

$$\inf_{u(\cdot)} \int_t^{t+T} (q(x(\tau)) + u^T(\tau)u(\tau))d\tau + \varphi(x(t+T))$$

$$\text{s.t.} \quad \dot{x} = f(x) + g(x)u$$

with the current measured state as the initial condition, and applying the optimizing solution $\hat{u}(\cdot)$ until a new state update is received. Repeating this calculation for each new state measurement yields a state-feedback control law. This type of implementation is commonly referred to as a receding or moving horizon and is the foundation of model predictive control [12]. As is evident from this sort of control scheme, obtaining a reduced value of the performance index is of utmost importance.

The philosophy behind receding horizon control can be summarized as follows. It exploits the simplicity of the Euler-Lagrange approach by repeatedly solving trajectory optimizations emanating from the current state. The solution to each optimization provides an approximation to the value function at the current state, as well as an accompanying open-loop control trajectory. The receding horizon approach converts these open-loop trajectories into the desired state-feedback. The key to this methodology is that it only requires that the optimal control problem be solved for the states encountered along the current trajectory, in this way avoiding the global nature of the HJB approach and its associated computational intractability.

4.1 Computational issues

Despite the computational advantages of an Euler-Lagrange approach over those of the HJB viewpoint, the on-line implementation of receding horizon control is still computationally demanding. In fact, the practical implementation of receding horizon control is often hindered by the computational burden of the on-line optimization which theoretically must be solved continuously. In reality, the optimization is solved at discrete sampling times and the corresponding control moves are applied until they can be updated at the next sampling instance. The choice of both the sampling time and horizon are largely influenced by the ability to solve the required optimization within the allowed time interval. These considerations often limit the application of receding horizon control to systems with sufficiently slow dynamics to be able to accommodate such on-line inter-sample computation.

For linear systems under quadratic objective functions, the on-line optimization is reduced to a tractable quadratic program, even in the presence of linear input and output constraints. This ability to incorporate constraints was the initial attraction of receding horizon implementations. For nonlinear systems the optimization is in general non-convex and hence has no efficient solution. Methods for fast solutions or approximations to solutions of these Euler-Lagrange type optimizations have occupied an entire research area by themselves and will not be expounded on here. (See [23,22,11]).

While these numerical and practically oriented issues are compelling, there are fundamental issues related to the theoretical foundations of receding horizon control that deserve equal scrutiny. The mot critical of these are well illustrated by considering the stability and performance properties of idealized receding horizon control.

4.2 Stability

While using a numerical optimization as an integral part of the control scheme allows great flexibility, especially concerning the incorporation of constraints, it complicates the analysis of stability and performance properties of receding horizon control immensely. The reason for the difficulty is quite transparent. Since the control action is determined through a numerical on-line optimization at every sampling point, there is often no convenient closed form expression for the controller nor for the closed-loop system.

The lack of a complete theory for a rigorous analysis of receding horizon stability properties in nonlinear systems often leads to the use of intuition in the design process. Unfortunately, this intuition can be misleading. Consider, for example, the statement that horizon length provides a tradeoff between the issues of computation and of stability and performance. A longer horizon, while being computationally more intensive for the on-line optimization,

will provide a better approximation to the infinite horizon problem and hence the controller will inherit the stability guarantees and performance properties enjoyed by the infinite horizon solution. While this intuition is correct in the limit as the horizon tends to infinity [24], for horizon lengths applied in practice the relationship between horizon and stability is much more subtle and often contradicts such seemingly reasonable statements. This is best illustrated by the example used previously in Section 3.3. Recall that the system dynamics were given by:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1(\frac{\pi}{2} + \arctan{(5x_1)}) - \frac{5x_1^2}{2(1 + 25x_1^2)} + 4x_2 + 3u \end{cases}$$

with performance index:

$$\int_0^\infty (x_2^2 + u^2)dt \ .$$

For simplicity we will consider receding horizon controllers with no terminal weight (i.e. $\varphi(x) = 0$) and use a sampling interval of 0.1. By investigating the relationship between horizon length and stability through simulations from the initial condition [3,–2], a puzzling phenomena is uncovered. Beginning from the shortest horizon simulated, $T = 0.2$, the closed-loop system is found to be unstable (see Fig. 3). As the horizon is increased to $T = 0.3$, the results change dramatically and near optimal performance is achieved by the receding horizon controller. At this point, one might be tempted to assume that a sufficient horizon for stability has been reached and longer horizons would only improve the performance. In actuality the opposite happens and as the horizon is in-
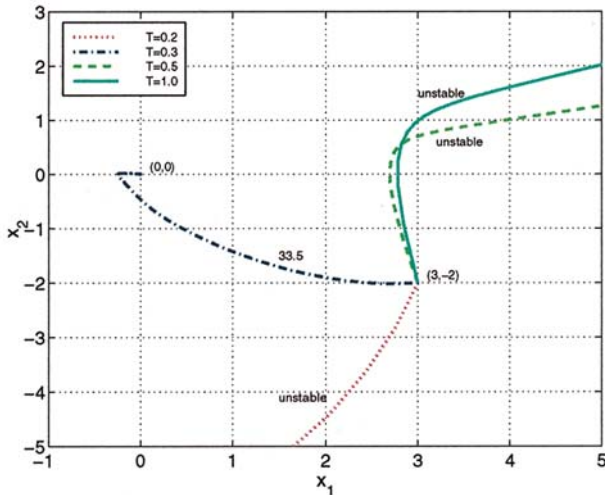
Table. 2. Comparison of controller performance from initial condition [3, –2].

Controller	Performance
$T = 0.2$: (dotted)	unstable
$T = 0.3$: (dash-dot)	33.5
$T = 0.5$: (dashed)	unstable
$T = 1.0$: (solid)	unstable

creased further the performance deteriorates and has returned to instability by a horizon of $T = 0.5$. This instability remains present even past a horizon of $T = 1.0$. The simulation results are summarized in Table 2 and Fig. 3.

It is important to recognize that the odd behavior we have encountered is not a nonlinear effect, nor the result of a cleverly chosen initial condition or sampling interval, but rather inherent to the receding horizon approach. In fact, the same phenomena takes place even for the linearized system:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{\pi}{2} & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 3 \end{bmatrix} u \ . \tag{15}$$

In this case, a more detailed analysis of the closed-loop system is possible due to the fact that the controller and closed-loop dynamics are linear and can be computed in closed form. Figure 4 shows the magnitude of the maximum eigenvalue of the discretized closed-loop system[2]
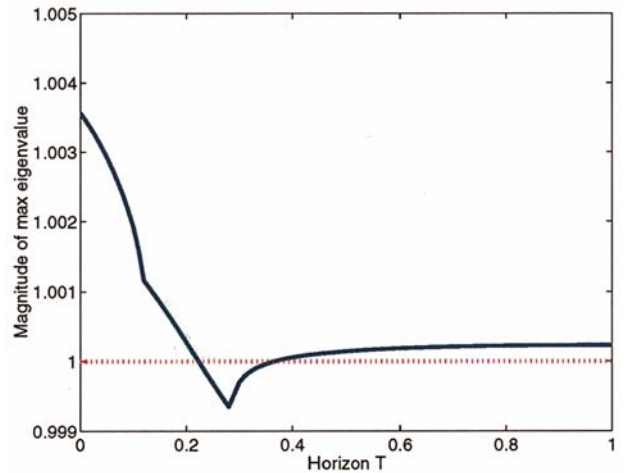

Fig. 4. Maximum eigenvalue versus horizon length for discretized linear system.

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[2] The receding horizon controller was computed by discretizing the continuous time system using a first order hold and time step of 0.001, and solving the Riccati difference equation of appropriate length, hence the eigenvalues correspond to a discrete-time system with stability occurring when the maximum eigenvalue has modulus less than 1.

Fig. 3. RHC for various horizon lengths,.

versus the horizon length of the receding horizon controller. This plot shows that stability is only achieved for a small range of horizons that include $T = 0.3$ and longer horizons lead to instability. It is not until a horizon of $T = 3.79$ that the controller becomes stabilizing once again.

These issues are not new in the receding horizon control community and related phenomena have been noted before by in the context of Riccati difference equations [2]. This delicate relationship between horizon length and stability has been addressed by various means. In fact, the majority of literature focuses on producing stabilizing formulations of receding horizon control. The first stabilizing approach was to employ an end constraint that required the final predicted state corresponding to the on-line optimization be identically zero. This guaranteed stability in both linear and nonlinear formulations at the expense of a restrictive and computationally burdensome equality constraint [18,19]. Other formulations, including contraction constraints [5], infinite horizon [26,4] and dual mode [21] have eased the dominating effect of an end constraint. Numerous other approaches have been developed, each deviating slightly from the formulation of receding horizon control that we have presented in order to provided stability guarantees.

Our focus is on the connection between receding horizon control and the classical Euler-Lagrange approach to optimal control. In essence, receding horizon techniques produce local approximations to the value function through Euler-Lagrange type on-line optimizations, and use these top produce a control law. It is also this lack of global information that can lead receding horizon control astray in terms of stability, and must be dealt with when stabilizing formulations are desired. On the other hand, the locality of the optimizations allow receding horizon control to overcome the computational intractability associated with the HJB approach.

V. COMBINING OFF-LINE ANALYSIS WITH ON-LINE COMPUTATION: CLFs AND RHC

The previous sections illustrated the underlying concepts and limitations in the control Lyapunov function and receding horizon methodologies. Table 5 summarizes some of the key properties of each approach.

When viewed together, these properties are seen to be complementary, suggesting that the control Lyapunov

Table. 5

CLF	RHC
Relies on global information	Relies on local information
Stability oriented	Performance oriented
Requires off-line analysis	Requires on-line computation
Connections to HJB formulation	Connections to E-L formulation

function and receding horizon methodologies are natural partners. Off-line analysis produces a control Lyapunov function, representing the best approximation to the value function, while receding horizon style computation can then be used on-line, optimizing over trajectories emanating from the current state, improving the solution by utilizing as must computation as is available. A proper combination of the properties of CLFs and receding horizon control should have the potential to overcome the limitations imposed by each technique individually.

In this final section we outline two approaches that blend the information from a CLF with on-line receding horizon style computation. The first extends the CLF based optimization in pointwise min-norm controllers to a receding horizon, while the second enhances the standard receding horizon scheme by using a CLF as a terminal weight. While this section only offers an initial glimpse at the opportunities available by viewing CLFs and receding horizon control as partners, we hope that it is sufficient to spur others to investigate these connections as well.

5.1 Receding horizon extensions of pointwise min-norm controllers

Pointwise min-norm controllers are formulated as an on-line optimization (see (12)), but in practice this optimization is solved off-line and in closed form. Hence, while benefiting from the stability properties of the underlying CLF, pointwise min-norm controllers fail to take advantage of on-line computing capabilities. Yet, by viewing the pointwise min-norm optimizations (12) as a "zero horizon" receding horizon optimization, it is actually possible to extend them to a receding horizon scheme. One implementation of this idea is as follows.

Let V be a CLF and let u_σ and x_σ denote the control and state trajectories obtained by solving the pointwise min-norm problem with parameter $\sigma(x)$ (cf. (12)-(13)). Consider the following receding horizon objective:

$$\inf_{u(\cdot)} \int_t^{t+T} (q(x)) + u^T u) d\tau$$

$$s.t. \quad \dot{x} = f(x) + g(x)u \tag{16}$$

$$\frac{\partial V}{\partial x}[f + gu(t)] \leq -\epsilon\sigma(x(t)) \tag{17}$$

$$V(x(t + T)) \leq V(x_\sigma(t + T)) \tag{18}$$

with $0 < \epsilon \leq 1$. This is a standard receding horizon scheme with two CLF based constraints. The first, (17), applies to all implemented control actions at the beginning of the horizon, and is a direct stability constraint similar to (13) in the pointwise min-norm problem, although relaxed by ϵ. On the other hand, the constraint (18) applies at the end of the horizon, and uses the pointwise min-norm solution to require that all allowable trajectories reach a final state

$x(t + T)$ that lies within the level curve of the CLF that passes through $x_\sigma(t + T)$.

We refer to this receding horizon scheme as an extension of pointwise min-norm controllers since in the limit as the horizon T tends to zero, this on-line optimization reduces to the pointwise min-norm optimization. Additionally, when the parameter σ_s (14) corresponding to Sontag's formula is used, the property that the controller will be optimal when the shape of the level curves from the CLF correspond to those of the value function is preserved. Hence, it even provides a corresponding extension of Sontag's formula [25].

The CLF constraints, while adding computational difficulty to the on-line optimizations, actually ease a number of implementation problems. Due to (17), which requires that all closed-loop trajectories are consistent with the CLF V, stability properties of this scheme are independent of the horizon, *T*. This allows horizon lengths to be varied on-line, extending when more on-line computation is available, and shrinking to zero (i.e. the pointwise min-norm controller) when no on-line computation is possible. Furthermore, stability does not depend on global or even local solutions to the on-line optimizations. Finally, one should note that the trajectory from the pointwise min-norm solution provides an initial feasible trajectory for each receding horizon optimization. Hence, this scheme truly builds upon the pointwise min-norm controller, exploiting all available on-line computation to improve upon this solution. More details about this scheme, including derivations of these properties, can be found in [25].

When this scheme is applied to our 2d nonlinear oscillator example using a small value for ϵ and the parameter σ_s corresponding to Sontag's formula, near optimal performance is recovered [25]. This is true even for relatively short horizons, demonstrating the beneficial mix of on-line computation coupled with the knowledge of a CLF. A more representative example can be found in [32] where a number of control techniques are compared on a ducted fan model.

5.2 CLF enhanced receding horizon schemes

The preceding scheme began from the CLF point of view, extending pointwise min-norm controllers to incorporate on-line optimization. The other point of view is also possible. Standard receding horizon formulations can be altered to benefit from the knowledge of a CLF. Below, we briefly present a scheme that begins from this premise.

A CLF is an obvious choice for a terminal weight in a receding horizon scheme. In this case, each receding horizon optimization takes the form:

$$\inf_{u(\cdot)} \int_t^{t+T} (q(x(\tau)) + u^T(\tau)u(\tau))d\tau + V(x(t+T))$$

$$\text{s.t.} \quad \dot{x} = f(x) + g(x)u \ .$$

If the CLF were actually the value function, V^*, then this optimization would be equivalent to the infinite horizon objective (2) and produce the optimal controller. In general, we can only assume the CLF is a rough approximation to the value function, in which case the receding horizon scheme based upon the above optimization cannot even be shown to be stabilizing. Fortunately, if the CLF possesses a slightly stronger property, specifically that

$$\inf_{u(\cdot)}[V_x(f(x) + g(x)u)] \leq -q(x) \ ,$$

then stability does result. CLFs that satisfy this condition arise from some standard control methodologies, including the so-called quasi-LPV approach [31], providing a starting point for the derivation of a CLF to be used in conjunction with this scheme.

Here, as before, the idea is to utilize the stability properties of CLFs, without sacrificing the performance advantages of receding horizon computation. Combining both into a single scheme allows access to their complementary properties, hopefully overcoming the limitations of each individually. Details of this approach are available in [14] and applications to a ducted fan model are given in [13].

The two CLF based receding horizon schemes presented in this section are by no means the only possible approaches. Our extension of pointwise min-norm controllers to a receding horizon scheme is not unique, and CLFs have the potential to couple with on-line computations in numerous ways. Nevertheless, we do hope to have illustrated some of the opportunities available from the combination of ideas from control Lyapunov functions and receding horizon style optimization, and encouraged others to pursue these connections and opportunities as well.

VI. SUMMARY

In this paper we explored two approaches to the nonlinear optimal control problem. Both have roots in standard optimal control techniques. Control Lyapunov functions are best interpreted in the context of Hamilton-Jacobi-Bellman equations, especially a variation of Sontag's formula that naturally arises from HJB equations and furthermore is a special case of a more general class of CLF based controllers known as pointwise min-norm controllers. Even with strong ties to the optimal control problem, CLF based approaches err on the side of stability and can result in poor performance when the CLF does not closely resemble the value function.

The second technique is that of receding horizon control which is based on the repeated on-line solution of open-loop optimal control problems which more closely relate to an Euler-Lagrange framework. The intractability of the HJB equations are overcome by solving for the optimal control only along the current trajectory through on-line computation. This approach chooses to err on the

side of performance and in its purest form lacks guaranteed stability properties. Stability and performance concerns become even more critical when short horizons must be used to accommodate the extensive on-line computation required.

While seemingly unrelated, when viewed in the context of the optimal control problem these two techniques are seen to possess very complementary properties derived from their approaches to the same underlying problem. This suggests the potential to combine approaches into hybrid schemes that blend the most useful aspects of each point of view. Finally, we outlined two such schemes, utilizing the ideas and techniques presented in this paper. In the end, it is our hope that this perspective encourages researchers to look beyond their own fields of expertise, and consider the advantages to be gained by coupling their techniques with the contibutions of others.

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